Università degli Studi di Napoli Federico II

Dottorato di Ricerca in Fisica
Ciclo XXXVI

Coordinatore: Prof. Vincenzo Canale

Spacetime noncommutativity and cold atoms

Settore Scientifico Disciplinare: FIS/02

Dottorando
Giuseppe Fabiano

Tutors
Prof. Giovanni Amelino-Camelia
Prof. Flavio Mercati

Anni: 2020/2024

Introduction

Currently, the most accurate depiction of Nature is through quantum theory, describing particles and interactions at the fundamental level, and general relativity, which predicts a variety of macroscopic phenomena. Remarkably, even before these theories attained their status as well-established frameworks, with predictions confirmed across numerous physical regimes, inherent incompatibilities between quantum theory and general relativity had been recognized. For instance, a famous argument by Bronstein [1, 2] shows a fundamental incompatibility between the equivalence and the uncertainty principles by investigating the localizability of the electromagnetic and gravitational fields. The uncertainty principle allows for arbitrarily sharp localization of fields, depending on the charge to mass ratio of the probe used to measure it. Since the electric charge and the mass are two distinct physical quantities, sharp localization of the electromagnetic field is viable. As for the gravitational field, the equivalence principle (still confirmed today with very high precision [3]) dictates that the gravitational charge and the mass are one and the same, thus providing a localization limit of the order of the Planck length $\ell_p = \sqrt{\frac{\hbar G}{c^3}} \approx 10^{-35}$ m [1].

If this argument is to be taken at face value, the classical notion of smooth spacetime should be replaced by that of a "quantized" spacetime, incorporating the idea of a localization limit. Moreover, the tools of Riemannian geometry would not be sufficient to describe objects that lack the notion of infinitesimal points. A first attempt at a model of quantum spacetime was put forward by Snyder, with the hope of regularizing the ultraviolet divergences of quantum electrodynamics [4]. In his original paper, he showed that a spacetime with a discrete spacing is indeed compatible with Lorentz invariance as long as the coordinate functions satisfy non-trivial commutation relations. In more formal advancements, it was understood that smooth spacetime manifolds could be completely characterized by the commutative algebra of functions on that manifold [5, 6]. Replacing that algebra with a noncommutative one leads to a quantization of the coordinate functions (*i.e.* they satisfy non-trivial commutation relations) and to the modern notion of noncommutative geometry [7].

In most of the modern approaches to quantum gravity, the issue of localizability is one of the guiding principles in laying the foundations of the theory. In String Theory, the intrinsic length scale of strings has been conjectured to prevent probing arbitrarily small distances [8] while in Loop Quantum Gravity the localization limit is encoded in the discreteness of the spectra of geometric operators [9]. For both of these proposals, there exists a regime in which the theories can be effectively described using the language of noncommutative geometry [10–12]. Another compelling example is that of 2+1-dimensional General Relativity. This theory lacks local propagating degrees of freedom and can be successfully quantized, contrary to its 3+1-dimensional counterpart. When the theory is coupled to matter and the gravitational degrees of freedom are integrated out, the result is a theory of matter propagating on a noncommutative spacetime [13, 14], where the Planck scale is the scale of coordinate noncommutativity.

These theoretical hints have encouraged investigations on noncommutative spacetime as an effective regime of a tentative full theory of quantum gravity, without referring to any particular approach. Especially relevant for physical applications is the fate of relativistic symmetries, which can be either broken or deformed upon the introduction of a noncommutativity scale. The former scenario provides a Lorentz-breaking preferred background with respect to which the noncommutativity of coordinates is specified. The latter case requires that such noncommutativity should be the same for every observer, typically yielding a generalization of the Poincaré group. In the context of noncommutative geometry, these symmetry deformations are described with the language of quantum groups and Hopf algebras [15–17], which can be thought of as nonlinear generalizations of Lie algebras.

In the early 2000s, these investigations culminated in a series of seminal papers advocating that quantum gravity effects may manifest themselves as observable deviations from special relativity [18–21] at the energies we are able to probe today, very far from the conjectured Planck scale. The idea of deformations of relativistic symmetries was also concretely implemented in a new physical framework known as Doubly Special Relativity [22–24], where the noncommutativity scale is promoted to a second relativistic invariant scale, on par with the speed of light c. These efforts gave birth to what is known today as quantum gravity phenomenology [25–28], a research program with the objective of developing falsifiable effective quantum gravity models and comparing the new physics effects to a variety of physical scenarios ranging from astrophysical to table-top experiments.

This thesis contains the analyses and the results of several research works I have coauthored during my doctoral studies, concerning both coneptual and phenomenological aspects of noncommutative spacetime. In chapter 1 I briefly review the basic mathematical concepts behind quantum groups and Hopf algebras, providing both historical and more recent examples, including some new results derived in [29]. Chapter 2 is dedicated to the development of noncommutative quantum field theory on the lightlike κ-Minkowski noncommutative sapcetime, mainly following [30]. The focus is on the definition of multiparticle states and on the fate of discrete symmetries when this particular type of noncommutativity is enforced. I move on to two still technical, but less formal chapters. In chapter 3, I mainly follow [31], where we study the role of quantum complementarity for observers living in a universe where standard rotational symmetry is replaced by symmetry under the $SU_q(2)$ quantum group. The results found suggest a conceptual reassessment of observers and reference frames when quantum properties of spacetime and its symmetries are taken into account. Chapter 4 focuses on the definition of Noether charges in noncommutative spacetime and is based on [32], where we find a strong connection between the conserved charges of a multiparticle system and their interacting potential. Last but not least, chapter 5 is dedicated to quantum gravity phenomenology in the infrared, and is based on a forthcoming publication. After briefly reviewing the present status of quantum gravity phenomenology, I show some preliminary results suggesting that effective quantum gravity models inspired by noncommutative spacetimes with IR/UV mixing mechanisms may lead to observable corrections in atom interferometry experiments.

Unless otherwise specified, the system of natural units in which $c = \hbar = 1$ is employed throughout the thesis.

List of publications on which the thesis is based

- G. Fabiano and F. Mercati, "Multiparticle states in braided lightlike κ-Minkowski noncommutative QFT", Phys.Rev.D 109 (2024) 4, 046011, [arXiv:2310.15063 [hep-th]].
- G. Fabiano, G. Gubitosi, F. Lizzi, L. Scala and P. Vitale, "Bicrossproduct vs. twist quantum symmetries in noncommutative geometries: the case of ϱ-Minkowski," JHEP 08 (2023), 220 [arXiv:2305.00526 [hep-th]].
- G. Amelino-Camelia, G. Fabiano and D. Frattulillo, "Total momentum and other Noether charges for particles interacting in a quantum spacetime," [arXiv:2302.08569 [hep-th]].
- G. Amelino-Camelia, V. D'Esposito, G. Fabiano, D. Frattulillo, P. A. Hoehn and F. Mercati, "Quantum Euler angles and agency-dependent spacetime," [arXiv:2211.11347 [gr-qc]] (accepted for publication in Progress of Theoretical and Experimental Physics).

Other research works I coauthored during my Ph.D.

- Alves Batista, R., *et al.* (COST CA18108) "White Paper and Roadmap for Quantum Gravity Phenomenology in the Multi-Messenger Era", [arXiv: 2312.00409 [gr-qc]].
- P. Bosso, G. Fabiano, D. Frattulillo and F. Wagner, "The fate of Galilean relativity in minimal-length theories", Phys.Rev.D 109 (2024) 4, 046016, [arXiv:2307.12109 [gr-qc]].
- A. D'Alise, G. Fabiano, D. Frattulillo, S. Hohenegger, D. Iacobacci, F. Pezzella and F. Sannino, "Positivity Conditions for Generalised Schwarzschild Space-Times," Phys. Rev. D 108 (2023) no.8, 084042 [arXiv:2305.12965 [gr-qc]].
- R. Loll, G. Fabiano, D. Frattulillo and F. Wagner, "Quantum Gravity in 30 Questions," PoS CORFU2021 (2022), 316 [arXiv:2206.06762 [hep-th]].
- A. D'Alise, G. De Nardo, M. G. Di Luca, G. Fabiano, D. Frattulillo, G. Gaudino, D. Iacobacci, M. Merola, F. Sannino and P. Santorelli, *et al.* "Standard model anomalies: lepton flavour non-universality, $g-2$ and W-mass," JHEP 08 (2022), 125 [arXiv:2204.03686 [hep-ph]].
- A. Addazi, *et al.* (COST CA18108 Collaboration) "Quantum gravity phenomenology at the dawn of the multi-messenger era—A review," Prog. Part. Nucl. Phys. 125 (2022), 103948 [arXiv:2111.05659 [hep-ph]].

Contents

Chapter 1

Quantum groups and Hopf Algebras

We give the formal definition of a Hopf Algebra by reviewing the relevant mathematical structures needed for its construction. The review is by no means exhaustive, and only serves the purpose of introducing the mathematical language employed in the subsequent chapters. We also give the definition of a quantum homogeneous space and discuss how its symmetries can be described by means of a Hopf Algebra. A more formal and complete treatment of the subject is contained in [15–17]. In closing the chapter, we give three examples of noncommutative spacetime models and their associated symmetries which are well studied in the literature.

1.1 Mathematical preliminaries

A Lie group $\mathcal{G}$ can be described in terms of the algebra of complex valued functions on the group, denoted by $\mathbb{C}[\mathcal{G}]$. The defining operations of the algebra are the commutative product and the sum between two functions, as well as the multiplication of a function by a scalar. Let $f, h : \mathbb{C}[\mathcal{G}] \to \mathbb{C}$ be functions on the Lie group and $z \in \mathbb{C}$; then, for every $g \in \mathcal{G}$, these operations may formally be written as

$$(f \cdot h)(g) = f(g) \cdot h(g), \qquad (f + h)(g) = f(g) + h(g), \qquad (z\, f)(g) = z\, f(g)\,. \tag{1.1}$$

The algebra is also unital, *i.e.* it posses the identity element $\mathbb{1}$, such that

$$\mathbb{1}(g) = 1 \qquad \forall g \in \mathcal{G}\,. \tag{1.2}$$

The operations relevant for group theory such as group product, group inverse and existence of the group identity can thus be understood as maps on $\mathbb{C}[\mathcal{G}]$ as follows. The group product is implemented by a map $\Delta : \mathbb{C}[\mathcal{G}] \to \mathbb{C}[\mathcal{G}] \otimes \mathbb{C}[\mathcal{G}] \sim \mathbb{C}[\mathcal{G}, \mathcal{G}]$, known as *coproduct*, acting as

$$\Delta[f](g, g') = f(gg') \qquad \forall g, g' \in \mathcal{G}\,. \tag{1.3}$$

The group inverse is encoded in the *antipode* $S : \mathbb{C}[\mathcal{G}] \to \mathbb{C}[\mathcal{G}]$:

$$S[f](g) = f(g^{-1}), \qquad \forall g \in \mathcal{G}\,, \tag{1.4}$$

while the identity element is implemented though the *counit* $\epsilon : \mathbb{C}[\mathcal{G}] \to \mathbb{C}$:

$$\epsilon[f] = f(e)\,, \tag{1.5}$$

where e is the identity of the Lie group. In turn, the Lie group axioms imply certain properties for the maps Δ, S, ϵ. The associativity of the group product translates to what is known as the co-associativity of the coproduct:

$$g(g'g'') = (gg')g'' \iff (\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta\,, \tag{1.6}$$

where $id : \mathbb{C}[\mathbb{G}] \to \mathcal{C}[\mathcal{G}]$ is the identity map. This relation can be easily shown in terms of Sweedler's notation for the coproduct. We write the expression for the coproduct of a function $f \in \mathbb{C}[\mathcal{G}]$ as

$$\Delta[f] \equiv f_{(1)} \otimes f_{(2)} \,, \tag{1.7}$$

where $f_{(1)}$ and $f_{(2)}$ are a shorthand notation for the components of the tensor product and summation over all terms of $\Delta[f]$ is implied. Applying a generic $f \in \mathbb{C}[\mathcal{G}]$ to the left hand-side of (1.6), we obtain

$$\begin{aligned}
&f(g(g'g'')) = f((gg')g'') \iff \Delta[f](g, g'g'') = \Delta[f](gg', g'') \\
&f_{(1)}(g) \otimes f_{(2)}(g'g'') = f_{(1)}(gg') \otimes f_2(g'') \\
&f_{(1)}(g) \otimes f_{(2)(1)}(g') \otimes f_{(2)(2)}(g'') = f_{(1)(1)}(g) \otimes f_{(1)(2)}(g') \otimes f_{(2)}(g'') \\
&(id \otimes \Delta) \circ \Delta[f](g, g', g'') = (\Delta \otimes id) \circ \Delta[f](g, g', g'') \,.
\end{aligned} \tag{1.8}$$

The existence of the identity element implies that the counit map is neutral with respect to the coproduct, namely

$$eg = ge = g \iff (id \otimes \epsilon) \circ \Delta = (\epsilon \otimes id) \circ \Delta = id \,, \tag{1.9}$$

while the existence of the inverse entails an analogous compatibility between product, coproduct, counit and anitpode:

$$gg^{-1} = g^{-1}g = e \iff \cdot \circ (S \otimes id) \circ \Delta = \cdot \circ (id \otimes S) \circ \Delta = \epsilon \,, \tag{1.10}$$

i.e., multiplying a group element by its inverse from the left or the right is the same thing, and yields the identity element. The proofs of these other two properties follow the same line of reasoning as (1.8). Let us appreciate how these unfamiliar structures are implemented, by considering $\mathbb{C}[GL(n)]$, where $GL(n)$ is the Lie group of invertible matrices of dimension n. The group multiplication, identity element and inverse are respectively given by

$$(MM')^i_j = M^i_k M'^k_j \,, \qquad e^i_j = \delta^i_j \,, \qquad (M^{-1})^i_j = \frac{adj^T(M)^i_j}{det(M)} \quad i, j = 1, \ldots, n \,, \tag{1.11}$$

where $M, M', M^{-1} \in \mathcal{G}$, $adj(M)$ and $det(M)$ are the matrix adjugate and the determinant of M, respectively. The group manifold can be coordinatized by N^2 coordinate functions $M^i_j : GL(n) \to \mathbb{C}$, which give the i, j-th component of the matrix representation of the group element, and the unit function $\mathbb{1} : GL(n) \to 1$. These coordinate functions, subject to the constraint $detM \neq 0$, provide a basis for the commutative algebra $\mathbb{C}[GL(n)]$. The group rules are codified by the following maps

$$\begin{aligned}
&\Delta[\mathbb{1}] = \mathbb{1} \otimes \mathbb{1} \,, \qquad \Delta[M^i_j] = M^i_k \otimes M^k_j \\
&S[M^i_j] = (M^{-1})^i_j \,, \qquad \epsilon[M^i_j] = \delta^i_j \,.
\end{aligned} \tag{1.12}$$

The above expressions can be used to prove the compatibility conditions (1.6),(1.9),(1.10). We have just shown, both in general and with an example, that Lie group structures can be expressed in algebraic language, through the use of the algebra of functions on the group $\mathbb{C}[\mathcal{G}]$. While this framework is certainly redundant for the description of Lie groups, it is ideal for the characterization of quantum groups, where the pointwise product defined in (1.1) becomes noncommutative. The ensuing algebraic structure, endowed with all the operations and compatibility relations defined in the examples above, is known as a Hopf algebra, which we will define shortly, after some technicalities are introduced.

A **unital associative algebra** $\mathcal{A}$ is a vector space over a field $\mathbb{K}$, with a product $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and a unit element $\mathbb{1}$, codified in a map $\eta : \mathbb{K} \to \mathcal{A}$, satisfying

$$\begin{aligned} &\mu \circ (\mu \otimes id) = \mu \circ (id \otimes \mu) \\ &\mu \circ (id \otimes \eta) = id = \mu \circ (\eta \otimes id)\,. \end{aligned} \tag{1.13}$$

The first property is just the associativity of the product while the second represents the two-sidedness of multiplication of an algebra element by a scalar.

A **coalgebra** $\mathcal{C}$ is a vector space over a field $\mathbb{K}$, with two $\mathbb{K}$-linear maps, the *coproduct* $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ and *counit* $\epsilon : \mathcal{C} \to \mathbb{K}$. Notice that the coproduct goes from a single copy of the vector space to a tensor product of $\mathcal{C}$ with itself, while the counit goes from the vector space to the field $\mathbb{K}$. These operations are in fact dual with respect to their algebraic counterparts, as shown in fig. 1.1. The coalgebra maps satisfy the following axioms

$$\begin{aligned} &(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta\,, \\ &(id \otimes \epsilon) \circ \Delta = id = (\epsilon \otimes id) \circ \Delta\,. \end{aligned} \tag{1.14}$$

These are just the coassociativity property and the neutrality of the counit map with respect

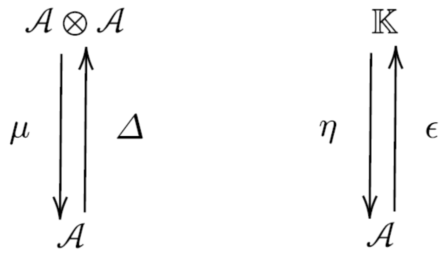

Figure 1.1: Duality of Algebra and Coalgebra structures: the domain of μ and η is the image of Δ and ϵ, respectively

to the coproduct we also found in the example for $\mathbb{C}[\mathcal{G}]$. The coalgebra axioms are dual with respect to the defining axioms of the algebra, in the way shown by the commutative diagrams in fig. 1.2. The duality is expressed by the fact that one diagram can be mapped into another by simply flipping the arrows and exchanging algebra and coalgebra maps.

Formally, two vector spaces X, Y over $\mathbb{K}$ are dual if there exists a nondegenerate sesquilinear form $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{K}$. So, given a unital algebra $\mathcal{A}$, equipped with multiplication μ and

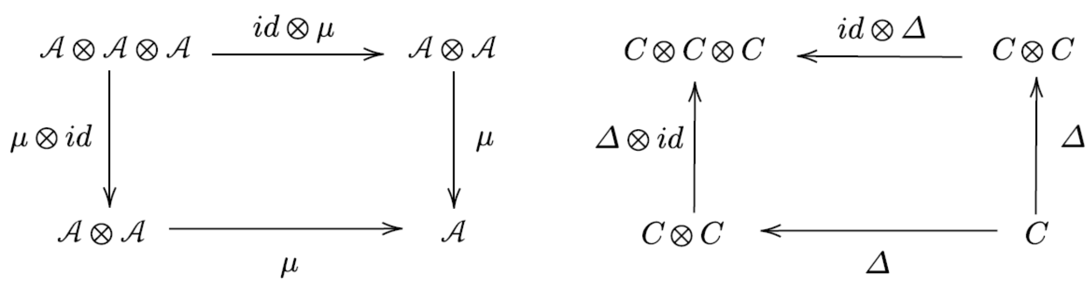

Figure 1.2: Commutative diagrams for associativity and coassociativity axioms. One diagram can be mapped onto another by flipping the direction of the arrows and exchanging algebraic operations with coalgebraic ones.

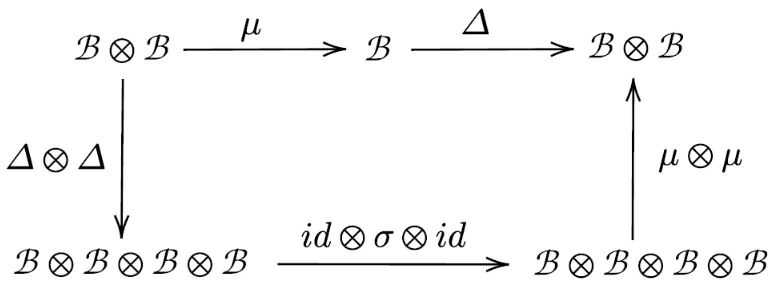

Figure 1.3: Commutative diagram representing the homomorphism property of the coproduct Δ with respect to the product μ. In the map above the bottom arrow, the flip operator $\sigma : \mathcal{B}\otimes\mathcal{B} \to \mathcal{B}\otimes\mathcal{B}$, with $\sigma(a\otimes b) = b\otimes a$ for $a, b \in \mathcal{B}$ is also present, to take into account the proper of the components of the tensor product.

unit element $\mathbb{1}$, and its dual vector space $\mathcal{A}^*$, one can define

$$\begin{aligned} \langle \mu(a,b), c\rangle &= \langle a\otimes b, \Delta[c]\rangle\,, \qquad \forall a,b\in\mathcal{A}\,, c\in\mathcal{A}^* \\ \langle \mathbb{1}, c\rangle &= \epsilon(c)\,. \end{aligned} \tag{1.15}$$

Assuming the algebraic properties for the multiplication and identity, the coalgebraic operations $\Delta : \mathcal{C} \to \mathcal{C}\otimes\mathcal{C}$ and $\epsilon : \mathcal{C} \to \mathbb{K}$ can be shown to satisfy the coalgebra axioms through (1.15), and the vice-versa also holds. We thus conclude that algebras and coalgebras are dual structures.

It is also possible to define a structure endowed with both algebraic and co-algebraic operations. A **bialgebra** $\mathcal{B}$ is a unital algebra and a coalgebra, whose coproduct Δ and counity ϵ are algebra homomorphisms with respect to the product μ (also denoted by $\cdot$) and the identity $\mathbb{1}$. In formulas:

$$\begin{aligned} \Delta(a\cdot b) =& \Delta(a)\cdot\Delta(b), \qquad \epsilon(a\cdot b) = \epsilon(a)\epsilon(b), \qquad \forall a,b\in\mathcal{B} \\ \Delta(\mathbb{1}) =& \mathbb{1}\otimes\mathbb{1}, \qquad \epsilon(\mathbb{1}) = 1, \qquad 1\in\mathbb{K}\,. \end{aligned} \tag{1.16}$$

The homomorphism properties are depicted in the commutative diagrams in fig. 1.3 and fig. 1.4. It is worth noticing that a bialgebra is a self-dual structure. Indeed, by reversing the role of multiplication, identity, coproduct and counit in the commutative diagram in fig. 1.3, one basically obtains the same commutative diagram. The diagrams in fig. 1.4 are not invariant under such exchange, but are self-dual. Overall, we may conclude that the dual of a bialgebra is another bialgebra. At this point, we have introduced the relevant structures that characterize the group multiplication and the identity element. To complete the algebraic description of a Lie group, we need something to play the role of the group inverse.

A **Hopf Algebra** $\mathcal{H}$ is a bialgebra over a field $\mathbb{K}$ equipped with a $\mathbb{K}$-linear map $S : \mathcal{H} \to \mathcal{H}$ called *antipode*, which satisfies the following compatibility condition with the other bialgebra structures:

$$\mu\circ(S\otimes id)\circ\Delta = \mu\circ(id\otimes S)\circ\Delta = \mathbb{1}\epsilon\,, \tag{1.17}$$

which, in the language of functions over a Lie group, expresses the fact that multiplying a group element by its inverse from the left or from the right yields the same result, the identity element. It can be shown [16] that this definition implies that S is unique and that it is an antihomomorphism and an anticoalgebra map, namely

$$\begin{aligned} &S(a\cdot b) = S(b)\cdot S(a), \qquad S(\mathbb{1}) = \mathbb{1} \\ &(S\otimes S)\circ\Delta = \sigma\circ\Delta\circ S, \qquad \epsilon\circ S(a) = \epsilon(a)\,, \end{aligned} \tag{1.18}$$

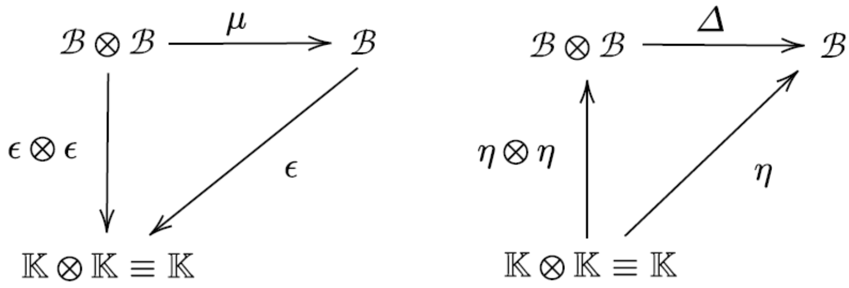

Figure 1.4: commutative diagrams for the homomorphism properties of the unit and counit. The diagrams are dual to each other, by exchanging the direction of the arrows and replacing the algebraic operations with their respective coalgebraic counterparts.

where $a \in \mathcal{H}$ and σ is the flip operator $\sigma : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ such that $\sigma(a \otimes b) = b \otimes a$, for $a, b \in \mathcal{H}$. It is easy to show that the commutative diagram depicting the Hopf Algebra property (1.17) is invariant under the inversion of all arrows, modulo the exchange $\Delta \leftrightarrow \mu$, $\epsilon \leftrightarrow \eta$, while S is left untouched. It is indeed the case that the dual of a Hopf Algebra $\mathcal{H}$ is another Hopf Algebra $\mathcal{H}^*$, whose maps are defined in terms of the dual pairing $\langle \cdot, \cdot \rangle$, as :

$$\begin{aligned} &\langle a \cdot b, c\rangle = \langle a \otimes b, \Delta(c)\rangle, \qquad \langle \mathbb{1}, c\rangle = \epsilon(c), \\ &\langle a, c \cdot d\rangle = \langle \Delta(a), c \otimes d\rangle, \qquad \langle a, \mathbb{1}\rangle = \epsilon(a), \\ &\langle S(a), c\rangle = \langle a, S(c)\rangle, \qquad \forall a, b \in \mathcal{H}^*,\ c, d \in \mathcal{H}\,. \end{aligned} \tag{1.19}$$

Basically, the structures of $\mathcal{H}$ define the structures of $\mathcal{H}^*$ and vice-versa. To further illustrate this concept with an example, let us consider the dual of the Hopf Algebra of functions on a Lie Group $\mathbb{C}[\mathcal{G}]$. Consider the coordinate functions $x^i \in \mathbb{C}[\mathcal{G}]$ for a neighbourhood of the identity e of the group, such that $x^i(e) = 0$. We can introduce the dually paired basis $t_i \in \mathbb{C}[\mathcal{G}]^*$ as

$$\begin{aligned} &\langle \mathbb{1}, \mathbb{1}\rangle = 1, \quad \langle \mathbb{1}, t_j\rangle = 0, \quad \langle x^i, \mathbb{1}\rangle = 0 \\ &\langle x^i, t_j\rangle = \delta^i_j, \quad \langle x^{i_1} \cdots x^{i_n}, t_j\rangle = 0\,, \quad \forall n > 1\,. \end{aligned} \tag{1.20}$$

Let g and g' be group elements inside a coordinate chart x^i that includes the identity. The coproduct map $\Delta[x^i](g, g') = x^i(g, g')$ gives the coordinates of the group element gg' in this chart. This can be thought of as a Taylor series around the identity

$$\Delta[x^i](g, g') = x^i(g, g') = \sum_{n,m=0}^{\infty} \sum_{\{i\},\{j\}} c_{i_1,\ldots,i_n,j_1,\ldots,j_m} x^{i_1}(g) \cdots x^{i_n}(g) x^{j_1}(g') \cdots x^{j_m}(g')\,. \tag{1.21}$$

The fact that $\Delta[x^i](g, e) = \Delta[x^i](e, g) = x^i(g)$ implies that the first terms of the series can be written as

$$\Delta[x^i](g, g') = x^i(g) + x^i(g') + \gamma^i_{jk} x^j(g) x^k(g') + \ldots\,. \tag{1.22}$$

Namely, the coproduct of the coordinate functions assumes the form

$$\Delta[x^i] = x^i \otimes \mathbb{1} + \mathbb{1} \otimes x^i + \gamma^i_{jk} x^j \otimes x^k + \ldots\,. \tag{1.23}$$

Using relations (1.20), it is possible to show that

$$\langle \mathbb{1}, t_j t_k\rangle = 0\,, \qquad \langle x^i, t_j t_k\rangle = \langle \Delta[x^i], t_j \otimes t_k\rangle = \gamma^i_{jk}\,, \qquad \langle x^i x^j, t_k t_l\rangle = \delta^i_k \delta^j_l + \delta^j_k \delta^i_l\,, \tag{1.24}$$

while all relations containing higher order terms in the coordinate functions are 0. By taking the antisymmetric part in the dual basis elements, we obtain

$$\langle x^i, [t_j, t_k]\rangle = \gamma^i_{jk} - \gamma^i_{kj} \tag{1.25}$$

while analogous relations with higher order monomials in x^i are all 0. This implies that the generators t_i close a Lie Algebra with structure constant $C^i_{jk} = \gamma^i_{jk} - \gamma^i_{kj}$. The construction for an associative algebra in which generators close a Lie algebra is unique and is none other than the universal enveloping algebra $\mathcal{U}[\mathfrak{g}]$ associated to the Lie algebra $\mathfrak{g}$ of the Lie group $\mathcal{G}$. With the techniques developed above it is also possible to find all the other relevant Hopf algebra structures for $\mathcal{U}[\mathfrak{g}]$, which are summarized below:

$$\begin{aligned} &[t_j, t_k] = c^i_{jk} t_i\,, \qquad \Delta[t_i] = t_i \otimes \mathbb{1} + \mathbb{1} \otimes t_i\,, \\ &\epsilon(t_i) = 0\,, \qquad S(t_i) = -t_i\,. \end{aligned} \tag{1.26}$$

When studying symmetries of noncommutative spacetimes, the above structures above will acquire non-linear deformations proportional to the noncommutativity scale, in general. This further strengthens the motivation behind the development of such abstract algebraic structures. In the context of Lie groups and Lie algebras, they are of course a redundant description. However, when dealing with non-linear deformations of relativistic symmetries, the description in terms of Hopf Algebras is inevitable, as Lie algebras and groups only allow for linear structures such as in (1.26).

In our discussion of noncommutative quantum field theory, we will focus on a particular class of Hopf Algebras known as *triangular* Hopf Algebras. A quasi-triangular Hopf Algebra posses an invertible element $R \in \mathcal{H} \otimes \mathcal{H}$ known as the R-matrix, which satisfies the following properties:

$$\begin{aligned} &R\Delta(a)R^{-1} = (\sigma \circ \Delta)(a) \\ &(\Delta \otimes \mathbb{1})(R) = R_{13}R_{23} \\ &(\mathbb{1} \otimes \Delta)(R) = R_{13}R_{12} \end{aligned} \tag{1.27}$$

where R_{ij} indicates that the two components of the R-matrix are placed in the i-th and j-th components of the tensor product, in order. As a consequence of the last two properties, the R-matrix can be shown to be the solution of the Yang-Baxter equation, which reads

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}\,. \tag{1.28}$$

If the R-matrix satisfies

$$R^{-1} = R_{21} \leftrightarrow RR_{21} = \mathbb{1} = R_{21}R\,, \tag{1.29}$$

where $R_{21} = (\sigma \otimes R)$, then the Hopf Algebra is said to be triangular. A quasitriangular Hopf-Algebra satisfies the Poincaré-Birkhoff-Witt property [33], particularily relevant for non-abelian Hopf Algebras. The property ensures that, once a finite set of generators of the algebra are specified, the ordered monomials in any given order are a basis for polynomials of fixed degree. In other words, all monomials of a specified order are independent of each other, which is a desirable property when dealing with noncommutative field theory, where the protagonists of the analysis are functions of noncommutative coordinates. In the case-study we will investigate, we will show that coordinate noncommutativity is also governed by the R-matrix, ensuring the Poincaré-Birkhoff-Witt property. Therefore, a noncommutative field and products of noncommutative fields can be expanded in series without ambiguity. It will also be shown that the R-matrix plays a fundamental role in characterizing the deformation of bosonic and fermionic statistics.

As discussed in the Introduction, Hopf Algebras are a useful tool to describe deformations of relativistic symmetries acting on a noncommutative spacetime, which may be realized as a quantum homogeneous space. In the abstract algebraic language we are developing, this requires the introduction of two dual notions: that of an algebra module and of a coalgebra comodule.

Given an algebra $\mathcal{A}$, a left- (right-) $\mathcal{A}$-module is a vector space $\mathcal{M}$ whose elements can be multiplied by $\mathcal{A}$ from the left (right), yielding another element of $\mathcal{M}$. As a familiar example, consider the $\mathfrak{so}(3)$ Lie algebra, specified by the commutators $[L_i, L_j] = \epsilon_{ijk}L_k$. A left $\mathfrak{so}(3)$-module is the vector space $\mathbb{R}^3$, coordinatized by x_i. The action of L_i on x_j can be defined as

$$L_i \triangleright x_l = \epsilon_{ijl}x_j \,, \tag{1.30}$$

and it needs to be compatible with the product of the $\mathfrak{so}(3)$ Lie algebra, given by the Lie bracket. Indeed, it can be explicitly verified that

$$L_{[i} \triangleright (L_{j]} \triangleright x_k) = \epsilon_{ijl}L_l \triangleright x_k \,. \tag{1.31}$$

The action $\triangleright : \mathfrak{so}(3) \times \mathbb{R}^3 \to \mathbb{R}^3$ transforms elements of $\mathbb{R}^3$ into other elements of $\mathbb{R}^3$. In general, if μ and $\mathbb{1}$ are the product and the identity of the algebra, then the compatibility condition can be written as

$$\begin{aligned} &\triangleright \circ (id \otimes \triangleright) = \triangleright \circ (\mu \otimes id)\,, \\ &\triangleright \circ (\eta \otimes id) = id\,, \end{aligned} \tag{1.32}$$

while the second simply ensures that acting with the identity of the algebra leaves the element of the vector space invariant. As usual when dealing with Hopf Algebras, it is also useful to introduce the dual notion of action, named *coaction*, where the Hopf Algebra comultiplication is involved.

Given a coalgebra $\mathcal{C}$, a left comodule is a vector space $\mathcal{X}$ whose elements can be comultiplied by $\mathcal{C}$ by means of a left coaction $' : \mathcal{X} \to \mathcal{C} \otimes \mathcal{X}$, which is compatible with the coproduct Δ in the following sense:

$$\begin{aligned} &(\Delta \otimes id) \circ {}' = (id \otimes {}') \circ {}'\,, \\ &(\epsilon \otimes id) \circ {}' = id\,. \end{aligned} \tag{1.33}$$

The left coaction is dual with respect to the action in the usual sense. By exchanging the coalgebraic operations with the algebraic ones in (1.33) and reversing the order of the map composition, we obtain exactly (1.32). As an example, recall the algebra of functions on the invertible matrices of dimension n, $\mathbb{C}[GL(n)]$, and consider the vector space $\mathbb{R}^n$, coordinatized by x^i. A left coaction of $\mathbb{C}[GL(n)]$ on $\mathbb{R}^n$ can be realized as

$$(x^i)' = M^i_j \otimes x^j \,, \tag{1.34}$$

which basically expresses the fact that matrices of the $GL(n)$ group transform vectors of $\mathbb{R}^n$ into vectors of $\mathbb{R}^n$ in an abstract way. The compatibility conditions (1.33) can be easily verified using expressions (1.12). Indeed, the first in (1.33) reads

$$(\Delta \otimes id)(M^i_j \otimes x^j) = \Delta[M^i_j] \otimes x^j = M^i_k \otimes M^k_j \otimes x^j = M^i_k \otimes (x^k)' = ((x^i)')' \,. \tag{1.35}$$

Given that a Hopf Algebra $\mathcal{H}$ is both an algebra and a coalgebra, it is possible to define both an action and a coaction when acting on a vector space $\mathcal{X}$. The latter is promoted to a quantum homogeneous space if a Hopf Algebra $\mathcal{H}$, equipped with an action $\triangleright$ and a coaction $'$, leaves $\mathcal{X}$ covariant in the sense that

$$\begin{aligned} &h \triangleright (a \cdot b) = (h_{(1)} \triangleright a) \cdot (h_{(2)} \triangleright b)\,, \\ &(a \cdot b)' = a' \cdot b' \qquad h \in \mathcal{H}\,,\; a, b \in \mathcal{X}\;, \end{aligned} \tag{1.36}$$

i.e. both the left action and left coaction are compatible with the product $\cdot$ of the homogeneous space $\mathcal{X}$. In the physical applications we are about to present, $\mathcal{X}$ is a non-commutative spacetime (in the sense that coordinates satisfy non-trivial commutation relations) and $\mathcal{H}$ is a Hopf Algebra describing a quantum version or the Poincaré group or the Poincaré algebra, which leave the commutators covariant. The idea is that the spacetime of different observers connected by relativistic transformations should experience the same type of noncommutativity.

1.2 θ-Minkowski

The simplest example of noncommutative spacetime, denoted by $\mathcal{X}_\theta$, is one in which the coordinates satisfy commutation relations of the form

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \tag{1.37}$$

where $\theta^{\mu\nu}$ is a constant, real valued matrix. This is known as θ-Minkowski or the canonical/Moyal noncommutative spacetime and has been intensively studied in the context of String Theory [10, 11] and of noncommutative quantum field theory [34, 35].

The algebra of functions on the Poincaré group is deformed accordingly and is denoted by $\mathbb{C}_\theta(ISO(3,1))$, commonly known as θ-Poincaré. This deformation of the Poincaré group is a symmetry of $\mathcal{X}_\theta$ if commutation relations (1.37) are covariant under the standard coaction:

$$x'^\mu = \Lambda^\mu_\nu \otimes x^\nu + a^\mu \otimes \mathbb{1} \;\rightarrow\; [x'^\mu, x'^\nu] = i\theta^{\mu\nu}\,, \tag{1.38}$$

where Λ^μ_ν and a^μ refer to the Lorentz and translation parameters of the θ-Poincaré transformation, respectively. The compatibility condition (1.38) implies that the quantum group parameters satisfy the following commutation relations

$$[a^\mu, a^\nu] = i(\delta^\mu_\rho \delta^\nu_\sigma - \Lambda^\mu_\rho \Lambda^\nu_\sigma)\theta^{\rho\sigma}\,, \quad [\Lambda^\mu_\rho, \Lambda^\nu_\sigma] = [\Lambda^\mu_\nu, a^\rho] = 0\,, \tag{1.39}$$

while the coalgebra sector and the antipode are left unchanged:

$$\begin{aligned} &\Delta[\Lambda^\mu_\nu] = \Lambda^\mu_\rho \otimes \Lambda^\rho_\nu\,, \quad \Delta[a^\mu] = \Lambda^\mu_\rho \otimes a^\rho + a^\mu \otimes \mathbb{1}\,, \\ &S[\Lambda^\mu_\nu] = (\Lambda^{-1})^\mu_\nu\,, \quad S[a^\mu] = -(\Lambda^{-1})^\mu_\rho a^\rho\,, \\ &\epsilon[\Lambda^\mu_\nu] = \delta^\mu_\nu\,, \quad \epsilon[a^\mu] = 0\,, \end{aligned} \tag{1.40}$$

and the Lorentz matrices still satisfy the orthogonality relations $\Lambda^\mu_\rho \Lambda^\nu_\sigma \eta_{\mu\nu} = \eta_{\rho\sigma}$, with $\eta_{\rho\sigma}$ being the usual Minkowski metric. The Hopf Algebra dual to $\mathbb{C}_\theta(ISO(3,1))$ is the θ-deformed universal enveloping algebra of the Poincaré group, denoted by $\mathcal{U}_\theta(\mathfrak{iso}(3,1))$, and is characterized by undeformed commutators

$$\begin{aligned} &[P_\mu, P_\nu] = 0\,, \qquad [M_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu)\,, \\ &[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho} M_{\nu\rho} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho})\,, \end{aligned} \tag{1.41}$$

where P_μ and $M_{\mu\nu}$ are the translations and Lorentz generators, respectively. The antipodes and counits are also undeformed, $S(P_\mu) = -P_\mu$, $S(M_{\mu\nu}) = -M_{\mu\nu}$, $\epsilon(P_\mu) = \epsilon(M_{\mu\nu}) = 0$; all the non-linearity is stored in the coproducts

$$\begin{aligned} \Delta[P_\mu] &= P_\mu \otimes \mathbb{1} + \mathbb{1} \otimes P_\mu \,, \\ \Delta[M_{\mu\nu}] &= M_{\mu\nu} \otimes \mathbb{1} + \mathbb{1} \otimes M_{\mu\nu} - \frac{1}{2}\theta^{\rho\sigma}\left(\eta_{\rho[\mu}P_{\nu]} \otimes P_\sigma + P_\rho \otimes \eta_{\sigma[\mu]}P_{\nu]}\right) . \end{aligned} \tag{1.42}$$

These can be obtained via the application of a "Drinfeld twist", starting from the undeformed coproducts. The twist, denoted by $\mathcal{F}$, is an invertible element of $\mathcal{U}_\theta(\mathfrak{iso}(3,1)) \otimes \mathcal{U}_\theta(\mathfrak{iso}(3,1))$, whose expression is given by

$$\mathcal{F} = \exp\left(\frac{i}{2}\theta^{\mu\nu}P_\mu \otimes P_\nu\right), \tag{1.43}$$

and is combined with the undeformed coproducts $\Delta^{(0)}(X)$, $X \in \mathcal{U}_\theta(\mathfrak{iso}(3,1))$, to obtain the deformed ones by the similarity transformation

$$\Delta(X) = \mathcal{F}^{-1}\Delta^{(0)}(X)\mathcal{F} \,. \tag{1.44}$$

Starting from the twist operator (1.43), it is possible to construct the R-matrix as $R = \mathcal{F}_{21}\mathcal{F}^{-1} = (\mathcal{F})^{-2}$ [36]. The θ-Poincaré algebra is triangular, since one can easily verify that $RR_{21} = \mathbb{1} \otimes \mathbb{1}$. The existence of the R-matrix allows the construction of the so-called "θ-Minkowski braided tensor product algebra" [37,38]. This concept will be formally introduced in chapter 2 and analyzed in detail for the lightlike κ-Minkowski spacetime. For the scope of the present section, it suffices to say that this algebra is necessary to define multi-local functions when constructing quantum field theory on the θ-Minkowski spacetime. The main idea is that the construction of N-point functions requires multiple copies of the noncommutative spacetime from an algebraic point of view. Coordinates belonging to different copies of the braided tensor product must satisfy non-trivial commutation relations if they are to be invariant under the action of the relevant quantum group.

For the θ-Minkowski case, the status of the corresponding noncommutative quantum field theory is still unclear. A first proposal advanced by Fiore & Wess [36] finds that all N-point functions are equal to the ones employed in the commutative case so that the theory is completely equivalent to the commutative one. A more recent approach based on Oeckl's algebraic definition of braided quantum field theory [39,40], finds that N-point functions for $N > 3$ are indeed deformed [41], leaving room for possible deviations from the standard theory. It is worth mentioning that a a 1+1-dimensional toy model based on θ-noncommutativity has been studied by Grosse and Wulkenhaar [42] and was proven to be finite at all energies. This is the only known example of an interacting quantum field theory that is well-defined at all scales, and realizes Snyder's original dream of keeping the divergences of quantum field theory under control via noncommutativity.

Apart from these formal developments, more phenomenological approaches to quantum field theory on θ-Minkowski predict violations of the Pauli exclusion principle as potential testable effects [43–45]. The idea is that fermionic construction operators satisfy a *braided* commutation relation given by [43]

$$a^\dagger(p)a^\dagger(q) + R^{-1}(q,p)a^\dagger(q)a^\dagger(p) = 0 \,, \tag{1.45}$$

where p, q are two four-momenta and $R^{-1}(q,p) = \exp(i\theta_{\mu\nu}q^\mu p^\nu)$. Underground experiments searching for forbidden Pauli transitions allow to put constraints on θ-noncommutativity parameters [44–46]. The tightest bounds on this noncommutativity scale been found in [46]:

expressing $\theta \sim 1/\Lambda^2$, where Λ is an energy scale in units of the Planck energy, the authors in [46] find that $\Lambda > 6.9 \cdot 10^{-2}$ if $\theta^{0i} = 0$ and $\Lambda > 2.6 \cdot 10^2$ otherwise.

Approaches to quantum field theory on θ-Minkowski with breaking of Lorentz symmetry have also been studied [47,48], yielding surprising results. When analyzing loop corrections to the propagator, the usual ultraviolet-infrared decoupling that forms the basis for conventional renormalization programs disappears. Instead, the ultraviolet and infrared degrees of freedom couple to each other when integrating over high momenta in loop integrals, via a mechanism known as IR/UV mixing. As it will be relevant in our cold atom phenomenology studies in chapter 5 (but under a very different form), we review the basic aspects of this mechanism following [48]. Corrections to the propagator in noncommutative field theories involve both planar and non-planar diagrams, the latter of which are exclusive to noncommutative scenarios. Upon introducing an ultraviolet momentum cutoff Λ, the planar tadpole diagram is characterized by a contribution of the form

$$\int_0^\Lambda dk \frac{k^3}{k^2+m^2} = \frac{1}{2}\Lambda^2 - \frac{1}{2}m^2 \ln\left(1+\frac{\Lambda^2}{m^2}\right), \tag{1.46}$$

while the corresponding nonplanar contribution is given by

$$\int_0^\Lambda dk \cos\left(\frac{1}{2}k\tilde{p}\right)\frac{k^3}{k^2+m^2} = \int_0^\Lambda dk \cos\left(\frac{k}{2\tilde{p}^{-1}}\right)\frac{k^3}{k^2+m^2}, \tag{1.47}$$

where $\tilde{p}_\mu \equiv \theta^{\mu\nu}p_\nu$ and p_μ is the external momentum. The nonplanar diagram is cut off by the smaller between Λ and $|\tilde{p}^{-1}|$. In fact, for $\Lambda \ll |\tilde{p}|^{-1}$, the θ-dependent cosine function approaches one, while in the other regime, for $\Lambda \gg |\tilde{p}|^{-1}$, the integrand oscillates rapidly in the integration region such that $k \gg |\tilde{p}|^{-1}$. In this latter case, (1.47) can be expanded as

$$\int_0^\Lambda dk \cos\left(\frac{k}{2\tilde{p}^{-1}}\right)\frac{k^3}{k^2+m^2} \simeq \frac{1}{2}\left(\frac{2}{|\tilde{p}|}\right)^2 - \frac{1}{2}m^2 \ln\left(1+\frac{\left(\frac{2}{|\tilde{p}|}\right)^2}{m^2}\right). \tag{1.48}$$

As in the commutative limit, the planar diagram diverges in the $\Lambda \to \infty$ limit. The non-planar contribution is instead independent of Λ, and is finite until $\tilde{p} \neq 0$. The UV portion of the loop integration introduces the dependence on $|\tilde{p}|^{-1}$, bringing about an IR singularity when $\tilde{p} \to 0$. This is the essence of the IR/UV mixing mechanism. The introduction of an UV noncommutativity scale brings about non-trivial effects also in the IR portion of the theory.

1.3 Timelike κ-Minkowski

The $3+1$-dimensional κ-Minkowski noncommutative spacetime is defined by commutations relations among coordinates of the form

$$[x^\mu, x^\nu] = \frac{i}{\kappa}(v^\mu x^\nu - v^\nu x^\mu), \quad \mu,\nu = 0,\dots,3, \tag{1.49}$$

where κ is a deformation parameter with the dimensions of energy and v^μ is a set of four real parameters. Contrary to the θ-Minkowski case, these commutators are linear in the coordinate functions, which is an appealing opportunity for quantum gravity phenomenology. If one is to associate the deformation parameter to the Planck energy, the leading order corrections appear at the first order in the deformation scale, as opposed to θ-Minkowski, where the leading order effects would be quadratic in the Planck energy.

