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Towards a p-Adic Model of Quantum Information Theory

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To the memory of my father,
a parent, mentor, and guiding light.

May the time and efforts invested in this work repay,
if only to a small extent, your precious teachings,
and the time and thought that, until the last of your days,
you chose to devote to your dear ones.

Grazie di tutto papà!

ABSTRACT

p-Adic quantum mechanics is a branch of modern theoretical physics which arose, at the end of the last century, from Volovich and Vladimirov's conjecture that the existence of a smallest measurable length — i.e., the so-called Planck length — entails a non-Archimedean character for the space-time at small distances. Since, up to isomorphisms, the only non-Archimedean field one can construct by completing the field of rational numbers is the field of p-adic numbers $\mathbb{Q}_p$, it is natural to take this field as the right candidate to model space-time coordinates at the Planck regime. There are actually two possible routes one can follow when trying to construct p-adic models of quantum mechanics. The first approach describes physical states of a p-adic quantum system by means of p-adic wave functions, i.e., square integrable functions from $\mathbb{Q}_p$ to the field of complex numbers $\mathbb{C}$. The second approach, instead, follows a more 'radical' p-adic point of view and postulates a non-Archimedean structure also for the carrier vector space (i.e., the Hilbert space) of physical states. In this dissertation, we aim at extending p-adic quantum mechanics towards quantum information theory, following both the two approaches to the p-adic quantization. In particular, pursuing the first route, we observe that a suitable model of a p-adic qubit is provided by considering two-dimensional irreducible projective representations of the group $\mathrm{SO}(3,\mathbb{Q}_p)$ — the special orthogonal group on $\mathbb{Q}_p$. Since this group is compact, according to the celebrated Peter-Weyl theorem, the determination of the Haar measure (or, the Haar integral, viewing such a measure as a functional) can be regarded as a preliminary step for studying its irreducible representations. Hence, as a starting point, we provide a general method for constructing the Haar measure on every p-adic Lie group; in particular, we also argue that this measure can be regarded as the measure naturally induced by the invariant volume form on the group, as it happens for standard real Lie groups. We then apply our general results to the special orthogonal groups over $\mathbb{Q}_p$ (in dimension $2, 3$ and 4), hence paving the way to the development of harmonic analysis on these groups, with potential applications in both p-adic quantum mechanics and p-adic quantum information theory. Following the second approach to the p-adic quantization, instead, we first introduce a suitable notion of a p-adic Hilbert space over a quadratic extension of $\mathbb{Q}_p$. Next, we characterize some classes of linear operators acting in it. Resorting to the algebraic definition of quantum states, we define a state for a p-adic quantum system as a suitable functional acting on the (ultrametric) Banach $*$-algebra of p-adic (bounded) observables. Eventually, to complete the statistical interpretation of the theory, we introduce the notion of a SOVM as a suitable p-adic counterpart to a POVM of the complex quantum theory.

Contents

List of Publications

This Ph.D. Dissertation is based on the following publications and preprints available online:

[1] P. Aniello, S. Mancini and V. Parisi, “Trace class operators and states in p-adic quantum mechanics”, *J. Math Phys.* **64**, 053506 (2023). (Contains results presented in Part III).

[2] P. Aniello, S. Mancini and V. Parisi, “A p-Adic Model of Quantum States and the p-Adic Qubit”, *Entropy* **25**, p. 86 (2023). (Contains results presented in Part III).

[3] P. Aniello, S. L’Innocente, S. Mancini, V. Parisi, I. Svampa and A. Winter, “Invariant measures on p-adic Lie groups: the p-adic quaternion algebra and the Haar integral on the p-adic rotation groups”, arXiv[math-ph]:2306.07110 (2023). (Contains results presented in Part II).

[4] P. Aniello, S. Mancini and V. Parisi, “Quantum mechanics on a p-adic Hilbert space: foundations and prospects”, *IJGMMP*, (2024). https://doi.org/10.1142/S0219887824400176. (Contains results presented in Part III, Chapter 6).

[5] P. Aniello, S. L’Innocente, S. Mancini, V. Parisi, I. Svampa and A. Winter, “Characterising the Haar measure on the p-adic rotation groups via inverse limits of measure spaces”, arXiv[math-ph]:2401.14298 (2024). (Contains results presented in Part I, Chapter 1).

List of symbols

Symbol Description

Symbol	Description
$\mathbb{Q}_p$	Field of p-adic numbers
$\mathbb{Z}_p$	Ring of p-adic integers
$\mathbb{Q}_{p,\mu}$	Quadratic extension of $\mathbb{Q}_p$
$\mathbb{Q}_p^*$	Multiplicative group of $\mathbb{Q}_p$
$\mathbb{Q}_{p,\mu}^*$	Multiplicative group of $\mathbb{Q}_{p,\mu}$
$\lvert\cdot\rvert_p$	p-Adic valuation
$B_r(a)$	Open ball in a with radius r
$\overline{B_r(a)}$	Closed ball in a with radius r
$\mathfrak{P}_p$	Valuation ideal of $\mathbb{Q}_p$
$\mathbb{K}/\mathbb{Q}_p$	Field extension of $\mathbb{Q}_p$
$[\mathbb{K}:\mathbb{Q}_p]$	Degree of the field extension
$\mathrm{N}_{\mathbb{K}/\mathbb{Q}_p}$	Norm function
$\upsilon_0(J,\mathbb{Q}_{p,\mu})$	Probability simplex
$\lvert\mathbb{Q}_p\rvert$	Valuation group of $\mathbb{Q}_p$
$\lvert\mathbb{Q}_{p,\mu}^*\rvert$	Valuation group of $\mathbb{Q}_{p,\mu}^*$
$c_0(I,X)$	Zero-converging sequences in X
$\ell^\infty(I,X)$	Bounded sequences in X
$\lVert\cdot\rVert_\infty$	Sup-norm
$\nu_{p,\mu}$	Isotropy index
$\mathbf{e}\equiv\{e_i\}$	Normal basis
$\langle\cdot,\cdot\rangle$	Non-Archimedean inner product
$\Phi\equiv\{\phi_i\}$	Orthonormal basis
$\mathrm{op}(\cdot)$	Matrix operator
$\mathcal{H}$	p-Adic Hilbert space
$\mathbb{H}(I)$	Coordinate p-adic Hilbert space
$\mathcal{H}'$	Topological dual of $\mathcal{H}$
$\mathcal{B}(\mathcal{H})$	Bounded operators in $\mathcal{H}$
$(\mathcal{H},\mathcal{H})_\Phi$	All-over matrix operators in $\mathcal{H}$
$\mathcal{B}_{\mathrm{ad}}(\mathcal{H})$	Adjointable operators in $\mathcal{H}$
$\mathcal{B}_{\mathrm{sa}}(\mathcal{H})$	Self-adjoint operators in $\mathcal{H}$
$\mathcal{U}(\mathcal{H})$	Unitary operators in $\mathcal{H}$
$\mathcal{B}(\mathcal{H})_{[1]}$	Unit sphere in $\mathcal{B}(\mathcal{H})$
$\mathcal{B}(\mathcal{H})_1$	Unit ball in $\mathcal{B}(\mathcal{H})$
$\mathcal{T}_\Phi(\mathcal{H})$	Traceable operators w.r.t. Φ
$\mathcal{T}(\mathcal{H})$	Trace class of $\mathcal{H}$
$\mathcal{T}_{\mathrm{w}}(\mathcal{H})$	Weak trace class operators in $\mathcal{H}$
$\mathscr{F}(\mathcal{H})$	Finite-rank operators in $\mathcal{H}$
$\mathcal{C}(\mathcal{H})$	Compact operators in $\mathcal{H}$
$\mathscr{B}_\Phi$	Block-finite operators in $\mathcal{H}$
$\mathrm{aco}_{\mathbb{Q}_p}(\cdot)$	Absolutely $\mathbb{Q}_p$-convex hull
$\mathrm{co}_{\mathbb{Q}_p}(\cdot)$	$\mathbb{Q}_p$-convex hull
$\mathrm{aff}_{\mathbb{Q}_p}(\cdot)$	$\mathbb{Q}_p$-affine hull
$\mathcal{S}(\mathcal{H})$	Algebraic states for $\mathcal{H}$
$\mathcal{T}_{\mathrm{st}}(\mathcal{H})$	Statistical operators in $\mathcal{H}$
$\mathcal{S}_{\mathrm{tr}}(\mathcal{H})$	Trace induced states in $\mathcal{H}$
$\mathcal{D}(\mathcal{H})$	Density operators in $\mathcal{H}$
μ	Haar measure
λ	Haar measure on $\mathbb{Q}_p^n$
$\mu_2^{(\kappa)}$	Haar measure on $\mathrm{SO}(2,\mathbb{Q}_p)$
μ_3	Haar measure on $\mathrm{SO}(3,\mathbb{Q}_p)$
μ_4	Haar measure on $\mathrm{SO}(4,\mathbb{Q}_p)$
$\Delta_G(\cdot)$	Modular function on G
$\mathrm{C}_c(G)$	Compactly supported continuous functions on G
$\mathrm{C}_c^+(G)$	Positive compactly supported continuous functions on G
$\mathbb{H}_p$	p-Adic quaternion algebra
$\mathbb{H}_p^\times$	Multiplicative group of invertible quaternions
$\mathrm{nrd}(\cdot)$	Reduced norm function
$\mathbf{H}_p$	p-Adic matrix quaternion algebra
$\mathbb{H}_p^0$	Pure quaternions
$\mu_{\mathbb{H}_p^\times}$	Haar measure on $\mathbb{H}_p^\times$

Introduction

Since the early stages of scientific inquiry, the concept of *'measurement'* has played a crucial role in the development of physical theories. In fact, in all its different branches, physics has primarily originated either from practical needs, or from particularly striking natural phenomena that, initially, determined its borders and established its fundamental definitions. Examples of this are provided by fields such as mechanics, acoustics, optics, or the theory of heat, where the 'physiological' element undoubtedly appears predominant. The essence of this phenomenological approach characterizing the early days of classical physics is perhaps captured, in its most significant and suggestive form, by Galilei's famous maxima: "Measure everything that is measurable, and make measurable everything that is not yet so".

This picture changed radically with the emergence of modern physics at the beginning of the 20th century. In fact, the observation of new physical processes and phenomena on length scales far from the immediate human experience has led, on the one hand, to the appearance of new branches of theoretical physics such as cosmology and quantum theory; on the other, it has also shown that the investigation and understanding of fundamental physical phenomena should undoubtedly rely on a completely different approach from the one characterizing the physics of the earlier centuries. In this regard, we believe that Dirac's enlightening words have captured, better than any other, this radical change in physical inquiry: "I learnt to distrust all physical concepts as the basis for a theory. Instead one should put one's trust in a mathematical scheme even if the scheme does not appear at first sight to be connected with physics. [...] The basic equations of the theory were worked out before their physical meaning was obtained. The physical meaning had to follow behind the mathematics" [1].

This new approach has culminated, nowadays, in the predominant effort in modern physics to formulate unified theoretical models (e.g., search for a standard model encompassing all the fundamental interactions) and to emancipate itself from those physiological and phenomenological elements that have characterized its early days (e.g., redefining quantum theory within a purely information-theoretic framework). In short, it seems that 'measurement' on the one hand (understood in its broadest sense of physical experimentation) and the rigorous study of mathematical models on the other hand (which encapsulate in a concise and precise way the physical content of fundamental natural processes) serve as the two guiding principles inspiring research in modern physics.

It is therefore not surprising that a 'measurement problem' has motivated, at the end of the last century, the emergence of a new branch of modern theoretical physics. In fact, general considerations in both quantum gravity and string theory suggest the existence of a smallest possible measurable length, i.e., the so-called *Planck length* $\ell_P = \sqrt{\hbar G/c^3} \sim 10^{-35} m$. Specifically, if Δx denotes an uncertainty in a length measurement, it should be true that

$$\Delta x \geq \ell_P,$$

essentially meaning that a length measurement smaller than the Planck length is forbidden, no matter if we increase the precision of the measurement. Otherwise said, this fundamental length poses a *natural unavoidable limitation* to the possible precision of any length measurement. Pursuing this observation to its logical conclusions, one is led to infer that, at a suitably small scale (comparable with the Planck length), space (or, in a relativistic framework, space-time) is not an infinitely divisible continuum, but, from a mathematical point of view, it should be described as a *totally disconnected topological space*. Therefore, in line with the general approach to modern theoretical physics, one can argue that the identification of the right mathematical framework for the description of the space-time geometry at a small length scale have to play a major role.

Based on these observations, Volovich and Vladimirov [2,3] postulated, in the late 1980s, that space-time should ultimately reveal a *non-Archimedean* micro-structure. In particular, starting from the observation that the only non-Archimedean field one can construct — up to isomorphism — by completing the field of rational numbers is the field $\mathbb{Q}_p$ of p-adic numbers (where p is a generic prime number), they proposed this field as a natural candidate for modeling space-time 'coordinates' at a very small length scale. This has then fueled the emergence of an entirely new branch of theoretical and mathematical physics, nowadays known as *p-adic quantum mechanics* [2–9].

At the best of our knowledge, there are two possible routes one can follow to describe a 'p-adic quantization'. The first route (and, actually, the most explored one), stems from the original ideas of Volovich and Vladimirov to exploit the field of p-adic numbers for space-time coordinates. Accordingly, in this model, a p-adic quantum system is essentially described by means of a triple

$$\{\mathrm{L}^2(\mathbb{Q}_p), W(z), U(t)\},$$

where $\mathrm{L}^2(\mathbb{Q}_p)$ is the state space, $W(z)$ is a unitary representation of the *Heisenberg-Weyl group* in $\mathrm{L}^2(\mathbb{Q}_p)$ — namely, the representation of the canonical commutation relations — and $U(t)$ is the unitary evolution operator associated with the p-adic Hamiltonian of the system. Here, the state space $\mathrm{L}^2(\mathbb{Q}_p)$ is simply the (complex) Hilbert space of square-integrable complex valued functions $\psi\colon \mathbb{Q}_p \to \mathbb{C}$ which, in this model, play the role of the 'p-adic wave functions'.

The second route one can follow, instead, concerns the carrier vector space of physical states itself and leads one, so to speak, to a more 'radical' p-adic point of view. Specifically, in this framework, physical states — in particular, wave functions — and observables live in a *p-adic Hilbert space*, and are tailored on a suitable model of p-adic probability theory.

Both these approaches have intrinsic strengths and limitations. For instance, referring to the first possible approach — namely, the route explored by Volovich and Vladimirov — there is no natural definition for the position and momentum operators. On the other hand, following the second possible route, some inconsistencies are observed related to the usual lattice structure of quantum mechanics [10].

In any case, regardless of the particular type of p-adic quantization exploited, over time there has been an increasing interest in application of p-adic numbers to quantum mechanics and to fundamental physical theories, ranging from quantum field theory and string theory (see [2, 5–7, 10–30], and references therein) to p-adic versions of both classical and quantum gravitational theory, along with unexpected cosmological applications related to the study of *dark energy* (see, e.g., [2–4, 19, 31–41]).

More recently, new and interesting implications of p-adic numbers — more oriented towards concrete applications — have begun to appear. In particular, p-adic models are currently also widely used to provide a useful mathematical tool for the characterization of complex systems showing a natural hierarchical structure. For instance, it has been proved that the ground state of *spin glasses* exhibits a natural ultrametric character: The so-called *Parisi matrix* admits a p-adic parametrization, as follows by observing that — once a suitable enumeration of the indices is adopted — its matrix elements can be expressed in terms of a (real-valued) function of the p-adic valuation of the difference of the matrix indices [42–47]. More recently, ultrametric models of statistical field theory have also been proposed [25,48–56], and, resorting to the well known correspondence between quantum field theory and neural networks, new hierarchical p-adic versions of the classical *restricted Boltzmann machines* have been obtained [57–60]. In this context, further interesting applications also appear in connection with *algebraic dynamical systems*; in particular, with problems arising from image analysis, image recognition, cryptography, compression of information and computer science [61–67].

Having described the distinctive features of p-adic quantum mechanics, and explored the possible applications of p-adic numbers in modelling physical phenomena, we aim at addressing, in the concluding part of this introductory chapter, three questions that may have naturally occurred to the reader who has reached this point.

The first question we try to answer — and, most likely, the first one the reader may ask — is: *What is the field of p-adic numbers $\mathbb{Q}_p$ from a purely mathematical point of view?*

Without claiming to be exhaustive and rigorous at this stage (for this purpose, in fact, we will devote an entire section in Part I of this dissertation), we try here to outline the fundamental properties of this field, in particular focusing on those aspects which, ultimately, justify its use in modelling space-time geometry at a small scale.

From an historical point of view, p-adic numbers were introduced by the German mathematician Kurt Hensel at the end of the XX century. What motivated Hensel was a thorough study of the analogies between the ring of integers $\mathbb{Z}$ — with its fraction field of rational numbers $\mathbb{Q}$ — and the ring $\mathbb{C}[X]$ of complex polynomials — with its fraction field of rational polynomials $\mathbb{C}(X)$. We remind that, in close analogy with the field of rational numbers $\mathbb{Q}$, where any rational number can be expressed as the quotient of two integers, a generic element $R(X)$ in $\mathbb{C}(X)$ is a rational function of the form

$$R(X) = \frac{F(X)}{G(X)}, \quad F(X), G(X) \in \mathbb{C}[X], \ \ G(X) \neq 0.$$

Moreover, $\mathbb{Z}$ and $\mathbb{C}[X]$ share some structural properties: They are both rings, and both admit a unique factorization property; namely, any integer can be expressed (uniquely) as ± 1 times the product of suitable powers of prime numbers, as well as any polynomial in $\mathbb{C}[X]$ can be written in the form

$$P(X) = z(X - \zeta_1)(X - \zeta_2) \dots (X - \zeta_n), \quad z, \zeta_1 \dots, \zeta_n \in \mathbb{C}.$$

It is essentially based on these observations that Hensel argued that primes $p \in \mathbb{Z}$ have to play a role 'analogous' to the one played by the linear polynomials $X - \zeta$ in $\mathbb{C}[X]$. Specifically, starting from the well known fact that any polynomial (actually, every rational function in $\mathbb{C}(X)$) can be expanded, in a unique way, in the form

$$R(X) = \frac{F(X)}{G(X)} = \sum_{i \geq i_0} z_i (X - \zeta)^i, \quad i_0 \in \mathbb{Z}, \ \ z_i, \zeta \in \mathbb{C}, \tag{$*$}$$

namely, in terms of a suitable (converging) Laurent series, pursuing further the analogy between the two rings (and the respective fraction fields), he argued that a representation like the one in ($*$) can be constructed also for rational numbers; namely, he tried to express every $q \in \mathbb{Q}$ by means of a suitable 'Laurent series' of the form

$$q = \frac{m}{n} = \sum_{i \geq i_0} c_i p^i, \quad m, n, c_i \in \mathbb{Z}. \tag{$**$}$$

We call an expression like the one in ($**$) a *p-adic expansion of q*. Clearly, a series as in ($**$) will not, in general, converge in $\mathbb{Q}$; it was precisely the attempt to define the convergence conditions of ($**$) what guided Hensel in the construction of the field of p-adic numbers $\mathbb{Q}_p$. Indeed, one way to characterize this field is to define it as the metric space where *all the p-adic expansions are convergent*. Its metric and topological properties — assuring the convergence of ($**$) — are noteworthy, and in sharp contrast with the ones observed in the fields of real or rational numbers. In fact, it turns out that $\mathbb{Q}_p$ is an *ultrametric space*, i.e., its topology is induced by an *ultrametric function* — a metric function satisfying a peculiar condition named *strong triangle inequality* — and a *totally disconnected* topological field. Both these properties, in particular, entail a non-Archimedean character for this field [68–76].

An astonishing result (nowadays known as Ostrowski's Theorem) proves that, starting from the field of rational numbers $\mathbb{Q}$, the field of p-adic numbers is the *only* field (up to isomorphism) one can construct with these peculiar characteristics. This observation is, ultimately, what led Volovich and Vladimirov to adopt $\mathbb{Q}_p$ as the 'right' field to capture the non-Archimedean character of space-time at the Planck regime.

Next, we now answer a second question the reader may have at this point: *What does this dissertation actually deal with?*

As previously observed, in fact, p-adic quantum mechanics has been widely studied and explored in its peculiar features and possible implications. In our opinion, however, it seems that at least one remarkable aspect of the p-adic theory of quantum mechanics has not been fully investigated yet. Indeed, the recent development of quantum technologies suggests exploring possible implications of p-adic numbers in quantum information theory as well. There are already promising indications that the distinctive properties of p-adic numbers (in particular, their fractal-like structure) may offer a new line of attack for notoriously hard problems in this context. For instance, it has been observed that p-adic numbers can be profitably used in the construction of the so-called mutually unbiased bases (MUBs) [77]. However, at the best of our knowledge, it seems that a systematic study of their possible application in this specific direction has not yet been adequately investigated. In this dissertation, we try to address this gap by setting the foundations for what may be considered a first step towards a p-adic model of quantum information theory. Our first aim is to appropriately define a notion of a qubit in this setting and to provide, at the same time, those technical tools necessary for a further development of the theory. In particular, following the first of the two approaches to the p-adic quantization (i.e., the model introduced by Volovich and Vladimirov), we argue that a suitable model of a p-adic qubit may be introduced by resorting to two-dimensional irreducible representations of the special orthogonal group $\mathrm{SO}(3, \mathbb{Q}_p)$ over $\mathbb{Q}_p$. Exploiting the well known Peter-Weyl theorem, these representations are found once a Haar measure for such a group is known. This issue is non-trivial since $\mathrm{SO}(3, \mathbb{Q}_p)$ — and, more in general, any p-adic Lie group — is locally isomorphic to $\mathbb{Q}_p^n$ and, as such, a totally disconnected topological space. While p-adic Lie groups have been extensively studied [78–80], we found no trace in the literature of a general construction for a Haar measure (or an invariant form) on such groups. We hence provide a solution to this problem by showing a general method for constructing an invariant measure on any p-adic Lie group and, more in general, on every totally disconnected topological group. We also observe that this measure can be regarded as the measure naturally induced by the invariant volume form on the group, as it happens for standard real Lie groups. We then apply our general results to the special orthogonal groups over $\mathbb{Q}_p$ (in dimension $2, 3$ and 4), hence paving the way to the development of harmonic analysis on these groups, with potential applications in both p-adic quantum mechanics and p-adic quantum information theory.

Following the second possible approach to the p-adic quantization, we try to provide a suitable definition of a qubit also in this setting. Once again, a preliminary problem we have to face with is to develop the 'right mathematical machinery' suitable for our purposes. Hence, as a first step, we propose a new notion of a p-adic Hilbert space (markedly different from other existing definitions in the literature). In particular, in our model of Hilbert space, the emphasis lies in the existence of an orthonormal basis for such a space. This not only allows us to conveniently define (Hilbert) subspaces (hence, paving the way for a thorough study of the logic of p-adic quantum mechanics), but also provides the right tool to characterize linear operators acting in p-adic Hilbert spaces. In particular, we introduce several new results concerning the theory of linear operators in the p-adic setting, with a special emphasis on the classes of unitary, compact and trace-class operators. Interestingly enough, the latter seems to play, in this p-adic setting, the role both of the trace class and of the Hilbert-Schmidt operators. Our investigations ultimately enables us to introduce a suitable notion of p-adic *statistical* and *density* states, as well as to define SOVMs (selfadjoint-operator-valued-measures) as a suitable tool for describing physical observables in the p-adic setting.

At this point, we find now natural to address one final question that the reader might reasonably pose: *Why choose to investigate both the two possible approaches to the p-adic quantization when formulating a p-adic model of information theory?*

To this concern, we can only reply that, lacking sufficient reasons to definitively favor one route over the other possible one, we have decided to explore, to the best of our capabilities, both of them, contenting us with that certainty which, in the last part of his life, Goethe claimed to be the greatest fortune a thinker may aspire to: "the certainty to have divined the comprehensible and calmly to revere the incomprehensible" [81].

Outline

Having discussed the core topics addressed in this dissertation, we now aim at providing a more detailed overview of its structure and contents. Specifically, we decided to organize this work into four distinct parts, each discussing a specific aspect of our final project aimed at setting the foundations of a p-adic quantum information theory. All the parts are preceded by a brief introduction — i.e., an abstract — which also serves as a 'declaration' specifying the main sources from which the material covered in that specific part is taken from. We hope that organizing in this way the material, will enhance the readability and the overall clarity of the exposition.

- **Part I : Preliminaries**. Here we collect all those results which, although well known in the literature, are necessary for the understanding of all the subsequent parts of this dissertation. In particular, we introduce the field $\mathbb{Q}_p$ of p-adic numbers — defined as a suitable metric completion of the field of rational numbers — and discuss its quadratic extensions $\mathbb{Q}_{p,\mu}$ with respect to a non-quadratic p-adic unit μ (i.e., $\mu \notin (\mathbb{Q}_p^*)^2$, and $|\mu|_p = 1$). The latter will be then exploited as the basis scalar field for the theory developed in Part III. We also introduce the notions of inverse limit and inverse system (of sets and topological groups), and provide an inverse limit description of the ring $\mathbb{Z}_p$ of *p-adic integers*. Finally, we characterize quadratic forms on $\mathbb{Q}_p$ and describe the classes of linear transformations which preserve them, namely, the p-adic special orthogonal groups in dimension two, three and four.

- **Part II : Studies on p-adic harmonic analysis**. In this part, we face the problem to construct a suitable model of p-adic qubit following the first approach to the p-adic quantization (namely, the one initiated by Volovich and Vladimirov). Our guiding observation is that, as argued in [82], a p-adic qubit can be defined by resorting to two-dimensional irreducible projective representations of the group of rotations $\mathrm{SO}(3, \mathbb{Q}_p)$ on $\mathbb{Q}_p^3$. The compactness of this group entails that all its irreducible unitary representations can be studied as subrepresentations of the regular representation, according to the well known *Peter-Weyl theorem* [83]. On the other hand, the study of the regular representation of compact groups, as well as several other central problems of abstract harmonic analysis, involve the construction of an invariant measure — i.e., the *Haar measure* — on such groups (or, the *Haar integral*, regarding such a measure as a *functional*) [83]. Moreover, whenever the group considered is compact, these irreducible (*projective*) representations are, in general, *square integrable* [84–87], and satisfy suitable orthogonality relations, involving, once again, the Haar measure. These observations motivate us to construct, in this part of the dissertation, the Haar measure for every p-adic Lie group, focusing then on the group $\mathrm{SO}(3,\mathbb{Q}_p)$ of our interest [88,89]. In particular, the strategy we adopt articulates in two main steps. First, by considering a suitable atlas of mutually disjoint charts, and resorting to the change-of-variables formula for multiple integrals on $\mathbb{Q}_p^n$, we provide a local expression for the Haar measure on any p-adic Lie group. Then, exploiting a suitable quaternionic representation of $\mathrm{SO}(3,\mathbb{Q}_p)$ and using the so-called Weil-Mackey-Bruhat formula [83,90,91], we construct the Haar integral of this group.

- **Part III : p-Adic quantum theory**. Here we address the problem to construct a suitable model of p-adic qubit following the second approach to the p-adic quantization — namely, the one where the carrier space itself of physical states is non-Archimedean [17, 92,93]. Our main concern in this part is, thus, to provide the basic mathematical tools for a general definition of physical states in the p-adic setting. In particular, by exploiting an algebraic approach, and adopting a p-adic model of probability theory [94,95], we are led to the conclusion that p-adic states should be defined as suitable linear functionals acting on a $*$-algebra of (bounded) p-adic observables. More precisely, we start our construction by preliminary introducing the notion of an Hilbert space $\mathcal{H}$ over a quadratic extension $\mathbb{Q}_{p,\mu}$ of $\mathbb{Q}_p$; next, we observe that linear operators in $\mathcal{H}$ are conveniently characterized in terms of suitable matrix operators. Specifically, we describe the classes of bounded, adjointable, unitary and trace class operators. We then characterize p-adic states, first, in their most abstract form as suitable functionals acting over the Banach $*$-algebra of adjointable operators in the carrier Hilbert space; then, we focus on the special class of the so-called *trace induced states*, hence characterizing statistical and density operators. Finally, we introduce the notion of SOVM (self-adjoint operator valued measure) as a convenient representation of a physical observable in the p-adic setting.

- **Part IV : Conclusion and Perspectives**. In this part, we briefly sum up the main results discussed in this dissertation, especially focusing on those achievements that are directly imputable to our personal contributions. We also highlight some natural continuations and extensions of the ideas presented.

Part I

Preliminaries

In this part of the thesis, we collect the basic notions and tools necessary for all our later derivations. Specifically, in Section 1.1 we introduce the field of p-adic numbers $\mathbb{Q}_p$ *and describe its most relevant topological properties. Next, in Subsection 1.1.2, we classify the quadratic extensions of this field. In particular, we first recall some basic notions concerning finite field extensions of* $\mathbb{Q}_p$*. Then, we focus on those of degree* 2*, hence addressing the quadratic ones. Section 1.2 deals with the inverse limit characterization of* $\mathbb{Z}_p$ *(and, hence of* $\mathbb{Q}_p$*). Eventually, we devote Chapter 2 to a thorough study of quadratic forms and the special orthogonal groups over* $\mathbb{Q}_p$*. None of the results presented here can be credited as our personal contribution.*

1

Basic Concepts

Throughout this work, we will constantly assume the field of p-adic numbers $\mathbb{Q}_p$ as the basis scalar field. Hence, for readers convenience, we find useful to recall in this first chapter some fundamental properties of $\mathbb{Q}_p$ and of its quadratic extensions. For more details, the reader may refer to, e.g., [6, 64, 68–70, 72–74, 83, 96–98].

1.1 Basics on p-adic numbers

Let $p \in \mathbb{N}$ be a prime number. According to the unique factorization theorem, every rational number $x \in \mathbb{Q}$ can be written (in a unique way) in the form $x = p^k m/n$, for some $k, m, n \in \mathbb{Z}$ and with $p \nmid m, n$. The so-called p-adic *absolute value* is defined as a map $|\cdot|_p : \mathbb{Q} \to \mathbb{R}^+$, with $|0|_p \equiv 0$ and

$$|x|_p := p^{-k}. \tag{1.1}$$

This map satisfies all the defining properties of an absolute value, or *valuation*; i.e., it is strictly positive on $\mathbb{Q}^* \equiv \mathbb{Q} \setminus \{0\}$, it factorizes under the product of two elements in $\mathbb{Q}$ and satisfies the triangle inequality. However, it also satisfies a more stringent condition, the so-called *ultrametric inequality* (or strong triangle inequality), namely,

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}, \quad \forall x, y \in \mathbb{Q}. \tag{1.2}$$

Moreover, from (1.2) it is also not difficult to prove (see, e.g., Proposition 2.3.4 in [68]) that, if $x, y \in \mathbb{Q}$, with $|x|_p \neq |y|_p$, then

$$|x + y|_p = \max\{|x|_p, |y|_p\}. \tag{1.3}$$

The inequality (1.2) represents the main difference w.r.t. the standard absolute value on $\mathbb{Q}$. A valuation on a certain field is called *non-Archimedean* or *ultrametric* if, like $|\cdot|_p$, it satisfies the strong triangle inequality; otherwise (e.g., in the case of the standard absolute value on $\mathbb{Q}$), it is called *Archimedean*.

A famous theorem due to Ostrowski shows that the standard absolute value $|\cdot|$ and the p-adic absolute value $|\cdot|_p$ — with p ranging over the prime numbers — exhaust all possible inequivalent valuations on $\mathbb{Q}$; namely, the following result holds [6, 68–70, 75]:

Theorem 1.1.1 (Ostrowski)**.** *Every non-trivial absolute value on $\mathbb{Q}$ is equivalent either to the standard absolute value $|\cdot|$, or to the p-adic absolute value $|\cdot|_p$, for some prime p in $\mathbb{N}$.*

Remark 1.1.1. It is worth observing that it the literature (see, e.g., [68]), the standard absolute value is sometimes denoted by $|\cdot|_\infty$. Adhering to this convention, and thinking to the symbol ∞ as to the "infinite prime", the Ostrowski theorem can be restated by saying that every non-trivial absolute value on $\mathbb{Q}$ is equivalent to one of the absolute values $|\cdot|_p$, where p is either a prime number or $p = \infty$.

Let us now introduce the map

$$\mathsf{d}_{|\cdot|_p}(x,y) \coloneqq |x-y|_p\,, \quad x,y\in\mathbb{Q}, \tag{1.4}$$

that is, the metric induced by $|\cdot|_p$. Obviously, the strong triangle inequality entails that $\mathsf{d}_{|\cdot|_p}$ satisfies

$$\mathsf{d}_{|\cdot|_p}(x,y) \leq \max\left\{\mathsf{d}_{|\cdot|_p}(x,z), \mathsf{d}_{|\cdot|_p}(z,y)\right\}, \quad x,y,z\in\mathbb{Q}. \tag{1.5}$$

In the mathematical literature, one refers to a space endowed with such a metric as an *ultrametric space*. Although the metric $\mathsf{d}_{|\cdot|_p}$, as well as the p-adic absolute value, differs from the standard metric on $\mathbb{Q}$ essentially for the ultrametric inequality, the consequences of this fact are noteworthy, e.g., from a topological point of view [6,68,70,73]. We will discuss the most relevant topological implications of (1.5) in the forthcoming subsection.

Let p in $\mathbb{N}$ be a prime number. By means of a standard procedure [68,75,98], the p-adic numbers $\mathbb{Q}_p$ can be defined as the field completion of $\mathbb{Q}$ w.r.t. the metric $\mathsf{d}_{|\cdot|_p}$, and then $\mathbb{Q}$ can be regarded as a dense subfield of the complete field $\mathbb{Q}_p$. We will denote by $\mathbb{Q}_p^* \equiv \mathbb{Q}_p \setminus \{0\}$ the multiplicative group of $\mathbb{Q}_p$.

Theorem 1.1.2 ([6, 64, 70, 75, 83])**.** *Every $x\in\mathbb{Q}_p^*$ admits a unique decomposition as a convergent series of the form*

$$x=\sum_{i=0}^{\infty} x_i\, p^{i+k} = p^k(x_0+x_1\,p+x_2\,p^2+\cdots), \quad k\in\mathbb{Z},\ x_i\in\{0,1,...,p-1\},\ x_0\neq 0, \tag{1.6}$$

and, conversely, every series of this form converges to some non-zero element of $\mathbb{Q}_p$. The continuous extension of the p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}$ to $\mathbb{Q}_p$ — extension which we still denote by the same symbol — is an ultrametric valuation on $\mathbb{Q}_p$. Explicitly, we have:

$$|x|_p = \left|\textstyle\sum_{i=0}^{\infty} x_i\, p^{i+k}\right|_p = p^{-k}, \quad \forall x\in\mathbb{Q}_p^*. \tag{1.7}$$

The field of p-adic numbers $\mathbb{Q}_p$ — being endowed with an ultrametric valuation — is called *ultrametric* or *non-Archimedean*. The topological peculiarities associated with ultrametricity justify the use of p-adic numbers when describing physics on length scales comparable to Planck's length l_{P} [6,7,99].

To conclude this subsection, we recall that the so-called *valuation ring* — w.r.t. $|\cdot|_p$ — of the non-Archimedean field $\mathbb{Q}_p$ is the ring of *p-adic integers* $\mathbb{Z}_p := \{\lambda\in\mathbb{Q}_p \mid |\lambda|_p\leq 1\} = \{\sum_{i=0}^{+\infty} a_i p^i \mid 0\leq a_i < p\}$, i.e., a subring of $\mathbb{Q}_p$ [68,75]. The set $\mathfrak{P}_p := \{\lambda\in\mathbb{Q}_p \mid |\lambda|_p < 1\} = p\mathbb{Z}_p \subset \mathbb{Z}_p$ is a maximal ideal in $\mathbb{Z}_p$ (actually, its unique maximal ideal) — the so-called *valuation ideal* of $\mathbb{Q}_p$ w.r.t. $|\cdot|_p$ — and every element of $\mathbb{Z}_p\setminus\mathfrak{P}_p$ is invertible. The quotient $\mathbb{Z}_p/\mathfrak{P}_p$ is called the *residue class field* of $\mathbb{Q}_p$ w.r.t. $|\cdot|_p$ (recall that the quotient of a ring by a maximal ideal is always a field); specifically, $\mathbb{Z}_p/\mathfrak{P}_p = \mathbb{Z}_p/p\mathbb{Z}_p$ is isomorphic to the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.

Remark 1.1.2. In the mathematical literature, the group of invertible elements in $\mathbb{Z}_p$ is usually denoted by $\mathbb{U}_p$, and its elements are called p-adic units [70,74,98,100]. Every $x\in\mathbb{Z}_p^*$, can be uniquely expressed in the form $x=p^n u(x)$, where $n\geq 0$ is an integer, and $u(x)\in\mathbb{U}_p$ is a p-adic unit (depending on x). Moreover, observing that $\mathbb{Q}_p=\mathbb{Z}_p[p^{-1}]$, it follows that also every x in $\mathbb{Q}_p^*$ can be written (uniquely) in the form $x=p^n u(x)$, for some $n\in\mathbb{Z}$, and $u(x)\in\mathbb{U}_p$. In particular, one has that $n\geq 0$ if and only if $x\in\mathbb{Z}_p$ (see Chap. II in [74]).

1.1.1 Topological properties of $\mathbb{Q}_p$

In this subsection, we collect the most relevant topological consequences of the ultrametric inequality (cf. (1.2)).

Let us start by explicitly noting the following fact. Indeed, from (1.3) it is easily shown that in $\mathbb{Q}_p$ — or, more generally, in any ultrametric space — all triangles are isosceles. To prove this claim, let x, y, z denote the three vertices of a 'triangle', and let $|x-y|_p$, $|x-z|_p$ and $|y-z|_p$ be the lengths of its sides. Since $|x-z|_p = |x-y+y-z|_p \leq \max\{|x-y|_p, |y-z|_p\}$, then either $|x-y|_p \neq |y-z|_p$ — in which case we have $|x-z|_p = \max\{|x-y|_p, |y-z|_p\}$ — or $|x-y|_p = |y-z|_p$. In either case, two of the three sides must be equal. This geometric feature of $\mathbb{Q}_p$ has profound topological implications, as we are now going to see. Let us now set the following:

Definition 1.1.1. Let $a \in \mathbb{Q}_p$, and let $r \in \mathbb{Z}$. The *open ball* $B_r(a)$ of center a and radius $p^r > 0$ is defined as the set

$$B_r(a) \coloneqq \{x \in \mathbb{Q}_p \mid |x-a|_p < p^r\}. \tag{1.8}$$

Similarly, the *closed ball* $\overline{B_r(a)}$ of radius $p^r > 0$, and center $a \in \mathbb{Q}_p$, is the set

$$\overline{B_r(a)} \coloneqq \{x \in \mathbb{Q}_p \mid |x-a|_p \leq p^r\}. \tag{1.9}$$

The following proposition encloses the main topological features of $\mathbb{Q}_p$ [68, 73, 76]

Proposition 1.1.1. *Let $a, b \in \mathbb{Q}_p$, and let $r, s \in \mathbb{Z}$. Then we have:*

- `(T1)` *If $b \in B_r(a)$ (resp. $\overline{B_r(a)}$), then $B_r(b) = B_r(a)$ (resp. $\overline{B_r(b)} = \overline{B_r(a)}$).*
- `(T2)` *The set $B_r(a)$ is both open and closed.*
- `(T3)` *The set $\overline{B_r(a)}$ is both open and closed.*
- `(T4)` *If $r \geq s > 0$, then either $B_r(a) \cap B_s(b) = \emptyset$, or $B_s(b) \subset B_r(a)$. Hence, any two balls in $\mathbb{Q}_p$ are either disjoint, or one is contained in the other.*

(Note: A set in a topological space is said to be *clopen* if it is both open and closed. Hence, `(T2)` and `(T3)` assert that open and closed balls in $\mathbb{Q}_p$ are clopen sets).

Proof. Let $a, b \in \mathbb{Q}_p$, and $r \in \mathbb{Z}$. Let us assume $b \in B_r(a)$, so that $|b-a|_p < p^r$. Let $x \in B_r(a)$. We have:

$$|x-b|_p = |x-a+a-b|_p \leq \max\{|x-a|_p, |a-b|_p\} < p^r, \tag{1.10}$$

i.e., $x \in B_r(b)$. This shows that $B_r(a) \subset B_r(b)$. Conversely, if $x \in B_r(b)$, then

$$|x-a|_p = |x-b+b-a|_p \leq \max\{|x-b|_p, |b-a|_p\} < p^r, \tag{1.11}$$

i.e., $B_r(b) \subset B_r(a)$. Hence, we proved $B_r(a) = B_r(b)$. By replacing "$<$" with "$\leq$", one can repeat a similar discussion for closed balls as well, thus proving `(T1)`.

It is clear that $B_r(a)$ is an open set (indeed, from `(T1)`, it follows that $B_r(a)$ contains the open balls around any of its points). We will now prove it is a closed set as well. Let us consider the set

$$B_r(a)^{\mathsf{c}} \coloneqq \{x \in \mathbb{Q}_p \mid |x-a|_p \geq p^r\}, \tag{1.12}$$

i.e., the *complement* in $\mathbb{Q}_p$ of the open ball $B_r(a)$. Let $y \in B_r(a)^{\mathsf{c}}$, and let $s \in \mathbb{Z}$ be such that $p^s < p^r$. If $z \in B_s(y)$ — i.e., the open ball of radius p^s around y — we have $|z-y|_p < p^s$. Then,

$$|z-a|_p = |z-y+y-a|_p = \max\{|z-y|_p, |y-a|_p\} = |y-a|_p \geq p^r. \tag{1.13}$$

Hence, $B_s(y) \subset B_r(a)^{\mathsf{c}}$. But this entails that $B_r(a)^{\mathsf{c}}$ is open, and, therefore, that $B_r(a)$

is closed. This proves `(T2)`. Similarly, one proves that any closed ball is open and, thus, that `(T3)` holds as well.

Lastly, let $r, s \in \mathbb{Z}$, with $r > s$ (and, hence, $p^r > p^s$). If $B_r(a) \cap B_s(b) \neq \emptyset$, there exists a point c in the intersection such that

$$B_s(b) = B_s(c) \subset B_r(c) = B_r(a), \tag{1.14}$$

i.e., $B_s(b) \subset B_r(a)$. Analogously, one proves the inclusion for closed balls, that is, `(T4)` follows. □

It is worth observing that `(T2)` and `(T3)` in Proposition 1.1.1 also entails that open and closed balls necessarily have *empty boundary*. This follows by observing that any ball centered around a boundary point of the open ball $B_r(a)$ contains points in both $B_r(a)$ and its complement $B_r(a)^c$. But since $B_r(a)$ and $B_r(a)^c$ are closed, we conclude that any boundary point must belong to $B_r(a) \cap B_r(a)^{\mathsf{c}} = \emptyset$.

The family of balls $\mathcal{B}(x) := \{B_{-n}(x)\}_{n\in\mathbb{N}}$ forms a base for the topology of $\mathbb{Q}_p$ at any given point $x \in \mathbb{Q}_p$. In particular, the set $\mathscr{B} := \bigcup_{x\in\mathbb{Q}_p} \mathcal{B}(x)$ forms a base for the (ultrametric) topology of $\mathbb{Q}_p$. In this topology, $\mathbb{Q}_p$ is easily seen to be locally compact, Hausdorff and separable.

Remark 1.1.3. It is not difficult to check that the field operations of $\mathbb{Q}_p$ are continuous functions w.r.t. the (metric) topology induced by $\mathsf{d}_{|\cdot|_p}$ (cf. (1.4)). This then entails that $\mathbb{Q}_p$ has a natural structure of *topological field*.

Remark 1.1.4. A Hausdorff topological space admitting a base $\mathcal{B}$ such that, for any pair of elements $A, B \in \mathcal{B}$, it is either $A \subset B$, or $B \subset A$, or $A \cap B = \emptyset$, is said to be a *non-Archimedean topological space*. In this case, it is possible to prove that all the elements of $\mathcal{B}$ are clopen [73]. A topological space X admitting a base consisting of clopen sets is usually called a *zero-dimensional* (topological) space. Hence, $\mathbb{Q}_p$, endowed with the metric topology, is a zero-dimensional topological field.

The field $\mathbb{Q}_p$, equipped with the ultrametric topology, is a *totally disconnected* set. We recall that a set A is *disconnected*, if there exists two (non-empty) open sets U_1, U_2 in A such that $A = U_1 \cup U_2$, and $U_1 \cap U_2 = \emptyset$. If $x \in A$, we define the *connected component* of x to be the union of all the connected sets in A containing x (equivalently, as the largest connected set in A which contains x). If the connected component of any x in A is the singleton $\{x\}$, we say that A is a *totally disconnected* set.

Every open (closed) ball in $\mathbb{Q}_p$ is a disconnected set: If $B_r(a)$ is an open ball, and if $x \in B_r(a)$, consider a smaller ball $B_\epsilon(x)$ centered in x and contained in $B_r(a)$. Then, $B_\epsilon(x)$ and $B_r(a) \backslash B_\epsilon(x)$ provides a disjoint partition of $B_r(a)$. Actually, one can prove the following result

Proposition 1.1.2 ([68])**.** *The field of p-adic numbers $\mathbb{Q}_p$ is a totally disconnected topological space.*

We do not discuss the topological properties of $\mathbb{Q}_p$ any further. For a detailed account, the reader may refer to, e.g., [6, 72, 73, 76, 101].

In the concluding part of this subsection, we address some relevant physical consequences stemming from the topological structure of $\mathbb{Q}_p$ [92]. In fact, the existence of a smallest measurable distance postulated by Volovich — namely, Planck's length ℓ_P — is consistent with a picture of space-time as a totally disconnected metrizable space, governed by an *ultrametric*; i.e., by a metric satisfying the strong triangle inequality. Such a space is modeled, in a natural way, by the field $\mathbb{Q}_p$ of p-adic numbers, rather than by the field of real numbers. With this picture in mind, one should also expect the existence of a connection between the elementary length ℓ_P and the balls of a certain radius in $\mathbb{Q}_p$. Specifically, we can think of each (say, closed) ball of radius of the order of ℓ_P as an elementary physical entity, whose single points cannot be identified, individually, with any (measurable) physical event. Note that, by the ultrametric inequality, the distance between two points picked arbitrarily form any fixed pair of disjoint balls is constant, whenever they belong to different balls; moreover, by properties `(T1)` and `(T4)` of Proposition 1.1.1, any point from each ball can be regarded as the center of that ball, and if we take two balls of the same radius, they either coincide or are disjoint. We therefore obtain a mathematically consistent *coarse-graining* of the measurable physical events.

1.1.2 Quadratic extensions of $\mathbb{Q}_p$

In part 3 of this work, we will explore a model of quantum mechanics over p-adic Hilbert spaces. Analogously to the standard case, where quantum states and observables are represented in terms of operators acting in complex Hilbert spaces, we expect that a p-adic model of quantum mechanics should necessarily involve Hilbert spaces defined over a suitable (quadratic) extension of $\mathbb{Q}_p$ (where a non-trivial involutive automorphism, i.e., a *conjugation*, can be defined). Hence, we find useful to discuss, at this point, some relevant facts concerning field extensions of $\mathbb{Q}_p$, in particular focusing on the quadratic ones.

Let us start by setting the following

Definition 1.1.2. We call any field $\mathbb{K}$ containing $\mathbb{Q}_p$ a *field extension* of $\mathbb{Q}_p$. We shall denote such an extension by $\mathbb{K}/\mathbb{Q}_p$.

If $\mathbb{K}/\mathbb{Q}_p$ is a field extension, then $\mathbb{K}$ has a natural vector space structure over $\mathbb{Q}_p$ and its dimension, $\dim_{\mathbb{Q}_p} \mathbb{K} \equiv \left[\mathbb{K} : \mathbb{Q}_p\right]$, is called the *degree* of $\mathbb{K}$ over $\mathbb{Q}_p$. In particular, in the case where $\left[\mathbb{K} : \mathbb{Q}_p\right] < \infty$, we say that $\mathbb{K}/\mathbb{Q}_p$ is a *finite field extension* of $\mathbb{Q}_p$.

In the remaining part of this subsection, we will essentially deal with the following two points:

(1) Construct a special class of field extensions of $\mathbb{Q}_p$ — namely, the so-called *quadratic extensions*;

(2) Extend the p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}_p$ to a suitable non-Archimedean valuation on the extended field $\mathbb{K}$.

We start by addressing point (1) first. Let us recall that every field with characteristic 0 admits a quadratic extension obtained by adjoining the square root of a non-quadratic element [68, 70, 98, 102]. Let $\mathbb{K}$ be an extension of $\mathbb{Q}_p$ of degree 2, that is, $\left[\mathbb{K} : \mathbb{Q}_p\right] = 2$, and let $\eta \in \mathbb{K}$ be an element of $\mathbb{K}$ not contained in $\mathbb{Q}_p$. Then, η is a solution of a polynomial of degree 2 over $\mathbb{Q}_p$ (see Proposition 12 in Chapt. 13 of [102]). Specifically, one has that the minimal polynomial of η, $q_\eta(x)$, is a *monic quadratic polynomial*, i.e.,

$$q_\eta(x) = x^2 + bx + c, \quad b, c \in \mathbb{Q}_p. \tag{1.15}$$

If $\mathbb{Q}_p(\eta)$ is the *smallest extension* of $\mathbb{Q}_p$ containing η, then we must have that $\mathbb{K} = \mathbb{Q}_p(\eta)$, as follows by observing that $\mathbb{Q}_p \subset \mathbb{Q}_p(\eta) \subset \mathbb{K}$, and that $\mathbb{K}$ has degree 2 over $\mathbb{Q}_p$. Now, the roots of the polynomial (1.15) are given by the usual quadratic formula, i.e.,

$$\eta = \frac{-b \pm \sqrt{b^2 - 4c}}{2}. \tag{1.16}$$

It is clear that $\sqrt{b^2 - 4c}$ is not a square in $\mathbb{Q}_p$ since, by assumption, $\eta \in \mathbb{K}$ is not contained in $\mathbb{Q}_p$. Moreover, $\sqrt{b^2 - 4c}$ is a root of the equation $x^2 - (b^2 - 4c) = 0$ in $\mathbb{K}$. Then, by (1.16), it follows that $\eta \in \mathbb{Q}_p(\sqrt{b^2 - 4c})$, i.e., $\mathbb{Q}_p(\eta) \subset \mathbb{Q}_p(\sqrt{b^2 - 4c})$. Conversely, since $\sqrt{b^2 - 4c} =$

$\mp(2\eta+b)$, we see that $\sqrt{b^2-4c} \in \mathbb{Q}_p(\eta)$ and, thus, that $\mathbb{Q}_p(\sqrt{b^2-4c}) \subset \mathbb{Q}_p(\eta)$. Concluding, it is shown that $\mathbb{Q}_p(\eta) = \mathbb{Q}_p(\sqrt{b^2-4c})$. Therefore, we can argue that every extension $\mathbb{K}$ of degree 2 over $\mathbb{Q}_p$ is of the form $\mathbb{Q}_p(\sqrt{\mu})$ (i.e., is a quadratic extension), for μ a non-quadratic element in $\mathbb{Q}_p$ and, conversely, every such extension is of degree 2 over $\mathbb{Q}_p$.

In the light of the above discussion, it is now natural to set the following:

Definition 1.1.3. Let $\mu \in \mathbb{Q}_p$ be a non-quadratic p-adic number, i.e., $\mu \notin (\mathbb{Q}_p^*)^2$. Introducing the symbol $\sqrt{\mu}$, the quadratic extension $\mathbb{Q}_{p,\mu} \equiv \mathbb{Q}_p(\sqrt{\mu})$ of $\mathbb{Q}_p$ associated with μ is the field

$$\mathbb{Q}_{p,\mu} := \big\{ z = x + y\sqrt{\mu} \mid x, y \in \mathbb{Q}_p \big\}. \tag{1.17}$$

In particular, $\mathbb{Q}_{p,\mu}$ is an extension of $\mathbb{Q}_p$ of degree $\big[\mathbb{Q}_{p,\mu} : \mathbb{Q}_p\big] = 2$.

Our next issue is now to classify all the (inequivalent) quadratic extensions of $\mathbb{Q}_p$. To this and, we first need to characterize the non-quadratic elements of this field. We start with the following fact:

Proposition 1.1.3 ([6])**.** *A p-adic number $x = \sum_{i=0}^{\infty} x_i\, p^{i+k} \in \mathbb{Q}_p^*$, $k \in \mathbb{Z}$, $x_i \in \{0,1,...,p-1\}$, $x_0 \neq 0$, is a quadratic element — i.e., $x \in (\mathbb{Q}_p^*)^2$ — if and only if the following conditions are satisfied:*

- *k is even;*
- *if $p \neq 2$, the equation $j^2 \equiv x_0 \pmod p$ admits a solution $j \in \mathbb{Z}$ — i.e., $x_0 \in \{1,...,p-1\}$ is a quadratic residue modulo p — whereas, if $p = 2$, $x_1 = x_2 = 0$.*

Let us denote by $\eta \in \mathbb{Q}_p^*$ a normalized element which is not a square, i.e., any p-adic number η such that $|\eta|_p = 1$ and $\eta \notin (\mathbb{Q}_p^*)^2$. The first condition implies that $\eta = \eta_0 + \eta_1\, p^1 + \eta_2\, p^2 + \cdots \neq p$. For $p \neq 2$ we can choose, in particular, any of the — exactly, $(p-1)/2$ — quadratic non-residues $(\bmod\, p)$ η contained in $\{2, \cdots, p-1\}$; e.g., for $p \equiv 3 \pmod 8$, or for $p \equiv 5 \pmod 8$, one can take $\eta = 2$ [6]. For $p = 2$, one must take η of the form $\eta = 1 + \eta_1 2 + \eta_2 2^2 + \cdots$, with $\eta_1, \eta_2 \in \{0,1\}$ and $\eta_1\eta_2 \neq 0$. At this point, by Proposition 1.1.3, it is clear that p and ηp do not belong to $(\mathbb{Q}_p^*)^2$, as well. In fact, $p = 1\,p^1$ entails that $k = 1$ does not satisfy the first condition therein. Similarly, for $\eta p = \eta_0\, p^1 + \eta_1\, p^2 + \cdots$.

Therefore, we have the following consequence of Proposition 1.1.3:

Corollary 1.1.1. *The following facts hold true:*

(a) *for $p \neq 2$, there is some $\eta \in \mathbb{Q}_p$ such that $\eta \notin (\mathbb{Q}_p^*)^2$ and $|\eta|_p = 1$, and ηp and p are not squares too;*

(b) *every $\mu \in \{2, \eta, 2\eta\}$ — with $\eta = 3, 5, 7$ — is not a quadratic element of $\mathbb{Q}_2$; namely, every $\mu \in \{2, 3, 5, 6, 7, 10, 14\}$ is not a square in $\mathbb{Q}_2$.*

Remark 1.1.5. The quotient group $\mathbb{Q}_p/(\mathbb{Q}_p^*)^2$ consists of four 'square classes', for $p \neq 2$, with representatives $\{1, \eta, p, \eta p\}$, where η is any normalized non-quadratic element of $\mathbb{Q}_p$; whereas, for $p = 2$, it consists of eight square classes, with representatives $\{1, 2, 3, 5, 6, 7, 10, 14\}$ (or, equivalently, $\{\pm 1, \pm 2, \pm 3, \pm 6\}$) [6, 97]. Therefore, Corollary 1.1.1 provides a classification of the square classes of $\mathbb{Q}_p^*$ different from $(\mathbb{Q}_p^*)^2$.

The forthcoming proposition classifies the possible quadratic extensions of $\mathbb{Q}_p$ for any prime $p \in \mathbb{N}$

Proposition 1.1.4 ([6, 97])**.** *The quadratic extensions of $\mathbb{Q}_p$ are classified as follows:*

(a) *if $p \neq 2$, there are precisely three non-isomorphic quadratic extensions of $\mathbb{Q}_p$, i.e., $\mathbb{Q}_{p,\mu}$, with $\mu \in \{\eta, p, \eta p\}$, for any $\eta \notin (\mathbb{Q}_p^*)^2$ such that $|\eta|_p = 1$;*

(b) *if $p = 2$, there are precisely seven non-isomorphic quadratic extensions of $\mathbb{Q}_p$, i.e., $\mathbb{Q}_{p,\mu}$, with $\mu \in \{2, \eta, 2\eta\}$ — for $\eta = 3, 5, 7$ — thus, with $\mu = 2, 3, 5, 6, 7, 10, 14$.*

Now, we turn our attention to point (2) above. Our concern here is to construct a suitable extension of the p-adic absolute value on $\mathbb{Q}_p$ to a (non-Archimedean) valuation on the extended field $\mathbb{K}/\mathbb{Q}_p$. We remind that by an extension of the p-adic valuation we mean a map $|\cdot|_{\mathbb{K}} : \mathbb{K} \to \mathbb{R}^+$ satisfying the following:

(a) $|x|_{\mathbb{K}} = 0$ iff $x = 0$;

(b) $|xy|_{\mathbb{K}} = |x|_{\mathbb{K}} |y|_{\mathbb{K}}$;

(c) $|x + y|_{\mathbb{K}} \leq \max\{|x|_{\mathbb{K}}, |y|_{\mathbb{K}}\}$,

and

(d) $|\alpha|_{\mathbb{K}} = |\alpha|_p$, $\forall \alpha \in \mathbb{Q}_p \subset \mathbb{K}$.

In the sequel, we will argue that a map verifying conditions (a)-(d) above can be defined on every field extension $\mathbb{K}/\mathbb{Q}_p$ of $\mathbb{Q}_p$. However, before getting into the technical details of the construction, we first prove the following result stating that whenever such an extension exists, it has to be necessarily unique.

Proposition 1.1.5. *Let $\mathbb{K}/\mathbb{Q}_p$ be a finite field extension of $\mathbb{Q}_p$. If $|\cdot|_{\mathbb{K}} : \mathbb{K} \to \mathbb{R}^+$ is an extension of the p-adic valuation on $\mathbb{K}$, then the following conditions are verified:*

(i) $\mathbb{K}$ is complete w.r.t. the metric induced by $|\cdot|_{\mathbb{K}}$;

(ii) If $|\cdot|'_{\mathbb{K}}$ is another extended valuation on $\mathbb{K}$, then it coincides *with $|\cdot|_{\mathbb{K}}$.*

Proof. Indeed $\mathbb{K}$, endowed with the absolute value $|\cdot|_{\mathbb{K}}$, is a finite-dimensional normed vector space over $\mathbb{Q}_p$. Hence, any two norms are equivalent, and $\mathbb{K}$ is complete w.r.t. the metric associated with any norm on it (cf. Theorem 6.2.1 in [68]). If $|\cdot|'_{\mathbb{K}}$ is another valuation on $\mathbb{K}$ extending $|\cdot|_p$, point (i) above entails that it is *equivalent*, as a norm, to $|\cdot|_{\mathbb{K}}$. In particular, Corollary 6.3.2 in [68] shows they are actually the same, hence proving (ii). □

Condition (ii) in Proposition 1.1.5 actually shows that there is at most one absolute value on $\mathbb{K}$ extending the p-adic valuation of $\mathbb{Q}_p$. This uniqueness condition entails an important feature of the extended valuation. Indeed, suppose that $\mathbb{K}$, $\mathbb{L}$ are two field extensions of $\mathbb{Q}_p$ such that $\mathbb{Q}_p \subset \mathbb{L} \subset \mathbb{K}$, and let $|\cdot|_{\mathbb{K}}$, $|\cdot|_{\mathbb{L}}$ be the extended valuations on $\mathbb{K}$ and $\mathbb{L}$ respectively. The restriction of $|\cdot|_{\mathbb{K}}$ to $\mathbb{L}$ is a valuation on $\mathbb{L}$ extending the p-adic absolute value. Hence, from (ii), it follows that

$$|x|_{\mathbb{L}} = |x|_{\mathbb{K}}, \quad \forall x \in \mathbb{K}/\mathbb{Q}_p, \tag{1.18}$$

that is, the valuation of any x — in the extension $\mathbb{K}/\mathbb{Q}_p$ — *does not depend* from the extension itself.

Having proved uniqueness, we now proceed to discuss existence. We first need to introduce the notion of a *normal extension* of $\mathbb{Q}_p$. Let $\mathbb{K}/\mathbb{Q}_p$ be a finite extension of $\mathbb{Q}_p$, and let $\mathbb{E}$ be an algebraically closed field containing $\mathbb{Q}_p$ — i.e., a field extension of $\mathbb{Q}_p$ containing the root of any polynomial with coefficients in $\mathbb{Q}_p$. Let $\omega\colon \mathbb{K} \to \mathbb{E}$ be a *field homomorphism* which induces the identity on $\mathbb{Q}_p$. We say that $\mathbb{K}/\mathbb{Q}_p$ is a *normal field extension* if all such ω have the same image in $\mathbb{E}$. Equivalently, by identifying $\mathbb{K}$ with one of its images in $\mathbb{E}$, we can say that $\mathbb{K}/\mathbb{Q}_p$ is normal if every field homomorphism $\omega\colon \mathbb{K} \to \mathbb{E}$ maps $\mathbb{K}$ to itself. In such a case, ω can be thought as an automorphism $\omega\colon \mathbb{K} \to \mathbb{K}$ which reduces to the identity when restricted to $\mathbb{Q}_p$. We call a map of this kind *an automorphism of the field extension* $\mathbb{K}/\mathbb{Q}_p$.

Remark 1.1.6. Given any finite extension $\mathbb{K}/\mathbb{Q}_p$ of $\mathbb{Q}_p$, it is always possible to construct a finite normal extension of $\mathbb{Q}_p$ containing $\mathbb{K}$. In particular, one refers to the smallest such one as to the *normal closure* of $\mathbb{K}$ [64, 68].

Let $\mathbb{K}/\mathbb{Q}_p$ be a finite field extension of $\mathbb{Q}_p$. We introduce the function

$$\mathrm{N}_{\mathbb{K}/\mathbb{Q}_p} \colon \mathbb{K} \to \mathbb{Q}_p, \tag{1.19}$$

called the *norm function* from $\mathbb{K}$ to $\mathbb{Q}_p$. There are several ways to define this map; here, we list three different (yet equivalent) characterizations [64, 68, 103, 104]:

(C1) Let $x \in \mathbb{K}$, and introduce the map M_x corresponding to the multiplication of the elements in $\mathbb{K}$ by the fixed point x. M_x is a $\mathbb{Q}_p$-linear map on $\mathbb{K}$ and, therefore, can be represented by the matrix $\mathsf{M}_x \equiv (\mathrm{M}_x)_{ij}$ w.r.t. any fixed basis of $\mathbb{K}$ (viewing $\mathbb{K}$ as a finite dimensional $\mathbb{Q}_p$-vector space). Then, the norm function is defined as $\mathrm{N}_{\mathbb{K}/\mathbb{Q}_p}(x) \coloneqq \det(\mathsf{M}_x)$.

(C2) Let x in $\mathbb{K}$, and consider $\mathbb{Q}_p(x)$ — namely, the smallest subfield of $\mathbb{K}$ containing both $\mathbb{Q}_p$ and x[1]. Let $d = [\mathbb{K} : \mathbb{Q}_p(x)]$, and let

$$f(Z) = Z^n + c_{n-1}Z^{n-1} + c_{n-2}Z^{n-2} + \ldots + c_1 Z + c_0 \tag{1.20}$$

be the *minimal* polynomial in $\mathbb{Q}_p[Z]$ such that $f(x) = 0$. Then, we can define $\mathrm{N}_{\mathbb{K}/\mathbb{Q}_p}(x) \coloneqq (-1)^{nd} c_d$.

(C3) Suppose that $\mathbb{K}/\mathbb{Q}_p$ is a normal extension of $\mathbb{Q}_p$. Then we have: $\mathrm{N}_{\mathbb{K}/\mathbb{Q}_p}(x) \coloneqq \prod_{\omega \in \Omega} \omega(x)$, where ω runs through the set Ω of all the automorphisms of $\mathbb{K}/\mathbb{Q}_p$.

Remark 1.1.7. It is possible to prove that the norm function is multiplicative, i.e.,

$$\mathrm{N}_{\mathbb{K}/\mathbb{Q}_p}(xy) = \mathrm{N}_{\mathbb{K}/\mathbb{Q}_p}(x)\mathrm{N}_{\mathbb{K}/\mathbb{Q}_p}(y), \tag{1.21}$$

for any $x, y \in \mathbb{K}$. This is particularly evident exploiting definition (C1) above, but it is not difficult to check that it holds also w.r.t. (C2) and (C3).

Remark 1.1.8. Let $\mathbb{K}/\mathbb{Q}_p$ be a finite normal extension of $\mathbb{Q}_p$. It is well known that, since $\mathbb{Q}_p$ is of characteristic 0, the set Ω of all the field automorphisms of $\mathbb{K}$ closes a finite group $\mathrm{Gal}(\mathbb{K}/\mathbb{Q}_p)$ — namely, the *Galois group* of $\mathbb{K}/\mathbb{Q}_p$ — of order $|\Omega| = |\mathrm{Gal}(\mathbb{K}/\mathbb{Q}_p)| = [\mathbb{K} : \mathbb{Q}_p]$ (cf. [64, 68]).

Let $\mathbb{K}/\mathbb{Q}_p$ be a finite normal extension of $\mathbb{Q}_p$, and let $|\cdot|_{\mathbb{K}}$ be a valuation in $\mathbb{K}$. If $\omega \colon \mathbb{K} \to \mathbb{K}$ is a field automorphism of $\mathbb{K}/\mathbb{Q}_p$, it is clear that the map $x \mapsto |\omega(x)|_p$ defines a valuation in $\mathbb{K}$. Hence, due to uniqueness (see (ii) in Proposition 1.1.5), it must be true that $|x|_{\mathbb{K}} = |\omega(x)|_p$, for every $x \in \mathbb{K}$. Multiplying over all the ω in Ω (and recalling that $|\Omega| = n = \left[\mathbb{K} : \mathbb{Q}_p\right]$, see Remark 1.1.8), we have:

$$|x|_{\mathbb{K}}^n = \left| \prod_{\omega \in \Omega} \omega(x) \right|_p . \tag{1.22}$$

On the other hand, exploiting definition (C3) of the norm function, we also have that

$$|x|_{\mathbb{K}} = \sqrt[n]{\left|\mathrm{N}_{\mathbb{K}/\mathbb{Q}_p}(x)\right|_p}, \tag{1.23}$$

i.e., from (1.23) we obtain an explicit formula through which express the absolute value on $\mathbb{K}$ in terms of the valuation on $\mathbb{Q}_p$. Actually, we can prove the following

[1] One refers to such a field as to the *field generated by* x over $\mathbb{Q}_p$ [102].

Theorem 1.1.3. *Let $\mathbb{K}/\mathbb{Q}_p$ be a finite extension of degree $n = \left[\mathbb{K} : \mathbb{Q}_p\right]$. Then, the map defined as*

$$\mathbb{K} \ni x \mapsto |x| := \sqrt[n]{\left|\mathrm{N}_{\mathbb{K}/\mathbb{Q}_p}(x)\right|_p} \in \mathbb{R}^+ \tag{1.24}$$

provides a non-Archimedean *valuation on $\mathbb{K}$ which extends the p-adic absolute value $|\cdot|_p$ of $\mathbb{Q}_p$.*

Proof. It is clear that $x \mapsto \sqrt[n]{\left|\mathrm{N}_{\mathbb{K}/\mathbb{Q}_p}(x)\right|_p}$ gives a valuation on $\mathbb{K}$: It factorizes under the product (cf. Remark 1.1.7), is positive definite, and $|x| = 0 \iff \mathrm{N}_{\mathbb{K}/\mathbb{Q}_p}(x) = 0 \iff x = 0$. Moreover, if $\alpha \in \mathbb{Q}_p$, then $\mathrm{N}_{\mathbb{K}/\mathbb{Q}_p}(\alpha) = \alpha^n$ (cf. definition (C1) of the norm function), and, hence,

$$|\alpha| = \sqrt[n]{|\alpha|_p^n} = |\alpha|_p\,. \tag{1.25}$$

i.e., $\sqrt[n]{\left|\mathrm{N}_{\mathbb{K}/\mathbb{Q}_p}(x)\right|_p}$ extends the p-adic valuation of $\mathbb{Q}_p$ to $\mathbb{K}$. Moreover, Theorem 6.3.5 in [68], also shows that $|\cdot|$ satisfies the strong triangle inequality. Concluding, the map (1.24) gives the (unique) non-Archimedean valuation on $\mathbb{K}$ which extends the p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}_p$. □

In what follows, whenever no confusion would arise with the ordinary Archimedean absolute value, we shall denote the extended valuation on $\mathbb{K}/\mathbb{Q}_p$ simply by $|\cdot|$.

Theorem 1.1.3 provides an explicit formula for the extension of the p-adic valuation on $\mathbb{K}/\mathbb{Q}_p$. Let us now particularize it to the special case of a quadratic extension $\mathbb{Q}_{p,\mu}$ of $\mathbb{Q}_p$. Here, we have only two field automorphisms on $\mathbb{Q}_{p,\mu}$, namely, the identity $\omega_{\mathsf{i}} \colon \mathbb{Q}_{p,\mu} \ni z \mapsto z \in \mathbb{Q}_{p,\mu}$, and the involutive automorphism ω_{c} defined as

$$\omega_{\mathsf{c}} \colon \mathbb{Q}_{p,\mu} \ni z = (x + \sqrt{\mu}y) \mapsto \overline{z} = (x - \sqrt{\mu}y) \in \mathbb{Q}_{p,\mu}. \tag{1.26}$$

Hence, exploiting the result in Theorem 1.1.3 we can argue that the (unique) extension of the p-adic valuation to $\mathbb{Q}_{p,\mu}$ is given by

$$|z| := \sqrt{\left|\mathrm{N}_{\mathbb{K}/\mathbb{Q}_p}(x)\right|_p} = \sqrt{|\omega_{\mathsf{i}}(z)\omega_{\mathsf{c}}(z)|_p} = \sqrt{|z\overline{z}|_p}, \tag{1.27}$$

for every $z \in \mathbb{Q}_{p,\mu}$. In what follows, we will call

$$x = \mathfrak{sc}(z) \equiv (z + \overline{z})/2 = \mathfrak{sc}(\overline{z}) \quad \text{and} \quad y = \mathfrak{ac}(z) \equiv (z - \overline{z})/2\sqrt{\mu} = -\mathfrak{ac}(\overline{z}) \tag{1.28}$$

the *selfconjugate* and the *anticonjugate* coordinate of $z \in \mathbb{Q}_{p,\mu}$, respectively.

Remark 1.1.9. Putting $\mathbb{Q}_{p,\mu}^* \equiv \mathbb{Q}_{p,\mu} \setminus \{0\}$, the set $\left|\mathbb{Q}_{p,\mu}^*\right| := \{|\alpha| \mid \alpha \in \mathbb{Q}_{p,\mu}^*\}$ is a *discrete subgroup* of the multiplicative group of all positive reals, called the *valuation group* of $\mathbb{Q}_{p,\mu}$. By relation (1.27), it is clear that $|\mathbb{Q}_{p,\mu}^*| \subset \{p^{k/2}\}_{k\in\mathbb{Z}}$. If we have a *strict* containment — i.e., if $|\mathbb{Q}_{p,\mu}^*| = \{p^k\}_{k\in\mathbb{Z}} = |\mathbb{Q}_p| \setminus \{0\}$ — then $\mathbb{Q}_{p,\mu}$ is said to be an *unramified extension* of $\mathbb{Q}_p$; otherwise (i.e., if $\left|\mathbb{Q}_{p,\mu}^*\right| = \{p^{k/2}\}_{k\in\mathbb{Z}}$), the quadratic extension $\mathbb{Q}_{p,\mu}$ is called (totally) *ramified*. It can be shown that, for every prime number p, there is — up to isomorphisms — exactly one unramified quadratic extension of $\mathbb{Q}_p$. E.g., $\mathbb{Q}_{2,5}$ is the only unramified quadratic extension of $\mathbb{Q}_2$ (up to isomorphisms). See Chapt. 7 of [98], Chapt. 5 of [68] and Chapt. 2 of [70].

Remark 1.1.10. It is worth noting that finite field extensions of $\mathbb{Q}_p$ show a structure that is much more involved than the one encountered in the case of the real numbers $\mathbb{R}$. Indeed, it is a well known result that all the (finite) field extensions of $\mathbb{R}$ actually coincide with the quadratic extension $\mathbb{C} \equiv \mathbb{R}(\sqrt{-1})$; in particular, $\mathbb{C}$ turns out to be algebraically closed (as a field) and complete (as a metric space). When switching to the p-adic setting, instead, this simple structure is no longer observed. In fact, finite extensions of $\mathbb{Q}_p$ are not reduced to a single quadratic extension: Quadratic extensions (and even considering extensions of higher orders) of $\mathbb{Q}_p$ are, in general, *not isomorphic* to each other. Further, non of the finite extensions of $\mathbb{Q}_p$ is algebraically closed. One can prove that the algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ is an extension of infinite degree (i.e., $[\overline{\mathbb{Q}}_p : \mathbb{Q}_p] = \infty$) which is, however, not complete as a metric space. A famous result due to Krasner shows that, by completing $\overline{\mathbb{Q}}_p$, one obtains a new complete field which also algebraically closed. It the literature, it is usually denoted by $\mathbb{C}_p$ and referred to as the field of *p-adic complex numbers*.

1.2 Inverse limit structure of $\mathbb{Q}_p$

In Subsection 1.1 we have defined the field of p-adic numbers $\mathbb{Q}_p$ as the metric completion of the field of rational numbers $\mathbb{Q}$ w.r.t. a suitable metric, namely, the metric induced by the p-adic valuation. Actually, this is not the only way this field can be defined. Another well known approach consists to define the ring of p-adic integers $\mathbb{Z}_p$ as a suitable inverse limit of an inverse family of rings, and then regard $\mathbb{Q}_p$ as the *fraction field* of $\mathbb{Z}_p$ [105, 106]. This is precisely the route we will follow in this subsection. In the meanwhile, this also gives us the opportunity to introduce the notions of *inverse limit* and *inverse family* of groups, rings and measure spaces which will play an important role in our later derivations.

Let $(\mathsf{N}, \leq)$ be a *(right-)directed partially ordered set*, i.e., a non-empty set N supplied with a reflexive, transitive and antisymmetric binary relation $\leq$, such that any subset of two elements of N is bounded (above). We first recall the definition of inverse family and inverse limit of sets [107], and topological groups [108] (see [98, 106, 109] for a more categorical approach).

Definition 1.2.1. Let $\{X_n\}_{n\in\mathsf{N}}$ be a family of sets (resp. topological groups), and $\{f_{nm}\colon X_m \to X_n\}_{n\leq m,\ n,m\in\mathsf{N}}$ a family of maps (resp. continuous group homomorphisms) such that

1. f_{nn} is the identity map on X_n, for every $n \in \mathsf{N}$,
2. $f_{nl} = f_{nm} \circ f_{ml}$, for every $n \leq m \leq l,\ \ n, m, l \in \mathsf{N}$.

We call $\big\{\{X_n\}_{n\in\mathsf{N}}, \{f_{nm}\colon X_m \to X_n\}_{n\leq m,\ n,m\in\mathsf{N}}\big\} \equiv \{X_n, f_{nm}\}_{\mathsf{N}}$ an *inverse family of sets* (resp. *of topological groups*). Let now $\prod_{n\in\mathsf{N}} X_n$ be the Cartesian product of the family of sets $\{X_n\}_{n\in\mathsf{N}}$. The *inverse* (or *projective*) *limit* of the inverse family of sets $\{X_n, f_{nm}\}_{\mathsf{N}}$ is

$$\begin{aligned}\varprojlim\{X_n, f_{nm}\}_{\mathsf{N}} &:= \left\{x \in \prod_{n\in\mathsf{N}} X_n \mid \mathrm{Pr}_n(x) = f_{nm} \circ \mathrm{Pr}_m(x), \text{for every } n \leq m,\ n, m \in \mathsf{N}\right\} \\ &= \left\{(x_n)_{n\in\mathsf{N}} \in \prod_{n\in\mathsf{N}} X_n \mid x_n = f_{nm}(x_m), \text{for every } n \leq m,\ n, m \in \mathsf{N}\right\} \subseteq \prod_{n\in\mathsf{N}} X_n, \qquad (1.29)\end{aligned}$$

where $\mathrm{Pr}_n\colon \prod_{n\in\mathsf{N}} X_n \to X_n,\ x = (x_n)_{n\in\mathsf{N}} \mapsto x_n$ is the canonical projection on the n-th component.

The *inverse limit* of the inverse family of topological groups $\{X_n, f_{nm}\}_{\mathsf{N}}$ is the subgroup of the direct product group $\prod_{n\in\mathsf{N}} X_n$ as in (1.29) endowed with the coarser topology for which Pr_n is continuous for every $n \in \mathsf{N}$, coinciding with the topology induced by the product topology of $\prod_{n\in\mathsf{N}} X_n$.

It is possible to prove that the inverse limit of an inverse family of sets or topological groups always exists (this would not be true, in general, in the broader setting of an inverse family in an arbitrary category), and unique [107, 108]. In particular, uniqueness of the inverse limit actually means that if X and X' are two inverse limits of a same inverse family (with projection maps $\{\mathrm{Pr}_n\}_n$ and $\{\mathrm{Pr}'_n\}_n$ respectively), then there exists a *unique* isomorphism $f\colon X \to X'$ such that $\mathrm{Pr}'_n \circ f = \mathrm{Pr}_n$ for every $n \in \mathsf{N}$.

Next, we recall the notion of inverse family of measure spaces

Definition 1.2.2. An *inverse family of measure spaces* is a family $\{(X_n, S_n, \mu_n), f_{nm}\}_{\mathsf{N}}$ of measure spaces such that

1. $\{X_n, f_{nm}\}_{\mathsf{N}}$ is an inverse family of sets,
2. f_{nm} is *measure preserving*, i.e., for $n < m$, $f_{nm}^{-1}(S_n) \subseteq S_m$ and for $E_n \in S_n$, $\mu_n(E_n) = \mu_m\left(f_{nm}^{-1}(E_n)\right)$.

A special case of our interest is the inverse family of measure spaces considered over a locally compact group G. Indeed, if G is a locally compact group, the inverse limit of left (resp. right) Haar measures on a suitable inverse family of quotient groups is proven to be the left (resp. right) Haar measure on the inverse limit group G [110]. As we will see, one can exploit this result to construct the Haar measure of the p-adic special orthogonal groups (cf. Remark 5.5.3).

We now focus on our main concern to give an inverse limit characterization to $\mathbb{Q}_p$. For any integer $n \geq 1$ and every fixed prime number $p \in \mathbb{N}$, let us consider the cyclic group $\mathbb{Z}/p^n\mathbb{Z}$ of order p^n (i.e., $\mathbb{Z}/p^n\mathbb{Z}$ is the ring of classes of integers mod p^n). An element of $\mathbb{Z}/p^n\mathbb{Z}$ naturally defines and element of $\mathbb{Z}/p^m\mathbb{Z}$ (for $m \leq n$) via the map $f_{mn}\colon \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}$ defined as

$$f_{mn}(1 + p^n\mathbb{Z}) = 1 + p^m\mathbb{Z}. \tag{1.30}$$

The maps f_{mn} are surjective homomorphisms such that $\ker(f_{mn}) = p^{m-1}\mathbb{Z}/p^m\mathbb{Z}$. The set $\{\mathbb{Z}/p^n\mathbb{Z},\ f_{mn}\}_{\mathbb{N}}$ defines an inverse system; in particular, we can set the following

Definition 1.2.3. The inverse limit $\varprojlim\{\mathbb{Z}/p^n\mathbb{Z},\ f_{mn}\}_{\mathbb{N}}$ of the inverse system $\{\mathbb{Z}/p^n\mathbb{Z},\ f_{mn}\}_{\mathbb{N}}$ is the ring of p-adic integers $\mathbb{Z}_p$, i.e.,

$$\mathbb{Z}_p = \varprojlim\{\mathbb{Z}/p^n\mathbb{Z},\ f_{mn}\}_{\mathbb{N}}. \tag{1.31}$$

By definition, any element of $\mathbb{Z}_p$ is a sequence $x = (\ldots, x_n, \ldots, x_1)$ with $x_n \in \mathbb{Z}/p^n\mathbb{Z}$. Addition and multiplication in $\mathbb{Z}_p$ are defined componentwise i.e., $\mathbb{Z}_p$ is a subring of $\Pi_n\mathbb{Z}/p^n\mathbb{Z}$.

Remark 1.2.1. If we endow $\mathbb{Z}/p^n\mathbb{Z}$ with the discrete topology and $\Pi_n\mathbb{Z}/p^n\mathbb{Z}$ with the product topology, then the topology inherited by $\mathbb{Z}_p$ turns it into a compact space.

Let $\mathrm{Pr}_n\colon \mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}$ denote the projection homomorphism. Then, we have that the sequence

$$0 \longrightarrow \mathbb{Z}_p \xrightarrow{p^n} \mathbb{Z}_p \xrightarrow{\mathrm{Pr}_n} \mathbb{Z}/p^n\mathbb{Z} \longrightarrow 0 \tag{1.32}$$

is an exact sequence of abelian groups [74]; hence, $\mathbb{Z}_p/p^n\mathbb{Z}_p \cong \mathbb{Z}/p^n\mathbb{Z}$. In particular, by noting that $\mathbb{Z}_p$ is an *integral domain*, we can then set the following

Definition 1.2.4. The field of p-adic numbers $\mathbb{Q}_p$ is the *field of fractions* of the ring $\mathbb{Z}_p$

Remark 1.2.2. It is worth observing that Definition 1.2.4 provides an inverse limit characterization of $\mathbb{Q}_p$ via the inverse limit definition of the ring of p-adic integers $\mathbb{Z}_p$. Actually, it is possible to prove that $\mathbb{Q}_p$ — as well as all its proper closed subgroups — can be *directly characterised* as the inverse limit of a suitable inverse family. Specifically, the following isomorphisms of topological groups hold

$$\mathbb{Q}_p \simeq \varprojlim\left\{\mathbb{Q}_p/p^n\mathbb{Z}_p,\ \phi_{nl}\right\}_{\mathbb{Z}_{>0}}, \qquad p^m\mathbb{Z}_p \simeq \varprojlim\left\{p^m\mathbb{Z}_p/p^n\mathbb{Z}_p,\ \phi_{nl}\right\}_{\mathbb{Z}_{>m}}, \quad m \in \mathbb{Z} \tag{1.33}$$

Here, we are assuming $\mathbb{Q}_p$ equipped with the p-adic ultrametric topology, $p^m\mathbb{Z}_p$ with the subspace topology, and the quotient groups with the quotient topologies (coinciding with the discrete topologies); the continuous group homomorphisms ϕ_{nl} are defined as

$$x_{(l)} + p^l\mathbb{Z}_p \mapsto x_{(l)} + p^n\mathbb{Z}_p, \tag{1.34}$$

for every $l \geq n > m$. For further details on this point, the reader may refer to, e.g., Proposition II.6 in [89].

2

Quadratic forms and the p-adic special orthogonal groups

We introduce quadratic forms over the field of p-adic numbers $\mathbb{Q}_p$, and construct the classes of linear transformations which preserve them, namely, the p-adic special orthogonal groups.

