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Hydrodynamic Stability of Boussinesq's Flows in Porous Media

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CONTENTS

So eine Arbeit wird eigentlich
nie fertig, man muß sie für fertig
erklären, wenn man nach Zeit
und Umständen das möglichste
getan hat*.

Goethe, *Italienische Reise*.

* A work like this is actually never finished; you have to declare it finished when you have done the best possible given the time and circumstances.

INTRODUCTION

It was on March 15th, 1901, when Henry Bénard defended at University of Paris, Sorbonne, his thesis entitled *"Les Tourbillons cellulaires dans une nappe liquide propageant de la chaleur par convection en régime permanent"*†, [23]. It is worth recalling that the French Physical Society organized a scientific conference to commemorate the centenary of Bénard thesis, and the meeting was honoured by the presence of Professor Pierre-Gilles de Gennes (Wolf Prize in Physics in 1990 and Nobel laureate in Physics in 1991) who gave the opening talk, [86]. Lectures were held by recognized scholars who have contributed to the development of Bénard work, i.e. J. E. Wesfreid [134], P. Manneville [24], M. Velarde [112], J. Gollub [115], M. Provansal [100], G. Nicolis [39], B. Castaing [33] and P. Coullet [1]. The results contained in this celebrated work have been the milestone for intensive research activities and the present doctoral thesis deals with some recent advances in this field. Indeed, the onset of thermal convection in fluids saturating porous media, as well as in clear fluids, is a thriving research field and it has been constantly investigated over the years because cellular structures are observed in many physicochemical systems far from equilibrium and the applications are numerous and remarkable, constituting a driving force for researchers to improve the mathematical models in this area. Examples of applications can be found in geological processes [138], in biological systems and biotechnology [129], and in engineering [95].
In this thesis, two typical situations bringing thermal instability in an horizontal layer will be considered. The first is the uniformly heating from below mechanism, which is the phenomenon responsible for the activation of the so-called *Rayleigh-Bénard convection* in a clear fluid at rest, [32]. The corresponding mathematical problem, called the Bénard Problem, was solved by Lord Rayleigh in 1916 [105]. Concerning fluid-saturated porous media, the Rayleigh-Bénard instability yields the formulation of the Horton-Rogers-Lapwood Problem [64, 78], and the specific interests for geophysical applications were highlighted in pioneering papers as those by Wooding [136] and Elder [42]. The second typical situation occurs when the fluid density attains a maximum in the interior of the layer. In this case, the process of thermal convection

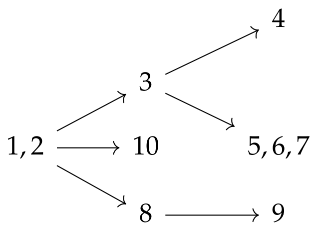
† Cellular vortices in a thin liquid layer propagating heat by convection in a stationary regime.

refers to the instability of a part of the layer, which will then penetrate into an upper stability-stratified region. This density inversion phenomenon is indeed responsible for the activation of the so-called *penetrative convection*. The mathematical formulation of this problem was addressed for the first time by George Veronis [131]. Since then, considerable attention has been devoted to penetrative convection, in both frameworks of clear fluids and porous media; the latter in particular for its applications in geophysics [53].
This thesis deals with both the situations stated right before and, as a consequence, the topics and the related results are twofold. In particular, after a discussion concerning the physical and mathematical background, the second part of the thesis is focused on the phenomenon of penetrative convection in porous media while the third part on the Rayleigh-Bénard convection for a class of fluids called Extended-Quasi-Thermal-Incompressible.
Despite the well-known difficulties arising in the resolution of partial differential equations' systems in fluid dynamics, it is always possible to explicitly find a particular class of solutions, namely the *basic state motions*. Once such a basic steady solution is found, i.e. heat propagates by conduction, then the onset of convective motions refers to the instant when the conduction solution becomes unstable. Intuition suggested to Lord Rayleigh that the onset of instability had to be related to a dimensionless number Ra, which would have been duly named after him. This number gives "a measure" of the destabilising effect of the buoyancy over the stabilising effect of molecular diffusion of momentum. It turns out that convective motions occurs once the Rayleigh number overcomes a critical value Ra_c. Accordingly, the aim the mathematical analysis of convection problems consists precisely in determining the critical thresholds of the Rayleigh number beyond which instability occurs, and/or in determining the conditions guaranteeing the stability of the basic steady solution. To this end, linear and nonlinear stability analyses are performed. In the former framework, sufficient conditions for instability to occur are provided, while in the latter framework sufficient conditions for the stability of the basic solution are determined.

In the following, a precise description of the thesis' articulation is summarized. Indeed, the contents are organized into four parts and ten chapters. **Part I** (Chapters 1-2) is devoted to the introductory definitions and theories concerned with hydrodynamic stability and thermodynamic consistency issues. **Part II** (Chapters 3-7) deals with recent stability results achieved in the contest of penetrative convection in porous media. **Part III** (Chapters 8-9) contains new well-posedness and stability results concerning Darcy porous convection for fluids which are slightly compressible and **Part IV** (Chapter 10) deals with the numerical technique employed in the thesis to solve some generalized eigenvalue problems. **Chapter 1** contains the fundamentals concerning modelling aspects (i.e. governing equations and thermodynamic consistency issues), while **Chapter 2** summarize the essential mathematical devices employed in the subsequent investigations. In **Chapter 3** the onset of penetrative convection in presence of inertia is analysed. Linear and nonlinear stability analyses of the conduction solution are performed and the thermodynamic consistency of the constitutive equation for the density is discussed. In particular, concerning the nonlinear analysis, both global and local results (without and with restrictions on the initial data, respectively) are addressed and the nonlinear thresholds in a suitable weighted L^2-norm are found. This chapter has a key role in the second part of the thesis since all the subsequent stability results are compared with the one obtained there. **Chapter 4** address the onset of thermal convection accounting for effect due to a variable gravity field. The remaining chapters of Part II, deal with more deep modification of the structure of the governing equations. Indeed, in **Chapter 5** the onset of penetrative convection in an inclined porous layer in investigated. Unlike the previous and subsequent analyses, in this framework the linear instability of the conduction solution is investigated for longitudinal (also called streamwise), transverse (also called spanwise), and full three-dimensional perturbations and the nonlinear stability analysis with respect to the longitudinal ones. If one is interested in investigating the effect of double porosity and temperature on the onset of thermal convection, one deals with *Bi-Disperse Porous Media* (BDPM) and convection in the hypothesis of *Local-Thermal-Non-Equilibrium* (LTNE), respectively. In particular, **Chapter 6** is devoted to the analysis of the onset of penetrative convection in a layer of BDPM, while **Chapter 7** to the LTNE hypotheses, employing a Darcy-Brinkman model since LTNE effect may be greater with a larger porosity. All the previous models and the related investigations are mainly motivated by the aim to derive more detailed and realistic description of the phenomenon of penetrative convection. The final chapters, deal with fluid motion in porous media which are Extended-Quasi-Thermal-Incompressible (EQTI). In **Chapter 8** a model for convection in EQTI fluid-saturated porous media is introduced and the existence, uniqueness and regularity of its solutions is studied, while in **Chapter 9** the hydrodynamic stability of the model is investigated. The thesis ends with **Chapter**

10 where a useful numerical method (Chebyshev-τ) is described in details and with a final remark that recaps all the results and the future perspectives.

This thesis should not be necessarily read in a linear fashion. The following chart shows the approximate interdependence of the various chapters.



Essential nomenclature

Symbol	Unit	Description
d	m	depth of the layer
t	s	time
T_L	K	lower temperature
T_U	K	upper temperature
μ	$kg/(m\,s)$	dynamic fluid viscosity
ρ	kg/m^3	fluid density
α	K^{-1}	thermal expansion coefficient
β	Pa^{-1}	compressibility factor
g	m/s^2	modulus of gravitational acceleration
c_p	$m^3/(s^2\,K)$	specific heat capacity at constant pressure
c_V	$m^3/(s^2\,K)$	specific heat capacity at constant volume
V	m^3	volume
k	m^2	permeability of the porous body
χ	$(kg\,m)/(s^3\,K)$	thermal conductivity
κ	m^2/s	thermal diffusivity
p	Pa	pressure field
p_0	Pa	reference pressure
$\mathbf{v}$	m/s	seepage velocity
T	K	temperature field
T_0	K	reference temperature
Φ		dimensionless stream function
ϕ		porosity
Ra		Rayleigh-Darcy number

Part I

PRELIMINARIES

1 CONVECTION IN POROUS MEDIA AND THERMODYNAMIC CONSISTENCY

1.1 THE BASIC GOVERNING EQUATIONS

On October 29th and 30th 1855, Henry Darcy carried out some experiments in Dijon, France, to determine "the laws of water flow through sand", [114]. Translating from his original report, his experiments showed that "for sand of comparable nature, one can assume that the discharge volume is directly proportional to the head and inversely proportional to the thickness of the layer traversed", [37]. Darcy, as well as Bénard, had discovered the secret of one of nature's laws, which is nowadays essential in almost every aspect of hydrogeology, soil science, petroleum engineering and other fields involving flow in porous media. This relationship of proportionality between the gradient of the pressure p and the so-called seepage velocity* $\mathbf{v}$, namely the fluid velocity in presence of a porous medium, reads

$$\mathbf{v} = -\frac{k}{\mu}\nabla p\,, \tag{1.1}$$

where k is the permeability of the porous medium (depending only on its geometry) and μ is the dynamic viscosity of the fluid. Let us remark that the Darcy's law can be direct derived from the Navier-Stokes equations through volume averaging procedure, [88, 135]. In 1947, in a *Letter to the Editor* received on June 4th [28], Brinkman proposed an alternative to the Darcy's law, for cases where the medium porosity is high. Let us recall that the porosity ϕ is defined as the ratio between the volume occupied by the empty space and the total volume of the medium. For mediums such that $\phi > 0.6$, the Brinkman's law reads

$$\nabla p = -\frac{\mu}{k}\mathbf{v} + \widetilde{\mu}\Delta\mathbf{v}\,, \tag{1.2}$$

where $\widetilde{\mu}$ is the effective viscosity (different from the dynamic viscosity μ). In a near subsequent paper [29], Brinkman wrote that his equation (1.2) "has the advantage of approximating (1.1) for low value of k and the Stokes equation neglecting inertial term [$\nabla p = \mu\Delta\mathbf{v}$, ndr] for high

* The seepage velocity is defined via the Dupuit-Forchheimer relation $\mathbf{v} = \phi\mathbf{V}$, where ϕ and $\mathbf{v}$ are the porosity of the medium and the intrinsic average velocity, respectively.

value of k". Brinkman's law will be considered in Chapter 7, in the contest of LTNE. Let us note that in (1.1) body forces as gravity are neglected. Since in this thesis the body force term, to be more precise the fluid's density appearing there, is the main character of the subsequent investigations, let us consider

$$\frac{\mu}{k}\mathbf{v} = -\nabla p + \rho_f \mathbf{g}, \tag{1.3}$$

where ρ_f is the fluid density and $\mathbf{g}$ is the gravity acceleration. Moreover, in Chapter 3, the following extension of (1.3) will be employed [90]

$$\rho_f c_a \frac{\partial \mathbf{v}}{\partial t} + \frac{\mu}{k}\mathbf{v} = -\nabla p + \rho_f \mathbf{g}, \tag{1.4}$$

where c_a is the acceleration coefficient, in order to take into account the inertia effect. The previous equations for the balance of momentum, are coupled with the conservation of mass law

$$\frac{\partial \rho_f}{\partial t} + \nabla \cdot (\rho_f \mathbf{v}) = 0, \tag{1.5}$$

and the conservation of energy

$$\rho_f c_p \left(\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = \chi \Delta T - p \nabla \cdot \mathbf{v} + \lambda (\nabla \cdot \mathbf{v})^2 + 2\mu \mathbf{D} : \mathbf{D}, \tag{1.6}$$

where c_p is the specific heat at constant pressure and χ is the thermal conductivity of the fluid. Note that in (1.6) the classical Fourier law is assumed.

1.2 THE BOUSSINESQ APPROXIMATION

1.2.1 *Heuristic derivation*

In non-isothermal processes, the temperature variations give rise to variations in the properties of the fluid, in the density and viscosity for example. An analysis including the full effects of these is so complicated that some approximation becomes essential, [128]. The equations are commonly used in a form known as the *Boussinesq*, or *Oberbeck-Boussinesq* (OB) *approximation*. In the following, several aspects of this assumption will be summarized.

In his 1903 Vol. II of the monumental monograph *Théorie Analitique de* *la Caleaur*, Joseph Valentine Boussinesq remark† that: "The variations of density can be ignored except where they are multiplied by the acceleration of gravity in equation of motion for the vertical component of the velocity vector", [27]. The amazing consequence of this observation, called in 1916 by Rayleigh *Boussinesq approximation*, is the possibility to work with a *quasi*-incompressible system of coupled dynamic (Darcy in our case) and thermal (Fourier) equations where buoyancy is the main driving force. It is worth observing that Boussinesq deduced the approximation essentially on the basis of pertinent physical considerations starting from the exact equations for a compressible and heat conductor fluid subject to gravity. The crucial observation is that small variations of density are approximately related to a constant pressure, ρ being in a such case a function of the temperature T. Then, Boussinesq observed that an important consequence of the heating is the reduction of weight of the particles during their ascending motion, and in this case the weight $g\rho$ is divided by $(1+\alpha T)$. As a consequence, he deduced the emergence of an additional gravity term, $\rho_0 g\alpha T$, proportional to T but directed from below in the upward direction, in the momentum equation for w. Finally, Boussinesq observed that, in thermal convection problems, the velocity change appreciably the form of the particles but without important modification on the volume (isochoric process), [139].

In other worlds, the above situation occur when the density variations in the temperature are of moderate amount. On closer inspection, the origin of the simplification in these cases is due to the smallness of the coefficient of thermal (volume) expansion. For liquids one shall be mostly be concerned with, α is in the range 10^{-3} to 10^{-4}, [32]. For variations in temperature not exceeding e.g. 10°, the variations in the density are at most 1%. That's the reason why variations of this small amount can be ignored with the only important exception: the variability of ρ in the body force therm due to gravity. This is because the acceleration resulting from

$$\delta\rho = \alpha\Delta T \tag{1.7}$$

where ΔT is a measure of the variations in temperature which occur, can be quite large. Accordingly, we can treat ρ constant in all the terms

† *"[...] il fallait encore observer que, dans la plupart des mouvements provoqués par la chaleur sur nos fluides pesants, les volumes ou les densités se conservent à très peu près, quoique la variation correspondante du poids de l'unité de volume soit justement la cause des phénomènes qu'il s'agit d'analyser. De là résulte la possibilité de négliger les variations de la densité, là où elles ne sont pas multipliées par la gravité g, tout en conservant, dans les calculs, leur produit par celle-ci."*, [27, pag. VII].

in the equation of motion *except* in the one in the external force. This is the Oberbeck-Boussinesq approximation. If we consider ρ_0 a reference density at a reference temperature T_0, then the equation of motion (1.3) becomes

$$\frac{\mu}{k}\mathbf{v} = -\nabla p + (\rho_0 + \delta\rho)\,\mathbf{g}\,, \tag{1.8}$$

where

$$\delta\rho = -\rho_0\alpha(T - T_0), \tag{1.9}$$

and hence

$$\rho_f(T) = \rho_0[1 - \alpha(T - T_0)]. \tag{1.10}$$

A fluid for which the Boussinesq approximation is adequate is sometimes called *Boussinesq fluid*, [85].

Summarising, to account for convective motions of an *essentially* isochoric kind, one can avoid the intractable problem associated with the exact compressible equations of motion by assuming that the flow is *as if* incompressible (i.e. $\rho = \rho_0$ and in particular $\nabla \cdot \mathbf{v} = 0$) except that density charges are not ignored in the body-force terms of the momentum equation, the density charges are induced by changes of temperature but not by pressure (i.e. $\rho = \rho(T)$ as in (1.10)) and the velocity gradients are sufficiently small so that the effect on the temperature of conversion of work to heat can be ignored (i.e. $\mathbf{D} : \mathbf{D} \approx 0$), [70]. As a direct consequence of the above assumption, equations (1.3), (1.5) and (1.6) becomes

$$\begin{cases} \dfrac{\mu}{k}\mathbf{v} = -\nabla p + \rho_0[1 - \alpha(T - T_0)]\mathbf{g}\,, \\ \nabla \cdot \mathbf{v} = 0\,, \\ \dfrac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \kappa \Delta T, \end{cases} \tag{1.11}$$

where $\kappa = \frac{\chi}{\rho_0 c_p}$ is the thermal diffusivity. System of equations (1.11) is known as *Darcy-Bénard problem* in the OB approximation, which is the prototypical model for convective motion in porous media. Let us remark that equation $(1.11)_2$ describes an incompressible flow, but this does not imply that the fluid itself is incompressible. Let us stress that in non-isothermal processes the world *incompressibility* has to be used carefully in the OB approximation framework. In a certain way, one can say, following [128], that "In part, the Boussinesq approximation is the counterpart for convection of the incompressibility approximation".
In conclusion, let us remark that traditionally the approximation under

exam generally attributed to Boussinesq was actually of earlier origin and was used by Oberbeck in 1879, [92]. It is probably because of Lord Rayleigh, who may have been unaware of the earlier papers, [70].

1.2.2 *Rigorous justification*

There have been many attempts providing a rigorous justification for the OB approximation, in order to describe the thermo-mechanical response of linear viscous fluids that can undergo isochoric motions in isothermal processes but can sustain motions that are not necessarily isochoric in non-isothermal processes. This fluids are said to be, roughly speaking, *mechanically incompressible* but *thermally compressible*. Concerning clear fluids, in [101] a concise but detailed survey on the most considerable effort expended to justify the OB approximation is presented. However, it is pointed out that all these attempts, e.g. [116, 63, 59], are flawed. Indeed, Rajagopal *et al.* in [101] and in a subsequent extended work [104] carried out a perturbation analysis to show that the OB approximation is not an approximation that gathers terms that are all of the same order in the perturbation parameter, [103]. Concerning flows through porous media, in the pure mechanical case the model that is used is the Darcy's equation (1.1) or modifications like (1.2)-(1.4). In order to accommodate thermal effects for flows through porous media, one usually tags on a term to the balance of the linear momentum that accounts for the buoyancy effect (1.3), and assume the OB approximation. However, this procedure of modifying Darcy's equation seems to be in a first glance too much *ad hoc*, requiring a careful justification. In [102] a systematic approach for obtaining an OB approximation for the flow of a fluid through a porous medium is carried on.

1.3 THERMODYNAMIC FRAMEWORK

In the context of isothermal mechanics, an *ideal* incompressible material is a medium that can only be deformed without any change in volume. However, as it probably becomes clear from the previous section, when the process is non-isothermal, the notion of incompressibility is not well defined. In presence of compressible fluids the pressure is a constitutive function while for incompressible fluids it is only a Lagrange multiplier associated with the constraint of incompressibility. Therefore, it will be convenient to choose the pressure (instead of the density) and the temperature as thermodynamic variables and let the other quantities, i.e. specific volume $V = 1/\rho$ and internal energy ε, being determined by constitutive equations in the form

$$V \equiv V(p,T), \quad \varepsilon \equiv \varepsilon(p,T). \tag{1.12}$$

There are two important parameters for a fluid, namely the *thermal expansion coefficient* α and the *compressibility factor* β, defined by

$$\alpha = \frac{1}{V}V_T, \quad \beta = -\frac{1}{V}V_p. \tag{1.13}$$

Experimental evidences reveal that for fluids considered as incompressible the volume changes little with the temperature and remains practically unchanged with the pressure. For this reason many authors consider as incompressible a fluid for which the specific volume does not vary with the pressure but varies only with the temperature, i.e.

$$V \equiv V(T). \tag{1.14}$$

As underlined by Ruggeri and Gouin [57], the first model of incompressibility was proposed by Ingo Müller, [87]. He required for a fluid to be considered incompressible that *all* the constitutive equations do not depend on pressure. The underlying reason stems from his using of variables ρ and T, since in the experiments the density of incompressible materials depends on temperature ad it is reasonable to assume this for all the constitutive functions. Accounting for this definitions, Müller proved that the only function $V \equiv V(T)$, i.e. $\rho \equiv \rho(T)$, compatible with the entropy principle is a constant as shown in the next section.

1.3.1 *The Müller paradox*

A *perfectly incompressible fluid*, or *incompressible* in the sense of Müller, is a medium whose constitutive equations do not depend on pressure, in particular

$$\rho \equiv \rho(T), \quad \varepsilon \equiv \varepsilon(T), \tag{1.15}$$

where ε is the internal energy. If S is the entropy density, the Gibbs relation

$$TdS = d\varepsilon + pdV, \tag{1.16}$$

can be rewritten using the chemical potential

$$\mu = \varepsilon + pV - TS, \tag{1.17}$$

in the following way

$$d\mu = Vdp - SdT. \tag{1.18}$$

From equations (1.17) and (1.18), it follows

$$\mu_p := \left(\frac{\partial \mu}{\partial p}\right)_T = V, \quad \mu_T := \left(\frac{\partial \mu}{\partial T}\right)_p = -S, \tag{1.19}$$

so (1.17) becomes

$$\varepsilon = \mu - p\mu_p - T\mu_T. \tag{1.20}$$

Then, as a consequence

$$\varepsilon_p = -pV_p - TV_T, \tag{1.21}$$

and according to the incompressibility assumption (1.15) we have $\varepsilon_p = 0$, $V_p = (\rho^{-1})_p = 0$, and so

$$V_T = 0, \tag{1.22}$$

which means that V is constant, or equivalently

$$\rho = \text{const.} \tag{1.23}$$

This conclusion is actually in disagreement with empirical observations, according to which fluids expand when heated, and the theoretical assumptions such as the OB approximation. For this reason, the previous contradiction is called *Müller paradox*.

1.3.2 *Overcoming the paradox*

It seems that the model of incompressibility proposed by of Müller is quite restrictive. For this reason, Ruggeri and Gouin [57], proposed a second model which requires that *the only* constitutive function independent of the pressure is the density

$$\rho \equiv \rho(T), \tag{1.24}$$

and as a result the Gibbs equation is satisfied. This material was named *quasi-thermal-incompressible medium*. From (1.24), integrating equation (1.19) it follows

$$\mu(p, T) = V(T)p + \widehat{\mu}(T), \tag{1.25}$$

and substituting in (1.20), we obtain

$$\varepsilon(p, T) = -TV'(T)p + e(T), \tag{1.26}$$

where

$$e(T) = \widehat{\mu}(T) - T\widehat{\mu}'(T), \tag{1.27}$$

being $\widehat{\mu}(T)$ a function only depending on T and $' := \frac{d}{dt}$. Moreover, equation (1.25) is equivalent to

$$\frac{\varepsilon(p,T)}{e(T)} = -\frac{TV'(T)}{e(T)}p + 1. \tag{1.28}$$

Therefore, from (1.28), a *quasi-thermal-incompressible fluid* tends to be *perfectly incompressible* if $\varepsilon(p,T)$ can be approximate with $e(T)$, i.e. if

$$p \ll \frac{e(T)}{|V'|T} = \frac{\rho^2 e(T)}{|\rho'|T}. \tag{1.29}$$

Let us consider the classical linear behaviour

$$V(T) = V_0[1 + \alpha(T - T_0)], \tag{1.30}$$

where α is the thermal expansion coefficient and $V_0 = 1/\rho_0$ is a reference volume value associated with a reference temperature T_0. If we consider the identity

$$h(p,T) = \varepsilon(p,T) + pV(T), \tag{1.31}$$

where $h = h(p,T)$ is the enthalpy and we assume $c_p \equiv h_T$ constant, then from equation (1.26) we have

$$e'(T) = \varepsilon_T(p,T) + V'(T)p + TV''(T)p. \tag{1.32}$$

Therefore, since we are assuming V being linear in T (1.30), by comparison with the derivative with respect to temperature of the enthalpy (1.31) we obtain

$$e(T) = c_p T, \tag{1.33}$$

i.e. (1.29) becomes

$$p \ll p_{cr} := \frac{c_p \rho_0}{\alpha}. \tag{1.34}$$

The critical pressure value (1.34) characterize the fact that a quasi-thermal-incompressible fluid is experimentally similar to a perfectly incompressible fluid and, in such a case, the Müller paradox is removed.

1.3.3 *Thermodynamic restrictions*

The quasi-thermal incompressibility is obtained as a limit process justifying the compatibility between incompressibility and and the Gibbs relation when inequality (1.34) holds. Let us stress that quasi-thermal incompressibility does not characterize a real compressible material for which the chemical potential μ must be a concave function of p and T. In case V depends only on T, the chemical potential is a linear function of p, see (1.25), and consequently it cannot be concave, [56]. For an actual real compressible material, the volume V needs to depends on p. As a consequence, quasi-thermal-incompressible materials can be considered as an approximation of incompressible materials when the pressure is sufficiently small such that (1.34) is satisfied. Some thermodynamic conditions verified by compressible fluids when the specific volume is governed by the following constitutive function

$$V \equiv V(p, T), \tag{1.35}$$

is now considered. Namely, the *entropy principle* and the *thermodynamic stability* must be satisfied. In local equilibrium, the entropy principle requires the validity of the Gibbs equation (1.16) and the choice of independent variables p and T induces the chemical potential (1.17) as a natural thermodynamic potential. The thermal equation of state (1.35) is determined by experiments while the chemical potential and the entropy density can be deduced as follows

$$\mu = \int V(p, T)\, dp + \widetilde{\mu}(T), \quad S = -\int V_T(p, T)\, dp - \widetilde{\mu}'(T). \tag{1.36}$$

Moreover, taking into account (1.36), from (1.20) we get

$$\varepsilon(p, T) = e(T) + \int V\, dp - pV - T \int V_T\, dp, \tag{1.37}$$

with

$$e(T) = \widetilde{\mu}(T) - T\widetilde{\mu}'(T). \tag{1.38}$$

In particular, the solution of the previous first order linear ODE is

$$\widetilde{\mu}(T) = -T \int \frac{e(T)}{T^2} dT. \tag{1.39}$$

Moreover, the specific heat c_p is defined as the partial derivative of the specific enthalpy (1.31) with respect to temperature at constant pressure p. As a consequence equation (1.37) yields

$$c_p = e'(T) - T \int V_{TT} dp. \tag{1.40}$$

Thermodynamic stability requires the chemical potential being a concave function of p and T. Indeed

$$\mu_{pp} = V_p < 0, \tag{1.41}$$

and from (1.36), (1.38) and (1.40) it follows

$$\begin{aligned}
\mu_{TT} &= -S_T \\
&= -\frac{\partial}{\partial T}\left(\int V_T(p,T)\,dp - \tilde{\mu}'(T)\right) \\
&= \int V_{TT} dp + \frac{e'(T)}{T} \\
&= -\frac{c_p}{T} \\
&< 0.
\end{aligned} \tag{1.42}$$

Moreover

$$\mu_{pp}\mu_{TT} - \mu_{Tp}^2 = -\frac{c_p V_p}{T} - V_T^2, \tag{1.43}$$

which is positive if and only if

$$V_p < -\frac{T V_T^2}{c_p}. \tag{1.44}$$

Therefore, by using (1.13), inequality (1.44) can be written in terms of the thermal (volume) expansion coefficient α and the compressibility factor β as follows

$$\beta > \beta_{cr} := \frac{\alpha^2 T V}{c_p} (> 0). \tag{1.45}$$

In conclusion we can summarize the above thermodynamic restrictions in the following two statements [57]:

- **Statement 1.** For any constitutive function $V \equiv V(p,T)$ and $e \equiv e(T)$, the entropy principle is satisfied if the chemical potential, entropy density and internal energy are given by $(1.36)_1$, $(1.36)_2$ and (1.37), together with (1.39).

- **Statement 2.** Thermodynamic stability requires that the state function $V \equiv V(p,T)$ and $e \equiv e(T)$ satisfy the inequality (1.44) (with $c_p > 0$). Consequently, there exists a lower bound limit β_{cr} of β such that if $\beta > \beta_{cr}$, then the material is stable.

1.3.4 *Extended-quasi-thermal-incompressible fluids*

Let us consider a reference state $\mathfrak{R} = (p_0, T_0, V_0)$. In a neighbourhood of $\mathfrak{R}$ let us chose a small non-dimensional parameter $\delta \ll 1$ such that

$$\delta = \alpha_0 T_0, \quad \beta_0 p_0 = O(\delta^2), \tag{1.46}$$

i.e. β_0 is of order δ^2, where α_0 and β_0 are the thermal expansion coefficient and the compressibility factor at the reference state $\mathfrak{R}$. Following Gouin and Ruggeri [57], a compressible fluid satisfying the thermodynamic conditions from the previous section (*Statement 1,2*) is called *extended-quasi-thermal-incompressible* (EQTI) if there exists $\widehat{V}(T)$ and $\widehat{\varepsilon}(T)$ such that

$$V(p,T) = \widehat{V}(T) + O(\delta^2), \tag{1.47}$$

with $\widehat{V}'(T) = O(\delta)$, and

$$\varepsilon(p,T) = \widehat{\varepsilon}(T) + O(\delta^2). \tag{1.48}$$

The previous definition tells us that an EQTI fluid is a stable compressible fluid that approximates an incompressible fluid to order δ^2 in the sense of Müller's definition. As a consequence of the previous definitions we have the following representation

$$V(p,T) = V_0 + \delta W(T) - \delta^2 U(p,T), \tag{1.49}$$

with $W(T)$ and $U(p,T)$ constitutive functions chosen in agreement with the conditions in Section 1.3.3. Moreover, from equation $(1.36)_1$ we deduce

$$\mu = pV_0 + \delta p W(T) + \delta^2 \widehat{\mu}(T) + \widetilde{\mu}(T). \tag{1.50}$$

with

$$\widehat{\mu}(T) = -\int U(p,T)\, dp. \tag{1.51}$$

Moreover, from (1.37) and (1.49) we have

$$\begin{aligned} \varepsilon(p,T) = e(T) + \delta^2 \widehat{\mu}(T) + \delta^2 U(p,T) \\ - T\delta W'(T)p + \delta^2 T \int U_T dp, \end{aligned} \tag{1.52}$$

i.e.

$$\varepsilon(p,T) = e(T) - T\delta W'(T)p + O(\delta^2). \tag{1.53}$$

Due to (1.53) in order to satisfy (1.48) we require that the pressure cannot exceed the following critical value or order δ^{-1}

$$p \ll p_{cr} := \frac{1}{\delta}\frac{e(T)}{TW'(T)}, \tag{1.54}$$

and as a consequence

$$\varepsilon(p,T) = e(T) + O(\delta^2). \tag{1.55}$$

From (1.13), (1.46) and (1.49) we have

$$\alpha = \delta \frac{W'(T)}{V_0} + O(\delta^2), \quad \beta = \delta^2 \frac{U_p(p,T)}{V_0}, \tag{1.56}$$

with $W'(T_0) = V_0/T_0$. Moreover from (1.40) and (1.45) it follows

$$\beta_{cr} = \delta^2 \frac{(W'(T))^2}{V_0 c_p}, \quad c_p \approx e'(T), \tag{1.57}$$

and from (1.44), the thermodynamic stability is guaranteed when

$$U_p(p,T) > \frac{T\,(W'(T))^2}{c_p}, \quad c_p \approx e'(T) > 0. \tag{1.58}$$

We can conclude that the following statement holds [57]:

- **Statement 3.** An EQTI fluid given by constitutive equations (1.49) and (1.55), satisfying (1.58), is a good approximation of an incompressible fluid as V and ε differ to order δ^2 from functions depending only on T, provided the pressure is smaller then a critical value p_{cr} given by (1.54) and the compressibility factor is greater than a critical value β_{cr} given by (1.57).

The most significant case is one can take into account is a linear dependence of V with respect of T and p, namely

$$V(p,T) = V_0 \left[1 + \alpha(T - T_0) - \beta(p - p_0)\right]. \tag{1.59}$$

with $e(T) = c_p T$, which is a particular case of (1.49) when

$$\delta = \alpha T_0, \quad W(T) = \frac{V_0}{T_0}(T - T_0), \quad U(p,T) = \frac{\beta}{\alpha^2}\frac{V_0}{T_0^2}(p - p_0). \tag{1.60}$$

Hence, a fluid is EQTI if

$$\delta \ll 1, \quad \beta > \beta_{cr}, \quad p \ll p_{cr}. \tag{1.61}$$

In conclusion, in this section we obtained that a fluid can be considered as incompressible if the pressure is smaller than a critical pressure of order α^{-1} and if the compressibility factor is greater than a critical value of order α^2. In the OB approximation when the fluid is slightly compressible, i.e. $\beta \neq 0$, the density function will contain an additional term, namely

$$\rho(p, T) = \rho_0[1 - \alpha(T - T_0) + \beta(p - p_0)]. \tag{1.62}$$

In Chapter 9 we revisited the Darcy-Bénard problem (1.11) in the OB approximation taking into account (1.62), in order to investigate the compressibility effects on the onset of Darcy-Bénard convection.

1.4 THE VERONIS LAW

When the phenomenon of the onset of convective motions is investigated under the OB approximation, one classically consider a linear decreasing density profile (1.10), since essentially all fluids expand in a linear manner when heated. An almost unique exception to this decreasing behaviour is given by water. It is well known that water exhibit many anomalous macroscopic property. One of these is that water maximise density at $4°C$, i.e. it is almost parabolic around its density inversion point. Since a linear dependence in T cannot detect this density inversion phenomenon, the constitutive density need to be revisited. The first model for quadratic density was proposed by Veronis in 1963, [131]:

$$\rho(T) = \rho_0[1 - \alpha(T - T_0)^2]. \tag{1.63}$$

As shown in Figure 1.1, the constitutive density law (1.63) by Veronis, when $T_0 = 4°C$, accurately catches the experimental data reported in Table 1.1. As anticipated in the introduction, this density inversion phe-

water density (kg/m^3)	$T_U(°C)$
999.880	0
999.960	2
999.972	4
999.922	6
999.812	8
999.647	10

Table 1.1: Experimental data of water density.

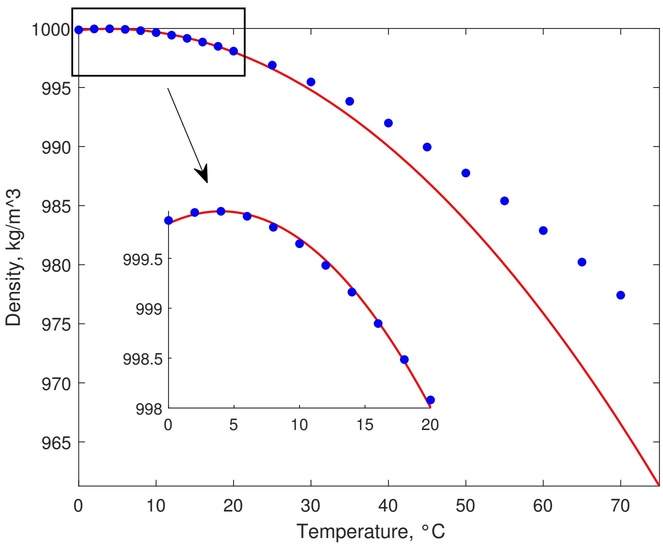


Figure 1.1: Comparison between experimental values of the water density under isobaric condition [132] and the Veronis' constitutive law (1.63) where $\alpha = 7.68 \times 10^{-6}\ °C^{-2}$ and $T_0 = 4°C$.

nomenon is responsible for the activation of the penetrative convection. Under different physical setups, the onset of penetrative convection will be the topic of Part II of the present thesis.

1.4.1 *Penetrative convection in porous media*

Unlike natural convection, penetrative convection occurs when the density of the fluid is not monotonically decreasing in temperature (as for the majority of fluids) but the fluid contracts, rather than expands, through heating. This anomalous behaviour, is exhibited in very few (even if remarkable) cases in nature, among which water represents the most important case. Other cases include graphene, some complex compounds, some iron alloys, and cubic zirconium tingstenate (ZrW_2O_8), [25]. In the following, we will consider a porous layer Σ saturated by water with the bottom plane maintained at $0°C$ and the top plane at a temperature greater than $4°C$, which is the temperature's value at which water has a maximum. It turns out that Σ is partitioned in two sub-regions: Σ_1 and Σ_2, as depicted in Figure 1.2. The former region is delimited by the lower plane and an intermediate plane where the fluid density attains its maximum, and the latter is delimited by the intermediate plane and the upper plane. Because of the above-described behaviour, the assumed physical setup implies that Σ_1 is a potentially unstable fluid region laying below a stably stratified region Σ_2. When convective motion occur in Σ_1, the fluid motion will then *penetrate* in the upper stable region Σ_2.

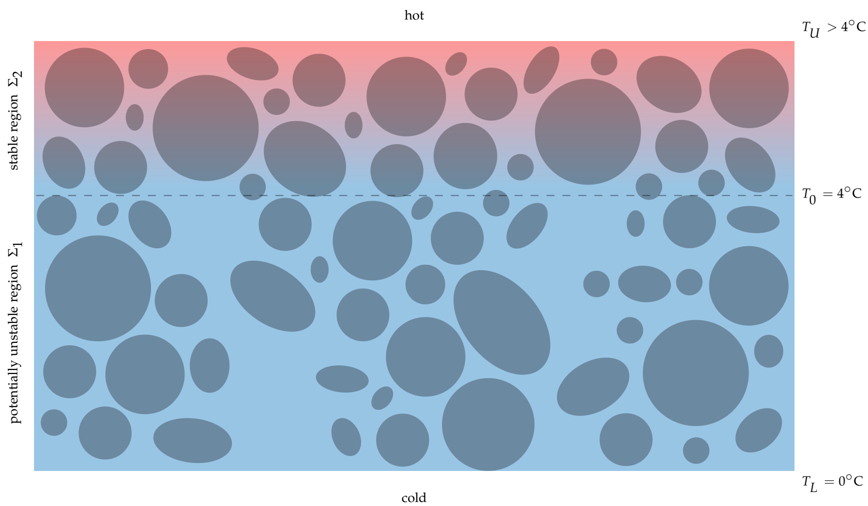


Figure 1.2: Physical setup for the penetrative convection problem.

2 TECHNICAL BACKGROUND

In this chapter the theoretical framework concerning the mathematical tools needed for the subsequent investigations is summarized. In particular, the basic definitions of the function spaces employed, the basic concepts concerning the weak formulation and the well-posedness are presented in Section 2.1. Moreover, Sections 2.3 and 2.4 are devoted to the description of the fundamental concepts regarding the stability analysis of the basic solution a dynamical system.

2.1 MATHEMATICAL TOOLS

2.1.1 *Lebesgue, Sobolev and Bochner spaces*

From now on, Ω will denote an open set of $\mathbb{R}^n$ and $u : \Omega \longrightarrow \mathbb{R}$ a real valued function. Let us recall that given $p \geq 1$ the **Lebesgue space** $L^p(\Omega)$ is the space of Lebesgue measurable functions, i.e. functions whose norm

$$\begin{aligned} \|u\|_{L^p(\Omega)} &= \left(\int_\Omega |u|^p dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \\ \|u\|_{L^\infty(\Omega)} &= \operatorname*{ess\,sup}_\Omega |u|, \end{aligned} \tag{2.1}$$

is finite. $L^p(\Omega)$ is a Banach space (separable in case $p < \infty$) and a Hilbert space if $p = 2$ with respect to the scalar product

$$(u, v)_{L^2(\Omega)} = \int_\Omega f g \, dx. \tag{2.2}$$

Let $\alpha \in \mathbb{N}_0^n$ and let us set $|\alpha| = \alpha_1 + \ldots + \alpha_n$. If $u \in C^k(\Omega)$, then we define

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}. \tag{2.3}$$

Moreover, let $\alpha \in \mathbb{N}_0^n$, $k \geq 1$ and $p \geq 1$, then the vector space

$$W^{k,p}(\Omega) = \{ u \in L^p(\Omega) \mid \exists\, D^\alpha u \in L^p(\Omega), \ \forall\, |\alpha| \leq k \}, \tag{2.4}$$

whose elements are L^p-integrable functions endowed with L^p-integrable weak derivatives $v (= D^\alpha u)$ for all $|\alpha| \leq k$, i.e. for all $\phi \in C_c^\infty(\Omega)$

$$\int_\Omega u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_\Omega v \phi \, dx, \tag{2.5}$$

is called **Sobolev space**. This space naturally admits a norms

$$\begin{aligned}\|u\|_{W^{k,p}(\Omega)} &= \left(\sum_{|\alpha|\leq k} \|D^{\alpha}u\|^{p}_{L^{p}(\Omega)}\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \\ \|u\|_{W^{k,\infty}(\Omega)} &= \sum_{|\alpha|\leq k} \|D^{\alpha}u\|_{L^{\infty}(\Omega)},\end{aligned} \tag{2.6}$$

with respect to which $W^{k,p}(\Omega)$ is a Banach space (separable in case $p < \infty$). Moreover, when $p = 2$ the space $W^{k,2}(\Omega)$ (usually denoted by $H^{k}(\Omega)$) is a Hilbert space with respect to the scalar product

$$(u, v)_{H^{k}(\Omega)} = \sum_{|\alpha|\leq k} (D^{\alpha}u, D^{\alpha}v)_{L^{2}(\Omega)}. \tag{2.7}$$

A sequence $u_k \to u$ in $W^{k,p}(\Omega)$ if $\lim_{k\to+\infty} \|u_k - u\|_{W^{k,p}(\Omega)} = 0$.

In the sequel we will need spaces such that $n = 2$, $k = 1, 2$ and $p = 2$. The space

$$W^{1,2}(\Omega) = \left\{u \in L^{2}(\Omega) \mid \exists\, u_x, u_y \in L^{2}(\Omega)\right\}, \tag{2.8}$$

with associate norm

$$\begin{aligned}\|u\|_{W^{1,2}(\Omega)} &= \left(\|u\|^{2}_{L^{2}(\Omega)} + \|u_x\|^{2}_{L^{2}(\Omega)} + \|u_y\|^{2}_{L^{2}(\Omega)}\right)^{\frac{1}{2}} \\ &= \left(\int_{\Omega} u^{2}dx + \int_{\Omega} \left[u_x^{2} + u_y^{2}\right] dx\right)^{\frac{1}{2}} \\ &= \left(\int_{\Omega} u^{2}dx + \int_{\Omega} |\nabla u|^{2}dx\right)^{\frac{1}{2}} \\ &= \left(\|u\|^{2}_{L^{2}(\Omega)} + \|\nabla u\|^{2}_{L^{2}(\Omega)}\right)^{\frac{1}{2}}\end{aligned} \tag{2.9}$$

is the space of all functions $u \in L^{2}(\Omega)$ whose gradient $\nabla u \in L^{2}(\Omega)$. Moreover, the space

$$W^{2,2}(\Omega) = \left\{u \in L^{2}(\Omega) \mid \exists\, u_x, u_y, u_{xy}, u_{xx}, u_{yy} \in L^{2}(\Omega)\right\}, \tag{2.10}$$

with associate norm

$$\begin{aligned}\|u\|_{W^{2,2}(\Omega)} &= \Big(\|u\|^2_{L^2(\Omega)} + \|u_x\|^2_{L^2(\Omega)} + \|u_y\|^2_{L^2(\Omega)} + \|u_{xy}\|^2_{L^2(\Omega)} \\ &\qquad + \|u_{xx}\|^2_{L^2(\Omega)} + \|u_{yy}\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}} \\ &= \Big(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \\ &\qquad + \|u_{xy}\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}}\end{aligned} \tag{2.11}$$

is the space of all functions $u \in L^2(\Omega)$ whose gradient ∇u, Laplacian Δu and mixed partial derivatives are in $L^2(\Omega)$.

We denote by $W_0^{k,p}(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$, and we interpret it as the space comprising those functions $u \in W^{k,p}(\Omega)$ such that "$D^\alpha u = 0$ on $\partial\Omega$" for all $|\alpha| \leq k-1$. It is customary to write $H_0^k(\Omega) := W_0^{k,2}(\Omega)$. It is important to have an explicit characterization of the dual space of $H_0^1(\Omega)$. Indeed, we denote by $H^{-1}(\Omega)$ the dual space to $H_0^1(\Omega)$. If $f \in H^{-1}(\Omega)$ we define the norm

$$\|f\|_{H^{-1}(\Omega)} := \Big\{\langle f, u\rangle \mid u \in H_0^1(\Omega),\ \|u\|_{H_0^1(\Omega)} \leq 1\Big\}, \tag{2.12}$$

where $\langle\ ,\ \rangle$ denotes the pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

It is possible to consider functions $u : T \longrightarrow X$ whose values are in a Banach space X (with associate norm $\|\ \|_X$). The **Bochner space** $L^p(T; X)$ is the space of Bochner measurable functions $u : T \longrightarrow X$ with values in the Banach space X, such that the corresponding norm

$$\begin{aligned}\|u\|_{L^p(T;X)} &= \left(\int_T \|u(t)\|_X^p\, dx\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \\ \|u\|_{L^\infty(T;X)} &= \operatorname*{ess\,sup}_T \|u(t)\|_X,\end{aligned} \tag{2.13}$$

is finite. $L^p(T; X)$ is a Banach space (separable is $p < \infty$ and X is separable) and if $p = 2$ and X is a Hilbert space, then it is also a Hilbert space with respect to the scalar product

$$(u, v)_{L^2(T;X)} = \int_T \langle u(t), v(t)\rangle_X\, dx. \tag{2.14}$$

2.1.2 *Hilbert's bases, weak formulation and well-posedness*

Given a Hilbert space H and a set $\widetilde{\mathcal{S}}$ of mutually orthogonal vectors in H, we can take the smallest closed linear subspace $\widetilde{H}$ of H containing $\widetilde{\mathcal{S}}$. Then, $\widetilde{\mathcal{S}}$ will be an orthogonal basis of $\widetilde{H}$ which may be smaller than H, being an incomplete orthogonal set, or be H itself when it is complete. Considering the linear span of $\widetilde{\mathcal{S}}$, denoted by

$$[\widetilde{\mathcal{S}}\,] = \left\{ \sum_{i=1}^{k} A_i \phi_i \;\middle|\; k \in \mathbb{N},\ \phi_i \in \widetilde{\mathcal{S}},\ A_i \in \mathbb{R} \right\}, \tag{2.15}$$

we have the set of all finite linear combinations of elements (vectors) of $\widetilde{\mathcal{S}}$. In the case of infinite $\widetilde{\mathcal{S}}$, infinite linear combinations (i.e. where a combination may involve an infinite sum, assuming that such sums are defined somehow as in, say, a Banach space) are excluded by the definition. In case we need to consider infinite sums, we may construct the *closure* of the linear span *with respect to a proper norm*, say $\|\ \|_{L^2}$. Sums are limit of sequences of partial sums, and limit is a topological dependent notion. Hence, considering the closure with respect to a norm implies convergence of the sequences of partial sums and this defines the sum of the infinite series. Therefore, if

$$u \in \widetilde{L}^2 = \overline{[\widetilde{\mathcal{S}}]}^{\|\ \|_{L^2}}, \tag{2.16}$$

then u can be written as a linear combination of elements of the basis $\widetilde{\mathcal{S}}$. Now, in this space we are writing functions in series convergent in L^2, but we don't know if the limit is differentiable. Therefore, for example, if one close with respect to the $W^{k,p}$ norm we are choosing the way in which the partial sums will converge.

Let us recall the formal passage from classical solution to weak solution of a partial differential equation. Consider the Poisson problem posed in a domain Ω supplemented with homogeneous Dirichlet boundary conditions

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.17}$$

A classical solution (or strong solution) of the previous problem is a function $u \in C^2(\Omega)$ satisfying the above equations. However, (2.17) can be reformulated so as to look for a solution in the distributional sense by testing the equation against smooth functions. Reformulating the problem amounts to relaxing the pointwise regularity (i.e. continuity) required to ensure the existence of the classical derivative to the (weaker) existence of the distributional derivative which regularity is to be interpreted in terms of Lebesgue spaces: the obtained problem is a *weak formulation* and a solution to this problem (i.e. in the distributional sense) is called *weak solution*.