One can introduce a (commutative) arbitrary constant metric tensor $g_{\mu\nu}$, and require that it is preserved by a quantum group of symmetries, which also acts covariantly on the coordinate functions, so that the form of (1.49) is the same for each observer. One obtains different quantum groups depending on the relationship between the parameters v^μ and the metric $g_{\mu\nu}$. If the parameters form a lightlike/null vector, *i.e.* $v^\mu v^\nu g_{\mu\nu} = 0$, one obtains a *triangular* Hopf algebra [49], which is the best-behaved case in terms of constructing a noncommutative quantum field theory (we will study it in detail in chapter 2). This quantum group has been discovered in [50–52]. The spacelike case has been discussed in [53,54], while the timelike one, which we will focus on in this subsection, was first introduced in [55–57] and is by far the most-studied one [58–82]. The symmetries of (1.49) are expressed in terms of the κ-Poincaré quantum group, denoted by $\mathbb{C}_\kappa[ISO(3,1)]$. The algebra sector reads [83] (all greek indices run in the set $\{0, ..., 3\}$)

$$
\begin{aligned}
&[\Lambda^\mu{}_\nu, \Lambda^\rho{}_\sigma] = 0, \quad [a^\mu, a^\nu] = \frac{i}{\kappa}(v^\mu a^\nu - v^\nu a^\mu) \\
&[a^\gamma, \Lambda^\mu{}_\nu] = \frac{i}{\kappa}[(\Lambda^\mu{}_\alpha v^\alpha - v^\mu)\Lambda^\gamma{}_\nu + (\Lambda^\alpha{}_\nu g_{\alpha\beta} - g_{\nu\beta})v^\beta g^{\mu\gamma}] \\
&\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta g^{\alpha\beta} = g^{\mu\nu}, \quad \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu g_{\rho\sigma} = g_{\mu\nu}\,.
\end{aligned}
\tag{1.50}
$$

The coproduct Δ, antipode S and counit ϵ, are given by

$$
\begin{aligned}
&\Delta[\Lambda^\mu{}_\nu] = \Lambda^\mu{}_\alpha \otimes \Lambda^\alpha{}_\nu, \quad \Delta[a_\mu] = \Lambda^\mu{}_\nu \otimes a^\nu + a^\mu \otimes 1 \\
&S[\Lambda^\mu{}_\nu] = (\Lambda^{-1})^\mu{}_\nu, \quad S[a^\mu] = -(\Lambda^{-1})^\mu{}_\nu a^\nu, \quad \epsilon[\Lambda^\mu{}_\nu] = \delta^\mu_\nu, \quad \epsilon[a^\mu] = 0.
\end{aligned}
\tag{1.51}
$$

Upon specifying $v^\mu = (1, \vec{0})$ and $g_{\alpha\beta} = diag(1, -1, -1, -1)$ the timelike case is characterized by commutation relations of the form

$$
[x^0, x^i] = \frac{i}{\kappa} x^i\,. \tag{1.52}
$$

One of the appeals of the timelike case is that, superficially, the algebra (1.52) appears spatially isotropic, and indeed it is invariant under commutative/undeformed spatial rotations. At an early time of investigation of the physics of quantum groups, when some phenomenological consequences were being conjectured, undeformed spatial isotropy seemed compelling, because before clarifying the difference between symmetry breaking and symmetry deformations, a non-isotropic model could be feared to be incompatible with very basic observations of the isotropy of empty space [84]. Moreover, this deformation of standard Lorentzian geometry also appears naturally in studies of $2+1$ quantum gravity [14,85–87].

The commutation relations (1.50) are thus specified to

$$
\begin{aligned}
&[a^\mu, a^\nu] = \frac{i}{\kappa}(\delta^\mu{}_0 a^\nu - \delta^\nu{}_0 a^\mu), \quad [\Lambda^\alpha{}_\beta, \Lambda^\gamma{}_\delta] = 0 \\
&[\Lambda^\alpha{}_\beta, a^\rho] = -\frac{i}{\kappa}((\Lambda^\alpha{}_0 - \delta^\alpha{}_0)\Lambda^\rho{}_\beta + (\Lambda_{0\beta} - g_{0\beta})g^{\alpha\rho})\,,
\end{aligned}
\tag{1.53}
$$

while the coproducts and antipode remain the same. The dual Hopf algebra in the timelike case in the so-called bicrossproduct basis is denoted by $\mathcal{U}_\kappa(\mathfrak{iso}(3,1)$ and was found in [88], characterized by commutation relations between the Poincaré generators given by

$$
\begin{aligned}
&[P_\mu, P_\nu] = 0\,, \qquad [R_i, P_j] = i\epsilon_{ijk} P_k\,, \\
&[R_i, P_0] = 0 \quad [N_i, P_0] = iP_i \\
&[N_i, P_j] = i\delta_{ij}\left(\frac{\kappa}{2}(1 - e^{-\frac{2P_0}{\kappa}}) + \frac{1}{2\kappa}P_k P_k\right) - \frac{i}{\kappa} P_i P_j\,,
\end{aligned}
\tag{1.54}
$$

where N_i, R_i are boosts and rotations, respectively, and the Lorentz subalgebra remains undeformed. The coproducts are given by

$$\begin{aligned}\Delta P_0 &= P_0 \otimes \mathbb{1} + \mathbb{1} \otimes P_0\,,\\ \Delta P_i &= P_i \otimes \mathbb{1} + e^{-\frac{P_0}{\kappa}} \otimes P_i\,,\\ \Delta R_i &= R_i \otimes \mathbb{1} + \mathbb{1} \otimes R_i\,,\\ \Delta N_i &= N_i \otimes \mathbb{1} + e^{-\frac{P_0}{\kappa}} \otimes N_i - \frac{1}{\kappa}\epsilon_{ijk} P_j \otimes N_k\,.\end{aligned} \tag{1.55}$$

To further clarify the definition of covariant Hopf Algebra action defined in (1.36), let us verify that commutation relations (1.52) are covariant under the application of the translation generators P_μ. We define said action as $P_\mu \triangleright x^\nu = -i\delta^\nu_\mu$. Focusing on the spatial translations P_i, when acting on products of coordinates we have

$$\begin{aligned}P_i \triangleright x^0 x^j =& \big(P_{i(1)} \triangleright x^0\big)\big(P_{i(2)} \triangleright x^j\big) = (P_i \triangleright x^0)(1 \triangleright x^j) + (e^{-\frac{P_0}{\kappa}} \triangleright x^0)(P_i \triangleright x^j) = \\ & - i\delta^j_i (x^0 + \frac{i}{\kappa})\,,\end{aligned} \tag{1.56}$$

$$\begin{aligned}P_i \triangleright x^j x^0 =& \big(P_{i(1)} \triangleright x^j\big)\big(P_{i(2)} \triangleright x^0\big) = (P_i \triangleright x^j)(1 \triangleright x^0) + (e^{-\frac{P_0}{\kappa}} \triangleright x^j)(P_i \triangleright x^0) = \\ & - i\delta^j_i x^0\,,\end{aligned} \tag{1.57}$$

so that

$$P_i \triangleright [x^0, x^i] = \delta^j_i\,. \tag{1.58}$$

From the right-hand-side of (1.52), we simply have $P_i \triangleright \frac{i}{\kappa} x^j = \frac{1}{\kappa}\delta^j_i$, thus proving covariance. With analogous techniques, the same can be shown for P_0 and for the Lorentz generators, upon defining

$$N_i \triangleright x^j = i x^0 \delta^i_j, \qquad N_i \triangleright x^0 = i x_i, \qquad M_i \triangleright x^j = \epsilon^j_{ik} x^k, \qquad M_i \triangleright x^0 = 0\,, \tag{1.59}$$

and using the coproducts (1.55).

The term "bicrossproduct" refers to the algebraic generalization of the semi-direct product. In the present case, $\mathcal{U}_\kappa(\mathfrak{iso}(3,1)$ is realized as the bicrossproduct between the undeformed Lorentz enveloping algebra in 3+1 dimensions and the deformed translation subalgebra spanned by P_μ. In terms of this basis, the noncommutative space time (1.52) can be realized in a natural way by duality with respect to the translation subalgebra, as discussed in detail in [88].

In this basis, the Casimir element of the κ-Poincaré algebra is given by

$$C = 4\kappa^2 \sinh^2\left(\frac{P_0}{2\kappa}\right) - e^{\frac{P_0}{\kappa}} P_i P_i\,, \tag{1.60}$$

and, as we will discuss in detail in chapter 5, has interesting phenomenological consequences if one is to interpret it as a deformation of the usual mass-shell relation of relativistic particles [26–28]. On the more technical side, several works have explored noncommutative quantum field theory and gauge theories on timelike κ-Minkowski [78, 89–97]. Only very recently, there has been a first step towards a connection with the phenomenology of particle physics [98]. One of the obstructions encountered in the construction of the full theory is a lack of a notion of multiparticle states of identical particles [99–103]. As we will see in chapter 2, this issue can be resolved for the lightlike κ-Minkowski case, thanks to the existence of a universal R-matrix.

1.4 ρ-Minkowski

The ρ-Minkowski noncommutative spacetime is an example of a class of Lie-algebra type quantum spacetimes, introduced in [104], whose quantum symmetries can be realized by means of a twist deformation. In [29], it was shown that the symmetries can also be realized in terms of a bicrossproduct construction, akin to what is done for timelike κ-Minkowski in [88]. The commutation relations among the coordinates are of the form

$$[x^0, x^1] = -i\rho x^2, \qquad [x^0, x^2] = i\rho x^1\,, \tag{1.61}$$

while the other commutators are 0; in particular, the x^3 coordinate is central. This may be referred to as a sort of angular noncommutativity, given that the time variable acts as the generator of rotations in the (x_1, x_2) plane. Indeed, (1.61) is just the Euclidean subalgebra involving the three spatial translations and the rotation about the 3-axis. As a consequence, when these coordinates are represented as operators on a Hilbert space, the spectrum of the time coordinate is discrete [105]. The associated ρ-Poincaré quantum group, denoted by $\mathcal{C}_\rho(P)$ in [29], is defined by the following commutation relations [105, 106]:

$$\begin{aligned}
&[a^\mu, a^\nu] = i\rho[\delta^\nu{}_0(a^2\delta^\mu{}_1 - a^1\delta^\mu{}_2) - \delta^\mu{}_0(a^2\delta^\nu{}_1 - a^1\delta^\nu{}_2)]\,, \quad [\Lambda^\mu{}_\nu, \Lambda^\rho{}_\sigma] = 0\,,\\
&[\Lambda^\mu{}_\nu, a^\rho] = i\rho\,[\Lambda^\rho{}_0(\Lambda^\mu{}_1 g_{2\nu} - \Lambda^\mu{}_2 g_{1\nu}) - \delta^\rho{}_0(\Lambda_{2\nu}\delta^\mu{}_1 - \Lambda_{1\nu}\delta^\mu{}_2)]\ .
\end{aligned} \tag{1.62}$$

Like in the previous examples and in the other cases of quantum Poincaré groups outlined in [104], the coalgebra sector and the antipodes have their expressions undeformed.

$$\begin{aligned}
&\Delta(\Lambda^\mu{}_\nu) = \Lambda^\mu{}_\alpha \otimes \Lambda^\alpha{}_\nu\,, \quad \Delta(a^\mu) = \Lambda^\mu{}_\nu \otimes a^\nu + a^\mu \otimes 1\,,\\
&\epsilon(\Lambda^\mu{}_\nu) = \delta^\mu{}_\nu\,, \quad \epsilon(a^\mu) = 0\,, \quad S(\Lambda^\mu{}_\nu) = (\Lambda^{-1})^\mu{}_\nu\,, \quad S(a^\mu) = -a^\nu(\Lambda^{-1})^\mu{}_\nu\,.
\end{aligned} \tag{1.63}$$

Using these quantum group transformations and the representations of the spacetime coordinates, the properties of observers and localization were investigated in [106]. The dual Hopf Algebra in the bicrossproduct basis found in [29] is denoted by $\mathcal{U}_\rho(\mathfrak{p})$ and is characterized by the usual Poincaré commutators, while the coproducts are ρ-deformed and read

$$\begin{aligned}
\Delta P_0 &= P_0 \otimes 1 + 1 \otimes P_0,\\
\Delta P_1 &= P_1 \otimes 1 + \cos(\rho P_0) \otimes P_1 - \sin(\rho P_0) \otimes P_2,\\
\Delta P_2 &= P_2 \otimes 1 + \cos(\rho P_0) \otimes P_2 + \sin(\rho P_0) \otimes P_1,\\
\Delta P_3 &= P_3 \otimes 1 + 1 \otimes P_3,\\
\Delta R_1 &= R_1 \otimes 1 + \cos(\rho P_0) \otimes R_1 - \sin(\rho P_0) \otimes R_2,\\
\Delta R_2 &= R_2 \otimes 1 + \cos(\rho P_0) \otimes R_2 + \sin(\rho P_0) \otimes R_1,\\
\Delta R_3 &= R_3 \otimes 1 + 1 \otimes R_3,\\
\Delta N_1 &= N_1 \otimes 1 + \cos(\rho P_0) \otimes N_1 - \sin(\rho P_0) \otimes N_2 + \rho P_1 \otimes R_3,\\
\Delta N_2 &= N_2 \otimes 1 + \cos(\rho P_0) \otimes N_2 + \sin(\rho P_0) \otimes N_1 + \rho P_2 \otimes R_3,\\
\Delta N_3 &= N_3 \otimes 1 + 1 \otimes N_3 + \rho P_3 \otimes R_3.
\end{aligned} \tag{1.64}$$

The counits are all 0 while the antipodes are given by

$$\begin{aligned}
&S(P_0) = -P_0,\\
&S(P_1) = -P_1\cos(\rho P_0) - P_2\sin(\rho P_0),\\
&S(P_2) = -P_2\cos(\rho P_0) + P_1\sin(\rho P_0),\\
&S(P_3) = -P_3,\\
&S(R_1) = -R_1\cos(\rho P_0) - R_2\sin(\rho P_0),\\
&S(R_2) = -R_2\cos(\rho P_0) + R_1\sin(\rho P_0),\\
&S(R_3) = -R_3,\\
&S(N_1) = -\cos(\rho P_0)N_1 - \sin(\rho P_0)N_2 + \rho\cos(\rho P_0)P_1R_3 + \rho\sin(\rho P_0)P_2R_3,\\
&S(N_2) = -\cos(\rho P_0)N_2 + \sin(\rho P_0)N_1 + \rho\cos(\rho P_0)P_2R_3 - \rho\sin(\rho P_0)P_1R_3,\\
&S(N_3) = -N_3 + \rho R_3P_3.
\end{aligned} \tag{1.65}$$

This Hopf Algebra structure is obtained by applying a Hopf Algebra isomorphism [29] to the algebra found in [107] via twist deformation. The isomorphism does not affect the generators P_0, R_3, so that the form of the R-matrix

$$R = \exp\left[i\rho(R_3 \otimes P_0 - P_0 \otimes R_3)\right], \tag{1.66}$$

is the same in the two bases. One can easily check that $RR_{21} = \mathbb{1}\otimes\mathbb{1}$, so that the Hopf Algebra is triangular. The existence of the R-matrix makes this model appealing for the construction of a quantum field theory following the prescriptions outlined in chapter 2, and will be done in the near future. The results are to be compared with the ones obtained using a different approach [108], where IR/UV mixing properties arise in loop corrections to 2- and 4-point functions.

Gaining inspiration from this noncommutative framework, phenomenological studies have found instances of relative locality [109] in models with ρ-deformed symmetries [110–113]. The effect found is known as dual lensing and predicts that if two particles of different energy travel parallel to each other for an observer at rest with respect to the source that emitted them, then the two particles will be seen under a non-zero angle for a boosted observer. In the not-so-far feature, there will be the possibility of putting bounds on these phenomenological models inspired by the ρ-noncommutative framework, thanks to the new generation of multisatellite telescopes [114, 115].

An analogous type of angular noncommutativity, known as λ-Minkowski, was studied in [116, 117]. The algebraic structure is almost identical to the one outlined for ρ-Minkowski, but the roles of x^0 and x^3 are exchanged. In this case, time is a commutative variable, so that the model avoids the difficulties in setting up an Hamiltonian analysis, crucial for the development of interacting quantum field theory. Investigations of this type are reported in [116], where the IR/UV mixing mechanism plays a role in loop corrections to the propagator.

Chapter 2

Braided κ-lightlike noncommutative Quantum Field Theory

Quantum field theory on noncommutative spacetimes has been studied for decades [34, 118, 119], with the original motivation of taming the divergences of the theory developed on classical spacetime. The status of quantum field theory on the well known θ- and timelike κ-Minkowski has been briefly reviewed in section 1.2 and section 1.3, respectively.

Recently, the very first steps have been taken towards the investigation of quantum field theory on the lightlike κ-Minkowski spacetime [30,120,121], symmetric under the "lightlike" κ-Poincaré quantum group [50–52]. The focus of [120] was the construction of N-point functions, for which the concept of braiding was realized to be crucial. In the commutative case, N-point functions are simply functions from several points on the spacetime manifold onto the complex or real numbers, and in terms of the (commutative) algebra of functions on the manifold, they can be simply formulated as elements of tensor products of copies of the same algebra of functions. In the noncommutative setting, simply taking the tensor product fails to produce a covariant structure: in other words, assuming that the coordinates of different points commute with each other is not covariant under the quantum group Poincaré transformations, as we will review shortly. Fortunately, there is a generalization of the concept of tensor product, called *braiding* [49], which allows one to identify a noncommutative algebra of N-points that is invariant under the relevant quantum group. In [120], the covariant braided N-point algebra for a general parametrization of κ-Minkowski-like noncommutative spacetimes was constructed, and it was proven that its associativity is only compatible with the lightlike model. Furthermore, the coordinate differences between different points (and therefore all N-point functions) were shown to be commutative, just like in the work of Wess and Fiore [36,122]. In [120], a proposal for a covariant Pauli-Jordan function was put forward, but a technical obstacle prevented the definition of general Lorentz-invariant N-point functions. Namely, the momentum space of the theory was observed to be not closed under Lorentz transformations, which practically meant that certain momentum space integrals would have a Lorentz-breaking upper bound related to the deformation energy scale. Thanks to a recent observation [96], this problem of the non-closure of momentum space under Lorentz transformations can be solved by enlarging the basis of noncommutative functions that are used in the Fourier expansion of fields, to plane waves that include a constant complex contribution to the frequency. This allows one to "double" momentum space into two halves that are connected to each other by Lorentz transformations, and together, are globally Lorentz invariant. This observation was used in the recent work [121] to define a free complex scalar field theory consistently (using covariant quantization based on a Pauli-Jordan function), and derive the associated deformed construction and annihilation operator algebra. Unfortunately, this algebra turned out to be extremely complicated, due to the presence of the additional region of momentum space, which, together with the mass shell, splits the commutator of two creation/annihilation operators into no less than *twenty* cases which need to be listed separately.

In the most recent results we will be reporting [30], we find a substantial simplification for the oscillator algebra. We are able to find a simple representation for our deformed creation and annihilation operators that can be expressed in one line, and is based on infinite nonlinear combinations of standard creation and annihilation operators. Such representation makes the unwieldy algebra of [121] treatable, and allows us to begin drawing some physical conclusions from the theory. First, the one-particle sector is completely undeformed and coincides with that of a commutative free complex scalar quantum field theory. Secondly, the charge conjugation operator is undeformed and Poincaré covariant. This was not the case in other approaches to quantum field theory on κ-Minkowski (in the case of timelike v^μ) [96]. In particular, the recent [94] shows that the charge conjugation operator sends a one-particle state into a one-antiparticle state with different momentum. This phenomenon is not present in our model. Regarding P and T symmetries, these are not symmetries of the commutation relations between coordinates, and this fact manifests itself already at the level of the one-particle sector: we are not able to introduce a P or a T operator that acts on the oscillator algebra or on the Fock space in the desired way. This, however, does not prevent PT symmetry from being realized: thanks to the antilinearity of the T operator, both the coordinate commutation relations and the free field theory can be shown to be PT-invariant. Having C and PT, the CPT invariance of the model is also guaranteed.

The nontriviality of the model manifests itself all in the multi-particle sector. Already at the level of two particles one sees that the total momentum depends nonlinearly on the momenta of the two particles, and the action of Lorentz transformations on two momenta becomes nonlinear and mixes the components of the momenta of the two particles (something dubbed "backreaction" in previous works [77,123]). Finally, we are able to introduce a "braided flip operator" that exchanges the momenta of two particles in a nonlinear way, which possesses all the properties that such an operator should: it is Lorentz covariant and is an involution (its square is the identity operator). This operator can be used to define symmetric and antisymmetric states, which are necessary to define the Fock space of bosonic and fermionic fields. Recent work by another group [103] showed that, in the case of the "timelike" κ-Minkowski spacetime, such a flip operator does not exist. The next best things are either non-Lorentz-covariant at all orders, or are not involutive [85, 100, 124, 125], which means that one can build an infinite tower of two-particle states that all share the same total momentum. The conclusion of the authors of [103] is that the very notion of identical particles, and (anti-)symmetrized multiparticle states loses meaning. These results do not apply to the model considered in the present chapter, as our flip operator is both involutive and Lorentz-covariant. This allows us to introduce a well-defined notion of multi-particle states, which is something that has eluded studies of quantum field theory on κ-Minkowski for decades. The deformed multi-particle states allow for a revision of the classical concepts of indistinguishability of identical particles and of the Pauli exclusion principle. We find that, given enough precision, particles of the same species which are described by a deformed (anti)-symmetric state can be distinguished by an experiment measuring their momenta. Moreover, the class of states prohibited by the Pauli Exclusion Principle is instead allowed in this setting, while another class of states not excluded by the standard Principle is instead prohibited.

2.1 Noncommutative geometry of lightlike κ-Minkowski

2.1.1 The lightlike κ-Minkowski spacetime and the κ-Poincaré quantum group

In the following, we will focus on a $1+1$-dimensional quantum field theory. Nevertheless, the construction of the braided tensor product algebra is valid in any number of dimensions. Therefore, we start with the definition of the $d+1$-dimensional κ-Minkowski spacetime and the associated κ-Poincaré quantum group $\mathbb{C}_\kappa[ISO(d,1)]$. The relevant expressions are just a trivial generalization of those written in the beginning of section 1.3. For simplicity, we choose to work in units in which $\kappa = 1$. The commutation relations between coordinates are

$$[x^\mu, x^\nu] = i(v^\mu x^\nu - v^\nu x^\mu)\,, \quad \mu = 0, 1, \ldots, d\,, \tag{2.1}$$

where v^μ is a set of $d+1$ real parameters. The algebra of functions on Minkowski spacetime is hence deformed into a noncommutative algebra $\mathcal{A}$, generated by x^μ and the identity 1, and equipped with a noncommutative product defined by (2.1). The relevant structures of $\mathbb{C}_\kappa[ISO(d,1)]$ are given by

$$\begin{aligned}
&[\Lambda^\mu{}_\nu, \Lambda^\rho{}_\sigma] = 0, \quad [a^\mu, a^\nu] = i(v^\mu a^\nu - v^\nu a^\mu) \\
&[a^\gamma, \Lambda^\mu{}_\nu] = i[(\Lambda^\mu{}_\alpha v^\alpha - v^\mu)\Lambda^\gamma{}_\nu + (\Lambda^\alpha{}_\nu g_{\alpha\beta} - g_{\nu\beta})v^\beta g^{\mu\gamma}] \\
&\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta g^{\alpha\beta} = g^{\mu\nu}, \quad \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu g_{\rho\sigma} = g_{\mu\nu},
\end{aligned} \tag{2.2}$$

with greek indices running in the set $\{0, .., d\}$ and $g_{\mu\nu}$ being a constant, invertible metric. The coproduct Δ, antipode S and counit ϵ, are given by

$$\begin{aligned}
&\Delta[\Lambda^\mu{}_\nu] = \Lambda^\mu{}_\alpha \otimes \Lambda^\alpha{}_\nu, \quad \Delta[a_\mu] = \Lambda^\mu{}_\nu \otimes a^\nu + a^\mu \otimes 1 \\
&S[\Lambda^\mu{}_\nu] = (\Lambda^{-1})^\mu{}_\nu, \quad S[a^\mu] = -(\Lambda^{-1})^\mu{}_\nu a^\nu, \quad \epsilon[\Lambda^\mu{}_\nu] = \delta^\mu_\nu, \quad \epsilon[a^\mu] = 0.
\end{aligned} \tag{2.3}$$

The Poincaré transformations of spacetime coordinates can be understood in terms of a left co-action operator $\cdot' : \mathcal{A} \to \mathbb{C}_\kappa[ISO(d,1)] \otimes \mathcal{A}$. We will write this coaction in a compact way as

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu\,, \tag{2.4}$$

where the product on the right-hand side is understood as the tensor product $\Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes 1$. In this notation, it is understood that $[\Lambda^\mu{}_\nu, x^\rho] = [a^\mu, x^\nu] = 0$. It is easy to check that, given the coordinate transformation (2.4) and the commutation rules (2.2), the commutator (2.1) is covariant, in the sense that

$$[x'^\mu, x'^\nu] = i(v^\mu x'^\nu - v^\nu x'^\mu)\,, \tag{2.5}$$

and the commutation relations appear identical to all inertial observers.

In the commutative limit, a two-point function is a function of two copies of Minkowski space $\mathbb{R}^{d,1} \times \mathbb{R}^{d,1}$. The commutative algebra of such functions is $\mathbb{C}[\mathbb{R}^{d,1} \times \mathbb{R}^{d,1}]$, which, under the canonical isomorphism, can be identified with the tensor product algebra $\mathbb{C}[\mathbb{R}^{d,1}] \otimes \mathbb{C}[\mathbb{R}^{d,1}]$, which is canonically defined as generated by the coordinate functions:

$$x_1^\mu = x^\mu \otimes 1\,, \quad x_2^\mu = 1 \otimes x^\mu\,, \tag{2.6}$$

which are such that $[x_1^\mu, x_2^\nu] = 0$. The extension of this construction to the noncommutative setting is not so straightforward. Adopting a similar prescription in the noncommutative case, the $\mathcal{A}^{\otimes 2} \equiv \mathcal{A} \otimes \mathcal{A}$ algebra is generated by $x^\mu_{1,2}$, where

$$[x_1^\mu, x_1^\nu] = i(v^\mu x_1^\nu - v^\nu x_1^\mu)\,, \quad [x_2^\mu, x_2^\nu] = i(v^\mu x_2^\nu - v^\nu x_2^\mu)\,, \quad [x_1^\mu, x_2^\nu] = 0\,. \tag{2.7}$$

Extending the κ-Poincaré left coaction (2.4) in the canonical way, we obtain

$$\begin{aligned} x_1'^{\mu} &= \Lambda^{\mu}_{\nu} \otimes x_1^{\nu} + a^{\mu} \otimes 1^{\otimes 2} = \Lambda^{\mu}_{\nu} \otimes x^{\nu} \otimes 1 + a^{\mu} \otimes 1 \otimes 1 \\ x_2'^{\mu} &= \Lambda^{\mu}_{\nu} \otimes x_2^{\nu} + a^{\mu} \otimes 1^{\otimes 2} = \Lambda^{\mu}_{\nu} \otimes 1 \otimes x^{\nu} + a^{\mu} \otimes 1 \otimes 1 \,, \end{aligned} \tag{2.8}$$

and one can immediately verify that

$$[x_1'^{\mu}, x_2'^{\nu}] = [\Lambda^{\mu}_{\rho}, a^{\nu}] \otimes (x_1^{\rho} - x_2^{\rho}) + [a^{\mu}, a^{\nu}] \otimes 1^{\otimes 2} \neq 0 \,. \tag{2.9}$$

The way to make the above commutator covariant is to relax the commutativity between x_1^{μ} and x_2^{ν}, giving rise to the algebraic structure known as "braided tensor product". Our algebra of two points still closes two κ-Minkowski subalgebras, and we want to find the form of the cross-commutation relations starting from the most general expression

$$[x_1^{\mu}, x_2^{\nu}] = i a^{\mu\nu}_{\rho\sigma} v^{\rho} v^{\sigma} + i v^{\rho} (b^{\mu\nu}_{\rho\sigma} x_1^{\sigma} + c^{\mu\nu}_{\rho\sigma} x_2^{\sigma}) \,, \tag{2.10}$$

which can be obtained by dimensional analysis and requiring it to be polynomial in the coordinates such that the commutative limit is restored when $v^{\mu} \to 0$. The invariance of the $[x_1^{\mu}, x_2^{\nu}]$ commutator implies that it should have the form

$$[x_1^{\mu}, x_2^{\nu}] = i[v^{\mu} x_1^{\nu} - v^{\nu} x_2^{\mu} - \eta^{\mu\nu} \eta_{\rho\sigma} v^{\rho} (x_1^{\sigma} - x_2^{\sigma})] \,. \tag{2.11}$$

The invariance is guaranteed even when considering more than two copies of $\mathcal{A}$, so we may generalize the notation to $[x_a, x_b]$, with labels a, b running over any set of more than two integers. Moreover, if we require the $\mathcal{A}^{\otimes N}$ algebra to be associative, the commutators also need to satisfy the Jacobi identity

$$[x_a^{\mu}, [x_b^{\nu}, x_c^{\rho}]] + [x_b^{\nu}, [x_c^{\rho}, x_a^{\mu}]] + [x_c^{\rho}, [x_a^{\mu}, x_b^{\nu}]] = 0 \,. \tag{2.12}$$

This yields the condition $v^{\alpha} v_{\alpha} = 0$, meaning that the only κ-deformation which allows the construction of a braided tensor product algebra is lightlike. This class of κ-noncommutative spacetimes is the only one on which the algebraic notion of multi-local functions can be defined. We will denote the braided tensor product algebra of N points as $\mathcal{A}^{\tilde{\otimes} N}$ tp distinguish it from the standard tensor product algebra $\mathcal{A}^{\otimes N}$.

We now restrict our attention to the lightlike κ-Minkowski spacetime in $1+1$ dimensions, with the following choices for the metric $g_{\mu\nu}$ and the constants v^{μ}:

$$v^{\mu} = (2, 0) \,, \qquad g_{\mu\nu} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \Rightarrow \qquad g_{\mu\nu} v^{\mu} v^{\nu} = 0 \,, \tag{2.13}$$

leading to the commutations relations between coordinates of the form

$$[x^+, x^-] = 2i\, x^- \,, \qquad x^{\pm} = x^0 \pm x^1 \,. \tag{2.14}$$

Using the fact that the Lorentz sector of the κ-Poincaré algebra is commutative and the orthogonality conditions with respect to the lightlike metric in (2.2), it is possible to show that

$$\Lambda^+_- = \Lambda^-_+ = 0 \,. \tag{2.15}$$

The algebra (2.2) is thus specified to

$$\begin{aligned} &[\Lambda^{\mu}{}_{\nu}, \Lambda^{\rho}{}_{\sigma}] = 0, \quad [a^+, a^-] = 2ia^- \\ &[a^+, \Lambda^+{}_+] = 2i[(\Lambda^+{}_+ - 1)\Lambda^+{}_+], \quad [a^+, \Lambda^-{}_-] = 2i(\Lambda^-{}_- - 1) \end{aligned} \tag{2.16}$$

while all the other commutators are 0. The coalgebra and the antipode do not depend on the specific values of v^μ and $g_{\mu\nu}$.

These symmetries can also be described in terms of the dual Hopf Algebra $\mathcal{U}_\kappa[\mathfrak{iso}(1,1)]$, which can be thought of as a noncommutative deformation of the universal enveloping algebra $\mathcal{U}[\mathfrak{iso}(1,1)]$ of the Poincaré Lie algebra $\mathfrak{iso}(1,1)$. To extract the relevant structures of $\mathcal{U}_\kappa[\mathfrak{iso}(1,1)]$, we apply a finite transformation on noncommutative plane waves, with a given ordering, and extract the action of the generators of the algebra by evaluating the first order of the transformation rules of plane waves. In this calculation and throughout the manuscript, we choose to work with the x^+ to-the-right ordering, and a transformed plane wave can be written as

$$e^{ik_- x'^-} e^{ik_+ x'^+} \,, \tag{2.17}$$

where x'^-, x'^+ can be read off from (2.4) and $k_\mu \in \mathbb{C}$.[1]. In our $1+1$-dimensional example, the Lorentz part of the transformation can be parametrized by a single operator τ, as follows:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} e^\tau & 0 \\ 0 & e^{-\tau} \end{pmatrix} . \tag{2.18}$$

From commutators (2.2), it is possible to show that

$$[a^+, \tau] = 2i(e^\tau - 1)\,, \qquad [a^-, \tau] = 0\,. \tag{2.19}$$

Using relations (2.19) and some algebra, we can write (2.17) as

$$e^{ik_- e^{-\tau} x^-} e^{\frac{i}{2}\log[1+e^\tau(e^{2k_+}-1)]x^+} e^{ik_- a^-} e^{ik_+ a^+}\,. \tag{2.20}$$

Focusing on the Lorentz sector of the transformation, we want to write

$$e^{ik_- e^{-\tau} x^-} e^{\frac{i}{2}\log[1+e^\tau(e^{2k_+}-1)]x^+} \approx (1 + i\tau N\triangleright) e^{ik_- x^-} e^{ik_+ x^+} + \mathcal{O}(\tau^2)\,, \tag{2.21}$$

where N is the boost operator in $\mathcal{U}_\kappa[\mathfrak{iso}(1,1)]$ and $\triangleright$ is a left action $\triangleright : \mathcal{U}_\kappa[\mathfrak{iso}(1,1)] \otimes \mathcal{A} \to \mathcal{A}$. Expanding the left hand-side at first order in τ, one finds

$$e^{ik_- e^{-\tau} x^-} e^{\frac{i}{2}\log[1+e^\tau(e^{2k_+}-1)]x^+} \approx (1 - i\tau x^- k_-) e^{ik_- x^-} \left(1 + i\tau x^+ \left(\frac{1-e^{-2k_+}}{2}\right)\right) e^{ik_+ x^+}\,. \tag{2.22}$$

This can be understood as a non-linear deformation of the action of the standard boost operator on a commutative plane-wave which is then mapped to a noncommutative one with a given ordering (in this case x^+ to the right), by means of a Weyl map $\Omega : \mathbb{C}[\mathbb{R}^2] \to \mathcal{A}$ [126], defined as (kx is a shorthand for $k_\mu x^\mu$):

$$\Omega(e^{ikx}) = e^{ik_- x^-} e^{ik_+ x^+}\,. \tag{2.23}$$

The action of the boost operator N can thus be written as

$$N \triangleright \Omega(e^{ikx}) = \Omega\left[\left((ix^-\partial_-) + x^+\left(\frac{1-e^{2i\partial_+}}{2}\right)\right) e^{ikx}\right]. \tag{2.24}$$

By inspecting the translation sector of the transformation, the action of the $\tilde{P}_\pm$ generators can instead be defined as

$$\tilde{P}_\pm \triangleright \Omega(e^{ikx}) = \Omega(-i\partial_\pm e^{ikx}) = k_\pm \Omega(e^{ikx}) \tag{2.25}$$

[1] Here and in the following, we consider ordered exponentials of the noncommutative coordinates with both real and complex parameters. The properties of the exponentials do not depend whether the parameters are real of complex, in general. For the sake of simplicity, we will refer to these functions as *plane waves*, even if the parameter has an imaginary component (see section 2.2.1)

Using expressions (2.24)-(2.25) and applying the generators in succession on a single plane waves, one finds the commutators:

$$[N, \tilde{P}_+] = i\left(\frac{1 - e^{-2\tilde{P}_+}}{2}\right) \qquad [N, \tilde{P}_-] = -i\tilde{P}_- \,, \tag{2.26}$$

which can be easily shown to satisfy the Jacobi identities. The coproducts encode the deviation from the Leibniz rule, and are found by applying the generators on products of plane waves. The antipode is obtained by acting on "inverse" plane waves, *i.e.* plane waves which multiplied by their standard counterpart give the identity. The counit codifies the action of the generators on plane waves with $k = 0$.

$$\begin{aligned} &\Delta[\tilde{P}_+] = \tilde{P}_+ \otimes 1 + 1 \otimes \tilde{P}_+, \quad \Delta[\tilde{P}_-] = \tilde{P}_- \otimes 1 + e^{-2\tilde{P}_+} \otimes \tilde{P}_- \\ &\Delta[N] = N \otimes 1 + e^{-2\tilde{P}_+} \otimes N, \quad S[N] = -Ne^{2\tilde{P}_+}, \quad S[\tilde{P}_-] = -\tilde{P}_- e^{2\tilde{P}_+} \\ &S[\tilde{P}_+] = -\tilde{P}_+, \quad \epsilon[N] = \epsilon[\tilde{P}_+] = \epsilon[\tilde{P}_-] = 0 \,. \end{aligned} \tag{2.27}$$

The procedures outlined above define a Hopf Algebra: all its axioms are satisfied, including the compatibility rules with the commutators (2.26) (*i.e.*, the homomorphism property of Δ, S and ϵ). The structures thus obtained define the lightlike κ-Poincaré Hopf algebra in the so-called bicrossproduct basis (characterized by momenta which close a Hopf subalgebra [52, 88]). In particular, expressions (2.24),(2.25) define the infinite-dimensional representation of $\mathcal{U}_\kappa[\mathfrak{iso}(1,1)]$ in the bicrossproduct basis. The mass Casimir element of this algebra is

$$C = \frac{1}{2}\tilde{P}_-(e^{2\tilde{P}_+} - 1) \,. \tag{2.28}$$

The action of the Weyl map is also useful to define generic noncommutative functions in $\mathcal{A}$, by means of a noncommutative Fourier transform

$$f(x) = \int \mathrm{d}^2k \, \tilde{f}(k)\Omega(e^{ikx}) \,. \tag{2.29}$$

For such generic functions, a κ-Poincaré transformation can be written as (id is the identity map, and the dots indicate all higher order monomials in the transformation parameters, with a given, specified ordering: in this case, τ is chosen to be to the right of a^+, which is, in turn, to the right of a^-):

$$f(x') = e^{i\,a^-\otimes\tilde{P}_-}e^{i\,a^+\otimes\tilde{P}_+}e^{i\,\tau\otimes N}(id\otimes\triangleright)f(x) = 1\otimes f(x) + i\,a^\mu\otimes\tilde{P}_\mu\triangleright f(x) + i\,\tau\otimes N\triangleright f(x) + \dots \tag{2.30}$$

with $\tilde{P}_\mu, N \in \mathcal{U}_\kappa[\mathfrak{iso}(1,1)]$ and the left action on coordinates is easily read from (2.24),(2.25),

$$\tilde{P}_\mu \triangleright x^\nu = -i\,\delta^\mu{}_\nu \,, \qquad N \triangleright x^\pm = \pm i x^\pm \,. \tag{2.31}$$

A peculiarity of the κ-lightlike Hopf algebra, which will prove to be useful in characterizing the physical results of this work is the fact it is quasi-triangular,[2] *i.e.*, it admits a quantum R-matrix. It has been derived in [52, 127, 128], by exploiting an isomorphism between $\mathcal{C}_\kappa(ISO(d,1))$ and $\mathcal{U}_\kappa(\mathfrak{iso}(d,1))$, where $d = 1, 2, 3$. In $1+1$ dimensions, the expression of the R-matrix is given by

$$R = e^{-2i\tilde{P}_+\otimes N}e^{2iN\otimes\tilde{P}_+} \,, \tag{2.32}$$

[2]In our specific case, a stronger condition holds: the R-matrix is *triangular*, meaning that $R^{BA}_{DC}R^{CD}_{EF} = \delta^A{}_E\,\delta^B{}_F$ [127].

and in terms of it, relations (2.2), specified by (2.13) can be written in a compact way as "RTT" relations, often used in the quantum group literature [33]. Relations (2.31) define a three-dimensional representation ρ^A_B, with $A, B = \{+, -, 2\}$ for $\tilde{P}_\mu, N$ acting on vectors of the form $X^A \equiv (x^\mu, 1)$:

$$\rho(\tilde{P}_+)^A{}_B = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \rho(\tilde{P}_-)^A{}_B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & 0 & 0 \end{pmatrix}, \qquad \rho(N)^A{}_B = \begin{pmatrix} -i & 0 & 0 \\ 0 & +i & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.33}$$

By noticing that the $\rho(\tilde{P}_\pm)^A_B$ matrices are nilpotent, in this representation(2.32) reduces to:

$$R = 1 \otimes 1 - 2i\tilde{P}_+ \otimes N + 2iN \otimes \tilde{P}_+ \,, \tag{2.34}$$

and by realizing the tensor product as the standard Kronecker product, in components we find

$$R^{AB}_{CD} = \delta^A_C \delta^B_D + 2\,i\,\left(\delta^A_+ \delta^B_+ \delta^2_C \delta^+_D - \delta^A_+ \delta^B_- \delta^2_C \delta^-_D - \delta^A_+ \delta^B_+ \delta^+_C \delta^2_D + \delta^A_- \delta^B_+ \delta^-_C \delta^2_D\right) \quad . \tag{2.35}$$

By defining

$$T^A{}_B = \begin{pmatrix} \Lambda^+{}_+ & \Lambda^+{}_- & a^+ \\ \Lambda^-{}_+ & \Lambda^-{}_- & a^- \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^\tau & 0 & a^+ \\ 0 & e^{-\tau} & a^- \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.36}$$

one can explicitly verify that the expression

$$T^A{}_C \, T^B{}_D \, R^{DC}_{EF} = R^{BA}_{CD} \, T^C{}_E \, T^D{}_F \,, \tag{2.37}$$

reproduces the commutation relations (2.16). Moreover, the commutation relations between coordinates can be written in a compact way as

$$X^A \, X^B = R^{BA}_{CD} \, X^C \, X^D \,. \tag{2.38}$$

Equivalently, these "RXX" relations can also be verified using the infinite-dimensional representation of $\mathcal{U}_\kappa(\mathfrak{iso}(d,1))$ and relations (2.31). For instance:

$$x^+ x^- = \mu \circ R \triangleright (x^- \otimes x^+) = x^- x^+ + 2i \,, \tag{2.39}$$

where $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the noncommutative multiplication of $\mathcal{A}$. This deformed flip property can also be extended to products of plane waves. Indeed, one can verify that

$$\mu \circ R \triangleright \left[\Omega(e^{iqx}) \otimes \Omega(e^{ikx})\right] = \Omega(e^{ikx})\Omega(e^{iqx}) \,, \tag{2.40}$$

where $k, q \in \mathbb{C}^2$. As we will see in the subsequent sections, the R-matrix proves to be a valuable instrument in switching plane waves even in the braided tensor product algebra introduced in [120, 121], so that the task of covariant quantization of the noncommutative scalar field becomes more feasible.

In what follows, we will work in a different basis for $\mathcal{U}_\kappa[\mathfrak{iso}(1,1)]$, which is connected to the bicrossproduct one by a redefinition of the + momentum, given by

$$P_+ = \frac{1}{2}\left(e^{2\tilde{P}_+} - 1\right) \,. \tag{2.41}$$

The $\mathcal{U}_\kappa[\mathfrak{iso}(1,1)]$ commutators are, in this basis, the undeformed ones of the Poincaré algebra,

$$[N, P_+] = iP_+ \qquad [N, P_-] = -iP_- \,. \tag{2.42}$$

All the non-linearity is moved to the coproducts and the antipodes, which now take the form

$$\begin{aligned}
&\Delta[P_+] = P_+ \otimes 1 + 1 \otimes P_+ + 2P_+ \otimes P_+, \qquad S(P_+) = -\frac{P_+}{1+2P_+},\\
&\Delta[P_-] = P_- \otimes 1 + 1 \otimes P_- - \frac{2P_+}{1+2P_+} \otimes P_-, \qquad S(P_-) = -P_-(1+2P_+),\\
&\Delta[N] = N \otimes 1 + 1 \otimes N - \frac{2P_+}{1+2P_+} \otimes N, \qquad S(N) = -N(1+2P_+)\,,
\end{aligned} \tag{2.43}$$

while the counits remain all zero. The Casimir element in these variables can be obtained by substituting (2.41) in (2.28) and is undeformed:

$$C = P_+ P_-\,, \tag{2.44}$$

in agreement with the linearity of commutators (2.42).

Nonlinear transformations of the translation generators lead to different bases for the $\mathcal{U}_\kappa[\mathfrak{iso}(1,1)]$ Hopf algebra, which, as we will see in the following, correspond to different coordinate systems on momentum space. The theory we are presenting in this chapter has the aspiration of being invariant under general coordinate transformations on momentum space. This would imply that the physical observables do not depend on the momentum coordinate systems used in their prediction (see the discussion in [129], Sec. II). The presence of such an invariance in our theory is supported by the preliminary results in [121] (sec. 2.3), which show that the two-point functions of the theory are the same regardless of the coordinate system on momentum space that was used to calculate them.

Even in a generally-covariant theory, certain situations are better described by certain choices of coordinates, *e.g.* Cartesian coordinates in Minkowski space are preferred because they transform covariantly under Lorentz transformations. In our model, the choice of coordinates $P_\pm$ has the same advantage: they transform in an undeformed fashion under boosts. For this reason, we find it convenient to work with them. As it turns out, plane waves are eigenfunctions of the momenta, and the $P_\pm$ basis has the following eigenvalues:

$$P_+\Omega(e^{ikx}) = \frac{1}{2}(e^{2k_+} - 1)\Omega(e^{ikx}) \qquad P_-\Omega(e^{ikx}) = k_-\Omega(e^{ikx})\,. \tag{2.45}$$

These relations inspire a redefinition of the momenta appearing in the plane waves:

$$\xi_- = k_-\,, \qquad \xi_+ = \frac{1}{2}(e^{2k_+} - 1)\,, \qquad \Rightarrow \qquad \Omega(e^{i\,kx}) = e^{i\xi_- x^-} e^{\frac{i}{2}\ln(1+2\xi_+)x^+}\,, \tag{2.46}$$

so that

$$P_\pm\Omega(e^{ikx}) = P_\pm \triangleright e^{i\xi_- x^-} e^{\frac{i}{2}\ln(1+2\xi_+)x^+} = \xi_\pm E[\xi]\,, \tag{2.47}$$

where we defined $E[\xi] \equiv e^{i\xi_- x^-} e^{\frac{i}{2}\ln(1+2\xi_+)x^+}$. The algebraic properties of noncommutative plane waves reflect the non-linear structure of momentum space. First of all, from redefinition (2.46), we notice that the ξ_+ component of the momentum is bounded from below, $\xi_+ > -1/2$, so that plane waves $E[\xi]$ only cover half of momentum space [120, 121]. Although, if the value of k is allowed to assume complex values, then ξ_+ ranges over all of the real numbers. We will come back to this issue in the subsequent sections. Products of plane waves define the deformed composition law for momentum (denoted by Δ with a slight abuse of notation) and the deformed inverse momenta (denoted with S), which mimick the structures of coproduct and antipode, respectively:

$$\begin{aligned}
\Delta(\xi,\eta) &= \left(\xi_- + \frac{\eta_-}{1+2\xi_+}, \xi_+ + \eta_+ + 2\xi_+\eta_+\right)\,,\\
S(\xi) &= \left(-\xi_-(1+2\xi_+), -\frac{\xi_+}{1+2\xi_+}\right)\,,
\end{aligned} \tag{2.48}$$

for $\xi, \eta \in \mathbb{C}^2$. Namely, we have that

$$\begin{aligned} E[\xi]E[\eta] &= E[\Delta(\xi,\eta)]\,, \\ E[\xi]E[S(\xi)] &= 1 \end{aligned} \tag{2.49}$$

The Hopf algebra properties then imply the following consistency relations between composition law and antipode, which can be checked explicitly using (2.48):

$$\begin{aligned} &\Delta[\xi,\Delta(\eta,\chi)] = \Delta[\Delta(\xi,\eta),\chi] = \Delta[\xi,\eta,\chi]\,, \\ &\Delta[\xi,S(\xi)] = \Delta[S(\xi),\xi] = 0\,, \\ &S[\Delta(\xi,\eta)] = \Delta[S(\eta),S(\xi)]\,, \end{aligned} \tag{2.50}$$

for $\xi, \eta, \chi \in \mathbb{C}^2$. The first relation implies the associativity of the composition law, the second the existence of a momentum inverse, and the third codifies the anti-homomorphism property of the antipode. The Casimir element (2.44) defines mass-shells in momentum-space through the constraint

$$m^2 = \xi_+\xi_-\,, \tag{2.51}$$

just as in the ordinary theory.