2.1 Quadratic forms on $\mathbb{Q}_p$

We start by reminding the general definition of a quadratic form over a $\mathbb{Q}_p$-vector space [71, 74, 98, 111, 112]:

Definition 2.1.1. Let V be a vector space over $\mathbb{Q}_p$. A function $\mathrm{Q}\colon V \to \mathbb{Q}_p$ is called a *quadratic form* on V if

(1) $\mathrm{Q}(\alpha x) = \alpha^2 \mathrm{Q}(x)$, for every $\alpha \in \mathbb{Q}_p$, $x \in V$;

(2) The map $(x, y) \mapsto \mathrm{Q}(x + y) - \mathrm{Q}(x) - \mathrm{Q}(y)$ is a *bilinear form*.

In what follows, we will always assume V of finite dimension. Let us now introduce the map $\langle \cdot\,, \cdot \rangle \colon V \times V \to \mathbb{Q}_p$ defined as

$$(x, y) \mapsto \langle x, y \rangle \coloneqq \frac{1}{2}\Big(\mathrm{Q}(x + y) - \mathrm{Q}(x) - \mathrm{Q}(y)\Big), \tag{2.1}$$

which is a *symmetric bilinear form* on V. In the literature, it is usually referred to as the *scalar product associated with the quadratic form* Q. Using (2) in Definition 2.1.1, one can easily verify that $\mathrm{Q}(x) = \langle x, x \rangle$; namely, quadratic forms on V are in bijective correspondence with symmetric bilinear forms.

Remark 2.1.1. It is worth observing that a relation like the one in (2.1) would not hold in a general field $\mathbb{F}$; e.g., in the case of a field $\mathbb{F}$ of characteristic 2, where 2 is not an invertible element in field.

Let V, W be two $\mathbb{Q}_p$-vector spaces, and Q, Q′ two quadratic forms defined on V and W respectively. A linear map $\sigma\colon V \to W$ such that

$$\mathrm{Q}' \circ \sigma = \mathrm{Q} \tag{2.2}$$

is called a *morphism* (or *metric morphism*) of V into W. In this case, we have:

$$\langle x, y \rangle = \langle \sigma(x), \sigma(y) \rangle\,, \quad \forall x, y \in V. \tag{2.3}$$

Let now $\mathbf{v} = \{v_i\}_{i=1}^n$ be a basis in V; If $x = \sum_{i=1}^n x_i v_i$, recalling that $\mathrm{Q}(x) = \langle x, x \rangle$, we have:

$$\mathrm{Q}(x) = \left\langle \sum_{i=1}^n x_i v_i, \sum_{j=1}^n x_j v_j \right\rangle = \sum_{i,j=1}^n x_i x_j \langle v_i, v_j \rangle = \sum_{i,j=1}^n x_i x_j \mathrm{Q}_{ij}, \tag{2.4}$$

where we set $\mathrm{Q}_{ij} \equiv \langle v_i, v_j \rangle$. The matrix $\widehat{\mathrm{Q}} := (\mathrm{Q}_{ij})_{i,j}$ — with entries $\mathrm{Q}_{ij} \equiv \langle v_i, v_j \rangle$ — is usually called the *matrix representation of the quadratic form* Q w.r.t. the basis $\mathbf{v}$. If $\mathbf{v}' = \{v'_i\}_{i=1}^n$ is a new basis of V, and if $U = (U_{ij})_{i,j}$ is an invertible matrix in $\mathrm{GL}(n, \mathbb{Q}_p)$ such that $v'_i = \sum_j U_{ij} v_j$, the matrix $\widehat{\mathrm{Q}}'$ of Q — w.r.t. the new basis $\mathbf{v}'$ — is given by

$$\widehat{\mathrm{Q}}' = U \widehat{\mathrm{Q}} U^{\intercal}, \tag{2.5}$$

where $U^{\intercal}$ denotes, as usual, the transpose of the matrix U. In particular, from this it follows that

$$\det(\widehat{\mathrm{Q}}') = \det(U \widehat{\mathrm{Q}} U^{\intercal}) = \det(U)^2 \det(\widehat{\mathrm{Q}}), \tag{2.6}$$

i.e., $\det(\widehat{\mathrm{Q}})$ is determined, in general, up to a multiplicative factor in $(\mathbb{Q}_p^*)^2$. Adhering to the standard terminology, we call $\det(\widehat{\mathrm{Q}})$ the *discriminant* of Q, and denote it by $\mathrm{d}(\mathrm{Q})$ [74].

Definition 2.1.2. Let V be an n-dimensional vector space over $\mathbb{Q}_p$, and let Q, Q′ be two quadratic forms on V. We say that Q is *equivalent* to Q′ — and we write $\mathrm{Q} \sim \mathrm{Q}'$ — if there exists an invertible matrix $U \in \mathrm{GL}(n, \mathbb{Q}_p)$ such that

$$\widehat{\mathrm{Q}}' = U \widehat{\mathrm{Q}} U^{\intercal}. \tag{2.7}$$

It is clear that condition (2.7) provides, indeed, an equivalence relation between quadratic forms on $\mathbb{Q}_p$. If Q is a quadratic form, we say that two vectors x, y in V are *orthogonal* w.r.t. Q — and we write $x \perp y$ — if

$$\langle x, y \rangle = 0, \tag{2.8}$$

namely, if they are orthogonal w.r.t. the scalar product associated with Q. If $H \subset V$ is a vector subspace of V, we denote by $H^\perp := \{x \in V \mid \langle x, h \rangle = 0,\ \forall h \in H\}$ the *orthogonal complement* of H in V; using bilinearity of $\langle \cdot, \cdot \rangle$, one can easily check that $H^\perp$ is a vector subspace of V. The orthogonal complement $V^\perp$ of V itself is called the *radical* of V, and denoted by $\mathrm{rad}(V)$; its *codimension* $\mathrm{codim}(\mathrm{rad}(V))$, defined as

$$\mathrm{codim}\big(\mathrm{rad}(V)\big) = \dim(V) - \dim(V^\perp), \tag{2.9}$$

is called the *rank* of Q, and denoted by $\mathrm{r}(\mathrm{Q})$. If $V^\perp = 0$, we say that Q is *nondegenerate* (in which case we have $\mathrm{r}(\mathrm{Q}) = n = \dim(V)$); this is equivalent to say that $\mathrm{d}(\mathrm{Q}) \neq 0$. In this case, recalling (2.6), we also have that $\mathrm{d}(Q) \in \mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$ is an element of the group of the square classes of $\mathbb{Q}_p$.

We call a vector x in V *isotropic* if $\mathrm{Q}(x) = 0$; accordingly, we say that a subspace $W \subset V$ is isotropic if all its elements are isotropic, namely, if $W \subset W^\perp$. If Q is nondegenerate, and if V contains a nonzero isotropic vector, then $\mathrm{Q}(V) = \mathbb{Q}_p$; that is, for every $\alpha \in \mathbb{Q}_p$ there exists a vector $v \in V$ such that $\mathrm{Q}(v) = \alpha$ (see Corollary of Chap. IV, n° 1.3 in [74]).

Let $\mathbf{e} \equiv \{e_i\}_{i=1}^n$ be a basis in V. We say that it is an *orthogonal basis* if its elements are pairwise orthogonal

$$\langle e_i, e_j \rangle = \delta_{ij}, \quad i, j = 1, \ldots, n, \tag{2.10}$$

where $\langle \cdot, \cdot \rangle \colon V \times V \to \mathbb{Q}_p$ is the bilinear form defined in (2.1).

If $\mathbf{e} \equiv \{e_i\}_{i=1}^n$ is an orthogonal basis in V, recalling (2.4), we see that the matrix representation of Q — w.r.t. $\mathbf{e}$ — is reduced to a diagonal form, namely,

$$\widehat{\mathrm{Q}} = \begin{pmatrix} \mathrm{Q}_{11} & 0 & \ldots & 0 \\ 0 & \mathrm{Q}_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathrm{Q}_{nn} \end{pmatrix}. \tag{2.11}$$

In such a case, we also see that, expressing $x \in V$ as $x = \sum_i x_i e_i$, we have:

$$\mathrm{Q}(x) = \mathrm{Q}_{11} x_1^2 + \mathrm{Q}_{22} x_2^2 + \cdots + \mathrm{Q}_{nn} x_n^2. \tag{2.12}$$

Remark 2.1.2. Given a $\mathbb{Q}_p$-vector space V, it is always possible to construct an orthogonal basis in it (cf. Theorem 1, Chap. IV, n° 1.4 in [74]). Therefore, without loss of generality, we can always assume a quadratic form on V to be written in a diagonal form.

Definition 2.1.2 provides, via condition (2.7), an equivalence relation of quadratic forms on V. Our next concern is now to characterize the equivalence classes of *n-variable* quadratic forms on $\mathbb{Q}_p$. To this end, we first discuss the broader setting of equivalent quadratic forms on a generic $\mathbb{Q}_p$-vector space V. Here, we shall see that the characterization of the equivalence classes is provided once a complete set of *invariants* of quadratic forms is known. Then, we switch to the special case where $V = \mathbb{Q}_p^n$, hence characterizing equivalent classes of n-variables quadratic forms.

For our purposes, we have to preliminary introduce the notions of *Legendre* and *Hilbert* symbols [74, 98, 100, 111]. Let p be an odd prime number, and let a be an integer prime to p (i.e., $a \in \mathbb{F}_p^*$). The *quadratic residue symbol* or (*Legendre symbol*) $\left(\frac{a}{p}\right)$ is defined as

$$\left(\frac{a}{p}\right) := \begin{cases} 1 & \text{if there exists a square root of } a \text{ in } \mathbb{F}_p^* \\ -1 & \text{otherwise.} \end{cases} \tag{2.13}$$

It is clear that the residue symbol provides an isomorphism between $\mathbb{F}_p^*/(\mathbb{F}_p^*)^2$ and $\{\pm 1\}$; i.e., $\left(\frac{a}{p}\right)$ is the image of the class of a in $\mathbb{F}_p^*/(\mathbb{F}_p^*)^2$ into $\{\pm 1\}$. Note that, this also entails that the Legendre symbol is multiplicative, i.e., if $a, b \in \mathbb{F}_p^*$, we have:

$$\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right). \tag{2.14}$$

Using the Legendre symbol, it is not difficult to prove the following facts (cf. Theorem 2.2 in [100]):

(a) If p, q are two odd primes such that $p \neq q$, then the residue symbols of p and q are related via the so-called *quadratic reciprocity law*, namely,

$$\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}}\left(\frac{p}{q}\right). \tag{2.15}$$

(b) -1 is a square in $\mathbb{F}_p^*$ if and only if $p \equiv 1 \bmod 4$, and it is not a square for $p \equiv 3 \bmod 4$. That is, the so-called *first supplementary law* is satisfied:

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} = \begin{cases} 1 & \text{iff } p \equiv 1 \bmod 4 \\ -1 & \text{iff } p \equiv 3 \bmod 4. \end{cases} \tag{2.16}$$

(c) 2 is a square in $\mathbb{F}_p^*$ if and only if $p \equiv 1, 7 \bmod 8$, and there is no square root of 2 in $\mathbb{F}_p^*$ whenever $p \equiv 3, 5 \bmod 8$; namely, the *second supplementary law* is verified

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}} = \begin{cases} 1 & \text{iff } p \equiv 1, 7 \bmod 8 \\ -1 & \text{iff } p \equiv 3, 5 \bmod 8. \end{cases} \tag{2.17}$$

(d) If a, b are two integers in $\mathbb{F}_p^*$, we have:

$$\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right) \tag{2.18}$$

whenever $a \equiv b \bmod p$.

Relations (a)-(d) provide useful tools for the explicit computation of the residue symbol. For instance, take $a = 162$ and $p = 17$. We have:

$$\left(\frac{162}{17}\right) = \left(\frac{2}{17}\right)\left(\frac{81}{17}\right) = \left(\frac{13}{17}\right) = \left(\frac{17}{13}\right) = \left(\frac{4}{13}\right) = 1. \tag{2.19}$$

Here, in the first equality, we have made use of property (2.14); in the second one, we have exploited (c) and (d) ($17 \equiv 1 \bmod 8$, and $81 \equiv 13 \bmod 17$), while the third follows from (a). Since $17 \equiv 4 \bmod 13$, condition (d) provides the fourth equality. Eventually, resorting again to (2.14), we have that $\left(\frac{4}{13}\right) = \left(\frac{2}{13}\right)\left(\frac{2}{13}\right) = 1$, thus arriving at the last equality.

We now turn our attention to the notion of *Hilbert symbol*.

Definition 2.1.3. Let $a, b \in \mathbb{Q}_p^*$. The Hilbert symbol $(a,b)_{\mathrm{H}}$ of a and b is defined as

$$(a,b)_{\mathrm{H}} := \begin{cases} 1 & \text{iff } z^2 - ax^2 - by^2 = 0 \text{ admits a solution } (x,y,z) \neq (0,0,0) \text{ in } \mathbb{Q}_p^3 \\ -1 & \text{otherwise.} \end{cases} \tag{2.20}$$

The following proposition collects the most relevant properties of the Hilbert symbol

Proposition 2.1.1. *Let $a, b, c \in \mathbb{Q}_p^*$. The following relations hold*

(H1) $(a,b)_{\mathrm{H}} = (b,a)_{\mathrm{H}}$, *and* $(a,c^2)_{\mathrm{H}} = 1$*;*

(H2) $(a,-a)_{\mathrm{H}} = 1$, *and, for* $a \neq -1$, $(a, 1-a)_{\mathrm{H}} = 1$*;*

(H3) $(a,bc)_{\mathrm{H}} = (a,b)_{\mathrm{H}}(a,c)_{\mathrm{H}}$.

Proof. See Proposition 2.4 in [100], and Proposition 2, Chap III, n° 1.1, in [74]. □

From Definition 2.1.3, it is clear that the Hilbert symbol is defined up to a quadratic element in $\mathbb{Q}_p^*$. Indeed, let $a, b \in \mathbb{Q}_p^*$ be such that $(a,b)_{\mathrm{H}} = 1$. This means that there exists $(x,y,z) \neq (0,0,0)$ in $\mathbb{Q}_p^3$ such that

$$z^2 - ax^2 - by^2 = 0. \tag{2.21}$$

On the other hand, if $c \in \mathbb{Q}_p^*$, also (cx, cy, cz) is a solution of (2.21), namely

$$z^2 - ax^2 - by^2 = 0 \quad \Longrightarrow \quad c^2z^2 - (ac^2)x^2 - (bc^2)y^2 = 0, \tag{2.22}$$

from which it follows that $(ac^2, bc^2) = 1$. Conversely, if $(ac^2, bc^2) = 1$, for $c \in \mathbb{Q}_p^*$, then

$$\begin{aligned} &z^2 - ac^2x^2 - bc^2y^2 = 0 \qquad (\text{for } (x,y,z) \neq (0,0,0) \text{ in } \mathbb{Q}_p^3) \\ &z^2 - a(cx)^2 - b(cy)^2 = 0 \qquad (\text{for } (cx,cy,z) \neq (0,0,0) \text{ in } \mathbb{Q}_p^3), \end{aligned} \tag{2.23}$$

i.e., $(a,b)_{\mathrm{H}} = 1$. A similar reasoning can be repeated also in case where $(a,b)_{\mathrm{H}} = -1$. In particular, this shows that the Hilbert symbol defines a map from $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2 \times \mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$ into $\{\pm 1\}$.

We recall that, as seen in Remark 1.1.2, every p-adic number can be expressed in the form $x = p^k u(x)$, for $k \in \mathbb{Z}$, and $u(x)$ a p-adic unit. The next result provides useful formulas to explicitly compute the Hilbert symbol

Theorem 2.1.1. *Let $a, b \in \mathbb{Q}_p^*$, and let $a = p^\alpha u(a)$ and $b = p^\beta u(b)$, for $\alpha, \beta \in \mathbb{Z}$, and $u(a), u(b)$ p-adic units in $\mathbb{U}_p$. We have:*

$$(a,b)_{\mathrm{H}} = (-1)^{\alpha\beta\frac{p-1}{2}} \left(\frac{\overline{u}(a)}{p}\right)^{\beta} \left(\frac{\overline{u}(b)}{p}\right)^{\alpha}, \tag{2.24}$$

for $p \neq 2$, and

$$(a,b)_{\mathrm{H}} = (-1)^{\frac{u(a)-1}{2}\cdot\frac{u(b)-1}{2}+\alpha\frac{u(b)^2-1}{8}+\beta\frac{u(a)^2-1}{8}}, \tag{2.25}$$

for $p = 2$. Here, $\overline{u}(a)$ and $\overline{u}(b)$ denote respectively the images of $u(a)$ and $u(b)$ via the homomorphism $\pi\colon \mathbb{U}_p \to \mathbb{F}_p^*$ of reduction $\bmod\ p$, i.e., $\overline{u}(a) = \pi(u(a))$ (resp. $\overline{u}(b) = \pi(u(b))$).

Proof. See Theorem 1, Chap. III, n° 1.2 in [74]. □

Let now V be a $\mathbb{Q}_p$-vector space, and let Q be a quadratic form with rank $\mathrm{r}(\mathrm{Q}) = n = \dim(V)$. If $\mathbf{e} \equiv \{e_i\}_{i=1}^n$ is an orthogonal basis in V, let $\widehat{\mathrm{Q}} = \mathrm{diag}(q_1, \ldots, q_n)$, $q_1, \ldots, q_n \in \mathbb{Q}_p^*$, be the matrix representation of Q w.r.t. $\mathbf{e}$, and let $\mathrm{d}(\mathrm{Q}) = \prod_{i=1}^n q_i$ be its discriminant. We can introduce the quantity $\epsilon(\mathrm{Q})$ defined as

$$\epsilon(\mathrm{Q}) \coloneqq \prod_{i<j}^{n} (q_i, q_j)_{\mathrm{H}} \quad (\text{in } \mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2). \tag{2.26}$$

From Definition 2.1.3, it is clear that $\epsilon(\mathrm{Q}) \in \{\pm 1\}$. A well known result shows that $\epsilon(\mathrm{Q})$ does not depend from the choice of the orthogonal basis in V. Then, recalling Definition 2.1.2, it is clear that $\epsilon(\cdot)$ provides an *invariant* of the equivalence classes of quadratic forms on V (see Theorem 5, Chap. IV, n° 2.1 in [74]). In the literature, it is often called the *Hasse invariant* of the quadratic form Q, and provides a first invariant of equivalent forms. Actually, to discriminate between the possible equivalence classes, one needs of a *maximal set* of invariants. In particular, we have the following result

Theorem 2.1.2. *Let V be a $\mathbb{Q}_p$-vector space. Then, the discriminant, the rank and the Hasse invariant form a complete set of invariants for the equivalence classes of quadratic forms on V. Specifically, if Q and Q′ are two quadratic forms on V, they are equivalent — in the sense of Definition 2.1.2 — if and only if*

$$\mathrm{d}(\mathrm{Q}) = \mathrm{d}(\mathrm{Q}'), \quad \mathrm{r}(\mathrm{Q}) = \mathrm{r}(\mathrm{Q}'), \quad \textit{and} \quad \epsilon(\mathrm{Q}) = \epsilon(\mathrm{Q}'), \tag{2.27}$$

i.e., if and only if they have same rank, discriminant and Hasse invariant.

Proof. See Theorem 7 n° 2.3, Chap. IV in [74]. □

Theorem 2.1.2 provides a method for characterizing the classes of equivalent quadratic forms on a $\mathbb{Q}_p$-vector space V. Our next concern is now to particularize this result to the spacial case where $V \equiv \mathbb{Q}_p^n$, namely, to classify *n-variables quadratic forms* on $\mathbb{Q}_p$.

Let us first remind that an *n-variables quadratic form* on $\mathbb{Q}_p$ is a polynomial F of the form

$$\mathrm{F}(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_{ij} x_i x_j, \quad a_{ij} \in \mathbb{Q}_p, \tag{2.28}$$

namely, an n-variables *homogeneous polynomial* of degree 2 over $\mathbb{Q}_p$ [74, 111]. We can rewrite (2.28) in a more symmetric form by setting

$$\mathrm{F}(x_1, \ldots, x_n) = \sum_{i,j} a_{ij} x_i x_j = 2\sum_{i=1}^{n} \mathrm{F}_{ii} x_i^2 + 2\sum_{i<j} \mathrm{F}_{ij} x_i x_j, \tag{2.29}$$

where $\mathrm{F}_{ij} \coloneqq 1/2(a_{ij} + a_{ji})$; we call $\widehat{\mathrm{F}} = (\mathrm{F}_{ij})_{i,j}$ the matrix representation of F. In particular, denoting by $\mathbf{x} = (x_1, \ldots, x_n)$ a vector in $V \equiv \mathbb{Q}_p^n$, we also see that

$$\mathrm{F}(\mathbf{x}) \equiv \mathrm{F}(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} a_{ij} x_i x_j = \mathbf{x}\widehat{\mathrm{F}}\mathbf{x}^{\mathsf{T}}. \tag{2.30}$$

Let now F and F′ be two n-variables quadratic forms on $\mathbb{Q}_p$. Recalling Definition 2.1.2, we say that F and F′ are equivalent if there exists $M \in \mathrm{GL}(n, \mathbb{Q}_p)$ such that $\mathrm{F}(\mathbf{x}) = \mathrm{F}'(\mathbf{x}M)$ for all $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n$. Indeed, from (2.30), we have that

$$\begin{aligned} \mathrm{F}(\mathbf{x}) &= \mathbf{x}\widehat{\mathrm{F}}\mathbf{x}^\intercal = (\mathbf{x}M)\widehat{\mathrm{F}}(\mathbf{x}M)^\intercal \\ &= \mathbf{x}(M\widehat{\mathrm{F}}M^\intercal)\mathbf{x}^\intercal \\ &= \mathbf{x}\widehat{F}'\mathbf{x}^\intercal, \end{aligned} \tag{2.31}$$

i.e., F and F′ are equivalent if and only if $\widehat{F} = M\widehat{\mathrm{F}}'M^\intercal$, for some $M \in \mathrm{GL}(n, \mathbb{Q}_p)$.

Any n-variables quadratic from F on $\mathbb{Q}_p$ naturally induces a map $\mathrm{Q}_\mathrm{F}\colon \mathbb{Q}_p^n \to \mathbb{Q}_p$ — the so-called *quadratic map* associated with F — defined as

$$\mathbb{Q}_p^n \ni (x_1, \ldots, x_n) \equiv \mathbf{x} \mapsto \mathrm{Q}_\mathrm{F}(\mathbf{x}) := \mathbf{x}\widehat{\mathrm{F}}\mathbf{x}^\intercal; \tag{2.32}$$

it satisfies the following remarkable properties [111]:

(1) For every α in $\mathbb{Q}_p$ and any $\mathbf{x}$ in $\mathbb{Q}_p^n$, $\mathrm{Q}_\mathrm{F}(\alpha\mathbf{x}) = \alpha^2\mathrm{Q}_\mathrm{F}(\mathbf{x})$;

(2) The map $\mathrm{B}_\mathrm{F}(\mathbf{x}, \mathbf{y}) := 1/2(\mathrm{Q}_\mathrm{F}(\mathbf{x} + \mathbf{y}) - \mathrm{Q}_\mathrm{F}(\mathbf{x}) - \mathrm{Q}_\mathrm{F}(\mathbf{y}))$, $\mathbf{x}, \mathbf{y} \in \mathbb{Q}_p^n$ is a *symmetric bilinear form* on $\mathbb{Q}_p^n \times \mathbb{Q}_p^n$;

(3) $\mathrm{B}_\mathrm{F}(\mathbf{x}, \mathbf{x}) = \mathrm{Q}_\mathrm{F}(\mathbf{x})$.

Conditions (1)–(3) above entails that Q_F is a quadratic form in the sense of Definition 2.1.1 (with $V \equiv \mathbb{Q}_p^n$). Hence, all the results previously discussed for quadratic forms on a generic $\mathbb{Q}_p$-vector space can be applied to Q_F — and, hence, to F — as well. In particular, every n-variables quadratic form F can be assumed to be diagonal, i.e., written in the form $\mathrm{F}(\mathbf{x}) = f_1x_1^2 + \ldots + f_nx_n^2$, for some $f_1, \ldots, f_n \in \mathbb{Q}_p$. In this case, one also has that the discriminant, $\mathrm{d}(\mathrm{F})$ of F is expressed as $\mathrm{d}(\mathrm{F}) = \prod_{i=1}^n f_i$, while its rank $\mathrm{r}(\mathrm{F})$ coincides with the cardinality of the set $\mathfrak{R} := \{i \in \{1, \ldots, n\} \mid f_i \neq 0\}$ of non-null matrix entries of $\widehat{\mathrm{F}}$. In particular, $\mathrm{r}(\mathrm{F}) = n$ if and only if $\mathrm{d}(\mathrm{F}) \neq 0$; equivalently, if and only if F is nondegenerate. Expressing F in a diagonal form, we can define its Hasse invariant by setting

$$\epsilon(\mathrm{F}) := \prod_{i<j}(f_i, f_j)_\mathrm{H} \in \{\pm 1\}. \tag{2.33}$$

The next result is a natural consequence of Theorem 2.1.2

Corollary 2.1.1 (of Theorem 2.1.2)**.** *Two n-variables quadratic forms* F, F′ *on* $\mathbb{Q}_p$ *are* equivalent *if and only if they have the same rank, discriminant, and Hasse invariant* ϵ*.*

Let F be an n-variable quadratic form on $\mathbb{Q}_p^n$. We say that F represents an element $a \in \mathbb{Q}_p$ it there exists $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n$, $\mathbf{x} \neq 0$, such that $\mathrm{F}(\mathbf{x}) = a$. Of special interest for us is the case where F represents 0 in $\mathbb{Q}_p$. In particular, the following result holds [74]

Proposition 2.1.2. *Let* F *be a non-degenerate n-variables quadratic form on* $\mathbb{Q}_p$*. For* F *to represent* 0 *it is necessary and sufficient that*

(i) $n = 2$ *and* $\mathrm{d}(\mathrm{F}) = -1$*;*

(ii) $n = 3$ *and* $(-1, -\mathrm{d}(\mathrm{F}))_\mathrm{H} = \epsilon(\mathrm{F})$*;*

(iii) $n = 4$ *and either* $\mathrm{d}(\mathrm{F}) \neq 1$*, or* $\mathrm{d}(\mathrm{F}) = 1$ *and* $(-1, -1)_\mathrm{H} = \epsilon(\mathrm{F})$*.*

For $n \geq 5$*, all forms represent* 0 *in a non-trivial way.*

If there is no $\mathbf{x} \in \mathbb{Q}_p^n$, $\mathbf{x} \neq 0$, such that $\mathrm{F}(\mathbf{x}) = 0$ — that is, if F does not represent 0 — then we say that F is a *definite quadratic form*. Definite quadratic forms will play a central role in our later derivations; indeed, in the next section, we shall prove that they lead to *compact* special orthogonal groups on $\mathbb{Q}_p$.

To complete this subsection, we now proceed to explicitly classifying the different classes of quadratic forms on $\mathbb{Q}_p$. We will discuss separately the cases $p = 2$ and $p \neq 2$. Moreover, since we are interested in definite non-degenerate quadratic forms, we will assume that $\mathrm{r}(\mathrm{F}) = \dim(\mathbb{Q}_p^n) = 2, 3, 4$.

Let us start from the case $p \neq 2$. We recall that the group of square classes of $\mathbb{Q}_p$ is the group $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2 = \{1, \eta, p, \eta p\}$, for η a non-quadratic unit in $\mathbb{Q}_p$ — i.e., $\eta \notin (\mathbb{Q}_p^*)^2$, and $|\eta|_p = 1$ (see Corollary 1.1.4). Assuming the quadratic form to be diagonal, and recalling that the discriminant function takes values in the group of square classes of $\mathbb{Q}_p$, we are naturally led to consider only (diagonal) quadratic forms with entries in the group $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$.

Let us consider first $\mathrm{r}(\mathrm{F}) = 2 = \dim(\mathbb{Q}_p^n)$. In view of Corollary 2.1.1, the different classes of quadratic forms on $\mathbb{Q}_p$ are classified according to the possible values assumed by the rank, the discriminant and the Hasse invariant. Since we have already set $\mathrm{r}(\mathrm{F}) = 2$, we now proceed to determining the Hasse invariant. To facilitate the discussion, we treat separately the cases $p \equiv 1 \bmod 4$ and $p \equiv 3 \bmod 4$. Let us consider the $p \equiv 1 \bmod 4$ case first. We have 16 possible (diagonal) quadratic forms we can construct with coefficients in $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$; the Hilbert symbols of their matrix entries are explicitly given by:

$$
\begin{aligned}
&(1,1)_{\mathrm{H}} = \left(\frac{1}{p}\right)^0 \left(\frac{1}{p}\right)^0 = 1, && (1,\eta)_{\mathrm{H}} = \left(\frac{1}{p}\right)^0 \left(\frac{\eta}{p}\right)^0 = 1,\\
&(1,p)_{\mathrm{H}} = \left(\frac{1}{p}\right)^1 \left(\frac{1}{p}\right)^0 = 1, && (1,\eta p)_{\mathrm{H}} = \left(\frac{1}{p}\right)^1 \left(\frac{\eta}{p}\right)^0 = 1,\\
&(\eta,1)_{\mathrm{H}} = (1,\eta)_{\mathrm{H}} = 1, && (\eta,\eta)_{\mathrm{H}} = \left(\frac{\eta}{p}\right)^0 \left(\frac{\eta}{p}\right)^0 = 1\\
&(\eta,p)_{\mathrm{H}} = \left(\frac{\eta}{p}\right)^1 \left(\frac{1}{p}\right)^0 = -1, && (\eta,\eta p)_{\mathrm{H}} = \left(\frac{\eta}{p}\right)^1 \left(\frac{\eta}{p}\right)^0 = -1,\\
&(p,1)_{\mathrm{H}} = (1,p)_{\mathrm{H}} = 1, && (p,\eta)_{\mathrm{H}} = (\eta,p)_{\mathrm{H}} = -1,\\
&(p,p)_{\mathrm{H}} = \left(\frac{1}{p}\right)^1 \left(\frac{1}{p}\right)^1 = 1, && (p,\eta p)_{\mathrm{H}} = \left(\frac{1}{p}\right)^1 \left(\frac{\eta}{p}\right)^1 = -1,\\
&(\eta p,1)_{\mathrm{H}} = (1,\eta p)_{\mathrm{H}} = 1, && (\eta p,\eta)_{\mathrm{H}} = (\eta,\eta p)_{\mathrm{H}} = -1,\\
&(\eta p,p)_{\mathrm{H}} = (p,\eta p)_{\mathrm{H}} = -1, && (\eta p,\eta p)_{\mathrm{H}} = \left(\frac{\eta}{p}\right)^1 \left(\frac{\eta}{p}\right)^1 = 1.
\end{aligned}
\tag{2.34}
$$

In deriving (2.34) we have used the symmetry property `(H1)` of the Hilbert symbol, and relation (2.24) (here, for $p \equiv 1 \bmod 4$, we have $(-1)^{\alpha\beta\cdot\frac{p-1}{2}} = 1$). Moreover, since $\mathrm{r}(\mathrm{F}) = 2$, it is clear that (2.34) also provides the possible values of the Hasse invariant.

Let us now pass to the case $p \equiv 3 \bmod 4$. Once again, we have 16 possible (diagonal) quadratic forms we can construct with coefficients in $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$. The explicit computation of the Hilbert symbols (and, hence, of the Hasse invariant) provides

$$
\begin{aligned}
&(1,1)_{\mathrm{H}} = (-1)^0 \left(\frac{1}{p}\right)^0 \left(\frac{1}{p}\right)^0 = 1, && (1,\eta)_{\mathrm{H}} = (-1)^0 \left(\frac{1}{p}\right)^0 \left(\frac{\eta}{p}\right)^0 = 1,\\
&(1,p)_{\mathrm{H}} = (-1)^{2k+1} \left(\frac{1}{p}\right)^0 \left(\frac{1}{p}\right)^1 = 1, && (1,\eta p)_{\mathrm{H}} = (-1)^0 \left(\frac{1}{p}\right)^1 \left(\frac{\eta}{p}\right)^0 = 1,\\
&(\eta,1)_{\mathrm{H}} = (-1)^0 \left(\frac{\eta}{p}\right)^0 \left(\frac{1}{p}\right)^0 = 1, && (\eta,\eta)_{\mathrm{H}} = (-1)^0 \left(\frac{\eta}{p}\right)^0 \left(\frac{\eta}{p}\right)^0 = 1,
\end{aligned}
$$

$$
\begin{aligned}
&(\eta,p)_{\mathrm{H}} = (-1)^0 \left(\frac{\eta}{p}\right)^1 \left(\frac{1}{p}\right)^0 = -1, && (\eta,\eta p)_{\mathrm{H}} = (-1)^0 \left(\frac{\eta}{p}\right)^1 \left(\frac{\eta}{p}\right)^0 = -1,\\
&(p,1)_{\mathrm{H}} = (1,p)_{\mathrm{H}} = 1, && (p,\eta)_{\mathrm{H}} = (\eta,p)_{\mathrm{H}} = -1,\\
&(p,p)_{\mathrm{H}} = (-1)^{2k+1} \left(\frac{1}{p}\right)\left(\frac{1}{p}\right) = -1, && (p,\eta p)_{\mathrm{H}} = (-1)^{2k+1} \left(\frac{1}{p}\right)^1 \left(\frac{\eta}{p}\right)^1 = 1,\\
&(\eta p,1)_{\mathrm{H}} = (1,\eta p)_{\mathrm{H}} = 1, && (\eta p,\eta)_{\mathrm{H}} = (\eta,\eta p)_{\mathrm{H}} = (-1),\\
&(\eta p,p)_{\mathrm{H}} = (p,\eta p)_{\mathrm{H}} = 1, && (\eta p,\eta p)_{\mathrm{H}} = (-1)^{2k+1} \left(\frac{\eta}{p}\right)^1 \left(\frac{\eta}{p}\right)^1 = -1, \quad (2.35)
\end{aligned}
$$

where, in (2.35), we have again exploited the symmetry property (H1) of the Hasse invariant, and relation (2.25). In particular, in (2.25), we now have $(-1)^{\alpha\beta\cdot\frac{p-1}{2}} = (-1)^{(2k+1)\alpha\beta}$, $k \in \mathbb{N}$, as follows by observing that $p \equiv 3 \bmod 4$ entails $\frac{p-1}{2} = 2k+1$, $k \in \mathbb{N}$.

To complete the classification of the two-dimensional quadratic forms for $p \equiv 1 \bmod 4$ and $p \equiv 3 \bmod 4$, it only remains to compute the possible values assumed by the discriminant function. This is easily done once observed that 1 is the only quadratic element in $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$. For the sake of clarity, we collect the possible values of $\mathrm{d}(\mathrm{F})$, (and of the Hilbert symbols) in the following tables:

(a)

$(\cdot,\cdot)_{\mathrm{H}}$	1	η	p	ηp
1	1	1	1	1
η	1	1	-1	-1
p	1	-1	1	-1
ηp	1	-1	-1	1

(b)

$(\cdot,\cdot)_{\mathrm{H}}$	1	η	p	ηp
1	1	1	1	1
η	1	1	-1	-1
p	1	-1	-1	1
ηp	1	-1	1	-1

(c)

d(F)	1	η	p	ηp
1	1	η	p	ηp
η	η	1	ηp	p
p	p	ηp	1	η
ηp	ηp	p	η	1

TABLE 2.1: (a): Hilbert symbols for $p \equiv 1 \bmod 4$; (b): Hilbert symbols for $p \equiv 3 \bmod 4$. (c): Possible values of the discriminant function.

Summing up the above discussion, we have proved the following

Proposition 2.1.3. *Let p be a prime number. Any* non-degenerate, two-dimensional *quadratic form* F *on $\mathbb{Q}_p$ is equivalent to one of the following pairwise inequivalent forms:*

- $x_0^2 + x_1^2 \sim \{\eta x_0^2 + \eta x_1^2,\ px_0 + px_1^2,\ \eta p x_0^2 + \eta p x_1^2\}$
- $x_0^2 + \eta x_1^2$, $x_0^2 + px_1^2$, $x_0^2 + \eta p x_1^2$, $\eta x_0^2 + px_1^2$, $\eta x_0^2 + \eta p x_1^2$, $px_0^2 + \eta p x_1^2$

in the case where $p \equiv 1 \bmod 4$, and to

- $x_0^2 + x_1^2 \sim \eta x_0^2 + \eta x_1^2$
- $px_0^2 + px_1^2 \sim \eta p x_0^2 + \eta p x_1^2$
- $x_0^2 + \eta x_1^2 \sim px_0^2 + \eta p x_1^2$
- $x_0^2 + px_1^2$, $x_0^2 + \eta p x_1^2$, $\eta x_0^2 + px_1^2$, $\eta x_0^2 + \eta p x_1^2$,

for $p \equiv 3 \bmod 4$.

Although all the forms in Proposition 2.1.3 are non-degenerate, they are not all definite. In particular, resorting to Proposition 2.1.2, it is easily checked that in the case $p \equiv 1 \bmod 4$ the quadratic form $x_0^2 + x_1^2$ — and all the forms equivalent to it — is *non definite*. Similarly, for $p \equiv 3 \bmod 4$ the quadratic forms $x_0^2 + \eta x_1^2$, $px_0^2 + \eta p x_1^2$ represent 0 non-trivially, hence, are non-definite too.

Let F and F′ be two n-variables quadratic forms on $\mathbb{Q}_p$. We recall that F′ is said to be a *scaling* for F if there exists $\lambda \in \mathbb{Q}_p^*$ such that $\mathrm{F}' = \lambda \mathrm{F}$. Equivalently, if the associated matrices $\widehat{\mathrm{F}}$ and $\widehat{\mathrm{F}}'$ are related by $\widehat{\mathrm{F}}' = \lambda \widehat{\mathrm{F}}$. Using the scaling equivalence, we can further collect the (definite) quadratic forms in Proposition 2.1.3 in the following three classes

$$\mathrm{F}_{-v}(x) \coloneqq x_0^2 - v x_1^2, \quad \mathrm{F}_p(x) \coloneqq x_0^2 + p x_1^2, \quad \mathrm{F}_{\frac{p}{u}}(x) \coloneqq \eta x_0^2 + p x_1^2, \tag{2.36}$$

for

$$v = \begin{cases} -1 & \text{if } p \equiv 3 \mod 4, \\ -\eta & \text{if } p \equiv 1 \mod 4. \end{cases} \tag{2.37}$$

We now move to the three-dimensional case, namely, we assume $\mathrm{r}(\mathrm{F}) = 3$. Let $p > 2$. Similarly to the previous situation, we discuss separately the cases $p \equiv 1 \bmod 4$ and $p \equiv 3 \bmod 4$. Specifically, for $p \equiv 1 \bmod 4$, an explicit computation provides the following table collecting the values of the discriminant function and of the Hasse invariant (here, we are denoting with $\mathrm{diag}(a, b, c)$ the possible diagonal matrices with coefficients in $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$):

F	d(F)	ϵ(F)
$\mathrm{diag}(1,1,\eta)$, $\mathrm{diag}(\eta,\eta,\eta)$, $\mathrm{diag}(\eta,p,p)$, $\mathrm{diag}(\eta,\eta p,\eta p)$	η	1
$\mathrm{diag}(1,1,1)$, $\mathrm{diag}(1,p,p)$, $\mathrm{diag}(1,\eta,\eta)$, $\mathrm{diag}(1,\eta p,\eta p)$	1	1
$\mathrm{diag}(1,1,\eta p)$, $\mathrm{diag}(\eta,\eta,\eta p)$, $\mathrm{diag}(\eta p,\eta p,\eta p)$, $\mathrm{diag}(p,p,\eta p)$	ηp	1
$\mathrm{diag}(1,1,p)$, $\mathrm{diag}(\eta,\eta,p)$, $\mathrm{diag}(p,p,p)$, $\mathrm{diag}(p,\eta p,\eta p)$	p	1
$\mathrm{diag}(1,\eta,p)$	ηp	-1
$\mathrm{diag}(1,p,\eta p)$	η	-1
$\mathrm{diag}(\eta,p,\eta p)$	1	-1
$\mathrm{diag}(1,\eta,\eta p)$	p	-1

TABLE 2.2: Values of the Hasse invariant and the discriminant function for rank 3 quadratic forms ($p \equiv 1 \bmod 4$).

In the table above, we have computed the values of the Hasse invariant according to (2.33) and relations (2.34). For instance, $\epsilon(\mathrm{diag}(1,1,\eta)) = (\eta,1)_{\mathrm{H}}(\eta,1)_{\mathrm{H}}(1,1)_{\mathrm{H}} = 1$, $\epsilon(\mathrm{diag}(1,p,p)) = (p,p)_{\mathrm{H}}(p,1)_{\mathrm{H}}(p,1)_{\mathrm{H}} = 1$, $\epsilon(\mathrm{diag}(\eta,\eta,p\eta)) = (p\eta,\eta)_{\mathrm{H}}(p\eta,\eta)_{\mathrm{H}}(\eta,\eta)_{\mathrm{H}} = 1$, $\epsilon(\mathrm{diag}(p,p\eta,p\eta)) = (p\eta,p\eta)_{\mathrm{H}}(p\eta,1)_{\mathrm{H}}(p\eta,1)_{\mathrm{H}} = 1$, $\epsilon(\mathrm{diag}(1,\eta,p)) = (p,\eta)_{\mathrm{H}}(p,1)_{\mathrm{H}}(\eta,1)_{\mathrm{H}} = -1$, $\epsilon(\mathrm{diag}(1,p,p\eta)) = (p\eta,p)_{\mathrm{H}}(p\eta,1)_{\mathrm{H}}(1,1)_{\mathrm{H}} = -1$, and similarly for all the other cases.

Let us assume now $p \equiv 3 \bmod 4$. Again, we can perform a similar computation thus arriving to the following table listing the possible values of the discriminant and the Hasse invariant:

F	d(F)	ϵ(F)
$\mathrm{diag}(1,1,\eta)$, $\mathrm{diag}(\eta,\eta,\eta)$, $\mathrm{diag}(1,p,p\eta)$	η	1
$\mathrm{diag}(1,1,1)$, $\mathrm{diag}(1,\eta,\eta)$, $\mathrm{diag}(\eta,p,p\eta)$	1	1
$\mathrm{diag}(p,p,p)$, $\mathrm{diag}(p,p\eta,p\eta)$, $\mathrm{diag}(1,\eta,p\eta)$	p	-1
$\mathrm{diag}(p\eta,p\eta,p\eta)$, $\mathrm{diag}(p,p,p\eta)$, $\mathrm{diag}(1,\eta,p)$	$p\eta$	-1
$\mathrm{diag}(1,1,1)$, $\mathrm{diag}(\eta,\eta,p)$	p	1
$\mathrm{diag}(1,1,p\eta)$, $\mathrm{diag}(\eta,\eta,p\eta)$	$p\eta$	1
$\mathrm{diag}(1,p,p)$, $\mathrm{diag}(1,p\eta,p\eta)$	1	-1
$\mathrm{diag}(\eta,p,p)$, $\mathrm{diag}(\eta,p\eta,p\eta)$	p	-1

TABLE 2.3: Values of the Hasse invariant and the discriminant function for rank 3 quadratic forms ($p \equiv 3 \bmod 4$).

Proposition 2.1.4. *Let p be a prime number. Then, for $p \equiv 1 \bmod 4$, any* non-degenerate, three-dimensional *quadratic form* F *on $\mathbb{Q}_p$ is equivalent to one of the following pairwise inequivalent forms:*

- $\mathrm{diag}(1,1,\eta) \sim \{\mathrm{diag}(\eta,\eta,\eta),\ \mathrm{diag}(\eta,p,p),\ \mathrm{diag}(\eta,\eta p,\eta p)\}$
- $\mathrm{diag}(1,1,1) \sim \{\mathrm{diag}(1,p,p),\ \mathrm{diag}(1,\eta,\eta),\ \mathrm{diag}(1,\eta p,\eta p)\}$
- $\mathrm{diag}(1,1,\eta p) \sim \{\mathrm{diag}(\eta,\eta,\eta p),\ \mathrm{diag}(\eta p,\eta p,\eta p),\ \mathrm{diag}(p,p,\eta p)\}$
- $\mathrm{diag}(1,1,p) \sim \{\mathrm{diag}(\eta,\eta,p),\ \mathrm{diag}(p,p,p),\ \mathrm{diag}(p,\eta p,\eta p)\}$
- $\mathrm{diag}(1,\eta,p),\ \mathrm{diag}(1,p,\eta p),\ \mathrm{diag}(\eta,p,\eta p),\ \mathrm{diag}(1,\eta,\eta p),$

and to

- $\mathrm{diag}(1,1,\eta) \sim \{\mathrm{diag}(\eta,\eta,\eta),\ \mathrm{diag}(1,p,p\eta)\}$
- $\mathrm{diag}(1,1,1) \sim \{\mathrm{diag}(1,\eta,\eta),\ \mathrm{diag}(\eta,p,p\eta)\}$
- $\mathrm{diag}(p,p,p) \sim \{\mathrm{diag}(p,p\eta,p\eta),\ \mathrm{diag}(1,\eta,p\eta)\}$
- $\mathrm{diag}(p\eta,p\eta,p\eta) \sim \{\mathrm{diag}(p,p,p\eta),\ \mathrm{diag}(1,\eta,p)\}$
- $\mathrm{diag}(1,1,1) \sim \mathrm{diag}(\eta,\eta,p),\ \mathrm{diag}(1,1,p\eta) \sim \mathrm{diag}(\eta,\eta,p\eta))$
- $\mathrm{diag}(1,p,p) \sim \mathrm{diag}(1,p\eta,p\eta)$
- $\mathrm{diag}(\eta,p,p) \sim \mathrm{diag}(\eta,p\eta,p\eta)$

for $p \equiv 3 \bmod 4$

Similarly to the two-dimensional case, not all the quadratic forms in Proposition 2.1.4 are definite. Using (iii) in Proposition 2.1.2, it is easily checked that the only rank-3 definite quadratic form on $\mathbb{Q}_p$ is, up to scaling,

$$\mathrm{F}_+(\mathbf{x}) \coloneqq x_0^2 - v x_1^2 + p x_2^2, \tag{2.38}$$

for v a parameter such that

$$v = \begin{cases} -1 & \text{if } p \equiv 3 \bmod 4 \\ -\eta & \text{if } p \equiv 1 \bmod 4. \end{cases} \tag{2.39}$$

The discussion above can be repeated, with a slight modification, in the case where $\mathrm{r(F)} = 4$. Once again, we have to compute the possible values of the discriminant and the Hasse invariant, and then collect all the forms with the same value of the two invariants. In a similar fashion, one also deals with the case $p = 2$; here the strategy is again to compute the discriminant and the Hasse invariant — w.r.t. a fixed value of the rank — but, with the only difference that now the group of square classes of $\mathbb{Q}_2$ consists of 8 elements, precisely $\mathbb{Q}_2^*/(\mathbb{Q}_2^*)^2 = \{\pm 1, \pm 2, \pm 5, \pm 10\}$. We do not discuss the computational details here, and refer the reader to [71, 113] for an explicit derivation. Summing up, we can state the following theorem which provides a complete characterization of the equivalence classes of (definite) quadratic forms for $p \geq 2$, and $\mathrm{r(F)} = 2, 3, 4$

Theorem 2.1.3. *For every prime $p > 2$, let η be a p-adic unit in $\mathbb{Q}_p$, and let $v \in \mathbb{U}_p$ be defined by*

$$v \coloneqq \begin{cases} -1 & \text{if } p \equiv 3 \mod 4, \\ -\eta & \text{if } p \equiv 1 \mod 4. \end{cases} \tag{2.40}$$

In the case where $p > 2$, there are (precisely) three definite quadratic forms on $\mathbb{Q}_p^2$, up to equivalence and scaling,

$$\mathrm{F}_{-v}(\mathbf{x}) = x_0^2 - vx_1^2, \quad \mathrm{F}_p(\mathbf{x}) = x_0^2 + px_1^2, \quad \mathrm{F}_{\frac{p}{u}}(\mathbf{x}) = ux_0^2 + px_1^2, \tag{2.41}$$

and there are seven on $\mathbb{Q}_2^2$, namely,

$$\begin{aligned} \mathrm{F}_1(\mathbf{x}) &= x_0^2 + x_1^2, \qquad \mathrm{F}_{\pm 2}(\mathbf{x}) = x_0^2 \pm 2x_1^2, \\ \mathrm{F}_{\pm 5}(\mathbf{x}) &= x_0^2 \pm 5x_1^2, \quad \mathrm{F}_{\pm 10}(\mathbf{x}) = x_0^2 \pm 10x_1^2. \end{aligned} \tag{2.42}$$

There is a unique definite quadratic form on $\mathbb{Q}_p^3$ (depending on p), up to equivalence and scaling, i.e.,

$$\mathrm{F}_+(\mathbf{x}) = \begin{cases} x_0^2 - vx_1^2 + px_2^2 & \text{if } p > 2, \\ x_0^2 + x_1^2 + x_2^2 & \text{if } p = 2, \end{cases} \tag{2.43}$$

as well as on $\mathbb{Q}_p^4$, i.e.,

$$\mathrm{F}_{(4)}(\mathbf{x}) = \begin{cases} x_0^2 - vx_1^2 + px_2^2 - pvx_3^2 & \text{if } p > 2, \\ x_0^2 + x_1^2 + x_2^2 + x_3^2 & \text{if } p = 2. \end{cases} \tag{2.44}$$

No quadratic form on $\mathbb{Q}_p^n$ is definite for $n \geq 5$.

2.2 p-Adic rotation groups

In this section, we characterize p-adic special orthogonal groups over the field of p-adic numbers. We start by recalling that the (real) special orthogonal groups $\mathrm{SO}(n, \mathbb{R})$, are defined, for every n, as the groups of linear transformations preserving the (definite) quadratic form on $\mathbb{R}^n$. In particular, the unique non-degenerate definite quadratic form on $\mathbb{R}^n$, for every $n \geq 2$, is given (up to equivalence and scaling) by $\mathrm{F}_{\mathbb{R}}(\mathbf{x}) = \sum_{i=0}^{n-1} x_i^2$. This is represented in the canonical basis — namely, the basis $\{e_i\}_{i \in \mathbb{N}}$ consisting of vectors of the form $e_i = (0, 0, \ldots, 1, 0, \ldots)$, with all but the i-*th* component equal to 0 — by the n-dimensional identity matrix I_n. Hence, we can set

$$\begin{aligned} \mathrm{SO}(n, \mathbb{R}) &= \{L \in \mathsf{M}(n, \mathbb{R}) \mid L^{\intercal} L = \mathrm{I}_n, \ \det(L) = 1\} && (2.45) \\ &= \{L \in \mathsf{M}(n, \mathbb{R}) \mid \langle L\mathbf{x}, L\mathbf{y}\rangle_{\mathbb{R}} = \langle \mathbf{x}, \mathbf{y}\rangle_{\mathbb{R}} \text{ for every } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \ \det(L) = 1\}, && (2.46) \end{aligned}$$

where $\langle \cdot\,, \cdot \rangle_{\mathbb{R}} \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the Euclidean scalar product on $\mathbb{R}^n$, and $\mathsf{M}(n, \mathbb{R})$ denotes the associative algebra of $n \times n$ matrices over $\mathbb{R}$. Hence, paralleling this situation in the p-adic case, we are naturally led to set the following

Definition 2.2.1. Let V be a $\mathbb{Q}_p$-vector space and let Q be a (non degenerate, definite) quadratic form on it. The *special orthogonal group* $\mathrm{SO}(V, \mathrm{Q})$ is defined as the set of linear maps on V that are symmetries of Q, i.e,

$$\begin{aligned} \mathrm{SO}(V, \mathrm{Q}) &= \{L \in \mathrm{End}(V) \mid \mathrm{Q}(Lx) = \mathrm{Q}(x), \det(L) = 1 \ \forall x \in V\} && (2.47) \\ &= \{L \in \mathrm{End}(V) \mid \langle Lx, Ly \rangle = \langle x, y \rangle, \det(L) = 1 \ \forall x, y \in V\} && (2.48) \\ &\cong \left\{L \in \mathsf{M}(n, \mathbb{Q}_p) \mid L^{\intercal} \widehat{\mathrm{Q}} L = \widehat{\mathrm{Q}}\right\}, && (2.49) \end{aligned}$$

where $\langle \cdot\,, \cdot \rangle$ is the inner product associated with the quadratic form Q, and $\widehat{\mathrm{Q}}$ denotes its matrix representation.

Using the results of Theorem 2.1.3, we can now characterize the classes of (compact) special orthogonal groups over $\mathbb{Q}_p$.

Proposition 2.2.1. *The p-adic special orthogonal groups associated with the definite quadratic forms on* $\mathbb{Q}_p^2$ *are (up to isomorphism)*

$$\mathrm{SO}(2,\mathbb{Q}_p)_\kappa = \{L \in \mathsf{M}(2,\mathbb{Q}_p) \mid \widehat{\mathrm{F}}_\kappa = L^\intercal \widehat{\mathrm{F}}_\kappa L,\ \det(L)=1\}, \tag{2.50}$$

where $\widehat{\mathrm{F}}_\kappa$ *denote the matrix representation, in the canonical basis of* $\mathbb{Q}_p^2$*, of the quadratic forms in* (2.41) *and* (2.42)*. The index* κ *ranges in* $\{-v, p, \frac{p}{u}\}$ *whenever* $p>2$*, while* $\kappa \in \{1, \pm 2, \pm 5, \pm 10\}$ *when* $p=2$*.*
For every $p \geq 2$*, the special orthogonal group associated with the definite quadratic form on* $\mathbb{Q}_p^3$ *is (up to isomorphisms)*

$$\mathrm{SO}(3,\mathbb{Q}_p) = \{L \in \mathsf{M}(3,\mathbb{Q}_p) \mid \widehat{\mathrm{F}}_+ = L^\intercal \widehat{\mathrm{F}}_+ L,\ \det(L)=1\}, \tag{2.51}$$

while the one on $\mathbb{Q}_p^4$ *is*

$$\mathrm{SO}(4,\mathbb{Q}_p) = \{L \in \mathsf{M}(4,\mathbb{Q}_p) \mid \widehat{\mathrm{F}}_{(4)} = L^\intercal \widehat{\mathrm{F}}_{(4)} L,\ \det(L)=1\}. \tag{2.52}$$

$\widehat{\mathrm{F}}_+$ *and* $\widehat{\mathrm{F}}_{(4)}$ *are the matrix representations in the canonical basis of* $\mathbb{Q}_p^3$ *and* $\mathbb{Q}_p^4$ *of the quadratic forms* (2.43) *and* (2.44) *respectively.*

Remark 2.2.1. It is worth noting that equivalent quadratic forms lead to isomorphic special orthogonal groups, while if two quadratic forms are related via a scaling relation, they actually define the same special orthogonal group. Namely, if V is a $\mathbb{Q}_p$-vector space, and Q, Q′ are two quadratic forms on it, then if $\mathrm{Q} \sim \mathrm{Q}'$, one has $\mathrm{SO}(\mathbb{Q}_p^n, \mathrm{Q}) \cong \mathrm{SO}(\mathbb{Q}_p^n, \mathrm{Q}')$ [74], while if $\mathrm{Q}' = \lambda \mathrm{Q}$, $\lambda \in \mathbb{Q}_p^*$, then $\mathrm{SO}(\mathbb{Q}_p^n, \mathrm{Q}) = \mathrm{SO}(\mathbb{Q}_p^n, \mathrm{Q}')$.

Any $\mathrm{SO}(\mathbb{Q}_p^n, \mathrm{F})$ is a topological group, once supplied with the subspace topology of $\mathsf{M}(n,\mathbb{Q}_p) \cong \mathbb{Q}_p^{n^2}$. In particular, the groups $\mathrm{SO}(n,\mathbb{Q}_p)$, $n=2,3,4$, are compact as subsets in $\mathbb{Q}_p^{n^2}$. Indeed, we can introduce a p-adic (non-Archimedean) norm on $\mathrm{SO}(n,\mathbb{Q}_p)$ by setting $\|M\|_p = \|(M_{ij})_{i,j}\|_p \coloneqq \max_{i,j=1,\dots,n} |M_{ij}|_p$. Hence, $\mathrm{SO}(n,\mathbb{Q}_p)$, $n=2,3,4$, turn into topological groups, whenever they are endowed with the natural topology generated by the open balls of the p-adic norm. We recall that a set $K \subset \mathbb{Q}_p^m$ is *compact* if and only if it is *closed* and *bounded* w.r.t. the ultrametric topology generated by (the open balls of) the p-adic norm $\mathrm{N}_p(\mathbf{x}) \coloneqq \max_{i=1,\dots,m} |x_i|_p$ of $\mathbb{Q}_p^m$ [73]. $\mathrm{SO}(n,\mathbb{Q}_p)$, $n=2,3,4$, are closed, as they are groups of solutions of a system of continuous (polynomial) equations, and bounded [89]. Hence, from this, we can argue that $\mathrm{SO}(n,\mathbb{Q}_p)$, for $n=2,3,4$, are compact topological groups.

Remark 2.2.2. We have used definite quadratic forms to define the p-adic special orthogonal groups. It turns out that those groups defined on indefinite quadratic forms are not bounded, whence, not compact [113].

The next theorem provides a parameterization of the compact p-adic special orthogonal groups in dimension two [113].

Theorem 2.2.1. *Any element of* $\mathrm{SO}(2,\mathbb{Q}_p)_\kappa$ *takes the following matrix form in the canonical basis of* $\mathbb{Q}_p^2$*:*

$$R_\kappa(\alpha) = \begin{pmatrix} \frac{1-\kappa\alpha^2}{1+\kappa\alpha^2} & -\frac{2\kappa\alpha}{1+\kappa\alpha^2} \\ \frac{2\alpha}{1+\kappa\alpha^2} & \frac{1-\kappa\alpha^2}{1+\kappa\alpha^2} \end{pmatrix}, \quad \alpha \in \mathbb{Q}_p \cup \{\infty\}, \tag{2.53}$$

where $R_\kappa(\infty) = -R_\kappa(0) = -\mathrm{I}_2$*, and* $\kappa \in \{-v, p, \frac{p}{u}\}$ *for* $p>2$*, while* $\kappa \in \{1, \pm 2, \pm 5, \pm 10\}$ *for* $p=2$*. The composition of two elements in* $\mathrm{SO}(2,\mathbb{Q}_p)_\kappa$*, for any fixed* κ*, is given by*

$$R_\kappa(\alpha) R_\kappa(\beta) = R_\kappa\left(\frac{\alpha+\beta}{1-\kappa\alpha\beta}\right), \tag{2.54}$$

for every $\alpha, \beta \in \mathbb{Q}_p \cup \{\infty\}$.

Remark 2.2.3. By choosing $\kappa = 1$, and taking $\alpha = \tan(\theta/2)$, a generic p-adic rotation, as given in (2.53), assumes the form

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{2.55}$$

i.e., it 'formally' reduces to a real planar rotation by an angle θ.

Part II

Studies on p-adic harmonic analysis

The second part of this dissertation addresses the problem to construct an invariant measure — that is, the Haar measure — for a p-adic Lie group. This is indeed a necessary tool for the study of the two-dimensional (projective) irreducible representations of $\mathrm{SO}(3,\mathbb{Q}_p)$ *which, ultimately, will provide a model of p-adic qubit. This part is organized as follows: In Subsection 3.1 we recall some basic notions concerning the Haar measure on locally compact groups and the lifts of Haar integrals on quotient groups. We then discuss p-adic manifolds and p-adic Lie groups, especially focusing on their topological properties. Section 4 provides a general formula for the Haar measure on a p-adic Lie group, eventually showing that it coincides with the measure associated with the (maximal-rank) invariant differential form defined on the group. In Section 4.2 we apply the (previously constructed) theory to the p-adic special orthogonal groups in dimensions two, three, and four. Specifically, in Subsection 4.2.1, we derive the Haar measure on* $\mathrm{SO}(2,\mathbb{Q}_p)$*. Next, in Section 5 we construct, for any prime number p, the p-adic quaternion algebra, and we highlight its relation with the groups* $\mathrm{SO}(3,\mathbb{Q}_p)$ *and* $\mathrm{SO}(4,\mathbb{Q}_p)$ *in Subsection 5.3. Eventually, by exploiting the Weil-Mackey-Bruhat formula, we construct the Haar integrals on* $\mathrm{SO}(3,\mathbb{Q}_p)$ *and* $\mathrm{SO}(4,\mathbb{Q}_p)$*. The material discussed here is in large part based on [88, 89]. Deviations from the published version mostly affect notations and typesetting.*

3

Basic notions and tools

In this chapter, we collect some basic results and tools which will be relevant for all our later derivations. We begin by recalling the notion of Haar measure on a locally compact group. Next, we introduce p-adic Lie groups and discuss some of their most relevant (topological) features. Eventually, we conclude by providing a brief outline of integration theory on a p-adic manifold.

3.1 The Haar measure on a locally compact group and the lifts of Haar integrals

Let G be a locally compact (Hausdorff) topological group; in short, a LC group. By a *left* (resp. *right*) *Haar measure* μ on G we mean a non-zero *Radon measure* for which the following condition holds:

$$\mu(g\mathcal{E}) = \mu(\mathcal{E}) \quad (\text{resp.}\ \mu(\mathcal{E}g) = \mu(\mathcal{E})), \tag{3.1}$$

for every Borel set $\mathcal{E} \subset G$, and $g \in G$ [83, 114]. We refer to (3.1) as to the *left-invariance* (resp. *right-invariance*) property of the measure.

It is worth recalling a remarkable characterization of the left (resp. right) Haar measure provided by a suitable left- (right-)invariance condition for a class of functionals on $\mathrm{C}_c(G)$ — the algebra of compactly-supported continuous complex-valued functions on G [83, 115].

Remark 3.1.1. We are adopting the convention that the *support*, $\mathrm{supp}(f)$, of a continuous function f is the closure of the open set $\{g \in G \mid f(g) \in \mathbb{C} \setminus \{0\}\}$.

Let μ be a fixed Radon measure on a LC group G. The map defined as

$$\mathrm{C}_c(G) \ni f \mapsto I(f) := \int_G f(g)\mathrm{d}\mu(g) \in \mathbb{C} \tag{3.2}$$

is easily seen to be a positive linear functional on $\mathrm{C}_c(G)$. On the other hand, the celebrated Riesz Representation Theorem (cf. Theorem 7.2 in [115]) assures that for *every* positive linear functional on $\mathrm{C}_c(G)$, there is a *unique* Radon measure μ on G such that I is represented as in (3.2). Exploiting this correspondence, a Radon measure μ is a left Haar measure if and only if the associated functional is left-invariant, i.e., if and only if the condition

$$\int_G (L_h f)(g)\mathrm{d}\mu(g) = \int_G f(g)\mathrm{d}\mu(g) \tag{3.3}$$

holds for every $f \in C_c(G)$. Here, the map L_h, for $h \in G$, of *left translation* on $\mathrm{C}_c(G)$ is defined as $(L_h f)(g) := f(h^{-1}g)$. By defining the *right translation* via $(R_h f)(g) = f(gh)$, we capture analogously right-invariance of the measure. In what follows, whenever μ is a Haar measure on G, we will refer to the integral in the r.h.s. of (3.2) as to the *Haar integral* associated with μ.

It is a well known result (see, e.g., Theorems 2.10 and 2.20 in [83]) that any LC group admits an *essentially uniquely defined* Haar measure. In particular, if μ and ν are left Haar measures on G, then there exists $c \in \mathbb{R}_*^+$ such that $\mu = c\nu$. If G is a LC group, its left and right Haar measures are related via the so-called *modular function* $\Delta\colon G \to \mathbb{R}_*^+$ [83]. In the case where $\Delta \equiv 1$ (as it happens for abelian and compact groups), G is called *unimodular*, meaning that left and right Haar measures coincide.

Remark 3.1.2. A locally compact group G has *finite* left (and right) Haar measure μ if and only if it is compact [83, 116]; in this case, it is possible (and customary) to normalize the Haar measure in such a way that $\mu(G) = 1$.

Example 3.1.1 (Haar measure on $\mathbb{Q}_p$)**.** The (additive) group of the field of p-adic numbers $\mathbb{Q}_p$ ($p \in \mathbb{N}$ prime) is a LC group once endowed with its standard *ultrametric* topology (namely, the topology induced by the non-trivially valued, *non-Archimedean absolute value* $|\cdot|_p$ on $\mathbb{Q}_p$). Therefore, it admits a left Haar measure λ. Since $(\mathbb{Q}_p, +)$ is abelian (hence, unimodular), λ is right-invariant as well, i.e.,

$$\lambda(\mathcal{E} + x) = \lambda(\mathcal{E}) = \lambda(x + \mathcal{E}) \tag{3.4}$$

holds for every Borel subset $\mathcal{E}$ in $\mathcal{B}_{\mathbb{Q}_p}$, and any $x \in \mathbb{Q}_p$. Since the subring $\mathbb{Z}_p$ of *p-adic integers* is a compact subset of $\mathbb{Q}_p$, we can normalize λ by setting

$$\lambda(\mathbb{Z}_p) = 1. \tag{3.5}$$

It is now not difficult to explicitly construct the measure λ. Indeed, let $\overline{B_r(x_0)} := \{x \in \mathbb{Q}_p \mid |x - x_0|_p \leq p^r\}$ be a ball centred in $x_0 \in \mathbb{Q}_p$ of radius $p^r > 0$. Since $\overline{B_1(0)} = \mathbb{Z}_p$, owing to the invariance condition (3.4) and the normalization (3.5), we get $\lambda\left(\overline{B_1(x)}\right) = 1$ for every $x \in \mathbb{Q}_p$. Moreover, the topological features of $\mathbb{Q}_p$ — i.e., any ball of radius p^k, $k > 0$, is a *disjoint* union of p^k balls of radius 1 — also entail that $\lambda\left(\overline{B_k(x)}\right) = p^k$ for every $k \in \mathbb{Z}$, $x \in \mathbb{Q}_p$. Hence, we get to the conclusion that the measure of every Borel set $\mathcal{E}$ of $\mathbb{Q}_p$ is given by

$$\lambda(\mathcal{E}) = \inf \left\{ \sum_{j \geq 1} p^{m_j} \mid \mathcal{E} \subset \bigcup_{j \geq 1} \overline{B_{m_j}(x_j)} \right\}, \tag{3.6}$$

analogous to the formula for the Lebesgue measure on the real line.

Example 3.1.2. The group $\mathbb{Q}_p^n = \mathbb{Q}_p \times \ldots \times \mathbb{Q}_p$ (n-times), endowed with the product topology, has a natural structure of (additive) LC group; hence, it admits a left (and right) Haar measure. To find it explicitly, it is enough to observe that, being $\mathbb{Q}_p$ a second countable LC group, there is no distinction between the standard product of measures and the Radon product (see Section 2.2 in [83]). Therefore, the Haar measure on $\mathbb{Q}_p^n$ is provided by the n-times product of the Haar measure on $\mathbb{Q}_p$, i.e.,

$$\lambda^n = \lambda \times \ldots \times \lambda \;\; (n\text{-times}), \quad \lambda \text{ Haar measure on } \mathbb{Q}_p. \tag{3.7}$$

With a slight abuse of notation, we will denote by λ the Haar measure on $\mathbb{Q}_p^n$ for every $n \in \mathbb{N}$, as the dimension n will be clear from the context.

Let G be a LC group, and let X be a LC Hausdorff space. We call X a (*transitive*) *G-space* whenever it is equipped with a (transitive) continuous left action $(\cdot)[\cdot]\colon G \times X \to X$ of G. If G is a locally compact second countable Hausdorff (in short, LCSC) group, and H a closed *normal* subgroup of G (e.g., the centre of G), let $X \equiv G/H$ denote the quotient (LCSC) group. Furthermore, let $q\colon G \to X$ be the *quotient map* (i.e., the projection homomorphism) which is an *open continuous* map. We can then define a natural continuous action $(\cdot)[\,\cdot\,]\colon G\times X\to X$ of G on X, i.e.,

$$g[x] := q(g)x, \qquad g\in G,\ x\in X. \tag{3.8}$$

This action is transitive and, hence, turns $X\equiv G/H$ into a transitive G-space. In the literature, one refers to such a G-space as to a *homogeneous space* [83, 90, 91, 114, 116, 117].

Let now μ_G, μ_H, μ_X denote the (left) Haar measures on G, H, $X\equiv G/H$ respectively, and let Δ_G, Δ_H be the modular functions on G and H. It is a standard fact that (since X admits a X-invariant, hence G-invariant, measure μ_X; see Theorem 2.51 of [83])

$$\Delta_G(h) = \Delta_H(h), \qquad \forall h\in H, \tag{3.9}$$

i.e., $\Delta_H = \Delta_G|_H$. Therefore, if G is unimodular, then H shares the same property.

Let $(X,\mathcal{B}_X)$, $(Y,\mathcal{B}_Y)$ be *(Borel) measurable spaces.* We recall that a map $\varphi\colon X\to Y$ is called a *Borel map* if, for every Borel set $\mathcal{E}\in\mathcal{B}_Y$, $\varphi^{-1}(\mathcal{E})\in\mathcal{B}_X$; it is called a *Borel isomorphism* if it is one-one, onto, and f^{-1} is a Borel map. If $X\equiv G/H$ is a quotient group, we also denote by $\mathtt{s}\colon X\to G$ a *Borel (cross) section* of X into G, i.e., a Borel map satisfying the condition $q(\mathtt{s}(x)) = x$, for every $x\in X$.

Proposition 3.1.1 (Lemma 6 of [85])**.** *For every Borel section $\mathtt{s}\colon X\to G$, the mapping*

$$\gamma_{\mathtt{s}}\colon X\times H\ni (x,h)\mapsto \mathtt{s}(x)h\in G \tag{3.10}$$

is a Borel isomorphism ($X\times H$ being endowed with the product topology).

For every $f\in \mathrm{C}_c(G)$, we put

$$\begin{aligned}(Pf)(x) &:= \int_H \mathrm{d}\mu_H(h)(f\circ\gamma_{\mathtt{s}})(x,h)\\ &= \int_H \mathrm{d}\mu_H(h) f(\mathtt{s}(x)h), \qquad x\in X.\end{aligned} \tag{3.11}$$

Remark 3.1.3. It is worth observing that the function $H\ni h\mapsto f(gh)\in\mathbb{C}$, for any $g\in G$, is in $\mathrm{C}_c(H)$ (in particular, $gh\in\mathrm{supp}(f)\implies h\in g^{-1}\mathrm{supp}(f)\bigcap H$), where $g^{-1}\mathrm{supp}(f)\bigcap H$ is a compact subset of G and, hence, of H). Therefore, the integral on the r.h.s. of (3.11) is well-defined.

Remark 3.1.4. Note that, by the left-invariance of μ_H, the integral $\int_H \mathrm{d}\mu_H(h)f(gh)$ is constant w.r.t. g varying in $q^{-1}(\{x\})$, for every $x\in X$. Hence, $(Pf)(x)\in\mathbb{C}$ does not depend on the choice of the cross section $\mathtt{s}$.

Theorem 3.1.1. *For every $f\in\mathrm{C}_c(G)$, the function*

$$X\ni x\mapsto (Pf)(x)\in\mathbb{C} \tag{3.12}$$

belongs to $\mathrm{C}_c(X)$, and the mapping $\mathrm{C}_c(G)\ni f\mapsto Pf\in\mathrm{C}_c(X)$ is surjective. Moreover, for every $f\in\mathrm{C}_c(G)$, we have that

$$\begin{aligned}\int_G \mathrm{d}\mu_G(g)f(g) &= \int_{X\times H}\mathrm{d}\mu_X\times\mu_H(x,h)f(\mathtt{s}(x)h)\\ &= \int_X \mathrm{d}\mu_X(x)\int_H \mathrm{d}\mu_H(h)f(\mathtt{s}(x)h)\\ &= \int_X \mathrm{d}\mu_X(x)(Pf)(x), \qquad \textit{(Weil-Mackey-Bruhat)},\end{aligned} \tag{3.13}$$

where the Haar measures μ_G, μ_H, μ_X are supposed to be suitably normalized and $\mathtt{s}\colon X\to G$ is any Borel cross section.

(Note: Since X, H are LCSC groups, in the first line of (3.13) it is not necessary to make a distinction between the standard product of measures and the Radon product [83].)

Proof. See Section 2.6 of [83]; in particular, Proposition 2.50 and Theorem 2.51. □

For every $\phi \in \mathrm{C}_c(X)$ and $\psi \in \mathrm{C}_c(G)$, we set

$$(\mathscr{L}_\psi \phi)(g) := \psi(g)\phi(q(g)), \quad g \in G. \tag{3.14}$$

It is easy to see that $\mathscr{L}_\psi \phi \in \mathrm{C}_c(G)$; in particular, we have that

$$\mathrm{supp}(\mathscr{L}_\psi \phi) \subset \mathrm{supp}(\psi) \bigcap q^{-1}(\mathrm{supp}(\phi)) \tag{3.15}$$

is a compact subset of G.

Lemma 3.1.1. *For every compact subset K of X, there exists a function $\psi \in \mathrm{C}_c^+(G)$ such that*

$$(P\psi)(x) = 1, \quad \forall x \in K. \tag{3.16}$$

Here and in the following, we set $\mathrm{C}_c^+(G) := \{f \in \mathrm{C}_c(G) \mid f \geq 0,\ f \not\equiv 0\}$.

Proof. Use Lemma 2.49 of [83]. □

By Lemma 3.1.1, for every nonempty compact subset K of X, we can define the following (nonempty) subset of $\mathrm{C}_c^+(G)$

$$\Psi_K := \{\psi \in \mathrm{C}_c^+(G) \mid (P\psi)(x) = 1,\ \forall x \in K\}. \tag{3.17}$$

By convention, we put $\Psi_\emptyset = \{\psi \equiv 0\}$.

Definition 3.1.1. Given any $\phi \in \mathrm{C}_c(X)$, for every $\psi \in \Psi_{\mathrm{supp}(\phi)}$, we call the function $\mathscr{L}_\psi \phi \in \mathrm{C}_c(G)$ a *Weil-Mackey-Bruhat (WMB) lift* — specifically, the ψ*-lift* — of ϕ.

Lemma 3.1.2. *For every $\phi \in \mathrm{C}_c(X)$, and every $\psi \in \Psi_{\mathrm{supp}(\phi)}$, we have that*

$$P(\mathscr{L}_\psi \phi) = \phi. \tag{3.18}$$

Proof. In fact, by Lemma 3.1.1, we have:

$$\begin{aligned}\big(P(\mathscr{L}_\psi \phi)\big)(x) &= \int_H \mathrm{d}\mu_H(h)\psi(\mathtt{s}(x)h)\phi(q(\mathtt{s}(x))h) \\ &= (P\psi)(x)\phi(x) = \phi(x), \qquad \forall x \in X,\end{aligned} \tag{3.19}$$

where $\mathtt{s}\colon X \to G$ is any Borel cross section ($q(\mathtt{s}(x)h) = x$). □

We are now able to express any Haar integral on X as a Haar integral on G:

Theorem 3.1.2. *Let ϕ be a function in $\mathrm{C}_c(X)$. Then, for every WMB lift $\mathscr{L}_\psi \phi \in \mathrm{C}_c(G)$ of ϕ* ($\psi \in \Psi_{\mathrm{supp}(\phi)}$), we have that

$$\int_X \mathrm{d}\mu_X(x)\phi(x) = \int_G \mathrm{d}\mu_G(g)(\mathscr{L}_\psi \phi)(g), \tag{3.20}$$

where a suitable (mutual) normalization of μ_X and μ_G is assumed.

Proof. In fact, by the second assertion of Theorem 3.1.1,

$$\int_G \mathrm{d}\mu_G(g)(\mathscr{L}_\psi \phi)(g) = \int_X \mathrm{d}\mu_X(x)\big(P(\mathscr{L}_\psi \phi)\big)(x) = \int_X \mathrm{d}\mu_X(x)\phi(x), \tag{3.21}$$

where, for the second equality, we have used Lemma 3.1.2. □

We will call the Haar integral on the r.h.s. of (3.20) a *lift* of the Haar integral on the l.h.s. of the same formula.

In our specific applications, $X = G/H$ will be a *compact* group. In this case, some of the previously discussed results admit a remarkable generalization. To start with, let us notice that, when X is compact, $\mathrm{C}_c(X)$ coincides with the set $\mathrm{C}(X)$ of all continuous functions on X. Let us put $\Psi \equiv \Psi_X$. From Theorem 3.1.2, we can immediately prove the following:

Corollary 3.1.1. *Let $X = G/H$ be compact. Then, for every $\psi \in \Psi$, we have that*

$$\int_X \mathrm{d}\mu_X(x)\phi(x) = \int_G \mathrm{d}\mu_G(g)(\mathscr{L}_\psi \phi)(g), \qquad \forall \phi \in \mathrm{C}(X), \tag{3.22}$$

where a suitable (mutual) normalization of μ_X and μ_G is assumed.

Remark 3.1.5. Fixed any $\psi \in \Psi \equiv \Psi_X$, the map $\mathscr{L}_\psi \colon \mathrm{C}(X) \to \mathrm{C}_c(G)$ is a right inverse of $P \colon \mathrm{C}_c(G) \to \mathrm{C}(X)$, i.e., it satisfies relation (3.18) for *all* $\phi \in \mathrm{C}(X)$.

Remark 3.1.6. Without any assumption of compactness of X, the map $P \colon \mathrm{C}_c(G) \to \mathrm{C}_c(X)$ can be extended to a (surjective) map $\widehat{P} \colon \mathrm{L}^1(G) \to \mathrm{L}^1(X)$, defined by

$$\big(\widehat{P} f\big)(x) := \int_H \mathrm{d}\mu_H(h) f(\mathtt{s}(x)h), \qquad x \in X,\ f \in \mathrm{L}^1(G); \tag{3.23}$$

see Lemma 7 of [85] and Theorem 3.4.6 of [90] ($\mathrm{L}^1(G) \equiv \mathrm{L}^1(G, \mu_G)$ denotes the set of complex-valued functions on G whose absolute value is integrable w.r.t. μ_G). Moreover, the *extended WMB formula* holds:

$$\int_G \mathrm{d}\mu_G(g) f(g) = \int_X \mathrm{d}\mu_X(x)\big(\widehat{P} f\big)(x), \qquad \forall f \in \mathrm{L}^1(G), \tag{3.24}$$

for a suitable (mutual) normalization of the Haar measures μ_X, μ_G.

The forthcoming Theorem will provide us with a suitable generalization of the results in Lemma 3.1.2 and in Theorem 3.1.2, tailored to the case where $X = G/H$ is a compact group. To this end, we find useful to preliminary recall the notion of *pushforward measure*.

Definition 3.1.2. Let $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$ be (Borel) measurable spaces. Let μ be a Borel measure on X. If $\varphi \colon X \to Y$ is a Borel map of X into Y, the *pushforward measure* $\varphi_*\mu$ of μ through φ is the measure on $(Y, \mathcal{B}_Y)$ defined by

$$\varphi_*\mu(\mathcal{E}) := \mu \circ \varphi^{-1}(\mathcal{E}), \tag{3.25}$$

for every Borel set $\mathcal{E}$ in $\mathcal{B}_Y$.

Remark 3.1.7. If $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$ are Borel measurable spaces, and if $f\colon Y \to \mathbb{R}$ is a Borel function on Y, the following (abstract) change-of-variables formula (C.O.V.F., in short) holds [118]:

$$\int_X (f \circ \varphi)\mathrm{d}\mu = \int_Y f\mathrm{d}(\varphi_* \mu). \tag{3.26}$$

Moreover, from (3.26), it is not difficult to prove the following relation [118]:

$$\varphi_*(g\mathrm{d}\mu) = g \circ \varphi^{-1}\mathrm{d}(\varphi_*\mu), \tag{3.27}$$

for every Borel function $g\colon X \to \mathbb{R}$. We shall constantly resort to this formula in our description of integration theory on $\mathbb{Q}_p$-manifolds.

We are now ready to prove the following result

Theorem 3.1.3. *Let $X = G/H$ be compact. Then, for every $\psi \in \Psi$, the map $\mathscr{L}_\psi\colon \mathrm{C}(X) \to \mathrm{C}_c(G)$ admits an extension — a so-called* extended WMB lift —

$$\widehat{\mathscr{L}_\psi}\colon \mathrm{L}^1(X) \to \mathrm{L}^1(G), \tag{3.28}$$

defined by

$$\big(\widehat{\mathscr{L}_\psi}\phi\big)(g) \coloneqq \psi(g)(\phi \circ q)(g), \tag{3.29}$$

that is a right inverse of $\widehat{P}$:

$$\widehat{P}\big(\widehat{\mathscr{L}_\psi}\phi\big) = \phi, \quad \forall \phi \in \mathrm{L}^1(X). \tag{3.30}$$

Moreover, for every $\phi \in L^1(X)$, we have that

$$\int_X \mathrm{d}\mu_X(x)\phi(x) = \int_G \mathrm{d}\mu_G(g)\big(\widehat{\mathscr{L}_\psi}\phi\big)(g), \tag{3.31}$$

for a suitable (mutual) normalization of μ_X, μ_G.

Proof. Let us first prove that, for every $\phi \in \mathrm{L}^1(X)$, the (Borel) function $\widehat{\mathscr{L}_\psi}\phi$ belongs to $\mathrm{L}^1(G)$. In fact, by Lemma 7 of [85], for any Borel section $\mathtt{s}\colon X \to G$, the pushforward measure $(\gamma_{\mathtt{s}})_*(\mu_X \times \mu_H)$ coincides (up to normalization) with μ_G. Hence, we have that

$$\begin{aligned}
\int_G \mathrm{d}\mu_G(g)\big|\big(\widehat{\mathscr{L}_\psi}\phi\big)(g)\big| &= \int_G \mathrm{d}\mu_G(g)\psi(g)|\phi(q(g))| \\
&= \int_G \mathrm{d}\big((\gamma_{\mathtt{s}})_*(\mu_X \times \mu_H)\big)(g)\psi(g)|\phi(q(g))| \\
&= \int_{X\times H} \mathrm{d}\mu_X \times \mu_H(x,h)\psi(\mathtt{s}(x)h)|\phi(x)| \\
&= \int_X \mathrm{d}\mu_X(x)\int_H \mathrm{d}\mu_H(h)\psi(\mathtt{s}(x)h)|\phi(x)|,
\end{aligned} \tag{3.32}$$

where the last equality is obtained by Tonelli's theorem. Therefore, we find that

$$\int_G \mathrm{d}\mu_G(g)\big|\big(\widehat{\mathscr{L}_\psi}\phi\big)(g)\big| = \int_X \mathrm{d}\mu_X(x)|\phi(x)| = \|\phi\|_{\mathrm{L}^1(X)}, \tag{3.33}$$

and $\widehat{\mathscr{L}_\psi}\phi \in \mathrm{L}^1(G)$. At this point, one easily proves (3.30) and (3.31). □

Remark 3.1.8. Relation (3.33) shows that $\widehat{\mathscr{L}_\psi}\colon \mathrm{L}^1(X) \to \mathrm{L}^1(G)$ is a (linear) isometry.