Let $u \in C^2(\overline{\Omega})$ be a classical solution to the Poisson problem (2.17) and let test the equation against any smooth function $\varphi \in C_c^\infty$:

$$-\int_\Omega \Delta u \varphi \, d\Omega = \int_\Omega f \varphi \, d\Omega. \tag{2.18}$$

Integrating by parts and recalling φ has compact support in Ω (it vanish on the boundary), the left-hand side reads

$$\begin{aligned} -\int_\Omega \Delta u \varphi \, d\Omega &= -\int_\Omega \nabla \cdot (\nabla u \varphi) \, d\Omega + \int_\Omega \nabla u \cdot \nabla \varphi \, d\Omega \\ &= -\int_{\partial\Omega} \nabla u \varphi \cdot \boldsymbol{\nu} \, d\sigma + \int_\Omega \nabla u \cdot \nabla \varphi \, d\Omega \\ &= \int_\Omega \nabla u \cdot \nabla \varphi \, d\Omega, \end{aligned} \tag{2.19}$$

$\boldsymbol{\nu}$ being the outward pointing unit normal at each point on the boundary. Hence, the distributional form formulation reads

$$\int_\Omega \nabla u \cdot \nabla \varphi \, d\Omega = \int_\Omega f \varphi \, d\Omega, \quad \forall \varphi \in C_c^\infty. \tag{2.20}$$

A weak formulation of the Poisson problem consists in finding $u \in H$, given f, such that

$$\int_\Omega \nabla u \cdot \nabla \varphi \, d\Omega = \int_\Omega f \varphi \, d\Omega, \quad \forall \varphi \in V, \tag{2.21}$$

in which H and V are a function spaces yet to be defined, both satisfying regularity constraints and for H boundary conditions constraint.

How to choose the *solution space H* and the *test function space V*? Sobolev spaces $W^{k,2}(\Omega)$ are natural choice to *measure* functions involved in the weak formulation of partial differential equations as the integrals relies on the fact that integrals of powers $|\ |^p$ of u and weak derivatives $D^\alpha u$ for some $1 \leq p < \infty$ exist. Since the weak Poisson problem involves first order derivatives, we should consider a solution in $W^{1,2}(\Omega)$, with the weak derivative Du being a function of $L^2(\Omega)$ which identifies with the classical derivative (if it exists) almost everywhere. Moreover, the solution should satisfy the boundary condition of the strong form of the problem. The homogeneous Dirichlet boundary condition is embedded in the function space of the solution: u vanishing on the boundary $\partial\Omega$ yields that we should seek u in $W_0^{1,2}(\Omega)$. We have seen that any weak solution *lives* in $W_0^{1,2}(\Omega)$ and it is possible to establish that the natural space for the test function is the same space, i.e. the weak formulation is satisfied if $\varphi \in W_0^{1,2}(\Omega)$.

In the usual sense, a problem is *well-posed* if it admits a unique solution which is bounded in norm by the data (forcing term, boundary conditions) which are independent on the solution and appear at the right-hand side of the equation.

2.2 LINEAR EVOLUTION EQUATIONS

In this section we outline the general framework concerning the existence, uniqueness and regularity of appropriately defined weak solutions of second-order parabolic PDEs, that will be investigated in Chapter 8.

2.2.1 *Second-order parabolic equations*

Let us assume Ω is an open set of $\mathbb{R}^n$, set $\Omega_T = \Omega \times (0, T]$ for some fixed time $T > 0$ and consider the following initial boundary value problem

$$\begin{cases} u_t + Lu = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times [0, T], \\ u = g & \text{on } \Omega \times \{t = 0\}, \end{cases} \tag{2.22}$$

where $f : \Omega_T \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$ are given, $u : \overline{\Omega}_T \to \mathbb{R}$ is the unknown

$$u = u(\mathbf{x}, t). \tag{2.23}$$

and L denotes for each time t a linear second-order partial differential operator

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij}(\mathbf{x}, t) u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(\mathbf{x}, t) u_{x_i} + c(\mathbf{x}, t) u, \tag{2.24}$$

for given coefficients a^{ij}, b^i and c $(i, j = 1, \ldots, n)$.

Let us temporary suppose that $u = u(\mathbf{x}, t)$ is a smooth solution of (2.22). We need to switch our viewpoint by associating with u a mapping

$$\mathrm{u} : [0, T] \longrightarrow W_0^{1,2}(\Omega), \tag{2.25}$$

defined by

$$[\mathrm{u}(t)](\mathbf{x}) := u(\mathbf{x}, t), \quad \mathbf{x} \in \Omega,\ 0 \leq t \leq T. \tag{2.26}$$

Indeed, we are considering u not as a function of $\mathbf{x}$ and t, but rather as a mapping $\mathbf{u}$ of t into the space $W_0^{1,2}(\Omega)$ of functions of $\mathbf{x}$. Let us similarly define

$$\mathrm{f} : [0, T] \longrightarrow L^2(\Omega) \tag{2.27}$$

by

$$[\mathrm{f}(t)](\mathbf{x}) := f(\mathbf{x}, t), \quad \mathbf{x} \in \Omega,\ 0 \leq t \leq T. \tag{2.28}$$

If we now fix a function $v \in W_0^{1,2}(\Omega)$, we can multiply $(2.22)_1$ by v and integrate by parts, to find

$$\langle \mathrm{u}', v \rangle + B[\mathrm{u}, v; t] = (\mathrm{f}, v), \tag{2.29}$$

for each $0 \leq t \leq T$, where $' = \frac{d}{dt}$ and B is the time-dependent bilinear form

$$B[u, v; t] := \int_\Omega \left[\sum_{i,j=1}^n a^{ij}(\cdot, t) u_{x_i} v_{x_j} + \sum_{i=1}^n b^i(\cdot, t) u_{x_i} v + c(\cdot, t) uv \right] dx, \tag{2.30}$$

for $u, v \in H_0^1(\Omega)$ and a.e. $0 \leq t \leq T$.

We say a function $\mathrm{u} \in L^2(0, T; H_0^1(\Omega))$, with $\mathrm{u}' \in L^2(0, T; H^{-1}(\Omega))$ is a *weak solution* of the parabolic problem (2.22) provided (2.29) for each $v \in H_0^1(\Omega)$ and a.e. time $0 \leq t \leq T$ and $\mathrm{u}(0) = g$, [43].

2.2.2 *Galerkin method*

In order to build a weak solution of the parabolic problem (2.22), we can first construct solutions of certain finite dimensional approximation to (2.22) and then passing to limit. This is the idea behind the so called *Galerkin method*. More specifically, assume $\{w_k\}_{k=1}^\infty$ is an orthogonal basis of smooth functions for $W_0^{1,2}(\Omega)$ and an orthonormal basis for $L^2(\Omega)$, and fix $m \in \mathbb{N}$. We will look for a function

$$\mathrm{u}_m(t) : [0, T] \longmapsto W_0^{1,2}(\Omega), \tag{2.31}$$

of the form

$$\mathrm{u}_m(t) := \sum_{k=1}^{m} d_m^k(t) w_k, \tag{2.32}$$

where we need to select the coefficients $d_m^k(t)$ $(0 \leq t \leq T, k = 1, \ldots, m)$ so that for all $k = 1, \ldots, m$

$$d_m^k(0) = (g, w_k) \tag{2.33}$$

and

$$\langle \mathrm{u}_m', w_k \rangle + B[\mathrm{u}_m, w_k; t] = (\mathrm{f}, w_k), \tag{2.34}$$

Thus, we look for a function u_m of the form (2.32) satisfying the "projection" (2.34) of problem (2.22) in the finite-dimensional subspace spanned by $\{w_k\}_{k=1}^m$. To this aim, the fist step consists in the *construction of approximate solutions*, i.e. finding a unique function u_m, for each integer $m \in \mathbb{N}$, of the form (2.32) satisfying (2.33) and (2.34). The subsequent step is to send m to infinity, providing *energy estimates* and showing that a subsequence to our solution u_m of the approximate problem converges to a (unique) weak solution of (2.22).

2.3 PRELIMINARIES ON STABILITY THEORY

Generally speaking, the prediction of the time evolution of a physical phenomenon is the core concept for real world applications and it can be achieved through a *qualitative analysis* of the mathematical model describing the phenomenon when the explicit solution cannot be explicitly determined. In the subsequent investigations, the critical Rayleigh number for the onset of convection is determined through the stability analysis of the basic state motion. This section is devoted to the description of the fundamental theoretical background concerning the linear and the nonlinear stability analysis.

Let $\mathcal{F}$ be a physical phenomenon whose description we are interested in and let $\Omega \subseteq \mathbb{R}$ be the domain in which the phenomenon takes place. Moreover, let $u_i(\mathbf{x}, t)$ ($i = 1, \ldots, n$ and $(\mathbf{x}, t) \in \Omega \times [0, T]$) be the relevant quantities describing the state of $\mathcal{F}$. The vector $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ is called *state vector* and one can model the physical phenomenon by the following system of partial differential equations

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{F}\left(t, \mathbf{x}, \mathbf{u}, \frac{\partial u_i}{\partial x_r}, \frac{\partial^2 u_j}{\partial x_r \partial x_s}, \ldots\right), \quad \text{in } \Omega \times (0, T), \tag{2.35}$$

with $i, j = 1, \dots, n$ and $r, s = 1, 2, 3$, whose initial and boundary conditions are

$$\begin{aligned} &\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \text{on } \Omega, \\ &A(\mathbf{u}, \nabla \mathbf{u}) = \widehat{\mathbf{u}}, \quad \text{on } \partial\Omega \times [0, T], \end{aligned} \tag{2.36}$$

where A is an assigned operator and $\widehat{\mathbf{u}}$ is prescribed. The problem (2.35)-(2.36) is a mathematical model describing the evolution of $\mathcal{F}$. Determining explicit solutions to an initial boundary value problem is not a trivial matter. Therefore, a qualitative analysis of the model is needed in order to capture information on the behaviour of a solution.

Let (X, d) be a metric linear space and $S_r(\mathbf{x})$ an open ball or radius $r > 0$ centred in $\mathbf{x} \in X$. A *dynamical system* on the metric space (X, d) is an application

$$f : (\mathbf{v}_0, t) \in X \times \mathbb{R}^+ \longmapsto f(\mathbf{v}_0, t) \in X, \tag{2.37}$$

such that $f(\mathbf{v}_0, 0) = \mathbf{v}_0$. A solution $\mathbf{u} = \mathbf{u}(\mathbf{u}_0, t)$ with $\mathbf{u}(\mathbf{u}_0, 0) = \mathbf{u}_0$ to the problem (2.35)-(2.36) is a dynamical system. A *motion of initial datum* $\mathbf{v}_0$ is a dynamical system $\mathbf{v}$ defined as:

$$\mathbf{v}(\mathbf{v}_0, \cdot) : t \in \mathbb{R}^+ \longmapsto \mathbf{v}(\mathbf{v}_0, t) \in X, \tag{2.38}$$

such that $\mathbf{v}(\mathbf{v}_0, 0) = \mathbf{v}_0$. If $\mathbf{v}(\mathbf{v}_0, t) = \mathbf{v}_0$ for all $t \in \mathbb{R}^+$, the motion $\mathbf{v}(\mathbf{v}_0, \cdot)$ is said to be *stationary*, and $\mathbf{v}_0$ is called *equilibrium point*. A motion $\mathbf{v}(\mathbf{v}_0, \cdot)$ *depends continuously on the initial data* if and only if

$$\begin{aligned} &\forall \varepsilon > 0,\ \forall T > 0,\ \exists \delta(\varepsilon, T) > 0 \text{ such that} \\ &\mathbf{v}_1 \in S_\delta(\mathbf{v}_0) \implies \mathbf{v}(\mathbf{v}_1, t) \in S_\varepsilon\big(\mathbf{v}(\mathbf{v}_0, t)\big),\ \forall t \in [0, T]. \end{aligned} \tag{2.39}$$

A motion $\mathbf{v}(\mathbf{v}_0, \cdot)$ is *stable in the sense of Ljapunov* if

$$\begin{aligned} &\forall \varepsilon > 0,\ \exists \delta(\varepsilon) > 0 \text{ such that} \\ &\mathbf{v}_1 \in S_\delta(\mathbf{v}_0) \implies \mathbf{v}(\mathbf{v}_1, t) \in S_\varepsilon\big(\mathbf{v}(\mathbf{v}_0, t)\big),\ \forall t \in \mathbb{R}^+. \end{aligned} \tag{2.40}$$

A motion $\mathbf{v}(\mathbf{v}_0, \cdot)$ is *unstable in the sense of Ljapunov* if

$$\begin{aligned} &\exists \varepsilon > 0 \text{ such that } \forall \delta(\varepsilon) > 0, \exists \hat{t} > 0 \text{ such that} \\ &\mathbf{v}_1 \in S_\delta(\mathbf{v}_0) \implies \mathbf{v}(\mathbf{v}_1, \hat{t}) \notin S_\varepsilon\big(\mathbf{v}(\mathbf{v}_0, \hat{t})\big). \end{aligned} \tag{2.41}$$

A motion $\mathbf{v}(\mathbf{v}_0, \cdot)$ is called *attractor over a set Y* if

$$\mathbf{v}_1 \in Y \implies \lim_{t \to +\infty} d\big[\mathbf{v}(\mathbf{v}_0, t), \mathbf{v}(\mathbf{v}_1, t)\big] = 0. \tag{2.42}$$

A motion $\mathbf{v}(\mathbf{v}_0, \cdot)$ is said to be *asymptotically stable* if it is stable and there exists $\delta_1 > 0$ such that $\mathbf{v}(\mathbf{v}_0, \cdot)$ is an attractor over $S_{\delta_1}(\mathbf{v}_0)$. Moreover, $\mathbf{v}(\mathbf{v}_0, \cdot)$ is said to be *exponentially stable* if

$$\begin{aligned} &\exists \delta_1 > 0,\ \lambda(\delta_1) > 0,\ M(\delta_1) > 0 \text{ such that} \\ &\mathbf{v}_1 \in S_{\delta_1}(\mathbf{v}) \implies d\big[\mathbf{v}(\mathbf{v}_1, t), \mathbf{v}(\mathbf{v}_0, t)\big] \leq M \exp(-\lambda t) d[\mathbf{v}_1, \mathbf{v}_0]. \end{aligned} \tag{2.43}$$

If $\delta_1 = +\infty$, then $\mathbf{v}(\mathbf{v}_0, \cdot)$ is asymptotically, exponentially, globally stable. Given a *basic motion* $\mathbf{v}^*(\mathbf{v}_0, t)$, its stability may be expressed in therms of the *perturbation*

$$\mathbf{u}(\mathbf{u}_0, t) = \mathbf{v}(\mathbf{v}_1, t) - \mathbf{v}^*(\mathbf{v}_0, t). \tag{2.44}$$

with $\mathbf{u}_0 = \mathbf{v}_1 - \mathbf{v}_0$. A basic motion is said to be *stable with respect to perturbations of the initial data* if

$$\begin{aligned} &\forall \varepsilon > 0,\ \exists \delta(\varepsilon) > 0 \text{ such that} \\ &\mathbf{u}_0 \in S_\delta(\mathbf{0}) \implies \mathbf{u}(\mathbf{u}_0, t) \in S_\varepsilon(\mathbf{0}),\ \forall t \in \mathbb{R}^+. \end{aligned} \tag{2.45}$$

In plain words, a given basic motion is stable if all perturbations that are small initially remain small for all time, and it is unstable if at least one small perturbation grows so much that it ceases to be small after some time. Since the previous definitions are intimately connected to the definition of the *distance*, it turns out that the notion of stability in the Ljapunov sense in *topologically dependent*, i.e. it depends on the topology of the space (X, d). More specifically, when one deals with problems with a *finite number* of degree of freedom, embedded in finite-dimensional spaces, the concept of stability is independent on the chosen norm since in $\mathbb{R}^n$ all the norms are equivalent. However, for phenomena with *infinite* number of degree of freedom, which are usually described by partial differential equations in infinite-dimensional spaces, stability will depend on the chosen norm. For a detailed discussion about topology dependent stability see [47].

2.3.1 *The Ljapunov direct method*

In this section, the fundamentals of the *Lyapunov direct method*, will be outlined. This method allows to get important information of solutions, although they cannot be obtained explicitly. The underlying idea, consists in determining the sign of the time derivative of an auxiliary function (called *Ljapunov function*) evaluated along solutions of the differential system under investigation.

Given a dynamical system (2.37) on the metric space (X, d) a function

$$V : I \subseteq X \longrightarrow \mathbb{R} \tag{2.46}$$

is said to be a *Lyapunov function* on I if $V \in C^1(I)$ and V is a non-increasing function in time along the solution of (2.37) with initial data in I. By virtue of (2.45), we introduce the method to investigate the stability of the null solution of a given basic state. Indeed, let X be a normed linear space and let $\mathcal{G}_r$, $r > 0$, be the set of functions $g : [0, r) \to \mathbb{R}^+$ that are continuous, strictly increasing and such that $g(0) = 0$. Therefore, the Ljapunov direct method can be summarized by the following statements.

Theorem 2.3.1. Let u be a dynamical system on a normed space X and let $\mathbf{0}$ be an equilibrium point. If V is a Ljapunov function on the open set $S_r(\mathbf{0})$ such that

i. $V(\mathbf{0}) = 0$,

ii. $\exists f \in \mathcal{G}_r$ such that

$$V(\mathbf{u}) \geq f(\|\mathbf{u}\|), \quad \forall \mathbf{u} \in S_r(\mathbf{0}),$$

then $\mathbf{0}$ is stable. Moreover, if

iii. $\exists g \in \mathcal{G}_r$ such that

$$\dot{V}(\mathbf{u}) \leq -g(\|\mathbf{u}\|), \quad \forall \mathbf{u} \in S_r(\mathbf{0}),$$

then $\mathbf{0}$ is asymptotically stable.

The following result holds as a particular case of Theorem 2.3.1.

Theorem 2.3.2 (Ljapunov)**.** Let $\mathbf{u}$ be a dynamical system on X and let $\mathbf{0}$ be an equilibrium point. If V is a Ljapunov function on $S_r(\mathbf{0})$ such that

$$V(\mathbf{0}) = 0, \quad V(\mathbf{u}) > 0, \quad \forall \mathbf{u} \neq \mathbf{0}, \tag{2.47}$$

then, the stability with respect to the measure V of perturbation is obtained. Moreover, if there exists a positive constant γ such that the following inequality holds along the solutions

$$\dot{V} \leq \gamma V, \tag{2.48}$$

then

$$V \leq V(\mathbf{u}_0) \exp(-\gamma t), \tag{2.49}$$

i.e. the asymptotic exponential stability with respect to the measure V is obtained.

The proofs of the previous results can be found in [47]. In the following Section, we will explain how a proper choice of the norm connects the linear and nonlinear stability analyses of the basic steady solution of a dynamical system.

2.4 STABILITY ANALYSIS

In the present Section, we summarize several classical arguments concerning the linear and nonlinear stability analysis. The following topics are extensively addressed in the monographs by Joseph [69], Georgescu [54] and Galdi-Rionero [48], as well as in several important articles by Prodi [99], Sattinger [111], Kirchgässner-Kielhöfer [74] and Galdi-Straughan [50].

2.4.1 *Linear theory and principle of exchange of stabilities*

Let H be a Hilbert space, $\mathbf{v}^*(\mathbf{v}_0, t)$ be a basic motion and $\mathbf{u} = \mathbf{v}(\mathbf{v}_1, t) - \mathbf{v}^*(\mathbf{v}_0, t)$ be the perturbation arising after perturbing the basic motion. Let us consider the following initial value problem

$$\begin{cases} \mathbf{u}_t + L\mathbf{u} + N(\mathbf{u}) = \mathbf{0}, \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(t), \end{cases} \tag{2.50}$$

with L a linear autonomous operator and N a non linear one such that $N(\mathbf{0}) = \mathbf{0}$, so that (2.50) admits the null solution. Let us assume that

i. the domain of L is dense in H and that L is compact with compact resolvent, i.e. $\rho(L) := \{\lambda \in \mathbb{C} \ : \ \exists (L - \lambda T)^{-1}\}$;

ii. the bilinear form associated with L is defined and bounded on a space H' which is compactly embedded in H.

In order to study the linear stability of the basic state, let us consider the linear version of (2.50):

$$\begin{cases} \mathbf{u}_t + L\mathbf{u} = \mathbf{0}, \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(t), \end{cases} \tag{2.51}$$

Under the hypothesis *i.* the following classical result holds.

Theorem 2.4.1. The spectrum of the operator L consists entirely of an at most denumerable number of eigenvalues $\{\sigma_n\}_{n \in \mathbb{N}}$ with finite multiplicity (both algebraic and geometric). Moreover, these eigenvalues can cluster only at infinity.

A proof of the previous theorem can be found in [73]. Since the operator L is autonomous we can consider solutions of the following type:

$$\mathbf{u}(\mathbf{x}, t) = \widehat{\mathbf{u}}(\mathbf{x}) \exp(-\sigma t). \tag{2.52}$$

As a result, the behaviour in time of the perturbation $\mathbf{u}(\mathbf{x}, t)$ is governed by the sign of the real part of σ. In particular, if $\mathrm{Re}(\sigma) > 0$ for all σ, then the zero solution is *linearly stable*. Let us note that the eigenvalues satisfying

$$L\varphi = \sigma\varphi, \tag{2.53}$$

are not necessarily real as long as the operator L is in general non-symmetric. Nevertheless, the eigenvalues can be ordered in the following way

$$\mathsf{Re}(\sigma_1) \leq \mathsf{Re}(\sigma_2) \leq \ldots \leq \mathsf{Re}(\sigma_n) \leq \ldots. \tag{2.54}$$

As a consequence, since for the linear stability we need all the eigenvalues to be with positive linear part, the linear stability analysis reduces to study the sign of the first eigenvalues. Indeed the zero solution of (2.50) is *linear stable* if

$$\mathrm{Re}(\sigma_1) > 0. \tag{2.55}$$

In the subsequent investigations σ_1 will depend on the dimensionless Rayleigh number Ra. Our aim is to determine the least value Ra_c for which $\mathrm{Ra}(\sigma_1) = 0$, namely for which instability sets in. One also expect that $\mathrm{Re}(\sigma_1) > 0$ if $\mathsf{Ra} < \mathsf{Ra}_c$ and $\mathrm{Re}(\sigma_1) < 0$ if $\mathsf{Ra} > \mathsf{Ra}_c$.

We need now to describe the connection between linear stability results and nonlinear stability ones. We will see that a strong connection is obtained not only when the linear operator is symmetric, but also when it is not. More specifically, we will see that the when a condition called *principle of exchange of stabilities* holds in a well defined sense, then the link between linear and nonlinear stability is provided. We say that the *principle of exchange of stabilities* (PES) holds if

$$\mathrm{Im}(\sigma_1) \neq 0 \implies \mathrm{Re}(\sigma_1) < 0. \tag{2.56}$$

Moreover, we say that PES holds in the *weak form* if

$$\mathrm{Re}(\sigma_1) = 0 \implies \mathrm{Im}(\sigma_1) = 0, \tag{2.57}$$

while it holds in *strong form* if all the eigenvalues of the operator L are real. If the operator L is symmetric, or symmetrizable, then PES holds in strong form. We say that, instability sets in a *secondary state motion* when

$\sigma_1 = 0$ at the criticality. On the other hand, if σ_1 is a pure imaginary number at the criticality, instability sets in as an *oscillatory motion*. In this case, PES doesn't holds and overstability is possible. Physically, the previous mathematical conditions can be interpreted as follows, following Landau [77] . Let a perturbation cause a fluid element to move upwards. This element will be surrounded by cooler fluid, and its temperature is reduced by conduction, even if it remains above that of the environment. The buoyancy force on it is therefore upwards and it continues to move in the same direction, more slowly or more quickly according to the relation between the temperature gradient and the coefficients of heat dissipation. In either case, there is no "restoring force" and therefore no oscillation.

2.4.2 *Nonlinear theory*

The aim of this section is to describe the connection between the linear stability results and the nonlinear ones. First of all, let us recall that the null solution of (2.50) is *nonlinearly stable* if

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \text{ such that } \|\mathbf{u}_0\| < \delta \implies \|\mathbf{u}(t)\| < \varepsilon, \tag{2.58}$$

and there exists $\gamma \in (0, +\infty]$ such that

$$\|\mathbf{u}_0\| < \gamma \implies \lim_{t \to +\infty} \|\mathbf{u}(t)\| = 0. \tag{2.59}$$

If $\gamma = +\infty$, then the null solution of (2.50) is said to be *unconditionally stable*, while if $\gamma < +\infty$ the solution is *conditionally stable*.

The operator L is generally non-symmetric. However, it can be always decomposed in the following way:

$$L = L_1 + L_2, \tag{2.60}$$

where L_1 and L_2 are the symmetric and skew-symmetric part of L, respectively. These two operators are such that $\mathrm{dom}(L_2) \supset \mathrm{dom}(L_1) = \mathrm{dom}(L)$. It turns out that L_1 satisfies the hypothesis of Theorem 2.4.1. Moreover, because of the symmetry, the eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}}$ associated with L_1 are all real and may be ordered as follows:

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots. \tag{2.61}$$

Let $L_1[\varphi, \varphi]$, with $\varphi \in H'$, be the bilinear form associated with the operator L_1, i.e.

$$\langle L_1 \varphi, \varphi \rangle = L_1[\varphi, \varphi], \quad \forall \varphi \in \mathrm{dom}(L_1). \tag{2.62}$$

It turns out that if φ is an eigenfunction related to the eigenvalue λ_1, then

$$\lambda_1 = \min_{\varphi \in H'} \frac{L_1[\varphi, \varphi]}{\|\varphi\|^2}. \tag{2.63}$$

As a consequence, the following result holds.

Theorem 2.4.2. If $\lambda_1 > 0$, then the zero solution of (2.50) is unconditionally nonlinearly stable.

A proof of the previous result can be found in [125]. From the above argument, it follows that while the linear stability reduces to studying the sign of eigenvalues related to the linear operator L, the nonlinear stability involves the eigenvalues of the symmetric part L_1, only. Indeed, we can say that *if the skew-symmetric part L_2 of L is zero, i.e. $L = L_1$, then the linear stability implies the nonlinear one and vice-versa.*

Part II

RECENT RESULTS ON PENETRATIVE CONVECTION IN POROUS MEDIA

3

DENSITY INVERSION PHENOMENON IN POROUS MEDIA

In the present chapter, the onset of penetrative convection in a porous medium is investigated. The results presented in the following are based on the paper [2], written in collaboration with Prof. F. Capone. The subsequent investigations extend the results obtained in a paper by Straughan *et al.* [53] where a conditional nonlinear stability analysis of the conduction solution for penetrative convection problem in porous medium (taking into account Darcy's law with inertia) was performed, and the patterned ground genesis was explained as a type of stability phenomenon. Indeed, the goal of this Chapter is to develop a nonlinear energy stability analysis which will yield *unconditional* stability results. The chapter is organized as follows. In Section 3.1 the mathematical model and the associated nondimentional perturbation equations are introduced and the conduction solution, whose linear instability and nonlinear stability we are focused on, is determined. In Section 3.2 the principle of exchange of stabilities (PES) is proved and the critical Rayleigh number, Ra_L, for the onset of steady convection is found. In Section 3.3, global nonlinear stability analysis, via weighted energy method, is performed and a global stability threshold $\mathsf{Ra}_W (< \mathsf{Ra}_L)$ is determined. In order to minimise the gap between the (linear) instability threshold and the nonlinear one, a suitable energy functional is introduced and a local nonlinear stability result is found. We developed a code on Matlab software in order to solve the generalized eigenvalue problems arising from the linear and nonlinear analysis. The Matlab code is based on the Chebyshev-τ spectral method coupled to the QZ algorithm. In Section 3.4 numerical simulations are performed in order to analyse the behaviour of the linear instability and the nonlinear stability thresholds with respect to the upper boundary plane temperature. The chapter ends with a concluding Section, 3.5, that recaps all the results.

3.1 FORMULATION OF THE PROBLEM AND THERMODYNAMIC CONSISTENCY

Let us consider a cartesian frame of reference $Oxyz$ with fundamental unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ (the latter pointing vertically upwards) and let L be a layer of an isotropic porous medium (horizontally unbounded and of vertical thickness $d > 0$), saturated by homogeneous incompressible fluid heated from above and confined between two impermeable heat conductive planes kept at constant and uniform temperature T_L and T_U.

As remarked in the introductory chapter, even though perfect incompressible media do not exist in nature, they can be considered as the limit case of compressible one. Indeed, an incompressible fluid can be defined as a medium whose constitutive equations do not depend on pressure, see [87]. However, Müller proved that this definition of incompressibility is compatible with the entropy principle only if the density is a constant function. This is actually in disagreement with empirical observations, according to which fluids expand when heated, and the theoretical assumptions such as the widely employed Boussinesq approximation. To fix this contradiction, Guoin *et al.*, in [56], introduced the definition of *quasi-thermal-incompressible fluid*: a medium for which the only equation independent of the pressure among all the constitutive equations is the density. Employing this definition the authors proved that a quasi thermal incompressible fluid tends to be (perfectly) incompressible when

$$p \ll \frac{c_p}{|V'|} = \frac{\rho^2 c_p}{|\rho'|}, \tag{3.1}$$

p, ρ and c_p being pressure, density and specific heat capacity at constant pressure. Therefore the incompressible model is compatible with the entropy principle only if (3.1) holds and the contradiction highlighted by Müller is fixed. Employing the Boussinesq approximation the density is assumed constant in all the terms of the governing equations except in the body force therm due to gravity. Since we are investigating the penetrative convection, in the body force term we assume the density having the quadratic dependence (1.63). It follows that the incompressibility model is consistent when

$$p \ll p_{cr} \quad \text{where} \quad p_{cr} = \frac{c_p \rho_0}{2\alpha|T - T_0|}. \tag{3.2}$$

In penetrative convection problem temperature values are considered in the parabolic region of the density function, hence $|T - T_0|$ is at most of order 10^1. In order to evaluate the order of magnitude of the critical pressure p_{cr} given in (3.2), at temperature $T_0 = 4°C$ we get $\rho_0 \simeq 10^3\, kg/m^3$ and $c_p \simeq 4.8 \times 10^3\, J/(kg\, K)$, see [80]. Therefore we deduce that the critical pressure assumes, at least, the value $p_{cr} \simeq 10^{10} Pa \simeq 10^5\, atm$. Notice that for temperature values that are far from the parabolic region, a linear dependence of the density function is assumed and the value of the critical pressure can be found in [56].

We assume that generalized Darcy's law, in the Boussinesq approximation, governs the fluid motion in the porous layer in order to include the inertia term in the momentum equation, namely, [90]

$$\rho_0 c_a \frac{\partial \mathbf{v}}{\partial t} + \frac{\mu}{k}\mathbf{v} = -\nabla p - g\rho \mathbf{k}, \tag{3.3}$$

where $\mathbf{v}$, p and ρ are the seepage velocity, pressure and density respectively, $\mathbf{k} = (0,0,1)$, g is the modulus of gravitational acceleration, μ is the dynamic viscosity of the fluid, k is the permeability of the porous medium and c_a is the acceleration coefficient. Being interested in penetrative convection, we assume the density ρ be given by (1.63). Therefore, equation (3.3) together with the incompressibility condition and the energy balance equation yields the following system of governing equations:

$$\begin{cases} \rho_0 c_a \dfrac{\partial \mathbf{v}}{\partial t} + \dfrac{\mu}{k}\mathbf{v} = -\nabla p - g\rho_0[1 - \alpha(T - T_0)^2]\mathbf{k}, \\ \nabla \cdot \mathbf{v} = 0, \\ \dfrac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \kappa \Delta T. \end{cases} \tag{3.4}$$

To system (3.4) the following boundary conditions are appended:

$$T(x,y,0,t) = T_L = 0°C, \quad T(x,y,d,t) = T_U \geq 4°C, \tag{3.5}$$

and:

$$\mathbf{v} \cdot \mathbf{n} = 0 \ \text{ on } \ z = 0, d, \tag{3.6}$$

where $\mathbf{n}$ is the unit outward normal to the impermeable horizontal planes bounding the layer.

The problem (3.4)-(3.6) admits the steady state (conduction solution):

$$m_b = (\mathbf{v}_b, p_b, T_b)\,, \tag{3.7}$$

with

$$
\begin{aligned}
&\mathbf{v}_b = (0,0,0),\\
&p_b = c - g\rho_0\left(1-\alpha T_0^2\right)z - g\rho_0\alpha T_0\frac{T_U}{d}z^2 + g\rho_0\alpha\frac{T_U^2}{3d^2}z^3,\\
&T_b = \frac{T_U}{d}z.
\end{aligned} \tag{3.8}
$$

In order to study the stability of m_b, let us introduce the following perturbation fields:

$$
\mathbf{v} = \mathbf{v}_b + \mathbf{u}, \quad p = p_b + \pi, \quad T = T_b + \theta, \tag{3.9}
$$

with $\mathbf{u} = \left(u, v, w\right)$ and the following non-dimensional parameters:

$$
\begin{gathered}
\mathbf{x} = d\mathbf{x}^*, \quad t = \tau t^*, \quad \mathbf{u} = U\mathbf{u}^*,\\
\pi = P\pi^*, \quad \theta = T\theta^*\\
\tau = \frac{d^2}{\kappa}, \qquad U = \frac{\kappa}{d}, \quad P = \frac{\mu\kappa}{k}, \quad T = \sqrt{\frac{\mu\kappa}{2g\rho_0\alpha k d}}
\end{gathered} \tag{3.10}
$$

Then the resulting non-dimensional perturbation equations, omitting all the asterisks, are

$$
\begin{cases}
\dfrac{1}{\mathsf{Va}}\dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u} = -\nabla\pi - \mathsf{Ra}(\zeta - z)\theta\mathbf{k} + \dfrac{\theta^2}{2}\mathbf{k},\\
\nabla\cdot\mathbf{u} = 0,\\
\dfrac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta + \mathsf{Ra}w = \Delta\theta,
\end{cases} \tag{3.11}
$$

where Ra and Va are, respectively, the thermal Rayleigh-Darcy number and the Vadasz number, given by

$$
\mathsf{Ra} = \sqrt{\frac{2g\rho_0\alpha k d T_U^2}{\mu\kappa}}, \quad \mathsf{Va} = \frac{d^2\mu}{\rho_0 c_a \kappa k}, \tag{3.12}
$$

and $\zeta = \frac{4}{T_U}$. To system (3.11), we append the following initial conditions:

$$
\begin{gathered}
\mathbf{u}(\mathbf{x}, t_0) = \mathbf{u}_0(\mathbf{x}), \quad \pi(\mathbf{x}, t_0) = \pi_0(\mathbf{x}),\\
\theta(\mathbf{x}, t_0) = \theta_0(\mathbf{x}),
\end{gathered} \tag{3.13}
$$

with $\nabla\cdot\mathbf{u}_0 = 0$, and the following boundary conditions:

$$
w = \theta = 0 \qquad z = 0, 1. \tag{3.14}
$$

In the subsequent analysis we will suppose that the perturbation fields are periodic in the x and y direction (of periods $2\pi/a_x$ and $2\pi/a_y$ respectively), and belong to $W^{1,2}(V)$, for all $t > t_0$, and denote by

$$V = \left[0, \frac{2\pi}{a_x}\right] \times \left[0, \frac{2\pi}{a_y}\right] \times [0,1]\,, \tag{3.15}$$

the periodicity cell, where a_x and a_y are the wave numbers in the x and y direction, respectively. Moreover, we will denote by $\langle \, , \, \rangle$ and $\| \; \|$, the usual scalar product and the related norm, respectively, in the functional space $L^2(V)$.

3.2 INSTABILITY ANALYSIS OF THE BASIC MOTION

In order to study the linear instability of m_b, let us consider the linear version of (3.11), i.e.

$$\begin{cases} \dfrac{1}{\mathsf{Va}}\dfrac{\partial \mathbf{u}}{\partial t} = -\mathbf{u} - \nabla \pi - \mathsf{Ra} M(z)\theta \mathbf{k}, \\ \nabla \cdot \mathbf{u} = 0, \\ \dfrac{\partial \theta}{\partial t} = -\mathsf{Ra} w + \Delta \theta, \end{cases} \tag{3.16}$$

with $M(z) = \zeta - z$. Since system (3.16) is linear and autonomous we are allowed to seek for solutions of the form

$$\begin{aligned} \mathbf{u}(\mathbf{x},t) &= \widehat{\mathbf{u}}(\mathbf{x})e^{\sigma t}, \\ \pi(\mathbf{x},t) &= \widehat{\pi}(\mathbf{x})e^{\sigma t}, \\ \theta(\mathbf{x},t) &= \widehat{\theta}(\mathbf{x})e^{\sigma t}, \end{aligned} \tag{3.17}$$

with $\sigma \in \mathbb{C}$. By virtue of (3.17), system (3.16) becomes

$$\begin{cases} \sigma \dfrac{1}{\mathsf{Va}}\widehat{\mathbf{u}} = -\widehat{\mathbf{u}} - \nabla \widehat{\pi} - \mathsf{Ra} M(z)\widehat{\theta} \mathbf{k}, \\ \nabla \cdot \widehat{\mathbf{u}} = 0, \\ \sigma \widehat{\theta} = -\mathsf{Ra} \widehat{w} + \Delta \widehat{\theta}. \end{cases} \tag{3.18}$$

3.2.1 *Principle of exchange of stabilities*

Let us consider the third component of the double curl of equation $(3.18)_1$, i.e.

$$\left(1 + \frac{\sigma}{\mathsf{Va}}\right) \Delta \widehat{w} = \mathsf{Ra} M(z) a^2 \widehat{\theta}, \tag{3.19}$$

where we are considering periodic perturbations in the x and y directions such that

$$\Delta_1\widehat{\theta} = -a^2\widehat{\theta}, \tag{3.20}$$

with $\Delta_1 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. From $(3.18)_3$ we have

$$\widehat{w} = -\frac{\sigma}{\mathsf{Ra}}\widehat{\theta} + \frac{1}{\mathsf{Ra}}\Delta\widehat{\theta},$$

and hence

$$\Delta\widehat{w} = -\frac{\sigma}{\mathsf{Ra}}\Delta\widehat{\theta} + \frac{1}{\mathsf{Ra}}\Delta\Delta\widehat{\theta}. \tag{3.21}$$

Substituting (3.21) in (3.19) one obtains

$$-\frac{\sigma}{\mathsf{Ra}}\left(1 + \frac{\sigma}{\mathsf{Va}}\right)\Delta\widehat{\theta} + \left(1 + \frac{\sigma}{\mathsf{Va}}\right)\frac{1}{\mathsf{Ra}}\Delta\Delta\widehat{\theta} = \mathsf{Ra}M(z)a^2\widehat{\theta}. \tag{3.22}$$

On multiplying (3.22) by $\widehat{\theta}^*$ (where the asterisks denote the complex conjugate) and integrating on the periodicity cell V, one obtains

$$\begin{aligned}\frac{\sigma}{\mathsf{Ra}}\left(1 + \frac{\sigma}{\mathsf{Va}}\right)\int_V |\nabla\widehat{\theta}|^2 dV + \left(1 + \frac{\sigma}{\mathsf{Va}}\right)\frac{1}{\mathsf{Ra}}\int_V (\Delta\widehat{\theta})^2 dV \\ = \mathsf{Ra}a^2\int_V M(z)\widehat{\theta}^2 dV.\end{aligned} \tag{3.23}$$

Since $\sigma = \sigma_R + i\sigma_I$, let us consider the imaginary part of (3.23)

$$\left(2\frac{\sigma_R\sigma_I}{\mathsf{RaVa}} + \frac{\sigma_I}{\mathsf{Ra}}\right)\int_V |\nabla\widehat{\theta}|^2 dV + \frac{\sigma_I}{\mathsf{RaVa}}\int_V (\Delta\widehat{\theta})^2 dV = 0, \tag{3.24}$$

i.e.

$$\sigma_I\left[\left(2\frac{\sigma_R}{\mathsf{Va}} + \frac{1}{\mathsf{Ra}}\right)\int_V |\nabla\widehat{\theta}|^2 dV + \frac{1}{\mathsf{RaVa}}\int_V (\Delta\widehat{\theta})^2 dV\right] = 0. \tag{3.25}$$

Hence, from (3.25) it follows that if $\sigma_I \neq 0$, then $\sigma_R < 0$ and this proves that it is sufficient to consider the stationary convection boundary $\sigma \equiv 0$, see [120].

3.2.2 *Normal mode analysis*

To find the critical Rayleigh number of linear theory, we concentrate on finding the lowest eigenvalue of the system

$$\begin{cases}\mathbf{0} = \widehat{\mathbf{u}} + \nabla\widehat{\pi} + \mathsf{Ra}M(z)\widehat{\theta}\mathbf{k}, \\ 0 = -\mathsf{Ra}\widehat{w} + \Delta\widehat{\theta}.\end{cases} \tag{3.26}$$

The double cur of $(3.26)_1$ is taken to remove the pressure term, where the third component is chosen. The resulting linearized governing set of equations are

$$\begin{cases} 0 = -\Delta\widehat{w} - \mathsf{Ra}M(z)\Delta_1\widehat{\theta}, \\ 0 = -\mathsf{Ra}\widehat{w} + \Delta\widehat{\theta}. \end{cases} \tag{3.27}$$

By virtue of periodicity of the perturbation fields in the horizontal directions x and y, taking into account the boundary conditions (3.14), we can separate the spatial variables, employing normal mode solutions

$$\begin{aligned} \widehat{w}(x,y,z) &= \widetilde{W}(z)e^{i\left(a_x x + a_y y\right)}, \\ \widehat{\theta}(x,y,z) &= \widetilde{\Theta}(z)e^{i\left(a_x x + a_y y\right)} \end{aligned} \tag{3.28}$$

for details see Appendix 3.6.1. Letting $D^k = d^k/dz^k$ $(k = 1,2)$ and $a^2 = a_x^2 + a_y^2$, from (3.27), in view of (3.28), one obtains

$$\begin{cases} (D^2 - a^2)\widetilde{W}(z) - \mathsf{Ra}M(z)a^2\widetilde{\Theta}(z) = 0, \\ \mathsf{Ra}\widetilde{W}(z) - (D^2 - a^2)\widetilde{\Theta}(z) = 0, \end{cases} \tag{3.29}$$

with the boundary conditions $\widetilde{W} = \widetilde{\Theta} = 0$, on $z = 0,1$. System (3.29) constitutes an eigenvalue system of ordinary differential equations, where the critical Rayleigh number for the onset of convection, is given by

$$\mathsf{Ra}_L = \min_{a^2 \in \mathbb{R}^+} \mathsf{Ra}^2(a^2). \tag{3.30}$$

The numerical results for the linear theory are presented in Section 3.5 to facilitate a direct comparison with those of nonlinear analysis we will develop in the following section.

3.3 NONLINEAR STABILITY

3.3.1 *Global nonlinear stability of the basic motion*

In order to study the global nonlinear stability of the conduction solution (3.7) let us introduce the following weighted energy function:

$$E(t) = \frac{1}{2\mathsf{Va}}\|\mathbf{u}\|^2 + \frac{1}{2}\left\langle g(z), \theta^2 \right\rangle, \tag{3.31}$$

where $g = g(z)$ is a regular real value positive function to be chosen suitably later. Along the solutions of (3.11), it tours out:

$$\begin{aligned}\frac{dE}{dt} = &- \|\mathbf{u}\|^2 - \mathsf{Ra}\,\langle M(z)\theta, w\rangle + \frac{1}{2}\left\langle w, \theta^2\right\rangle + \frac{1}{2}\left\langle g'(z)w, \theta^2\right\rangle \\ &- \mathsf{Ra}\,\langle g(z)\theta, w\rangle - \langle \theta_z, \theta g'(z)\rangle - \left\langle g(z), |\nabla\theta|^2\right\rangle,\end{aligned} \tag{3.32}$$

In order to control the cubic nonlinear term $\langle (1 + g'(z))w, \theta^2\rangle$ we choose

$$g(z) = \mu - z, \tag{3.33}$$

being μ a parameter to be optimally selected at our disposal. Hereafter, we will assume $\mu > 1$ in such a way $g(z) > 0$ for all $z \in (0, 1)$. Setting

$$\begin{aligned} I &= -\left\langle \Big(\mu + \zeta - 2z\Big)\theta, w\right\rangle, \\ D &= \|\mathbf{u}\|^2 + \left\langle \mu - z, |\nabla\theta|^2\right\rangle, \end{aligned} \tag{3.34}$$

we find out

$$\frac{dE}{dt} = -D\left(1 - \mathsf{Ra}\frac{I}{D}\right) \leq -D\left(1 - \frac{\mathsf{Ra}}{\mathsf{Ra}_W}\right), \tag{3.35}$$

where $1/\mathsf{Ra}_W = \max_{\mathcal{H}}(I/D)$, see [108] and for a discussion concerning the existence of maxima see Appendix 3.6.2, and $\mathcal{H}$ is the space of kinematically admissible perturbations, namely

$$\begin{aligned}\mathcal{H} = &\Big\{(\mathbf{u}, \theta) \in W^{1,2}(V) \mid \nabla\cdot\mathbf{u} = 0, \text{ periodic in } x \text{ and } y \\ &\qquad \text{directions, with period } \tfrac{2\pi}{a_x}, \tfrac{2\pi}{a_y}, \text{ satisfying (3.14)}\Big\}.\end{aligned} \tag{3.36}$$

By virtue of Poincaré and weighted Poincaré inequalities [47], since $z \in (0, 1)$, one can show that

$$\frac{1}{2}\left\langle (\mu - z), \theta^2\right\rangle \leq b\left\langle (\mu - z), |\nabla\theta|^2\right\rangle, \tag{3.37}$$

where $b = \max\left\{\frac{\gamma}{2}, 2\right\}$, being $\gamma = \gamma(V)$ the Poincaré constant. Therefore, from (3.31) and (3.37), one obtains $E \leq \beta D$, where $\beta = \max\left\{\frac{1}{2\mathsf{Va}}, b\right\}$. Thus, letting $a = \beta^{-1}\left(\frac{\mathsf{Ra}_W - \mathsf{Ra}}{\mathsf{Ra}_W}\right) > 0$, we use (3.35) to show that

$$E(t) \leq E(0)e^{-at}. \tag{3.38}$$

We can conclude that, provided $\mathsf{Ra} < \mathsf{Ra}_W$, $E(t)$ decreases at least exponentially to zero as t goes to infinity, thus Ra_W represents a threshold for global (that is for all initial data) nonlinear stability.

As concerns the variational problem

$$1/\mathsf{Ra}_W = \max_{\mathcal{H}}(I/D), \tag{3.39}$$

the Euler-Lagrange equations are:

$$\begin{cases} \mathbf{u} + \mathsf{Ra}_W F(z)\theta\mathbf{k} = \nabla\varpi, \\ -\mathsf{Ra}_W F(z)w - \theta_z + g(z)\Delta\theta = 0, \end{cases} \tag{3.40}$$

where $F(z) = \frac{\mu}{2} + \frac{\zeta}{2} - z$ and ϖ is a Lagrange multiplier. Taking the double curl of $(3.40)_1$ where the third component is chosen, one obtains:

$$\begin{cases} \Delta w + \mathsf{Ra}_W F(z)\Delta_1\theta = 0, \\ \mathsf{Ra}_W F(z)w + \theta_z - g(z)\Delta\theta = 0. \end{cases} \tag{3.41}$$

By employing normal modes, system (3.41) becomes

$$\begin{cases} (D^2 - a^2)\widetilde{W}(z) - \mathsf{Ra}_W F(z)a^2\widetilde{\Theta}(z) = 0, \\ \mathsf{Ra}_W F(z)\widetilde{W}(z) + D\widetilde{\Theta}(z) - g(z)(D^2 - a^2)\widetilde{\Theta}(z) = 0, \end{cases} \tag{3.42}$$

with the boundary conditions $\widetilde{W} = \widetilde{\Theta} = 0$, on $z = 0, 1$. System (3.42) is a fourth-order generalized eigenvalue problem for the critical Rayleigh number Ra_W, which is given by:

$$\mathsf{Ra}_W = \max_{\mu>1} \min_{a^2\in\mathbb{R}^+} \mathsf{Ra}^2(a^2, \mu). \tag{3.43}$$

In Figure 3.1 the linear instability thresholds and the global nonlinear stability thresholds are represented, with particular regard to their behaviour with respect to the upper boundary plane temperature T_U.