The proof of (2.49) simply follows from the definition of plane waves (2.47) and the noncommutativity relations (2.14). The plane wave product can be written explicitly as

$$E[\xi]E[\eta] = e^{i\xi_- x^-} e^{\frac{i}{2}\ln(1+2\xi_+)x^+} e^{i\eta_- x^-} e^{\frac{i}{2}\ln(1+2\eta_+)x^+}\,. \tag{2.52}$$

We want to write this as a single plane wave $E[\chi]$ and show that $\chi = \Delta(\xi,\eta)$. In order to do this, we want to bring all of the exponentials with x^+ to the right. It is convenient to notice that

$$\begin{aligned} &x^+x^- = x^-(2i + x^+)\,, \\ &(x^+)^2x^- = x^-(2i + x^+)^2\,, \\ &\vdots \\ &(x^+)^n x^- = x^-(2i + x^+)^n\,. \end{aligned} \tag{2.53}$$

Therefore,

$$e^{ik_+x^+}x^- = \sum_n \frac{(ik_+)^n}{n!}(x^+)^n x^- = x^- \sum_n \frac{(ik_+)^n}{n!}(2i + x^+)^n = x^- e^{-2k_+} e^{ik_+x^+}\,, \tag{2.54}$$

so that the left multiplication of an x^+ exponential by an x^- exponential can be reordered as

$$e^{ik_+x^+}e^{iq_-x^-} = e^{ik_+x^+}\sum_m \frac{(iq_-)^m}{m!}(x^-)^m = \sum_m \frac{(iq_-e^{-2k_+}x^-)^m}{m!}e^{ik_+x^+} = e^{ie^{-2k_+}q_-x^-}e^{ik_+x^+}\,. \tag{2.55}$$

Applying this result to (2.52) with $k_+ = \frac{1}{2}\ln(1+2\xi_+)$ and $q_- = \eta_-$ yields

$$E[\xi]E[\eta] = e^{i\left(\xi_- + \frac{\eta_-}{1+2\xi_+}\right)x^-} e^{\frac{i}{2}\ln(1+2(\xi_+ + 2\eta_+\xi_+))x^+} = E[\Delta(\xi,\eta)]\,, \tag{2.56}$$

thus proving the first property in (2.49). The second one is just a consequence of the first when $\eta = S(\xi)$.

Making use of property (2.55), we can now also prove property (2.40) adapted to our linear momentum redefinition. The braided tensor product version of this equation, which we will prove in a subsequent section, is going to be invaluable in order to swiftly implement the covariant quantization scheme for our complex scalar field. In terms of the linear momentum variable P_+ the expression for the R-matrix (2.32) can be rewritten as

$$R = e^{-i\ln(1+2P_+)\otimes N} e^{iN\otimes\ln(1+2P_+)} \tag{2.57}$$

Let us apply this operator on the product of plane waves, *i.e.* we want to compute

$$\mu \circ R \triangleright (E[\eta] \otimes E[\xi]) = \mu \circ e^{-i\ln(1+2P_+)\otimes N} e^{iN\otimes\ln(1+2P_+)} \triangleright (E[\eta] \otimes E[\xi])\,. \tag{2.58}$$

We start by applying the rightmost exponential figuring in the expression for the R-matrix to the tensor product of plane waves. We can expand said exponential in a formal series of operators and exploit the fact that P_+ acts diagonally on plane waves to obtain

$$\begin{aligned} e^{iN\otimes\ln(1+2P_+)} \triangleright (E[\eta] \otimes E[\xi]) &= \sum_n \frac{i^n}{n!}(N^n \triangleright E[\eta] \otimes \ln(1+2P_+)^n \triangleright E[\xi]) = \\ =&\left(\sum_n \frac{(i\ln(1+2\xi_+))^n}{n!} N^n \triangleright E[\eta]\right) \otimes E[\xi] = E\left[\frac{\eta_-}{1+2\xi_+}, \eta_+ + 2\eta_+\xi_+\right] \otimes E[\xi]\,. \end{aligned} \tag{2.59}$$

The application of the lefmost exponential in the R-matrix now yields

$$\begin{aligned} &e^{-i\ln(1+2P_+)\otimes N} \triangleright \left(E\left[\frac{\eta_-}{1+2\xi_+}, \eta_+ + 2\eta_+\xi_+\right] \otimes E[\xi]\right) = \\ =&\sum_n \frac{(-i)^n}{n!}\left(\ln(1+2P_+)^n E\left[\frac{\eta_-}{1+2\xi_+}, \eta_+ + 2\eta_+\xi_+\right] \otimes N^n E[\xi]\right) = \\ =&E\left[\frac{\eta_-}{1+2\xi_+}, \eta_+ + 2\eta_+\xi_+\right] \otimes \sum_n \frac{(-i)^n}{n!} \ln(1+2(\eta_+ + 2\eta_+\xi_+))^n N^n \triangleright E[\xi] = \\ =&E\left[\frac{\eta_-}{1+2\xi_+}, \eta_+ + 2\eta_+\xi_+\right] \otimes E\left[(1+2\eta_+ + 4\eta_+\xi_+)\xi_-, \frac{\xi_+}{1+2\eta_+ + 4\eta_+\xi_+}\right]. \end{aligned} \tag{2.60}$$

Multiplying the resulting plane waves in the last line of the above equation and using (2.55) to bring the "η" plane wave to the right, we obtain

$$E\left[\frac{\eta_-}{1+2\xi_+}, \eta_+ + 2\eta_+\xi_+\right] E\left[(1+2\eta_+ + 4\eta_+\xi_+)\xi_-, \frac{\xi_+}{1+2\eta_+ + 4\eta_+\xi_+}\right] = E[\xi]E[\eta]\,. \tag{2.61}$$

So we have shown that

$$\mu \circ R \triangleright (E[\eta] \otimes E[\xi]) = E[\xi] \otimes E[\eta]\,, \tag{2.62}$$

As can be inferred by the computations above, the action of the R-matrix simply consists in boosting the plane waves in the tensor product with two different parameters which depend on their momenta.

This concludes the analysis of the lightlike κ-Minkowski noncommutative spacetime $\mathcal{A}$ and its symmetries, as long as a single copy of $\mathcal{A}$ is concerned. We have extensively reviewed the properties of the κ-Poincaré quantum group, described in terms of $\{\Lambda^\mu_\nu, a^\mu\}$, and of its dual, the κ-deformation of the enveloping algebra $\mathcal{U}(\mathfrak{iso}(1,1)$, described in terms of $P_\pm, N$. We have found the infinite-dimensional representation for these operators and studied the properties of plane waves of a single copy of $\mathcal{A}$, highlighting the role of the R-matrix, only available in the lightlike versions of κ-deformations of Minkowski spacetime. Later on, we will see that the set of operators P_+, P_-, N can also be represented in terms of creation-annihilation operators of standard quantum field theory, and their action on well defined (multi)-particle states follows the Hopf algebraic structures displayed above. Before diving into such considerations, we continue with some mathematical preliminaries needed to construct a consistent quantum field theory on the $1+1$D lightlike κ-Minkowski quantum spacetime.

2.1.2 Braided N-point algebra and its representations

The defining commutation relations for the $1+1$D braided lightlike κ-Minkowski algebra $\mathcal{A}^{\tilde{\otimes}N}$ are given by (2.11), specified by (2.13):

$$[x_a^+, x_b^+] = 2i(x_a^+ - x_b^+), \quad [x_a^+, x_b^-] = 2ix_b^-, \quad [x_a^-, x_b^-] = 0\,, \tag{2.63}$$

with $a, b = 1, ..., N$. These relations can equivalently be written in terms of center of mass and relative coordinates:

$$x_{cm}^\mu = \frac{1}{N}\sum_{a=1}^{N} x_a^\mu \qquad y_a^\mu = x_a^\mu - x_{cm}^\mu\,, \tag{2.64}$$

so that (2.63) becomes

$$[x_{cm}^+, x_{cm}^-] = 2ix_{cm}^-, \quad [x_{cm}^+, y_a^\pm] = \mp 2iy_a^\pm, \quad [x_{cm}^-, y_a^\pm] = 0. \tag{2.65}$$

It is easy to check that the coordinate differences $\Delta x_{ab}^\mu := x_a^\mu - x_b^\nu$ are commutative (a feature also shared by the braided tensor product of the θ-Moyal noncommutative spacetime [36]):

$$[\Delta x_{ab}^\mu, \Delta x_{cd}^\nu] = 0, \quad a, b, c, d = 1, \ldots, N \quad \text{and } \mu, \nu = +, -. \tag{2.66}$$

This, combined with the fact that κ-Poincaré invariant N-point functions depend solely on coordinate differences (proved in [120]), immediately tells us that κ-Poincaré invariant N-point functions are commutative themselves. This greatly simplifies the interpretation of the theory, given that all physical information should be encoded in N-point functions. Once again, commutation relations (2.63) can be written in terms of an R-matrix [33] as

$$X_a^A\, X_b^B = R^{BA}_{CD}\, X_b^C\, X_a^D\,, \tag{2.67}$$

where $X_a^A = (x_a^\mu, 1)$, and the operators appearing in the R-matrix act in the same way on the x_a^μ coordinates whatever the value of a.

In [120], a representation for the center of mass and relative coordinates has been found, and reads

$$\hat{x}_{cm}^+ = 2ix_{cm}^-\frac{\partial}{\partial x_{cm}^-} + i + 2i\sum_{a=1}^{N-1}\left(y_a^+\frac{\partial}{\partial y_a^+} - y_a^-\frac{\partial}{\partial y_a^-}\right), \qquad \hat{x}_{cm}^- = x_{cm}^-, \quad \hat{y}_a^+ = y_a^+, \quad \hat{y}_a^- = y_a^-, \tag{2.68}$$

and is such that $x_{cm}^\pm, y_a^\pm$ are Hermitian. For purposes which shall be clear once we discuss noncommutative plane waves in more detail, we will consider a more general, one-parameter class of representations, given by

$$\hat{x}_{cm}^+ = 2i\left(x_{cm}^-\frac{\partial}{\partial x_{cm}^-} + s\right) + 2i\sum_{a=1}^{N-1}\left(y_a^+\frac{\partial}{\partial y_a^+} - y_a^-\frac{\partial}{\partial y_a^-}\right), \qquad \hat{x}_{cm}^- = x_{cm}^-, \quad \hat{y}_a^+ = y_a^+, \quad \hat{y}_a^- = y_a^-, \tag{2.69}$$

where $0 < s < 1$, and the Hermitian representation is regained with $s = 1/2$. When analyzing plane waves, in the subsequent sections, it is useful to study the action of operators of the type $e^{itx_{cm}^+}$ on functions of the braided tensor product algebra. Using (2.69), it is easy to check that, for any complex t,

$$e^{it\hat{x}_{cm}^+} f(x_{cm}^-, y_a^+, y_a^-) = e^{-t\,a} f(e^{-2t}x_{cm}^-, e^{2t}y_a^+, e^{-2t}y_a^-)\,. \tag{2.70}$$

Then, exploiting the fact that

$$e^{itx_a^+} = e^{it(x_{cm}^+ + y_a^+)} = e^{i\left(\frac{e^{2t}-1}{2}\right)y_a^+} e^{itx_{cm}^+}\,, \tag{2.71}$$

we obtain

$$e^{-\frac{\pi}{2}x_a^+} = e^{i\left(\frac{e^{\pi i}-1}{2}\right)y_a^+} e^{-\frac{\pi}{2}x_{cm}^+} = e^{-i\,y_a^+} e^{-\frac{\pi}{2}x_{cm}^+}\,, \tag{2.72}$$

so that

$$e^{-\frac{\pi}{2}\hat{x}_a^+} f(x_{cm}^-, y_a^+, y_a^-) = e^{-iy_a^+} e^{-i\pi\, s} f(e^{-i\pi}x_{cm}^-, e^{i\pi}y_a^+, e^{-i\pi}y_a^-) = e^{-i\pi s} e^{-iy_a^+} f(-x_{cm}^-, -y_a^+, -y_a^-)\,. \tag{2.73}$$

The square of this operator is then simply

$$e^{-\pi\,\hat{x}_a^+} f(x_{cm}^-, y_a^+, y_a^-) = e^{-2i\pi\, s} f(x_{cm}^-, y_a^+, y_a^-)\,. \tag{2.74}$$

and thus

$$e^{-n\pi x_a^+} f(x_{cm}^-, y_a^+, y_a^-) = e^{-2i\pi\, s\, n} f(x_{cm}^-, y_a^+, y_a^-)\,. \tag{2.75}$$

Having introduced the one-parameter family of representations (2.69), we would like to find a condition that fixes the parameter s. This will be identified in the next Section, in order to eliminate a sign ambiguity that emerges when introducing a certain type of noncommutative plane waves (first introduced in [96, 121]) that are necessary to ensure the covariance of the theory. In the meantime, we need to briefly discuss the Hermiticity/self-adjointness properties of the N-point coordinate operators x_a^μ. This will be necessary, as later we will need to introduce an involution that sends a noncommutative plane wave into its inverse, which is necessary in order to discuss field theory. What we would like is an involutive, anti-linear anti-homomorphism which sends $E[\xi]$ (from Eq. (2.47)) into $E^\dagger[\xi]$ such that $E^\dagger[\xi]E[\xi] = E[\xi]E^\dagger[\xi] = 1$. We start by defining a putative operator $*$ as the "naive" Hermitian conjugation on operators, such that its action on $\hat{x}_{cm}^+$ is given by

$$(\hat{x}_{cm}^+)^* = \hat{x}_{cm}^+ + 2i(2s-1) \tag{2.76}$$

where, as expected, we obtain that $(\hat{x}_{cm}^+)^* = \hat{x}_{cm}^+$ only when $s = 1/2$, which corresponds to the symmetric ordering for the representation (2.69). We can now define $\dagger$ as the operator that leaves $\hat{x}_{cm}^+$ invariant for any choice of s, $(\hat{x}_{cm}^+)^\dagger = \hat{x}_{cm}^+$, so that its relation with $*$ is simply given by

$$(\hat{x}_{cm}^+)^\dagger = (\hat{x}_{cm}^+)^* - 2i(2s-1) \tag{2.77}$$

The $(\cdot)^*$ operator is the Hermitian conjugate with respect to the standard inner product of $L^2(\mathbb{R}^{2N-1})$:

$$\int_{\mathbb{R}^{2N-1}} \bar{\psi}\;\varphi\; dx_{cm}^- dy_1^- \dots dy_{N-1}^- dy_1^+ \dots dy_{N-1}^+\,, \tag{2.78}$$

where ψ, ϕ are square-integrable functions on $\mathbb{R}^{2N-1}$. The $\dagger$ operation is the Hermitian conjugate with respect to a different inner product:

$$\int_{\mathbb{R}^{2N-1}} (x_{cm}^-)^{2s-1}\,\bar{\psi}\,\varphi\, dx_{cm}^- dy_1^- \dots dy_{N-1}^- dy_1^+ \dots dy_{N-1}^+\,, \tag{2.79}$$

where, in this case, the space of functions that have a finite norm is different from $L^2(\mathbb{R}^{2N-1})$. For $s > 1/2$, it includes $L^2(\mathbb{R}^{2N-1})$, and also functions that diverge sufficiently slowly in $x_{cm}^- \to 0$. For $s < 1/2$, the space is smaller than $L^2(\mathbb{R}^{2N-1})$, as the functions need to go to zero sufficiently fast at $x_{cm}^- \to 0$. The fact that the representations of $\mathcal{A}$ and the related braided algebras require different inner products for the self-adjointness of the generators has been already noticed in [63, 64, 130]. From now on, we will use the $\dagger$ operator to conjugate plane waves.

2.2 Braided lightlike κ-deformed quantum field theory

2.2.1 Old and new-type noncommutative plane waves and momentum space

We have introduced plane waves for a single copy of the $\mathcal{A}$ in section 2.1 using the linear momentum parametrization. As it will be relevant for what follows, we add a label indicating in which copy of the braided tensor product algebra $\mathcal{A}^{\tilde{\otimes} N}$ the plane wave is defined. Also, for the time being, we restric our definition of $E_a[\xi]$ to values of $\xi_+ > -1/2$:

$$E_a[\xi] := e^{i\xi_- x_a^-} e^{\frac{i}{2}\ln(1+2\xi_+)x_a^+}\,, \qquad \xi_+ > -1/2. \tag{2.80}$$

Under the involution that leaves x^+ invariant, the above transforms as

$$E_a^\dagger[\xi] = E_a[S(\xi)]\,, \tag{2.81}$$

where $S(\xi)$ is the antipode defined in (2.48), while the product of two plane waves gives a composition law compatible with the coproducts in (2.43):

$$E_a[\xi]E_a[\eta] = E_a[\Delta(\xi,\eta)]\,, \tag{2.82}$$

as anticipated in section 2.1. The spacetime coordinates $x^\pm$ close the Lie algebra (2.14) of the affine group of the line, $\mathfrak{aff}(1)$. Thus, when focusing on values of $\xi_+ > -1/2$, plane waves (2.80) span the connected component of the identity of the corresponding Lie group, Aff(1). This is just a semiplane of 1+1-dimensional Minkowski space, bounded by a straigth line [120, 121]. The boundary is given by the reality constraint for the logarithm term, $\xi_+ > -1/2$ and implies that such plane waves only cover half of the Minkowski momentum space, an issue already pointed out in [120]. There, it was shown that a field theory built from plane waves (2.80) spoils κ-Poincaré invariance. This can easily be seen by considering a κ-Poincaré transformation of (2.80):

$$E_a'[\xi] = e^{ie^{-\tau}\xi_- x^-} e^{\frac{i}{2}\ln(1+2e^\tau\xi_+)x^+} e^{i\xi_- a^-} e^{\frac{i}{2}\ln(1+2\xi_+)a^+}\,. \tag{2.83}$$

Notice that in this linear parametrization, the boost simply acts as a dilation on $\xi_\pm$, given the linear structure of the commutators (2.42). For any value of τ, a positive value of ξ_+ remains positive, and we obtain a different group element connected to the identity. When ξ_+ is negative, an excessively large boost may result in $e^\tau\xi_+ < -1/2$, so that the argument of the logarithm in (2.80) becomes negative and we obtain a group element not connected to the identity, which we can think of as a plane wave of the form (2.80) with a complex "+" argument. In [121], these "new type" plane waves were identified as the missing piece of the puzzle needed to construct a consistent κ-Poincaré invariant field theory.

Suppose we boost a plane wave of the form (2.80), such that $e^\tau\xi_+ < -1/2$; then, the logarithm term can be written as

$$\ln[-|1+2e^\tau\xi_+|] = i\,\pi + \ln|1+2e^\tau\xi_+| + 2\,n\,\pi\,i\,. \tag{2.84}$$

Focusing on the Lorentz transformation sector of (2.83):

$$e^{i\,e^{-\tau}\xi_- x_a^-} e^{\frac{i}{2}\ln[-|1+2e^\tau\xi_+|]\,x_a^+} = e^{i\,e^{-\tau}\xi_- x_a^-} e^{\frac{i}{2}\ln|1+2e^\tau\xi_+|\,x_a^+} e^{-\frac{\pi}{2}\,x_a^+} e^{-n\,\pi\,x_a^+}. \tag{2.85}$$

We now come to an issue not discussed in [121]. There, using representation (2.68), a sign ambiguity emerges in (2.85), due to the fact that $e^{-n\pi x_a^+} \equiv (-1)^n$. In our novel parametric representation (2.69), using the identification $e^{-n\pi x_a^+} \equiv e^{-2i\pi s n}$ from (2.75), (2.85) becomes

$$e^{i\,e^{-\tau}\xi_- x_a^-} e^{\frac{i}{2}\ln[-|1+2e^\tau\xi_+|]\,x_a^+} = e^{i\,e^{-\tau}\xi_- x_a^-} e^{\frac{i}{2}\ln|1+2e^\tau\xi_+|\,x_a^+} e^{-\frac{\pi}{2}\,x_a^+} e^{-2i\,\pi s\,n}. \tag{2.86}$$

To avoid the aforementioned sign ambiguity, we may choose $s = 1$. Notice that this implies that the coordinates are only Hermitian with respect to the inner product (2.79). Nevertheless, the physical quantities characterizing our quantum field theory (two point functions) are not affected by this choice. From now on, whenever we refer to the (braided or not) κ-Minkowski coordinate algebra we mean representation (2.69) with the choice $s = 1$, and the Hermitian conjugate operator $\dagger$ defined in Eq. (2.77). Having solved the sign ambiguity, we have singled out one new type plane wave, among the infinite possibilities arising from crossing the momentum space boundary with a too large boost:

$$E_a[\xi] \to \mathcal{E}\left[e^{-\tau}\xi_-, \frac{1}{2}\ln|1+2e^{\tau}\xi_+|\right], \tag{2.87}$$

where, as in [121], we have defined

$$\mathcal{E}_a[\xi] \coloneqq E_a[\xi]e^{-\frac{\pi}{2}x_a^+} \tag{2.88}$$

Notice that the new type plane wave (2.88) can be written as

$$\mathcal{E}_a[\xi] = e^{i\xi_- x^-}e^{\frac{i}{2}\ln(1+2\xi_+)x^+}e^{-\frac{\pi}{2}} = e^{i\xi_- x^-}e^{\frac{i}{2}\ln(-1-2\xi_+)x^+} = e^{i\eta_- x^-}e^{\frac{i}{2}\ln(1+2\eta)x^+} \tag{2.89}$$

where

$$\eta_- = \xi_- \qquad \eta_+ = -1 - \xi_+\,, \tag{2.90}$$

If ξ_+ is in the range $]-1/2;\infty[$, then η_+ is in the range $]-\infty;-1/2[$. Therefore, we may formally extend the definition of $E_a[\xi]$ to all real values of ξ_+ with a slight abuse of notation:

$$\mathcal{E}_a[\xi] \coloneqq E_a\left[\xi_-, \frac{1}{2}\ln(1+2\xi_+)\right], \qquad \xi_+ < -1/2\,, \tag{2.91}$$

The advantage of this redefinition is that one can than exploit the properties found for old type plane waves and products of old type plane waves, since they only depend on the algebraic properties of the braided tensor product algebra. Since the new variables in (2.91) cover the other half of momentum space, as expected, the on-shell relation is still (2.51), also implied by the linearity of the commutators of $\mathcal{U}_\kappa(\mathfrak{iso}(1,1))$.

2.2.2 Pauli-Jordan function

As is well known, a quantum field theory on a commutative spacetime (in particular Minkowski) is entirely defined in terms of N-point functions [131]. In the noncommutative setting, these functions are replaced with elements of the braided tensor product algebra of N-points, which should also be invariant under κ-Poincaré transformations. In [54], it was shown that N-point functions only depend on coordinate differences, which close an abelian subalgebra of $\mathcal{A}^{\otimes N}$, so that the N-point functions themselves are simply commutative functions. This hugely simplifies the process of covariant quantization, in which the Pauli-Jordan function, written in terms of the two-point function, intervenes in the defining commutation relations of the quantum field theory. The two-point function should also satisfy the κ-Klein Gordon equation, which simply reduces to the standard Klein-Gordon equation in linear momentum variables. The first step is identifying products of two plane waves that depend solely on coordinate differences. Using the $\mathcal{A}^{\tilde{\otimes}2}$ algebra in (2.63), one can compute the most general plane wave product, which has the form

$$\begin{aligned} E_1[\xi]E_2[\eta] = &\exp\left\{i\left(\xi_- x_1^- + \frac{\eta_-}{1+2\xi_+}x_2^-\right)\right\}\times \\ &\times\exp\left\{i\frac{\ln[(1+2\xi_+)(1+2\eta_+)]}{2(\eta_+ + \xi_+ + 2\eta_+\xi_+)}\left(\xi_+ x_1^+ + \eta_+(1+2\xi_+)x_2^+\right)\right\}. \end{aligned} \tag{2.92}$$

In the above, the variables $\xi_\pm, \eta_\pm$ range over all real numbers, in order to cover both old and new type plane waves. Indeed, (2.92) only depends on the algebraic properties dictated by the commutation relations (2.63), and not by the specific value of the arguments. Requiring that (2.92) only depends on coordinate differences yields $\eta = S(\xi)$, so that the product of the two plane waves becomes

$$E_1[\xi]E_2[\eta] = E_1[\xi]E_2[S(\xi)] = E_1[\xi]E_2^\dagger[\xi] = e^{i\xi_-(x_1^- - x_2^-)}e^{i\xi_+(x_1^+ - x_2^+)}\,, \tag{2.93}$$

as requested. To write down the two-point function, an ordering prescription for plane waves living in different spaces of the braided tensor product must be specified. We choose to put the coordinates of the second point to the right, and no physical quantities will be affected by such prescription, given that the dependence of the function is solely through the coordinate difference. The most general form for the two-point function is thus

$$\Delta(x_1 - x_2) = F(x_1 - x_2) + H(x_1 - x_2)\,, \tag{2.94}$$

with F being related to products of old type plane waves and H being related to products of new type plane waves as follows

$$\begin{aligned} F(x_1 - x_2) &= \int_{-1/2}^{\infty} d\xi_+ \int_{-\infty}^{\infty} d\xi_-\, E_1[\xi]E_2^\dagger[\xi]f(\xi)\delta(\xi_+\xi_- - m^2) \\ H(x_1 - x_2) &= \int_{-\infty}^{-1/2} d\xi_+ \int_{-\infty}^{\infty} d\xi_- \mathcal{E}_1[\xi]\mathcal{E}_2^\dagger[\xi]h(\xi)\delta(\xi_+\xi_- - m^2)\,. \end{aligned} \tag{2.95}$$

It can be checked explicitly that the products $E_1[\xi]E_2^\dagger[\xi]$, $E_1^\dagger[\xi]E_2[\xi]$, $\mathcal{E}_1[\xi]\mathcal{E}_2^\dagger[\xi]$ and $\mathcal{E}_1^\dagger[\xi]\mathcal{E}_2[\xi]$ are the only independent plane wave products that depend on coordinate differences; all other plausible combinations (for example mixing old and new type plane waves) result in one of the above. Choosing one or the other alternative for the specific class of plane waves is irrelevant, since they both give the same contribution to the two-point function (modulo a constant factor) [121]. The functions $f(\xi)$ and $h(\xi)$ are supposed to be Lorentz-invariant functions of the momenta. As in the commutative case, we can assume them to be constants in momentum space, namely

$$f(\xi) = f_+\Theta(\xi_+) + f_-\Theta(-\xi_+), \qquad h(\xi) = h_-\Theta(-\xi_+)\,, \tag{2.96}$$

where $h(\xi)$ does not possess a h_+ term given that the new-type plane waves are only defined for negative values of ξ_+, in the sense of (2.91).

Enforcing the on-shell relation dictated by the δ-functions, the first of the integrals in (2.95) can be rewritten as

$$\begin{aligned} F(x_1 - x_2) = &\int_0^\infty \frac{d\xi_+}{2\xi_+} E_1\left[\frac{m^2}{\xi_+}, \frac{1}{2}\ln(1+2\xi_+)\right] E_2^\dagger\left[\frac{m^2}{\xi_+}, \frac{1}{2}\ln(1+2\xi_+)\right] f_+ + \\ &- \int_{-1/2}^0 \frac{d\xi_+}{2\xi_+} E_1\left[\frac{m^2}{\xi_+}, \frac{1}{2}\ln(1+2\xi_+)\right] E_2^\dagger\left[\frac{m^2}{\xi_+}, \frac{1}{2}\ln(1+2\xi_+)\right] f_- \end{aligned} \tag{2.97}$$

For convenience, we introduce the shorthand notation

$$e_a(\xi_+) := \exp\left[i\frac{m^2}{\xi_+}x_a^-\right] \exp\left[\frac{i}{2}\ln(1+2\xi_+)x_a^+\right] \qquad \xi_+ > -\frac{1}{2}\,, \tag{2.98}$$

for on-shell plane waves with momenta greater than $-1/2$. In the first integral of (2.97), we perform the following change of variables

$$\xi_+ = \sqrt{p^2 + m^2} + p\,, \tag{2.99}$$

and recalling (2.93), along with the fact that $x_a^{\pm} = x_a^0 \pm x_a^1$, it can be rewritten as

$$\int_{-\infty}^{\infty} \frac{dp}{2\sqrt{p^2+m^2}} e^{2i[\sqrt{p^2+m^2}(x_1^0-x_2^0)+p(x_1^2-x_2^1)]} f_+ \tag{2.100}$$

For the second integral in (2.97), the change to cartesian components of the momentum is

$$\xi_+ = -\sqrt{p^2+m^2}+p\,, \tag{2.101}$$

to take into account the negative frequency modes. Summing up the two contributions, the expression for $F(x_1-x_2)$ can be rewritten as

$$\begin{aligned} F(x_1-x_2) = & \int_{-\infty}^{\infty} \frac{dp}{2\sqrt{p^2+m^2}} e^{2i[\sqrt{p^2+m^2}(x_1^0-x_2^0)+p(x_1^2-x_2^1)]} f_+ + \\ & -\int_{-\infty}^{-m^2+1/4} \frac{dp}{2\sqrt{p^2+m^2}} e^{-2i[\sqrt{p^2+m^2}(x_1^0-x_2^0)+p(x_1^2-x_2^1)]} f_- \,. \end{aligned} \tag{2.102}$$

The first integral is identical to the commutative two-point function while the second-one is manifestly non-Lorentz invariant, given the presence of the integration boundary $\left(-m^2+1/4\right)$. This issue does not affect the definition of the Wightman function, the positive frequency part of the two-point function which, up to constant, is simply given by

$$\Delta_W(x_1-x_2) = \int_{-\infty}^{\infty} \frac{dp}{2\sqrt{p^2+m^2}} e^{2i[\sqrt{p^2+m^2}(x_1^0-x_2^0)+p(x_1^2-x_2^1)]} \,. \tag{2.103}$$

However, the Pauli-Jordan function, defined as the anti-Hermitian part of the Wightman function *i.e.*,

$$\Delta_{PJ}(x_1-x_2) = \Delta_W(x_1-x_2) - \Delta_W^{\dagger}(x_1-x_2)\,, \tag{2.104}$$

cannot be constructed since there is no way to write $\Delta_W^{\dagger}(x_1-x_2)$ in terms of old type plane waves with x_2 to the right. Indeed, let us write

$$\Delta_W^{\dagger}(x_1-x_2) = \int_0^{\infty} \frac{d\xi_+}{2\xi_+} e_2(\xi_+) e_1^{\dagger}(\xi_+) = \int_0^{\infty} \frac{d\xi_+}{2\xi_+} e_2(\xi_+) e_1(S(\xi_+))\,, \tag{2.105}$$

and try to re-order the plane waves by putting all the x_2 coordinates to the right. As explained in section 2.1.1, this can be done with the aid of the R-matrix. We have that

$$e_2(\xi_+)e_1(S(\xi_+)) = \mu \circ R \triangleright (e_1(S(\xi_+)) \otimes e_2(\xi_+))\,. \tag{2.106}$$

Recalling the expression (2.57) for the R-matrix, this results in

$$e_2(\xi_+)e_1(S(\xi_+)) = e_1(-\xi_+) e_2\left(\frac{\xi_+}{1-2\xi_+}\right). \tag{2.107}$$

For ξ_+ in the range $]0;\infty[$, the plane wave of point x_1 is not always of old type, thus proving our statement. This further strengthens the claim that a well-defined Pauli Jordan function necessitates of the contributions from the new type plane waves. Indeed, let us return to the $H(x_1-x_2)$ function and check whether we can rewrite it in a way that is complementary to the one found in $F(x_1-x_2)$. Recalling the definition of $h(\eta)$ in (2.96), we can write

$$H(x_1-x_2) = \int_{-\infty}^{-1/2} \frac{d\eta_+}{2\eta_+} \epsilon_1(\xi_+)\epsilon_2^{\dagger}(\xi_+) h_-\,, \tag{2.108}$$

where we have introduced another shorthand notation

$$\epsilon_a(\xi_+) := \exp\left[i\frac{m^2}{\xi_+}x_a^-\right]\exp\left[\frac{i}{2}\ln(1+2\xi_+)x_a^+\right] \qquad \xi_+ < -\frac{1}{2}\,, \tag{2.109}$$

Using the change of variables (2.101), (2.108) can further be rewritten as

$$H(x_1-x_2) = -\frac{1}{2}\int_{-m^2+1/4}^{\infty}\frac{dp}{\sqrt{p^2+m^2}}e^{-2i[\sqrt{p^2+m^2}(x_1^0-x_2^0)+p(x_1^2-x_2^1)]}h_-\,. \tag{2.110}$$

Notice that by setting $h_- = f_-$, we obtain an integral which is complementary to the second integral in (2.102). By setting $f_+ = 2A$ and $h_+ = h_- = f_- = -2B$, we obtain a κ-Poincaré invariant result, which is just the commutative Poincaré invariant two-point function employed in standard quantum field theory:

$$\begin{aligned}\Delta(x_1-x_2) =& A\int_{-\infty}^{\infty}\frac{dp}{\sqrt{p^2+m^2}}e^{2i[\sqrt{p^2+m^2}(x_1^0-x_2^0)+p(x_1^2-x_2^1)]}+\\ &+B\int_{-\infty}^{\infty}\frac{dp}{\sqrt{p^2+m^2}}e^{-2i[\sqrt{p^2+m^2}(x_1^0-x_2^0)+p(x_1^2-x_2^1)]}\,.\end{aligned} \tag{2.111}$$

The Wightman function corresponds to the positive frequency part, selected by $A=-1, B=0$. The Pauli-Jordan function is constructed as its anti-Hermitian part and reads

$$\Delta_{PJ}(x_1-x_2) = -\int_0^{\infty}\frac{d\xi_+}{2\xi_+}\Big(e_1(\xi_+)e_2^\dagger(\xi_+) - e_2(\xi_+)e_1^\dagger(\xi_+)\Big) \tag{2.112}$$

Recalling equation (2.107), we can rewrite the second part of the integral by bringing the plane wave of point 2 to the right, as follows

$$\begin{aligned}&\int_0^{\infty}\frac{d\xi_+}{2\xi_+}e_2(\xi_+)e_1^\dagger(\xi_+) =\\ &=\int_0^{\infty}\frac{d\xi_+}{2\xi_+}\exp\left[-i\frac{m^2}{\xi_+}x_1^-, i\frac{1}{2}\ln(1-2\xi_+)x_1^+\right]\exp\left[-i\frac{m^2}{S(-\xi_+)}x_2^-, i\frac{1}{2}\ln(1+2S(-\xi_+))x_2^+\right] =\\ &=\int_0^{1/2}\frac{d\xi_+}{2\xi_+}e_1(-\xi_+)e_2(S(-\xi)) + \int_{1/2}^{\infty}\frac{d\xi_+}{2\xi_+}\epsilon_1(-\xi_+)\epsilon_2(S(-\xi_+)) =\\ &=\int_0^{-1/2}\frac{d\xi_+}{2\xi_+}e_1(\xi_+)e_2(S(\xi)) - \int_{-\infty}^{-1/2}\frac{d\xi_+}{2\xi_+}\epsilon_1(\xi_+)\epsilon_2(S(\xi_+)) =\\ &=\int_0^{\infty}\frac{d\xi_+}{2\xi_+(1+2\xi_+)}e_1^\dagger(\xi_+)e_2(\xi_+) - \int_{-\infty}^{-1/2}\frac{d\xi_+}{2\xi_+}\epsilon_1(\xi_+)\epsilon_2^\dagger(\xi_+)\,,\end{aligned} \tag{2.113}$$

where in the third equality we have performed the $\xi_+ \to S(\xi_+)$ change of variables and recognized that $e_a(S(\xi_+)) = e_a^\dagger(\xi_+)$ (the analogous property also applies for $\epsilon_a(\xi_+)$, as hinted in the last equality). The final expression for the Pauli-Jordan function is thus

$$\begin{aligned}\Delta_{PJ}(x_1-x_2) = &-\int_0^{\infty}\frac{d\xi_+}{2\xi_+}e_1(\xi_+)e_2^\dagger(\xi_+) + \int_0^{\infty}\frac{d\xi_+}{2\xi_+(2\xi_++1)}e_1^\dagger(\xi_+)e_2(\xi_+)+\\ &-\int_{-\infty}^{-1/2}\frac{d\xi_+}{2\xi_+}\epsilon_1(\xi_+)\epsilon_2^\dagger(\xi_+)\,.\end{aligned} \tag{2.114}$$

Notice that the last integral can also be rewritten as

$$-\int_{-\infty}^{-1/2}\frac{d\xi_+}{2\xi_+}\epsilon_1(\xi_+)\epsilon_2^\dagger(\xi_+) = \int_{-\infty}^{-1/2}\frac{d\xi_+}{2\xi_+(1+2\xi_+)}\epsilon_1^\dagger(\xi_+)\epsilon_2(\xi_+)\,, \tag{2.115}$$

by performing the $\xi_+ \to S(\xi_+)$ change of variables, which maps the $]-\infty;-1/2[$ region onto itself. Although not explicitly evident in (2.114), the overall integration is performed over all momentum space, as is evident in (2.111). In particular, the expression contains only one between the combinations $\epsilon_1(\xi_+)\epsilon_2^\dagger(\xi_+)$ or $\epsilon_1^\dagger(\xi_+)\epsilon_2(\xi_+)$ in contrast with their old type counterparts, which are both present. As already hinted above, this is due to the fact the antipode leaves the $]-\infty;-1/2[$ region invariant, so an expression containing both new type plane waves combinations would overcount these new Fourier modes.

2.3 Covariant quantization and oscillator algebra

We can expand a scalar field $\phi(x_a)$ in terms of old type and new type plane waves and require that it satisfies the κ-Klein-Gordon equation [121], which is just the usual Klein-Gordon equation in linear momentum variables. The expression for the scalar field is simply given by

$$\begin{aligned}\phi(x_a) = &\int_{-\infty}^{\infty} d\xi_- \int_{-1/2}^{\infty} d\xi_+\, \delta(\xi_+\xi_- - m^2)\tilde{\phi}_1(\xi)E_a[\xi]+\\ &+\int_{-\infty}^{\infty} d\xi_- \int_{-\infty}^{-1/2} d\xi_+\, \delta(\xi_+\xi_- - m^2)\tilde{\phi}_2(\xi)\mathcal{E}_a[\xi]\,.\end{aligned} \tag{2.116}$$

Enforcing the on-shell constraints the field can be expressed as

$$\begin{aligned}\phi(x_a) = &-\int_{-1/2}^{0} \frac{d\xi_+}{2\xi_+}\tilde{\phi}_1(\xi_+)E_a\left[\frac{m^2}{\xi_+}, \frac{1}{2}\ln(1+2\xi_+)\right] + \int_{0}^{\infty} \frac{d\xi_+}{2\xi_+}\tilde{\phi}_1(\xi_+)E_a\left[\frac{m^2}{\xi_+}, \frac{1}{2}\ln(1+2\xi_+)\right] +\\ &-\int_{-\infty}^{-1/2} \frac{d\xi_+}{2\xi_+}\tilde{\phi}_2(\xi_+)\mathcal{E}_a\left[\frac{m^2}{\xi_+}, \frac{1}{2}\ln\left(1+2\xi_+\right)\right]\,,\end{aligned} \tag{2.117}$$

In terms of on-shell plane waves introduced in the previous subsection, this becomes

$$\begin{aligned}\phi(x_a) = &\int_{0}^{\infty} \frac{d\xi}{2\xi}\left[\frac{1}{2\xi+1}\tilde{\phi}_1\left(S(\xi)\right)e_a^\dagger(\xi) + \tilde{\phi}_1(\xi)e_a(\xi)\right] +\\ &+\int_{-\infty}^{-\frac{1}{2}} \frac{d\xi}{2\xi(1+2\xi)}\tilde{\phi}_2\left(S(\xi)\right)\epsilon_a^\dagger(\xi)\,.\end{aligned} \tag{2.118}$$

Hereafter, we will only focus on on-shell plane waves. Therefore, to further simplify the notation, we remove the + subscript from linear momentum and implicitly refer to the + component of momenta unless otherwise stated. As was the case in the construction of the Pauli-Jordan function, we notice that the region $\xi < -1/2$ is mapped onto itself via the application of S. This suggests that, while the integral containing old type plane waves may be customarily expanded in terms of both old type plane waves *and* their Hermitian conjugates, for the integral containing new type plane waves, only one between $\mathcal{E}$ and $\mathcal{E}^\dagger$ is needed, otherwise one would be overcounting Fourier modes. Taking inspiration from the undeformed quantum field theory, we define

$$\begin{cases}\tilde{\phi}_1(S(\xi)) = a(\xi) & \xi > 0\\ \tilde{\phi}_1(\xi) = \bar{b}(\xi) & \xi > 0\\ \tilde{\phi}_2(\xi) = \alpha(\xi) & \xi < -\frac{1}{2}\,,\end{cases} \tag{2.119}$$

where the bar indicates complex conjugation. Upon quantization, $a(\xi)$ will play the role of a particle annihilation operator while $b^\dagger(\xi)$ will play the role of an anti-particle construction operator. The newly introduced operator $\alpha(\xi)$ is defined "across" the momentum space border and introduces a relation between operators $a(\xi)$ and $b^\dagger(\xi)$ when the involved momenta also lie across the border. Of course, this feature is unique to the noncommutative scenario and is washed away when taking the $\kappa \to \infty$ limit. The expression for the scalar field is thus rewritten as

$$\begin{aligned}\phi(x_a) = &\int_0^\infty \frac{d\xi}{2\xi}\left[\frac{1}{2\xi+1}a(\xi)e_a^\dagger(\xi) + \bar{b}(\xi)e_a(\xi)\right] + \\ &+ \int_{-\infty}^{-\frac{1}{2}} \frac{d\xi}{2\xi(1+2\xi)}\alpha(\xi)\epsilon_a^\dagger(\xi)\,.\end{aligned} \tag{2.120}$$

We now promote the Fourier coefficients $a(\xi), b(\xi), \alpha(\xi)$ and indicate their Hermitian conjugates with the † symbol[3]. We finally have all the ingredients needed to implement the covariant quantization approach. The Pauli-Jordan function is the one found in (2.114), which we report for completeness:

$$\begin{aligned}\Delta_{\mathrm{PJ}}(x_1 - x_2) = &-\int_0^{+\infty}\frac{d\xi}{2\xi}e_1(\xi)e_2^\dagger(\xi) + \int_0^{+\infty}\frac{d\xi}{2\xi}\frac{1}{2\xi+1}e_1^\dagger(\xi)e_2(\xi) + \\ &-\int_{-\infty}^{-\frac{1}{2}}\frac{d\eta}{2\eta}\epsilon_1(\eta)\epsilon_2^\dagger(\eta)\,.\end{aligned} \tag{2.121}$$

The quantum scalar field (2.118) is denoted as $\hat{\phi}$ and is treated as an element of $\mathcal{A}\otimes\mathcal{O}(\mathcal{H})$, where $\mathcal{O}(\mathcal{H})$ is the set of operators acting on the (anti-)particle Hilbert space. The expression for $\hat{\phi}$ is simply

$$\begin{aligned}\hat{\phi}(x_a) = &\int_0^\infty \frac{d\xi}{2\xi}\left[\frac{1}{2\xi+1}a(\xi)e_a^\dagger(\xi) + b^\dagger(\xi)e_a(\xi)\right] + \\ &+ \int_{-\infty}^{-\frac{1}{2}} \frac{d\xi}{2\xi(1+2\xi)}\alpha(\xi)\epsilon_a^\dagger(\xi)\,,\end{aligned} \tag{2.122}$$

and the covariant quantization rules are

$$[\hat{\phi}(x_1), \hat{\phi}^\dagger(x_2)] = \Delta_{\mathrm{PJ}}(x_1 - x_2), \quad [\hat{\phi}(x_1), \hat{\phi}(x_2)] = [\hat{\phi}^\dagger(x_1), \hat{\phi}^\dagger(x_2)] = 0. \tag{2.123}$$

Commutators (2.123) then involve products of creation and annihilation operators as well as products of noncommutative plane waves. In explicitly writing these expressions, we must reorder products of plane waves with the x_2 to-the-right ordering previously specified. Then, the quantization rules (2.123) imply commutation relations for the creation and annihilation operators. The general strategy to perform this calculation is based on the fact that all plane waves, regardless of their type, can be exchanged by making use of the R-matrix. By formally indicating both old- and new-type plane waves by e_a, the switch reads:

$$e_2(\eta)e_1(\xi) = \mu\circ R\triangleright e_1(\xi)\otimes e_2(\eta) = e_1(\xi + 2\xi\eta)e_2\left(\frac{\eta}{1+2\xi+4\xi\eta}\right), \tag{2.124}$$

where R is defined in (2.57). Whether the resulting waves are of old or new type depends on the specific values of ξ, η. This procedure leads to a splitting of the commutation relations between creation and annihilation in various regions of momentum space. The resulting list of commutation relations is still rather involved, but the overall picture is much simpler than the one presented in [121], thanks to the linear momentum redefinition. From the first commutator in (2.123), we obtain:

[3] Although we will indicate the Hermitian conjugates of these operators with the usual † symbol, as is also the case with plane waves, it is important to keep in mind that they act on different Hilbert spaces.