To conclude this section, we state the following remarkable consequence of Theorem 3.1.1

Theorem 3.1.4. *Let us suppose that H is compact. Then, for a suitable normalization of μ_G and μ_X, $q_*(\mu_G) = \mu_X$.*

Proof. Since H is compact, $q_*(\mu_G)$ is a Radon measure on X (for every compact $E \subset X$, $q^{-1}(E) = KH$, with $K \subset G$ compact by Lemma 2.48 of [83]; hence, KH is compact too). Then, for every $\phi \in \mathrm{C}_c(X)$, by the WMB formula we have that

$$\begin{aligned}\int_X \mathrm{d}q_*(\mu_G)(x)\phi(x) &= \int_G \mathrm{d}\mu_G(g)\phi(q(g)) \\ &= \int_X \mathrm{d}\mu_X(x) \int_H \mathrm{d}\mu_H(h)\phi(x) = \int_X \mathrm{d}\mu_X(x)\phi(x), \end{aligned} \tag{3.34}$$

where we have assumed that $\mu_H(H) = 1$ and μ_G,μ_H are suitably normalized. Hence, $q_*(\mu_G) = \mu_X$. □

3.2 p-Adic Lie groups

In this section, we discuss the main features of p-adic manifolds and p-adic Lie groups [78, 79, 119]. As in the standard real setting, the starting point is to introduce a suitable notion of chart on a Hausdorff space.

Definition 3.2.1. Let $\mathscr{X}$ be a Hausdorff space. A *chart* on $\mathscr{X}$ is a triple $(\mathscr{U}, \varphi, \mathbb{Q}_p^n)$, where $\mathscr{U} \subset \mathscr{X}$ is an open subset and $\varphi\colon \mathscr{U} \to \mathbb{Q}_p^n$ is a map such that $\varphi\colon \mathscr{U} \to \varphi(\mathscr{U})$ is a homeomorphism. We refer to $\mathscr{U}$ as the *domain* of the chart, and to $n \in \mathbb{N}$ as its dimension.

If $x \in \mathscr{U} \subset \mathscr{X}$, we say that $(\mathscr{U}, \varphi, \mathbb{Q}_p^n)$ is a *chart around* x. In the following, we will set $\overleftarrow{\varphi}\colon \varphi(\mathscr{U}) \to \mathscr{U}$ to be the inverse map of φ on its range.

Definition 3.2.2. If U is an open subset of $\mathbb{Q}_p^n$, a function $f\colon U \to \mathbb{Q}_p$ is said to be a $\mathbb{Q}_p$-*analytic function*, if it is expressed by a convergent power series in a neighborhood of every x in U. A map $f = (f_1, \ldots, f_m)\colon U \to \mathbb{Q}_p^m$ is said to be a $\mathbb{Q}_p$-*analytic map*, if every f_i, $i = 1, \ldots, m$, is a $\mathbb{Q}_p$-analytic function.

Definition 3.2.3. Two charts $(\mathscr{U}_1, \varphi_1, \mathbb{Q}_p^{n_1})$ and $(\mathscr{U}_2, \varphi_2, \mathbb{Q}_p^{n_2})$ on $\mathscr{X}$ are *compatible*, if both $\varphi_2 \circ \overleftarrow{\varphi}_1\colon \varphi_1(\mathscr{U}_1 \cap \mathscr{U}_2) \to \varphi_2(\mathscr{U}_1 \cap \mathscr{U}_2)$ and $\varphi_1 \circ \overleftarrow{\varphi}_2\colon \varphi_2(\mathscr{U}_1 \cap \mathscr{U}_2) \to \varphi_1(\mathscr{U}_1 \cap \mathscr{U}_2)$ are $\mathbb{Q}_p$-analytic maps.

If $(\mathscr{U}_1, \varphi_1, \mathbb{Q}_p^{n_1})$ and $(\mathscr{U}_2, \varphi_2, \mathbb{Q}_p^{n_2})$ are compatible charts on $\mathscr{X}$ such that $\mathscr{U}_1 \cap \mathscr{U}_2 \neq \emptyset$, then one can prove that, necessarily, it is $n_1 = n_2$ [78].

Definition 3.2.4. An *atlas* $\mathcal{A}$ for $\mathscr{X}$ is a family $\{(\mathscr{U}_\alpha, \varphi_\alpha, \mathbb{Q}_p^{n_\alpha})\}_{\alpha \in A}$ of pairwise compatible charts which cover $\mathscr{X}$, i.e., $\mathscr{X} = \bigcup_{\alpha \in A} \mathscr{U}_\alpha$. An atlas $\mathcal{A}$ for $\mathscr{X}$ is called n-dimensional if all the charts in $\mathcal{A}$ have dimension n.

Similarly to the standard real case, it is now natural to set the following:

Definition 3.2.5. A Hausdorff space, $\mathscr{X}$, together with a maximal (w.r.t. inclusion) atlas $\mathcal{A}$ is called a $\mathbb{Q}_p$-*analytic manifold*. The manifold is called n-dimensional if the atlas $\mathcal{A}$ is n-dimensional.

For notational convenience, in what follows we shall denote an n-dimensional atlas on $\mathscr{X}$ as $\mathcal{A} = \{(\mathscr{U}_\alpha, \varphi_\alpha)\}_{\alpha\in A}$; moreover, we will refer to a '$\mathbb{Q}_p$-analytic manifold' simply as to a '$\mathbb{Q}_p$-manifold'. If $\mathscr{X}$, $\mathscr{Y}$ are two $\mathbb{Q}_p$-manifolds of dimension m and n respectively, we shall say that a map f from $\mathscr{X}$ to $\mathscr{Y}$ is $\mathbb{Q}_p$-*analytic* if, for every $x \in \mathscr{X}$, there exist a chart $(\mathscr{U}, \varphi, \mathbb{Q}_p^m)$ on $\mathscr{X}$ around x, and a chart $(\mathscr{V}, \psi, \mathbb{Q}_p^n)$ on $\mathscr{Y}$ around $f(x)$, such that $f(\mathscr{U}) \subset \mathscr{V}$, and $\psi \circ f \circ \overleftarrow{\varphi}\colon \varphi(\mathscr{U}) \to \mathbb{Q}_p^n$ is a $\mathbb{Q}_p$-analytic map.

Remark 3.2.1. Every $\mathbb{Q}_p$-manifold $\mathscr{X}$ is both totally disconnected and locally compact (TDLC in short). In particular, the latter condition entails that for every point x of $\mathscr{X}$, the set $\mathcal{T}_x$ of all compact open subsets in $\mathscr{X}$ containing x forms a base at x (see Lemma 7.1.1 in [120]). Therefore, the set $\mathcal{T}(\mathscr{X}) = \bigcup_{x\in\mathscr{X}} \mathcal{T}_x$ of all the compact open subsets of $\mathscr{X}$ forms a basis for the topology of $\mathscr{X}$.

Analytic differential forms on $\mathbb{Q}_p$-manifolds are defined in a similar fashion to the standard real setting (see Chapter 2 in [120] for a thorough discussion). Indeed, let $\mathscr{X}$ be a $\mathbb{Q}_p$-manifold of dimension n, and let $\mathcal{A} = \{(\mathscr{U}_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ be an atlas on $\mathscr{X}$. If Θ is a *differential form of degree $k < n$* on $\mathscr{X}$, its restriction $\Theta_\alpha \coloneqq \Theta|_{\mathscr{U}_\alpha}$ — in the local coordinates of $(\mathscr{U}_\alpha, \varphi_\alpha)$ — is given by

$$\Theta_\alpha(u) = \sum_{j_1<\ldots<j_k} \theta^\alpha_{j_1\ldots j_k}(u) dx_{j_1} \wedge \ldots \wedge dx_{j_k}, \tag{3.35}$$

where $\theta^\alpha_{j_1\ldots j_k}$ are $\mathbb{Q}_p$-valued functions on $\mathscr{U}_\alpha$, and where we set $\varphi_\alpha(u) = (x_1, \ldots, x_n)$ to denote the local coordinates of u in $\mathscr{U}_\alpha$. If, for every $\alpha \in A$, the maps $\theta^\alpha_{j_1,\ldots,j_k}$ are all $\mathbb{Q}_p$-analytic functions on $\mathscr{U}_\alpha$, we say that Θ is a $\mathbb{Q}_p$-*analytic differential k-form* on $\mathscr{X}$. If Ω is a $\mathbb{Q}_p$-analytic differential n-form on $\mathscr{X}$ (i.e., of maximal degree equal to the dimension n of $\mathscr{X}$), its local expression $\Omega_\alpha \coloneqq \Omega|_{\mathscr{U}_\alpha}$ can be written as

$$\Omega_\alpha(u) = \omega_\alpha(u) \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n, \tag{3.36}$$

for $\omega_\alpha\colon \mathscr{U}_\alpha \to \mathbb{Q}_p$ a $\mathbb{Q}_p$-analytic function. In what follows, we shall abbreviate '$\mathbb{Q}_p$-analytic differential k-form' to 'differential k-form'.

Let $F\colon \mathscr{X} \to \mathscr{Y}$ be a $\mathbb{Q}_p$-analytic map between n-dimensional $\mathbb{Q}_p$-manifolds $\mathscr{X}$, $\mathscr{Y}$, and let Ξ be a differential n-form on $\mathscr{Y}$. The *pullback* $F^*\Xi$ of Ξ through F is a well-defined differential n-form on $\mathscr{X}$ [120]; specifically, if $(\mathscr{U}_\alpha, \varphi_\alpha)$ is a chart in $\mathscr{X}$, and $(\mathscr{V}_\beta, \psi_\beta)$ is a chart in $\mathscr{Y}$, then, on $\mathscr{U}_\alpha \cap F^{-1}(\mathscr{V}_\beta)$ one has:

$$F^*\Xi_\beta = F^*(\xi_\beta\, \mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}y_n) = (\xi_\beta \circ F)(\det \mathrm{D}F) \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n, \tag{3.37}$$

where $(x_i)_{i=1}^n$ and $(y_j)_{j=1}^n$ denote the systems of local coordinates of $\mathscr{U}_\alpha$ and $\mathscr{V}_\beta$ respectively, and where $\mathrm{D}F$ is the *Jacobian matrix* of the transformation F.

To conclude this subsection, we now discuss the principal object of our investigations:

Definition 3.2.6. A *p-adic Lie group* G is a $\mathbb{Q}_p$-manifold which is also a group, and such that the multiplication map

$$G \times G \ni (g, h) \mapsto gh \in G \tag{3.38}$$

is $\mathbb{Q}_p$-analytic.

From Definition 3.2.6, it follows that the inverse map, $G \ni g \mapsto g^{-1} \in G$, is a $\mathbb{Q}_p$-analytic map. Moreover, it is clear that every p-adic Lie group is a TDLC Hausdorff space (see Definition 3.2.5 and Remark 3.2.1).

Remark 3.2.2. Let G be a p-adic Lie group. For $h \in G$, the map ℓ_h of *left translation* by h is defined as:

$$G \ni g \mapsto \ell_h(g) \coloneqq hg \in G. \tag{3.39}$$

This map is the composition of the map $G \ni g \mapsto (h, g) \in G \times G$, and the multiplication map defined in (3.38); hence, it is $\mathbb{Q}_p$-analytic (the composition of $\mathbb{Q}_p$-analytic maps is a $\mathbb{Q}_p$-analytic map; see Lemma 8.4 in [78]). Similarly, one can define the map of right translation, r_h, on G, which is $\mathbb{Q}_p$-analytic as well.

Remark 3.2.3. A classical result by Van Dantzig (see Theorem 7.7 in [121]) states that a TDLC group admits a base at the identity consisting of compact open subgroups (and vice versa). This result provides a peculiar characterization of the topology of p-adic Lie groups.

Since a p-adic Lie group, G, is a $\mathbb{Q}_p$-manifold, we can clearly define differential k-forms on it. In particular, we say that a differential k-form Θ on G is *left-invariant* if $\ell_h^*\Theta = \Theta$ for any $h \in G$, i.e., if

$$\ell_h^*\Theta(g) = \Theta(h^{-1}g) \tag{3.40}$$

holds for every g and h in G. Right-invariant differential n-forms are defined similarly with ℓ_h replaced by r_h. By taking $h \equiv g^{-1}$ and $g \equiv e$ in (3.40), we also see that

$$\ell_{g^{-1}}^*\Theta(e) = \Theta(g), \tag{3.41}$$

that is, if Θ is left-invariant on G, its value at every point on G is determined by the value Θ assumes at the identity e in G. In the next subsection, we shall prove that a left-invariant n-form on G can always be constructed, and that it naturally induces the left-invariant Haar measure on G. For the moment, we want to stress a relevant topological feature of p-adic Lie groups which will turn out to be central in our later derivations. We recall that a Hausdorff space $\mathscr{X}$ is called *paracompact*, if every open covering of $\mathscr{X}$ can be refined into a *locally finite* open covering. We say that $\mathscr{X}$ is *strictly paracompact* if every open cover of $\mathscr{X}$ admits a refinement consisting of pairwise disjoint open sets.

Proposition 3.2.1. *Let G be a second countable p-adic Lie group. Then, G is a strictly paracompact space.*

Proof. By assumption, G is locally compact, second countable and Hausdorff, hence σ-compact (i.e., union of countably many compact subspaces). Every σ-compact space is Lindelöf, and, therefore, paracompact (cf. Theorem 5.1.11. in [122]). Then, the proposition follows by the equivalence of the points i and ii of Proposition 8.7 in [78]. □

Remark 3.2.4. From Proposition 3.2.1 it follows that any second countable p-adic Lie group G can always be endowed with an atlas consisting of pairwise disjoint charts. Indeed, since every atlas is an open covering, it admits a refinement consisting of pairwise disjoint open sets. Then, the restriction of the coordinate maps of the initial atlas to the sets in the refinement provides a new system of charts for G.

Remark 3.2.5. It is well known that a LC group G is *Polish* if and only if it admits a second countable topology (see Theorem 5.3 in [121]). Therefore, a second countable p-adic Lie group is also a Polish group.

In this work, p-adic Lie groups will always be assumed to be second countable (as so are the most important examples); hence, they are LCSC Polish groups.

3.3 Integration on p-adic manifolds

This section deals with integration theory on p-adic manifolds [120]. For our purposes, we will need a p-adic counterpart of the well known change-of-variables formula for multiple integrals on $\mathbb{R}^n$. Therefore, we start with the following:

Theorem 3.3.1 (Change-of-variables formula)**.** *Let $a \in \mathbb{Q}_p^n$ and let $\xi = (\xi_1, \ldots, \xi_n)\colon U \subset \mathbb{Q}_p^n \to V \subset \mathbb{Q}_p^n$ be a $\mathbb{Q}_p$-analytic isomorphism between an open neighborhood U of a, and an open neighborhood V of $\xi(a)$, such that*

$$\det\left(\frac{\partial \xi_i}{\partial x_j}(a)\right) \neq 0. \tag{3.42}$$

Then, for every integrable function f on V, the following formula holds:

$$\int_V f(x)\mathrm{d}\lambda_{|V}(x) = \int_U f(\xi(x)) \left|\det\left(\frac{\partial \xi_i}{\partial x_j}(x)\right)\right|_p \mathrm{d}\lambda_{|U}(x), \tag{3.43}$$

where λ is the Haar measure on $\mathbb{Q}_p^n$.

Proof. Formula (3.43) is actually a special case of the abstract C.O.V.F. (see (3.26) in Remark 3.1.7) specialized to the case where the pushforward of the measure on U is realized via a $\mathbb{Q}_p$-analytic map. See Proposition 7.4.1 in [120] for the technical details. □

Let $\mathscr{X}$ be a second countable n-dimensional $\mathbb{Q}_p$-manifold, and Ω a differential n-form on $\mathscr{X}$. If $\mathcal{A} = \{(\mathscr{U}_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ is an atlas on $\mathscr{X}$, Ω is expressed as in (3.36) in the local coordinates of each chart in $\mathcal{A}$. Then, we can associate a Radon measure μ_Ω with Ω by setting

$$\mu_\Omega(\mathscr{C}) := \int_{\mathscr{C}} |\omega_\alpha(u)|_p \mathrm{d}\big((\overleftarrow{\varphi}_\alpha)_*\lambda\big)(u), \tag{3.44}$$

for every compact (open) subset $\mathscr{C} \subset \mathscr{U}_\alpha$ of $\mathscr{X}$, and where $\mathrm{d}((\overleftarrow{\varphi}_\alpha)_*\lambda)(u)$ denotes the pushforward of the Haar measure λ by $\overleftarrow{\varphi}_\alpha$. It is not difficult to verify that this measure is well-defined: If $(\mathscr{U}_\beta, \varphi_\beta)$ is another chart in $\mathcal{A}$ containing $\mathscr{C}$ (i.e., $\mathscr{C} \subset \mathscr{U}_\beta \cap \mathscr{U}_\alpha$), then

$$\int_{\mathscr{C}} |\omega_\alpha(u)|_p \mathrm{d}\big((\overleftarrow{\varphi}_\alpha)_*\lambda\big)(u) = \int_{\mathscr{C}} |\omega_\beta(u)|_p \mathrm{d}\big((\overleftarrow{\varphi}_\beta)_*\lambda\big)(u), \tag{3.45}$$

that is, $\mu_\Omega(\mathscr{C})$ does not depend on the considered chart containing $\mathscr{C}$. We shall give the proof of this result in Remark 3.3.2 below.

Remark 3.3.1. Since a second countable $\mathbb{Q}_p$-manifold is σ-compact, the measure (7.34) is *regular* (cf. Theorem 7.8 in [115]). Then, the measure of a Borel set $\mathcal{E} \subset \mathscr{U}_\alpha$ of $\mathscr{X}$ is given, by *inner regularity*, by the supremum of the measures of the compact (open) sets contained in $\mathcal{E}$.

If $f \in \mathrm{C}_c(\mathscr{X})$ is such that $\mathrm{supp}(f) \subset \mathscr{C} \subset \mathscr{U}_\alpha$, its integral w.r.t. μ_Ω is also well-defined, and is given by

$$\int_{\mathscr{X}} f\Omega := \int_{\mathscr{U}_\alpha} f(u)|\omega_\alpha(u)|_p \mathrm{d}\big((\overleftarrow{\varphi}_\alpha)_*\lambda\big)(u). \tag{3.46}$$

Let now $\mathscr{C}$ be an arbitrary compact (open) subset of $\mathscr{X}$. Its measure w.r.t. μ_Ω can be defined as follows. First, we can decompose $\mathscr{C}$ as

$$\mathscr{C} = \bigsqcup_i \mathscr{C}_i, \qquad \mathscr{C}_i \subset \mathscr{U}_\alpha, \text{ for some } \alpha \in A, \tag{3.47}$$

i.e., as a disjoint union of compact (open) subsets $\mathscr{C}_i$, each contained in some $\mathscr{U}_\alpha$. Then, the measure of $\mathscr{C}$ is given by

$$\mu_\Omega(\mathscr{C}) = \sum_i \mu_\Omega(\mathscr{C}_i). \tag{3.48}$$

Similarly, we can then extend the measure (3.48) to arbitrary Borel sets $\mathcal{E}$ in $\mathscr{X}$ (see Remark 3.3.1). Exploiting (3.46), it is then not difficult to define the integral of an arbitrary function $f \in \mathrm{C}_c(\mathscr{X})$ w.r.t. μ_Ω, as well.

We can consider the pushforward of the measure μ_Ω via φ_α to a measure on $\mathbb{Q}_p^n$. This allows us to treat the integration theory on a manifold $\mathscr{X}$ via integrals on $\mathbb{Q}_p^n$. Indeed, using formula (3.27), we have:

$$\mathrm{d}\big((\varphi_\alpha)_*\mu_\Omega\big)(x) = (\varphi_\alpha)_*\Big(|\omega_\alpha(u)|_p \mathrm{d}\big((\overleftarrow{\varphi}_\alpha)_*\lambda\big)(u)\Big) = |(\omega_\alpha \circ \overleftarrow{\varphi}_\alpha)(x)|_p \mathrm{d}\lambda(x), \tag{3.49}$$

where $\varphi_\alpha(u) = (x_1, \dots, x_n) =: x$ denotes the coordinate representation of the point $u \in \mathscr{X}$. Hence, using the (abstract) C.O.V.F. (cf. relation (3.26) in Remark 3.1.7) with $f = \chi_{\mathscr{C}}$, we obtain

$$\mu_\Omega(\mathscr{C}) = \int_{\mathscr{C}} |\omega_\alpha(u)|_p \mathrm{d}\big((\overleftarrow{\varphi}_\alpha)_*\lambda\big)(u) = \int_{\varphi_\alpha(\mathscr{C})} |(\omega_\alpha \circ \overleftarrow{\varphi}_\alpha)(x)|_p \mathrm{d}\lambda(x). \tag{3.50}$$

Furthermore, if $f \in \mathrm{C}_c(\mathscr{X})$ is a function with $\mathrm{supp}(f) \subset \mathscr{C} \subset \mathscr{U}_\alpha$, it is clear that

$$\int_{\mathscr{U}_\alpha} f(u)|\omega_\alpha(u)|_p \mathrm{d}\big((\overleftarrow{\varphi}_\alpha)_*\lambda\big)(u) = \int_{\varphi_\alpha(\mathscr{U}_\alpha)} (f \circ \overleftarrow{\varphi}_\alpha)(x)|(\omega_\alpha \circ \overleftarrow{\varphi}_\alpha)(x)|_p \mathrm{d}\lambda(x). \tag{3.51}$$

With the above discussion, we get to the following two conclusions. First, from (3.44), we see that

$$\mu_\Omega|_{\mathscr{U}_\alpha} \ll (\overleftarrow{\varphi}_\alpha)_*\lambda, \qquad \frac{\mathrm{d}\mu_\Omega|_{\mathscr{U}_\alpha}}{\mathrm{d}(\overleftarrow{\varphi}_\alpha)_*\lambda} = |\omega_\alpha|_p, \tag{3.52}$$

i.e., $\mu_\Omega|_{\mathscr{U}_\alpha}$ is an *absolutely continuous measure* w.r.t. $(\overleftarrow{\varphi}_\alpha)_*\lambda$, with *Radon-Nikodym derivative* given by $|\omega_\alpha|_p \colon \mathscr{U}_\alpha \to \mathbb{R}_*^+$. Secondly, (3.50) entails that $(\varphi_\alpha)_*\mu_\Omega \ll \lambda|_{\varphi_\alpha(\mathscr{U}_\alpha)}$ as well, with Radon-Nikodym derivative $|\omega_\alpha \circ \overleftarrow{\varphi}_\alpha|_p \colon \varphi_\alpha(\mathscr{U}_\alpha) \to \mathbb{R}_*^+$. The latter condition means that the pushforward of the measure μ_Ω on $\mathscr{X}$ via the maps φ_α, for every $\alpha \in A$, provides an absolutely continuous measure w.r.t. the Haar measure λ on $\mathbb{Q}_p^n$. Their Radon-Nikodym derivative — which, for notational convenience, hereafter we will simply denote by η — is globally defined, and it is *uniquely defined* up to a set of points of null measure (any other Radon-Nikodym derivative is equal to η *almost everywhere*). Accordingly, we shall denote by $\eta_\alpha := |\omega_\alpha \circ \overleftarrow{\varphi}_\alpha|_p$ the restriction of η on $\varphi_\alpha(\mathscr{U}_\alpha) \subset \mathbb{Q}_p^n$, for every $\mathscr{U}_\alpha$ in the covering atlas $\mathcal{A}$ of $\mathscr{X}$. Exploiting this notation, and recalling condition (3.50), we can then write

$$\mu_\Omega(\mathscr{C}) = \int_{\varphi_\alpha(\mathscr{C})} \eta_\alpha(x) \mathrm{d}\lambda(x), \tag{3.53}$$

for every compact (open) subset $\mathscr{C} \subset \mathscr{U}_\alpha \subset \mathscr{X}$. Similarly, we can express the integral w.r.t. Ω of every function $f \in \mathrm{C}_c(\mathscr{X})$ — with $\mathrm{supp}(f) \subset \mathscr{C} \subset \mathscr{U}_\alpha$ — as

$$\int_{\mathscr{X}} f\Omega = \int_{\mathscr{U}_\alpha} f(u)|\omega_\alpha(u)|_p \mathrm{d}\big((\overleftarrow{\varphi}_\alpha)_*\lambda\big)(u) = \int_{\varphi_\alpha(\mathscr{U}_\alpha)} (f \circ \overleftarrow{\varphi}_\alpha)(x)\eta_\alpha(x) \mathrm{d}\lambda(x). \tag{3.54}$$

Remark 3.3.2. Using the local representation (3.53), it is now not difficult to prove the equality of integrals in (3.45). Indeed, let $\mathscr{C} \subset \mathscr{U}_\alpha \cap \mathscr{U}_\beta$ be a compact (open) set in $\mathscr{X}$. Then, we want to show that

$$\int_{\varphi_\alpha(\mathscr{C})} \eta_\alpha(x) \mathrm{d}\lambda(x) = \int_{\varphi_\beta(\mathscr{C})} \eta_\beta(y) \mathrm{d}\lambda(y). \tag{3.55}$$

We first consider the change of variable $y = (\varphi_\beta \circ \overleftarrow{\varphi}_\alpha)(x)$ in the r.h.s. of (3.55). Then, using Theorem 3.3.1, we see that (3.55) holds if and only if

$$\eta_\alpha(x) = (\eta_\beta \circ \varphi_\beta \circ \overleftarrow{\varphi}_\alpha)(x)|\det \mathrm{D}(\varphi_\beta \circ \overleftarrow{\varphi}_\alpha)(x)|_p, \tag{3.56}$$

where $\det \mathrm{D}(\varphi_\beta \circ \overleftarrow{\varphi}_\alpha)(x)$ denotes the Jacobian of the transformation $\varphi_\beta \circ \overleftarrow{\varphi}_\alpha$. On the other hand, the pullback formula (3.37) also shows that

$$\begin{aligned}(\omega_\alpha \circ \overleftarrow{\varphi}_\alpha)(x)\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n &= (\overleftarrow{\varphi}_\alpha)^*\Omega \\ &= (\varphi_\beta \circ \overleftarrow{\varphi}_\alpha)^*(\overleftarrow{\varphi}_\beta)^*\Omega \\ &= (\varphi_\beta \circ \overleftarrow{\varphi}_\alpha)^*(\omega_\beta \circ \overleftarrow{\varphi}_\beta)(y)\mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}y_n \\ &= (\omega_\beta \circ \overleftarrow{\varphi}_\beta \circ \varphi_\beta \circ \overleftarrow{\varphi}_\alpha)(x) \det \mathrm{D}(\varphi_\beta \circ \overleftarrow{\varphi}_\alpha)(x)\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n.\end{aligned} \tag{3.57}$$

Therefore, taking the p-adic absolute value of the l.h.s. and of the last equality in (3.57) entails that (3.56) (and, hence, (3.55)) holds.

To conclude this subsection, we prove that it is always possible to construct an essentially unique — i.e., uniquely defined up to a multiplicative constant — (left-)invariant differential n-form on every n-dimensional p-adic Lie group. We will then show that it is naturally associated with the (left) Haar measure on the group. This will draw a parallel with the standard theory of (real) Lie groups [123, 124].

Let us first note that, also in the p-adic setting the tangent space T_eG to G at $e \in G$ has a natural structure of *Lie algebra* $\mathfrak{g}$, whenever the elements $X \in \mathrm{T}_eG$ are identified with the corresponding *left-invariant vector fields* $\widetilde{X}$ on G [119]. Let $X_1, \ldots, X_n$ be a basis of T_eG, and let $\widetilde{X}_1, \ldots, \widetilde{X}_n$ be the corresponding left-invariant vector fields in $\mathfrak{g}$. We can now define, for all g in G, the 1-forms $\omega_1, \ldots, \omega_n$ on G via the condition

$$(\omega_i)_g \left((\widetilde{X}_j)_g \right) = \delta_{ij}, \quad j = 1, \ldots, n. \tag{3.58}$$

By construction, $\omega_1, \ldots, \omega_n$ are left-invariant 1-forms on G, as follows by observing that

$$(\ell_g^* \omega_i)(\tilde{X}_j) = \omega_i(\ell_g^* \tilde{X}_j) = \omega_i(\tilde{X}_j). \tag{3.59}$$

In particular, this also entails that $\omega_1, \ldots, \omega_n$ form a basis of the dual space of T_gG for every $g \in G$. Therefore, the differential form Ω_{inv} defined as

$$\Omega_{\mathrm{inv}} \coloneqq \omega_1 \wedge \cdots \wedge \omega_n, \tag{3.60}$$

is a (nowhere vanishing) left-invariant n-form on G. Indeed, since the pullback ℓ_g^* commutes with $\wedge$, we have:

$$\ell_g^* \Omega_{\mathrm{inv}} = \ell_g^*(\omega_1 \wedge \ldots \wedge \omega_n) = \ell_g^* \omega_1 \wedge \ldots \wedge \ell_g^* \omega_n = \omega_1 \wedge \ldots \wedge \omega_n = \Omega_{\mathrm{inv}}, \tag{3.61}$$

that is, Ω_{inv} is left-invariant. It is clear that any constant multiple of Ω_{inv} is a left-invariant n-form as well. Conversely, if $\breve{\Omega}$ is another left-invariant n-form on G, there must exist $c \in \mathbb{Q}_p$ such that $\breve{\Omega}(e) = c\Omega_{\mathrm{inv}}(e)$. But then, the left-invariance condition (3.41) entails that $\breve{\Omega}(g) = c\Omega_{\mathrm{inv}}(g)$ for every g in G.

We want now to show that if Ω_{inv} is the left-invariant differential n-form on G, its induced measure, $\mu_{\Omega_{\mathrm{inv}}}$, is the left Haar measure on G (up to multiplicative constants). Indeed, we already know that $\mu_{\Omega_{\mathrm{inv}}}$ is a Radon measure. To conclude that it is a Haar measure, we have to show that it is left-invariant. Let $\mathscr{C}$ be a compact (open) set in G. From the left-invariance of Ω_{inv} we see that

$$\omega_\alpha \circ \overleftarrow{\varphi}_\alpha = (\overleftarrow{\varphi}_\alpha)^* \Omega_{\mathrm{inv}} = (\overleftarrow{\varphi}_\alpha)^* L_{g^{-1}}^* \Omega_{\mathrm{inv}}, \tag{3.62}$$

for every $g \in G$. This entails that $\mu_{\Omega_{\mathrm{inv}}}(g\mathscr{C}) = \mu_{\Omega_{\mathrm{inv}}}(\mathscr{C})$, for every compact (open) set $\mathscr{C} \subset G$, and g in G (see (3.50)). Moreover, since G is second countable, $\mu_{\Omega_{\mathrm{inv}}}$ is regular. In particular, inner regularity entails that

$$\mu_{\Omega_{\mathrm{inv}}}(\mathcal{E}) = \sup\{\mu_{\Omega_{\mathrm{inv}}}(K) \mid K \subset \mathcal{E} \text{ compact}\}$$

$$
\begin{aligned}
&= \sup\{\mu_{\Omega_{\mathrm{inv}}}(gK) \mid K \subset \mathcal{E} \text{ compact}\} \\
&= \mu_{\Omega_{\mathrm{inv}}}(g\mathcal{E}), \qquad (3.63)
\end{aligned}
$$

for every Borel set $\mathcal{E}$ in G, and every $g \in G$. Concluding, we proved that $\mu_{\Omega_{\mathrm{inv}}}$ is a left-invariant Radon measure on G, and since the Haar measure is essentially uniquely defined, it must coincide with the Haar measure on G up to a multiplicative constant.

4

The Haar measure on p-adic Lie groups

In this chapter, we show how to construct a left Haar measure μ on a (second countable) p-adic Lie group G. Our approach exploits the peculiar topological features of p-adic Lie groups, and relies on the possibility to construct a *quasi-invariant measure* for G. Eventually, we will prove that the measure thus constructed coincides with the measure induced by the left-invariant differential n-form Ω_{inv} on G (see Subsection 3.3).

4.1 Construction of the Haar measure

We begin by recalling the notion of a quasi-invariant measure [83]. Let G be a p-adic Lie group, and let ν be a Radon measure on it. For $h \in G$, we can define the left translation ν^h, of ν by h, as

$$\nu^h(\mathcal{E}) \coloneqq \nu(h\mathcal{E}), \tag{4.1}$$

for every Borel set $\mathcal{E} \in \mathcal{B}_G$. We say that ν is *quasi-invariant* if the measures ν^h are all equivalent, i.e., mutually absolutely continuous [115]. In such a case, we have:

$$\mathrm{d}\nu^h(g) = \eta(h,g)\mathrm{d}\nu(g), \tag{4.2}$$

where $\eta\colon G \times G \to \mathbb{R}_*^+$ is a positive map on $G \times G$. The function η is the Radon-Nikodym derivative $\mathrm{d}\nu^h/\mathrm{d}\nu$. For $h, h' \in G$, since $\nu^{hh'} = (\nu^h)^{h'}$, the chain rule for the Radon-Nikodym derivative entails the following *cocycle formula*:

$$\eta(hh', g) = \eta(h, h'g)\eta(h', g), \tag{4.3}$$

for every $g \in G$. In particular, using (4.3) it is not difficult to prove the following result.

Lemma 4.1.1. *Let G be a p-adic Lie group, and let ν be a quasi-invariant measure on it. The measure defined as*

$$\mathrm{d}\mu(g) \coloneqq \eta(g,e)^{-1}\mathrm{d}\nu(g) \tag{4.4}$$

— where e denotes the identity element in G — is a left Haar measure on G.

Proof. Let μ^h be the left translation, by h in G, of the measure μ, as defined in (4.1). For every Borel set $\mathcal{E}$ in $\mathcal{B}_G$, we have:

$$\mu^h(\mathcal{E}) = \int_{h\mathcal{E}} \eta(g,e)^{-1}\mathrm{d}\nu(g) = \int_{\mathcal{E}} \eta(hg,e)^{-1}\mathrm{d}\nu^h(g), \tag{4.5}$$

where in the last equality we have used the change of variable $h^{-1}g \mapsto g$. Then, taking into account condition (4.2) for quasi-invariant measures, and exploiting the cocycle formula (4.3), we have:

$$\eta(hg,e) = \eta(h,g)\eta(g,e), \qquad \mathrm{d}\nu^h(g) = \eta(h,g)\mathrm{d}\nu(g), \tag{4.6}$$

which yield

$$\int_{\mathcal{E}} \eta(hg,e)^{-1} \mathrm{d}\nu^h(g) = \int_{\mathcal{E}} \eta(h,g)^{-1}\eta(g,e)^{-1}\eta(h,g)\mathrm{d}\nu(g) = \int_{\mathcal{E}} \eta(g,e)^{-1}\mathrm{d}\nu(g) = \mu(\mathcal{E}). \quad (4.7)$$

Therefore, the first equality in (4.5) and the last one in (4.7) give the desired result. □

From Lemma 4.1.1 we see that it is always possible to construct a left Haar measure on a p-adic Lie group G once known a quasi-invariant measure on it. Hence, our next step is to show how to explicitly construct a quasi-invariant measure on G.

Let $\mathcal{A} = \{(\mathscr{U}_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ be a *disjoint atlas* on G (cf. Proposition 3.2.1). We can construct a (regular) Radon measure ν on G as follows. First, in every chart $(\mathscr{U}_\alpha, \varphi_\alpha)$ in $\mathcal{A}$, we define a measure ν_α on $\mathscr{U}_\alpha$ by setting

$$\nu_\alpha \coloneqq (\overleftarrow{\varphi}_\alpha)_* \lambda_\alpha, \qquad \lambda_\alpha = \lambda|_{\varphi_\alpha(\mathscr{U}_\alpha)}, \quad (4.8)$$

that is, ν_α is the pushforward measure, via $\overleftarrow{\varphi}_\alpha \colon \varphi_\alpha(\mathscr{U}_\alpha) \to \mathbb{Q}_p^n$, of the restricted Haar measure $\lambda|_{\varphi_\alpha(\mathscr{U}_\alpha)}$ on $\mathbb{Q}_p^n$. Note that since ν_α is finite on compact sets, it is a Radon measure. In this way, we have constructed a Radon measure on every chart $(\mathscr{U}_\alpha, \varphi_\alpha)$ in $\mathcal{A}$. To obtain a Radon measure ν on the whole group G, we can then act as follows. Given any Borel set $\mathcal{E}$ in $\mathcal{B}_G$, we express it as the disjoint union $\mathcal{E} = \bigsqcup_{\alpha \in A} \mathcal{E}_\alpha$, where $\mathcal{E}_\alpha \coloneqq \mathcal{E} \cap \mathscr{U}_\alpha$, and set

$$\nu(\mathcal{E}) \coloneqq \sum_{\alpha \in A} \nu_\alpha(\mathcal{E}_\alpha). \quad (4.9)$$

Since A is countable, the series in (4.9) contains a countable number of non-null terms. It is now easily proved that the measure defined in (4.9) is a (regular) Radon measure on G. Indeed, ν takes values in $[0, +\infty]$ as so do all the ν_αs, and $\nu(\emptyset) = 0$. If $\{\mathcal{E}_i\}_i$ is a countable family of Borel sets in G, then $\nu(\cup_i \mathcal{E}_i) = \sum_i \nu(\mathcal{E}_i)$, as follows by observing that the ν_αs are σ-additive, and that the summation order can be exchanged in the double series $\sum_\alpha \sum_i \nu_\alpha(\mathcal{E}_i \cap \mathscr{U}_\alpha)$ by positivity of the ν_αs. Moreover, ν is clearly finite on compact sets, and since G is second countable, we can conclude that ν is a regular, and hence Radon, measure on G (cf. Theorem 7.8 in [115]).

Our next step is to show that this measure is quasi-invariant. To this end, let $h \in G$ be some fixed point, and let us set, for any $\alpha, \beta \in A$,

$$\mathscr{U}^h_{\alpha,\beta} \coloneqq \{g \in \mathscr{U}_\alpha \mid hg \in \mathscr{U}_\beta\} = h^{-1}\Big((h\mathscr{U}_\alpha) \cap \mathscr{U}_\beta)\Big) = \mathscr{U}_\alpha \cap (h^{-1}\mathscr{U}_\beta). \quad (4.10)$$

Note that $\mathscr{U}^h_{\alpha,\beta} \subset \mathscr{U}_\alpha$ is an *open* set and

$$\mathscr{U}_\alpha = \bigsqcup_\beta \mathscr{U}^h_{\alpha,\beta}. \quad (4.11)$$

Assuming that $\mathscr{U}^h_{\alpha,\beta} \neq \emptyset$, for every $j = 1, \ldots, n$, and at given $h \in G$, we put

$$\varphi_\alpha(\mathscr{U}^h_{\alpha,\beta}) \ni x \mapsto \zeta_{\beta,j}(h;x) \coloneqq \varphi_{\beta,j}(h\overleftarrow{\varphi}_\alpha(x)) \in \mathbb{Q}_p, \quad (4.12)$$

where $\varphi_{\beta,j}$ is the j-th vector component of $\varphi_\beta \colon \mathscr{U}_\beta \to \mathbb{Q}_p^n$, i.e., $\varphi_\beta = (\varphi_{\beta,1}, \ldots, \varphi_{\beta,j}, \ldots, \varphi_{\beta,n})$. Moreover, the definition of $\zeta_{\beta,j}(h;\,\cdot\,)$ can be extended to the whole open set $\varphi_\alpha(\mathscr{U}_\alpha) = \bigsqcup_\beta \varphi_\alpha(\mathscr{U}^h_{\alpha,\beta})$ by varying β in A. In this way, we obtain a map $\zeta_{\beta,j}(h;\,\cdot\,) \colon \varphi_\alpha(\mathscr{U}_\alpha) \to \mathbb{Q}_p$ (for suitable labels β depending on the charts as in (4.12)), for any given $h \in G$. We can then define a function

$$\rho_\beta(h;\,\cdot\,) \colon \varphi_\alpha(\mathscr{U}_\alpha) \to \mathbb{R}^+_*, \quad \rho_\beta(h;x) \coloneqq \left| \det \left[\frac{\partial \zeta_{\beta,j}}{\partial x_k}(h;x) \right]_{1 \le j,k \le n} \right|_p. \quad (4.13)$$

Eventually, we obtain a function $\eta\colon G \times G \to \mathbb{R}_*^+$, defined as follows:

$$\eta(h,g) \coloneqq \rho_\beta(h;\varphi_\alpha(g)), \quad g \in \mathscr{U}_\alpha,\ \alpha \in A. \tag{4.14}$$

Let us now define a (regular) Radon measure μ_h on G by setting

$$\mathrm{d}\mu_h(g) = \eta(h,g)\mathrm{d}\nu(g). \tag{4.15}$$

We want to prove that ν is quasi-invariant and, moreover, $\mu_h = \nu^h$, so that

$$\frac{\mathrm{d}\nu^h}{\mathrm{d}\nu}(g) = \eta(h,g). \tag{4.16}$$

Since ν is a regular measure, then ν^h and μ_h are *regular* measures. Hence, by outer regularity, it is sufficient to show that

$$\mu_h(\mathcal{O}) = \nu^h(\mathcal{O}), \tag{4.17}$$

for every open set $\mathcal{O} \subset G$. Actually, since

$$\mathcal{O} = \bigsqcup_\alpha (\mathcal{O} \cap \mathscr{U}_\alpha), \tag{4.18}$$

it is sufficient to prove (4.17) on every open subset $\mathcal{O} \equiv \mathcal{O}_\alpha$ of $\mathscr{U}_\alpha$, for $\alpha \in A$. Moreover, since for any open $\mathcal{O}_\alpha \subset \mathscr{U}_\alpha$,

$$\mathcal{O}_\alpha = \bigsqcup_\beta (\mathcal{O}_\alpha \cap \mathscr{U}^h_{\alpha,\beta}) = \bigsqcup_\beta \mathcal{O}^h_{\alpha,\beta}, \qquad \mathcal{O}^h_{\alpha,\beta} \coloneqq \mathcal{O}_\alpha \cap \mathscr{U}^h_{\alpha,\beta}, \tag{4.19}$$

it is enough to prove (4.17) on every open subset $\mathcal{O} \equiv \mathcal{O}^h_{\alpha,\beta}$ of $\mathscr{U}^h_{\alpha,\beta}$. Assuming that $\mathcal{O}^h_{\alpha,\beta} \neq \emptyset$ (otherwise there is nothing to prove), we have:

$$\begin{aligned}
\nu^h(\mathcal{O}^h_{\alpha,\beta}) &= \int_{h\mathcal{O}^h_{\alpha,\beta}} \mathrm{d}\nu(g) \\
&= \int_{\varphi_\beta(h\mathcal{O}^h_{\alpha,\beta})} \mathrm{d}\lambda(x) && (\text{since } h\mathcal{O}^h_{\alpha,\beta} \subset \mathscr{U}_\beta) \\
&= \int_{\varphi_\alpha(\mathcal{O}^h_{\alpha,\beta})} \rho(h;x)\mathrm{d}\lambda(x) && (\text{by C.O.V.F. (3.43)}) \\
&= \int_{\mathcal{O}^h_{\alpha,\beta}} \eta(h,g)\mathrm{d}\nu(g) && (\text{by (4.8)-(4.9) and (4.14)}) \\
&= \mu_h(\mathcal{O}^h_{\alpha,\beta}).
\end{aligned} \tag{4.20}$$

In conclusion, we have $\nu^h = \mu_h$. Therefore, ν^h and ν are mutually absolutely continuous, for every $h \in G$ — namely, ν is quasi-invariant — and $\frac{\mathrm{d}\nu^h}{\mathrm{d}\nu}(g) = \eta(h,g)$.

As a direct consequence of Lemma 4.1.1, the left Haar measure μ on G is of the form

$$\mathrm{d}\mu(g) = \eta(g,e)^{-1}\mathrm{d}\nu(g). \tag{4.21}$$

With the above construction, we have proved the following result:

Theorem 4.1.1. *Let G be a p-adic Lie group, and let $\mathcal{A} = \{(\mathscr{U}_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ be a disjoint atlas on G. If μ is the left Haar measure on G then, for every Borel set $\mathcal{E}$ in $\mathcal{B}_G$, and any $\mathscr{U}_\alpha$ in $\mathcal{A}$, the following equality holds:*

$$\mu(\mathcal{E} \cap \mathscr{U}_\alpha) = \int_{\varphi_\alpha(\mathcal{E}\cap\mathscr{U}_\alpha)} \left| \det \left[\frac{\partial \zeta_{\alpha,j}}{\partial x_k} (\overleftarrow{\varphi}_\alpha(y); \varphi_0(e)) \right]_{1\le j,k\le n} \right|_p^{-1} \mathrm{d}\lambda(y), \tag{4.22}$$

where $(\mathscr{U}_0, \varphi_0)$ is the chart around the identity $e \in G$, $(x_k)_{k=1}^n$ denotes a system of local coordinates w.r.t. $(\mathscr{U}_0, \varphi_0)$, and ζ_j is the map defined in (4.12).

Remark 4.1.1. In (4.22), the functions $\zeta_{\alpha,j}$ are correctly labelled by α. In fact, their derivatives are performed in a neighborhood of $x = \varphi_0(e)$, and, thus, $\zeta_{\alpha,j}(\overleftarrow{\varphi}_\alpha(y); x) = \varphi_{\alpha,j}(\overleftarrow{\varphi}_\alpha(y)\overleftarrow{\varphi}_0(x))$ whenever $\overleftarrow{\varphi}_0(x) \in \mathscr{U}_0$ is 'sufficiently close to e' such that $\overleftarrow{\varphi}_\alpha(y)\overleftarrow{\varphi}_0(x) \in \mathscr{U}_\alpha$.

Now, we prove that Theorem 4.1.1 still holds in the case of an atlas including possibly overlapping charts. Indeed, let $\mathcal{A} = \{(\mathscr{U}_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ be an arbitrary atlas on G. Since G is strictly paracompact (see Proposition 3.2.1), we can always find a refinement $\mathcal{A}'$ of $\mathcal{A}$ consisting of pairwise disjoint charts. Then, Theorem 4.1.1 provides us with a left Haar measure on (every chart of) $\mathcal{A}'$. To show that this measure is well-defined on $\mathcal{A}$ as well, we have to prove that for every Borel set $\mathcal{E}$ in $\mathcal{B}_G$ contained in the intersection of two charts in $\mathcal{A}$, the value of the integral in (4.22) is the *same* w.r.t. the local coordinates of the two charts; that is, we want to prove

$$\mu(\mathcal{E} \cap \mathscr{U}_\alpha) = \mu(\mathcal{E} \cap \mathscr{U}_\beta), \tag{4.23}$$

for every Borel set $\mathcal{E}$ in G such that $\mathcal{E} \subset \mathscr{U}_\alpha \cap \mathscr{U}_\beta$, $\mathscr{U}_\alpha, \mathscr{U}_\beta \in \mathcal{A}$. To start with, the r.h.s. of (4.23) explicitly is

$$\mu(\mathcal{E} \cap \mathscr{U}_\beta) = \int_{\varphi_\beta(\mathcal{E}\cap\mathscr{U}_\beta)} \left| \det \left[\frac{\partial \zeta_{\beta,j}}{\partial x_k} (\overleftarrow{\varphi}_\beta(z); \varphi_0(e)) \right]_{1\le j,k\le n} \right|_p^{-1} \mathrm{d}\lambda(z), \tag{4.24}$$

where we have denoted by y the local coordinates in the chart $(\mathscr{U}_\beta, \varphi_\beta)$. Then, the change of variable $z = \varphi_\beta \circ \overleftarrow{\varphi}_\alpha(y)$ immediately yields

$$\begin{aligned}
&\int_{\varphi_\beta(\mathcal{E}\cap\mathscr{U}_\beta)} \left| \det \left[\frac{\partial \zeta_{\beta,j}}{\partial x_k} (\overleftarrow{\varphi}_\beta(z); \varphi_0(e)) \right] \right|_p^{-1} \mathrm{d}\lambda(z) \\
&= \int_{\varphi_\alpha(\mathcal{E}\cap\mathscr{U}_\alpha)} \left| \det \left[\frac{\partial \zeta_{\beta,j}}{\partial x_k} \Big((\overleftarrow{\varphi}_\beta \circ \varphi_\beta \circ \overleftarrow{\varphi}_\alpha)(y); \varphi_0(e) \Big) \right] \right|_p^{-1} \left| \det \left[\frac{\partial (\varphi_\beta \circ \overleftarrow{\varphi}_\alpha)_j}{\partial y_k}(y) \right] \right|_p \mathrm{d}\lambda(y) \\
&= \int_{\varphi_\alpha(\mathcal{E}\cap\mathscr{U}_\alpha)} \left| \det \left[\frac{\partial \zeta_{\alpha,j}}{\partial x_k} (\overleftarrow{\varphi}_\alpha(y); \varphi_0(e)) \right] \right|_p^{-1} \left| \det \left[\frac{\partial (\varphi_\beta \circ \overleftarrow{\varphi}_\alpha)_j}{\partial y_k}(y) \right] \right|_p^{-1} \left| \det \left[\frac{\partial (\varphi_\beta \circ \overleftarrow{\varphi}_\alpha)_j}{\partial y_k}(y) \right] \right|_p \mathrm{d}\lambda(y) \\
&= \int_{\varphi_\alpha(\mathcal{E}\cap\mathscr{U}_\alpha)} \left| \det \left[\frac{\partial \zeta_{\alpha,j}}{\partial x_k} (\overleftarrow{\varphi}_\alpha(y); \varphi_0(e)) \right] \right|_p^{-1} \mathrm{d}\lambda(y) = \mu(\mathcal{E} \cap \mathscr{U}_\alpha),
\end{aligned} \tag{4.25}$$

where, for notational convenience, we have omitted $1 \le j, k \le n$ in the Jacobians. Note that in the second equality of (4.25), we have used the C.O.V.F. for multiple integrals in $\mathbb{Q}_p^n$ (cf. Theorem 3.3.1). Moreover, in the third equality, we have used the fact that $\zeta_{\beta,j}$ and $\zeta_{\alpha,j}$ are related via the condition $\zeta_{\alpha,j} = \varphi_{\alpha,j} \circ \overleftarrow{\varphi}_{\beta,j} \circ \zeta_{\beta,j}$, and then we have exploited the usual chain rule for the Jacobian of a composite function. Therefore, (4.23) shows that the (left) Haar measure in Theorem 4.1.1 is well-defined over overlapping charts; that is, it does not depend on the particular chosen chart in $\mathcal{A}$. Concluding, we have the following Corollary of Theorem 4.1.1.

Corollary 4.1.1. *Let G be a p-adic Lie group, and let $\mathcal{A} = \{(\mathscr{U}_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ be a (not necessarily disjoint) atlas on G. The left Haar measure μ on G is expressed, in the local coordinates of any given chart $(\mathscr{U}_\alpha, \varphi_\alpha)$ in $\mathcal{A}$, as*

$$\mu(\mathcal{E} \cap \mathscr{U}_\alpha) = \int_{\varphi_\alpha(\mathcal{E}\cap\mathscr{U}_\alpha)} \left| \det \left[\frac{\partial \zeta_{\alpha,j}}{\partial x_k} (\overleftarrow{\varphi}_\alpha(y); \varphi_0(e)) \right]_{1\le j,k\le n} \right|_p^{-1} \mathrm{d}\lambda(y), \tag{4.26}$$

for every Borel set $\mathcal{E} \in \mathcal{B}_G$, where $(\mathscr{U}_0, \varphi_0)$ is the chart around $e \in G$, and $(x_k)_{k=1}^n$ denotes a system of local coordinates w.r.t. $(\mathscr{U}_0, \varphi_0)$.

To conclude this section, we now show that the Haar measure (4.26) *coincides* with the measure on G associated with the left-invariant differential n-form $\Omega_{\rm inv}$ on G, as constructed in Subsection 3.3. Indeed, let us denote with $\breve{\Omega}$ the differential n-form on G whose local expression $\breve{\Omega}_\alpha$, in every chart $\mathscr{U}_\alpha$ in $\mathcal{A}$, is given by

$$\breve{\Omega}_\alpha(g) = \det[\mathrm{D}\zeta_\alpha(g;\varphi_0(e))]^{-1}\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n, \tag{4.27}$$

where, as usual, $\mathrm{D}\zeta_\alpha$ denotes the Jacobian matrix of $\zeta_\alpha = (\zeta_{\alpha,j})_{j=1}^n$, and where we set $\varphi_\alpha(g) = (x_1,\ldots,x_n)$. It is clear that the measure $\mu_{\breve{\Omega}}$, associated with $\breve{\Omega}$ via relation (3.50), coincides with the Haar measure in (4.26). (It is worth noting that, from Corollary 4.1.1, it follows that $\breve{\Omega}$ does not depend on the particular chosen chart on G, i.e., it is a well-defined differential n-form on G). To prove that the form (4.27) *coincides* with the left-invariant differential n-form $\Omega_{\rm inv}$ on G, it is enough to show that condition (3.40) holds, i.e., $\ell_h^*\breve{\Omega}(hg) = \breve{\Omega}(g)$, for every h, g in G. Indeed, this will prove that $\breve{\Omega}$ is a left-invariant differential n-form on G, and due to its essential uniqueness, we can then conclude that it coincides with $\Omega_{\rm inv}$ (up to a multiplicative constant). In fact, we have:

$$\begin{aligned}\ell_h^*\breve{\Omega}(hg) &= \ell_h^*\Big(\det[\mathrm{D}\zeta_\beta(\,\cdot\,;\varphi_0(e))]^{-1}(hg)\mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}y_n\Big)\\ &= \det[\mathrm{D}(\zeta_\beta(\,\cdot\,;\varphi_0(e)) \circ \ell_h)]^{-1}(hg)\det[\mathrm{D}\ell_h]\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n\\ &= \det[\mathrm{D}\zeta_\alpha(h^{-1}hg;\varphi_0(e))]^{-1}\det[\mathrm{D}\ell_h]^{-1}\det[\mathrm{D}\ell_h]\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n\\ &= \breve{\Omega}(g),\end{aligned} \tag{4.28}$$

where we set $\varphi_\alpha(hg) = (y_1,\ldots,y_n)$. Note that, in the second equality we have used the pullback formula (3.37) for differential forms, while in the third equality we have used the formula for the Jacobian of a composite function, taking into account the relation $\zeta_{\alpha,j} = \varphi_{\alpha,j} \circ \overleftarrow{\varphi}_{\beta,j} \circ \zeta_{\beta,j}$ between $\zeta_{\alpha,j}$ and $\zeta_{\beta,j}$. Hence, since the Haar measure is essentially uniquely defined, we get to the conclusion that the left Haar measure (4.26) must *coincide* (up to a multiplicative constant) with the measure $\mu_{\Omega_{\rm inv}}$ induced by the left-invariant differential n-form $\Omega_{\rm inv}$ on G.

Remark 4.1.2. Let us clarify how the local formula (4.26) for the Haar measure μ on G allows us to globally integrate a function on G. Given $f \in \mathrm{C}_c(G)$, its Haar integral $\int_G f(g)\mathrm{d}\mu(g)$ can be computed by splitting f as a sum of its components on local supports contained in the domains of the charts in an atlas for G. This is done by making use of a partition of unity $\{\chi_\alpha\}_{\alpha\in A}$ under an atlas $\{(\mathscr{U}_\alpha,\varphi_\alpha)\}_{\alpha\in A}$ of G. Then, the following relations hold:

$$\int_G f(g)\mathrm{d}\mu(g) = \int_G \sum_{\alpha\in A}\chi_\alpha f(g)\mathrm{d}\mu(g) = \sum_{\alpha\in A}\int_{\mathscr{U}_\alpha}\chi_\alpha f(g)\mathrm{d}\mu(g). \tag{4.29}$$

Each integral in the summation can be computed by using the local formulas (4.26).

4.2 Applications

As observed in Subsection 2.2 of Part I, the groups $\mathrm{SO}(n,\mathbb{Q}_p)$, $n = 2,3,4$, are compact. Hence, they admit a (left and right) Haar measure, which is essentially uniquely defined, i.e., unique up to a normalization constant factor. The construction of the Haar measure on $\mathrm{SO}(2,\mathbb{Q}_p)_\kappa$ immediately follows by formula (4.26). On the other hand, we will explicitly construct the Haar integrals for $\mathrm{SO}(3,\mathbb{Q}_p)$ and $\mathrm{SO}(4,\mathbb{Q}_p)$. A fruitful approach is to introduce a suitable *p-adic quaternion algebra*, $\mathbb{H}_p$, and exploit its relations with the p-adic special orthogonal groups in dimension three and four. Then, their Haar integrals are expressed as lifts to Haar integrals on suitable groups of p-adic quaternions, whose Haar measures are found, once again, by means of a direct application of (4.26).

4.2.1 The Haar measure on $\mathrm{SO}(2,\mathbb{Q}_p)_\kappa$

In this subsection, we explicitly construct a left and right Haar measure on every $\mathrm{SO}(2,\mathbb{Q}_p)_\kappa$ — characterized as in Proposition 2.2.1.

According to parameterization (2.53), $\mathrm{SO}(2,\mathbb{Q}_p)_\kappa$ is homeomorphic to the p-adic projective line, and it is covered by two disjoint charts. One coordinate map, say $\varphi_{(\kappa)}$, is defined on $\mathrm{SO}(2,\mathbb{Q}_p)_\kappa\backslash\{-\mathrm{I}\}$ to $\mathbb{Q}_p$, and it is such that $\overleftarrow{\varphi}_{(\kappa)}(x)\equiv R(\alpha)$ (cf. Theorem 2.2.1); the other one maps $-\mathrm{I}\in\mathrm{SO}(2,\mathbb{Q}_p)_\kappa$ to ∞. Since the groups $\mathrm{SO}(2,\mathbb{Q}_p)_\kappa$ are compact and infinite (uncountable), the singleton $\{-\mathrm{I}\}$ has *zero* Haar measure. The Jacobian in (4.26) is now easily computed: By recalling the composition law (2.54), we find

$$\Big[\frac{\partial\zeta_{(\kappa)}}{\partial\beta}(\overleftarrow{\varphi}_{(\kappa)}(\alpha);\beta)\Big]\equiv\frac{\mathrm{d}}{\mathrm{d}\beta}\left(\frac{\alpha+\beta}{1-\kappa\alpha\beta}\right)=\frac{1+\kappa\alpha^2}{(1-\kappa\alpha\beta)^2}. \tag{4.30}$$

(Note that $-\kappa$ is never a square [113], i.e., $1+\kappa\alpha^2\neq 0$ for every $\alpha\in\mathbb{Q}_p$). Therefore, an application of (4.26) — with $\beta=\varphi_{(\kappa)}(\mathrm{I})=0$ — immediately yields the Haar measure of every Borel subset $\mathcal{E}$ in $\mathrm{SO}(2,\mathbb{Q}_p)_\kappa$:

$$\mu_2^{(\kappa)}(\mathcal{E})=\int_{\varphi_{(\kappa)}(\mathcal{E})}\frac{1}{|1+\kappa\alpha^2|_p}\mathrm{d}\lambda(\alpha), \tag{4.31}$$

with $\mathrm{d}\lambda(\alpha)$ the Haar measure on $\mathbb{Q}_p$.

Remark 4.2.1. One can directly verify that the measure in (4.31) is a Haar measure, i.e., left- and right-invariant. Indeed, let us consider the functional invariance condition in (3.3):

$$\int\limits_{\mathrm{SO}(2,\mathbb{Q}_p)_\kappa}L_gf(x)\mathrm{d}\mu_2^{(\kappa)}(x)=\int\limits_{\alpha\in\mathbb{Q}_p}L_{R_\kappa(\beta)}f(R_\kappa(\alpha))\frac{\mathrm{d}\lambda(\alpha)}{|1+\kappa\alpha^2|_p}=\int\limits_{\alpha\in\mathbb{Q}_p}f(R_\kappa(-\beta)R_\kappa(\alpha))\frac{\mathrm{d}\lambda(\alpha)}{|1+\kappa\alpha^2|_p}, \tag{4.32}$$

for $f\in\mathrm{C}\big(\mathrm{SO}(2,\mathbb{Q}_p)_\kappa\big)$ a compactly supported function on $\mathrm{SO}(2,\mathbb{Q}_p)_\kappa$ (recall that $\mathrm{C}_c(X)=\mathrm{C}(X)$, whenever X is compact), and where $g=R_\kappa(\beta)$, for some $\beta\in\mathbb{Q}_p$. In the last integral, we have also used the fact that $L_gf(x)=f(g^{-1}x)$ (i.e., the left translation of functions on $\mathrm{SO}(2,\mathbb{Q}_p)_\kappa$), together with $R_\kappa(\beta)^{-1}=R_\kappa(-\beta)$. Recalling formula (2.54), we have:

$$\int\limits_{\mathrm{SO}(2,\mathbb{Q}_p)_\kappa}L_gf(x)\mathrm{d}\mu_2^{(\kappa)}(x)=\int\limits_{\alpha\in\mathbb{Q}_p}f\left(R_\kappa\left(\frac{\alpha-\beta}{1+\kappa\alpha\beta}\right)\right)\frac{1}{|1+\kappa\alpha^2|_p}\mathrm{d}\lambda(\alpha). \tag{4.33}$$

Let us now set $\varpi=(\alpha-\beta)/(1+\kappa\alpha\beta)$. We have:

$$\alpha=\frac{\varpi+\beta}{1-\kappa\beta\varpi},\qquad \mathrm{d}\lambda(\alpha)=\frac{1+\kappa\beta^2}{(1-\kappa\varpi\beta)^2}\mathrm{d}\varpi, \tag{4.34}$$

and, by inserting (4.34) into (4.33), we obtain

$$\int\limits_{\varpi\in\mathbb{Q}_p}f(\varpi)\frac{|1-\kappa\varpi\beta|_p^2}{|(1-\kappa\varpi\beta)^2+\kappa(\varpi+\beta)^2|_p}\frac{|1+\kappa\beta^2|_p}{|1-\kappa\varpi\beta|_p^2}\mathrm{d}\varpi=\int\limits_{\varpi\in\mathbb{Q}_p}f(\varpi)\frac{|1+\kappa\beta^2|_p}{|1+\kappa\varpi^2+\kappa\beta^2+\kappa^2\varpi^2\beta^2|_p}\mathrm{d}\varpi$$
$$=\int\limits_{\varpi\in\mathbb{Q}_p}f(\varpi)\frac{|1+\kappa\beta^2|_p}{|(1+\kappa\varpi^2)(1+\kappa\beta^2)|_p}\mathrm{d}\varpi=\int\limits_{\varpi\in\mathbb{Q}_p}f(\varpi)\frac{1}{|1+\kappa\varpi^2|_p}\mathrm{d}\varpi=\int\limits_{\mathrm{SO}(2,\mathbb{Q}_p)_\kappa}f(x)\mathrm{d}\mu_2^{(\kappa)}(x). \tag{4.35}$$

This shows the left-invariance of the measure. On the other hand, since the group is compact, this also entails the right-invariance of the measure (4.31).

Remark 4.2.2. The Haar measure of any Borel subset $\mathcal{F}$ of $\mathrm{SO}(2,\mathbb{R})$ is given by

$$\mu(\mathcal{F}) = \lambda(\varphi(\mathcal{F})), \tag{4.36}$$

where λ denotes the Haar measure on $\mathbb{R}$, and the coordinate map on $\mathrm{SO}(2,\mathbb{R})$ is given by $\varphi\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} = \theta \in [0, 2\pi[$. On the other hand, with $\kappa = 1$ and $\alpha = \tan\left(\frac{\theta}{2}\right)$, an element of $\mathrm{SO}(2,\mathbb{Q}_p)_1$ becomes formally identical to an element of $\mathrm{SO}(2,\mathbb{R})$ (cf. Remark 2.2.3). Therefore, one may expect that such a 'reduction' applies also for the Haar measure. Indeed, using the C.O.V.F. for p-adic integrals (see Theorem 3.3.1) we have:

$$\mu_2^{(\kappa)}(\mathcal{E}) = \int_{\varphi_{(\kappa)}(\mathcal{E})} \frac{1}{|1+\kappa\alpha^2|_p} \mathrm{d}\lambda(\alpha) \to \tag{4.37}$$

$$\int_{\varphi(\mathcal{F})} \left|\frac{1}{1+\tan^2(\theta/2)}\right|_p \left|\frac{1}{\cos^2(\theta/2)}\right|_p \mathrm{d}\lambda(\theta) = \int_{\varphi(\mathcal{F})} \mathrm{d}\lambda(\theta),$$

i.e., the Haar measure on $\mathrm{SO}(2,\mathbb{Q}_p)_\kappa$ reduces to that on $\mathrm{SO}(2,\mathbb{R})$, up to the normalization constant factor.

5

The quaternion algebra $\mathbb{H}_p$

The study of real quaternions was originally motivated by their property to model Euclidean orthogonal transformations of $\mathbb{R}^3$ and $\mathbb{R}^4$ [112]. It turns out that this familiar picture keeps some of its main futures — but also requires some essential modifications — when switching from the real to the p-adic setting. In what follows, we will describe the *quaternion algebra* $\mathbb{H}_p$ over the field $\mathbb{Q}_p$ of p-adic numbers [125], in a way that closely mimics its real counterpart (briefly reminded in Appendix A); later, (cf. Section 5.3), we shall clarify its relations with the p-adic special orthogonal groups in dimension three and four. The cases where $p > 2$ and $p = 2$ will be discussed separately.

5.1 Case $p > 2$

In the standard real case, the quaternion algebra $\mathbb{H}$ is the vector space $\mathbb{R}^4 \cong \mathbb{R} \times \mathbb{R}^3$ equipped with a suitable standard basis, namely, the one consisting of the vectors $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ in $\mathbb{R}^4$ satisfying the commutation rules (A.3) of Appendix A. From this, one can then define an isomorphism which realizes $\mathbb{H}$ as a subalgebra of $\mathsf{M}(2, \mathbb{C})$. Switching to the p-adic setting, it is then natural to set the following

Definition 5.1.1. Let $p > 2$ be an odd prime. By a *p-adic quaternion algebra* we mean a four-dimensional vector space $\mathbb{H}_p \cong \mathbb{Q}_p \times \mathbb{Q}_p^3$ over $\mathbb{Q}_p$ which is a $\mathbb{Q}_p$-algebra, and satisfies the following conditions:

(a) There exist $\mathbf{i}, \mathbf{j}$ in $\mathbb{H}_p$ such that, denoting by 1 the multiplicative identity in $\mathbb{H}_p$, the set $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k} \coloneqq \mathbf{ji}\}$ is a $\mathbb{Q}_p$-basis in $\mathbb{H}_p$.

(b) The basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in $\mathbb{H}_p$ satisfy the following commutation rules:

$$\mathbf{i}^2 = v, \quad \mathbf{j}^2 = -p, \quad \mathbf{k}^2 = pv, \quad \mathbf{ji} = -\mathbf{ij}, \quad \mathbf{ki} = -\mathbf{ik} = v\mathbf{j}, \quad \mathbf{kj} = -\mathbf{jk} = p\mathbf{i}, \tag{5.1}$$

for $v \in \mathbb{Q}_p$ a non-quadratic p-adic unit.

Remark 5.1.1. By means of a direct calculation, one verifies that the centre of the quaternion algebra $\mathbb{H}_p$ coincides with the base field $\mathbb{Q}_p$. This is reminiscent, to some extent, of the standard real case where, similarly, one shows that the field of real numbers $\mathbb{R}$ is the centre of the real quaternion algebra $\mathbb{H}$.

On the quaternion algebra $\mathbb{H}_p$, we can define a natural *involutive anti-automorphism* by setting

$$\mathbb{H}_p \ni \xi = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \mapsto \overline{\xi} \coloneqq q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3, \quad \xi \in \mathbb{H}_p. \tag{5.2}$$

Then, it is easily checked that, for every $\xi \in \mathbb{H}_p$, the product of ξ and $\overline{\xi}$ results into

$$\xi\overline{\xi} = \mathrm{F}_{(4)}(q_0, q_1, q_2, q_3) = q_0^2 - vq_1^2 + pq_2^2 - pvq_3^2, \tag{5.3}$$

that is, the unique (up to linear equivalence and scaling) four-dimensional definite quadratic form over $\mathbb{Q}_p$, for $p > 2$ (cf. (2.44) in Theorem 2.1.3). Therefore, we can express the inverse ξ^{-1} of every (non-null) p-adic quaternion as

$$\xi^{-1} = \frac{\overline{\xi}}{\mathrm{F}_{(4)}(q_0, q_1, q_2, q_3)}. \tag{5.4}$$

In what follows, we shall denote by

$$\mathbb{H}_p^{\times} \coloneqq \{\xi \in \mathbb{H}_p \mid \xi \neq 0\} = \{\xi = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \in \mathbb{H}_p \mid \mathrm{F}_{(4)}(q_0, q_1, q_2, q_3) \neq 0\} \tag{5.5}$$

the multiplicative group of *invertible quaternions.*

Remark 5.1.2. In the literature (e.g., see [112]), the *reduced norm* is defined as the map

$$\mathbb{H}_p \ni \xi \mapsto \mathrm{nrd}(\xi) \coloneqq \xi\overline{\xi} = \mathrm{F}_{(4)}(q_0, q_1, q_2, q_3) \in \mathbb{Q}_p. \tag{5.6}$$

It is easily checked that nrd is a multiplicative map; namely, $\mathrm{nrd}(\xi\eta) = \xi\eta\overline{\xi\eta} = \xi\eta\overline{\eta}\overline{\xi} = \xi\mathrm{nrd}(\eta)\overline{\xi} = \mathrm{nrd}(\eta)\xi\overline{\xi} = \mathrm{nrd}(\xi)\mathrm{nrd}(\eta)$, for every $\xi, \eta \in \mathbb{H}_p$. Moreover, for every $\alpha \in \mathbb{Q}_p$ and $\xi \in \mathbb{H}_p$, $\mathrm{nrd}(\alpha\xi) = \alpha^2\mathrm{nrd}(\xi)$, and $\mathrm{nrd}(\overline{\xi}) = \mathrm{F}_{(4)}(q_0, -q_1, -q_2, -q_3) = \mathrm{F}_{(4)}(q_0, q_1, q_2, q_3) = \mathrm{nrd}(\xi)$. In what follows, we shall denote by $\overline{\xi}/\mathrm{nrd}(\xi)$ the inverse element (5.4) of a quaternion $\xi \in \mathbb{H}_p^{\times}$.

In the group of invertible quaternions $\mathbb{H}_p^{\times}$, it is possible to single out the subgroup of the so-called *unit quaternions*, namely, the group:

$$\mathrm{U}(\mathbb{H}_p) \coloneqq \{\xi \in \mathbb{H}_p^{\times} \mid \xi^{-1} = \overline{\xi}\} \equiv \{\xi \in \mathbb{H}_p \mid \mathrm{nrd}(\xi) = 1\}. \tag{5.7}$$

We want now to show that, as in the standard real case, $\mathbb{H}_p$ can be realized as a suitable matrix algebra. To begin with, we recall that in the quadratic form $\mathrm{F}_{(4)}(\mathbf{x}) = x_0^2 - vx_1^2 + px_2^2 - pvx_3^2$ on $\mathbb{Q}_p$, $v \in \mathbb{Q}_p$ is a non-quadratic p-adic unit, i.e., $v \notin (\mathbb{Q}_p^*)^2$ and $|v|_p = 1$. Accordingly, we can consider the quadratic field extension $\mathbb{Q}_{p,v}$ of $\mathbb{Q}_p$ by $\sqrt{v}$. Let $\mathsf{M}(2, \mathbb{Q}_{p,v})$ denote the algebra of two-dimensional matrices over $\mathbb{Q}_{p,v}$, and let $\mathbf{H}_p$ be the subalgebra of the matrices M in $\mathsf{M}(2, \mathbb{Q}_{p,v})$ of the form

$$M = \begin{pmatrix} x_0 + \sqrt{v}x_1 & -x_2 + \sqrt{v}x_3 \\ p(x_2 + \sqrt{v}x_3) & x_0 - \sqrt{v}x_1 \end{pmatrix}, \tag{5.8}$$

where $x_i \in \mathbb{Q}_p$, $i = 0, 1, 2, 3$. It is easily checked that $\mathbf{H}_p$ is a (unital) $\mathbb{Q}_p$-*division algebra*, where the inverse of every non-null element $M \in \mathbf{H}_p$ is

$$M^{-1} = \frac{1}{\det(M)} \begin{pmatrix} x_0 - \sqrt{v}x_1 & x_2 - \sqrt{v}x_3 \\ -p(x_2 + \sqrt{v}x_3) & x_0 + \sqrt{v}x_1 \end{pmatrix}, \tag{5.9}$$

and where $\det(M) = \mathrm{F}_{(4)}(x_0, x_1, x_2, x_3)$. Let us now introduce the matrices $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in $\mathsf{M}(2, \mathbb{Q}_{p,v})$ defined as

$$\mathbf{i} \coloneqq \begin{pmatrix} \sqrt{v} & 0 \\ 0 & -\sqrt{v} \end{pmatrix}, \quad \mathbf{j} \coloneqq \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}, \quad \mathbf{k} \coloneqq \begin{pmatrix} 0 & \sqrt{v} \\ p\sqrt{v} & 0 \end{pmatrix}. \tag{5.10}$$

It is clear that every M in $\mathbf{H}_p$ can be expressed as follows:

$$M = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3, \quad x_0, x_1, x_2, x_3 \in \mathbb{Q}_p, \tag{5.11}$$

(here, we are omitting the identity matrix I_2 multiplying x_0); that is, $\mathbf{H}_p$ coincides with the $\mathbb{Q}_p$-*linear span* of the set $\{\mathrm{I}_2, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Moreover, $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfies the following commutation rules:

$$\mathbf{i}^2 = v\mathrm{I}_2, \quad \mathbf{j}^2 = -p\mathrm{I}_2, \quad \mathbf{k}^2 = pv\mathrm{I}_2, \quad \mathbf{ji} = -\mathbf{ij} = \mathbf{k}, \quad \mathbf{ki} = -\mathbf{ik} = v\mathbf{j}, \quad \mathbf{kj} = -\mathbf{jk} = p\mathbf{i}, \tag{5.12}$$

from which we can argue that $\mathbf{H}_p$ is a *non-commutative* $\mathbb{Q}_p$-division algebra.

Remark 5.1.3. As in the complex case, the subset of invertible elements in $\mathbf{H}_p$ forms a group

$$\mathbf{H}_p^\times \coloneqq \{M \in \mathbf{H}_p \mid M \neq 0_2\} = \{M \in \mathbf{H}_p \mid \det(M) \neq 0\}, \tag{5.13}$$

where 0_2 denotes the null 2×2 matrix on $\mathbb{Q}_{p,v}$. Moreover, we can single out the subgroup $\mathrm{U}(\mathbf{H}_p)$ of elements in $\mathbf{H}_p^\times$ having unit determinant, i.e.,

$$\mathrm{U}(\mathbf{H}_p) \coloneqq \{M \in \mathbf{H}_p^\times \mid \det(M) = 1\}, \tag{5.14}$$

which provides the p-adic counterpart of (A.9) in Appendix A.

In the light of the discussion above, it is now not difficult to prove the following result:

Proposition 5.1.1. *For every prime $p > 2$, the p-adic quaternion algebra $\mathbb{H}_p$ is isomorphic to the $\mathbb{Q}_p$-division subalgebra $\mathbf{H}_p$ of $\mathsf{M}(2, \mathbb{Q}_{p,v})$.*

Proof. Let us consider the map

$$\theta_p : \mathbb{H}_p \ni \xi = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \mapsto \theta_p(\xi) \coloneqq \begin{pmatrix} q_0 + \sqrt{v}q_1 & -q_2 + \sqrt{v}q_3 \\ p(q_2 + \sqrt{v}q_3) & q_0 - \sqrt{v}q_1 \end{pmatrix} \in \mathbf{H}_p. \tag{5.15}$$

It is clear that θ_p is one-one, onto and linear, i.e., it is an isomorphism of vector spaces. Also, θ_p is a ring homomorphism, since $\theta_p(\xi\eta) = \theta_p(\xi)\theta_p(\eta)$ for every $\xi, \eta \in \mathbb{H}_p$. Hence, it defines an *algebra isomorphism* from $\mathbb{H}_p$ to $\mathbf{H}_p$. □

The algebra isomorphism θ_p identifies the basis vectors $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ of $\mathbb{H}_p$ with I_2 and the matrices (5.10) in the spanning set of $\mathbf{H}_p$, respectively. This then also justifies our abuse of notation in using the same symbols for the basis elements of both $\mathbb{H}_p$ and $\mathbf{H}_p$.

Remark 5.1.4. Exploiting the algebra isomorphism θ_p, one can easily check that

$$\mathrm{nrd}(\xi) = \det(\theta_p(\xi)) = \mathrm{F}_{(4)}(q_0, q_1, q_2, q_3). \tag{5.16}$$

Therefore, we can interchangeably use $\mathrm{nrd}(\xi)$, $\det(\theta_p(\xi))$ and $\mathrm{F}_{(4)}(q_0, q_1, q_2, q_3)$ to denote the reduced norm of $\xi = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$ in $\mathbb{H}_p$.

Remark 5.1.5. Using the isomorphism θ_p, it is clear that the subgroups $\mathrm{U}(\mathbb{H}_p)$ and $\mathbb{H}_p^\times$ of $\mathbb{H}_p$ are isomorphic, respectively, to the subgroups $\mathrm{U}(\mathbf{H}_p)$ and $\mathbf{H}_p^\times$ of $\mathbf{H}_p$ (cf. Remark 5.1.3).

5.2 Case $p = 2$

As for the $p > 2$ case, we start by giving the following

Definition 5.2.1. Let $p = 2$. By a 2*-adic quaternion algebra* we mean a four-dimensional vector space $\mathbb{H}_2 \cong \mathbb{Q}_2 \times \mathbb{Q}_2^3$ over $\mathbb{Q}_2$ which is a $\mathbb{Q}_2$-algebra, and satisfies the following conditions:

(c) There exist $\mathbf{i}, \mathbf{j}$ in $\mathbb{H}_2$ such that, denoting by 1 the multiplicative identity in $\mathbb{H}_2$, the set $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is a $\mathbb{Q}_2$-basis in $\mathbb{H}_2$.

(d) The basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy the following commutation rules:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}. \tag{5.17}$$

We can endow $\mathbb{H}_2$ with the involution (5.2), thus turning it into an involutive algebra. Then, the inverse ξ^{-1} of every non-null 2-adic quaternion ξ can be expressed as

$$\xi^{-1} = \frac{\overline{\xi}}{\mathrm{nrd}(\xi)}. \tag{5.18}$$

Moreover, we can single out the subgroup $\mathbb{H}_2^\times \leq \mathbb{H}_2$ of invertible 2-adic quaternions by putting

$$\mathbb{H}_2^\times = \{\xi \in \mathbb{H}_2 \mid \xi \neq 0\} \equiv \{\xi \in \mathbb{H}_2 \mid \mathrm{nrd}(\xi) \neq 0\}, \tag{5.19}$$

as well as the subgroup $\mathrm{U}(\mathbb{H}_2) \leq \mathbb{H}_2^\times$ of unit quaternions defined as

$$\mathrm{U}(\mathbb{H}_2) = \{\xi \in \mathbb{H}_2^\times \mid \mathrm{nrd}(\xi) = 1\}. \tag{5.20}$$

We want now prove that $\mathbb{H}_2$ can be made in a one to one correspondence with a suitable matrix algebra. To this end, we recall that the definite quadratic form of $\mathbb{Q}_2^4$ is now given by (2.44); moreover, since -1 is not a square in $\mathbb{Q}_2$, we can consider the quadratic extension $\mathbb{Q}_{2,-1}$ of $\mathbb{Q}_2$ by -1. Let $\mathsf{M}(2, \mathbb{Q}_{2,-1})$ denote the algebra of two-dimensional matrices on $\mathbb{Q}_{2,-1}$, and let $\mathbf{H}_2 \subset \mathsf{M}(2, \mathbb{Q}_{2,-1})$ be the subalgebra of matrices M defined by

$$M := \begin{pmatrix} x_0 + \sqrt{-1}x_1 & x_2 + \sqrt{-1}x_3 \\ -x_2 + \sqrt{-1}x_3 & x_0 - \sqrt{-1}x_1 \end{pmatrix}, \quad x_i \in \mathbb{Q}_2, \ \forall i = 0, \ldots, 3. \tag{5.21}$$

By construction, we have that $\det(M) = \mathrm{F}_{(4)}(x_0, x_1, x_2, x_3) = x_0^2 + x_1^2 + x_2^2 + x_3^2$. Hence, every non-zero $M \in \mathbf{H}_2$ is invertible, with inverse given by

$$M^{-1} = \frac{1}{\det(M)} \begin{pmatrix} x_0 - \sqrt{-1}x_1 & -x_2 - \sqrt{-1}x_3 \\ x_2 - \sqrt{-1}x_3 & x_0 + \sqrt{-1}x_1 \end{pmatrix}; \tag{5.22}$$

i.e., $\mathbf{H}_2$ is an associative (unital) $\mathbb{Q}_2$-division algebra. Next, let us introduce the matrices $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in $\mathsf{M}(2, \mathbb{Q}_{2,-1})$ defined by

$$\mathbf{i} := \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad \mathbf{j} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} := \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}. \tag{5.23}$$

Every M in $\mathbf{H}_2$ can be expressed as $M = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$ (we have omitted the identity I_2 multiplying x_0); that is, $\mathbf{H}_2$ can be realized as the $\mathbb{Q}_2$-linear span of $\{\mathrm{I}_2, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. By further noting that $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy the commutation rules

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathrm{I}_2, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \tag{5.24}$$

we also see that $\mathbf{H}_2$ is a *non-commutative* $\mathbb{Q}_2$-algebra.

Remark 5.2.1. As in the $p > 2$ case, we can introduce the group

$$\mathbf{H}_2^\times := \{M \in \mathbf{H}_2 \mid M \neq 0_2\} \equiv \{M \in \mathbf{H}_2 \mid \det(M) \neq 0\} \tag{5.25}$$

of the invertible matrices in $\mathbf{H}_2$, as well as the subgroup

$$\mathrm{U}(\mathbf{H}_2) = \left\{M \in \mathbf{H}_2^\times \mid \det(M) = 1\right\}. \tag{5.26}$$

The following result is a straightforward adaptation of Proposition 5.1.1

Proposition 5.2.1. *Let $p = 2$. Then, the 2-adic quaternion algebra $\mathbb{H}_2$ is isomorphic with the subalgebra $\mathbf{H}_2$ of $\mathsf{M}(2, \mathbb{Q}_{2,-1})$.*

Proof. It suffices to consider the map

$$\theta_2\colon \mathbb{H}_2 \ni \xi = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \mapsto \theta_2(\xi) := \begin{pmatrix} q_0 + \sqrt{-1}q_1 & q_2 + \sqrt{-1}q_3 \\ -q_2 + \sqrt{-1}q_3 & q_0 - \sqrt{-1}q_1 \end{pmatrix} \in \mathbf{H}_2, \quad (5.27)$$

and observe that it provides the desired algebra isomorphism. □

Remark 5.2.2. The quaternion algebra $\mathbb{H}_2$ shares some analogies with the standard real quaternion algebra $\mathbb{H}$. In particular, the matrix representation of a 2-adic quaternion is 'essentially the same' as in the standard case (just set $\sqrt{-1} := i$ for the square root of the non quadratic element $-1 \in \mathbb{Q}_2$). This is what one expects upon considering the 'formal equivalence' of the real four-dimensional quadratic form $\mathrm{F}_{\mathbb{R}}$ with the four-dimensional quadratic form $\mathrm{F}_{(4)}$ on $\mathbb{Q}_2$. However, the analogies between standard and p-adic quaternion algebras cannot be pursued too far. Indeed, a fundamental difference between $\mathbb{H}_p$, for every prime $p \geq 2$, and $\mathbb{H}$ is the following. For the latter, we have that $\mathrm{F}_{\mathbb{R}}(q_0, q_1, q_2, q_3) = \|(q_0, q_1, q_2, q_3)\|^2_{\mathbb{R}^4}$, i.e., the definite quadratic form $\mathrm{F}_{\mathbb{R}}$ on $\mathbb{R}^4$ *coincides* with the squared Euclidean norm of $\mathbb{R}^4$. (This also entails that the reduced norm of $\mathbb{H}$ is equivalent to the (square of) the Euclidean norm of $\mathbb{R}^4$. See Remarks 5.1.2 and A.0.1). On the other hand, in the p-adic setting, we only have the equivalence $\mathrm{F}_{(4)} \equiv \mathrm{nrd}$, i.e., the reduced norm of $\mathbb{H}_p$ *does not coincide* with the square of the p-adic norm of $\mathbb{Q}_p^4$.