3.3.2 *Local nonlinear stability of m_b*

In order to reduce the gap existing between the linear instability threshold and the global nonlinear one, let us consider the following energy function

$$E^*(t) = \frac{1}{2\mathsf{Va}}\|\mathbf{u}\|^2 + \frac{\eta}{2}\|\theta\|^2 + \frac{1}{4}\|\theta^2\|^2, \tag{3.44}$$

where $\eta > 0$ a parameter to be optimally selected in order to maximise the stability threshold. Along the solutions of (3.11), it turns out

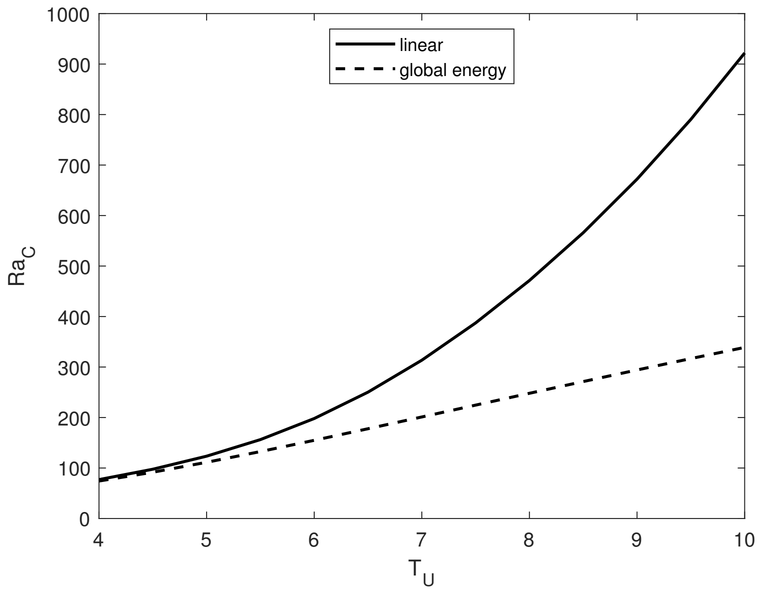


Figure 3.1: Linear instability (solid line) and global nonlinear stability (dashed line) thresholds comparison against T_U.

$$\begin{aligned}\frac{dE^*}{dt} =& - \|\mathbf{u}\|^2 - \mathsf{Ra}\left\langle M(z)\theta, w\right\rangle + \frac{1}{2}\left\langle w, \theta^2\right\rangle - \mathsf{Ra}\eta\left\langle w, \theta\right\rangle \\ & - \mathsf{Ra}\left\langle w, \theta^3\right\rangle - \eta\|\nabla\theta\|^2 - \frac{3}{4}\|\nabla\theta^2\|^2.\end{aligned} \tag{3.45}$$

Setting

$$\begin{aligned} I^* = - \left\langle \left(\eta + \zeta - z\right)\theta, w\right\rangle, \quad D^* = \|\mathbf{u}\|^2 + \eta\|\nabla\theta\|^2, \\ N^* = \frac{1}{2}\left\langle w, \theta^2\right\rangle - \mathsf{Ra}\left\langle w, \theta^3\right\rangle, \end{aligned} \tag{3.46}$$

we find out

$$\begin{aligned}\frac{dE^*}{dt} &= -D^*\left(1 - \mathsf{Ra}\frac{I^*}{D^*}\right) - \frac{3}{4}\|\nabla\theta^2\|^2 + N^* \\ &\le -D_1^* + N^*,\end{aligned} \tag{3.47}$$

where

$$D_1^* = \delta D^* + \frac{3}{4}\|\nabla\theta^2\|^2, \tag{3.48}$$

with $1/\mathrm{Ra}_W = \max_{\mathcal{H}}(I^*/D^*)$ and $\delta = (\mathrm{Ra}_E - \mathrm{Ra})/\mathrm{Ra}_E$. Now, assuming that $\mathrm{Ra} < \mathrm{Ra}_E$, in order to prove the exponential decay of E^*, we must suitably control the nonlinear terms. Setting $\widehat{\delta} = \delta^{-\frac{1}{2}}$ it follows from Cauchy-Schwartz inequality that

$$N^* \leq \widehat{\delta}\, D_1^{*\frac{1}{2}} \left(\frac{1}{2} \|\theta^2\|^{\frac{1}{2}} \|\theta^2\|^{\frac{1}{2}} + \mathrm{Ra} \|\theta^4\|^{\frac{1}{2}} \|\theta^2\|^{\frac{1}{2}} \right). \tag{3.49}$$

We need the following Sobolev embedding inequality

$$\int_V \psi^4 \, dV \leq k \left(\int_V |\nabla \psi|^2 \, dV \right)^2, \tag{3.50}$$

with $k = k(V)$ a positive constants depending only on the geometry of the domain V. By using (3.50) it follows from (3.49) that

$$\begin{aligned} N^* &\leq \widehat{\delta} D_1^{*\frac{1}{2}} \left(\frac{1}{2} \sqrt[4]{k} \frac{1}{\sqrt{\eta}} \widehat{\delta} D_1^{*\frac{1}{2}} \sqrt{2} E^{*\frac{1}{4}} + \mathrm{Ra} \sqrt[4]{k} \frac{2}{\sqrt{3}} D_1^{\frac{1}{2}} \sqrt{2} E^{*\frac{1}{4}} \right) \\ &= \varepsilon D_1^* E^{*\frac{1}{4}}, \end{aligned} \tag{3.51}$$

where

$$\varepsilon = \widehat{\delta} \sqrt[4]{k} \left(\frac{1}{\sqrt{2\eta}} \sqrt{\frac{\mathrm{Ra}_E}{\mathrm{Ra}_E - \mathrm{Ra}}} + 2\mathrm{Ra} \sqrt{\frac{2}{3}} \right) > 0. \tag{3.52}$$

Therefore, from (3.47) and (3.51) we get

$$\frac{dE^*}{dt} \leq -D_1^* \left[1 - \varepsilon E^{*\frac{1}{4}} \right]. \tag{3.53}$$

From (3.53) we may now find sufficient conditions to ensure that E^* decays monotonically to zero, [49]. To this end, we suppose that

$$E^*(0) < \varepsilon^{-4}, \tag{3.54}$$

therefore from (3.53) it follows $\frac{dE^*}{dt} \leq 0$ in some neighbourhood of 0. This implicates that E^* cannot exceed its initial value for all $t > 0$ and so (3.53) yields

$$\frac{dE^*}{dt} \leq -A D_1^*, \tag{3.55}$$

with $A = 1 - \varepsilon E^{*\frac{1}{4}}(0)$. By virtue of Poincaré inequality, one obtains $E^* \leq c D_1^*$, where $c = \max\left\{ \frac{1}{2\mathrm{Va}\delta}, \frac{\gamma}{2\delta}, \frac{\gamma}{3} \right\}$. Thus, letting $\ell = c^{-1} A > 0$, we find out

$$E^*(t) \leq E^*(0) e^{-\ell t}, \tag{3.56}$$

from which it follows that, provided $\text{Ra} < \text{Ra}_E$, E^* decreases at least exponentially to zero as t goes to infinity, thus Ra_E represent a threshold for local (that is for initial data such that $E^*(0) < \varepsilon^{-4}$) nonlinear stability. Of course, the rate of decay is dependent on how close $E^{*\frac{1}{4}}$ is to ε and also on how close is Ra to the energy limit Ra_E.

As concerns the variational problem

$$1/\text{Ra}_E = \max_{\mathcal{H}}(I^*/D^*), \tag{3.57}$$

the Euler-Lagrange equations are

$$\begin{cases} (D^2 - a^2)\widetilde{W}(z) - \text{Ra}_E G(z) a^2 \widetilde{\Theta}(z) = 0, \\ \text{Ra}_E G(z)\widetilde{W}(z) - \eta(D^2 - a^2)\widetilde{\Theta}(z) = 0, \end{cases} \tag{3.58}$$

where $G(z) = \frac{\eta}{2} + \frac{\zeta}{2} - \frac{z}{2}$, together with boundary conditions $\widetilde{W} = \widetilde{\Theta} = 0$, on $z = 0, 1$. System (3.58) is a fourth-order generalized eigenvalue problem for the critical Rayleigh number Ra_E, which is given by

$$\text{Ra}_E = \max_{\eta > 0} \min_{a^2 \in \mathbb{R}^+} \text{Ra}^2(a^2, \eta). \tag{3.59}$$

3.4 NUMERICAL RESULTS

Systems (3.29), (3.42) and (3.58) were solved using the spectral Chebyshev-τ method, [40, 126]. The method essentially consists in transforming the spatial domain $(0, 1)$ onto the Chebyshev domain $(-1, 1)$ and expand the solutions $\widetilde{W}$ and $\widetilde{\Theta}$ in truncated Chebyshev series

$$\widetilde{W}(z) = \sum_{k=0}^{N+2} w_k T_k(z), \quad \widetilde{\Theta}(z) = \sum_{k=0}^{N+2} \theta_k T_k(z), \tag{3.60}$$

where T_k is the k-th Chebyshev polynomial of the first kind, and $\{T_k\}_{k \geq 0}$ forms an orthogonal system of $L^2(-1, 1)$ with respect to the weight function $\mathrm{w}(z) = 1/(\sqrt{1 - z^2})$, [82]. The inner product with T_k, $k = 0, \ldots, N+2$, is taken to form a generalized eigenvalue problem for the Rayleigh number and the QZ algorithm is then used to solve it. Practically this procedure consists in having to solve the following matrices systems, respectively related to the boundary value problem (3.29), (3.42) and (3.58)

$$\begin{pmatrix} \mathtt{D}^2 - a^2\mathtt{I} & 0 \\ 0 & \mathtt{D}^2 - a^2\mathtt{I} \end{pmatrix} \begin{pmatrix} \widetilde{W} \\ \widetilde{\Theta} \end{pmatrix} = \mathsf{Ra} \begin{pmatrix} 0 & M(\mathtt{Z})a^2\mathtt{I} \\ \mathtt{I} & 0 \end{pmatrix} \begin{pmatrix} \widetilde{W} \\ \widetilde{\Theta} \end{pmatrix} \tag{3.61a}$$

$$\begin{pmatrix} \mathtt{D}^2 - a^2\mathtt{I} & 0 \\ 0 & \mathtt{D} - g(\mathtt{Z})(\mathtt{D}^2 - a^2\mathtt{I}) \end{pmatrix} \begin{pmatrix} \widetilde{W} \\ \widetilde{\Theta} \end{pmatrix} = \mathsf{Ra}_W \begin{pmatrix} 0 & F(\mathtt{Z})a^2 \\ F(\mathtt{Z}) & 0 \end{pmatrix} \begin{pmatrix} \widetilde{W} \\ \widetilde{\Theta} \end{pmatrix} \tag{3.61b}$$

$$\begin{pmatrix} \mathtt{D}^2 - a^2\mathtt{I} & 0 \\ 0 & \eta(\mathtt{D}^2 - a^2\mathtt{I}) \end{pmatrix} \begin{pmatrix} \widetilde{W} \\ \widetilde{\Theta} \end{pmatrix} = \mathsf{Ra}_E \begin{pmatrix} 0 & G(\mathtt{Z})a^2 \\ G(\mathtt{Z}) & 0 \end{pmatrix} \begin{pmatrix} \widetilde{W} \\ \widetilde{\Theta} \end{pmatrix} \tag{3.61c}$$

where $(\widetilde{W}, \widetilde{\Theta})^\mathsf{T}$ denotes the vector $(w_0, \dots, w_{N+2}, \theta_0, \dots, \theta_{N+2})$ and $\mathtt{Z}$, $\mathtt{D}$ and $\mathtt{D}^2$ are the Chebyshev matrices arising from the representation of the function z, the differential operator D and D^2, respectively. The systems are solved for the eigenvalues $(\mathsf{Ra}_L)_n$, $(\mathsf{Ra}_W)_n$ and $(\mathsf{Ra}_E)_n$ and the corresponding eigenvectors. Therefore, once the smallest positive eigenvalues are selected (as well as the corresponding eigenvectors) the perturbations (3.60) are completely determined.

In Figure 3.2 the linear, local nonlinear and global nonlinear critical thermal Rayleigh numbers are represented as functions of the upper boundary plane temperature T_U, with the aim to graphically analyse the values shown in Table 3.1. Moreover, in Figure 3.4 marginal stability and instability curves are plotted for different values of the upper plane temperature. Figure 3.2 and Table 3.1 show the stabilizing effect of the upper boundary plane temperature T_U on the onset of convection, in agreement with findings of Hill *et al.* [61] in absence of both inertia and throughflow. Figure 3.3 shows the Chebyshev eigenfunction $W(z)$ for several values of T_U, in particular the change in sign indicates the existence of multiple rotating convective cells (as pointed out in [131, 79]) and therefore the onset of penetrative convection. Finally, in Figure 3.5 roll perturbations profiles, see [89], are depicted in correspondence of three different upper boundary plane temperature T_U for which there is no change in sign ($T_U = 4°C$) and for which a single and double change in sign is present ($T_U = 8°C$ and $T_U = 12°C$, respectively).

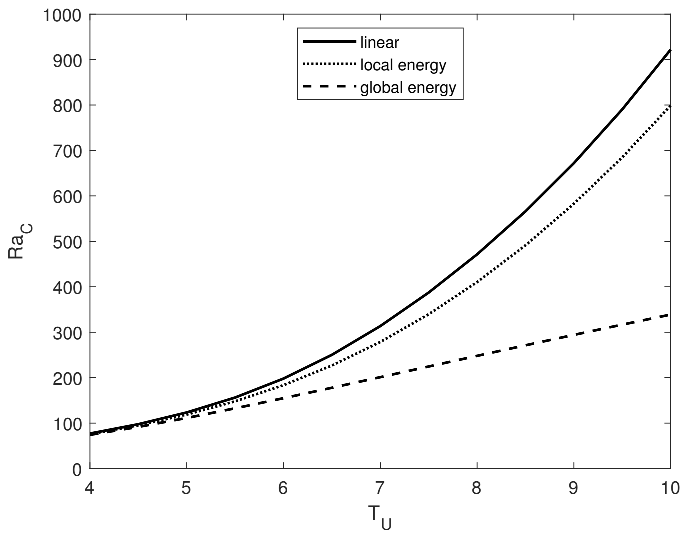


Figure 3.2: Linear instability (solid line), local nonlinear stability (dotted line) and global nonlinear stability (dashed line) thresholds comparison against T_U.

3.5 FINAL REMARK

The onset of penetrative convection in a horizontal porous layer uniformly heated from below, has been analysed, according to generalized Darcy's law. In particular, it has been proved that:

- the principle of exchange of stabilities holds, and hence convection sets in through a stationary motion;

Global Energy			Local Energy			Linear		
Ra_W	μ	a^2	Ra_E	η	a^2	Ra_L	a^2	$T_U(°C)$
74.219	1.119	9.786	75.825	0.541	10.256	77.079	10.209	4
91.685	1.056	9.787	95.195	0.440	10.471	97.624	10.421	4.5
111.343	1.025	9.860	118.891	0.363	10.791	123.462	10.767	5
132.591	1.016	10.086	147.966	0.304	11.261	156.288	11.344	5.5
154.873	1.012	10.387	183.561	0.260	11.927	198.030	12.314	6
177.813	1.010	10.736	226.782	0.227	12.845	250.284	13.881	6.5
201.129	1.0091	11.121	278.565	0.202	14.059	313.547	16.138	7
224.589	1.0083	11.515	339.597	0.183	15.586	387.347	18.893	7.5
248.009	1.0077	11.915	410.364	0.169	17.376	471.384	21.868	8
271.244	1.0072	12.315	491.255	0.158	19.400	566.045	24.888	8.5
294.182	1.0069	12.714	582.666	0.149	21.617	672.119	27.984	9
316.739	1.0065	13.097	685.022	0.139	24.195	790.475	31.192	9.5
338.852	1.0062	13.471	798.913	0.133	26.682	921.929	34.549	10

Table 3.1: Critical values of thermal Rayleigh and wave numbers and the parameters μ and η against the upper layer temperature T_U.

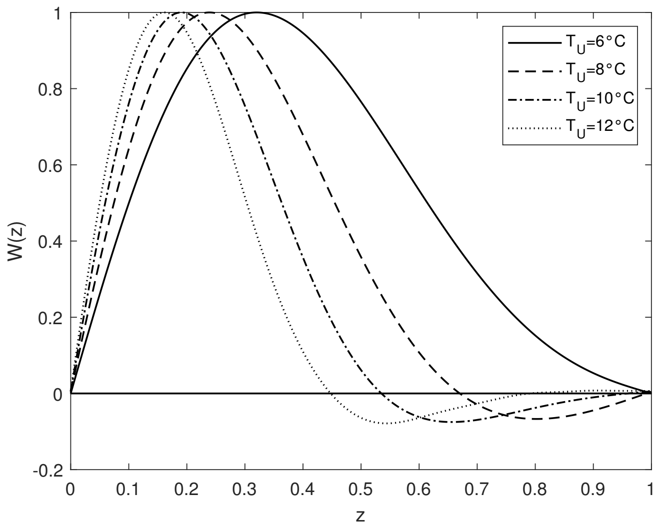
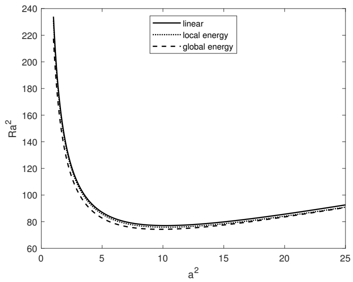
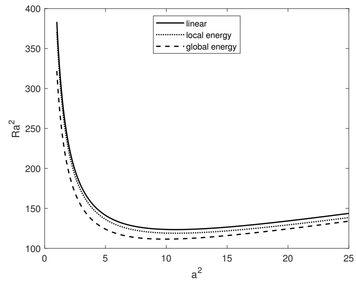


Figure 3.3: Profiles of $\widetilde{W}(z)$ normalized over the spatial layer.

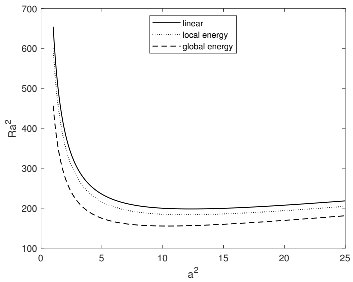
- global nonlinear stability thresholds in the L^2-norm guarantees the global (i.e., for all initial data) nonlinear stability of the conductive solution;
- local nonlinear stability thresholds reduce the gap between the linear and nonlinear critical Rayleigh numbers;
- the Vadasz number, arising from the inertia term appearing in the extended Darcy's law, does not affect the onset of convection.



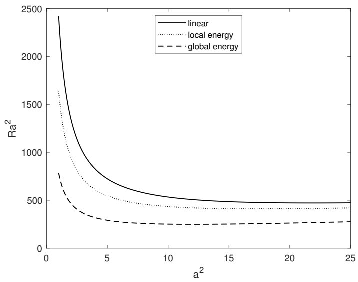
(a): $T_U = 4°C$



(b): $T_U = 5°C$



(c): $T_U = 6°C$



(d): $T_U = 8°C$

Figure 3.4: Plot of the marginal stability and instability curves for several values of the upper plane temperature.

3.6 APPENDIX

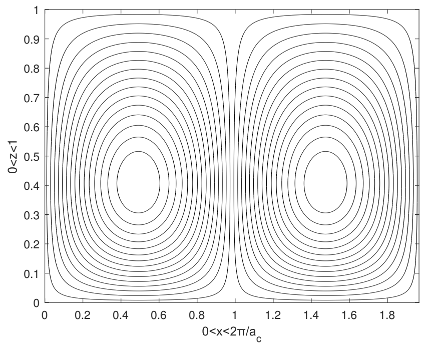
3.6.1 *Normal mode solutions*

System (3.27) can be written, setting $\widehat{\mathbf{W}}(\mathbf{x}) = \left(\widehat{\theta}(\mathbf{x}), \widehat{w}(\mathbf{x})\right)$, in the equivalent form

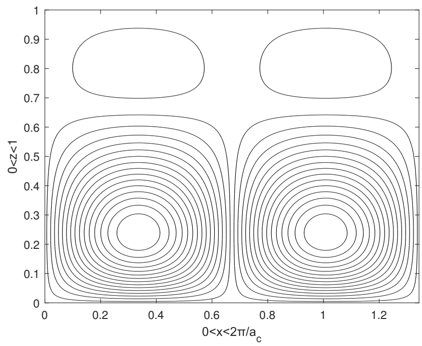
$$\mathbf{0} = L\widehat{\mathbf{W}}, \tag{3.62}$$

where

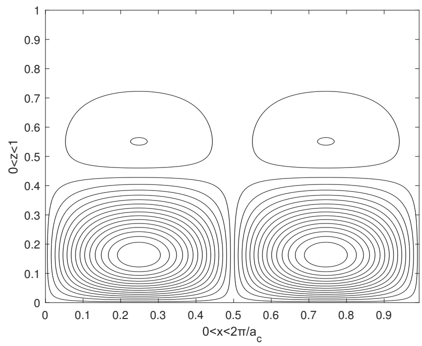
$$L = \begin{pmatrix} -\mathsf{Ra}M(z)\Delta_1 & -\Delta \\ \Delta & -\mathsf{Ra} \end{pmatrix}. \tag{3.63}$$



(a): $T_U = 4°C$



(b): $T_U = 8°C$



(c): $T_U = 12°C$

Figure 3.5: Roll perturbations profiles.

L can be decomposed, setting $-\mathsf{Ra}M(z) = F(z)$, as follows:

$$L = \begin{pmatrix} F(z)(\partial_{xx} + \partial_{yy}) & -\partial_{xx} - \partial_{yy} \\ \partial_{xx} + \partial_{yy} & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\partial_{zz} \\ \partial_{zz} & -\mathsf{Ra} \end{pmatrix}, \tag{3.64}$$

that is, in components

$$L_{ij} = H_{ij}(\partial_{xx} + \partial_{yy}) + J_{ij}, \tag{3.65}$$

where

$$H = \begin{pmatrix} F(z) & -1 \\ 1 & 0 \end{pmatrix} \qquad J = \begin{pmatrix} 0 & -\partial_{zz} \\ \partial_{zz} & -\mathsf{Ra} \end{pmatrix}. \tag{3.66}$$

Therefore system (3.62) can be written as

$$0 = L_{ij}\widehat{W}_j = H_{ij}(\partial_{xx} + \partial_{yy})\widehat{W}_j + J_{ij}\widehat{W}_j. \tag{3.67}$$

If we now separate the variables in the following way:

$$\widehat{W}_j(x, y, z) = K_j(x, y)Z_j(z), \tag{3.68}$$

then equation (3.67) becomes

$$0 = H_{ij}(\partial_{xx} + \partial_{yy})K_j(x, y)Z_j(z) + J_{ij}K_j(x, y)Z_j(z), \tag{3.69}$$

i.e.

$$H_{ij}Z_j(z)(\partial_{xx} + \partial_{yy})K_j(x, y) = -K_j(x, y)J_{ij}Z_j(z). \tag{3.70}$$

As a consequence, we have that

$$\frac{(\partial_{xx} + \partial_{yy})K_j(x, y)}{K_j(x, y)} = -\frac{J_{ij}Z_j(z)}{H_{ij}Z_j(z)} = c. \tag{3.71}$$

Now from (3.71) it follows that

$$\partial_{xx}K_j(x, y) + \partial_{yy}K_j(x, y) = cK_j(x, y). \tag{3.72}$$

Now let's separate the variables again

$$K_j(x, y) = X_j(x)Y_j(y), \tag{3.73}$$

so that equation (3.72) becomes

$$\begin{aligned} Y_j(y)\partial_{xx}X_j(x) + X_j(x)\partial_{yy}Y_j(y) &= cX_j(x)Y_j(y) \\ &= \tfrac{c}{2}X_j(x)Y_j(y) + \tfrac{c}{2}X_j(x)Y_j(y), \end{aligned} \tag{3.74}$$

and

$$Y_j(y)\left(\partial_{xx}X_j(x) - \tfrac{c}{2}X_j(x)\right) = X_j(x)\left(\tfrac{c}{2}Y_j(y) - \partial_{yy}Y_j(y)\right), \tag{3.75}$$

or equivalently

$$\frac{\partial_{xx}X_j(x) - \frac{c}{2}X_j(x)}{X_j(x)} = \frac{\frac{c}{2}Y_j(y) - \partial_{yy}Y_j(y)}{Y_j(y)} = \mu. \tag{3.76}$$

From equation (3.76) we can conclude that

$$\partial_{xx}X_j(x) - \tfrac{c}{2}X_j(x) = \mu X_j(x) \quad \text{i.e.} \quad \partial_{xx}X_j(x) = \left(\mu + \tfrac{c}{2}\right)X_j(x), \tag{3.77}$$

and

$$\tfrac{c}{2}Y_j(y) - \partial_{yy}Y_j(y) = \mu Y_j(y) \quad \text{i.e.} \quad \partial_{yy}Y_j(y) = \left(\tfrac{c}{2} - \mu\right)Y_j(x). \tag{3.78}$$

Let us note that because of the periodicity in x and y direction (with spatial frequency given by wave numbers a_x and a_y) we have that equations (3.77) and (3.78) admit non trivial periodic solution if and only if, respectively, $\mu + \frac{c}{2} < 0$ (i.e. $\mu < -\frac{c}{2}$) and $\frac{c}{2} - \mu < 0$ (i.e. $\mu > \frac{c}{2}$) which is true if and only if $c < 0$, therefore

$$X_j(x) = c_1 e^{ia_x x} \quad \text{and} \quad Y_j(y) = c_2 e^{ia_y y}. \tag{3.79}$$

Hence, going back to (3.73) we have

$$K_j(x,y) = \bar{c}e^{ia_x x}e^{ia_y y} = \bar{c}e^{i\left(a_x x + a_y y\right)}, \tag{3.80}$$

and from equation (3.68) we have

$$\widehat{W}_j(x,y,z) = \widetilde{\Xi}(z)e^{i\left(a_x x + a_y y\right)}, \tag{3.81}$$

i.e. $\widehat{\theta}$ and $\widehat{w}$ can be written in the form (3.28), namely

$$\begin{aligned} \widehat{w}(x,y,z) &= \widetilde{W}(z)e^{i\left(a_x x + a_y y\right)}, \\ \widehat{\theta}(x,y,z) &= \widetilde{\Theta}(z)e^{i\left(a_x x + a_y y\right)}. \end{aligned} \tag{3.82}$$

3.6.2 *A note concerning the existence of maxima*

Let us consider the functional

$$\mathcal{F}(\mathbf{u},\theta) = \frac{I}{D} = \frac{-\left\langle (\mu+\zeta-2z)\theta, w\right\rangle}{\|\mathbf{u}\|^2 + \langle \mu - z, |\nabla\theta|^2\rangle} = \frac{-\displaystyle\int_V (\mu+\zeta-2z)\theta w\, dV}{\displaystyle\int_V |\mathbf{u}|^2 dV + \int_V (\mu - z)|\nabla\theta|^2 dV}. \tag{3.83}$$

We want to prove that $\max_{\mathcal{H}} \mathcal{F}(\mathbf{u},\theta)$ exists. Accounting for [108], first of all we need to prove that (3.83) is bounded. Indeed, recalling that $\mu > 1$, $\zeta \leq 1$ and $z \in (0,1)$ therefore $\mu - z > \mu - 1 = c_2 > 0$, then it follows from the Cauchy-Schwartz, Poincaré and generalized Young inequalities that

$$\begin{aligned}\mathcal{F}(\mathbf{u},\theta) &= \frac{\left\langle (\mu+\zeta-2z)\theta, w\right\rangle}{\|\mathbf{u}\|^2 + \langle \mu - z, |\nabla\theta|^2\rangle} \leq \frac{\|(\mu+\zeta-2z)\theta\|\,\|w\|}{\|\mathbf{u}\|^2 + \langle \mu - z, |\nabla\theta|^2\rangle} \\ &\leq \frac{c_1\|\theta\|\,\|w\|}{\|\mathbf{u}\|^2 + c_2\|\nabla\theta\|^2} \leq \frac{c_1\|\theta\|\,\|\mathbf{u}\|}{\|\mathbf{u}\|^2 + c_2\|\nabla\theta\|^2} \\ &\leq \frac{c_1\left(\frac{\|\theta\|^2}{2\varepsilon} + \frac{\varepsilon\|\mathbf{u}\|^2}{2}\right)}{\|\mathbf{u}\|^2 + c_2\|\nabla\theta\|^2} \leq \frac{\frac{c_1}{2}\left(\frac{\gamma\|\nabla\theta\|^2}{\varepsilon} + \varepsilon\|\mathbf{u}\|^2\right)}{\|\mathbf{u}\|^2 + c_2\|\nabla\theta\|^2} \\ &= \frac{\frac{c_1}{2}\left(\frac{\gamma}{c_2}\|\mathbf{u}\|^2 + c_2\|\nabla\theta\|^2\right)}{\|\mathbf{u}\|^2 + c_2\|\nabla\theta\|^2} \leq \frac{c_1\lambda}{2} < +\infty,\end{aligned} \tag{3.84}$$

where we set $\varepsilon = \frac{\gamma}{c_2}$ and $\lambda = \max\left\{\frac{\gamma}{c_2}, 1\right\}$. Since the functional is bounded, then $\exists \sup \mathcal{F}(\mathbf{u},\theta) < +\infty$. We note now that the key point in order to prove the existence of maximum is that functional is invariant under rescaling [108], i.e. $\forall t > 0$

$$\begin{aligned}\mathcal{F}(t\mathbf{u}, t\theta) &= \frac{\left\langle (\mu+\zeta-2z)t\theta, tw\right\rangle}{\|t\mathbf{u}\|^2 + \langle \mu - z, |\nabla(t\theta)|^2\rangle} \\ &= \frac{t^2\left\langle (\mu+\zeta-2z)\theta, w\right\rangle}{t^2\left(\|\mathbf{u}\|^2 + \langle \mu - z, |\nabla\theta|^2\rangle\right)} \\ &= \frac{\left\langle (\mu+\zeta-2z)\theta, w\right\rangle}{\|\mathbf{u}\|^2 + \langle \mu - z, |\nabla\theta|^2\rangle} \\ &= \mathcal{F}(\mathbf{u},\theta).\end{aligned} \tag{3.85}$$

Let now $\{(\mathbf{u}_k, \theta_k)\}_k \subset \mathcal{H}$ be a sequence such that

$$\mathcal{F}(\mathbf{u}_k, \theta_k) \longrightarrow \sup \mathcal{F}(\mathbf{u}, \theta). \tag{3.86}$$

We want to show that it exists a subsequence $\{(\mathbf{u}_{k_n}, \theta_{k_n})\}_k \subset \mathcal{H}$ such that $(\mathbf{u}_{k_n}, \theta_{k_n}) \to (\widetilde{\mathbf{u}}, \widetilde{\theta}) \in \mathcal{H}$. To this aim, let us define a sequence $\{(\widetilde{\mathbf{u}}_k, \widetilde{\theta}_k)\}_k \subset \mathcal{H}$ such that

$$\widetilde{\mathbf{u}}_k = \frac{\mathbf{u}_k}{\langle(\mu-\zeta-2z)\theta_k, w_k\rangle^{1/2}} \quad \text{and} \quad \widetilde{\theta}_k = \frac{\theta_k}{\langle(\mu-\zeta-2z)\theta_k, w_k\rangle^{1/2}}. \tag{3.87}$$

As a consequence we have that

$$\begin{aligned}
\mathcal{F}(\widetilde{\mathbf{u}}_k, \widetilde{\theta}_k) &= \frac{\left\langle (\mu+\zeta-2z)\frac{\theta_k}{\langle(\mu-\zeta-2z)\theta_k, w_k\rangle^{1/2}}, \frac{w_k}{\langle(\mu-\zeta-2z)\theta_k, w_k\rangle^{1/2}} \right\rangle}{\|\widetilde{\mathbf{u}}_k\|^2 + \left\langle \mu - z, \left|\nabla\widetilde{\theta}_k\right|^2 \right\rangle} \\
&= \frac{1}{\|\widetilde{\mathbf{u}}_k\|^2 + \left\langle \mu - z, \left|\nabla\widetilde{\theta}_k\right|^2 \right\rangle} > 0.
\end{aligned} \tag{3.88}$$

Therefore, for all $k \in \mathbb{N}$ we have that $\|\widetilde{\mathbf{u}}_k\| \le c_1$ and $\|\nabla\widetilde{\theta}_k\| \le c_2$ and it follows from Rellich-Kondrakov and Banach-Alaoglu theorems that $\widetilde{\mathbf{u}}_{k_n} \rightharpoonup \widetilde{\mathbf{u}}$ in L^2 and $\widetilde{\theta}_{k_n} \to \widetilde{\theta}$ in L^2 so $\nabla\widetilde{\theta}_{k_n} \rightharpoonup \nabla\widetilde{\theta}$, respectively. Now from the previous weak limit (lower semicontinuity) it follows (recalling that if $f_k \rightharpoonup f$ then $\liminf_k \|f_k\| \ge \|f\|$ implicates that $\limsup_k \|f_k\|^{-1} \le \|f\|^{-1}$), that

$$\begin{aligned}
\sup_n \mathcal{F}(\widetilde{\mathbf{u}}_{k_n}, \widetilde{\theta}_{k_n}) &= \limsup_n \mathcal{F}(\widetilde{\mathbf{u}}_{k_n}, \widetilde{\theta}_{k_n}) = \liminf_n \left[\mathcal{F}(\widetilde{\mathbf{u}}_{k_n}, \widetilde{\theta}_{k_n})\right]^{-1} \\
&= \liminf_n \left(\|\widetilde{\mathbf{u}}_{k_n}\| + \left\langle \mu - z, |\nabla\widetilde{\theta}_{k_n}| \right\rangle\right) \\
&\le \|\widetilde{\mathbf{u}}\| + \langle \mu - z, |\nabla\theta| \rangle
\end{aligned} \tag{3.89}$$

Moreover, it is immediate to prove that $\nabla \cdot \widetilde{\mathbf{u}} = 0$, i.e. that for all $\varphi \in C_c^\infty$

$$\int_V \langle \widetilde{\mathbf{u}}, \nabla\varphi \rangle \, dV = 0. \tag{3.90}$$

We can notice in conclusion that

$$\left[\liminf_n \mathcal{F}(\widetilde{\mathbf{u}}_{k_n}, \widetilde{\theta}_{k_n})^{-1}\right]^{-1} \ge \left[\mathcal{F}(\widetilde{\mathbf{u}}, \widetilde{\theta})^{-1}\right]^{-1}, \tag{3.91}$$

therefore

$$\limsup_n \mathcal{F}(\widetilde{\mathbf{u}}_{k_n}, \widetilde{\theta}_{k_n}) \leq \mathcal{F}(\widetilde{\mathbf{u}}, \widetilde{\theta}), \tag{3.92}$$

and the existence of the maximum of the functional (3.83) is eventually proved.

4 THE EFFECT OF A VARIABLE GRAVITY FIELD ON THE ONSET OF PENETRATIVE CONVECTION

The aim of this chapter is to investigate the phenomenon of penetrative convection in presence of variable gravity fields, which adds an intriguing layer of complexity. The topic studied is the content of a future publication in collaboration with Prof. F. Capone, Dott. G. Massa and F. Iovanna. Pradhan and Samal [98], Straughan [121] and Chen and Chen [34] engaged in a detailed analysis of the effects of a variable gravity field on thermal convection. Moreover, Rionero and Straughan [109] determined the thresholds of linear and nonlinear stability for porous convection exposed to internal heat sources under variable gravity fields. The above studies illuminate the intricate interplay between convection dynamics, penetrative behaviour, and the compelling influence of variable gravity fields within porous media.
The chapter is organized as follows. In Section 4.1 the governing equations describing the onset of penetrative porous convection accounting for a variable gravity field are presented, together with the conduction solution and the associated perturbation equations. In Section 4.2 the principle of exchange of stabilities is proved – hence penetrative convection can arise only through steady motions – and the linear instability analysis of the thermal conduction solution is performed. In Section 4.3 the nonlinear stability analysis of the conduction solution is performed via the weighted energy method. In Section 4.4 the numerical investigations of the stability results are performed in order to numerically determined the linear and nonlinear critical Rayleigh numbers for the onset of penetrative porous convection, and in order to describe the effect of the variable gravity field on the onset of penetrative stationary porous convection. The chapter ends with a Section that recaps all the obtained results.

4.1 GOVERNING EQUATIONS

Introducing a reference frame $Oxyz$ fundamental unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ (with $\mathbf{k}$ pointing vertically upward), let us consider a horizontal porous layer $L = \mathbb{R}^2 \times [0, d]$ filled by a homogeneous viscous fluid at rest. Employing the Oberbeck-Boussinesq approximation, we assume the quadratic density (1.63). Moreover, in this chapter, we consider a gravity field varying with the spatial coordinate z, i.e. $\mathbf{g} = -g(z)\mathbf{k}$ with

$$g(z) = g_0[1 + \epsilon h(z)], \tag{4.1}$$

positive definite and g_0 being the modulus of the gravity acceleration. Therefore, the governing equations describing the onset of penetrative porous convection accounting for a variable gravity field, are:

$$\begin{cases} \dfrac{\mu}{K}\mathbf{v} = -\nabla p - g(z)\rho_0[1 - \alpha(T - T_0)^2]\mathbf{k}, \\ \nabla \cdot \mathbf{v} = 0, \\ \dfrac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = k_m \Delta T, \end{cases} \tag{4.2}$$

where $\mathbf{x} = (x, y, z)$, $\mathbf{v}$ is the seepage velocity, p is the pressure, μ is the fluid viscosity, K is the permeability, k_m is the thermal diffusivity. To system (4.2) we append the following boundary conditions

$$\mathbf{v} \cdot \mathbf{n} = 0 \text{ on } z = 0, d, \quad T(z = 0) = 0, \quad T(z = d) = T_U \geq 4°C. \tag{4.3}$$

System (4.2) is equivalent to

$$\begin{cases} -\dfrac{\mu}{K}\mathbf{v} - \nabla \tilde{p} - 2g(z)\rho_0\alpha T_0 T\mathbf{k} + g(z)\rho_0\alpha T^2\mathbf{k} = \mathbf{0}, \\ \nabla \cdot \mathbf{v} = 0, \\ \dfrac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = k_m \Delta T, \end{cases} \tag{4.4}$$

where $\tilde{p} = p + G(z)\rho_0(1 - \alpha T_0^2)$ is the reduced pressure, $G(z)$ being defined such that $G'(z) = g(z)$.

4.1.1 *Thermal conduction solution and dimensionless perturbation equations*

System (4.4) under boundary conditions (4.3) admits the thermal conduction solution $m_0 = (\hat{\mathbf{v}}, \hat{p}, \hat{T})$, where

$$\begin{aligned} &\hat{\mathbf{v}} = \mathbf{0}, \ \hat{T}(z) = \frac{T_U}{d} z, \\ &\hat{p}(z) = p_0 + c_1 \int g(z) z \, dz + c_2 \int g(z) z^2 \, dz - G(z)\rho_0(1 - \alpha T_0^2), \end{aligned} \tag{4.5}$$

with $c_1 = -2\rho_0\alpha T_0 T_U d^{-1}$, $c_2 = \rho_0\alpha T_U^2 d^{-2}$ and p_0 a prescribed value arising from the appropriate boundary conditions on p. To analyse the stability of the conduction solution m_0, introducing a generic perturbation $\{\mathbf{u} = (u, v, w), \pi, \theta\}$, let us consider the perturbation equations:

$$\begin{cases} \dfrac{\mu}{K}\mathbf{u} = -\nabla\pi - 2g(z)\rho_0\alpha T_0\theta\mathbf{k} + 2g(z)\rho_0\alpha\dfrac{T_U}{d}z\theta\mathbf{k} + g(z)\rho_0\alpha\theta^2\mathbf{k}, \\ \nabla\cdot\mathbf{u} = 0, \\ \dfrac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = -\dfrac{T_U}{d}w + k_m\Delta\theta. \end{cases} \tag{4.6}$$

Accounting for the following non-dimensional parameters

$$\mathbf{x}^* = \frac{\mathbf{x}}{d},\ t^* = \frac{t}{\tau},\ \theta^* = \frac{\theta}{T^\#},\ \mathbf{u}^* = \frac{\mathbf{u}}{U},\ \pi^* = \frac{\pi}{P},$$

where the scales are defined as follows

$$U = \frac{k_m}{d},\ \tau = \frac{d^2}{k_m},\ P = \frac{k_m\mu}{K},\ T^\# = \sqrt{\frac{\mu k_m}{2Kg_0\rho_0\alpha d}},$$

the dimensionless perturbation equations (omitting all the asterisks for notational convenience) are given by

$$\begin{cases} \mathbf{u} = -\nabla\pi - \mathsf{Ra}H(z)M(z)\theta\mathbf{k} + H(z)\dfrac{\theta^2}{2}\mathbf{k}, \\ \nabla\cdot\mathbf{u} = 0, \\ \dfrac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = -\mathsf{Ra}w + \Delta\theta, \end{cases} \tag{4.7}$$

where the Rayleigh number Ra is defined as

$$\mathsf{Ra} = \sqrt{\frac{2T_U^2 d\rho_0\alpha g_0 K}{\mu k_m}},$$

and the function H and M are given by

$$H(z) = 1 + \epsilon h(z) \quad \text{and} \quad M(z) = \zeta - z, \quad \zeta = \frac{T_0}{T_U}. \tag{4.8}$$

To system (4.7) we append the following initial and boundary conditions:

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \pi(\mathbf{x}, 0) = \pi_0(\mathbf{x}), \quad \theta(\mathbf{x}, 0) = \theta_0(\mathbf{x}),$$

with $\nabla\cdot\mathbf{u}_0 = 0$, and

$$w = \theta = 0 \quad \text{on } z = 0, 1, \tag{4.9}$$

respectively. We assume the perturbation fields to be periodic in the horizontal directions x and y with period $\frac{2\pi}{a_x}$ and $\frac{2\pi}{a_y}$, respectively, and define the periodicity cell

$$V = \left[0, \frac{2\pi}{a_x}\right] \times \left[0, \frac{2\pi}{a_y}\right] \times [0,1].$$

4.2 LINEAR THEORY

In this Section, we perform the instability analysis of the thermal conduction solution m_0 considering the linear version of (4.7), i.e.

$$\begin{cases} \mathbf{u} = -\nabla\pi - \mathsf{Ra}H(z)M(z)\theta\mathbf{k}, \\ \nabla\cdot\mathbf{u} = 0, \\ \dfrac{\partial\theta}{\partial t} = -\mathsf{Ra}w + \Delta\theta. \end{cases} \tag{4.10}$$

Since system (4.10) is autonomous, we assume the following solutions

$$\Phi(\mathbf{x}, t) = e^{\sigma t}\,\bar{\Phi}(\mathbf{x}), \quad \forall\Phi \in \{\mathbf{u}, \theta, \pi\}, \tag{4.11}$$

with $\sigma \in \mathbb{C}$, obtaining

$$\begin{cases} \bar{\mathbf{u}} = -\nabla\bar{\pi} - \mathsf{Ra}H(z)M(z)\bar{\theta}\mathbf{k}, \\ \nabla\cdot\bar{\mathbf{u}} = 0, \\ \sigma\bar{\theta} = -\mathsf{Ra}\bar{w} + \Delta\bar{\theta}. \end{cases} \tag{4.12}$$

Let us consider the third component of the double curl of $(4.12)_1$. therefore, together with the boundary conditions (4.9), we obtain the following boundary value problem in w and θ:

$$\begin{cases} -\Delta\bar{w} = \mathsf{Ra}H(z)M(z)\Delta_1\bar{\theta}, \\ \sigma\bar{\theta} = -\mathsf{Ra}\bar{w} + \Delta\bar{\theta}, \end{cases} \tag{4.13}$$

with $\Delta_1 = \partial_{xx} + \partial_{yy}$ the horizontal Laplacian. Since we are considering periodic perturbations in the x and y directions, let us consider the following solutions

$$\bar{\Phi}(x, y, z) = \tilde{\Phi}(z)e^{i(a_x x + a_y y)}, \quad \forall\Phi \in \{w, \theta\}. \tag{4.14}$$

Hence, substituting (4.14) in (4.13), one gets

$$\begin{cases} (D^2 - a^2)\tilde{w} - \mathsf{Ra}H(z)M(z)a^2\tilde{\theta} = 0, \\ \sigma\tilde{\theta} = -\mathsf{Ra}\tilde{w} + (D^2 - a^2)\tilde{\theta}, \end{cases} \tag{4.15}$$

where $a^2 = a_x^2 + a_y^2$ is the wavenumber and $D = \frac{d}{dz}$. The following theorem holds.

Theorem 4.2.1. The strong form of the principle of exchange of stabilities holds for system (4.15) – (4.9), therefore, penetrative convection can occur only via steady motions.

Proof. Let us apply the operator $D^2 - a^2$ to $(4.15)_2$, hence by virtue of $(4.15)_1$, we obtain

$$\sigma(D^2 - a^2)\tilde{\theta} = -\mathsf{Ra}^2 H(z)M(z)a^2\tilde{\theta} + (D^2 - a^2)^2\tilde{\theta}. \tag{4.16}$$

Multiplying (4.16) by the complex conjugate $\tilde{\theta}^*$ of $\tilde{\theta}$, and integrating over $[0,1]$, we get

$$\begin{aligned}\sigma\left[\|D\tilde{\theta}\|^2 + a^2\|\tilde{\theta}\|^2\right] = \mathsf{Ra}^2 a^2 \left\langle H(z)M(z), \tilde{\theta}^2\right\rangle \\ - \|D^2\tilde{\theta}\|^2 - a^4\|\tilde{\theta}\|^2 - 2a^2\|D\tilde{\theta}\|^2.\end{aligned} \tag{4.17}$$

From (4.17) it follows that $\sigma \in \mathbb{R}$, hence the principle of exchange of stabilities holds. □

By virtue of Theorem 4.2.1, we assume $\sigma = 0$ in (4.15) to determine the linear critical Rayleigh number for the onset of (steady) penetrative convection, hence we get

$$\begin{cases}(D^2 - a^2)\tilde{w} = \mathsf{Ra}H(z)M(z)a^2\tilde{\theta}, \\ (D^2 - a^2)\tilde{\theta} = \mathsf{Ra}\tilde{w}.\end{cases} \tag{4.18}$$

To determine the instability threshold for the onset of penetrative convection, which is

$$\mathsf{Ra}_L = \min_{a^2 \in \mathbb{R}^+} \{\mathsf{Ra}^2(a^2) |\ \mathsf{Ra} \text{ verifies } (4.18)\}, \tag{4.19}$$

we will numerically solve the generalized eigenvalue problem (4.18) in Section 4.4.

4.3 NONLINEAR STABILITY OF THE CONDUCTION SOLUTION

In this Section, we perform the nonlinear stability analysis of the conduction solution to system (4.7)–(4.9), choosing as Lyapunov function

$$E(t) = \frac{1}{2}\left\langle f(z), \theta^2\right\rangle, \tag{4.20}$$

where $f(z)$ is a positive and continuous weight function to be suitably chosen later. Multiplying $(4.7)_1$ by $\mathbf{u}$ and $(4.7)_3$ by $f(z)\theta$, integrating over the periodicity cell V and adding the resulting equations, we obtain

$$\begin{aligned}\frac{dE}{dt} =& -\|\mathbf{u}\|^2 - \left\langle f(z), |\nabla\theta|^2 \right\rangle - \mathsf{Ra}\,\langle (f(z) + H(z)M(z))\theta, w\rangle \\ &+ \frac{1}{2}\left\langle (f'(z) + H(z))w, \theta^2 \right\rangle + \frac{1}{2}\left\langle f''(z), \theta^2 \right\rangle.\end{aligned} \tag{4.21}$$

With the aim of obtaining a global stability result, we choose $f(z)$ such that

$$f'(z) = -H(z), \tag{4.22}$$

so that we remove the cubic nonlinear terms in equation (4.21), which becomes

$$\frac{dE}{dt} = I - D \leq D\,(\Lambda - 1)\,, \tag{4.23}$$

where we set

$$\begin{aligned} I &= -\mathsf{Ra}\,\langle (f(z) + H(z)M(z))\theta, w\rangle + \frac{1}{2}\left\langle f''(z), \theta^2 \right\rangle, \\ D &= \|\mathbf{u}\|^2 + \left\langle f(z), |\nabla\theta|^2 \right\rangle, \end{aligned} \tag{4.24}$$

and

$$\Lambda = \max_{\mathcal{H}} \frac{I}{D}, \tag{4.25}$$

$\mathcal{H}$ being the space of kinematically admissible perturbations defined as

$$\begin{aligned}\mathcal{H} = \{&(w,\theta) \in (H^1(V))^2 | w = \theta = 0 \text{ on } z = 0,1; \text{periodic in } x, y \\ &\text{with periods } 2\pi/a_x, 2\pi/a_y\}.\end{aligned}$$

4.3.1 *The stability theorem*

Before proving the main stability theorem, let us recall the following a priori estimate of the perturbation $\theta(\mathbf{x}, t)$ to the temperature field.