- In the region $\xi \in]0;+\infty[$, $\eta \in]0;\frac{1}{4\xi}[$

$$b^\dagger(\xi)b(\eta) - \frac{1}{1-4\xi\eta} b\left(\frac{\eta+2\xi\eta}{1-4\xi\eta}\right) b^\dagger\left(\frac{\xi+2\xi\eta}{1-4\xi\eta}\right) = -2\xi\delta(\xi-\eta) \tag{2.125}$$

- In the region $\xi \in]0;+\infty[$, $\eta \in]\frac{1}{4\xi};\infty[$

$$b^\dagger(\xi)b(\eta) + \frac{1}{1-4\xi\eta} \alpha^\dagger\left(-\frac{\eta+2\xi\eta}{1+2\eta}\right) \alpha\left(-\frac{\xi+2\xi\eta}{1+2\xi}\right) = -2\xi\delta(\xi-\eta) \tag{2.126}$$

- In the region $\eta \in]0;+\infty[$, $\xi \in]0;\infty[$

$$b^\dagger(\xi)a^\dagger(\eta) = \frac{1+2\eta}{1+2\eta+2\xi} a^\dagger(\eta+2\xi\eta) b^\dagger\left(\frac{\xi}{1+2\xi+4\xi\eta}\right) \tag{2.127}$$

$$a(\xi)b(\eta) = \frac{1+2\xi}{1+2\xi+4\xi\eta} b\left(\frac{\eta}{1+2\xi+4\xi\eta}\right) a\left(\xi+2\xi\eta\right) \tag{2.128}$$

$$a(\xi)a^\dagger(\eta) - \frac{(1+2\xi)(1+2\eta)}{1+2\xi+2\eta} a^\dagger\left(\frac{\eta}{1+2\xi}\right) a\left(\frac{\xi}{1+2\eta}\right) = 2\eta(1+2\eta)\delta(\xi-\eta) \tag{2.129}$$

- In the region $\eta \in]-\infty;-\frac{1}{2}[$, $\xi \in]-\infty;-\frac{1}{2}[$

$$\alpha(\xi)\alpha^\dagger(\eta) + \frac{(1+2\xi)(1+2\eta)}{1+2\xi+2\eta} a^\dagger\left(\frac{\eta}{1+2\xi}\right) a\left(\frac{\xi}{1+2\eta}\right) = 2\eta(1+2\eta)\delta(\xi-\eta) \tag{2.130}$$

- In the region $\eta \in]-\infty;-\frac{1}{2}[$, $\xi \in]0;\infty[$

$$b^\dagger(\xi)\alpha^\dagger(\eta) = \frac{1+2\eta}{1+2\eta+4\eta\xi} \alpha^\dagger(\eta+2\xi\eta) a\left(-\frac{\xi}{1+2\eta+2\xi+4\xi\eta}\right) \tag{2.131}$$

- In the region $\eta \in]0;+\infty[$, $\xi \in]-\infty;-\frac{1}{2}[$

$$\alpha(\xi)b(\eta) = \frac{1+2\xi}{1+2\xi+4\xi\eta} a^\dagger\left(-\frac{\eta}{1+2\eta+2\xi+4\xi\eta}\right) \alpha(\xi+2\eta\xi) \tag{2.132}$$

- In the region $\eta \in]0;-\frac{1}{2}-\xi[$, $\xi \in]-\infty;-\frac{1}{2}[$

$$\alpha(\xi)a^\dagger(\eta) = \frac{(1+2\eta)(1+2\xi)}{1+2\eta+2\xi} b\left(-\frac{\eta}{1+2\eta+2\xi}\right) \alpha\left(\frac{\xi}{1+2\eta}\right) \tag{2.133}$$

- In the region $\eta \in]-\frac{1}{2}-\xi;\infty[$, $\xi \in]-\infty;-\frac{1}{2}[$

$$\alpha(\xi)a^\dagger(\eta) = -\frac{(1+2\eta)(1+2\xi)}{1+2\eta+2\xi} a^\dagger\left(\frac{\eta}{1+2\xi}\right) b^\dagger\left(-\frac{\xi}{1+2\xi+2\eta}\right) \tag{2.134}$$

- In the region $\eta \in]-\frac{1}{2}-\xi;-\frac{1}{2}[$, $\xi \in]0;+\infty[$

$$a(\xi)\alpha^\dagger(\eta) = -\frac{(1+2\eta)(1+2\xi)}{1+2\eta+2\xi} b\left(-\frac{\eta}{1+2\eta+2\xi}\right) \alpha\left(\frac{\xi}{1+2\eta}\right) \tag{2.135}$$

- In the region $\eta \in]-\infty;-\frac{1}{2}-\xi[$, $\xi \in]0;+\infty[$

$$a(\xi)\alpha^\dagger(\eta) = \frac{(1+2\eta)(1+2\xi)}{1+2\eta+2\xi} \alpha^\dagger\left(\frac{\eta}{1+2\xi}\right) b^\dagger\left(-\frac{\xi}{1+2\xi+2\eta}\right) \tag{2.136}$$

From the $[\hat{\phi}(x_1), \hat{\phi}(x_2)] = 0$ commutator, the resulting relations are:

- In the region $\eta \in]0;+\infty[$, $\xi \in]0;\infty[$

$$b^\dagger(\xi)b^\dagger(\eta) = b^\dagger\left(\eta + 2\xi\eta\right) b^\dagger\left(\frac{\xi}{1+2\eta+4\xi\eta}\right) \quad (2.137)$$

$$a(\xi)b^\dagger(\eta) = b^\dagger\left(\frac{\eta}{1+2\xi}\right) a\left(\frac{\xi}{1+2\eta}\right) \quad (2.138)$$

$$a(\xi)a(\eta) = a\left(\frac{\eta}{1+2\xi+4\xi\eta}\right) a\left(\xi + 2\xi\eta\right) \quad (2.139)$$

- In the region $\xi \in]0;+\infty[$, $\eta \in]0;\frac{1}{4\xi}[$

$$b^\dagger(\xi)a(\eta) = a\left(\frac{\eta+2\xi\eta}{1-4\xi\eta}\right) b^\dagger\left(\frac{\xi+2\xi\eta}{1-4\xi\eta}\right) \quad (2.140)$$

- In the region $\xi \in]0;+\infty[$, $\eta \in]\frac{1}{4\xi};\infty[$

$$b^\dagger(\xi)a(\eta) = \alpha\left(\frac{\eta+2\xi\eta}{1-4\xi\eta}\right) \alpha\left(-\frac{\xi+2\xi\eta}{1+2\xi}\right) \quad (2.141)$$

- In the region $\xi \in]0;+\infty[$, $\eta \in]-\infty;-\frac{1}{2}[$

$$b^\dagger(\xi)\alpha(\eta) = \alpha\left(\frac{\eta+2\xi\eta}{1-4\xi\eta}\right) a\left(-\frac{\xi+2\xi\eta}{1+2\xi}\right) \quad (2.142)$$

- In the region $\xi \in]0;+\infty[$, $\eta \in]-\frac{1+2\xi}{4\xi};-\frac{1}{2}[$

$$a(\xi)\alpha(\eta) = \alpha\left(\frac{\eta}{1+2\xi+4\xi\eta}\right) b^\dagger\left(-\frac{\xi+2\xi\eta}{1+2\xi+4\xi\eta}\right) \quad (2.143)$$

- In the region $\xi \in]0;+\infty[$, $\eta \in]-\infty;-\frac{1+2\xi}{4\xi}[$

$$a(\xi)\alpha(\eta) = a\left(\frac{\eta}{1+2\xi+4\xi\eta}\right) \alpha\left(\xi + 2\eta\xi\right) \quad (2.144)$$

- In the region $\xi \in]-\infty;-\frac{1}{2}[$, $\eta \in]0;-\frac{1}{2}-\xi[$

$$\alpha(\xi)b^\dagger(\eta) = a\left(-\frac{\eta}{1+2\xi+2\eta}\right) \alpha\left(\frac{\xi}{1+2\eta}\right) \quad (2.145)$$

- In the region $\xi \in]-\infty;-\frac{1}{2}[$, $\eta \in]-\frac{1}{2}-\xi;+\infty[$

$$\alpha(\xi)b^\dagger(\eta) = \alpha\left(-\frac{\eta}{1+2\xi+2\eta}\right) b^\dagger\left(-\frac{\xi}{1+2\eta+2\xi}\right) \quad (2.146)$$

- In the region $\xi \in]-\infty;-\frac{1}{2}[$, $\eta \in]0;+\infty[$

$$\alpha(\xi)a(\eta) = b^\dagger\left(-\frac{\eta}{1+2\eta+2\xi+4\xi\eta}\right) \alpha(\xi + 2\xi\eta) \quad (2.147)$$

- In the region $\xi \in]-\infty;-\frac{1}{2}[$, $\eta \in]-\infty;-\frac{1}{2}[$

$$\alpha(\xi)\alpha(\eta) = b^\dagger\left(-\frac{\eta}{1+2\xi+2\eta+4\eta\xi}\right) a\left(\xi + 2\eta\xi\right) \quad (2.148)$$

Commutation relations for the $[\hat{\phi}^\dagger(x_1), \hat{\phi}^\dagger(x_2)] = 0$ commutator can be obtained by taking the Hermitian conjugate of the commutators stemming from $[\hat{\phi}(x_1), \hat{\phi}(x_2)] = 0$. We will explicitly show how to obtain relations (2.125) and (2.126) as an example. For the first, consider the integral

$$\int_0^\infty \frac{d\eta}{2\eta}\int_0^\infty \frac{d\xi}{2\xi}\, b(\eta)b^\dagger(\xi)e_2^\dagger(\eta)e_1(\xi)\,, \tag{2.149}$$

stemming from the $\hat\phi^\dagger(x_2)\hat\phi(x_1)$ term. The plane waves are re-ordered with the R-matrix as follows

$$\begin{aligned} e_2^\dagger(\eta)e_1(\xi) =&\mu\circ R\triangleright[e_1(\xi)\otimes e_2(S(\eta))] = e_1\left(\frac{\xi}{1+2\eta}\right)e_2\left(-\frac{\eta}{1+2\eta+2\xi}\right) = \\ =&e_1\left(\frac{\xi}{1+2\eta}\right)e_2^\dagger\left(\frac{\eta}{1+2\xi}\right). \end{aligned} \tag{2.150}$$

The integral in (2.149) can thus be rewritten as

$$\begin{aligned} &\int_0^\infty \frac{d\eta}{2\eta}\int_0^\infty \frac{d\xi}{2\xi}\, b(\eta)b^\dagger(\xi)e_1\left(\frac{\xi}{1+2\eta}\right)e_2^\dagger\left(\frac{\eta}{1+2\xi}\right) = \\ =&\int_0^\infty d\xi\int_0^{1/4\xi} d\eta\,\frac{1}{4\xi\eta(1-4\xi\eta)}\, b\left(\frac{\eta+2\eta\xi}{1-4\xi\eta}\right)b^\dagger\left(\frac{\xi+2\eta\xi}{1-4\xi\eta}\right)e_1(\xi)e_2^\dagger(\eta)\,. \end{aligned} \tag{2.151}$$

Notice that the waves obtained in (2.150) via application of the R-matrix are still of old type since both $\frac{\xi}{1+2\eta}$ and $\frac{\eta}{1+2\xi}$ are both greater than $-1/2$. The integral in (2.151) is to be compared with the corresponding terms stemming from $\hat\phi(x_1)\hat\phi^\dagger(x_2)$ and $\Delta_{PJ}(x_1-x_2)$, yielding commutation relation (2.125) with the appropriate range of validity. The complementary part of this commutation relation is found by inspecting the term of the type

$$\int_{-\infty}^{-1/2}\frac{d\eta}{2\eta(1+2\eta)}\int_{-\infty}^{-1/2}\frac{d\xi}{2\xi(1+2\xi)}\,\alpha^\dagger(\eta)\alpha(\xi)\epsilon_2^\dagger(\eta)\epsilon_1(\xi)\,, \tag{2.152}$$

which, with techniques analogous to the previous case, can be rewritten as

$$\begin{aligned} &\int_{-\infty}^{-1/2}\frac{d\eta}{2\eta(1+2\eta)}\int_{-\infty}^{-1/2}\frac{d\xi}{2\xi(1+2\xi)}\,\alpha^\dagger(\eta)\alpha(\xi)\epsilon_1\left(\frac{\xi}{1+2\eta}\right)\epsilon_2^\dagger\left(\frac{\eta}{1+2\xi}\right) = \\ =-&\int_0^\infty d\xi\int_{1/4\xi}^\infty d\eta\,\frac{1}{4\xi\eta(1-4\xi\eta)}\,\alpha^\dagger\left(\frac{\eta+2\eta\xi}{1-4\xi\eta}\right)\alpha\left(\frac{\xi+2\eta\xi}{1-4\xi\eta}\right)e_1(\xi)e_2^\dagger(\eta)\,. \end{aligned} \tag{2.153}$$

In this case, the product of two new type plane waves has turned into a product of two old type plane waves upon x_2 to-the-right ordering. Again, comparing this term with the appropriate ones figuring in the $\hat\phi(x_1)\hat\phi^\dagger(x_2)$ term and in $\Delta_{PJ}(x_1-x_2)$ we obtain the complement of rule (2.125), given by (2.126). All of the other commutation relations can be obtained using this same strategy.

2.3.1 Representation of the deformed oscillator algebra

A useful technique employed in studies of quantum field theories on noncommutative spacetime is to represent the creation and annihilation operators of the deformed theory in terms of the ones of the standard theory [43, 132]. We introduce operators c and $c^\dagger$ which satisfy the standard bosonic commutation relations (in lightcone coordinates [133]):

$$[c(\xi),c^\dagger(\eta)] = 2\xi\delta(\xi-\eta) \qquad [c(\xi),c(\eta)] = [c^\dagger(\xi),c^\dagger(\eta)] = 0\,, \tag{2.154}$$

for any real value of ξ,η. These operators act on the usual Fock space employed in quantum field theory. The vacuum state $|0\rangle$ is annihilated by $c(\xi)$ and $c^\dagger(-\xi)$, for $\xi>0$. Then, single particle states are defined as excitations of the vacuum state as

$$c^\dagger(\xi)\,|0\rangle = |\xi\rangle_P \qquad \xi>0\,, \tag{2.155}$$

while single anti-particle states are instead defined by

$$c(-\xi)\,|0\rangle = |\xi\rangle_{AP} \qquad \xi > 0\,. \tag{2.156}$$

This parametrization of the standard oscillator algebra might not be familiar to the reader: it is a compact way of expressing the bosonic algebra of a complex scalar field in terms of a single infinite one-parameter set of operators. This is possible because, when expressed in lightcone coordinates, the creation and annihilation operators for particles and antiparticles depend on a single positive parameter, $\xi > 0$, which is the lightcone momentum (fig. 2.1). Instead of having different symbols for the particle and antiparticle operators, we define $c(\xi)$ on negative values of ξ too, and identify $c(\xi)$ for negative ξ with the creation operator for the antiparticles. The corresponding annihilation operators will be the Hermitian conjugates of those. This choice just amounts to a relabeling of the Fourier coefficients of our scalar fields, and makes the notation more compact.

We recall the expressions for the Poincaré charges in terms of these operators. The boost operator reads

$$N = -i\int_0^\infty \frac{d\xi}{2\xi}\xi\left(\frac{dc^\dagger(\xi)}{d\xi}c(\xi) + \frac{dc(-\xi)}{d\xi}c^\dagger(-\xi)\right)\,, \tag{2.157}$$

while the translations generators are given by

$$P_+ = \int_0^\infty \frac{d\xi}{2}(c^\dagger(\xi)c(\xi) + c(-\xi)c^\dagger(-\xi)), \qquad P_- = \int_0^\infty m^2\frac{d\xi}{2\xi^2}(c^\dagger(\xi)c(\xi) + c(-\xi)c^\dagger(-\xi))\,. \tag{2.158}$$

These generators close the standard Poincaré algebra given that they are undeformed. Using (2.154) and (2.157), it is also easy to show that

$$[N, c(\xi)] = -i\xi\frac{dc(\xi)}{d\xi} \qquad [N, c^\dagger(\xi)] = -i\xi\frac{dc^\dagger(\xi)}{d\xi}\,, \tag{2.159}$$

for every ξ, and hence

$$\begin{aligned} e^{ixN}c(\xi)e^{-ixN} &= c(e^x\xi), \\ e^{ixN}c^\dagger(\xi)e^{-ixN} &= c^\dagger(e^x\xi)\,. \end{aligned} \tag{2.160}$$

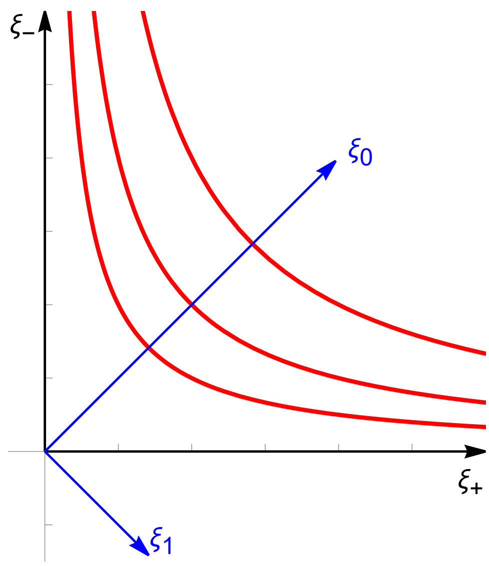

Figure 2.1: The mass-shells (in red) in light-cone coordinates are all confined in the $\xi_+ > 0$, $\xi_- > 0$ region.

For what follows, it is convenient to introduce the shorthand notation for the following finite boost transformation with momentum-dependent rapidity:

$$e^{i\ln(1+2\xi)N} := B_\xi \,. \tag{2.161}$$

It allows to write a representation for $a(\xi), b^\dagger(\xi)$ in a compact way, as follows (for $\xi > 0$)

$$\begin{aligned} a(\xi) &= \frac{1}{\sqrt{1+2S(\xi)}} B_{S(\xi)} c(\xi) = \frac{1}{\sqrt{1+2S(\xi)}} c(-S(\xi)) B_{S(\xi)} \\ b^\dagger(\xi) &= \frac{1}{\sqrt{1+2\xi}} c(-\xi) B_\xi = \frac{1}{\sqrt{1+2\xi}} B_\xi c(S(\xi)) \\ a^\dagger(\xi) &= \frac{1}{\sqrt{1+2S(\xi)}} c^\dagger(\xi) B_\xi = \frac{1}{\sqrt{1+2S(\xi)}} B_\xi c^\dagger(-S(\xi)) \\ b(\xi) &= \frac{1}{\sqrt{1+2\xi}} B_{S(\xi)} c^\dagger(-\xi) = \frac{1}{\sqrt{1+2\xi}} c^\dagger(S(\xi)) B_{S(\xi)} \,. \end{aligned} \tag{2.162}$$

When $\xi < -1/2$, the commutators involving $\alpha(\xi), \alpha^\dagger(\xi)$ impose the constraints

$$\begin{aligned} \alpha(\xi) &= \frac{1}{\sqrt{1+2S(\xi)}} B_{S(\xi)} c(\xi) = a(\xi) = b^\dagger(S(\xi)) \\ \alpha^\dagger(\xi) &= \frac{1}{\sqrt{1+2S(\xi)}} c^\dagger(\xi) B_\xi = a^\dagger(\xi) = b(S(\xi)) \end{aligned} \tag{2.163}$$

which can be obtained by extending the definitions in (2.162) to negative ξ. Notice that these constraints also identify $a(\xi)$ with $b^\dagger(S(\xi))$, which is consistent with (2.162) and with the commutative limit. Indeed, when $\kappa \to \infty$, the momentum space boundary $\xi = -\kappa/2$ vanishes, so that a and $b^\dagger$ are not constrained anymore, as it should be in the commutative quantum field theory of the complex scalar field.

The presence of the particular operator B_ξ in these representations is by no means incidental. It can be traced back to the plane wave flip governed by the R-matrix (2.124), which is explicitly dependent on N. When changing variables in the integrals appearing in the covariant quantization procedure, the *braiding* of the momenta in plane waves is then reflected in the arguments of the creation and annihilation operators. The deformed harmonic oscillator algebra (2.163) is quite different from the one found in [134] for θ-Moyal noncommutative quantum field theory. There, the arguments of the creation and annihilation operators are left untouched, but the commutation relations are deformed by multiplication of a phase, dependent on $p_\mu \theta^{\mu\nu} q_\nu$, with p, q being the momenta involved and $\theta^{\mu\nu}$ the antisymmetric matrix controlling the noncommutativity between coordinates.

Having represented the deformed creation and annihilation operators in terms of ordinary ones, we can now define (anti)-particle states of the deformed theory making use of the ordinary operators c, $c^\dagger$ on the standard Fock space.

2.3.2 1-particle Fock state and C, P, T symmetries

We start by exploring the 1-particle states of the deformed theory. They are elements of the 1-particle Hilbert space $\mathcal{H}$. From representations (2.162), it is immediate to see the the vacuum of the ordinary theory, $|0\rangle$ is also annihilated by the annihilation operators of the deformed theory

$$a(\xi)\,|0\rangle = b(\xi)\,|0\rangle = 0 \,. \tag{2.164}$$

Single particle states are then defined as excitations of the vacuum

$$\frac{a^\dagger(\xi)}{\sqrt{1+2\xi}}\,|0\rangle = c^\dagger(\xi)B_\xi\,|0\rangle = c^\dagger(\xi)\,|0\rangle = |\xi\rangle_P\ , \tag{2.165}$$

where we used the fact that $B_\xi\,|0\rangle = |0\rangle$ for every ξ and the square root factor in the denominator guarantees normalization. Single anti-particle states are instead given by

$$\frac{b^\dagger(\xi)}{\sqrt{1+2S(\xi)}}\,|0\rangle = c(-\xi)B_\xi\,|0\rangle = c(-\xi)\,|0\rangle = |\xi\rangle_{AP}\ . \tag{2.166}$$

The single (anti)-particle states are thus identical to the ones defined in the commutative quantum field theory. As a consequence of this, the action of the momentum operators $P_\pm$ defined in (2.158) give the standard results

$$\begin{aligned} P_+\,|\xi\rangle_P &= \xi\,|\xi\rangle_P\ , & P_+\,|\xi\rangle_{AP} &= \xi\,|\xi\rangle_{AP}\ , \\ P_-\,|\xi\rangle_P &= \frac{m^2}{\xi}\,|\xi\rangle_P\ , & P_-\,|\xi\rangle_{AP} &= \frac{m^2}{\xi}\,|\xi\rangle_{AP}\ . \end{aligned} \tag{2.167}$$

What about the $\alpha(\xi)$ operator? By letting it act on the vacuum, it is easy to see that $\alpha(\xi)\,|0\rangle = \alpha^\dagger(\xi)\,|0\rangle = 0$. So we see that on the one-particle states, the $\alpha(\xi)$ leave no observable trace.

We attempt to define the charge conjugation operator as is ordinarily done in standard quantum field theory. We require that

$$\mathcal{C}\hat{\phi}(x^-,x^+)\mathcal{C}^{-1} = \hat{\phi}^\dagger(x^-,x^+)\,. \tag{2.168}$$

Recalling the expression for the scalar field (2.120), the above constraint yields, for $\xi > 0$

$$\mathcal{C}\frac{a(\xi)}{1+2\xi}\mathcal{C}^{-1} = b(\xi) \qquad \mathcal{C}\,b^\dagger(\xi)\,\mathcal{C}^{-1} = \frac{a^\dagger(\xi)}{1+2\xi}\,, \tag{2.169}$$

while for $\xi < -1/2$, we have

$$\mathcal{C}\frac{\alpha(\xi)}{1+2\xi}\mathcal{C}^{-1} = -\alpha^\dagger(S(\xi))\,. \tag{2.170}$$

For single particle states, (2.169) yields simply

$$\mathcal{C}\,|\xi\rangle_{AP} = |\xi\rangle_P\ , \tag{2.171}$$

as is the case in the undeformed quantum field theory. As a result, we can write the charge conjugation operator as

$$\mathcal{C} = \int_0^\infty d\xi\Big[c^\dagger(\xi)c^\dagger(-\xi) + c(\xi)c(-\xi)\Big]\,, \tag{2.172}$$

which is just the usual expression one obtains also in commutative quantum field theory. Using the above and representations (2.162), (2.163) for the creation and annihilation operators, properties (2.169) and (2.170) can be explicitly verified.

A remark on the consequences of Eq. (2.172): as can be seen from Eq. (2.167), the one-particle state and the one-antiparticle state associated to it through the charge conjugation operator carry the same momentum. This departs from what was recently found in the timelike κ-Minkowski case in [94], where it appears that the charge conjugation operator sends a one-particle state into a one-antiparticle state with different momentum. This led to an interesting phenomenology and the possibility of putting bounds to the noncommutativity parameters only a few order of magnitude lower than the Planck energy [135, 136].

Regarding parity (P) and time revesal (T), **in the commutative case, in lightcone coordinates,** these operators are introduced as, respectively:

$$P: \, x^{\pm} \to x^{\mp} \,, \qquad T: \, x^{\pm} \to -x^{\mp} \,, \tag{2.173}$$

which are mapped to two involutive operators $\mathcal{P}$ and $\mathcal{T}$ acting on the creation and annihilation operators, defined by

$$\mathcal{P}\hat{\phi}(x^-, x^+)\mathcal{P}^{-1} = \hat{\phi}(x^+, x^-) \,, \qquad \mathcal{T}\hat{\phi}(x^-, x^+)\mathcal{T}^{-1} = \overline{\hat{\phi}(-x^+, -x^-)} \,, \tag{2.174}$$

where the operator over the quantum field on the right hand side of the action of the $\mathcal{T}$ operator is a complex conjugate, as opposed to a Hermitian conjugate, as it acts only on the plane waves in the Fourier expansion of the fields, and leaves the construction and annihilation operators unchanged. It is necessary to compose the naïve time reversal operator with this complex conjugate, thereby making the operator *antilinear,* in order to have a well-behaved transformation on the Fock space (the naïve operator $\hat{\phi}(x^-, x^+) \to \hat{\phi}(-x^+, -x^-)$ would be unacceptable, as it would end up annihilating all one-particle states [131]). Replacing in the above the expansion of an on-shell quantum field in creation and annihilation operators [*i.e.* the commutative equivalent of Eq. (2.120)], one gets the following action of $\mathcal{P}$ and $\mathcal{T}$:

$$\begin{aligned} \mathcal{P}a(\xi)\mathcal{P}^{-1} &= \pm a\left(\frac{m^2}{\xi}\right), \quad \mathcal{P}b(\xi)\mathcal{P}^{-1} = \pm b\left(\frac{m^2}{\xi}\right), \\ \mathcal{P}a^\dagger(\xi)\mathcal{P}^{-1} &= \pm a^\dagger\left(\frac{m^2}{\xi}\right), \quad \mathcal{P}b^\dagger(\xi_+)\mathcal{P}^{-1} = \pm b^\dagger\left(\frac{m^2}{\xi}\right), \end{aligned} \tag{2.175}$$

and

$$\begin{aligned} \mathcal{T}a(\xi)\mathcal{T}^{-1} &= a\left(\frac{m^2}{\xi}\right), \quad \mathcal{T}b(\xi)\mathcal{T}^{-1} = b\left(\frac{m^2}{\xi}\right), \\ \mathcal{T}a^\dagger(\xi)\mathcal{T}^{-1} &= a^\dagger\left(\frac{m^2}{\xi}\right), \quad \mathcal{T}b^\dagger(\xi)\mathcal{T}^{-1} = b^\dagger\left(\frac{m^2}{\xi}\right), \end{aligned} \tag{2.176}$$

where the $\pm$ sign depends on the parity of the particle and the overall phase for the time-reversal was omitted since it has no effect on our discussion. Acting on the vacuum with the left- and right-hand sides of the equations above, one gets:

$$\begin{aligned} \mathcal{P}\,|\xi\rangle_P &= \pm\left|\frac{m^2}{\xi}\right\rangle_P, \quad \mathcal{P}\,|\xi\rangle_{AP} = \pm\left|\frac{m^2}{\xi}\right\rangle_{AP}, \\ \mathcal{T}\,|\xi\rangle_P &= \left|\frac{m^2}{\xi}\right\rangle_P, \quad \mathcal{T}\,|\xi\rangle_{AP} = \left|\frac{m^2}{\xi}\right\rangle_{AP}. \end{aligned} \tag{2.177}$$

One could imagine to extend this analysis to the noncommutative case, exactly like what we did in Subsec. 2.3.2 for the charge conjugation operator. However, an obstacle immediately manifests itself: there is no sense in which the coordinate commutation relations (2.14) can be invariant under parity and time-reversal transformations. In the noncommutative case, we have to choose what these operators do to the noncommutative product between coordinates: they may leave it unchanged, meaning that they are *homomorphisms* for this product, or they may exchange the product order, in which case they are *anti-homomorphisms.* This distinction is absent in the commutative case, precisely because the products are commutative. So, for consistency with the commutative limit, we need a linear P operator and an antilinear T operator, however we are free to choose either of them as homomorphisms or anti-homomorphisms. Regardless of what we choose, since the commutation relations (2.14) have x^- on the right-hand side, an operator that sends x^- to x^+ can never leave them invariant.

If we insist on introducing a P and a T operator as in (2.174), acting on our on-shell noncommutative quantum fields (2.120), then our on-shell plane waves are sent to off-shell ones. For example, choosing P to be a homomorphism one gets the following transformation rule for a noncommutative plane wave:

$$e[\xi] \to e^{i\frac{m^2}{\xi}x^+} e^{\frac{i}{2}\ln(1+2\xi)x^-} = e^{\frac{i}{2}\ln(1+2\xi)e^{-2\frac{m^2}{\xi}}x^-} e^{i\frac{m^2}{\xi}x^+} , \tag{2.178}$$

and the pair of momentum components that appear on the right hand side:

$$\left(\frac{1}{2}\ln(1+2\xi)e^{-2\frac{m^2}{\xi}}, \frac{m^2}{\xi}\right) , \tag{2.179}$$

does not satisfy the on-shell relation anymore. The same happens for the other on-shell waves in the field expansion, including those of "new type". If we chose P to be an anti-homomorphism:

$$e[\xi] \to e^{\frac{i}{2}\ln(1+2\xi)x^-} e^{i\frac{m^2}{\xi}x^+} , \tag{2.180}$$

we end up with the following a pair of momentum components:

$$\left(\frac{1}{2}\ln(1+2\xi), \frac{m^2}{\xi}\right) , \tag{2.181}$$

which again does not satisfy the on-shell relation (the ξ_- and ξ_+ components are in the wrong order). Analogous calculations can also be done for the time reversal operator $\mathcal{T}$, and still result in an off-shell plane wave.

This is just a manifestation of the non-invariance of the basic commutation relations (2.14), which are the starting point of the whole model. This theory is parity- and time-reversal-breaking. However, the theory can still be said to preserve combined PT invariance: if both P and T are chosen to have the same behaviour with respect to the noncommutative product, *i.e.* they are both homomorphisms or anti-homomorphisms, and P is assumed linear while T is assumed antilinear, then the coordinate commutation relations (2.14) turn our to be invariant. Such a PT operator would leave also the on-shell plane waves $e[\xi]$ invariant, however the new-type waves could, in principle, change: looking at Eq. (2.88), $\mathcal{E}_a[\xi] = E_a[\xi]e^{-\frac{\pi}{2}x_a^+}$, it is clear that, an antilinear homomorphism like our PT operator would leave $E_a[\xi]$ invariant, while changing the $e^{-\frac{\pi}{2}x_a^+}$ term into $e^{+\frac{\pi}{2}x_a^+}$. This, however, is harmless, as we can easily prove that $e^{-\frac{\pi}{2}x_a^+} = e^{+\frac{\pi}{2}x_a^+}$ in our representation, when acting on functions of a single variable. Thus, the new-type plane waves are also left invariant by PT.

It appears that PT transformations are still a symmetry of our theory. In particular, PT acts like the identity on the scalar field Fock space (this is true in the commutative case too for spin-zero fields [131]). Finally, CPT is preserved too.

2.3.3 Braided flip operator and multiparticle states

We now begin exploring the multi-particle sector of the theory. To get a feeling of the novelties introduced by our noncommutative framework, let us start by focusing on two particle states (the conclusions drawn will be analogous for anti-particle states):

$$|\xi\rangle_P \otimes |\eta\rangle_P = \frac{a^\dagger(\xi)}{\sqrt{1+2\xi}}\,|0\rangle \otimes \frac{a^\dagger(\eta)}{\sqrt{1+2\eta}}\,|0\rangle , \tag{2.182}$$

which are elements of the tensor product of two copies of the 1-particle Hilbert space $\mathcal{H}$. The total momentum for this two-particle state is obtained by acting with the coproducts of the translation generators, as dictated by Hopf Algebra axioms when acting on the tensor product of its representation. To obtain the total + component of the momentum, we apply the coproduct (2.43) for P_+,

$$\begin{aligned}\Delta[P_+](|\xi\rangle_P \otimes |\eta\rangle_P) = P_+ |\xi\rangle_P \otimes |\eta\rangle_P + |\xi\rangle_P \otimes P_+ |\eta\rangle_P + 2P_+ |\xi\rangle_P \otimes P_+ |\eta\rangle_P = \\ =(\xi + \eta + 2\xi\eta)(|\xi\rangle_P \otimes |\eta\rangle_P) = \Delta[\xi,\eta]_+(|\xi\rangle_P \otimes |\eta\rangle_P)\,,\end{aligned} \tag{2.183}$$

where the $\Delta[\xi,\eta]$ operation for liner momentum was defined in (2.48). In a similar fashion, we can calculate the − component for the total momentum, yielding

$$\begin{aligned}\Delta[P_-](|\xi\rangle_P \otimes |\eta\rangle_P) = P_- |\xi\rangle_P \otimes |\eta\rangle_P + |\xi\rangle_P \otimes P_- |\eta\rangle_P - \frac{2P_+}{1+2P_+} |\xi\rangle_P \otimes P_- |\eta\rangle_P = \\ =\left(\frac{m^2}{\xi} + \frac{m^2}{\eta} - \frac{2\xi}{1+2\xi}\frac{m^2}{\eta}\right)(|\xi\rangle_P \otimes |\eta\rangle_P) = \Delta[\xi,\eta]_-(|\xi\rangle_P \otimes |\eta\rangle_P)\,,\end{aligned} \tag{2.184}$$

The same line of reasoning can be applied to anti-particle states, obtaining the same results for the total momenta.

In ordinary quantum field theory, multi-particle states live in symmetrized or anti-symmetrized tensor-products of single-particle states, which characterize the notion of identical particles. The key ingredient is the ordinary flip operator σ, which is an involutive operation on the tensor product of Hilbert spaces of single particle states, where the multi-particle states are defined. In general, an analogous construction of the multi-particle Fock space is not so straightforward for quantum field theories on noncommutative spacetime [103]. The main reason for this is that the standard flip operation applied to a two-particle state yields another two-particle state carrying different total momentum, due to the noncommutative nature of the coproducts. In our specific model, this simply follows from observing that

$$\Delta[P_-](|\xi\rangle_P \otimes |\eta\rangle_P) \neq \Delta[P_-](|\eta\rangle_P \otimes |\xi\rangle_P)\,. \tag{2.185}$$

The way out of this *empasse* is to define a deformed notion of particle exchange. This is possible, for example, in noncommutative quantum field theory on the θ-Moyal noncommutative spacetime [36,43], due to the properties of the twist operator, linked to the existence of an R-matrix [137]. For the much-studied timelike κ-Minkowski case, several works [100,103,124] have tried to identify a *braiding* of single-particle states in order to construct a deformed notion of symmetric and anti-symmetric states. These approaches all present some shortcomings: either the braiding is not involutive, or it is not covariant when constructing the theory at all orders in κ. The recent [103] finds that, accepting a non-involutive flip operator as the physical one, the notion of identical particles has to be abandoned. The lack of involutivity of the flip operator leads, in fact, to an infinite tower of states characterized by the same total momentum. In the present work, we find that the κ-lightlike framework, although characterized by the same non-abelian momentum Lie-group structure as the timelike case, admits a well-defined notion of identical particles, thanks to the existence of the universal R-matrix. Consider, for instance, the two-particle state defined as

$$\tilde{R}(|\xi\rangle_P \otimes |\eta\rangle_P) \coloneqq R \circ \sigma\ (|\xi\rangle_P \otimes |\eta\rangle_P) = R(|\eta\rangle_P \otimes |\xi\rangle_P)\,, \tag{2.186}$$

i.e., we act with the flip operator σ, where $\sigma\ (|\xi\rangle_P \otimes |\eta\rangle_P) = |\eta\rangle_P \otimes |\xi\rangle_P$, and then with the

R-matrix. In detail, we have

$$\begin{aligned}\tilde{R}(|\xi\rangle_P \otimes |\eta\rangle_P) =& e^{-2i\ln(1+2P_+)\otimes N} e^{2iN\otimes \ln(1+2P_+)}(|\eta\rangle_P \otimes |\xi\rangle_P) = \\ =& e^{-2i\ln(1+2P_+)\otimes N}\left(B_\xi \,|\eta\rangle_P \otimes |\xi\rangle_P\right) = \\ =& \,|\eta + 2\xi\eta\rangle_P \otimes B_{S(\eta+2\xi\eta)}\,|\xi\rangle_P = \\ =& \,|\eta + 2\eta\xi\rangle_P \otimes \left|\frac{\xi}{1+2\eta+4\eta\xi}\right\rangle_P .\end{aligned} \tag{2.187}$$

The structure of the new obtained two-particle states mimicks the structure of the Hermitian conjugate of commutation relation (2.139), where the deformation emerges from applying the R-matrix to exchange plane waves upon performing covariant quantization, as discussed in section 2.3. By acting with the momentum coproducts (2.43), it is now easy to check that

$$\Delta[P_\pm]\Bigg(|\eta+2\eta\xi\rangle_P \otimes \left|\frac{\xi}{1+2\eta+4\eta\xi}\right\rangle_P\Bigg) = \Delta(\xi,\eta)_\pm\Bigg(|\eta+2\eta\xi\rangle_P \otimes \left|\frac{\xi}{1+2\eta+4\eta\xi}\right\rangle_P\Bigg), \tag{2.188}$$

so the deformed symmetric state (2.186) has the same total momentum as (2.182), thus being a suitable candidate for our construction of deformed (anti-)symmetric states. It is also easy to show that $\tilde{R}$ is an involutive operator, *i.e.* ($\tilde{R}^2 = 1$). Indeed, repeating the same analysis as in (2.187), one can show that

$$\tilde{R}\left(|\eta+2\eta\xi\rangle_P \otimes \left|\frac{\xi}{1+2\eta+4\eta\xi}\right\rangle_P\right) = |\xi\rangle_P \otimes |\eta\rangle_P \,. \tag{2.189}$$

This last property makes $\tilde{R}$ and ideal candidate for constructing a deformed symmetrization operator, useful in defining deformed symmetric states in our field theory. We can define it as:

$$\mathcal{S}^+ := \frac{1}{2}(1\otimes 1 + \tilde{R})\,, \tag{2.190}$$

which is such that $(\mathcal{S}^+)^2 = \mathcal{S}^+$, *i.e.* $\mathcal{S}^+$ is idempotent. Then, we can define deformed symmetric two-particle states simply as

$$\sqrt{2}\,\mathcal{S}^+\left(|\xi\rangle_P \otimes |\eta\rangle_P\right)\,, \tag{2.191}$$

where the $\sqrt{2}$ factor is introduced for normalization. In an analogous way, we can define the antisymmetrization operator $\mathcal{S}^-$:

$$\mathcal{S}^- := \frac{1}{2}(1\otimes 1 - \tilde{R})\,, \tag{2.192}$$

which is also idempotent and can be used to define antisymmetric multi-particle states.[4]

So far, we have shown that there exists an involutive braiding that suggests the definition of deformed symmetric two-particle states in lightlike κ-Minkowski quantum field theory. We now show the covariance of such braiding, in order to complete the picture.

A single particle state transforms under a finite boost of parameter τ as

$$|\xi\rangle_P \to e^{i\tau N}\,|\xi\rangle_P = |e^\tau \xi\rangle_P \tag{2.193}$$

When acting on the tensor product of single-particle states, the boost coproduct (2.43) needs to be taken into account. For a finite transformation, using the commutation relations (2.42), we can prove that

$$e^{i\tau\,\Delta[N]} = e^{i\tau N\otimes \mathbb{1}} e^{i\ln\left(\frac{(1+2P_+)e^\tau}{1+2e^\tau P_+}\right)\otimes N} . \tag{2.194}$$

[4]Although so far we only worked out the quantization of a scalar field, we can already say something about fermionic fields and their deformed Fock space, just by analyzing the general properties of the R-matrix.

For our two particle state (2.182), this yields

$$|\xi\rangle_P \otimes |\eta\rangle_P \to e^{i\tau N}\,|\xi\rangle_P \otimes e^{i\ln\left(\frac{(1+2\xi)e^{\tau}}{1+2\xi e^{\tau}}\right)N}\,|\eta\rangle_P = |e^{\tau}\xi\rangle_P \otimes \left|\frac{(1+2\xi)}{1+2e^{\tau}\xi}e^{\tau}\eta\right\rangle_P . \tag{2.195}$$

The deformed flipped state (2.187) is instead mapped into

$$|\eta+2\eta\xi\rangle_P \otimes \left|\frac{\xi}{1+2\eta+4\eta\xi}\right\rangle_P \to |e^{\tau}\eta(1+2\xi)\rangle_P \otimes \left|\frac{e^{\tau}\xi}{1+2e^{\tau}\eta(1+2\xi)}\right\rangle_P . \tag{2.196}$$

Conversely, by first boosting the two-particle state (2.182) and then flipping it with $\tilde{R}$, the result is

$$\tilde{R}\left(|e^{\tau}\xi\rangle_P \otimes \left|\frac{(1+2\xi)}{1+2e^{\tau}\xi}e^{\tau}\eta\right\rangle_P\right) = |e^{\tau}\eta(1+2\xi)\rangle_P \otimes \left|\frac{e^{\tau}\xi}{1+2e^{\tau}\eta(1+2\xi)}\right\rangle_P , \tag{2.197}$$

which is identical to the right-hand side of (2.196). We have thus proved that

$$\tilde{R}e^{i\tau\Delta[N]} = e^{i\tau\Delta[N]}\tilde{R}\,. \tag{2.198}$$

Basically, our deformed flip operator $\tilde{R}$ commutes with all the Hopf Algebra generators $P_\pm, N$, also given its compatibility with the momenta coproducts shown above. Therefore, relativistic covariance is guaranteed.

2.4 Physical interpretation of deformed multi-particle states

2.4.1 On the indistinguishability of identical particles

In quantum mechanics, two particles of the same species are described by a symmetric or anti-symmetric state [138], defined as

$$\sqrt{2}\left(\frac{1\pm\sigma}{2}\right)|p\rangle\otimes|q\rangle = \frac{|p\rangle\otimes|q\rangle \pm |q\rangle\otimes|p\rangle}{\sqrt{2}} , \tag{2.199}$$

where σ is the standard flip operator and p, q are the linear momenta of the particles. Operationally, the indistinguishability of the two particles may be understood as follows. Suppose we have a calorimeter that can measure the energy of one particle at a time, from which we can deduce the corresponding momentum (we are in 1+1 dimensions). According to state (2.199), the calorimeter can measure either p or q. If our calorimeter measures momentum p, for example, we have no way of knowing if the measured particle is the one in the first or second place of the tensor product. This indistinguishability simply follows from the (anti)-symmetric property of the quantum mechanical state describing the two-particle system. What happens then if the two-particle state is instead defined by the deformed (anti)-symmetrization operators $\mathcal{S}^\pm$ in (2.190)? We reintroduce the dimensional parameter κ, for clarity. Consider a decay process of an initial particle of mass M with momentum $\Pi_\mu = (\Pi, M^2/\Pi)$ (in light-cone coordinates). The particle decays into two identical particles of mass m and momenta $\xi_\mu = (\xi, m^2/\xi)$, $\eta_\mu = (\eta, m^2/\eta)$. We will call ξ the momentum of the particle that enters the coproduct (2.48), in the deformed momentum conservation law, from the left, while η is the label of the momentum on the right-hand side of the coproduct. Notice that this labeling choice has nothing to do with the placement of the particle momenta in the tensor product, and has no physical consequences: we could choose the opposite convention and nothing would change in the calculations. The deformed momentum conservation law dictates:

$$\begin{cases} \Pi = \xi + \eta + 2\xi\,\eta\,, \\ \dfrac{M^2}{\Pi} = \dfrac{m^2}{\xi} + \left(1 - \dfrac{2\,\xi}{\kappa+2\xi}\right)\dfrac{m^2}{\eta}\,, \end{cases} \tag{2.200}$$

the above two equations can be solved with respect to ξ and η, and they have two solutions (recall that all the on-shell momenta, Π, ξ and η are positive-definite):

$$\begin{aligned} \xi &= F_1(\Pi, M, m)\,, \quad \eta = G_1(\Pi, M, m, \kappa)\,, \\ \xi &= F_2(\Pi, M, m)\,, \quad \eta = G_2(\Pi, M, m, \kappa)\,, \end{aligned} \tag{2.201}$$

where

$$\begin{aligned} F_1(\Pi, M, m) &= \frac{\Pi}{2}\left(1+\sqrt{1-\frac{4m^2}{M^2}}\right), \qquad F_2(\Pi, M, m) = \frac{\Pi}{2}\left(1-\sqrt{1-\frac{4m^2}{M^2}}\right), \\ G_1(\Pi, M, m, \kappa) &= \frac{\kappa\Pi\left(M(\kappa+2\Pi)\sqrt{M^2-4m^2}-4m^2\Pi+M^2(\kappa+2\Pi)\right)}{8m^2\Pi^2+2\kappa M^2(\kappa+2\Pi)}\,, \\ G_2(\Pi, M, m, \kappa) &= \frac{\kappa\Pi\left(-M(\kappa+2\Pi)\sqrt{M^2-4m^2}-4m^2\Pi+M^2(\kappa+2\Pi)\right)}{8m^2\Pi^2+2\kappa M^2(\kappa+2\Pi)}\,. \end{aligned} \tag{2.202}$$

If we choose the first solution, the final state will be the following:

$$|\psi_1\rangle = \sqrt{2}\,\mathcal{S}^+\big(\,|F_1\rangle \otimes |G_1\rangle\,\big) = \frac{1}{\sqrt{2}}\Big(\,|F_1\rangle \otimes |G_1\rangle + \tilde{R}\,[|F_1\rangle \otimes |G_1\rangle]\,\Big)\,, \tag{2.203}$$

while if we choose the second:

$$|\psi_2\rangle = \sqrt{2}\,\mathcal{S}^+\big(\,|F_2\rangle \otimes |G_2\rangle\,\big) = \frac{1}{\sqrt{2}}\Big(\,|F_2\rangle \otimes |G_2\rangle + \tilde{R}\,[|F_2\rangle \otimes |G_2\rangle]\,\Big)\,. \tag{2.204}$$

However, as it turns out, the two states are identical. In fact, it is possible to show that

$$\tilde{R}\,[|F_1\rangle \otimes |G_1\rangle] = |F_2\rangle \otimes |G_2\rangle\;, \qquad \tilde{R}\,[|F_2\rangle \otimes |G_2\rangle] = |F_1\rangle \otimes |G_1\rangle\;, \tag{2.205}$$

implying $|\psi_1\rangle = |\psi_2\rangle$, just like in the undeformed theory. This is a nontrivial rigidity of the noncommutative framework, consequence of the Hopf-algebraic constraints that entail its relativistic nature. The compatibility between the deformed momentum composition law and the flip operator is what is behind it. The final state, $|\psi_1\rangle = |\psi_2\rangle$ is proportional the sum of the following two kets (at first order in κ^{-1}):

$$\begin{aligned} |F_1\rangle \otimes |G_1\rangle \qquad &= |F_1\rangle \otimes \left| F_2 - \frac{2m^2\Pi^2}{\kappa\, M^2} + \mathcal{O}(\kappa^{-2}) \right\rangle\,, \\ \tilde{R}\,[|F_1\rangle \otimes |G_1\rangle] \;\; &= |F_2\rangle \otimes \left| F_1 - \frac{2m^2\Pi^2}{\kappa\, M^2} + \mathcal{O}(\kappa^{-2}) \right\rangle\,. \end{aligned} \tag{2.206}$$

The result is consistent with the commutative limit $\kappa \to \infty$. However, when the κ-deformation is switched on, the qualitative features of this multi-particle state are completely different from their undeformed counterpart. If the calorimeter measures the momentum of one of the particles, we can obtain one of the following four results:

$$\begin{aligned} &\text{Unflipped, Left:} && \frac{\Pi}{2}\left(1+\sqrt{1-\frac{4m^2}{M^2}}\right), \\ &\text{Flipped, Left:} && \frac{\Pi}{2}\left(1-\sqrt{1-\frac{4m^2}{M^2}}\right), \\ &\text{Flipped, Right:} && \frac{\Pi}{2}\left(1+\sqrt{1-\frac{4m^2}{M^2}}\right) - \frac{2m^2\Pi^2}{\kappa\, M^2} + \mathcal{O}(\kappa^{-2})\,, \\ &\text{Unflipped, Right:} && \frac{\Pi}{2}\left(1-\sqrt{1-\frac{4m^2}{M^2}}\right) - \frac{2m^2\Pi^2}{\kappa\, M^2} + \mathcal{O}(\kappa^{-2})\,, \end{aligned} \tag{2.207}$$

according to whether we are measuring the left- or right-hand side of the tensor product of the unflipped state, $|F_1\rangle \otimes |G_1\rangle$, or of the flipped state, $\tilde{R}\,[|F_1\rangle \otimes |G_1\rangle]$. In the noncommutative theory, the four momenta (2.207) are all different. Measuring the momentum of one particle allows us to identify which side of the tensor product it came from, and from which state (flipped or unflipped). Therefore, there is a sense in which the indistinguishability of identical particles is lost when the κ-deformation is taken into account. There is no avoiding this if we want to construct a relativistic theory. As already stressed in section 2.3.3, a state of the type (2.199) would not be covariant under the κ-Poincaré transformations, which exhibit all their non-trivial behaviour on multi-particle states, given that the coproduct is involved.

2.4.2 On the Pauli exclusion principle

We now explore the consequences of the deformed permutation symmetry on the Pauli exclusion principle. In standard quantum theory, a state describing two fermions with the same quantum numbers is annihilated by the undeformed anti-symmetrizer:

$$\left(\frac{1-\sigma}{2}\right)|\xi\rangle \otimes |\xi\rangle = 0\,. \tag{2.208}$$

This is the essence of the Pauli exclusion principle, which has been confirmed in a variety of experiments searching for classically prohibited transitions to states of the form $|\xi\rangle \otimes |\xi\rangle$. It is then natural to ask what is the fate of the Pauli exclusion principle in our κ-deformed framework. Assuming that 2-fermion states are left invariant by $\mathcal{S}^-$, what is the class of states annihilated by this operator? We require that:

$$\mathcal{S}^-(|\xi\rangle_P \otimes |\eta\rangle_P) = 0\,, \tag{2.209}$$

which uniquely selects the two particle states

$$|\xi\rangle_P \otimes |-S(\xi)\rangle_P = |\xi\rangle_P \otimes |\xi/(1+2\xi)\rangle_P\;. \tag{2.210}$$

The same holds for $|\eta/(1-2\eta)\rangle \otimes |\eta\rangle$, which is the same state as (2.210), just parametrized with respect to the momentum of the particle on the right hand side of the tensor product. In the undeformed case, the solution $\xi = \eta$ would have been selected, in agreement with (2.208). In light of this reasoning, we can visualize the Pauli principle in a simple manner. In the (ξ, η) plane, which contains admissible pairs of momenta that can be attributed to fermions, the Pauli principle excludes a one-dimensional subset of the (ξ, η) plane: the pairs lying on a curve $\eta = f(\xi)$. In the commutative case, $f(\xi) = \xi$ (the bisector of the plane), while the κ-deformed version is $f(\xi) = -S(\xi)$ (see fig. 2.2) .