5.3 Relation between p-adic quaternions and special orthogonal groups

This section clarifies the relations between p-adic quaternions and the p-adic groups of rotations $\mathrm{SO}(3, \mathbb{Q}_p)$ and $\mathrm{SO}(4, \mathbb{Q}_p)$, for every $p \geq 2$. We begin with $\mathrm{SO}(3, \mathbb{Q}_p)$. Let us consider the action by *conjugation* of the group $\mathbb{H}_p^\times$ of invertible quaternions on $\mathbb{H}_p$; namely, the map

$$\mathbb{H}_p \ni \eta \mapsto \xi\eta\xi^{-1} \in \mathbb{H}_p, \quad (5.28)$$

where $\xi \in \mathbb{H}_p^\times$, and $p \geq 2$. This map is an *isometric linear transformation* of $\mathbb{H}_p$, since it preserves the reduced norm of every quaternion η in $\mathbb{H}_p$:

$$\begin{aligned} \mathrm{nrd}(\xi\eta\xi^{-1}) &= \mathrm{nrd}(\xi)\mathrm{nrd}(\eta)\mathrm{nrd}(\xi^{-1}) \\ &= \mathrm{nrd}(\xi)\mathrm{nrd}(\xi^{-1})\mathrm{nrd}(\eta) \\ &= \mathrm{nrd}(\xi\xi^{-1})\mathrm{nrd}(\eta) \\ &= \mathrm{nrd}(\eta); \end{aligned} \quad (5.29)$$

equivalently, the action by conjugation of $\mathbb{H}_p^\times$ preserves the definite quadratic form of $\mathbb{Q}_p^4$. Moreover, the operation $\eta \mapsto \xi\eta\xi^{-1}$ leaves the centre $\mathbb{Q}_p$ of $\mathbb{H}_p$ pointwise fixed and, hence, also leaves the orthogonal subspace $\mathbb{Q}_p^3$ invariant.

(Note: Here, we refer to the orthogonality w.r.t. the inner product induced by the definite quadratic form of $\mathbb{Q}_p^4$, as defined in (2.1)).

Let us now consider the restriction of (5.28) to the subset $\mathbb{H}_p^0 := \{\nu \in \mathbb{H}_p \mid \nu = \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3\}$ of *pure imaginary quaternions* in $\mathbb{H}_p$; that is, let us consider the map

$$\kappa_p(\xi)\colon \mathbb{H}_p^0 \ni \nu \mapsto \kappa_p(\xi)\nu := \xi\nu\xi^{-1}, \quad \xi \in \mathbb{H}_p^\times. \quad (5.30)$$

By noting that $\mathbb{H}_p^0 \cong \mathbb{Q}_p^3$, and reminding that the action (5.28) is an isometric transformation of $\mathbb{H}_p$, we deduce that $\kappa_p(\xi)$ preserves the restriction of $\mathrm{F}_{(4)}$ to $\mathbb{Q}_p^3$, i.e., the (equivalent) quadratic form F_+. Hence, we deduce that $\kappa_p(\xi) \in \mathrm{O}(3,\mathbb{Q}_p) \cong \{L \in \mathrm{End}(\mathbb{Q}_p^3) \mid \mathrm{F}_+(L\mathbf{x}) = \mathrm{F}_+(\mathbf{x}),\ \forall \mathbf{x} \in \mathbb{Q}_p^3\}$ represents an orthogonal transformation in $\mathbb{Q}_p^3$. Next, by observing that, for every $\xi, \rho \in \mathbb{H}_p$ and $\nu \in \mathbb{H}_p^0$, the equalities $\kappa_p(\xi\rho)\nu = (\xi\rho)\nu(\xi\rho)^{-1} = \xi\big(\rho\nu\rho^{-1}\big)\xi^{-1} = \kappa_p(\xi)\kappa_p(\rho)\nu$ hold, we can conclude that $\kappa_p : \mathbb{H}_p^\times \to \mathrm{O}(3,\mathbb{Q}_p)$ provides a group homomorphism.

Let us now explicitly derive its action on a pure imaginary quaternion ν in $\mathbb{H}_p^0$. If $\xi = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \in \mathbb{H}_p^\times$ and $\nu = \mathbf{i}s_1 + \mathbf{j}s_2 + \mathbf{k}s_3 \in \mathbb{H}_p^0$, the action of $\kappa_p(\xi)$ on ν is given by

$$\xi\nu\xi^{-1} = (q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3)(\mathbf{i}s_1 + \mathbf{j}s_2 + \mathbf{k}s_3)(q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3)\frac{1}{\mathrm{nrd}(\xi)}, \tag{5.31}$$

where we have used the fact that $\xi^{-1} = \overline{\xi}/\mathrm{nrd}(\xi)$ (see Remark 5.1.2). Expanding the above products, one sees that the scalar part vanishes, as expected, and, by collecting the terms in $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$, we get

$$\kappa_p(\xi) = \frac{1}{\mathrm{nrd}(\xi)}\begin{pmatrix} q_0^2 - vq_1^2 - pq_2^2 + pvq_3^2 & 2p(q_0q_3 + q_1q_2) & -2p(q_0q_2 + vq_1q_3) \\ 2v(q_0q_3 - q_1q_2) & q_0^2 + vq_1^2 + pq_2^2 + pvq_3^2 & -2v(q_0q_1 + pq_2q_3) \\ 2(q_0q_2 - vq_1q_3) & 2(-q_0q_1 + pq_2q_3) & q_0^2 + vq_1^2 - pq_2^2 - pvq_3^2 \end{pmatrix} \tag{5.32}$$

for $p > 2$, and

$$\kappa_2(\xi) = \frac{1}{\mathrm{nrd}(\xi)}\begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_3q_0) & 2(q_2q_0 + q_3q_1) \\ 2(q_1q_2 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_1q_0) \\ 2(q_1q_3 - q_2q_0) & 2(q_1q_0 + q_2q_3) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix} \tag{5.33}$$

for $p = 2$. A direct calculation shows that the transformations (5.32) and (5.33) have unit determinant, i.e.,

$$\det(\kappa_p(\xi)) = \frac{1}{\mathrm{nrd}(\xi)^3}\mathrm{F}_{(4)}(q_0, q_1, q_2, q_3)^3 = 1. \tag{5.34}$$

Therefore, we get to the conclusion that, for every prime $p \geq 2$, and every $\xi \in \mathbb{H}_p^\times$, $\kappa_p(\xi) \in \mathrm{SO}(3,\mathbb{Q}_p)$ is a *three-dimensional p-adic rotation.*

The above discussion shows that $\kappa_p(\mathbb{H}_p^\times) \subseteq \mathrm{SO}(3,\mathbb{Q}_p)$. We are now going to prove that, actually, also the reverse inclusion $\mathrm{SO}(3,\mathbb{Q}_p) \subseteq \kappa_p(\mathbb{H}_p^\times)$ holds. Indeed, let us first introduce the map $\tau_\rho\colon \mathbb{H}_p^0 \to \mathbb{H}_p^0$ defined, for every $\rho \in \mathbb{H}_p^\times \cap \mathbb{H}_p^0$, as

$$\tau_\rho(\nu) \coloneqq \nu - \frac{2\,\langle \nu, \rho\rangle}{\mathrm{nrd}(\rho)}\rho, \tag{5.35}$$

where $\langle\cdot,\cdot\rangle$ denotes the bilinear form associated with the quadratic form F_+ in $\mathbb{Q}_p^3$ (cf. (2.1)). It is easily shown that this map satisfies the conditions $\tau_\rho(\rho) = -\rho$ and $\mathrm{F}_+(\tau_\rho(\nu)) = \mathrm{F}_+(\nu)$, for any $\nu \in \mathbb{H}_p^0 \cong \mathbb{Q}_p^3$; namely, $\tau_\rho \in \mathrm{O}(3,\mathbb{Q}_p) \setminus \mathrm{SO}(3,\mathbb{Q}_p)$ defines a hyperplane reflection (w.r.t. ρ) in $\mathbb{H}_p^0$. Moreover, by taking into account the defining properties of $\langle\cdot,\cdot\rangle$, nrd and $\mathbb{H}_p^0$, and recalling that, for a pure imaginary quaternion ν in $\mathbb{H}_p^0$, one has $\overline{\nu} = -\nu$, we see that the reflection (5.35) is explicitly given by $\tau_\rho(\nu) = -\rho\nu\rho^{-1}$, i.e., $\tau_\rho \equiv -\kappa_p(\rho)$. On the other hand, by a classical theorem of Cartan and Dieudonné (cf. Theorem 4.5.7. in [112]), every special orthogonal transformation in $\mathrm{SO}(3,\mathbb{Q}_p)$ can be written as the composition of two such reflections, i.e., $g = \tau_{\rho_1}\tau_{\rho_2}$, for all $g \in \mathrm{SO}(3,\mathbb{Q}_p)$, and suitable $\rho_1, \rho_2 \in \mathbb{H}_p^\times \cap \mathbb{H}_p^0$. Therefore, every p-adic rotation in $\mathrm{SO}(3,\mathbb{Q}_p)$ is expressed by

$$g = \tau_{\rho_1}\tau_{\rho_2} = (-\tau_{\rho_1})(-\tau_{\rho_2}) = \kappa_p(\rho_1)\kappa_p(\rho_2) = \kappa_p(\rho_1\rho_2) = \kappa_p(\xi), \tag{5.36}$$

for $\xi \coloneqq \rho_1\rho_2 \in \mathbb{H}_p^\times$. This then shows that $\kappa_p(\mathbb{H}_p^\times) = \mathrm{SO}(3,\mathbb{Q}_p)$, i.e., that κ_p is surjective.

The following result is now straightforward and crucial for our purposes.

Theorem 5.3.1. *The group* $\mathrm{SO}(3,\mathbb{Q}_p)$ *is isomorphic to the quotient of the group* $\mathbb{H}_p^\times$ *of invertible quaternions, and the multiplicative group* $\mathbb{Q}_p^*$ *of non-null elements in* $\mathbb{Q}_p$*, namely*

$$\mathrm{SO}(3,\mathbb{Q}_p) \cong \mathbb{H}_p^\times/\mathbb{Q}_p^*. \tag{5.37}$$

Proof. To prove the group isomorphism (5.37), we can equivalently show that the following

$$1 \to \mathbb{Q}_p^* \hookrightarrow \mathbb{H}_p^\times \xrightarrow{\kappa_p} \mathrm{SO}(3,\mathbb{Q}_p) \to 1 \tag{5.38}$$

is a short exact sequence. We already know that κ_p is surjective. Furthermore, the kernel of κ_p, $\ker(\kappa_p)$, coincides with the image $\mathbb{Q}_p^*$ of the embedding in the short sequence:

$$\begin{aligned}\ker(\kappa_p) &= \{\xi \in \mathbb{H}_p^\times \mid \kappa_p(\xi) = \mathrm{I} \in \mathrm{SO}(3,\mathbb{Q}_p)\}\\ &= \{\xi \in \mathbb{H}_p^\times \mid \kappa_p(\xi)\nu = \nu \text{ for every } \nu \in \mathbb{H}_p^0\}\\ &= \{\xi \in \mathbb{H}_p^\times \mid \xi\nu = \nu\xi \text{ for every } \nu \in \mathbb{H}_p^0\}\\ &= \{\xi \in \mathbb{H}_p^\times \mid \xi\rho = \rho\xi \text{ for every } \rho \in \mathbb{H}_p\} = \mathbb{Q}_p^*,\end{aligned} \tag{5.39}$$

as $\mathbb{Q}_p^*$ is the centre of $\mathbb{H}_p^\times$ (see Remark 5.1.1). □

The exact sequence (5.38) is reminiscent, to some extent, of the exact sequence

$$1 \to \{\pm 1\} \hookrightarrow \mathrm{U}(\mathbb{H}) \cong \mathrm{SU}(2,\mathbb{C}) \twoheadrightarrow \mathrm{SO}(3,\mathbb{R}) \to 1, \tag{5.40}$$

of the standard real case (cf. the isomorphism (A.13) in Appendix A.0.2). Here, the main difference with the sequence (5.38) is provided by the fact that the groups $\mathrm{U}(\mathbb{H})$ and $\mathbb{F}_2 = \{\pm 1\}$ are replaced, in the p-adic setting, by the groups $\mathbb{H}_p^\times$ and $\mathbb{Q}_p^*$ respectively. The reason for this discrepancy is related to the peculiar features of the base field $\mathbb{Q}_p$. Indeed, it is possible to prove [111, 112] that a sequence as in (5.40) is exact if and only if $\mathrm{nrd}(\mathbb{H}^\times) \subset (\mathbb{F}^*)^2$, namely, if and only if the reduced norm of every invertible quaternion is a quadratic element of the field. In the case where $\mathbb{F} = \mathbb{R}$, this is certainly true. Instead, in the p-adic setting, $\mathrm{nrd}(\mathbb{H}_p^\times) \subset (\mathbb{Q}_p^*)^2$ is *never* true.

We want now to show that $\mathrm{SO}(3,\mathbb{Q}_p)$ and $\mathbb{H}_p^\times/\mathbb{Q}_p^*$ are homeomorphic. This fact will indeed play a fundamental role in our construction of the lift of the Haar integrals on $\mathrm{SO}(3,\mathbb{Q}_p)$ to $\mathbb{H}_p^\times$.

Let us preliminary recall that every LCSC Hausdorff space is a *standard Borel space* once endowed with its Borel σ-algebra. Accordingly, one calls a space X a *standard Borel G-space* if X is a G-space (cf. Subsection 3.1), its Borel structure is standard, and if the action of G on X is a *Borel map*. If X is a standard Borel G-space, and $x \in X$ is a fixed point, let $G_x := \{g \in G \mid g[x] = x\}$ be the *stability subgroup* at x. One can show (cf. Corollary 5.8 in [117]) that G_x is a closed subgroup of G. Moreover, denoting by $q\colon G \to G/G_x$ the projection homomorphism, the map

$$G/G_x \ni q(g) \mapsto g[x] \in X \tag{5.41}$$

is a Borel isomorphism, and it is a homeomorphism whenever X is LCSC (cf. Theorem 5.11 in [117]). Therefore, in such a case, $X \cong G/G_x$ are homeomorphic spaces in a natural way. We are now ready to prove the following result.

Proposition 5.3.1. *The group isomorphism* (5.37) *between* $\mathrm{SO}(3,\mathbb{Q}_p)$ *and* $\mathbb{H}_p^\times/\mathbb{Q}_p^*$ *is also an isomorphism of topological groups.*

Proof. The proof we give here is based on general measure-theoretical arguments on G-spaces; for a more specific proof, involving the reduced norm of p-adic quaternions, see Appendix B.

As a vector space, $\mathbb{H}_p \cong \mathbb{Q}_p \times \mathbb{Q}_p^3 = \mathbb{Q}_p^4$, and we can provide $\mathbb{H}_p$ with the product topology (on $\mathbb{Q}_p$, we consider the natural (ultra-)metric topology generated by the p-adic absolute value). Similarly, $\mathbb{H}_p^\times$ and $\mathbb{Q}_p^4 - \{0\}$ are homeomorphic topological spaces whenever they are equipped with the induced topology as subspaces of $\mathbb{H}_p$ and $\mathbb{Q}_p^4$ respectively. The continuity of the group operations (multiplication and inverse) of $\mathbb{H}_p^\times$ is inherited from the continuity of the addition, inner multiplication (according to the commutation relations among the basis elements) and multiplication by scalars of $\mathbb{Q}_p^4 - \{0\}$; therefore $\mathbb{H}_p^\times$ is a topological group. Also, $\mathbb{H}_p^\times$ is LCSC, as $\mathbb{Q}_p^4 - \{0\}$ is so (being an open subspace of the locally compact Hausdorff space $\mathbb{Q}_p^4$). We have already observed that $\mathrm{SO}(3,\mathbb{Q}_p)$ is a compact second countable Hausdorff group, once endowed with the topology introduced in Section 3. Hence, $\mathrm{SO}(3,\mathbb{Q}_p)$, supplied with its Borel σ-algebra is a standard Borel space. We want now to show that $\mathrm{SO}(3,\mathbb{Q}_p)$ is a standard Borel $\mathbb{H}_p^\times$-space. To this end, we have to find a Borel action of $\mathbb{H}_p^\times$ on $\mathrm{SO}(3,\mathbb{Q}_p)$.

Let us introduce the map from $\mathbb{H}_p^\times \times \mathrm{SO}(3,\mathbb{Q}_p)$ to $\mathrm{SO}(3,\mathbb{Q}_p)$ defined as

$$\mathbb{H}_p^\times \times \mathrm{SO}(3,\mathbb{Q}_p) \ni (\xi, R) \mapsto \xi[R] := \kappa_p(\xi) R \in \mathrm{SO}(3,\mathbb{Q}_p). \tag{5.42}$$

It is easily shown that the map (5.42) provides a continuous left action of $\mathbb{H}_p^\times$ on $\mathrm{SO}(3,\mathbb{Q}_p)$. Indeed, continuity follows from that of κ_p and of the matrix multiplication in $\mathrm{SO}(3,\mathbb{Q}_p)$. Next, we have that $\xi[\nu[R]] = \kappa_p(\xi)\big(\kappa_p(\nu)R\big) = \big(\kappa_p(\xi)\kappa_p(\nu)\big)R = \kappa_p(\xi\nu)R = (\xi\nu)[R]$, for every $\xi,\nu \in \mathbb{H}_p^\times$, $R \in \mathrm{SO}(3,\mathbb{Q}_p)$. Moreover $R \mapsto \xi[R]$ is a homeomorphism for every fixed $\xi \in \mathbb{H}_p^\times$, as follows by observing that $R \mapsto \xi[R]$ is surjective (since the multiplication in $\mathrm{SO}(3,\mathbb{Q}_p)$ by the matrix $\kappa_p(\xi)$ is so), and injective (since if $\kappa_p(\xi)R_1 = \kappa_p(\xi)R_2$, then $R_1 = R_2$ by the invertibility of $\kappa_p(\xi) \in \mathrm{SO}(3,\mathbb{Q}_p)$), and both the map and its inverse are continuous (as they are just matrix multiplications and inverses). This shows that (5.42) is a continuous (actually, Borel) left action of $\mathbb{H}_p^\times$ on $\mathrm{SO}(3,\mathbb{Q}_p)$. This action is also transitive, since it exists an element $R \in \mathrm{SO}(3,\mathbb{Q}_p)$ such that its orbit $\{\kappa_p(\xi)R \mid \xi \in \mathbb{H}_p^\times\}$ is the whole space $\mathrm{SO}(3,\mathbb{Q}_p)$ (it is enough to consider $R = \mathrm{I}$, and the surjectivity of κ_p). Therefore, we can argue that $\mathrm{SO}(3,\mathbb{Q}_p)$ is a standard Borel (transitive) $\mathbb{H}_p^\times$-space. On the other hand, the stability subgroup at every $R \in \mathrm{SO}(3,\mathbb{Q}_p)$ is given by $\{\xi \in \mathbb{H}_p^\times \mid \xi[R] = R\} = \{\xi \in \mathbb{H}_p^\times \mid \kappa_p(\xi)R = R\} = \{\xi \in \mathbb{H}_p^\times \mid \kappa_p(\xi) = \mathrm{I}\} = \ker(\kappa_p) = \mathbb{Q}_p^*$; hence, we can conclude that $\mathrm{SO}(3,\mathbb{Q}_p)$ and $\mathbb{H}_p^\times/\mathbb{Q}_p^*$ are homeomorphic spaces. In particular, the homeomorphism is as in (5.41) with, for instance, the stability subgroup at $\mathrm{I} \in \mathrm{SO}(3,\mathbb{Q}_p)$. Explicitly, the homeomorphism is $\mathbb{H}_p^\times/\mathbb{Q}_p^* \ni \xi\mathbb{Q}_p^* \mapsto \kappa_p(\xi) \in \mathrm{SO}(3,\mathbb{Q}_p)$. This is, indeed, the same map providing the isomorphism in Theorem 5.3.1. □

Proposition 5.3.1 concludes our discussion on the relations between p-adic quaternions and rotations in $\mathrm{SO}(3,\mathbb{Q}_p)$. Now, we carry out a similar analysis to clarify the relation between quaternions and the elements in $\mathrm{SO}(4,\mathbb{Q}_p)$. To begin with, let us introduce the left action of $\mathbb{H}_p^\times \times \mathbb{H}_p^\times$ on $\mathbb{H}_p$ defined by

$$\mathbb{H}_p \ni \eta \mapsto \xi\eta\varrho^{-1} \in \mathbb{H}_p, \quad (\xi,\varrho) \in \mathbb{H}_p^\times \times \mathbb{H}_p^\times. \tag{5.43}$$

This action is by *similarities*, as follows by noting that

$$\mathrm{nrd}(\xi\eta\varrho^{-1}) = \mathrm{nrd}(\xi)\mathrm{nrd}(\eta)\mathrm{nrd}(\varrho^{-1}) = \frac{\mathrm{nrd}(\xi)}{\mathrm{nrd}(\varrho)}\mathrm{nrd}(\eta). \tag{5.44}$$

In particular, the action is by *isometries* whenever $\mathrm{nrd}(\xi) = \mathrm{nrd}(\varrho)$. Hence, let us introduce the group

$$\mathbb{P}(\mathbb{H}_p^\times) := \{(\xi,\varrho) \in \mathbb{H}_p^\times \times \mathbb{H}_p^\times \mid \mathrm{nrd}(\xi) = \mathrm{nrd}(\varrho)\}. \tag{5.45}$$

The restriction of the action (5.43) to a pair $(\xi,\varrho)\in\mathbb{P}(\mathbb{H}_p^\times)\leq\mathbb{H}_p^\times\times\mathbb{H}_p^\times$ is denoted by $\kappa_p'(\xi,\varrho)$; namely, we set

$$\kappa_p'(\xi,\varrho):\mathbb{H}_p\ni\eta\mapsto\kappa_p'(\xi,\varrho)\eta:=\xi\eta\varrho^{-1}\in\mathbb{H}_p,\quad(\xi,\varrho)\in\mathbb{P}(\mathbb{H}_p^\times).\tag{5.46}$$

Since this action is by isometries, and $\mathbb{H}_p\cong\mathbb{Q}_p^4$, then $\kappa_p'(\xi,\varrho)\in\mathrm{O}(4,\mathbb{Q}_p)\cong\{L\in\mathrm{End}(\mathbb{Q}_p^4)\mid \mathrm{F}_{(4)}(Lx)=\mathrm{F}_{(4)}(x),\text{ for every }x\in\mathbb{Q}_p^4\}$. It can be easily checked that, as done for the maps $\kappa_p(\xi)$ in the three-dimensional case, $\kappa_p'(\xi,\varrho)\in\mathrm{SO}(4,\mathbb{Q}_p)$, for every $(\xi,\varrho)\in\mathbb{P}(\mathbb{H}_p^\times)$. Also, $\kappa_p'\colon\mathbb{P}(\mathbb{H}_p^\times)\to\mathrm{SO}(4,\mathbb{Q}_p)$ is a group homomorphism, and we get to the following result:

Theorem 5.3.2. *The group* $\mathrm{SO}(4,\mathbb{Q}_p)$ *is isomorphic to the quotient of the group* $\mathbb{P}(\mathbb{H}_p^\times)$ *and the multiplicative group* $\mathbb{Q}_p^*$ *of non-null p-adic numbers:*

$$\mathrm{SO}(4,\mathbb{Q}_p)\cong\mathbb{P}(\mathbb{H}_p^\times)/\mathbb{Q}_p^*.\tag{5.47}$$

Proof. Since $\mathrm{char}(\mathbb{Q}_p)\neq 2$, the isomorphism (5.47) follows from Proposition 4.5.17. in [112]. In particular, to prove (5.47), it suffices to show that the following

$$1\to\mathbb{Q}_p^*\hookrightarrow\mathbb{P}(\mathbb{H}_p^\times)\xrightarrow{\kappa_p'}\mathrm{SO}(4,\mathbb{Q}_p)\to 1\tag{5.48}$$

is a short exact sequence. This is done similarly to the proof of Theorem 5.3.1: Surjectivity of the map $\kappa_p'\colon\mathbb{P}(\mathbb{H}_p^\times)\to\mathrm{SO}(4,\mathbb{Q}_p)$ again follows by the Cartan-Dieudonné Theorem (cf. Theorem 4.5.7. in [112]), and its kernel is

$$\begin{aligned}\ker(\kappa_p')&=\{(\xi,\varrho)\in\mathbb{P}(\mathbb{H}_p^\times)\mid\kappa_p'(\xi,\varrho)=\mathrm{I}\in\mathrm{SO}(4,\mathbb{Q}_p)\}\\&=\{(\xi,\varrho)\in\mathbb{P}(\mathbb{H}_p^\times)\mid\kappa_p'(\xi,\varrho)\eta=\eta\text{ for every }\eta\in\mathbb{H}_p\}\\&=\{(\xi,\varrho)\in\mathbb{P}(\mathbb{H}_p^\times)\mid\xi\eta=\eta\varrho\text{ for every }\eta\in\mathbb{H}_p\}.\end{aligned}\tag{5.49}$$

In particular, the last condition must hold for $\eta=1\in\mathbb{H}_p$, providing the necessary condition $\xi=\varrho$; hence,

$$\begin{aligned}\ker(\kappa_p')&=\{(\xi,\xi)\in\mathbb{P}(\mathbb{H}_p^\times)\mid\xi\eta=\eta\xi\text{ for every }\eta\in\mathbb{H}_p\}\\&\cong\{\xi\in\mathbb{H}_p^\times\mid\xi\eta=\eta\xi\text{ for every }\eta\in\mathbb{H}_p\}=\mathbb{Q}_p^*.\end{aligned}\tag{5.50}$$

That is, the kernel of κ_p' is the diagonally embedded $\mathbb{Q}_p^*\cong\mathbb{Q}_p^*(1,1)$ in $\mathbb{P}(\mathbb{H}_p^\times)$. □

Remark 5.3.1. The short exact sequences (5.48) is the p-adic counterpart of the following sequence for the standard real setting:

$$1\to\{\pm1\}\hookrightarrow\mathrm{U}(\mathbb{H})\times\mathrm{U}(\mathbb{H})\twoheadrightarrow\mathrm{SO}(4,\mathbb{R})\to 1,\tag{5.51}$$

where $\mathrm{U}(\mathbb{H})$ denotes the group of unit quaternions (see (A.4) in Appendix A). This then entails the well known group isomorphism (A.14). The main difference with the p-adic case is provided by the fact that $\mathrm{U}(\mathbb{H})\times\mathrm{U}(\mathbb{H})$ and $\mathbb{F}_2=\{\pm1\}$ are now replaced by $\mathbb{P}(\mathbb{H}_p^\times)$ and $\mathbb{Q}_p^*$ respectively. Once again, this discrepancy is a consequence of the fact that in the p-adic setting, $\mathrm{nrd}(\mathbb{H}_p^\times)\not\subset(\mathbb{Q}_p^*)^2$.

Similarly to what we did for $\mathrm{SO}(3,\mathbb{Q}_p)$, we are interested in proving that $\mathrm{SO}(4,\mathbb{Q}_p)$ and $\mathbb{P}(\mathbb{H}_p^\times)/\mathbb{Q}_p^*$ are homeomorphic; this will allow us to consider the lift of the Haar integrals on $\mathrm{SO}(4,\mathbb{Q}_p)$ to that on $\mathbb{P}(\mathbb{H}_p^\times)$.

Proposition 5.3.2. *The group isomorphism* (5.47) *between* $\mathrm{SO}(4,\mathbb{Q}_p)$ *and* $\mathbb{P}(\mathbb{H}_p^\times)/\mathbb{Q}_p^*$ *is also an isomorphism of topological groups.*

Proof. Consider the group $\mathbb{P}(\mathbb{H}_p^\times)$ with the subspace topology induced by $\mathbb{Q}_p^8$ (the latter, being endowed with p-adic topology). The group operations are continuous, hence $\mathbb{P}(\mathbb{H}_p^\times)$ is a topological group. It is also Hausdorff and second countable, being a subspace of the Hausdorff second countable space $\mathbb{Q}_p^8$. In addition, $\mathbb{P}(\mathbb{H}_p^\times)$ is a closed subspace of the locally compact Hausdorff space $\mathbb{Q}_p^8$, hence it is locally compact as well. We are now going to show that, actually, $\mathrm{SO}(4,\mathbb{Q}_p)$ is a standard Borel $\mathbb{P}(\mathbb{H}_p^\times)$-space. The group $\mathrm{SO}(4,\mathbb{Q}_p)$ with p-adic topology is compact, second countable and Hausdorff. Thus, $\mathrm{SO}(4,\mathbb{Q}_p)$ along with its Borel σ-algebra is a standard Borel space. Let us introduce the map

$$\mathbb{P}(\mathbb{H}_p^\times)\times\mathrm{SO}(4,\mathbb{Q}_p)\ni\big((\xi,\rho),\,R\big)\mapsto(\xi,\rho)[R]:=\kappa_p'(\xi,\rho)R\in\mathrm{SO}(4,\mathbb{Q}_p).\tag{5.52}$$

This map is continuous, and such that $(\xi,\rho)\big[(\nu,\eta)[R]\big]=\kappa_p'(\xi,\rho)\big(\kappa_p'(\nu,\eta)R\big)=\kappa_p'\big((\xi,\rho)(\nu,\eta)\big)R=\big((\xi,\rho)(\nu,\eta)\big)[R]$, for every $(\xi,\rho),(\nu,\eta)\in\mathbb{P}(\mathbb{H}_p^\times)$, $R\in\mathrm{SO}(4,\mathbb{Q}_p)$ (here, we have used the fact that κ_p' is a homomorphism). Moreover, the map $R\mapsto(\xi,\rho)[R]$ is a homeomorphism, for every fixed $(\xi,\rho)\in\mathbb{P}(\mathbb{H}_p^\times)$. Therefore, the map (5.52) is an action of $\mathbb{P}(\mathbb{H}_p^\times)$ on $\mathrm{SO}(4,\mathbb{Q}_p)$, which is transitive by surjectivity of κ_p'. Actually, it is also a Borel map and, hence, $\mathrm{SO}(4,\mathbb{Q}_p)$ is a standard Borel $\mathbb{P}(\mathbb{H}_p^\times)$-space. Now, we observe that the stability subgroup at any $R\in\mathrm{SO}(4,\mathbb{Q}_p)$ is $\{(\xi,\rho)\in\mathbb{P}(\mathbb{H}_p^\times)\mid(\xi,\rho)[R]=R\}=\{(\xi,\rho)\in\mathbb{P}(\mathbb{H}_p^\times)\mid\kappa_p'(\xi,\rho)R=R\}=\{(\xi,\rho)\in\mathbb{P}(\mathbb{H}_p^\times)\mid\kappa_p'(\xi,\rho)=\mathrm{I}\}=\ker(\kappa_p')=\mathbb{Q}_p^*$. Thus, we can argue that $\mathbb{P}(\mathbb{H}_p^\times)/\mathbb{Q}_p^*$ and $\mathrm{SO}(4,\mathbb{Q}_p)$ are homeomorphic, the homeomorphism being provided, once again, by (5.41). In particular, if we consider the stability subgroup, for instance, at $\mathrm{I}\in\mathrm{SO}(4,\mathbb{Q}_p)$, the homeomorphism is explicitly given by $\mathbb{P}(\mathbb{H}_p^\times)/\mathbb{Q}_p^*\ni(\xi,\rho)\mathbb{Q}_p^*\mapsto\kappa_p'(\xi,\rho)\in\mathrm{SO}(4,\mathbb{Q}_p)$, and coincides with the isomorphism of Theorem 5.3.2. □

5.4 The Haar integral on $\mathrm{SO}(3,\mathbb{Q}_p)$

The construction of the Haar integral on $\mathrm{SO}(3,\mathbb{Q}_p)$ can be conveniently carried out by exploiting the conclusions of Theorem 3.1.3 and Proposition 5.3.1. In particular, this will bring us along two main steps: First, we shall construct the Haar measure on $\mathbb{H}_p^\times$ and, hence, its associated Haar integral. Then, owing to the result in Theorem 3.1.3, we will show that there is a natural lift of the Haar integral on $\mathrm{SO}(3,\mathbb{Q}_p)$ to that of $\mathbb{H}_p^\times$.

To begin with, let us notice that, since $\mathbb{H}_p^\times$ is locally compact, it admits a left Haar measure.

Proposition 5.4.1. *The group $\mathbb{H}_p^\times$ of invertible quaternions is unimodular.*

Proof. We exploit the well known result that a locally compact group is unimodular whenever there exists a compact neighborhood of the identity element which is invariant under the inner automorphisms of the group (see Chapter V in [116]). In the present case, $1\in\mathbb{H}_p^\times$ is an element of $\mathbb{Q}_p^*\leq\mathbb{H}_p^\times$. Since $\mathbb{Q}_p^*$ is the centre of $\mathbb{H}_p^\times$, every subset of $\mathbb{Q}_p^*$ is invariant under the inner automorphisms of $\mathbb{H}_p^\times$. $\mathbb{Z}_p^*$ provides an instance of such an invariant compact neighborhood of the identity 1. □

As a consequence of Proposition 5.4.1, the left and the right Haar measures on $\mathbb{H}_p^\times$ coincide, and we can construct it by directly exploiting formula (4.26). In particular, since $\mathbb{H}_p^\times\cong\mathbb{Q}_p^4-\{0\}$ as topological spaces, the map φ defining this homeomorphism provides us a *global* coordinate map for the elements of $\mathbb{H}_p^\times$. Specifically, if $\xi=q_0+\mathbf{i}q_1+\mathbf{j}q_2+\mathbf{k}q_3\in\mathbb{H}_p^\times$, its coordinates are given by $\varphi(\xi):=(q_0,q_1,q_2,q_3)$. Therefore, the density function η (cf. Section 4) characterizing the Haar measure on $\mathbb{H}_p^\times$ will be globally defined on the whole $\mathbb{H}_p^\times$. We will discuss the cases $p>2$ and $p=2$ separately.

Let us assume $p > 2$ first. If $\xi = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$, and $\chi = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$ are two quaternions in $\mathbb{H}_p^\times$, their composition will result in another quaternion $\zeta = z_0 + \mathbf{i}z_1 + \mathbf{j}z_2 + \mathbf{k}z_3$; namely

$$\begin{aligned}\zeta = z_0 + \mathbf{i}z_1 + \mathbf{j}z_2 + \mathbf{k}z_3 &= (q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3)(x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3)\\ &= q_0x_0 + \mathbf{i}q_0x_1 + \mathbf{j}q_0x_2 + \mathbf{k}q_0x_3 + \mathbf{i}q_1x_0 + vq_1x_1\\ &- \mathbf{k}q_1x_2 - \mathbf{j}vq_1x_3 + \mathbf{j}q_2x_0 + \mathbf{k}q_2x_1 - pq_2x_2 - \mathbf{i}pq_2x_3\\ &+ \mathbf{k}q_3x_0 + \mathbf{j}vq_3x_1 + \mathbf{i}pq_3x_2 + pvq_3x_3,\end{aligned} \tag{5.53}$$

from which we argue that

$$\begin{aligned} z_0 &= q_0x_0 + vq_1x_1 - pq_2x_2 + pvq_3x_3, & z_1 &= q_0x_1 + q_1x_0 - pq_2x_3 + pq_3x_2,\\ z_2 &= q_0x_2 + q_2x_0 - vq_1x_3 + vq_3x_1, & z_3 &= q_0x_3 + q_3x_0 - q_1x_2 + q_2x_1.\end{aligned} \tag{5.54}$$

We can now compute the function $\frac{\partial \zeta_j}{\partial x_k}\big(\overleftarrow{\varphi}(q);\varphi(e)\big)$, where the vectors of coordinates of e and ξ are $(1,0,0,0)$ and $(q_0,q_1,q_2,q_3) \coloneqq q$ respectively:

$$\frac{\partial \zeta_j}{\partial x_k}\big(\overleftarrow{\varphi}(q);\varphi(e)\big) = \begin{pmatrix} \frac{\partial\zeta_0}{\partial x_0} & \frac{\partial\zeta_0}{\partial x_1} & \frac{\partial\zeta_0}{\partial x_2} & \frac{\partial\zeta_0}{\partial x_3}\\ \frac{\partial\zeta_1}{\partial x_0} & \frac{\partial\zeta_1}{\partial x_1} & \frac{\partial\zeta_1}{\partial x_2} & \frac{\partial\zeta_1}{\partial x_3}\\ \frac{\partial\zeta_2}{\partial x_0} & \frac{\partial\zeta_2}{\partial x_1} & \frac{\partial\zeta_2}{\partial x_2} & \frac{\partial\zeta_2}{\partial x_3}\\ \frac{\partial\zeta_3}{\partial x_0} & \frac{\partial\zeta_3}{\partial x_1} & \frac{\partial\zeta_3}{\partial x_2} & \frac{\partial\zeta_3}{\partial x_3}\end{pmatrix}\Bigg|_{\substack{x_0=1,\\ x_1,x_2,x_3=0}} = \begin{pmatrix} q_0 & vq_1 & -pq_2 & pvq_3\\ q_1 & q_0 & pq_3 & -pq_2\\ q_2 & vq_3 & q_0 & -vq_1\\ q_3 & q_2 & -q_1 & q_0\end{pmatrix}. \tag{5.55}$$

This yields

$$\det\left(\frac{\partial \zeta_j}{\partial x_k}\big(\overleftarrow{\varphi}(q);\varphi(e)\big)\right) = (q_0^2 - vq_1^2 + pq_2^2 - pvq_3^2)^2, \tag{5.56}$$

which, as anticipated, is globally defined on $\mathbb{H}_p^\times$. Then, using (4.26), we can conclude that the Haar measure of any Borel subset $\mathcal{E}$ of $\mathbb{H}_p^\times$ is

$$\mu_{\mathbb{H}_p^\times}(\mathcal{E}) = \int_{\varphi(\mathcal{E})} \frac{1}{|q_0^2 - vq_1^2 + pq_2^2 - pvq_3^2|_p^2}\mathrm{d}\lambda(q), \tag{5.57}$$

where we recall that $\mathrm{d}\lambda(q)$ denotes the Haar measure on $\mathbb{Q}_p^4$ (cf. Example 3.1.2).

For the $p = 2$ case, a similar discussion to the one leading from (5.53) to (5.56) shows that

$$\det\left(\frac{\partial \zeta_i}{\partial x_j}\big(\overleftarrow{\varphi}(q);\varphi(e)\big)\right) = \det\begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3\\ q_1 & q_0 & -q_3 & q_2\\ q_2 & q_3 & q_0 & -q_1\\ q_3 & -q_2 & q_1 & q_0\end{pmatrix} = (q_0^2 + q_1^2 + q_2^2 + q_3^2)^2. \tag{5.58}$$

Then, using (4.26), the Haar measure of any Borel subset $\mathcal{E}$ of $\mathbb{H}_2^\times$ is

$$\mu_{\mathbb{H}_2^\times}(\mathcal{E}) = \int_{\varphi(\mathcal{E})} \frac{1}{|q_0^2 + q_1^2 + q_2^2 + q_3^2|_2^2}\mathrm{d}\lambda(q). \tag{5.59}$$

We summarize the above results with the following

Proposition 5.4.2. *Let $p \geq 2$ be a prime number, and let $\mathbb{H}_p^\times$ be the group of invertible p-adic quaternions. The Haar measure $\mu_{\mathbb{H}_p^\times}$ on $\mathbb{H}_p^\times$ is given by*

$$\mu_{\mathbb{H}_p^\times}(\mathcal{E}) = \int_{\varphi(\mathcal{E})} \frac{1}{|\mathrm{F}_{(4)}(q_0,q_1,q_2,q_3)|_p^2}\mathrm{d}\lambda(q), \tag{5.60}$$

for every Borel set $\mathcal{E} \subset \mathbb{H}_p^\times$, where $\xi = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$, $\varphi(\xi) = (q_0,q_1,q_2,q_3)$, $\mathrm{d}\lambda(q) = \mathrm{d}q_0\mathrm{d}q_1\mathrm{d}q_2\mathrm{d}q_3$ is the Haar measure on $\mathbb{Q}_p^4$, and $\mathrm{F}_{(4)}$ is the definite quadratic form of $\mathbb{Q}_p^4$ (see (2.44) in Theorem 2.1.3).

Exploiting the results of Theorem 3.1.3, we shall now prove that there exists a one-one correspondence between the Haar integrals on $\mathrm{SO}(3,\mathbb{Q}_p)$ and those of $\mathbb{H}_p^\times$. Indeed, let us consider the quotient group $\mathbb{H}_p^\times/\mathbb{Q}_p^*$. According to the results in Subsection 3.1, denoting by λ the Haar measure on $\mathbb{Q}_p$ (see Example 3.1.1), and with $\mathtt{s}\colon \mathbb{H}_p^\times/\mathbb{Q}_p^* \to \mathbb{H}_p^\times$ a Borel cross section, the map $\widehat{P}\colon \mathrm{L}^1(\mathbb{H}_p^\times) \to \mathrm{L}^1(\mathbb{H}_p^\times/\mathbb{Q}_p^*)$ defined as

$$(\widehat{P}f)(x) \coloneqq \int_{\mathbb{Q}_p} \mathrm{d}\lambda(\alpha) f(\mathtt{s}(x)\alpha), \qquad x \in \mathbb{H}_p^\times/\mathbb{Q}_p^*,\ f \in \mathrm{L}^1(\mathbb{H}_p^\times) \tag{5.61}$$

is a well-defined *surjection* of $\mathrm{L}^1(\mathbb{H}_p^\times)$ onto $\mathrm{L}^1(\mathbb{H}_p^\times/\mathbb{Q}_p^*)$ (cf. Remark 3.1.6). For K a compact subset of $\mathbb{H}_p^\times$, we can then define the set (cf. (3.17)):

$$\Psi_K \coloneqq \{\psi \in \mathrm{C}_c^+(\mathbb{H}_p^\times) \mid (P\psi)(q) = 1,\ \forall q \in K\}. \tag{5.62}$$

In particular, adhering to the notation used in Subsection 3.1, we set $\Psi \equiv \Psi_{\mathbb{H}_p^\times/\mathbb{Q}_p^*}$. Moreover, for every $\psi \in \Psi$, we also denote by $\widehat{\mathscr{L}_\psi}\colon \mathrm{L}^1(\mathbb{H}_p^\times/\mathbb{Q}_p^*) \to \mathrm{L}^1(\mathbb{H}_p^\times)$ — i.e., the extended WMB lift — the right inverse of $\widehat{P}$, as defined in (3.29). We are now ready to prove the following

Theorem 5.4.1. *Let μ_3 and $\mu_{\mathbb{H}_p^\times}$ be the Haar measures on $\mathrm{SO}(3,\mathbb{Q}_p)$ and $\mathbb{H}_p^\times$ respectively. For every prime $p \geq 2$, and any $\phi \in \mathrm{L}^1\big(\mathrm{SO}(3,\mathbb{Q}_p)\big)$, the following equality holds:*

$$\int_{\mathrm{SO}(3,\mathbb{Q}_p)} \mathrm{d}\mu_3(R)\phi(R) = \int_{\mathbb{H}_p^\times} \mathrm{d}\mu_{\mathbb{H}_p^\times}(q)\big(\widehat{\mathscr{L}_\psi}\phi\big)(q), \tag{5.63}$$

where $\widehat{\mathscr{L}_\psi}\phi \in \mathrm{L}^1(\mathbb{H}_p^\times)$ is the (extended) WMB lift of the map ϕ.

Proof. From Proposition 5.3.1, $\mathrm{SO}(3,\mathbb{Q}_p)$ is homeomorphic to $\mathbb{H}_p^\times/\mathbb{Q}_p^*$, hence this two spaces are Borel isomorphic. In particular, by exploiting the homeomorphism between $\mathrm{SO}(3,\mathbb{Q}_p)$ and $\mathbb{H}_p^\times/\mathbb{Q}_p^*$, it is clear that, for any $\phi \in \mathrm{L}^1\big(\mathrm{SO}(3,\mathbb{Q}_p)\big)$, the Haar integral in the l.h.s. of (5.63) can be expressed in terms of a Haar integral of the function ϕ on the homogeneous space $\mathbb{H}_p^\times/\mathbb{Q}_p^*$. On the other hand, the same homeomorphism also entails that $\mathbb{H}_p^\times/\mathbb{Q}_p^*$ is a compact group. But then, the equality in (5.63) directly follows from (3.31) in Theorem 3.1.3. □

Remark 5.4.1. We stress that the result of Theorem 5.4.1 provides an *equivalence of Haar integrals*. In other terms, computing the Haar integral of a function ϕ in $\mathrm{L}^1\big(\mathrm{SO}(3,\mathbb{Q}_p)\big)$ is equivalent to performing the integration of the function $\widehat{\mathscr{L}_\psi}\phi$, namely

$$\int\limits_{\mathrm{SO}(3,\mathbb{Q}_p)} \mathrm{d}\mu_3(R)\phi(R) = \int\limits_{\mathbb{H}_p^\times} \frac{(\widehat{\mathscr{L}_\psi}\phi)(q_0,q_1,q_2,q_3)}{|\mathrm{F}_{(4)}(q_0,q_1,q_2,q_3)|_p^2} \mathrm{d}q_0\mathrm{d}q_1\mathrm{d}q_2\mathrm{d}q_3, \tag{5.64}$$

where we have used the explicit form of the Haar measure (5.60).

5.5 The Haar integral on $\mathrm{SO}(4,\mathbb{Q}_p)$

Here we extend to $\mathrm{SO}(4,\mathbb{Q}_p)$ the results of the last subsection. In particular, in complete analogy to what was done for $\mathrm{SO}(3,\mathbb{Q}_p)$, we will provide a suitable lift of the Haar integral on $\mathrm{SO}(4,\mathbb{Q}_p)$ to that on $\mathbb{P}(\mathbb{H}_p^\times)$.

We start by observing that the group $\mathbb{P}(\mathbb{H}_p^\times)$ is locally compact and, hence, it admits a left Haar measure. Moreover, $\mathbb{P}(\mathbb{H}_p^\times)$ is unimodular, since it is a subgroup of the unimodular group $\mathbb{H}_p^\times \times \mathbb{H}_p^\times$ (being the direct product of the unimodular groups $\mathbb{H}_p^\times$) [116]. Since the measure on every chart covering $\mathbb{P}(\mathbb{H}_p^\times)$ can be obtained by translating the measure around its identity element e, it is enough to explicitly provide the latter by exploiting formula (4.26).

Consider the pairs of quaternions (α,β), $(\gamma,\delta)\in\mathbb{P}(\mathbb{H}_p^\times)$. From the defining condition of the group $\mathbb{P}(\mathbb{H}_p^\times)$, it must be true that $\mathrm{nrd}(\alpha)=\mathrm{nrd}(\beta)$ and $\mathrm{nrd}(\gamma)=\mathrm{nrd}(\delta)$. Explicitly, let $\alpha,\beta,\gamma,\delta$ be given by

$$\begin{aligned} \alpha &= a_0+\mathbf{i}a_1+\mathbf{j}a_2+\mathbf{k}a_3, & \beta &= b_0+\mathbf{i}b_1+\mathbf{j}b_2+\mathbf{k}b_3,\\ \gamma &= c_0+\mathbf{i}c_1+\mathbf{j}c_2+\mathbf{k}c_3, & \delta &= d_0+\mathbf{i}d_1+\mathbf{j}d_2+\mathbf{k}d_3. \end{aligned} \tag{5.65}$$

We shall denote the composition of the two elements $(\alpha,\beta),(\gamma,\delta)$ in $\mathbb{P}(\mathbb{H}_p^\times)$ by $\zeta\coloneqq(\zeta_1,\zeta_2)=(\alpha\gamma,\beta\delta)$, where $\zeta_1=z_0+\mathbf{i}z_1+\mathbf{j}z_2+\mathbf{k}z_3$ and $\zeta_2=z_0'+\mathbf{i}z_1'+\mathbf{j}z_2'+\mathbf{k}z_3'$. Clearly, ζ is a function of the parameters a_i,b_i,c_i,d_i for $i=0,1,2,3$. Now, to construct the Haar measure on $\mathbb{P}(\mathbb{H}_p^\times)$, we have first to compute the Jacobian of the function ζ. In particular, we shall consider (α,β) as fixed, and (γ,δ) as variables. In what follows, we will treat separately the cases $p>2$ and $p=2$.

Assume $p>2$ at first. The components z_i and z_i', $i=0,1,2,3$, of ζ_1 and ζ_2, can be computed in the same way in which we found (5.54):

$$\begin{aligned} z_0 &= a_0c_0+va_1c_1-pa_2c_2+pva_3c_3, & z_0' &= b_0d_0+vb_1d_1-pb_2d_2+pvb_3d_3\\ z_1 &= a_0c_1+a_1c_0-pa_2c_3+pa_3c_2, & z_1' &= b_0d_1+b_1d_0-pb_2d_3+pb_3d_2\\ z_2 &= a_0c_2+a_2c_0-va_1c_3+va_3c_1, & z_2' &= b_0d_2+b_2d_0-vb_1d_3+vb_3d_1\\ z_3 &= a_0c_3+a_3c_0-a_1c_2+a_2c_1, & z_3' &= b_0d_3+b_3d_0-b_1d_2+b_2d_1. \end{aligned} \tag{5.66}$$

As anticipated, z_i and z_i' are functions of the parameters c_i,d_i, $i=0,1,2,3$ (we are assuming a_i,b_i to be fixed). The defining condition $\mathrm{nrd}(\gamma)=\mathrm{nrd}(\delta)$ of the group $\mathbb{P}(\mathbb{H}_p^\times)$ is equivalent to $c_0^2=\mathrm{F}_{(4)}(d_0,d_1,d_2,d_3)+vc_1^2-pc_2^2+pvc_3^2$, and imposes a constraint on the variables c_i,d_i. Actually, this condition allows us to only consider $(c_1,c_2,c_3,d_0,d_1,d_2,d_3)$ as *independent* parameters in a neighborhood of $e\in\mathbb{P}(\mathbb{H}_p^\times)$: As the forthcoming remark will clarify, in such a neighborhood $\mathrm{F}_{(4)}(d_0,d_1,d_2,d_3)+vc_1^2-pc_2^2+pvc_3^2$ is a quadratic element, and its square root will then provide c_0 up to a sign.

Remark 5.5.1. The identity element of $\mathbb{P}(\mathbb{H}_p^\times)$ is identified in $\mathbb{Q}_p^8$ by the vector $(c_0,c_1,c_2,c_3,d_0,d_1,d_2,d_3)=(1,0,0,0,1,0,0,0)$. Now, consider an open ball in $\mathbb{Q}_p^8$ of centre $(1,0,0,0,1,0,0,0)$ and radius 1, in the usual p-adic topology. Here, $d_0=1+py_0$, $d_i=py_i$, $c_i=px_i$ with $y_0,y_i,x_i\in\mathbb{Z}_p$, $i=1,2,3$. Then, $\mathrm{F}_{(4)}(d_0,d_1,d_2,d_3)=1+pt$ and $\mathrm{F}_{(4)}(d_0,d_1,d_2,d_3)+vc_1^2-pc_2^2+pvc_3^2=1+pt'$, where $t\coloneqq 2y_0+p\mathrm{F}_{(4)}(y_0,y_1,y_2,y_3)$, $t'\coloneqq t+p(vx_1^2-px_2^2+pvx_3^2)\in\mathbb{Z}_p$. We can now resort to Hensel's Lemma (Chapter II, Section 2.2 of [74]) to show that, actually, $1+pz$, $z\in\mathbb{Z}_p$, is a square in $\mathbb{Z}_p$, i.e. that $f(x)\coloneqq x^2-1-pz$ admits roots in $\mathbb{Z}_p$. First, $f(x)=x^2-1 \mod p$ has roots $x=\pm1 \mod p$, where the derivative $\frac{df(x)}{dx}=2x$ takes values $\pm2\neq0 \mod p$. Then, Hensel's Lemma allows us to (uniquely) lift each of these simple roots to a root of the same function modulo p^n, $n\in\mathbb{Z}_{>1}$, converging to a p-adic root. This proves that $\mathrm{F}_{(4)}(d_0,d_1,d_2,d_3)$ and $\mathrm{F}_{(4)}(d_0,d_1,d_2,d_3)+vc_1^2-pc_2^2+pvc_3^2$ are squares. Hence, we can write

$$c_0=\pm\sqrt{\mathrm{F}_{(4)}(d_0,d_1,d_2,d_3)+vc_1^2-pc_2^2+pvc_3^2}, \tag{5.67}$$

at least for $(c_0,c_1,c_2,c_3,d_0,d_1,d_2,d_3)$ in a ball in $\mathbb{Q}_p^8$ centred in $(1,0,0,0,1,0,0,0)$ of radius 1.

From Remark 5.5.1 above, we know that the domain of definition of the square root (5.67) is non empty, and contains a neighborhood of the coordinates of the identity element of $\mathbb{P}(\mathbb{H}_p^\times)$, where $c_1, c_2, c_3, d_0, d_1, d_2, d_3$ can be assumed as independent variables. Here, we are referring to the coordinate map on such a neighborhood of $e \in \mathbb{P}(\mathbb{H}_p^\times)$ as

$$\mathbb{P}(\mathbb{H}_p^\times) \ni (\gamma, \delta) \mapsto \varphi_0((\gamma, \delta)) := (c_1, c_2, c_3, d_0, d_1, d_2, d_3) \in \mathbb{Q}_p^7, \tag{5.68}$$

where γ and δ are as in (5.65). The same argument can be repeated for the condition $\mathrm{nrd}(\zeta_1) = \mathrm{nrd}(\zeta_2)$, to express z_0 in terms of the other independent coordinates z_0', z_i, z_i', $i = 1, 2, 3$. In conclusion, we are left with the following 7×7 Jacobian matrix

$$\left.\frac{\partial \zeta_{0,j}\left(\overleftarrow{\varphi}_0(a_i, b_i); \varphi_0(e)\right)}{\partial x_k}\right|_{1\le j,k\le 7} = \left.\begin{pmatrix} \frac{\partial z_1}{\partial c_1} & \frac{\partial z_1}{\partial c_2} & \frac{\partial z_1}{\partial c_3} & \frac{\partial z_1}{\partial d_0} & \frac{\partial z_1}{\partial d_1} & \frac{\partial z_1}{\partial d_2} & \frac{\partial z_1}{\partial d_3} \\ \frac{\partial z_2}{\partial c_1} & \frac{\partial z_2}{\partial c_2} & \frac{\partial z_2}{\partial c_3} & \frac{\partial z_2}{\partial d_0} & \frac{\partial z_2}{\partial d_1} & \frac{\partial z_2}{\partial d_2} & \frac{\partial z_2}{\partial d_3} \\ \frac{\partial z_3}{\partial c_1} & \frac{\partial z_3}{\partial c_2} & \frac{\partial z_3}{\partial c_3} & \frac{\partial z_3}{\partial d_0} & \frac{\partial z_3}{\partial d_1} & \frac{\partial z_3}{\partial d_2} & \frac{\partial z_3}{\partial d_3} \\ \frac{\partial z_0'}{\partial c_1} & \frac{\partial z_0'}{\partial c_2} & \frac{\partial z_0'}{\partial c_3} & \frac{\partial z_0'}{\partial d_0} & \frac{\partial z_0'}{\partial d_1} & \frac{\partial z_0'}{\partial d_2} & \frac{\partial z_0'}{\partial d_3} \\ \frac{\partial z_1'}{\partial c_1} & \frac{\partial z_1'}{\partial c_2} & \frac{\partial z_1'}{\partial c_3} & \frac{\partial z_1'}{\partial d_0} & \frac{\partial z_1'}{\partial d_1} & \frac{\partial z_1'}{\partial d_2} & \frac{\partial z_1'}{\partial d_3} \\ \frac{\partial z_2'}{\partial c_1} & \frac{\partial z_2'}{\partial c_2} & \frac{\partial z_2'}{\partial c_3} & \frac{\partial z_2'}{\partial d_0} & \frac{\partial z_2'}{\partial d_1} & \frac{\partial z_2'}{\partial d_2} & \frac{\partial z_2'}{\partial d_3} \\ \frac{\partial z_3'}{\partial c_1} & \frac{\partial z_3'}{\partial c_2} & \frac{\partial z_3'}{\partial c_3} & \frac{\partial z_3'}{\partial d_0} & \frac{\partial z_3'}{\partial d_1} & \frac{\partial z_3'}{\partial d_2} & \frac{\partial z_3'}{\partial d_3} \end{pmatrix}\right|_{\substack{c_1=c_2=c_3=0\\ d_0=1, d_1=d_2=d_3=0}}, \tag{5.69}$$

where, in the l.h.s., $\varphi_0(e) = (0, 0, 0, 1, 0, 0, 0)$ and $(x_k)_{k=1}^7 = (c_1, c_2, c_3, d_0, d_1, d_2, d_3)$.

By using (5.67), the partial derivatives of the dependent variable c_0 w.r.t. the independent ones are

$$\left.\frac{\partial c_0}{\partial c_i}\right|_{\substack{c_1=c_2=c_3=0\\ d_0=1, d_1=d_2=d_3=0}} = \left.\frac{\partial c_0}{\partial d_i}\right|_{\substack{c_1=c_2=c_3=0\\ d_0=1, d_1=d_2=d_3=0}} = 0, \quad \text{for } i = 1, 2, 3, \tag{5.70}$$

and

$$\left.\frac{\partial c_0}{\partial d_0}\right|_{\substack{c_1=c_2=c_3=0\\ d_0=1, d_1=d_2=d_3=0}} = \pm 1. \tag{5.71}$$

Hence, using the expressions in (5.66), the Jacobian matrix (5.69) is straightforwardly computed and reads:

$$\left.\frac{\partial \zeta_{0,j}\left(\overleftarrow{\varphi}_0(a_i, b_i); \varphi_0(e)\right)}{\partial x_k}\right|_{1\le j,k\le 7} = \begin{pmatrix} a_0 & pa_3 & -pa_2 & \pm a_1 & 0 & 0 & 0 \\ va_3 & a_0 & -va_1 & \pm a_2 & 0 & 0 & 0 \\ a_2 & -a_1 & a_0 & \pm a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_0 & vb_1 & -pb_2 & pvb_3 \\ 0 & 0 & 0 & b_1 & b_0 & pb_3 & -pb_2 \\ 0 & 0 & 0 & b_2 & vb_3 & b_0 & -vb_1 \\ 0 & 0 & 0 & b_3 & b_2 & -b_1 & b_0 \end{pmatrix}. \tag{5.72}$$

The p-adic absolute value of the determinant of such a matrix is

$$\begin{aligned} \left|\det \frac{\partial \zeta_{0,j}\left(\overleftarrow{\varphi}_0(a_i, b_i); \varphi_0(e)\right)}{\partial x_k}\right|_p &= \left|a_0(a_0^2 - va_1^2 + pa_2^2 - pva_3^2)(b_0^2 - vb_1^2 + pb_2^2 - pvb_3^2)^2\right|_p \\ &= \left|\sqrt{\mathrm{F}_{(4)}(b_0, b_1, b_2, b_3) + va_1^2 - pa_2^2 + pva_3^2}\ \mathrm{F}_{(4)}(b_0, b_1, b_2, b_3)^3\right|_p. \end{aligned} \tag{5.73}$$

For the last equality, we used again the condition $\mathrm{nrd}(\xi) = \mathrm{nrd}(\rho)$ in a suitable neighborhood for the coordinates of the identity of $\mathbb{P}(\mathbb{H}_p^\times)$, where $a_0 = \pm\sqrt{\mathrm{F}_{(4)}(b_0, b_1, b_2, b_3) + va_1^2 - pa_2^2 + pva_3^2}$ is well-defined.

Let us now switch to the case where $p = 2$. The components of ζ_1 and ζ_2 are now given by:

$$\begin{aligned} z_0 &= a_0c_0 - a_1c_1 - a_2c_2 - a_3c_3, & z_0' &= b_0d_0 - b_1d_1 - b_2d_2 - b_3d_3 \\ z_1 &= a_0c_1 + a_1c_0 + a_2c_3 - a_3c_2, & z_1' &= b_0d_1 + b_1d_0 + b_2d_3 - b_3d_2 \\ z_2 &= a_0c_2 + a_2c_0 - a_1c_3 + a_3c_1, & z_2' &= b_0d_2 + b_2d_0 - b_1d_3 + b_3d_1 \\ z_3 &= a_0c_3 + a_3c_0 + a_1c_2 - a_2c_1, & z_3' &= b_0d_3 + b_3d_0 + b_1d_2 - b_2d_1. \end{aligned} \tag{5.74}$$

The defining condition $\mathrm{nrd}(\nu) = \mathrm{nrd}(\varrho)$ is equivalent to $c_0^2 = \mathrm{F}_{(4)}(d_0, d_1, d_2, d_3) - c_1^2 - c_2^2 - c_3^2$. Then, analogously to the case where $p > 2$, it is not difficult to prove that, in a suitable neighborhood of $(1, 0, 0, 0, 1, 0, 0, 0)$ in $\mathbb{Q}_p^8$, c_0 can be expressed in terms of the independent variables $c_1, c_2, c_3, d_0, d_1, d_2, d_3$, as the forthcoming Remark will clarify.

Remark 5.5.2. Consider an open ball in $\mathbb{Q}_2^8$ of centre $(1, 0, 0, 0, 1, 0, 0, 0)$ and radius $1/2$, in which $d_0 = 1 + 4y_0$, $d_i = 4y_i$, $c_i = 4x_i$, with $y_0, y_i, x_i \in \mathbb{Z}_2$, $i = 1, 2, 3$. In this case, $\mathrm{F}_{(4)}(d_0, d_1, d_2, d_3) = (1 + 4y_0)^2 + (4y_1)^2 + (4y_2)^2 + (4y_3)^2 = 1 + 8\big(y_0 + 2\mathrm{F}_{(4)}(y_0, y_1, y_2, y_3)\big)$ and $\mathrm{F}_{(4)}(d_0, d_1, d_2, d_3) - c_1^2 - c_2^2 - c_3^2 = 1 + 8\big[y_0 + 2\big(\mathrm{F}_{(4)}(y_0, y_1, y_2, y_3) - x_1^2 - x_2^2 - x_3^2\big)\big]$ are squares in $\mathbb{Z}_2$, as they are congruent to 1 modulo 8. Therefore, we can write

$$c_0 = \pm\sqrt{\mathrm{F}_{(4)}(d_0, d_1, d_2, d_3) - c_1^2 - c_2^2 - c_3^2}, \tag{5.75}$$

at least for $(c_0, c_1, c_2, c_3, d_0, d_1, d_2, d_3)$ in an open ball in $\mathbb{Q}_2^8$ centred in $(1, 0, 0, 0, 1, 0, 0, 0)$, and of radius $1/2$.

As a consequence, the coordinate map on a suitable neighborhood of $e \in \mathbb{P}(\mathbb{H}_2^\times)$ to $\mathbb{Q}_2^7$ is as in (5.68). An analogous discussion can be carried out for the condition $\mathrm{nrd}(\zeta_1) = \mathrm{nrd}(\zeta_2)$, to show that z_0 can be expressed as a function of the (independent) variables z_0', z_i, z_i', $i = 1, 2, 3$, in a suitable neighborhood of the identity. It follows that the Jacobian matrix for $p = 2$ is of the same form (5.69) for $p > 2$ and, as one can easily check, the partial derivatives of the dependent variable c_0 are again given by (5.70) and (5.71). Thus, we obtain

$$\left.\frac{\partial \zeta_{0,j}\big(\overleftarrow{\varphi}_0(a_i, b_i); \varphi_0(e)\big)}{\partial x_k}\right|_{1 \le j,k \le 7} = \begin{pmatrix} a_0 & -a_3 & a_2 & \pm a_1 & 0 & 0 & 0 \\ a_3 & a_0 & -a_1 & \pm a_2 & 0 & 0 & 0 \\ -a_2 & a_1 & a_0 & \pm a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_0 & -b_1 & -b_2 & -b_3 \\ 0 & 0 & 0 & b_1 & b_0 & -b_3 & b_2 \\ 0 & 0 & 0 & b_2 & b_3 & b_0 & -b_1 \\ 0 & 0 & 0 & b_3 & -b_2 & b_1 & b_0 \end{pmatrix}, \tag{5.76}$$

which then yields

$$\begin{aligned} \left|\det \frac{\partial \zeta_{0,j}\big(\overleftarrow{\varphi}_0(a_i, b_i); \varphi_0(e)\big)}{\partial x_k}\right|_2 &= \left|a_0(a_0^2 + a_1^2 + a_2^2 + a_3^2)(b_0^2 + b_1^2 + b_2^2 + b_3^2)^2\right|_2 \\ &= \left|\sqrt{\mathrm{F}_{(4)}(b_0, b_1, b_2, b_3) - a_1^2 - a_2^2 - a_3^2}\; \mathrm{F}_{(4)}(b_0, b_1, b_2, b_3)^3\right|_2. \end{aligned} \tag{5.77}$$

Once more, this expression is valid in the domain of definition of the p-adic square root, containing a neighborhood for the coordinates of the identity of $\mathbb{P}(\mathbb{H}_2^\times)$.

By exploiting (4.26), the p-adic absolute values (5.73) and (5.77) immediately yield the Haar measure on a neighborhood of the identity element of $\mathbb{P}(\mathbb{H}_p^\times)$.

Proposition 5.5.1. *Let $p \geq 2$ be a prime number, and let $\mathbb{P}(\mathbb{H}_p^\times)$ be the group of p-adic quaternion pairs* (5.45)*. For any Borel subset $\mathcal{E}$ of $\mathbb{P}(\mathbb{H}_p^\times)$, the following equalities hold:*

$$\mu_{\mathbb{P}(\mathbb{H}_p^\times)}(\mathcal{E}\cap\mathscr{U}_0) = \int\limits_{\varphi_0(\mathcal{E}\cap\mathscr{U}_0)} \left|\sqrt{\mathrm{F}_{(4)}(b_0,b_1,b_2,b_3)+va_1^2-pa_2^2+pva_3^2}\,\mathrm{F}_{(4)}(b_0,b_1,b_2,b_3)^3\right|_p^{-1} \mathrm{d}\lambda(q), \tag{5.78}$$

for $p > 2$, while for $p = 2$,

$$\mu_{\mathbb{P}(\mathbb{H}_2^\times)}(\mathcal{E}\cap\mathscr{U}_0) = \int\limits_{\varphi_0(\mathcal{E}\cap\mathscr{U}_0)} \left|\sqrt{\mathrm{F}_{(4)}(b_0,b_1,b_2,b_3)-a_1^2-a_2^2-a_3^2}\,\mathrm{F}_{(4)}(b_0,b_1,b_2,b_3)^3\right|_2^{-1} \mathrm{d}\lambda(q). \tag{5.79}$$

Here, $\mu_{\mathbb{P}(\mathbb{H}_p^\times)}$ and $\mathrm{d}\lambda(q) = da_1da_2da_3db_0db_1db_2db_3$ denote the Haar measure on $\mathbb{P}(\mathbb{H}_p^\times)$ and on $\mathbb{Q}_p^7$ respectively, $\mathrm{F}_{(4)}$ denotes the definite quadratic form of $\mathbb{Q}_p^4$, and $\mathscr{U}_0$ is a suitable neighborhood of the identity element $e \in \mathbb{P}(\mathbb{H}_p^\times)$ where the coordinate map φ_0 (cf. (5.68)*) is defined.*

Since, by translation invariance, one can 'move' the measure on the fixed chart around e all over the group, Proposition 5.5.1 is enough to compute any Haar integral on the whole $\mathbb{P}(\mathbb{H}_p^\times)$.

At this point, we are now ready to construct the Haar integral on $\mathrm{SO}(4,\mathbb{Q}_p)$. Indeed, as done for $\mathrm{SO}(3,\mathbb{Q}_p)$, we can define a suitable (surjective) map $\widehat{P}\colon \mathrm{L}^1\big(\mathbb{P}(\mathbb{H}_p^\times)\big) \to \mathrm{L}^1\big(\mathbb{P}(\mathbb{H}_p^\times)/\mathbb{Q}_p^*\big)$ such that

$$(\widehat{P}f)(x)\colon \int_{\mathbb{Q}_p} \mathrm{d}\lambda(\alpha) f(\mathtt{s}(x)\alpha), \quad x \in \mathbb{P}(\mathbb{H}_p^\times/\mathbb{Q}_p^*),\ f \in \mathrm{L}^1\big(\mathbb{P}(\mathbb{H}_p^\times)\big), \tag{5.80}$$

where $\mathtt{s}\colon \mathbb{P}(\mathbb{H}_p^\times)/\mathbb{Q}_p^* \to \mathbb{P}(\mathbb{H}_p^\times)$ is any Borel cross section of $\mathbb{P}(\mathbb{H}_p^\times)/\mathbb{Q}_p^*$ onto $\mathbb{P}(\mathbb{H}_p^\times)$. Again, for any compact subset K of $\mathbb{P}(\mathbb{H}_p^\times)$, we define

$$\Psi_K \coloneqq \{\psi \in \mathrm{C}_c^+\big(\mathbb{P}(\mathbb{H}_p^\times)\big) \mid (P\psi)(q) = 1,\ \forall q \in K\}, \tag{5.81}$$

and set $\Psi \equiv \Psi_{\mathbb{P}(\mathbb{H}_p^\times)/\mathbb{Q}_p^*}$. Then, for every $\psi \in \Psi$, the map $\widehat{\mathscr{L}_\psi}\colon \mathrm{L}^1\big(\mathbb{P}(\mathbb{H}_p^\times)/\mathbb{Q}_p^*\big) \to \mathrm{L}^1\big(\mathbb{P}(\mathbb{H}_p^\times)\big)$ — i.e., the extended WMB lift — provides a right inverse of $\widehat{P}$.

In the light of the above discussion, the following result is now clear:

Theorem 5.5.1. *Let μ_4 and $\mu_{\mathbb{P}(\mathbb{H}_p^\times)}$ be the Haar measures on $\mathrm{SO}(4,\mathbb{Q}_p)$, and $\mathbb{P}(\mathbb{H}_p^\times)$ respectively. For every prime $p \geq 2$, and any $\phi \in \mathrm{L}^1\big(\mathrm{SO}(4,\mathbb{Q}_p)\big)$, the following equality holds:*

$$\int\limits_{\mathrm{SO}(4,\mathbb{Q}_p)} \mathrm{d}\mu_4(R)\phi(R) = \int\limits_{\mathbb{P}(\mathbb{H}_p^\times)} \mathrm{d}\mu_{\mathbb{P}(\mathbb{H}_p^\times)}(q)(\widehat{\mathscr{L}_\psi}\phi)(q), \tag{5.82}$$

where $\widehat{\mathscr{L}_\psi}\phi \in \mathrm{L}^1\big(\mathbb{P}(\mathbb{H}_p^\times)\big)$ is the (extended) WMB lift of the map ϕ.

Proof. By Proposition 5.3.2, $\mathrm{SO}(4,\mathbb{Q}_p)$ and $\mathbb{P}(\mathbb{H}_p^\times)/\mathbb{Q}_p^*$ are homeomorphic and, then, Borel isomorphic. This then entails that, for any given function $\phi \in \mathrm{L}^1\big(\mathrm{SO}(4,\mathbb{Q}_p)\big)$, we can express its Haar integral (w.r.t. the Haar measure μ on $\mathrm{SO}(4,\mathbb{Q}_p)$) as an Haar integral on $\mathbb{P}(\mathbb{H}_p^\times)/\mathbb{Q}_p^*$. Moreover, the same homeomorphism also implies that $\mathbb{P}(\mathbb{H}_p^\times)/\mathbb{Q}_p^*$ is a compact group. But then, the equality in (5.82) directly follows from (3.31) of Theorem 3.1.3. □

Remark 5.5.3. It is worth noting that the method we have followed here for constructing the Haar measure on the p-adic special orthogonal groups is not the only possible one. In fact, in [89], it was recently showed that the invariant measure on $\mathrm{SO}(n, \mathbb{Q}_p)$, for $n = 2, 3, 4$, can be obtained by using an *inverse limit approach*. Specifically, since such groups are (Hausdorff) compact and totally disconnected, they are *profinite groups*. This entails that they are isomorphic to the inverse limit of a suitable inverse family of finite groups (see Subsection 1.2). On the other hand, once an inverse measure system was constructed for this family, and observing that, an invariant measure for the groups in family is given by the counting measure, one can prove that its projective limit (as a measure space) provides the Haar measure on the inverse limit groups, namely, on $\mathrm{SO}(n, \mathbb{Q}_p)$, for $n = 2, 3, 4$.

Part III

p-Adic quantum theory

In this part of the dissertation, we focus on the description of p-adic quantum states *and* observables *in a p-adic Hilbert space. We begin in Chapter 6 by discussing p-adic normed and Banach spaces over* $\mathbb{Q}_p$*; by further introducing a suitable notion of an inner product, we also provide a definition of a* Hilbert space *in the p-adic setting. In Chapter 7 we study bounded and adjointable operators, mainly described in terms of* matrix operators *acting over p-adic Hilbert spaces. In Section 7.2, the p-adic* unitary operators $\mathcal{U}(\mathcal{H})$ *are introduced, and their complete characterization is then provided in Theorem 7.2.3. Section 7.3 deals with the* trace class $\mathcal{T}(\mathcal{H})$ *of a p-adic Hilbert space. We prove that this set is a* left ideal *in the space of bounded operators* $\mathcal{B}(\mathcal{H})$*, and a* two-sided $*$-ideal *in the Banach* $*$*-algebra of bounded adjointable operators* $\mathcal{B}_{\mathrm{ad}}(\mathcal{H})$*. Eventually, we show that* $\mathcal{T}(\mathcal{H})$ *also has the structure of a p-adic Hilbert space, the so-called* p-adic Hilbert-Schmidt space*. In Chapter 8, we discuss physical states in the p-adic setting. We first provide an algebraic definition of a p-adic state tailored on a suitable p-adic model of probability theory; then, we focus on the special classes of the so-called* statistical *and* density states*. Finally, we introduce the notion of a SOVM as a suitable p-adic counterpart to the POVMs of the standard complex quantum mechanics. The material discussed here is in large part based on [17, 92, 93]. Deviations from the published version mostly affect notations and typesetting.*

6

p-adic Banach and Hilbert spaces

Until today, there seems to be no universally accepted model of a Hilbert space over the field of p-adic numbers or its quadratic extensions [9, 18, 126–128]. Having in mind applications to quantum mechanics, in this chapter we introduce notions of p-adic Banach and Hilbert spaces that are suitable for our purposes.

6.1 p-adic Banach spaces

We start with the following:

Definition 6.1.1. By a *normed vector space* over $\mathbb{Q}_{p,\mu}$ we mean a pair $(X, \|\cdot\|)$, where X is a vector space over $\mathbb{Q}_{p,\mu}$ and $\|\cdot\|$ is an *ultrametric norm* defined on X; i.e., a map $\|\cdot\| : X \to \mathbb{R}^+$ such that

- $\|x\| = 0$ iff $x = 0$;
- $\|\alpha x\| = |\alpha|\ \|x\|$, for all $\alpha \in \mathbb{Q}_{p,\mu}$ and all $x \in X$;
- $\|x + y\| \leq \max(\|x\|, \|y\|)$, for all $x, y \in X$.

Remark 6.1.1. In the literature [9, 70, 72, 101, 128], the pair $(X, \|\cdot\|)$ would be called an *ultrametric* (or non-Archimedean) normed space. Putting

$$\|X\| := \{\|x\| \mid x \in X\} \quad \text{and} \quad |\mathbb{Q}_{p,\mu}| := \{|\alpha| \mid \alpha \in \mathbb{Q}_{p,\mu}\}, \tag{6.1}$$

the sets $\|X\|$ and $|\mathbb{Q}_{p,\mu}|$ are not, in general, related by any inclusion relation. However, the inclusion $\|X\| \subset |\mathbb{Q}_{p,\mu}|$ entails the existence of unit vectors in X and also implies the reverse inclusion, in such a way that, actually, $\|X\| = |\mathbb{Q}_{p,\mu}|$. In the case where this condition is satisfied, $\|X\| \setminus \{0\}$ coincides with the valuation group $|\mathbb{Q}^*_{p,\mu}|$ (recall Remark 1.1.9). Since the valuation group $|\mathbb{Q}^*_{p,\mu}|$ is discrete, then, by Theorem 3 in [129], the field $\mathbb{Q}_{p,\mu}$ is *spherically complete*, i.e., every nest of closed balls in $\mathbb{Q}_{p,\mu}$ has a non-empty intersection.

Remark 6.1.2. In the following, we will mainly deal with *separable* (p-adic) normed and Banach spaces, and we will consider separable (p-adic) Hilbert spaces only. In this regard, note that, since $\mathbb{Q}_{p,\mu}$ is separable, we do not need to use the — in this case equivalent — notion of a normed space *of countable type* [73, 76]. We will consider some non-separable ultrametric Banach spaces in Section 6.4.

Any p-adic normed space is a (ultra)-metric space. Thus, it can be completed, so resulting into a *p-adic Banach space* (i.e., an ultrametric Banach space over $\mathbb{Q}_{p,\mu}$).

Proposition 6.1.1 ([69, 73, 101])**.** *Let $(X, \|\cdot\|)$ be a p-adic Banach space. A series $\sum_i x_i$ in X is convergent if and only if $\lim_i x_i = 0$. In particular, (regarding $\mathbb{Q}_{p,\mu}$ as a complete p-adic normed space) a series $\sum_i x_i$ in $\mathbb{Q}_{p,\mu}$ converges if and only if $\lim_i x_i = 0$.*

We now consider a class of p-adic Banach spaces that will be central for our purposes.

Let I be a *countable* index set (in the case where this set is finite, we will put $I = \{1, 2, \ldots, n\}$, for some $n \in \mathbb{N}$; otherwise we put $I = \mathbb{N}$), and let X be a p-adic Banach space. We introduce the space $c_0(I, X)$ of *zero-converging* — in the case where $I = \mathbb{N}$ — sequences in X:

$$c_0(I, X) := \big\{x = \{x_i\}_{i \in I} \mid x_i \in X, \ \lim_i \|x_i\| = 0\big\}. \tag{6.2}$$

In particular, with $X = \mathbb{Q}_{p,\mu}$ (regarded as a one-dimensional vector space, endowed with the norm $|\cdot|$), we obtain the sequence space

$$c_0(I, \mathbb{Q}_{p,\mu}) := \big\{x = \{x_i\}_{i \in I} \mid x_i \in \mathbb{Q}_{p,\mu}, \ \lim_i |x_i| = 0\big\}. \tag{6.3}$$

Remark 6.1.3. In order to include the case where I is finite, here and in the following we set: $\lim_i x_i \equiv 0$ and $\lim_i \|x_i\| \equiv 0$, for I finite.

The space $c_0(I, X)$, endowed with the *sup-norm* $\|\cdot\|_\infty$ defined by

$$\|x\|_\infty := \sup_{i \in I} \|x_i\| = \max_{i \in I} \|x_i\|, \quad \forall x \in c_0(I, X), \tag{6.4}$$

is a (ultrametric) normed space over $\mathbb{Q}_{p,\mu}$. Moreover, it is possible to prove that $c_0(I, X)$ is complete w.r.t. the sup-norm, and, thus, the pair $(c_0(I, X), \|\cdot\|_\infty)$ is a p-adic Banach space [70].

Remark 6.1.4. The p-adic Banach space $(c_0(I, \mathbb{Q}_{p,\mu}), \|\cdot\|_\infty)$ is a particular case of the ultrametric Banach space $c_0(I, \mathbb{K})$ — where $\mathbb{K}$ is a complete, non-trivially valued, non-Archimedean field [69, 128–130] and

$$c_0(I, \mathbb{K}) := \big\{x = \{x_i\}_{i \in I} \mid x_i \in \mathbb{K}, \ \lim_i |x_i|_{\mathbb{K}} = 0\big\} \tag{6.5}$$

— endowed with the norm

$$\|x\|_\infty := \sup_{i \in I} |x_i|_{\mathbb{K}} = \max_{i \in I} |x_i|_{\mathbb{K}}. \tag{6.6}$$

Next, we introduce the notion of norm-orthogonal system of vectors in a p-adic normed space [69, 73, 76, 101, 131]:

Definition 6.1.2. Let $(X, \|\cdot\|)$ be a p-adic normed space. Two vectors $x, y \in X$ are said to be (mutually) *norm-orthogonal* if, for any $\alpha \in \mathbb{Q}_{p,\mu}$, $\|x\| \leq \|x + \alpha y\|$, or, equivalently, if $\|\alpha x + \beta y\| = \max\{\|\alpha x\|, \|\beta y\|\}$, for all $\alpha, \beta \in \mathbb{Q}_{p,\mu}$. More generally, a finite set $\{x_1, \ldots, x_n\}$ in X is said to be *norm-orthogonal* if

$$\left\|\sum_{i=1}^{n} \alpha_i x_i\right\| = \max_i |\alpha_i| \, \|x_i\|, \ \text{ for all } \ \{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{Q}_{p,\mu}. \tag{6.7}$$

An arbitrary set $\mathfrak{B} \subset X$ is called *norm-orthogonal* if every finite subset of $\mathfrak{B}$ is. In particular, a norm-orthogonal set $\mathfrak{B} \subset X$ is said to be *normal* if $\|x\| = 1$, for all $x \in \mathfrak{B}$.

Definition 6.1.3. Let $(X, \|\cdot\|)$ be a (separable) p-adic Banach space. A countable subset $\mathfrak{B}$ of $X \setminus \{0\}$ is said to be a *norm-orthogonal (normal)* basis if

(B1) $\mathfrak{B}$ is a norm-orthogonal (normal) set;

(B2) for each $x \in X$, there exists a map $c_x : \mathfrak{B} \to \mathbb{Q}_{p,\mu}$ such that

$$x = \sum_{b \in \mathfrak{B}} c_x(b) \, b. \tag{6.8}$$

Note that a normal basis in X is, in particular, a (normalized) Schauder basis [96, 131]; also see point (iii) in the proposition below. We will denote such a basis by $\boldsymbol{e} \equiv \{e_i\}_{i\in I}$.

Proposition 6.1.2 ([69]). *Let $(X, \|\cdot\|)$ be a p-adic Banach space, let $\{e_i\}_{i\in I}$ be a normal basis and let $x = \sum_{i\in I} \alpha_i e_i$, with $\alpha_1, \alpha_2, \dots \in \mathbb{Q}_{p,\mu}$. Then, the following facts hold:*

(i) *in the case where $I = \mathbb{N}$, $\lim_i \alpha_i = 0$;*

(ii) *$\|x\| = \max_{i\in I} |\alpha_i|$;*

(iii) *if, for some $\lambda_1, \lambda_2, \dots \in \mathbb{Q}_{p,\mu}$, $\sum_{i\in I} \lambda_i e_i = x$, then $\alpha_i = \lambda_i$, $\forall i \in I$; namely the expansion of every vector in X w.r.t. the basis $\{e_i\}_{i\in I}$ is unique.*

Remark 6.1.5. By the unconditional convergence of a series in a p-adic Banach space, every permutation of a normal basis is a normal basis too.