Lemma 4.3.1. Let us define the sets

$$V_1 = \{\mathbf{x} \in V \mid \theta(\mathbf{x}, t) > \xi^* - \hat{T}(z)\}, \tag{4.26}$$

$$V_2 = \{\mathbf{x} \in V \mid \theta(\mathbf{x}, t) \leq \xi^* - \hat{T}(z)\}. \tag{4.27}$$

where $\xi^* = \frac{T_U}{T^\#}$. If

$$\theta_0(\mathbf{x}) \in W^{2,2}(V), \tag{4.28}$$

then, for almost every $\mathbf{x} \in V_1$

$$\theta(\mathbf{x}, t) + \hat{T}(z) - \xi^* \leq \bar{\theta}_0, \tag{4.29}$$

with

$$\bar{\theta}_0 = \operatorname*{ess\,sup}_{V_1}[(\theta_0(\mathbf{x}) + \hat{T}(z) - \xi^*)_+](< +\infty). \tag{4.30}$$

Proof. Estimate (4.29) can be proved following step by step the procedure in [31]. □

Theorem 4.3.1. Condition $\Lambda < 1$ guarantees the global nonlinear exponential stability of the conduction solution in the E-norm.

Proof. Since the weight function f is *positive and bounded* in $[0, 1]$, one has

$$\inf \frac{\int_V f(z)|\nabla\theta|^2 dV}{\int_V f(z)\theta^2 dV} \geq \frac{m}{M} \inf \frac{\int_V |\nabla\theta|^2 dV}{\int_V \theta^2 dV} \geq \frac{m\pi^2}{M}, \tag{4.31}$$

m and M being the minimum and the maximum values of f, respectively. Setting $c = \frac{M}{m\pi^2}$, from (4.31) it follows

$$\left\langle f(z), \theta^2 \right\rangle \leq c \left\langle f(z), |\nabla\theta|^2 \right\rangle. \tag{4.32}$$

By virtue of (4.32), from (4.23) one gets

$$\frac{dE}{dt} \leq D\,(\Lambda - 1) \leq 2c\,(\Lambda - 1)\,E \implies E(t) \leq E(0)e^{2c(\Lambda-1)t}, \tag{4.33}$$

i.e., if $\Lambda < 1$ the Lyapunov function E decays exponentially to zero. Moreover, setting $c_1 = \max_{[0,1]} |H(z)M(z)|$ and $c_2 = \max_{[0,1]} |H(z)|$, let us multiply $(6.6)_1$ by $\mathbf{u}$ and integrate over the periodicity cell V. By virtue of the generalized Cauchy inequality, we get

$$\|\mathbf{u}\|^2 \leq 2\mathsf{Ra}^2 c_1^2 \|\theta\|^2 + c_2^2 \|\theta^2\|^2. \tag{4.34}$$

In particular, by virtue of the a priori estimate (4.29), it follows that

$$\|\theta^2\|^2 \leq 4M_1 \|\theta\|^2, \tag{4.35}$$

where

$$M_1 = \max\left\{\bar{\theta}_0^2, \left[\max_{[0,1]} |\hat{T}(z) - \xi^*|\right]^2\right\}.$$

Therefore, accounting for (4.33), (4.34) and (4.35), condition $\Lambda < 1$ guarantees the global nonlinear exponential asymptotic stability of the conduction solution in the E-norm (4.21). □

Let us remark that, defining Ra_E as the critical value of the Rayleigh number such that $\Lambda = 1$, the condition $\Lambda < 1$ is equivalent to $\mathsf{Ra} < \mathsf{Ra}_E$. The Euler-Lagrange equations associated with the variational problem (4.25) are

$$\begin{cases} 2(D^2 - a^2)\tilde{w} = \mathsf{Ra}_E[f(z) + H(z)M(z)]a^2\tilde{\theta}, \\ 2f(z)(D^2 - a^2)\tilde{\theta} = \mathsf{Ra}_E[f(z) + H(z)M(z)]\tilde{w} - f''(z)\tilde{\theta} - 2f'(z)D\tilde{\theta}, \end{cases} \tag{4.36}$$

and the global nonlinear stability threshold for the onset of penetrative convection is

$$\mathsf{Ra}_E = \max_{\lambda} \min_{a^2 \in \mathbb{R}^+} \{\mathsf{Ra}^2(a^2) |\ \mathsf{Ra} \text{ verifies } (4.36)\}, \tag{4.37}$$

where λ is the constant of integration arising from $f(z) = -\int H(z)dz$.

4.4 NUMERICAL INVESTIGATIONS

This Section deals with the numerical investigations of the stability results we found in the previous Sections, in particular, we will analyze the behaviour of Ra_L and Ra_E with respect to the parameters ϵ and the upper layer temperature T_U, with the aim of describing the effect of the variable gravity field on the onset of penetrative stationary convection. The linear instability threshold (4.19) and the global nonlinear stability threshold (4.37) are determined by solving the ordinary differential eigenvalue problems (4.18) and (4.36), respectively, by a user-written Matlab code based on a combination of the Shooting method and the Newton-Raphson method, whose accuracy and convergence are extensively discussed in the literature, see e.g. [94, 133].

According to [109], as gravitational laws we consider

(i) $g(z) = -g_0(1 - \epsilon z), \quad \epsilon \in [0, 1],$

(ii) $g(z) = -g_0(1 - \epsilon z^2), \quad \epsilon \in [0, 1],$

(iii) $g(z) = -g_0(1 - \epsilon(e^z - 1)), \quad \epsilon \in \left[0, \frac{1}{e-1}\right].$

Hence, accounting for $(4.8)_1$ and (4.22), it follows that in the case

- (i), $f(z) = -z + \epsilon \dfrac{z^2}{2} + \lambda_1$, with $\lambda_1 > 1 - \dfrac{\epsilon}{2}$,
- (ii), $f(z) = -z + \epsilon \dfrac{z^3}{3} + \lambda_2$, with $\lambda_2 > 1 - \dfrac{\epsilon}{3}$,

- (iii), $f(z) = -(1+\epsilon)z + \epsilon e^z + \lambda_3$, with $\lambda_3 > 1 - \epsilon(e-1)$.

Let us remark that, as shown in Tables 4.1–4.3, for $\epsilon = 0$ — i.e. when the gravity field is assumed constant $\mathbf{g} = -g_0\mathbf{k}$ — we recovered the stability results found in [2], where the onset of penetrative convection in a horizontal porous layer, with constant gravity field, has been analysed.

In Figure 4.1(a) we plot the gravitational laws for $\epsilon = 0.2$ and $\epsilon = 0.58$, in Figure 4.1(b) the linear instability thresholds are depicted for $T_U = 4$ as functions of ϵ for the linear, quadratic and exponential decreasing gravity laws, while in 4.1(c) the linear instability thresholds are depicted for $T_U = 8$. The critical Rayleigh number is an increasing function with respect to ϵ, meaning that the decrease of gravity has a stabilizing effect on the onset of convection. Physically, this is motivated by the fact that the buoyancy effects are stronger at the bottom of the layer and decrease with height. Among the gravitational laws taken into account, the exponential one leads to the fastest decrease of gravity, therefore the onset of convection *is enhanced the least* by the gravity law $\mathbf{g} = -g_0[1 - \epsilon(e^z - 1)]\mathbf{k}$. This behaviour is also shown in Figures 4.2 and in Figure 4.3: in Figures 4.2 the linear instability thresholds are plotted for quoted values of ϵ and for $T_U = 4$ for the linear, quadratic and exponential decreasing gravity laws, while in Figure 4.3 the linear and nonlinear critical Rayleigh numbers are depicted as functions of ϵ for $T_U = 4$, for all the gravitational laws we considered.

From Tables 4.1–4.3 and from Figure 4.4 – where the linear threshold Ra_L as function of T_U for all the gravitational laws we considered is depicted – one can notice the stabilizing effect of the upper layer temperature T_U on the onset of convection since the Rayleigh number is an increasing function with respect to T_U. Physically the increase of the upper plane temperature leads to a slimming of the potentially unstable fluid region, in which the temperature of the fluid goes from $0°C$ to $4°C$. Hence, this region struggles to penetrate the stably stratified one, and this leads to a lag in the onset of penetrative convection.
Let us note that $T_U = 4°C$ represents a limit case where the stably stratified region, in which the convective currents penetrate, vanishes, i.e. the entire porous layer becomes potentially unstable and the instability thresholds for the onset of convective motions are reported in Tables 4.1(a), 4.2(a) and 4.3(a). In this regard, we can note a continuity between the limit case and the case $T_U > 4$ in which penetrative convection can occur, see Figure 4.4.

If T_U is very large there will be a thinning of the potentially unstable fluid region. As a consequence: (i) we will move away from the parabolic neighbourhood where the density ρ_f attains a maximum and in which it is physically meaningful to investigate penetrative convection due to a quadratic temperature profile of the fluid density; (ii) the vertical amplitude of the unstable stratified region will be so small that it will be insensitive to variations of gravity. Moreover, we can see that $T_U \in o(R_L(T_U))$ as depicted in 4.4.

Since we obtained that the linear and the nonlinear thresholds do not coincide (as one can notice from Tables 4.1–4.3 and from Figure 4.3), there exists a region of *subcritical instabilities*. In particular, for increasing T_U, we recover an expected behaviour characterizing the onset of penetrative convection: as T_U increases, the gap between the linear and the nonlinear thresholds increases. Regarding the influence of ϵ on the linear and the nonlinear stability thresholds, for small values of T_U and ϵ, the gap between Ra_L and Ra_E is small.

In Figures 4.6 and 4.5, we plot the convective rolls. Since we defined the convective cell as $V = [0, 2\pi/a_x] \times \left[0, 2\pi/a_y\right] \times [0,1]$, recalling that the wavenumber is $a^2 = a_x^2 + a_y^2$, the convective rolls are defined as $V = [0, 2\pi/a_c] \times [0, +\infty] \times [0,1]$, i.e. $a_y \longrightarrow 0$ and a_c^2 being the critical wavenumber.
In Figures 4.6 we plot the convective rolls for upper plane temperature $T_U = 4$, for increasing quoted values of ϵ and for all the gravitational laws. As ϵ increases, one can notice that the convective cell (gradually) loses the symmetry which characterizes the case $\epsilon = 0$, due to the decrease of gravity. In Figures 4.5 we plot the convective rolls for upper plane temperature $T_U = 8$, for increasing quoted values of ϵ and for all the gravitational laws. Moreover, in Figures 4.5 one can notice the presence of a secondary cell, which is consistent with the experimental observations [131], and, for all admissible $\epsilon > 0$, the cell is confined near the lower surface, where the gravity is stronger.
Finally, regarding the behaviour of the critical wavenumber a^2, from Tables 4.1–4.3 one can notice that a^2 increases as T_U increases, this means that the convective cells become *narrower*, as also shown in Figures 4.6–4.5.

λ_1	Ra_E	a^2	Ra_L	a^2	ϵ
1.119	74.2715	9.786	77.0797	10.209	0
0.998	83.4036	9.916	85.9413	10.299	0.25
0.883	94.7766	10.100	97.0006	10.431	0.5
0.775	109.1338	10.372	111.1255	10.633	0.75
0.678	127.4534	10.764	129.6446	10.963	1

(a) $\zeta = 1(T_U = 4)$

λ_1	Ra_E	a^2	Ra_L	a^2	ϵ
1.013	154.8721	10.388	198.0301	12.314	0
0.887	175.5029	10.510	214.2652	12.368	0.25
0.761	201.1480	10.648	233.3305	12.438	0.5
0.638	232.2681	10.839	256.0109	12.533	0.75
0.538	266.4904	11.358	283.3975	12.662	1

(b) $\zeta = 2/3(T_U = 6)$

λ_1	Ra_E	a^2	Ra_L	a^2	ϵ
1.008	248.1558	11.890	471.4321	21.827	0
0.882	286.1510	12.106	498.0429	21.413	0.25
0.756	336.6519	12.431	527.5292	20.727	0.5
0.629	405.1913	12.794	560.1067	19.632	0.75
0.507	489.0353	12.830	595.0978	18.043	1

(c) $\zeta = 0.5(T_U = 8)$

Table 4.1: Linear and nonlinear critical Rayleigh numbers Ra_L and Ra_E and critical λ_1 for quoted values of ϵ and of the upper layer temperature T_U, for linear decreasing gravity, i.e. $h(z) = -z$.

λ_1	Ra_E	a^2	Ra_L	a^2	ϵ
1.119	74.2713	9.787	77.0797	10.209	0
1.052	78.6626	9.902	81.0883	10.283	0.25
0.989	83.4031	10.041	85.4864	10.347	0.5
0.933	88.5003	10.210	90.3227	10.488	0.75
0.882	93.9590	10.394	95.6514	10.630	1

(a) $\zeta = 1 (T_U = 4)$

λ_1	Ra_E	a^2	Ra_L	a^2	ϵ
1.015	154.7322	10.396	198.0301	12.314	0
0.931	166.1466	10.470	203.1012	12.308	0.25
0.8464	178.2511	10.510	208.4136	12.302	0.5
0.767	190.3911	10.528	213.9822	12.297	0.75
0.706	201.115	10.712	219.8228	12.294	1

(b) $\zeta = 2/3 (T_U = 6)$

λ_1	Ra_E	a^2	Ra_L	a^2	ϵ
1.008	248.1558	11.890	471.4321	21.827	0
0.924	272.4476	12.092	476.3860	21.494	0.25
0.839	301.0849	12.320	481.1004	21.047	0.5
0.755	334.3416	12.523	485.3637	20.478	0.75
0.675	367.7618	12.260	488.8668	19.759	1

(c) $\zeta = 0.5 (T_U = 8)$

Table 4.2: Linear and nonlinear critical Rayleigh numbers Ra_L and Ra_E and critical λ_3 for quoted values of ϵ and of the upper layer temperature T_U, for exponential decreasing gravity, i.e. $h(z) = -z^2$.

λ_1	Ra_E	a^2	Ra_L	a^2	ϵ
1.119	74.2715	9.790	77.0797	10.209	0
0.709	86.5956	10.012	88.8612	10.362	0.25
0.316	102.6988	10.372	104.5633	10.621	0.5
0.195	108.8519	10.530	110.7169	10.743	0.58

(a) $\zeta = 1(T_U = 4)$

λ_1	Ra_E	a^2	Ra_L	a^2	ϵ
1.013	154.9721	10.384	198.0301	12.314	0
0.582	184.0259	10.544	217.5126	12.257	0.25
0.162	219.4924	10.750	241.0885	12.420	0.5
0.038	231.0761	10.908	249.7052	12.447	0.58

(b) $\zeta = 2/3(T_U = 6)$

λ_1	Ra_E	a^2	Ra_L	a^2	ϵ
1.008	248.1558	11.890	471.4321	21.827	0
0.577	306.3430	12.284	500.4766	21.166	0.25
0.145	391.9845	12.668	531.3187	20.063	0.5
0.011	422.3020	12.560	541.1906	19.553	0.58

(c) $\zeta = 0.5(T_U = 8)$

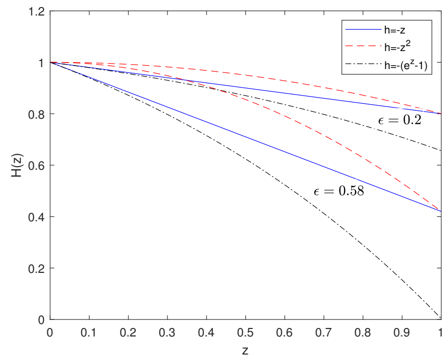
Table 4.3: Linear and nonlinear critical Rayleigh numbers Ra_L and Ra_E and critical λ_2 for quoted values of ϵ and of the upper layer temperature T_U, for quadratic decreasing gravity, i.e. $h(z) = -(e^z - 1)$.

4.5 CONCLUSIONS

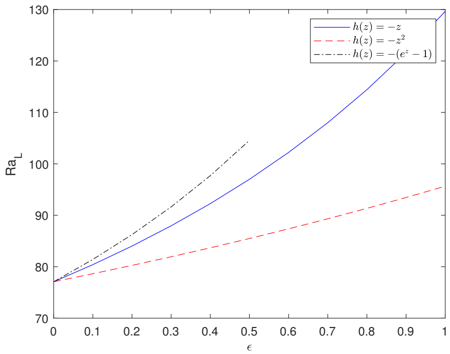
In this chapter, we analysed the effect of a variable gravity field on the onset of penetrative convection in a horizontal fluid-saturated porous layer. We proved the principle of exchange of stabilities, therefore penetrative convection can arise only through steady motions. We performed the linear instability and the nonlinear stability analyses (via the weighted energy method) of the thermal conduction solution. Via numerical simulations, we numerically determined the linear and nonlinear critical Rayleigh numbers Ra_L and Ra_E for the onset of penetrative porous steady convection. In particular, in order to describe the effect of the variable gravity field on the onset of penetrative convection, we analysed the behaviour of Ra_L and Ra_E with respect to ϵ and the upper layer temperature T_U, founding that

i. there is no coincidence between the linear and the nonlinear thresholds, hence there exists a region of subcritical instabilities, whose thickness is accentuated as T_U and ϵ simultaneously increase,

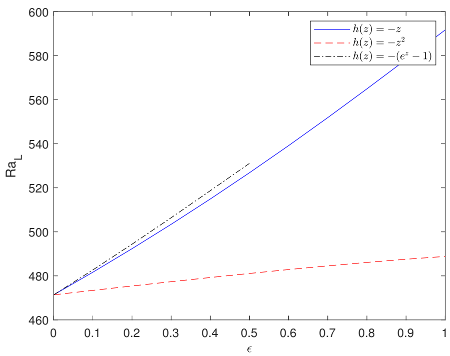
ii. the critical Rayleigh number is an increasing function with respect to ϵ, hence, the decrease of gravity has a stabilizing effect on the onset of convection.



(a)

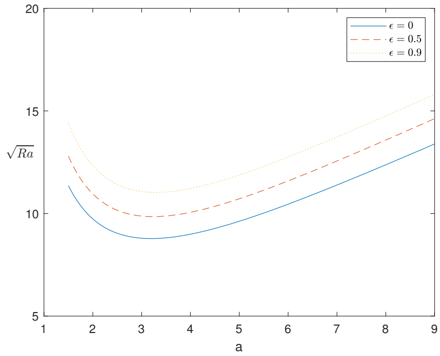


(b): $T_U = 4$

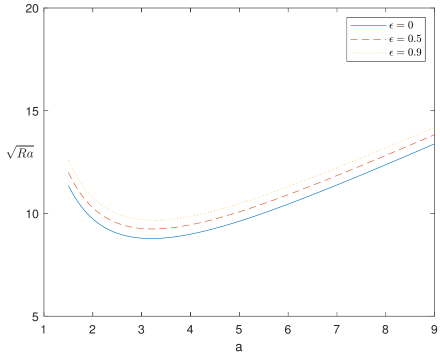


(c): $T_U = 8$

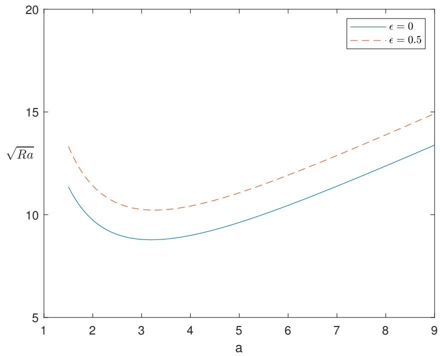
Figure 4.1: (a): Gravitational laws for quoted values of ϵ. (b): Linear instability thresholds as functions of ϵ for linear, quadratic and exponential decreasing gravity laws, for $T_U = 4$. (c): Linear instability thresholds as functions of ϵ for linear, quadratic and exponential decreasing gravity laws, for $T_U = 8$.



(a): $h(z) = -z$

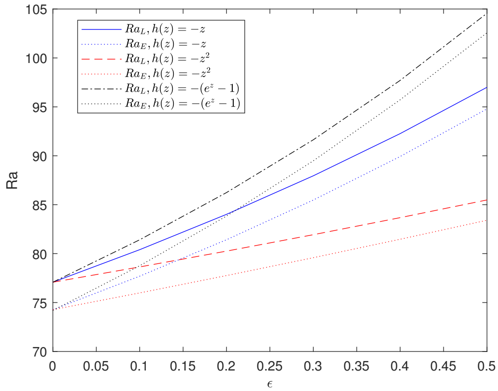


(b): $h(z) = -z^2$



(c): $h(z) = -(e^z - 1)$

Figure 4.2: Linear instability thresholds for quoted values of ϵ and for $T_U = 4$, for (a) linear decreasing gravity (b) quadratic decreasing gravity (c) exponential decreasing gravity.



(a): $T_U = 4$

Figure 4.3: Linear and nonlinear thresholds Ra_L and Ra_E as functions of ϵ for $T_U = 4$ and for all the gravitational laws we considered.

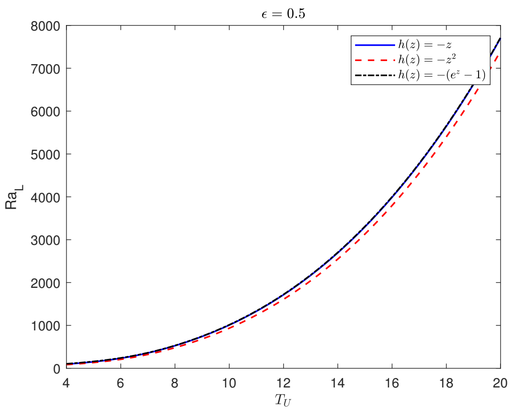
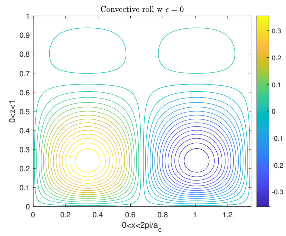
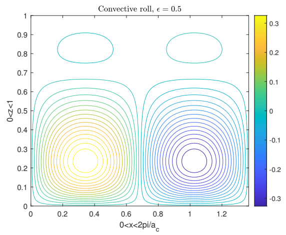


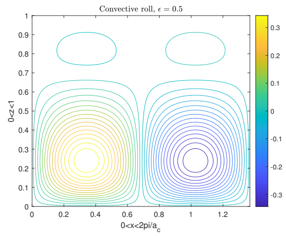
Figure 4.4: Linear threshold Ra_L as function of T_U for $\epsilon = 0.5$ and for all the gravitational laws we considered.



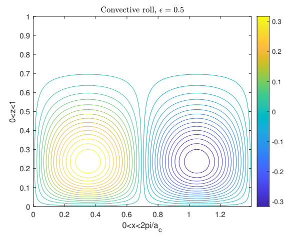
(a): constant gravity



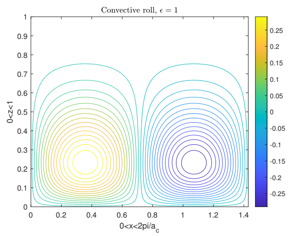
(b): $h(z) = -z$



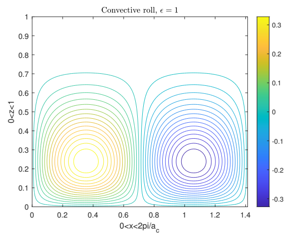
(c): $h(z) = -z^2$



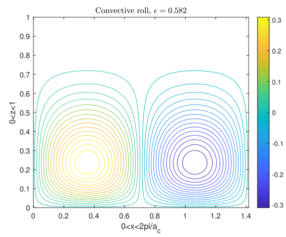
(d): $h(z) = -(e^z - 1)$



(e): $h(z) = -z$

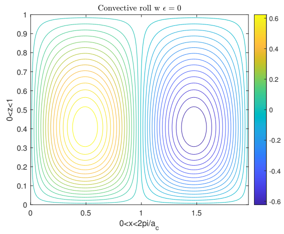


(f): $h(z) = -z^2$

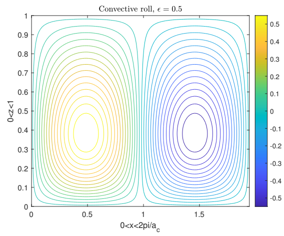


(g): $h(z) = -(e^z - 1)$

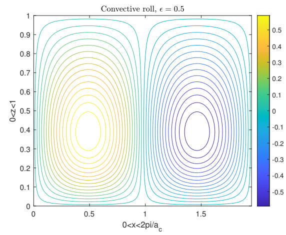
Figure 4.5: Convective rolls for quoted value of ϵ and for $T_U = 8$.



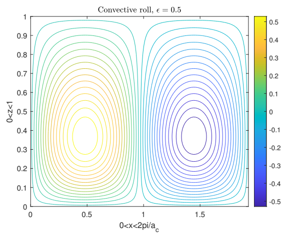
(a): constant gravity



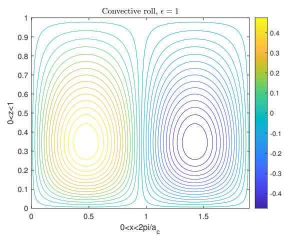
(b): $h(z) = -z$



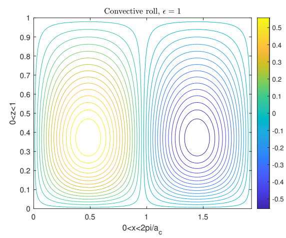
(c): $h(z) = -z^2$



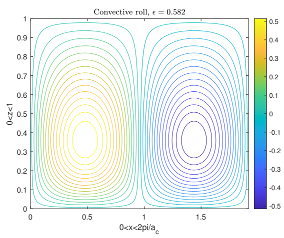
(d): $h(z) = -(e^z - 1)$



(e): $h(z) = -z$



(f): $h(z) = -z^2$



(g): $h(z) = -(e^z - 1)$

Figure 4.6: Convective rolls for quoted value of ϵ and for $T_U = 4$.

5 THE ONSET OF PENETRATIVE CONVECTION IN AN INCLINED POROUS LAYER

Among all physical setups, the inclination of the layer plays a relevant role in the thermal convection of fluid-saturated porous medium, and it applies in many environmental circumstances. In this regard, it has been noticed that thermal convection has a role in the diffusion of pollutants in the underground [68, 137] or in land deformation involving thermal gradient, see [46, and references therein]. Because of the aforementioned applications both experimental [72, 71] and theoretical studies [106, 21, 19, 45, 118, 44] have been conducted in recent years. The novelty of the present research, see [15] written in collaboration with Prof. F. Capone, Dott. G. Cantini e Prof. M. Carnevale, is the development of a theoretical investigation of penetrative convection in a fluid-saturated inclined porous medium. The authors addressed the present study with the aim of a deep understanding of the combined mechanism of penetrative convection and inclination of the porous layer, which gives an explanation of several phenomena like patterned ground formations [53] and thawing subsea permafrost [66]. We will study the linear instability of the conduction solution for longitudinal (also called streamwise), transverse (also called spanwise), and full three-dimensional perturbations and the nonlinear stability analysis with respect to the longitudinal ones. In particular, the numerical results for the linear instability, obtained via the Chebyshev-τ method, show that the most destabilizing perturbations are the longitudinal ones and, as expected, the transverse ones exhibit a peculiar phenomenon: they destabilize only up to a certain critical inclination angle of the layer. This means, in other words, that above this critical angle, there will be a preferred orientation for the perturbations at the onset of the secondary flow. Indeed, from the numerical analysis of the three-dimensional perturbations, we show that the spanwise ones are the most destabilizing not only with respect to the streamwise but also with respect to any general roll perturbation. Hence, this numerically shows the validity of a Squire-like theorem, for details see [117, 41].

Summing up, the chapter is organized as follows. In Section 5.1 the mathematical model is introduced and the Darcy-Rayleigh number is defined in the non-dimensional framework. Then, the steady-state solution is computed and the perturbed non-dimensional system is derived. Section 5.2 deals with the linear instability analysis. In particular, the longitudinal and transverse perturbations cases are analyzed separately. Moreover, for the former case, the *principle of exchange of stabilities* is proved and the linear critical Rayleigh numbers, for the onset of steady convection are found to be the same as those for the horizontal layer up to a scaling factor. In Section 5.3 we give some nonlinear stability results of the longitudinal perturbations. More specifically, by introducing a suitable Lyapunov functional, we find the non-linear critical Rayleigh numbers be solving the Euler-Lagrange equations arising from a maximum problem. In Section 5.4, the employed numerical method (Chebyshev-τ) is explained, and the chapter ends with a concluding Section 5.5 in which all the numerical results are shown and commented on and the obtained results are summarized.

5.1 PROBLEM FRAMING

5.1.1 *Mathematical model*

Let $Oxyz$ be a Cartesian reference frame with fundamental unit vector $\mathbf{i}, \mathbf{j}, \mathbf{k}$ (the latter pointing vertically upwards) and let L be an isotropic porous medium with thickness $d > 0$, inclined of an angle $\varphi \in \left[0, \frac{\pi}{2}\right]$ with respect to the horizontal plane. The layer L is saturated by a homogeneous Newtonian fluid confined between two parallel impermeable planes kept at uniform and constant temperatures T_L and T_U. In this chapter, the above physical setup is considered. We assume that Darcy's law models the momentum balance equation and we adopt the Boussinesq approximation scheme and the Veronis density law (1.63), namely:

$$\frac{\mu}{k}\mathbf{v} = -\nabla p - g\rho(T)\mathbf{r}, \tag{5.1}$$

where $\mathbf{v}$, p and ρ are the seepage velocity, pressure and density respectively, $\mathbf{r} = (\sin\varphi, 0, \cos\varphi)^T$, g is the modulus of gravitational acceleration, μ is the dynamic viscosity and k the permeability of the porous medium. Equation (5.1), together with the mass conservation law and the energy balance equation governing the behavior of the temperature field in local thermal equilibrium, yields the following system of governing equations:

$$\begin{cases} \dfrac{\mu}{k}\mathbf{v} = -\nabla\tilde{p} - 2g\rho_0\alpha T_0 T\mathbf{r} + g\rho_0\alpha T^2\mathbf{r}, \\ \nabla\cdot\mathbf{v} = 0, \\ \dfrac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T = \kappa\Delta T, \end{cases} \tag{5.2}$$

where κ is the thermal diffusivity and

$$\tilde{p} = p + g\rho_0(1 - \alpha T_0^2)(x\sin\varphi + z\cos\varphi). \tag{5.3}$$

For the sake of brevity, but without loss of generality, we omit the tilde on the pressure function p. We complete system (5.2) with the following boundary conditions:

$$T(x, y, 0, t) = T_L, \quad T(x, y, d, t) = T_U, \tag{5.4}$$

and

$$\mathbf{v}\cdot\mathbf{n} = 0 \quad \text{on} \quad z = 0, d, \tag{5.5}$$

where $\mathbf{n}$ is the unit outward normal to the planes bounding the layer. In particular, for the problem under examination, we assume:

$$T_L = 0°C, \quad T_0 = 4°C, \quad T_U \geq 4°C. \tag{5.6}$$

5.1.2 *Steady state solution and perturbation equations*

In order to rewrite the system (5.2) in non-dimensional form let us introduce the following non-dimensional parameters:

$$\mathbf{x} = d\mathbf{x}^*, \quad t = \tau t^*, \quad \mathbf{v} = V\mathbf{v}^*, \quad p = Pp^*, \quad T = T_U T^*, \tag{5.7}$$

with

$$\tau = \frac{d^2}{\kappa}, \quad V = \frac{\kappa}{d}, \quad P = \frac{\mu\kappa}{k}. \tag{5.8}$$

Then the resulting non-dimensional equations of motion, omitting all the asterisks, are the following:

$$\begin{cases} \mathbf{v} = -\nabla p - \mathsf{Ra}\left(\zeta T - \dfrac{T^2}{2}\right)\mathbf{r}, \\ \nabla\cdot\mathbf{v} = 0, \\ \dfrac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T = \Delta T, \end{cases} \tag{5.9}$$

where the dimensionless parameter Ra (the thermal Darcy-Rayleigh number) and ζ are respectively given by:

$$\mathsf{Ra} = \frac{2g\rho_0\alpha k d T_U^2}{\mu\kappa}, \qquad \zeta = \frac{T_0}{T_U}. \tag{5.10}$$

Note that the boundary conditions on the temperature, in the non-dimensional framework, are the following:

$$T(x,y,0,t) = 0, \quad T(x,y,1,t) = 1. \tag{5.11}$$

We now seek stationary and laminar basic solutions

$$m_b = (\mathbf{v}_b, p_b, T_b), \tag{5.12}$$

with $\mathbf{v}_b = (v_b(z), 0, 0)$, whose instability and stability are going to be the core of the present investigations. From $(5.9)_3$ one easily finds:

$$T_b(z) = z. \tag{5.13}$$

By substituting (5.13) in $(5.9)_1$ we obtain

$$\begin{aligned} v_b(z) &= -\frac{\partial p_b}{\partial x} - \mathsf{Ra}\left(\zeta z - \frac{z^2}{2}\right)\sin\varphi, \\ 0 &= -\frac{\partial p_b}{\partial y}, \\ 0 &= -\frac{\partial p_b}{\partial z} - \mathsf{Ra}\left(\zeta z - \frac{z^2}{2}\right)\cos\varphi. \end{aligned} \tag{5.14}$$

Moreover, from equations $(5.14)_{2,3}$ it follows that

$$p_b = p_b(x,z) = g(x) - \mathsf{Ra}\left(\zeta\frac{z^2}{2} - \frac{z^3}{6}\right)\cos\varphi, \tag{5.15}$$

and substituting in $(5.14)_1$ we have

$$v_b(z) + \mathsf{Ra}\left(\zeta z - \frac{z^2}{2}\right)\sin\varphi = -\frac{d}{dx}g(x). \tag{5.16}$$

The previous ordinary differential equation must be satisfied identically in $\mathbb{R}\times[0,1]$. As a consequence, that there exist a constant η, a pressure gradient along x and a constant c, see [45], such that

$$g(x) = \eta x + c. \tag{5.17}$$

Hence

$$v_b(z) + \mathsf{Ra}\left(\zeta z - \frac{z^2}{2}\right)\sin\varphi = -\eta. \tag{5.18}$$

For simplicity we set $\eta = 0$, and, recalling (5.3), we have

$$p_b(x,z) = C - \mathsf{Ra}\left(\zeta\frac{z^2}{2} - \frac{z^3}{6}\right)\cos\varphi - A(1-\alpha T_0^2)(x\sin\varphi + z\cos\varphi), \tag{5.19}$$

with $A = \frac{g\rho_0 dk}{\mu\kappa}$ a non-dimensional constant. Therefore, the complete steady state solution is fully computed

$$\begin{aligned} m_b = \Bigg(\Big(&-\mathsf{Ra}\Big(\zeta z - \frac{z^2}{2}\Big)\sin\varphi, 0, 0\Big), \\ &c - \mathsf{Ra}\Big(\zeta\frac{z^2}{2} - \frac{z^3}{6}\Big) - A(1-\alpha T_0^2)(x\sin\varphi + z\cos\varphi), z\Bigg). \end{aligned} \tag{5.20}$$

Note that if $\varphi = 0$, m_b coincides with the steady state motion for penetrative convection in a horizontal layer found in [2].

In order to study the stability of m_b, we introduce the following perturbation fields

$$\mathbf{v} = \mathbf{v}_b + \mathbf{u}, \quad p = p_b + \pi, \quad T = T_b + \theta, \tag{5.21}$$

with $\mathbf{u} = (u, v, w)$. Eventually, the resulting non-dimensional perturbation equations are

$$\begin{cases} \mathbf{u} = -\nabla\pi - \mathsf{Ra}M(z)\theta\mathbf{r} + \dfrac{\mathsf{Ra}}{2}\theta^2\mathbf{r}, \\ \nabla\cdot\mathbf{u} = 0, \\ \dfrac{\partial\theta}{\partial t} + v_b(z)\theta_x + w + \mathbf{u}\cdot\nabla\theta = \Delta\theta, \end{cases} \tag{5.22}$$

with $M(z) = \zeta - z$, together with initial conditions

$$\mathbf{u}(\mathbf{x}, t_0) = \mathbf{u}_0(\mathbf{x}), \quad \pi(\mathbf{x}, t_0) = \pi_0(\mathbf{x}), \quad \theta(\mathbf{x}, t_0) = \theta_0(\mathbf{x}) \tag{5.23}$$

and boundary conditions

$$u = v = w = \theta = 0 \quad \text{on} \quad z = 0, 1. \tag{5.24}$$

In the sequel, we will suppose that the $\mathbf{u}$, π and θ are periodic functions in x and y direction, of period $\frac{2\pi}{a_x}$ and $\frac{2\pi}{a_y}$ respectively, and denote the periodicity cell by

$$V = \left[0, \frac{2\pi}{a_x}\right] \times \left[0, \frac{2\pi}{a_y}\right] \times [0, 1]\,. \tag{5.25}$$

5.2 INSTABILITY ANALYSIS

In order to investigate the linear instability of (5.20) let us consider the linear version of system (5.22)

$$\begin{cases} \mathbf{u} = -\nabla \pi - \mathrm{Ra} M(z) \theta \mathbf{r}, \\ \nabla \cdot \mathbf{u} = 0, \\ \frac{\partial \theta}{\partial t} + v_b(z)\theta_x + w = \Delta \theta. \end{cases} \quad (5.26)$$

By taking the third component of the double curl of $(5.26)_1$, one easily obtains

$$\begin{cases} \Delta w = \mathrm{Ra} \Big[-\sin\varphi \theta_x + M(z) \sin\varphi \theta_{xz} - \cos\varphi M(z) \Delta_1 \theta \Big], \\ \dfrac{\partial \theta}{\partial t} + v_b(z)\theta_x + w = \Delta \theta. \end{cases} \quad (5.27)$$

Now by virtue of the periodicity and the fact that the system is linear and autonomous, we are allowed to seek (normal mode) solutions of the form

$$w(\mathbf{x}, t) = e^{i\left(a_x x + a_y y\right)} \widetilde{W}(z) e^{\sigma t} \text{ and } \theta(\mathbf{x}, t) = e^{i\left(a_x x + a_y y\right)} \widetilde{\Theta}(z) e^{\sigma t}, \quad (5.28)$$

with $\sigma \in \mathbb{C}$. By virtue of (5.28), (5.27) becomes

$$\begin{cases} (D^2 - a^2)\widetilde{W} = \mathrm{Ra} \left[-\sin\varphi i a_x \widetilde{\Theta} + (\zeta - z) \sin\varphi i a_x D\widetilde{\Theta} + (\zeta - z) \cos\varphi a^2 \widetilde{\Theta} \right], \\ (D^2 - a^2 - \sigma)\widetilde{\Theta} - \widetilde{W} = -\mathrm{Ra} \left(\zeta z - \dfrac{z^2}{2} \right) \sin\varphi i a_x \widetilde{\Theta}, \end{cases} \quad (5.29)$$

where $D = \frac{d}{dz}$, together with boundary conditions $\widetilde{W} = \widetilde{\Theta} = 0$, on $z = 0, 1$.

5.2.1 *Analysis of the longitudinal perturbations*

If we assume that the perturbations are longitudinal this means that they do not depend on x and from (10.110) we obtain

$$\begin{cases} (D^2 - a_y^2)\widetilde{W} = \mathrm{Ra}(\zeta - z) \cos\varphi a_y^2 \widetilde{\Theta}, \\ (D^2 - a_y^2 - \sigma)\widetilde{\Theta} - \widetilde{W} = 0. \end{cases} \quad (5.30)$$

It is not difficult to show that σ is a real number. Indeed, from $(5.30)_2$, one has

$$(D^2 - a_y^2)^2 \widetilde{\Theta} - \sigma (D^2 - a_y^2) \widetilde{\Theta} - (D^2 - a_y^2) \widetilde{W} = 0, \quad (5.31)$$

then, from $(5.30)_1$, one gets

$$(D^2 - a_y^2)^2\widetilde{\Theta} - \sigma(D^2 - a_y^2)\widetilde{\Theta} - \mathsf{Ra}(\zeta - z)\cos\varphi a_y^2\widetilde{\Theta} = 0. \tag{5.32}$$

Multiplying (5.32) by the complex conjugate of $\widetilde{\Theta}$ and integrating on the periodicity cell V, one obtains

$$\sigma\left(\int_0^1 (D\widetilde{\Theta})^2 dz + a_y^2\int_0^1 \widetilde{\Theta}^2 dz\right) = -\int_0^1 (D^2\widetilde{\Theta})^2 dz \\ - 2a_y^2\int_0^1 (D\widetilde{\Theta})^2 dz - a_y^4\int_0^1 \widetilde{\Theta}^2 dz + \mathsf{Ra}\cos\varphi a_y^2\int_0^1 (\zeta - z)\widetilde{\Theta}^2 dz. \tag{5.33}$$

This shows that $\sigma \in \mathbb{R}$ and the principle of exchange of stability holds, i.e. convection can occur only via a stationary motion. As a consequence, we can set $\sigma = 0$ in (5.30)

$$\begin{cases} (D^2 - a_y^2)\widetilde{W} = \mathsf{Ra}(\zeta - z)\cos\varphi a_y^2\widetilde{\Theta}, \\ (D^2 - a_y^2)\widetilde{\Theta} - \widetilde{W} = 0, \end{cases} \tag{5.34}$$

in order to find the critical linear Rayleigh number for the onset of stationary convective motions with respect to the longitudinal perturbations, for any fixed inclination angle and upper layer temperature:

$$\mathsf{Ra}_L^{\mathcal{L}} = \min_{a_y^2 \in \mathbb{R}^+} \mathsf{Ra}(a_y^2), \tag{5.35}$$

where the superscript $\mathcal{L}$ stays for Longitudinal. Moreover, it turns out that:

$$\mathsf{Ra}_L^{\mathcal{L}}(\varphi) = \frac{\mathsf{Ra}_L(0)}{\cos\varphi}, \tag{5.36}$$

with $\mathsf{Ra}_L(0)$ the critical linear Rayleigh number in the case of a horizontal layer $\varphi = 0$, see [2].

5.2.2 *Analysis of the transverse perturbations*

The system of transverse perturbations, i.e. independent on y, is introduced in this Section. Setting $a_y = 0$ in (10.110) one obtains:

$$\begin{cases} (D^2 - a_x^2)\widetilde{W} = \mathsf{Ra}\left[-\sin\varphi i a_x\widetilde{\Theta} + (\zeta - z)\sin\varphi i a_x D\widetilde{\Theta} + (\zeta - z)\cos\varphi a_x^2\widetilde{\Theta}\right], \\ (D^2 - a_x^2 - \sigma)\widetilde{\Theta} - \widetilde{W} = -\mathsf{Ra}\left(\zeta z - \dfrac{z^2}{2}\right)\sin\varphi i a_x\widetilde{\Theta}. \end{cases} \tag{5.37}$$

It is worth observing that system (5.37) constitutes an eigenvalue system of ordinary differential equations

$$A\mathbf{x} = \sigma B\mathbf{x}, \tag{5.38}$$

with

$$A = \begin{pmatrix} D^2 - a_x^2 & \begin{matrix} \mathsf{Ra}\sin\varphi i a_x - \mathsf{Ra}(\zeta - z)\sin\varphi i a_x D \\ -\mathsf{Ra}(\zeta - z)\cos\varphi a_x^2 \end{matrix} \\ -I & D^2 - a_x^2 + \mathsf{Ra}\left(\zeta z - \frac{z^2}{2}\right)\sin\varphi i a_x \end{pmatrix} \tag{5.39}$$

and

$$B = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}. \tag{5.40}$$

The determination of the critical linear Rayleigh numbers with respect to the transverse perturbations, $\mathsf{Ra}_L^{\mathcal{T}}$ (where the superscript $\mathcal{T}$ stays for Transverse), and the related stability results are delegated to Section 5.5. In particular, the numerical method employed for the resolution of the generalized eigenvalue problem (5.38) is described in Section 5.4.

5.3 NONLINEAR FRAMEWORK

In this Section, we perform the nonlinear stability analysis by applying the weighted energy method, see [48], to the system for nonlinear longitudinal perturbations in order to obtain the critical energy Rayleigh numbers from a maximum problem. Let us consider the system for nonlinear longitudinal perturbations

$$\begin{cases} u = -\mathsf{Ra}M(z)\theta\sin\varphi + \dfrac{\mathsf{Ra}}{2}\theta^2\sin\varphi \\ v = -\pi_y \\ w = -\pi_z - \mathsf{Ra}M(z)\theta\cos\varphi + \dfrac{\mathsf{Ra}}{2}\theta^2\cos\varphi \\ v_y + w_z = 0 \\ \theta_t + w + v\theta_y + w\theta_z = \Delta\theta \end{cases} \tag{5.41}$$

in particular we can reduce to study subsystem $(5.41)_{2-5}$, see [45]. Multiplying $(5.41)_2$ by v, $(5.41)_3$ by w, integrate each over V and adding the result, one obtains

$$\|w\|^2 = -\mathsf{Ra}\cos\varphi\,\langle M(z)\theta, w\rangle + \frac{\mathsf{Ra}\cos\varphi}{2}\left\langle w, \theta^2\right\rangle. \tag{5.42}$$

and similarly, multiplying $(5.41)_5$ by $g(z)\theta$ (setting $g(z) = \mathrm{Ra}\cos\varphi(\mu - z)$ in order to get rid of the cubic nonlinear terms, with $\mu > 1$ being a parameter) and integrating each, one obtains, adding the result with (5.42):

$$\frac{\mathrm{Ra}\cos\varphi}{2}\frac{d}{dt}\left\langle \mu - z, \theta^2 \right\rangle = -\|w\|^2 - \mathrm{Ra}\cos\varphi\,\langle(\mu + \zeta - 2z)\theta, w\rangle - \mathrm{Ra}\cos\varphi\left\langle \mu - z, |\nabla\theta|^2 \right\rangle. \tag{5.43}$$

Let us then consider the following weighted Lyapunov functional:

$$E(t) = \frac{\mathrm{Ra}\cos\varphi}{2}\left\langle \mu - z, \theta^2 \right\rangle \tag{5.44}$$

and set

$$\begin{aligned} I(t) &= -\mathrm{Ra}\cos\varphi\,\langle(\mu + \zeta - 2z)\theta, w\rangle, \\ D(t) &= \|w\|^2 + \mathrm{Ra}\cos\varphi\left\langle \mu - z, |\nabla\theta|^2 \right\rangle. \end{aligned} \tag{5.45}$$

Therefore, from (5.43) it follows that

$$\frac{dE}{dt} = I - D \le D(m-1), \tag{5.46}$$

where

$$m = \max_{\mathcal{H}} \frac{I}{D}, \tag{5.47}$$

and $\mathcal{H}$ is the space of kinematically admissible perturbations, namely

$$\mathcal{H} = \{(w,\theta) \in [W^{1,2}(V)]^2 \mid w = \theta = 0 \text{ on } z = 0,1,\ w_z = 0, \text{ periodic in } y \text{ direction with period } 2\pi/a_y\}. \tag{5.48}$$

From Poincaré and weighted Poincaré inequalities, one obtains, see [48]

$$\frac{1}{2}\left\langle \mu - z, \theta^2 \right\rangle \le \xi \left\langle \mu - z, |\nabla\theta|^2 \right\rangle, \tag{5.49}$$

where $\xi = \max\left\{\frac{c_p}{2}, 2\right\}$, and $c_p = c_p(V)$ is the Poincaré constant and, as a consequence, $D \ge \xi^{-1}E$. Hence, if $m < 1$ from (5.46) it follows

$$\frac{dE}{dt} \le D(m-1) \le \xi^{-1}(m-1)E, \tag{5.50}$$

i.e.

$$E(t) \le E(0)\exp\left(\xi^{-1}(m-1)t\right). \tag{5.51}$$

The energy estimate (5.51) proves that, provided $m < 1$, E decreases at least exponentially to zero as time goes to infinity. Finally, by using the generalized Cauchy-Schwartz and triangular inequalities we have

$$\left(1 - \frac{1}{2\varepsilon_1} - \frac{\varepsilon_2}{2}\right) \|w\|^2 \leq \mathsf{Ra}^2 \cos^2\varphi \left(\frac{\varepsilon_1}{2}\|\theta\|^2 + \frac{1}{8\varepsilon_2}\|\theta^2\|^2\right), \quad (5.52)$$

and setting $\varepsilon_1 = 2$ and $\varepsilon_2 = \frac{1}{2}$, we get

$$\frac{1}{2}\|w\|^2 \leq \mathsf{Ra}^2 \cos^2\varphi \left(\|\theta\|^2 + \frac{1}{4}\|\theta^2\|^2\right). \quad (5.53)$$

Estimate (5.53) implies that the condition $m < 1$ guarantees the exponential decay of the third component of the perturbed seepage velocity and the global nonlinear stability of the conduction solution with respect to the E-norm (5.44) is provided.