Notice that our deformed identical-particles states are Lorentz-covariant,

$$\Delta(e^{i\tau N})\,|\xi\rangle_P \otimes |-S(\xi)\rangle_P = |e^{\tau}\xi\rangle_P \otimes |-S(e^{\tau}\xi)\rangle_P\;, \tag{2.211}$$

in the sense that identical-particles states are sent to boosted identical-particles states by a Lorentz transformation. This is due to the fact that the curve $\eta = -S(\xi)$ lays on an orbit of the Lorentz group.

In light of the previous observations, we notice that the state $|\xi\rangle_P \otimes |\xi\rangle_P$ is not annihilated by $\mathcal{S}^-$, contrary to the commutative case. Notice, however, that this form of the state is not preserved by Lorentz transformations: if an observer attributes the same momentum ξ to two particles, by the action of the finite boost generator (2.194) on the state $|\xi\rangle_P \otimes |\xi\rangle_P$, a boosted observer would attribute different momenta to them:

$$|\xi\rangle_P \otimes |\xi\rangle_P \to |e^{\tau}\xi\rangle_P \otimes \left|\frac{e^{\tau}\xi(1+2\xi)}{1+2e^{\tau}\xi}\right\rangle_P\,, \tag{2.212}$$

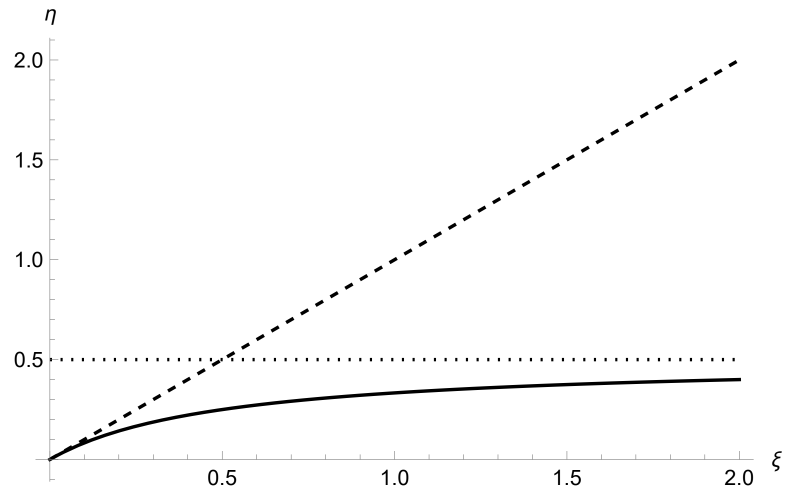

Figure 2.2: The +-momentum space of momentum-pairs for two-particle states, in units of κ. The dashed line represents the pairs excluded by the undeformed Pauli exclusion principle. The thick curve represents the pairs excluded by the deformed version of the exclusion principle when noncommutativity is taken into account. For large ξ, the curve saturates at $1/2\kappa$, which is the dotted asymptote in the plot.

with τ being the boost parameter.

The discussion above highlights the fact that, in our model, states of the form $|\xi\rangle_P \otimes |\xi\rangle_P$ are not excluded by the antisymmetrization operator, because they are not the true "identical particle" states of the theory. This could potentially lead to new physical phenomena, which could be interpreted as departures from the Pauli Exclusion Principle (PEP). Notice that the detection of a state of the form $|\xi\rangle_P \otimes |\xi\rangle_P$ would not necessarily imply that the PEP, and all of its physical consequences, is violated in our theory: for example, it might well be the case that the theory keeps forbidding more than two electrons to share the same atomic orbital, in view of the aforementioned existence of classes of states that are excluded by the antisymmetrization operator. At any rate, considering that there now are stringent bounds on PEP violations [139–141], it would be interesting to study the basic physical processes that are tested by these experiments, within the context of our theory, and investigate possible observable consequences of noncommutativity. Notice that such modelization would require significant further development of the theory (at the very least, interacting quantum field theorys with Dirac fields). Results obtained through rudimentary/simplistic methodologies, especially if they compromise Poincaré invariance, fall short of the necessary rigor and hold no significance within the context of our theoretical framework.

Discussion and outlook

Building on the results of [120] and [121], we were able to define a quantum field theory on the κ-Minkowski noncommutative spacetime in the same spirit as [36, 122]: the coordinates of N different points cannot belong to the simple tensor product algebra, otherwise it would not be κ-Poincaré covariant. One needs to introduce a *braiding,* which requires a quantum R matrix for the κ-Poincaré group. This exists only in the lightlike case, *i.e.* when the commutators between the coordinates (2.1) are described by a vector v^μ that is lightlike, or null, with respect to the metric $g_{\mu\nu}$ that is preserved by the κ-Poincaré group (2.2). Within this framework, one can define consistently covariant N-point functions, which are the backbone of quantum field theory. The striking advantage of the approach of [36, 122], which is shared by our model, as proven in [120], is that the translation-invariant combinations of different coordinates (*i.e.* the coordinate differences) are commutative, which implies that the N-point functions are all commutative. This hugely simplifies the physical interpretation of the theory, as we do not have to deal with noncommutative correlation functions, whose meaning would be rather obscure. A similar conceptual simplification is achieved in many other approaches to noncommutative quantum field theory by using a star product, and defining a path integral over commutative functions in which the action is turned into a nonlocal, infinite-derivative functional of the fields. Then, the correlation functions are commutative objects, simply obtained as expectation values or functional variations of the partition function. However, there is no sense in which these commutative N-point functions can be invariant under the quantum group of isometries of the noncommutative spacetime they are supposed to live in. In our approach, we have a way of writing commutative N-point functions which are κ-Poincaré invariant, and we believe that this is a key advantage of the approach based on braiding. As shown in [36, 122] in the case of θ-Moyal noncommutative spacetimes, with the braided structures one can define a quantum field theory built upon the Wightman axioms, and the quantization of a free complex scalar field can be performed with the introduction of a covariant Pauli–Jordan function. In the case of θ-Moyal spacetimes, the quantum field theorys thus defined turned out to be completely indistinguishable from their commutative counterparts, as all the N-point functions of the free theory, as well as the perturbative expansion of the N-point functions of an interacting theory, turn out to be undeformed. In our lightlike κ-Minkowski case, we find that, although the Pauli-Jordan and two-point functions are undeformed, a dependence on the deformation parameter appears at the level of multiparticle states already in the free theory. The momentum, boost and charge conjugation operators are undeformed, however the creation and annihilation operators can be written, in a key advancement obtained in this research area for the first time, as an infinite nonlinear combination of undeformed creation and annihilation operators. The deformed creation operators act in a trivial way on the vacuum, and the one-particle sector looks undeformed. However, as soon as we create more than one particle we start seeing a dependence on the noncommutativity parameter: the momentum of two particles is a nonlinear combination of the two single-particle momenta. The way that two particle momenta boost under Lorentz transformations is nonlinear and mixes the momenta of the two particles. We can introduce a covariant and involutive flip operator, which acts nonlinearly on the momenta of the two particles, changing them in a more complicated way than simply exchanging them. This flip is used to define two covariant and idempotent symmetrization and antisimmetrization operators, whose image is the Fock space of bosonic and, respectively, fermionic fields. The situation is substantially simpler compared to the attempts at defining a quantum field theory on the *timelike* κ-Minkowski spacetime: in this case, the absence of a quantum R matrix makes it impossible to define a flip operator that is both involutive and Lorentz-covariant [85, 100, 103, 124, 125], which implies that the notion of identical particles and (anti-)symmetrized multiparticle states loses meaning [103]. We proved that our theory is C-, PT- and CPT-invariant, however P and T symmetries do not hold separately. This can already be seen at the level of the coordinate commutation relations, which break P and T symmetry. The theory allows for the existence of states which are excluded by the Pauli principle in the classical setting. This opens up the interesting phenomenological opportunity of setting bounds on the model by experimental results searching for evidence of transitions into such states.

The noncommutative quantum field theory defined in [120, 121] and completed here seems in healthy shape, and motivates interest in several future research directions. The simplest one is to write the N-point functions of the free theory for N larger than two, to check whether they are undeformed too, or perhaps the nontriviality of the multiparticle sector manifests into a dependence of higher correlators on the noncommutativity parameter. A further issue to consider is that the model studied so far is in 1+1 spacetime dimensions, and its generalization to 3+1 dimensions seems straightforward, but it has not been worked out explicitly and might yet hide some surprises. The next natural step is to introduce an interaction, which is where the theory has the highest chances of providing some predictions that depart from standard quantum field theory on commutative Minkowski space. Further down the road, gauge theories and fermions might be explored, and perhaps a possible connection with CP violation in the Standard Model. Finally, it would be interesting to develop the theory in terms of the approach developed in [41, 142–145], which works for any quantum group deformation of the Poincaré group that can be expressed as a twist, and compare the results with the ones reported in this chapter.

Chapter 3

Quantum Euler Angles and agency-dependent spacetime

In chapter 1, we explored the mathematical frameworks of quantum groups and Hopf algebras, suitable for investigating deformation of relativistic symmetries. We have only dealt with the properties of the relevant mathematical structures, but have yet not discussed the implications on the nature of observers and reference frames. In this chapter we set the stage for exploring how quantum gravity induced deformations of classical symmetries could introduce some quantum aspects in the description of observers and reference frames.

Observers play a somewhat "external" role in both general relativity and quantum theory. In the former, they are assumed to exert a negligible backreaction on spacetime, which they can thus probe without influencing it. In quantum theory, it is the Heisenberg cut [146, 147] that separates them from the observed system. The observers' knowledge and choices are described classically, however, there are mutually-incompatible measurements that they can choose to perform (complementarity). Quantum mechanics introduces therefore an important novelty: *the operationally-meaningful properties of the observed systems depend on the choices made by the observer.* Through the impact of these choices, observers are "agents" for what concerns measurements in quantum mechanics.

When quantum theory and general relativity are combined in a quantum theory of gravity, one ends up considering spacetime as a quantum object. We may then contemplate the possibility that the agency-dependence of quantum mechanics might challenge the notion of an objective spacetime that all observers agree upon: how spacetime reveals itself to observers may depend on some of their choices. Conversely, the quantum properties of spacetime will likely affect the spectrum of possible operations available to an agent. These observations suggest a picture in which spacetime and the agency of observers affect each other inextricably, so much so that the "externality" idealization, a good working hypothesis in general relativity and quantum theory, will have to be abandoned in favor of a notion of "internal" observers in quantum gravity [148].

Consider the example of two purely misaligned reference systems anchored to two observers, Alice and Bob. In standard quantum theory, Alice and Bob can align their reference frames sharply, by exchanging an infinite number of physical systems, such as qubits (see fig. 3.1). While the choices of the two observers affect the single quantum mechanical measurement due to quantum complementarity, as they exchange an infinite number of physical systems, they will recover an infinitely precise description of the rotation matrix that maps one reference frame into the other. Suppose now that standard rotational invariance, implemented as invariance under the $SU(2)$ group, is replaced by invariance under the $SU_q(2)$ quantum group,

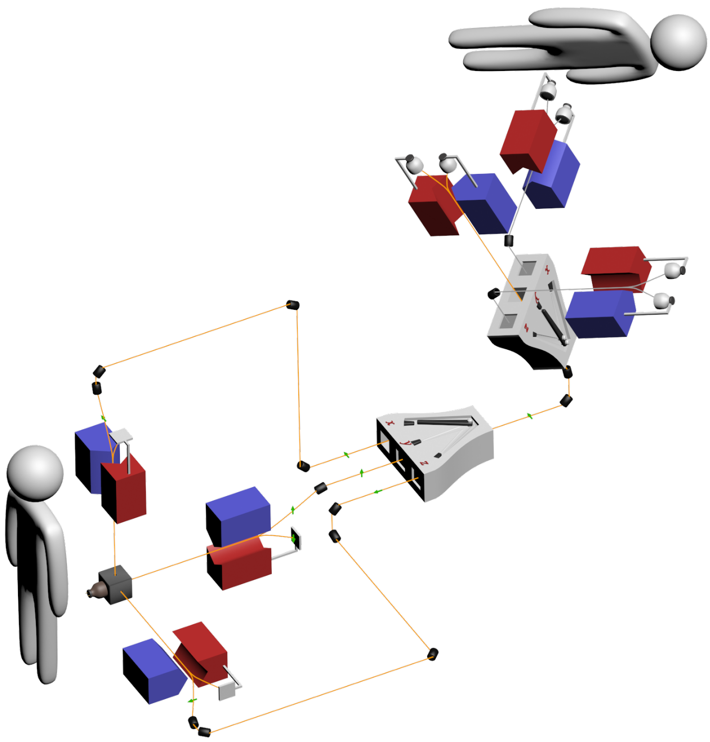

Figure 3.1: Alice (on the left) prepares a set of N qubits (e.g., electron spins) in the spin-up eigenstate of her x axis (e.g., by passing unpolarized electrons through a x-oriented Stern–Gerlach apparatus, and selecting only the ones that emerge with their spins up). She then sends these electrons to Bob, whose laboratory is rotated by an unknown amount with respect to hers. Bob divides these N qubits in three groups, and sends each group through a machine that measures the spin along one of his three orthogonal axes (e.g., three perpendicular Stern–Gerlach apparata - in the picture, he is passing the electrons through a y-oriented machine). He then counts the number of spin-up and spin-down measurements that each machine reads, and calculates the expectation value of the corresponding observable. Repeating the experiment for a set of N qubits that Alice selected to be polarized along the $y-$, and, respectively, $z-$axes allows Bob to build a statistics of the expectation values of the nine observables associated to each pair of choices of axes made by him and Alice. In the large-N limit, these expectation values tend to the nine components of the rotation matrix R that connects Alice's reference frame and Bob's. Notice that, in this illustrative picture the electron beams (in orange) are manipulated with some "mirrors" (the black cylinders) which are assumed not to have any effect on the qubit states.

where group parameters are promoted to quantum operators and q is a dimensionless parameter characterizing the deformation. Given that they satisfy non-trivial commutation relations, quantum complementarity suggests that there will be an intrinsic limit on how precise we are able to determine the (possibly deformed) rotation matrix that connects the two observers. In an alignment procedure as the one described in fig. 3.1, we might then expect that even after the exchange of an infinite number of qubits, there will be a non-zero, unavoidable intrinsic uncertainty characterizing the transformation connecting the two reference frames. Moreover, the magnitude and type of uncertainty will depend on the choices of the observers, as suggested by quantum complementarity. In this sense, the transition from classical to quantum symmetries (motivated by quantum gravity) promotes observers to agents whose choices reflect the structure of the spacetime they observe.

In exploring this uncharted territory, $SU_q(2)$ is one of the easiest quantum groups to handle, given the vast mathematical literature behind it and its relationship with the familiar rotation group. Moreover, this quantum group is largely studied also in some formal approaches to quantum gravity. In loop quantum gravity with a cosmological constant, the local group of dreibeins is $SU_q(2)$, where q depends on a dimensionless ratio between the Planck length and the Hubble length associated to the cosmological constant [149]. In this context, it has been argued that the combined effect of introducing a minimal length (the Planck length) and a maximal radius (the Hubble length) results in a minimal possible resolution to angular measurements [150]. We now move on to recall some technical features of $SU_q(2)$.

3.1 Quantum rotation matrices in $SU_q(2)$

3.1.1 $SU_q(2)$ and its homomorphism with $SO_q(3)$

Before diving into its deformed counterpart, let us recall some basic properties of the $SU(2)$ group, in order to set the notation. $SU(2)$ is the group of unitary matrices of dimension 2 with unit determinant. In the defining representation, a generic $U \in SU(2)$ can be written as

$$U = \begin{pmatrix} a & -c^* \\ c & a^* \end{pmatrix} , \tag{3.1}$$

where a and c are two complex numbers satisfying the unitarity condition $aa^* + cc^* = 1$. An $SU(2)$ matrix is thus specified with three real numbers, which may be organized into three angular degrees of freedom, as follows:

$$a = e^{i\eta} \cos\frac{\theta}{2} \qquad c = e^{i\delta} \sin\frac{\theta}{2} , \tag{3.2}$$

with $\eta, \delta \in [0, 2\pi)$ and $\theta \in [0, \pi)$. $SU(2)$ is the double cover of the 3D space rotation group $SO(3)$. Given $U \in SU(2)$, the corresponding $SO(3)$ element can be computed via the canonical homomorphism:

$$R_{ij} = \frac{1}{2} \operatorname{Tr}\left\{U \sigma_j U^\dagger \sigma_i\right\} , \tag{3.3}$$

where σ^i are the Pauli matrices. In terms of variables a,c this 3×3 matrix can be written as

$$R = \begin{pmatrix} \frac{1}{2}(a^2 - c^2 + (a^*)^2 - (c^*)^2) & \frac{i}{2}(-a^2 + c^2 + (a^*)^2 - (c^*)^2) & (a^*c + c^*a) \\ \frac{i}{2}(a^2 + c^2 - (a^*)^2 - (c^*)^2) & \frac{1}{2}(a^2 + c^2 + (a^*)^2 + (c^*)^2) & -i(a^*c - c^*a) \\ -(ac + c^*a^*) & i(ac - c^*a^*) & 1 - 2cc^* \end{pmatrix} . \tag{3.4}$$

The double covering nature of $SU(2)$ is explicitly evident in the above, since the map $a \to -a$ and $c \to -c$ leaves the rotation matrix invariant. Using parametrization (3.2), the rotation matrix (3.4) can also be rewritten in terms of angular variables in a more "recognizable" fashion. Indeed, the angles of (3.2) are a just a linear redefinition of the well-known Euler angles:

$$\theta = \beta \qquad \eta = \frac{\alpha + \gamma}{2} \qquad \delta = \frac{\pi}{2} - \frac{\alpha - \gamma}{2} , \tag{3.5}$$

in terms of which a rotation matrix is written as $R(\alpha, \beta, \gamma) = R_z(\alpha) R_x(\beta) R_z(\gamma)$, where R_x,R_z are rotations around the x and z axis, respectively.

The quantum group $SU_q(2)$ (also denoted by $\mathbb{C}_q(SU(2))$) is defined as the algebra generated by operators a, c, subject to the following commutation relations

$$\begin{aligned} &ac = qca \qquad ac^* = qc^*a \qquad cc^* = c^*c \\ &c^*c + a^*a = \mathbb{1} \qquad aa^* - a^*a = (1-q^2)c^*c\,. \end{aligned} \tag{3.6}$$

Here, $\mathbb{1}$ refers to the identity element of the algebra and q is the deformation parameter assumed to be close to 1 and in particular $q \lesssim 1$. Indeed, in the $q \to 1$ limit, we obtain the commutative limit and recover the classical description of $SU(2)$. In passing, we mention that it suffices to consider $q \lesssim 1$ since, if $q > 1$, the mapping $a \mapsto a^*$, $c \mapsto qc^*$ sends the $SU_q(2)$ algebra to the $SU_{q^{-1}}(2)$ one. The Hopf Algebra description can be completed in terms of the following maps

$$\begin{cases} \Delta(a) = a\otimes a - qc^*\otimes c \\ \Delta(a^*) = a^*\otimes a^* - qc\otimes c^* \\ \Delta(c) = c\otimes a + a^*\otimes c \\ \Delta(c^*) = c^*\otimes a^* + a\otimes c^* \end{cases}, \qquad \begin{cases} S(a) = a^* \\ S(a^*) = a \\ S(c) = -q^{-1}c \\ S(c^*) = -qc^* \end{cases}, \qquad \begin{cases} \epsilon(a) = 1 \\ \epsilon(a^*) = 1 \\ \epsilon(c) = 0 \\ \epsilon(c^*) = 0 \end{cases}. \tag{3.7}$$

To establish a first link with the classical picture, we present the generalization of the 1/2-spin representation (3.1), given by [151]

$$\begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} \qquad a, c \in C(SU_q(2)) \quad q \in (0,1)\,. \tag{3.8}$$

In order to describe a deformation of 3D rotation matrices, we may construct an analogous of the $SU(2)$ to $SO(3)$ homomorphism, by promoting (3.3) to its quantum counterpart, by simply replacing U with U_q:

$$(R_q)_{ij} = \frac{1}{2}\,\mathrm{Tr}\left\{\sigma_j\, U_q^\dagger\, \sigma_i\, U_q\right\}. \tag{3.9}$$

Computing the quadratic elements explicitly, we obtain:

$$R_q = \begin{pmatrix} \frac{1}{2}(a^2 - qc^2 + (a^*)^2 - q(c^*)^2) & \frac{i}{2}(-a^2 + qc^2 + (a^*)^2 - q(c^*)^2) & \frac{1}{2}(1+q^2)(a^*c + c^*a) \\ \frac{i}{2}(a^2 + qc^2 - (a^*)^2 - q(c^*)^2) & \frac{1}{2}(a^2 + qc^2 + (a^*)^2 + q(c^*)^2) & -\frac{i}{2}(1+q^2)(a^*c - c^*a) \\ -(ac + c^*a^*) & i(ac - c^*a^*) & 1-(1+q^2)cc^* \end{pmatrix}, \tag{3.10}$$

and one can check that this reduces to the standard rotation matrix in the limit $q \to 1$. It is worth noticing that in (3.9), the cyclic property of the trace does not hold, leading to a "quantization ambiguity" in writing the rotation matrix as $\frac{1}{2}\,\mathrm{Tr}\left\{\sigma_j\, U_q^\dagger\, \sigma_i\, U_q\right\}$ or $\frac{1}{2}\,\mathrm{Tr}\left\{\sigma_i\, U_q\, \sigma_j\, U_q^\dagger\right\}$. However, it can be shown that the two deformation proposals are linked by a similarity transformation which does not affect the qualitative nature of our results. As such, we will not dwell with this issue further and refer to [31] for an in-depth discussion.

The classical analog of (3.10) describes all possible elements of $SO(3)$ when varying the complex numbers a and c with continuity. Each of these classical matrices describes a possible relative orientation between observers A and B. In the quantum case (3.10) has no physical meaning when taken alone, but only when paired with a certain state $|\psi\rangle \in \mathcal{H}$, where $\mathcal{H}$ is the Hilbert space on which a and c act. In the classical case, a and c codify information about the alignment of two reference frames, via their angular parametrization. For this reason, when being promoted to operators, we interpret the states on which they act as the ones codifying the relative orientation between the reference frames of observers A and B. The physical information about this orientation can be extracted by computing the expectation values $\langle\psi|(R_q)_{ij}|\psi\rangle$ with their relative uncertainties stemming from the noncommutative nature of these matrix elements, given by

$$\Delta_{ij} = \sqrt{\langle\psi|(R_q)^2_{ij}|\psi\rangle - \langle\psi|(R_q)_{ij}|\psi\rangle^2}\,. \tag{3.11}$$

In this framework, the Δ_{ij} do not vanish simultaneously, in general, for a given state $|\psi\rangle$ introducing a "fuzziness" in the alignment procedure. When this occurs, the latter is affected by an intrinsic uncertainty which cannot be eliminated: A and B are not able to sharply align their reference frames anymore. In light of these arguments, it is crucial to identify $\mathcal{H}$ and study the representations of operators a and c on it.

3.1.2 $SU_q(2)$ representations and quantum Euler angles

The representations of the algebra (3.6) have been thoroughly studied in [152]. The Hilbert space containing the two unique irreducible representations of the $SU_q(2)$ algebra, $q \in (0,1)$, is $\mathcal{H} = \mathcal{H}_\pi \oplus \mathcal{H}_\rho$ where $\mathcal{H}_\pi = \ell^2 \otimes L^2(S^1) \otimes L^2(S^1)$ and $\mathcal{H}_\rho = L^2(S^1)$. If $\chi, \phi \in [0, 2\pi[$ are coordinates on S^1 and $|n\rangle$ is the canonical basis of ℓ^2, the algebra of functions on $SU_q(2)$ is represented as

$$\rho(a)\,|\chi\rangle = e^{i\chi}\,|\chi\rangle \qquad \rho(a^*)\,|\chi\rangle = e^{-i\chi}\,|\chi\rangle \qquad \rho(c)\,|\chi\rangle = \rho(c^*)\,|\chi\rangle = 0 \tag{3.12}$$

$$\begin{aligned} \pi(a)\,|n,\phi,\chi\rangle &= e^{i\chi}\sqrt{1-q^{2n}}\,|n-1,\phi,\chi\rangle \qquad \pi(c)\,|n,\phi,\chi\rangle = e^{i\phi}q^n\,|n,\phi,\chi\rangle \\ \pi(a^*)\,|n,\phi,\chi\rangle &= e^{-i\chi}\sqrt{1-q^{2n+2}}\,|n+1,\phi,\chi\rangle \qquad \pi(c^*)\,|n,\phi,\chi\rangle = e^{-i\phi}q^n\,|n,\phi,\chi\rangle\,. \end{aligned} \tag{3.13}$$

The fact that the quantum number χ appears in both representations ρ and π is by no means incidental. In [31], it is shown that representation ρ can be obtained as a limit of representation π when acting on a certain class of states, in agreement with the fact that in the classical limit the most general rotations are specified by three independent parameters. According to (3.12), only the $SO(1)$ subgroup is spanned by the ρ representation. Therefore, we can formally work exclusively with representation π, but practically it is easier to work with states of representation ρ for a specific class of rotations.

As is done in the classical case, we would like to give physical meaning to the quantum parameters describing the $SU_q(2)$ group. A way to do this is by recalling expression (3.2) and inspecting (3.13). We notice that the phases χ, ϕ are just continuous parameters, so we can make the identification $\chi \equiv \eta$, $\phi \equiv \delta$. Then, exploiting the fact that c is a diagonal operator, we are led to a significant result

$$q^n = \sin\left(\frac{\theta(n)}{2}\right) \iff \theta(n) = 2\arcsin(q^n)\,. \tag{3.14}$$

Namely, the Euler angle θ becomes quantized. At this point, a clarification is in order. The definition of $\theta(n)$ via (3.14) is consistent with the definition of $cos\left(\frac{\theta(n)}{2}\right)$ in terms of $\pi(a^*a)$, which also acts diagonally. Indeed, on a generic basis state $|n,\phi,\chi\rangle$, we have that

$$\pi(a^*a)\,|n,\phi,\chi\rangle = (1-q^{2n})\,|n,\phi,\chi\rangle = \cos^2\left(\frac{\theta(n)}{2}\right)|n,\phi,\chi\rangle\,. \tag{3.15}$$

Furthermore, it is consistent with the commutation relations in (3.6), like $a^*a + c^*c = 1$, for example. Nevertheless, one could also define the $\theta(n)$ angle starting from $\pi(aa^*)$, which acts diagonally but is different from $\pi(a^*a)$, since a and a^* do not commute. As can be easily checked, with this definition we would have the same discrete values of $\theta(n)$ as in the previous definition, with the exception of $\theta = \pi$. Indeed, for this other ordering, the first equation in (3.14) would be written as $q^{n+1} = sin\left(\frac{\theta(n)}{2}\right)$. It will be evident in what follows that this angle redefinition does not affect the nature of the qualitative results of our analysis. The quantization of the θ angle is captured in fig. 3.2, where its qualitative features are discussed.

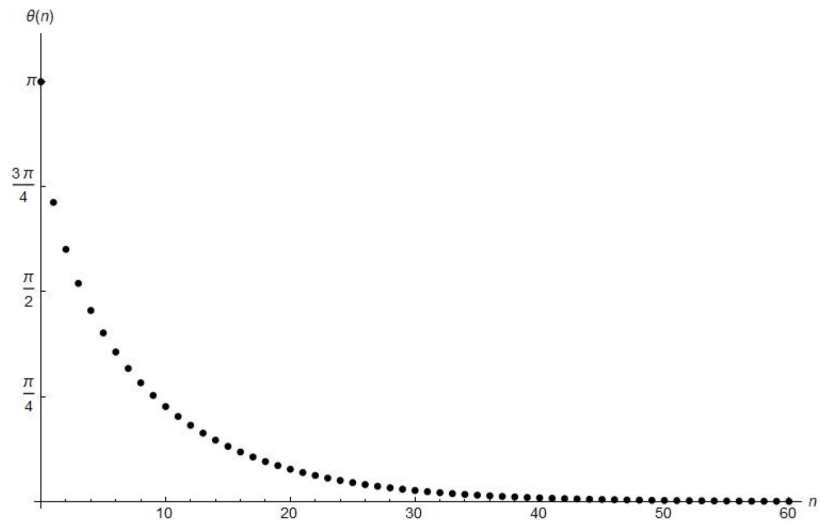

Figure 3.2: Discretized angles (3.14) computed with $q = 0.9$. $\theta(n)$ is decreasing with n starting from π and approaching 0 for large values of n. Interestingly, the step between consecutive angles decreases as n gets very larger, and for very large n we can approximate the angular distribution as being continuous. This can easily be verified analytically by computing $\Delta\theta(n) = \theta(n+1) - \theta(n)$ and taking the limit for $n \to \infty$. Another feature that gives robustness to our proposal is that the angular steps get smaller as q is closer to 1.

3.1.3 Semi-classical rotations

To gain further intuition from the classical picture, where the rotation axis and the angle by which the rotation is performed are specified by three Euler angles, we henceforth focus on "semi-classical" rotations, specified by the three quantum numbers $\theta(n), \phi, \chi$, that describe small deformations of classical rotations defined by these angles. More precisely, the states $|\psi(\theta, \phi, \chi)\rangle$ yielding such deformed rotations should satisfy

$$\begin{cases} \langle\psi(\theta,\phi,\chi)|(R_q)_{ij}|\psi(\theta,\phi,\chi)\rangle = R_{ij}(\theta,\phi,\chi) + O(1-q) \\ \Delta^2_{ij} = O(1-q) \end{cases} \quad \forall\, i, j\ , \tag{3.16}$$

where $R_{ij} \in \mathrm{SO}(3)$ and where θ is one of the allowed values in (3.14). This prescription constrains the quantum rotations we can construct to the ones in which the θ Euler angle can only take the allowed values in (3.14), for a fixed q. Namely, we focus only on those states which reproduce $SO(3)$ matrices in the limit $q \to 1$, which is not the case for all states of the Hilbert space. In order to clarify the meaning of this prescription, we provide some examples.

A first class of states that trivially satisfies requirement (3.16) is the set of eigenstates $|\chi\rangle \in \mathcal{H}_\rho$. It is easy to see that the expectation value of the quantum rotation matrix (3.10) in such states describes a rotation of angle 2χ around the z-axis, namely:

$$\langle\chi|R_q|\chi\rangle = \begin{pmatrix} \cos(2\chi) & -\sin(2\chi) & 0 \\ \sin(2\chi) & \cos(2\chi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.17}$$

with all the Δ_{ij} being zero. Thus, rotations around the z-axis are classical: two observers, A and B, whose relative orientation is described by $|\chi\rangle \in \mathcal{H}_\rho$, can align themselves sharply.

The relative simplicity of rotations around the z-axis is not representative of the richness of structure of other rotations, and this is mainly due to the fact that for generic basis states of the form $|n,\phi,\chi\rangle$ the first condition in (3.16) is not satisfied, forcing us to consider superpositions of such states. Indeed, consider the basis state $|n,\phi,\chi\rangle$: the expectation value of the quantum rotation matrix on it reads

$$\langle n,\phi,\chi|\, R_q\, |n,\phi,\chi\rangle = \begin{pmatrix} -q^{2n+1}\cos(2\phi) & -q^{2n+1}\sin(2\phi) & 0 \\ -q^{2n+1}\sin(2\phi) & q^{2n+1}\cos(2\phi) & 0 \\ 0 & 0 & 1-q^{2n}(1+q^2) \end{pmatrix} . \tag{3.18}$$

As is clear, there is no possibility of ever obtaining non-zero elements for the $\{13, 23, 31, 32\}$ entries of the rotation matrix and the dependence on χ is never accounted for. In general, such a state does not reproduce a rotation of angles $\{\theta(n), \phi, \chi\}$ in the commutative limit and thus does not satisfy (3.16). Since single basis states do not work, we are led to consider superpositions of the form

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n\, |n,\phi,\chi\rangle \quad , \quad \sum_{n=0}^{\infty} |c_n|^2 = 1 \ , \tag{3.19}$$

since ϕ and χ are identified with their classical counterparts. Among the states of this form that satisfy (3.16), we need to find those for which the uncertainties are kept under control. The best way to do this is to demand that the coefficients $\{c_n\}$ minimize the functional

$$S\Big[\{c_n\}_{n=0}^{\infty}, \mu\Big] = \sum_{i,j} \Delta_{ij}^2 - \mu\big(\langle\psi|\psi\rangle - 1\big) , \tag{3.20}$$

where μ is a Lagrange multiplier enforcing normalization of $|\psi\rangle$. In general, solving the minimization problem (3.20) is a daunting task computationally. Therefore, we invoke physical intuition to construct these states. Fixing a value for q, we build superpositions of states $|n,\phi,\chi\rangle$ centered around a certain $\bar{n} \in \mathbb{N}$. We compute the expectation values and uncertainties relative to such states numerically and check that they reproduce deformations of classical rotation matrices specified by angles $(\theta(\bar{n}), \phi, \chi)$, in the sense of (3.16). This will be described in detail in the next subsection.

It is worth noticing that there is a special class of such semi-classical rotations for which some simplifications arise. This is the case of rotations with $\theta = \pi$ around axes of the $x-y$ plane. The classical theory predicts that $\chi = 0$ and a generic ϕ select a direction in the $x-y$ plane ($\phi = \pi/2$ selects a rotation around the x-axis, while $\phi = 0$ selects one around the y-axis) around which we rotate of an angle θ. As emphasized before, since ϕ, χ behave classically, the quantum numbers associated to these angles specify the axis of rotation also in the quantum case. For the case $\theta(0) = \pi$, building a superposition of basis states is not necessary to satisfy (3.16) and the basis state $|0,\phi,0\rangle$ is sufficient to describe the corresponding semi-classical rotation around an axis in the $x - y$ plane, specified by ϕ. The expectation value of R_q on such state is

$$\langle 0,\phi,0|R_q|0,\phi,0\rangle = \begin{pmatrix} -q\cos(2\phi) & -q\sin(2\phi) & 0 \\ -q\sin(2\phi) & q\cos(2\phi) & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{3.21}$$

while the Δ_{ij} are non-zero and do not depend on the specific value of ϕ:

$$\Delta R_q = \begin{pmatrix} \frac{1}{2}\sqrt{(1-q^2)(1-q^4)} & \frac{1}{2}\sqrt{(1-q^2)(1-q^4)} & \frac{(1+q^2)}{2}\sqrt{(1-q^2)} \\ \frac{1}{2}\sqrt{(1-q^2)(1-q^4)} & \frac{1}{2}\sqrt{(1-q^2)(1-q^4)} & \frac{(1+q^2)}{2}\sqrt{(1-q^2)} \\ \sqrt{q^2-q^4} & \sqrt{q^2-q^4} & 0 \end{pmatrix}. \tag{3.22}$$

From the above matrices, it can be verified analytically that the state $|0,\phi,0\rangle$ satisfies (3.16).

For a generic axis of rotation, the quantum Euler angle θ will play a non-trivial role both in the determination of the axis itself and in the determination of the rotation angle.

3.1.4 Numerical construction of semi-classical states

As anticipated in the previous subsection, we resort to numerical computations to construct some of the states describing semi-classical rotations. We focus on the case $q = 0.99$ as an illustrative example. However, by increasing the value of q closer to 1, it is possible to check that the states presented below satisfy (3.16) with increasing precision.

We will focus on rotations in the (y, z)-plane, thus setting $\chi = 0$, $\phi = \frac{\pi}{2}$. We want to construct states that semi-classically describe a deformed rotation of a certain angle in this plane. Since n defines the only angle left, θ, we choose to build superpositions (3.19) centered on particular values of n, dubbed $\bar{n}$, with coefficients c_n multiplying the states $|n,\phi,\chi\rangle$ rapidly decreasing as $\theta(n)$ deviates from $\theta(\bar{n})$. Our ansatz is that these coefficients have the form of a discretized Gaussian distribution.

The variance is then chosen in the following way. Recalling that

$$\theta(n) = 2\arcsin q^n \tag{3.23}$$

we can define $\Delta\bar{n}$ as

$$\Delta\bar{n} := \left.\frac{dn}{d\theta}\right|_{\theta(\bar{n})} \Delta\theta\,. \tag{3.24}$$

We take $\Delta\theta = \frac{\pi}{2} - \arcsin q$, which is just half of the value of the maximum angular deviation and weigh it with the rate of change of n with respect to θ, approximated as the derivative. Therefore, the value of the variance depends on the central value $\bar{n}$. This approximation becomes more and more accurate with increasing values of n for which $\theta(n)$ becomes quasi-continuous.

We then define our superposition coefficients c_n as

$$c_n = \frac{e^{-\frac{(\bar{n}-n)^2}{2\Delta\bar{n}^2}}}{\sum_{n=0}^{\infty} e^{-\frac{(\bar{n}-n)^2}{2\Delta\bar{n}^2}}}\,. \tag{3.25}$$

For computational reasons, we do not consider the full superposition going from $n = 0$ to $n = \infty$ but we truncate the series by considering the 3-σ range of our Gaussian. Namely, the sum goes from $n_{\min}$ to $n_{\max}$, where

$$n_{\min} - \bar{n} = -3\Delta\bar{n} \qquad n_{\max} - \bar{n} = 3\Delta\bar{n}\,. \tag{3.26}$$

For relatively small values of $\bar{n}$, the value for n_{min} might be negative when considering the 3-σ range. When this happens, we simply truncate the Gaussian and start the series from $n = 0$. We have explicitly verified that this doesn't affect our results in a significant way.

We have used this algorithm to construct the states used for the numerical analysis of table 3.1. In what follows, we show the results for the computation of the expectation values of the matrix elements $(R_q)_{ij}$ with their relative uncertainties for states centered around $n = 95, 34, 7, 0$, corresponding to angles $\theta = 45.275°, 90.560°, 137.518°, \theta = 180°$.

- $n = 95,\ \theta = 45.275°$

$$\langle\psi(45.275°)|R_q|\psi(45.275°)\rangle = \begin{pmatrix} 0.997 & 0.000 & 0.000 \\ 0.000 & 0.695 & 0.708 \\ 0.000 & -0.708 & 0.698 \end{pmatrix}$$
$$\Delta R_q = \begin{pmatrix} 0.005 & 0.071 & 0.030 \\ 0.071 & 0.073 & 0.069 \\ 0.030 & 0.069 & 0.073 \end{pmatrix} \tag{3.27}$$

- $n = 34,\ \theta = 90.560°$

$$\langle\psi(90.560°)|R_q|\psi(90.560°)\rangle = \begin{pmatrix} 0.990 & 0.000 & 0.000 \\ 0.000 & -0.015 & 0.990 \\ 0.000 & -0.990 & -0.005 \end{pmatrix}$$
$$\Delta R_q = \begin{pmatrix} 0.015 & 0.100 & 0.100 \\ 0.100 & 0.099 & 0.004 \\ 0.100 & 0.004 & 0.100 \end{pmatrix} \tag{3.28}$$

- $n = 7,\ \theta = 137.518°$

$$\langle\psi(137.518°)|R_q|\psi(137.518°)\rangle = \begin{pmatrix} 0.982 & 0.000 & 0.000 \\ 0.000 & -0.739 & 0.663 \\ 0.000 & -0.663 & -0.721 \end{pmatrix}$$
$$\Delta R_q = \begin{pmatrix} 0.025 & 0.067 & 0.173 \\ 0.067 & 0.067 & 0.074 \\ 0.173 & 0.074 & 0.067 \end{pmatrix} \tag{3.29}$$

- n=0, $\theta = 180°$

$$\langle\psi(180°)|R_q|\psi(180°)\rangle = \begin{pmatrix} 0.99 & 0 & 0 \\ 0 & -0.99 & 0 \\ 0 & 0 & -0.99 \end{pmatrix}$$
$$\Delta R_q = \begin{pmatrix} 0.014 & 0.014 & 0.140 \\ 0.014 & 0.014 & 0.140 \\ 0.140 & 0.140 & 0 \end{pmatrix}. \tag{3.30}$$

To obtain the states describing q-deformations of rotations of negative angles $\zeta < 0$, we just have to consider the same states but with $\phi = -\frac{\pi}{2}$. It is straightforward to show that for states of the form (3.19) with $\chi = 0$ having real superposition coefficients like (3.25), the map $\phi \mapsto -\phi$ exchanges $(R_q)_{23}$ with $(R_q)_{32}$, leaving all the other elements unchanged, and yields the same uncertainty matrix.

For the $\theta = 0$ case, we use the state in the representation ρ which gives the identity matrix in computing the expectation values which is simply given by $|\chi = 0\rangle \in \mathcal{H}_\rho$. From (3.17) we thus have

$$\langle\psi(0^\circ)|R_q|\psi(0^\circ)\rangle = \mathbb{1}_{3\times 3} \qquad \Delta R_q = \mathbf{0}_{3\times 3}\,. \tag{3.31}$$

3.2 Same stars, different skies

Our results admit an interesting interpretation regarding the relationship between observers and the spacetime they observe, namely that the choices made by observers in setting up their reference frames affect the spacetime properties they observe.

We shall argue for this by first advocating Einstein's operational notion of spacetime, whose points have physical meaning only in as much as they label an event there occurring. We use as reference example a network of sources emitting photons ("stars"): each photon emission is a physical point of spacetime. These points of spacetime will be labeled by measured coordinates and uncertainties on those coordinates. Given one reasonable operational assumption, our key observation will be that, in our model, these uncertainties are not intrinsic properties of the spacetime points but rather depend on the choices made by the observer. Put another way, different skies may originate from the observation of the same stars.

3.2.1 Agency-dependent space

We will apply our formalism to draw a connection between the q-deformation of the $SU(2)$ group and a conceivable agency-dependence of space. While this is not a necessary conclusion, we will now argue by means of a thought experiment that this property of space does follow from a physically reasonable assumption. Thereafter, we illustrate it with a concrete numerical example.

Consider two observers, Alice and Bob, each equipped with their own set of telescopes, who want to map the starry sky. Each of them chooses their reference frame (in particular their z-axis) to assign coordinates to the stars they observe. We assume that their origins coincide but their z-axes do not. Without loss of generality we can focus on the (y, z)-plane so that, for each observer, any telescope is mapped onto another by a rotation about the x-axis. In the classical case, Alice and Bob can compare their experimental results by rotating their data with a classical rotation matrix. Formally, there exists an element of the $SO(3)$ group that sharply describes the relative orientation between Alice and Bob.

We now analyze how the situation changes in the noncommutative framework. As in the classical case, we shall assume that the theory is spatially relativistic in the sense that it does not contain any *a priori* preferred direction in space. In this light, it is reasonable to further assume – as we henceforth do – that Alice and Bob can choose the direction along which points are sharp independently from one another.

For definiteness, we focus on observer Alice first. In her task of mapping the starry sky, she focuses on a particular star, which she observes with one of her telescopes. This instrument is generally misaligned with respect to the z-axis she has chosen. In our framework, this means that there exists a state $|\psi_A\rangle$ connecting the relative orientation between the aforementioned telescope and the z-axis. Following our physical interpretation, $|\psi_A\rangle$ determines a quantum rotation matrix through the expectation value of R_q and an uncertainty matrix defined by (3.11). This means that Alice cannot appreciate the relative orientation between the telescope and her z-axis with arbitrary precision since there will always be some intrinsic uncertainty given by the deformation of the $SU(2)$ group. As a consequence of this, she will not be able to

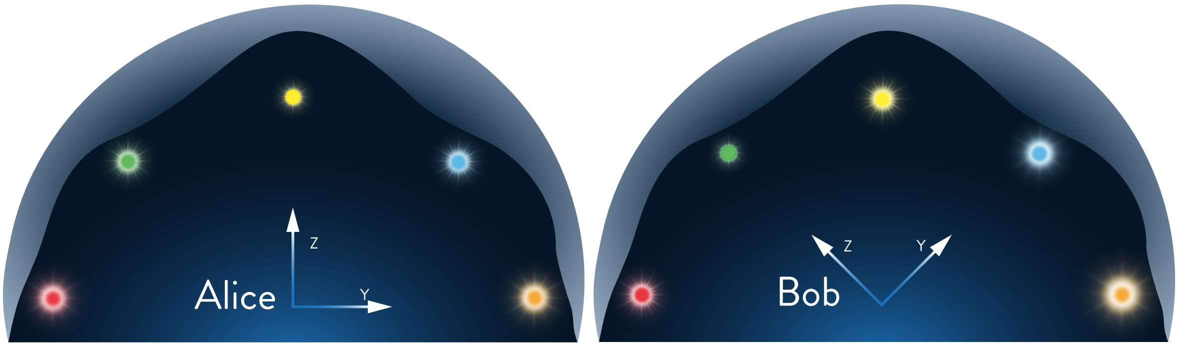

Figure 3.3: A semi-quantitative description of the starry skies observed by Alice and Bob. Alice's z-axis is aligned with the yellow star while Bob's z-axis is aligned with the green star, so that the yellow star is sharp for Alice while the green star is sharp for Bob. For Alice (Bob) the fuzziness increases as the angular deviation from the yellow (green) star increases and the same stars are observed with different fuzziness because of the relative orientation of the z-axes of the two agents.

sharply deduce the position of this star in the sky. For that same star, this line of reasoning also applies to Bob but the state $|\psi_B\rangle$ will generally be different from $|\psi_A\rangle$, implying a different degree of fuzziness. In turn, this procedure can be applied to any star in the sky so that Alice and Bob each have their own picture of the celestial sphere, as can be seen in fig. 3.3.

Given our background assumptions, the result is that the uncertainty associated with each star is not an intrinsic property of spacetime but depends on the choices made by the observers. In particular, different choices of the z-axis give rise to different pictures of the starry sky. Accordingly, there is no way for Alice and Bob to define an objective celestial sphere, meaning that the definition of space itself cannot be independent of the observer who reconstructs it. In this sense, we say that Alice and Bob are agents: we are abandoning the idea of objective space, replacing it with a notion of *observed space* from which we cannot subtract the choices of the observer who infers it.

As mentioned earlier, it is important to note that, even though the z-axis is a preferred direction for observers (only its points can be sharp), our framework produces an isotropic description of space. Indeed, spatial rotations are a (q-deformed) symmetry. There is no direction which is preferred *a priori*, and the special role played by the z-axis in the reference frame of a given observer is only the result of the choices made by that observer in setting up their frame. The oberver chooses freely their preferred z-direction. This should be contrasted with the case of standard spatial anisotropy, in which different directions have different properties *a priori*, independently of the choices made by observers, and invariance under spatial rotations is lost.