Theorem 6.1.1. *Let $(X, \|\cdot\|)$ be a (separable) p-adic Banach space over $\mathbb{Q}_{p,\mu}$. Then, it admits a norm-orthogonal basis. Moreover, X admits a normal basis if and only if $\|X\| = |\mathbb{Q}_{p,\mu}|$* (equivalently, if and only if $\|X\| \subset |\mathbb{Q}_{p,\mu}|$; see Remark 6.1.1). *If the last condition is satisfied, the mapping*

$$c_0(I, \mathbb{Q}_{p,\mu}) \ni \{x_i\}_{i\in I} \mapsto \sum_{i\in I} x_i e_i \in X \tag{6.9}$$

defines a surjective isometry of $c_0(I, \mathbb{Q}_{p,\mu})$ onto X.

Proof. The first assertion of the theorem follows from Theorem 50.8 in [69], taking into account the fact that every finite extension of $\mathbb{Q}_p$ is locally compact. Alternatively, one can use the fact that $\mathbb{Q}_{p,\mu}$ is spherically complete, X is separable (equivalently, of countable type) and Lemma 5.5 in [73]. The second assertion is (the separable version of) the Monna-Fleischer Theorem; see Section 4.4.5 in [70], taking into account the fact that the valuation group $|\mathbb{Q}^*_{p,\mu}|$ is a discrete subgroup of the multiplicative group of all positive reals. For the final assertion of the theorem, see Proposition 3 in Section 4.4.2 of [70]. □

In the light of the previous result, we set the following:

Definition 6.1.4. We say that a p-adic Banach space $(X, \|\cdot\|)$ over $\mathbb{Q}_{p,\mu}$ is *normal* if $\|X\| = |\mathbb{Q}_{p,\mu}|$; equivalently, if it admits a normal basis.

Remark 6.1.6. Let $(X, \|\cdot\|)$ be a (separable) p-adic Banach space. We define $\dim(X)$ as the countable cardinality of any norm-orthogonal basis in X. In the finite-dimensional case, $\dim(X)$ coincides with the algebraic dimension of X. In the infinite-dimensional case, we simply put $\dim(X) = \infty$.

6.2 p-adic inner product Banach spaces

We now aim at introducing a suitable notion of *p-adic Hilbert space* over $\mathbb{Q}_{p,\mu}$. The first step is to provide a convenient notion of p-adic inner product (Banach) space:

Definition 6.2.1. Let $(X, \|\cdot\|)$ be a p-adic Banach space over $\mathbb{Q}_{p,\mu}$. By a *non-Archimedean inner product* on X we mean a map $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{Q}_{p,\mu}$ such that

(i) $\langle\cdot,\cdot\rangle$ is a *sesquilinear form*, i.e., it is linear in its second argument and conjugate-linear in its first argument (w.r.t. the conjugation in $\mathbb{Q}_{p,\mu}$ introduced in Section 3);

(ii) $\langle\cdot,\cdot\rangle$ is *Hermitian*, i.e., $\langle x,y\rangle = \overline{\langle y,x\rangle}$;

(iii) the *Cauchy-Schwarz inequality* holds, i.e., $|\langle x,y\rangle| \leq \|x\|\,\|y\|$.

We call the triple $(X, \|\cdot\|, \langle\cdot,\cdot\rangle)$, where $\langle\cdot,\cdot\rangle$ is a non-Archimedean inner product, an *inner product p-adic Banach space*.

Given an inner product p-adic Banach space $(X, \|\cdot\|, \langle\cdot,\cdot\rangle)$, we say that $\langle\cdot,\cdot\rangle$ is *non-degenerate* if, moreover, $\langle x,y\rangle = 0$, for all $y \in X$, implies that $x = 0$.

Remark 6.2.1. Note that here, in general, $\|x\| \neq \sqrt{|\langle x,x\rangle|}$.

Remark 6.2.2. It is worth noting that the Cauchy-Schwarz inequality immediately implies that a non-Archimedean inner product $\langle\cdot,\cdot\rangle$ on a p-adic Banach space is *continuous* w.r.t. both its arguments (separately), and also *jointly* continuous (i.e., as a map from $X \times X$, endowed with the product topology, into $\mathbb{Q}_{p,\mu}$), where the topology on X is the one induced by the norm.

Example 6.2.1. Let us provide an example of an inner product p-adic Banach space. Given a normal p-adic Banach space X, let us consider the (non-degenerate, Hermitian) sesquilinear form defined by

$$X \times X \ni (x,y) \mapsto \langle x,y\rangle := \sum_{i\in I} \overline{x_i}\, y_i, \tag{6.10}$$

where, for some *normal basis* $\{e_i\}_{i\in I}$, $x = \sum_{i\in I} x_i e_i$ and $y = \sum_{i\in I} y_i e_i$. Clearly, we have that

$$|\langle x,y\rangle| = \left|\sum_{i\in I} \overline{x_i}\, y_i\right| \leq \max_{i\in I} |x_i|\,|y_i| \leq \max_{i\in I} |x_i| \, \max_{j\in I} |y_j| = \|x\|_\infty \|y\|_\infty, \tag{6.11}$$

i.e., the Cauchy-Schwarz inequality is satisfied. We call this inner product the *canonical inner product* in X associated with the normal basis $\{e_i\}_{i\in I}$.

Remark 6.2.3. A normal inner product p-adic Banach space $(X, \|\cdot\|, \langle\cdot,\cdot\rangle)$ — with $\dim(X) \geq 2$, and even assuming that $\langle\cdot,\cdot\rangle$ is non-degenerate — may contain *isotropic vectors*, i.e., non-null vectors x such that $\langle x,x\rangle = 0$. Let us construct an example of such a vector. Suppose that $\langle\cdot,\cdot\rangle$ is the canonical inner product associated with a normal basis $\{e_i\}_{i\in I}$, of Example 6.2.1. For $p \equiv 1 \pmod 4$ and for μ a non-quadratic element of $\mathbb{Q}_p$, taking into account the fact that -1 is a square in $\mathbb{Q}_p$, let $x \in X$ be given by $x = (1-\sqrt{\mu})e_1 + (\sqrt{-1} + \sqrt{-1}\sqrt{\mu})e_2$; i.e., the components $x_i \in \mathbb{Q}_{p,\mu}$ of the vector x — w.r.t. the fixed normal basis $\{e_i\}_{i\in I}$ of X — are given by $x_1 = 1-\sqrt{\mu}$, $x_2 = \sqrt{-1}+\sqrt{-1}\sqrt{\mu}$, and $x_i = 0$, for $i \geq 3$. Then, we have that $\langle x,x\rangle = \overline{x_1}\,x_1 + \overline{x_2}\,x_2 = (1-\mu^2) + (-1+\mu^2) = 0$, namely, x is a (non-null) isotropic vector.

In an inner product p-adic Banach space $(X, \|\cdot\|, \langle\cdot,\cdot\rangle)$, we have two (distinct) natural notions of orthogonality: the — previously introduced — norm-orthogonality and the *inner-product-orthogonality* (IP-orthogonality). Clearly, we say that two vectors $x, y \in X$ are IP-orthogonal — and we write $x \perp y$ — if $\langle x,y\rangle = 0$.

Definition 6.2.2. Let $(X, \|\cdot\|, \langle\cdot,\cdot\rangle)$ be a normal inner product p-adic Banach space. A (finite or denumerable) sequence of vectors $\Phi \equiv \{\phi_i\}_{i\in I}$ is said to be an *orthonormal basis* in X, if the following conditions hold:

(O1) Φ is a normal basis in $(X, \|\cdot\|)$;

(O2) $\langle\phi_i,\phi_j\rangle = \delta_{ij}$, for all $i,j \in I$.

Let $(X, \|\cdot\|, \langle\cdot\,,\cdot\rangle)$ be a normal inner product p-adic Banach space, and suppose that X admits an orthonormal basis $\Phi \equiv \{\phi_i\}_{i\in I}$. By providing explicit examples, we now show that the construction of a *new* orthonormal basis $\Psi \equiv \{\psi_i\}_{i\in I}$ in X, starting from the given orthonormal basis $\{\phi_i\}_{i\in I}$, is not a trivial task as it would be, say, in an ordinary separable complex Hilbert space.

Assume, at first, that $\dim(X) = 2$. In order to construct a new orthonormal basis in X, suppose that $z \in \mathbb{Q}_{p,\mu}$ is such that $z\overline{z} = 2$ and $|z| = \sqrt{|z\overline{z}|_p} = 1$ ($\iff |2|_p = 1 \iff p \neq 2$).

Example 6.2.2. Let us give a few explicit examples where, for $p \neq 2$, the condition $z\overline{z} = 2$ is realized.

1. Let us take $p = 3$ and $\mu = 5$ (note that $5 = 2 + 1 \cdot 3$ is a quadratic non-residue mod 3). Then, $z = \sqrt{7}^3 - \sqrt{5}$ — where $\sqrt{7}^3$ is one of the 3-adic square roots of $7 = 1 + 2\cdot 3 \in (\mathbb{Q}_3^*)^2$, i.e.,
$$\sqrt{7}^3 = \begin{cases} 1 + 1\cdot 3 + 1 \cdot 3^2 + 0 \cdot 3^3 + 2 \cdot 3^4 + \cdots \\ 2 + 1\cdot 3 + 1 \cdot 3^2 + 2 \cdot 3^3 + 0 \cdot 3^4 + \cdots \end{cases} \tag{6.12}$$
— verifies the condition $z\overline{z} = 7 - 5 = 2$, and $|z| = \sqrt{|2|_3} = 1$.

2. Let $p = 3$ and $\mu = 2 \notin (\mathbb{Q}_3^*)^2$. Then, $z = 2 + \sqrt{2}$ is such that $z\overline{z} = 4 - 2 = 2$, where $|2|_3 = 1$.

3. Let $p = 5$ and $\mu = 3$ (3 is a quadratic non-residue mod 5). We set $z = \sqrt{29}^5 + 3\sqrt{3}$, where $\sqrt{29}^5$ is one of the 5-adic square roots of $29 = 4 + 0 \cdot 5 + 1 \cdot 5^2 \in (\mathbb{Q}_5^*)^2$, i.e.,
$$\sqrt{29}^5 = \begin{cases} 2 + 0\cdot 5 + 4 \cdot 5^2 + 3 \cdot 5^3 + 4 \cdot 5^4 + \cdots \\ 3 + 4\cdot 5 + 0 \cdot 5^2 + 1 \cdot 5^3 + 0 \cdot 5^4 + \cdots \end{cases} . \tag{6.13}$$
Then, we have that $z\overline{z} = 29 - 27 = 2$, where $|2|_5 = 1$.

4. Let us take $p = 7$ (and, say, $\mu = 7$). We set $z = \sqrt{2}^7$ $(2 \equiv 3^2 \pmod 7)$, with
$$\sqrt{2}^7 = \begin{cases} 3 + 1\cdot 7 + 2 \cdot 7^2 + 6 \cdot 7^3 + 1 \cdot 7^4 + \cdots \\ 4 + 5\cdot 7 + 4 \cdot 7^2 + 0 \cdot 7^3 + 5 \cdot 7^4 + \cdots \ , \end{cases} \tag{6.14}$$
so that $z\overline{z} = 2$, where $|2|_7 = 1$.

Now, given an orthonormal basis $\{\phi_1, \phi_2\}$ in X ($\dim(X) = 2$, $p \neq 2$), the set $\{\psi_1, \psi_2\}$, where
$$\psi_1 = \frac{1}{z}(\phi_1 + \phi_2), \quad \psi_2 = \frac{1}{z}(\phi_1 - \phi_2), \tag{6.15}$$
with $z \in \mathbb{Q}_{p,\mu}$ being chosen as above, is an orthonormal basis for X. Indeed, we have that $\langle\psi_1, \psi_2\rangle = 0$ and $\langle\psi_1, \psi_1\rangle = 2/z\overline{z} = 1 = \langle\psi_2, \psi_2\rangle$. Moreover: $\|\psi_1\| = \|\psi_2\| = 1/|z| = 1$. *But*, it remains to show that $\{\psi_1, \psi_2\}$ is a *norm-orthogonal* set, as well. To clarify this point, let us first prove the following fact:

Lemma 6.2.1. *If $x_1, x_2 \in \mathbb{Q}_{p,\mu}$ and $p \neq 2$ — equivalently, $|2|_p = 1$ — then*
$$\max\{|x_1|, |x_2|\} = \max\{|x_1 + x_2|, |x_1 - x_2|\}. \tag{6.16}$$

Proof. We can suppose, without loss of generality, that $|x_1| \geq |x_2|$. Then, we have:
$$|x_1| = |2x_1| = |(x_1 + x_2) + (x_1 - x_2)| \leq \max\{|x_1 + x_2|, |x_1 - x_2|\} \leq \max\{|x_1|, |x_2|\} = |x_1|, \tag{6.17}$$
so that (6.16) holds true. □

We can now conclude that the vectors in (6.15) form an orthonormal set. Indeed, let $x = x_1\psi_1 + x_2\psi_2$ be any vector in $X = \operatorname{span}\{\psi_1, \psi_2\} = \operatorname{span}\{\phi_1, \phi_2\}$. We have:

$$x = \frac{1}{z}(x_1 + x_2)\phi_1 + \frac{1}{z}(x_1 - x_2)\phi_2. \tag{6.18}$$

Since $\{\phi_1, \phi_2\}$ is a norm-orthogonal set and $|z| = 1$, we have that

$$\|x\| = \max\left\{\frac{1}{|z|}|x_1 + x_2|, \frac{1}{|z|}|x_1 - x_2|\right\} = \max\{|x_1 + x_2|, |x_1 - x_2|\} = \max\{|x_1|, |x_2|\}, \tag{6.19}$$

where the last equality holds by Lemma 6.2.1. Therefore, the set

$$\{\psi_1 = z^{-1}(\phi_1 + \phi_2),\ \psi_2 = z^{-1}(\phi_1 - \phi_2)\} \tag{6.20}$$

is norm-orthogonal too. Clearly, if $\dim(X) > 2$, then

$$\psi_1 = \frac{1}{z}(\phi_1 + \phi_2), \quad \psi_2 = \frac{1}{z}(\phi_1 - \phi_2), \quad \psi_3 = \phi_3, \quad \dots \tag{6.21}$$

is again an orthonormal basis in X.

Let us now consider the case where $\dim(X) = \infty$ and $p \neq 2$. If $z \in \mathbb{Q}_{p,\mu}$ is such that $z\overline{z} = 2$, then $\Psi \equiv \{\psi_1, \psi_2, \psi_3, \dots\}$ — with

$$\psi_1 = \frac{1}{z}(\phi_1 + \phi_2), \quad \psi_2 = \frac{1}{z}(\phi_1 - \phi_2), \quad \psi_3 = \frac{1}{z}(\phi_3 + \phi_4), \quad \psi_4 = \frac{1}{z}(\phi_3 - \phi_4), \quad \dots \tag{6.22}$$

— is an orthonormal basis. Indeed, if

$$x = \sum_{j\in\mathbb{N}} x_j\psi_j = \frac{1}{z}\sum_{j \text{ odd}} ((x_j + x_{j+1})\phi_j + (x_j - x_{j+1})\phi_{j+1}), \tag{6.23}$$

then, since $|z| = 1$,

$$\|x\| = \max_{j \text{ odd}}\{|x_j + x_{j+1}|,\, |x_j - x_{j+1}|\}. \tag{6.24}$$

Hence, by Lemma 6.2.1,

$$\|x\| = \max_{j \text{ odd}}\{|x_j|,\, |x_{j+1}|\} = \max_{j\in\mathbb{N}}\{|x_j|\}, \tag{6.25}$$

so that Ψ is a norm-orthogonal (and IP-orthogonal) set, and an orthonormal basis in X, because $\operatorname{span}\{\psi_j\}_{j\in\mathbb{N}} = \operatorname{span}\{\phi_j\}_{j\in\mathbb{N}}$ so that $\overline{\operatorname{span}\{\psi_j\}_{j\in\mathbb{N}}}^{\|\cdot\|} = \overline{\operatorname{span}\{\phi_j\}_{j\in\mathbb{N}}}^{\|\cdot\|} = X$.

6.3 p-adic Hilbert spaces

To the best of our knowledge, the existence of an orthonormal basis in a generic inner product p-adic Banach space is not guaranteed (even assuming that the inner product is non-degenerate). Therefore, it is natural to set the following:

Definition 6.3.1. Let $(X, \|\cdot\|, \langle\cdot,\cdot\rangle)$ be an inner product p-adic Banach space (over $\mathbb{Q}_{p,\mu}$). We say that X is a *p-adic Hilbert space* if it admits an orthonormal basis $\{\phi_i\}_{i\in I}$ (in the sense of Definition 6.2.2). We will typically denote (the carrier space of) a p-adic Hilbert space by $\mathcal{H}$.

Let $x \in \mathcal{H}$, and let $\Phi \equiv \{\phi_i\}_{i\in I}$ be an orthonormal basis in $\mathcal{H}$. By the first condition in Definition 6.2.2, we can express x — in a unique way — as $x = \sum_{i\in I} x_i\phi_i$, for some set of coefficients $\{x_i\}_{i\in I}$ in $\mathbb{Q}_{p,\mu}$. Moreover, by the second condition in the same definition, and taking into account the continuity, w.r.t. each of its arguments, of the non-Archimedean inner product (see Remark 6.2.2), we have that

$$\langle\phi_j, x\rangle = \langle\phi_j, \textstyle\sum_{i\in I} x_i\phi_i\rangle = \displaystyle\sum_{i\in I} x_i\,\langle\phi_j,\phi_i\rangle = x_j, \quad \forall j \in I. \tag{6.26}$$

Thus, we see that any $x \in \mathcal{H}$ is expressed — w.r.t. the fixed orthonormal basis Φ in $\mathcal{H}$ — as

$$x = \sum_{i\in I}\langle\phi_i, x\rangle\phi_i, \tag{6.27}$$

from which we deduce the *non-Archimedean Parseval* identity

$$\|x\| = \max_{i\in I} |\langle\phi_i, x\rangle|. \tag{6.28}$$

Proposition 6.3.1. *Let $\mathcal{H}$ be a p-adic Hilbert space over $\mathbb{Q}_{p,\mu}$. Then, $\mathcal{H}$ is normal — i.e., $\|\mathcal{H}\| = |\mathbb{Q}_{p,\mu}|$ — and the non-Archimedean inner product $\langle\cdot,\cdot\rangle$ defined on it is non-degenerate, i.e.,*

$$\langle x, y\rangle = 0,\ \ \forall y \in \mathcal{H} \quad\Longrightarrow\quad x = 0. \tag{6.29}$$

Proof. Since $\mathcal{H}$ admits a (ortho-)normal basis, then, by the second assertion of Theorem 6.1.1, $\|\mathcal{H}\| = |\mathbb{Q}_{p,\mu}|$. Let $\Phi \equiv \{\phi_i\}_{i\in I}$ be an orthonormal basis of $\mathcal{H}$. If $\langle x, y\rangle = 0$, $\forall y \in \mathcal{H}$, it must be true that $\langle x, \phi_i\rangle = 0$, $\forall i \in I$. But then, by (6.27) and by the uniqueness of the decomposition of a vector w.r.t. a (ortho-)normal basis in $\mathcal{H}$, it follows that $x = 0$. □

Example 6.3.1. Let us consider the normal p-adic Banach space $(c_0(I, \mathbb{Q}_{p,\mu}), \|\cdot\|_\infty)$ (see Section 6.1), where $\|x\|_\infty = \max_{i\in I}|x_i|$ ($x = \{x_i\}_{i\in I}$). Let $\boldsymbol{e} \equiv \{e_i\}_{i\in I}$ be the set of all sequences in $c_0(I, \mathbb{Q}_{p,\mu})$ whose elements are of the form

$$e_1 = (1, 0, 0, \cdots), \quad e_2 = (0, 1, 0, \cdots), \quad e_3 = (0, 0, 1, \cdots), \quad \ldots \quad . \tag{6.30}$$

Clearly, the set $\{e_i\}_{i\in I}$ is a normal basis — the *standard basis* — and $\dim(c_0(I, \mathbb{Q}_{p,\mu})) = \mathrm{card}(I)$. Let us endow $(c_0(I, \mathbb{Q}_{p,\mu}), \|\cdot\|_\infty)$ with the *canonical inner product*, associated with $\{e_i\}_{i\in I}$, introduced in Example 6.2.1:

$$c_0(I, \mathbb{Q}_{p,\mu}) \times c_0(I, \mathbb{Q}_{p,\mu}) \ni (x, y) \mapsto \langle x, y\rangle := \sum_{i\in I} \overline{x_i}\, y_i \in \mathbb{Q}_{p,\mu}. \tag{6.31}$$

By construction, we have that

$$\langle e_i, e_j\rangle = \delta_{ij}, \quad \forall i, j \in I, \tag{6.32}$$

i.e., $\{e_i\}_{i\in I}$ is an orthonormal basis for $c_0(I, \mathbb{Q}_{p,\mu})$. Therefore, the p-adic Banach space $(c_0(I, \mathbb{Q}_{p,\mu}), \|\cdot\|_\infty)$, endowed with the inner product in (6.31), and admitting the orthonormal basis $\{e_i\}_{i\in I}$, is a p-adic Hilbert space. In the literature [9, 101], this p-adic Hilbert space is sometimes called *coordinate p-adic Hilbert space*, and denoted by $\mathbb{H}(I)$. More generally, given a normal p-adic Banach space X and a normal basis $\{e_i\}_{i\in I}$ in X, we can endow this space with the sesquilinear form defined by (6.10), so that $\{e_i\}_{i\in I}$ becomes an orthonormal basis and X a p-adic Hilbert space.

In the light of Remark 6.2.3 about the existence of isotropic vectors in an inner product p-adic Banach space, we set the following:

Definition 6.3.2. For every quadratic extension $\mathbb{Q}_{p,\mu}$ of $\mathbb{Q}_p$, we define the *isotropy index* $\nu_{p,\mu} \in \mathbb{N}$ as

$$\nu_{p,\mu} := \min\{\mathrm{card}(\mathrm{supp}(x)) \mid x \in c_0(\mathbb{N}, \mathbb{Q}_{p,\mu}),\ \langle x, x\rangle = 0\}, \tag{6.33}$$

where $\mathrm{supp}(x) \subset \mathbb{N}$ denotes the *support* of the sequence $x = \{x_i\}_{i\in\mathbb{N}} \in c_0(\mathbb{N}, \mathbb{Q}_{p,\mu})$; namely,

$$\mathrm{supp}(x) := \{i \in \mathbb{N} \mid x_i \neq 0\}. \tag{6.34}$$

Proposition 6.3.2. *Given a quadratic extension $\mathbb{Q}_{p,\mu}$ of $\mathbb{Q}_p$, $\nu_{p,\mu} \in \{2, 3\}$.*

Proof. It is clear that $\nu_{p,\mu} > 1$. Let us show that, in particular, either $\nu_{p,\mu} = 2$ or $\nu_{p,\mu} = 3$. Assume, at first, that $p \neq 2$. By Lemma 54.6 in [75], there exist numbers $\alpha, \beta, \gamma \in \mathbb{Q}_p$ such that $\alpha \neq 0 \neq \beta$ and $\alpha^2 + \beta^2 + \gamma^2 = 0$. Therefore, putting $x = \alpha e_1 + \beta e_2 + \gamma e_3$, where $\{e_i\}_{i\in\mathbb{N}}$ is the standard basis in $c_0(\mathbb{N}, \mathbb{Q}_{p,\mu})$, we have that $\langle x, x\rangle = \alpha^2 + \beta^2 + \gamma^2 = 0$; i.e., $\nu_{p,\mu} \in \{2, 3\}$. For $p = 2$, the same result can be achieved by means of a direct calculation. E.g., for $\mu = 2$, we can put $x = (1 + \sqrt{2})e_1 + e_2$ (so that $\langle x, x\rangle = (1 - 2) + 1 = 0$). For $\mu = 3$, we can take $x = (1 + \sqrt{3})e_1 + e_2 + e_3$. For $\mu = 5$, we take $x = (1 + \sqrt{5})e_1 + 2e_2$. The remaining cases ($p = 2$ and $\mu = 6, 7, 10, 14$) are similar and, once again, it turns out that $\nu_{p,\mu} \in \{2, 3\}$. □

It is worth observing that the coordinate p-adic Hilbert space $\mathbb{H}(I)$ (Example 6.3.1) plays a role analogous to the role played by $\ell^2(I)$ for the (separable) complex Hilbert spaces: There exists an isomorphism of p-adic Hilbert spaces between $\mathcal{H}$ and $\mathbb{H}(I)$, where $\dim(\mathcal{H}) = \mathrm{card}(I)$. Here, we are assuming the following:

Definition 6.3.3. Given p-adic Hilbert spaces $(\mathcal{H}, \|\cdot\|, \langle\cdot,\cdot\rangle)$ and $(\mathcal{K}, \|\cdot\|, \langle\cdot,\cdot\rangle)$ over the same quadratic extension $\mathbb{Q}_{p,\mu}$ of $\mathbb{Q}_p$, a linear map $W\colon \mathcal{H} \to \mathcal{K}$ is called an *isomorphism of p-adic Hilbert spaces* (an *automorphism*, in the case where $\mathcal{H} = \mathcal{K}$) if

(I1) W is an isometry: $\|Wx\| = \|x\|$, $\forall x \in \mathcal{H}$;

(I2) $W\mathcal{H} = \mathcal{K}$;

(I3) $\langle Wx, Wy\rangle = \langle x, y\rangle$, $\forall x, y \in \mathcal{H}$.

Otherwise stated, W is an isomorphism of p-adic Banach spaces — a surjective isometry — preserving the inner product.

Then, let $\mathcal{H}$ be a p-adic Hilbert space, and let $\Phi \equiv \{\phi_i\}_{i\in I}$ be an orthonormal basis in $\mathcal{H}$. By Theorem 6.1.1, the map

$$W_\Phi : \mathcal{H} \ni x = \sum_{i\in I} \langle \phi_i, x\rangle\, \phi_i \mapsto \breve{x} \equiv \{\langle \phi_i, x\rangle\}_{i\in I} \in \mathbb{H}(I) \tag{6.35}$$

is an isomorphism of $\mathcal{H}$ onto $\mathbb{H}(I)$, since it is a surjective isometry and, by the continuity of the inner product (see Remark 6.2.2),

$$\begin{aligned}\langle x, y\rangle &= \langle \textstyle\sum_{i\in I} \langle \phi_i, x\rangle\, \phi_i, \sum_{j\in I} \langle \phi_j, y\rangle\, \phi_j\rangle \\ &= \sum_{i\in I}\sum_{j\in I} \overline{\langle \phi_i, x\rangle}\, \langle \phi_j, y\rangle\, \langle \phi_i, \phi_j\rangle \\ &= \sum_{i\in I} \overline{\langle \phi_i, x\rangle}\, \langle \phi_i, y\rangle = \langle \breve{x}, \breve{y}\rangle,\end{aligned} \tag{6.36}$$

i.e., the inner product is preserved. Note that, $W_\Phi \phi_i = e_i$, $\forall i \in I$, where $\{e_i\}_{i\in I}$ is the standard basis in $\mathbb{H}(I)$, and $\dim(\mathcal{H}) = \mathrm{card}(I) = \dim(\mathbb{H}(I))$. Therefore, two p-adic Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, over the same quadratic extension of $\mathbb{Q}_p$, are isomorphic if and only if $\dim(\mathcal{H}) = \dim(\mathcal{K})$, in complete analogy w.r.t. separable complex Hilbert spaces.

However, the analogies between the complex and the p-adic Hilbert spaces cannot be pursued too far. E.g., in a complex Hilbert space the norm stems directly from the scalar product, and the closed subspaces — endowed with the subset inclusion and with the orthogonal complementation — form an (orthomodular) orthocomplemented lattice [132]; in particular, a relation of the form (6.38) below holds true, whereas, for a p-adic Hilbert space, we have the following:

Proposition 6.3.3. *Let $\mathcal{H}$ be a p-adic Hilbert space over $\mathbb{Q}_{p,\mu}$, with $\dim(\mathcal{H}) \geq \nu_{p,\mu} \in \{2,3\}$. Then, the mapping*

$$\mathcal{H} \ni x \mapsto \sqrt{|\langle x, x\rangle|} \in \mathbb{R}^+ \tag{6.37}$$

is not *a norm. Moreover, in general, it is* not *true that*

$$\emptyset \neq \mathcal{V} \subset \mathcal{H} \quad \text{and} \quad \mathcal{V} = \mathcal{V}^{\perp\perp} \quad \Longrightarrow \quad \mathcal{V} + \mathcal{V}^{\perp} = \mathcal{H}, \tag{6.38}$$

where $\mathcal{V}^{\perp} := \{x \in \mathcal{H} \mid \langle x, y\rangle = 0, \forall y \in \mathcal{V}\}$; i.e., there exists some non-empty subset $\mathcal{V}$ of $\mathcal{H}$ such that $\mathcal{V} = \mathcal{V}^{\perp\perp}$ and violating the relation $\mathcal{V} + \mathcal{V}^{\perp} = \mathcal{H}$.

(Note: For every non-empty subset $\mathcal{V}$ of $\mathcal{H}$, $\mathcal{V}^{\perp}$ is a norm-closed linear subspace of $\mathcal{H}$.)

Proof. For every orthonormal basis $\Phi = \{\phi_1, \phi_2, \ldots\}$ in $\mathcal{H}$, the mapping

$$\mathcal{H} \ni x = \sum_k x_k \phi_k \mapsto \breve{x} = \{\breve{x}_1, \breve{x}_2, \ldots\} \in c_0(\mathbb{N}, \mathbb{Q}_{p,\mu}), \tag{6.39}$$

— where $\breve{x}_k = x_k$, for $k \leq \dim(\mathcal{H})$, and $\breve{x}_k = 0$, otherwise — is an isometry preserving the inner product ($c_0(\mathbb{N}, \mathbb{Q}_{p,\mu})$ being endowed with the canonical inner product associated with its standard basis). Therefore, by Proposition 6.3.2, if $\dim(\mathcal{H}) \geq \nu_{p,\mu}$, then $\mathcal{H}$ admits (nonzero) isotropic vectors: $\exists x \in \mathcal{H}$, $x \neq 0$, such that $\sqrt{|\langle x, x\rangle|} = 0$. This observation proves the first assertion. To prove the second one, we now provide a counterexample to implication (6.38). Let us first show that, for every $0 \neq x \in \mathcal{H}$, $\mathbb{Q}_{p,\mu}\, x = (\mathbb{Q}_{p,\mu}\, x)^{\perp\perp}$. In fact, since $\langle \cdot, \cdot \rangle$ is non-degenerate, there is some $y \in \mathcal{H}$ such that $\langle x, y\rangle \neq 0$. Then, for every $z \in \mathcal{H}$, the vector $\tilde{z} = z - \langle x, z\rangle \langle x, y\rangle^{-1} y$ is IP-orthogonal to x:

$$\langle x, \tilde{z}\rangle = \langle x, z\rangle - \langle x, z\rangle \langle x, y\rangle^{-1} \langle x, y\rangle = 0. \tag{6.40}$$

Therefore, for every $z \in \mathcal{H}$, $\tilde{z} \in (\mathbb{Q}_{p,\mu}\, x)^{\perp}$. Thus, given any $w \in (\mathbb{Q}_{p,\mu}\, x)^{\perp\perp}$, we have that $\langle w, \tilde{z}\rangle = 0$, $\forall z \in \mathcal{H}$; i.e.,

$$\langle w, z - \alpha y\rangle = 0, \quad \forall z \in \mathcal{H}, \tag{6.41}$$

where $\alpha = \langle x, z\rangle \langle x, y\rangle^{-1}$. This condition is equivalent to

$$\langle w - \overline{\beta} x, z\rangle = 0, \quad \forall z \in \mathcal{H}, \tag{6.42}$$

where $\beta = \langle x, y\rangle^{-1} \langle w, y\rangle$ ($\alpha \langle w, y\rangle = \beta \langle x, z\rangle$). Hence, as the inner product is non-degenerate, $w - \overline{\beta} x = 0$; i.e., $w = \overline{\beta} x \in \mathbb{Q}_{p,\mu}\, x$, so that $(\mathbb{Q}_{p,\mu}\, x)^{\perp\perp} \subset \mathbb{Q}_{p,\mu}\, x$. But, for any $\emptyset \neq \mathcal{V} \subset \mathcal{H}$, it is always true that $\mathcal{V} \subset \mathcal{V}^{\perp\perp}$; therefore, actually, $\mathbb{Q}_{p,\mu}\, x = (\mathbb{Q}_{p,\mu}\, x)^{\perp\perp}$. At this point, observe that, given $0 \neq x \in \mathcal{H}$, we have:

$$\begin{aligned} \langle x, x\rangle = 0 \ &\Longrightarrow \ \mathbb{Q}_{p,\mu}\, x \subset (\mathbb{Q}_{p,\mu}\, x)^{\perp} \\ &\Longrightarrow \ \mathbb{Q}_{p,\mu}\, x + (\mathbb{Q}_{p,\mu}\, x)^{\perp} = (\mathbb{Q}_{p,\mu}\, x)^{\perp} \neq \mathcal{H}. \end{aligned} \tag{6.43}$$

Here, the relation $(\mathbb{Q}_{p,\mu}\, x)^{\perp} \neq \mathcal{H}$ must hold, because the assumption that $(\mathbb{Q}_{p,\mu}\, x)^{\perp} = \mathcal{H}$ would imply

$$\mathbb{Q}_{p,\mu}\, x = (\mathbb{Q}_{p,\mu}\, x)^{\perp\perp} = \mathcal{H}^{\perp} = \{0\}; \tag{6.44}$$

i.e., we would have a contradiction. In conclusion, if $\dim(\mathcal{H}) \geq \nu_{p,\mu}$, there exists a (nonzero) isotropic vector $x \in \mathcal{H}$, so that

$$\mathbb{Q}_{p,\mu}\, x = (\mathbb{Q}_{p,\mu}\, x)^{\perp\perp} \quad \text{and} \quad \mathbb{Q}_{p,\mu}\, x + (\mathbb{Q}_{p,\mu}\, x)^{\perp} \neq \mathcal{H}; \tag{6.45}$$

i.e., implication (6.38) is violated. □

Remark 6.3.1. In the case where $\dim(\mathcal{H}) = \infty$, the second assertion of the previous proposition can be regarded as a manifestation of Solèr's celebrated theorem [133], according to which a vector space over a division ring, endowed with a (non-degenerate) Hermitian form satisfying a relation of the form (6.38), and admitting an infinite orthonormal sequence, must be real, complex or quaternionic. Note that Solèr calls an Hermitian space where a relation of the type (6.38) is satisfied an *orthomodular space*. This is due to the fact that the canonical orthocomplemented lattice of (form-closed) subspaces of a orthomodular space turns out to be an orthomodular lattice; see, e.g., Theorem 2.8 in [134].

In the next section, we will also argue that a further distinguishing mark of an infinite-dimensional p-adic Hilbert space $\mathcal{H}$, versus a complex Hilbert space, is that it *cannot* be identified with its (topological) dual $\mathcal{H}'$.

6.3.1 Subspaces of a p-adic Hilbert space

Defining a suitable notion of subspace of a p-adic Hilbert space is not as straightforward as in the complex setting. As usual, we will consider the standard complex case merely as a useful model, and then highlight the (non-trivial) differences when dealing with the p-adic case.

We therefore start by recalling a few basic facts concerning the subspaces of a complex Hilbert space $\mathcal{K}$ [135–138]. We say that $\mathcal{W} \subset \mathcal{K}$ is a *Hilbert subspace* of $\mathcal{K}$, if it is a norm-closed linear subspace (in such a case, $\mathcal{W}$ is in a natural way a Hilbert space itself). If $\mathcal{W}$ is a Hilbert subspace of $\mathcal{K}$, then its *orthogonal complement* $\mathcal{W}^{\perp} := \{x \in \mathcal{K} \mid \langle x, w \rangle = 0, \ \ \forall w \in \mathcal{W}\}$, is a Hilbert subspace too, and $\mathcal{W} \cap \mathcal{W}^{\perp} = \emptyset$. Moreover, every vector $x \in \mathcal{K}$ can be expressed as the sum of two unique mutually orthogonal vectors, one lying in $\mathcal{W}$ (the *projection* of x onto $\mathcal{W}$) and the other lying in $\mathcal{W}^{\perp}$. Otherwise stated, $\mathcal{K}$ is the *orthogonal sum* of $\mathcal{W}$ and $\mathcal{W}^{\perp}$, and we write $\mathcal{K} = \mathcal{W} \oplus \mathcal{W}^{\perp}$.

Now, let $\mathcal{H}$ be a p-adic Hilbert space. We want to provide a suitable definition of a Hilbert subspace $\mathcal{V}$. To this end, let us consider the following points:

1. If $\dim(\mathcal{H}) \geq \nu_{p,\mu} \in \{2, 3\}$ — where $\nu_{p,\mu}$ is the *isotropy index* of $\mathcal{H}$ (see Definition 6.3.2) — then there exist isotropic vectors in $\mathcal{H}$, namely, non-zero vectors x such that $\langle x, x \rangle = 0$. If $x \neq 0$ is such a vector and we consider the one dimensional linear subspace $\mathcal{V} := \mathbb{Q}_{p,\mu} x$, then it is clear that, putting

$$\mathcal{V}^{\perp} := \{x \in \mathcal{H} \mid \langle x, v \rangle = 0, \ \ \forall v \in \mathcal{V}\}, \tag{6.46}$$

   we have that $\mathcal{V} \cap \mathcal{V}^{\perp} \neq \emptyset$.

2. If now $\mathcal{V}$ is any norm-closed linear subspace of $\mathcal{H}$, then, in general, it will not admit an orthonormal basis. Otherwise stated, $\mathcal{V}$ is not, in general, a p-adic Hilbert space itself.

By the previous observations, we are naturally led to the following:

Definition 6.3.4. Let $\mathcal{H}$ be a p-adic Hilbert space, and let $\mathcal{V} \subset \mathcal{H}$ be a norm-closed subspace. We say that $\mathcal{V}$ is a *Hilbert subspace* of $\mathcal{H}$ if (either $\mathcal{V} = \{0\}$ or) $\mathcal{V}$ is a p-adic Hilbert space itself.

Definition 6.3.4 is equivalent to saying that a norm-closed subspace $\mathcal{V}$ of $\mathcal{H}$ is a Hilbert subspace if it admits an orthonormal basis so that $(\mathcal{V}, \|\cdot\|, \langle \cdot , \cdot \rangle)$ is a p-adic Hilbert space.

Remark 6.3.2. From Definition 6.3.4, it follows that if $\mathcal{Z} \subset \mathcal{V} \subset \mathcal{H}$, where $\mathcal{V}$ is a Hilbert subspace of $\mathcal{H}$ and $\mathcal{Z}$ is a Hilbert subspace of $\mathcal{V}$, then $\mathcal{Z}$ is a Hilbert subspace of $\mathcal{H}$; i.e., the relation characterizing a Hilbert subspace is transitive.

One can easily check that

Proposition 6.3.4. *Let $\mathcal{H}$ be a p-adic Hilbert space, and let $\mathcal{V} \subset \mathcal{H}$ be a Hilbert subspace. Then, $\mathcal{V}^\perp$ is a closed subspace of $\mathcal{H}$ admitting the following equivalent definitions:*

(i) $\mathcal{V}^\perp$ is the linear subspace of $\mathcal{H}$ defined by

$$\mathcal{V}^\perp := \{x \in \mathcal{H} \mid \langle x, v\rangle = 0, \ \ \forall v \in \mathcal{V}\}. \tag{6.47}$$

(ii) $\mathcal{V}^\perp$ is the linear subspace of $\mathcal{H}$ defined by

$$\mathcal{V}^\perp := \{x \in \mathcal{H} \mid \langle x, \psi_j\rangle = 0, \ \forall \psi_j \in \Psi_J\}, \tag{6.48}$$

w.r.t. some (hence, any) orthonormal basis $\Psi_J \equiv \{\psi_j\}_{j\in J}$ in $\mathcal{V}$.

Given a Hilbert subspace $\mathcal{V}$ of a Hilbert space $\mathcal{H}$, the closed subspace $\mathcal{V}^\perp$ will *not* be, in general, a Hilbert subspace of $\mathcal{H}$; in fact, the existence of an orthonormal basis in $\mathcal{V}^\perp$ is not guaranteed. This observation motivates us to introduce a further notion of subspace of a p-adic Hilbert space. Specifically, we set the following:

Definition 6.3.5. We say that a Hilbert subspace $\mathcal{V}$ of a p-adic Hilbert space $\mathcal{H}$ is *regular* if some (hence, any) orthonormal basis in $\mathcal{V}$ can be extended to an orthonormal basis of $\mathcal{H}$.

Proposition 6.3.5. *Let $\mathcal{V}$ be a Hilbert subspace of the p-adic Hilbert space $\mathcal{H}$. The following facts hold:*

- *If $\mathcal{V}$ is regular, then $\mathcal{V}^\perp$ is a regular Hilbert subspace too.*
- *If $\mathcal{V}^\perp$ is a Hilbert subspace of $\mathcal{H}$ and, moreover,*

$$\|v + u\| = \max\{\|v\|, \|u\|\}, \quad \forall v \in \mathcal{V}, \forall u \in \mathcal{V}^\perp, \tag{6.49}$$

then $\mathcal{V}$ and $\mathcal{V}^\perp$ are regular Hilbert subspaces.

Proof. Let us prove the first point. If $\mathcal{V}$ is regular, then there exists an orthonormal basis $\Psi_I \equiv \{\psi_i\}_{i\in I}$ of $\mathcal{H}$ such that, for some subset J of I — putting $\Psi_J \equiv \{\psi_j\}_{j\in J}$ — we have that Ψ_J is an orthonormal basis in $\mathcal{V}$ (equivalently, $\overline{\mathrm{span}}(\Psi_J) = \mathcal{V}$). It is immediate to check that, setting $\Psi_{I\setminus J} \equiv \Psi_I \setminus \Psi_J = \{\psi_i\}_{i\in I\setminus J}$, we have:

$$\mathcal{V}^\perp = \overline{\mathrm{span}}(\Psi_{I\setminus J}). \tag{6.50}$$

By relation (6.50), $\Psi_{I\setminus J}$ is an orthonormal basis in $\mathcal{V}^\perp$, which is then a regular Hilbert subspace.

Let us prove the second point. If $\mathcal{V}^\perp$ is a Hilbert subspace too, then there are orthonormal bases $\Phi_I = \{\phi_i\}_{i\in I}$ and $\Psi_J \equiv \{\psi_j\}_{j\in J}$ of $\mathcal{V}$ and $\mathcal{V}^\perp$, respectively. Given any $\eta \in \mathcal{H}$, we put

$$\eta_\Phi := \sum_i \langle \phi_i, \eta\rangle\, \phi_i \text{ and } \eta_\Psi := \sum_j \langle \psi_j, \eta\rangle\, \psi_j. \tag{6.51}$$

We have that

$$\langle \phi_i, \eta_\Phi\rangle = \langle \phi_i, \eta\rangle \implies \langle \phi_i, \eta - \eta_\Phi\rangle = 0, \tag{6.52}$$

i.e., $\eta - \eta_\Phi \in \mathcal{V}^\perp$, and, since $\langle \phi_i, \psi_j \rangle = 0$,

$$\langle \psi_j, \eta_\Phi \rangle = 0. \tag{6.53}$$

Hence,

$$\eta = \eta_\Phi + (\eta - \eta_\Phi), \text{ where } \eta - \eta_\Phi \in \mathcal{V}^\perp, \tag{6.54}$$

so that

$$\begin{aligned} \eta - \eta_\Phi = \sum_j \langle \psi_j, \eta - \eta_\Phi \rangle \, \psi_j &\overset{(6.53)}{=} \sum_j \langle \psi_j, \eta \rangle \, \psi_j \\ &= \eta_\Psi. \end{aligned} \tag{6.55}$$

In conclusion, $\eta = \eta_\Phi + \eta_\Psi$ and, by condition (6.49),

$$\|\eta\| = \max \big\{ \|\eta_\Phi\| , \|\eta_\Psi\| \big\} = \max \big\{ |\langle \phi_i, \eta \rangle| , |\langle \psi_j, \eta \rangle| \mid i \in I, \, j \in J \big\}. \tag{6.56}$$

Therefore, $\Phi_I \cup \Psi_J$ is an orthonormal basis in $\mathcal{H}$, and $\mathcal{V}, \mathcal{V}^\perp$ are regular Hilbert subspaces of $\mathcal{H}$. □

Remark 6.3.3. Note that condition (6.49) is equivalent to the fact that $\mathcal{V}$ and $\mathcal{V}^\perp$ are norm-orthogonal subspaces of $\mathcal{H}$. In fact, we have:

$$\begin{aligned} \mathcal{V} \overset{\|\cdot\|}{\perp} \mathcal{V}^\perp \overset{\text{def}}{\iff} & \; v \overset{\|\cdot\|}{\perp} u, \quad \forall v \in \mathcal{V}, \quad \forall u \in \mathcal{V}^\perp \\ \iff & \; \|\alpha v + \beta u\| = \max\{\|\alpha v\| , \|\beta u\|\}, \quad \forall \alpha, \beta \in \mathbb{Q}_{p,\mu}, \forall v \in \mathcal{V}, \forall u \in \mathcal{V}^\perp \\ \iff & \; \|v + u\| = \max\{\|v\| , \|u\|\}, \quad \forall v \in \mathcal{V}, \forall u \in \mathcal{V}^\perp. \end{aligned} \tag{6.57}$$

6.4 Linear operators between p-adic normed spaces

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be two p-adic normed spaces, and let $L : X \to Y$ be a linear operator from X to Y. In the p-adic setting, as in the complex case, linear operators are *continuous* precisely when they are *bounded* [70, 72, 73, 76, 101]; specifically, L is bounded if

$$\|L\| := \sup_{x \neq 0} \frac{\|Lx\|_Y}{\|x\|_X} < \infty. \tag{6.58}$$

(A bounded conjugate-linear operator and its norm are defined analogously). We denote the space of bounded — equivalently, continuous — linear operators $L : X \to Y$, by $\mathcal{B}(X, Y)$, and we refer to the norm in (6.58) as the *operator norm*.

Theorem 6.4.1. *Let X, Y be p-adic normed spaces over $\mathbb{Q}_{p,\mu}$ — with Y complete and normal, i.e., $\|Y\|_Y = |\mathbb{Q}_{p,\mu}|$ — and let X_0 be a linear subspace of X. Then, every linear operator $L_0 \in \mathcal{B}(X_0, Y)$ admits an extension $L \in \mathcal{B}(X, Y)$ — a so-called* Hahn-Banach extension *of L_0 — such that $\|L\| = \|L_0\|$. If X_0 is dense in X, the bounded extension L of L_0 is unique.*

Proof. By Lemma 2.4 in [73] (or by Proposition 20.2 in [69]) the normed space Y, being complete and normal (and the valuation group $|\mathbb{Q}^*_{p,\mu}|$ discrete), is spherically complete. Then, the first assertion of the theorem follows from the non-Archimedean Hahn-Banach theorem (i.e., Ingleton's theorem; see Theorem 4.8 in [73]). The second assertion is clear. □

From now on, we will remove subscripts X and Y from the associated norms, since it will be clear from the context to which space they refer.

Proposition 6.4.1 ([70]). *The space $\mathcal{B}(X, Y)$, endowed with the operator norm, is an ultrametric normed space over $\mathbb{Q}_{p,\mu}$, which is complete (hence, an ultrametric Banach space) whenever Y is.*

As usual, in the case where $X = Y$, we simply write $\mathcal{B}(X)$ rather than $\mathcal{B}(X, X)$. Moreover, since $\|ST\| \leq \|S\| \, \|T\|$, for all $S, T \in \mathcal{B}(X)$, if X is a p-adic Banach space, then $\mathcal{B}(X)$ is a *unital ultrametric Banach algebra* [72].

As in the standard complex case, we define the (topological) *dual* of a p-adic Banach space X as the ultrametric Banach space $X' := \mathcal{B}(X, \mathbb{Q}_{p,\mu})$. Denoting by X'' the *bidual* of X, the linear map $\mathcal{I}_X : X \to X''$ defined by $\big(\mathcal{I}_X(x)\big)(\xi) := \xi(x)$, for all $x \in X$ and $\xi \in X'$, is continuous (see Chapter 3 of [73]). If $\dim(X) < \infty$, then X is *reflexive* (i.e., $\mathcal{I}_X$ is a surjective isometry); otherwise, since $\mathbb{Q}_{p,\mu}$ is spherically complete, by a classical result of Fleischer — see Theorem 4.16 in [73] — X is *not* reflexive. Nevertheless, in the case where X is infinite-dimensional, it may be 'pseudoreflexive':

Definition 6.4.1. A p-adic Banach space X is said to be *pseudoreflexive* if the linear map $\mathcal{I}_X : X \to X''$ is an isometry.

Remark 6.4.1. Let $\mathcal{H}$ be a p-adic Hilbert space, with $\dim(\mathcal{H}) = \infty$. Then, its dual $\mathcal{H}'$ is *not* a p-adic Hilbert space isomorphic to $\mathcal{H}$ (like in the complex case), but a p-adic Banach space isomorphic to $\ell^\infty(\mathbb{N}, \mathbb{Q}_{p,\mu})$; see the forthcoming definition (6.59) and Proposition 6.4.3 below.

In order to describe the dual of the p-adic Banach space $c_0(I, X)$, where $I = \{1, 2, \ldots\}$ is a countable index set, we recall that, for a p-adic Banach space X, the space $\ell^\infty(I, X)$ is defined as follows:

$$\ell^\infty(I, X) := \{\xi = \{\xi_i\}_{i \in I} \mid \xi_i \in X,\ \xi \text{ bounded sequence in } X\}. \tag{6.59}$$

This space, equipped with the norm

$$\|\xi\|_\infty := \sup_{i \in I} \|\xi_i\|, \tag{6.60}$$

is a p-adic Banach space [70, 73]. In particular, for $X = \mathbb{Q}_{p,\mu}$, we obtain the p-adic Banach space

$$\ell^\infty(I, \mathbb{Q}_{p,\mu}) := \{\xi = \{\xi_i\}_{i \in I} \mid \xi_i \in \mathbb{Q}_{p,\mu},\ \xi \text{ bounded sequence in } \mathbb{Q}_{p,\mu}\}, \tag{6.61}$$

endowed with the norm $\|\xi\|_\infty := \sup_{i \in I} |\xi_i|$.

Remark 6.4.2. The p-adic Banach space $\ell^\infty(\mathbb{N}, \mathbb{Q}_{p,\mu})$ — differently from $c_0(\mathbb{N}, \mathbb{Q}_{p,\mu})$ — is *not* separable; equivalently — recall Remark 6.1.2 — it is not of countable type. In fact, for every $J \subset \mathbb{N}$, let $1_J \in \ell^\infty(\mathbb{N}, \mathbb{Q}_{p,\mu})$ be defined by

$$(1_J)_i = \begin{cases} 1 & \text{if } i \in J \\ 0 & \text{if } i \notin J \end{cases}. \tag{6.62}$$

Clearly, $\|1_J - 1_K\|_\infty = 1$, whenever $J \neq K \subset \mathbb{N}$. Let us put

$$\mathscr{D}_J := \big\{\xi \in \ell^\infty(\mathbb{N}, \mathbb{Q}_{p,\mu}) \mid \|\xi - 1_J\|_\infty \leq p^{-1}\big\}. \tag{6.63}$$

Thus, $\{\mathscr{D}_J\}_{J \subset \mathbb{N}}$ is a countably infinite set of balls in $\ell^\infty(\mathbb{N}, \mathbb{Q}_{p,\mu})$. Note that these balls are mutually disjoint because, if $\xi \in 1_J$ and $J \neq K \subset \mathbb{N}$, then

$$1 = \|1_J - 1_K\|_\infty \leq \max\{\|1_J - \xi\|_\infty, \|\xi - 1_K\|_\infty\} = \max\big\{p^{-1}, \|\xi - 1_K\|_\infty\big\}, \tag{6.64}$$

so that $\|\xi - 1_K\|_\infty \geq 1$ and $\xi \notin \mathscr{D}_K$. Now, let $\mathscr{E}$ be any dense subset of $\ell^\infty(\mathbb{N}, \mathbb{Q}_{p,\mu})$. Each ball in $\{\mathscr{D}_J\}_{J\subset\mathbb{N}}$ must contain at least one element of $\mathscr{E}$, and such an element is not contained in any other ball in $\{\mathscr{D}_J\}_{J\subset\mathbb{N}}$. It follows that there is an uncountable subset of $\mathscr{E}$, so that $\mathscr{E}$ itself is uncountable and, hence, $\ell^\infty(\mathbb{N}, \mathbb{Q}_{p,\mu})$ is not separable. It is worth observing that, more generally, $\ell^\infty(\mathbb{N}, \mathbb{K})$ — where $\mathbb{K}$ is any complete ultrametric field, with a non-trivial valuation — is not of countable type; see Theorem 2.5.15 in [76].

Proposition 6.4.2 ([70, 76, 129]). *Let X be a p-adic Banach space over $\mathbb{Q}_{p,\mu}$. The topological dual of the space $c_0(I, X)$ is isomorphic, as a p-adic Banach space, to $\ell^\infty(I, X')$. The identification of $c_0(I, X)'$ with $\ell^\infty(I, X')$ is given via the bilinear pairing*

$$\ell^\infty(I, X') \times c_0(I, X) \ni (\xi, y) \mapsto \sum_{i\in I} \xi_i(y_i) =: \xi(y) \in \mathbb{Q}_{p,\mu}. \tag{6.65}$$

Remark 6.4.3. Note that the norm of any $x \in c_0(I, \mathbb{Q}_{p,\mu}) \subset \ell^\infty(I, \mathbb{Q}_{p,\mu})$ coincides with the norm of x regarded as an element of $\ell^\infty(I, \mathbb{Q}_{p,\mu})$ (equivalently, of $c_0(I, \mathbb{Q}_{p,\mu})'$). Moreover, by suitably composing the pairing (6.65) with the conjugate-linear isometry $\{\xi_i\}_{i\in I} \mapsto \{\overline{\xi_i}\}_{i\in I}$ of $\ell^\infty(I, \mathbb{Q}_{p,\mu})$ onto itself, we obtain the *sesquilinear pairing*

$$\ell^\infty(I, \mathbb{Q}_{p,\mu}) \times c_0(I, \mathbb{Q}_{p,\mu}) \ni (\xi, y) \mapsto \sum_{i\in I} \overline{\xi_i}\, y_i = \overline{\xi}(y) \equiv \langle \xi, y\rangle \in \mathbb{Q}_{p,\mu}. \tag{6.66}$$

This pairing determines a conjugate-linear isometry

$$c_0(I, \mathbb{Q}_{p,\mu}) \ni x \mapsto \langle x, \cdot\rangle \in c_0(I, \mathbb{Q}_{p,\mu})'. \tag{6.67}$$

Also note that $c_0(I, \mathbb{Q}_{p,\mu})$ is pseudoreflexive, because the mapping

$$c_0(I, \mathbb{Q}_{p,\mu}) \ni x \mapsto \big\langle \overline{(\cdot)}, x\big\rangle \in c_0(I, \mathbb{Q}_{p,\mu})'' \tag{6.68}$$

— where $\big\langle \overline{(\cdot)}, x\big\rangle \colon \ell^\infty(I, \mathbb{Q}_{p,\mu}) \ni \xi \mapsto \langle \overline{\xi}, x\rangle = \xi(x) = \sum_{i\in I} \xi_i x_i$ — is a linear isometry ($\|\langle x, \cdot\rangle\| = \|x\|_\infty$). Clearly, we have: $\langle \xi, x\rangle = \big(\mathcal{I}_X(x)\big)(\overline{\xi})$, with $X = c_0(I, \mathbb{Q}_{p,\mu})$. Observe that, with a slight abuse, we are using the same symbol $\langle \cdot, \cdot\rangle$ for the inner product (6.31) and for the sesquilinear pairing (6.66).

Proposition 6.4.3. *Let $\mathcal{H}$ be a p-adic Hilbert space over $\mathbb{Q}_{p,\mu}$, and let $\Phi \equiv \{\phi_i\}_{i\in I}$ be an orthonormal basis in $\mathcal{H}$. The mapping*

$$\mathcal{J}_{\mathcal{H}} : \mathcal{H} \ni \psi \mapsto \langle \psi, \cdot\rangle \in \mathcal{H}' \tag{6.69}$$

is a conjugate-linear isometry of $\mathcal{H}$ into its dual $\mathcal{H}'$, that is surjective if and only if $\dim(\mathcal{H}) < \infty$. The p-adic Banach space $\mathcal{H}'$ is isomorphic to $\ell^\infty(I, \mathbb{Q}_{p,\mu})$ — with $\mathrm{card}(I) = \dim(\mathcal{H}')$ — and this isomorphism is implemented by the surjective isometry

$$\mathcal{L}_\Phi \colon \ell^\infty(I, \mathbb{Q}_{p,\mu}) \ni \xi = \{\xi_i\}_{i\in I} \mapsto \sum_{i\in I} \xi_i \langle \phi_i, \cdot\rangle \in \mathcal{H}', \tag{6.70}$$

where, if $I = \mathbb{N}$, the series converges w.r.t. the weak-topology.*

Finally, $\mathcal{H}$ is reflexive if and only if $\dim(\mathcal{H}) < \infty$; in the case where $\dim(\mathcal{H}) = \infty$, $\mathcal{H}$ is pseudoreflexive, because the mapping

$$\mathcal{I}_{\mathcal{H}} : \mathcal{H} \ni \psi \mapsto \Big(\mathcal{H}' \ni \phi' \mapsto \phi'(\psi) \in \mathbb{Q}_{p,\mu}\Big) \in \mathcal{H}'' \tag{6.71}$$

is an isometry of $\mathcal{H}$ into its bidual $\mathcal{H}''$.

Proof. The mapping (6.69) is a (conjugate-linear) isometry, because

$$\begin{aligned}\|\mathcal{J}_{\mathcal{H}}\psi\| = \sup_{\eta\neq 0}\frac{|\langle\psi,\eta\rangle|}{\|\eta\|} &= \sup_{\eta\neq 0}\frac{|\sum_{i\in I}\langle\psi,\phi_i\rangle\langle\phi_i,\eta\rangle|}{\|\eta\|}\\ &= \sup_{\eta\neq 0}\frac{\max_{i\in I}|\langle\psi,\phi_i\rangle|\,|\langle\phi_i,\eta\rangle|}{\max_{i\in I}|\langle\phi_i,\eta\rangle|}\\ &= \max_{i\in I}|\langle\phi_i,\psi\rangle| = \|\psi\|. \end{aligned} \tag{6.72}$$

Since $\mathcal{H}$, as a p-adic Banach space, is isomorphic to $c_0(I,\mathbb{Q}_{p,\mu})$ ($\mathrm{card}(I)=\dim(\mathcal{H})$) via the mapping $\mathcal{H}\ni x=\sum_{i\in I}\langle\phi_i,x\rangle\phi_i\mapsto\{\langle\phi_i,x\rangle\}_{i\in I}\in c_0(I,\mathbb{Q}_{p,\mu})$, then, by suitably composing this linear isometry with the bilinear pairing (6.65), we see that the map $\mathcal{L}_\Phi\colon \ell^\infty(I,\mathbb{Q}_{p,\mu})\to\mathcal{H}'$ — $\big(\mathcal{L}_\Phi\big)(\psi):=\sum_{i\in I}\xi_i\langle\phi_i,\psi\rangle$ — is an isomorphism of p-adic Banach spaces. Therefore, we can write $\mathcal{L}_\Phi=\sum_{i\in I}\xi_i\langle\phi_i,\cdot\rangle$, where, for $I=\mathbb{N}$, the series converges pointwise, namely, w.r.t. the weak*-topology (see, e.g., Section 7.3 of [76]). Also note that $\mathcal{J}_{\mathcal{H}}(\mathcal{H})=\mathcal{L}_\Phi(c_0(I,\mathbb{Q}_{p,\mu}))$; hence, $\mathcal{J}_{\mathcal{H}}$ is surjective if and only if $\dim(\mathcal{H})<\infty$. Finally, we have already observed that, if $\dim(\mathcal{H})=\infty$, then $\mathcal{H}$ is not reflexive. Nevertheless, $\mathcal{H}$ is pseudoreflexive, because the mapping

$$\mathcal{I}_{\mathcal{H}}\colon \mathcal{H}\ni\psi=\sum_{i\in I}x_i\phi_i\mapsto\Big(\mathcal{H}'\ni\phi'=\sum_{j\in I}\xi_j\langle\phi_j,\cdot\rangle\mapsto\phi'(\psi)=\sum_{i\in I}\xi_i x_i\in\mathbb{Q}_{p,\mu}\Big)\in\mathcal{H}'' \tag{6.73}$$

is a linear isometry. □

Remark 6.4.4. With regard to the isometry (6.70), note that — since, for every $\xi\equiv\{\xi_i\}_{i\in I}\in\ell^\infty(I,\mathbb{Q}_{p,\mu})$ and every finite subset I_0 of I,

$$\big\|\textstyle\sum_{i\in I_0}\xi_i\langle\phi_i,\cdot\rangle\big\| = \max_{i\in I_0}|\xi_i|\,\|\mathcal{J}_{\mathcal{H}}\phi_i\| = \max_{i\in I_0}|\xi_i| \tag{6.74}$$

— in the case where $I=\mathbb{N}$, the (pointwise converging) series $\sum_{i\in I}\xi_i\langle\phi_i,\cdot\rangle$ converges w.r.t. the norm topology if and only if $\xi\in c_0(I,X)$.

As a consequence of the first assertion of Proposition 6.4.3, we have the following:

Corollary 6.4.1. *If $A\in\mathcal{B}(\mathcal{H})$ and $B\in\mathcal{B}(\mathcal{H}')$ satisfy the intertwining relation*

$$\mathcal{J}_{\mathcal{H}}\circ A = B\circ\mathcal{J}_{\mathcal{H}}, \tag{6.75}$$

then $B\,\mathcal{J}_{\mathcal{H}}(\mathcal{H})\subset\mathcal{J}_{\mathcal{H}}(\mathrm{ran}(A))\subset\mathcal{J}_{\mathcal{H}}(\mathcal{H})$ and $\|A\|=\|B_0\|\le\|B\|$, where B_0 is the restriction of B to the closed subspace $\mathcal{J}_{\mathcal{H}}(\mathcal{H})$ of $\mathcal{H}'$.

Proof. If relation (6.75) holds, then, since $\mathcal{J}_{\mathcal{H}}\colon\mathcal{H}\to\mathcal{H}'$ is an isometry, we have

$$\begin{aligned}\|A\| = \sup_{\psi\neq 0}\frac{\|A\psi\|}{\|\psi\|} &= \sup_{\psi\neq 0}\frac{\|\mathcal{J}_{\mathcal{H}}(A\psi)\|}{\|\mathcal{J}_{\mathcal{H}}(\psi)\|}\\ &= \sup_{\psi\neq 0}\frac{\big\|B\big(\mathcal{J}_{\mathcal{H}}(\psi)\big)\big\|}{\|\mathcal{J}_{\mathcal{H}}(\psi)\|}\\ &= \sup_{0\neq\phi'\in\mathcal{J}_{\mathcal{H}}(\mathcal{H})}\frac{\|B\phi'\|}{\|\phi'\|} = \|B_0\|\le\|B\|, \end{aligned} \tag{6.76}$$

where B_0 is the restriction of B to $\mathcal{J}_{\mathcal{H}}(\mathcal{H})$, which, by (6.75), is a (closed) subspace of $\mathcal{H}'$, stable under the action of B. □

Definition 6.4.2. If $A\in\mathcal{B}(\mathcal{H})$ and $B\in\mathcal{B}(\mathcal{H}')$ satisfy the intertwining relation (6.75) and, moreover, $\|A\|=\|B\|$, we say that B is a *dual Hahn-Banach extension* of A.

For every bounded linear operator $A \in \mathcal{B}(\mathcal{H})$, one can define its *Banach space adjoint* — or *generalized adjoint* — $A' : \mathcal{H}' \to \mathcal{H}'$ by setting $A'(\phi') := \phi' \circ A$, namely,

$$\big(A'\phi'\big)(\psi) := \phi'(A\psi), \quad \forall \psi \in \mathcal{H}, \forall \phi' \in \mathcal{H}'. \tag{6.77}$$

In particular, $A'(\langle \phi, \cdot \rangle) = \langle \phi, \cdot \rangle \circ A = \langle \phi, A(\cdot) \rangle$, for all $\phi \in \mathcal{H}$; namely,

$$\big(A' \circ \mathcal{J}_{\mathcal{H}}\big)(\phi) = \mathcal{J}_{\mathcal{H}}(\phi) \circ A, \quad \forall \phi \in \mathcal{H}. \tag{6.78}$$

Proposition 6.4.4. *If $A \in \mathcal{B}(\mathcal{H})$, then $A' \in \mathcal{B}(\mathcal{H}')$ and $\|A'\| = \|A\|$.*

Proof. See Section 4F of [73]. □

7

Linear operators in a p-adic Hilbert space

In this chapter, we derive some useful characterizations of operators acting in p-adic Hilbert spaces. Specifically, we will be concerned with the classes of bounded, adjointable, trace class and Hilbert-Schmidt operators. We will focus on the case where $\dim(\mathcal{H}) = \infty$, because in the finite-dimensional setting most of the subsequent results become trivial. Accordingly, we will identify the index set I of the previous chapter with $\mathbb{N}$. For the sake of conciseness, we put $c_0 \equiv c_0(\mathbb{N}, \mathbb{Q}_{p,\mu})$ and $\ell^\infty \equiv \ell^\infty(\mathbb{N}, \mathbb{Q}_{p,\mu})$.

7.1 Bounded and adjointable operators in a p-adic Hilbert space

Let (A_{mn}), $m, n \in \mathbb{N}$, be an infinite matrix with entries in $\mathbb{Q}_{p,\mu}$. We denote the set of all such matrices by $\mathsf{M}_\infty(\mathbb{Q}_{p,\mu})$. The matrix (A_{mn}) determines a linear operator $\operatorname{op}(A_{mn})$ in c_0 by putting

$$\operatorname{dom}(\operatorname{op}(A_{mn})) := \Big\{x = \{x_n\}_{n\in\mathbb{N}} \in c_0 \mid \text{the series } \textstyle\sum_n A_{mn}x_n \text{ converges, } \forall m \in \mathbb{N}, \text{ and } \textstyle\sum_m \sum_n A_{mn}x_n \text{ converges too, i.e., } \Big\{ \textstyle\sum_n A_{mn}x_n \Big\}_{m\in\mathbb{N}} \in c_0 \Big\}, \tag{7.1}$$

$$\operatorname{op}(A_{mn})\, x := \Big\{ \textstyle\sum_n A_{mn}x_n \Big\}_{m\in\mathbb{N}}, \quad x = \{x_n\}_{n\in\mathbb{N}} \in \operatorname{dom}(\operatorname{op}(A_{mn})). \tag{7.2}$$

Clearly, the *matrix operator* $\operatorname{op}(A_{mn})$ will be — in general — unbounded, and here we are assuming that it is defined on its *maximal domain* $\operatorname{dom}(\operatorname{op}(A_{mn}))$.

We further introduce the following set of linear operators in c_0:

$$(c_0, c_0) := \{\operatorname{op}(A_{mn}) \mid (A_{mn}) \in \mathsf{M}_\infty(\mathbb{Q}_{p,\mu}),\ \operatorname{dom}(\operatorname{op}(A_{mn})) = c_0\}. \tag{7.3}$$

By Theorem 65 in [129], $\operatorname{op}(A_{mn}) \in (c_0, c_0)$ if and only if

(i) $\lim_m A_{mn} = 0$, $\forall n \in \mathbb{N}$;

(ii) $\sup_m(\sup_n |A_{mn}|) < \infty$.

Remark 7.1.1. It is worth stressing the following points:

- Since ($\overline{\lim} \equiv \limsup$)

$$\overline{\lim_m}\Big(\sup_n |A_{mn}|\Big) \le \sup_m \Big(\sup_n |A_{mn}|\Big) < \infty, \tag{7.4}$$

  then the further condition that — see Theorem 65 of [129] —

$$\lim_l \frac{1}{l}\, \overline{\lim_m}\Big(\sup_n |A_{mn}|\Big) = 0 \tag{7.5}$$

  is redundant in the case we are considering.

- By the Principle of the Iterated Suprema, if for $r_{mn} \geq 0$, $m, n \in \mathbb{N}$, either $\sup_{m,n} r_{mn} < \infty$, or $\sup_m \sup_n r_{mn} < \infty$, or $\sup_n \sup_m r_{mn} < \infty$, then

$$\sup_{m,n} r_{mn} = \sup_m \sup_n r_{mn} = \sup_n \sup_m r_{mn} < \infty. \tag{7.6}$$

Thus, condition (ii) above can be replaced with

(ii)′ $\sup_{m,n} |A_{mn}| < \infty$.

We next switch from c_0 to an infinite-dimensional p-adic Hilbert space $\mathcal{H}$ over $\mathbb{Q}_{p,\mu}$. We say that a linear operator A in $\mathcal{H}$ is *all-over* if

$$\mathrm{dom}(A) = \mathcal{H}. \tag{7.7}$$

Since, by Theorem 6.4.1, a densely defined *bounded* linear operator in a p-adic Hilbert space admits a *unique* bounded linear extension to the whole space, with the same norm — precisely as it happens in the standard complex setting; see, e.g., Theorem 4.5 of [136] — we will tacitly assume bounded operators to be all-over, i.e., to be defined on the whole $\mathcal{H}$, unless otherwise specified. As in Section 6.4, we denote the space of all such operators by $\mathcal{B}(\mathcal{H})$.

Keeping in mind the isomorphism of p-adic Hilbert spaces $W_\Phi \colon \mathcal{H} \to \mathbb{H}$ — see (6.35) — where $\mathbb{H} \equiv \mathbb{H}(\mathbb{N})$ is the Banach space c_0, endowed with its canonical inner product (6.31), we can now consider *matrix operators* in $\mathcal{H}$. An infinite matrix $(A_{mn}) \in \mathsf{M}_\infty(\mathbb{Q}_{p,\mu})$ — together with an orthonormal basis $\Phi \equiv \{\phi_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ — determines a linear operator $\mathrm{op}_\Phi(A_{mn})$ in $\mathcal{H}$ as follows:

$$\begin{aligned}\mathrm{dom}(\mathrm{op}_\Phi(A_{mn})) := \big\{\psi = \textstyle\sum_n x_n\phi_n = \sum_n \langle\phi_n,\psi\rangle\phi_n \mid \sum_n A_{mn}x_n \text{ converges, } \forall m \in \mathbb{N}, \text{ and}\\ \textstyle\sum_m(\sum_n A_{mn}x_n)\phi_m \text{ converges too, i.e., } \big\{\sum_n A_{mn}x_n\big\}_{m\in\mathbb{N}} \in c_0\big\},\end{aligned} \tag{7.8}$$

$$\mathrm{op}_\Phi(A_{mn})\,\psi := \textstyle\sum_m \big(\sum_n A_{mn}\langle\phi_n,\psi\rangle\big)\phi_m, \quad \psi \in \mathrm{dom}(\mathrm{op}_\Phi(A_{mn})). \tag{7.9}$$

Taking into account definitions (7.1), (7.2), (7.8) and (7.9), we see that

$$\mathrm{dom}(\mathrm{op}(A_{mn})) = W_\Phi\ \mathrm{dom}(\mathrm{op}_\Phi(A_{mn})) \quad \text{and} \quad \mathrm{op}(A_{mn}) \circ W_\Phi = W_\Phi \circ \mathrm{op}_\Phi(A_{mn}). \tag{7.10}$$

For every orthonormal basis $\Phi \equiv \{\phi_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$, we put

$$(\mathcal{H},\mathcal{H})_\Phi := \{\mathrm{op}_\Phi(A_{mn}) \mid (A_{mn}) \in \mathsf{M}_\infty(\mathbb{Q}_{p,\mu}),\ \mathrm{dom}(\mathrm{op}_\Phi(A_{mn})) = \mathcal{H}\}; \tag{7.11}$$

i.e., $(\mathcal{H},\mathcal{H})_\Phi$ is the set of the all-over matrix operators in $\mathcal{H}$ associated with Φ. It is clear that $(\mathcal{H},\mathcal{H})_\Phi$ is, in a natural way, a linear space over $\mathbb{Q}_{p,\mu}$. Actually, we will show that its definition does not depend on the choice of Φ.

To this end, first note that every bounded operator $A \in \mathcal{B}(\mathcal{H})$ belongs to $(\mathcal{H},\mathcal{H})_\Phi$ (whatever the orthonormal basis $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}$ is) because, putting

$$A_{mn} \equiv \langle\phi_m, A\phi_n\rangle, \qquad (\lim_m A_{mn} = 0,\ \forall n \in \mathbb{N},\ \sup_{m,n}|A_{mn}| \leq \|A\|) \tag{7.12}$$

we have that

$$A = \mathrm{op}_\Phi(A_{mn}) = \sum_m\sum_n A_{mn}\langle\phi_n,\cdot\rangle\phi_m = \sum_n\sum_m A_{mn}\langle\phi_n,\cdot\rangle\phi_m, \tag{7.13}$$

where $(\langle\phi_n,\cdot\rangle\phi_m)\,\psi := \langle\phi_n,\psi\rangle\phi_m$ (i.e., $\langle\phi_n,\cdot\rangle\phi_m \equiv |\phi_m\rangle\langle\phi_n|$, in Dirac's notation) and both the iterated series converge — as can be easily checked — w.r.t. the strong operator topology in $\mathcal{B}(\mathcal{H})$ (i.e., the initial topology induced by the family of maps $\{\mathscr{E}_\psi \colon \mathcal{B}(\mathcal{H}) \to \mathcal{H}\}_{\psi\in\mathcal{H}}$, where $\mathscr{E}_\psi(A) := A\psi$). Thus, $\mathcal{B}(\mathcal{H}) \subset (\mathcal{H},\mathcal{H})_\Phi$; precisely:

Theorem 7.1.1. *For every orthonormal basis $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}$ in $\mathcal{H}$, we have that*

$$\mathcal{B}(\mathcal{H}) = (\mathcal{H},\mathcal{H})_\Phi = \{\mathrm{op}_\Phi(A_{mn}) \mid \lim_m A_{mn} = 0,\ \forall n, \text{ and } \sup_{m,n} |A_{mn}| < \infty\}. \tag{7.14}$$

Moreover, for every $A = \mathrm{op}_\Phi(A_{mn}) \in \mathcal{B}(\mathcal{H})$,

$$\|A\| = \sup_{m,n} |A_{mn}| = \sup_n \|A\phi_n\|. \tag{7.15}$$

Proof. We have already observed that $\mathcal{B}(\mathcal{H}) \subset (\mathcal{H},\mathcal{H})_\Phi$. By relations (7.10), we have:

$$(\mathcal{H},\mathcal{H})_\Phi = \{\mathrm{op}_\Phi(A_{mn}) \mid \mathrm{op}\,(A_{mn}) \in (c_0, c_0)\}, \tag{7.16}$$

where the set (c_0, c_0) is completely characterized by conditions (i) and (ii). Hence, the second equality in (7.14) holds true.

It remains to prove the inclusion $(\mathcal{H},\mathcal{H})_\Phi \subset \mathcal{B}(\mathcal{H})$ — so that, actually, $(\mathcal{H},\mathcal{H})_\Phi = \mathcal{B}(\mathcal{H})$ — and, moreover, to show that

$$\|A\| = \sup_{m,n} |A_{mn}| = \sup_n \|A\phi_n\|, \quad \forall A = \mathrm{op}_\Phi(A_{mn}) \in \mathcal{B}(\mathcal{H}). \tag{7.17}$$

Let $A \in (\mathcal{H},\mathcal{H})_\Phi$, with $A = \mathrm{op}_\Phi(A_{mn})$. For every $\psi \in \mathcal{H}$, we have that

$$A\psi = \sum_m \left(\sum_n A_{mn}\langle\phi_n,\psi\rangle\right)\phi_m; \tag{7.18}$$

hence:

$$\|A\psi\| = \sup_m \left|\sum_n A_{mn}\langle\phi_n,\psi\rangle\right| \le \sup_m \sup_n |A_{mn}|\,|\langle\phi_n,\psi\rangle| \ \le \|\psi\| \sup_m \sup_n |A_{mn}|. \tag{7.19}$$

Therefore, A is bounded and

$$\|A\| \le \sup_m \sup_n |A_{mn}| = \sup_{m,n} |A_{mn}|. \tag{7.20}$$

It is then shown that $(\mathcal{H},\mathcal{H})_\Phi \subset \mathcal{B}(\mathcal{H})$ too, and therefore, actually, the two sets coincide.

Now, let $A = \mathrm{op}_\Phi(A_{mn}) \in (\mathcal{H},\mathcal{H})_\Phi = \mathcal{B}(\mathcal{H})$. Note that

$$\|A\phi_n\| = \left\|\sum_m \langle\phi_m, A\phi_n\rangle\phi_m\right\| = \sup_m |\langle\phi_m, A\phi_n\rangle| = \sup_m |A_{mn}|, \quad \forall n \in \mathbb{N}. \tag{7.21}$$

From (7.20) and (7.21) it follows that

$$\sup_{m,n} |A_{mn}| = \sup_n \sup_m |A_{mn}| = \sup_n \|A\phi_n\| \le \|A\| \le \sup_{m,n} |A_{mn}|. \tag{7.22}$$

Hence, actually, $\|A\| = \sup_{m,n} |A_{mn}| = \sup_n \|A\phi_n\|$, for any $A = \mathrm{op}_\Phi(A_{mn}) \in \mathcal{B}(\mathcal{H})$. □

Summarizing, it is proven that $\mathcal{B}(\mathcal{H})$ coincides with the linear space $(\mathcal{H},\mathcal{H})_\Phi$ of the *all-over matrix operators* in $\mathcal{H}$ w.r.t. an orthonormal basis $\Phi \equiv \{\phi_n\}_{n\in\mathbb{N}}$ (independently of the choice of Φ); moreover, one can give a complete characterization of the infinite matrices in $\mathsf{M}_\infty(\mathbb{Q}_{p,\mu})$ that correspond to bounded operators (w.r.t. any orthonormal basis in $\mathcal{H}$). Furthermore, the norm of a bounded operator is given by the supremum, in valuation, of its matrix elements (again, w.r.t. any orthonormal basis).

At this point, we move on to discuss the notion of 'proper adjoint' of a bounded operator in a p-adic Hilbert space $\mathcal{H}$. In fact, as argued in Section 6.4, since, in the infinite-dimensional setting, $\mathcal{H}$ cannot be identified with its dual $\mathcal{H}'$, with every bounded operator $A \in \mathcal{B}(\mathcal{H})$

is associated a generalized adjoint $A' \in \mathcal{B}(\mathcal{H}')$. Nevertheless, one can single out a suitable class of bounded operators admitting a genuine 'Hilbert space adjoint' (the proper adjoint).

Given any $A \in \mathcal{B}(\mathcal{H})$, we first associate with A a linear operator $A^\dagger$ in $\mathcal{H}$ — the so called *pseudo-adjoint* of A — as follows. We start with defining its domain as

$$\mathrm{dom}(A^\dagger) := \{\phi \in \mathcal{H} \mid \langle \phi, A\psi\rangle = \langle \eta, \psi\rangle, \text{ for some } \eta \in \mathcal{H}, \text{ and for all } \psi \in \mathcal{H}\}. \tag{7.23}$$

It is clear that $\mathrm{dom}(A^\dagger)$ is a linear subspace of $\mathcal{H}$.

Remark 7.1.2. The linear subspace $\mathrm{dom}(A^\dagger)$ can be regarded as the set of all vectors $\phi \in \mathcal{H}$ such that the bounded functional $\langle \phi, A(\cdot)\rangle \in \mathcal{H}'$ can be identified, via the conjugate-linear isometry $\mathcal{J}_{\mathcal{H}}$, with an element η of $\mathcal{H}$; i.e.,

$$\phi \in \mathrm{dom}(A^\dagger) \iff \langle \phi, A(\cdot)\rangle = \langle \eta, \cdot\rangle = \mathcal{J}_{\mathcal{H}}\, \eta, \tag{7.24}$$

for some $\eta \in \mathcal{H}$.

Proposition 7.1.1. *The condition*

$$\big(\mathcal{J}_{\mathcal{H}}\, \phi\big)(A\psi) = \langle \phi, A\psi\rangle = \langle A^\dagger\phi, \psi\rangle = \big(\mathcal{J}_{\mathcal{H}}(A^\dagger\phi)\big)(\psi), \ \ \forall \phi \in \mathrm{dom}(A^\dagger), \ \forall \psi \in \mathcal{H}, \tag{7.25}$$

uniquely determines a bounded linear operator $A^\dagger \colon \mathrm{dom}(A^\dagger) \to \mathcal{H}$*, whose domain* $\mathrm{dom}(A^\dagger)$ *is given by* (7.23)*, and such that* $\|A^\dagger\| \le \|A\|$*. Moreover, we have the following dichotomy: either* $\mathrm{dom}(A^\dagger) = \mathcal{H}$*, or* $\mathrm{dom}(A^\dagger)$ *is a closed subspace of* $\mathcal{H}$*, with* $\mathrm{dom}(A^\dagger) \subsetneq \mathcal{H}$*.*

Proof. For every $\phi \in \mathrm{dom}(A^\dagger)$, the relation $(\mathcal{J}_{\mathcal{H}}\, \phi) \circ A = \mathcal{J}_{\mathcal{H}}\, \eta = \mathcal{J}_{\mathcal{H}}(A^\dagger\phi)$, determines a linear operator $A^\dagger \colon \mathrm{dom}(A^\dagger) \to \mathcal{H}$, where the vector $\eta =: A^\dagger\phi \in \mathcal{H}$ is unique because $\mathcal{J}_{\mathcal{H}}$ is a (conjugate-linear) isometry. Equivalently, one can use the fact that the sesquilinear form $\langle \cdot, \cdot\rangle$ is non-degenerate; see Proposition 6.3.1. (The linearity of $A^\dagger$ is evident.) Moreover, since

$$\|\mathcal{J}_{\mathcal{H}}(A^\dagger\phi)\| = \|A^\dagger\phi\| = \|(\mathcal{J}_{\mathcal{H}}\, \phi) \circ A\| \le \|\mathcal{J}_{\mathcal{H}}\, \phi\| \|A\| = \|\phi\| \|A\|, \quad \phi \in \mathrm{dom}(A^\dagger), \tag{7.26}$$

$A^\dagger$ is bounded, with $\|A^\dagger\| \le \|A\|$.