Concerning the variational problem (5.47) the associated Euler-Lagrange equations are the following

$$\begin{cases} 2mw\mathbf{k} + \mathsf{Ra}\cos\varphi(\mu + \zeta - 2z)\theta\mathbf{k} = \nabla\varpi \\ -(\mu + \zeta - 2x)w - m\theta_z + m(\mu - z)\Delta\theta = 0 \end{cases} \quad (5.54)$$

where ϖ is a Lagrange multiplier. Let us remark that the nonlinear stability condition $m < 1$ is equivalent to the condition $\mathsf{Ra} < \mathsf{Ra}_E$, where Ra_E is the critical nonlinear Rayleigh number. Therefore, the criticality is reached when $\mathsf{Ra} = \mathsf{Ra}_E$ in system (5.54) in correspondence of $m = 1$. Therefore, taking the third component of the double curl of $(5.54)_1$ and employing normal modes representation in (5.54), we obtain:

$$\begin{cases} (D^2 - a_y^2)W - \mathsf{Ra}\cos\varphi\left(\frac{\mu}{2} + \frac{\zeta}{2} - z\right)a_y^2\Theta = 0 \\ \left(\frac{\mu}{2} + \frac{\zeta}{2} - z\right)W + D\Theta - (\mu - z)(D^2 - a_y^2)\Theta = 0 \end{cases} \quad (5.55)$$

together with boundary conditions $W = \Theta = 0$, on $z = 0, 1$. System (5.55) is a fourth-order generalized eigenvalue problem for the critical Rayleigh number Ra_E, which is given by

$$\mathsf{Ra}_E^{\mathcal{L}} = \max_{\mu > 1} \min_{a^2 \in \mathbb{R}^+} \mathsf{Ra}(a_y^2, \mu) \quad (5.56)$$

Moreover, it turns out that:

$$\mathsf{Ra}_E^{\mathcal{L}}(\varphi) = \frac{\mathsf{Ra}_E(0)}{\cos\varphi}, \quad (5.57)$$

with $\mathsf{Ra}_E(0)$ the critical non-linear Rayleigh number in the case of a horizontal layer $\varphi = 0$, see [2].

5.4 NUMERICAL TECHNIQUE

In this section the eigenvalue system (5.38)-(5.40) is solved using the Chebyschev-τ method developed by Dongarra et al in [40]. In order to ensure a deeper comprehensibility and repeatability of the employed method, let us start by considering the following generalized eigenvalue problem with homogeneous boundary conditions

$$\begin{cases} Lu(x) = \sigma M u(x) & x \in [-1, 1] \\ B_1 u(1) = 0 & C_1 u(-1) = 0 \\ \vdots & \vdots \\ B_{\frac{\Gamma}{2}} u(1) = 0 & C_{\frac{\Gamma}{2}} u(-1) = 0 \end{cases} \tag{5.58}$$

where L and M are two arbitrary differential operators of order Γ, and B_i, C_i are the operators defining the set of boundary conditions at $x = 1$ and $x = -1$, respectively.
On multiplying the equation and the boundary conditions in (5.58) by the i-th ($i = 0, \ldots, N$) Chebyschev polynomial under the usual product $\int_{-1}^{1} f(x)g(x)\frac{dx}{\sqrt{1-x^2}}$, one obtains:

$$\sum_{k=0}^{N}\sum_{j=0}^{N+\Gamma} L_{kj}u_j = \sigma \sum_{k=0}^{N}\sum_{j=0}^{N+\Gamma} M_{kj}u_j, \tag{5.59}$$

$$\begin{aligned} Bu(1) = 0 \quad &\Rightarrow \sum_{k=0}^{N+\Gamma}\sum_{j=0}^{N+\Gamma} B_{kj}u_j = 0, \\ Cu(-1) = 0 \quad &\Rightarrow \sum_{k=0}^{N+\Gamma}\sum_{j=0}^{N+\Gamma} (-1)^k C_{kj}u_j = 0, \end{aligned} \tag{5.60}$$

Therefore, the system can be written as

$$\begin{pmatrix} L \\ \mathbf{b}_i^T \\ \mathbf{c}_i^T \end{pmatrix} \mathbf{u} = \sigma \begin{pmatrix} M \\ \mathbf{0}_i^T \\ \mathbf{0}_i^T \end{pmatrix} \mathbf{u}. \tag{5.61}$$

Since the boundary conditions establish a relation between the last Γ coefficients and the others, the last Γ rows of the left-side matrix can be used to erase the last Γ columns of the first $N + 1$ row of the matrices belonging to the right and left side of the equation. The following form is now obtained

$$\begin{pmatrix} L' & 0 & \dots & 0 \\ & \mathbf{b}_i^T & & \\ & \mathbf{c}_i^T & & \end{pmatrix} \mathbf{u} = \sigma \begin{pmatrix} M' & 0 & \dots & 0 \\ & \mathbf{0}_i^T & & \\ & \mathbf{0}_i^T & & \end{pmatrix} \mathbf{u}, \tag{5.62}$$

where L' and M' are the operator matrices after this reduction. The equation can be now limited to the first $N+1$ coefficient

$$L'\mathbf{u}' = \sigma M'\mathbf{u}', \tag{5.63}$$

where $\mathbf{u}'$ is the vector containing the first $N+1$ expansion coefficients. Since the matrix M' is not singular, the problem can be reduced to a standard eigenvalue problem and it can be solved with standard methods.

In order to apply the above method to the eigenvalue problem (5.38)-(5.40), the equations and the boundary conditions need to be changed into a suitable framework. With this aim, applying the $\left(D^2 - a^2\right)$ operator to $(10.110)_2$ we obtain

$$\begin{aligned} \left(D^2 - a^2\right)\widetilde{W} = {} & \left(D^2 - a^2\right)^2 \widetilde{\Theta} - \sigma \left(D^2 - a^2\right) \widetilde{\Theta} \\ & + \mathsf{Ra} \sin\varphi i a_x \left[\left(\zeta z - \frac{z^2}{2}\right) \left(D^2 - a^2\right) + 2\left(\zeta - z\right) D - 1 \right] \widetilde{\Theta}, \end{aligned} \tag{5.64}$$

and, substituting it into $(10.110)_1$, we get

$$\begin{aligned} \Big\{ \left(D^2 - a^2\right)^2 + & \mathsf{Ra} \sin\varphi i a_x \Big[\left(\zeta z - \frac{z^2}{2}\right) \left(D^2 - a^2\right) \\ & + \left(\zeta - z\right) D \Big] - \mathsf{Ra} \cos\varphi a^2 \Big\} \widetilde{\Theta} = \sigma \left(D^2 - a^2\right) \widetilde{\Theta}. \end{aligned} \tag{5.65}$$

Let us notice that, in order to change the boundaries from $[0,1]$ to $[-1,1]$ the change of variable $z = \frac{\hat{z}}{2} + \frac{1}{2}$ is performed. Then, defining $\widehat{\mathsf{Ra}} = \frac{\mathsf{Ra}}{4}$, $\hat{a}_x = \frac{a_x}{2}$, $\hat{a}_y = \frac{a_y}{2}$, $\widehat{D} = \frac{d}{d\hat{z}}$, we have

$$\begin{cases} L\widetilde{\Theta} = \sigma \left(\widehat{D}^2 - \hat{a}^2\right) \widetilde{\Theta} \\ \widetilde{\Theta}(z) = 0, \quad z = \pm 1, \\ \widehat{D}^2 \widetilde{\Theta}(z) = 0, \quad z = \pm 1, \end{cases} \tag{5.66}$$

with

$$L = \left(\widehat{D}^2 - \widehat{a}^2\right) + \widehat{\mathsf{Ra}} \sin\varphi i \widehat{a_x} \Big[P_2(\widehat{z}) \left(\widehat{D}^2 - \widehat{a}^2\right) + P_1(\widehat{z}) \widehat{D} \Big] - \widehat{\mathsf{Ra}} \cos\varphi \widehat{a}^2, \tag{5.67}$$

where we set

$$P_1(\widehat{z}) = \zeta(\widehat{z}+1) - \frac{(\widehat{z}+1)^2}{4}, \quad P_2(\widehat{z}) = \zeta(\widehat{z}+1) - \frac{\widehat{z}+1}{2}. \tag{5.68}$$

Since the problem has been reduced to a form like (5.58) the method can be employed in order to find the eigenvalues through Shur's decomposition. Indeed, in the spectrum of the found eigenvalues the one with the largest real part is selected, and, for every fixed wave number, the Rayleigh number is varied until its real part changes its sign. The range of the Rayleigh number and wave number has been chosen for the purpose of individuating the locus of the minimum of the marginal stability curve. This sign changing from negative(positive) to positive(negative) marks the transition from stability(instability) to instability(stability). The calculation has been then repeated for different inclination angles until none of this kind of point is found. This procedure has been repeated for several temperature values of the upper layer, $T_U = 4, 6, 8$ $^\circ C$ as long as we want to investigate penetrative convection in the parabolic neighborhood of the maximum density value. The linear critical Rayleigh numbers with respect to the transverse perturbations are reported in Tabel 5.1.

In conclusion, let us report that a preliminary study of the method's performance has been conducted. First of all, a sensitivity analysis has been conducted to investigate the behavior of the solver on the employed number of polynomials. Moreover, simulations with different numbers of polynomials have been launched calculating the eigenvalues of the investigated equation for the following parameters:

$$T_U = 4^\circ C, \quad a_x = 2, \quad a_y = 0, \quad \mathsf{Ra} = 400, \quad \phi = 30^\circ$$

The results are shown in Figure 5.1: the values of both imaginary and real part of the calculated most unstable eigenvalues are stable for a number of polynomials between 8 and 120. This kind of analysis is the same conducted by Orszag [93], showing the convergence of the Chebyschev τ-method for the penetrative convection problem discussed above. With a larger number of polynomials, the results are strongly unstable. This is due to the truncation error in the derivative matrices as stressed by Dongarra et al. [40]. From this analysis, the choice of 16 polynomials to perform the calculations seemed reasonable, ensuring desired precision and avoiding truncation errors and time-consuming simulations.

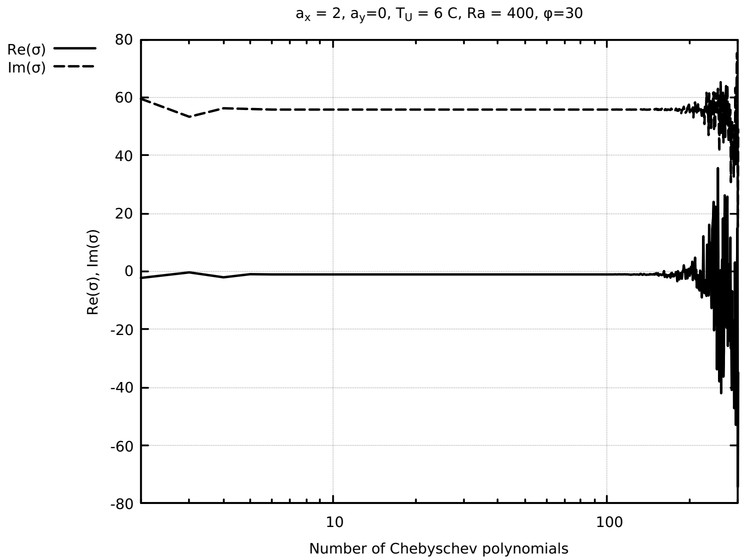


Figure 5.1: Sensitivity of eigenvalue calculations on number of polynomials.

5.5 NUMERICAL RESULTS AND CONCLUSIONS

In this section, we present the results from the numerical solution of the generalized eigenvalue problems (5.34), (5.37) and (5.55), arising from the linear analysis of longitudinal and transverse perturbations and the non-linear analysis of the longitudinal perturbations, respectively. To evaluate the effect of the layer inclination and of the upper-layer temperature on the onset of convective motions, we performed numerical simulations for quoted values of φ and T_U, respectively.
Concerning the linear analysis of the transverse perturbations, in Figure 5.2 we can observe the marginal stability curves for quoted values of the layer inclination and the upper-layer temperature, and the corresponding critical Rayleigh numbers are reported in Table 5.1. These curves are obtained by selecting, for a fixed wave number, the linear

Rayleigh number at which the real part of the eigenvalue changes its sign. Figure 5.2 shows clearly the stabilizing effect of inclination on the onset of penetrative convection. Moreover, regarding the upper-layer temperature, we can observe that:

- since $\zeta = \frac{T_0}{T_U}$, with $T_0 = 4°C$, is inversely proportional to the temperature of the upper layer, as the temperature of the upper layer increases, given a fixed inclination, instabilities arise at a higher Rayleigh number, showing the stabilizing effect of the upper layer temperature;

- on the other hand, higher temperatures allows instability to arise at higher inclination.

The latter behavior can be seen in a clearer way in Figure 5.3 (a), where the critical Rayleigh numbers are plotted against the inclination at different temperatures. The continuous lines represent the critical Rayleigh numbers for longitudinal perturbations while the dashed ones represent the transverse ones. It can be noticed that the transversal perturbations are more stable than the longitudinal ones for *all* the inclinations. Moreover, temperature increases shift to higher values of the critical angle as it is shown in Table 5.2. Let us remark that, as it is expected, in the case of a horizontal layer, i.e. $\varphi = 0$, the found critical thresholds coincides with the ones found in [2].
Moreover, from the analysis of the three-dimensional perturbations, see e.g. Figure 5.4, we saw that the longitudinal perturbations are the most destabilizing not only with respect to the transversal perturbations but also with respect to any rolls in the plane (a_x, a_y), proving a Squire-like theorem for the problem under examination. Lastly, nonlinear analysis of longitudinal perturbations has been performed with the weighted energy method and in Figure 5.3(b) the critical thresholds are plotted against inclinations for quoted values of upper layer temperature, comparing them for the corresponding threshold obtained through the linear analysis.
In summary, our results show that:

- in the limit case $\varphi \to 0$, i.e. horizontal layers, the instability thresholds coincide with the ones found in [2], [125] and [61];

- the inclination of the layer has a stabilizing effect on the onset of convection;

- the most destabilizing perturbations are the longitudinal ones;

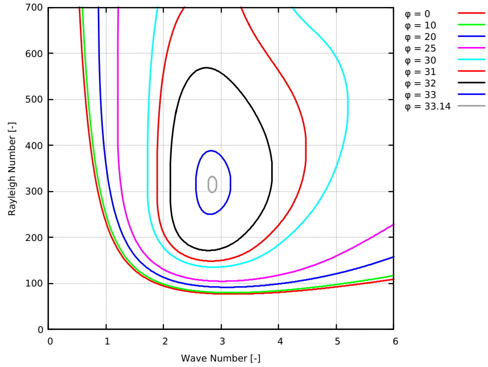
$\zeta = 1\,(T_U = 4)$			$\zeta \simeq 0.6\,(T_U = 6)$			$\zeta = 0.5\,(T_U = 8)$			
$\mathrm{Ra}_E^{\mathcal{L}}$	$\mathrm{Ra}_L^{\mathcal{L}}$	$\mathrm{Ra}_L^{\mathcal{T}}$	$\mathrm{Ra}_E^{\mathcal{L}}$	$\mathrm{Ra}_L^{\mathcal{L}}$	$\mathrm{Ra}_L^{\mathcal{T}}$	$\mathrm{Ra}_E^{\mathcal{L}}$	$\mathrm{Ra}_L^{\mathcal{L}}$	$\mathrm{Ra}_L^{\mathcal{T}}$	$\varphi(deg)$
74.219	77.0797	77.0797	154.873	198.031	198.031	248.009	471.483	471.483	0
74.502	77.3741	77.8129	155.464	199.7874	199.617	248.956	473.284	475.651	5
75.364	78.2687	80.1363	157.262	201.0859	205.441	251.835	478.756	488.752	10
76.837	79.7987	84.4853	160.336	205.0167	215.832	256.758	488.115	512.864	15
78.982	82.0265	91.8193	164.812	210.7402	233.725	263.956	501.742	552.685	20
81.891	85.04804	104.945	170.883	218.5030	261.563	273.648	520.224	618.801	25
85.701	89.0039	135.478	178.832	228.6665	317.971	286.376	544.422	742.793	30
90.605	94.0969	*	189.065	241.7512	511.990	302.763	575.574	1115.460	35

Table 5.1: Critical Rayleigh numbers for quoted values of the upper layer temperature and inclination angles.

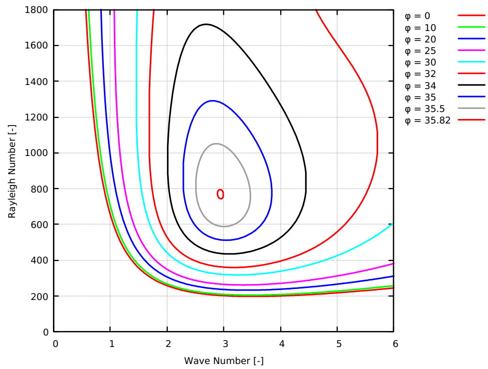
	$\zeta = 1\,(T_U = 4)$	$\zeta \simeq 0.6\,(T_U = 6)$	$\zeta = 0.5\,(T_U = 8)$
φ_{cr}	$33.16°$	$35.84°$	$36.30°$

Table 5.2: Critical values of φ

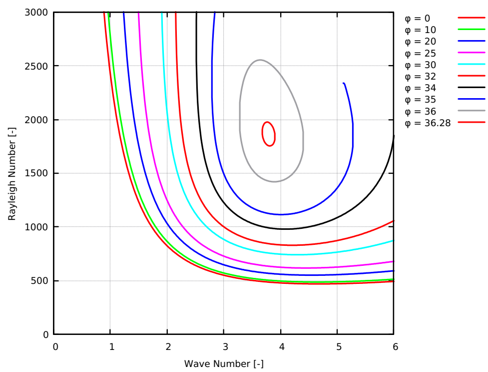
- a Squire's-like theorem is numerically proved, which means that the longitudinal perturbations are the most dangerous with respect to any general roll perturbation;
- the principle of exchange of stabilities has been proved in the case of longitudinal perturbations;
- the transverse perturbations destabilize only up to a certain inclination's angle and the critical angles are found for quoted values of the upper plane temperature;
- the nonlinear stability analysis for the longitudinal perturbations, with the weighted energy method, is performed.



(a): $\zeta = 1\,(T_U = 4°C)$.

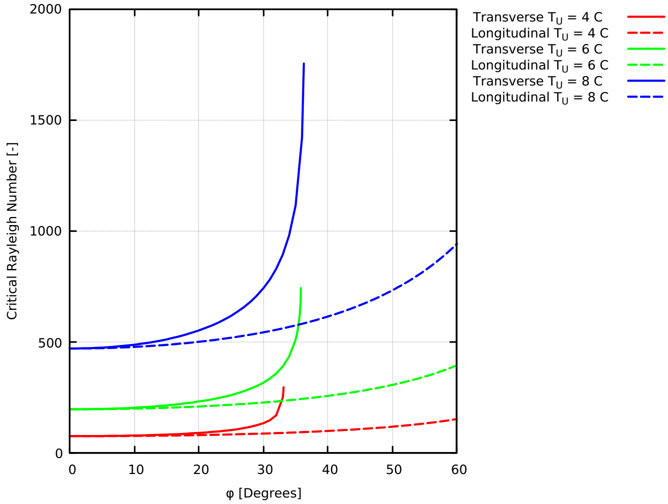


(b): $\zeta \simeq 0.6\,(T_U = 6°C)$.

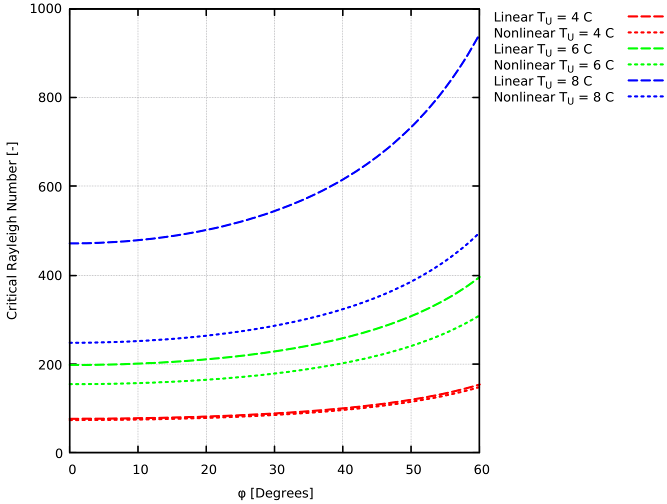


(c): $\zeta = 0.5\,(T_U = 8°C)$.

Figure 5.2: Marginal stability curves for quoted values of the upper layer temperature, related to the linear transverse perturbations.



(a): Critical longitudinal and transverse Rayleigh numbers as functions of the layer's inclination.



(b): Critical linear and non-linear longitudinal Rayleigh numbers as functions of the layer's inclination.

Figure 5.3: Comparison of marginal stability curves.

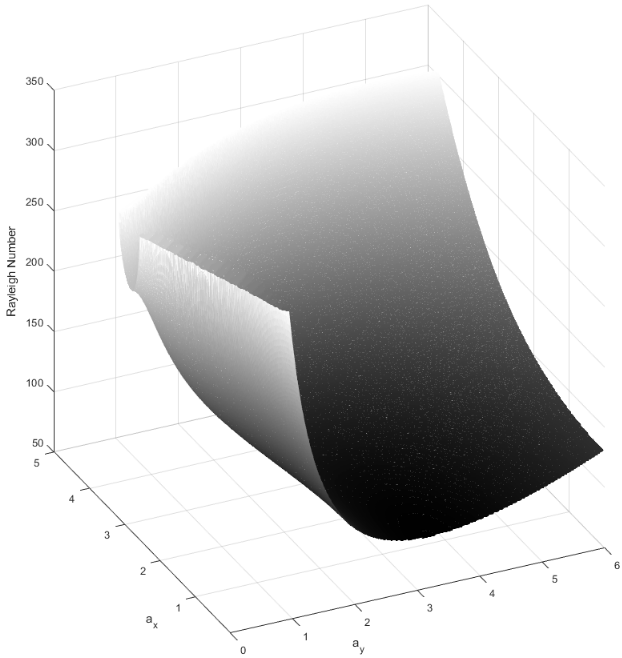


Figure 5.4: Critical surface in correspondence of $\zeta = 1$ ($T_U = 4°C$) and $\varphi = 33°$.

6

CONVECTIVE CURRENTS IN BI-DISPERSE POROUS MEDIA

6.1 DUAL-POROSITY MATERIALS

Regarding dual-porosity materials, bi-disperse porous media have a large number of industrial and geophysical applications (e.g. to model stockpiled pieces of coal, [16]). As defined in [35], a bi-disperse porous medium (BDPM) is characterized by a solid matrix with an interconnected void, but the solid skeleton has cracks or fissures in it, so a BDPM is a compound of clusters of large particles that are themselves agglomerations of smaller particles. Therefore, these materials are characterized by two types of pores and two different porosity: the macropores are the pores between the clusters, while the micropores are the pores within the clusters, so let ϕ be the porosity associated to macropores and ϵ the porosity associated to micropores. Moreover, the macropores are referred to as f-phase, while the remainder of the structure is referred to as p-phase [91]. Through numerical simulations, Imani and Hooman in [67] showed that when the macropores are relatively large compared to the micropores then one may assume the temperatures of the solid skeleton match those of the fluid in the macro and micropores, hence the local thermal equilibrium hypothesis can be assumed. Accounting for the discussion made in [67], in this chapter we focus the attention on a single-temperature bi-disperse porous medium, i.e. $T = T^f = T^p$.

The novelty of this chapter is the investigation of penetrative convection in bi-disperse porous media, and it is based on a joint paper [14] written in collaboration with Prof. F. Capone, Prof. R. De Luca and Dott. G. Massa. This investigation is motivated by the aim to derive a better model for pattern ground formation [53], since the porous structure of soils can be subjected to cracks due to thermal stresses and fractured porous media can be efficiently modelled by bi-disperse porous media. The chapter is organized as follows. In Section 6.1.1 the mathematical set-up is described and the dimensionless equations describing the evolutionary behaviour of the perturbation to the conduction solution are derived. In Section 6.2, the strong form of the principle of exchange of stabilities is proved and the linear instability analysis of the conduction solution is performed, in order to determine the instability threshold for the onset of penetrative convection. In Section 6.3, via weighted energy method, the nonlinear critical Rayleigh number is determined and compared to the linear one. In Section 6.3.2, the gap between linear and nonlinear critical Rayleigh numbers is reduced by introducing additional conditions on the initial data. In Section 6.4 the linear instability threshold and the local and global stability thresholds are compared and studied with respect to the physical parameters of the problem. The chapter ends with a concluding Section in which all the results are collected.

6.1.1 *Governing equations*

Let us consider a reference frame $Oxyz$ with fundamental unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ (with $\mathbf{k}$ pointing vertically upward) and a plane layer $L = \mathbb{R}^2 \times [0, d]$ uniformly heated from below. Let $\mathbf{v}^s, T^s, p^s$ be the kinematic, temperature and pressure fields, respectively, for $s = \{f, p\}$ (f and p referring to f-phase and p-phase). A single temperature BDPM fills the layer L, i.e. the fluid temperature in the f-phase and the fluid temperature in the p-phase coincide ($T^f = T^p = T$). Moreover, a Boussinesq approximation is applied. In this chapter, we assume the fluid density ρ in the buoyancy force term as (1.63). The governing equations are, cf. [51],

$$\begin{cases} -\dfrac{\mu}{K_f}\mathbf{v}^f - \xi(\mathbf{v}^f - \mathbf{v}^p) - \nabla p^f - g\rho(T)\mathbf{k} = \mathbf{0}, \\ -\dfrac{\mu}{K_p}\mathbf{v}^p - \xi(\mathbf{v}^p - \mathbf{v}^f) - \nabla p^p - g\rho(T)\mathbf{k} = \mathbf{0}, \\ \nabla \cdot \mathbf{v}^f = 0, \\ \nabla \cdot \mathbf{v}^p = 0, \\ (\rho c)_m \dfrac{\partial T}{\partial t} + (\rho c)_F(\mathbf{v}^f + \mathbf{v}^p) \cdot \nabla T = k_m \Delta T, \end{cases} \tag{6.1}$$

where $\mathbf{x} = (x, y, z)$, ξ = interaction coefficient between the f-phase and the p-phase, $\mathbf{g} = -g\mathbf{k}$ = gravity, μ = fluid viscosity, c = specific heat, $(\rho c)_m = (1-\phi)(1-\epsilon)(\rho c)_{sol} + \phi(\rho c)_f + \epsilon(1-\phi)(\rho c)_p$, $k_m = (1-\phi)(1-\epsilon)k_{sol} + \phi k_f + \epsilon(1-\phi)k_p$ = thermal conductivity (the subscript *sol* is referred to the solid skeleton). Since we are confining ourselves to the case of a single temperature BDPM, the macropores and micropores are saturated by the same fluid, so we assume $(\rho c)_f = (\rho c)_p = (\rho c)_F$

[46], hence $(\rho c)_m = (1-\phi)(1-\epsilon)(\rho c)_{sol} + [\phi + \epsilon(1-\phi)](\rho c)_F$. System (6.1) is subject to the following boundary conditions

$$\begin{aligned} &\mathbf{v}^s \cdot \mathbf{n} = 0 \text{ on } z = 0, d \quad \text{for } s = \{f, p\} \\ &T(x,y,0) = 0, \quad T(x,y,d) = T_U \geq 4°C \end{aligned} \tag{6.2}$$

Employing (1.63) in (6.1), one obtains

$$\begin{cases} -\dfrac{\mu}{K_f}\mathbf{v}^f - \xi(\mathbf{v}^f - \mathbf{v}^p) - \nabla p^f - 2g\rho_F\alpha T_0 T\mathbf{k} + g\rho_F\alpha T^2\mathbf{k} = \mathbf{0}, \\ -\dfrac{\mu}{K_p}\mathbf{v}^p - \xi(\mathbf{v}^p - \mathbf{v}^f) - \nabla p^p - 2g\rho_F\alpha T_0 T\mathbf{k} + g\rho_F\alpha T^2\mathbf{k} = \mathbf{0}, \\ \nabla \cdot \mathbf{v}^f = 0, \\ \nabla \cdot \mathbf{v}^p = 0, \\ (\rho c)_m \dfrac{\partial T}{\partial t} + (\rho c)_F(\mathbf{v}^f + \mathbf{v}^p) \cdot \nabla T = k_m \Delta T. \end{cases} \tag{6.3}$$

System (6.3) under the boundary conditions (6.2) admits the thermal conduction solution

$$\hat{\mathbf{v}}^f = \mathbf{0}, \ \hat{\mathbf{v}}^p = \mathbf{0}, \ \hat{T}(z) = \frac{T_U}{d} z \tag{6.4}$$

Introducing a perturbation $\{\mathbf{u}^f, \mathbf{u}^p, \pi^f, \pi^p, \theta\}$ to the conduction solution, with $\mathbf{u}^s = (u^s, v^s, w^s)$ for $s = \{f, p\}$, the perturbation equations arising from (6.3) are

$$\begin{cases} -\dfrac{\mu}{K_f}\mathbf{u}^f - \xi(\mathbf{u}^f - \mathbf{u}^p) - \nabla \pi^f - 2g\rho_F\alpha T_0\theta\mathbf{k} \\ \qquad\qquad + 2g\rho_F\alpha \dfrac{T_U}{d} z\theta\mathbf{k} + g\rho_F\alpha\theta^2\mathbf{k} = \mathbf{0}, \\ -\dfrac{\mu}{K_p}\mathbf{u}^p - \xi(\mathbf{u}^p - \mathbf{u}^f) - \nabla \pi^p - 2g\rho_F\alpha T_0\theta\mathbf{k} \\ \qquad\qquad + 2g\rho_F\alpha \dfrac{T_U}{d} z\theta\mathbf{k} + g\rho_F\alpha\theta^2\mathbf{k} = \mathbf{0}, \\ \nabla \cdot \mathbf{u}^f = 0, \\ \nabla \cdot \mathbf{u}^p = 0, \\ (\rho c)_m \dfrac{\partial \theta}{\partial t} + (\rho c)_F(\mathbf{u}^f + \mathbf{u}^p) \cdot \nabla\theta = -(\rho c)_F \dfrac{T_U}{d}(w^f + w^p) + k_m \Delta\theta. \end{cases} \tag{6.5}$$

To derive the dimensionless equations describing the evolutionary behaviour of the perturbation fields, let us introduce the following non-dimensional parameters:

$$\begin{aligned} &\mathbf{x}^* = \frac{\mathbf{x}}{d}, \ t^* = \frac{t}{\tau}, \ \theta^* = \frac{\theta}{T^\#}, \ \mathbf{u}^{s*} = \frac{\mathbf{u}^s}{U}, \\ &\pi^{s*} = \frac{\pi^s}{P^\#}, \ \text{for } s = \{f, p\}, \ \gamma = \frac{K_f\xi}{\mu}, \ K_r = \frac{K_f}{K_p}, \end{aligned}$$

where the scales are given by

$$U = \frac{k_m}{(\rho c)_F d}, \; \tau = \frac{d^2(\rho c)_m}{k_m}, \; P^\# = \frac{k_m \mu}{(\rho c)_F K_f}, \; T^\# = \sqrt{\frac{\mu k_m}{2K_f g \rho_F \alpha d (\rho c)_F}},$$

and let us define the Rayleigh number Ra

$$\mathsf{Ra} = \sqrt{\frac{2T_U^2 d(\rho c)_F \rho_F \alpha g K_f}{\mu k_m}}.$$

Therefore, the non-dimensional perturbation equations, dropping all the asterisks, are

$$\begin{cases} -\mathbf{u}^f - \gamma(\mathbf{u}^f - \mathbf{u}^p) - \nabla \pi^f - \mathsf{Ra} N(z)\theta \mathbf{k} + \dfrac{\theta^2}{2}\mathbf{k} = \mathbf{0}, \\ -K_r \mathbf{u}^p - \gamma(\mathbf{u}^p - \mathbf{u}^f) - \nabla \pi^p - \mathsf{Ra} N(z)\theta \mathbf{k} + \dfrac{\theta^2}{2}\mathbf{k} = \mathbf{0}, \\ \nabla \cdot \mathbf{u}^f = 0, \\ \nabla \cdot \mathbf{u}^p = 0, \\ \dfrac{\partial \theta}{\partial t} + (\mathbf{u}^f + \mathbf{u}^p) \cdot \nabla \theta = -\mathsf{Ra}(w^f + w^p) + \Delta \theta, \end{cases} \tag{6.6}$$

where

$$N(z) = \zeta - z, \quad \zeta = \frac{T_0}{T_U}. \tag{6.7}$$

Let us introduce the periodic cell

$$V = \left[0, \frac{2\pi}{a_x}\right] \times \left[0, \frac{2\pi}{a_y}\right] \times [0, 1].$$

The perturbation fields are supposed to belong to $W^{1,2}(V)$ and to be periodic in the horizontal directions x and y with period $\frac{2\pi}{a_x}$ and $\frac{2\pi}{a_y}$, respectively.

6.2 PRINCIPLE OF EXCHANGE OF STABILITIES

In order to determine the instability threshold for the onset of convection, let us consider the linear version of (6.6):

$$\begin{cases} -\mathbf{u}^f - \gamma(\mathbf{u}^f - \mathbf{u}^p) - \nabla \pi^f - \mathsf{Ra} N(z)\theta \mathbf{k} = \mathbf{0}, \\ -K_r \mathbf{u}^p - \gamma(\mathbf{u}^p - \mathbf{u}^f) - \nabla \pi^p - \mathsf{Ra} N(z)\theta \mathbf{k} = \mathbf{0}, \\ \nabla \cdot \mathbf{u}^f = 0, \\ \nabla \cdot \mathbf{u}^p = 0, \\ \dfrac{\partial \theta}{\partial t} = -\mathsf{Ra}(w^f + w^p) + \Delta \theta, \end{cases} \tag{6.8}$$

and, since (6.8) is autonomous, let us separate the variables as follows:

$$f(\mathbf{x},t) = \bar{f}(\mathbf{x})e^{\sigma t}, \quad \text{with } \sigma \in \mathbb{C}.$$

Hence, we get

$$\begin{cases} -\mathbf{u}^f - \gamma(\mathbf{u}^f - \mathbf{u}^p) - \nabla \pi^f - \mathsf{Ra}N(z)\theta\mathbf{k} = \mathbf{0}, \\ -K_r\mathbf{u}^p - \gamma(\mathbf{u}^p - \mathbf{u}^f) - \nabla \pi^p - \mathsf{Ra}N(z)\theta\mathbf{k} = \mathbf{0}, \\ \nabla \cdot \mathbf{u}^f = 0, \\ \nabla \cdot \mathbf{u}^p = 0, \\ \sigma\theta = -\mathsf{Ra}(w^f + w^p) + \Delta\theta. \end{cases} \tag{6.9}$$

Taking the third component of double curl of $(6.9)_1$ and $(6.9)_2$, one gets the following boundary value problem in w^f, w^p, θ

$$\begin{cases} (\gamma+1)\Delta w^f - \gamma\Delta w^p + \mathsf{Ra}N(z)\Delta_1\theta = 0, \\ (\gamma+K_r)\Delta w^p - \gamma\Delta w^f + \mathsf{Ra}N(z)\Delta_1\theta = 0, \\ \sigma\theta = -\mathsf{Ra}(w^f + w^p) + \Delta\theta, \end{cases} \tag{6.10}$$

where $\Delta_1 = \partial_{xx} + \partial_{yy}$, under the boundary conditions:

$$w^f = w^p = \theta = 0, \quad \text{on } z = 0, 1. \tag{6.11}$$

System (6.10) is equivalent to:

$$\begin{cases} \Delta w^f = -\mathsf{Ra}N(z)c_1\Delta_1\theta, \\ \Delta w^p = -\mathsf{Ra}N(z)c_2\Delta_1\theta, \\ \Delta\theta - \mathsf{Ra}(w^f + w^p) = \sigma\theta, \end{cases} \tag{6.12}$$

where we set:

$$c_1 = \frac{2\gamma + K_r}{\gamma + K_r + \gamma K_r} \qquad \text{and} \qquad c_2 = \frac{2\gamma + 1}{\gamma + K_r + \gamma K_r}. \tag{6.13}$$

Theorem 6.2.1. The strong form of the principle of exchange of stabilities holds for system (6.12) - (6.11), therefore, if convection occurs, it can occur only via stationary motions.

Proof. Applying the laplacian operator to $(6.12)_3$, by virtue of $(6.12)_1$ and $(6.12)_2$, one gets:

$$\Delta\Delta\theta + \mathsf{Ra}^2N(z)\Delta_1\theta(c_1 + c_2) = \sigma\Delta\theta. \tag{6.14}$$

Since we are considering periodic perturbations in the x and y directions such that:

$$\Delta_1\theta = -a^2\theta, \tag{6.15}$$

equation (6.14) becomes:

$$\Delta\Delta\theta - \mathsf{Ra}^2 N(z) a^2 \theta (c_1 + c_2) = \sigma \Delta\theta. \tag{6.16}$$

Let us multiply (6.16) by the complex conjugate of θ, θ^*, and integrate over the periodicity cell V, so one obtains:

$$\|\nabla\nabla\theta\|^2 - a^2 \mathsf{Ra}^2 (c_1 + c_2) \left\langle N(z), \theta^2 \right\rangle = -\sigma \|\nabla\theta\|^2. \tag{6.17}$$

Since $\sigma = \sigma_R + i\sigma_I$, the imaginary part of (6.17) is $\sigma_I \|\nabla\theta\|^2 = 0$. Hence, $\sigma_I = 0$ and $\sigma \in \mathbb{R}$. □

In order to determine the instability threshold for the onset of stationary convection, let us set $\sigma = 0$ in (6.12) and let us employ normal solutions

$$f(x, y, z) = \hat{f}(z) e^{i(a_x a + a_y y)}, \quad \forall f \in \{w^f, w^p, \theta\} \tag{6.18}$$

obtaining:

$$\begin{cases} (D^2 - a^2)\widehat{w}^f = \mathsf{Ra} a^2 N(z) c_1 \widehat{\theta}, \\ (D^2 - a^2)\widehat{w}^p = \mathsf{Ra} a^2 N(z) c_2 \widehat{\theta}, \\ (D^2 - a^2)\widehat{\theta} = \mathsf{Ra}(\widehat{w}^f + \widehat{w}^p), \end{cases} \tag{6.19}$$

where $a^2 = a_x^2 + a_y^2$ is the wavenumber and $D = \frac{d}{dz}$. The critical linear Rayleigh number for onset of stationary convection is given by:

$$\mathsf{Ra}_L = \min_{a^2 \in \mathbb{R}^+} \mathsf{Ra}^2(a^2), \tag{6.20}$$

and it will be found by numerical simulations computed applying the Chebyshev-τ method to

$$A\mathbf{U} = \mathsf{Ra} B \mathbf{U} \tag{6.21}$$

where $\mathbf{U} = (\hat{w}^f, \hat{w}^p, \hat{\theta})^T$ and:

$$A = \begin{pmatrix} D^2 - a^2 & 0 & 0 \\ 0 & D^2 - a^2 & 0 \\ 0 & 0 & D^2 - a^2 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 0 & N(z) c_1 a^2 \\ 0 & 0 & N(z) c_2 a^2 \\ 1 & 1 & 0 \end{pmatrix}. \tag{6.22}$$

6.3 STABILITY RESULTS

6.3.1 *Global stability analysis*

In this Section we will perform the nonlinear stability analysis applying the weighted energy method to the following full nonlinear system:

$$\begin{cases} -\mathbf{u}^f - \gamma(\mathbf{u}^f - \mathbf{u}^p) - \nabla\pi^f - \mathsf{Ra}N(z)\theta\mathbf{k} + \dfrac{\theta^2}{2}\mathbf{k} = \mathbf{0}, \\ -K_r\mathbf{u}^p - \gamma(\mathbf{u}^p - \mathbf{u}^f) - \nabla\pi^p - \mathsf{Ra}N(z)\theta\mathbf{k} + \dfrac{\theta^2}{2}\mathbf{k} = \mathbf{0}, \\ \nabla\cdot\mathbf{u}^f = 0, \\ \nabla\cdot\mathbf{u}^p = 0, \\ \dfrac{\partial\theta}{\partial t} + (\mathbf{u}^f + \mathbf{u}^p)\cdot\nabla\theta = -\mathsf{Ra}(w^f + w^p) + \Delta\theta. \end{cases} \tag{6.23}$$

Let us consider as Lyapunov functional:

$$E(t) = \frac{1}{2}\langle g(z), \theta^2\rangle, \tag{6.24}$$

whose time derivative along the solutions of system (6.23) is:

$$\begin{aligned} \frac{dE}{dt} =& \frac{1}{2}\left\langle g'(z)(w^f + w^p), \theta^2\right\rangle - \mathsf{Ra}\left\langle w^f + w^p, \theta g(z)\right\rangle \\ & - \langle\theta_z, \theta g'(z)\rangle - \left\langle g(z), |\nabla\theta|^2\right\rangle - \|\mathbf{u}^f\|^2 - \gamma\|\mathbf{u}^f - \mathbf{u}^p\|^2 \\ & - K_r\|\mathbf{u}^p\|^2 - \mathsf{Ra}\left\langle N(z)(w^f + w^p), \theta\right\rangle + \frac{1}{2}\left\langle\theta^2, w^f + w^p\right\rangle. \end{aligned} \tag{6.25}$$

In order to get rid of the cubic nonlinear terms, let as choose as weight function $g(z) = \mu - z$. The parameter μ has to be chosen such that g is positive $\forall z \in (0,1)$, i.e. condition $\mu > 1$ is required. Hence, from (6.25) we get the following energy equation:

$$\frac{dE}{dt} = \mathsf{Ra}I - D, \tag{6.26}$$

where, since $N(z) = \zeta - z$, we set:

$$\begin{aligned} I &= -\left\langle\zeta + \mu - 2z, (w^f + w^p)\theta\right\rangle \\ D &= \|\mathbf{u}^f\|^2 + \gamma\|\mathbf{u}^f - \mathbf{u}^p\|^2 + K_r\|\mathbf{u}^p\|^2 + \left\langle\mu - z, |\nabla\theta|^2\right\rangle \end{aligned} \tag{6.27}$$

Setting:

$$\frac{1}{\mathsf{Ra}_E} = \max_{\mathcal{H}} \frac{I}{D}, \tag{6.28}$$

with $\mathcal{H}$ being the class of the kinematically admissible perturbations:

$$\mathcal{H} = \Big\{(\mathbf{u}^f, \mathbf{u}^p, \theta) \in (W^{1,2}(V))^3 \mid \nabla \cdot \mathbf{u}^s = 0,\ s = \{f, p\},$$
$$x, y - \text{periodic with period } 2\pi/a_x, 2\pi/a_y,\ \text{verifying (6.11)}\Big\} \tag{6.29}$$

the following theorem holds.

Theorem 6.3.1. If $\mathsf{Ra} < \mathsf{Ra}_E$, then the global nonlinear asymptotic exponential stability of the conduction solution (6.4) in the E-norm (6.24) is guaranteed.

Proof. Since $z \in (0,1)$, by virtue of the Poincaré and the weighted Poincaré inequality, we get:

$$\frac{1}{2}\left\langle \mu - z, \theta^2 \right\rangle \leq c\left\langle \mu - z, |\nabla\theta|^2 \right\rangle \tag{6.30}$$

where $c = \max\left\{\frac{c_p}{2}, 2\right\}$, $c_p = c_p(V)$ being the Poincaré constant. By virtue of (6.30), from (6.26) it follows:

$$\frac{dE}{dt} = \mathsf{Ra}I - D \leq -D\left(1 - \frac{\mathsf{Ra}}{\mathsf{Ra}_E}\right) \leq -cE\left(\frac{\mathsf{Ra}_E - \mathsf{Ra}}{\mathsf{Ra}_E}\right), \tag{6.31}$$

and this implies that

$$E(t) \leq E(0)\exp\left(-c\left(\frac{\mathsf{Ra}_E - \mathsf{Ra}}{\mathsf{Ra}_E}\right)t\right). \tag{6.32}$$

Therefore, if $\mathsf{Ra} < \mathsf{Ra}_E$, $E = \frac{1}{2}\langle \mu - z, \theta^2\rangle$ decays exponentially to zero (without additional restrictions on the initial data). Moreover, multiplying $(6.23)_1$ by $\mathbf{u}^f$ and $(6.23)_2$ by $\mathbf{u}^p$ and by virtue of the generalized Cauchy inequality, we obtain:

$$\|\mathbf{u}^f\|^2 + K_r\|\mathbf{u}^p\|^2 \leq \mathsf{Ra}^2\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right)\|\theta\|^2 + \left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right)\|\theta^2\|^2 \tag{6.33}$$

where ε_i and η_i, for $i = 1, 2$, are positive constants such that $\max\{\varepsilon_1, \eta_1\} = 1$ and $\max\{\varepsilon_2, \eta_2\} = K_r$. The exponential decay of E guarantees the exponential decay of $\|\mathbf{u}^f\|$ and $\|\mathbf{u}^p\|$, therefore $\mathsf{Ra} < \mathsf{Ra}_E$ guarantees the global nonlinear asymptotic exponential stability of the conduction solution (6.4) in the E-norm (6.24). □

In order to determine the nonlinear stability threshold, let us compute the Euler-Lagrange equations associated to the variational problem (6.28), namely:

$$\begin{cases} 2\Delta w^f = -\mathsf{Ra}_E(\zeta + \mu - 2z)c_1\Delta_1\theta, \\ 2\Delta w^p = -\mathsf{Ra}_E(\zeta + \mu - 2z)c_2\Delta_1\theta, \\ 2(\mu - z)\Delta\theta = (\zeta + \mu - 2z)\mathsf{Ra}_E(w^f + w^p) + 2\theta_{,z}. \end{cases} \tag{6.34}$$

Employing solutions (6.18) in (6.34), one obtains

$$\begin{cases} 2(D^2 - a^2)\widehat{w}^f = \mathsf{Ra}_E a^2(\zeta + \mu - 2z)c_1\widehat{\theta}, \\ 2(D^2 - a^2)\widehat{w}^p = \mathsf{Ra}_E a^2(\zeta + \mu - 2z)c_2\widehat{\theta}, \\ 2(\mu - z)(D^2 - a^2)\widehat{\theta} = (\zeta + \mu - 2z)\mathsf{Ra}_E(\widehat{w}^f + \widehat{w}^p) + 2D\widehat{\theta}. \end{cases} \tag{6.35}$$

The nonlinear critical Rayleigh number is given by:

$$\mathsf{Ra}_E = \max_{\mu > 1} \min_{a^2 \in \mathbb{R}^+} \mathsf{Ra}^2(a^2, \mu) \tag{6.36}$$

and it will be determined via numerical computations, in order to get a comparison between the linear instability threshold (6.20) and the nonlinear stability threshold (6.36).
System (6.35) can be written as

$$A\mathbf{U} = \mathsf{Ra}B\mathbf{U} \tag{6.37}$$

where $\mathbf{U} = (\hat{w}^f, \hat{w}^p, \hat{\theta})^T$ and:

$$A = \begin{pmatrix} 2(D^2 - a^2) & 0 & 0 \\ 0 & 2(D^2 - a^2) & 0 \\ 0 & 0 & 2(\mu - z)(D^2 - a^2) - 2D \end{pmatrix}, \tag{6.38}$$

and

$$B = \begin{pmatrix} 0 & 0 & (\zeta + \mu - 2z)c_1a^2 \\ 0 & 0 & (\zeta + \mu - 2z)c_2a^2 \\ \zeta + \mu - 2z & \zeta + \mu - 2z & 0 \end{pmatrix}. \tag{6.39}$$

Solving system (6.37) by mean of the Chebyshev-τ method, we get Figure 6.1 — where Ra_L and Ra_E are depicted as functions of the upper plane temperature T_U. A region of subcritical instability is found, since there is no coincidence between the linear instability threshold and the global nonlinear stability one.