3.2.2 Numerical analysis

To gain insight into these conceptual novelties and to better grasp the meaning of fig. 3.3, we now show some numerical examples in support of our claims. We start with Alices's point of view. Let her z-axis be aligned with a certain star (α) and consider another star (β) she wants to observe from a telescope, whose relative orientation with respect to the first star is described by the following state:

$$|\psi_A\rangle = \left|n = 0; \phi = \frac{\pi}{2}; \chi = 0\right\rangle . \tag{3.32}$$

This state, already introduced in section section 3.1.3 and which satisfies the "classicality" conditions (3.16), can be seen as a q-deformed rotation of π around the x-axis. The expectation value is obtained by substituting $\phi = \frac{\pi}{2}$ in (3.21) while the variances are the ones in (3.22)

$$\langle\psi_A|R_q|\psi_A\rangle = \begin{pmatrix} q & 0 & 0 \\ 0 & -q & 0 \\ 0 & 0 & -q^2 \end{pmatrix} \qquad \Delta R_q = \begin{pmatrix} \frac{1}{2}\sqrt{(1-q^2)(1-q^4)} & \frac{1}{2}\sqrt{(1-q^2)(1-q^4)} & \frac{(1+q^2)}{2}\sqrt{(1-q^2)} \\ \frac{1}{2}\sqrt{(1-q^2)(1-q^4)} & \frac{1}{2}\sqrt{(1-q^2)(1-q^4)} & \frac{(1+q^2)}{2}\sqrt{(1-q^2)} \\ \sqrt{q^2-q^4} & \sqrt{q^2-q^4} & 0 \end{pmatrix} \tag{3.33}$$

which at first order in $(1-q)$ give

$$\begin{aligned} \langle\psi_A|R_q|\psi_A\rangle &= \begin{pmatrix} 1-(1-q) & 0 & 0 \\ 0 & -1+(1-q) & 0 \\ 0 & 0 & -1+2(1-q) \end{pmatrix} , \\ \Delta R_q &= \begin{pmatrix} \sqrt{2}(1-q) & \sqrt{2}(1-q) & \sqrt{2(1-q)} \\ \sqrt{2}(1-q) & \sqrt{2}(1-q) & \sqrt{2(1-q)} \\ \sqrt{2(1-q)} & \sqrt{2(1-q)} & 0 \end{pmatrix} . \end{aligned} \tag{3.34}$$

Identifying Alice's z-axis with the vector $v = (0, 0, 1)$ and applying these matrices on v, the transformed vector $v' = (v'_1, v'_2, v'_3)$ will lie in the range

$$-\sqrt{2(1-q)} \leq v'_1 \leq \sqrt{2(1-q)} \qquad -\sqrt{2(1-q)} \leq v'_2 \leq \sqrt{2(1-q)} \qquad v'_3 = -1 + 2(1-q) . \tag{3.35}$$

This quantitatively shows what we mean by fuzziness: the q-rotated vector v' lies in a cone with aperture given by

$$\Delta\alpha \approx 2\sqrt{2(1-q)} . \tag{3.36}$$

The agency feature of the model can be well understood if we now compare these results with the ones obtained if Alice chose to align her z-axis with star β. In this case she would have seen β sharply and α under a cone with aperture (3.36). This line of reasoning can be extended when considering multiple stars: different states describing the relative orientation of these stars with respect to Alice's z-axis will produce different uncertainties in determining their direction. In turn, this is characterized by different cone apertures under which the stars are seen. In table 3.1 we exhibit some numerical examples expressing this feature.

From this analysis, it is worth noticing that the fuzziness grows as our quantized Euler angle increases, resulting in a starry sky inferred by Alice, similar to what is depicted in fig. 3.3. Of course, the analysis can also be repeated for Bob, who chooses a different z-axis, in principle, and observes the same stars as Alice. The states describing the relative orientation between these stars and his z-axis will be different from the ones characterizing Alice's frame. The degree of fuzziness he observes for a particular star will be different from the one assigned to it by Alice, resulting in a different inference of the starry sky, as shown in fig. 3.3.

Discussion and outlook

Through the example shown in this chapter, it is clear that by taking the quantum symmetries hypothesis (motivated by quantum gravity) at face value, we are in need of a conceptual

$\theta(n)$	Aperture
$0°$	0
$\pm 45.275°$	0.210
$\pm 90.560°$	0.284
$\pm 137.518°$	0.402
$\pm 180°$	0.442

Table 3.1: Dependence of the aperture on the observation angle $\theta(n)$ using $q = 0.99$. The aperture is monotonically increasing as the angles grow in absolute value. The states used to obtain these results are constructed numerically in section 3.1.4. For consistency, the aperture for $180°$ is not truncated at first order in $(1-q)$ as in (3.36), but is computed numerically with eq. (3.33) with $q = 0.99$. The aperture of the cone is the same for opposite angles since the uncertainty matrix remains the same for opposite angles, as discussed in section 3.1.4.

reassessment of observers and reference frames. Quantum mechanics forces the introduction of a notion of *complementarity* in describing a physical system, when transitioning from classical to quantum observables. The above analysis shows that this is also to be expected when transitioning from classical to quantum symmetries. The mathematical details leading to this physical interpretation are nonetheless similar in nature. The promotion of smooth functions describing observables (reference frame transformations) to operators satisfying non-trivial commutation relations inevitably introduces uncertainty relations between the associated physical quantities (group transformation parameters) which cannot be subtracted from the analysis. As a result, the information regarding a physical system (spacetime) depends on the choices of the observers themselves. The semi-quantitative results obtained in this example supports the claim anticipated in the beginning of the chapter. A quantum communication protocol that takes into account the quantum features of symmetry deformations is likely to deny arbitrary sharp alignment between two reference frames. This is indeed one of the next directions to explore using the quantum group framework. If invariance under $SU_q(2)$ is invoked, two observers would exchange q-spinors (the generalization of spinors) as physical systems to deduce the relative orientation between their reference frames, with the expectation that these cannot be determined with arbitrary precision. The benefits of the analysis of this novel quantum communication protocol would be twofold: the origin of the fuzziness would be understood from an *operational* point of view and we would have a first example of a fuzzy communication protocol inspired by a quantum gravity scenario, strengthening the link between quantum gravity and quantum foundations and information. A preliminary step to take towards this direction is to fully enforce the quantum spacetime and symmetries hypothesis, extending the analysis presented in the previous sections. In fact, it should be stressed that in this preparatory work, we have performed a semi-classical analysis, only focusing on the quantum features of the symmetry transformations, while leaving spacetime classical. A complete treatment should also take the quantum properties of spacetime into account (the homogeneous space on which $SU_q(2)$ acts is also a quantum space). We conclude by also mentioning a further possible development along this line of research; that of investigating agency-dependence also when translations and boosts are taken into account. The target example would be the κ-Poincaré quantum group, largely employed in studies of quantum gravity phenomenology. We may anticipate observations similar to the ones obtained in our $SU_q(2)$ example, as suggested by results in [63].

Chapter 4

Noether charges and interactions in quantum spacetime

Deformed relativistic symmetries emerging from noncommutative spacetime models have been intensively studied in various contexts these past three decades [29, 88, 126, 137], but the associated conserved charges are still poorly understood. The missing piece of the puzzle is a full generalization of the Noether theorem, which has been found in some studies [153, 154] pertaining only to free theories, which have no physical counterpart. Central to the issue of shedding light on the nature of Noether charges in noncommutative spacetime is the fact that substantial progress has been made in studying *empty* noncommutative spacetime and its associated symmetries but the consequences of introducing particles and particles interactions on a noncommutative spacetime are still unclear. Particularly noteworthy in this context is a study [155] focusing on the renowned Snyder spacetime, characterized by commutation relations

$$[x_\mu, x_\nu] = i\lambda^2 M_{\mu\nu}\,, \tag{4.1}$$

where λ is a length scale and $M_{\mu\nu}$ are the Lorentz generators. As shown in [155] and anticipated in Snyder's original paper [156], the Snyder model leads to a description of space in Cartesian coordinates as a cubical lattice of spacing λ. This feature can indeed be uncovered by studying the properties of the *kinematical* Hilbert space on which the coordinate operators are represented. However, already for a non-interacting model, the introduction of particles requires the enforcement of some Hamiltonian constraint, defining a *physical* Hilbert space, which is a restriction of the aforementioned kinematical Hilbert space. The Cartesian coordinates do not leave the Hamiltonian constraint invariant and so novel self-adjoint operators which commute with the Hamiltonian constraint need to be introduced as position observables. The covariant quantum mechanics analysis conducted in [155] shows that the enforcement of the Hamiltonian constraint leaves no observable trace of space discretization. Moreover, the result is generalized and shows that all of observables compatible with the Hamiltonian constraint are undeformed, trivializing the theory. From these considerations, the take home message is that questions regarding the physical consequences of living in a noncommutative spacetime should be answered in a setting where the presence of particles and interactions is fully enforced. The analysis we will briefly show follows this philosophy, shedding light on the nature of Noether charges in a noncommutative spacetime in a a first quantization setting, where a spatial version of the well-studied timelike κ-Minkowski spacetime is considered. The results show a strong connection between the form of the conserved charges and that of the interaction potential.

4.1 Preliminaries

Before getting to our novel results, we devote this section to a short review of properties of spatial 2D κ-Minkowski and to a general perspective on possible noncommutative-spacetime generalizations of harmonic-oscillator-type Hamiltonians (the type of Hamiltonians for which, in the following sections, we shall derive Noether charges).

4.1.1 Spatial 2D κ-Minkowski

The most studied variant of κ-Minkowski noncommutativity is a case of space/time noncommutativity (spatial coordinates commute among themselves but do not commute with the time coordinate), which in the 2D case is characterized by the following commutator between time and spatial coordinate [88]

$$[x^0, x^1] = i\ell x^1 \,, \tag{4.2}$$

where ℓ (often rewritten as $1/\kappa$) is a length scale usually assumed to be of the order of the Planck length. It is well established [55, 60, 88] that the symmetries of 2D space/time κ-Minkowski noncommutativity are described by the 2D κ-Poincaré Hopf Algebra.

In the study we are here reporting we follow Ref. [157] by focusing on a scenario with a time coordinate which is fully commutative and two spatial coordinates governed by κ-Minkowski noncommutativity

$$[x_2, x_1] = i\ell x_1 \,. \tag{4.3}$$

All the results establishd in a wide literature on the 2D space/time κ-Minkowski of Eq.(4.2) and its Hopf-algebra symmetries are easily converted into results for our 2D spatial κ-Minkowski of Eq.(4.3) and its Hopf-algebra symmetries, by the replacement of coordinates $x^0 \to ix_2$, a replacement of noncommutativity parameter $\ell \to i\ell$, and then replacing the time-translator generator with a suitable generator of translations along the x_2 direction, $P_0 \to -iP_2$ while the boost generator of 2D space/time κ-Minkowski is replaced by the rotation generator of 2D spatial κ-Minkowski, $N \to -iR$. This leads to a description of the translation and rotation symmetries of 2D spatial κ-Minkowski such that

$$[P_2, P_1] = 0\,, \qquad [R, P_2] = -iP_1\,, \qquad [R, P_1] = \frac{i}{2\ell}(1 - e^{-2\ell P_2}) + i\frac{\ell}{2}P_1^2\,, \tag{4.4}$$

which is a deformation of the Euclidean algebra in 2 dimensions. A central element of this algebra, which will be a crucial ingredient for the construction of our Hamiltonians, is given by

$$\mathcal{C} = \frac{4}{\ell^2}\sinh^2(\ell P_2/2) + e^{\ell P_2}P_1^2\,. \tag{4.5}$$

This is a deformation of the $P_1^2 + P_2^2$ Casimir element of the Euclidean algebra.

We shall introduce interactions among particles within a Hamiltonian setup and be satisfied showing our results to order ℓ^2. We note here some commutation relations which shall be valuable in those Hamiltonian analyses:

$$[x_1, P_1] = i\,, \quad [x_1, P_2] = 0\,, \quad [x_2, P_1] = -i\ell P_1\,, \quad [x_2, P_2] = i\,, \tag{4.6}$$

$$\begin{aligned} &[R, x_1] = ix_2\,, \\ &[R, x_2] = -i\big(x_1 - \ell x_1 P_2 + \frac{\ell}{2}x_2 P_1 + \frac{\ell}{2}P_1 x_2 + \ell^2 x_1 P_2^2 + \frac{\ell^2}{4}(x_1 P_1^2 + P_1^2 x_1)\big)\,, \end{aligned} \tag{4.7}$$

which satisfy Jacobi identities.

The nonlinearity of the commutators (4.4), typical of Hopf-algebra symmetries, produce the difficulties for Noether charges which are the main focus of the study we are here reporting. For free particles it has been shown [72, 153] that the charges associated to P_1, P_2 and R are conserved (but of course any nonlinear function of a conserved quantity is also conserved). For interacting particles it is unclear which combinations of the charges should be conserved in particle reactions. In particular, for a process $A + B \to C + D$ it is clear that $P_1^A + P_1^B = P_1^C + P_1^D$ is not an acceptable conservation law because of the nonlinearity of $[R, P_1]$ (*i.e.* $P_1^A + P_1^B = P_1^C + P_1^D$ would not be covariant). So it is clear that the total momentum of a system composed of particles A and B cannot have the component $P_1^A + P_1^B$, but it is not clear which nonlinear combination of the momenta gives the total momentum of a system (and would be therefore conserved in particle reactions). A popular way to guess the momentum-composition formula is based on the so-called "coproduct" [68, 158, 159], which for our purposes it is sufficient to introduce in terms of the properties of suitably ordered products of plane waves: for two plane waves of momenta k and q one has that, as a result of the noncommutativity (4.3),

$$e^{ik_1x^1}e^{ik_2x^2}e^{iq_1x^1}e^{iq_2x^2} = e^{i(k\oplus_\kappa q)_1x^1}e^{i(k\oplus_\kappa q)_2x^2}\,, \tag{4.8}$$

where

$$\begin{aligned}(k \oplus_\kappa q)_1 &= k_1 + e^{-\ell k_2}q_1\,,\\ (k \oplus_\kappa q)_2 &= k_2 + q_2\,.\end{aligned} \tag{4.9}$$

In order for these quantities to close the single-particle algebra (4.4), rotations should also combine non-linearly

$$(R_k \oplus_\kappa R_q) = R_k + e^{-\ell k_2}R_q\,. \tag{4.10}$$

In particular, in [157], it was pointed out that this tentative way of computing the total-momentum, as in (4.8), is inspired by the request that fields describing several particles should be multiplied in the same quantum spacetime point. However, the conserved charges when dynamics is involved are shown to be different than the coproduct-inspired ones.

Alternative ways for guessing the momentum-composition formula have also been proposed. As an alternative to the "κ-coproduct composition law" of Eqs.(4.9)-(4.10) we shall also consider the "proper-dS composition law",

$$\begin{aligned}\mathcal{P}_1 = (p^A \oplus_{dS} p^B)_1 &= p_1^A + p_1^B - \ell(p_2^A p_1^B + p_1^A p_2^B) +\\ &\quad + \frac{\ell^2}{2}\left[(p_2^A p_1^B + p_1^A p_2^B)(p_2^A + p_2^B) - p_1^A (p_1^B)^2 - (p_1^A)^2 p_1^B\right]\\ \mathcal{P}_2 = (p^A \oplus_{dS} p^B)_2 &= p_2^A + p_2^B + \ell p_1^A p_1^B - \frac{\ell^2}{2}\left[-p_1^B p_1^A (p_2^B + p_2^A) + p_2^A (p_1^B)^2 + (p_1^A)^2 p_2^B\right]\\ \mathcal{R} = (R^A \oplus_{dS} R^B) &= R^A + R^B\end{aligned} \tag{4.11}$$

which was motivated using some geometric arguments (one can show that with these choices of composition laws momentum space acquires the geometrical structure of de Sitter space [160]).

4.1.2 Deformations of harmonic-oscillator Hamiltonians

Our next task is to introduce the class of Hamiltonians on which we shall focus our search of Noether charges. Their core ingredient is the harmonic oscillator potential in 2 spatial dimensions. We shall consider deformations of the Hamiltonian

$$H_0^{AB} = \frac{(\vec{p}^A)^2}{2m} + \frac{(\vec{p}^B)^2}{2m} + \frac{1}{2}g(\vec{q}^A - \vec{q}^B)^2 \tag{4.12}$$

where g is the coupling constant, the labels A and B refer to the two particles interacting, $\vec{q}^J$ ($J \in \{A, B\}$) are ordinary commutative spatial coordinates, and $\vec{p}^J$ are the corresponding momenta, with standard Heisenberg commutators ($[q_j^J, p_k^K] = i\,\delta^{JK}\delta_{jk}$, with $J, K \in \{A, B\}$ and $j, k = 1, 2$). The total momentum and total angular momentum defined through

$$\vec{P} = \vec{p}^A + \vec{p}^B \qquad R_0 = R_0^A + R_0^B \tag{4.13}$$

are conserved charges since they commute with the Hamiltonian, $[H_0^{AB}, \vec{P}] = 0$ and $[H_0^{AB}, R_0] = 0$. Both the total generators $\{P_i, R_0\}$ and the single particle generators $\{p_i^I, R_0^I\}$ close the undeformed Galilean algebra.

For reasons which shall soon be clear, we want to test our approach also for interactions among more than two particles, and for that purpose our starting point is the 3-particle Hamiltonian

$$H_0^{ABC} = \frac{(\vec{p}^A)^2}{2m} + \frac{(\vec{p}^B)^2}{2m} + \frac{(\vec{p}^C)^2}{2m} + \frac{1}{2}g(\vec{q}^A - \vec{q}^B)^2 + \frac{1}{2}g(\vec{q}^A - \vec{q}^C)^2 + \frac{1}{2}g(\vec{q}^B - \vec{q}^C)^2 \tag{4.14}$$

This is of interest to us particularly because the interacting potential $V_3(\vec{q}^A, \vec{q}^B, \vec{q}^C)$ can be split into the sum $V_2(\vec{q}^A, \vec{q}^B) + V_2(\vec{q}^A, \vec{q}^C) + V_2(\vec{q}^B, \vec{q}^C)$ with V_2 having the same functional form for each pair of particles: in the case studies for which we perfomed our Noether-charge analyses this property cannot be maintained in presence of noncommutativity of coordinates.

Evidently, the Hamiltonian (4.14) commutes with the total charges defined as $\vec{P} = \vec{p}^A + \vec{p}^B + \vec{p}^C$ and $R_0 = R_0^A + R_0^B + R_0^C$.

A key ingredient of our deformed Hamiltonians will be of course the kinetic term, for which we adopt the form

$$H_K \equiv \frac{\mathcal{C}}{2m} \approx \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \ell\frac{p_1^2 p_2}{2m} + \ell^2\frac{p_1^2 p_2^2}{4m} + \ell^2\frac{p_2^4}{24m} \tag{4.15}$$

obtained from the Casimir element $\mathcal{C}$ of our Eq.(4.4) (to order ℓ^2).

We will look for suitable interaction potentials within some rather broad parametrizations. We parametrize the two-particle case as follows

$$V^{AB} = V(\vec{x}^A, \vec{x}^B) = \frac{1}{2}g(\vec{x}^A - \vec{x}^B)^2 + \ell g \sum \alpha_{ijk}^{IJK} p_i^I x_j^J x_k^K + \ell^2 g \sum \beta_{ijkh}^{IJKH} p_i^I p_j^J x_k^K x_h^H \tag{4.16}$$

where α_{ijk}^{IJK} and β_{ijkh}^{IJKH} are numerical coefficients and the sum extends both to spatial indices (lower case letters) and particle indices (upper case letters).

Similarly, for the three-particle case our *ansatz* is given by

$$\begin{aligned} V^{ABC} = V(\vec{x}^A, \vec{x}^B, \vec{x}^C) =& \frac{1}{2}g(\vec{x}^A - \vec{x}^B)^2 + \frac{1}{2}g(\vec{x}^B - \vec{x}^C)^2 + \frac{1}{2}g(\vec{x}^C - \vec{x}^A)^2 + \\ &+ \ell g \sum \tilde{\alpha}_{ijk}^{IJK} p_i^I x_j^J x_k^K + \ell^2 g \sum \tilde{\beta}_{ijkh}^{IJKH} p_i^I p_j^J x_k^K x_h^H \end{aligned} \tag{4.17}$$

where $\tilde{\alpha}_{ijk}^{IJK}$ and $\tilde{\beta}_{ijkh}^{IJKH}$ are other sets of numerical coefficients and the particle indices run over $\{A, B, C\}$.

4.2 Charges with proper-dS composition

The debate on the alternative ways to combine charges in a κ-Minkowski setup has mainly relied on naturalness arguments based on the properties of free particles in κ-Minkowski. As announced in our opening remarks, we here intend to show that there is no notion of "naturalness" at stake here: how charges should combine depends on the form of the laws of interaction among particles (so evidently goes beyond the scopes of the description of free particles) and different composition laws can emerge from different descriptions of the interactions. We shall establish our case relying on Hamiltonian theories within first-quantized quantum mechanics, where the relevant issues can be seen in particularly vivid fashion.

We choose as our first task the one of exhibiting a Hamiltonian (within first-quantized quantum mechanics) which selects uniquely the proper-dS composition law, which we already reviewed in Eq.(4.11) and we show again here for convenience:

$$\begin{aligned}
\mathcal{P}_1 = (p^A \oplus_{dS} p^B)_1 = p_1^A + p_1^B - \ell(p_2^A p_1^B + p_1^A p_2^B) + \\
+ \frac{\ell^2}{2}\left[(p_2^A p_1^B + p_1^A p_2^B)(p_2^A + p_2^B) - p_1^A (p_1^B)^2 - (p_1^A)^2 p_1^B\right] \\
\mathcal{P}_2 = (p^A \oplus_{dS} p^B)_2 = p_2^A + p_2^B + \ell p_1^A p_1^B - \frac{\ell^2}{2}\left[- p_1^B p_1^A (p_2^B + p_2^A) + p_2^A (p_1^B)^2 + (p_1^A)^2 p_2^B\right] \\
\mathcal{R} = (R^A \oplus_{dS} R^B) = R^A + R^B
\end{aligned} \tag{4.18}$$

One can easily verify that $\mathcal{P}_1\,, \mathcal{P}_2\,, \mathcal{R}$ close the algebra (4.4) up to order ℓ^2, which we also rewrite here for convenience

$$[\mathcal{P}_2, \mathcal{P}_1] = 0 \qquad [\mathcal{R}, \mathcal{P}_2] = -i\mathcal{P}_1 \qquad [\mathcal{R}, \mathcal{P}_1] = i(\mathcal{P}_2 - \ell\mathcal{P}_2^2 + \frac{\ell}{2}\mathcal{P}_1^2 + \frac{2\ell^2\mathcal{P}_2^3}{3}) \tag{4.19}$$

We start by showing that for the case of two particles interacting there is a Hamiltonian H_{dS}^{AB}, deformation of the H_0^{AB} of Eq.(4.12), such that $[\vec{\mathcal{P}}, H_{dS}^{AB}] = 0$ and $[\mathcal{R}, H_{dS}^{AB}] = 0$. As anticipated in Subsection 4.1.2, our Hamiltonian H_{dS}^{AB} will be of the form

$$H_{dS}^{AB} = H_K^A + H_K^B + V_{dS}^{AB} \tag{4.20}$$

where H_K is fixed to be the one of Eq.(4.15), while V_{dS}^{AB} must be specified consistently with Eq.(4.16), for some choice of the parameters that Eq.(4.16) leaves to be determined.

We work partly by reverse engineering: we use $[\vec{\mathcal{P}}, H_{dS}^{AB}] = 0$ and $[\mathcal{R}, H_{dS}^{AB}] = 0$ as conditions that must be satisfied by the parameters of Eq.(4.16), and then, once we have such an acceptable V_{dS}^{AB}, we show that the resulting Hamiltonian H_{dS}^{AB} uniquely selects the proper-dS charges (4.18) as its conserved charges.

We find that in particular the following choice of V_{dS}^{AB}

$$\begin{aligned}
V_{dS}^{AB} =& \frac{g}{2}\Big[(\vec{x}^A - \vec{x}^B)^2 + \\
&+2\ell\Big(- p_2^A (x_1^A)^2 + \frac{1}{2}p_1^A x_1^A x_2^A + \frac{1}{2}x_2^A x_1^A p_1^A + p_2^A x_1^A x_1^B - x_2^A p_1^A x_1^B + (A \leftrightarrow B)\Big) + \\
&+\frac{1}{2}\ell^2\Big((p_1^B)^2(-2(x_2^A)^2 + 6x_2^A x_2^B - 2(x_2^B)^2) + 4p_1^B p_2^B x_1^A x_2^A - p_1^A p_1^B x_2^A x_2^B + p_1^B p_2^A x_1^A x_2^B + \\
&-6p_1^B x_1^A x_2^B p_2^B - 2p_1^B x_1^B (p_2^A x_2^A - x_2^B p_2^B) - 2(p_2^B)^2((x_1^A)^2 - x_1^A x_1^B - (x_1^B)^2) + \\
&+p_2^B p_1^A x_2^A x_1^B - 2p_2^B p_2^A x_1^A x_1^B - 3p_2^B x_2^A x_1^B p_1^B + 2x_1^A p_1^A (p_1^A x_1^A - p_1^A x_1^B + p_2^A x_2^A + \\
&-\frac{3}{2}p_2^A x_2^B + \frac{3}{2}x_2^B p_2^B) + x_2^A p_2^A x_1^B p_1^B + (A \leftrightarrow B)\Big)\Big]
\end{aligned} \tag{4.21}$$

is indeed such that $[\vec{\mathcal{P}}, H_K^A + H_K^B + V_{dS}^{AB}] = 0$ and $[\mathcal{R}, H_K^A + H_K^B + V_{dS}^{AB}] = 0$.

We observe that our V_{dS}^{AB} is symmetric under exchange of the particles (this is not always the case, see later). Most importantly, we find that indeed the Hamiltonian $H_K^A + H_K^B + V_{dS}^{AB}$ uniquely selects the proper-dS charges (4.18) as its conserved charges. In order to see this we start from a general parametrization of the two-particle charges

$$\begin{aligned} P_1^{tot} &= \sum p_1^I + \ell\gamma_{ij}^{IJ} p_i^I p_j^J + \ell^2 \Gamma_{ijk}^{IJK} p_i^I p_j^J p_k^K \\ P_2^{tot} &= \sum p_2^I + \ell\theta_{ij}^{IJ} p_i^I p_j^J + \ell^2 \Theta_{ijk}^{IJK} p_i^I p_j^J p_k^K \\ R^{tot} &= \sum R^I + \ell\phi_i^{IJ} p_i^I R^J + \ell^2 \Phi_{ij}^{IJK} p_i^I p_j^J R^K \end{aligned} \tag{4.22}$$

where $\gamma, \theta, \phi, \Gamma, \Theta, \Phi$ are sets of real coefficients and the sum is intended over particle indices I, J, K (which take values in $\{A, B\}$) and over the spatial indices i, j, k. We also require that no terms with all particle indices equal to each other are present, so that we recover the definition of single particle charge when the charges of the other particles are zero.

By requesting that these charges commute with $H_K^A + H_K^B + V_{dS}^{AB}$ the parameters in Eq.(4.22) get fully fixed, giving indeed the proper-dS charges (4.18).

Next we turn to the corresponding three-particle case, for which the proper-dS composition leads to the following formulas for the charges:

$$\begin{aligned} \tilde{\mathcal{P}}_1 = ((p^A \oplus_{dS} p^B) \oplus_{dS} p^C)_1 = p_1^A + p_1^B + p_1^C - \ell\big(p_2^B(p_1^C + p_1^A) + p_2^C(p_1^B + p_1^A) + \\ + p_2^A(p_1^C + p_1^B)\big) + \frac{\ell^2}{2}\big(-2p_1^A p_1^B p_1^C - (p_1^B)^2 p_1^A + 2p_1^A p_2^B p_2^C - p_1^B (p_1^A)^2 - (p_1^C)^2 (p_1^B + p_1^A) + \\ - (p_1^C)(p_1^B + p_1^A)^2 + 2p_2^C p_1^B p_2^A + (p_2^B + p_2^A)(p_2^B p_1^A + p_2^A p_1^B) + \\ + (p_2^A + p_2^B + p_2^C)(p_2^C(p_1^B + p_1^A) + p_1^C(p_2^A + p_2^B))\big) \end{aligned}$$

$$\begin{aligned} \tilde{\mathcal{P}}_2 = ((p^A \oplus_{dS} p^B) \oplus_{dS} p^C)_2 = p_2^A + p_2^B + p_2^C + \ell\big(p_1^B p_1^A + p_1^C p_1^B + p_1^A p_1^C\big) + \\ - \frac{\ell^2}{2}\big(p_2^C(p_1^A + p_1^B)^2 - p_1^C(p_2^C(p_1^B + p_1^A) + (p_1^B - p_1^A)(p_2^B - p_2^A)) + \\ + (p_1^C)^2(p_2^B + p_2^A) + (p_1^B - p_1^A)(-p_2^B p_1^A + p_1^B p_2^A)\big) \end{aligned}$$

$$\tilde{\mathcal{R}} = (R^A \oplus_{dS} R^B) \oplus_{dS} R^C = R^A + R^B + R^C \tag{4.23}$$

Evidently we must find a Hamiltonian H_{dS}^{ABC}, deformation of the H_0^{ABC} of Eq.(4.14), such that $[\vec{\tilde{\mathcal{P}}}, H_{dS}^{ABC}] = 0$ and $[\tilde{\mathcal{R}}, H_{dS}^{ABC}] = 0$. As anticipated in Subsection 4.1.2, our Hamiltonian H_{dS}^{ABC} will be of the form

$$H_{dS}^{ABC} = H_K^A + H_K^B + H_K^C + V_{dS}^{ABC} \tag{4.24}$$

where H_K is again fixed to be the one of Eq.(4.15), while V_{dS}^{ABC} must be specified consistently with Eq.(4.17), for some choice of the parameters that Eq.(4.17) leaves to be determined.

A natural first guess is that the three-particle potential V_{dS}^{ABC} be given (see Eq(4.14)) by a combination of our two-particle potentials given in Eq.(4.21), *i.e.* $V_{dS}^{ABC} = V_{dS}^{AB} + V_{dS}^{BC} + V_{dS}^{AC}$, but one can easily check that this does not commute with the three-particle proper-dS charges (4.23). What does work is adding an extra term:

$$V_{dS}^{ABC} = V_{dS}^{AB} + V_{dS}^{BC} + V_{dS}^{AC} + V_{dS(\bigstar)}^{ABC} \tag{4.25}$$

with

$$\begin{aligned} V^{ABC}_{dS(\bigstar)} = &\frac{g\ell^2}{2}\Big(p_1^C p_2^B (x_1^C x_2^A - x_2^C x_1^A) - p_1^C p_2^A x_2^C x_1^B - p_2^C p_1^B (x_1^C x_2^A - x_2^C x_1^A) - p_2^C p_1^A x_1^C x_2^B + \\ &+ p_1^B p_1^A x_2^C (2x_2^C - x_2^A - x_2^B) - p_1^B p_2^A x_2^C (2x_1^C - x_1^B) - 2p_2^B p_1^A x_1^C x_2^C + \\ &+ p_2^B p_2^A x_1^C (2x_1^C - x_1^A - x_1^B) + p_1^A x_1^C p_2^B x_2^B + x_1^C p_1^C p_2^A x_2^B + \\ &+ x_2^C p_2^C p_1^A x_1^B + x_1^A p_1^A p_2^B x_2^C + x_2^A p_2^A p_1^B x_1^C\Big) \end{aligned} \tag{4.26}$$

One can easily check that the H^{ABC}_{dS} of Eqs.(4.24), (4.25), (4.26) commutes with the proper-dS charges (4.23). Most importantly we find that indeed our Hamiltonian H^{ABC}_{dS} uniquely selects the proper-dS charges (4.23) as its conserved charges. In order to see this we start from a general parametrization of the three-particle charges

$$\begin{aligned} \tilde{P}_1^{tot} &= \sum p_1^I + \ell\tilde{\gamma}_{ij}^{IJ} p_i^I p_j^J + \ell^2\tilde{\Gamma}_{ijk}^{IJK} p_i^I p_j^J p_k^K \\ \tilde{P}_2^{tot} &= \sum p_2^I + \ell\tilde{\theta}_{ij}^{IJ} p_i^I p_j^J + \ell^2\tilde{\Theta}_{ijk}^{IJK} p_i^I p_j^J p_k^K \\ \tilde{R}^{tot} &= \sum R^I + \ell\tilde{\phi}_i^{IJ} p_i^I R^J + \ell^2\tilde{\Phi}_{ij}^{IJK} p_i^I p_j^J R^K \end{aligned} \tag{4.27}$$

which shares the same properties outlined for the two-particle ansatz (4.22) (the particle indices run over $\{A, B, C\}$ and $\tilde{\gamma}, \tilde{\theta}, \tilde{\phi}, \tilde{\Gamma}, \tilde{\Theta}, \tilde{\Phi}$ are sets of real coefficients).

We find that by requesting that these charges commute with our $H_K^A + H_K^B + V_{dS}^{AB}$ the parameters in Eq.(4.27) get fully fixed, giving indeed the proper-dS charges (4.23).

We leave to future studies the task of exploring the meaning of the extra term $V^{ABC}_{dS(\bigstar)}$. Whereas the potential in the original three-particle Hamiltonian H_0^{ABC} of Eq.(4.14) was just a sum of two-particle potentials, we found that the potential in its correct "proper-dS deformation" H^{ABC}_{dS} must include the extra term $V^{ABC}_{dS(\bigstar)}$ which is cubic in the observables of the three particles and is made of all terms involving simultaneously observables of all the three particles.

Also noteworthy is that for the three-particle case the proper-dS composition gives charges which are not symmetric under particle exchange (see (4.23)) and accordingly our Hamiltonian H^{ABC}_{dS} also is not symmetric under particle exchange. We do not see any objective problem with this lack of particle-exchange symmetry, but still it is a bit unsettling. This made us interested in investigating which charges would be conserved if we adopted a particle-exchange symmetrized version of our Hamiltonian H^{ABC}_{dS}

$$H^{ABC}_{dS(sym)} = H_K^A + H_K^B + H_K^C + V_{dS}^{AB} + V_{dS}^{BC} + V_{dS}^{AC} + \frac{1}{6}\sum_{\pi(A,B,C)} V^{\pi(ABC)}_{dS(\bigstar)} \tag{4.28}$$

i.e. the Hamiltonian obtained by summing over all the possible particle permutations, $\pi(ABC)$, of the extra term.

We then ask for which choices of the parameters of our Eq.(4.27) the Hamiltonian $H^{ABC}_{dS(sym)}$ commutes with the charges parametrized in our Eq.(4.27), and we find that $H^{ABC}_{dS(sym)}$ uniquely selects as its conserved charges the following ones

$$\begin{aligned}
\mathcal{P}_1^{dS(sym)} &= \frac{1}{3}[(p^A \oplus_{dS} p^B) \oplus_{dS} p^C + p^A \oplus_{dS} (p^B \oplus_{dS} p^C) + (p^A \oplus_{dS} p^C) \oplus_{dS} p^B]_1 = \\
&= p_1^A + p_1^B + p_1^C - \ell(p_1^A p_2^B + p_1^B p_2^A + p_1^A p_2^C + p_1^C p_2^A + p_1^B p_2^C + p_1^C p_2^B) + \\
&+ \frac{\ell^2}{2}\big(p_1^A((p_2^B)^2 + (p_2^C)^2) + (p_1^B + p_1^C)(p_2^A)^2 - p_1^A(p_1^B)^2 - p_1^B(p_1^A)^2 + \\
&- p_1^C(p_1^B)^2 - p_1^B(p_1^C)^2 + p_1^B(p_2^C)^2 + p_1^C(p_2^B)^2 + + p_1^C p_2^C p_2^B + p_1^B p_2^B p_2^C + \\
&- p_1^A(p_1^C)^2 - p_1^C(p_1^A)^2 - 4p_1^A p_1^B p_1^C + \frac{8}{3}(p_1^A p_2^B p_2^C + p_1^B p_2^A p_2^C + p_1^C p_2^B p_2^A) + \\
&+ p_2^A(p_1^C p_2^C + p_1^B p_2^C + p_1^A p_2^C + p_1^A p_2^B)\big)
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}_2^{dS(sym)} &= \frac{1}{3}[(p^A \oplus_{dS} p^B) \oplus_{dS} p^C + p^A \oplus_{dS} (p^B \oplus_{dS} p^C) + (p^A \oplus_{dS} p^C) \oplus_{dS} p^B]_2 = \\
&= p_2^A + p_2^B + p_2^C + \ell(p_1^C(p_1^B + p_1^A) + p_1^B p_1^A) + \\
&- \frac{1}{6}\ell^2(3(p_1^C)^2(p_2^B + p_2^A) - p_1^C(3p_2^C(p_1^B + p_1^A) + 3p_1^B p_2^B - 4p_1^B p_2^A - 4p_2^B p_1^A + 3p_1^A p_2^A) + \\
&+ p_2^C(3(p_1^B)^2 + 4p_1^B p_1^A + 3(p_1^A)^2) + 3(p_1^B - p_1^A)(p_1^B p_2^A - p_2^B p_1^A))
\end{aligned}$$

$$\mathcal{R}^{dS(sym)} = R^A + R^B + R^C \tag{4.29}$$

which are indeed symmetric under particle exchange. Moreover, these charges $\mathcal{P}_1^{dS(sym)}$, $\mathcal{P}_2^{dS(sym)}$, $\mathcal{R}^{dS(sym)}$ close the algebra (4.4).

4.3 Charges with κ-coproduct composition

We now move on to applying the same strategy of the analysis to the coproduct composition law of Eqs.(4.9)-(4.10), which we rewrite here (at order ℓ^2) for convenience

$$\begin{aligned}
\mathcal{P}_1 &= (p^A \oplus_\kappa p^B)_1 = p_1^A + p_1^B - \ell p_2^A p_1^B + \frac{\ell^2}{2}(p_2^A)^2 p_1^B \\
\mathcal{P}_2 &= (p^A \oplus_\kappa p^B)_2 = p_2^A + p_2^B \\
\mathcal{R} &= R^A \oplus_\kappa R^B = R^A + R^B - \ell p_2^A R^B + \frac{\ell^2}{2}(p_2^A)^2 R^B \,,
\end{aligned} \tag{4.30}$$

As done for the proper-dS case, our first objective is to find a Hamiltonian H_κ^{AB}, deformation of the H_0^{AB} of Eq.(4.12), such that $[\vec{\mathcal{P}}, H_\kappa^{AB}] = 0$ and $[\mathcal{R}, H_\kappa^{AB}] = 0$. Applying the same strategy of the previous section, we find that the Hamiltonian $H_\kappa^{AB} = H_K^A + H_K^B + V_\kappa^{AB}$ with

$$\begin{aligned}
V_\kappa^{AB} &= \frac{g}{2}\Big[(\vec{x}^A - \vec{x}^B)^2 + \\
2\ell&\Big(-p_2^A(x_1^A)^2 + x_1^A p_2^A x_1^B + \frac{1}{2}x_2^A p_1^A x_1^A + \frac{1}{2}x_2^A x_1^A p_1^A - x_2^B p_1^A x_1^A - \frac{1}{2}x_2^B p_1^A x_1^A\Big) \\
2\ell^2&\Big((p_2^A)^2(x_1^A)^2 + \frac{1}{2}x_1^A(p_1^A)^2 x_1^A - \frac{1}{2}x_1^A(p_2^A)^2 x_1^B\Big)\Big]
\end{aligned} \tag{4.31}$$

is such that indeed $[\vec{\mathcal{P}}, H_K^A + H_K^B + V_\kappa^{AB}] = 0$ and $[\mathcal{R}, H_K^A + H_K^B + V_\kappa^{AB}] = 0$. And we find that the Hamiltonian $H_K^A + H_K^B + V_\kappa^{AB}$ uniquely selects the κ-coproduct charges (4.30) as its conserved charges. This is easily shown by starting again from the general charge ansatz (4.22) and requiring that they commute with $H_K^A + H_K^B + V_\kappa^{AB}$: this requirement fully fixes all the parameters in Eq.(4.22), giving indeed the κ-coproduct charges (4.30).

It is noteworthy that the κ-coproduct charges (4.30) are not symmetric under the exchange of particles A and B, and accordingly also our Hamiltonian H_κ^{AB} is not symmetric (because the potential V_κ^{AB} of (4.31) is not symmetric). We found that the analogous issue of lacking particle-exchange symmetry that we encountered in our analysis of the proper-dS composition law could be "fixed" by resorting to a symmetrized version of the Hamiltonian, but for the κ-coproduct composition law this is not the case: if one considers the symmetrized Hamiltonian

$$H_{\kappa(sym)}^{AB} = \frac{H_\kappa^{AB} + H_\kappa^{BA}}{2} \tag{4.32}$$

then one finds that no choice of the parameters in (4.22) leads to charges that commute with $H_{AB}^{\kappa(sym)}$.

For the three-particle case the κ-coproduct composition law gives

$$\begin{aligned}
\tilde{\mathcal{P}}_1 = (p^A \oplus_\kappa p^B \oplus_\kappa p^C)_1 = p_1^A + p_1^B + p_1^C + \ell\left(-p_1^B p_2^A - p_1^C(p_2^A + p_2^B)\right) + \\
+\ell^2\left(\frac{p_1^B (p_2^A)^2}{2} + \frac{1}{2}p_1^C (p_2^A + p_2^B)^2\right)
\end{aligned}$$

$$\tilde{\mathcal{P}}_2 = (p^A \oplus_\kappa p^B \oplus_\kappa p^C)_2 = p_2^A + p_2^B + p_2^C \tag{4.33}$$

$$\begin{aligned}
\tilde{\mathcal{R}} = (R^A \oplus_\kappa R^B \oplus_\kappa R^C) = R^A + R^B + R^C + \ell\left(-p_2^A\left(R^B\right) - R^C\left(p_2^A + p_2^B\right)\right) + \\
+\ell^2\left(\frac{1}{2}R^C\left(p_2^A + p_2^B\right)^2 + \frac{1}{2}(p_2^A)^2 R^B\right)
\end{aligned}$$

Using the same procedure of Section 4.2 one finds that the Hamiltonian

$$H_\kappa^{ABC} = H_K^A + H_K^B + H_K^C + V_\kappa^{AB} + V_\kappa^{BC} + V_\kappa^{AC} + V_{\kappa(\bigstar)}^{ABC}\,, \tag{4.34}$$

with

$$\begin{aligned}
V_{\kappa(\bigstar)}^{ABC} =& g\ell\big(p_1^B(x_1^C x_2^A - x_2^C x_1^A + x_1^A x_2^B - x_1^C x_2^B) + p_2^B x_1^B(x_1^C - x_1^A)\big) + \\
&+ g\frac{\ell^2}{2}\big((p_1^B)^2(x_1^C x_1^A - x_1^B x_1^C + x_1^B x_1^A) - (p_2^B)^2(x_1^C x_1^A - 2x_1^A x_1^B) + p_1^B x_1^C(p_1^A x_1^A - p_2^B x_2^B) + \\
&+ p_1^B p_2^A(x_2^C x_1^A - x_1^A x_2^B) + p_2^B(p_1^B x_1^A x_2^C + p_2^A x_1^A x_1^B)\big)\,,
\end{aligned} \tag{4.35}$$

commutes with $\vec{\tilde{\mathcal{P}}}$ and $\tilde{\mathcal{R}}$. It is noteworthy that the κ-coproduct extra term $V_{\kappa(\bigstar)}^{ABC}$, besides involving terms that depend simultaneously on observables of all three particles, also involves terms that depend only on two of the particles (and these additional terms cannot be re-absorbed in a redefinition of the potentials $\tilde{V}_\kappa^{IJ}$ since they are different for different pairs of particles).

Also in this case we find that the Hamiltonian H_κ^{ABC} of our Eq.(4.34) uniquely selects the κ-coproduct charges (4.33) as its conserved charges: by requesting that the parametrized charges of Eq.(4.27) commute with H_κ^{ABC} the parameters in Eq.(4.27) get fully fixed, giving indeed the κ-coproduct charges (4.33).

H_κ^{ABC} is not symmetric under particle exchange and its symmetrized version,

$$H_{\kappa(sym)}^{ABC} = H_K^A + H_K^B + H_K^C + \frac{1}{6}\sum_{\pi(A,B,C)} V_\kappa^{\pi(ABC)}\,, \tag{4.36}$$

is not a viable alternative since it does not have any conserved charges: there is no choice of the parameters in Eq.(4.27) such that the parametrized charges of Eq.(4.27) commute with $H^{ABC}_{\kappa(sym)}$.

Discussion and outlook

Inevitably, the physics community is approaching the challenge of understanding the deformed relativistic symmetries of some quantum spacetimes from a perspective which is mainly informed by our experience with special relativity, but a price can be payed when we unknowingly make inferences based on the linearity of most special-relativistic laws. In particular, the way in which special relativity governs how free-particle charges combine in conservation laws applicable when particles interact is completely governed by the linearity of transformation laws, so that charges inevitably combine linearly. Working within special relativity one does not even fully appreciate how the chosen form of interaction could affect the conservation laws, because the linearity of transformation laws imposes that in all cases charges combine linearly, independently of the type of interactions being considered. This is probably the reasons why, before the study we here reported, the debate on total charges for quantum spacetimes had not contemplated a possible role for the interactions, and instead relied on one or another "naturalness argument" based on the form of the relativistic properties of free particles.

We here showed, using the toy model of spatial 2D κ-Minkowski, that the nonlinearity of deformed-relativistic transformation laws is such that the correct notion of total charge depends strongly on how one introduces interactions among particles. We found that, starting from the same description of free particles, for interacting particles one can have at least three different ways for obtaining total charges: the one based on the proper-dS composition law, the one based on the κ-coproduct composition law, and the one obtained by symmetrizing the proper-dS composition law. Interestingly, we also found that it is instead not possible to introduce interactions such that conservation laws are obtained by symmetrizing the κ-coproduct composition law.

We are confident that the lessons learned within the spatial 2D κ-Minkowski toy model apply also to other quantum spacetimes. Where we suspect that the specific form of spacetime quantization might affect the analysis is the required level of complexity of Hamiltonians. The Hamiltonians we here exhibited, the ones that do enjoy deformed relativistc invariance, are not very simple. To the human eye they appear to be unpleasantly complex, and it would be surprising (though of course possible) that Nature would choose such complex ways to describe interactions among particles. It is therefore natural to wonder if some ways to quantize spacetime with deformed relativistic symmetries could produce simpler descriptions of interactions among particles. If such an aspect of simplicity was found for a certain scheme of spacetime quantization it might provide encoragement for the studies of other aspects of that quantum spacetime.

Chapter 5

Quantum gravity phenomenology with cold atoms

About thirty years ago, the legitimacy of quantum gravity as a genuine branch of physics was strongly challenged [161]. Based on purely heuristic arguments, it was believed that the only relevant regime in which quantum gravity effects could be studied directly would be in the immediate post big-bang era of the Universe. Indeed, it was argued that if the characteristic scale of quantum gravity is the Planck energy $E_P \approx 10^{28}$ eV, we would never have any chance of setting up laboratory-based experiments to test quantum gravity effects in the foreseeable future.

Today, we are at the dawn of quantum gravity phenomenology [25, 27], and it is not because there has been a huge leap in human technology which has brought us to contemplate experimental situations where the relevant energy scale is 10^{28} eV nor have we gained access to the immediate post big-bang era of the Universe. The birth of this new season of physics research is due to a change of perspective with respect to the arguments presented in [161].

Physical predictions pertaining to a model or theory characterized by some physical scale can be of either *smooth* or *steep* onset. Steep onset effects manifest themselves when the physical quantities characterizing the system under study are in the vicinity of the scale relevant for the effect under investigation. If the relevant scale of the effect is some energy scale and the energy of the system under study is below that scale, there will be no observable traces of that target effect. As soon as the target energy is reached by the system, the desired effect switches on and can be observed. This mechanism is relevant, for example, when considering energy thresholds for interaction processes. If the energy of the system is below the threshold, the process is not kinematically allowed and there will be no chance to observe it. When the threshold energy is reached, the process becomes kinematically allowed and there will be some probability to observe it based on its specific mechanisms.

On the other hand, smooth onset effects manifest themselves even when the physical quantities characterizing a system are far from the target scale. In the example involving an energy scale, these effects would typically depend on the ratio between the energy of the system and the target energy scale, so they would leave observable traces even if the energies are many orders of magnitude apart. Consider for example the discovery of the Higgs boson. While the existence of this particle was only recognized when reaching the desired energy to observe processes involving the particle directly [162], its presence could already be deduced at lower energies due to its intervention in higher order corrections of other processes [163].