Let us now show that $\mathrm{dom}(A^\dagger)$ is a closed subspace of $\mathcal{H}$. In fact, by Theorem 6.4.1, there is a bounded operator B in $\mathcal{H}$ that agrees with $A^\dagger$ on $\mathrm{dom}(A^\dagger)$, and such that $\|B\| = \|A^\dagger\| \le \|A\|$. Therefore, for every sequence $\{\chi_n\}_{n\in\mathbb{N}} \subset \mathrm{dom}(A^\dagger)$, with $\chi_n \to \chi \in \overline{\mathrm{dom}(A^\dagger)}^{\|\cdot\|}$, we have that

$$\langle B\chi, \psi\rangle = \lim_n \langle B\chi_n, \psi\rangle = \lim_n \langle A^\dagger\chi_n, \psi\rangle = \lim_n \langle \chi_n, A\psi\rangle = \langle \chi, A\psi\rangle, \quad \forall \psi \in \mathcal{H}, \tag{7.27}$$

where we have used the continuity of B and of the inner product $\langle \cdot, \cdot\rangle$; hence: $\chi \in \mathrm{dom}(A^\dagger) = \overline{\mathrm{dom}(A^\dagger)}^{\|\cdot\|}$ and $B = A^\dagger$. In conclusion, either $\mathrm{dom}(A^\dagger) = \mathcal{H}$, or $\mathrm{dom}(A^\dagger)$ is a closed subspace of $\mathcal{H}$, strictly contained in $\mathcal{H}$. □

Remark 7.1.3. By the definition of $A^\dagger$ and by the dichotomy in Proposition 7.1.1, it is clear that, if $\mathrm{dom}(A^\dagger) \subsetneq \mathcal{H}$, then, given any bounded extension $B \in \mathcal{B}(\mathcal{H})$ of $A^\dagger$, we have:

$$\begin{cases} \langle B\phi, \psi\rangle = \langle \phi, A\psi\rangle, \ \ \forall \phi \in \mathrm{dom}(A^\dagger) = \overline{\mathrm{dom}(A^\dagger)}^{\|\cdot\|}, \ \forall \psi \in \mathcal{H}, \\ \langle B\phi, \psi\rangle \neq \langle \phi, A\psi\rangle, \ \ \forall \phi \notin \mathrm{dom}(A^\dagger), \ \forall \psi \notin \mathscr{S}_\phi, \end{cases} \tag{7.28}$$

where the set $\mathscr{S}_\phi := \{\psi \in \mathcal{H} \mid \langle B\phi, \psi\rangle = \langle \phi, A\psi\rangle\}$ is a non-dense, closed linear subspace of $\mathcal{H}$ depending on $\phi \notin \mathrm{dom}(A^\dagger)$.

Definition 7.1.1. We say that $A \in \mathcal{B}(\mathcal{H})$ is *adjointable* if the pseudo-adjoint $A^\dagger$ of A is all-over, i.e., if $\mathrm{dom}(A^\dagger) = \mathcal{H}$. In such a case, we put $A^* \equiv A^\dagger$ and we call the all-over linear operator A^* the (proper) *adjoint* of $A \in \mathcal{B}(\mathcal{H})$.

It is clear that the collection of all adjointable operators in $\mathcal{H}$ — denoted hereafter by $\mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ — is, in a natural way, a linear space over $\mathbb{Q}_{p,\mu}$ (a linear subspace of $\mathcal{B}(\mathcal{H})$). We are now going to characterize it.

Theorem 7.1.2. *If $A \in \mathcal{B}(\mathcal{H})$ is adjointable, then its (proper) adjoint A^* is a bounded operator. Given any orthonormal basis $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}$ in $\mathcal{H}$, if $A = \mathrm{op}_\Phi(A_{mn}) \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$, then*

$$A^* = \mathrm{op}_\Phi(A^*_{mn}) \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H}), \tag{7.29}$$

with $A^*_{mn} = \overline{A_{nm}}$. Therefore, if $A = \mathrm{op}_\Phi(A_{mn}) \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$, then

(A1) $\sup_{m,n} |A_{mn}| < \infty$,

(A2) $\lim_m A_{mn} = 0$, $\forall n \in \mathbb{N}$,

(A3) $\lim_n A_{mn} = 0$, $\forall m \in \mathbb{N}$,

and, moreover,

$$\|A^*\| = \sup_{m,n} |A_{mn}| = \|A\|. \tag{7.30}$$

Conversely, if, for some orthonormal basis $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}$, $A = \mathrm{op}_\Phi(A_{mn})$ — i.e., if A is a matrix operator associated with Φ — where the entries of the matrix $(A_{mn}) \in \mathsf{M}_\infty(\mathbb{Q}_{p,\mu})$ are supposed to satisfy conditions (A1)–(A3) *above, then $A \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$.*

Notation 7.1.1. Taking into account the mapping (6.70), for every $\xi \in \ell^\infty$, we write symbolically

$$\xi_\Phi = \langle \textstyle\sum_n \xi_n \phi_n, \cdot \rangle \equiv \mathcal{L}_\Phi(\overline{\xi}) = \displaystyle\sum_n \overline{\xi_n}\, \langle \phi_n, \cdot \rangle \in \mathcal{H}'. \tag{7.31}$$

Here, the (pointwise converging) series $\sum_{i\in I} \overline{\xi_i}\, \langle \phi_i, \cdot \rangle$ converges w.r.t. the norm topology too if and only if $\xi \in c_0$ (Remark 6.4.4). With this notation, if, in particular, $\xi \in c_0 \subset \ell^\infty$, the functional $\langle \sum_n \xi_n \phi_n, \cdot \rangle$ can be directly identified with the element $\sum_n \xi_n \phi_n$ of $\mathcal{H}$ (with norm $\| \sum_n \xi_n \phi_n \| = \|\xi\|_\infty = \sup_n |\xi_n| = \max_n |\xi_n|$):

$$\xi_\Phi(\psi) = \langle \textstyle\sum_n \xi_n \phi_n, \psi \rangle = \displaystyle\sum_n \overline{\xi_n}\, \langle \phi_n, \psi \rangle. \tag{7.32}$$

Proof of Theorem 7.1.2. Given an orthonormal basis $\Phi \equiv \{\phi_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ and a bounded operator $A \in \mathcal{B}(\mathcal{H})$, $A = \mathrm{op}_\Phi(A_{mn})$, by the second series expansion in (7.13) — converging w.r.t. the strong operator topology — we have that

$$\langle \psi, A\chi \rangle = \sum_n \sum_m A_{mn} \langle \psi, \phi_m \rangle \langle \phi_n, \chi \rangle = \sum_n \left(\sum_m \overline{\overline{A_{mn}}\, \langle \phi_m, \psi \rangle} \right) \langle \phi_n, \chi \rangle, \tag{7.33}$$

for all $\psi, \chi \in \mathcal{H}$, where (by the arbitrariness of $\chi \in \mathcal{H}$)

$$\left\{ \xi^\psi_n \equiv \textstyle\sum_m \overline{A_{mn}}\, \langle \phi_m, \psi \rangle \right\}_{n\in\mathbb{N}} \in \ell^\infty. \tag{7.34}$$

Using Notation 7.1.1, we can write

$$\langle \psi, A\chi \rangle = \sum_n \overline{\xi^\psi_n}\, \langle \phi_m, \chi \rangle \equiv \langle \textstyle\sum_n \xi^\psi_n \phi_m, \chi \rangle. \tag{7.35}$$

Here, in general, we have a pairing between an element of $\mathcal{H}'$ (i.e., $\langle\psi, A(\cdot)\rangle = \langle\sum_n \xi_n^\psi \phi_m, \cdot\rangle$) and the vector $\chi \in \mathcal{H}$.

Assume now that $A \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$. It follows that, for some $\eta = A^*\psi \in \mathcal{H}$,

$$\langle\textstyle\sum_n \xi_n^\psi \phi_m, \chi\rangle = \langle\psi, A\chi\rangle = \langle A^*\psi, \chi\rangle, \quad \forall\chi \in \mathcal{H}. \tag{7.36}$$

Hence, by the first assertion of Proposition 6.4.3, we must have that

$$\{\xi_n^\psi\}_{n\in\mathbb{N}} \in c_0 \quad \text{and} \quad \sum_n \xi_n^\psi \phi_n = A^*\psi \in \mathcal{H}. \tag{7.37}$$

We stress that here $\sum_n \xi_n^\psi \phi_n$ should be regarded as a *bona fide* expansion in $\mathcal{H}$. Recalling (7.34), we see that

$$A^* = \mathrm{op}_\Phi(A^*_{mn}), \tag{7.38}$$

where $A^*_{mn} = \overline{A_{nm}}$.

Therefore, the adjoint A^* of A — that, by definition, is defined on the whole Hilbert space $\mathcal{H}$ and, by Proposition 7.1.1, is bounded — is an all-over matrix operator and, by Theorem 7.1.1, we conclude that

(i) $\sup_{m,n} |A^*_{mn}| < \infty$ ($\Longleftrightarrow \sup_{m,n} |A_{mn}| < \infty$),

(ii) $\lim_m A^*_{mn} = 0$, $\forall n \in \mathbb{N}$ ($\Longleftrightarrow \lim_n A_{mn} = 0$, $\forall m \in \mathbb{N}$),

and, moreover, since $A \in \mathcal{B}(\mathcal{H})$, also

(iii) $\lim_n A^*_{mn} = 0$, $\forall m \in \mathbb{N}$ ($\Longleftrightarrow \lim_m A_{mn} = 0$, $\forall n \in \mathbb{N}$).

It is clear that $\|A^*\| = \sup_{m,n} |A^*_{mn}| = \sup_{m,n} |A_{mn}| = \|A\|$.

Now, let $A = \mathrm{op}_\Phi(A_{mn})$ — with $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}$ any orthonormal basis in $\mathcal{H}$ — be a matrix operator. By relation (7.14) in Theorem 7.1.1, if $\sup_{m,n} |A_{mn}| < \infty$ and $\lim_m A_{mn} = 0$, for every $n \in \mathbb{N}$, then A is a bounded operator so that, for all $\psi, \chi \in \mathcal{H}$, relation (7.33) holds true. Next, if, moreover, $\lim_n A_{mn} = 0$, for every $m \in \mathbb{N}$, then — putting $A^*_{mn} \equiv \overline{A_{nm}}$ — we also have that $\mathrm{op}(A^*_{mn}) \in (c_0, c_0)$. Hence,

$$\{\textstyle\sum_n A^*_{mn} x_n\}_{m\in\mathbb{N}} = \{\sum_n \overline{A_{nm}}\, x_n\}_{m\in\mathbb{N}} \in c_0, \quad \forall x = \{x_n\}_{n\in\mathbb{N}} \in c_0, \tag{7.39}$$

and, for every $\psi \in \mathcal{H}$, the series

$$\sum_m \sum_n A^*_{mn} \langle\phi_n, \psi\rangle \phi_m, \tag{7.40}$$

must converge to some vector $\eta = \mathrm{op}_\Phi(A^*_{mn})\,\psi \in \mathcal{H}$, where $\mathrm{op}_\Phi(A^*_{mn}) \in (\mathcal{H}, \mathcal{H})_\Phi = \mathcal{B}(\mathcal{H})$. In conclusion, for every $\psi \in \mathcal{H}$, there is some $\eta = \mathrm{op}_\Phi(A^*_{mn})\,\psi \in \mathcal{H}$ such that, given any $\chi \in \mathcal{H}$, by (7.33) (with the indices m, n merely re-named) we have:

$$\langle\psi, A\chi\rangle = \sum_m \left(\sum_n \overline{\overline{A_{nm}}\,\langle\phi_n, \psi\rangle}\right)\langle\phi_m, \chi\rangle = \sum_m \left(\sum_n \overline{A^*_{mn}\,\langle\phi_n, \psi\rangle}\right)\langle\phi_m, \chi\rangle = \langle\eta, \chi\rangle. \tag{7.41}$$

Otherwise stated, $A = \mathrm{op}_\Phi(A_{mn})$ — with the matrix (A_{mn}) that satisfies conditions `(A1)`–`(A3)` in the statement of the theorem — is adjointable, and $A^* = \mathrm{op}_\Phi(A^*_{mn})$, because the relation $\langle\psi, A\chi\rangle = \langle A^*\psi, \chi\rangle$, for all $\psi, \chi \in \mathcal{H}$ determines A^* uniquely. □

We will now derive some direct consequences of Theorem 7.1.2.

Corollary 7.1.1. *If $A \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$, then $A^* \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ too and*

$$(A^*)^* = A. \tag{7.42}$$

Moreover, for all $A, B \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ and all $\alpha \in \mathbb{Q}_{p,\mu}$, we have that ($\alpha A, A+B \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ and) $AB, \mathrm{Id} \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$, together with

$$(\alpha A)^* = \overline{\alpha} A, \quad (A+B)^* = A^* + B^*, \quad (AB)^* = B^* A^*. \tag{7.43}$$

Proof. If $A = \mathrm{op}_\Phi(A_{mn}) \in \mathcal{B}(\mathcal{H})$ is adjointable, then the matrix elements of (A_{mn}) satisfy conditions (A1)–(A3) in Theorem 7.1.2, and clearly, the matrix elements of A^* w.r.t. the orthonormal basis $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}$ — recall that $A^* = \mathrm{op}_\Phi(A^*_{mn})$, with $A^*_{mn} = \overline{A_{nm}}$ — satisfy these conditions too. Hence, A^* is adjointable too and $(A^*)^* = \mathrm{op}_\Phi(A_{mn}) = A$. We have already observed that $\mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ is a linear subspace of $\mathcal{B}(\mathcal{H})$. The remaining facts are clear from the definition of the adjoint of a bounded operator in a p-adic Hilbert space. □

Corollary 7.1.2. *If $A \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$, then its generalized adjoint $A' \in \mathcal{B}(\mathcal{H}')$ is a dual Hahn-Banach extension of its adjoint $A^* \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$. Moreover, if $A \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ and, for some all-over linear operator B in $\mathcal{H}$, the intertwining relation $A' \circ \mathcal{J}_{\mathcal{H}} = \mathcal{J}_{\mathcal{H}} \circ B$ holds, then $B \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$, A' is a dual Hahn-Banach extension of B and $B = A^*$.*

Proof. We have already observed that the generalized adjoint A' of a bounded operator A satisfies the intertwining relation (6.78); i.e., for every $\phi \in \mathcal{H}$, $\big(A' \circ \mathcal{J}_{\mathcal{H}}\big)(\phi) = \mathcal{J}_{\mathcal{H}}(\phi) \circ A$. If $A \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$, by Proposition 7.1.1, the proper adjoint A^* is (uniquely) determined by the condition that $\mathcal{J}_{\mathcal{H}}(\phi) \circ A = \big(\mathcal{J}_{\mathcal{H}} \circ A^*\big)(\phi)$, for all $\phi \in \mathcal{H}$. Therefore, we have that

$$\big(A' \circ \mathcal{J}_{\mathcal{H}}\big)(\phi) = \mathcal{J}_{\mathcal{H}}(\phi) \circ A = \big(\mathcal{J}_{\mathcal{H}} \circ A^*\big)(\phi), \quad \forall \phi \in \mathcal{H}; \tag{7.44}$$

i.e., the conjugate-linear isometry $\mathcal{J}_{\mathcal{H}}$ intertwines A^* with A'. Thus, A' is a dual Hahn-Banach extension of A^*, because, by relation (7.30) in Theorem 7.1.2 and by Proposition 6.4.4, we also have that $\|A^*\| = \|A\| = \|A'\|$.

Moreover, if $A \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ and, for some all-over operator B in $\mathcal{H}$, $A' \circ \mathcal{J}_{\mathcal{H}} = \mathcal{J}_{\mathcal{H}} \circ B$, then

$$\big(\mathcal{J}_{\mathcal{H}} \circ A^*\big)(\phi) = \mathcal{J}_{\mathcal{H}}(\phi) \circ A = \big(A' \circ \mathcal{J}_{\mathcal{H}}\big)(\phi) = \big(\mathcal{J}_{\mathcal{H}} \circ B\big)(\phi), \quad \forall \phi \in \mathcal{H}, \tag{7.45}$$

so that $B = A^* \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ (Corollary 7.1.1), because the relation $\mathcal{J}_{\mathcal{H}}(\phi) \circ A = \big(\mathcal{J}_{\mathcal{H}} \circ A^*\big)(\phi)$, satisfied for all $\phi \in \mathcal{H}$, uniquely determines the operator A^* (Proposition 7.1.1). □

Notation 7.1.2. Given an infinite matrix $(A_{mn}) \in \mathsf{M}_\infty(\mathbb{Q}_{p,\mu})$, by writing $\lim_{m+n} A_{mn} = \alpha$, for some $\alpha \in \mathbb{Q}_{p,\mu}$, we mean that

$$\forall \epsilon > 0, \ \mathrm{card}(\{(m,n) \in \mathbb{N} \times \mathbb{N} \mid |A_{mn} - \alpha| \geq \epsilon\}) < \infty. \tag{7.46}$$

Equivalently, we mean that

$$\forall \epsilon > 0, \ \exists \mathtt{N} \in \mathbb{N}, \text{ such that, if } \max\{m,n\} > \mathtt{N}, \text{ then } |A_{mn} - \alpha| < \epsilon, \tag{7.47}$$

or, also, that

$$\forall \epsilon > 0, \ \exists \mathtt{N} \in \mathbb{N}, \text{ such that, if } m+n > \mathtt{N}, \text{ then } |A_{mn} - \alpha| < \epsilon. \tag{7.48}$$

Corollary 7.1.3. *Let $\Phi = \{\phi_n\}_{n\in\mathbb{N}}$ be any orthonormal basis in $\mathcal{H}$, and let $A = \mathrm{op}_\Phi(A_{mn})$ be a matrix operator. If the elements of the matrix (A_{mn}) satisfy the condition that*

$$\lim_{m+n} A_{mn} = 0, \tag{7.49}$$

then $A \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$.

Proof. Condition (7.49) implies:

(i) $\sup_{m,n} |A_{mn}| < \infty$, because for every $\epsilon > 0$ we have that the set $\{m, n \in \mathbb{N}\times\mathbb{N} \mid |A_{mn}| \geq \epsilon\}$ is finite.

(ii) $\lim_m A_{mn} = 0$, $\forall n \in \mathbb{N}$, and $\lim_n A_{mn} = 0$, $\forall m \in \mathbb{N}$.

By the final assertion on Theorem 7.1.2, it follows that $A \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$. □

Definition 7.1.2. We say that a bounded operator $A \in \mathcal{B}(\mathcal{H})$ is *self-adjoint* if

$$A \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H}) \quad \text{and} \quad A^* = A. \tag{7.50}$$

From Theorem 7.1.2 we also immediately derive the following:

Corollary 7.1.4. *Let $\Phi = \{\phi_n\}_{n\in\mathbb{N}}$ be any orthonormal basis in $\mathcal{H}$. An adjointable bounded operator $A \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ — $A = \mathrm{op}_\Phi(A_{mn})$ — is self-adjoint if and only if*

$$A_{mn} = \overline{A_{nm}}, \quad \forall m, n \in \mathbb{N}. \tag{7.51}$$

Therefore, a matrix operator $\mathrm{op}_\Phi(A_{mn})$ is self-adjoint if and only if

(S1) $A_{mn} = \overline{A_{nm}}$, $\forall m, n \in \mathbb{N}$,

(S2) $\sup_{m,n} |A_{mn}| < \infty$,

(S3) $\lim_m |A_{mn}| = 0$, $\forall n \in \mathbb{N}$.

It is clear that the set of all self-adjoint bounded operators in $\mathcal{H}$ — denoted hereafter by $\mathcal{B}_{\mathrm{sa}}(\mathcal{H})$ — is a $\mathbb{Q}_p$-linear subspace of $\mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ (by field restriction).

We conclude this section by observing that $\mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ is a Banach $*$-algebra.

Proposition 7.1.2. *The linear space $\mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ is a p-adic Banach space and a (unital) Banach subalgebra of $\mathcal{B}(\mathcal{H})$. Therefore, $\mathcal{B}_{\mathrm{ad}}(\mathcal{H})$, endowed with the adjoining operation $A \mapsto A^*$, is a p-adic Banach $*$-algebra.*

Proof. By Corollary 7.1.1, the linear subspace $\mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ of $\mathcal{B}(\mathcal{H})$ is actually a subalgebra of $\mathcal{B}(\mathcal{H})$, containing the identity Id, and the mapping $\mathcal{B}_{\mathrm{ad}}(\mathcal{H}) \ni A \mapsto A^* \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ is an involution. Therefore, the only thing to be shown is that $\mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ is closed in $\mathcal{B}(\mathcal{H})$. In fact, let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{B}_{\mathrm{ad}}(\mathcal{H})$, converging in $\mathcal{B}(\mathcal{H})$: $\lim_n A_n = A \in \mathcal{B}(\mathcal{H})$ (in the norm topology). Since the adjoining operation is an *isometric* involution, then $\lim_n A_n^* = B$, for some bounded operator $B \in \mathcal{B}(\mathcal{H})$ ($\{A_n^*\}_{n\in\mathbb{N}}$ being a Cauchy sequence in $\mathcal{B}(\mathcal{H})$). It follows that

$$\langle \phi, A\psi\rangle = \lim_n \langle \phi, A_n\psi\rangle = \lim_n \langle A_n^*\phi, \psi\rangle = \langle B\phi, \psi\rangle, \quad \forall \phi, \psi \in \mathcal{H}. \tag{7.52}$$

Thus, A is adjointable (and $B = A^*$); i.e., $\mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ is a p-adic Banach space. □

7.2 Unitary operators in a p-adic Hilbert space

The definition of a unitary operator in a p-adic Hilbert space is not as simple as in the complex case. Clearly, this is due to the fact that the relation between the norm and the inner product is not the 'standard one'. As in the complex setting, one can actually consider various (equivalent) definitions. Since, orthonormal bases and matrix operators turn out to play a central role in the p-adic setting, we will introduce unitary operators as matrix operators relating any pair of orthonormal bases; see Definition 7.2.4 below. Eventually, it will be shown that a unitary operator is nothing but an automorphism of a p-adic Hilbert space (Definition 6.3.3).

In order to prove the main result of this section, we first need to collect a few preliminary facts. We will work in a p-adic Hilbert space $\mathcal{H}$ over $\mathbb{Q}_{p,\mu}$, with $\dim(\mathcal{H}) = \mathsf{N}$, where $\mathsf{N} \in \mathbb{N}$ or $\mathsf{N} = \infty$; accordingly, we will put $\mathbb{N}_{\leq \mathsf{N}} := \{n \in \mathbb{N} \mid n \leq \mathsf{N}\}$ (i.e., $\mathbb{N}_{\leq \mathsf{N}} \equiv \mathbb{N}$, for $\mathsf{N} = \infty$). As in Section 7.1, we will assume bounded operators to be all-over, and as in the previous sections, the term 'isometry' will stand for 'norm-isometry'. However, since the inner product will play a major role here, for the sake of clarity it is worth starting with the following:

Definition 7.2.1. A *linear operator* A in $\mathcal{H}$ is called an *isometry* if

$$\mathrm{dom}(A) = \mathcal{H} \quad \text{and} \quad \|A\phi\| = \|\phi\|, \quad \forall \phi \in \mathcal{H}; \tag{7.53}$$

i.e., if it is all-over and norm-preserving (N-preserving).

Lemma 7.2.1. *Let A be a linear operator in $\mathcal{H}$. The following facts are equivalent:*

(i) *A is a surjective isometry;*

(ii) *A is bounded, admits a bounded inverse A^{-1} (with $\mathrm{dom}(A^{-1}) = \mathcal{H}$) and*

$$\|A\| = 1 = \|A^{-1}\|. \tag{7.54}$$

Proof. Suppose that (i) holds. Then, by (7.53), A is bounded and $\|A\| = 1$. Moreover, A is bijective and the linear operator A^{-1} is an isometry too, because, for every $\psi \in \mathcal{H}$, there is some $\phi \in \mathcal{H}$ such that $\psi = A\phi$ and $\|\psi\| = \|A\phi\| = \|\phi\|$. Therefore, $\|A^{-1}\psi\| = \|A^{-1}A\phi\| = \|\phi\| = \|\psi\|$; whence, A^{-1} is bounded and $\|A^{-1}\| = 1 = \|A\|$.

Conversely, suppose that (ii) holds (in particular, $\ker(A) = \{0\}$). Assume that A is *not* an isometry. Then, there exists some $\phi \in \mathcal{H}$, $\phi \neq 0$, such that

$$0 \neq \|\phi\| \neq \|A\phi\| \neq 0. \tag{7.55}$$

Hence, we have:

$$0 \neq \frac{\|A\phi\|}{\|\phi\|} \neq 1. \tag{7.56}$$

Now, if $\|A\phi\|/\|\phi\| > 1$, then we would have $\|A\| > 1$, which would contradict one of the hypotheses in (ii). Instead, if $0 < \|A\phi\|/\|\phi\| < 1$, then we would have $\|A^{-1}\| > 1$, because, in such a case, we should conclude that

$$1 < \frac{\|\phi\|}{\|A\phi\|} = \frac{\|A^{-1}\psi\|}{\|AA^{-1}\psi\|} = \frac{\|A^{-1}\psi\|}{\|\psi\|}, \quad \text{for some } \psi \in \mathcal{H} \setminus \{0\}. \tag{7.57}$$

This, as well, would contradict one of the hypotheses of (ii). Therefore, the bijection A must be an isometry. □

Remark 7.2.1. By the Bounded Inverse Theorem (see, e.g., Corollary 3.6 in [73], or Subsect. 2.8 of [96]), if A is a *bijective* bounded operator in $\mathcal{H}$ — in particular, $\mathrm{dom}(A) = \mathrm{ran}(A) = \mathcal{H}$ — then A^{-1} is bounded too. Thus, point (ii) in Lemma 7.2.1 can be reformulated as follows:

(ii)′ A is bounded and bijective, and $\|A\| = 1 = \|A^{-1}\|$.

Definition 7.2.2. We say that a linear operator A in $\mathcal{H}$ is *inner-product-preserving* (in short, IP-preserving) if

$$\langle A\phi, A\psi \rangle = \langle \phi, \psi \rangle, \quad \forall\, \phi, \psi \in \mathrm{dom}(A). \tag{7.58}$$

Lemma 7.2.2. *If A is an IP-preserving, bounded operator in $\mathcal{H}$, admitting a bounded inverse A^{-1} (equivalently, an IP-preserving, bijective bounded operator), then A^{-1} is IP-preserving too:*

$$\langle A^{-1}\phi, A^{-1}\psi\rangle = \langle \phi, \psi\rangle, \quad \forall \phi, \psi \in \mathcal{H}. \tag{7.59}$$

Proof. For every pair of vectors $\phi, \psi \in \mathcal{H}$, we have that $\phi = A\eta$, $\psi = A\chi$, for some $\eta, \chi \in \mathcal{H}$, because A is surjective, and

$$\langle A^{-1}\phi, A^{-1}\psi\rangle = \langle A^{-1}A\eta, A^{-1}A\chi\rangle = \langle \eta, \chi\rangle = \langle A\eta, A\chi\rangle = \langle \phi, \psi\rangle,$$

where we have used the fact that A is IP-preserving. □

Remark 7.2.2. From the previous proof it is clear that, if A is any IP-preserving, injective operator in $\mathcal{H}$, then

$$\langle A^{-1}\phi, A^{-1}\psi\rangle = \langle \phi, \psi\rangle, \qquad \forall \phi, \psi \in \mathrm{dom}(A^{-1}) = \mathrm{ran}(A). \tag{7.60}$$

Remark 7.2.3. A bounded operator admitting a bounded inverse is often called — and we will indeed call it — a *top-linear isomorphism*. Thus, Lemma 7.2.2 can be rephrased as follows: If A is an IP-preserving top-linear isomorphism, then A^{-1} is an IP-preserving top-linear isomorphism too.

Proposition 7.2.1. *A densely defined, IP-preserving operator is injective. If a bounded operator $A \in \mathcal{B}(\mathcal{H})$ is IP-preserving, then $\|A\| \geq 1$; in particular, if A is an IP-preserving top-linear isomorphism, then*

$$\|A\|, \|A^{-1}\| \geq 1. \tag{7.61}$$

Proof. If a linear operator A in $\mathcal{H}$ is IP-preserving, given any $\psi \in \ker(A)$, we have that

$$0 = \langle A\psi, A\phi\rangle = \langle \psi, \phi\rangle, \qquad \forall \phi \in \mathrm{dom}(A). \tag{7.62}$$

Thus, if A is densely defined — i.e., if $\overline{\mathrm{dom}(A)}^{\|\cdot\|} = \mathcal{H}$ — then by the continuity of the inner product (w.r.t. each of its arguments), we conclude that, actually, $\langle \psi, \phi\rangle = 0$, $\forall \phi \in \mathcal{H}$; hence, the Hermitian sesquilinear form $\langle \cdot\,, \cdot\rangle$ being non-degenerate, $\psi = 0$. Therefore, $\ker(A) = \{0\}$ and A is injective.

Now, suppose that A is bounded and IP-preserving. Applying the latter property and the Cauchy-Schwarz inequality, we find that

$$|\langle \phi, \psi\rangle| = |\langle A\phi, A\psi\rangle| \leq \|A\phi\|\,\|A\psi\| \leq \|A\|^2\,\|\phi\|\,\|\psi\|, \tag{7.63}$$

for all $\phi, \psi \in \mathcal{H}$. Setting $\psi = \phi$ in relation (7.63), and choosing this vector ϕ in such a way that $|\langle \phi, \phi\rangle| = 1 = \|\phi\|$ (e.g. an element of an orthonormal basis in $\mathcal{H}$), we conclude that $\|A\|^2 \geq 1$; hence $\|A\| \geq 1$. Finally, if A is an IP-preserving top-linear isomorphism, then, by Lemma 7.2.2, A^{-1} enjoys the same property, so that both inequalities in (7.61) hold true. □

We will now prove that, under mild conditions, an IP-preserving operator is a top-linear isomorphism.

Theorem 7.2.1. *A surjective, IP-preserving, all-over operator A in $\mathcal{H}$ is an adjointable top-linear isomorphism and $A^* = A^{-1}$; moreover,*

$$\|A\| = \|A^*\| = \|A^{-1}\| \geq 1. \tag{7.64}$$

Proof. Since A is IP-preserving and $\mathrm{dom}(A) = \mathcal{H}$, by Proposition 7.2.1 it is injective. Let us prove that A is bounded (equivalently continuous). By the Closed Graph Theorem (see, e.g., Theorem 3.5 in [73], or Subsect. 2.8 of [96]), it is sufficient to show that A is a closed operator. Let $\{\chi_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ such that

$$\lim_n \chi_n = 0 \text{ and } \lim_n A\chi_n = \phi, \text{ for some } \phi \in \mathcal{H}. \tag{7.65}$$

In order to conclude that A is closed, we need to show that $\phi = 0$. Indeed, by the continuity of the scalar product (w.r.t. each of its arguments) and by the fact that A is IP-preserving, we have:

$$\langle \phi, A\psi\rangle = \langle \mathrm{lim}_n A\chi_n, A\psi\rangle = \lim_n \langle A\chi_n, A\psi\rangle = \lim_n \langle \chi_n, \psi\rangle = \langle \mathrm{lim}_n \chi_n, \psi\rangle = 0, \quad \forall \psi \in \mathcal{H}. \tag{7.66}$$

Since A is surjective, we conclude that $\langle \phi, \eta\rangle = 0$, $\forall \eta \in \mathcal{H}$; hence, the Hermitian sesquilinear form $\langle\cdot\,,\cdot\rangle$ being non-degenerate, $\phi = 0$, so that A is closed.

Summarizing, a surjective, IP-preserving, all-over operator A is bijective and bounded; hence, by the Bounded Inverse Theorem, a top-linear isomorphism.

Let us now show that $A \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ and $A^* = A^{-1}$. In fact, by Lemma 7.2.2, the bounded operator A^{-1} is IP-preserving too; hence:

$$\langle \phi, A\psi\rangle = \langle A^{-1}\phi, A^{-1}A\psi\rangle = \langle A^{-1}\phi, \psi\rangle, \qquad \forall \phi, \psi \in \mathcal{H}. \tag{7.67}$$

Therefore, A is adjointable and $A^* = A^{-1}$, so that $\|A\| = \|A^*\| = \|A^{-1}\|$ and, by (7.61) in Proposition 7.2.1, relation (7.64) holds true. □

Notation 7.2.1. Given two vectors $\phi, \psi \in \mathcal{H}$, by writing

$$\phi \overset{\|\cdot\|}{\perp} \psi, \tag{7.68}$$

we mean that ϕ and ψ are *norm-orthogonal* to each other (recall from Section 6.2 that $\phi \perp \psi$ means that ϕ and ψ are IP-orthogonal, instead); i.e., that $\|\alpha\phi + \beta\psi\| = \max\{\|\alpha\phi\|, \|\beta\psi\|\}$, for all $\alpha, \beta \in \mathbb{Q}_{p,\mu}$.

Definition 7.2.3. A linear operator A in $\mathcal{H}$ is said to be *norm-orthogonality-preserving* (in short, NO-preserving) if

$$\phi, \psi \in \mathrm{dom}(A),\ \phi \overset{\|\cdot\|}{\perp} \psi \quad \implies \quad A\phi \overset{\|\cdot\|}{\perp} A\psi. \tag{7.69}$$

Theorem 7.2.2. *Every all-over, NO-preserving operator in $\mathcal{H}$ is bounded. Specifically, every all-over, NO-preserving operator in $\mathcal{H}$ is a nonzero scalar multiple of an isometry and, conversely, a nonzero scalar multiple of an isometry is NO-preserving. In particular, a linear operator A in $\mathcal{H}$ is an isometry if and only if A is an all-over, NO-preserving (hence, bounded) operator such that $\|A\| = 1$.*

Proof. Since $\|\mathcal{H}\| := \{\|\phi\| \mid \phi \in \mathcal{H}\} = |\mathbb{Q}_{p,\mu}|$, the 'ramification index' of $\mathcal{H}$ is equal to 1, so that we can apply Corollary 1.3 in [139] (actually, the first assertion of the theorem follows from Corollary 1.1 *ibidem*, and does not require the mentioned property of $\mathcal{H}$). □

We can now introduce the unitary operators in the p-adic setting and provide a suitable characterization of this class of operators.

Definition 7.2.4. A matrix operator in $\mathcal{H}$ of the form

$$U = \mathrm{op}_\Phi(\langle \phi_m, \psi_n\rangle) \tag{7.70}$$

— where $\Phi \equiv \{\phi_m\}_{m=1}^{\mathsf{N}}$, $\Psi \equiv \{\psi_n\}_{n=1}^{\mathsf{N}}$ are orthonormal bases in $\mathcal{H}$ — is called a *unitary operator*. We will denote the set of all such operators in $\mathcal{H}$ by $\mathcal{U}(\mathcal{H})$.

The set $\mathcal{U}(\mathcal{H})$ is characterized by the following result:

Theorem 7.2.3. *Given a linear operator U in $\mathcal{H}$, the following facts are equivalent:*

(U1) *U is a unitary operator — i.e., $U = \mathrm{op}_\Phi(\langle\phi_m, \psi_n\rangle)$ — for some pair of orthonormal bases $\Phi \equiv \{\phi_m\}_{m=1}^{\mathsf{N}}$ and $\Psi \equiv \{\psi_n\}_{n=1}^{\mathsf{N}}$ in $\mathcal{H}$;*

(U2) *$U \in \mathcal{B}(\mathcal{H})$ and, for some pair of orthonormal bases $\Phi \equiv \{\phi_m\}_{m=1}^{\mathsf{N}}$ and $\Psi \equiv \{\psi_n\}_{n=1}^{\mathsf{N}}$, $U\phi_k = \psi_k$, $\forall k \in \mathbb{N}_{\leq \mathsf{N}}$;*

(U3) *$U \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$, $\|U\| = 1$ and $UU^* = \mathrm{Id} = U^*U$;*

(U4) *U is a surjective IP-preserving, all-over (hence, bounded) operator and $\|U\| = 1$;*

(U5) *U is an IP-preserving top-linear isomorphism and $\|U\| = 1 = \|U^{-1}\|$;*

(U6) *U is an automorphism of the p-adic Hilbert space $\mathcal{H}$, namely, an IP-preserving surjective isometry;*

(U7) *U is a surjective, IP-preserving, NO-preserving, all-over operator;*

(U8) *U is bounded and transforms orthonormal bases into orthonormal bases.*

Proof. We will first show that (U1) $\iff$ (U2) $\implies$ (U3).

Note that

$$|\langle\phi_m, \psi_n\rangle| \leq \|\phi_m\|\,\|\psi_n\| = 1,\ \forall m, n \in \mathbb{N}_{\leq \mathsf{N}},\ \text{and}\ \lim_m \langle\phi_m, \psi_n\rangle = 0,\ \forall n \in \mathbb{N}_{\leq \mathsf{N}}. \tag{7.71}$$

Hence, by Theorem 7.1.1, we have that $\mathrm{op}_\Phi(\langle\phi_m, \psi_n\rangle) \in \mathcal{B}(\mathcal{H})$ and, moreover,

$$\mathrm{op}_\Phi(\langle\phi_m, \psi_n\rangle)\phi_k = \sum_m \langle\phi_m, \psi_k\rangle\phi_m = \psi_k, \quad \forall k \in \mathbb{N}_{\leq \mathsf{N}}. \tag{7.72}$$

Therefore, (U1) $\implies$ (U2).

Conversely, if U satisfies (U2), then

$$U = \mathrm{op}_\Phi(\langle\phi_m, U\phi_n\rangle) = \mathrm{op}_\Phi(\langle\phi_m, \psi_n\rangle), \tag{7.73}$$

where in the first equality, we have used the expression of a bounded matrix operator (w.r.t. any orthonormal basis). Thus, (U1) holds true.

Now, given a unitary operator $U = \mathrm{op}_\Phi(\langle\phi_m, \psi_n\rangle) \in \mathcal{U}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$, since $\lim_n\langle\phi_m, \psi_n\rangle = 0$, for all $m \in \mathbb{N}_{\leq \mathsf{N}}$, by the last assertion of Theorem 7.1.2 we conclude that U is adjointable; i.e., $\mathcal{U}(\mathcal{H}) \subset \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ as well. Moreover, as previously shown, the unitary operator $U = \mathrm{op}_\Phi(\langle\phi_m, \psi_n\rangle)$ is completely determined by condition (U2); therefore:

$$\langle\phi_m, U^*\psi_k\rangle = \langle U\phi_m, \psi_k\rangle = \langle\psi_m, \psi_k\rangle = \delta_{mk}. \tag{7.74}$$

Thus, $U^*\psi_k = \phi_k$, $\forall k \in \mathbb{N}_{\leq \mathsf{N}}$, and hence — noting that: $A \in \mathcal{B}(\mathcal{H})$, $A\phi_m = \phi_m$, $\forall m \in \mathbb{N}_{\leq \mathsf{N}}$ (where $\{\phi_m\}_{m=1}^{\mathsf{N}}$ is any orthonormal basis) $\implies$ $A = \mathrm{Id}$ — we have:

$$U^*U = \mathrm{Id} = UU^*;\ \text{i.e., } U^* = U^{-1}. \tag{7.75}$$

Also note that $\sup_m\langle\phi_m, \psi_n\rangle = \|\psi_n\| = 1$, $\forall n \in \mathbb{N}_{\leq \mathsf{N}}$; hence:

$$1 = \sup_{m,n} \langle\phi_m, \psi_n\rangle = \|U\| = \|U^*\| = \|U^{-1}\|. \tag{7.76}$$

Thus, if $U \in \mathcal{U}(\mathcal{H})$, then U satisfies condition (U3).

Next, it is clear that (U3) $\implies$ (U4), because, if the conditions in (U3) are satisfied, then U is a surjective (adjointable) bounded operator and

$$\langle U\chi, U\eta\rangle = \langle U^*U\chi, \eta\rangle = \langle \chi, \eta\rangle, \qquad \forall \chi, \eta \in \mathcal{H}; \tag{7.77}$$

i.e., U is IP-preserving. Moreover, by Theorem 7.2.1, (U4) $\implies$ (U5). Also, if U satisfies (U5), then, by Lemma 7.2.1, U is a surjective isometry (and IP-preserving); i.e., (U5) $\implies$ (U6).

Let us now prove that (U6) $\iff$ (U7). In fact, by the second assertion of Theorem 7.2.2, if U is an isometry, then it is a NO-preserving (all-over) operator; hence: (U6) $\implies$ (U7). Conversely, if U is a NO-preserving, all-over operator, then (again by the second assertion of Theorem 7.2.2) U is a non-zero scalar multiple of an isometry: $U = zJ$, with $z \in \mathbb{Q}_{p,\mu} \setminus \{0\}$. Now, if, moreover, U is IP-preserving and surjective, then by Theorem 7.2.1, $\|U\| = \|U^{-1}\|$. Thus, J is a surjective isometry and

$$|z| = \|zJ\| = \|U\| = \|U^{-1}\| = \|z^{-1}J^{-1}\| = |z|^{-1} \quad \implies \quad |z| = 1. \tag{7.78}$$

Therefore, $U = zJ$ is an IP-preserving, surjective isometry; i.e., (U7) $\implies$ (U6), as well.

At this point, let us observe that (U6) implies (U8). Indeed, if U is an IP-preserving, surjective isometry, then, given any orthonormal basis $\{\phi_m\}_{m=1}^{\mathsf{N}}$ in $\mathcal{H}$, and, putting $\psi_n = U\phi_n$, $\forall n \in \mathbb{N}_{\leq \mathsf{N}}$, we obtain another orthonormal basis $\{\psi_n\}_{n=1}^{\mathsf{N}}$, because

$$\langle \psi_j, \psi_k\rangle = \langle U\phi_j, U\phi_k\rangle = \langle \phi_j, \phi_k\rangle = \delta_{jk}, \tag{7.79}$$

and, for every (finite or denumerable) set $\{z_n\}_{n=1}^{\mathsf{N}} \subset \mathbb{Q}_{p,\mu}$ — converging to 0, if $\mathsf{N} = \infty$ — and every $\chi \in \mathcal{H}$,

$$\left\|\textstyle\sum_n z_n\psi_n\right\| = \left\|U^{-1}\textstyle\sum_n z_n\psi_n\right\| = \left\|\textstyle\sum_n z_n\phi_n\right\| = \max_n |z_n|, \tag{7.80}$$

where we have used the fact that U^{-1} is an isometry, and

$$\chi = U(U^{-1}\chi) = \sum_n \langle \phi_n, U^{-1}\chi\rangle\, U\phi_n = \sum_n \langle U\phi_n, UU^{-1}\chi\rangle\, U\phi_n = \sum_n \langle \psi_n, \chi\rangle \psi_n. \tag{7.81}$$

Thus, by (7.80) and (7.81), $\{\psi_n\}_{n=1}^{\mathsf{N}}$ is a normal basis, and specifically, by (7.79), it is orthonormal.

Finally, it is obvious that (U8) $\implies$ (U2), and this observation completes the proof, since overall we have shown that: (U1) $\iff$ (U2) $\implies$ (U3) $\implies$ (U4) $\implies$ (U5) $\implies$ (U6) $\implies$ (U8) $\implies$ (U2), and, moreover, (U6) $\iff$ (U7). □

Remark 7.2.4. One can easily check that

$$U := \mathrm{op}_\Phi(\langle \phi_m, \psi_n\rangle) = \sum_k |\psi_k\rangle\langle\phi_k| = \mathrm{op}_\Psi(\langle \phi_m, \psi_n\rangle), \tag{7.82}$$

— where, if $\mathsf{N} = \infty$, the series converges w.r.t. the strong operator topology — and

$$U^* = \mathrm{op}_\Phi(\langle \psi_m, \phi_n\rangle) = \sum_k |\phi_k\rangle\langle\psi_k| = \mathrm{op}_\Psi(\langle \psi_m, \phi_n\rangle) = U^{-1} \in \mathcal{U}(\mathcal{H}). \tag{7.83}$$

Remark 7.2.5. By the characterization (U6) of $\mathcal{U}(\mathcal{H})$, it is clear that $\mathcal{U}(\mathcal{H})$ is, in a natural way, a group. In fact, the product (composition) of two unitary operators is unitary and $\mathrm{Id} \in \mathcal{U}(\mathcal{H})$. Moreover, by (7.83) — or, say, by (U6) and Lemma 7.2.2 (if U is an IP-preserving surjective isometry, then U^{-1} shares the same property) — if $U \in \mathcal{U}(\mathcal{H})$, then $U^{-1} = U^* \in \mathcal{U}(\mathcal{H})$ too. Let us also observe that the *unitary group* of the p-adic Hilbert space $\mathcal{H}$ is the intersection of two other remarkable groups, i.e.,

$$\mathcal{U}(\mathcal{H}) = \mathscr{I}(\mathcal{H}) \cap \mathscr{N}(\mathcal{H}), \tag{7.84}$$

where:

- $\mathscr{I}(\mathcal{H}) \subset \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ is the group of all surjective, IP-preserving, all-over operators (note that, by Theorem 7.2.1 and Lemma 7.2.2, if $A \in \mathscr{I}(\mathcal{H})$, then $A^{-1} = A^* \in \mathscr{I}(\mathcal{H})$ too);
- $\mathscr{N}(\mathcal{H})$ is the group of all surjective, NO-preserving, all-over operators — equivalently, the group of all non-zero scalar multiples of surjective isometries (Theorem 7.2.2).

Remark 7.2.6. In the case where $\mathcal{H}$ is finite-dimensional — $\dim(\mathcal{H}) = \mathtt{N} \in \mathbb{N}$; hence, $\mathcal{B}(\mathcal{H}) = \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ is just the set $\mathrm{Lin}(\mathcal{H})$ of all linear operators in $\mathcal{H}$, and $UU^* = \mathrm{Id}$ if and only if $U^*U = \mathrm{Id}$ — the characterization (U3) of $\mathcal{U}(\mathcal{H})$ provides a simple description of the unitary group of $\mathcal{H}$ as a matrix group, i.e.,

$$\mathcal{U}(\mathcal{H}) = \left\{\mathrm{op}_\Phi(U_{mn}) \mid \textstyle\sum_{n=1}^{\mathtt{N}} U_{ln}\overline{U_{mn}} = \delta_{lm},\ \max_{m,n}|U_{mn}| = 1\right\}, \tag{7.85}$$

where $\Phi = \{\phi_m\}_{m=1}^{\mathtt{N}}$ is any orthonormal basis in $\mathcal{H}$. It is worth observing that here the condition

$$\|\mathrm{op}_\Phi(U_{mn})\| = \max_{m,n}|U_{mn}| = 1 \tag{7.86}$$

cannot be dispensed with (unlike the complex case). We illustrate this point by an explicit example.

Assume that $\mathcal{H}$ is a p-adic Hilbert space, with $p \neq 2$ and $\dim(\mathcal{H}) = 4$. As shown in the proof of Proposition 5.3 in [128], there exists a solution $x_1, \ldots, x_4$ of the equation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = p^{2\mathtt{K}}, \qquad x_1, \ldots, x_4 \in \mathbb{Z}, \tag{7.87}$$

— for any $\mathtt{K} \in \mathbb{N}$ — satisfying the condition that

$$\max_i |x_i| = 1. \tag{7.88}$$

Consider, then, the matrix (with rational coefficients)

$$(A_{mn}) = \frac{1}{p^{\mathtt{K}}}\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ -x_2 & x_1 & -x_4 & x_3 \\ -x_4 & -x_3 & x_2 & x_1 \\ -x_3 & x_4 & x_1 & -x_2 \end{pmatrix}, \tag{7.89}$$

where $x_1, \ldots, x_4$ is the aforementioned solution of (7.87)–(7.88). Clearly, we have:

$$\textstyle\sum_{n=1}^4 A_{ln}\overline{A_{mn}} = \sum_{n=1}^4 A_{ln}A_{mn} = \delta_{lm};\ \text{but } \max_{m,n}|A_{mn}| = p^{\mathtt{K}} \neq 1. \tag{7.90}$$

Thus, $A = \mathrm{op}_\Phi(A_{mn}) \in \mathscr{I}(\mathcal{H})$ (because $A^* = A^{-1}$, hence, A is IP-preserving), *but* $A \notin \mathscr{N}(\mathcal{H})$, because

$$\|A\| = \|A^*\| = \|A^{-1}\| = p^{\mathtt{K}} > 1, \tag{7.91}$$

so that A cannot be a non-zero scalar multiple of an isometry (in such a case, we should have that $\|A^{-1}\| = \|A\|^{-1}$). Therefore, $A \notin \mathcal{U}(\mathcal{H})$. Otherwise stated, A cannot be unitary, since it preserves the inner product, but not the norm-orthogonality.

Remark 7.2.7. Let us observe explicitly that the group $\mathscr{I}(\mathcal{H})$ admits a further characterization; namely,

$$\mathscr{I}(\mathcal{H}) = \left\{A \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H}) \mid A \text{ bijective and } A^* = A^{-1}\right\}. \tag{7.92}$$

In fact, by Theorem 7.2.1, $\mathscr{I}(\mathcal{H})$ is contained in the set defined on the right hand side of (7.92). Conversely, it is clear that every bijective operator $A \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$, such that $A^* = A^{-1}$, is IP-preserving:

$$\langle A\eta, A\chi\rangle = \langle A^*A\eta, \chi\rangle = \langle \eta, \chi\rangle, \qquad \forall \eta, \chi \in \mathcal{H}. \tag{7.93}$$

Therefore, relation (7.92) holds true. As a consequence, we obtain a simple description of the unitary group $\mathcal{U}(\mathcal{H})$. Indeed, note that, by (7.92) and by the characterization `(U3)` of a unitary operator, we have:

$$\mathcal{U}(\mathcal{H}) = \mathscr{I}(\mathcal{H}) \cap \mathcal{B}(\mathcal{H})_{[1]} = \mathscr{I}(\mathcal{H}) \cap \mathcal{B}(\mathcal{H})_1, \tag{7.94}$$

where $\mathcal{B}(\mathcal{H})_{[1]}$ and $\mathcal{B}(\mathcal{H})_1$ are, respectively, the *unit sphere* and that *unit ball* in $\mathcal{B}(\mathcal{H})$; i.e.,

$$\mathcal{B}(\mathcal{H})_{[1]} := \{A \in \mathcal{B}(\mathcal{H}) \mid \|A\| = 1\}, \quad \mathcal{B}(\mathcal{H})_1 := \{A \in \mathcal{B}(\mathcal{H}) \mid \|A\| \leq 1\}. \tag{7.95}$$

The first equality in (7.94) corresponds to the characterization `(U4)` of a p-adic unitary operator, and the second equality follows from (7.64) in Theorem 7.2.1, according to which $\|\mathscr{I}(\mathcal{H})\| \subset [1, \infty)$.

Remark 7.2.8. Considering again the case where $\mathcal{H}$ is finite-dimensional — $\dim(\mathcal{H}) = \mathsf{N} \in \mathbb{N}$ and $\mathcal{B}(\mathcal{H}) = \mathrm{Lin}(\mathcal{H}) = \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ — using elementary methods of matrix analysis one can prove that, in this case, the group $\mathscr{N}(\mathcal{H})$ admits the following further characterization:

$$\mathscr{N}(\mathcal{H}) = \big\{A \in \mathrm{Lin}(\mathcal{H}) \mid \|A\| = 1 = |\det(A)|\big\}, \tag{7.96}$$

where $\det(A)$ is the determinant of the representative matrix of A w.r.t. any basis in the finite-dimensional vector space $\mathcal{H}$. Clearly, if $A \in \mathcal{U}(\mathcal{H})$, then $|\det(A)| = 1$ automatically, because $|\det(AA^*)| = \big|\det(A)\,\overline{\det(A)}\,\big| = |\det(A)|^2$.

Example 7.2.1. Let us consider the case where $p = 2$ and $\mu = 14$; i.e., $\mathcal{H}$ is a p-adic Hilbert space over $\mathbb{Q}_2(\sqrt{14})$. Let us assume that $\dim(\mathcal{H}) = 2$, and, given an orthonormal basis $\Phi = \{\phi_1, \phi_2\}$ in $\mathcal{H}$, let us consider a linear operator $U = \mathrm{op}_\Phi(U_{mn})$. By (7.85), U is unitary if and only if

$$(U_{mn}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ where: } a\overline{a} + b\overline{b} = 1 = c\overline{c} + d\overline{d},\ a\overline{c} + b\overline{d} = 0,\ \max\{|a|, |b|, |c|, |d|\} = 1. \tag{7.97}$$

To satisfy this condition, we can put, e.g., $a = \sqrt{-7}^{\,2}$, $b = \frac{2}{a}\sqrt{14}$, $c = b$ and $d = a$, where $\sqrt{-7}^{\,2}$ is any of the two 2-adic square roots of $-7 = 1 + 0\cdot 2 + 0\cdot 2^2 + 1\cdot 2^3 + 1\cdot 2^4 + \cdots \in (\mathbb{Q}_2^*)^2$. Therefore, $\Psi \equiv \{\psi_1 = a\phi_1 + b\phi_2, \psi_2 = b\phi_1 + a\phi_2\}$ is another orthonormal basis in $\mathcal{H}$.

7.3 The trace class of a p-adic Hilbert space

In this section, we will introduce a suitable notion of trace class operator in a p-adic Hilbert space $\mathcal{H}$. As in Section 7.1, we will assume that $\dim(\mathcal{H}) = \infty$ (and we will use the notations adopted therein), because in the finite-dimensional case the notion of trace introduced here becomes completely analogous to the notion of trace of a linear operator in a finite-dimensional complex Hilbert space and the results of this section hold true with obvious modifications.

7.3.1 Traceable operators

We start with the following:

Definition 7.3.1. Let $\Phi = \{\phi_m\}_{m\in\mathbb{N}}$ be an orthonormal basis in $\mathcal{H}$, and let T be a (densely defined) linear operator in $\mathcal{H}$ such that $\Phi \subset \mathrm{dom}(T)$. We say that the operator T is *traceable* w.r.t. Φ if the series

$$\sum_m \langle \phi_m, T\phi_m \rangle \tag{7.98}$$

is convergent. Namely, if $\lim_m \langle \phi_m, T\phi_m \rangle = 0$ (see Proposition 6.1.1).

Proposition 7.3.1. *A matrix operator $T = \mathrm{op}_\Phi(T_{mn})$ is (such that $\Phi \subset \mathrm{dom}(T)$ and) traceable w.r.t. Φ if and only if*

$$\lim_m T_{mn} = 0, \ \ \forall n \in \mathbb{N}, \ \ \textit{and} \ \ \lim_m T_{mm} = 0. \tag{7.99}$$

Proof. It is easy to see that $\Phi \subset \mathrm{dom}(T)$, with $T = \mathrm{op}_\Phi(T_{mn})$, if and only if $\lim_m T_{mn} = 0$, $\forall n \in \mathbb{N}$. Indeed, recalling (7.8) and (7.9), if $\phi_n \in \mathrm{dom}(T)$, then $T\phi_n = \sum_m T_{mn}\phi_m$ (hence, $\{T_{mn}\}_{m\in\mathbb{N}} \in c_0$) and, conversely, if $\lim_m T_{mn} = 0$, then $\phi_n \in \mathrm{dom}(T)$.

Moreover, if $\phi_n \in \mathrm{dom}(T)$, then ($T\phi_n = \sum_m T_{mn}\phi_m$ and)

$$\langle \phi_n, T\phi_n \rangle = \sum_m T_{mn} \langle \phi_n, \phi_m \rangle = T_{nn}. \tag{7.100}$$

Therefore, if $\Phi \subset \mathrm{dom}(T)$, then $\lim_m T_{mn} = 0$, $\forall n$, and, if, moreover, the series (7.98) is convergent, then

$$\lim_m T_{mm} = \lim_m \langle \phi_m, T\phi_m \rangle = 0. \tag{7.101}$$

Conversely, if both conditions in (7.99) hold true, then $\Phi \subset \mathrm{dom}(T)$, and $\lim_m \langle \phi_m, T\phi_m \rangle = \lim_m T_{mm} = 0$, so that the series (7.98) converges; i.e., $T = \mathrm{op}_\Phi(T_{mn})$ is traceable w.r.t. Φ. □

Remark 7.3.1. By Proposition 7.3.1 and by the characterization of matrix elements of a bounded operator (see (7.14) in Theorem 7.1.1), it is clear that one can construct matrix operators in $\mathcal{H}$ ($\dim(\mathcal{H}) = \infty$) that are traceable w.r.t. a given orthonormal basis in $\mathcal{H}$, but *not* necessarily bounded. Precisely, one has to take any matrix operator $T = \mathrm{op}_\Phi(T_{mn})$ satisfying both conditions in (7.99) and such that $\sup_{m,n} |T_{mn}| = \infty$.

The previous remark motivates us to consider a smaller class of matrix operators for the definition of the trace class of $\mathcal{H}$.

Definition 7.3.2. Let $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}$ be an orthonormal basis in $\mathcal{H}$. We introduce the following set of matrix operators:

$$\mathcal{T}_\Phi(\mathcal{H}) := \big\{\mathrm{op}_\Phi(T_{mn}) \mid T_{mn} \in \mathsf{M}_\infty(\mathbb{Q}_{p,\mu}) \text{ s.t. } \lim_{m+n} T_{mn} = 0\big\}. \tag{7.102}$$

Remark 7.3.2. Recalling Notation 7.1.2, the limit $\lim_{m+n} T_{mn} = 0$ means that

$$\forall \epsilon > 0, \, \mathrm{card}(\{(m,n) \in \mathbb{N} \times \mathbb{N} \mid |T_{mn}| \geq \epsilon\}) < \infty. \tag{7.103}$$

Equivalently, $\lim_{m+n} T_{mn} = 0$ means that

$$\forall \epsilon > 0, \, \exists \mathtt{N} \in \mathbb{N}, \text{ such that, if } \max\{m,n\} > \mathtt{N}, \text{ then } |T_{mn}| < \epsilon, \tag{7.104}$$

or, also, that

$$\forall \epsilon > 0, \, \exists \mathtt{N} \in \mathbb{N}, \text{ such that, if } m+n > \mathtt{N}, \text{ then } |T_{mn}| < \epsilon. \tag{7.105}$$

Moreover, conditions (7.103)–(7.105) are equivalent to assuming that the double series $\sum_{m,n} T_{mn}$ is convergent, where

$$\sum_{m,n} T_{mn} = \lim_{\mathtt{N}\to\infty} \left(\sum_{m=1}^{\mathtt{N}} \sum_{n=1}^{\mathtt{N}} T_{mn} \right). \tag{7.106}$$

It is a remarkable fact that, given a double sequence $\{x_{mn}\}_{m,n\in\mathbb{N}}$ in $\mathbb{Q}_{p,\mu}$, if $\lim_{m+n} x_{mn} = 0$, then both the iterated series

$$\sum_m \sum_n x_{mn} \quad \text{and} \quad \sum_n \sum_m x_{mn} \tag{7.107}$$

converge and

$$\sum_m \sum_n x_{mn} = \sum_n \sum_m x_{mn} = \sum_{m,n} x_{mn}. \tag{7.108}$$

Therefore, if $\lim_{m+n} T_{mn} = 0$, then the convergent double series (7.106) can be expressed as an iterated series.

Another useful fact is that, given a double sequence $\{x_{mn}\}_{m,n\in\mathbb{N}} \subset \mathbb{Q}_{p,\mu}$,

$$\lim_{m+n} x_{mn} = 0 \quad \iff \quad \begin{cases} \lim_m x_{mn} = 0,\ \forall n \in \mathbb{N}, \quad \lim_n x_{mn} = 0,\ \forall m \in \mathbb{N}, \\ \text{and } \lim_{m,n} x_{mn} = 0 \quad \text{(Pringsheim limit)}^1 \end{cases} \tag{7.109}$$

$$\iff \quad \lim_m x_{mn} = 0,\ \forall n \in \mathbb{N}, \text{ and } \lim_n x_{mn} = 0, \textit{ uniformly} \text{ in } m \in \mathbb{N}. \tag{7.110}$$

In relation (7.110), the expression "$\lim_n x_{mn} = 0$, *uniformly* in $m \in \mathbb{N}$" means: for every $\epsilon > 0$, $\exists \mathtt{N} \in \mathbb{N}$ such that, for $n > \mathtt{N}$ and all $m \in \mathbb{N}$, $|x_{mn}| < \epsilon$.

For the previous claims, see p. 62 of [69], Exercise 23.B, and Chapter 8 of [130].

If a linear operator T in $\mathcal{H}$ is traceable w.r.t. an orthonormal basis $\Phi = \{\phi_m\}_{m\in\mathbb{N}}$, we denote the sum of the series (7.98) by the symbol $\mathrm{tr}_\Phi(T)$.

Proposition 7.3.2. *If $T = \mathrm{op}_\Phi(T_{mn}) \in \mathcal{T}_\Phi(\mathcal{H})$, then it is traceable w.r.t. Φ — in particular, $\Phi \subset \mathrm{dom}(T)$ — and*

$$\mathrm{tr}_\Phi(T) := \sum_m \langle \phi_m, T\phi_m \rangle = \sum_m T_{mm}. \tag{7.111}$$

Proof. Observe that

$$\mathrm{card}(\{m \in \mathbb{N} \mid |T_{mm}| \geq \epsilon\}) \leq \mathrm{card}(\{(m,n) \in \mathbb{N}\times\mathbb{N} \mid |T_{mn}| \geq \epsilon\}). \tag{7.112}$$

Thus, recalling Remark 7.3.2, we argue that

$$T = \mathrm{op}_\Phi(T_{mn}) \in \mathcal{T}_\Phi(\mathcal{H}) \quad \stackrel{\text{def}}{\iff} \quad \lim_{m+n} T_{mn} = 0 \quad \Longrightarrow \lim_m T_{mm} = 0, \tag{7.113}$$

and

$$\lim_{m+n} T_{mn} = 0 \quad \Longrightarrow \quad \lim_m T_{mn} = 0, \ \ \forall n \in \mathbb{N}. \tag{7.114}$$

Thus, if $T = \mathrm{op}_\Phi(T_{mn})$ belongs to $\mathcal{T}_\Phi(\mathcal{H})$, then $\Phi \subset \mathrm{dom}(T)$ (see the proof of Proposition 7.3.1) and T is traceable w.r.t. Φ. Moreover, $\langle \phi_m, T\phi_m \rangle = T_{mm}$, for every $m \in \mathbb{N}$ (see (7.100)); hence, relation (7.111) holds true. □

We now provide a more precise characterization of the set of matrix operators $\mathcal{T}_\Phi(\mathcal{H})$.

[1] Namely, $\forall \epsilon > 0$, $\exists \mathtt{N} \in \mathbb{N}$ such that, if $m, n > \mathtt{N}$, then $|x_{mn}| < \epsilon$.

Proposition 7.3.3. *Let $\Phi \equiv \{\phi_n\}_{n\in\mathbb{N}}$ be any orthonormal basis in $\mathcal{H}$. Then, the following facts are equivalent:*

(T1) $T \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ *and* $\lim_{m+n}\langle\phi_m, T\phi_n\rangle = 0$*;*

(T2) $T \in \mathcal{B}(\mathcal{H})$ *and* $\lim_{m+n}\langle\phi_m, T\phi_n\rangle = 0$*;*

(T3) $T \in \mathcal{T}_\Phi(\mathcal{H})$*.*

Proof. It is obvious that (T1) $\Longrightarrow$ (T2). Also, if $T \in \mathcal{B}(\mathcal{H})$, then $T = \mathrm{op}_\Phi(T_{mn})$, where $T_{mn} = \langle\phi_m, T\phi_n\rangle$; hence (T2) $\Longrightarrow$ (T3). Next, if $T \in \mathcal{T}_\Phi(\mathcal{H})$, then by Corollary 7.1.3, we have that $T \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$. Moreover, since T is bounded, $T = \mathrm{op}_\Phi(\langle\phi_m, T\phi_n\rangle)$. Hence, (T3) $\Longrightarrow$ (T1), and the proof is complete. □

Corollary 7.3.1. *$\mathcal{T}_\Phi(\mathcal{H})$ is a linear subspace of $\mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ and*

$$T \in \mathcal{T}_\Phi(\mathcal{H}) \implies T^* \in \mathcal{T}_\Phi(\mathcal{H}). \tag{7.115}$$

Proof. Since (T1) $\Longleftrightarrow$ (T3) in Proposition 7.3.3 — therefore, $\mathcal{T}_\Phi(\mathcal{H}) \subset \mathcal{B}_{\mathrm{ad}}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ — it is sufficient to note that, for $S, T \in \mathcal{B}(\mathcal{H})$, $\phi, \psi \in \mathcal{H}$ and scalars $a, b \in \mathbb{Q}_{p,\mu}$, the following estimate holds:

$$|\langle\phi, (aS + bT)\psi\rangle| \le \max\{|a|\,|\langle\phi, S\psi\rangle|, |b|\,|\langle\phi, T\psi\rangle|\}. \tag{7.116}$$

It follows that

$$\lim_{m+n}\langle\phi_m, S\phi_n\rangle = 0 = \lim_{m+n}\langle\phi_m, T\phi_n\rangle \implies \lim_{m+n}\langle\phi_m, (aS + bT)\phi_n\rangle = 0, \tag{7.117}$$

namely, $S, T \in \mathcal{T}_\Phi(\mathcal{H}) \implies aS + bT \in \mathcal{T}_\Phi(\mathcal{H})$. Moreover, if $T \in \mathcal{T}_\Phi(\mathcal{H}) \subset \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$, then $\langle\phi_m, T\phi_n\rangle = \overline{\langle\phi_n, T^*\phi_m\rangle}$. Hence, $\lim_{m+n}\langle\phi_m, T\phi_n\rangle = 0 \implies \lim_{m+n}\langle\phi_m, T^*\phi_n\rangle = 0$, i.e., the implication (7.115) holds true. □

7.3.2 The trace class

Our next task is to show that, actually, the definition of the linear subspace $\mathcal{T}_\Phi(\mathcal{H})$ of $\mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ does not depend on the choice of Φ; i.e., given any pair of orthonormal bases $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}$ and $\Psi \equiv \{\psi_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$, we have that

$$\mathcal{T}_\Phi(\mathcal{H}) = \mathcal{T}_\Psi(\mathcal{H}). \tag{7.118}$$

To prove this important fact, we need to establish a further relevant property of $\mathcal{T}_\Phi(\mathcal{H})$. To this aim, we will use the following technical result:

Lemma 7.3.1. *Let $(A_{lm}), (T_{mn}) \in \mathsf{M}_\infty(\mathbb{Q}_{p,\mu})$ be any pair of infinite matrices satisfying the following conditions:*

(a) $\alpha \equiv \sup_{l,m}|A_{lm}| < \infty$*,*

(b) $\lim_l A_{lm} = 0$*, $\forall m \in \mathbb{N}$, and*

(c) $\lim_{m+n} T_{mn} = 0$*.*

Then, for every $(l, n) \in \mathbb{N}\times\mathbb{N}$, the series $\sum_m A_{lm}T_{mn}$ converges to some $S_{ln} \in \mathbb{Q}_{p,\mu}$ and $\lim_{l+n} S_{ln} = 0$.

Proof. By condition (a), the sequence $\{A_{lm}\}_{m\in\mathbb{N}}$ belongs to ℓ^∞, for every $l \in \mathbb{N}$, and, by condition (c), the sequence $\{T_{mn}\}_{m\in\mathbb{N}}$ belongs to c_0, for every $n \in \mathbb{N}$ (see relation (7.109) in Remark 7.3.2). Hence, the series $\sum_m A_{lm}T_{mn}$ is convergent, for all $l, n \in \mathbb{N}$, and we can put

$$S_{ln} = \sum_m A_{lm}T_{mn} \in \mathbb{Q}_{p,\mu}. \tag{7.119}$$

Moreover, condition (c) also entails that

(d) $\tau \equiv \sup_{m,n} |T_{mn}| < \infty$.

Let us assume that $\alpha, \tau > 0$ (otherwise there is nothing to prove), and let us take any $\epsilon > 0$.

Now, by (b), there exists some $\mathtt{L} \in \mathbb{N}$ such that, if $m \leq \mathtt{N}$ and $l > \mathtt{L}$, $|A_{lm}| < \epsilon/\tau$ (because we are considering a *finite* set $\{A_{l1}\}_{l\in\mathbb{N}}, \ldots, \{A_{l\mathtt{N}}\}_{l\in\mathbb{N}}$ of sequences converging to 0). Next, by (c), there is some $\mathtt{N} \in \mathbb{N}$ such that, if $\max\{m, n\} > \mathtt{N}$, then $|T_{mn}| < \epsilon/\alpha$.

Summarizing, we have found the following additional conditions:

(e) $l > \mathtt{L},\ m \leq \mathtt{N} \implies |A_{lm}| < \epsilon/\tau$,

(f) $\max\{m, n\} > \mathtt{N} \implies |T_{mn}| < \epsilon/\alpha$.

Therefore, eventually we obtain the following estimates:

(E1) By (a) and (f) — for all $l \in \mathbb{N}$ and all $n > \mathtt{N}$ — we have that

$$|S_{ln}| = |\textstyle\sum_m A_{lm}T_{mn}| \leq \sup_m |A_{lm}|\,|T_{mn}| < \alpha\,\frac{\epsilon}{\alpha} = \epsilon.$$

(E2) By (d) and (e), and by (a) and (f) — for all $l > \mathtt{L}$ and all $n \leq \mathtt{N}$ — we have:

$$|S_{ln}| \leq \sup_m |A_{lm}|\,|T_{mn}| = \max\left\{\max_{m\leq\mathtt{N}}\{|A_{lm}|\,|T_{mn}|\}, \sup_{m>\mathtt{N}} |A_{lm}|\,|T_{mn}|\right\} < \max\left\{\frac{\epsilon}{\tau}\,\tau,\ \alpha\,\frac{\epsilon}{\alpha}\right\} = \epsilon.$$

In conclusion, by the estimates (E1) and (E2),

$$l > \mathtt{L} \text{ and/or } n > \mathtt{N} \implies |S_{ln}| < \epsilon \tag{7.120}$$

(and, *a fortiori*, if $\max\{l, n\} > \mathtt{M} \equiv \max\{\mathtt{L}, \mathtt{N}\}$, then $|S_{ln}| < \epsilon$).

Eventually, we have shown that, for every $\epsilon > 0$, the set

$$\{(l, n) \in \mathbb{N}\times\mathbb{N} \mid |S_{ln}| > \epsilon\} \tag{7.121}$$

is finite. Equivalently, for every $\epsilon > 0$, there exists some $\mathtt{M} \in \mathbb{N}$ such that, if $\max\{l, n\} > \mathtt{M}$, then $|S_{ln}| < \epsilon$; namely, $\lim_{l+n} S_{ln} = 0$. □

Theorem 7.3.1. *Given any orthonormal basis* Φ*, the linear subspace* $\mathcal{T}_\Phi(\mathcal{H})$ *of* $\mathcal{B}_{\mathrm{ad}}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ *is a* left ideal *in* $\mathcal{B}(\mathcal{H})$*, i.e.,*

$$AT \in \mathcal{T}_\Phi(\mathcal{H}), \quad \forall A \in \mathcal{B}(\mathcal{H}), \forall T \in \mathcal{T}_\Phi(\mathcal{H}). \tag{7.122}$$

Moreover, $\mathcal{T}_\Phi(\mathcal{H})$ *is a* two sided $*$-ideal *in* $\mathcal{B}_{\mathrm{ad}}(\mathcal{H})$*, i.e.,*

$$T^* \in \mathcal{T}_\Phi(\mathcal{H}), \quad AT,\, TA \in \mathcal{T}_\Phi(\mathcal{H}),\ \ \forall T \in \mathcal{T}_\Phi(\mathcal{H}),\ \ \forall A \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H}). \tag{7.123}$$

Proof. Let us prove property (7.122) of $\mathcal{T}_\Phi(\mathcal{H})$. Since $A \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{T}_\Phi(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$, we have that $S = AT \in \mathcal{B}(\mathcal{H})$ and

$$A = \mathrm{op}_\Phi(A_{lm}),\, T = \mathrm{op}_\Phi(T_{mn}) \quad \Longrightarrow \quad S = \mathrm{op}_\Phi(S_{ln}), \tag{7.124}$$

where $S_{ln} = \sum_m A_{lm} T_{mn}$. Here, the infinite matrix (A_{lm}) satisfies conditions (a) and (b) in Lemma 7.3.1, because A is bounded (Theorem 7.1.1). Moreover, (T_{mn}) satisfies condition (c) (by the definition of $\mathcal{T}_\Phi(\mathcal{H})$). Hence, by the same lemma, $\lim_{l+n} S_{ln} = 0$; i.e., $S = AT \in \mathcal{T}_\Phi(\mathcal{H})$.

Let us now prove that $\mathcal{T}_\Phi(\mathcal{H})$ satisfies properties (7.123), as well. We have already shown that, if $T \in \mathcal{T}_\Phi(\mathcal{H})$, then $T^* \in \mathcal{T}_\Phi(\mathcal{H})$ too (Corollary 7.3.1).

Next, if, additionally, $A \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$, then $AT, A^*T^* \in \mathcal{T}_\Phi(\mathcal{H})$, so that

$$TA = (A^*T^*)^* \in \mathcal{T}_\Phi(\mathcal{H}). \tag{7.125}$$

The proof is complete. □

We will now derive two remarkable consequences of Theorem 7.3.1; the most important one is the following:

Corollary 7.3.2. *For every pair of orthonormal bases $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}$ and $\Psi \equiv \{\psi_n\}_{n\in\mathbb{N}}$, we have that*

$$\mathcal{T}_\Phi(\mathcal{H}) = \mathcal{T}_\Psi(\mathcal{H}) \equiv \mathcal{T}(\mathcal{H}). \tag{7.126}$$

Proof. Let U be the unitary operator determined by condition (U2) in Theorem 7.2.3, i.e.,

$$U = \mathrm{op}_\Phi(\langle \phi_m, U\phi_n\rangle) = \mathrm{op}_\Phi(\langle \phi_m, \psi_n\rangle). \tag{7.127}$$

Recalling Proposition 7.3.3, we have:

$$\begin{aligned} T \in \mathcal{T}_\Phi(\mathcal{H}) &\iff U^*TU \in \mathcal{T}_\Phi(\mathcal{H}) \quad ((7.123) \text{ in Theorem 7.3.1}, \;\; U^* = U^{-1} \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})) \\ &\iff 0 = \lim_{m+n} \langle \phi_m, U^*TU\phi_n\rangle = \lim_{m+n} \langle \psi_m, T\psi_n\rangle \\ &\overset{\mathrm{def}}{\iff} T \in \mathcal{T}_\Psi(\mathcal{H}). \end{aligned} \tag{7.128}$$

Therefore, $\mathcal{T}_\Psi(\mathcal{H}) = \mathcal{T}_\Phi(\mathcal{H})$, for any pair of orthonormal bases Φ, Ψ in $\mathcal{H}$. □

Definition 7.3.3. We call an operator belonging to the two-sided $*$-ideal $\mathcal{T}(\mathcal{H})$ of $\mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ — whose definition does not depend on the choice of an orthonormal basis in $\mathcal{H}$ (by Corollary 7.3.2) — a *trace class* operator. The linear space $\mathcal{T}(\mathcal{H})$ itself will be called the *the trace class of* $\mathcal{H}$.

We next obtain a second remarkable consequence of Theorem 7.3.1:

Corollary 7.3.3. *Given a linear operator T in $\mathcal{H}$, the following facts are equivalent:*

(i) $T \in \mathcal{T}(\mathcal{H})$*;*

(ii) $T \in \mathcal{B}(\mathcal{H})$ *and, for* some *pair $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}, \Psi \equiv \{\psi_n\}_{n\in\mathbb{N}}$ of orthonormal bases, satisfies the condition that*

$$\lim_{m+n} \langle \phi_m, T\psi_n\rangle = 0; \tag{7.129}$$

(iii) $T \in \mathcal{B}(\mathcal{H})$ *and, for* every *pair $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}, \Psi \equiv \{\psi_n\}_{n\in\mathbb{N}}$ of orthonormal bases in $\mathcal{H}$, satisfies condition* (7.129)*.*

Proof. Clearly, (iii) $\implies$ (ii). Let us prove that (ii) $\implies$ (i).

Assume that (ii) holds, and let U be the unitary operator determined by

$$U\phi_k = \psi_k,\ \forall k \in \mathbb{N};\ \text{i.e.,}\ U = \mathrm{op}_\Phi(\langle \phi_m, \psi_n \rangle). \tag{7.130}$$

By (7.129) we have:

$$0 = \lim_{m+n} \langle \phi_m, T\psi_n \rangle = \lim_{m+n} \langle \phi_m, TU\phi_n \rangle. \tag{7.131}$$

Therefore, $TU \in \mathcal{T}(\mathcal{H})$ and, by Theorem 7.3.1, $T = (TU)U^* \in \mathcal{T}(\mathcal{H})$ too; i.e., (ii) $\implies$ (i).

It is then sufficient to show that (i) $\implies$ (iii), as well. Let $T \in \mathcal{T}(\mathcal{H})$ and let $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}$, $\Psi \equiv \{\psi_n\}_{n\in\mathbb{N}}$ be *any* pair of orthonormal bases in $\mathcal{H}$. We have:

$$\lim_{m+n} \langle \phi_m, T\psi_n \rangle = \lim_{m+n} \langle \phi_m, TU\phi_n \rangle = 0. \tag{7.132}$$

Here, U is the unitary operator determined by (7.130), and we have used the fact that $TU \in \mathcal{T}(\mathcal{H})$ (Theorem 7.3.1). □

Remark 7.3.3. Recalling Remark 7.3.2, the condition that $T \in \mathcal{B}(\mathcal{H})$ satisfies (7.129) is equivalent to the condition that the series

$$\sum_{m,n} \langle \phi_m, T\psi_n \rangle \tag{7.133}$$

be convergent. This is reminiscent of the fact that, in a (infinite-dimensional, separable) *complex Hilbert space* $\mathcal{K}$,

$$T \in \mathcal{T}(\mathcal{K}) \iff \sum_m \sum_n |\langle \eta_m, T\chi_n \rangle| < \infty, \tag{7.134}$$

for some — equivalently, for any — pair $\{\eta_m\}_{m\in\mathbb{N}}, \{\chi_n\}_{n\in\mathbb{N}}$ of orthonormal bases in $\mathcal{K}$. This is probably the tightest connection that one can establish between the p-adic and the complex trace class. Recall indeed that, for a complex Hilbert space $\mathcal{K}$, one usually first defines the trace of a *positive* bounded operator (that may be finite or infinite). Then, the trace class $\mathcal{T}(\mathcal{K})$ is introduced as the set of all bounded operators T such that their absolute value $|T|$ (the unique positive square root of T^*T) has a *finite* trace; see, e.g., [135]. But this route cannot be pursued in the p-adic setting, because there is no natural notion of positivity for a bounded operator.

Having shown that the definition of the trace class $\mathcal{T}(\mathcal{H})$ does not depend on the choice of an orthonormal basis in $\mathcal{H}$, we now want to prove that, for every $T = \mathrm{op}_\Phi(T_{mn}) \in \mathcal{T}(\mathcal{H})$, the *trace* itself of T — i.e., the quantity (recall Proposition 7.3.2)

$$\mathrm{tr}_\Phi(T) := \sum_m \langle \phi_m, T\phi_m \rangle = \sum_m T_{mm} \in \mathbb{Q}_{p,\mu} \tag{7.135}$$

— *does not* depend on the orthonormal basis $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}$, so that we can call it the *trace of the operator* $T \in \mathcal{T}(\mathcal{H})$.

We first need to establish a technical fact.

Lemma 7.3.2. *Given double sequences* $\{x_{mn}\}_{m,n\in\mathbb{N}}$, $\{y_{mn}\}_{m,n\in\mathbb{N}}$ *in* $\mathbb{Q}_{p,\mu}$, *the following facts hold true:*

(i) $\lim_{m+n} x_{mn} = 0$ *and* $|y_{mn}| \leq \alpha \in \mathbb{R}^+$, $\forall m, n \in \mathbb{N} \implies \lim_{m+n} x_{mn}y_{mn} = 0$.

(ii) *If* $\{x_{mn}\}_{m,n\in\mathbb{N}}$ *is of the form* $x_{mn} = y_m z_n$, *where* $y_m \to 0$ *and* $z_n \to 0$, *then*

$$\lim_{m+n} x_{mn} = 0. \tag{7.136}$$

Proof. Claim (i) is obvious. Let us prove (ii).

Both the sequences $\{y_m\}_{m\in\mathbb{N}}, \{z_n\}_{n\in\mathbb{N}}$ converge to zero; hence:

$$\lim_m x_{mn} = \lim_m y_m z_n = 0, \text{ for all } n \in \mathbb{N}, \text{ and } \lim_n y_m z_n = 0, \text{ for all } m \in \mathbb{N}. \tag{7.137}$$

Moreover,

$$\lim_{m,n} x_{mn} = 0, \qquad \text{(Pringsheim limit)} \tag{7.138}$$

because, $\forall \epsilon > 0$, $\exists \mathtt{N} \in \mathbb{N}$ such that, if $m, n > \mathtt{N}$, then

$$|y_m|, |z_n| < \sqrt{\epsilon}, \tag{7.139}$$

so that $|x_{mn}| = |y_m|\,|z_n| < \epsilon$. As recalled in Remark 7.3.2, conditions (7.137) and (7.138) together entail that

$$\lim_{m+n} x_{mn} = 0, \tag{7.140}$$

which proves (ii). □

Theorem 7.3.2. *If $T \in \mathcal{T}(\mathcal{H})$, then, for any pair of orthonormal bases $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}$ and $\Psi \equiv \{\psi_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$, we have that*

$$\mathrm{tr}_\Phi(T) = \mathrm{tr}_\Psi(T) \equiv \mathrm{tr}(T). \tag{7.141}$$

Proof. Indeed, first note that

$$\mathrm{tr}_\Phi(T) := \sum_m \langle \phi_m, T\phi_m\rangle = \sum_m \sum_n \langle \phi_m, \psi_n\rangle\langle \psi_n, T\phi_m\rangle, \tag{7.142}$$

where we have used the expansion $\phi_m = \sum_n \langle \psi_n, \phi_m\rangle \psi_n$ and the continuity of the inner product. Since $|\langle \phi_m, \psi_n\rangle| \le 1$, for all $m, n \in \mathbb{N}$, then, by the implication (i) $\Longrightarrow$ (ii) in Corollary 7.3.3, and by claim (i) of Lemma 7.3.2, we can argue that

$$\begin{aligned} T \in \mathcal{T}(\mathcal{H}) \quad &\Longrightarrow \quad \lim_{m+n} \langle \psi_n, T\phi_m\rangle = 0 \qquad \text{(Corollary 7.3.3)} \\ &\Longrightarrow \quad \lim_{m+n} \langle \phi_m, \psi_n\rangle\langle \psi_n, T\phi_m\rangle = 0 \qquad \text{(Lemma 7.3.2).} \end{aligned}$$

Thus, as recalled in Remark 7.3.2 (see (7.108)), we can exchange the sums on the r.h.s. of (7.142), so obtaining

$$\begin{aligned} \mathrm{tr}_\Phi(T) &= \sum_n \sum_m \langle \phi_m, \psi_n\rangle\langle \psi_n, T\phi_m\rangle \\ &= \sum_n \sum_m \langle \phi_m, \psi_n\rangle\langle \psi_n, T(\textstyle\sum_k \langle \psi_k, \phi_m\rangle \psi_k)\rangle \\ &= \sum_n \sum_m \sum_k \langle \psi_k, \phi_m\rangle\langle \phi_m, \psi_n\rangle\langle \psi_n, T\psi_k\rangle, \end{aligned} \tag{7.143}$$

where, for the second equality, we have used the fact that $T\sum_k \langle \psi_k, \phi_m\rangle \psi_k = \sum_k \langle \psi_k, \phi_m\rangle (T\psi_k)$ (T being bounded) and, once again, the continuity of the inner product.