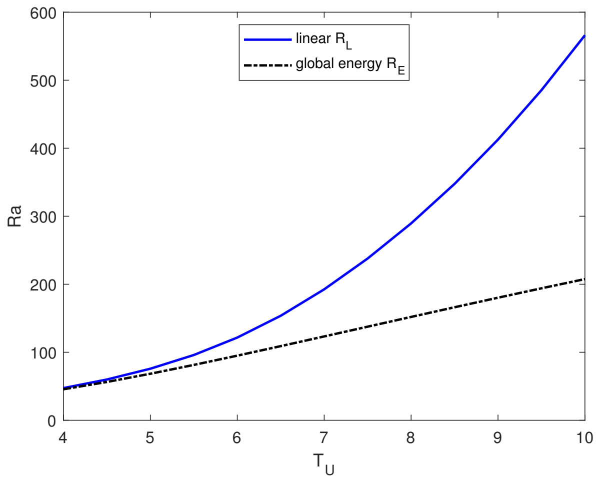


Figure 6.1: Linear instability threshold Ra_L and global stability threshold Ra_E as functions of the upper temperature T_U, for $\{\gamma = 0.8,\ K_r = 1.5\}$.

6.3.2 *Local stability analysis*

To reduce the gap between the linear and the nonlinear thresholds (see Figure 6.1), let us consider the following Lyapunov functional:

$$E_N(t) = \frac{\eta}{2}\|\theta\|^2 + \frac{1}{4}\|\theta^2\|^2, \tag{6.40}$$

where η is a positive coupling parameter. Multiplying $(6.23)_1$ by $\mathbf{u}^f$, $(6.23)_2$ by $\mathbf{u}^p$ and $(6.23)_5$ by $\eta\theta + \theta^3$, integrating over the periodicity cell and adding the resulting equations, one gets

$$\begin{aligned} \frac{dE_N}{dt} =& - \mathsf{Ra}\eta \left\langle w^f + w^p, \theta \right\rangle - \mathsf{Ra} \left\langle w^f + w^p, \theta^3 \right\rangle - \eta\|\nabla\theta\|^2 \\ & - \frac{3}{4}\|\nabla\theta^2\|^2 - \|\mathbf{u}^f\|^2 - \gamma\|\mathbf{u}^f - \mathbf{u}^p\|^2 - K_r\|\mathbf{u}^p\|^2 \\ & - \mathsf{Ra} \left\langle N(z)\theta, w^f + w^p \right\rangle + \frac{1}{2}\left\langle \theta^2, w^f + w^p \right\rangle. \end{aligned} \tag{6.41}$$

Setting

$$\begin{aligned} I_N &= - \left\langle w^f + w^p, \theta(\zeta + \eta - z) \right\rangle, \\ D_N &= \|\mathbf{u}^f\|^2 + \gamma\|\mathbf{u}^f - \mathbf{u}^p\|^2 + K_r\|\mathbf{u}^p\|^2 + \eta\|\nabla\theta\|^2, \\ N &= -\mathsf{Ra} \left\langle w^f + w^p, \theta^3 \right\rangle + \frac{1}{2}\left\langle \theta^2, w^f + w^p \right\rangle, \end{aligned} \tag{6.42}$$

from (6.41) we get the following energy equations:

$$\frac{dE_N}{dt} = \mathsf{Ra} I_N - D_N - \frac{3}{4}\|\nabla\theta^2\|^2 + N \tag{6.43}$$

Let us set

$$\frac{1}{\mathsf{Ra}_N} = \max_{\mathcal{H}} \frac{I_N}{D_N}, \tag{6.44}$$

hence, the following theorem holds.

Theorem 6.3.2. If $\mathsf{Ra} < \mathsf{Ra}_N$ and $E_N(0) < \varepsilon^{-4}$ (ε being a suitable positive constant), then the local nonlinear asymptotic exponential stability of the conduction solution (6.4) in the E_N-norm (6.40) is guaranteed.

Proof. From the energy equation (6.43), it follows

$$\begin{aligned} \frac{dE_N}{dt} &= -D_N\left(1 - \mathsf{Ra}\frac{I_N}{D_N}\right) - \frac{3}{4}\|\nabla\theta^2\|^2 + N \\ &\leq -D_N\left(1 - \frac{\mathsf{Ra}}{\mathsf{Ra}_N}\right) - \frac{3}{4}\|\nabla\theta^2\|^2 + N \\ &\leq -D_1 + N, \end{aligned} \tag{6.45}$$

where we set

$$D_1 = \xi D_N + \frac{3}{4}\|\nabla\theta^2\|^2 \quad \text{and} \quad \xi = 1 - \frac{\mathsf{Ra}}{\mathsf{Ra}_N}. \tag{6.46}$$

In order to estimate the time derivative of E_N, let us obtain an estimate on the nonlinear term N. With this aim in mind, by virtue of the Cauchy-Schwartz and the triangular inequalities, we get

$$N \leq \left(\|w^f\| + \|w^p\|\right)\left(\frac{1}{2}\|\theta^2\| + \mathsf{Ra}\|\theta^3\|\right). \tag{6.47}$$

Let us underline that under the assumption $\mathsf{Ra} < \mathsf{Ra}_N$, ξ is positive. Therefore, since

$$\begin{aligned} \|w^f\|^2 &\leq \|\mathbf{u}^f\|^2 \leq D_N \leq \xi^{-1} D_1, \\ \|w^p\|^2 &\leq \|\mathbf{u}^p\|^2 \leq K_r^{-1} D_N \leq K_r^{-1}\xi^{-1} D_1, \end{aligned} \tag{6.48}$$

and

$$\|\theta^3\| = \left\langle \theta^2, \theta^4 \right\rangle^{\frac{1}{2}} \leq \|\theta^2\|^{\frac{1}{2}}\|\theta^4\|^{\frac{1}{2}}, \tag{6.49}$$

from (6.47) it follows:

$$N \leq \xi^{-\frac{1}{2}}(1 + K_r^{-\frac{1}{2}}) D_1^{\frac{1}{2}} \left(\frac{1}{2}\|\theta^2\|^{\frac{1}{2}}\|\theta^2\|^{\frac{1}{2}} + \mathsf{Ra}\|\theta^2\|^{\frac{1}{2}}\|\theta^4\|^{\frac{1}{2}}\right). \tag{6.50}$$

Let us observe that by virtue of Sobolev embedding inequality one obtains:

$$\begin{aligned} \|\theta^2\|^2 \leq k_1 \|\nabla\theta\|^4 \quad &\Longrightarrow \quad \|\theta^2\|^{\frac{1}{2}} \leq \sqrt[4]{k_1}\|\nabla\theta\| \leq \sqrt[4]{k_1}\eta^{-\frac{1}{2}}\xi^{-\frac{1}{2}}D_1^{\frac{1}{2}}, \\ \|\theta^4\|^2 \leq k_2 \|\nabla\theta^2\|^4 \quad &\Longrightarrow \quad \|\theta^4\|^{\frac{1}{2}} \leq \sqrt[4]{k_2}\|\nabla\theta^2\| \leq \sqrt[4]{k_2}\frac{2}{\sqrt{3}}D_1^{\frac{1}{2}}, \end{aligned} \tag{6.51}$$

with k_1 and k_2 positive constants depending on the cell V. Moreover,

$$\|\theta^2\|^{\frac{1}{2}} \leq \sqrt{2}E_N^{\frac{1}{4}}. \tag{6.52}$$

By virtue of (6.51) and (6.52), from (6.50) one gets:

$$\begin{aligned} N &\leq \xi^{-\frac{1}{2}}(1+K_r^{-\frac{1}{2}})D_1^{\frac{1}{2}}\left(\frac{1}{2}\sqrt[4]{k_1}\eta^{-\frac{1}{2}}\xi^{-\frac{1}{2}}D_1^{\frac{1}{2}}\sqrt{2}E_N^{\frac{1}{4}} + \mathsf{Ra}\sqrt{2}E_N^{\frac{1}{4}}\sqrt[4]{k_2}\frac{2}{\sqrt{3}}D_1^{\frac{1}{2}}\right) \\ &= \xi^{-\frac{1}{2}}(1+K_r^{-\frac{1}{2}})\sqrt{2}\left(\frac{1}{2}\sqrt[4]{k_1}\eta^{-\frac{1}{2}}\xi^{-\frac{1}{2}} + \mathsf{Ra}\sqrt[4]{k_2}\frac{2}{\sqrt{3}}\right)D_1E_N^{\frac{1}{4}}. \end{aligned} \tag{6.53}$$

Since ξ is positive, let us introduce the following positive constant:

$$\varepsilon = \xi^{-\frac{1}{2}}(1+K_r^{-\frac{1}{2}})\sqrt{2}\left(\frac{1}{2}\sqrt[4]{k_1}\eta^{-\frac{1}{2}}\xi^{-\frac{1}{2}} + \mathsf{Ra}\sqrt[4]{k_2}\frac{2}{\sqrt{3}}\right). \tag{6.54}$$

Hence (6.53) becomes

$$N \leq \varepsilon\, D_1 E_N^{\frac{1}{4}}, \tag{6.55}$$

therefore from (6.45) one obtains:

$$\frac{dE_N}{dt} \leq -D_1(1-\varepsilon\, E_N^{\frac{1}{4}}). \tag{6.56}$$

Hence, to guarantee the exponential decay of E_N, the following additional condition on the initial data is required:

$$E_N(0) < \varepsilon^{-4}. \tag{6.57}$$

Provided (6.57), from (6.56) it follows that $\frac{dE_N}{dt}$ is negative in a neighbourhood of zero, therefore $E_N(t)$ cannot exceed its initial value for all $t > 0$. So, defining the positive constant $\Gamma = 1 - \varepsilon\, E_N^{\frac{1}{4}}(0)$, (6.56) yields

$$\frac{dE_N}{dt} \leq -\Gamma D_1. \tag{6.58}$$

Moreover, from the Poincaré inequality it follows

$$E_N \le \frac{\eta c_{p'}}{2}\|\nabla\theta\|^2 + \frac{c_{p''}}{4}\|\nabla\theta^2\|^2 \le \left(\frac{\xi^{-1}c_{p'}}{2} + \frac{c_{p''}}{3}\right) D_1. \tag{6.59}$$

Setting $c = \left(\frac{\xi^{-1}c_{p'}}{2} + \frac{c_{p''}}{3}\right)^{-1}$, from (6.58) one gets

$$\frac{dE_N}{dt} \le -c\Gamma E_N. \tag{6.60}$$

Integrating (6.60), one finally finds

$$E_N(t) \le E_N(0)\exp\left(-c\Gamma t\right) \tag{6.61}$$

and the exponential decay of E_N is guaranteed, provided $\mathsf{Ra} < \mathsf{Ra}_N$. Finally, let us observe from (6.33) it follows

$$\|\mathbf{u}^f\|^2 + K_r\|\mathbf{u}^p\|^2 \le \left[\mathsf{Ra}^2\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right)\frac{2}{\eta} + 4\left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right)\right] E_N, \tag{6.62}$$

so, the exponential decay of E_N guarantees the local stability (i.e. for initial data such that $E_N(0) < \varepsilon^{-4}$) of the conduction solution. □

To determine the local stability threshold for the onset of convection, the variational problem (6.44) has to be solved, so let us derive the associated Euler-Lagrange equations which are, employing solutions (6.18), the following:

$$\begin{cases} (D^2 - a^2)\widehat{w}^f = \mathsf{Ra}_N a^2 \frac{1}{2}(\zeta + \eta - z)c_1\widehat{\theta}, \\ (D^2 - a^2)\widehat{w}^p = \mathsf{Ra}_N a^2 \frac{1}{2}(\zeta + \eta - z)c_2\widehat{\theta}, \\ \eta(D^2 - a^2)\widehat{\theta} = \frac{1}{2}(\zeta + \mu - z)\mathsf{Ra}_N(\widehat{w}^f + \widehat{w}^p). \end{cases} \tag{6.63}$$

The local nonlinear critical Rayleigh number is given by:

$$\mathsf{Ra}_N = \max_{\eta>0}\min_{a^2\in\mathbb{R}^+} \mathsf{Ra}^2(a^2,\mu) \tag{6.64}$$

and its behaviour with respect to the upper plane temperature will be investigated in the following, applying the Chebychev-τ Method to

$$A\mathbf{U} = \mathsf{Ra}B\mathbf{U} \tag{6.65}$$

where $\mathbf{U} = (\hat{w}^f, \hat{w}^p, \hat{\theta})^T$ and:

$$A = \begin{pmatrix} D^2 - a^2 & 0 & 0 \\ 0 & D^2 - a^2 & 0 \\ 0 & 0 & \eta(D^2 - a^2) \end{pmatrix}, \tag{6.66}$$

and

$$B = \begin{pmatrix} 0 & 0 & \frac{1}{2}(\zeta+\mu-z)c_1a^2 \\ 0 & 0 & \frac{1}{2}(\zeta+\mu-z)c_2a^2 \\ \frac{1}{2}(\zeta+\mu-z) & \frac{1}{2}(\zeta+\mu-z) & 0 \end{pmatrix}. \qquad (6.67)$$

6.4 NUMERICAL RESULTS

In this Section, we will compare the instability and stability results obtained in the previous Sections applying the Chebychev-τ method to the generalized eigenvalue problems (6.21), (6.37) and (6.65), with the aim to underline how penetrative convection arises and how the double porosity structure of the medium affects the onset of penetrative convection. For the following numerical simulations, we fixed the set of parameters $\{\gamma = 0.8,\ K_r = 1.5\}$ (see [51]). In Table 6.1 the critical values of Rayleigh number Ra, wavenumber a^2 and coupling parameters μ and η are collected for quoted values of the upper layer temperature T_U. The behaviour of Ra for increasing values of T_U can be visualized in Figure 6.2, where the linear instability threshold Ra_L, local stability threshold Ra_N and global stability threshold Ra_E are depicted as functions of the upper temperature T_U. Let us observe that

- the upper plane temperature has a stabilizing effect on the onset of penetrative convection, since the critical Rayleigh numbers Ra_L, Ra_E and Ra_N are increasing functions with respect to T_U;

- the local nonlinear stability threshold actually reduces the gap between the linear instability and the global nonlinear stability thresholds (see also Figure 6.3, where the neutral curves are plotted for the fixed upper temperature T_U, in particular, 6.3(a) $T_U = 4$, 6.3(b) $T_U = 5$, 6.3(c) $T_U = 6$, 6.3(d) $T_U = 8$);

- when $K_p \to 0$, $\xi \to 0$ and $\epsilon \to 0$, i.e. if $c_1 \to 1$ and $c_2 \to 0$, the bi-disperse porous medium approximates a monodispersive one. In the limit case for $c_1 \to 1$ and $c_2 \to 0$, we recovered the results found in [2], where the Authors analysed the onset of penetrative convection in a single porosity material.

To evaluate the role of the double porosity structure, we perform numerical simulations for quoted values of the dimensionless permeability ratio and the dimensionless momentum transfer coefficient, namely

$$K_r = \frac{K_f}{K_p} \quad \text{and} \quad \gamma = \frac{\xi K_f}{\mu}. \tag{6.68}$$

Accounting for the discussion made in [52], we analysed the behaviour of the linear instability threshold Ra_L, local stability threshold Ra_N and global stability threshold Ra_E with respect to $K_r \in [0, 10]$ or very large K_r and $\gamma \in [0, 1]$ or very small γ, and for fixed upper plane temperature $T_U = 4$ and $T_U = 6$ (see Figure 6.4 and 6.5). The parameters K_r and γ have both a stabilizing effect on the onset of instability, so if the physical properties of the bi-disperse porous medium become more pronounced, the onset of penetrative convection is delayed. This result is coherent with the existing literature, since it has been proven by Nield and Kuznetsov in [91] that the critical Rayleigh number for the onset of bi-disperse convection is higher than the critical Rayleigh number for the onset of convection in a single porosity material, so the heat transfer due to the convective fluid motion is delayed in double porosity materials.
Let us also underline the peculiar interaction between K_r and γ on the critical Rayleigh number. As shown in Figure 6.5, γ has a *linear* stabilizing effect on Ra when K_r is very large (see Figure 6.5(a)).
Furthermore, in Figures 6.6(a) and 6.6(b) the kinematic eigenfunctions $\widehat{w}^p$ and $\widehat{w}^f$ are depicted for increasing permeability ratio K_r, such that

$$K_r \gg 1 \quad \text{or} \quad K_r \ll 1. \tag{6.69}$$

In particular:

- if $K_r \gg 1$ then $K_f \gg K_p$, so the fluid motion in the f-phase is facilitated,
- if $K_r \ll 1$ then $K_f \ll K_p$, so the fluid motion in the p-phase is facilitated,

hence the eigenfunction $\widehat{w}^p$ decreases for increasing K_r (see Figure 6.6(a)), while the eigenfunction $\widehat{w}^f$ increases for increasing K_r (see Figure 6.6(b)). Moreover, note that the change in sign of the kinematic eigenfunctions indicates the presence of a counter-rotating secondary cell, which is typical in penetrative convection. This mathematical result obtained via numerical simulations is actually consistent with the experimental observations [131].

Global Energy			Local Energy			Linear		
Ra_E	a^2	μ	Ra_N	a^2	η	Ra_L	a^2	$T_U(^\circ C)$
45.6052	9.793	1.119	46.5592	10.256	0.54	47.3296	10.209	4
68.4154	9.884	1.028	73.0029	10.785	0.365	75.8101	10.767	5
95.0110	10.400	1.015	112.7126	11.927	0.260	121.5974	12.314	6
123.2762	11.119	1.012	171.0490	14.059	0.202	192.5290	16.138	7
151.9521	11.884	1.010	251.9789	17.375	0.169	289.4467	21.857	8
180.2210	12.665	1.0090	357.8099	21.667	0.148	412.7047	27.986	9
207.5922	13.401	1.0083	490.9588	26.504	0.133	566.0971	34.548	10

Table 6.1: Critical values of thermal Rayleigh numbers, wavenumbers and coupling parameters for quoted values of the upper layer temperature T_U.

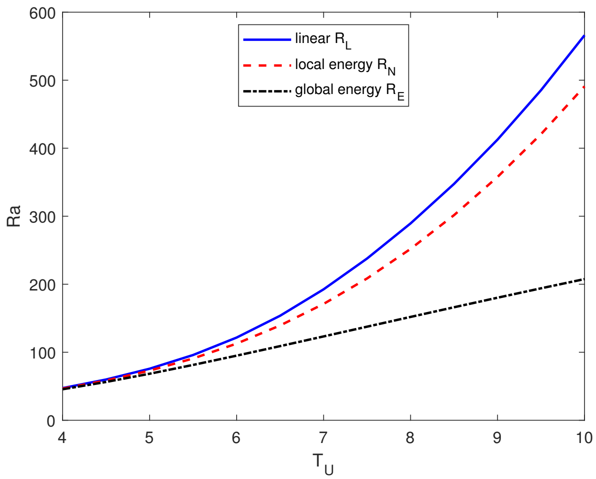
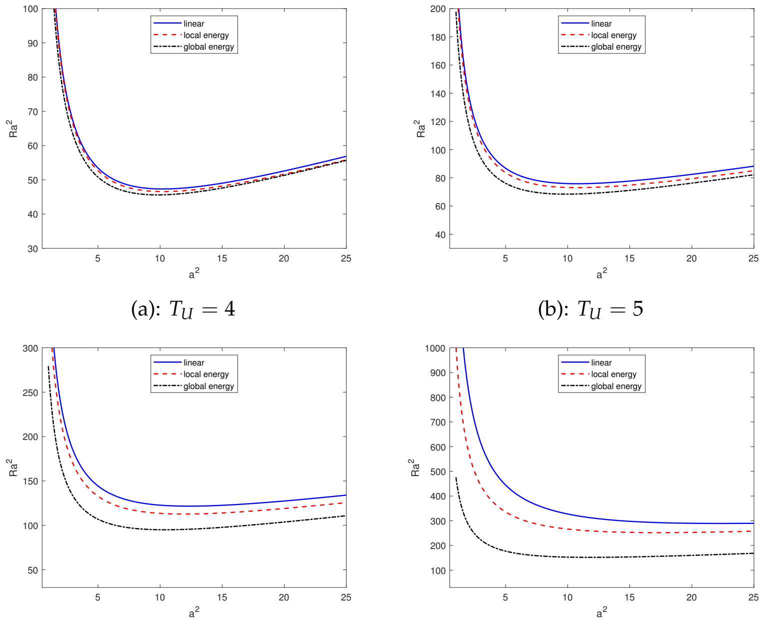


Figure 6.2: Linear instability threshold Ra_L, local stability threshold Ra_N and global stability threshold Ra_E as functions of the upper temperature T_U.

6.5 CONCLUDING REMARK

The onset of penetrative convection in a horizontal layer of fluid-saturated bi-disperse porous medium has been analysed. In particular, the penetrative convection has been modeled employing a quadratic $\rho(T)$. Since we proved the validity of the strong form of the principle of exchange of stabilities, i.e. convection can occur only via steady motions, we performed the linear instability analysis of the conduction solution in order to determine the instability threshold for the onset of stationary convection. The linear instability analysis led to a generalized eigenvalue problem for Ra that we numerically solved via the Chebychev-τ method, determining the linear critical Rayleigh number for the onset of steady convection. Via nonlinear stability analysis, we found a region of subcritical instability, since there is no coincidence between the linear and the global nonlinear critical Rayleigh numbers, Ra_L and Ra_E respectively. In order to reduce the gap between the linear and the nonlinear thresholds, we performed a local stability analysis: we determined a stability threshold that requires additional conditions on the initial data. In particular, we found $\mathsf{Ra}_E < \mathsf{Ra}_N < \mathsf{Ra}_L$, Ra_N being the local stability threshold. Moreover, via numerical computations, we analysed the behaviour of the critical Rayleigh numbers with respect to the upper plane temperature T_U, proving that T_U has a stabilizing

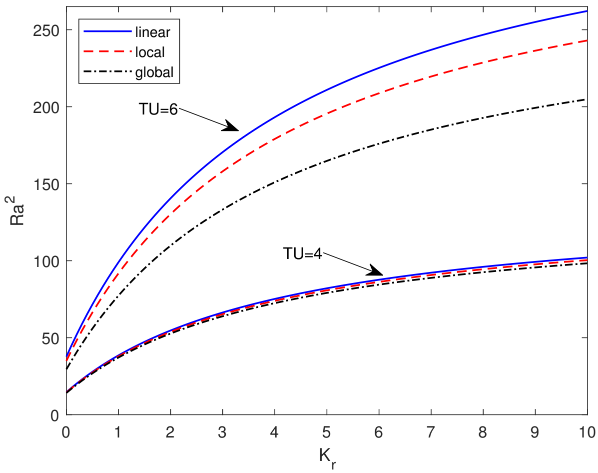


(a): $T_U = 4$ (b): $T_U = 5$

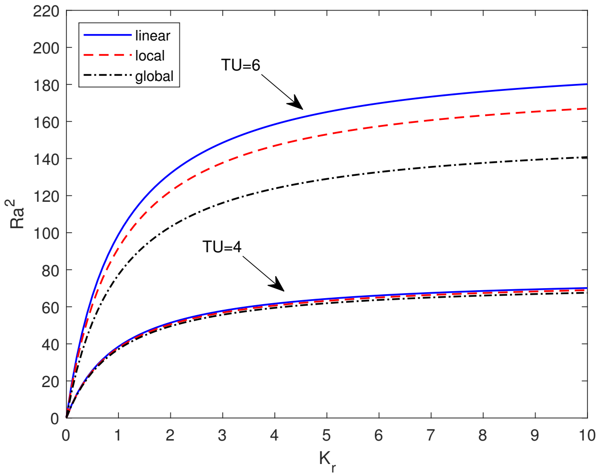
(c): $T_U = 6$ (d): $T_U = 8$

Figure 6.3: Linear instability threshold Ra_L, local stability threshold Ra_N and global stability threshold Ra_E for fixed upper temperature: (a) $T_U = 4$, (b) $T_U = 5$, (c) $T_U = 6$, (d) $T_U = 8$.

effect on the onset of convective instabilities. Moreover, the effect of the double porosity structure on the onset of penetrative convection has been analysed.

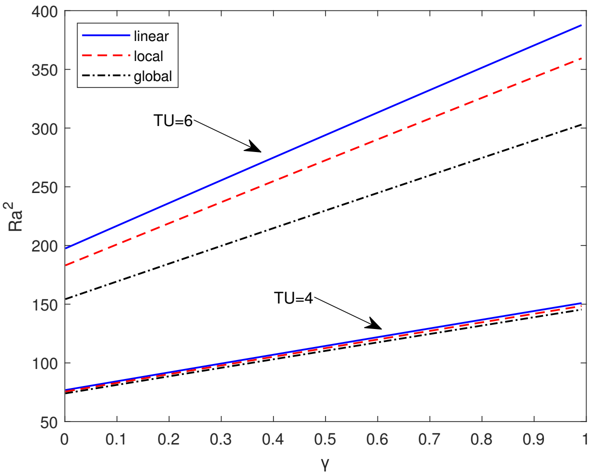


(a): $\gamma = 0.8$

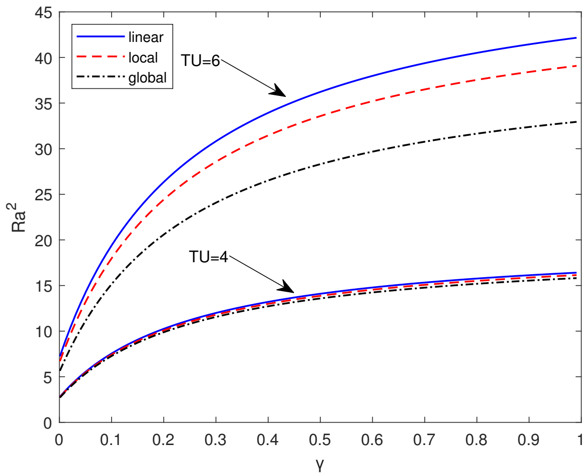


(b): $\gamma = 10^{-3}$

Figure 6.4: Linear instability threshold Ra_L, local stability threshold Ra_N and global stability threshold Ra_E as functions of permeability ratio K_r, for fixed $\gamma = 0.8$ in (a) and $\gamma = 10^{-3}$ in (b), and for upper plane temperature $T_U = 4$ and $T_U = 6$.

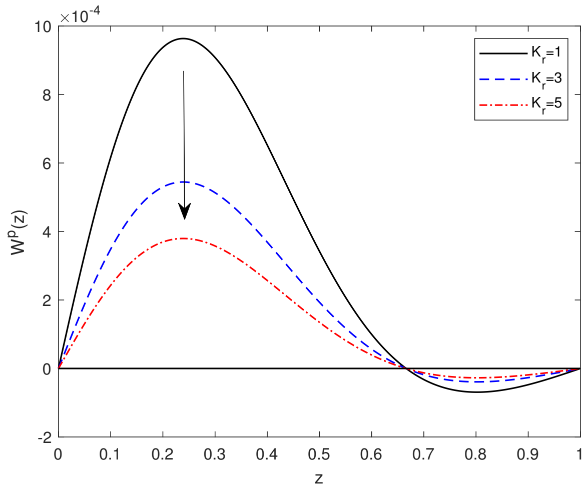


(a): $K_r = 263.16$

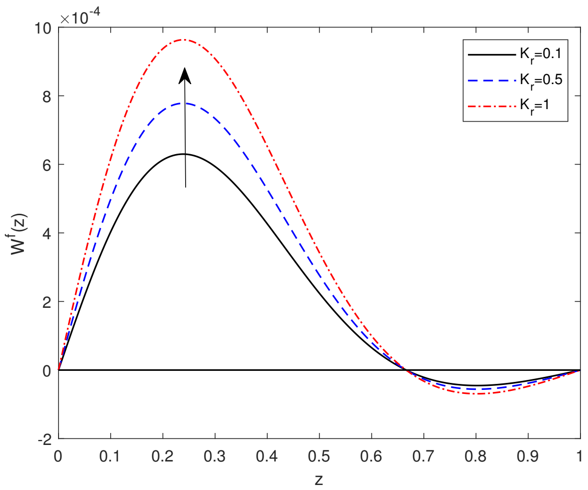


(b): $K_r = 0.037$

Figure 6.5: Linear instability threshold Ra_L, local stability threshold Ra_N and global stability threshold Ra_E as functions of momentum transfer coefficient γ, for fixed $K_r = 263.16$ in (a) and $K_r = 0.037$ in (b), and for upper plane temperature $T_U = 4$ and $T_U = 6$.



(a)



(b)

Figure 6.6: (a): Kinematic eigenfunctions $\widehat{w}^p$ for quoted values of K_r. (b): Kinematic eigenfunctions $\widehat{w}^f$ for quoted values of K_r. The other parameters are: $\gamma = 0.8$ and $T_U = 8$.

7

A DARCY-BRINKMAN MODEL IN LOCAL THERMAL NON-EQUILIBRIUM

7.1 DOUBLE TEMPERATURE MODEL

In the present chapter, we analyse the onset of penetrative convective currents in a high-porosity porous medium in local thermal non-equilibrium (LTNE). The topic studied is the content of a future publication in collaboration with Prof. F. Capone and Dott. J. A. Gianfrani. The previous problems we exposed in this thesis model the penetrative convection phenomenon in porous media under the hypothesis of local thermal equilibrium (LTE), by defining a single temperature for both fluid and solid phases, which are in thermal equilibrium. From the modeling viewpoint, one temperature equation is required. The novelty of the present investigation is the development of a theoretical investigation of a two-field penetrative convection model which involves two energy equations, one for the fluid phase and one for the solid one. These equations are coupled by means of a term representing loss or gain of heat from the other phase. In literature, assuming that heat exchanges between fluid and solid phases are allowed is well-known as LTNE hypothesis. The two-temperature model was first introduced by Banu and Rees with a groundbreaking paper [17] in 2002. Since then, many researchers have devoted their attention to thermal instability phenomena in porous media in LTNE [20, 18, 75]. Straughan in 2006 [124] was the first to study the nonlinear stability in a porous medium in LTNE.
The chapter is organised as follows. In Section 7.1.1 the mathematical model describing the phenomenon of penetrative convection in the hypothesis of LTNE is described and the non-dimensional equations governing the evolution of the perturbations of the conduction solution are derived. In Section 7.2 the strong form of the principle of exchange of stabilities is proved, and the instability analysis of the linearised model is performed in order to determine the instability thresholds for the onset of penetrative convection, via the Chebyshev-τ method. In Section 7.3, weighted energy analysis is performed and the nonlinear critical Rayleigh numbers are determined via the shooting method. In Section 7.4 numerical simulations are performed, commented on, and

summarised. The chapter ends with an Appendix containing some technical details about a useful employed a priori estimate in the context of nonlinear analysis.

7.1.1 *Mathematical Model*

Let $Oxyz$ be a Cartesian frame of reference where the z-axis is vertically upward and let us consider a horizontal isotropic porous layer delimited by two impervious planes ($z = 0$ and $z = d$) and saturated by a Newtonian fluid at rest. Planes confining the layer are kept at a constant temperature at any time in such a way that the fluid-saturated porous medium is heated from above. As a consequence, a uniform gradient of temperature is imposed and maintained constant across the medium. Moreover, let us assume the LTNE between solid matrix and fluid, which implies that heat exchanges between solid and fluid phases occur. This assumption requires us to define two unknown fields: the fluid temperature T^f and the solid temperature T^s. Hence, let T_L be the uniform temperature on the lower plane ($z = 0$) and let T_U be the uniform temperature on the upper plane ($z = d$). The following boundary conditions hold for the problem at stake:

$$T^s = T^f = T_L \text{ on } z = 0, \qquad T^s = T^f = T_U \text{ on } z = d, \tag{7.1}$$

where $T_U > T_L$. In penetrative convection problems, the planes' temperatures are prescribed in such a way that fluid density $\rho_f(T^f)$ may achieve a maximum value in $[T_L, T_U]$. Specifically, as shown by Veronis [131], fluid density exhibits a parabolic behaviour in that temperature interval when it is described by the quadratic function

$$\rho_f(T^f) = \rho_0 \left[1 - \alpha(T^f - T_0)^2\right], \tag{7.2}$$

where T_0 is a reference temperature, ρ_0 the corresponding reference fluid density and α is the thermal expansion coefficient. We now restrict our attention to when the fluid saturating porous medium is *water*. Let us introduce a very last assumption regarding the porous layer. In order for the LTNE hypothesis to hold strongly we assume that the porous medium exhibits very high porosity [122]. As a consequence, within a representative elementary volume, the fraction of void spaces is considerably greater than the total volume.

It is well-known in literature that in order to better describe the fluid motion within a porous medium with high porosity the Darcy-Brinkman law is the best candidate, [90]. According to this law, the momentum equation is

$$\frac{\mu}{k}\mathbf{v} = -\nabla p - g\rho_f(T^f)\mathbf{k} + \widetilde{\mu}\Delta\mathbf{v}, \tag{7.3}$$

where $\mathbf{v}$, p and ρ_f are the seepage velocity, pressure and fluid density respectively, $\mathbf{k} = (0,0,1)$, g is the modulus of gravity acceleration, k is the permeability of the porous medium, μ is the fluid dynamic viscosity, while $\widetilde{\mu}$ is the effective viscosity ($\mu \neq \widetilde{\mu}$). Along with the continuity equation and the energy balance equations for both fluid and solid phases, Eq. (7.3) provides the following model

$$\begin{cases} \frac{\mu}{k}\mathbf{v} = -\nabla\widetilde{p} - 2g\rho_0\alpha T_0 T^f\mathbf{k} + g\rho_0\alpha(T^f)^2\mathbf{k} + \widetilde{\mu}\Delta\mathbf{v}, \\ \nabla\cdot\mathbf{v} = 0, \\ \varepsilon(\rho c)_f\frac{\partial T^f}{\partial t} + (\rho c)_f\mathbf{v}\cdot\nabla T^f = \varepsilon k_f\Delta T^f + h(T^s - T^f), \\ (1-\varepsilon)(\rho c)_s\frac{\partial T^s}{\partial t} = (1-\varepsilon)k_s\Delta T^s - h(T^s - T^f), \end{cases} \tag{7.4}$$

where $\widetilde{p}$ is defined as follows

$$\widetilde{p} = p + g\rho_0(1 - \alpha T_0^2)z, \tag{7.5}$$

and ε, ρ_i, c_i and k_i ($i = f, s$) are medium porosity, density, specific heat and thermal conductivity, respectively, of fluid and solid phases. In the present framework, the boundary conditions (7.1) become the following

$$\begin{aligned} &T^s = T^f = T_L = 0°\mathrm{C}, \quad z = 0 \\ &T^s = T^f = T_U \geq 4°\mathrm{C}, \quad z = d \\ &\mathbf{v}\cdot\mathbf{n} = 0, \quad z = 0, d \end{aligned} \tag{7.6}$$

where Eq. $(7.6)_3$ models the impervious planes, being $\mathbf{n}$ the outward unit normal to planes $z = 0, d$. In system (7.4) the Oberbeck-Boussinesq approximation is considered.

7.1.2 *Steady state motion and nondimentional perturbed equations*

Model (7.4) is now non-dimensionalised by introducing the following set of variables, where the asterisks denote nondimensional fields

$$\begin{aligned} &\mathbf{x} = d\mathbf{x}^*, \quad t = \tau t^* \\ &\mathbf{v} = V\mathbf{v}^*, \quad \widetilde{p} = Pp^*, \quad T^f = T^\#(T^f)^*, \quad T^s = T^\#(T^s)^*, \end{aligned} \tag{7.7}$$

where

$$\begin{aligned}
&\tau = \frac{(\rho c)_f d^2}{k_f}, \\
&V = \frac{\varepsilon k_f}{(\rho c)_f d}, \quad P = \frac{\mu \varepsilon k_f}{k(\rho c)_f}, \quad T^{\#} = \sqrt{\frac{\mu \varepsilon k_f}{2 g \rho_0 \alpha k (\rho c)_f d}}.
\end{aligned} \tag{7.8}$$

By substituting (7.7)-(7.8) into (7.4), we end up with the following nondimensional model (where asterisks have been dropped out of notation's convenience):

$$\begin{cases}
\mathbf{v} = -\nabla p - \mathsf{Ra}\zeta T^f \mathbf{k} + \dfrac{(T^f)^2}{2}\mathbf{k} + \mathsf{Da}\Delta \mathbf{v}, \\
\nabla \cdot \mathbf{v} = 0, \\
\dfrac{\partial T^f}{\partial t} + \mathbf{v} \cdot \nabla T^f = \Delta T^f + H(T^s - T^f), \\
A\dfrac{\partial T^s}{\partial t} = \Delta T^s - H\gamma(T^s - T^f),
\end{cases} \tag{7.9}$$

being

$$H = \frac{h d^2}{\varepsilon k_f}, \quad A = \frac{(\rho c)_s k_f}{(\rho c)_f k_s}, \quad \gamma = \frac{\varepsilon k_f}{(1-\varepsilon) k_s}, \tag{7.10}$$

interaction heat transfer coefficient, diffusivity ratio, weighted conductivity ratio, respectively. While the non-dimensional parameters Ra (the thermal Darcy-Rayleigh number), Da (the Darcy number) and ζ are respectively given by

$$\mathsf{Ra} = \sqrt{\frac{2 g \rho_0 \alpha k (\rho c)_f d T_U^2}{\mu \varepsilon k_f}}, \quad \mathsf{Da} = \frac{\widetilde{\mu} k}{d^2 \mu}, \quad \zeta = \frac{4}{T_U}. \tag{7.11}$$

It is worth remarking that, by definition,

$$\mathsf{Ra} = \frac{T_U}{T^{\#}}, \tag{7.12}$$

therefore, boundary conditions (7.6) become

$$\begin{aligned}
&T^s = T^f = 0, \quad z = 0, \\
&T^s = T^f = \mathsf{Ra}, \quad z = 1, \\
&\mathbf{v} \cdot \mathbf{n} = 0, \quad z = 0, 1.
\end{aligned} \tag{7.13}$$

Model (7.9) and (7.13) admits the following steady solution as basic motion (conduction solution), according to which fluid is at rest and heat spreads by conduction:

$$m_b = (\mathbf{v}_b, p_b, T_b^f, T_b^s) \tag{7.14}$$

with

$$\begin{aligned} &\mathbf{v}_b = (0,0,0), \\ &p_b = cost. - g\rho_0 dP^{-1}(1-\alpha T_0^2)z - \frac{\mathsf{Ra}\ \zeta}{2}z^2 + \frac{\mathsf{Ra}^2}{6}z^3, \\ &T_b^f = \mathsf{Ra}\ z, \\ &T_b^s = \mathsf{Ra}\ z. \end{aligned} \tag{7.15}$$

In order to study the stability of m_b, let us introduce the perturbation fields $(\mathbf{u}, \pi, \theta, \phi)$ to velocity, pressure, fluid temperature and solid temperature, respectively. The following solution of (7.9) originates once perturbations on initial data are applied:

$$\mathbf{v} = \mathbf{v}_b + \mathbf{u}, \quad p = p_b + \pi, \quad T^f = T_b^f + \theta, \quad T^s = T_b^s + \phi. \tag{7.16}$$

Substituting (7.16) into (7.9), we get

$$\begin{cases} \mathbf{u} = -\nabla\pi - \mathsf{Ra}M(z)\theta\mathbf{k} + \dfrac{\theta^2}{2}\mathbf{k} + \mathsf{Da}\Delta\mathbf{u}, \\ \nabla\cdot\mathbf{u} = 0, \\ \dfrac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta + \mathsf{Ra}w = \Delta\theta + H(\phi-\theta), \\ A\dfrac{\partial\phi}{\partial t} = \Delta\phi - H\gamma(\phi-\theta), \end{cases} \tag{7.17}$$

with $M(z) = \zeta - z$ and with the *stress-free* boundary conditions

$$u_z = v_z = w = \phi = \theta = 0, \quad z = 0,1, \tag{7.18}$$

and initial conditions

$$\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x}), \quad \theta(\mathbf{x},0)=\theta_0(\mathbf{x}), \quad \phi(\mathbf{x},0)=\phi_0(\mathbf{x}), \quad \pi(\mathbf{x},0)=\pi_0(\mathbf{x}), \tag{7.19}$$

where $\nabla\cdot\mathbf{u}_0 = 0$. In the following, motivated by the physics of the problem, we assume that perturbations are periodic in the x and y direction with periods $\frac{2\pi}{k_x}$ and $\frac{2\pi}{k_y}$, respectively. Moreover, $\forall g \in \{\mathbf{u}, \pi, \theta, \phi\}$

$$g\ :(\mathbf{x},t)\in\Omega\times\mathbb{R}^+ \to g(\mathbf{x},t)\in\mathbb{R} \text{ and } g\in W^{2,2}(\Omega)\ \forall t\in\mathbb{R}^+, \tag{7.20}$$

where we denote by Ω the periodicity cell

$$\Omega = \left[0, \frac{2\pi}{k_x}\right] \times \left[0, \frac{2\pi}{k_y}\right] \times [0,1] \tag{7.21}$$

and g can be expanded in a Fourier series uniformly convergent in Ω.

7.2 INSTABILITY RESULTS

In this section, we are going to study the instability of the basic motion m_b, with the aim of determining the critical Rayleigh number beyond which thermal instability occurs. Hence, by neglecting nonlinear terms in (7.17), we get

$$\begin{cases} \mathbf{u} = -\nabla\pi - \mathsf{Ra}M(z)\theta\mathbf{k} + \mathsf{Da}\Delta\mathbf{u}, \\ \nabla \cdot \mathbf{u} = 0, \\ \dfrac{\partial\theta}{\partial t} + \mathsf{Ra}w = \Delta\theta + H(\phi - \theta), \\ A\dfrac{\partial\phi}{\partial t} = \Delta\phi - H\gamma(\phi - \theta), \end{cases} \tag{7.22}$$

with boundary conditions

$$u_z = v_z = w = \phi = \theta = 0, \quad z = 0, 1. \tag{7.23}$$

System (7.22) is autonomous, therefore we can look for solutions such that

$$f(\mathbf{x}, t) = \widehat{f}(\mathbf{x})e^{\sigma t} \quad \forall f \in \{\mathbf{u}, \pi, \theta, \phi\}, \tag{7.24}$$

where $\sigma \in \mathbb{C}$ is the temporal growth rate of perturbation. Let us now apply the double curl to Eq. $(7.22)_1$ and retain only the third component of the resulting equation. Substitution of (7.24) into the resulting model originating from (7.22) will lead to

$$\begin{cases} \Delta\widehat{w} = -\mathsf{Ra}M(z)\Delta_1\widehat{\theta} + \mathsf{Da}\Delta\Delta\widehat{w}, \\ \sigma\widehat{\theta} + \mathsf{Ra}\widehat{w} = \Delta\widehat{\theta} + H(\widehat{\phi} - \widehat{\theta}), \\ \sigma\dfrac{A}{\gamma}\widehat{\phi} = \dfrac{1}{\gamma}\Delta\widehat{\phi} - H(\widehat{\phi} - \widehat{\theta}). \end{cases} \tag{7.25}$$

By defining the operator $\mathcal{L}$ as follows

$$\mathcal{L} := \Delta - \mathsf{Da}\Delta\Delta, \tag{7.26}$$

and by applying this operator to $(7.25)_2$ and $(7.25)_3$, we get

$$\begin{cases} \mathcal{L}\widehat{w} = -\mathsf{Ra}M(z)\Delta_1\widehat{\theta}, \\ \sigma\mathcal{L}\widehat{\theta} + \mathsf{Ra}\mathcal{L}\widehat{w} = \mathcal{L}\Delta\widehat{\theta} + H(\mathcal{L}\widehat{\phi} - \mathcal{L}\widehat{\theta}), \\ \sigma\dfrac{A}{\gamma}\mathcal{L}\widehat{\phi} = \dfrac{1}{\gamma}\mathcal{L}\Delta\widehat{\phi} - H(\mathcal{L}\widehat{\phi} - \mathcal{L}\widehat{\theta}), \end{cases} \tag{7.27}$$

from which, easily,

$$\begin{cases} \sigma\mathcal{L}\widehat{\theta} - \mathsf{Ra}^2M(z)\Delta_1\widehat{\theta} = \mathcal{L}\Delta\widehat{\theta} + H(\mathcal{L}\widehat{\phi} - \mathcal{L}\widehat{\theta}), \\ \sigma\dfrac{A}{\gamma}\mathcal{L}\widehat{\phi} = \dfrac{1}{\gamma}\mathcal{L}\Delta\widehat{\phi} - H(\mathcal{L}\widehat{\phi} - \mathcal{L}\widehat{\theta}). \end{cases} \tag{7.28}$$

Let us multiply $(7.28)_1$ by θ^* and $(7.28)_2$ by ϕ^*, where the asterisk denotes the complex conjugate and integrate over the periodicity cell Ω. Hence, we obtain

$$\begin{cases} \sigma \left\langle \mathcal{L}\widehat{\theta}, \widehat{\theta}^* \right\rangle = \mathsf{Ra}^2 \left\langle M(z)\Delta_1 \widehat{\theta}, \widehat{\theta}^* \right\rangle + \left\langle \mathcal{L}\Delta\widehat{\theta}, \widehat{\theta}^* \right\rangle \\ \qquad\qquad + H\left(\left\langle \mathcal{L}\widehat{\phi}, \widehat{\theta}^* \right\rangle - \left\langle \mathcal{L}\widehat{\theta}, \widehat{\theta}^* \right\rangle \right), \\ \sigma \dfrac{A}{\gamma} \langle \mathcal{L}\widehat{\phi}, \widehat{\phi}^* \rangle = \dfrac{1}{\gamma} \langle \mathcal{L}\Delta\widehat{\phi}, \widehat{\phi}^* \rangle - H\left(\langle \mathcal{L}\widehat{\phi}, \widehat{\phi}^* \rangle - \left\langle \mathcal{L}\widehat{\theta}, \widehat{\phi}^* \right\rangle \right). \end{cases} \tag{7.29}$$

By virtue of periodicity assumption and boundary conditions (7.23), we get

$$\begin{aligned} \sigma\left(-\|\nabla\widehat{\theta}\|^2 - \mathsf{Da}\|\nabla\nabla\widehat{\theta}\|^2 \right) &= -\,\mathsf{Ra}^2 \left\langle M(z), |\nabla_1 \widehat{\theta}|^2 \right\rangle - \|\nabla\nabla\widehat{\theta}\|^2 \\ &\quad - \mathsf{Da}\|\nabla\nabla\nabla\widehat{\theta}\|^2 + H\left(\left\langle \mathcal{L}\widehat{\phi}, \widehat{\theta}^* \right\rangle - \|\nabla\widehat{\theta}\|^2 - \mathsf{Da}\|\nabla\nabla\widehat{\theta}\|^2 \right), \end{aligned}$$

$$\begin{aligned} \sigma\left(-\frac{A}{\gamma}\|\nabla\widehat{\phi}\|^2 - \frac{A\mathsf{Da}}{\gamma}\|\nabla\nabla\widehat{\phi}\|^2 \right) &= -\,\frac{1}{\gamma}\|\nabla\nabla\widehat{\phi}\|^2 - \frac{\mathsf{Da}}{\gamma}\|\nabla\nabla\nabla\widehat{\theta}\|^2 \\ &\quad - H\left(\|\nabla\widehat{\phi}\|^2 - \mathsf{Da}\|\nabla\nabla\widehat{\phi}\|^2 - \left\langle \mathcal{L}\widehat{\theta}, \widehat{\phi}^* \right\rangle \right). \end{aligned} \tag{7.30}$$

Adding the two previous equations, it turns out that

$$\sigma \in \mathbb{R}. \tag{7.31}$$

This is immediate since the following chain of equivalences holds

$$\left\langle \mathcal{L}\widehat{\phi}, \widehat{\theta}^* \right\rangle = \left\langle \widehat{\phi}, \mathcal{L}(\widehat{\theta}^*) \right\rangle = \left\langle \widehat{\phi}, (\mathcal{L}\widehat{\theta})^* \right\rangle = \left\langle \mathcal{L}\widehat{\theta}, \widehat{\phi}^* \right\rangle^*, \tag{7.32}$$

therefore,

$$\left\langle \mathcal{L}\widehat{\theta}, \widehat{\phi}^* \right\rangle + \left\langle \mathcal{L}\widehat{\theta}, \widehat{\phi}^* \right\rangle^* \in \mathbb{R}. \tag{7.33}$$

As a consequence, we can claim the validity of the *strong form of the principle of exchange of stabilities*. Consequently, a new stable steady configuration is reached. In this context, it is said that thermal convection occurs *via steady motions*.