If we are to eventually observe some quantum gravity effects, it is clear that the smooth onset scenario is the most favorable one, but it is not enough, in general. The huge gap between energies we currently probe, E, and the quantum gravity scale (supposedly of the order of the Planck energy) E_{QG}, would still be a limitation for direct observation of effects that depend on the ratio E/E_{QG}, unless the effects are *amplified* by some mechanism. Smooth onset effects enhanced by amplifiers are the best chance we have to observe some quantum gravity signatures, a possibility not realized in [161]. We may think of amplifiers as functions of the relevant physical quantities at play that compensate for the small leading order dependence of the effect, which in the quantum gravity case is some power of E/E_{QG}. The concept of amplification is crucial in quantum gravity research, but it is not new and has already been exploited to make new physics discoveries in the past century. It is sufficient to think about Brownian motion, the detection of which has been decisive evidence of the fact that macroscopic matter is made up of atoms. In that case, the observable effect is the collision between a dust particle and the smaller particles of some fluid. Of course, it would have been unthinkable of deducing the atomic nature of the fluid by observing the collision of the dust particle with a single fluid particle. Instead, the observation of the collective effect of a huge number of collisions, resulting in a random walk of the dust particle, is what made the discovery possible. The huge number of particles (the amplifier) enhances the tiny displacement caused by the collisions to a macroscopic effect observable within the sensitivities of the microscopes available at the time.

The hope is that a similar phenomenological situation might also apply for quantum gravity. For this reason, several effective quantum gravity models have been put forward during the last two decades, in order to capture signatures of the yet unknown Planck scale physics at the physical regimes available to us today. In the perspective of searching for some observable effects, most studies have focused on astrophysical phenomena, where highly energetic probes are involved. This line of investigation was also motivated by the advent of multi-messenger astronomy, characterized by an increase of the quality and quantity of experimental data obtained by the detection of various cosmic messengers (photons, neutrinos, cosmic rays and gravitational waves) from numerous and diverse sources [27]. Only a handful of works [164,165] have investigated the possibility that quantum gravity effects could also be relevant in table-top experiments with infrared probes (like cold atoms) and that they could be detected with present day sensitivities. Thanks to experimental advancements, this trend is changing in recent years (see for example [166–168]). After briefly reviewing the most studied effective quantum gravity models and the status of quantum gravity phenomenology in the multi-messenger approach, we will show how an effective model inspired by the lightlike κ-Minkowski introduces corrections in atom interferometry experiments that can be detected with present day sensitivities.

5.1 Lorentz Invariance Violation vs. Doubly Special Relativity

If quantum gravity phenomena admit an effective description at low energies through the introduction of E_{QG}, two possiblities arise: either Lorentz symmetry is broken due to the introduction of the new energy scale or a new set of symmetries must be identified to accomodate it. In the early stages of quantum gravity phenomenology, it was suggested that E_{QG} could play a role in modifications of the in vacuo dispersion relation [18], as follows

$$m^2 = E^2 - p^2 + f(E, p, m, E_{QG}) \,, \tag{5.1}$$

where f is an unknown function with the dimensions of mass squared depending on the kinematical quantities at play. For phenomenological studies, the focus is on a class of models expanded at the first order in the quantum gravity scale where, for example, the dispersion relation can be written as

$$m^2 \approx E^2 - p^2 + f_1 \frac{E^3}{E_{QG}} + f_2 \frac{p^2 E}{E_{QG}}\,, \tag{5.2}$$

where f_1, f_2 are two real constants. It is straightforward to check that this expression is not invariant under the usual Lorentz boost transformations for non-zero values of f_1, f_2. Without any additional ingredient, phenomenological models with modified dispersion relations of the form (5.1) break Lorentz invariance and are known as Lorentz Invariance Violating (LIV) models. They introduce a preferred class of reference frames where the on-shell relation is valid so that the quantum gravity scale itself becomes observer dependent. By contrast, Doubly Special Relativity (DSR) theories preserve the principle of relativity and require that the quantum gravity scale should be an invariant energy scale, in addition to the speed of light [22–24]. The relativity principle ensures that the modified dispersion relation (5.1) is the same for every inertial observer, in contrast with the LIV models. As we will now show with an example, this inevitably requires the deformation of special relativistic symmetries and of the energy-momentum conservation laws. We gain inspiration from the timelike κ-Minkowski framework presented in section 1.3 and restrict it to $1+1$ dimensions for simplicity (the extension to the $3+1$ case is straightforward). At the first order in the deformation parameter, the Casimir element inspires a modified dispersion relation of the form

$$m^2 = E^2 - p^2 - \ell E p^2\,, \tag{5.3}$$

where ℓ is a shorthand notation for $1/E_{QG}$. If we allow a deformation of the boost transformations such that

$$[N, E] = ip \qquad [N, p] = i\left(E - \ell E^2 - \frac{\ell}{2} p^2\right), \tag{5.4}$$

then (5.3) is invariant under the deformed Lorentz transformation at first order in ℓ. Indeed

$$\begin{aligned} [N, m^2] =& [N, E^2 - p^2 - \ell E p^2] = \\ =& i(2Ep - 2pE + 2\ell p E^2 + \ell p^3 - \ell p^3 - 2\ell p E^2) = 0\,. \end{aligned} \tag{5.5}$$

When more particles are involved, it can be easily shown that the standard energy-momentum composition law is not compatible with the deformed boost transformations (5.4). Consider a scattering process involving two particles, labeled by A, B interacting with another pair of particles, labeled by C, D. The standard energy-momentum conservation law reads

$$E_A + E_B = E_C + E_D\,, \qquad p_A + p_B = p_C + p_D\,. \tag{5.6}$$

Employing the deformed transformations (5.4), it can be shown that

$$[N_A + N_B, p_A + p_B] \neq [N_C + N_D, p_C + p_D]\,, \tag{5.7}$$

spoiling relativistic covariance. The way out of this issue is to require that both the momentum and boost composition laws are deformed. This time, gaining inspiration from the timelike κ-Poincaré coproducts in section 1.3, one can verify that the composition laws

$$\begin{aligned} (E_A \oplus E_B) = E_A + E_B \qquad (p_A \oplus p_B) = p_A + p_B - \ell E_A p_B \\ (N_A \oplus N_B) = N_A + N_B - \ell E_A N_B \end{aligned} \tag{5.8}$$

guarantee that the novel energy-momentum conservation law is covariant. Indeed, it can be shown that if

$$(E_A \oplus E_B) = (E_C \oplus E_D) \qquad (p_A \oplus p_B) = (p_C \oplus p_D)\,, \tag{5.9}$$

then, the deformed boost composition law yields

$$\begin{aligned} [N_A \oplus N_B, E_A \oplus E_B] &= [N_C \oplus N_D, E_C \oplus E_D] \\ [N_A \oplus N_B, p_A \oplus p_B] &= [N_C \oplus N_D, p_C \oplus p_D] \,, \end{aligned} \tag{5.10}$$

at first order in ℓ. As stressed in chapter 4 for the spatial counterpart of the timelike κ-Minkowski model, and first observed in [160], the deformed boost transformation (5.4) is not exclusively compatible with the composition laws (5.8). That they would work is merely a consequence of the Hopf algebra axioms of the timelike κ-Poincaré algebra. In [160], investigations of the momentum space structure of the timelike κ-Poincaré model lead to the realization that the composition laws

$$\begin{aligned} (E_A \oplus E_B) &= E_A + E_B - \ell p_A p_B \\ (p_A \oplus p_B) &= p_A + p_B - \ell(E_A p_B + E_B p_A) \\ (N_A \oplus N_B) &= N_A + N_B \,, \end{aligned} \tag{5.11}$$

are also compatible with the algebra (5.4). These are none other than the Loentzian version of the dS composition laws introduced in (4.11).

The two phenomenological models presented are fully relativistic, differing only in their multi-particle sector, and there is no reason to prefer one over the other. In this bottom up approach where the first order corrections to special relativity are contemplated, it is important to explore all the phenomenological possibilities. If ever, only data will guide us towards the framework that best describes the relevant physical phenomena [169].

The structural difference between DSR and Special Relativity resides in the geometry of momentum space [160]. In the former, the momentum space geometry presents non-linearities that vanish in the limit $\ell \to 0$, where we recover the flat and linear momentum space of Special Relativity. The main idea is that the deformed dispersion relation is linked to the metric on momentum space, the deformed symmetries are the Killing vectors associated to that metric and the composition laws depend on the connection [160]. In the case of the timelike κ-Poincaré model, the momentum space is identified with the de-Sitter manifold AN(3) in 3+1 dimensions [62, 69, 70], and the two different composition laws correspond to different choices for the connection: the noncommutative but associative composition laws (5.8) result from a choice of a torsionful connection while the commutative but non-associative composition laws (5.11) stem from the metric connection [160], which is torsionless.

In passing, it is worth mentioning that some first steps have been taken towards the generalization of relativistic theories with an invariant energy scale when also spacetime curvature is taken into account [170–172]. Some phenomenological analyses have found interesting physical effects that depend on the interplay between the Planck scale and the cosmological scale [173–176]. More formal developments have found that a pivotal role might be played by Finsler geometry [177, 178], a generalization of Riemannian geometry in which the metric acquires a velocity dependence. It was also shown that flat DSR models can be embedded within this framework [179–181].

In the next section we will review some of the most studied predictions of DSR and LIV models, relevant for astrophysical experiments, discussing analogies and differences between the new physics effects stemming from the two frameworks.

5.2 Quantum gravity phenomenology in the multi-messenger approach

An immediate consequence of the modification of the in-vacuo dispersion relation is an energy-dependent correction to the speed of massless particles [18]. At leading order in $1/E_{QG}$, this modification can be written in the form

$$E^2 = p^2 + sE^2 \frac{E}{E_{QG}} + \mathcal{O}\left(\frac{E}{E_{QG}}\right)^2 , \tag{5.12}$$

where $s = \pm 1$ is a sign distinguishing different models. Assuming that the velocity of massless particles is calculated as $v = dE/dp$, we obtain

$$v \approx \left(1 + s\frac{E}{E_{QG}}\right) . \tag{5.13}$$

The case $s = 1$ allows for subluminal propagation of massless particles while the case $s = -1$ allows for superluminal propagation instead. These tiny velocity corrections lead to potentially observable effects in the following scenario. Consider a cosmological source emitting two photons of different energies nearly simultaneously. When propagating for cosmological distances to reach a telescope on Earth, the small velocity difference translates to an observable time delay between the arrival of the two photons [18]. When taking into account spacetime curvature according to the standard cosmological model, the expression for the time delay can be written as [182]

$$\Delta t = s\frac{E_2 - E_1}{E_{QG}} \int_0^z \frac{1+z'}{H(z')} dz' , \tag{5.14}$$

where $H(z') = H_0\sqrt{\Omega_m(1+z')^3 + \Omega_\Lambda}$, Ω_m, Ω_Λ are the matter and dark energy fractions in the Universe, respectively, and H_0 is the Hubble constant. This is known as the Jacob-Piran formula [182] and provides an example of smooth onset quantum gravity effect which depends on the ratio between the energy of the probes involved and the quantum gravity scale. The part of the amplifier is played by the cosmological distance.

In [18], Gamma Ray Bursts (GRBs) were identified as a promising candidate sources to study this effect. GRBs are a transient astrophysics phenomena characterized by very bright emissions of gamma rays. The first observations suggested the existence of two classes of GRBs, short and long, based on the duration of the prompt phase, with a boundary around 2s. Particularly relevant for investigations on the time delay effect are short GRBs, given the 2s upper bound on the intrinsic time lag between hard and soft photons produced in the explosion. Indeed, for a GRB of redshift $z = 1$, the prediction is that two photons with an energy difference of about 100 GeV will arrive with a time delay of about 4s (with $E_{QG} \approx E_p$), well within the reach of current experimental sensitivities.

So far, several strategies of analysis have been identified to set bounds on the quantum gravity scale by testing the time delay effect using gamma ray data coming from GRBs [183–189], the majority of which yield a lower bound on E_{QG} of the order of 10^{19} GeV. The major challenge for this type of studies is to discern between possible delays due to the GRB emission mechanism and quantum gravity induced time delays.

In this perspective, a complementary experimental opportunity lies in the observation of cosmological neutrinos. These were widely expected to be emitted from GRBs [190], but after several years of activity, the IceCube collaboration found no conclusive evidence of neutrino-GRB association [191, 192]. It was soon realized that the absence of such associations could be due to quantum gravity corrections to the propagation speed of neutrinos, stemming from modifications to the in-vacuo dispersion relation [193]. Typical cosmological neutrino energies lie in the TeV-PeV range, which translates into a time delay up to the order of days, so that the intrinsic GRB emission mechanisms can be safely neglected.

Some first analyses [112,193,194] using IceCube data found preliminary evidence that GRB-neutrino pairs could exist when taking into account the time delay effect, but were inconclusive regarding the super- or sub-luminal nature of the propagation. Whether this is to be considered a weak point of those analyses depends on the specific model under consideration. Some effective quantum gravity models predict that all half-integer spin particles should be affected by the same corrections [26] while some other predict that the sign of the time delay should depend on the helicity, which however IceCube does not measure [195]. After IceCube revised the angular data of the detected neutrinos, a new analysis [196] was conducted providing encouraging, but still preliminary evidence of quantum gravity induced time delays due to subluminal propagation of the very high energy neutrinos. To get a feeling of how these phenomenological analyses are conducted, we give a brief review of the main steps performed to reach this last outstanding result, following [196].

- The objective is to test the correlation between times of arrival of neutrinos (with respect to the prompt emission of their GRBs) and their redshift rescaled energy:

$$\Delta t = \eta D(1) \frac{\mathcal{K}(E,z)}{M_p}\,, \tag{5.15}$$

  where η is a dimensionless parameter and $\mathcal{K}(E,z) \equiv ED(z)/D(1)$, with

$$D(z) = \int_0^z dz' \frac{1+z'}{H_0\sqrt{\Omega_\Lambda + (1+z')^3\Omega_m}}\,, \tag{5.16}$$

  being the redshift function present in the Jacob-Piran formula introduced in (5.14).

- The focus is on events with energies between 60 TeV and 500 TeV and a temporal window of 3 days, prior and post the prompt emission of the GRB. A wider energy range, say up to 1 PeV, would allow for a temporal window of up to 6 days, which would increase the number of cases where multiple GRB partners are associated to a single neutrino, greatly increasing the computational complexity when data simulations have to be performed (see last step).

- Angular associations between neutrinos and GRBs are performed by requiring that the angular distance between the netrinos and the source is less than 3σ, where $\sigma = \sqrt{\sigma_\nu^2 + \sigma_{GRB}^2}$, with σ_ν,σ_{GRB} being the uncertainties on the angular position of the neutrino and the GRB, respectively.

- The selection criteria discussed above yield a list of N neutrinos, each associated to multiple GRBs, in general. Among these, the dataset with the N GRB-neutrino pairs leading to the highest correlation between Δt and $\mathcal{K}(E,z)$ is selected.

- The robustness of the result is judged by computing a False Alarm Probability (FAP), a statistical indicator signaling how likely it would be that background events might exhibit the correlation found using the available data. This step is performed with "fake" data, realized by randomizing the times of arrival of the neutrinos, while keeping their energies and directions fixed. The FAP is then calculated by estimating how likely it would be that a set of N background GRB-neutrino candidates would produce a correlation greater or equal to the highest correlation found in the previous step.

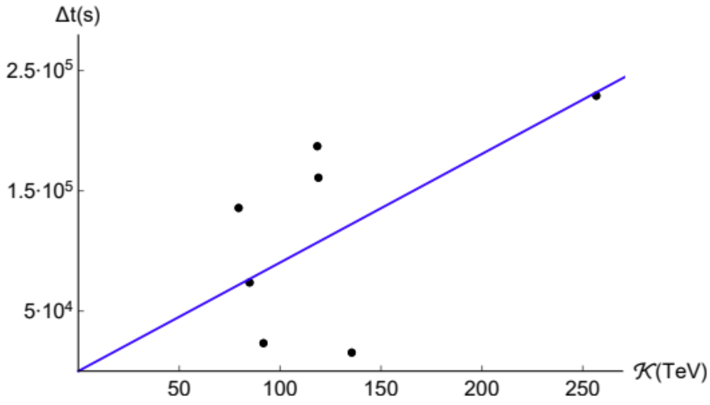

Figure 5.1: The seven late GRB neutrino candidates selected with the criteria outlined above. The blue line is obtained by performing a best fit using (5.15). Figure taken from [196].

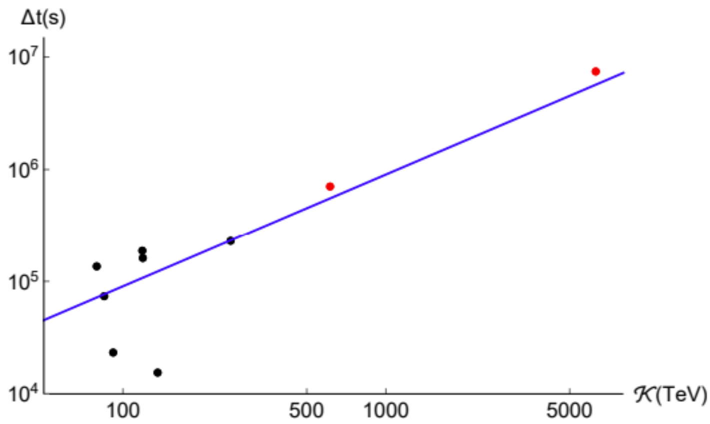

Figure 5.2: The two late GRB-neutrino candidates with energy greater than 500 TeV are added in red to the plot of fig. 5.1. Figure taken from [196].

In [196], seven late neutrino-GRB candidates have been identified, with a highest correlation of 0.56. The seven pairs with the highest correlation yield $\eta = 21.7 \pm 9.0$ (see fig. 5.1) with a FAP of only 0.7%. The early neutrinos are found to be compatible with background events. To avoid the computational complications discussed above, neutrinos with energies above 500 TeV are treated differently. These are selected if they satisfy the same angular criterion for the previous GRB-neutrino candidates and if their time of arrival lies within the range $|\Delta t - \eta\, \mathcal{K}(E,z)| \leq 2\, \delta\eta\, \mathcal{K}(E,z)$, when Δt is positive, where $\delta\eta$ is the uncertainty on η. In [196], two such neutrinos are selected, so that fig. 5.1 is updated to fig. 5.2. By dealing with the multiple GRB-neutrino candidates as done previosuly, the highest correlation increases to 0.9997. Upon performing the time randomization for these more energetic neutrinos, the FAP of finding two neutrino-GRB pairs with energy greater than 500 TeV compatible with the previous 7 candidates is of only 0.005%. This outstanding result is not conclusive evidence of quantum gravity induced time delays of subluminal nature, but it encourages to proceed with further analyses, possibly using the same logic, once more data is acquired.

The Jacob-Piran formula employed in time delay analyses mentioned above is by far the most used in phenomenological studies, but is by no means the only possibility to contemplate. In the context of LIV models, it has been shown that in principle any arbitrary form of redshift dependence could be allowed [174,197]. In DSR theories, the constraints coming from relativistic symmetries only allow a linear combination of three distinct time delay terms with different redshift dependence [173, 176, 198].

Another notable opportunity for this type of astrophysical phenomenology is the modification of energy thresholds. In effective quantum gravity models these can be altered due to the changes in the kinematics of particle processes arising from modified dispersion relations and modified energy-momentum conservation laws. In this scenario, the difference between LIV and DSR becomes all the more important, given that the latter models are supplemented with with modified energy-momentum conservation laws, yielding very different results with respect to their symmetry breaking counterpart. A relevant example is electron-positron pair production from the interaction of very-high energy photons with low-energy photons, such as those from the CMB and extragalactic background light (EBL). Following [199], the modified energy threshold for the high-energy photon can be written in the LIV and DSR scenario as

$$\begin{aligned} E_{\rm th}^{\rm LIV} &\approx \frac{m_e^2}{\epsilon}\left(1+\alpha\frac{m_e^4}{\epsilon^3 E_{QG}}\right)=\left(1+\alpha\frac{m_e^2}{\epsilon^2}\frac{E_{\rm th}^{\rm SR}}{E_{QG}}\right)E_{\rm th}^{\rm SR}, \\ E_{\rm th}^{\rm DSR} &\approx \frac{m_e^2}{\epsilon}\left(1+\beta\frac{m_e^2}{\epsilon E_{QG}}\right)=\left(1+\beta\frac{E_{\rm th}^{\rm SR}}{E_{QG}}\right)E_{\rm th}^{\rm SR}, \end{aligned} \tag{5.17}$$

where m_e is the mass of the electron, ϵ is the energy of the background photon, α and β are dimensionless parameters characterizing the deformation. This result is computed at first order in the inverse of E_{QG} and predicts that the universe is more transparent to high-energy radiation for positive values of α, β while the converse is true for negative values of these coefficients. For the LIV scenario, the amplifier is given by $\frac{m_e^2}{\epsilon^2}$, which ranges from 10^{11} to for visible EBL photons to 10^{18} for CMB photons while in the DSR case the effect is much attenuated, due to the absence of this ratio. For phenomenology, the basic idea is that observation of very-high energy photons allows to set limits on the quantum gravity scale using (5.17). More refined analyses evaluate corrections to the mean free path of the very-high energy photons by also taking into account the spectral density of background photons [200]. Going beyond the deformation of the kinematics of the process, one should further consider corrections to the cross-sections of the relevant processes, which in the LIV case is handled by adopting the LIV extension of the Standard Model [201], while in the DSR context such technology is still lacking, although some first formulas have been proposed in [202].

In the context of LIV models, considerably more sensible to modifications of energy thresholds than their DSR counterparts, a further phenomenological distinction is in order when particle processes are considered. Indeed, superluminal LIV models allow for in-vacuo photon decay ($\gamma \to e^+e^-$) and neutrino electron-positron pair emission ($\nu \to \nu e^+e^-$) which are forbidden processes in subluminal LIV and in DSR models. For example, the energy threshold for photon decay when the dispersion relation is modified as in (5.12) is given by $E_\gamma = \sqrt[3]{4m_e^2 E_{QG}}$ [203], leading to an estimate of the lower bound given by

$$E_{QG} > \frac{E_\gamma^3}{4m_e^2}\,, \tag{5.18}$$

when a high energy photon is observed. Once kinematically allowed, there is a high probability for the decay to occur, so this allows to put very stringent bounds on the superluminal LIV scale upon the observation of high energy photons. The latest bounds come from the observation of ultra-high energy gamma rays by LHAASO. In [188] the collaboration estimated that the superluminal first order LIV scale should be higher than 10^5 times the Planck scale, improving previous limits by an order of magnitude.

5.3 Quantum gravity phenomenology in the infrared

The phenomenology discussed in the previous section may be referred to as "UV phenomenology", given that highly energetic probes are involved in the relevant physical scenarios. We will now present the main results obtained in the complementary "IR quantum gravity phenomenology" program, much less explored than its UV counterpart. The focus is on infrared probes, in particular cold atoms, so we will be interested in the Galilean-relativistic limit ($p \ll m$) of modified dispersion relations. In a pioneering study [164], a LIV dispersion relation of the form

$$E \simeq m + \frac{p^2}{2m} + \frac{1}{2M_P}\left(\xi_1 m p + \xi_2 p^2 + \xi_3 \frac{p^3}{m}\right), \tag{5.19}$$

was considered, where ξ_1, ξ_2, ξ_3 are three dimensionless parameters. The main idea developed in [164] was to quantify corrections to the ratio h/m, measured with high accuracy in atom interferometry experiments, where m is the mass of the atom. As they will also be relevant for the original results of the next subsection, we review the basic concepts regarding atom interferometry and report the results of [164]. For obvious reasons, we restore the units of Planck's constant in what follows.

Atom interferometers measure the difference in phase between atomic matter waves traveling along different paths, which are usually controlled with laser beams. The phase shifts are measured with extremely high accuracy, yielding high precision measurements useful in fundamental physics, like tests of the equivalence principle [3] and measurements of the fine-structure constant [204, 205]. Of interest for us is the measurement of the h/m ratio of an atom subject to a two-photon Raman transition [206], which is a way of imparting momentum to an atom through a process involving the absorption of a photon of frequency ν and the stimulated emission (in the opposite direction) of a photon of frequency ν'. Upon the absorption of a photon, the atom transitions from its ground state $|g\rangle$ to an excited state $|e\rangle$ with energy separation given by $h\nu_*$. From energy conservation we have

$$\frac{p^2}{2m} + h\nu = h\nu' + \frac{p_f^2}{2m}, \tag{5.20}$$

where p is the initial momentum of the atom and p_f is its final momentum. The momentum conservation reads

$$p_f = p + h(\nu + \nu'), \tag{5.21}$$

and the lasers are fine-tuned such that

$$h(\nu + \nu') = 2h\nu_*. \tag{5.22}$$

Combining the two conservation laws, we can obtain an estimate of the h/m ratio:

$$\frac{\Delta\nu}{2\nu_*(\nu_* + p/h)} = \frac{h}{m}, \tag{5.23}$$

where $\Delta\nu = \nu - \nu'$. In cases where the quantities $\Delta\nu, \nu_*$ and p are well controlled in atom interferometry experiments [207], the ratio $\frac{h}{m}$ can be estimated with a sensitivity as high as a part in 10^{10}. When introducing Planck-scale corrections, we follow the same logic as in the undeformed case, with the exception that the kinetic energy is given by (5.19). We start by only switching on only coefficient ξ_1 in (5.19), yielding the following expression for $\Delta\nu$, at first order in $1/M_p$:

$$\Delta\nu \simeq \frac{2\nu_*(h\nu_* + p)}{m} + \xi_1 \frac{m}{M_p}\nu_*. \tag{5.24}$$

The estimate for h/m now contains a ξ_1-dependent correction, and reads

$$\frac{\Delta\nu}{2\nu_*(\nu_* + p/h)}\left[1 - \xi_1\left(\frac{m}{2M_p}\right)\left(\frac{m}{h\nu_* + p}\right)\right] = \frac{h}{m}\,. \tag{5.25}$$

As could be expected when introducing a mass scale in the corrections for the dispersion relation, the quantum gravity correction exhibits a suppression factor m/Mp, which is of the order of 10^{-17} for the Caesium and Rubidium atoms typically employed in the interferometry experiments [204, 205]. However, the above expression also contains a factor $m/(h\nu_* + p)$ which plays the role of the amplifier in this case. Indeed, this dimensionless combination of mass, momentum and resonance frequency can be of the order of 10^9 in typical experiments [207]. It turns out that such an amplification yields a sizeable correction that falls within the sensitivities of atom interferometry measurements. Using data collected from the Caesium atom interferometry experiment conducted at the time [207], [164] reported an estimate for ξ_1 of -1.8 ± 2.1 at 95% confidence level, thus showing for the first time that quantum gravity effects in infrared physics might be as detectable as their ultraviolet counterpart, enriching the number of phenomenological possibilities at our disposal. The quantum gravity effects proportional to ξ_2 and ξ_3 turn out to be less interesting. When switching on only the correction term proportional to p^2 in (5.19), we obtain

$$\frac{\Delta\nu}{2\nu_*(\nu_* + p/h)}\left[1 - \xi_2\left(\frac{m}{2M_p}\right)\right] = \frac{h}{m}\,. \tag{5.26}$$

Compared to its ξ_1 counterpart, this type of correction loses a factor 10^9 due to the absence of the amplification term, thus leading to less stringent constraints. In this perspective, the situation is even worse for the ξ_3 term, which produces an expression of the form

$$\frac{\Delta\nu}{2\nu_*(\nu_* + p/h)}\left[1 - \xi_3\left(\frac{4h^2\nu_*^2 + 6h\nu_* p + 3p^2}{2M_p(h\nu_* + p)}\right)\right] = \frac{h}{m}\,, \tag{5.27}$$

so it is characterized by a suppression of the order $\sim p/M_p \ll m/M_p$. The fact that the ξ_1 term produces the highest correction is not unexpected and may be understood qualitatively as follows. The dispersion relation (5.19) is valid in the infrared regime $p/m \ll 1$ and the introduction of the Planck mass as an UV scale triggers a *kinematical IR/UV mixing* mechanism. When the relevant momenta involved are very small with respect to the mass of the atom, as is the case in atom interferometry experiments, the correction term linear in p becomes of leading order in the dispersion relation (except for the constant mass term) and results in an enhancement of the kinetic energy thanks to the amplifier m/p. In this context, the mixing mechanism is said to be kinematical since it arises from the modifications to the dispersion relation, in contrast with the IR/UV mixing in θ-Poincaré mentioned in section 1.2, which is triggered when analyzing the dynamics of the theory. In formulas, (5.19) can be rewritten as

$$E \simeq m + \frac{p^2}{2m}\left[1 + 2\xi_1\frac{m}{M_p}\frac{m}{p} + 2\xi_2\frac{m}{M_p} + 2\xi_3\frac{p}{M_p}\right]. \tag{5.28}$$

The corrections to the kinetic energy in the square brackets reflect the functional form of the corrections obtained for the ratio h/m, which is computed from an energy difference. In (5.28) it is even clearer that in the "small momenta" regime, the kinetic energy receives a major correction from the ξ_1 term.

5.4 Cold atom phenomenology in lightlike κ-Minkowski

As mentioned in section 5.2, it is of theoretical and phenomenological interest to distinguish between LIV and DSR effects. In the context of cold atom phenomenology, a first DSR computation of the corrections to h/m has been performed in [208], yielding corrections of the "ξ_2 type", as in (5.26). In [208], the DSR model is inspired from a particular basis of the timelike κ-Poincaré algebra [209], in which the dispersion relation is unmodified and all the non-triviality is stored in the energy-momentum conservation. We will now show that a DSR model inspired by the lightlike κ-Minkowski noncommutative framework exhibits a dispersion relation with a kinematical IR/UV mixing mechanism analogous to the one produced by the ξ_1 term in (5.19), yielding measurable corrections to the kinematics of cold atoms in interferometry experiments.

5.4.1 Mathematical preliminaries

We start by considering the 3+1-dimensional κ-lightlike noncommutative framework following [210], characterized by commutation relations among the coordinates of the form:

$$[x^+, x^-] = i\sqrt{2}\ell x^- , \qquad [x^+, x^i] = i\sqrt{2}\ell x^i , \qquad i = 2, 3 . \tag{5.29}$$

In the above and in the subsequent equations, ℓ is a deformation parameter which we have defined as $\ell = 1/\sqrt{2}\kappa$ with respect to the conventions adopted in [210]. The complete Hopf Algebra structure describing the set of deformed symmetries for this noncommutative spacetime can be found in [51, 52, 210]and is written in the so-called bicrossproduct basis, which differs from the one employed in the quantum field theory chapter, as explained in chapter 2. For the purposes of the present study we will only be interested in the coproducts of the translation generators:

$$\begin{aligned} \Delta(P_+) &= P_+ \otimes 1 + 1 \otimes P_+ , \\ \Delta(P_I) &= P_I \otimes 1 + e^{-\sqrt{2}\ell P_+} \otimes P_I , \end{aligned} \tag{5.30}$$

where $I \in \{-, 2, 3\}$, and on the deformed Casimir given by

$$C = \frac{4}{\sqrt{2}\ell} P_- e^{\frac{\ell P_+}{\sqrt{2}}} \sinh\left(\ell \frac{P_+}{\sqrt{2}}\right) - (P_2^2 + P_3^2) e^{\sqrt{2}\ell P_+} . \tag{5.31}$$

To gain a more direct physical interpretation of the physical quantities characterizing the cold-atom kinematics, let us switch to Cartesian coordinates using the maps

$$x^\pm = \frac{x^0 \pm x^1}{\sqrt{2}} \qquad P_\pm = \frac{P_0 \pm P_1}{\sqrt{2}} , \tag{5.32}$$

while the 2, 3 components are left unchanged. The coordinate commutation relations (5.29) can thus be rewritten as

$$[x^0, x^1] = i\ell(x^1 - x^0) , \quad [x^0, x^i] = i\ell x^i , \quad [x^1, x^i] = i\ell x^i , \quad i = 2, 3 \tag{5.33}$$

and the coproducts (5.30) assume the form

$$\begin{aligned} \Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0 + (1 - e^{-\ell(P_0+P_1)}) \otimes \frac{P_1 - P_0}{2} , \\ \Delta(P_1) &= P_1 \otimes 1 + 1 \otimes P_1 + (1 - e^{-\ell(P_0+P_1)}) \otimes \frac{P_0 - P_1}{2} , \\ \Delta(P_2) &= P_2 \otimes 1 + e^{-\ell(P_0+P_1)} \otimes P_2 , \\ \Delta(P_3) &= P_3 \otimes 1 + e^{-\ell(P_0+P_1)} \otimes P_3 . \end{aligned} \tag{5.34}$$

while the mass-Casimir is

$$C = \frac{2}{\ell}(P_0 - P_1)e^{\frac{\ell(P_0+P_1)}{2}} \sinh\left(\ell\frac{P_0+P_1}{2}\right) - (P_2^2 + P_3^2)e^{\ell(P_0+P_1)}. \tag{5.35}$$

In the equations presented above, it is clear that direction x_1 plays a special role in lightlike κ-Minkowski. Along the lines of what is discussed in chapter 3, this feature does not imply that there is a physical preferred direction in space. The relativistic consistency relations of this noncommutative framework deform rotation symmetry, as can be appreciated by inspecting the commutators and coproducts involving the rotation generators in [210]. The idea is that each observer connected by a deformed symmetry transformation experiences different effects along the x^1 direction with respect to their x^2, x^3 directions. We will comment on the phenomenological consequences of this feature in the subsequent paragraphs.

The key elements of our DSR model are obtained in the standard way [26]: the deformed mass-shell relation is inspired by the Casimir (5.35) while the energy-momentum composition laws are inspired by the coproducts (5.34). For phenomenological purposes, we expand these quantities at first order in ℓ and compute all relevant quantities up to this order. The composition of two momenta, k_μ, q_μ, thus reads

$$\begin{aligned}
(k \oplus q)_0 &= k_0 + q_0 - \frac{\ell}{2}(k_0 + k_1)(q_0 - q_1)\,, \\
(k \oplus q)_1 &= k_1 + q_1 + \frac{\ell}{2}(k_0 + k_1)(q_0 - q_1)\,, \\
(k \oplus q)_2 &= k_2 + q_2\,(1 - \ell(k_0 + k_1))\,, \\
(k \oplus q)_3 &= k_3 + q_3\,(1 - \ell(k_0 + k_1))\,,
\end{aligned} \tag{5.36}$$

while the dispersion relation for a particle of mass m and momentum p_μ reads

$$m^2 = \left(p_0^2 - p_1^2\right)\left(1 + \ell\left(\frac{p_0 + p_1}{2}\right)\right) - (p_2^2 + p_3^2)\left(1 + \ell(p_0 + p_1)\right). \tag{5.37}$$

A perturbative solution of the on-shell relation above, at first order in ℓ, is:

$$p_0 = \sqrt{m^2 + p_1^2 + p_2^2 + p_3^2} + \ell\,\frac{\left(-m^2 + p_2^2 + p_3^2\right)\left(\sqrt{m^2 + p_1^2 + p_2^2 + p_3^2} + p_1\right)}{4\sqrt{m^2 + p_1^2 + p_2^2 + p_3^2}}\,. \tag{5.38}$$

Since our objective is to describe the deformed kinematics of cold atoms, we expand (5.38) up to first order in m^{-1}, obtaining

$$p_0 = m - \ell\,\frac{m^2}{4} + \frac{p_1^2 + p_2^2 + p_3^2}{2m} - \frac{\ell}{4}\left[m\,p_1 - p_2^2 - p_3^2 - \frac{p_1}{2m}\left(p_1^2 + 3\,p_2^2 + 3\,p_3^2\right)\right]. \tag{5.39}$$

Inspecting (5.39), one can recognize some similarities with the isotropic LIV dispersion relation (5.19), with the structural difference that our DSR formula spoils the usual notion of isotropy. The term proportional to mp_1 is a byproduct of the kinematical IR/UV mixing mechanism. It will play a role analogous to the "ξ_1" term of the LIV case, amplifying our result thanks to the small momenta of the atoms. The p^2 and p^3 terms in the deformations are analogous to the "ξ_2" and "ξ_3" terms of (5.19) and will produce subleading corrections with respect to mp_1. It is also worth mentioning that in (5.39), the mass m is not defined as the rest energy of a particle, since $p_0 = m - \ell\frac{m^2}{4}$, when the spatial components of the momentum vanish. If one insists on the definition of mass as the rest energy of a particle, then the physical mass m' should be defined in terms of m as $m' = m - \ell\frac{m^2}{4}$. Nevertheless, one can show that this redefinition does not affect the leading order of our results, so for all practical purposes we can keep using (5.39) as the expression for our mass-shell relation. The IR/UV mixing term produced in (5.39) is unavailable in the timelike κ-Poincaré scenario. Indeed, insisting on the standard notion of isotropy, the corresponding "ξ_1" term would be proportional to $m|\vec{p}|$, which is absent in the frameworks inspired by Hopf Algebras, due to the fact that all relevant quantities are analytic in P_μ.

Since we want to study interactions between cold atoms and photons, we are also interested in the on-shell relation for massless particles which reads:

$$p_0 = \sqrt{p_1^2 + p_2^2 + p_3^2} + \frac{\ell p_2^2}{4} + \frac{\ell p_3^2}{4} + \frac{\ell p_1}{4\sqrt{p_1^2 + p_2^2 + p_3^2}}(p_2^2 + p_3^2)\,, \tag{5.40}$$

at first order in ℓ. As expected, the IR/UV mixing mechanism plays no role in this case, given that the mass is 0. Moreover, the role of the special direction x^1 can be clearly appreciated. When propagating along the x^1 axis, the photon energy receives no quantum gravity corrections.

5.4.2 Planck scale corrections along a generic direction

A complete atom-interferomety sequence can be well approximated as the propagation of an atomic wave in 1 spatial dimension, which interacts with photons traveling along the same direction.

We can model the Raman transition as a two-body into two-body interaction, involving an atom and a photon both in the initial and in the final state. Since the composition law is noncommutative, we have a total of four possible orderings choices for the momenta entering the kinematical description of the interaction. A full-fledged interaction theory on lightlike κ-Minkowski would also allow us to weigh the various momenta configurations in computing the overall correction for this process. In absence of this, we have no way of preferring one configuration over the other, so the final result will be the average of the four possible interactions.

The idea is similar to the one discussed for the LIV model, with the additional complication of the deformed energy-momentum conservation laws. Let p^i_μ, p^f_μ be the initial and final momentum of the atom, respectively, and let k_μ, k'_μ be the momenta associated to the photon in the initial and final state, respectively. For the $A + \gamma \to A' + \gamma'$ ordering, where A and γ refer to the atom and the photon in the initial state, respectively, and primed quantities refer to the final state, the energy momentum conservation reads

$$(p^i \oplus k)_\mu = (p^f \oplus k')_\mu\,, \quad \mu = 0, ..., 3\,, \tag{5.41}$$

where the $\oplus$ is defined in (5.36) for all the components. Upon enforcing the on-shell relation, the ensuing system of equations can be solved to obtain corrections to the quantity h/m at first order in ℓ. We parametrize them via

$$\frac{\Delta\nu}{2\nu_*(\nu_* + p^i/h)}[1 + \ell\alpha] = \frac{h}{m}\,, \tag{5.42}$$

where p^i is the modulus of the spatial part of the initial momentum of the atom and α has dimensions of energy and is a function of the momenta involved. Expressing the spatial part of p^i_μ as $\vec{p^i} = p^i(\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$, where $0 < \theta < \pi$ and $0 < \phi < 2\pi$ are the usual polar coordinates, the resulting list of leading order corrections for the various particle orderings is given by

- $A + \gamma \to A' + \gamma'$

$$\alpha = -\frac{1}{(p^i + h\nu^*)}\frac{3m^2}{4}\cos(\phi)\sin(\theta)\,, \tag{5.43}$$

- $\gamma + A \to \gamma' + A'$

$$\alpha = \frac{m^2}{4(p^i + h\nu^*)}\cos(\phi)\sin(\theta)\,, \tag{5.44}$$

- $\gamma + A \to +A' + \gamma'$

$$\alpha = -m^2\frac{\cos(\phi)\sin(\theta)}{4(p^i + h\nu^*)}\,, \tag{5.45}$$

- $A + \gamma \to +\gamma' + A'$

$$\alpha = -m^2\frac{\cos(\phi)\sin(\theta)}{4(p^i + h\nu^*)}\,. \tag{5.46}$$

All the correction terms are amplified by the $m/(p^i + h\nu_*)$ factor, which is analogous to the one obtained in (5.25) for the LIV scenario. Such an amplification factor is a byproduct of the IR/UV mixing term in the on-shell relation and is absent in the timelike κ-Poincaré framework where the leading order correction is of the order $\ell\alpha \propto \ell m$ [208], thus being a first of its kind in a DSR framework.

We now come to a crucial point for our phenomenological discussion. To appreciate the corrections along a generic direction, it is necessary to define a reference system and identify the propagation direction of the atom wave according to that reference frame. In this DSR framework in which one of the spatial directions plays a special role, operationally defining such a reference system is not so straightforward. To better understand the issue, let us think about the corresponding situation in a LIV context, where only spatial isotropy is broken so that there is a *fixed* special direction in space. In that case, the operational procedure of setting up a reference system is more clear. Ideally, one would have to perform an experiment in which the effect under study is is sensible to the direction of propagation of the system. Repeating the experiment along different directions and analyzing the distribution of outcomes, one would then identify a characteristic feature in the data, signaling which is the preferred direction in space. The analogous procedure in the DSR context adds a layer of complexity from the interpretive point of view. Repeating the procedure described above for the LIV scenario, each observer would in principle identify a special direction in their reference frame, but this special direction is bound to be different for different observers, given the relativistic nature of the theory. The missing piece of the puzzle for a satisfactory physical interpretation lies one step before performing the experimental procedure to determine the special direction. If the special direction is unique to each observer, how is it selected in the first place? What is the operational procedure that allows each observer to select direction x^1 if they were to live in a world governed by the lightlike κ-Poincaré symmetries? Recall that a similar issue is also present in chapter 3, where the reconstruction of spacetime as a collection of fuzzy events depends on each observer's choice of the z-axis. In that context too, the operational procedure needed to perform the choice of the special axis requires further investigation.

At our current understanding of fully relativistic models with deformed isotropy, the phenomenological scheme we propose to obtain our physical prediction is to average over the momenta orderings and the spatial configurations. The correction terms we have obtained are of the form

$$\alpha_i = \frac{m^2}{p^i + h\nu^*} k_i \cos(\phi) \sin(\theta) \,, \tag{5.47}$$

for numerical coefficients k_i running in the set $\{-\frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}\}$. Given the particular angular dependence of the α_i, it is easy to see that the average correction $\langle\alpha\rangle$ is

$$\langle\alpha\rangle = \frac{1}{16\pi} \int_{S^2} d\Omega \sum_i \alpha_i = 0 \,, \tag{5.48}$$

so that there is no effective correction to the mean value of $\frac{h}{m}$. Nevertheless, the variance of the correction is non-trivial, yielding

$$(\Delta\alpha)^2 = \frac{1}{16\pi} \int_{S^2} d\Omega \sum_i \alpha_i^2 = \frac{3\pi}{128} \left(\frac{m^2}{(p^i + h\nu^*)} \right)^2 , \tag{5.49}$$

which is to be compared to the variance obtained from the data in an atom interferometry experiment where measurements are performed for an experimental setup oriented in various configurations.

Discussion and outlook

Over the past 20 years, quantum gravity phenomenology has progressed so much that it has been established as its own research area in frontier physics [27, 28]. The development of effective quantum gravity models is mature enough that new physics predictions can be derived in a variety of physical scenarios ranging from highly energetic particles travelling through the cosmos to cold atoms accurately controlled in table-top experiments.

By now, the astrophysics sector is quite advanced, thanks to the synergy between theorists developing the effective models and experimentalists devising accurate statistical analyses to test the potential quantum gravity effects. Fundamental for the cause is also the parallel advancement of multi-messenger astronomy [27, 28].

The infrared, table-top sector is still largely unexplored and does not yet benefit from the aforementioned synergy between theory and experiment. Among the most essential objectives, the theoretical side requires further comprehension and systematization of models of the IR/UV mixing mechanism. In this chapter, for the first time, we have exposed the kinematical IR/UV mixing mechanism for the κ-lightlike noncommutative spacetime, known for more than 20 years [51, 52]. The result is the first of its kind in the context of theories with deformed relativistic symmetries. Other instances of IR/UV mixing have been found when studying quantum field theories on noncommutative spacetime [108, 116], yielding infrared divergences when the dynamics of the theory is investigated. Also paramount for setting meaningful bounds to the quantum gravity scale is to connect the theoretical predictions to what is directly measured in atom interferometry experiments, which are differential phase shifts. In the latest measurements of the h/m ratio using Caesium and Rubidium atoms [204, 205], the experimental setups involve several transitions of different nature to impart momentum to the atom. The atom waves are split and recombined using a system of lasers (akin to what is done for laser interferometry using mirrors) yielding differential phase shifts that can be measured with high precision, so that the ratio h/m is estimated with high sensitivity. A meaningful bound on a deformation parameter can only be set once a reliable computation of the quantum gravity corrections to the differential phase shift is performed. We plan to do this in the near future, by analyzing the interferometric sequences employed in the latest measurements of the h/m ratio. Nevertheless, this is not the end of the story. Due to the interpretative challenge posed by the deformed spatial isotropy of the model, any computation involving the lightlike κ-Minkwoski model is likely to require an angular average. A direct comparison between theoretical prediction and experimental evidence will thus inevitably require that the experiment itself is conducted in various configurations, by rotating the relevant apparatuses. This is currently not the case for the latest measurements mentioned above [204, 205]. These issues suggest a complementary, phenomenology-oriented research direction, which is to design interferometry configurations that may be more sensible to the quantum gravity effects predicted by effective and noncommutative spacetime models of the IR/UV mechanism.

Summary and conclusions

In this thesis we have explored some conceptual and phenomenological aspects of spacetime noncommutativity models, motivated by the very early stages of research development in quantum gravity. Throughout the various works on which this thesis is based, we have appreciated the fact that both formal and toy models can provide us with insights on the novelties introduced by the quantization of spacetime.

In chapter 2 we set up the quantum field theory for a complex scalar field in the lightlike κ-Minkowski noncommutative spacetime. After realizing that the lightlike deformation is the only viable one for constructing multilocal functions that serve as the basis for defining κ-Poincaré invariant N-point functions, we investigated the Fock space of the theory. The single-particle sector is identical to its undeformed counterpart, with the only difference that parity and time reversal are not symmetries of the theory, as can also be inferred by the defining commutators of the noncommutative spacetime. For the first time in the literature of κ-deformed quantum field theories, we proposed a well-defined notion of multiparticle states, thanks to the existence of the universal R-matrix, unique to the lightlike case. The R-matrix is the fundamental ingredient in constructing a deformed flip operator, the generalization of the usual one employed in standard quantum field theory to define bosonic and fermionic states. Gaining inspiration from the scalar case, we analyzed multi-fermionic states, finding that the concept of indistinguishability of particles of the same species is lost upon introducing the invariant energy scale. Moreover, the theory presents deviations from the Pauli exclusion principle, allowing for states containing two fermions with the same quantum numbers while excluding a different class of states. Especially this last novelty deserves further investigation in light of possible phenomenological applications to searches for Pauli exclusion principle violations in underground experiments [139–141].

Chapter 3 focuses on a toy model where the usual notion of isotropy is replaced by symmetry under the $SU_q(2)$ quantum group. Classical group parameters are replaced by operators acting on a Hilbert space whose states are interpreted as the ones describing the relative orientation between two reference frames. Upon investigating the Hilbert space of the theory, we found that one of the classical Euler angles becomes quantized, while the remaining two are continuous as their classical counterparts. Appropriately defining a quantum version of the $SO(3)$ rotation matrix, we found that rotations around a certain axis can be defined sharply, while all other rotations are inevitably affected by uncertainties on their angle parameters, due to quantum complementarity. We exploited a novel qualitative feature of quantum symmetries, that of agency-dependent spacetime. Spacetime, seen as a collections of events, is dependent on the choices of the observers in setting up their reference frame. Different observers will associate a different degree of fuzziness to a same event, due to the different states connecting their axes to the direction of that event.