Next, since $|\langle \psi_k, \phi_m\rangle| \le 1$, for all $k, m \in \mathbb{N}$, $\lim_m \langle \phi_m, \psi_n\rangle = 0$, for all $n \in \mathbb{N}$, and

$$\lim_k \langle \psi_n, T\psi_k\rangle = \lim_k \langle T^*\psi_n, \psi_k\rangle = 0, \quad \forall n \in \mathbb{N}, \tag{7.144}$$

T being (of trace class, hence) adjointable, we have:

$$\lim_{m+k} \langle \phi_m, \psi_n\rangle\langle \psi_n, T\psi_k\rangle = 0, \quad \forall n \in \mathbb{N}, \qquad \text{(by point (ii) of Lemma 7.3.2)}$$

$$\Longrightarrow \lim_{m+k} \langle \psi_k, \phi_m\rangle\langle \phi_m, \psi_n\rangle\langle \psi_n, T\psi_k\rangle = 0, \quad \forall n \in \mathbb{N}. \qquad \text{(by point (i) of Lemma 7.3.2)}$$

Therefore, it is further possible to exchange the sums over m and k in the last line of (7.143).

Eventually, we obtain that

$$\begin{aligned} \mathrm{tr}_\Phi(T) &= \sum_n \sum_k \left(\sum_m \langle \psi_k, \phi_m\rangle\langle \phi_m, \psi_n\rangle \right) \langle \psi_n, T\psi_k\rangle \\ &= \sum_n \sum_k \langle \psi_k, \psi_n\rangle\langle \psi_n, T\psi_k\rangle \\ &= \sum_n \sum_k \delta_{nk}\langle \psi_n, T\psi_k\rangle \\ &= \sum_n \langle \psi_n, T\psi_n\rangle =: \mathrm{tr}_\Psi(T). \end{aligned} \tag{7.145}$$

Here, for obtaining the second equality we have exploited the (very familiar, in a complex Hilbert space) relation $\sum_m \langle \psi_k, \phi_m\rangle\langle \phi_m, \psi_n\rangle = \langle \psi_k, \psi_n\rangle$, which, in the p-adic setting, is provided by the last line of (6.36). □

The trace $\mathrm{tr}\colon \mathcal{T}(\mathcal{H}) \to \mathbb{Q}_{p,\mu}$ enjoys the following remarkable properties:

Proposition 7.3.4. *Given two trace class operators $S, T \in \mathcal{T}(\mathcal{H})$, we have:*

(P1) $\mathrm{tr}(S+T) = \mathrm{tr}(S) + \mathrm{tr}(T)$, *and* $\mathrm{tr}(\alpha T) = \alpha\, \mathrm{tr}(T)$, *for all $\alpha \in \mathbb{Q}_{p,\mu}$, i.e., the trace is a linear map;*

(P2) $\mathrm{tr}(T^*) = \overline{\mathrm{tr}(T)}$*;*

(P3) *for every unitary operator $U \in \mathcal{U}(\mathcal{H})$,* $\mathrm{tr}(UTU^*) = \mathrm{tr}(T)$.

Proof. The linearity of the trace is clear. Let us prove property (P2).

Let $\Phi \equiv \{\phi_n\}_{n\in\mathbb{N}}$ be any orthonormal basis in $\mathcal{H}$. Then, we have:

$$\mathrm{tr}(T^*) = \sum_n \langle \phi_n, T^*\phi_n\rangle = \sum_n \langle T\phi_n, \phi_n\rangle = \sum_n \overline{\langle \phi_n, T\phi_n\rangle} = \overline{\sum_n \langle \phi_n, T\phi_n\rangle} = \overline{\mathrm{tr}(T)}. \tag{7.146}$$

Moreover, for every unitary operator $U \in \mathcal{U}(\mathcal{H})$,

$$\mathrm{tr}(UTU^*) = \sum_n \langle \phi_n, UTU^*\phi_n\rangle = \sum_n \langle \psi_n, T\psi_n\rangle = \mathrm{tr}(T), \tag{7.147}$$

where we have used the fact that U^* is a unitary operator too, so that, by point (U8) of Theorem 7.2.3, $\Psi \equiv \{\psi_n\}_{n\in\mathbb{N}} = \{U^*\phi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis. □

7.3.3 The cyclic property

The reader will have noticed that in Proposition 7.3.4 are listed all the main properties of the trace — say, in a complex Hilbert space — *except* the 'cyclic property'. We are now going to show that the p-adic trace possesses this important property too, *provided that* the domain of the map $\mathrm{tr}(\cdot)$ be suitably extended (for the sake of simplicity, we will denote the extended map by the same symbol).

In fact, recalling the first assertion of Theorem 7.3.1, $\mathcal{T}(\mathcal{H})$ is a *left* — but not a right — ideal in $\mathcal{B}(\mathcal{H})$ ($\dim(\mathcal{H}) = \infty$). Let us better clarify this point by means of an explicit example.

Example 7.3.1. Let $B \in \mathcal{B}(\mathcal{H})$ be a bounded operator that is *not* adjointable, and let $\chi \in \mathcal{H}$ be a (nonzero) vector such that $\chi \notin \mathrm{dom}(B^\dagger)$, where $B^\dagger$ is the pseudo-adjoint of B. For every $\phi \in \mathcal{H}$, such that $\langle \phi, \phi \rangle = 1$ (e.g., an element of an orthonormal basis), we can consider the trace class operator $T = |\phi\rangle\langle\chi| \in \mathcal{T}(\mathcal{H})$. Let us show that the bounded operator $TB \in \mathcal{B}(\mathcal{H})$ is *not* a trace class operator. Indeed, for every $\psi \in \mathcal{H}$, we have: $\langle \phi, TB\psi \rangle = \langle \chi, B\psi \rangle$. Now, since $\chi \notin \mathrm{dom}(B^\dagger)$, *there is no vector* $\eta \in \mathcal{H}$ such that $\langle \eta, \psi \rangle = \langle \chi, B\psi \rangle = \langle \phi, TB\psi \rangle$, for all $\psi \in \mathcal{H}$; otherwise stated, $\phi \notin \mathrm{dom}((TB)^\dagger)$. Therefore, $TB \notin \mathcal{B}_{\mathrm{ad}}(\mathcal{H}) \supset \mathcal{T}(\mathcal{H})$, and this fact entails that $\mathcal{T}(\mathcal{H})$ is *not* a right ideal in $\mathcal{B}(\mathcal{H})$.

By the previous discussion, in order to derive, in the p-adic setting, the cyclic property of the trace, we need to introduce a new class of operators that we will call the *weak trace class*.

Definition 7.3.4. We say that a bounded operator $A \in \mathcal{B}(\mathcal{H})$ is *uniformly traceable* if it is traceable with respect to every orthonormal basis in $\mathcal{H}$ and, moreover,

$$\mathrm{tr}_\Phi(A) = \mathrm{tr}_\Psi(A) \equiv \mathrm{tr}(A) \tag{7.148}$$

for every pair of orthonormal bases $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}$ and $\Psi \equiv \{\psi_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$.

It is clear that the set of all uniformly traceable operators in $\mathcal{H}$ form a linear subspace $\mathcal{T}_{\mathrm{w}}(\mathcal{H})$ of $\mathcal{B}(\mathcal{H})$, which is precisely the weak trace class of $\mathcal{H}$.

Remark 7.3.4. Let $\mathcal{K}$ be a (separable) complex Hilbert space, with $\dim(\mathcal{K}) = \infty$. It is well known that a bonded operator $A \in \mathcal{B}(\mathcal{K})$ is of trace class if and only if it is uniformly traceable; i.e., if and only if the series

$$\sum_m \langle \chi_m, A\chi_m \rangle \tag{7.149}$$

converges to a *unique* limit for every orthonormal basis $\{\chi_m\}_{m\in\mathbb{N}}$ in $\mathcal{K}$ (see, e.g., Proposition 4.42 of [137]). It turns out that — see Remark 7.4.4 below — this property does not hold true for a p-adic Hilbert space; namely, $\mathcal{T}_{\mathrm{w}}(\mathcal{H}) \supsetneq \mathcal{T}(\mathcal{H})$ ($\dim(\mathcal{H}) = \infty$).

Proposition 7.3.5 (Cyclic property of the trace)**.** *For every bounded operator $B \in \mathcal{B}(\mathcal{H})$ and for every trace class operator $T \in \mathcal{T}(\mathcal{H})$, we have that*

$$BT \in \mathcal{T}(\mathcal{H}) \subset \mathcal{T}_{\mathrm{w}}(\mathcal{H}) \quad \text{and} \quad TB \in \mathcal{T}_{\mathrm{w}}(\mathcal{H}). \tag{7.150}$$

Moreover, we have:

$$\mathrm{tr}(BT) = \mathrm{tr}(TB). \tag{7.151}$$

Proof. We have already shown that $BT \in \mathcal{T}(\mathcal{H})$, because $\mathcal{T}(\mathcal{H})$ is a left ideal in $\mathcal{B}(\mathcal{H})$ (see Theorem 7.3.1, where $\mathcal{T}_\Phi(\mathcal{H}) \equiv \mathcal{T}(\mathcal{H})$). Then, we have:

$$\begin{aligned}\mathrm{tr}(BT) &= \sum_m \langle \phi_m, BT\phi_m \rangle \\ &= \sum_m \langle \phi_m, B(\textstyle\sum_n \langle \psi_n, T\phi_m \rangle\, \psi_n) \rangle \qquad (\text{because } T\phi_m = \textstyle\sum_n \langle \psi_n, T\phi_m \rangle \psi_n) \\ &= \sum_m \sum_n \langle \phi_m, B\psi_n \rangle \langle \psi_n, T\phi_m \rangle, \qquad (\text{continuity of } B \text{ and of the inner product})\end{aligned} \tag{7.152}$$

where $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}$, $\Psi \equiv \{\psi_n\}_{n\in\mathbb{N}}$ is any pair of orthonormal bases in $\mathcal{H}$.

Note that

$$|\langle \phi_m, B\psi_n \rangle| \le \|B\|, \ \forall m, n \in \mathbb{N}, \text{ and } \lim_{m+n} \langle \psi_n, T\phi_m \rangle = 0. \tag{7.153}$$

Hence, by point (i) of Lemma 7.3.2, we have:

$$\lim_{m+n} \langle \phi_m, B\psi_n\rangle \langle \psi_n, T\phi_m\rangle = 0. \tag{7.154}$$

By (7.154), we can exchange the sums in the last line of (7.152), so obtaining

$$\begin{aligned} \mathrm{tr}(BT) &= \sum_n \sum_m \langle \psi_n, T\phi_m\rangle \langle \phi_m, B\psi_n\rangle \\ &= \sum_n \langle \psi_n, T(\textstyle\sum_m \langle \phi_m, B\psi_n\rangle \phi_m)\rangle \\ &= \sum_n \langle \psi_n, TB\psi_n\rangle = \mathrm{tr}_\Psi(TB). \end{aligned} \tag{7.155}$$

By the arbitrariness of the orthonormal basis Ψ in $\mathcal{H}$, we conclude that $TB \in \mathcal{T}_{\mathrm{w}}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ and $\mathrm{tr}(TB) = \mathrm{tr}_\Psi(TB) = \mathrm{tr}(BT)$. □

Remark 7.3.5. The inclusion relation

$$TB \subset \mathcal{T}_{\mathrm{w}}(\mathcal{H}), \quad \forall\, T \in \mathcal{T}(\mathcal{H}),\ \forall B \in \mathcal{B}(\mathcal{H}), \tag{7.156}$$

is a manifestation of the fact that TB is a *compact operator*; see Corollary 7.4.2 below. In particular, the (non-adjointable, bounded) operator TB constructed in Example 7.3.1 is compact.

Proposition 7.3.6. *For every bounded operator $B \in \mathcal{B}(\mathcal{H})$ and for every trace class operator $T \in \mathcal{T}(\mathcal{H})$, we have that*

$$|\mathrm{tr}(BT)| = |\mathrm{tr}(TB)| \le \|B\|\,\|T\| \quad \textit{and} \quad |\mathrm{tr}(T)| \le \|T\|. \tag{7.157}$$

Proof. In fact, given any orthonormal basis $\{\phi_m\}_{m\in\mathbb{N}}$ in $\mathcal{H}$, $|\mathrm{tr}(BT)| = |\sum_m \langle \phi_m, BT\phi_m\rangle| \le \max_{m\in\mathbb{N}} |\langle \phi_m, BT\phi_m\rangle| \le \max_{m\in\mathbb{N}} \|BT\phi_m\| \le \|B\|\,\|T\|$. In particular, putting $B = \mathrm{Id}$, we obtain also the second inequality in (7.157). □

Corollary 7.3.4. *The map $\mathrm{tr}(\cdot)\colon \mathcal{T}(\mathcal{H}) \ni T \mapsto \mathrm{tr}(T) \in \mathbb{Q}_{p,\mu}$ is a bounded linear functional and $\|\mathrm{tr}(\cdot)\| = 1$.*

Proof. By the second inequality in (7.157) the functional $\mathrm{tr}(\cdot)$ is bounded and $\|\mathrm{tr}(\cdot)\| \le 1$. If ϕ is an element of an orthonormal basis in $\mathcal{H}$, then $|\mathrm{tr}(|\phi\rangle\langle\phi|)| = 1 = \|\,|\phi\rangle\langle\phi|\,\|$, so that the previous inequality is saturated. □

7.4 Trace class operators as compact operators

As is well known, the trace class operators in a (infinite-dimensional, separable) *complex* Hilbert space $\mathcal{K}$ form a Banach space, when endowed with the trace norm. This space is embedded in the Hilbert space of all Hilbert-Schmidt operators in $\mathcal{K}$ (endowed with the Hilbert-Schmidt product). The closure — w.r.t. the operator norm — of these spaces coincides with the closure of the linear space of all finite rank operators in $\mathcal{K}$, namely, with the Banach space of compact operators, which is the only proper closed two-sided ideal in the Banach algebra of bounded operators. In particular, the trace class of $\mathcal{K}$ is *not* closed w.r.t. the operator norm ($\dim(\mathcal{K}) = \infty$). See, e.g., the standard references [135–137, 140].

As the reader may expect, this familiar picture keeps some of its main features — but also requires some essential modification — when switching to a (infinite-dimensional) p-adic Hilbert space $\mathcal{H}$.

As above, for the sake of simplicity, we will assume that $\dim(\mathcal{H}) = \infty$, *but* all subsequent results (and their proofs) remain valid — with obvious adaptations, and even if possibly getting trivial — in the finite-dimensional setting. E.g., the 'p-adic singular value decomposition' — see Corollary 7.4.3 below — holds true in the case where $\dim(\mathcal{H}) < \infty$ and $\mathcal{T}(\mathcal{H}) = \mathcal{B}_{\mathrm{ad}}(\mathcal{H}) = \mathcal{B}(\mathcal{H})$ is just the space $\mathrm{Lin}(\mathcal{H})$ of all linear operators in $\mathcal{H}$.

Definition 7.4.1. An all-over linear operator C in $\mathcal{H}$ is said to be *compact* if $C\,\mathcal{H}_1$ — where $\mathcal{H}_1$ is the unit ball in $\mathcal{H}$: $\mathcal{H}_1 := \{\psi \in \mathcal{H} \mid \|\psi\| \leq 1\}$ — is a precompact subset of $\mathcal{H}$ (namely, if $C\,\mathcal{H}_1$ has a compact closure).

Remark 7.4.1. In formulating the previous definition, we have taken into account the fact that $\mathbb{Q}_{p,\mu}$ is locally compact, because, in this case, the *compactoid* subsets of $\mathcal{H}$ coincide with the precompact subsets. See Chapter 4 of [73]; in particular, Section 4.S and the subsequent definition of a compact operator in the non-Archimedean setting (also see the seminal paper [141]).

Remark 7.4.2. From Definition 7.4.1 it is clear that the linear space $\mathscr{F}(\mathcal{H})$ of all *finite-rank operators* — the linear operators in $\mathcal{H}$ having finite-dimensional range spaces — consists of compact operators.

Theorem 7.4.1. *The set $\mathcal{C}(\mathcal{H})$ of all compact operators in $\mathcal{H}$ is a closed linear subspace of $\mathcal{B}(\mathcal{H})$. Specifically, $\mathcal{C}(\mathcal{H})$ is the closure of the linear subspace $\mathscr{F}(\mathcal{H})$ of all finite rank operators in $\mathcal{H}$. Moreover, $\mathcal{C}(\mathcal{H})$ is the only proper closed two-sided ideal in $\mathcal{B}(\mathcal{H})$.*

Given any orthonormal basis $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}$ in $\mathcal{H}$, a bounded operator $A = \mathrm{op}_\Phi(A_{mn}) \in \mathcal{B}(\mathcal{H})$ — $A_{mn} = \langle \phi_m, A\phi_n\rangle$ — is compact if and only if

$$\lim_m \left(\sup_{n\in\mathbb{N}} |A_{mn}|\right) = 0. \tag{7.158}$$

Every compact operator $C \in \mathcal{C}(\mathcal{H})$ can be expressed as

$$C = \sum_{j\in J} \lambda_j\, e_j \odot \widetilde{e}_j, \tag{7.159}$$

where $J = \{1, 2, \ldots\}$ is a countable index set and

- *$\{\lambda_j\}_{j\in J} \subset \mathbb{Q}_{p,\mu}$ — for $C \neq 0$, we assume that $\{\lambda_j\}_{j\in J} \subset \mathbb{Q}^*_{p,\mu} \equiv \mathbb{Q}_{p,\mu} \setminus \{0\}$ — and, if $J = \mathbb{N}$, $\lim_j \lambda_j = 0$;*
- *$\{e_j\}_{j\in J}$ and $\{\widetilde{e}_j\}_{j\in J}$ are contained in $\mathcal{H}$ and $\mathcal{H}'$, respectively, with $\|e_j\| = \|\widetilde{e}_j\| = 1$;*
- *$\{e_j\}_{j\in J}$ is a (normalized) norm-orthogonal system in $\mathcal{H}$;*
- *$e_j \odot \widetilde{e}_j \colon \mathcal{H} \to \mathcal{H}$ is the bounded operator defined by $(e_j \odot \widetilde{e}_j)\,\psi := \widetilde{e}_j(\psi)\, e_j$, and the sum in (7.159) — whenever J is not finite — converges w.r.t. the norm topology.*

In particular, the norm-orthogonal system $\{e_j\}_{j\in J}$ can be chosen to be contained in an orthonormal basis in $\mathcal{H}$.

Conversely, every operator C of the previous form — i.e., such that $C\psi = \sum_{j\in J} \lambda_j\, \widetilde{e}_j(\psi)\, e_j$, for all $\psi \in \mathcal{H}$, with $\{\lambda_j\}_{j\in J}$, $\{e_j\}_{j\in J}$ and $\{\widetilde{e}_j\}_{j\in J}$ as specified above — is compact.

Proof. For the first two assertions, see Chapter 4 of [73]; in particular, Theorem 4.39 and the subsequent discussion. For the third assertion, since the valuation group $|\mathbb{Q}^*_{p,\mu}|$ is discrete, we can apply Theorem 5 and Corollary 6 of [142].

Let us prove the fourth assertion. Let $A = \mathrm{op}_\Phi(A_{mn})$ be a bounded operator. By the first series expansion in (7.13), for every vector $\psi \in \mathcal{H}$, we have:

$$A\psi = \sum_m \big(\sum_n A_{mn}\langle\phi_n, \psi\rangle\big)\phi_m, \quad \text{where } \xi^{(m)} \equiv \{A_{mn}\}_{n\in\mathbb{N}} \in \ell^\infty, \text{ for all } n \in \mathbb{N}. \tag{7.160}$$

Therefore, putting

$$\widetilde{f}_m = \mathcal{L}_\Phi(\xi^{(m)}) = \sum_{n\in\mathbb{N}} A_{mn}\langle\phi_n, \cdot\rangle \tag{7.161}$$

— where $\mathcal{L}_\Phi \colon \ell^\infty \to \mathcal{H}'$ is the surjective isometry defined by (6.70), and convergence of the series w.r.t. the weak*-topology is understood — we obtain that, for every $\psi \in \mathcal{H}$, $A\psi = \sum_m \widetilde{f}_m(\psi)\phi_m$, with $\|\widetilde{f}_m\| = \|\xi^{(m)}\|_\infty = \sup_{n\in\mathbb{N}} |A_{mn}|$. Moreover, by the corollary after Proposition 4 of [141], we conclude that A is compact if and only if $0 = \lim_m \|\widetilde{f}_m\| = \lim_m \big(\sup_{n\in\mathbb{N}}|A_{mn}|\big)$.

Decomposition (7.159) of a compact operator is essentially the equivalence of points (α) and (ε) in Theorem 4.40 of [73], but with some improvement that requires a suitable modification of the proof therein. We outline the modified proof.

Let C be a compact operator in $\mathcal{H}$, and let us assume that $C \neq 0$ (otherwise, there is nothing to prove). By the preceding part of the proof, we argue that C can be expressed in the form

$$C\psi = \sum_{j\in J} \widetilde{f}_j(\psi)\, e_j, \quad \forall\psi \in \mathcal{H}, \tag{7.162}$$

where $J = \{1, 2, \ldots\}$ is a countable index set, $\widetilde{f}_j \colon \mathcal{H} \to \mathbb{Q}_{p,\mu}$, $j \in J$, is a *nonzero* bounded linear functional — if $J = \mathbb{N}$, such that $\lim_j \|\widetilde{f}_j\| = 0$ — and $\{e_j\}_{j\in J}$ is a normalized norm-orthogonal system (in particular, it can be chosen to be contained in an orthonormal basis). Taking into account that $\|\mathcal{H}\| = |\mathbb{Q}_{p,\mu}|$, there is a subset $\{\lambda_j\}_{j\in J}$ of $\mathbb{Q}_{p,\mu}$ such that $0 < |\lambda_j| = \|\widetilde{f}_j\|$. It is then sufficient to put

$$\widetilde{e}_j := \frac{1}{\lambda_j}\widetilde{f}_j \tag{7.163}$$

— where $\{\widetilde{e}_j\}_{j\in J}$ is a set of normalized functionals in $\mathcal{H}'$ and, if $J = \mathbb{N}$, $\lim_j \lambda_j = 0$ — to obtain decomposition (7.159) from (7.162).

We stress that, for $J = \mathbb{N}$, since $\|e_j \odot \widetilde{e}_j\| \leq 1$ — and, hence, $\lim_j |\lambda_j|\, \|e_j \odot \widetilde{e}_j\| = 0$ — the series in (7.159) converges not only w.r.t. the strong operator topology, but also w.r.t. the norm topology.

Conversely, every linear operator C of the form $C = \sum_{j\in J} \lambda_j\, e_j \odot \widetilde{e}_j$ — with $\{\lambda_j\}_{j\in J}$, $\{e_j\}_{j\in J}$ and $\{\widetilde{e}_j\}_{j\in J}$ as above (in particular, $\|e_j\| = \|\widetilde{e}_j\| = 1$ and, if $J = \mathbb{N}$, $\lim_j |\lambda_j| = 0$) — is compact, because it is the norm-limit of a sequence of finite rank operators. □

From Theorem 7.4.1 we derive three important consequences.

Corollary 7.4.1. *Given any orthonormal basis* $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}$*, a matrix operator* $A = \mathrm{op}_\Phi(A_{mn})$ *in* $\mathcal{H}$ *is compact if and only if*

`(C1)` $\sup_{m,n} |A_{mn}| < \infty$,

`(C2)` $\lim_m A_{mn} = 0$, $\forall n \in \mathbb{N}$,

`(C3)` $\lim_{m,n} A_{mn} = 0$ *(Pringsheim limit).*

Proof. If $A = \mathrm{op}_\Phi(A_{mn})$ is compact, then it is bounded, so that, by Theorem 7.1.1, it satisfies conditions `(C1)` and `(C2)`. Moreover, it must satisfy condition (7.158) in Theorem 7.4.1, as well. The latter condition is easily shown to be equivalent to the pair of conditions formed by `(C2)` (once again) and `(C3)`. Conversely, if $A = \mathrm{op}_\Phi(A_{mn})$ satisfies conditions `(C1)`–`(C3)`, then it is bounded and verifies condition (7.158), as well; hence, by the fourth assertion of Theorem 7.4.1, it is compact. □

Corollary 7.4.2. *Every compact operator C in $\mathcal{H}$ is uniformly traceable — i.e., $\mathcal{C}(\mathcal{H}) \subset \mathcal{T}_{\mathrm{w}}(\mathcal{H})$ — and, if $\{\lambda_j\}_{j\in J}$, $\{e_j\}_{j\in J}$ and $\{\widetilde{e}_j\}_{j\in J}$ are as in Theorem 7.4.1,*

$$C = \sum_{j\in J} \lambda_j \, e_j \odot \widetilde{e}_j \implies \mathrm{tr}(C) = \sum_{j\in J} \lambda_j \, \widetilde{e}_j(e_j). \tag{7.164}$$

Proof. For any orthonormal basis $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}$ in $\mathcal{H}$, we have

$$\lim_{m+j} \lambda_j \, \langle \phi_m, e_j\rangle \, \widetilde{e}_j(\phi_m) = 0, \tag{7.165}$$

because ($\lim_m \langle \phi_m, e_j\rangle = 0 \implies$) $\lim_m \lambda_j \, \langle \phi_m, e_j\rangle \, \widetilde{e}_j(\phi_m) = 0$, for all $j \in \mathbb{N}$, and, moreover, $\lim_j \lambda_j \, \langle \phi_m, e_j\rangle \, \widetilde{e}_j(\phi_m) = 0$ *uniformly* in $m \in \mathbb{N}$ ($|\lambda_j \, \langle \phi_m, e_j\rangle \, \widetilde{e}_j(\phi_m)| \leq |\lambda_j|$); see relation (7.110) in Remark 7.3.2. It follows that, if $J = \mathbb{N}$, the double series $\sum_{j,m\in\mathbb{N}} \lambda_j \, \langle \phi_m, e_j\rangle \, \widetilde{e}_j(\phi_m)$ is convergent and its sum coincides with the sum of both the iterated series (Remark 7.3.2); hence:

$$\begin{aligned} \sum_{j\in\mathbb{N}} \lambda_j \, \widetilde{e}_j(e_j) &= \sum_{j\in\mathbb{N}} \lambda_j \, \widetilde{e}_j \big(\textstyle\sum_{m\in\mathbb{N}} \langle \phi_m, e_j\rangle \, \phi_m\big) \\ &= \sum_{j\in\mathbb{N}} \sum_{m\in\mathbb{N}} \lambda_j \, \langle \phi_m, e_j\rangle \, \widetilde{e}_j(\phi_m) \\ &= \sum_{j,m\in\mathbb{N}} \lambda_j \, \langle \phi_m, e_j\rangle \, \widetilde{e}_j(\phi_m) = \sum_{m\in\mathbb{N}} \sum_{j\in\mathbb{N}} \lambda_j \, \langle \phi_m, e_j\rangle \, \widetilde{e}_j(\phi_m) = \mathrm{tr}_\Phi(C). \end{aligned} \tag{7.166}$$

Here, the second equality follows from the continuity of the functional $\widetilde{e}_j$, whereas for the last equality we have used the decomposition of C (converging w.r.t. the norm topology) and the continuity of the inner product. Clearly, in the case where $J = \{1, 2, \ldots\}$ is a finite subset of $\mathbb{N}$, one can freely exchange the finite sum with the series, so obtaining the same result.

Now, since the quantity $\sum_{j\in J} \lambda_j \, \widetilde{e}_j(e_j)$ does not depend on the choice of the orthonormal basis $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}$, it turns out that every compact operator C in $\mathcal{H}$ is uniformly traceable and relation (7.164) holds true. □

Corollary 7.4.3. *Every compact operator C in $\mathcal{H}$, belonging to the closed subspace*

$$\mathcal{C}_{\mathrm{ad}}(\mathcal{H}) := \mathcal{C}(\mathcal{H}) \cap \mathcal{B}_{\mathrm{ad}}(\mathcal{H}) \tag{7.167}$$

of $\mathcal{C}(\mathcal{H})$, can be expressed in the form

$$C = \sum_{j\in J} \lambda_j \, |e_j\rangle\langle f_j|, \tag{7.168}$$

where $J = \{1, 2, \ldots\}$ is a countable index set and

- *$\{\lambda_j\}_{j\in J} \subset \mathbb{Q}_{p,\mu}$ — for $C \neq 0$, we assume that $\{\lambda_j\}_{j\in J} \subset \mathbb{Q}^*_{p,\mu}$ — and, if $J = \mathbb{N}$, $\lim_j \lambda_j = 0$;*
- *both the sets $\{e_j\}_{j\in J}$ and $\{f_j\}_{j\in J}$ are contained in $\mathcal{H}$, with $\|e_j\| = \|f_j\| = 1$;*
- *$\{e_j\}_{j\in J}$ — or $\{f_j\}_{j\in J}$ — is a (normalized) norm-orthogonal system in $\mathcal{H}$;*
- *$|e_j\rangle\langle f_j| \colon \mathcal{H} \to \mathcal{H}$ is the bounded operator defined by $(|e_j\rangle\langle f_j|) \, \psi := \langle f_j, \psi\rangle \, e_j$, and the sum in (7.159) — whenever J is not finite — converges w.r.t. the norm topology.*

In particular, the norm-orthogonal set $\{e_j\}_{j\in J}$ — alternatively, the norm-orthogonal set $\{f_j\}_{j\in J}$ — can be chosen to be contained in an orthonormal basis in $\mathcal{H}$.

Conversely, every operator C of the previous form — i.e., such that $C\psi = \sum_{j\in J} \lambda_j \langle f_j, \psi\rangle e_j$, for all $\psi \in \mathcal{H}$, with $\{\lambda_j\}_{j\in J}$, $\{e_j\}_{j\in J}$ and $\{f_j\}_{j\in J}$ as specified above — belongs to $\mathcal{C}_{\mathrm{ad}}(\mathcal{H})$.

Finally, $\mathcal{C}_{\mathrm{ad}}(\mathcal{H})$ is a two-sided $*$-ideal in $\mathcal{B}_{\mathrm{ad}}(\mathcal{H})$.

Proof. Let C be an adjointable compact operator in $\mathcal{H}$. Then, by Theorem 7.4.1, we have that $C = \sum_{j\in J} \lambda_j\, e_j \odot \widetilde{e}_j$, with $\{\lambda_j\}_{j\in J}$, $\{e_j\}_{j\in J}$ and $\{\widetilde{e}_j\}_{j\in J}$ as specified therein.

It is easy to check that the generalized adjoint $C' \in \mathcal{B}(\mathcal{H}')$ is given by

$$C' = \sum_{j\in J} \lambda_j\, \widetilde{e}_j \odot (\mathcal{I}_{\mathcal{H}} e_j), \qquad (\mathcal{I}_{\mathcal{H}} e_j \in \mathcal{H}''). \tag{7.169}$$

Here, $\mathcal{I}_{\mathcal{H}}\colon \mathcal{H} \to \mathcal{H}''$ is the isometry defined by (6.71) and the operator $\widetilde{e}_j \odot (\mathcal{I}_{\mathcal{H}} e_j)\colon \mathcal{H}' \to \mathcal{H}'$ is of the form $(\widetilde{e}_j \odot (\mathcal{I}_{\mathcal{H}} e_j))\,\phi' = ((\mathcal{I}_{\mathcal{H}} e_j)(\phi'))\,\widetilde{e}_j = \phi'(e_j)\,\widetilde{e}_j$, for all $\phi' \in \mathcal{H}'$; moreover, for $J = \mathbb{N}$, the series converges in $\mathcal{B}(\mathcal{H}')$ w.r.t. the norm topology (because $\|\widetilde{e}_j \odot (\mathcal{I}_{\mathcal{H}} e_j)\| = 1$ and, hence, $\lim_j |\lambda_j|\, \|\widetilde{e}_j \odot (\mathcal{I}_{\mathcal{H}} e_j)\| = 0$). Since C is adjointable, by Corollary 7.1.2, C' is a dual Hahn-Banach extension of the proper adjoint C^* of C.

Let us assume that, in particular, the set $\{e_j\}_{j\in J}$ is contained in an orthonormal basis. It follows that, for every $k \in J$,

$$\big(C' \circ \mathcal{J}_{\mathcal{H}}\big)(e_k) = \sum_{j\in J} \lambda_j\, \langle e_k, e_j\rangle\, \widetilde{e}_j = \lambda_k\, \widetilde{e}_k \in \mathcal{J}_{\mathcal{H}}(\mathrm{ran}(C^*)) \subset \mathcal{J}_{\mathcal{H}}(\mathcal{H}). \tag{7.170}$$

(Recall that $\mathcal{J}_{\mathcal{H}}\colon \mathcal{H} \to \mathcal{H}'$ is the conjugate-linear isometry defined by (6.69) and the intertwining relation $A' \circ \mathcal{J}_{\mathcal{H}} = \mathcal{J}_{\mathcal{H}} \circ C^*$ holds.) Therefore, $\{\widetilde{e}_j\}_{j\in J} \subset \mathrm{ran}(\mathcal{J}_{\mathcal{H}})$.

Observe now that, defining $f_j \in \mathcal{H}$ by

$$\mathcal{J}_{\mathcal{H}} f_j = \widetilde{e}_j \in \mathrm{ran}(\mathcal{J}_{\mathcal{H}}), \quad \forall j \in J \tag{7.171}$$

— where the vectors $\{f_j\}_{j\in J} \subset \mathcal{H}$ are uniquely determined by the functionals $\{\widetilde{e}_j\}_{j\in J}$ and $\|f_j\| = \|\widetilde{e}_j\| = 1$, because $\mathcal{J}_{\mathcal{H}}$ is a (conjugate-linear) isometry — we can introduce the bounded operator

$$D = \sum_{j\in J} \overline{\lambda_j}\, |f_j\rangle\langle e_j|. \tag{7.172}$$

Here, if $J = \mathbb{N}$, the series converges w.r.t. the norm topology, and D is both compact and adjointable (being the norm-limit of a sequence of *adjointable* finite rank operators). Then, we have:

$$\big(C' \circ \mathcal{J}_{\mathcal{H}}\big)(\psi) = \sum_{j\in J} \lambda_j\, \langle \psi, e_j\rangle\, \widetilde{e}_j = \sum_{j\in J} \lambda_j\, \overline{\langle e_j, \psi\rangle}\, \mathcal{J}_{\mathcal{H}} f_j = \big(\mathcal{J}_{\mathcal{H}} \circ D\big)(\psi), \quad \forall \psi \in \mathcal{H}. \tag{7.173}$$

Hence, by the second assertion of Corollary 7.1.2, $\mathcal{C}_{\mathrm{ad}}(\mathcal{H}) \ni D = C^*$ and, using the continuity of the adjoining operation in $\mathcal{B}_{\mathrm{ad}}(\mathcal{H})$, we find that

$$C = D^* = \sum_{j\in J} \big(\overline{\lambda_j}\, |f_j\rangle\langle e_j|\big)^* = \sum_{j\in J} \lambda_j\, |e_j\rangle\langle f_j|, \tag{7.174}$$

where $\{\lambda_j\}_{j\in J}$, $\{e_j\}_{j\in J}$ are as specified in Theorem 7.4.1 — in particular, the norm-orthogonal set $\{e_j\}_{j\in J}$ can be chosen to be contained in an orthonormal basis — and $\{f_j\}_{j\in J}$ is a set of normalized vectors.

Observe also that, by the first part of the proof, if C is compact and adjointable, then its adjoint is (both adjointable and) compact; i.e., $C \in \mathcal{C}_{\mathrm{ad}}(\mathcal{H}) \implies C^* \in \mathcal{C}_{\mathrm{ad}}(\mathcal{H})$ (and, of course, $C^{**} = C$), so that

$$\mathcal{C}_{\mathrm{ad}}(\mathcal{H})^* = \{C^* \mid C \in \mathcal{C}_{\mathrm{ad}}(\mathcal{H})\} \subset \mathcal{C}_{\mathrm{ad}}(\mathcal{H}) \tag{7.175}$$

and

$$\mathcal{C}_{\mathrm{ad}}(\mathcal{H}) = \mathcal{C}_{\mathrm{ad}}(\mathcal{H})^{**} \subset \mathcal{C}_{\mathrm{ad}}(\mathcal{H})^{***} = \mathcal{C}_{\mathrm{ad}}(\mathcal{H})^*. \tag{7.176}$$

Hence, actually, $\mathcal{C}_{\mathrm{ad}}(\mathcal{H}) = \mathcal{C}_{\mathrm{ad}}(\mathcal{H})^*$.

Therefore, every operator $D \in \mathcal{C}_{\mathrm{ad}}(\mathcal{H})$ is of the form $D = C^*$, for some $C \in \mathcal{C}_{\mathrm{ad}}(\mathcal{H})$, and, again by the first part of the proof, we know that it can be written as $\sum_{j\in J} \gamma_j \, |f_j\rangle\langle e_j|$, where: $\{\gamma_j \equiv \overline{\lambda_j}\}_{j\in J} \subset \mathbb{Q}_{p,\mu}$ and, if $J = \mathbb{N}$, $\lim_j \gamma_j = 0$; $\|e_j\| = \|f_j\| = 1$; the norm-orthogonal system $\{e_j\}_{j\in J}$ can be chosen to be contained in an orthonormal basis in $\mathcal{H}$. Thus, we have an alternative option for the choice of the norm-orthogonal system in decomposition (7.168) (where the vectors $\{e_j\}_{j\in J}$ now play the role of functionals, i.e., $\{\langle e_j, \cdot\rangle\}_{j\in J}$).

The third assertion of the corollary is clear, since, if C is of the form (7.168), then it is both compact and adjointable (being the norm-limit of a sequence of adjointable finite rank operators). Finally, $\mathcal{C}_{\mathrm{ad}}(\mathcal{H}) = \mathcal{C}(\mathcal{H}) \cap \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ is a two-sided $*$-ideal in $\mathcal{B}_{\mathrm{ad}}(\mathcal{H})$, because $\mathcal{C}(\mathcal{H})$ is a two-sided ideal in $\mathcal{B}(\mathcal{H})$, $\mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ is an algebra (w.r.t. composition of operators) and, as previously argued, $\mathcal{C}_{\mathrm{ad}}(\mathcal{H}) = \mathcal{C}_{\mathrm{ad}}(\mathcal{H})^*$. □

Remark 7.4.3. The expressions (7.159) and (7.168) may be regarded as a p-adic counterpart of the *singular value decomposition* [135–137, 140] of a compact operator in a separable complex Hilbert space. Decompositions (7.159) and (7.168) are easily seen to be not unique.

Definition 7.4.2. Any (non-unique) decomposition of the form (7.159) (or of the form (7.168)) will be called *a canonical decomposition* of the compact operator $C \in \mathcal{C}(\mathcal{H})$ (respectively, of the adjointable compact operator $C \in \mathcal{C}_{\mathrm{ad}}(\mathcal{H})$); in particular, we will assume that $\{e_j\}_{j\in J}$ — or $\{f_j\}_{j\in J}$, in the case where $C \in \mathcal{C}_{\mathrm{ad}}(\mathcal{H})$ — is a (normalized) norm-orthogonal system in $\mathcal{H}$. In the case where $\{e_j\}_{j\in J}$ — alternatively, $\{f_j\}_{j\in J}$, for $C \in \mathcal{C}_{\mathrm{ad}}(\mathcal{H})$ — is chosen to be contained in an orthonormal basis, we will call *orthonormal* the associated canonical decomposition.

We will now show that the trace class operators in $\mathcal{H}$ form a suitable class of compact operators.

Definition 7.4.3. We say that a linear operator A in $\mathcal{H}$ is *block-finite* w.r.t. an orthonormal basis $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}$ if it is of the form $A = \mathrm{op}_\Phi(A_{mn})$, where, for some $k \in \mathbb{N}$,

$$\max\{m, n\} > k \implies A_{mn} = 0. \tag{7.177}$$

We say that A is *block-finite* (*tout court*) if it is block-finite w.r.t. some orthonormal basis in $\mathcal{H}$.

Clearly, all block-finite operators are of *finite rank* (hence, compact), and the set of all block-finite operators w.r.t. an orthonormal basis $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}$ is a linear subspace $\mathscr{B}_\Phi$ of $\mathcal{C}(\mathcal{H})$.

Theorem 7.4.2. *$\mathcal{T}(\mathcal{H})$ is a closed linear subspace of $\mathcal{C}(\mathcal{H})$. Specifically, $\mathcal{T}(\mathcal{H})$ is the closure of the linear subspace $\mathscr{B}_\Phi$ of $\mathcal{C}(\mathcal{H})$, for every orthonormal basis $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}$ in $\mathcal{H}$.*

Proof. Let us first show that, for every orthonormal basis $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}$ in $\mathcal{H}$, $\mathcal{T}(\mathcal{H}) \subset \overline{\mathscr{B}_\Phi}^{\|\cdot\|}$. In fact, given any trace class operator $T = \mathrm{op}_\Phi(T_{mn}) \in \mathcal{T}(\mathcal{H})$, we can define the matrix operator ${}^kT := \mathrm{op}_\Phi\big({}^kT_{mn}\big)$, $k \in \mathbb{N}$, where

$${}^kT_{mn} = \begin{cases} T_{mn}, \text{ if } \max\{m,n\} \le k, \\ 0, \text{ otherwise.} \end{cases} \tag{7.178}$$

Clearly, ${}^kT \in \mathscr{B}_\Phi$ and, by construction,

$$\left\| T - {}^kT \right\| = \sup_{m,n} \left| T_{mn} - {}^kT_{mn} \right| = \sup\{|T_{mn}| \mid \max\{m,n\} > k\}. \tag{7.179}$$

Now, since T is of trace class, for every $\epsilon > 0$, there is some $j_\epsilon \in \mathbb{N}$ such that

$$\max\{m,n\} > j_\epsilon \implies |T_{mn}| < \epsilon; \tag{7.180}$$

hence, for every $k \geq j_\epsilon$,

$$\left\| T - {}^kT \right\| = \sup\{|T_{mn}| \mid \max\{m,n\} > k\} \leq \sup\{|T_{mn}| \mid \max\{m,n\} > j_\epsilon\} \leq \epsilon. \tag{7.181}$$

Therefore, $\lim_k \left\| T - {}^kT \right\| = 0$, hence, $\mathcal{T}(\mathcal{H})$ is contained in the norm-closure of $\mathscr{B}_\Phi$.

Let us next prove that $\mathcal{T}(\mathcal{H}) \supset \overline{\mathscr{B}_\Phi}^{\|\cdot\|}$, as well, so that, actually, $\mathcal{T}(\mathcal{H}) = \overline{\mathscr{B}_\Phi}^{\|\cdot\|}$. Indeed, let us now suppose that $\left\{{}^kC = \mathrm{op}_\Phi({}^kC_{mn})\right\}_{k\in\mathbb{N}}$ is a sequence in $\mathscr{B}_\Phi$ converging, in norm, to some (necessarily compact) operator $C = \mathrm{op}_\Phi(C_{mn})$. Then, for every $\epsilon > 0$, there is some $j_\epsilon \in \mathbb{N}$ such that

$$k > j_\epsilon \implies \left\| C - {}^kC \right\| < \epsilon. \tag{7.182}$$

Moreover, for every $k \in \mathbb{N}$, since ${}^kC$ is block-finite w.r.t. $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}$, there is some $l_k \in \mathbb{N}$ such that

$$\max\{m,n\} > l_k \implies {}^kC_{mn} = 0. \tag{7.183}$$

Therefore, for every $\epsilon > 0$, there is some $j_\epsilon \in \mathbb{N}$ such that

$$\begin{aligned} k > j_\epsilon \implies \epsilon > \left\| C - {}^kC \right\| &= \sup_{m,n} \left| C_{mn} - {}^kC_{mn} \right| \\ &= \max\left\{ \sup_{\max\{m,n\}\leq l_k} \left| C_{mn} - {}^kC_{mn} \right|, \sup_{\max\{m,n\}>l_k} |C_{mn}| \right\}. \end{aligned} \tag{7.184}$$

In conclusion, for every $\epsilon > 0$, there is some $l \in \mathbb{N}$ — say, $l \equiv l_k$, for any $k > j_\epsilon$ — such that

$$\max\{m,n\} > l \implies |C_{mn}| < \epsilon. \tag{7.185}$$

Otherwise stated, $\lim_{m+n} |C_{mn}| = 0$, so that $C = \mathrm{op}_\Phi(C_{mn}) = \lim_k {}^kC$ is a trace class operator and — by the arbitrariness of the converging sequence $\left\{{}^kC\right\}_{k\in\mathbb{N}} \subset \mathscr{B}_\Phi$ — $\mathcal{T}(\mathcal{H}) \supset \overline{\mathscr{B}_\Phi}^{\|\cdot\|}$.

Eventually, it is shown that $\mathcal{T}(\mathcal{H}) = \overline{\mathscr{B}_\Phi}^{\|\cdot\|}$ and the proof is complete. □

We will next prove a remarkable characterization of the trace class of $\mathcal{H}$.

Theorem 7.4.3. *The following characterization of the trace class of $\mathcal{H}$ holds true:*

$$\mathcal{T}(\mathcal{H}) = \mathcal{C}_{\mathrm{ad}}(\mathcal{H}) := \mathcal{C}(\mathcal{H}) \cap \mathcal{B}_{\mathrm{ad}}(\mathcal{H}). \tag{7.186}$$

Moreover, given any canonical decomposition of a trace class operator $T \in \mathcal{T}(\mathcal{H}) = \mathcal{C}_{\mathrm{ad}}(\mathcal{H})$ — i.e., $T = \sum_{j\in J} \lambda_j \, |e_j\rangle\langle f_j|$, where $\{\lambda_j\}_{j\in J}$, $\{e_j\}_{j\in J}$ and $\{f_j\}_{j\in J}$ are as specified in Corollary 7.4.3; in particular, $\{e_j\}_{j\in J}$, or $\{f_j\}_{j\in J}$, is a normlized norm-orthogonal system in $\mathcal{H}$ — we have that

$$\mathrm{tr}(T) = \sum_{j\in J} \lambda_j \, \langle f_j, e_j \rangle, \tag{7.187}$$

and the following estimate holds:

$$|\mathrm{tr}(T)| \leq \max_{j\in J} |\lambda_j| \, |\langle f_j, e_j\rangle| \leq \|T\| = \max_{j\in J} |\lambda_j|. \tag{7.188}$$

(Compare with the second inequality in (7.157).)

Proof. We already know that $\mathcal{T}(\mathcal{H}) \subset \mathcal{C}(\mathcal{H}) \cap \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$. Let us show that this inclusion is, actually, an equality. In fact, given any orthonormal basis $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}$ in $\mathcal{H}$, by Corollary 7.4.1 a compact operator $A = \mathrm{op}_\Phi(A_{mn}) \in \mathcal{C}(\mathcal{H})$ must verify conditions `(C1)`–`(C3)` therein. If, in addition, A is adjointable, then, by Theorem 7.1.2, we also have that

`(C4)` $\lim_n A_{mn} = 0$, $\forall m \in \mathbb{N}$.

By relation (7.109) in Remark 7.3.2, conditions `(C2)`–`(C4)` are equivalent to $\lim_{m+n} A_{mn} = 0$ (and condition `(C1)` becomes redundant). Hence, A is a trace class operator.

The proof of relation (7.187) is similar to the proof of Corollary 7.4.2: For any orthonormal basis $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}$ in $\mathcal{H}$,

$$\begin{aligned}\mathrm{tr}(T) = \mathrm{tr}\big(\textstyle\sum_{j\in J}\lambda_j\,|e_j\rangle\langle f_j|\big) &= \sum_{m\in\mathbb{N}}\sum_{j\in J}\lambda_j\,\langle\phi_m, e_j\rangle\langle f_j,\phi_m\rangle \\ &= \sum_{j\in J}\sum_{m\in\mathbb{N}}\lambda_j\,\langle f_j,\phi_m\rangle\langle\phi_m, e_j\rangle = \sum_{j\in J}\lambda_j\,\langle f_j, e_j\rangle. \end{aligned} \tag{7.189}$$

Here, if $J=\mathbb{N}$, exchanging the sums is justified by the fact that $\lim_{m+j}\lambda_j\,\langle\phi_m,e_j\rangle\langle f_j,\phi_m\rangle = 0$, because $\lim_m \lambda_j\langle\phi_m,e_j\rangle\langle f_j,\phi_m\rangle = 0$, for all $j\in\mathbb{N}$, and $\lim_j \lambda_j\,\langle\phi_m,e_j\rangle\langle f_j,\phi_m\rangle = 0$ *uniformly* in $m\in\mathbb{N}$ ($|\lambda_j\,\langle\phi_m,e_j\rangle\langle f_j,\phi_m\rangle| \le |\lambda_j|$).

Let us now prove that $\|T\| = \max_{j\in J}|\lambda_j|$. Since $\|T\| = \|T^*\|$, in the following we can assume, without loss of generality, that $\{e_j\}_{j\in J}$ (rather than $\{f_j\}_{j\in J}$) is a normalized norm-orthogonal system in $\mathcal{H}$. Hence, for every vector $\psi\in\mathcal{H}$, we have that $\|T\psi\| = \|\sum_{j\in J}\lambda_j\,\langle f_j,\psi\rangle\,e_j\| = \sup_{j\in J}|\lambda_j|\,|\langle f_j,\psi\rangle|$, and

$$\|T\| = \sup_{\psi\neq 0}\frac{\|T\psi\|}{\|\psi\|} = \sup_{j\in J}\left(|\lambda_j|\,\sup_{\psi\neq 0}\frac{|\langle f_j,\psi\rangle|}{\|\psi\|}\right) = \sup_{j\in J}|\lambda_j|\,\|\mathcal{J}_{\mathcal{H}}f_j\| = \max_{j\in J}|\lambda_j|, \tag{7.190}$$

where we have used the fact that $\mathcal{J}_{\mathcal{H}}$ is a (conjugate-linear) isometry. Then, since $|\langle f_j,e_j\rangle| \le 1$, the estimate (7.188) holds true. □

Remark 7.4.4. By Theorem 7.4.1, Corollary 7.4.1 and Corollary 7.4.3, it is clear that *not* every compact operator is adjointable and then $\mathcal{T}(\mathcal{H}) = \mathcal{C}_{\mathrm{ad}}(\mathcal{H}) \subsetneq \mathcal{C}(\mathcal{H}) \subset \mathcal{T}_{\mathrm{w}}(\mathcal{H})$ ($\dim(\mathcal{H}) = \infty$). For instance, the bounded operator TB in Example 7.3.1 is compact but not adjointable.

In a infinite-dimensional separable *complex* Hilbert space, the product of two trace class operators is of trace class too, *but* not every trace class operator is the product of two trace class operators (instead, it can expressed as the product of two Hilbert-Schmidt operators); see [135–137, 140]. In a p-adic Hilbert space $\mathcal{H}$, putting

$$\mathcal{T}(\mathcal{H})^2 := \{S\,T \mid S,T\in\mathcal{T}(\mathcal{H})\}, \tag{7.191}$$

we have that $\mathcal{T}(\mathcal{H})^2 \subset \mathcal{T}(\mathcal{H})$, because $\mathcal{T}(\mathcal{H})$ is a two-sided ideal in $\mathcal{B}_{\mathrm{ad}}(\mathcal{H})$; moreover, from Theorem 7.4.3 we derive the following:

Corollary 7.4.4. $\mathcal{T}(\mathcal{H})^2 = \mathcal{T}(\mathcal{H})$. *In particular, every trace class operator* $R\in\mathcal{T}(\mathcal{H})$ *can be expressed in the form* $R = SB$*, for some* $S,T\in\mathcal{T}(\mathcal{H})$.

Proof. We only need to prove that $\mathcal{T}(\mathcal{H})^2 \supset \mathcal{T}(\mathcal{H})$; i.e., that every $R\in\mathcal{T}(\mathcal{H})$ is of the form $R = ST$, for suitable $S,T\in\mathcal{T}(\mathcal{H})$. Since $\mathcal{T}(\mathcal{H}) = \mathcal{C}_{\mathrm{ad}}(\mathcal{H})$, we can write $R = \sum_{j\in J}\lambda_j\,|e_j\rangle\langle f_j|$, with $J = \{1,2,\ldots\}\subset\mathbb{N}$, and $\{\lambda_j\}_{j\in J}$, $\{e_j\}_{j\in J}$, $\{f_j\}_{j\in J}$ as specified in Corollary 7.4.3. Now, given any orthonormal basis $\Phi\equiv\{\phi_m\}_{m\in\mathbb{N}}$ in $\mathcal{H}$, let us set $S = \sum_{j\in J}\kappa_j\,|e_j\rangle\langle\phi_j|$, $T = \sum_{j\in J}\nu_j\,|\phi_j\rangle\langle f_j|$, where $\kappa_j\nu_j = \lambda_j$ and, if $J=\mathbb{N}$, $\lim_j\kappa_j = 0 = \lim_j\nu_j$. The existence of suitable sets $\{\kappa_j\}_{j\in J}$, $\{\nu_j\}_{j\in J}$ in $\mathbb{Q}_{p,\mu}$ satisfying the previous conditions is guaranteed by Lemma 8.1.5 in [76]. Therefore, $\{\kappa_j\}_{j\in J}$, $\{e_j\}_{j\in J}$ and $\{\phi_j\}_{j\in J}$ — and, analogously $\{\nu_j\}_{j\in J}$, $\{\phi_j\}_{j\in J}$ and $\{f_j\}_{j\in J}$ — are as prescribed in Corollary 7.4.3. Hence, $S,T\in\mathcal{C}_{\mathrm{ad}}(\mathcal{H})=\mathcal{T}(\mathcal{H})$ (note that Theorem 7.4.3 is essential here), and, by construction, $S\,T=\sum_{j\in J}\kappa_j\nu_j\,|e_j\rangle\langle f_j|=\sum_{j\in J}\lambda_j\,|e_j\rangle\langle f_j|=R$. □

7.5 The p-adic Hilbert-Schmidt space

In the light of the results of the previous section, it should not be surprising that, in the p-adic setting, $\mathcal{T}(\mathcal{H})$ actually plays a two-fold role: the trace class and the Hilbert-Schmidt space.

In fact, let us introduce the sesquilinear form

$$\mathcal{T}(\mathcal{H})\times\mathcal{T}(\mathcal{H})\ni(S,T)\mapsto\mathrm{tr}(S^*T)=:\langle S,T\rangle_{\mathcal{T}(\mathcal{H})}\in\mathbb{Q}_{p,\mu},\tag{7.192}$$

which is Hermitian, because

$$\langle S,T\rangle_{\mathcal{T}(\mathcal{H})}:=\mathrm{tr}(S^*T)=\overline{\mathrm{tr}(T^*S)}=\overline{\langle T,S\rangle}_{\mathcal{T}(\mathcal{H})}.\tag{7.193}$$

Notice that, here, for obtaining the second equality, we have used property `(P2)` of the trace (see Proposition 7.3.4).

We will call the Hermitian sesquilinear form $\langle\cdot,\cdot\rangle_{\mathcal{T}(\mathcal{H})}$ in $\mathcal{T}(\mathcal{H})$ the *p-adic Hilbert-Schmidt product*.

Given any orthonormal basis $\Phi\equiv\{\phi_m\}_{m\in\mathbb{N}}$ in $\mathcal{H}$, we can also consider the family of matrix operators $\left\{{}^{jk}E^{\Phi}\right\}_{j,k\in\mathbb{N}}$ defined by

$${}^{jk}E^{\Phi}:=\mathrm{op}_{\Phi}\big({}^{jk}E^{\Phi}_{mn}\big),\ \text{ where } {}^{jk}E^{\Phi}_{mn}=\delta_{jm}\delta_{kn};\tag{7.194}$$

namely, in the usual Dirac notation, ${}^{jk}E^{\Phi}=|\phi_j\rangle\langle\phi_k|$ (i.e., ${}^{jk}E^{\Phi}_{mn}\,\psi=\langle\phi_k,\psi\rangle\phi_j$). Note that, for every trace class operator $T:=\mathrm{op}_{\Phi}(T_{mn})$, we have:

$$\big\langle{}^{jk}E^{\Phi},T\big\rangle_{\mathcal{T}(\mathcal{H})}=\mathrm{tr}(|\phi_k\rangle\langle\phi_j|\,T)=T_{jk}.\tag{7.195}$$

It follows that the Hermitian sesquilinear form $\langle\cdot,\cdot\rangle_{\mathcal{T}(\mathcal{H})}$ is non-degenerate, because

$$\big\langle T,{}^{jk}E^{\Phi}\big\rangle_{\mathcal{T}(\mathcal{H})}=0,\ \forall j,k\in\mathbb{N}\implies T=0.\tag{7.196}$$

Theorem 7.5.1. *The p-adic Banach space $\mathcal{T}(\mathcal{H})$ — endowed with the p-adic Hilbert-Schmidt product $\langle\cdot,\cdot\rangle_{\mathcal{T}(\mathcal{H})}$ — becomes an inner product p-adic Banach space. Moreover, for every orthonormal basis $\Phi\equiv\{\phi_m\}_{m\in\mathbb{N}}$ in $\mathcal{H}$, $\{{}^{jk}E^{\Phi}\}_{j,k\in\mathbb{N}}$ is an orthonormal basis in $\mathcal{T}(\mathcal{H})$. Therefore, the triple $(\mathcal{T}(\mathcal{H}),\|\cdot\|,\langle\cdot,\cdot\rangle_{\mathcal{T}(\mathcal{H})})$ is, actually, a p-adic Hilbert space.*

Proof. We have already shown that $\mathcal{T}(\mathcal{H})$, endowed with the operator norm, is a p-adic Banach space, and that the sesquilinear form $\langle\cdot,\cdot\rangle_{\mathcal{T}(\mathcal{H})}$ is both Hermitian and non-degenerate.

Observe now that, for all $S,T\in\mathcal{T}(\mathcal{H})$, we have:

$$|\langle S,T\rangle_{\mathcal{T}(\mathcal{H})}|=|\mathrm{tr}(S^*T)|=|\textstyle\sum_m\langle S\phi_m,T\phi_m\rangle|\le\max_m|\langle S\phi_m,T\phi_m\rangle|\le\|S\|\,\|T\|;\tag{7.197}$$

i.e., $\langle\cdot,\cdot\rangle_{\mathcal{T}(\mathcal{H})}$ satisfies the Cauchy-Schwarz inequality, as well. Therefore, $\langle\cdot,\cdot\rangle_{\mathcal{T}(\mathcal{H})}$ is an inner product, and $\mathcal{T}(\mathcal{H})$, endowed with this sesquilinear form, is an inner product p-adic Banach space.

It remains to show that $\{{}^{jk}E^{\Phi}\}_{j,k\in\mathbb{N}}$ is an orthonormal basis in $\mathcal{T}(\mathcal{H})$. Since it is clear that

$$\langle {}^{jk}E^{\Phi}, {}^{rs}E^{\Phi}\rangle_{\mathcal{T}(\mathcal{H})} = \mathrm{tr}(|\phi_k\rangle\langle\phi_j|\,|\phi_r\rangle\langle\phi_s|) = \langle\phi_j,\phi_r\rangle\langle\phi_s,\phi_k\rangle = \delta_{jr}\delta_{ks}, \tag{7.198}$$

we only need to prove that $\{{}^{jk}E^{\Phi}\}_{j,k\in\mathbb{N}}$ is a *normal* basis. In fact, for every finite subset I of $\mathbb{N}\times\mathbb{N}$ and every finite subset $\{\alpha_{jk}\}_{j,k\in I}$ of $\mathbb{Q}_{p,\mu}$, we have:

$$\left\|\sum\nolimits_{j,k\in I}\alpha_{jk}\,{}^{jk}E^{\Phi}\right\| = \max_{j,k\in I}|\alpha_{jk}|. \tag{7.199}$$

Moreover, for every trace class operator $T := \mathrm{op}_{\Phi}(T_{mn})$, we have that

$$T = \lim_{l}{}^{l}T, \text{ where } {}^{l}T := \sum\nolimits_{\max\{j,k\}\le l} T_{jk}\,{}^{jk}E^{\Phi}. \tag{7.200}$$

This fact is a consequence of the estimate

$$\|T - {}^{l}T\| = \sup_{m,n}|T_{mn} - {}^{l}T_{mn}| = \sup\{|T_{mn}| \mid \max\{m,n\} > l\}, \tag{7.201}$$

together with the same argument used in the first part of the proof of Theorem 7.4.2, which shows that — T being of trace class — $\lim_l \|T - {}^{l}T\| = 0$.

In conclusion, $\{{}^{jk}E^{\Phi}\}_{j,k\in\mathbb{N}}$ is an orthonormal basis in the inner product p-adic Banach space $\mathcal{T}(\mathcal{H})$, which is then a p-adic Hilbert space. □

The p-adic Hilbert space $(\mathcal{T}(\mathcal{H}), \|\cdot\|, \langle\cdot,\cdot\rangle_{\mathcal{T}(\mathcal{H})})$ will be called the *p-adic Hilbert-Schmidt space.*

7.6 Selfadjoint trace class operators

Let us now consider the $\mathbb{Q}_p$-linear space $\mathcal{T}_{\mathrm{sa}}(\mathcal{H}) := \mathcal{T}(\mathcal{H}) \cap \mathcal{B}_{\mathrm{sa}}(\mathcal{H}) = \mathcal{C}_{\mathrm{ad}}(\mathcal{H}) \cap \mathcal{B}_{\mathrm{sa}}(\mathcal{H})$ of all selfadjoint trace class operators in the p-adic Hilbert space $\mathcal{H}$, that is closed in $\mathcal{T}(\mathcal{H})$, because the mapping $\mathcal{T}(\mathcal{H}) \ni T \mapsto T^* \in \mathcal{T}(\mathcal{H})$ is a (conjugate-linear) isometry and, hence, continuous (thus, $\mathcal{T}_{\mathrm{sa}}(\mathcal{H})$, endowed with the operator norm, is an ultrametric Banach space over $\mathbb{Q}_p$).

Proposition 7.6.1. *Every selfadjoint trace class operator $T \in \mathcal{T}_{\mathrm{sa}}(\mathcal{H})$ can be expressed in the form*

$$T = \sum_{j\in J}(\sigma_j\,|e_j\rangle\langle f_j| + \overline{\sigma_j}\,|f_j\rangle\langle e_j|), \tag{7.202}$$

where $J = \{1,2,\ldots\}$ is a countable index set and

- $\{\sigma_j\}_{j\in J} \subset \mathbb{Q}_{p,\mu}$ *— for $T \neq 0$, we assume that $\{\sigma_j\}_{j\in J} \subset \mathbb{Q}^*_{p,\mu}$ — and, if $J = \mathbb{N}$, $\lim_j \sigma_j = 0$;*
- $\{e_j\}_{j\in J}$ *is a normalized norm-orthogonal system in $\mathcal{H}$, and $\|f_j\| = 1$, for all $j \in J$;*
- *the sum in* (7.202) *— whenever J is not finite — converges w.r.t. the norm topology.*

In particular, the norm-orthogonal system $\{e_j\}_{j\in J}$ can be chosen to be contained in an orthonormal basis in $\mathcal{H}$.

Conversely, every linear operator T *of the previous form belongs to* $\mathcal{T}_{\mathrm{sa}}(\mathcal{H})$*, and*

$$\mathrm{tr}(T) = 2\sum_{j\in J} \mathfrak{sc}(\sigma_j \langle f_j, e_j\rangle) = \sum_{j\in J}(\sigma_j \langle f_j, e_j\rangle + \overline{\sigma_j}\langle e_j, f_j\rangle) \in \mathbb{Q}_p; \tag{7.203}$$

moreover, $|\mathrm{tr}(T)| \leq \|T\| \leq \max_{j\in J} |\sigma_j|$.

Proof. Clearly, a trace class operator $T \in \mathcal{T}(\mathcal{H})$ is selfadjoint if and only if it is of the form $T = A + A^*$, form some $A \in \mathcal{T}(\mathcal{H}) = \mathcal{C}_{\mathrm{ad}}(\mathcal{H})$. Then, by Corollary 7.4.3, $A = \sum_{j\in J} \sigma_j |e_j\rangle\langle f_j|$ — with $\{\sigma_j\}_{j\in J}$, $\{e_j\}_{j\in J}$ and $\{f_j\}_{j\in J}$ as above — so that $T \in \mathcal{T}_{\mathrm{sa}}(\mathcal{H})$ if and only if it is of the form (7.202), and then formula (7.203) follows immediately from (7.187). Moreover, by the estimate (7.188), we have that $|\mathrm{tr}(T)| \leq \|T\| = \|A + A^*\| \leq \max\{\|A\|, \|A^*\|\} = \|A\| = \max_{j\in J} |\sigma_j|$. □

Definition 7.6.1. Given a selfadjoint trace class operator $T \in \mathcal{T}_{\mathrm{sa}}(\mathcal{H})$, an expression of the form (7.202) will be called *a symmetric decomposition* of T. In the case where $\{e_j\}_{j\in J}$ is chosen to be contained in an orthonormal basis, we will call *orthonormal* the associated symmetric decomposition.

8

A p-adic model of quantum states

Building on the foundations laid down in the preceding sections, we will now attempt at achieving a general definition of a quantum state in the p-adic setting. As usual, the standard complex case will provide us with a useful road map, *but*, when dealing with a p-adic Hilbert space, the emergence of non-trivial peculiarities should be expected.

8.1 The complex setting in a nutshell

Since we do expect that the general lines, rather than the peculiar features, of the theory will be preserved when switching from the complex to the p-adic case, it may be sensible to consider, as a starting point, the most abstract formulation of standard quantum mechanics, i.e., the so-called *algebraic formulation* [137, 143–145]. This formulation relies on the following set of fundamental assumptions:

- A quantum system can be described by means of two main classes of objects — *states* and *observables* — mutually related by means of a natural *pairing* map. By suitably exploiting these two kinds of objects, one can then construct all other parts of the theory: measurements, symmetry transformations, dynamics etc.
- The (bounded) observables of the system are supposed to form the self-adjoint part $\mathfrak{A}_{\mathrm{sa}}$ of an abstract *non-commutative unital C^*-algebra* $\mathfrak{A}$.
- A generic *state* ω (of $\mathfrak{A}$) is defined as a *normalized positive functional* on $\mathfrak{A}$; i.e., as a functional $\omega\colon \mathfrak{A} \to \mathbb{C}$ satisfying the conditions

$$\omega(A^*A) \geq 0, \quad \forall A \in \mathfrak{A}, \qquad \omega(\mathrm{Id}) = 1. \tag{8.1}$$

  Here, the *positive elements* A^*A of $\mathfrak{A}$ form a convex cone.
- Denoting by $\mathfrak{S}(\mathfrak{A})$ the set of all states of the C^*-algebra $\mathfrak{A}$, the pairing between observables and states is provided by the evaluation map $\mathfrak{A}_{\mathrm{sa}} \times \mathfrak{S}(\mathfrak{A}) \ni (A, \omega) \mapsto \omega(A)$.

From these assumptions, one can then derive the following main facts:

1. Every state $\omega\colon \mathfrak{A} \to \mathbb{C}$ is automatically continuous (i.e., bounded, as a linear functional); specifically, it turns out that $\|\omega\| = \omega(\mathrm{Id}) = 1$.
2. $\mathfrak{S}(\mathfrak{A})$ is a convex subset of the (complex) Banach space of bounded functionals on $\mathfrak{A}$.
3. For every $A \in \mathfrak{A}$ and every state $\omega \in \mathfrak{S}(\mathfrak{A})$, $\omega(A^*) = \overline{\omega(A)}$.
4. In particular, for every observable $A \in \mathfrak{A}_{\mathrm{sa}}$ and every state $\omega \in \mathfrak{S}(\mathfrak{A})$, the *real* quantity $\omega(A)$ — i.e., the pairing of A with ω — can be interpreted as the *expectation value* of the observable A when the physical system is in the state ω.

5. By the celebrated *Gelfand-Naimark theorem* [137, 143, 144], $\mathfrak{A}$ can be realized as — i.e., is isometrically $*$-isomorphic to — a C^*-subalgebra $\mathfrak{C}$ of the C^*-algebra of all bounded operators $\mathcal{B}(\mathcal{K})$ in a complex Hilbert space $\mathcal{K}$. For the sake of simplicity, we will suppose henceforth that $\mathcal{K}$ is separable and $\mathfrak{C} = \mathcal{B}(\mathcal{K})$ (this is the case of ordinary quantum mechanics).

6. By the previous identification of $\mathfrak{A}$ with $\mathcal{B}(\mathcal{K})$, we can single out a remarkable class of states — the so-called *trace induced states* $\mathfrak{S}_{\mathrm{tr}}(\mathfrak{A})$ — that can be defined by

$$\omega \in \mathfrak{S}_{\mathrm{tr}}(\mathfrak{A}) \overset{\mathrm{def}}{\iff} \mathrm{tr}((\cdot)\rho_\omega) \colon \mathfrak{A} \to \mathbb{C}, \text{ for some } \rho_\omega \in \mathcal{D}(\mathcal{K}), \tag{8.2}$$

where $\mathcal{D}(\mathcal{K}) \subset \mathcal{T}(\mathcal{K})$ is the convex set of all unit-trace positive trace class operators in $\mathcal{K}$, the so-called *density* or *statistical* operators.

7. It is worth stressing that, in the case where $\dim(\mathcal{K}) = \infty$, $\mathfrak{S}_{\mathrm{tr}}(\mathfrak{A}) \subsetneq \mathfrak{S}(\mathfrak{A})$. There is a remarkable characterization of trace induced states as those states that are σ-*additive* [137, 146]. Identifying the abstract algebra $\mathfrak{A}$ with $\mathcal{B}(\mathcal{K})$, a state $\omega \in \mathfrak{S}(\mathcal{B}(\mathcal{K}))$ is σ-additive if

$$\omega\big(\textstyle\sum_{j\in J} P_j\big) = \displaystyle\sum_{j\in J} \omega(P_j), \tag{8.3}$$

for every (countable) family $\{P_j\}_{j\in J}$ of pairwise orthogonal projections in $\mathcal{K}$, where the possibly infinite sum $\sum_{j\in J} P_j$ is supposed to converge w.r.t. the weak operator topology (it actually converges w.r.t. the strong operator topology, as well). Therefore, $\omega \in \mathfrak{S}(\mathcal{B}(\mathcal{K}))$ is σ-additive if and only if $\omega = \mathrm{tr}((\cdot)\rho_\omega)$, for some density operator $\rho_\omega \in \mathcal{D}(\mathcal{K})$. The role, the meaning and the relevance of those states that are *not* completely additive is controversial [146], and one often restricts to the trace induced ones; equivalently, to density operators. This is analogous to restricting to σ-additive probability measures in classical statistical mechanics.

8. The *spectral decomposition* $A = \int_{\mathbb{R}} \lambda \, \mathrm{d}P_A(\lambda)$ of a selfadjoint operator in $\mathcal{K}$ — where P_A is the spectral measure uniquely associated with A — allows one to complete the probabilistic interpretation of the theory. In particular, it shows that every (bounded or unbounded) observable can be expressed in terms of the *lattice of projections* $\mathcal{P}(\mathcal{K}) \subset \mathcal{B}_{\mathrm{sa}}(\mathcal{K})$, whose elements are then regarded as the *elementary propositions* of the theory [146].

9. Eventually, one is led in a natural way to describe the observables of a quantum system in terms of PVMs (projection-valued measures) or, more generally, of POVMs (positive-operator-valued measures, also called "semispectral measures") [147–149]. The — both conceptually and mathematically transparent — generalization of PVMs into POVMs has a remarkable physical interpretation related to the theory of open quantum systems (Naimark's dilation theorem [147]).

8.2 Convexity and probability in the p-adic setting

Quantum probability theory is tailored on classical probability theory, of which it can be regarded as a non-commutative counterpart. This is not surprising because the outcome of a quantum measurement process must be, ultimately, a classical probability distribution. In particular, both theories share essentially the same notion of convexity.

Clearly, the basic rules of the game must be re-written when switching to the p-adic setting. We start with briefly introducing the p-adic (or, more generally, non-Archimedean) notion of convexity. Our treatment will be rather sketchy; for further details, the reader may refer to Section 2.5 of [72] and Section 3.1 of [76]. Moreover, we will adapt the main definitions and results to the special case that will be considered in the next subsection.

Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{Q}_{p,\mu}$. By field restriction, we can regard it as a vector space over $\mathbb{Q}_p$ and consider a notion of $\mathbb{Q}_p$-convexity (rather than $\mathbb{Q}_{p,\mu}$-convexity). We will keep trace of this choice — essentially motivated by our objectives — in the notation that will be adopted.

Definition 8.2.1. A subset $\mathscr{A}$ of X is said to be *absolutely* $\mathbb{Q}_p$*-convex* if ($0 \in \mathscr{A}$ and) $\lambda x + \mu y \in \mathscr{A}$, for all $x, y \in \mathscr{A}$ and all $\lambda, \mu \in \mathbb{Z}_p$, where $\mathbb{Z}_p = \{\lambda \in \mathbb{Q}_p \mid |\lambda| \leq 1\}$ is the ring of p-adic integers. Given any subset $\mathscr{X}$ of X, its *absolutely* $\mathbb{Q}_p$*-convex hull* $\mathrm{aco}_{\mathbb{Q}_p}(\mathscr{X})$ is defined as the intersection of all absolutely $\mathbb{Q}_p$-convex sets containing $\mathscr{X}$.

We have that $\mathrm{aco}_{\mathbb{Q}_p}(\emptyset) = \emptyset$ and, if $\mathscr{X} \neq \emptyset$,

$$\mathrm{aco}_{\mathbb{Q}_p}(\mathscr{X}) = \{\lambda_1 x_1 + \cdots + \lambda_n x_n \mid n \in \mathbb{N},\ x_1, \ldots, x_n \in \mathscr{X},\ \lambda_1, \ldots, \lambda_n \in \mathbb{Z}_p\}. \tag{8.4}$$

We will denote by $\overline{\mathrm{aco}}_{\mathbb{Q}_p}(\mathscr{X})$ the (norm-)closure of the set $\mathrm{aco}_{\mathbb{Q}_p}(\mathscr{X})$.

Definition 8.2.2. A subset $\mathscr{C}$ of X is said to be $\mathbb{Q}_p$*-convex* if it is either empty or of the form $x + \mathscr{A}$, for some $x \in X$ and some (nonempty) absolutely $\mathbb{Q}_p$-convex subset $\mathscr{A}$ of X. Given any subset $\mathscr{X}$ of X, its $\mathbb{Q}_p$*-convex hull* $\mathrm{co}_{\mathbb{Q}_p}(\mathscr{X})$ is defined as the intersection of all $\mathbb{Q}_p$-convex sets containing $\mathscr{X}$.

We will denote by $\overline{\mathrm{co}}_{\mathbb{Q}_p}(\mathscr{X})$ the closure of the set $\mathrm{co}_{\mathbb{Q}_p}(\mathscr{X})$. Given a p-adic Hilbert space $\mathcal{H}$ and an orthonormal basis $\Phi \equiv \{\phi_m\}_{m=1}^{\mathsf{N}}$ (where $\mathsf{N} \in \mathbb{N}$ or $\mathsf{N} = \infty$) in $\mathcal{H}$, the closed $\mathbb{Q}_p$-convex hull $\overline{\mathrm{co}}_{\mathbb{Q}_p}(\Phi)$ is said to be a $\mathbb{Q}_p$*-simplex* in $\mathcal{H}$.

One can easily check the following facts:

- A $\mathbb{Q}_p$-convex subset of X is absolutely $\mathbb{Q}_p$-convex if and only if it contains 0.
- $\mathrm{co}_{\mathbb{Q}_p}(\mathscr{X}) = \big\{\lambda_1 x_1 + \cdots + \lambda_n x_n \mid n \in \mathbb{N},\ x_1, \ldots, x_n \in \mathscr{X},\ \lambda_1, \ldots, \lambda_n \in \mathbb{Z}_p,\ \sum_{k=1}^{n} \lambda_k = 1\big\}$.

Definition 8.2.3. A map $g\colon X \to Y$ — where Y is a vector space over $\mathbb{Q}_{p,\mu}$ — is said to be $\mathbb{Q}_p$*-convex* if

$$g(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 g(x_1) + \lambda_2 g(x_2),\ \forall x_1, x_2 \in X,\ \forall \lambda_1, \lambda_2 \in \mathbb{Z}_p \text{ such that } \lambda_1 + \lambda_2 = 1. \tag{8.5}$$

Theorem 8.2.1. *Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{Q}_{p,\mu}$. For $p \neq 2$, a subset $\mathscr{C}$ of X is $\mathbb{Q}_p$-convex if and only if*

$$\lambda_1 x_1 + \lambda_2 x_2 \in \mathscr{C},\ \forall x_1, x_2 \in \mathscr{C},\ \forall \lambda_1, \lambda_2 \in \mathbb{Z}_p \text{ such that } \lambda_1 + \lambda_2 = 1, \tag{8.6}$$

whereas this condition is (necessary but) not *sufficient in the case where $p = 2$.*

For $p = 2$, a subset $\mathscr{C}$ of X is $\mathbb{Q}_p$-convex if and only if

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 \in \mathscr{C},\ \forall x_1, x_2, x_3 \in \mathscr{C},\ \forall \lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}_p \text{ such that } \lambda_1 + \lambda_2 + \lambda_3 = 1. \tag{8.7}$$

Proof. That a convex combination of three elements of a set belongs to this set is the standard sufficient condition ensuring convexity in a non-Archimedean normed space; see Theorem 3.1.15 in [76]. Then, condition (8.7) is (necessary and) sufficient for the $\mathbb{Q}_p$-convexity of X. Moreover, in the case where $p \neq 2$, the residue class field $\mathbb{F}_p$ of $\mathbb{Q}_p$ consists of $p \geq 3$ elements, so that, by Theorem 3.1.17 in [76], the milder condition (8.6) is sufficient too, whereas this condition is *not* sufficient for $p = 2$. (We stress that we can apply the previously mentioned results in non-Archimedean convexity because, by field restriction, X can be regarded as a vector space over $\mathbb{Q}_p$.) ☐

Corollary 8.2.1. *The range $g(X)$ of a $\mathbb{Q}_p$-convex map $g\colon X \to Y$ — where Y is a vector space over $\mathbb{Q}_{p,\mu}$ — is a $\mathbb{Q}_p$-convex subset of Y.*

Proof. Just note that, if g is $\mathbb{Q}_p$-convex, then also

$$g(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3) = \lambda_1 g(x_1) + \lambda_2 g(x_2) + \lambda_3 g(x_3), \tag{8.8}$$

for all $x_1, x_2, x_3 \in X$, and for all $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}_p$ such that $\lambda_1 + \lambda_2 + \lambda_3 = 1$ (assuming that, say, $\lambda_1 + \lambda_2 \neq 0$, write $\lambda_1 x_1 + \lambda_2 x_2 = (\lambda_1 + \lambda_2)(\lambda_1 x_1/(\lambda_1 + \lambda_2) + \lambda_2 x_2/(\lambda_1 + \lambda_2))$, where $\lambda_1/(\lambda_1 + \lambda_2), \lambda_2/(\lambda_1 + \lambda_2) \in \mathbb{Z}_p$), and apply Theorem 8.2.1. □

Definition 8.2.4. A nonempty subset $\mathscr{H}$ of X is said to be $\mathbb{Q}_p$-*affine* if it is of the form $x + \mathscr{L}$, for some $x \in X$ and some $\mathbb{Q}_p$-linear subspace $\mathscr{L}$ of X. Given any nonempty subset $\mathscr{X}$ of X, its $\mathbb{Q}_p$-*affine hull* $\operatorname{aff}_{\mathbb{Q}_p}(\mathscr{X})$ is the intersection of all $\mathbb{Q}_p$-affine sets containing $\mathscr{X}$.

Clearly, a $\mathbb{Q}_p$-affine subset of X is also $\mathbb{Q}_p$-convex, because every $\mathbb{Q}_p$-linear subspace of X is a $\mathbb{Q}_p$-absolutely convex subset of X. We will denote by $\overline{\operatorname{aff}}_{\mathbb{Q}_p}(\mathscr{X})$ the closure of the set $\operatorname{aff}_{\mathbb{Q}_p}(\mathscr{X})$. Given a p-adic Hilbert space $\mathcal{H}$ and an orthonormal basis $\Phi \equiv \{\phi_m\}_{m=1}^{\mathsf{N}}$ in $\mathcal{H}$, the closed $\mathbb{Q}_p$-affine hull $\overline{\operatorname{aff}}_{\mathbb{Q}_p}(\Phi)$ is said to be a $\mathbb{Q}_p$-*hyperplane* in $\mathcal{H}$.

One can easily prove the following facts:

- A $\mathbb{Q}_p$-affine subset of X is a $\mathbb{Q}_p$-linear subspace if and only if it contains 0.
- A nonempty subset $\mathscr{H}$ of X is $\mathbb{Q}_p$-affine if and only if, for every pair of vectors $x, y \in \mathscr{H}$, the $\mathbb{Q}_p$-*line* $\{x + \alpha(y - x)\}_{\alpha \in \mathbb{Q}_p} = \{y + \alpha(x - y)\}_{\alpha \in \mathbb{Q}_p}$ through x and y is contained in $\mathscr{H}$; namely, if and only if $\alpha x + (1 - \alpha) y \in \mathscr{H}$, for all $x, y \in \mathscr{H}$ and $\alpha \in \mathbb{Q}_p$.
- if $g\colon X \to Y$, where Y is a vector space over $\mathbb{Q}_{p,\mu}$, is $\mathbb{Q}_p$-*affine* — $g(\alpha x + (1 - \alpha) y) = \alpha g(x) + (1 - \alpha) g(y)$, for all $x, y \in X$, and $\alpha \in \mathbb{Q}_p$ — then $g(X)$ is a $\mathbb{Q}_p$-affine subset of Y.
- $\operatorname{aff}_{\mathbb{Q}_p}(\mathscr{X}) = \big\{ \sum_{i=1}^n \pi_i x_i \mid n \in \mathbb{N},\ x_1, \ldots, x_n \in \mathscr{X},\ \pi_1, \ldots, \pi_n \in \mathbb{Q}_p,\ \sum_{k=1}^n \pi_k = 1 \big\}$.

Proposition 8.2.1. *Given a map $g\colon X \to Y$ — where Y is a vector space over $\mathbb{Q}_{p,\mu}$ — the following facts are equivalent:*

`(A1)` *g is $\mathbb{Q}_p$-convex;*

`(A2)` *g is $\mathbb{Q}_p$-affine;*

`(A3)` *g is of the form $g(x) = g(0) + h(x)$, for some $\mathbb{Q}_p$-linear map $h\colon X \to Y$.*

Proof. Clearly, property `(A2)` implies `(A1)`. Let us prove that `(A1)` implies `(A2)`, as well. In fact, if g is $\mathbb{Q}_p$-convex, it is sufficient to show that $g(\alpha x + (1 - \alpha) y) = \alpha g(x) + (1 - \alpha) g(y)$, $\forall x, y \in X$, $\forall \alpha \in \mathbb{Q}_p \setminus \mathbb{Z}_p$, i.e., for $|\alpha| > 1$. Let us write y as a $\mathbb{Q}_p$-convex combination

$$y = \frac{1}{1 - \alpha}\,(\alpha x + (1 - \alpha) y) + \Big(1 - \frac{1}{1 - \alpha}\Big) x = \lambda(\alpha x + (1 - \alpha) y) + (1 - \lambda) x, \tag{8.9}$$

where $\lambda \equiv 1/(1 - \alpha), (1 - \lambda) \in \mathbb{Z}_p$, because $|\lambda| = |1/(1 - \alpha)| = 1/|(1 - \alpha)| = 1/|\alpha| < 1$ and $|1 - \lambda| = 1$. Observe, now, that we have:

$$\begin{aligned} g(y) &= g(\lambda(\alpha x + (1 - \alpha) y) + (1 - \lambda) x) \\ &= \lambda\, g(\alpha x + (1 - \alpha) y) + (1 - \lambda) g(x) = \frac{1}{1 - \alpha}\, g(\alpha x + (1 - \alpha) y) - \frac{\alpha}{1 - \alpha}\, g(x). \end{aligned} \tag{8.10}$$

In conclusion: $g(\alpha x + (1 - \alpha) y) = \alpha g(x) + (1 - \alpha) g(y)$, for $|\alpha| > 1$, and, hence, for all $\alpha \in \mathbb{Q}_p$ (g being $\mathbb{Q}_p$-convex).