Turning our attention back to system (7.25), because of periodicity of perturbation fields, we can write solutions as

$$\widehat{f}(x,y,z) = \sum_{n=1}^{+\infty} \widetilde{F}_n(x,y,z) \qquad \widehat{f} \in \{\mathbf{u}, \theta, \phi\}, \tag{7.34}$$

where

$$\Delta_1 \widetilde{F}_n(x,y,z) = -k^2 \widetilde{F}_n(z), \qquad k^2 = k_x^2 + k_y^2. \tag{7.35}$$

Once (7.34)-(7.35) is plugged into (7.25), linearity of the model allows to retain only the n-th component. Hence, dropping the subscript, system (7.25) becomes

$$\begin{cases} (D^2 - a^2)\widetilde{W} = \mathsf{Ra}M(z)k^2\widetilde{\Theta} + \mathsf{Da}(D^2 - k^2)^2\widetilde{W}, \\ \sigma\widetilde{\Theta} + \mathsf{Ra}\widetilde{W} = (D^2 - k^2)\widetilde{\Theta} + H(\widetilde{\Phi} - \widetilde{\Theta}), \\ \sigma A\widetilde{\Phi} = (D^2 - k^2)\widetilde{\Phi} - H\gamma(\widetilde{\Phi} - \widetilde{\Theta}), \end{cases} \tag{7.36}$$

where we denote by $D^2 = \frac{d^2}{dz^2}$, with boundary conditions

$$D^2\widetilde{W} = \widetilde{W} = \widetilde{\Theta} = \widetilde{\Phi} = 0, \quad z = 0, 1. \tag{7.37}$$

System (7.36)-(7.37) is a differential eigenvalue problem of this kind

$$\mathcal{A}\mathbf{X} = \sigma\mathcal{B}\mathbf{X} \qquad \mathbf{X} = (\widetilde{W}, \widetilde{\Theta}, \widetilde{\Phi}). \tag{7.38}$$

The presence of z-dependent coefficients makes the problem demanding from the analytical view point, therefore we implement and employ the Chebyshev-τ method. The idea behind this numerical procedure involves the discretisation of differential operators $\mathcal{A}$ and $\mathcal{B}$ by mean of Chebyshev polynomials, taking advantage of their several good properties. Once the problem has been reduced to an algebraic eigenvalue problem, the common MatLab routine `eig` is employed.

It is worth remarking that, having proved the strong form of the principle of exchange of stabilities, we expect the numerical method to provide only real values for σ. This check has been undertaken and verified, proving the goodness of numerical results.

In this framework, it is well known that $\sigma = 0$ allows us to determine the neutral stability curve, which delimits the instability region. Therefore, we are interested in determining first those couples (k, Ra) such that $\sigma(k, \mathsf{Ra}) = 0$ and then, among them, finding the couple (k_c, Ra_L) that solves the following minimum problem

$$\min_{k^2 \in \mathbb{R}^+} \mathsf{Ra}^2. \tag{7.39}$$

7.3 ENERGY STABILITY ANALYSIS

In this section we are going to study the nonlinear stability of the conduction solution m_b (7.14) via the well-established energy method [125, 48]. Let us introduce the following weighted energy functional

$$E(t) = \frac{1}{2}\left\langle g(z), \theta^2 \right\rangle + \frac{A}{2\gamma}\left\langle g(z), \phi^2 \right\rangle, \tag{7.40}$$

where $g(z)$ is a positive real function to be suitably chosen and $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ are the real scalar product on $L^2(\Omega)$ and the related norm, respectively. If Eq. $(7.17)_1$ is multiplied by $\mathbf{u}$ and integrated over Ω, Eq. $(7.17)_3$ is multiplied by $g(z)\theta$ and integrated over Ω, Eq. $(7.17)_4$ is multiplied by $g(z)\phi$ and integrated over Ω, the sum of the resulting equations can be written as

$$\begin{aligned}
\frac{dE}{dt} = &\frac{1}{2}\left\langle g'(z)w, \theta^2 \right\rangle - \mathsf{Ra}\left\langle g(z)w, \theta \right\rangle - \left\langle \theta_z, \theta g'(z) \right\rangle - \left\langle g(z), |\nabla\theta|^2 \right\rangle \\
&+ 2H\left\langle g(z)\phi, \theta \right\rangle - H\left\langle g(z), \theta^2 \right\rangle - \frac{1}{\gamma}\left\langle \phi_z, \phi g'(z) \right\rangle - \frac{1}{\gamma}\left\langle g(z), |\nabla\phi|^2 \right\rangle \\
&- H\left\langle g(z), \phi^2 \right\rangle + H\left\langle g(z)\phi, \theta \right\rangle - \|\mathbf{u}\|^2 - \mathsf{Ra}\left\langle M(z)\theta, w \right\rangle \\
&- \frac{1}{2}\left\langle w, \theta^2 \right\rangle - \mathsf{Da}\|\nabla\mathbf{u}\|^2.
\end{aligned} \tag{7.41}$$

Now, in order to handle the cubic term $\left\langle g'(z)w, \theta^2 \right\rangle - \left\langle w, \theta^2 \right\rangle$, we choose

$$g(z) = \mu - z, \qquad \mu > 1, \tag{7.42}$$

where μ is a parameter to be optimally chosen later. In such a way, (7.41) becomes

$$\frac{dE}{dt} = \mathsf{Ra}I - D, \tag{7.43}$$

where

$$\begin{aligned}
I = &- \left\langle (\mu + \zeta - 2z)w, \theta \right\rangle, \\
D = &\|\mathbf{u}\|^2 + \mathsf{Da}\|\nabla\mathbf{u}\|^2 + \left\langle \mu - z, |\nabla\theta|^2 \right\rangle \\
&+ \frac{1}{\gamma}\left\langle \mu - z, |\nabla\phi|^2 \right\rangle + H\left\langle \mu - z, |\theta - \phi|^2 \right\rangle
\end{aligned} \tag{7.44}$$

are respectively the production term and the dissipation term. Hence, by denoting

$$\frac{1}{\mathsf{Ra}_E} = \max_{\mathcal{H}} \frac{I}{D}, \tag{7.45}$$

where the space of kinematically admissible functions is defined as:

$$\mathcal{H} = \Big\{(\mathbf{u}, \theta, \phi) \in W^{1,2}(\Omega) \mid \nabla \cdot \mathbf{u} = 0,\ x, y \text{ periodic} \\ \text{with period } 2\pi/k_x, 2\pi/k_y,\ \text{verifying (7.18)}\Big\}, \tag{7.46}$$

from (7.43), we obtain

$$\frac{dE}{dt} = \mathsf{Ra} I - D = -D\left(1 - \mathsf{Ra}\frac{I}{D}\right) \leq -\left(\frac{\mathsf{Ra}_E - \mathsf{Ra}}{\mathsf{Ra}_E}\right) D. \tag{7.47}$$

Looking at definition $(7.44)_2$, by applying the weighted Poincaré inequality for which

$$\left\langle \mu - z, \theta^2 \right\rangle \leq c_P \left\langle \mu - z, |\nabla\theta|^2 \right\rangle, \quad \left\langle \mu - z, \phi^2 \right\rangle \leq c_P \left\langle \mu - z, |\nabla\phi|^2 \right\rangle, \tag{7.48}$$

we get from (7.47)

$$\frac{dE}{dt} \leq -c\left(\frac{\mathsf{Ra}_E - \mathsf{Ra}}{\mathsf{Ra}_E}\right) E, \tag{7.49}$$

where

$$c = \max\left\{\frac{2}{Ac_P}, \frac{2}{c_P}\right\}. \tag{7.50}$$

Hence, if $\mathsf{Ra} < \mathsf{Ra}_E$,

$$E(t) \leq E(0)e^{-\alpha t} \tag{7.51}$$

where

$$\alpha = c\frac{\mathsf{Ra}_E - \mathsf{Ra}}{\mathsf{Ra}_E}. \tag{7.52}$$

We have recovered the exponential decay in time of the energy $E(t)$, Eq. (7.40), when $\mathsf{Ra} < \mathsf{Ra}_E$. As a consequence, as long as $\mathsf{Ra} < \mathsf{Ra}_E$, perturbations on fluid and solid temperature tend to zero exponentially as $t \to +\infty$. In order to discuss the nonlinear stability of m_b within the regime for $\mathsf{Ra} < \mathsf{Ra}_E$, we need to determine the faith of the seepage velocity norm. We are able to show that we can control $\|\mathbf{u}\|$ with $\|\theta\|$.

Indeed, let us multiply $(7.17)_1$ by $\mathbf{u}$ and integrate over Ω. The resulting equation will be

$$\|\mathbf{u}\|^2 + \mathsf{Da}\|\nabla\mathbf{u}\|^2 = |\mathsf{Ra}\,\langle g(z)\theta, w\rangle\,| + \left|\frac{1}{2}\left\langle \theta^2, w\right\rangle\right|. \tag{7.53}$$

The Cauchy-Schwartz inequality and the Poincaré inequality lead to

$$\left(\frac{1}{4} + \frac{\mathsf{Da}}{c_p}\right)\|\mathbf{u}\|^2 \leq \frac{M^2\mathsf{Ra}^2}{2}\|\theta\|^2 + \frac{1}{4}\|\theta^2\|^2, \tag{7.54}$$

where $M := \max_{z\in[0,1]} |\zeta - z|$. The following Lemma comes to help on estimating and controlling $\|\theta^2\|^2$:

Lemma 7.3.1. Let Ω_1 and Ω_2 be sets that partition the periodicity cell Ω such that

$$\begin{aligned} \Omega_1 &= \{\mathbf{x} \in \Omega \ : \ \theta(\mathbf{x}, t) > \mathsf{Ra} - T_b\}, \\ \Omega_2 &= \{\mathbf{x} \in \Omega \ : \ \theta(\mathbf{x}, t) \le \mathsf{Ra} - T_b\}. \end{aligned} \tag{7.55}$$

If

$$\theta_0(\mathbf{x}), \phi_0(\mathbf{x}) \in W^{2,2}(\Omega) \cap L^\infty(\Omega), \tag{7.56}$$

then, there exists a positive constant Γ such that

$$\theta(\mathbf{x}, t) + T_b(z) - \mathsf{Ra} \le \Gamma, \tag{7.57}$$

with

$$\Gamma = \begin{cases} \bar{\theta}_0 & \text{if} \quad \phi_0 \le \mathsf{Ra} - T_b(z), \\ \bar{\theta}_0 + \bar{\phi}_0 & \text{otherwise,} \end{cases} \tag{7.58}$$

and

$$\begin{aligned} \bar{\theta}_0 &= \operatorname*{ess\,sup}_{\Omega_1} \left\{ (\theta_0(\mathbf{x}) + T_b(z) - \mathsf{Ra})_+ \right\}, \\ \bar{\phi}_0 &= \operatorname*{ess\,sup}_{\Omega} \left\{ (\phi_0(\mathbf{x}) + T_b(z) - \mathsf{Ra})_+ \right\}. \end{aligned} \tag{7.59}$$

Proof. Since the proof is lengthy, we delegate it to Appendix 7.5. □

Consequently, recalling the inequality $(a+b)^2 \le 2(a^2+b^2)$ and applying Lemma 7.3.1, one obtains

$$\begin{aligned} \|\theta^2\|^2 &\le 2\int_{\Omega_1} (\theta + T_b - \mathsf{Ra})^2 \theta^2 d\Omega + 2\int_\Omega (-T_b + \mathsf{Ra})^2 \theta^2 d\Omega \\ &\le \Upsilon \|\theta\|^2 \end{aligned} \tag{7.60}$$

where

$$\Upsilon = \max\left\{ 4\left[\bar{\theta}_0 + \bar{\phi}_0\right]^2, 2\max_{z\in[0,1]}\left[(-T_b + \mathsf{Ra})^2\right]\right\} \tag{7.61}$$

and

$$\left(\frac{1}{4} + \frac{\mathsf{Da}}{c_p}\right)\|\mathbf{u}\|^2 \le \frac{M^2 \mathsf{Ra}^2}{2}\|\theta\|^2 + \frac{\Upsilon}{4}\|\theta\|^2. \tag{7.62}$$

In conclusion, we recover the following stability theorem

Theorem 7.3.1. If $\mathsf{Ra} < \mathsf{Ra}_E$, then the conduction solution m_b is globally nonlinearly stable.

The critical threshold Ra_E is determined by solving the variational problem (7.45). The Euler-Lagrange equations are

$$\begin{cases} \mathbf{u} - \mathsf{Da}\Delta\mathbf{u} + \mathsf{Ra}_E F(z)\theta\mathbf{k} = \nabla l, \\ \mathsf{Ra}_E F(z)w + \theta_z - g(z)\Delta\theta + Hg(z)(\theta - \phi) = 0, \\ \phi_z - g(z)\Delta\phi - H\gamma g(z)(\theta - \phi) = 0, \end{cases} \tag{7.63}$$

where $F(z) = \frac{\mu}{2} - \frac{\zeta}{2} - z$ and l is a Lagrange multiplier.

By retaining the third component of the double curl of $(7.63)_1$ and employing (7.34)-(7.35) to write the solution, we get

$$\begin{cases} (D^2 - k^2)\widetilde{W} - \widetilde{Y} = 0, \\ -\widetilde{Y} + \mathsf{Da}(D^2 - k^2)\widetilde{Y} + \mathsf{Ra}_E F(z)k^2\widetilde{\Theta} = 0, \\ \mathsf{Ra}_E F(z)\widetilde{W} + D\widetilde{\Theta} - g(z)(D^2 - k^2)\widetilde{\Theta} + Hg(z)(\widetilde{\Theta} - \widetilde{\Phi}) = 0, \\ D\widetilde{\Phi} - g(z)(D^2 - k^2)\widetilde{\Phi} - H\gamma g(z)(\widetilde{\Theta} - \widetilde{\Phi}) = 0, \end{cases} \tag{7.64}$$

with the usual boundary conditions:

$$\widetilde{Y} = \widetilde{W} = \widetilde{\Theta} = \widetilde{\Phi} = 0, \quad z = 0, 1. \tag{7.65}$$

In order to solve the 8th order differential eigenvalue problem (7.64)-(7.65), with non-constant coefficients, we implement a shooting method coupled with a Newton-Raphson scheme. For additional details on the method we refer to [107, 106].

To ensure that Eqs. (7.64)-(7.65) provide nonzero solution, we need to add one extra boundary condition:

$$D\widetilde{\Theta} = 1, \text{ on } z = 0 \tag{7.66}$$

and, consequently, one extra equation

$$D\mathsf{Ra}_E = 0, \tag{7.67}$$

which enforces that Ra_E is constant. Moreover, the critical Rayleigh number for nonlinear stability will be provided by solving the following problem

$$\mathsf{Ra}_{E,c}^2 = \max_{\mu>1} \min_{k^2\in\mathbb{R}^+} \mathsf{Ra}_E^2. \tag{7.68}$$

As a result, system (7.64), (7.65), (7.66), (7.67) is replaced by the following 18th order system

$$
\begin{cases}
(D^2-k^2)\widetilde{W}-\widetilde{Y}=0,\\
-\widetilde{Y}+\mathsf{Da}(D^2-k^2)\widetilde{Y}+\mathsf{Ra}_E F(z)k^2\widetilde{\Theta}=0,\\
\mathsf{Ra}_E F(z)\widetilde{W}+D\widetilde{\Theta}-g(z)(D^2-k^2)\widetilde{\Theta}+Hg(z)(\widetilde{\Theta}-\widetilde{\Phi})=0,\\
D\widetilde{\Phi}-g(z)(D^2-k^2)\widetilde{\Phi}-H\gamma g(z)(\widetilde{\Theta}-\widetilde{\Phi})=0,\\
D\mathsf{Ra}_E=0,\\
(D^2-k^2)\widetilde{W}_k-2k\widetilde{W}-\widetilde{Y}_k=0,\\
-\widetilde{Y}_k+\mathsf{Da}(D^2-k^2)\widetilde{Y}_k-2k\widetilde{Y}+\mathsf{Ra}_E F(z)k^2\widetilde{\Theta}_k+2\mathsf{Ra}_E F(z)k\widetilde{\Theta}=0,\\
\mathsf{Ra}_E F(z)\widetilde{W}_k+D\widetilde{\Theta}_k-g(z)(D^2-k^2)\widetilde{\Theta}_k+2g(z)k\widetilde{\Theta}+Hg(z)(\widetilde{\Theta}_k-\widetilde{\Phi}_k)=0,\\
D\widetilde{\Phi}_k-g(z)(D^2-k^2)\widetilde{\Phi}_k+2g(z)k\widetilde{\Phi}-H\gamma g(z)(\widetilde{\Theta}_k-\widetilde{\Phi}_k)=0,\\
Dk=0,
\end{cases}
\tag{7.69}
$$

where the following boundary conditions are added to (7.65)-(7.66)

$$
\begin{aligned}
&\widetilde{Y}_k=\widetilde{W}_k=\widetilde{\Theta}_k=\widetilde{\Phi}_k=0,\ \text{on } z=0,1,\\
&D\widetilde{\Theta}_k=0,\ \text{on } z=0,
\end{aligned}
\tag{7.70}
$$

and where the subscript k denotes the partial derivative with respect to the wavenumber k. Finally, system (7.65), (7.66), (7.69), (7.70) is solved for each choice of $\mu > 1$.

7.4 NUMERICAL SIMULATIONS AND CONCLUSIONS

This section is devoted to numerical results obtained from the application of Chebyshev-τ method and shooting method to differential eigenvalue problems (7.36), (7.37) and (7.69), (7.65), (7.66), (7.70), respectively. We discuss the effect of parameters characterising the studied physical problem on the onset of thermal instability. First of all, let us remark that, given the difficulty in assigning a precise value to the interaction heat transfer coefficient H, we set an interval where H can vary, following the choice of [58]. Therefore, we set $H \in (10^{-4}, 10^4)$. This choice will allow us to have a better insight of the effect of parameters on the critical Rayleigh number Ra_L^2, defined as a function of $\log_{10} H$. In Figure 7.1, the neutral stability curves for quoted values of the upper plane temperature T_U are depicted. The stabilising effect of T_U on the occurrence of penetrative convection is highlighted. This behaviour is expected from a physical viewpoint given that an increasing upper plane temperature results in a thinning of the potentially unstable fluid

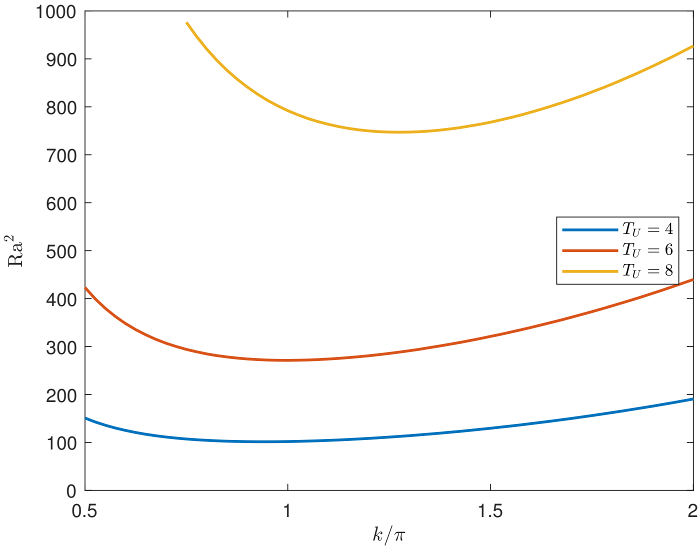


Figure 7.1: Neutral stability curves for quoted values of the upper plane temperature T_U, with Da $= 0.01, \gamma = 10$ and $H = 100$.

region, which is the part of the fluid where its temperature is in the interval $(0, 4)^\circ C$. Consequently, this fluid portion struggles to penetrate the upper stable fluid region, resulting in a delay of the onset of penetrative convection. The stabilising effect of T_U is also evident in Figure 7.2, where, unlike the previous case, H is varying in the aforementioned interval. Hence, Figure 7.2 shows the variation of the critical Rayleigh number from linear analysis with respect to $\log_{10} H$ for four values of T_U. It is worth remarking the monotonic behaviour of Ra_L with H, which is a typical trend in problems in the LTNE regime [17]. Moreover, let us notice that the critical Rayleigh number tends asymptotically to constant values in the limit for both large and small H. Physically, as H approaches 0 the fluid acts independently from the solid phase as they are not exchanging heat, whereas as $H \to \infty$, the solid and fluid phases exchange heat so fast that they reach thermal equilibrium immediately and they can be considered as a single phase. In both cases, we recover the LTE regime. As a result, the respective mathematical problems are identical except for a rescaling of Ra_L. The rescaling factor has been determined analytically in [17]. Let us remark that, following definition

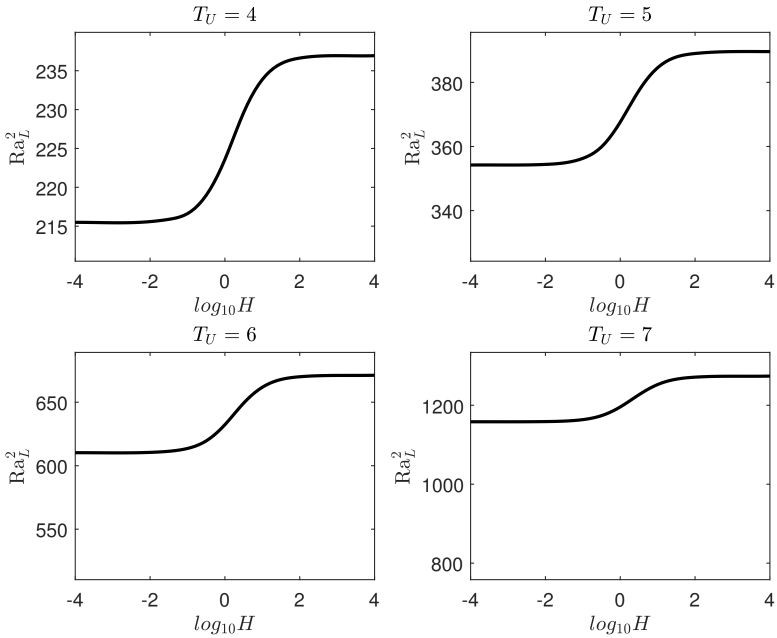


Figure 7.2: Critical linear Rayleigh number as function of $\log_{10} H$ for quoted values of the upper plane temperature T_U, with $\mathsf{Da} = 0.1$ and $\gamma = 10$.

(7.11), Ra^2 is the Rayleigh number based on fluid properties, while the rescaled Rayleigh number is based on the porous medium properties:

$$\frac{\gamma}{\gamma+1}\mathsf{Ra}^2 = \frac{2g\rho_0\alpha k(\rho c)_f d T_U^2}{\mu\left[\varepsilon k_f + (1-\varepsilon)k_s\right]} \tag{7.71}$$

In Figure 7.3 we report a comparison between the critical Rayleigh number and the rescaled critical Rayleigh number. As expected, the rescaled Rayleigh number when $H \to \infty$ approaches Ra_L in the limit for small H values. This agreement is also evident in Table 7.1, where results obtained for this problem are compared with findings in [2]. Table 7.1 shows a very good agreement, confirming the goodness of the numerical procedure employed in the present chapter. Figure 7.4 shows the destabilising effect of γ on the onset of penetrative convection. This behaviour is well-known in literature and it is physically reasonable. Large γ implies that the weighted fluid thermal conductivity is higher than the solid one. Therefore, heat diffusion is quicker within the fluid

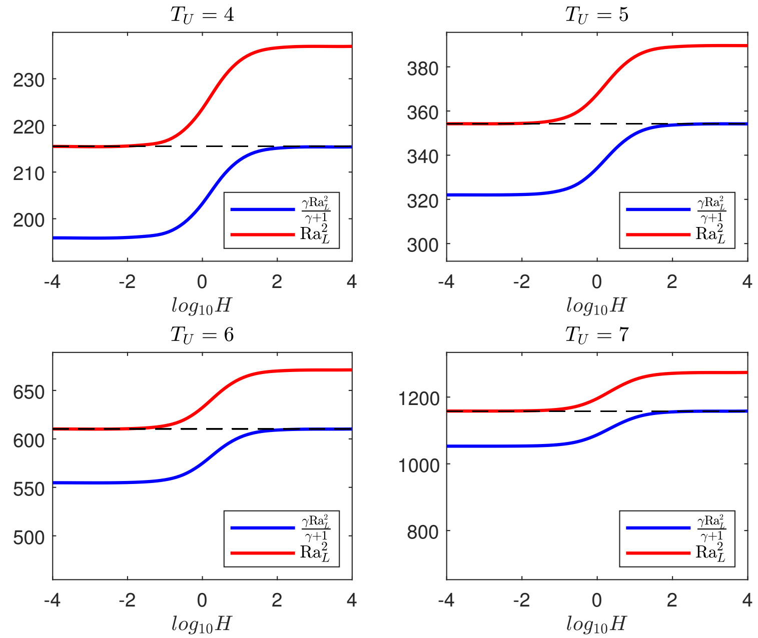


Figure 7.3: Comparison between critical Rayleigh number and rescaled critical Rayleigh number as functions of $\log_{10} H$ for quoted values of the upper plane temperature T_U, with Da $= 0.1$ and $\gamma = 10$.

phase, resulting in a easier occurrence of thermal instability.

Regarding the effect of high porosity on the onset of instability, Figure 7.5 describes the behaviour of neutral stability curves for prescribed values of the Darcy number. As expected, increasing Da leads to higher critical Rayleigh number. Table 7.2 and Figure 7.6 are meant to compare results obtained from linear and nonlinear stability analyses. In Figure 7.6, the neutral stability curves from the linear analysis are plotted together with curves obtained solving the Euler-Lagrange equations coming from a weighted energy analysis.

In summary, our results show that

- in the limit cases, $H \to 0$ and $H \to \infty$, i.e. in LTE, the instability thresholds coincide with the ones found in [2];
- the principle of exchange of stabilities has been rigorously proved;

Ra_L^2[2]	Ra_L^2 $(H=0)$	$\frac{\gamma \mathsf{Ra}_L^2}{\gamma+1}$ $(H=\infty)$	$T_U(°C)$
77.079	77.071	77.065	4
123.462	123.447	123.450	5
198.030	198.009	198.026	6
313.547	313.531	313.527	7
471.384	471.331	471.338	8
672.119	672.072	672.050	9
921.929	921.882	921.850	10

Table 7.1: Critical Rayleigh numbers in the limit of LTE compared with the ones obtained in [2], with $\mathsf{Da}=0$ and $\gamma=1$, for quoted values of T_U.

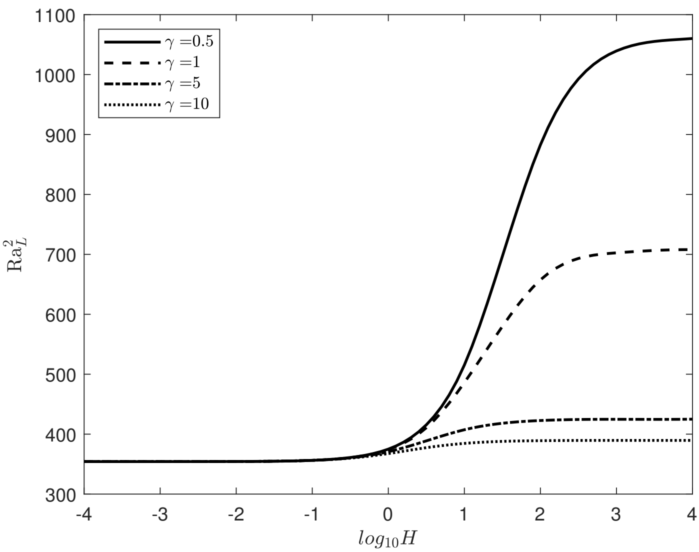


Figure 7.4: Critical linear Rayleigh number as function of $\log_{10} H$ for quoted values of γ, with $\mathsf{Da}=0.1$ and $T_U=5$.

- the upper bounding plane temperature and Darcy's number have both a stabilising effect on the onset of penetrative convection;

$\zeta = 1\,(T_U = 4)$			$\zeta = 0.8\,(T_U = 5)$			$\zeta \simeq 0.6\,(T_U = 6)$			
μ	$\mathrm{Ra}^2_{E,c}$	Ra^2_L	μ	$\mathrm{Ra}^2_{E,c}$	Ra^2_L	μ	$\mathrm{Ra}^2_{E,c}$	Ra^2_L	Da
1.110	136.089	140.583	1.018	203.914	223.555	1.004	282.691	352.754	0
1.118	163.836	169.692	1.022	247.388	273.790	1.005	347.809	447.179	0.01
1.132	384.144	400.464	1.031	584.823	657.492	1.008	837.449	1102.020	0.1

Table 7.2: Comparison between linear and nonlinear critical Rayleigh numbers for quoted values of T_U and Da, with $H = 100$ and $\gamma = 1$.

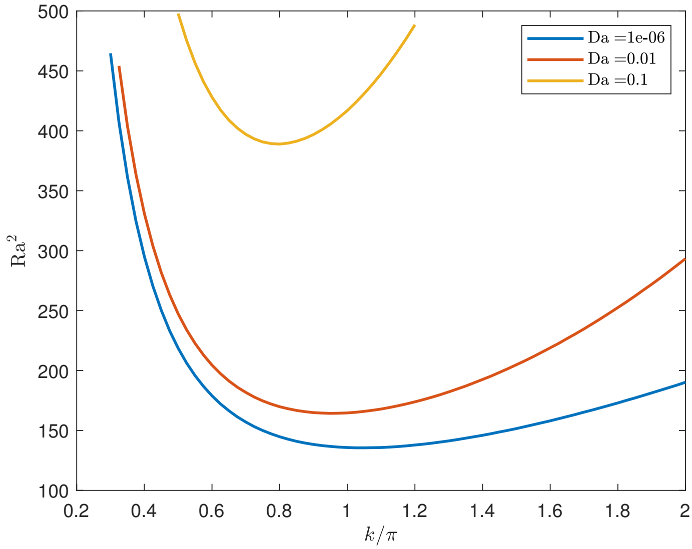


Figure 7.5: Neutral stability curves for quoted values of the Darcy number Da, with $T_U = 5, \gamma = 10$ and $H = 100$.

- in LTNE the instability thresholds show a monotonic trend when H increases;
- the nonlinear stability analysis, via weighted energy method has been performed.

7.5 APPENDIX

Let us consider the following system of partial differential equations

$$\begin{cases} \dfrac{\partial T^f}{\partial t} + \mathbf{v} \cdot \nabla T^f = \Delta T^f + H(T^s - T^f), \\ A\dfrac{\partial T^s}{\partial t} = \Delta T^s - H\gamma(T^s - T^f), \\ \nabla \cdot \mathbf{v} = 0, \end{cases} \tag{7.72}$$

and the following periodic boundary conditions

$$\begin{aligned} &T^s = T^f = 0 \quad z = 0, \\ &T^s = T^f = \mathrm{Ra} \quad z = 1, \\ &\mathbf{v} \cdot \mathbf{n} = 0 \quad z = 0, 1, \end{aligned} \tag{7.73}$$

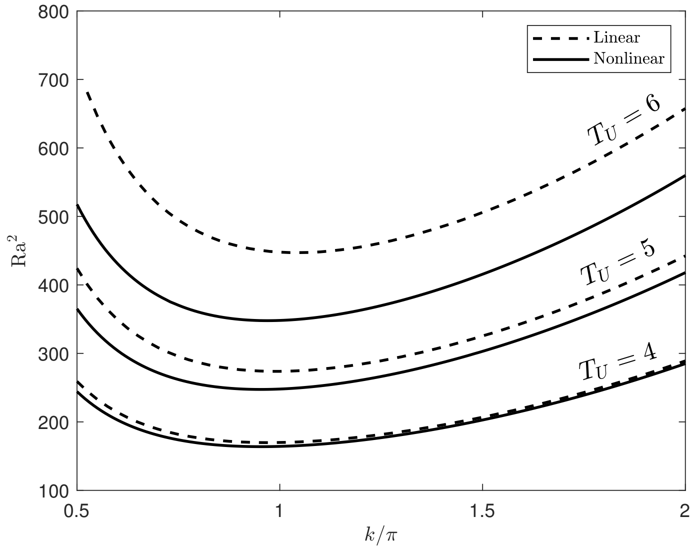


Figure 7.6: Linear and nonlinear marginal stability curves for quoted values of the upper plane temperature T_U with $\mathsf{Da} = 0.01$, $H = 100$ and $\gamma = 1$.

and initial conditions

$$\begin{aligned} T^f(\mathbf{x},0) &= T_0^f(\mathbf{x}), \quad \mathbf{x} \in \Omega, \\ T^s(\mathbf{x},0) &= T_0^s(\mathbf{x}), \quad \mathbf{x} \in \Omega, \\ \mathbf{v}(\mathbf{x},0) &= \mathbf{v}_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \end{aligned} \tag{7.74}$$

with $\nabla \cdot \mathbf{v}_0 = 0$. Let $r \in \{f, s\}$, and let us introduce the following truncated function

$$(T^r(\mathbf{x},t) - \mathsf{Ra})_+ = \begin{cases} T^r(\mathbf{x},t) - \mathsf{Ra} & \text{if} \quad T^r(\mathbf{x},t) > \mathsf{Ra}, \\ 0 & \text{otherwise.} \end{cases} \tag{7.75}$$

By multiplying $(7.72)_1$ by $[(T^f - \mathsf{Ra})_+]^{2n-1}$ and integrate over the periodicity cell Ω, we get

$$\begin{aligned} \int_\Omega \frac{\partial T^f}{\partial t}[(T^f - \mathsf{Ra})_+]^{2n-1}\, d\Omega + \int_\Omega \mathbf{v} \cdot \nabla T^f [(T^f - \mathsf{Ra})_+]^{2n-1}\, d\Omega \\ = \int_\Omega \Delta T^f [(T^f - \mathsf{Ra})_+]^{2n-1}\, d\Omega \\ + H \int_\Omega (T^s - T^f)[(T^f - \mathsf{Ra})_+]^{2n-1}\, d\Omega. \end{aligned} \tag{7.76}$$

The boundary conditions permit us to write (7.76) as

$$\frac{1}{2n}\frac{d}{dt}\int_\Omega [(T^f-\mathsf{Ra})_+]^{2n}\,d\Omega = -(2n-1)\int_\Omega [(T^f-\mathsf{Ra})_+]^{2n-2}(\nabla T^f)^2\,d\Omega$$
$$\underline{+H\int_\Omega T^s[(T^f-\mathsf{Ra})_+]^{2n-1}\,d\Omega - H\int_\Omega T^f[(T^f-\mathsf{Ra})_+]^{2n-1}\,d\Omega.} \tag{7.77}$$

Following the same ideas, if we multiply equation $(7.72)_2$ by $[(T^s-\mathsf{Ra})_+]^{2n-1}$ and integrate over the periodicity cell Ω, we obtain

$$\frac{A}{\gamma}\frac{1}{2n}\frac{d}{dt}\int_\Omega [(T^s-\mathsf{Ra})_+]^{2n}\,d\Omega = -\frac{1}{\gamma}(2n-1)\int_\Omega [(T^s-\mathsf{Ra})_+]^{2n-2}(\nabla T^s)^2\,d\Omega$$
$$\underline{-H\int_\Omega T^s[(T^s-\mathsf{Ra})_+]^{2n-1}\,d\Omega + H\int_\Omega T^f[(T^s-\mathsf{Ra})_+]^{2n-1}\,d\Omega.} \tag{7.78}$$

If we now sum the underlined terms in Eqs. (7.77) and (7.78), we notice, recalling the useful inequality $(|a|^{p-2}a-|b|^{p-2}b)(a-b)\geq 0$, that

$$\begin{aligned}
&H\int_\Omega [T^s-T^f]\left([(T^f-\mathsf{Ra})_+]^{2n-1}-[(T^s-\mathsf{Ra})_+]^{2n-1}\right)d\Omega\\
&\quad= -H\int_\Omega [T^f-T^s]\left([(T^f-\mathsf{Ra})_+]^{2n-1}-[(T^s-\mathsf{Ra})_+]^{2n-1}\right)d\Omega\\
&\quad= -H\int_\Omega [(T^f-\mathsf{Ra})-(T^s-\mathsf{Ra})]\left([(T^f-\mathsf{Ra})_+]^{2n-1}-[(T^s-\mathsf{Ra})_+]^{2n-1}\right)d\Omega\\
&\quad= -H\int_\Omega [(T^f-\mathsf{Ra})_+-(T^s-\mathsf{Ra})_+]\left([(T^f-\mathsf{Ra})_+]^{2n-1}-[(T^s-\mathsf{Ra})_+]^{2n-1}\right)d\Omega\\
&\quad\leq 0,
\end{aligned} \tag{7.79}$$

and, as a consequence, setting

$$\mathcal{F}(T^f,T^s) := \int_\Omega \left\{[(T^f-\mathsf{Ra})_+]^{2n}+\frac{A}{\gamma}[(T^s-\mathsf{Ra})_+]^{2n}\right\}d\Omega, \tag{7.80}$$

if we sum Eq. (7.77) to Eq. (7.78), we have

$$\begin{aligned}
\frac{1}{2n}\frac{d\mathcal{F}}{dt} &= -(2n-1)\int_\Omega [(T^f-\mathsf{Ra})_+]^{2n-2}(\nabla T^f)^2\,d\Omega\\
&\qquad -\frac{1}{\gamma}(2n-1)\int_\Omega [(T^s-\mathsf{Ra})_+]^{2n-2}(\nabla T^s)^2\,d\Omega\\
&\quad -H\int_\Omega [(T^f-\mathsf{Ra})_+-(T^s-\mathsf{Ra})_+]\left([(T^f-\mathsf{Ra})_+]^{2n-1}-[(T^s-\mathsf{Ra})_+]^{2n-1}\right)d\Omega\\
&\quad\leq 0.
\end{aligned} \tag{7.81}$$

Inequality (7.81) shows that

$$\frac{d}{dt}\mathcal{F}^{\frac{1}{2n}} = \frac{1}{2n}\frac{d\mathcal{F}}{dt}\mathcal{F}^{\frac{1}{2n}-1} \leq 0, \tag{7.82}$$

hence

$$\left[\mathcal{F}(T^f, T^s)\right]^{\frac{1}{2n}} \leq \left[\mathcal{F}(T_0^f, T_0^s)\right]^{\frac{1}{2n}}. \tag{7.83}$$

Recalling that $\lim_{p\to\infty}\left(\int_\Omega |f|^p + |g|^p \, d\Omega\right)^{\frac{1}{p}} \leq \|f\|_\infty + \|g\|_\infty$, from (7.83) we have, assuming $T_0^f, T_0^s \in L^\infty$, that

$$\begin{aligned}\operatorname*{ess\,sup}_\Omega \left\{(T^f - \mathsf{Ra})_+\right\} \leq \operatorname*{ess\,sup}_\Omega \left\{(T_0^f - \mathsf{Ra})_+\right\} \\ + \frac{A}{\gamma}\operatorname*{ess\,sup}_\Omega \left\{(T_0^s - \mathsf{Ra})_+\right\} (< \infty).\end{aligned} \tag{7.84}$$

Let us now consider $T^f(\mathbf{x}, t) = \theta(\mathbf{x}, t) + T_b(z)$, $T^s(\mathbf{x}, t) = \phi(\mathbf{x}, t) + T_b(z)$, with associated initial data $T_0^f(\mathbf{x}) = \theta_0(\mathbf{x}) + T_b(z)$, $T_0^s(\mathbf{x}) = \phi_0(\mathbf{x}) + T_b(z)$. Therefore, inequality (7.84) becomes

$$\begin{aligned}\operatorname*{ess\,sup}_\Omega \left\{(\theta + T_b - \mathsf{Ra})_+\right\} \leq \operatorname*{ess\,sup}_\Omega \left\{(\theta_0 + T_b - \mathsf{Ra})_+\right\} \\ + \frac{A}{\gamma}\operatorname*{ess\,sup}_\Omega \left\{(\phi_0 + T_b - \mathsf{Ra})_+\right\},\end{aligned} \tag{7.85}$$

and setting

$$\begin{aligned}\bar{\theta}_0 &:= \operatorname*{ess\,sup}_\Omega \left\{(\theta_0 + T_b - \mathsf{Ra})_+\right\} (< \infty), \\ \bar{\phi}_0 &:= \frac{A}{\gamma}\operatorname*{ess\,sup}_\Omega \left\{(\phi_0 + T_b - \mathsf{Ra})_+\right\} (< \infty),\end{aligned} \tag{7.86}$$

we can conclude that

$$\|\theta + T_b - \mathsf{Ra}\|_{L^\infty(\Omega_1)} \leq \Gamma, \tag{7.87}$$

i.e.

$$\theta(\mathbf{x}, t) + T_b(z) - \mathsf{Ra} \leq \Gamma, \qquad \text{a.e. on } \Omega_1, \tag{7.88}$$

where

$$\Gamma = \begin{cases} \bar{\theta}_0 & \text{if} \quad \phi_0 \leq \mathsf{Ra} - T_b, \\ \bar{\theta}_0 + \bar{\phi}_0 & \text{otherwise,} \end{cases} \tag{7.89}$$

and the Lemma is proved.

Part III

RECENT RESULTS ON EXTENDED-QUASI-THERMAL-INCOMPRESSIBLE FLUID'S FLOW IN POROUS MEDIA

8 AN EXISTENCE AND UNIQUENESS RESULT

As already remarked in the introductory part, Gouin and Ruggeri in [57], enforcing some essential thermodynamic conditions (namely *entropy principle* and *thermodynamic stability*) introduce the class of *extended-quasi-thermal-incompressible fluids*, and they proposed as a significant case to take into account the following density law:

$$\rho(p, T) = \rho_0[1 - \alpha(T - T_0) + \beta(p - p_0)], \tag{8.1}$$

where p and T are the pressure and temperature fields, respectively, ρ_0, T_0 and p_0 are the reference density, temperature and pressure value, respectively, and α and β are the thermal expansion coefficient, and compressibility factor, defined respectively by

$$\alpha = \frac{1}{V} V_T, \quad \beta = -\frac{1}{V} V_p. \tag{8.2}$$

One consequence of introducing this more comprehensive scheme is that the well-posedness of the corresponding mathematical model requires the pressure to be treated as an *independent* unknown instead as a Lagrange multiplier associated with the incompressibility constraint. For this reason the pressure will satisfy a suitable elliptic problem and will be subjected to Robin boundary conditions. The well-posedness and stability of the nonlinear perturbation system in the classical Bénard problem for slightly compressible fluids is addressed in [36, 38, 96]. Moreover, let us note that in recent time, the constitutive density (8.1) attracted considerable attention in the framework of hydrodynamic stability, in particular because the extra buoyancy term which gives a more accurate description of fluids which are slightly compressible. From recent investigations addressed in the contest of thermal convection problems in clear fluids and fluid-saturated porous media, see e.g. [97, 36], it is proved that the compressibility factor has a *destabilizing effect* on the onset of convection.
The objective of the present chapter is to investigate the well-posedness of the initial-boundary value problem modelling the nonlinear bi-dimensional perturbation of the steady state solution of a slightly compressible fluid-saturated porous medium. More specifically, the chapter is organized as follows. In section 8.1 the Darcy-Bénard problem for

slightly incompressible fluid-saturated porous media is introduced, and the steady state solution as well as the perturbed non-dimensional system are computed. Moreover, the Poisson pressure equation is introduced, together with the corresponding Robin boundary conditions, and a suitable change of variable is employed. In section 8.2, after after recalling some previous findings and known inequalities, we face with the proof of the existence and uniqueness of the nonlinear perturbed problem, taking advantage of several usual analytical techniques, e.g. derivation of *a priori* estimates and use of the Galerkin method with a suitable basis. Moreover, we prove that for "sufficiently small" Rayleigh numbers all solutions must decay exponentially to zero as time increase, proving the nonlinear stability stability of the conduction solution.

8.1 FORMULATION OF THE INITIAL BOUNDARY VALUE PROBLEM

Let us consider a reference frame $Oxyz$ with fundamental unit vectors $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and a horizontal layer $\Omega = \mathbb{R}^2 \times [0, d]$ of fluid-saturated porous medium, whose boundary will be indicated by $\partial\Omega = \Gamma = \Gamma_L \cup \Gamma_U$. To derive the governing equations for the seepage velocity $\mathbf{v}$, the temperature field T and the pressure field p, let us employ the modified Oberbeck-Boussinesq approximation, [57]:

- the fluid density ρ is constant in all terms of the governing equations (i.e. $\rho = \rho_0$), except in the bouyancy term;
- in the body force term, the fluid density is (8.1);
- $\nabla \cdot \mathbf{v} = 0$ and $\mathbf{D} : \mathbf{D} \approx 0$.

Now, the mathematical model, according to the Darcy's law, is given by

$$\begin{cases} \dfrac{\mu}{K}\mathbf{v} = -\nabla p - \rho_0[1 - \alpha(T - T_0) + \beta(p - p_0)]g\mathbf{k} \\ \nabla \cdot \mathbf{v} = 0 \\ \rho c_V \Big(\dfrac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T\Big) = \chi \Delta T \end{cases} \tag{8.3}$$

where μ, K, c_a, χ, c_V are fluid viscosity, permeability of the porous body, acceleration coefficient, thermal diffusivity and specific heat at constant volume, respectively. To system (8.3) the following initial conditions

$$\mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}), \quad T(\mathbf{x}, 0) = T_0(\mathbf{x}), \quad p(\mathbf{x}, 0) = p_0(\mathbf{x}), \tag{8.4}$$

and boundary conditions:

$$\begin{aligned} \mathbf{v}\cdot\mathbf{k} = 0 &\quad \text{on } z = 0, d \\ T = T_L &\quad \text{on } z = 0 \\ T = T_U &\quad \text{on } z = d \end{aligned} \tag{8.5}$$

with $T_L > T_U$, are appended. Conditions $(8.5)_1$ tells us that the boundaries are impermeable, while condition $(8.5)_{2,3}$ that the boundaries are isothermal and we assume that the boundaries are isobaric too. Moreover, we can assume free boundary condition (i.e. boundary free from tangential stress).

8.1.1 *Steady state and perturbed non-dimensional formulation*

System (8.3)-(8.5) admits the stationary conduction solution

$$\begin{aligned} &\mathbf{v}_b = \mathbf{0},\ T_b(z) = T_L - \frac{T_L - T_U}{d} z, \\ &p_b(z) = p_0 + \bar{p} e^{-\rho_0 g \beta z} + \frac{1}{\beta^2} \frac{\alpha(T_L - T_U)}{\rho_0 g d} \left(1 - e^{-\rho_0 g \beta z}\right) \\ &\qquad - \frac{1}{\beta}\left(\frac{\alpha(T_L - T_U)}{d} z + 1 - e^{-\rho_0 g \beta z}\right). \end{aligned} \tag{8.6}$$

where p_0 is the gauge pressure and $\bar{p}$ is a prescribed value arising from the appropriate boundary conditions on p.