In chapter 4, we focused on interactions between particles living in a Euclidean version of the timelike κ-Minkowski in 2 dimensions. We defined Noether charges (a concept still unclear in the context of spacetime noncommutativity) as those quantities that commute with a given Hamiltonian, thus establishing a strong link between the structure of the Noether charges and that of the interaction potential between particles. It turns out that for the Euclidean κ-Minkowski model, the total charges inspired by the coproducts of the theory are not the only choice available to define the total linear and angular momentum for a two- and three-particle system. An alternative choice, based on studies of the de-Sitter momentum space associated to κ-Minkowski, turns out to be compatible with the algebra sector of the model and yields a well-defined notion of conserved charges, on par with the coproduct inspired ones. We then investigated whether these charges could be "symmetrized", in the perspective of eliminating the ambiguities stemming from the noncommutativity and nonassociativity of the coproduct inspired and momentum space inspired composition laws, respectively. We found that for the former, such a symmetrization is not viable given that there exists no form of the interacting potential which commutes with the symmetrized charges, while for the latter this possibility is allowed.

The final chapter 5 is devoted to quantum gravity phenomenology in the infrared. A different basis of the lightlike κ-Minkowski framework adopted in chapter 2 inspired a DSR model with potential applications to cold-atom interferometry. Upon performing a nonrelativistic expansion of the deformed on-shell relation, we found an instance of a kinematical IR/UV mixing mechanism. In the infrared limit of very small momenta (with respect to the mass of the atom), a correction term to the kinetic energy, induced by the introduction of the ultraviolet noncommutativity scale, is the dominant one. By taking into account the deformed dispersion relation and the deformed energy-momentum conservation laws, we computed corrections to the kinematics describing a process in which momentum is imparted to a cold atom via absorption and stimulated emission of photons. The computation yields a correction to the uncertainty of the h/m ratio which can be measured with very high sensitivity in interferometry experiments. For the first time, we found a highly amplified correction in the DSR context, thanks to the mechanism of IR/UV mixing. So far, such a sizeable correction had only been found in LIV models in [164]. This preliminary result encourages further studies of applications of models with IR/UV mixing to cold atom interferometry experiments, with the hope that when all the details of the experimental sequence are taken into account, we will be able to put stringent limits on the noncommutativity scale.

While the topics treated in the various chapters are rather diverse, there is a common thread linking them, associated to a largely unexplored venue of research in deformed relativistic symmetries. The models employed in deriving the main results of this thesis are equipped, in one form or another, with a deformed notion of spatial isotropy. The first studies on spacetime noncommutativity with deformed relativistic symmetries all focused on isotropic models like timelike κ-Poincaré, with longitudinal deformations only affecting the boost sector of Lorentz symmetry. Expected features like spacetime fuzziness enter the picture only in the longitudinal sector and have been treated quite satisfactorily, also for what concerns the transformations between reference frames [63,64,211]. Analogous techniques can also be employed in the cases of *transverse* spacetime noncommutativity, as suggested by the preliminary results obtained for ρ-Minkowski [106]. Nevertheless, from the physical point of view, we are still far from a clear interpretation of models with deformed isotropy in which there is a special direction unique to each observer, in contrast with Lorentz-breaking models in which the preferred direction is the same for all observers. The results presented in this thesis do not address this question directly, but are certainly a good starting point for further investigations of the issue, with the hope of also finding novel phenomenological consequences for quantum gravity.

Bibliography

[1] G. E. Gorelik, "Matvei bronstein and quantum gravity: 70th anniversary of the unsolved problem," *Physics-Uspekhi*, vol. 48, p. 1039, oct 2005.

[2] M. Bronstein, "Quantum theory of weak gravitational fields," *Gen. Rel. Grav.*, vol. 44, pp. 267–283, 2012.

[3] P. Asenbaum, C. Overstreet, M. Kim, J. Curti, and M. A. Kasevich, "Atom-Interferometric Test of the Equivalence Principle at the 10^{-12} Level," *Phys. Rev. Lett.*, vol. 125, no. 19, p. 191101, 2020, 2005.11624.

[4] H. S. Snyder, "Quantized space-time," *Phys. Rev.*, vol. 71, pp. 38–41, Jan 1947.

[5] I. Gelfand and M. Neumark, "On the imbedding of normed rings into the ring of operators in hilbert space," *Mat. Sbornik.*, vol. 12, no. 2, pp. 197–217, 1943.

[6] I. E. Segal, "Irreducible representations of operator algebras," *Bull. Amer. Math. Soc.*, vol. 53, no. 2, pp. 73–88, 1947.

[7] A. Connes, *Non-commutative geometry.* Academic Press, 1994.

[8] G. Veneziano, "A stringy nature needs just two constants," *Europhysics Letters*, vol. 2, no. 3, p. 199, 1986.

[9] C. Rovelli and S. Speziale, "Reconcile planck-scale discreteness and the lorentz-fitzgerald contraction," *Physical Review D*, vol. 67, mar 2003.

[10] E. Witten, "Non-commutative geometry and string field theory," *Nuclear Physics B*, vol. 268, no. 2, pp. 253–294, 1986.

[11] N. Seiberg and E. Witten, "String theory and noncommutative geometry," *JHEP*, vol. 09, p. 032, 1999, hep-th/9908142.

[12] G. Amelino-Camelia, M. M. da Silva, M. Ronco, L. Cesarini, and O. M. Lecian, "Spacetime-noncommutativity regime of Loop Quantum Gravity," *Phys. Rev. D*, vol. 95, no. 2, p. 024028, 2017, 1605.00497.

[13] H.-J. Matschull and M. Welling, "Quantum mechanics of a point particle in (2+1)-dimensional gravity," *Class. Quant. Grav.*, vol. 15, pp. 2981–3030, 1998, gr-qc/9708054.

[14] L. Freidel and E. R. Livine, "3D Quantum Gravity and Effective Noncommutative Quantum Field Theory," *Phys. Rev. Lett.*, vol. 96, p. 221301, 2006, hep-th/0512113.

[15] V. Chari and A. Pressley, *A Guide to Quantum Groups.* Cambridge University Press, 1995.

[16] S. Majid, *A Quantum Groups Primer*. London Mathematical Society Lecture Note Series, Cambridge University Press, 2002.

[17] C. Kassel, *Quantum Groups*. Graduate Texts in Mathematics, Springer New York, 2012.

[18] G. Amelino-Camelia, J. R. Ellis, N. E. Mavromatos, D. V. Nanopoulos, and S. Sarkar, "Tests of quantum gravity from observations of gamma-ray bursts," *Nature*, vol. 393, pp. 763–765, 1998, astro-ph/9712103.

[19] G. Amelino-Camelia, "An Interferometric gravitational wave detector as a quantum gravity apparatus," *Nature*, vol. 398, pp. 216–218, 1999, gr-qc/9808029.

[20] D. Mattingly, "Modern tests of Lorentz invariance," *Living Rev. Rel.*, vol. 8, p. 5, 2005, gr-qc/0502097.

[21] G. Amelino-Camelia and T. Piran, "Planck scale deformation of Lorentz symmetry as a solution to the UHECR and the TeV gamma paradoxes," *Phys. Rev. D*, vol. 64, p. 036005, 2001, astro-ph/0008107.

[22] G. Amelino-Camelia, "Relativity in space-times with short distance structure governed by an observer independent (Planckian) length scale," *Int. J. Mod. Phys. D*, vol. 11, pp. 35–60, 2002, gr-qc/0012051.

[23] J. Magueijo and L. Smolin, "Lorentz invariance with an invariant energy scale," *Phys. Rev. Lett.*, vol. 88, p. 190403, 2002, hep-th/0112090.

[24] G. Amelino-Camelia, "Doubly special relativity," *Nature*, vol. 418, pp. 34–35, 2002, gr-qc/0207049.

[25] G. Amelino-Camelia, "Are we at the dawn of quantum gravity phenomenology?," *Lect. Notes Phys.*, vol. 541, pp. 1–49, 2000, gr-qc/9910089.

[26] G. Amelino-Camelia, "Quantum-Spacetime Phenomenology," *Living Rev. Rel.*, vol. 16, p. 5, 2013, 0806.0339.

[27] A. Addazi *et al.*, "Quantum gravity phenomenology at the dawn of the multi-messenger era—A review," *Prog. Part. Nucl. Phys.*, vol. 125, p. 103948, 2022, 2111.05659.

[28] R. Alves Batista *et al.*, "White Paper and Roadmap for Quantum Gravity Phenomenology in the Multi-Messenger Era," 12 2023, 2312.00409.

[29] G. Fabiano, G. Gubitosi, F. Lizzi, L. Scala, and P. Vitale, "Bicrossproduct vs. twist quantum symmetries in noncommutative geometries: the case of ϱ-Minkowski," *JHEP*, vol. 08, p. 220, 2023, 2305.00526.

[30] G. Fabiano and F. Mercati, "Multiparticle states in braided lightlike κ-Minkowski noncommutative QFT," 10 2023, 2310.15063.

[31] G. Amelino-Camelia, V. D'Esposito, G. Fabiano, D. Frattulillo, P. A. Hoehn, and F. Mercati, "Quantum Euler angles and agency-dependent spacetime," 11 2022, 2211.11347.

[32] G. Amelino-Camelia, G. Fabiano, and D. Frattulillo, "Total momentum and other Noether charges for particles interacting in a quantum spacetime," 2 2023, 2302.08569.

[33] J. Wess, "q-deformed heisenberg algebras," 1999, math-ph/9910013.

[34] R. J. Szabo, "Quantum field theory on noncommutative spaces," *Phys. Rept.*, vol. 378, pp. 207–299, 2003, hep-th/0109162.

[35] A. Addazi and A. Marcianò, "A modern guide to θ-Poincaré," *Int. J. Mod. Phys. A*, vol. 35, no. 32, p. 2042003, 2020, 1811.06425.

[36] G. Fiore and J. Wess, "On full twisted Poincare' symmetry and QFT on Moyal-Weyl spaces," *Phys. Rev. D*, vol. 75, p. 105022, 2007, hep-th/0701078.

[37] P. Aschieri, M. Dimitrijevic, F. Meyer, and J. Wess, "Noncommutative geometry and gravity," *Class. Quant. Grav.*, vol. 23, pp. 1883–1912, 2006, hep-th/0510059.

[38] A. P. Balachandran, T. R. Govindarajan, G. Mangano, A. Pinzul, B. A. Qureshi, and S. Vaidya, "Statistics and UV-IR mixing with twisted Poincare invariance," *Phys. Rev. D*, vol. 75, p. 045009, 2007, hep-th/0608179.

[39] R. Oeckl, "Braided quantum field theory," *Commun. Math. Phys.*, vol. 217, pp. 451–473, 2001, hep-th/9906225.

[40] R. Oeckl, "Untwisting noncommutative R^d and the equivalence of quantum field theories," *Nucl. Phys. B*, vol. 581, pp. 559–574, 2000, hep-th/0003018.

[41] M. Dimitrijević Ćirić, N. Konjik, V. Radovanović, and R. J. Szabo, "Braided quantum electrodynamics," *JHEP*, vol. 08, p. 211, 2023, 2302.10713.

[42] H. Grosse and R. Wulkenhaar, "Renormalization of phi**4 theory on noncommutative R**2 in the matrix base," *JHEP*, vol. 12, p. 019, 2003, hep-th/0307017.

[43] A. P. Balachandran, G. Mangano, A. Pinzul, and S. Vaidya, "Spin and statistics on the Groenwald-Moyal plane: Pauli-forbidden levels and transitions," *Int. J. Mod. Phys. A*, vol. 21, pp. 3111–3126, 2006, hep-th/0508002.

[44] A. Addazi, P. Belli, R. Bernabei, and A. Marciano, "Testing noncommutative spacetimes and violations of the Pauli Exclusion Principle through underground experiments," *Chin. Phys. C*, vol. 42, no. 9, p. 094001, 2018, 1712.08082.

[45] A. Addazi and R. Bernabei, "Tests of Pauli exclusion principle violations from noncommutative quantum gravity," *Int. J. Mod. Phys. A*, vol. 35, no. 32, p. 2042001, 2020, 1901.00390.

[46] K. Piscicchia *et al.*, "Experimental test of noncommutative quantum gravity by VIP-2 Lead," *Phys. Rev. D*, vol. 107, no. 2, p. 026002, 2023, 2212.04669.

[47] S. Minwalla, M. Van Raamsdonk, and N. Seiberg, "Noncommutative perturbative dynamics," *JHEP*, vol. 02, p. 020, 2000, hep-th/9912072.

[48] G. Amelino-Camelia, G. Mandanici, and K. Yoshida, "On the IR / UV mixing and experimental limits on the parameters of canonical noncommutative space-times," *JHEP*, vol. 01, p. 037, 2004, hep-th/0209254.

[49] S. Majid, *Foundations of Quantum Group Theory.* Cambridge University Press, 1995.

[50] A. Ballesteros, F. J. Herranz, M. A. del Olmo, and M. Santander, "4-D quantum affine algebras and space-time q symmetries," *J. Math. Phys.*, vol. 35, pp. 4928–4940, 1994, hep-th/9310140.

[51] A. Ballesteros, F. J. Herranz, M. A. del Olmo, and M. Santander, "A New 'null plane' quantum Poincare algebra," *Phys. Lett. B*, vol. 351, pp. 137–145, 1995, q-alg/9502019.

[52] A. Ballesteros, F. J. Herranz, and C. M. Pereña, "Null-plane quantum universal R-matrix," *Phys. Lett. B*, vol. 391, pp. 71–77, 1997, q-alg/9607009.

[53] A. Blaut, M. Daszkiewicz, J. Kowalski-Glikman, and S. Nowak, "Phase spaces of doubly special relativity," *Phys. Lett. B*, vol. 582, pp. 82–85, 2004, hep-th/0312045.

[54] F. Lizzi, M. Manfredonia, and F. Mercati, "The momentum spaces of κ-minkowski noncommutative spacetime," *Nucl. Phys.*, vol. B958, p. 115117, 2020, 1811.08409.

[55] J. Lukierski, A. Nowicki, and H. Ruegg, "New quantum Poincare algebra and k deformed field theory," *Phys. Lett. B*, vol. 293, pp. 344–352, 1992.

[56] J. Lukierski, A. Nowicki, and H. Ruegg, "Real forms of complex quantum anti-De Sitter algebra U-q(Sp(4:C)) and their contraction schemes," *Phys. Lett. B*, vol. 271, pp. 321–328, 1991, hep-th/9108018.

[57] S. Zakrzewski, "Quantum poincare group related to the kappa -poincare algebra," *J. Phys. A*, vol. 27, p. 2075, mar 1994.

[58] A. Pachol, *κ-Minkowski spacetime: Mathematical formalism and applications in Planck scale physics*. PhD thesis, Wroclaw U., 2011, 1112.5366.

[59] J. Kowalski-Glikman, "A short introduction to κ-deformation," *Int. J. Mod. Phys. A*, vol. 32, no. 35, p. 1730026, 2017, 1711.00665.

[60] M. Arzano and J. Kowalski-Glikman, *Deformations of Spacetime Symmetries: Gravity, Group-Valued Momenta, and Non-Commutative Fields*, vol. 986 of *Lecture Notes in Physics*. 6 2021.

[61] N. Franco and J.-C. Wallet, "Quantum causality constraints on kappa-Minkowski spacetime," *J. Phys. A*, vol. 55, no. 41, p. 414007, 2022, 2201.04057.

[62] M. Arzano and J. Kowalski-Glikman, "A group theoretic description of the κ-Poincaré Hopf algebra," *Phys. Lett. B*, vol. 835, p. 137535, 2022, 2204.09394.

[63] F. Lizzi, M. Manfredonia, F. Mercati, and T. Poulain, "Localization and Reference Frames in κ-Minkowski Spacetime," *Phys. Rev. D*, vol. 99, no. 8, p. 085003, 2019, 1811.08409.

[64] F. Lizzi, M. Manfredonia, and F. Mercati, "Localizability in κ-Minkowski spacetime," *Int. J. Geom. Meth. Mod. Phys.*, vol. 17, no. supp01, p. 2040010, 2020, 1912.07098.

[65] J. Lukierski and H. Ruegg, "Quantum kappa Poincare in any dimension," *Phys. Lett. B*, vol. 329, pp. 189–194, 1994, hep-th/9310117.

[66] J. Kowalski-Glikman and S. Nowak, "Noncommutative space-time of doubly special relativity theories," *Int. J. Mod. Phys. D*, vol. 12, pp. 299–316, 2003, hep-th/0204245.

[67] A. Agostini, G. Amelino-Camelia, and M. Arzano, "Dirac spinors for doubly special relativity and kappa Minkowski noncummutative space-time," *Class. Quant. Grav.*, vol. 21, pp. 2179–2202, 2004, gr-qc/0207003.

[68] A. Agostini, F. Lizzi, and A. Zampini, "Generalized Weyl systems and kappa Minkowski space," *Mod. Phys. Lett. A*, vol. 17, pp. 2105–2126, 2002, hep-th/0209174.

[69] J. Kowalski-Glikman, "De sitter space as an arena for doubly special relativity," *Phys. Lett. B*, vol. 547, pp. 291–296, 2002, hep-th/0207279.

[70] J. Kowalski-Glikman and S. Nowak, "Doubly special relativity and de Sitter space," *Class. Quant. Grav.*, vol. 20, pp. 4799–4816, 2003, hep-th/0304101.

[71] A. Agostini, "kappa-Minkowski representations on Hilbert spaces," *J. Math. Phys.*, vol. 48, p. 052305, 2007, hep-th/0512114.

[72] G. Amelino-Camelia, G. Gubitosi, A. Marciano, P. Martinetti, and F. Mercati, "A No-pure-boost uncertainty principle from spacetime noncommutativity," *Phys. Lett. B*, vol. 671, pp. 298–302, 2009, 0707.1863.

[73] G. Amelino-Camelia, G. Gubitosi, A. Marciano, P. Martinetti, F. Mercati, D. Pranzetti, and R. A. Tacchi, "First results of the Noether theorem for Hopf-algebra spacetime symmetries," *Prog. Theor. Phys. Suppl.*, vol. 171, pp. 65–78, 2007, 0710.1219.

[74] J. M. Carmona, J. L. Cortes, D. Mazon, and F. Mercati, "About Locality and the Relativity Principle Beyond Special Relativity," *Phys. Rev. D*, vol. 84, p. 085010, 2011, 1107.0939.

[75] J. M. Carmona, J. L. Cortes, and F. Mercati, "Relativistic kinematics beyond Special Relativity," *Phys. Rev. D*, vol. 86, p. 084032, 2012, 1206.5961.

[76] G. Amelino-Camelia, N. Loret, G. Mandanici, and F. Mercati, "Gravity in quantum spacetime," *Int. J. Mod. Phys. D*, vol. 19, pp. 2385–2392, 2010, 1007.0851.

[77] G. Gubitosi and F. Mercati, "Relative Locality in κ-Poincaré," *Class. Quant. Grav.*, vol. 30, p. 145002, 2013, 1106.5710.

[78] F. Mercati, "Quantum κ-deformed differential geometry and field theory," *Int. J. Mod. Phys. D*, vol. 25, no. 05, p. 1650053, 2016, 1112.2426.

[79] S. Meljanac, D. Meljanac, F. Mercati, and D. Pikutić, "Noncommutative spaces and Poincaré symmetry," *Phys. Lett. B*, vol. 766, pp. 181–185, 2017, 1610.06716.

[80] N. Loret, S. Meljanac, F. Mercati, and D. Pikutić, "Vectorlike deformations of relativistic quantum phase-space and relativistic kinematics," *Int. J. Mod. Phys. D*, vol. 26, no. 11, p. 1750123, 2017, 1610.08310.

[81] J. Lukierski, "Kappa-Deformations: Historical Developments and Recent Results," *J. Phys. Conf. Ser.*, vol. 804, no. 1, p. 012028, 2017, 1611.10213.

[82] F. Mercati and M. Sergola, "Physical Constraints on Quantum Deformations of Spacetime Symmetries," *Nucl. Phys. B*, vol. 933, pp. 320–339, 2018, 1802.09483.

[83] F. Mercati *to appear on the arXiv*, 2023.

[84] G. Amelino-Camelia, J. Kowalski-Glikman, G. Mandanici, and A. Procaccini, "Phenomenology of doubly special relativity," *Int. J. Mod. Phys. A*, vol. 20, pp. 6007–6038, 2005, gr-qc/0312124.

[85] M. Arzano, J. Kowalski-Glikman, and T. Trześniewski, "Beyond Fock space in three dimensional semiclassical gravity," *Class. Quant. Grav.*, vol. 31, no. 3, p. 035013, 2014, 1305.6220.

[86] M. Arzano, D. Latini, and M. Lotito, "Group Momentum Space and Hopf Algebra Symmetries of Point Particles Coupled to 2+1 Gravity," *SIGMA*, vol. 10, p. 079, 2014, 1403.3038.

[87] F. Cianfrani, J. Kowalski-Glikman, D. Pranzetti, and G. Rosati, "Symmetries of quantum spacetime in three dimensions," *Phys. Rev. D*, vol. 94, no. 8, p. 084044, 2016, 1606.03085.

[88] S. Majid and H. Ruegg, "Bicrossproduct structure of kappa Poincare group and noncommutative geometry," *Phys. Lett. B*, vol. 334, pp. 348–354, 1994, hep-th/9405107.

[89] P. Kosinski, J. Lukierski, and P. Maslanka, "Local D = 4 field theory on kappa deformed Minkowski space," *Phys. Rev. D*, vol. 62, p. 025004, 2000, hep-th/9902037.

[90] P. Kosinski, J. Lukierski, and P. Maslanka, "kappa deformed Wigner construction of relativistic wave functions and free fields on kappa-Minkowski space," *Nucl. Phys. B Proc. Suppl.*, vol. 102, pp. 161–168, 2001, hep-th/0103127.

[91] P. Kosinski, P. Maslanka, J. Lukierski, and A. Sitarz, "Generalized kappa deformations and deformed relativistic scalar fields on noncommutative Minkowski space," in *Conference on Topics in Mathematical Physics, General Relativity, and Cosmology on the Occasion of the 75th Birthday of Jerzy F. Plebanski*, pp. 255–277, 7 2003, hep-th/0307038.

[92] M. Dimitrijevic, F. Meyer, L. Moller, and J. Wess, "Gauge theories on the kappa Minkowski space-time," *Eur. Phys. J. C*, vol. 36, pp. 117–126, 2004, hep-th/0310116.

[93] F. Mercati and M. Sergola, "Pauli-Jordan function and scalar field quantization in κ-Minkowski noncommutative spacetime," *Phys. Rev.*, vol. D98, no. 4, p. 045017, 2018, 1801.01765.

[94] M. Arzano, A. Bevilacqua, J. Kowalski-Glikman, G. Rosati, and J. Unger, "κ-deformed complex fields and discrete symmetries," *Phys. Rev. D*, vol. 103, no. 10, p. 106015, 2021, 2011.09188.

[95] K. Hersent and J.-C. Wallet, "Gauge Theories on kappa-Minkowski spaces: Results and prospects.," *PoS*, vol. CORFU2021, p. 286, 2022, 2203.12706.

[96] A. Bevilacqua, J. Kowalski-Glikman, and W. Wislicki, "κ-deformed complex scalar field: Conserved charges, symmetries, and their impact on physical observables," *Phys. Rev. D*, vol. 105, no. 10, p. 105004, 2022, 2201.10191.

[97] A. Bevilacqua, "κ-deformed scalar field," 10 2023, 2311.00014.

[98] A. Bevilacqua, J. Kowalski-Glikman, and W. Wislicki, "Quantum Gravity Phenomenology and Particle Physics," 10 2023, 2310.05080.

[99] M. Arzano and A. Marciano, "Fock space, quantum fields and kappa-Poincare symmetries," *Phys. Rev. D*, vol. 76, p. 125005, 2007, 0707.1329.

[100] T. R. Govindarajan, K. S. Gupta, E. Harikumar, S. Meljanac, and D. Meljanac, "Deformed Oscillator Algebras and QFT in kappa-Minkowski Spacetime," *Phys. Rev. D*, vol. 80, p. 025014, 2009, 0903.2355.

[101] M. Daszkiewicz, J. Lukierski, and M. Woronowicz, "Towards quantum noncommutative kappa-deformed field theory," *Phys. Rev. D*, vol. 77, p. 105007, 2008, 0708.1561.

[102] C. A. S. Young and R. Zegers, "Covariant particle statistics and intertwiners of the kappa-deformed Poincare algebra," *Nucl. Phys. B*, vol. 797, pp. 537–549, 2008, 0711.2206.

[103] M. Arzano and J. Kowalski-Glikman, "Quantum particles in noncommutative spacetime: An identity crisis," *Phys. Rev. D*, vol. 107, no. 6, p. 065001, 2023, 2212.03703.

[104] J. Lukierski and M. Woronowicz, "New Lie-algebraic and quadratic deformations of Minkowski space from twisted Poincare symmetries," *Phys. Lett. B*, vol. 633, pp. 116–124, 2006, hep-th/0508083.

[105] F. Lizzi and P. Vitale, "Time Discretization From Noncommutativity," *Phys. Lett. B*, vol. 818, p. 136372, 2021, 2101.06633.

[106] F. Lizzi, L. Scala, and P. Vitale, "Localization and observers in ϱ-Minkowski spacetime," *Phys. Rev. D*, vol. 106, no. 2, p. 025023, 2022, 2205.10862.

[107] M. D. Ćirić, N. Konjik, and A. Samsarov, "Noncommutative scalar quasinormal modes of the Reissner–Nordström black hole," *Class. Quant. Grav.*, vol. 35, no. 17, p. 175005, 2018, 1708.04066.

[108] K. Hersent and J.-C. Wallet, "Field theories on ρ-deformed Minkowski space-time," *JHEP*, vol. 07, p. 031, 2023, 2304.05787.

[109] G. Amelino-Camelia, L. Freidel, J. Kowalski-Glikman, and L. Smolin, "Relative locality: A deepening of the relativity principle," *Gen. Rel. Grav.*, vol. 43, pp. 2547–2553, 2011, 1106.0313.

[110] G. Amelino-Camelia, L. Barcaroli, and N. Loret, "Modeling transverse relative locality," *Int. J. Theor. Phys.*, vol. 51, pp. 3359–3375, 2012, 1107.3334.

[111] G. Amelino-Camelia, M. Arzano, S. Bianco, and R. J. Buonocore, "The DSR-deformed relativistic symmetries and the relative locality of 3D quantum gravity," *Class. Quant. Grav.*, vol. 30, p. 065012, 2013, 1210.7834.

[112] G. Amelino-Camelia, L. Barcaroli, G. D'Amico, N. Loret, and G. Rosati, "Quantum-gravity-induced dual lensing and IceCube neutrinos," *Int. J. Mod. Phys. D*, vol. 26, no. 08, p. 1750076, 2017, 1609.03982.

[113] G. Amelino-Camelia, L. Barcaroli, S. Bianco, and L. Pensato, "Planck-scale dual-curvature lensing and spacetime noncommutativity," *Adv. High Energy Phys.*, vol. 2017, p. 6075920, 2017, 1708.02429.

[114] L. Burderia *et al.*, "GrailQuest & HERMES: Hunting for Gravitational Wave Electromagnetic Counterparts and Probing Space-Time Quantum Foam," in *SPIE Astronomical Telescopes + Instrumentation 2020*, 1 2021, 2101.07119.

[115] F. Fiore *et al.*, "The HERMES-Technologic and Scientific Pathfinder," in *SPIE Astronomical Telescopes + Instrumentation 2020*, 1 2021, 2101.03078.

[116] M. Dimitrijevic Ciric, N. Konjik, M. A. Kurkov, F. Lizzi, and P. Vitale, "Noncommutative field theory from angular twist," *Phys. Rev. D*, vol. 98, no. 8, p. 085011, 2018, 1806.06678.

[117] G. Gubitosi, F. Lizzi, J. J. Relancio, and P. Vitale, "Double quantization," *Phys. Rev. D*, vol. 105, no. 12, p. 126013, 2022, 2112.11401.

[118] P. Aschieri, M. Dimitrijevic, P. Kulish, F. Lizzi, and J. Wess, *Noncommutative Spacetimes: Symmetries in Noncommutative Geometry and Field Theory.* Lecture Notes in Physics, Springer Berlin Heidelberg, 2011.

[119] A. Chamseddine, C. Consani, N. Higson, M. Khalkhali, H. Moscovici, and G. Yu, *Advances in Noncommutative Geometry: On the Occasion of Alain Connes' 70th Birthday.* Springer International Publishing, 2020.

[120] F. Lizzi and F. Mercati, "κ-Poincaré-comodules, Braided Tensor Products and Noncommutative Quantum Field Theory," *Phys. Rev. D*, vol. 103, p. 126009, 2021, 2101.09683.

[121] M. G. Di Luca and F. Mercati, "New class of plane waves for κ-noncommutative quantum field theory," *Phys. Rev. D*, vol. 107, no. 10, p. 105018, 2023, 2211.11627.

[122] G. Fiore, "On the consequences of twisted Poincare' symmetry upon QFT on Moyal noncommutative spaces," in *Quantum field theory and beyond: Essays in honor of Wolfhart Zimmermann's 80th birthday (Ringberg Symposium)*, pp. 64–84, 9 2008, 0809.4507.

[123] S. Majid, "Algebraic approach to quantum gravity. II. Noncommutative spacetime," pp. 466–492, 4 2006, hep-th/0604130.

[124] M. Arzano and D. Benedetti, "Rainbow statistics," *Int. J. Mod. Phys. A*, vol. 24, pp. 4623–4641, 2009, 0809.0889.

[125] M. Arzano, "Fun from none: Deformed symmetries and Fock space," *AIP Conf. Proc.*, vol. 1196, no. 1, p. 1, 2009, 0909.4873.

[126] A. Agostini, G. Amelino-Camelia, and F. D'Andrea, "Hopf algebra description of noncommutative space-time symmetries," *Int. J. Mod. Phys. A*, vol. 19, pp. 5187–5220, 2004, hep-th/0306013.

[127] A. Ballesteros, E. Celeghini, F. J. Herranz, M. A. del Olmo, and M. Santander, "Universal r-matrices for non-standard (1+1) quantum groups," *J. Phys. A*, vol. 28, pp. 3129–3138, jun 1995.

[128] A. Ballesteros and F. J. Herranz, "(2+1) null-plane quantum poincaré group from a factorized universal r-matrix," *Mod. Phys. Lett. A*, vol. 11, pp. 1745–1755, jul 1996.

[129] G. Amelino-Camelia, L. Freidel, J. Kowalski-Glikman, and L. Smolin, "The principle of relative locality," *Phys. Rev. D*, vol. 84, p. 084010, 2011, 1101.0931.

[130] L. Dabrowski and G. Piacitelli, "Canonical k-Minkowski Spacetime," 4 2010, 1004.5091.

[131] M. E. Peskin and D. V. Schroeder, *An Introduction to quantum field theory.* Reading, USA: Addison-Wesley, 1995.

[132] E. Joung and J. Mourad, "QFT with Twisted Poincare Invariance and the Moyal Product," *JHEP*, vol. 05, p. 098, 2007, hep-th/0703245.

[133] T. Heinzl, "Light cone quantization: Foundations and applications," *Lect. Notes Phys.*, vol. 572, pp. 55–142, 2001, hep-th/0008096.

[134] G. Fiore, "On second quantization on noncommutative spaces with twisted symmetries," *J. Phys. A*, vol. 43, p. 155401, 2010, 0811.0773.

[135] M. Arzano, J. Kowalski-Glikman, and W. Wislicki, "A bound on Planck-scale deformations of CPT from muon lifetime," *Phys. Lett. B*, vol. 794, pp. 41–44, 2019, 1904.06754.

[136] M. Arzano, J. Kowalski-Glikman, and W. Wislicki, "Planck-scale deformation of CPT and particle lifetimes," 9 2020, 2009.03135.

[137] P. Aschieri, F. Lizzi, and P. Vitale, "Twisting all the way: From Classical Mechanics to Quantum Fields," *Phys. Rev. D*, vol. 77, p. 025037, 2008, 0708.3002.

[138] J. J. Sakurai and J. Napolitano, *Modern Quantum Mechanics.* Cambridge University Press, 2 ed., 2017.

[139] H. Shi *et al.*, "Experimental search for the violation of Pauli Exclusion Principle," *Eur. Phys. J. C*, vol. 78, no. 4, p. 319, 2018, 1804.04446.

[140] "New experimental limits on the pauli-forbidden transitions in ^{12}C nuclei obtained with 485 days borexino data," *Phys. Rev. C*, vol. 81, p. 034317, Mar 2010.

[141] R. Bernabei *et al.*, "New search for processes violating the Pauli exclusion principle in sodium and in iodine," *Eur. Phys. J. C*, vol. 62, pp. 327–332, 2009.

[142] H. Nguyen, A. Schenkel, and R. J. Szabo, "Batalin–Vilkovisky quantization of fuzzy field theories," *Lett. Math. Phys.*, vol. 111, no. 6, p. 149, 2021, 2107.02532.

[143] M. Dimitrijević Ćirić, G. Giotopoulos, V. Radovanović, and R. J. Szabo, "Braided L_∞-algebras, braided field theory and noncommutative gravity," *Lett. Math. Phys.*, vol. 111, no. 6, p. 148, 2021, 2103.08939.

[144] G. Giotopoulos and R. J. Szabo, "Braided symmetries in noncommutative field theory," *J. Phys. A*, vol. 55, no. 35, p. 353001, 2022, 2112.00541.

[145] D. Bogdanović, M. Dimitrijević Ćirić, V. Radovanović, and R. J. Szabo, "BV quantization of braided scalar field theory," in *CORFU2022: 22th Hellenic School and Workshops on Elementary Particle Physics and Gravity*, 4 2023, 2304.14073.

[146] W. Heisenberg, *The physical principles of the quantum theory.* Courier Corporation, 1949.

[147] H. M. Wiseman and G. J. Milburn, *Quantum measurement and control.* Cambridge university press, 2009.

[148] P. A. Höhn, "Reflections on the information paradigm in quantum and gravitational physics," *J. Phys. Conf. Ser.*, vol. 880, no. 1, p. 012014, 2017, 1706.06882.

[149] L. Smolin, "Quantum gravity with a positive cosmological constant," 2002, hep-th/0209079.

[150] E. Bianchi and C. Rovelli, "A Note on the geometrical interpretation of quantum groups and non-commutative spaces in gravity," *Phys. Rev. D*, vol. 84, p. 027502, 2011, 1105.1898.

[151] U. Franz, A. Skalski, and R. Tomatsu, "Idempotent states on compact quantum groups and their classification on

\

mathrm {U} _q (2) ,\mathrm {SU} _q (2) ,*and*\mathrm {SO} _q (3) ,″ *JournalofNoncommutativeGeometry*, *vol.* 7, *no.* 1, *pp.* 221 − −254, 2013.

[152] P. Podlés, "Symmetries of quantum spaces. subgroups and quotient spaces of quantum su(2) and so(3) groups," *Communications in Mathematical Physics*, vol. 170, pp. 1–20, May 1995.

[153] G. Amelino-Camelia, F. Briscese, G. Gubitosi, A. Marciano, P. Martinetti, and F. Mercati, "Noether analysis of the twisted Hopf symmetries of canonical noncommutative spacetimes," *Phys. Rev. D*, vol. 78, p. 025005, 2008, 0709.4600.

[154] G. Amelino-Camelia, G. Gubitosi, F. Mercati, and G. Rosati, "Conserved charges and quantum-group transformations in noncommutative field theories," 9 2010, 1009.3264.

[155] G. Amelino-Camelia and V. Astuti, "Misleading inferences from discretization of empty spacetime: Snyder-noncommutativity case study," *Int. J. Mod. Phys. D*, vol. 24, no. 10, p. 1550073, 2015, 1404.4773.

[156] H. S. Snyder, "Quantized space-time," *Phys. Rev.*, vol. 71, pp. 38–41, 1947.

[157] G. Amelino-Camelia, "Planck-scale soccer-ball problem: a case of mistaken identity," *Entropy*, vol. 19, no. 8, p. 400, 2017, 1407.7891.

[158] P. Kosinski, J. Lukierski, and P. Maslanka, "Local field theory on kappa Minkowski space, star products and noncommutative translations," *Czech. J. Phys.*, vol. 50, pp. 1283–1290, 2000, hep-th/0009120.

[159] S. Majid, "Meaning of noncommutative geometry and the Planck scale quantum group," *Lect. Notes Phys.*, vol. 541, pp. 227–276, 2000, hep-th/0006166.

[160] G. Amelino-Camelia, G. Gubitosi, and G. Palmisano, "Pathways to relativistic curved momentum spaces: de Sitter case study," *Int. J. Mod. Phys. D*, vol. 25, no. 02, p. 1650027, 2016, 1307.7988.

[161] C. J. Isham, "Structural issues in quantum gravity," in *14TH INTERNATIONAL CONFERENCE ON GENERAL RELATIVITY AND GRAVITATION (GR14)*, pp. 167–209, 8 1995, gr-qc/9510063.

[162] G. Aad *et al.*, "Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC," *Phys. Lett. B*, vol. 716, pp. 1–29, 2012, 1207.7214.

[163] "Precision Electroweak Measurements and Constraints on the Standard Model," 12 2010, 1012.2367.

[164] G. Amelino-Camelia, C. Laemmerzahl, F. Mercati, and G. M. Tino, "Constraining the Energy-Momentum Dispersion Relation with Planck-Scale Sensitivity Using Cold Atoms," *Phys. Rev. Lett.*, vol. 103, p. 171302, 2009, 0911.1020.

[165] F. Mercati, D. Mazon, G. Amelino-Camelia, J. M. Carmona, J. L. Cortes, J. Indurain, C. Laemmerzahl, and G. M. Tino, "Probing the quantum-gravity realm with slow atoms," *Class. Quant. Grav.*, vol. 27, p. 215003, 2010, 1004.0847.

[166] G. M. Tino, "Testing gravity with cold atom interferometry: Results and prospects," *Quantum Sci. Technol.*, vol. 6, no. 2, p. 024014, 2021, 2009.01484.

[167] R. Howl, V. Vedral, D. Naik, M. Christodoulou, C. Rovelli, and A. Iyer, "Non-Gaussianity as a signature of a quantum theory of gravity," *PRX Quantum*, vol. 2, p. 010325, 2021, 2004.01189.

[168] R. Howl, N. Cooper, and L. Hackermüller, "Gravitationally-induced entanglement in cold atoms," 4 2023, 2304.00734.

[169] L. Freidel and L. Smolin, "Gamma ray burst delay times probe the geometry of momentum space," 3 2011, 1103.5626.

[170] J. Magueijo and L. Smolin, "Generalized Lorentz invariance with an invariant energy scale," *Phys. Rev. D*, vol. 67, p. 044017, 2003, gr-qc/0207085.

[171] C. Pfeifer and J. J. Relancio, "Deformed relativistic kinematics on curved spacetime: a geometric approach," *Eur. Phys. J. C*, vol. 82, no. 2, p. 150, 2022, 2103.16626.

[172] J. J. Relancio, "Relativistic deformed kinematics: From flat to curved spacetimes," *Int. J. Geom. Meth. Mod. Phys.*, vol. 19, no. 09, p. 2230004, 2022, 2207.08471.

[173] G. Rosati, G. Amelino-Camelia, A. Marciano, and M. Matassa, "Planck-scale-modified dispersion relations in FRW spacetime," *Phys. Rev. D*, vol. 92, no. 12, p. 124042, 2015, 1507.02056.

[174] C. Pfeifer, "Redshift and lateshift from homogeneous and isotropic modified dispersion relations," *Phys. Lett. B*, vol. 780, pp. 246–250, 2018, 1802.00058.

[175] G. Amelino-Camelia, G. Rosati, and S. Bedić, "Phenomenology of curvature-induced quantum-gravity effects," *Phys. Lett. B*, vol. 820, p. 136595, 2021, 2012.07790.

[176] G. Amelino-Camelia, D. Frattulillo, G. Gubitosi, G. Rosati, and S. Bedić, "Phenomenology of DSR-relativistic in-vacuo dispersion in FLRW spacetime," 7 2023, 2307.05428.

[177] C. Pfeifer and M. N. R. Wohlfarth, "Finsler geometric extension of Einstein gravity," *Phys. Rev. D*, vol. 85, p. 064009, 2012, 1112.5641.

[178] C. Pfeifer, "Finsler spacetime geometry in Physics," *Int. J. Geom. Meth. Mod. Phys.*, vol. 16, no. supp02, p. 1941004, 2019, 1903.10185.

[179] G. Amelino-Camelia, L. Barcaroli, G. Gubitosi, S. Liberati, and N. Loret, "Realization of doubly special relativistic symmetries in Finsler geometries," *Phys. Rev. D*, vol. 90, no. 12, p. 125030, 2014, 1407.8143.

[180] I. P. Lobo, C. Pfeifer, P. H. Morais, R. A. Batista, and V. B. Bezerra, "Two-body decays in deformed relativity," *JHEP*, vol. 09, p. 003, 2022, 2112.12172.

[181] P. H. Morais, I. P. Lobo, C. Pfeifer, R. Alves Batista, and V. B. Bezerra, "Modified particle lifetimes as a signature of deformed relativity," *Phys. Lett. B*, vol. 848, p. 138380, 2024, 2308.03716.

[182] U. Jacob and T. Piran, "Lorentz-violation-induced arrival delays of cosmological particles," *JCAP*, vol. 01, p. 031, 2008, 0712.2170.

[183] J. R. Ellis, N. E. Mavromatos, D. V. Nanopoulos, and A. S. Sakharov, "Quantum-gravity analysis of gamma-ray bursts using wavelets," *Astron. Astrophys.*, vol. 402, pp. 409–424, 2003, astro-ph/0210124.

[184] M. Martinez and M. Errando, "A new method to study energy-dependent arrival delays on photons from astrophysical sources," *Astropart. Phys.*, vol. 31, pp. 226–232, 2009, 0803.2120.

[185] V. Vasileiou, A. Jacholkowska, F. Piron, J. Bolmont, C. Couturier, J. Granot, F. W. Stecker, J. Cohen-Tanugi, and F. Longo, "Constraints on Lorentz Invariance Violation from Fermi-Large Area Telescope Observations of Gamma-Ray Bursts," *Phys. Rev. D*, vol. 87, no. 12, p. 122001, 2013, 1305.3463.

[186] S. Zhang and B.-Q. Ma, "Lorentz violation from gamma-ray bursts," *Astropart. Phys.*, vol. 61, pp. 108–112, 2015, 1406.4568.

[187] G. Amelino-Camelia, G. D'Amico, F. Fiore, S. Puccetti, and M. Ronco, "In Vacuo Dispersion-Like Spectral Lags in Gamma-Ray Bursts," *Symmetry*, vol. 13, no. 4, p. 541, 2021, 1707.02413.

[188] Z. Cao *et al.*, "Exploring Lorentz Invariance Violation from Ultrahigh-Energy γ Rays Observed by LHAASO," *Phys. Rev. Lett.*, vol. 128, no. 5, p. 051102, 2022, 2106.12350.

[189] J. Bolmont *et al.*, "First Combined Study on Lorentz Invariance Violation from Observations of Energy-dependent Time Delays from Multiple-type Gamma-Ray Sources. I. Motivation, Method Description, and Validation through Simulations of H.E.S.S., MAGIC, and VERITAS Data Sets," *Astrophys. J.*, vol. 930, no. 1, p. 75, 2022, 2201.02087.

[190] T. Piran, "Gamma-ray bursts: A puzzle being resolved," *Phys. Rept.*, vol. 333, pp. 529–553, 2000, astro-ph/9907392.

[191] M. G. Aartsen *et al.*, "An All-Sky Search for Three Flavors of Neutrinos from Gamma-Ray Bursts with the IceCube Neutrino Observatory," *Astrophys. J.*, vol. 824, no. 2, p. 115, 2016, 1601.06484.

[192] M. G. Aartsen *et al.*, "Extending the search for muon neutrinos coincident with gamma-ray bursts in IceCube data," *Astrophys. J.*, vol. 843, no. 2, p. 112, 2017, 1702.06868.

[193] G. Amelino-Camelia, D. Guetta, and T. Piran, "Icecube neutrinos and lorentz invariance violation," *The Astrophysical Journal*, vol. 806, p. 269, jun 2015.

[194] G. Amelino-Camelia, L. Barcaroli, G. D'Amico, N. Loret, and G. Rosati, "IceCube and GRB neutrinos propagating in quantum spacetime," *Phys. Lett. B*, vol. 761, pp. 318–325, 2016, 1605.00496.

[195] F. W. Stecker, S. T. Scully, S. Liberati, and D. Mattingly, "Searching for Traces of Planck-Scale Physics with High Energy Neutrinos," *Phys. Rev. D*, vol. 91, no. 4, p. 045009, 2015, 1411.5889.

[196] G. Amelino-Camelia, M. G. Di Luca, G. Gubitosi, G. Rosati, and G. D'Amico, "Could quantum gravity slow down neutrinos?," *Nature Astron.*, vol. 7, no. 8, pp. 996–1001, 2023, 2209.13726.

[197] M. Rodriguez Martinez and T. Piran, "Constraining Lorentz violations with gamma-ray bursts," *JCAP*, vol. 04, p. 006, 2006, astro-ph/0601219.

[198] G. Amelino-Camelia, A. Marciano, M. Matassa, and G. Rosati, "Deformed Lorentz symmetry and relative locality in a curved/expanding spacetime," *Phys. Rev. D*, vol. 86, p. 124035, 2012, 1206.5315.

[199] J. M. Carmona, J. L. Cortés, L. Pereira, and J. J. Relancio, "Bounds on Relativistic Deformed Kinematics from the Physics of the Universe Transparency," *Symmetry*, vol. 12, no. 8, p. 1298, 2020, 2008.10251.

[200] J. M. Carmona, J. L. Cortés, J. J. Relancio, M. A. Reyes, and A. Vincueria, "Modification of the mean free path of very high-energy photons due to a relativistic deformed kinematics," *Eur. Phys. J. Plus*, vol. 137, no. 7, p. 768, 2022, 2109.08402.

[201] R. C. Myers and M. Pospelov, "Ultraviolet modifications of dispersion relations in effective field theory," *Phys. Rev. Lett.*, vol. 90, p. 211601, 2003, hep-ph/0301124.

[202] G. Albalate, J. M. Carmona, J. L. Cortés, and J. J. Relancio, "Twin Peaks: A possible signal in the production of resonances beyond special relativity," *Symmetry*, vol. 10, p. 432, 2018, 1809.08167.

[203] T. Jacobson, S. Liberati, and D. Mattingly, "Lorentz violation at high energy: Concepts, phenomena and astrophysical constraints," *Annals Phys.*, vol. 321, pp. 150–196, 2006, astro-ph/0505267.

[204] R. H. Parker, C. Yu, W. Zhong, B. Estey, and H. Müller, "Measurement of the fine-structure constant as a test of the standard model," *Science*, vol. 360, no. 6385, pp. 191–195, 2018.

[205] L. Morel, Z. Yao, P. Cladé, and S. Guellati-Khélifa, "Determination of the fine-structure constant with an accuracy of 81 parts per trillion," *Nature*, vol. 588, no. 7836, pp. 61–65, 2020.

[206] M. Kasevich and S. Chu, "Atomic interferometry using stimulated raman transitions," *Phys. Rev. Lett.*, vol. 67, pp. 181–184, Jul 1991.

[207] A. Wicht, J. M. Hensley, E. Sarajlic, and S. Chu, "A preliminary measurement of the fine structure constant based on atom interferometry," *Physica Scripta*, vol. 2002, p. 82, jan 2002.

[208] M. Arzano, J. Kowalski-Glikman, and A. Walkus, "A Bound on Planck-scale modifications of the energy-momentum composition rule from atomic interferometry," *EPL*, vol. 90, no. 3, p. 30006, 2010, 0912.2712.

[209] J. Kowalski-Glikman and A. Walkus, "Star product and interacting fields on kappa-Minkowski space," *Mod. Phys. Lett. A*, vol. 24, pp. 2243–2250, 2009, 0904.4036.

[210] A. Blaut, M. Daszkiewicz, and J. Kowalski-Glikman, "Doubly special relativity with light cone deformation," *Mod. Phys. Lett. A*, vol. 18, p. 1711, 2003, hep-th/0302157.

[211] G. Amelino-Camelia, V. Astuti, and G. Rosati, "Relative locality in a quantum spacetime and the pregeometry of κ-Minkowski," *Eur. Phys. J. C*, vol. 73, no. 8, p. 2521, 2013, 1206.3805.