The equivalence between `(A2)` and `(A3)` can be shown by a standard argument. We leave the details to the reader (to prove that `(A2)` implies `(A3)`, it is enough to show that the map $h\colon X \to Y$, $h(x) := g(x) - g(0)$, is $\mathbb{Q}_p$-homogeneous and, hence, also additive). □

It turns out that p-adic probability theory differs significantly from classical probability theory, because it mainly involves *affine* — rather than convex — structures. Nevertheless, both theories arise in a natural way from a common conceptual background. Again, our exposition will be sketchy; for further details and examples, see [94,95,150,151], and references therein.

The statistical output of a concrete experiment consists of (relative) frequencies of the form $n/\mathtt{N}$, where $\mathtt{N}$ is the total number of measurements performed during the experiment and $n \leq \mathtt{N}$ counts the number of measurements providing a fixed experimental outcome. Therefore, the possible statistical outputs of each experiment take values in the following subset of the field of rational numbers: $\mathscr{O}_{\mathbb{Q}} = \{q \in \mathbb{Q} \mid 0 \leq q \leq 1\}$. Assuming that a principle of *statistical stabilization* of frequencies holds (for $\mathtt{N} \to \infty$), it follows that the closure $\mathrm{cl}(\mathscr{O}_{\mathbb{Q}})$ of $\mathscr{O}_{\mathbb{Q}}$, in the completion of $\mathbb{Q}$ w.r.t. some suitable topology, should provide the set where all experimental statistical distributions take their values. Usually, one assumes that this topology is the one induced by the standard valuation on $\mathbb{Q}$, so obtaining $\mathrm{cl}(\mathscr{O}_{\mathbb{Q}}) = [0,1] \subset \mathbb{R}$. It is a remarkable fact that, if one considers the topology induced by the p-adic valuation, instead, then $\mathrm{cl}(\mathscr{O}_{\mathbb{Q}}) = \mathbb{Q}_p$; see Theorem 1.2 in Chapter VI of [150].

Therefore, we can set the following:

Definition 8.2.5. A (discrete) p-adic probability distribution is a countable set $\{\pi_j\}_{j\in J} \subset \mathbb{Q}_p$ such that $\sum_{j\in J} \pi_j = 1$.

It is worth observing the following simple facts:

- The set $\{1, 2, -1, -1\}$ is a legitimate p-adic probability distribution, whereas it is not a standard probability distribution.
- For every pair $\{\pi_j\}_{j\in J}$, $\{\widetilde{\pi}_k\}_{k\in K}$ of p-adic probability distributions, $\{\pi_j\, \widetilde{\pi}_k\}_{j\in J,\, k\in K}$ is a p-adic probability distribution too.
- For every p-adic probability distribution $\{\pi_j\}_{j\in J}$, $\max_{j\in J} |\pi_j| \geq 1$. Indeed, we have that $1 = \big|\sum_{j\in J} \pi_j\big| \leq \max_{j\in J} |\pi_j|$.
- For every quadratic extension $\mathbb{Q}_{p,\mu}$ of $\mathbb{Q}_p$, the collection of all probability distributions indexed by J can be identified, in a natural way, with a subset of $c_0(J, \mathbb{Q}_{p,\mu})$, i.e.,

  $$\varpi_0(J, \mathbb{Q}_{p,\mu}) := \big\{\{\pi_j\}_{j\in J} \in c_0(J, \mathbb{Q}_{p,\mu}) \mid \pi_j \in \mathbb{Q}_p,\ \forall j \in J,\ \textstyle\sum_{j\in J} \pi_j = 1\big\}. \tag{8.11}$$

  Note that $\varpi_0(J, \mathbb{Q}_{p,\mu})$ is a $\mathbb{Q}_p$-affine subset of $c_0(J, \mathbb{Q}_{p,\mu})$ — called the *probability hyperplane* of $c_0(J, \mathbb{Q}_{p,\mu})$ — that, apart from the trivial case where $\mathrm{card}(J) = 1$, is an *unbounded* subset of $c_0(J, \mathbb{Q}_{p,\mu})$, because it is a translate of the $\mathbb{Q}_p$-linear subspace $\big\{\{x_j\}_{j\in J} \in c_0(J, \mathbb{Q}_{p,\mu}) \mid x_j \in \mathbb{Q}_p,\ \forall j \in J,\ \sum_{j\in J} x_j = 0\big\}$. E.g., if $\mathrm{card}(J) \geq 2$, for every $n \in \mathbb{N}$, the probability distribution $\pi = \big\{p^{-n}, 1 - p^{-n}, 0, 0, \ldots\big\} \subset c_0(J, \mathbb{Q}_{p,\mu})$ is such that $\|\pi\|_\infty = p^n$.
- The probability hyperplane $\varpi_0(J, \mathbb{Q}_{p,\mu})$ contains a distinguished $\mathbb{Q}_p$-convex subset $\upsilon_0(J, \mathbb{Q}_{p,\mu})$ of $c_0(J, \mathbb{Q}_{p,\mu})$ — namely,

  $$\upsilon_0(J, \mathbb{Q}_{p,\mu}) := \big\{\{\pi_j\}_{j\in J} \in c_0(J, \mathbb{Q}_{p,\mu}) \mid \pi_j \in \mathbb{Z}_p,\ \forall j \in J,\ \textstyle\sum_{j\in J} \pi_j = 1\big\} \tag{8.12}$$

  — that is a *bounded* closed subset of $c_0(J, \mathbb{Q}_{p,\mu})$, called the *probability simplex* of $c_0(J, \mathbb{Q}_{p,\mu})$. E.g., the sequence $\big\{\pi_n = p^{n-1}(1-p)\big\}_{n\in\mathbb{N}}$ belongs to $\upsilon_0(\mathbb{N}, \mathbb{Q}_{p,\mu})$.

8.3 States in p-adic quantum mechanics

In the spirit of the algebraic approach to quantum mechanics and taking into account the peculiar features of p-adic probability theory, we now define a state of the unital Banach $*$-algebra $\mathcal{B}_{\rm ad}(\mathcal{H})$ as a suitable element of $\mathcal{B}_{\rm ad}(\mathcal{H})'$, where $\mathcal{H}$ is a p-adic Hilbert space over a quadratic extension $\mathbb{Q}_{p,\mu}$ of $\mathbb{Q}_p$.

Definition 8.3.1. A *state for the p-adic Hilbert space* $\mathcal{H}$ is a linear functional

$$\Omega\colon \mathcal{B}_{\rm ad}(\mathcal{H}) \to \mathbb{Q}_{p,\mu} \tag{8.13}$$

satisfying the following conditions:

(S1) Ω is a *bounded functional*, i.e., $\|\Omega\| := \sup_{\|A\|\neq 0} |\Omega(A)|/\|A\| < \infty$.

(S2) Ω is *involution-preserving*, i.e., $\Omega(A^*) = \overline{\Omega(A)}$, for all $A \in \mathcal{B}_{\rm ad}(\mathcal{H})$.

(S3) Ω is *normalized*, i.e., $\Omega(\mathrm{Id}) = 1$.

By comparison with the complex setting, it is clear that a distinguishing feature of the p-adic case is contained in condition (S2) (recall that an analogous property follows from the positivity condition in the complex case). Also note that, since every selfadjoint operator $A \in \mathcal{B}_{\rm sa}(\mathcal{H})$ can be written as $A = B + B^*$, for some adjointable operator $B \in \mathcal{B}_{\rm ad}(\mathcal{H})$, we have:

$$\Omega(A) = \Omega(B + B^*) = \Omega(B) + \overline{\Omega(B)} \in \mathbb{Q}_p, \quad \forall A \in \mathcal{B}_{\rm sa}(\mathcal{H}). \tag{8.14}$$

Proposition 8.3.1. *Given any state* $\Omega\colon \mathcal{B}_{\rm ad}(\mathcal{H}) \to \mathbb{Q}_{p,\mu}$ *for* $\mathcal{H}$, *there is a bounded functional* $\Omega_{\rm ext}\colon \mathcal{B}(\mathcal{H}) \to \mathbb{Q}_{p,\mu}$ *such that*

(E1) $\Omega_{\rm ext}$ *agrees with* Ω *on* $\mathcal{B}_{\rm ad}(\mathcal{H})$, *i.e.*, $\Omega_{\rm ext}(A) = \Omega(A)$, *for all* $A \in \mathcal{B}_{\rm ad}(\mathcal{H})$; *whence*, $\Omega_{\rm ext}$ *is involution-preserving on* $\mathcal{B}_{\rm ad}(\mathcal{H})$ — $\Omega_{\rm ext}(A^*) = \overline{\Omega_{\rm ext}(A)}$, $A \in \mathcal{B}_{\rm ad}(\mathcal{H})$ — *and* $\Omega_{\rm ext}(\mathrm{Id}) = 1$.

(E2) $\|\Omega_{\rm ext}\| = \|\Omega\|$, *where the norms are defined on* $\mathcal{B}(\mathcal{H})'$ *and* $\mathcal{B}_{\rm ad}(\mathcal{H})'$, *respectively.*

Proof. Given any state Ω for $\mathcal{H}$, by Theorem 6.4.1, it is sufficient to take a Hahn-Banach extension $\Omega_{\rm ext}\colon \mathcal{B}(\mathcal{H}) \to \mathbb{Q}_{p,\mu}$ of the bounded functional $\Omega\colon \mathcal{B}_{\rm ad}(\mathcal{H}) \to \mathbb{Q}_{p,\mu}$ (see Theorem 6.4.1). □

Let us denote by $\mathcal{S}(\mathcal{H})$ the set of all states for $\mathcal{H}$. By the preceding result, $\mathcal{S}(\mathcal{H})$ can be identified with the quotient $\mathcal{S}_{\rm ext}(\mathcal{H})/\sim$ of the set

$$\mathcal{S}_{\rm ext}(\mathcal{H}) := \{\Theta \in \mathcal{B}(\mathcal{H})' \mid \Theta \text{ is involution-preserving on } \mathcal{B}_{\rm ad}(\mathcal{H}) \text{ and } \Omega_{\rm ext}(\mathrm{Id}) = 1\} \tag{8.15}$$

w.r.t. the equivalence relation defined by

$$\Theta_1 \sim \Theta_2 \overset{\text{def}}{\iff} \Theta_1(A) = \Theta_2(A),\ \forall A \in \mathcal{B}_{\rm ad}(\mathcal{H}). \tag{8.16}$$

Moreover, in each equivalence class of $\mathcal{S}_{\rm ext}(\mathcal{H})$ modulo $\sim$, there is a functional $\Theta \in \mathcal{B}(\mathcal{H})'$ such that $\|\Theta\| = \|\Theta|_{\mathcal{B}_{\rm ad}(\mathcal{H})}\|$.

Proposition 8.3.2. $\mathcal{S}(\mathcal{H})$ *and* $\mathcal{S}_{\rm ext}(\mathcal{H})$ *are* $\mathbb{Q}_p$*-affine subsets of* $\mathcal{B}_{\rm ad}(\mathcal{H})'$ *and* $\mathcal{B}(\mathcal{H})'$, *respectively.*

Proof. Just observe that, by Definition 8.3.1, if $\Omega_1, \Omega_2 \in \mathcal{S}(\mathcal{H})$, then $\alpha\Omega_1 + (1-\alpha)\Omega_2 \in \mathcal{S}(\mathcal{H})$, for all $\alpha \in \mathbb{Q}_p$, and, by (8.15), an analogous property holds for $\mathcal{S}_{\rm ext}(\mathcal{H})$ too. □

8.4 The trace induced states

Just like in the standard complex setting, the algebraically defined p-adic states are somewhat too general and vaguely characterized objects to be useful for most practical applications, and it is natural to restrict to the much more concrete class of *tracial* states.

For the sake of notational simplicity, we will assume that $\dim(\mathcal{H}) = \infty$, but the subsequent results and their proofs remain valid — with obvious adaptations — in the finite-dimensional case (say, neglecting the trivial case where $\dim(\mathcal{H}) = 1$, for $2 \leq \dim(\mathcal{H}) < \infty$). In particular, if $\dim(\mathcal{H}) < \infty$, $\mathcal{T}(\mathcal{H}) = \mathcal{B}_{\rm ad}(\mathcal{H}) = \mathcal{B}(\mathcal{H})$ and all states for $\mathcal{H}$ are tracial.

Let us first consider the subset $\mathcal{T}_{\rm st}(\mathcal{H})$ (where the subscript stands for 'statistical') of $\mathcal{T}(\mathcal{H})$ defined by

$$\mathcal{T}_{\rm st}(\mathcal{H}) := \{S \in \mathcal{T}_{\rm sa}(\mathcal{H}) \mid \operatorname{tr}(S) = 1\}. \tag{8.17}$$

We endow $\mathcal{T}_{\rm st}(\mathcal{H})$ with the relative topology w.r.t. the p-adic Hilbert-Schmidt space $\mathcal{T}(\mathcal{H})$ (the norm-topology).

Theorem 8.4.1. *$\mathcal{T}_{\rm st}(\mathcal{H})$ is a closed $\mathbb{Q}_p$-affine subset of $\mathcal{T}(\mathcal{H})$. A linear operator S belongs to $\mathcal{T}_{\rm st}(\mathcal{H})$ if and only if it is of the form*

$$S = \sum_{j\in J} (\sigma_j \, |e_j\rangle\langle f_j| + \overline{\sigma_j} \, |f_j\rangle\langle e_j|), \tag{8.18}$$

where $J = \{1, 2, \ldots\}$ is a countable index set and

- *$\{\sigma_j\}_{j\in J} \subset \mathbb{Q}^*_{p,\mu}$ and, if $J = \mathbb{N}$, $\lim_j \sigma_j = 0$;*
- *$\{e_j\}_{j\in J}$ is a normalized norm-orthogonal system in $\mathcal{H}$, and $\|f_j\| = 1$, for all $j \in J$;*
- *$\sum_{j\in J}(\sigma_j \, \langle f_j, e_j\rangle + \overline{\sigma_j} \, \langle e_j, f_j\rangle) = 1$;*
- *the sum in (8.18) — whenever J is not finite — converges w.r.t. the norm topology.*

Every $S \in \mathcal{T}_{\rm st}(\mathcal{H})$ admits a decomposition of the previous form where, in particular, the norm-orthogonal system $\{e_j\}_{j\in J}$ is contained in an orthonormal basis in $\mathcal{H}$.

For every $S \in \mathcal{T}_{\rm st}(\mathcal{H})$, the functional $\operatorname{tr}((\cdot)S) \colon \mathcal{B}_{\rm ad}(\mathcal{H}) \to \mathbb{Q}_{p,\mu}$ is a state for $\mathcal{H}$, and the map

$$\tau_{\mathcal{H}} \colon \mathcal{T}_{\rm st}(\mathcal{H}) \ni S \mapsto \big(\operatorname{tr}((\cdot)S) \colon \mathcal{B}_{\rm ad}(\mathcal{H}) \to \mathbb{Q}_{p,\mu}\big) \in \mathcal{S}(\mathcal{H}) \tag{8.19}$$

is a continuous $\mathbb{Q}_p$-affine injection of $\mathcal{T}_{\rm st}(\mathcal{H})$ into $\mathcal{S}(\mathcal{H})$, where $\mathcal{S}(\mathcal{H})$ is endowed with the relative topology w.r.t. $\mathcal{B}_{\rm ad}(\mathcal{H})'$; moreover, $\|\tau_{\mathcal{H}}(S)\| = \|S\|$, for all $S \in \mathcal{T}_{\rm st}(\mathcal{H})$.

Proof. The fact that $\mathcal{T}_{\rm st}(\mathcal{H})$ is a $\mathbb{Q}_p$-affine subset of $\mathcal{T}(\mathcal{H})$ is straightforward from (8.17). By Corollary 7.3.4, the linear functional $\operatorname{tr}(\cdot) \colon \mathcal{T}(\mathcal{H}) \to \mathbb{Q}_{p,\mu}$ is bounded, so that $\mathcal{T}_{\rm st}(\mathcal{H})$ — which is the intersection of the closed subset $\mathcal{T}_{\rm sa}(\mathcal{H})$ of $\mathcal{T}(\mathcal{H})$ with the (closed) pre-image w.r.t. $\operatorname{tr}(\cdot)$ of the singleton set $\{1\} \subset \mathbb{Q}_{p,\mu}$ — is closed in $\mathcal{T}(\mathcal{H})$.

The second and the third assertion of the statement follow directly from the definition of $\mathcal{T}_{\rm st}(\mathcal{H})$ and Proposition 7.6.1.

Moreover, given any $A \in \mathcal{B}_{\rm ad}(\mathcal{H})$ and $T \in \mathcal{T}(\mathcal{H})$, the estimate

$$|\operatorname{tr}(AT)| \leq \|A\| \, \|T\|, \tag{8.20}$$

see the first inequality in (7.157), shows that $\operatorname{tr}((\cdot)T) \colon \mathcal{B}_{\rm ad}(\mathcal{H}) \to \mathbb{Q}_{p,\mu}$ is a bounded functional. In particular, for every $S \in \mathcal{T}_{\rm st}(\mathcal{H}) \subset \mathcal{T}_{\rm sa}(\mathcal{H})$, the bounded functional $\operatorname{tr}((\cdot)S)$ is both involution preserving — $\operatorname{tr}(A^*S) = \overline{\operatorname{tr}(SA)} = \overline{\operatorname{tr}(AS)}$, where we have used relation `(P2)` in Proposition 7.3.4 and the cyclic property of the trace — and normalized. Therefore, $\mathrm{tr}((\cdot)S)\colon \mathcal{B}_{\mathrm{ad}}(\mathcal{H}) \to \mathbb{Q}_{p,\mu}$ is a state for $\mathcal{H}$.

It is clear that the map $\tau_{\mathcal{H}}$ is $\mathbb{Q}_p$-affine. Consider, next, the linear map

$$\widetilde{\tau}_{\mathcal{H}}\colon \mathcal{T}(\mathcal{H}) \ni T \mapsto \big(\mathrm{tr}((\cdot)T)\colon \mathcal{B}_{\mathrm{ad}}(\mathcal{H}) \to \mathbb{Q}_{p,\mu}\big) \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})'. \tag{8.21}$$

Here, as previously noted, for every $T \in \mathcal{T}(\mathcal{H})$ the linear functional $\widetilde{\tau}_{\mathcal{H}}(T) = \mathrm{tr}((\cdot)T)$ is bounded and, by (8.20), $\|\widetilde{\tau}_{\mathcal{H}}(T)\| \leq \|T\|$. Now, taking $A \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ of the form $A = |\phi\rangle\langle\psi|$, where ϕ, ψ are arbitrary vectors in $\mathcal{H}$, we have that $\mathrm{tr}(AT) = \mathrm{tr}(|\phi\rangle\langle\psi|T) = \mathrm{tr}(|T\phi\rangle\langle\psi|) = \langle\psi, T\phi\rangle$. Thus, given any orthonormal basis $\{\phi_m\}_{m\in\mathbb{N}}$, we also have: $\|T\| = \sup_{m,n\in\mathbb{N}} |\langle\phi_m, T\phi_n\rangle| = \sup_{m,n\in\mathbb{N}} |\mathrm{tr}(|\phi_m\rangle\langle\phi_n|T)| \leq \|\widetilde{\tau}_{\mathcal{H}}(T)\|$ ($\|\,|\phi_m\rangle\langle\phi_n|\,\| = 1$). We conclude that $\|\widetilde{\tau}_{\mathcal{H}}(T)\| = \|T\|$; i.e., $\widetilde{\tau}_{\mathcal{H}}$ is a linear isometry, and then the $\mathbb{Q}_p$-affine map $\tau_{\mathcal{H}}$, which can be regarded as a restriction of $\widetilde{\tau}_{\mathcal{H}}$, is injective and continuous. Moreover, obviously, $\|\tau_{\mathcal{H}}(S)\| = \|S\|$, for all $S \in \mathcal{T}_{\mathrm{st}}(\mathcal{H})$. ☐

Definition 8.4.1. We call the operators in the $\mathbb{Q}_p$-affine subset $\mathcal{T}_{\mathrm{st}}(\mathcal{H})$ of $\mathcal{T}(\mathcal{H})$ the *statistical operators* in $\mathcal{H}$. Moreover, we call the states in the set $\mathcal{S}_{\mathrm{tr}}(\mathcal{H}) := \tau_{\mathcal{H}}(\mathcal{T}_{\mathrm{st}}(\mathcal{H})) \subset \mathcal{S}(\mathcal{H})$ the *trace induced states* for $\mathcal{H}$.

Since the map $\tau_{\mathcal{H}}$ is $\mathbb{Q}_p$-affine, then $\mathcal{S}_{\mathrm{tr}}(\mathcal{H})$ is a $\mathbb{Q}_p$-affine subset of $\mathcal{S}(\mathcal{H})$ that, by the final assertion of Theorem 8.4.1, can be isometrically identified with $\mathcal{T}_{\mathrm{st}}(\mathcal{H})$. Note that, by the second estimate in (7.157) (also see (7.188)), for every tracial state $\Omega = \tau_{\mathcal{H}}(S)$, $S \in \mathcal{T}_{\mathrm{st}}(\mathcal{H})$, we have:

$$\Omega(\mathrm{Id}) = 1 = \mathrm{tr}(S) \leq \|S\| = \|\Omega\|. \tag{8.22}$$

This is a further difference w.r.t. the complex case, where every state ω satisfies $\|\omega\| = \omega(\mathrm{Id}) = 1$.

Let us then have a closer look at the structure of the $\mathbb{Q}_p$-affine set $\mathcal{T}_{\mathrm{st}}(\mathcal{H})$ of all statistical operators. To highlight this affine structure, let us first observe that — introducing the set

$$\mathcal{T}_{\mathrm{sa}}(\mathcal{H})_0 := \{T \in \mathcal{T}_{\mathrm{sa}}(\mathcal{H}) \mid \mathrm{tr}(T) = 0\}, \tag{8.23}$$

which is a (closed) $\mathbb{Q}_p$-linear subspace of $\mathcal{T}(\mathcal{H})$ — for $S \in \mathcal{T}_{\mathrm{st}}(\mathcal{H})$ and $T \in \mathcal{T}_{\mathrm{sa}}(\mathcal{H})_0$,

$$S + T \in \mathcal{T}_{\mathrm{st}}(\mathcal{H}), \text{ and, if } \|S\| < \|T\|, \text{ then } \|S + T\| = \|T\|. \tag{8.24}$$

We will call $S + T$ a *zero-trace perturbation* of the statistical operator S (by T), and, by the previous argument, it is easy to see that there exists a zero-trace perturbation of S of arbitrarily large norm ($\mathcal{T}_{\mathrm{sa}}(\mathcal{H})_0$ being a $\mathbb{Q}_p$-linear subspace of $\mathcal{T}(\mathcal{H})$). Moreover, considering a symmetric decomposition (8.18) of S, there is a natural partition $J = J_0 \sqcup J_1$ of the index set J, where

$$J_0 := \big\{j \in J \mid \langle e_j, f_j\rangle = 0\big\} \quad \text{and} \quad J_1 := \big\{j \in J \mid \langle e_j, f_j\rangle \neq 0\big\}, \tag{8.25}$$

and we can write $S = S_0 + S_1$, with

$$S_0 := \sum_{j\in J_0} (\sigma_j\,|e_j\rangle\langle f_j| + \overline{\sigma_j}\,|f_j\rangle\langle e_j|) \in \mathcal{T}_{\mathrm{sa}}(\mathcal{H})_0,\ S_1 := \sum_{j\in J_1} (\sigma_j\,|e_j\rangle\langle f_j| + \overline{\sigma_j}\,|f_j\rangle\langle e_j|) \in \mathcal{T}_{\mathrm{st}}(\mathcal{H}). \tag{8.26}$$

(If $J_0 = \emptyset$, then we put $S_0 \equiv 0$.) Here, we have used formula (7.203) to conclude that $\mathrm{tr}(S_0) = 0$, whereas $\mathrm{tr}(S_1) = \mathrm{tr}(S) = 1$. Therefore, S is a zero-trace perturbation of S_1 by S_0.

We will now derive a useful refinement of a symmetric decomposition of a statistical operator. To this end, we will fruitfully adopt the following convenient notation:

Notation 8.4.1. For every pair of *nonzero vectors* $\phi, \psi \in \mathcal{H}$ — $\phi \neq 0 \neq \psi$ — and every $\sigma \in \mathbb{Q}_{p,\mu}^* \equiv \mathbb{Q}_{p,\mu} \setminus \{0\}$, we set

$$\widehat{\phi\psi}(\sigma) := \begin{cases} (\sigma + \overline{\sigma})^{-1}(\sigma\,|\phi\rangle\langle\psi| + \overline{\sigma}\,|\psi\rangle\langle\phi|) \in \mathcal{T}_{\mathrm{sa}}(\mathcal{H})_0 & \text{if } \langle\phi,\psi\rangle = 0 \\ (\sigma\,\langle\psi,\phi\rangle + \overline{\sigma}\,\langle\phi,\psi\rangle)^{-1}(\sigma\,|\phi\rangle\langle\psi| + \overline{\sigma}\,|\psi\rangle\langle\phi|) \in \mathcal{T}_{\mathrm{st}}(\mathcal{H}) & \text{if } \langle\phi,\psi\rangle \neq 0 \end{cases} . \tag{8.27}$$

Here, we stress that $\widehat{\phi\psi}(\sigma) \in \mathcal{T}_{\mathrm{sa}}(\mathcal{H})$, and $\mathrm{tr}\big(\widehat{\phi\psi}(\sigma)\big) = 1$, if $\langle\phi,\psi\rangle \neq 0$, whereas $\mathrm{tr}\big(\widehat{\phi\psi}(\sigma)\big) = 0$, otherwise. Moreover, we introduce the sets

$$\widehat{\mathcal{T}}_{\mathrm{st}}(\mathcal{H}) := \big\{S \in \mathcal{T}_{\mathrm{st}}(\mathcal{H}) \mid S = \widehat{\phi\psi}(\sigma),\ \phi,\psi \in \mathcal{H}\setminus\{0\},\ \langle\phi,\psi\rangle \neq 0,\ \sigma \in \mathbb{Q}_{p,\mu}^*\big\}, \tag{8.28}$$

$$\widehat{\mathcal{T}}_{\mathrm{sa}}(\mathcal{H})_0 := \big\{S \in \mathcal{T}_{\mathrm{sa}}(\mathcal{H})_0 \mid S = \widehat{\phi\psi}(\sigma),\ \phi,\psi \in \mathcal{H}\setminus\{0\},\ \langle\phi,\psi\rangle = 0,\ \sigma \in \mathbb{Q}_{p,\mu}^*\big\}. \tag{8.29}$$

We also put $\widehat{\phi\psi} \equiv \widehat{\phi\psi}(1)$. Note that $\widehat{\phi\psi}(\sigma) = \widehat{\psi\phi}(\overline{\sigma})$. Moreover, $\widehat{\phi\psi}(\alpha\sigma) = \widehat{\phi\psi}(\sigma)$, for all $\alpha \in \mathbb{Q}_p$, and — in the case where $\langle\phi,\psi\rangle \neq 0$ — an analogous equality holds, if we map ϕ or ψ into $\alpha\phi$ or $\alpha\psi$, respectively.

In particular, given a nonzero vector ψ, we have that $\widehat{\psi\psi}(\sigma) = \widehat{\psi\psi} = |\psi\rangle\langle\psi| \in \widehat{\mathcal{T}}_{\mathrm{sa}}(\mathcal{H})_0$, if the vector ψ is *isotropic*, and $\widehat{\psi\psi}(\sigma) = \widehat{\psi\psi} = \langle\psi,\psi\rangle^{-1}|\psi\rangle\langle\psi| \in \widehat{\mathcal{T}}_{\mathrm{st}}(\mathcal{H})$, otherwise. In the latter case, the statistical operator $\widehat{\psi\psi}$ is a (selfadjoint) *rank-one projection*: $\widehat{\psi\psi}\,\widehat{\psi\psi} = \widehat{\psi\psi}$.

Definition 8.4.2. Let $\phi, \psi \in \mathcal{H}$ be a pair of *nonzero vectors*, and let $\sigma \in \mathbb{Q}_{p,\mu}^*$. If $\langle\phi,\psi\rangle \neq 0$, we say that $\widehat{\phi\psi}(\sigma) \in \widehat{\mathcal{T}}_{\mathrm{st}}(\mathcal{H})$ is a *simple statistical operator*. If, instead, $\langle\phi,\psi\rangle = 0$, we say that $\widehat{\phi\psi}(\sigma) \in \widehat{\mathcal{T}}_{\mathrm{sa}}(\mathcal{H})_0$ is a *simple zero-trace operator*.

By the previously introduced notation, we can suitably re-write the symmetric decomposition $S = \sum_{j\in J}(\sigma_j\,|e_j\rangle\langle f_j| + \overline{\sigma_j}\,|f_j\rangle\langle e_j|) = S_0 + S_1$ of the statistical operator S. In fact, by construction, we have that

$$S_0 = \sum_{j\in J_0}(\sigma_j + \overline{\sigma_j})\,\widehat{e_j f_j}(\sigma_j) \equiv \sum_{j\in J_0}\gamma_j\,\widehat{e_j f_j}(\sigma_j) \in \mathcal{T}_{\mathrm{sa}}(\mathcal{H})_0, \quad \text{where: } \gamma_j \in \mathbb{Q}_p,\ j \in J_1, \tag{8.30}$$

while

$$\begin{aligned} S_1 &= \sum_{j\in J_1}(\sigma_j\,\langle f_j,e_j\rangle + \overline{\sigma_j}\,\langle e_j,f_j\rangle)\,\widehat{e_j f_j}(\sigma_j) \\ &\equiv \sum_{j\in J_1}\pi_j\,\widehat{e_j f_j}(\sigma_j) \in \mathcal{T}_{\mathrm{st}}(\mathcal{H}), \quad \text{where: } \pi_j \in \mathbb{Q}_p,\ j \in J_1, \text{ and } \textstyle\sum_{j\in J_1}\pi_j = \mathrm{tr}(S) = 1. \end{aligned} \tag{8.31}$$

Therefore, actually, $S_0 \in \overline{\mathrm{span}}_{\mathbb{Q}_p}\big(\widehat{\mathcal{T}}_{\mathrm{sa}}(\mathcal{H})_0\big)$ and $S_1 \in \overline{\mathrm{aff}}_{\mathbb{Q}_p}\big(\widehat{\mathcal{T}}_{\mathrm{st}}(\mathcal{H})\big)$.

Remark 8.4.1. Let us observe explicitly that $\overline{\mathrm{span}}_{\mathbb{Q}_p}\big(\widehat{\mathcal{T}}_{\mathrm{sa}}(\mathcal{H})_0\big) \subset \mathcal{T}_{\mathrm{sa}}(\mathcal{H})_0$, since the $\mathbb{Q}_p$-linear space $\mathcal{T}_{\mathrm{sa}}(\mathcal{H})_0$ is closed in $\mathcal{T}(\mathcal{H})$. Moreover, we have that $\mathrm{aff}_{\mathbb{Q}_p}\big(\widehat{\mathcal{T}}_{\mathrm{st}}(\mathcal{H})\big) \subset \mathcal{T}_{\mathrm{st}}(\mathcal{H})$, and hence $\overline{\mathrm{aff}}_{\mathbb{Q}_p}\big(\widehat{\mathcal{T}}_{\mathrm{st}}(\mathcal{H})\big) \subset \mathcal{T}_{\mathrm{st}}(\mathcal{H})$ too, because $\mathcal{T}_{\mathrm{st}}(\mathcal{H})$ is a *closed* $\mathbb{Q}_p$-affine subset of $\mathcal{T}(\mathcal{H})$.

It is clear that, conversely, every linear operator in $\mathcal{H}$ of the form $S = S_0 + S_1$ — where $S_0 \in \overline{\mathrm{span}}_{\mathbb{Q}_p}\big(\widehat{\mathcal{T}}_{\mathrm{sa}}(\mathcal{H})_0\big)$ and $S_1 \in \overline{\mathrm{aff}}_{\mathbb{Q}_p}\big(\widehat{\mathcal{T}}_{\mathrm{st}}(\mathcal{H})\big)$ — is a statistical operator, because in such a case S is a zero-trace perturbation of a statistical operator S_1 (by S_0).

We eventually get to the following result:

Theorem 8.4.2. *Every statistical operator S in $\mathcal{H}$ can be expressed as a zero-trace perturbation of a statistical operator S_1 — contained in the closed $\mathbb{Q}_p$-affine hull $\overline{\mathrm{aff}}_{\mathbb{Q}_p}\big(\widehat{\mathcal{T}}_{\mathrm{st}}(\mathcal{H})\big) \subset \mathcal{T}_{\mathrm{st}}(\mathcal{H})$ generated by all simple statistical operators — by an operator $S_0 \in \overline{\mathrm{span}}_{\mathbb{Q}_p}\big(\widehat{\mathcal{T}}_{\mathrm{sa}}(\mathcal{H})_0\big) \subset \mathcal{T}_{\mathrm{sa}}(\mathcal{H})_0$. Conversely, every zero-trace perturbation of a statistical operator — in particular, of an operator contained in $\overline{\mathrm{aff}}_{\mathbb{Q}_p}\big(\widehat{\mathcal{T}}_{\mathrm{st}}(\mathcal{H})\big)$ by an operator in $\overline{\mathrm{span}}_{\mathbb{Q}_p}\big(\widehat{\mathcal{T}}_{\mathrm{sa}}(\mathcal{H})_0\big)$ — is a statistical operator too.*

Therefore, we have that

$$\mathcal{T}_{\mathrm{st}}(\mathcal{H}) = \mathcal{T}_{\mathrm{sa}}(\mathcal{H})_0 + \mathcal{T}_{\mathrm{st}}(\mathcal{H}) = \mathcal{T}_{\mathrm{sa}}(\mathcal{H})_0 + \overline{\mathrm{aff}}_{\mathbb{Q}_p}\big(\widehat{\mathcal{T}}_{\mathrm{st}}(\mathcal{H})\big) = \overline{\mathrm{span}}_{\mathbb{Q}_p}\big(\widehat{\mathcal{T}}_{\mathrm{sa}}(\mathcal{H})_0\big) + \overline{\mathrm{aff}}_{\mathbb{Q}_p}\big(\widehat{\mathcal{T}}_{\mathrm{st}}(\mathcal{H})\big). \tag{8.32}$$

Moreover, for every $T \in \mathcal{T}_{\mathrm{st}}(\mathcal{H})$, we have:

$$\mathcal{T}_{\mathrm{st}}(\mathcal{H}) = T + \mathcal{T}_{\mathrm{sa}}(\mathcal{H})_0; \tag{8.33}$$

otherwise stated, $\mathcal{T}_{\mathrm{st}}(\mathcal{H})$ coincides with the (closed) $\mathbb{Q}_p$-affine subset $T + \mathcal{T}_{\mathrm{sa}}(\mathcal{H})_0$ of $\mathcal{T}(\mathcal{H})$.

Proof. The first assertion follows form our previous discussion; in particular, for any $S \in \mathcal{T}_{\mathrm{st}}(\mathcal{H})$, from the decomposition $S = S_0 + S_1$, with S_0, S_1 expressed as in (8.30) and (8.31), respectively. For the final assertion, just note that every statistical operator $S \in \mathcal{T}_{\mathrm{st}}(\mathcal{H})$ can be expressed in the form $S = T + (S - T)$, where T is some (fixed) statistical operator and $(S - T) \in \mathcal{T}_{\mathrm{sa}}(\mathcal{H})_0$. □

We have already noted that $\mathcal{T}_{\mathrm{st}}(\mathcal{H})$ — being $\mathbb{Q}_p$-affine — is an *unbounded* subset of $\mathcal{T}(\mathcal{H})$, in sharp contrast w.r.t. the complex case. It is natural to ask: Is there any p-adic parallel for the *bounded* convex set of density operators in a separable *complex* Hilbert space?

Let us consider the following set of statistical operators:

$$\mathcal{D}_r(\mathcal{H}) := \{S \in \mathcal{T}_{\mathrm{st}}(\mathcal{H}) \mid \|T\| \le r\}, \quad r \in \|\mathcal{T}(\mathcal{H})\| \setminus \{0\} = |\mathbb{Q}_{p,\mu}^*|. \tag{8.34}$$

Here, note that

- for $r < 1$, $\mathcal{D}_r(\mathcal{H}) = \{S \in \mathcal{T}_{\mathrm{st}}(\mathcal{H}) \mid \|T\| \le r\} = \emptyset$, because every statistical operator has norm not smaller than 1;
- by the previous point, for $r \ge 1$, $\mathcal{D}_r(\mathcal{H}) = \{S \in \mathcal{T}_{\mathrm{st}}(\mathcal{H}) \mid 1 \le \|T\| \le r\}$;
- for every for $s \in |\mathbb{Q}_{p,\mu}^*|$, $s \ge 1$, $\cup_{s \le r \in |\mathbb{Q}_{p,\mu}^*|} \mathcal{D}_r(\mathcal{H}) = \mathcal{T}_{\mathrm{st}}(\mathcal{H})$;
- expressing a statistical operator $S \in \mathcal{T}_{\mathrm{st}}(\mathcal{H})$ as a matrix operator — $S = \mathrm{op}_\Phi(S_{mn})$, for some orthonormal basis $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}$ — it is easy to see, by means of explicit examples, that $\mathcal{D}_1(\mathcal{H}) = \{T \in \mathcal{T}_{\mathrm{st}}(\mathcal{H}) \mid \|T\| = 1\} \neq \emptyset$ and, for $r, s \in |\mathbb{Q}_{p,\mu}^*|$, with $r > s \ge 1$, $\mathcal{D}_r(\mathcal{H}) \supsetneq \mathcal{D}_s(\mathcal{H})$, because $\|\mathcal{D}_r(\mathcal{H})\| = \{t \in |\mathbb{Q}_{p,\mu}^*| \mid 1 \le t \le r\}$ (note, by the way, that $\|\mathcal{T}_{\mathrm{st}}(\mathcal{H})\| = \{t \in |\mathbb{Q}_{p,\mu}^*| \mid t \ge 1\}$);
- by Theorem 8.2.1 and by the strong triangle inequality for the operator norm, for every $r \in |\mathbb{Q}_{p,\mu}^*|$, $r \ge 1$, $\mathcal{D}_r(\mathcal{H})$ is a $\mathbb{Q}_p$-convex subset of $\mathcal{T}(\mathcal{H})$.

Definition 8.4.3. We call *density operators* those statistical operators in the p-adic Hilbert space $\mathcal{H}$ belonging to the bounded subset $\mathcal{D}(\mathcal{H}) \equiv \mathcal{D}_1(\mathcal{H})$ of $\mathcal{T}(\mathcal{H})$ defined by

$$\mathcal{D}(\mathcal{H}) := \{T \in \mathcal{T}_{\mathrm{st}}(\mathcal{H}) \mid \|T\| \le 1\} = \{T \in \mathcal{T}_{\mathrm{st}}(\mathcal{H}) \mid \|T\| = 1\}. \tag{8.35}$$

Example 8.4.1. If $\psi \in \mathcal{H}$, is a non-isotropic nonzero vector, then the statistical operator $\widehat{\psi\psi} = \langle\psi,\psi\rangle^{-1}|\psi\rangle\langle\psi| \in \widehat{\mathcal{T}}_{\mathrm{st}}(\mathcal{H})$ is a density operator if and only if $|\langle\psi,\psi\rangle| = \|\psi\|^2$. For every orthonormal basis $\{\phi_m\}_{m\in\mathbb{N}}$ and every probability distribution $\{\pi_m\}_{m\in\mathbb{N}}$ contained in the probability simplex $\upsilon_0(\mathbb{N}, \mathbb{Q}_{p,\mu})$, $\sum_{m\in\mathbb{N}} \pi_m\, |\phi_m\rangle\langle\phi_m| \in \mathcal{D}(\mathcal{H})$. E.g., $\sum_{m\in\mathbb{N}} p^{m-1}(1 - p)\, |\phi_m\rangle\langle\phi_m|$ is a density operator.

Proposition 8.4.1. *$\mathcal{D}(\mathcal{H})$ is a $\mathbb{Q}_p$-convex closed subset of $\mathcal{T}(\mathcal{H})$. For every trace class operator $T \in \mathcal{T}(\mathcal{H})$, the following facts are equivalent:*

(D1) *$T \in \mathcal{D}(\mathcal{H})$.*

(D2) *$T \in \mathcal{T}_{\rm sa}(\mathcal{H})$ and admits a canonical decomposition of the form $T = \sum_{j\in J} \lambda_j |e_j\rangle\langle f_j|$, where $\{\lambda_j\}_{j\in J}$ is contained in the valuation ring $\mathfrak{V}_{p,\mu} \equiv \mathbb{Q}_{p,\mu,1} := \{z \in \mathbb{Q}_{p,\mu} \mid |z| \leq 1\}$ of $\mathbb{Q}_{p,\mu}$ and $\sum_{j\in J} \lambda_j \langle f_j, e_j\rangle = 1$.*

(D3) *$T \in \mathcal{T}_{\rm sa}(\mathcal{H})$ and admits a canonical decomposition of the form $T = \sum_{j\in J} \lambda_j |e_j\rangle\langle f_j|$, where $\max_{j\in J} |\lambda_j| = 1$ and $\sum_{j\in J} \lambda_j \langle f_j, e_j\rangle = 1$.*

If $p \neq 2$, conditions (D1)–(D3) *are equivalent to the following:*

(D4) *$T \in \mathcal{T}_{\rm sa}(\mathcal{H})$ and admits a symmetric decomposition of the form $T = \sum_{j\in J}(\sigma_j |e_j\rangle\langle f_j| + \overline{\sigma_j} |f_j\rangle\langle e_j|)$, where $\{\sigma_j\}_{j\in J} \subset \mathfrak{V}_{p,\mu}$ and $\sum_{j\in J}(\sigma_j \langle f_j, e_j\rangle + \overline{\sigma_j} \langle e_j, f_j\rangle) = 1$.*

Proof. As previously noted, by Theorem 8.2.1 and by the strong triangle inequality for the operator norm, $\mathcal{D}(\mathcal{H})$ is a $\mathbb{Q}_p$-convex subset of $\mathcal{T}(\mathcal{H})$. Moreover, $\mathcal{D}(\mathcal{H})$ is the intersection of the closed unit ball (or of the unit sphere) in $\mathcal{T}(\mathcal{H})$ with the closed subset $\mathcal{T}_{\rm st}(\mathcal{H})$ of $\mathcal{T}(\mathcal{H})$; hence, it is closed in $\mathcal{T}(\mathcal{H})$.

By relations (7.187) and (7.188), given any canonical decomposition $T = \sum_{j\in J} \lambda_j |e_j\rangle\langle f_j|$ of a trace class operator $T \in \mathcal{T}(\mathcal{H})$, we have that $\mathrm{tr}(T) = \sum_{j\in J} \lambda_j \langle f_j, e_j\rangle$ and $\|T\| = \max_{j\in J} |\lambda_j|$ (so that $\|T\| \leq 1 \iff \{\lambda_j\}_{j\in J} \subset \mathbb{Z}_p$). Therefore, conditions (D1) and (D2) are equivalent, and, by the second equality in (8.35), conditions (D2) and (D3) are equivalent too.

Moreover, if condition (D2) is satisfied, writing $T = \frac{1}{2}(T + T^*)$, we obtain the symmetric decomposition $T = \sum_{j\in J}(\sigma_j |e_j\rangle\langle f_j| + \overline{\sigma_j} |f_j\rangle\langle e_j|)$, where $\sum_{j\in J}(\sigma_j \langle f_j, e_j\rangle + \overline{\sigma_j} \langle e_j, f_j\rangle) = \mathrm{tr}(T) = \sum_{j\in J} \lambda_j \langle f_j, e_j\rangle = 1$ and $\{2\sigma_j = \lambda_j\}_{j\in J} \subset \mathfrak{V}_{p,\mu}$; in particular, if $p \neq 2$, then $|2\sigma_j| = |\sigma_j|$ and $\{\sigma_j\}_{j\in J} \subset \mathfrak{V}_{p,\mu}$. Thus, (D2) implies (D4).

Finally, if (D4) holds, then $T \in \mathcal{T}_{\rm sa}(\mathcal{H})$ is such that $\mathrm{tr}(T) = \sum_{j\in J}(\sigma_j \langle f_j, e_j\rangle + \overline{\sigma_j} \langle e_j, f_j\rangle) = 1$ and $\|T\| = \|\sum_{j\in J}(\sigma_j |e_j\rangle\langle f_j| + \overline{\sigma_j} |f_j\rangle\langle e_j|)\| \leq \max_{j\in J} \|\sigma_j |e_j\rangle\langle f_j|\| = \max_{j\in J} |\sigma_j| \leq 1$, so that condition (D1) is verified too. □

8.5 The SOVMs and the statistical interpretation

The statistical interpretation of trace induced states is ensured by defining the observables as a suitable p-adic counterpart of the POVMs:

Definition 8.5.1. A (discrete) *selfadjoint-operator-valued measure* (in short, a SOVM) in $\mathcal{H}$ is a norm-bounded countable family $\{A_i\}_{i\in I} \subset \mathcal{B}_{\rm sa}(\mathcal{H})$ such that $\sum_{i\in I} A_i = \mathrm{Id}$. Here, if the index set I is not finite, the series is supposed to converge w.r.t. the weak operator topology (i.e., the initial topology induced by the family of maps $\{\mathscr{E}_{\phi,\psi}\colon \mathcal{B}(\mathcal{H}) \to \mathbb{Q}_{p,\mu}\}_{\phi,\psi\in\mathcal{H}}$, $\mathscr{E}_{\phi,\psi}(A) := \langle \phi, A\psi\rangle$). We call a SOVM $\{A_i\}_{i\in I} \subset \mathcal{B}_{\rm sa}(\mathcal{H})$ *contractive* if, in particular, $\|A_i\| \leq 1$, for all $i \in I$; we say that it is *tracial* if $\{A_i\}_{i\in I} \subset \mathcal{T}_{\rm sa}(\mathcal{H})$.

Lemma 8.5.1. *Let $\{A_i\}_{i\in I} \subset \mathcal{B}_{\rm sa}(\mathcal{H})$ be a SOVM, where the index set I is (countably) infinite. Then, for every pair of vectors $\phi, \psi \in \mathcal{H}$,*

$$\lim_i \langle \phi, A_i\psi\rangle = 0. \tag{8.36}$$

Proof. In fact, we have that

$$\sum_{i\in I} A_i = \mathrm{Id} \text{ (w.r.t. the weak op. topology)} \implies \langle\phi,\psi\rangle = \langle\phi,(\textstyle\sum_{i\in I}A_i)\psi\rangle = \displaystyle\sum_{i\in I}\langle\phi,A_i\psi\rangle$$
$$\implies \lim_i\langle\phi,A_i\psi\rangle = 0, \tag{8.37}$$

where the second implication holds by Proposition 6.1.1. □

Proposition 8.5.1. *Let $\{A_i\}_{i\in I}\subset\mathcal{B}_{\rm sa}(\mathcal{H})$ be a SOVM. Then, for every trace induced state $\Omega\in\mathcal{S}_{\rm tr}(\mathcal{H})$, $\{\Omega(A_i)\}_{i\in I}$ is a p-adic probability distribution. In particular, if Ω is a* density state *— i.e., $\Omega=\tau_{\mathcal{H}}(S)$, for some $S\in\mathcal{D}(\mathcal{H})$ — and $\{A_i\}_{i\in i}$ is contractive, then $\{\Omega(A_i)\}_{i\in I}$ is contained in the probability simplex $\upsilon_0(I,\mathbb{Q}_{p,\mu})$. Finally, $\{A_i\}_{i\in I}$ is contractive if and only if* $\max_{i\in I}\|A_i\|=1$.

Proof. By relation (8.14), $\{\Omega(A_i)\}_{i\in I}\subset\mathbb{Q}_p$ and, if the index set I is finite, the first assertion follows directly from the condition that $\sum_{i\in I}A_i=\mathrm{Id}$. Thus, we will henceforth assume that the index set I is (countably) infinite.

Given any trace induced state $\Omega\in\mathcal{S}_{\rm tr}(\mathcal{H})$, we have that $\Omega=\tau_{\mathcal{H}}(S)$, for some $S\in\mathcal{T}_{\rm st}(\mathcal{H})$, and, taking any canonical decomposition $S=\sum_{j\in J}\lambda_j\,|e_j\rangle\langle f_j|$ of the trace class operator S, we have that

$$\Omega(A_i)=\mathrm{tr}(A_iS)=\mathrm{tr}\big(A_i\textstyle\sum_{j\in J}\lambda_j\,|e_j\rangle\langle f_j|\big)=\mathrm{tr}\big(\sum_{j\in J}\lambda_j\,|A_ie_j\rangle\langle f_j|\big)$$
$$=\sum_{j\in J}\lambda_j\,\mathrm{tr}\big(|A_ie_j\rangle\langle f_j|\big)=\sum_{j\in J}\lambda_j\,\langle f_j,A_ie_j\rangle. \tag{8.38}$$

Here, the third equality follows from the fact that the linear map $\mathcal{T}(\mathcal{H})\ni T\mapsto A_iT\in\mathcal{T}(\mathcal{H})$ is bounded, and for obtaining the fourth equality we have used the fact that $\mathrm{tr}(\cdot)\colon\mathcal{T}(\mathcal{H})\to\mathbb{Q}_{p,\mu}$ is a bounded functional and, if $J=\mathbb{N}$, the series $\sum_{j\in J}\lambda_j\,|A_ie_j\rangle\langle f_j|$ converges w.r.t. the norm topology.

Observe now that, by Lemma 8.5.1, $\lim_i\lambda_j\,\langle f_j,A_ie_j\rangle=0$, for all $j\in J$. Moreover, in the case where $J=\mathbb{N}$, since $\lim_j\lambda_j=0$ and $\alpha\equiv\sup_{i\in I}\|A_i\|<\infty$ (by the definition of a SOVM), then

$$|\langle f_j,A_ie_j\rangle|\le\alpha\|f_j\|\,\|e_j\|=\alpha,\ \forall i,j\in I \implies \lim_j\lambda_j\,\langle f_j,A_ie_j\rangle=0,\ \textit{uniformly}\text{ in } i\in I. \tag{8.39}$$

Therefore, we can freely exchange the sums, in the following calculation:

$$\begin{aligned}\sum_{i\in I}\Omega(A_i)&=\sum_{i\in I}\sum_{j\in J}\lambda_j\,\langle f_j,A_ie_j\rangle\\&=\sum_{j\in J}\sum_{i\in I}\lambda_j\,\langle f_j,A_ie_j\rangle\\&=\sum_{j\in J}\lambda_j\,\langle f_j,(\textstyle\sum_{i\in I}A_i)e_j\rangle=\displaystyle\sum_{j\in J}\lambda_j\,\langle f_j,e_j\rangle=\mathrm{tr}(S)=1.\end{aligned} \tag{8.40}$$

Here, we have used relation (8.38) and the fact that $\sum_{i\in I}A_i=\mathrm{Id}$ (w.r.t. the weak operator topology). In conclusion, it is proven that $\{\Omega(A_i)=\mathrm{tr}(A_iS)\}_{i\in I}\subset\mathbb{Q}_p$ is a p-adic probability distribution.

For the second assertion, just note that, if $\|A_i\|\le1$, then, for every canonical decomposition $S=\sum_{j\in J}\lambda_j\,|e_j\rangle\langle f_j|$ of the statistical operator $S\in\mathcal{T}_{\rm st}(\mathcal{H})$, we have that $|\Omega(A_i)|=|\mathrm{tr}(A_iS)|=|\sum_{j\in J}\lambda_j\,\langle f_j,A_ie_j\rangle|\le\max_{j\in J}|\lambda_j\,\langle f_j,A_ie_j\rangle|\le\max_{j\in J}|\lambda_j|$. Therefore, if $S\in\mathcal{D}(\mathcal{H})$, then, by the equivalence of conditions `(D1)` and `(D2)` in Proposition 8.4.1, $|\Omega(A_i)|\le1$.

Regarding the final assertion, we only need to show that if $\{A_i\}_{i\in I}$ is a contractive SOVM, then $\max_{i\in I}\|A_i\| = 1$. Indeed, if $\{A_i\}_{i\in I}$ is a contractive SOVM, we must have: $1 = \|\mathrm{Id}\| = \|\sum_{i\in I} A_i\| \le \max_{i\in I}\|A_i\| \le 1$. □

Example 8.5.1. Given a trace induced state $\mathrm{tr}((\cdot)S) \in \mathcal{S}_{\mathrm{tr}}(\mathcal{H})$, with every symmetric decomposition $S = \sum_{j\in J}(\sigma_j\,|e_j\rangle\langle f_j| + \overline{\sigma_j}\,|f_j\rangle\langle e_j|)$ — $J = \{1,2,\ldots\}$ — of the statistical operator S is associated, in a natural way, the p-adic probability distribution $\{\pi_0 = 0, \pi_1 = \sigma_1\,\langle f_1, e_1\rangle + \overline{\sigma_1}\,\langle e_1, f_1\rangle, \ldots\}$ and the SOVM

$$\{A_0 = \mathrm{Id} - S = \mathrm{Id} - \textstyle\sum_{j\in J}(\sigma_j\,|e_j\rangle\langle f_j| + \overline{\sigma_j}\,|f_j\rangle\langle e_j|), A_1 = \sigma_1\,|e_1\rangle\langle f_1| + \overline{\sigma_1}\,|f_1\rangle\langle e_1|, \ldots\}. \tag{8.41}$$

Note that, for $p \neq 2$, $\{A_0, A_1, \ldots\}$ is contractive if $S \in \mathcal{D}(\mathcal{H})$, because $\|A_0\| \le \max\{1, \|S\|\} = 1$, and $\|A_j\| = \|\sigma_j\,|e_j\rangle\langle f_j|\,\| = |\sigma_j| \le 1$, for all $j \in J$ (see Proposition 8.4.1). But (for every prime number p, and for $\dim(\mathcal{H}) = \infty$) it is *not* tracial because $A_0 = \mathrm{Id} - S \notin \mathcal{T}(\mathcal{H})$. For every orthonormal basis $\{\phi_m\}_{m\in\mathbb{N}}$ in $\mathcal{H}$, $\{|\phi_m\rangle\langle\phi_m|\}_{m\in\mathbb{N}}$ is a contractive tracial SOVM.

Part IV

Conclusion and Perspectives

The final part of this dissertation serves the dual role of collecting, on the one hand, the main results and achievements attained in this work and, on the other, of providing a preliminary exploration of the potential extensions to the ideas presented in the previous parts.

9

Summary and Outlook

The material presented in this dissertation is mostly based on our research works [17, 88, 89, 92, 93] aimed at setting the stage for a p-adic model of quantum information theory. Our initial motivation is the intriguing idea, originally proposed by Volovich, that the existence in nature of a shortest measurable length, i.e., the so-called *Planck's length*, entails that at a Planckian regime, physics may be ruled by the *strong triangle inequality*, that is, it should ultimately reveal a *non-Archimedean* character. In particular, below a scale comparable to Planck's length, space-time does not consist of infinitely divisible intervals, but only of isolated points, what we can imagine as a sort of 'space-time atoms'. Since, up to isomorphisms, the only *non-Archimedean field* one can construct by completing the field of rational numbers is the field of *p-adic numbers* $\mathbb{Q}_p$, one is led to take this field as the right candidate to model space-time coordinates at very short distances.

There are two possible routes one can follow when trying to construct a p-adic model of quantum mechanics. The first one adopts the point of view wherein physical states are described in terms of complex valued functions on $\mathbb{Q}_p$. The second approach, instead, moves in a more radical p-adic direction, and suggests that states and observables themselves have to live in a p-adic Hilbert space. Our first concern was to provide a suitable definition of a qubit — in the p-adic setting — exploring both the two possible directions. In particular, the main results we found can be synthesized as follows:

- In Part II, we started from the idea to define a p-adic qubit as a two-dimensional irreducible projective representation of the group of rotations $\mathrm{SO}(3,\mathbb{Q}_p)$ in $\mathbb{Q}_p^3$. This required us to construct, as a preliminary step, the Haar measure on this group. We first provided a general construction for an invariant measure on every p-adic Lie group. Here, we used heavily the topological properties of these groups; indeed, since they have a natural structure of a p-adic manifold — and, as such, are totally disconnected topological spaces locally homeomorphic to $\mathbb{Q}_p^n$ — they admit an open cover consisting of disjoint open sets. Then, our strategy was to 'locally import' the Haar measure of $\mathbb{Q}_p^n$ on any given fixed chart in the group, and then to exploit the change-of-variable formula for multiple integrals on $\mathbb{Q}_p^n$ to 'move' the invariant measure thus constructed all over the group. This provided us with an explicit formula yielding the local representation (i.e., w.r.t. the coordinates of a fixed chart) of the Haar measure on every p-adic Lie group. We then specialized this result to the groups $\mathrm{SO}(n,\mathbb{Q}_p)$ in dimensions $n = 2, 3, 4$, namely, to the *only* compact special orthogonal groups. For $\mathrm{SO}(3,\mathbb{Q}_p)$ and $\mathrm{SO}(4,\mathbb{Q}_p)$ we proved that a more convenient approach is first to realize these groups as suitable quotients of the multiplicative group of p-adic quaternions; then we found their Haar integrals by exploiting first our general formula, and then the well known Weil-Mackey-Bruhat lift.

- In Part III, we explored the second route to the p-adic quantization. Our first step here was to develop an abstract approach to quantum mechanics and quantum information theory over a quadratic extension of the field $\mathbb{Q}_p$ of p-adic numbers. We started by introducing a suitable notion of a Hilbert space $\mathcal{H}$ over $\mathbb{Q}_{p,\mu}$ (a quadratic extension of $\mathbb{Q}_p$). In our definition, a key element is the existence of an orthonormal basis for such spaces; this, in fact, allowed us to properly define Hilbert subspace and to extend, to the p-adic setting, some of the common geometric features of the standard complex Hilbert spaces. We then described some relevant classes of operators acting in a p-adic Hilbert space. In particular, we first observed that a far-reaching approach in this non-Archimedean setting is to define operators in terms of a suitable matrix representation. The first class of operators we described is the (ultrametric) Banach space $\mathcal{B}(\mathcal{H})$ of bounded operators. In particular, we observed that not every element of $\mathcal{B}(\mathcal{H})$ admits a *proper adjoint* and, in fact, the *adjointable* operators in $\mathcal{H}$ form an ultrametric Banach $*$-algebra $\mathcal{B}_{\mathrm{ad}}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$. In our theory, we identified this algebra with the algebra of (bounded) observables of a p-adic quantum system. Next, we characterized the class of unitary operators $\mathcal{U}(\mathcal{H}) \subset \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$, defined as those matrix operators relating any pair of orthonormal bases in $\mathcal{H}$. From this, we then proved some interesting properties of the unitary group $\mathcal{U}(\mathcal{H})$. In particular, we showed that it actually coincides with the intersection of the group $\mathscr{I}(\mathcal{H})$ of all surjective, IP-preserving, all-over operators, and the group $\mathscr{N}(\mathcal{H})$ of all surjective, NO-preserving, all-over operators. We passed then to study the trace class $\mathcal{T}(\mathcal{H}) \subset \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ of $\mathcal{H}$ again resorting to a suitable matrix representation. We showed that, similarly to the complex case, $\mathcal{T}(\mathcal{H})$ is a two-sided $*$-ideal in $\mathcal{B}_{\mathrm{ad}}(\mathcal{H})$. However, unlikely the standard setting, $\mathcal{T}(\mathcal{H})$ coincides with the Hilbert space of the Hilbert-Schmidt operators; moreover, $\mathcal{T}(\mathcal{H})^2 = \mathcal{T}(\mathcal{H})$, whence $\mathcal{T}(\mathcal{H})^n = \mathcal{T}(\mathcal{H})$, for all $n \in \mathbb{N}$, that is, $\mathcal{T}(\mathcal{H})$ alone plays the role that the trace ideals have in the standard complex case. With this machinery at hand, we then passed to characterize physical states and observables. Specifically, adopting the point of view of the algebraic formulation of (standard) quantum mechanics, we defined physical states in p-adic quantum mechanics as (suitably normalized) *involution-preserving* bounded functionals on the unital $*$-algebra $\mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ (the algebra of bounded physical observables). In particular, the role played by the σ-additive states in the standard complex case is now played by the *tracial states* $\mathcal{S}_{\mathrm{tr}}(\mathcal{H})$, induced, via the trace functional, by the *statistical operators* $\mathcal{T}_{\mathrm{st}}(\mathcal{H})$. Here, we observed again a peculiarity of the p-adic setting w.r.t. the standard quantum theory. Indeed, we showed that $\mathcal{T}_{\mathrm{st}}(\mathcal{H})$ is a $\mathbb{Q}_p$-*affine* — whence an *unbounded* — subset of $\mathcal{T}(\mathcal{H})$ (thus, in contrast with the convex set of density operators of a complex quantum system). Nevertheless, one can still define a $\mathbb{Q}_p$-*convex* (and norm-bounded) subset $\mathcal{D}(\mathcal{H})$ of *density operators* in $\mathcal{T}(\mathcal{H})$; but, the important point here is that these states are only a special instance of the much larger class of statistical operators. Eventually, we completed the statistical interpretation of the (new) theory by suitably defining the *observables* in p-adic quantum mechanics. In particular, we argued that the *selfadjoint-operator-valued measures* (SOVMs) may provide a suitable p-adic counterpart of the usual POVMs associated with a complex Hilbert space.

At this point, we want to briefly discuss some natural continuations and extensions of the ideas presented in this dissertation. Within the first approach to the p-adic quantization, it is clear that, having derived a general expression for the Haar measure for a p-adic Lie group, the potential applications involve extending to p-adic groups all the usual harmonic analysis techniques of the standard real case. A natural next step is, in particular, to study — via the Peter-Weyl theorem — the two-dimensional irreducible projective representations of the group $\mathrm{SO}(3, \mathbb{Q}_p)$. This will then provide us with an explicit model of a p-adic qubit. Another class of problems where the Haar measure plays a central role is related to the phase-space formulation of quantum mechanics [152]. Here, one starts by first noting that the usual group of translations on phase space (with its genuinely projective representations) is replaced, in the p-adic setting, by a compact p-adic Lie group. This group will admit square integrable representations, which can be constructed once the Haar measure for it is known. Exploiting these representations it is then possible to define *generalized Wigner transforms* mapping quantum-mechanical operators into complex valued functions on the relevant group [84–87].

For what concerns the second route to the p-adic quantization, there are several aspects of the theory that still need to be studied, or further clarified. Indeed, a first point which deserves a better investigation is the characterizations of the *symmetry transformations* in p-adic quantum mechanics. Recently, symmetry transformations of a p-adic quantum system have been defined as maps preserving the natural affine structure of the space of physical states [153]. In particular, it has also been observed that there exist symmetry transformations mapping a suitable density operator into a state of arbitrarily large norm [153]. The existence of these 'non-canonical' symmetry transformations is a further remarkable feature of p-adic quantum mechanics. Hence, a central problem is to provide a suitable classification of these symmetry transformations. This issue, however, is non-trivial as it requires to preliminary construct a suitable theory of 'p-adic projective representations'.

Another point that also needs more attention is the definition of *orthogonal projections* operators in p-adic Hilbert spaces and, accordingly, the characterization of the *logic* structure of a p-adic quantum system. Tensor products and entanglement are central in quantum information theory, and we expect that they will be central in the p-adic setting too. Hence, another interesting problem is to further investigate the tensor product structure of p-adic Hilbert spaces, in particular addressing the characterization of separable or entangled physical *states*, and of the corresponding physical *observables*. This will ultimately allow us to describe the physics of *composite* p-adic quantum systems. Finally, an interesting line of study — more oriented towards possible applications in p-adic quantum information theory — is the investigation of dynamical maps and dynamical (semi-)groups, which can be profitably used to characterize p-adic quantum channels and instruments.

Appendices

A

The real quaternion algebra and its relations with $\mathrm{SO}(3,\mathbb{R})$ *and* $\mathrm{SO}(4,\mathbb{R})$

We devote this appendix to a brief account on the real quaternion algebra $\mathbb{H}$, along with a discussion of the quaternionic realization of the elements in $\mathrm{SO}(3,\mathbb{R})$ and $\mathrm{SO}(4,\mathbb{R})$. This will also give us the opportunity to highlight analogies and differences with the p-adic case of Section 5 of Part II.

A.0.1 The real quaternion algebra $\mathbb{H}$

There are several ways of describing the real quaternion algebra $\mathbb{H}$ [83,105]. As a real vector space $\mathbb{H} \cong \mathbb{R} \times \mathbb{R}^3$, and any element in $\mathbb{H}$ is written as $\xi = (a, \mathbf{x})$, with $a \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^3$. The multiplication law is given by:

$$(a, \mathbf{x})(b, \mathbf{y}) = (ab - \mathbf{x} \cdot \mathbf{y}, b\mathbf{x} + a\mathbf{y} + \mathbf{x} \times \mathbf{y}), \tag{A.1}$$

where $\mathbf{x} \cdot \mathbf{y}$ and $\mathbf{x} \times \mathbf{y}$ are respectively the usual inner product and vector product between vectors in $\mathbb{R}^3$. It is easily verified that the product (A.1) is associative. The centre of $\mathbb{H}$ is given by the subspace $\mathbb{R} \times \{\mathbf{0}\} \cong \mathbb{R}$. Likewise, we identify $\{0\} \times \mathbb{R}^3$ with $\mathbb{R}^3$, in such a way that every element in $\mathbb{H}$ can be expressed as $\xi = a + \mathbf{x}$, $a \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^3$. Denoting by $\mathbf{i}, \mathbf{j}, \mathbf{k}$ the canonical basis vectors of $\mathbb{R}^3$, ξ can be expressed as

$$\xi = a + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}, \tag{A.2}$$

where $a, x_1, x_2, x_3 \in \mathbb{R}$. Then, the multiplication law between quaternions is given by specifying the products between the basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ [83,154]:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i} \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}. \tag{A.3}$$

It is straightforward to realize that $\mathbb{H}$ is a non-abelian algebra.

$\mathbb{H}$ is an involutive algebra, as the map $\mathbb{H} \ni \xi = a + \mathbf{x} \mapsto \overline{\xi} := a - \mathbf{x} \in \mathbb{H}$ is an involutive anti-automorphism. Moreover, $\xi\overline{\xi} = |\xi|^2 = a^2 + |\mathbf{x}|^2 = \mathrm{F}_{\mathbb{R}}(a, x_1, x_2, x_3)$, where $\mathrm{F}_{\mathbb{R}}$ denotes the definite quadratic form of $\mathbb{R}^4$. Thus, every non-zero element in $\mathbb{H}$ is invertible, with $\xi^{-1} = \overline{\xi}/|\xi|^2$, and so $\mathbb{H}$ is a division algebra. Those elements in $\mathbb{H}$ for which $|\xi| = 1$ are called *unit quaternions*. They form a group in $\mathbb{H}$, denoted by $\mathrm{U}(\mathbb{H})$:

$$\mathrm{U}(\mathbb{H}) = \{\xi \in \mathbb{H} \mid |\xi| = 1\} = \{\xi \in \mathbb{H} \mid \xi^{-1} = \overline{\xi}\}. \tag{A.4}$$

Remark A.0.1. In the literature (e.g., [112]), the quantity $\xi\overline{\xi}$ is referred to as the *reduced norm* of ξ in $\mathbb{H}$, and denoted by $\mathrm{nrd}(\xi)$ (see Remark 5.1.2). From the definition, it is clear that the reduced norm on the real quaternion algebra $\mathbb{H}$ is equivalent to the square of the Euclidean norm on $\mathbb{R}^4$ (since the definite quadratic form on $\mathbb{R}^4$ induces the Euclidean inner product, and vice versa). However, this is not the case when one defines a quaternion algebra over a generic field w.r.t. some quadratic form (see, for instance [111,112]): The latter does not necessarily induce the considered norm on the vector-space structure of that algebra.

There is another (yet equivalent) way in which $\mathbb{H}$ can be described [105]. Let us consider the subset $\mathbf{H}$ of the algebra $\mathsf{M}_2(\mathbb{C})$ of complex 2×2 matrices of the form

$$M = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} = \begin{pmatrix} q_0 + iq_1 & q_2 + iq_3 \\ -q_2 + iq_3 & q_0 - iq_1 \end{pmatrix}, \quad q_j \in \mathbb{R}, \tag{A.5}$$

for every $j = 0, 1, 2, 3$, where i denotes the imaginary unit. One can easily verify that $\mathbf{H}$ is a subalgebra (actually, a *division* algebra) in $\mathsf{M}_2(\mathbb{C})$. In particular, that every non-null element $M \in \mathbf{H}$ is invertible easily follows by observing that

$$\det(M) = \det \begin{pmatrix} q_0 + iq_1 & q_2 + iq_3 \\ -q_2 + iq_3 & q_0 - iq_1 \end{pmatrix} = q_0^2 + q_1^2 + q_2^2 + q_3^2 \tag{A.6}$$

is the (non-degenerate) four-dimensional definite quadratic form over $\mathbb{R}$. From (A.5), we also see that every element $M \in \mathbf{H}$ can be written as $M = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$, where

$$\mathbf{i} := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{A.7}$$

and where we omitted the identity matrix I_2 multiplying q_0. Moreover, it can be easily checked that $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ obey commutation relations which are analogous to the quaternion commutation relations (A.3). Indeed, the correspondence

$$\theta := \mathbb{H} \ni \xi = (q_0, q_1, q_2, q_3) \mapsto \theta(\xi) := \begin{pmatrix} q_0 + iq_1 & q_2 + iq_3 \\ -q_2 + iq_3 & q_0 - iq_1 \end{pmatrix} \in \mathbf{H} \tag{A.8}$$

defines an *algebra isomorphism* from the quaternions $\mathbb{H}$ to the algebra of complex matrices $\mathbf{H}$ [83]. In particular, unit quaternions are identified in $\mathbf{H}$ by

$$\mathrm{U}(\mathbf{H}) = \{M \in \mathbf{H} \mid \det(M) = 1\}. \tag{A.9}$$

A.0.2 Relations between real quaternions and rotations

Here we recall the relation between $\mathbb{H}$ and $\mathrm{SO}(3, \mathbb{R})$. Let $\xi \in \mathrm{U}(\mathbb{H})$ be a unit quaternion. The map $\mathbb{H} \ni \eta \mapsto \xi\eta\xi^{-1} \in \mathbb{H}$ is an *isometric linear transformation* of $\mathbb{H}$, which leaves the centre $\mathbb{R}$ of $\mathbb{H}$ pointwise fixed and, therefore, also leaves the orthogonal subspace $\mathbb{R}^3$ invariant. Hence, the restriction of this map to $\mathbb{R}^3$ is an element of $\mathrm{O}(3, \mathbb{R})$, which we denote by $\kappa(\xi)$:

$$\kappa(\xi)\mathbf{x} := \xi\mathbf{x}\xi^{-1}, \quad \mathbf{x} \in \mathbb{R}^3. \tag{A.10}$$

Furthermore, $\kappa(\xi\nu) = \kappa(\xi)\kappa(\nu)$, i.e., $\kappa \colon \mathrm{U}(\mathbb{H}) \to \mathrm{O}(3, \mathbb{R})$ is a homomorphism. Let us derive the explicit form of $\kappa(\xi)$. For $\mathbb{H} \ni \xi = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$, and $\mathbb{R}^3 \ni \mathbf{x} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$, we have:

$$\begin{aligned} \xi\mathbf{x}\xi^{-1} =& \mathbf{i}\big(x(q_0^2 + q_1^2 - q_2^2 - q_3^2) + 2y(q_1q_2 - q_3q_0) + 2z(q_2q_0 + q_3q_1)\big), \\ & \mathbf{j}\big(2x(q_1q_2 + q_0q_3) + y(q_0^2 - q_1^2 + q_2^2 - q_3^2) + 2z(q_2q_3 - q_1q_0)\big), \\ & \mathbf{k}\big(2x(q_1q_3 - q_2q_0) + 2y(q_1q_0 + q_2q_3) + z(q_0^2 - q_1^2 - q_2^2 + q_3^2)\big), \end{aligned} \tag{A.11}$$

from which we deduce that $\kappa(\xi)$ is given by

$$\kappa(\xi) = \begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_3q_0) & 2(q_2q_0 + q_3q_1) \\ 2(q_1q_2 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_1q_0) \\ 2(q_1q_3 - q_2q_0) & 2(q_1q_0 + q_2q_3) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix}. \tag{A.12}$$

A direct calculation shows that $\det(\kappa(\xi)) = 1$, i.e., $\kappa(\xi) \in \mathrm{SO}(3, \mathbb{R})$.

Remark A.0.2. The fact that $\kappa(\xi) \in \mathrm{SO}(3,\mathbb{R})$ also follows by observing that $\mathrm{U}(\mathbb{H})$ is *connected* and $\kappa\colon \mathrm{U}(\mathbb{H}) \to \mathrm{SO}(3,\mathbb{R})$ is continuous [83].

In the light of the discussion above, every unit quaternion $\xi = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \in \mathrm{U}(\mathbb{H})$ is associated with a rotation R in $\mathrm{SO}(3,\mathbb{R})$. In particular, κ is the group homomorphism $\mathrm{U}(\mathbb{H}) \to \mathrm{SO}(3,\mathbb{R})$ in the short exact sequence (5.40) [111, 112]. This then yields the group isomorphism

$$\mathrm{SO}(3,\mathbb{R}) \cong \mathrm{U}(\mathbb{H})/\mathbb{F}_2. \tag{A.13}$$

With a similar argument, based on the isometries $\mathbb{H} \ni \eta \mapsto \xi\eta\varrho^{-1} \in \mathbb{H}$ for $(\xi,\varrho) \in \mathrm{U}(\mathbb{H}) \times \mathrm{U}(\mathbb{H})$, one can prove the following isomorphism [112]:

$$\mathrm{SO}(4,\mathbb{R}) \cong (\mathrm{U}(\mathbb{H}) \times \mathrm{U}(\mathbb{H}))/\mathbb{F}_2. \tag{A.14}$$

(A.13) and (A.14) become homeomorphism, considering the standard topology for the involved spaces, providing double coverings for $\mathrm{SO}(3,\mathbb{R})$ and $\mathrm{SO}(4,\mathbb{R})$.

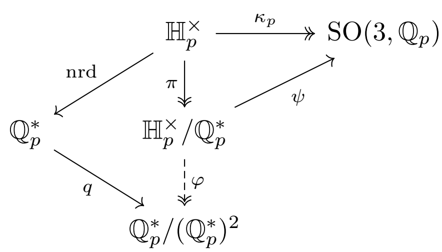
B

An alternative proof of Proposition 5.3.1

In Section 5.3 of Part II, we showed that the group isomorphism $\psi\colon \mathbb{H}_p^\times/\mathbb{Q}_p^* \to \mathrm{SO}(3,\mathbb{Q}_p)$ given in Theorem 5.3.1 is a homeomorphism. The proof of Proposition 5.3.1 relies on measure-theoretical results; here we provide an alternative proof which shows more explicitly the relation between p-adic rotations and quaternions, depending on their reduced norm.

As already argued in the proof of Proposition 5.3.1, the groups $\mathbb{H}_p^\times$ and $\mathrm{SO}(3,\mathbb{Q}_p)$ are locally compact, once supplied with their p-adic topology. The map κ_p is continuous, as $\kappa_p(\xi)$ is a rational function on the parameters q_0, q_1, q_2, q_3 of $\xi = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$, with denominator $\mathrm{nrd}(\xi) \neq 0$ for every $\xi \neq 0$. Therefore, κ_p redefined on the quotient of $\mathbb{H}_p^\times$ modulo $\ker(\kappa_p)$ is continuous, i.e. ψ is continuous. We are left to prove that also the inverse map ψ^{-1} is continuous, or equivalently that ψ is a closed function (it maps closed subsets of $\mathbb{H}_p^\times/\mathbb{Q}_p^*$ to closed subsets of $\mathrm{SO}(3,\mathbb{Q}_p)$). To ease this, we want to deal with compact spaces, rather than just locally compact ones.

Notice that $\mathrm{nrd} : \mathbb{H}_p^\times \to \mathbb{Q}_p^*$ is a homomorphism (it is multiplicative), as well as the quotient map $q : \mathbb{Q}_p^* \to \mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$, therefore $q \circ \mathrm{nrd} : \mathbb{H}_p^\times \to \mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$ is a homomorphism. Its kernel is $\ker(q\circ\mathrm{nrd}) = \left\{\xi \in \mathbb{H}_p^\times \mid \mathrm{nrd}(\xi) \in (\mathbb{Q}_p^*)^2\right\}$, and $\mathbb{Q}_p^*$ is a normal subgroup of $\ker(q\circ \mathrm{nrd})$ and $\mathbb{H}_p^\times$ (being its centre). It follows, by the fundamental homomorphism theorem, that there exists a unique group homomorphism $\varphi : \mathbb{H}_p^\times/\mathbb{Q}_p^* \to \mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$ such that $q\circ\mathrm{nrd} = \varphi\circ\pi$: This map is $\varphi(\xi\mathbb{Q}_p^*) = \mathrm{nrd}(\xi) \mod (\mathbb{Q}_p^*)^2$. In fact, given two distinct representatives of the same class, i.e. $\nu \neq \xi$ such that $\nu\mathbb{Q}_p^* = \xi\mathbb{Q}_p^*$, we have $\nu \in \xi\mathbb{Q}_p^*$ and hence $\varphi(\nu\mathbb{Q}_p^*) = \mathrm{nrd}(\nu) \mod (\mathbb{Q}_p^*)^2 = \mathrm{nrd}(\xi) \mod (\mathbb{Q}_p^*)^2 = \varphi(\xi\mathbb{Q}_p^*)$.

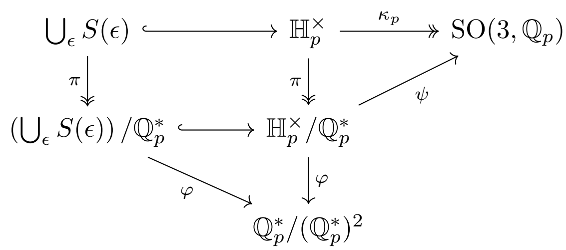


Actually, by the isomorphism theorem, we have $\mathbb{H}_p^\times/\ker(q\circ\mathrm{nrd}) \cong \mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$. For $\ker(q\circ\mathrm{nrd})$ is stable under multiplication by scalars in $\mathbb{Q}_p^*$, we have the induced quotient $\ker(q\circ\mathrm{nrd})/\mathbb{Q}_p^*$, such that $(\mathbb{H}_p^\times/\mathbb{Q}_p^*)/(\ker(q\circ\mathrm{nrd})/\mathbb{Q}_p^*) \cong \mathbb{H}_p^\times/\ker(q\circ\mathrm{nrd}) \cong \mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$. We observe that ξ and $\lambda\xi$ have same image w.r.t. $\varphi\circ\pi$ (or equivalently $q\circ\mathrm{nrd}$), for every $\xi \in \mathbb{H}_p^\times$, $\lambda \in \mathbb{Q}_p^*$. Therefore, to have a surjective map onto $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$, once chosen a representative for each equivalence class of this quotient group, it is enough to consider the restriction of $\mathbb{H}_p^\times$ to the set of quaternions having reduced norm exactly equal to those representatives. A similar argument applies to $\mathbb{H}_p^\times/\mathbb{Q}_p^*$. We recall [71,74] that

$$\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2 \cong \begin{cases}\{1,u,p,up\} \cong \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}, & p>2,\\ \{\pm1,\pm2,\pm5,\pm10\} \cong \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}, & p=2,\end{cases} \tag{B.1}$$

where $u \in \mathbb{U}_p$ is a non-squared invertible p-adic integer. We thus define $S(\epsilon) \coloneqq \{\xi \in \mathbb{H}_p^\times \mid$

$\mathrm{nrd}(\xi) = \epsilon\}$, by varying ϵ in the set of p-adic integer representatives of the equivalence classes of $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$, i.e. for $\epsilon = 1, u, p, up$ when $p > 2$ and for $\epsilon = \pm1, \pm2, \pm5, \pm10$ when $p = 2$. Now we have the following diagram where, by abuse of notation, we also denote by π and φ the homonyms maps redefined on $\bigcup_\epsilon S(\epsilon)$ and $\left(\bigcup_\epsilon S(\epsilon)\right)/\mathbb{Q}_p^*$ respectively, and where the injective maps are simply the (closed, continuous) canonical embeddings of $\bigcup_\epsilon S(\epsilon)$ and $\left(\bigcup_\epsilon S(\epsilon)\right)/\mathbb{Q}_p^*$ in $\mathbb{H}_p^\times$ and $\mathbb{H}_p^\times/\mathbb{Q}_p^*$ respectively.



The sets $S(\epsilon)$ are pairwise disjoint. Moreover, it can be shown that $S(\epsilon) \subset \mathbb{Z}_p^4$ is compact, in a similar fashion to the proof of the fact that the entries of $\mathrm{SO}(4, \mathbb{Q}_p)$ are p-adic integers (see Theorem 5 in [113]). As a consequence, $\bigcup_\epsilon S(\epsilon)$ is a compact subset of $\mathbb{H}_p^\times$, being the finite union of compact sets. Now that we can consider just the compact subspace $\bigcup_\epsilon S(\epsilon)$ of $\mathbb{H}_p^\times$, the proof of the fact that ψ is closed is straightforward. Consider a closed subset C in $\left(\bigcup_\epsilon S(\epsilon)\right)/\mathbb{Q}_p^* \subset \mathbb{H}_p^\times/\mathbb{Q}_p^*$. Its preimage $\pi^{-1}(C) \subset \bigcup_\epsilon S(\epsilon) \subset \mathbb{H}_p^\times$ is closed, since π is continuous. Closed subsets of the compact $\bigcup_\epsilon S(\epsilon)$ are compact, in particular $\pi^{-1}(C)$ is compact. The map $\kappa'_p \coloneqq \kappa_p|_{\bigcup_\epsilon S(\epsilon)} : \bigcup_\epsilon S(\epsilon) \to \mathrm{SO}(3, \mathbb{Q}_p)$ is continuous, as a restriction of the continuous map κ_p. The continuous image $\kappa'_p\left(\pi^{-1}(C)\right)$ of the compact set $\pi^{-1}(C)$ is compact. In turn, $\kappa'_p\left(\pi^{-1}(C)\right)$ is closed, being a compact subset of the compact Hausdorff group $\mathrm{SO}(3, \mathbb{Q}_p)$. This proves that κ'_p is a closed map. Finally, this implies that ψ is closed, and hence it is a homeomorphism.

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