Let $(\mathbf{u}, \theta, \pi)$ be a perturbation to the basic solution ($\mathbf{u} = \mathbf{v} - \mathbf{v}_b$, $\theta = T - T_b$ and $\pi = p - p_b$), with $\mathbf{u} = (u, v, w)$. Then, the equations governing the perturbation fields are

$$\begin{cases} \dfrac{\mu}{K}\mathbf{u} = -\nabla\pi + \rho_0 \alpha g \theta \mathbf{k} - \rho_0 \beta g \pi \mathbf{k} \\ \nabla\cdot\mathbf{u} = 0 \\ \dfrac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = \dfrac{T_L - T_U}{d}\mathbf{u}\cdot\mathbf{k} + k\Delta\theta \end{cases} \tag{8.7}$$

where $k = \frac{\chi}{\rho c_V}$. To system (8.7) the following initial conditions

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \theta(\mathbf{x}, 0) = \theta_0(\mathbf{x}), \quad \pi(\mathbf{x}, 0) = \pi_0(\mathbf{x}), \tag{8.8}$$

and boundary conditions are appended:

$$\begin{aligned} \mathbf{u}\cdot\mathbf{k} = 0 &\quad \text{on } z = 0, d \\ \theta = 0 &\quad \text{on } z = 0, d. \end{aligned} \tag{8.9}$$

Then system (8.3) is fulfilled by the null perturbation. Let us introduce the following scales

$$\pi = P\pi^*, \quad \mathbf{u} = U\mathbf{u}^*, \quad \theta = T^\#\theta^*, \quad t = \tau t^*, \quad x = dx^*,$$

where

$$P = \frac{\mu k}{K}, \quad U = \frac{k}{d}, \quad T^\# = T_L - T_U, \quad \tau = \frac{d^2}{k}.$$

Therefore, the corresponding dimensionless system of equations, omitting all the stars, is the following:

$$\begin{cases} \mathbf{u} = -\nabla\pi + \mathsf{Ra}\theta\mathbf{k} - \widehat{\beta}\pi\mathbf{k} \\ \nabla\cdot\mathbf{u} = 0 \\ \dfrac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = w + \Delta\theta \end{cases} \tag{8.10}$$

where

$$\mathsf{Ra} = \frac{\rho_0\alpha g d(T_L - T_U)K}{\mu k}, \quad \widehat{\beta} = \rho_0 g d\beta$$

are the Darcy-Rayleigh number and the dimensionless compressibility factor, respectively. To system (8.10) the following boundary conditions are appended:

$$w = \theta = 0 \quad \text{on } z = 0, 1. \tag{8.11}$$

8.1.2 *The Poisson pressure equation*

It is well known that the pressure field in case of incompressible flow can be recognized as the Lagrangian constraint variable that enforces the divergence-free constraint. In this new scheme, the well-posedness of the associated mathematical problem requires now the pressure field to be treated as an *independent unknown*, satisfying a suitable elliptic problem and subject to Robin boundary conditions. Indeed, by taking the divergence of $(8.10)_1$ we have

$$\nabla\cdot\mathbf{u} = -\nabla\cdot\nabla\pi + \mathsf{Ra}\nabla\cdot\theta\mathbf{k} - \widehat{\beta}\nabla\cdot\pi\mathbf{k},$$

hence

$$\Delta\pi + \widehat{\beta}\frac{\partial\pi}{\partial z} = \mathsf{Ra}\frac{\partial\theta}{\partial z}, \tag{8.12}$$

whose boundary conditions are, following [110]

$$\frac{\partial p}{\partial\mathbf{n}}\Big|_\Gamma = \nabla\pi\cdot\mathbf{n} = \nabla\pi\cdot\mathbf{k} = (-\mathbf{u} + \mathsf{Ra}\theta\mathbf{k} - \widehat{\beta}\pi\mathbf{k})\cdot\mathbf{k}\Big|_\Gamma,$$

hence the *natural* boundary conditions for the pressure Poisson problem (8.12) are of Robin type

$$\frac{\partial \pi}{\partial z} + \widehat{\beta}\pi = 0. \tag{8.13}$$

Therefore, we replace the continuity constraint (8.10) with the derived pressure Poisson equation (8.12), so we obtain

$$\begin{cases} \mathbf{u} = -\nabla\pi + \mathsf{Ra}\theta\mathbf{k} - \widehat{\beta}\pi\mathbf{k} \\ \Delta\pi + \widehat{\beta}\dfrac{\partial \pi}{\partial z} = \mathsf{Ra}\dfrac{\partial \theta}{\partial z} \\ \dfrac{\partial \theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = w + \Delta\theta \end{cases} \tag{8.14}$$

with associated boundary conditions

$$w = \theta = \frac{\partial \pi}{\partial z} + \widehat{\beta}\pi = 0 \quad \text{on } z = 0, 1. \tag{8.15}$$

If we consider the change of variable $\Pi = e^{\widehat{\beta}z}\pi$, then system (8.14) turns into

$$\begin{cases} \mathbf{u} = -e^{-\widehat{\beta}z}\nabla\Pi + \mathsf{Ra}\theta\mathbf{k} \\ \Delta\Pi - \widehat{\beta}\dfrac{\partial \Pi}{\partial z} = \mathsf{Ra}e^{-\widehat{\beta}z}\dfrac{\partial \theta}{\partial z} \\ \dfrac{\partial \theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = w + \Delta\theta \end{cases} \tag{8.16}$$

with initial conditions

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \theta(\mathbf{x}, 0) = \theta_0(\mathbf{x}), \quad \Pi(\mathbf{x}, 0) = \Pi_0(\mathbf{x}), \tag{8.17}$$

and the Robin condition for the pressure turned into Neumann conditions

$$w = \theta = \frac{\partial \Pi}{\partial z} = 0 \quad \text{on } z = 0, 1. \tag{8.18}$$

We are now in a full coupling contest and note that t is just a parameter for the elliptic problem $(8.16)_2$ and (8.18).

In the sequel we will focus on bi-dimensional perturbations in the plane (x, z) and we assume the periodicity of the perturbation in the x-direction.

8.2 WELL-POSEDNESS OF THE PROBLEM

Let us consider the full-nonlinear system

$$\begin{cases} \Delta\Pi - \widehat{\beta}\Pi_z = \mathsf{Ra}e^{\widehat{\beta}z}\theta_z \\ \mathbf{u} = -e^{-\widehat{\beta}z}\nabla\Pi + \mathsf{Ra}\theta\mathbf{k} \\ \theta_t + \mathbf{v}\cdot\nabla\theta = w + \Delta\theta \end{cases} \tag{8.19}$$

together with boundary conditions (8.18). In the following sections we will give some notations and preliminaries in order step by step prove the existence and uniqueness of weak solutions of (8.19).

8.2.1 *Some notation and preliminaries*

In the following, since we are interested in bi-dimensional flow, we will denote

$$\Omega_0 = \{(x, y) \in [0, 1] \times [0, 1]\}.$$

Let us consider $\mathcal{B} = \{\phi^i_{m,n}\}_{m,n\geq 0}$ such that

$$\phi^i_{m,n}(x, z) = \begin{cases} \cos(2\pi mx)\cos(\pi nz), & i = 1, \\ \sin(2\pi mx)\cos(\pi nz), & i = -1, \end{cases} \tag{8.20}$$

and $\mathcal{D} = \{\xi^i_{m,n}\}_{m,n\geq 0}$ such that

$$\xi^i_{m,n}(x, z) = \begin{cases} \cos(2\pi mx)\sin(\pi nz), & i = 1, \\ \sin(2\pi mx)\sin(\pi nz), & i = -1. \end{cases} \tag{8.21}$$

Note that the functions in the basis $\mathcal{B}$ have mean value zero, so we can write p in this basis. Moreover, for temperature field θ and the stream function Φ associated with $\mathbf{v}$ by $\mathbf{u} = (-\Phi_z, \Phi_x)$ we can use the basis $\mathcal{D}$. We can construct Sobolev space from basis (8.20) and (8.21). In particular, we denote by $\widetilde{W}^{k,2}(\Omega_0)$ the closure with respect to the $W^{k,2}$-norm of the finite combinations of elements of the basis $\mathcal{B}$ and by $\widehat{W}^{k,2}(\Omega_0)$ the closure with respect to the $W^{k,2}$-norm of the finite combinations of elements of the basis $\mathcal{D}$, $k = 1, 2$. Moreover, we denote by $\widehat{\mathcal{W}}^{1,2}(\Omega_0)$ the closure of the linear hull of the vectorial divergence-free functions obtained from $\mathcal{D}$. Let us recall the Poincaré inequality for Π and θ:

$$\|\Pi\|_{L^2} \leq \frac{1}{2\pi}\|\nabla\Pi\|_{L^2}, \quad \|\theta\|_{L^2} \leq \frac{1}{\sqrt{5}\pi}\|\nabla\theta\|_{L^2},$$

as well as the Ladyzhenskaya's inequality

$$\|\mathbf{v}\|_{L^4} \leq c\|\mathbf{v}\|^{1/2}_{L^2}\|\nabla\mathbf{v}\|^{1/2}_{L^2}$$

with c positive constant, and the following equivalence of norms, [38]

$$\frac{1}{16}\|\Delta\mathbf{v}\|_{L^2} \leq \|D^2\mathbf{v}\|_{L^2} \leq \frac{1}{4}\|\Delta\mathbf{v}\|_{L^2}.$$

8.2.2 *Existence and uniqueness for the reduced problem*

As a fist result, let us recall the following theorem proving existence and uniqueness of the reduced problem for the pressure.

Theorem 8.2.1. Let $f \in \widetilde{L}^2(\Omega_0)$ and assume $0 \leq \widehat{\beta} < 2\pi$. Then problem

$$\begin{cases} \Delta\Pi - \widehat{\beta}\Pi_z = e^{\widehat{\beta}z} f, & \text{in } \Omega \\ \Pi_z(x,0) = \Pi_z(x,1) = 0, & \text{in } x \in \mathbb{R}, \end{cases} \tag{8.22}$$

admits a unique solution $\Pi \in \widetilde{W}^{2,2}(\Omega_0)$. Moreover, Π satisfies the following estimates

$$\|\nabla\Pi\|_{L^2} \leq \frac{1}{2\pi - \widehat{\beta}} \left\| e^{\widehat{\beta}z} f \right\|_{L^2}, \quad \|\Delta\Pi\|_{L^2} \leq \frac{2\pi}{2\pi - \widehat{\beta}} \left\| e^{\widehat{\beta}z} f \right\|_{L^2}. \tag{8.23}$$

Proof. A proof of the following theorem can be found in [38]. However, we will recover the proof in order to let the content of the present chapter as clear and complete as possible. Let us consider the test function space $H = \{\varphi \in W^{1,2}(\Omega_0) \ : \ \langle\varphi\rangle = 0\}$. Then, multiplying both sides of $(8.19)_1$ by φ and integrating we obtain

$$B(\Pi, \varphi) := (\nabla\Pi, \nabla\varphi) + \widehat{\beta}(\Pi_z, \varphi) = -(e^{-\widehat{\beta}z} f, \varphi). \tag{8.24}$$

We can immediately note that

$$\begin{aligned} |B(\Pi, \varphi)| &\leq |(\nabla\Pi, \nabla\varphi)| + |\widehat{\beta}(\Pi_z, \varphi)| \\ &\leq \|\nabla\Pi\|_{L^2} \|\nabla\varphi\|_{L^2} + \widehat{\beta}\|\Pi_z\|_{L^2} \|\varphi\|_{L^2} \\ &\leq c\|\Pi\|_{W^{1,2}} \|\varphi\|_{W^{1,2}}, \end{aligned} \tag{8.25}$$

and

$$\begin{aligned} B(\Pi, \Pi) = \|\nabla\Pi\|_{L^2}^2 + \widehat{\beta}(\Pi_z, \Pi) &\geq \|\nabla\Pi\|_{L^2}^2 - \widehat{\beta}(\Pi_z, \Pi) \\ &\geq \|\nabla\Pi\|_{L^2}^2 - \frac{\widehat{\beta}}{2\pi} \|\nabla\Pi\|_{L^2}^2 \\ &= \left(1 - \frac{\widehat{\beta}}{2\pi}\right) \|\nabla\Pi\|_{L^2}^2, \end{aligned} \tag{8.26}$$

and $1 - \widehat{\beta}/(2\pi) > 0$ if and only if $\widehat{\beta} < 2\pi$. Then, from the Lax-Milgram theorem we have that (8.22) admits a unique solution in $\widetilde{W}^{2,2}_{\text{loc}}(\Omega_0)$. Concerning the estimate $(8.23)_1$ we have that

$$\begin{aligned}
\|\nabla\Pi\|^2_{L^2} &= -\widehat{\beta}(\Pi_z, \Pi) - (e^{\widehat{\beta}z} f, \Pi) \\
&\leq \widehat{\beta}\|\Pi_z\|_{L^2}\|\Pi\|_{L^2} + \left\|e^{\widehat{\beta}z} f\right\|_{L^2} \|\Pi\|_{L^2} \\
&= \left(\widehat{\beta}\|\Pi_z\|_{L^2} + \left\|e^{\widehat{\beta}z} f\right\|_{L^2}\right) \|\Pi\|_{L^2} \\
&\leq \frac{1}{2\pi}\left(\widehat{\beta}\|\Pi_z\|_{L^2} + \left\|e^{\widehat{\beta}z} f\right\|_{L^2}\right) \|\nabla\Pi\|_{L^2} \\
&\leq \frac{1}{2\pi}\left(\widehat{\beta}\|\nabla\Pi\|_{L^2} + \left\|e^{\widehat{\beta}z} f\right\|_{L^2}\right) \|\nabla\Pi\|_{L^2},
\end{aligned} \tag{8.27}$$

hence

$$\|\nabla\Pi\|_{L^2} \leq \frac{1}{2\pi - \widehat{\beta}} \left\|e^{\widehat{\beta}z} f\right\|_{L^2}. \tag{8.28}$$

Regarding $(8.23)_2$, we have that

$$\begin{aligned}
\|\Delta\Pi\|_{L^2} &\leq \|\widehat{\beta}\Pi_z + e^{\widehat{\beta}z} f\|_{L^2} \\
&\leq \widehat{\beta}\|\nabla\Pi\| + \left\|e^{\widehat{\beta}z} f\right\|_{L^2} \\
&\leq \frac{\widehat{\beta}}{2\pi - \widehat{\beta}} \left\|e^{\widehat{\beta}z} f\right\|_{L^2} + \left\|e^{\widehat{\beta}z} f\right\|_{L^2} \\
&= \frac{2\pi}{2\pi - \widehat{\beta}} \left\|e^{\widehat{\beta}z} f\right\|_{L^2} + \left\|e^{\widehat{\beta}z} f\right\|_{L^2}.
\end{aligned} \tag{8.29}$$

In conclusion, the du Bois-Reymond lemma tells us that

$$\Delta\Pi - \widehat{\beta}\Pi_z = e^{\widehat{\beta}z} f, \tag{8.30}$$

for a.e. $(x, z) \in \mathbb{R} \times [0, 1]$. This fact, together with estimate $(8.23)_2$, tells us that $\Pi \in \widetilde{W}^{2,2}(\Omega_0)$. □

8.2.3 *Basic* a priori *estimate*

In this section we will prove a fundamental *a priori* estimate. Let us set

$$E(t) := \frac{\mathsf{Ra}}{2} \frac{d}{dt} \|\theta(t)\|^2_{L^2}. \tag{8.31}$$

Then the following proposition holds:

Proposition 8.2.1. The following *a priori* estimate holds:

$$\frac{\mathsf{Ra}}{2}\frac{d}{dt}\|\theta\|^2 + \mathsf{Ra}\|\nabla\theta\|_{L^2}^2 - A\|\mathbf{u}\|_{L^2}^2 - B\|\nabla\Pi\|_{L^2}^2 \le c_0(\mathsf{Ra},\widehat{\beta})\|\theta\|_{L^2}^2, \tag{8.32}$$

where $c_0 > 0$, $A, B < 0$, and in particular

$$E(t) \le E(0)e^{c_0 t}. \tag{8.33}$$

Moreover, if Ra is sufficiently small, then $E(t)$ decays exponentially.

Proof. Multiplying $(8.19)_1$ by Π, $(8.19)_2$ by $\mathbf{u}$ and $(8.19)_3$ by $\mathsf{Ra}\theta$, after integrating over the periodicity cell, applying Cauchy-Schwartz and generalized Young inequalities and summing the resulting equations we get

$$\begin{aligned}\frac{\mathsf{Ra}}{2}\frac{d}{dt}\|\theta\|^2 = -\|\nabla\Pi\|_{L^2}^2 - \widehat{\beta}(\Pi_z,\Pi) - \mathsf{Ra}(e^{\widehat{\beta}z}\theta_z,\Pi)\\ - \|\mathbf{u}\|^2 + 2\mathsf{Ra}(\theta,w) - (e^{-\widehat{\beta}z}\nabla\Pi,\mathbf{u}) - \mathsf{Ra}\|\nabla\theta\|_{L^2}^2,\end{aligned} \tag{8.34}$$

and it follows from (8.34) that

$$\frac{\mathsf{Ra}}{2}\frac{d}{dt}\|\theta\|^2 + \mathsf{Ra}\|\nabla\theta\|_{L^2}^2 - A\|\mathbf{u}\|_{L^2}^2 - B\|\nabla\Pi\|_{L^2}^2 \le c_0(\mathsf{Ra},\widehat{\beta})\|\theta\|_{L^2}^2 \tag{8.35}$$

with

$$\begin{aligned}A &:= -1 + \frac{1}{2M_1} + \frac{M_2}{2},\\ B &:= -1 + \frac{\widehat{\beta}}{2\pi} + \frac{1}{2M}\left(\frac{1}{2\pi}+1\right) + \frac{1}{2M_2},\\ c_0 &:= \frac{\mathsf{Ra}^2 e^{2\widehat{\beta}}}{2}(\widehat{\beta}^2+1)M + 2\mathsf{Ra}^2 M_1.\end{aligned} \tag{8.36}$$

If we ask $A < 0$ and $B < 0$, the following restrictions hold

$$\begin{aligned}M_1 &> \frac{1}{2-M_2},\\ M_2 &\in \left(\frac{\pi}{2\pi-\widehat{\beta}},2\right),\\ M &> \frac{(2\pi+1)M_2}{2(2\pi-\widehat{\beta}-2\pi)}.\end{aligned} \tag{8.37}$$

As a consequence, from (8.35), we have

$$\frac{d}{dt}E(t) \leq c_0(\mathsf{Ra}, \widehat{\beta})E(t). \tag{8.38}$$

and the Grönwall inequality gives us (8.33). Moreover, setting

$$\hat{c}_0(\mathsf{Ra}, \widehat{\beta}) := \mathsf{Ra}\left(\frac{\mathsf{Ra}e^{2\widehat{\beta}}}{2}(\widehat{\beta}^2+1)M + 2\mathsf{Ra}M_1\right), \tag{8.39}$$

from (8.35) it follows

$$\begin{aligned}\frac{\mathsf{Ra}}{2}\frac{d}{dt}\|\theta(t)\|^2 &\leq -\mathsf{Ra}\|\nabla\theta\|^2_{L^2} + c_0(\widehat{\beta}, \mathsf{Ra})\|\theta\|^2_{L^2}\\ &\leq -10\pi^2\frac{\mathsf{Ra}}{2}\|\theta\|^2_{L^2} + 2\hat{c}_0(\mathsf{Ra}, \widehat{\beta})\frac{\mathsf{Ra}}{2}\|\theta\|^2_{L^2}\\ &= (-\widetilde{c}_1 + \widetilde{c}_0)\,\|\theta\|^2_{L^2},\end{aligned} \tag{8.40}$$

where $\widetilde{c}_1 = 10\pi^2$ and $\widetilde{c}_0 = 2\hat{c}_0$, i.e.

$$\frac{d}{dt}E(t) \leq (\widetilde{c}_0 - \widetilde{c}_1)E(t). \tag{8.41}$$

In conclusion, from a Grönwall type inequality we have

$$E(t) \leq E(0)^{(\widetilde{c}_0-\widetilde{c}_1)t}, \tag{8.42}$$

and if Ra is sufficiently small so that $\widetilde{c}_0 - \widetilde{c}_1 < 0$, i.e. if

$$\mathsf{Ra} < \frac{10\pi^2}{e^{2\widehat{\beta}}(\widehat{\beta}^2+1) + 4M_1}, \tag{8.43}$$

then E decays exponentially, and this completes the proof. □

8.2.4 *Preliminary results*

In this section we will give several preliminary results, which will make the final proof easier to demonstrate.

Lemma 8.2.1. The sequence $\{\theta^N\}_N$ is bounded in $L^\infty\big(0, T; \widetilde{L}^2(\Omega_0)\big)$, i.e. it exists $c_1 > 0$ such that for all N

$$\operatorname{ess\,sup}_t \|\theta^N(t)\|^2_{L^2} \leq c_1.$$

Proof. From the *a priori* estimates given in Proposition 8.2.1, it follows

$$E^N(t) \leq E^N(0)e^{c_0 t}, \tag{8.44}$$

hence

$$\frac{\mathsf{Ra}}{2}\|\theta^N(t)\|^2_{L^2} \leq \frac{\mathsf{Ra}}{2}\|\theta^N(0)\|^2_{L^2}e^{c_0 t} \leq \frac{\mathsf{Ra}}{2}\|\theta(0)\|^2_{L^2}e^{c_0 T} = c_1. \tag{8.45}$$

Then, passing to the ess sup the Lemma is proved. □

Lemma 8.2.2. The sequence $\{\theta^N\}_N$ is bounded in $L^2\big(0,T;\widetilde{L}^2(\Omega_0)\big)$

Proof. It directly follows from Lemma 8.2.1 that

$$\|\theta^N\|^2_{L^2(0,T;\widetilde{L}^2(\Omega_0))} = \int_0^T \|\theta^N(t)\|^2_{L^2}dt \leq Tc_1.$$

□

Lemma 8.2.3. The sequence $\{\nabla\theta^N\}_N$ is bounded in $L^2\big(0,T;\widetilde{L}^2(\Omega_0)\big)$

Proof. From Proposition 8.2.1, we have

$$\frac{\mathsf{Ra}}{2}\dot{E}^N(t) + \mathsf{Ra}\|\nabla\theta^N(t)\|^2_{L^2} \leq c_0(\mathsf{Ra},\widehat{\beta})\|\theta^N(t)\|^2_{L^2}. \tag{8.46}$$

As a consequence

$$\int_0^T \mathsf{Ra}\|\nabla\theta^N(t)\|^2_{L^2}dt \leq \int_0^T c_0\|\theta^N(t)\|^2_{L^2}dt - \frac{\mathsf{Ra}}{2}\int_0^T \dot{E}^N(t)\,dt \tag{8.47}$$

then, from Lemma 8.2.2, it follows

$$\begin{aligned}
\mathsf{Ra}\|\nabla\theta^N\|^2_{L^2(0,T;\widetilde{L}^2(\Omega_0))} &\leq Tc_0c_1 - \frac{\mathsf{Ra}}{2}\big(E^n(t) - E^n(0)\big)\\
&\leq Tc_0c_1 + \frac{\mathsf{Ra}}{2}E^N(0)\\
&\leq \mathsf{Ra}\left(\frac{T}{\mathsf{Ra}}c_0c_1 + \frac{1}{2}\|\theta(0)\|^2_{L^2}\right)\\
&= \mathsf{Ra}c_2
\end{aligned} \tag{8.48}$$

i.e.

$$\|\nabla\theta^N\|^2_{L^2(0,T;\widetilde{L}^2(\Omega_0))} = c_2. \tag{8.49}$$

□

Lemma 8.2.4. Let $\gamma > 0$. Then, the following estimate holds:

$$\|w^N(t)\|_{L^2}^2 \le \gamma \|\theta^N(t)\|_{W^{1,2}}^2. \tag{8.50}$$

Proof. It follows directly from Theorem 8.2.1, setting $\gamma = \max\left\{\frac{e^{2\widehat{\beta}}}{2\pi-\widehat{\beta}}, \mathsf{Ra}\right\}$, that

$$\begin{aligned}\|w^N(t)\|_{L^2}^2 &\le \|\nabla\Pi^N\|_{L^2}^2 + \mathsf{Ra}\|\theta^N\|_{L^2}^2 \\ &\le \frac{1}{2\pi-\widehat{\beta}}\left\|e^{\widehat{\beta}z}\theta_z^2(t)\right\|_{L^2}^2 + \mathsf{Ra}\|\theta^N(t)\|_{L^2}^2 \\ &\le \frac{e^{2\widehat{\beta}}}{2\pi-\widehat{\beta}}\left\|\nabla\theta^2(t)\right\|_{L^2}^2 + \mathsf{Ra}\|\theta^N(t)\|_{L^2}^2 \\ &\le \gamma\|\theta^N(t)\|_{W^{1,2}}^2.\end{aligned} \tag{8.51}$$

□

Lemma 8.2.5. The sequence $\{\dot{\theta}^N\}_N$ is bounded in $L^2\big(0,T;(\widetilde{W}^{1,2}(\Omega_0))^*\big)$

Proof. Let $v \in \widetilde{W}^{1,2}(\Omega_0)$, such that $v = v_1 + v_2$ with $v_2 \perp \{\phi^i_{mn}, \ |\mu| < N\} = \bar{V}_N$ and $v_1 \perp v_2$. Then from Lemma 8.2.4 and Lemmas 8.2.2, 8.2.3 it follows that

$$\begin{aligned}(\dot{\theta}^N, v) &= (\dot{\theta}^N, v_1) \\ &= (w^N, v_1) + (\Delta\theta^N, v_1) \\ &= (w^N, v_1) - (\nabla\theta^N, \nabla v_1) \\ &\le \|w^N\|_{L^2}\|v_1\|_{L^2} + \|\nabla\theta^N\|_{L^2}\|\nabla v_1\| \\ &\le \gamma'\|\theta^N\|_{W^{1,2}}\|v_1\|_{L^2} + \|\nabla\theta^N\|_{L^2}\|\nabla v_1\| \\ &\le c_3\|v_1\|_{W^{1,2}}.\end{aligned} \tag{8.52}$$

Therefore, we have

$$\frac{(\dot{\theta}^N, v)}{\|v_1\|_{W^{1,2}}} \le c_3, \tag{8.53}$$

and passing to the sup we obtain

$$\|\dot{\theta}^N(t)\|_{(\widetilde{W}^{1,2}(\Omega_0))^*} = \sup \frac{(\dot{\theta}^N, v)}{\|v_1\|_{W^{1,2}}} \le c_3. \tag{8.54}$$

Finally, the thesis follows integrating the previous equation, i.e.

$$\|\dot{\theta}^N\|_{L^2\left(0,T;(\widetilde{W}^{1,2}(\Omega_0))^*\right)} = \int_0^T \|\dot{\theta}^N(t)\|_{(\widetilde{W}^{1,2}(\Omega_0))^*}dt \le c_3 T. \tag{8.55}$$

□

We are now ready to prove the first main preliminary result.

Proposition 8.2.2. The sequences $\{\theta^N\}_N$ and $\{\dot\theta^N\}_N$ are weakly relatively sequentially compact.

Proof. First of all, it follows from Lemma 8.2.2 that $\{\theta^N\}_N$ is bounded in $L^2(0,T;\widetilde{W}^{1,2}(\Omega_0))$ and from Lemma 8.2.5 that $\{\dot\theta^N\}_N$ is bounded in the space $L^2(0,T;(\widetilde{W}^{1,2}(\Omega_0))^*)$. Hence, the Banach-Alaoglu theorem guarantees that $\{\theta^N\}_N$ and $\{\dot\theta^N\}_N$ are weakly-* compact in $L^2(0,T;\widetilde{W}^{1,2}(\Omega_0))$ (which is Hilbert). Therefore, the two sequences are weakly compact, i.e. there exist two subsequences $\{\theta^{N_j}\}_j$ and $\{\dot\theta^{N_j}\}_j$ such that

$$\theta^{N_j} \rightharpoonup \theta \quad\text{and}\quad \dot\theta^{N_j} \rightharpoonup \psi. \tag{8.56}$$

□

Corollary 8.2.1. It results $\psi = \dot\theta$.

Proof. Let us consider $f(t) \in C_c^\infty(0,T)$ and $w \in \widetilde{W}^{1,2}(\Omega_0)$, so the product $fw \in L^2\big(0,T;\widetilde{W}^{1,2}(\Omega_0)\big)$. It follows from the Tonelli-Fubini theorem that

$$\begin{aligned} A^N := &\int_0^T \left(\int_{\Omega_0} \dot\theta^N(t) f(t) w \, d\Omega\right) dt = \int_{\Omega_0}\left(\int_0^T \dot\theta^N(t) f(t) w\, dt\right) d\Omega \\ &= -\int_{\Omega_0}\left(\int_0^T \theta^N(t)\dot f(t) w\, dt\right) d\Omega = -\int_0^T\left(\int_{\Omega_0}\theta^N(t)\dot f(t) w\, d\Omega\right) dt \\ &=: B^N. \end{aligned} \tag{8.57}$$

Hence

$$\lim_N A^N = \int_0^T \langle \psi(t), f(t) w\rangle = -\int_0^T \big(\theta(t), \dot f(t) w\big)\, dt = \lim_N B^N, \tag{8.58}$$

i.e., by definition

$$\psi = \dot\theta. \tag{8.59}$$

□

Let us prove now some other preliminary Lemmas.

Lemma 8.2.6. The sequence $\{\mathbf{u}^N\}_N$ is bounded in $L^2\big(0,T;\widehat{L}^2(\Omega_0)\big)$.

Proof. It is easy to see that

$$\begin{aligned}\|\mathbf{u}^N(t)\|_{L^2}^2 &\leq \|\nabla\Pi^N\|_{L^2}^2 + \mathsf{Ra}^2\|\theta^N\|_{L^2}^2 \\ &\leq c\|\theta_z^N(t)\|_{L^2}^2 + \mathsf{Ra}\|\theta^N(t)\|_{L^2}^2 \\ &\leq \hat{c}\|\theta^N(t)\|_{W^{1,2}}^2\end{aligned} \tag{8.60}$$

and integrating, recalling Lemmas 8.2.2, 8.2.3 we obtain

$$\int_0^T \|\mathbf{u}^N(t)\|_{L^2}^2 dt \leq \hat{c}\int_0^T \|\theta^N(t)\|_{W^{1,2}}^2 dt \leq c_4. \tag{8.61}$$

□

Lemma 8.2.7. The sequence $\{\nabla\Pi^N\}_N$ is bounded in $L^2\big(0,T;\widetilde{L}^2(\Omega_0)\big)$.

Proof. The proof immediately follows from the fact that

$$\|\nabla\Pi^N(t)\|_{L^2}^2 \leq c\|\nabla\theta^N(t)\|_{L^2}^2, \tag{8.62}$$

and after integrating and exploiting Lemma 8.2.3. □

Corollary 8.2.2. The sequence $\{\Pi^N\}_N$ is bounded in $L^2\big(0,T;\widetilde{L}^2(\Omega_0)\big)$.

Proof. Since $\langle\Pi\rangle = 0$, then the proof follows from Lemma 8.2.7 and the Poincaré inequality. □

We are now ready to prove the second main preliminary result.

Proposition 8.2.3. The sequences $\{\mathbf{u}^N\}_N$ and $\{\Pi^N\}_N$ are weakly relatively sequentially compact.

Proof. The result can be proved analogously as in Proposition 8.2.2. Indeed, we obtain that there exists two subsequences

$$\mathbf{u}^{N_j} \rightharpoonup \mathbf{u} \ \text{ and } \ \Pi^{N_j} \rightharpoonup \Pi, \tag{8.63}$$

weakly convergent in $L^2\big(0,T;\widehat{L}^2(\Omega_0)\big)$ and $L^2\big(0,T;\widetilde{W}^{1,2}(\Omega_0)\big)$ respectively. □

The following result concern the existence and uniqueness os solutions for the linear version of (8.19).

Proposition 8.2.4. $(\theta,\mathbf{u},\Pi)$ is a weak solution of the linear version of system (8.19).

Proof. Let us consider

$$v = \sum_{|\mu| \le M} v^i_{mn}(t)\phi^i_{mn} \in \widetilde{W}^{1,2}(\Omega_0). \tag{8.64}$$

Let $N > M$, then we have that

$$\int_{\Omega_0} \dot{\theta}^N(t)\phi^i_{mn} d\Omega = \int_{\Omega_0} w^N \phi^i_{mn} d\Omega - \int_{\Omega_0} \nabla\theta^N \nabla\phi^i_{mn} d\Omega \tag{8.65}$$

and integrating we obtain

$$\int_0^T \left(\int_{\Omega_0} \dot{\theta}^N(t)v(t) \right) dt = \int_0^T \left(\int_{\Omega_0} w^n(t)v(t)d\Omega \right) dt - \int_0^T \left(\int_{\Omega_0} \nabla\theta^N(t) \cdot \nabla v(t) d\Omega \right) dt. \tag{8.66}$$

By Proposition 8.2.1-8.2.2, it follow that

$$\begin{aligned}
\lim_N \langle \dot{\theta}^N, v\rangle &= \lim_N \int_0^T \left(\int_{\Omega_0} \dot{\theta}^N(t)v(t)d\Omega \right) dt = \langle \dot{\theta}, v\rangle, \\
\lim_N (w^N, v) &= \lim_N \int_0^T \left(\int_{\Omega_0} w^N(t)v(t)d\Omega \right) dt = (w, v), \\
\lim_N (\nabla\theta^N, \nabla v) &= \lim_N \int_0^T \left(\int_{\Omega_0} \nabla\theta^N \cdot \nabla v d\Omega \right) dt = (\nabla\theta, \nabla v).
\end{aligned} \tag{8.67}$$

Therefore, $\forall M, \forall v \in L^2(0, T, \widetilde{W}^{1,2}(\Omega_0))$ we have that

$$\langle \dot{\theta}, v\rangle = (w, v) - (\nabla\theta, \nabla v) \tag{8.68}$$

Let us choose $v = \varphi g$, $\varphi \in C^\infty_c(0, T)$, $g \in \widetilde{W}^{1,2}(\Omega_0)$, hence (8.68) becomes

$$\langle \dot{\theta}, \varphi g\rangle = (w, \varphi g) - (\nabla\theta, \nabla(\varphi g)) = (w, \varphi g) - (\nabla\theta, \varphi\nabla g) \tag{8.69}$$

If we integrate we obtain

$$\int_0^T \langle \dot{\theta}(t), \varphi(t)g\rangle dt = \int_0^T (w(t), \varphi(t)g)dt - \int_0^T (\nabla\theta(t), \varphi(t)\nabla g)dt \tag{8.70}$$

i.e.

$$\int_0^T \left[\langle \dot{\theta}(t), g\rangle - (w(t), g) + (\nabla\theta(t), \nabla g) \right] \varphi(t)dt = 0 \tag{8.71}$$

It follows from the Fundamental Lemma of Calculus of Variations that

$$\langle \dot{\theta}, g\rangle - (w(t), g) + (\nabla\theta(t), \nabla g) = 0, \tag{8.72}$$

for a.e. $t \in (0,T)$ and for all $g \in \widetilde{W}^{1,2}(\Omega_0)$.
We can apply the same procedure to $(8.19)_1$ and $(8.19)_2$. Now, recalling that $\theta \in L^2(0,T,\widetilde{W}^{1,2}(\Omega_0))$ and $\dot{\theta} \in L^2(0,T,(\widetilde{W}^{1,2}(\Omega_0))^*)$ it follows that, [43, Thm, 5.9.3]

$$\theta \in C([0,T], \widetilde{L}^2(\Omega_0)), \tag{8.73}$$

At this point, we have that exists $\theta(\mathbf{x},0)$. We need now to prove that it is indeed θ_0. To this aim, let us consider $v \in C^1([0,T], \widetilde{W}^{1,2}(\Omega_0))$ such that $v(T) = 0$. Hence

$$\int_0^T \langle \dot{\theta}^N, v\rangle dt = \int_0^T (w^N(t), v(t))dt - \int_0^T (\nabla\theta^N(t), \nabla v(t))dt, \tag{8.74}$$

so

$$\begin{aligned} -\int_0^T \left(\int_{\Omega_0} \theta^N(t,x)\dot{v}(t,x)d\Omega\right)dt - \int_{\Omega_0}\theta^N(0)v(0)d\Omega \\ = \int_0^T (w^N(t), v(t))dt - \int_0^T (\nabla\theta^N(t), \nabla v(t)). \end{aligned} \tag{8.75}$$

Then, if we take the limit we obtain

$$\begin{aligned} \lim_N \left(-\int_0^T \left(\int_{\Omega_0} \theta^N(t)\dot{v}(t)d\Omega\right)dt\right) &= -\int_0^T (\theta(t), \dot{v}(t)) - (\theta_0, v(0)) \\ &= \int_0^T (w(t), v(t))dt - \int_0^T (\nabla\theta(t), \nabla v(t)). \end{aligned} \tag{8.76}$$

Moreover, from (8.68), we have $\langle \dot{\theta}, v\rangle = (w,v) - (\nabla\theta, \nabla v)$, but since

$$(\theta, \dot{v}) - (\theta(0), \dot{v}) - (\theta(0), v(0)) = (w,v) - (\nabla\theta, \nabla v), \tag{8.77}$$

from (8.76) and (8.77), we have

$$(\theta(0) - \theta_0, v_0) = 0, \tag{8.78}$$

i.e.

$$\theta(0) = \theta_0. \tag{8.79}$$

Equation (8.72) together with (8.79) θ tells us that θ is a weak solution of $(8.19)_3$. We can apply the same procedure to equations $(8.19)_1$ and $_2$.

Concerning the uniqueness, let us suppose that $(\theta_1, \mathbf{u}_1, \pi_1)$ and $(\theta_2, \mathbf{u}_2, \pi_2)$ are two solutions to system (8.19) such that

$$\theta = \theta_1 - \theta_2, \quad \mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2, \quad \pi = \pi_1 - \pi_2. \tag{8.80}$$

Since $(\theta, \mathbf{u}, \pi)$ solve the system, with zero initial data, from (8.33) we have that

$$E(t) = \frac{\mathsf{Ra}}{2}\|\theta(t)\|^2_{L^2} \leq \frac{\mathsf{Ra}}{2}\|\theta_0\|^2_{L^2} e^{c_0 t} = 0. \tag{8.81}$$

Therefore, the following chain of equivalence holds:

$$\|\theta(t)\|^2_{L^2} = 0 \iff \theta(t) = 0 \iff \theta_1 - \theta_2 = 0 \iff \theta_1 = \theta_2. \tag{8.82}$$

Form Theorem 8.2.1 we immediately have that $\Pi_1 = \Pi_2$ and finally

$$\mathbf{u}_1 = -e^{-\hat{\beta} z}\nabla\Pi_1 + R\theta_1\mathbf{k} = -e^{-\hat{\beta} z}\nabla\Pi_2 + R\theta_2\mathbf{k} = \mathbf{u}_2. \tag{8.83}$$

□

The following preliminary results are needed in order to prove the existence and uniqueness of the full nonlinear system (8.19).

Lemma 8.2.8. The following estimate holds

$$\|\nabla\mathbf{u}\|_{L^2} \leq (c(\widehat{\beta}) + \mathsf{Ra})\|\nabla\theta\|_{L^2} \tag{8.84}$$

Proof. Let us consider the divergence of $(8.19)_2$, i.e.

$$\nabla\mathbf{u} = \nabla\left(-e^{-\widehat{\beta} z}\nabla\Pi\right) + \nabla(\mathsf{Ra}\theta\mathbf{k}), \tag{8.85}$$

in particular

$$\nabla\mathbf{u} = \begin{pmatrix} 0 \\ \widehat{\beta} e^{-\widehat{\beta} z}\nabla\Pi \end{pmatrix} - e^{-\widehat{\beta} z} D^2\Pi + \mathsf{Ra}\begin{pmatrix} 0 \\ \nabla\theta \end{pmatrix}. \tag{8.86}$$

We can then estimate as follows

$$\begin{aligned} \|\nabla\mathbf{u}\|_{L^2} &\leq \widehat{\beta}\|\nabla\Pi\|_{L^2} + e^{\widehat{\beta}}\|D^2\Pi\| + \mathsf{Ra}\|\nabla\theta\|_{L^2} \\ &\leq \frac{\widehat{\beta} e^{\widehat{\beta}}}{2\pi - \widehat{\beta}}\|\nabla\theta\|_{L^2} + \frac{e^{\widehat{\beta}}}{4}\|\nabla\Pi\|_{L^2} + \mathsf{Ra}\|\nabla\theta\|_{L^2} \\ &\leq \frac{\widehat{\beta} e^{\widehat{\beta}}}{2\pi - \widehat{\beta}}\|\nabla\theta\|_{L^2} + \frac{\pi e^{2\widehat{\beta}}}{2(2\pi - \widehat{\beta})}\|\nabla\theta\|_{L^2} + \mathsf{Ra}\|\nabla\theta\|_{L^2} \\ &= (c(\widehat{\beta}) + \mathsf{Ra})\|\nabla\theta\|_{L^2} \end{aligned} \tag{8.87}$$

□

Proposition 8.2.5. For all $\varphi \in \widetilde{W}^{1,2}(\Omega_0)$, we have that

$$\lim_N \int_0^T \left(\mathbf{u}^N \cdot \nabla\theta^N - \mathbf{u} \cdot \nabla\theta, \varphi\right) dt = 0 \tag{8.88}$$

Proof. Let us notice that from the Ladyzhenskaya inequality and Lemmas 8.2.3, 8.2.6, 8.2.8 it follows that

$$\begin{aligned} \|(\mathbf{u}^N - \mathbf{u})\varphi\|_{L^2}\|\nabla\theta^N\|_{L^2} &\leq \|\mathbf{u}^N - \mathbf{u}\|_{L^4}\|\varphi\|_{L^4}\|\nabla\theta^N\|_{L^2} \\ &\leq \|\mathbf{u}^N - \mathbf{u}\|_{L^2}^{1/2}\|\mathbf{u}^N - \mathbf{u}\|_{L^2}^{1/2}\|\varphi\|_{L^4}\|\nabla\theta^N\|_{L^2}, \end{aligned} \tag{8.89}$$

in particular from the the Rellich-Kondrachov theorem we have that $\mathbf{u}^N \to \mathbf{u}$ in L^2, and hence from (8.89) we have that

$$\lim_N \|(\mathbf{u}^N - \mathbf{u})\varphi\|_{L^2} = 0. \tag{8.90}$$

Moreover it is easy to see that

$$\lim_N |(\nabla(\theta^N - \theta), \varphi\mathbf{u})| = 0, \tag{8.91}$$

since $\theta^N \rightharpoonup \theta$ in $L^2(0, T; \widetilde{W}^{1,2}(\Omega_0))$. Now since

$$\begin{aligned} &|((\mathbf{u}^N - \mathbf{u}) \cdot \nabla\theta^N, \varphi)| + |(\mathbf{u} \cdot \nabla(\theta^N - \theta), \varphi)| \\ &\qquad \leq \|(\mathbf{u}^N - \mathbf{u})\varphi\|_{L^2}\|\nabla\theta^N\|_{L^2} + |(\nabla(\theta^N - \theta), \varphi\mathbf{u})|, \end{aligned} \tag{8.92}$$

we deduce from (8.90), (8.91) and (8.92) that

$$\begin{aligned} &\lim_N \int_0^T \left(\mathbf{u}^N \cdot \nabla\theta^N - \mathbf{u} \cdot \nabla\theta, \varphi\right) dt \\ &\qquad = \lim_N \int_0^T \left((\mathbf{u}^N - \mathbf{u}) \cdot \nabla\theta^N + \mathbf{u} \cdot \nabla(\theta^N - \theta), \varphi\right) dt = 0. \end{aligned} \tag{8.93}$$

□

8.2.5 *The main theorem*

The following final Theorem eventually prove the existence and uniqueness of the nonlinear perturbation system's (8.19) solutions.

Theorem 8.2.2. $(\theta, \mathbf{u}, \Pi)$ is a weak solution of the full nonlinear system (8.19).

Proof. Let un consider (8.19) in stream the function Φ formulation

$$\begin{cases} \Delta\Pi - \widehat{\beta}\Pi_z = \mathsf{Ra}e^{\widehat{\beta}z}\theta_z, \\ \Delta\Phi = -\widehat{\beta}e^{-\widehat{\beta}z}\Pi_x + \mathsf{Ra}\theta_x, \\ \theta_t + \mathbf{u}\cdot\nabla\theta = \Phi_x + \Delta\theta. \end{cases} \tag{8.94}$$

together with boundary conditions

$$\theta = \Pi_z = \Delta\Phi = 0, \quad z = 0, 1, \tag{8.95}$$

where $\mathbf{u} = (-\Phi_z, \Phi_x)$, and let us set

$$\begin{aligned} \theta^N(x,z,t) &= \sum_{\substack{i=\pm1 \\ m,n=0}}^{N} A^i_{m,n}(t)\xi^i_{mn}(x,z), \\ \Pi^N(x,z,t) &= \sum_{\substack{i=\pm1 \\ m,n=0}}^{N} B^i_{m,n}(t)\phi^i_{mn}(x,z), \\ \Phi^N(x,z,t) &= \sum_{\substack{i=\pm1 \\ m,n=0}}^{N} C^i_{m,n}(t)\xi^i_{mn}(x,z). \end{aligned} \tag{8.96}$$

Following the same procedure as in [38], substituting (8.96) in (8.94), it is possible to find a non-linear system of ODEs whose solution is unique from the Peano's theorem. Once the existence and uniqueness of the truncated solution (8.96) is proved, then the passage to the limit as $N \to +\infty$ is guaranteed by the existence of converging subsequences provided by the previous preliminary results. □

9

COMPRESSIBILITY EFFECT ON DARCY POROUS CONVECTION

To the best of our knowledge, there is a lack of investigations on the onset of convective motions in porous media assuming the definition of extended-quasi-thermal-incompressible fluid discussed in Section 1.3.4. This lack motivated the present investigation whose results are based on [13], joint work with Prof. F. Capone, Prof. R. De Luca and Dott. G. Massa.
In Section 9.1 we derive the mathematical model describing the onset of convection for the Darcy-Bénard problem, while in Section 9.2 we perform a linear instability analysis of the thermal conduction solution and we analyse the asymptotic behaviour of the critical Rayleigh-Darcy number Ra with respect to the dimensionless compressibility factor $\widehat{\beta}$, proving the destabilizing effect of $\widehat{\beta}$ on the onset of convective instabilities. The chapter ends with a concluding Section that recaps all the results.

9.1 FORMULATION OF THE PROBLEM

Let us consider a reference frame $Oxyz$ with fundamental unit vectors $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ ($\mathbf{k}$ pointing vertically upwards) and a horizontal layer $L = \mathbb{R}^2 \times [0, d]$ of fluid-saturated porous medium. To derive the governing equations for the seepage velocity $\mathbf{v}$, the temperature field T and the pressure field p, let us employ the Boussinesq approximation.The mathematical model, according to Darcy's law, is the following

$$\begin{cases} \dfrac{\mu}{K}\mathbf{v} = -\nabla p - \rho_0[1 - \alpha(T - T_0) + \beta(p - p_0)]g\mathbf{k}, \\ \nabla \cdot \mathbf{v} = 0, \\ \rho_0 c_V \left(\dfrac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T\right) = \chi \Delta T, \end{cases} \tag{9.1}$$

where μ, K, χ, c_V are fluid viscosity, permeability of the porous body, thermal conductivity and specific heat at constant volume, respectively. To system (9.1) the boundary conditions are appended, i.e.:

$$\begin{aligned} \mathbf{v}\cdot\mathbf{k} = 0 &\quad \text{on } z = 0, d \\ T = T_L &\quad \text{on } z = 0 \\ T = T_U &\quad \text{on } z = d \end{aligned} \tag{9.2}$$

with $T_L > T_U$, since the layer is heated from below. Assuming the reference temperature $T_0 = T_L$, system (9.1)-(9.2) admits the following stationary conduction solution

$$\begin{aligned} &\mathbf{v}_b = \mathbf{0},\ T_b(z) = T_L - \frac{T_L - T_U}{d} z, \\ &p_b(z) = \frac{1}{\beta d} + \left[\frac{1}{\beta} - \frac{\alpha(T_L - T_U)}{\beta^2 \rho_0 g d}\right](e^{-\rho_0 g \beta z} - 1) - \frac{\alpha(T_L - T_U)}{\beta d} z. \end{aligned} \tag{9.3}$$

Let $(\mathbf{u}, \theta, \pi)$ be a perturbation to the basic solution, so the equations governing the perturbation fields are

$$\begin{cases} \dfrac{\mu}{K}\mathbf{u} = -\nabla\pi + \rho_0\alpha g\theta\mathbf{k} - \rho_0\beta g\pi\mathbf{k}, \\ \nabla\cdot\mathbf{u} = 0, \\ \dfrac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = \dfrac{T_L - T_U}{d}\mathbf{u}\cdot\mathbf{k} + k\Delta\theta, \end{cases} \tag{9.4}$$

where $k = \frac{\chi}{\rho_0 c_V}$ is the thermal diffusivity. Let us introduce the following scales

$$\pi = P\pi^*, \quad \mathbf{u} = U\mathbf{u}^*, \quad \theta = T^\#\theta^*, \quad t = \tau t^*, \quad x = dx^*,$$

$$P = \frac{\mu k}{K}, \quad U = \frac{k}{d}, \quad T^\# = T_L - T_U, \quad \tau = \frac{d^2}{k}.$$

Therefore, the corresponding dimensionless system of equations, omitting all the asterisks, is the following:

$$\begin{cases} \mathbf{u} = -\nabla\pi + \mathsf{Ra}\theta\mathbf{k} - \widehat{\beta}\pi\mathbf{k}, \\ \nabla\cdot\mathbf{u} = 0, \\ \dfrac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = w + \Delta\theta, \end{cases} \tag{9.5}$$

where $u = \mathbf{u}\cdot\mathbf{i}$ and $w = \mathbf{u}\cdot\mathbf{k}$ and

$$\mathsf{Ra} = \frac{\rho_0\alpha g d(T_L - T_U)K}{\mu k}, \quad \widehat{\beta} = \rho_0 d g\beta,$$

are the Rayleigh-Darcy number and the dimensionless compressibility factor, respectively.

To system (9.5) we add the following boundary and initial conditions

$$w = \theta = 0, \qquad \text{on } z = 0, 1 \tag{9.6}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \pi(\mathbf{x}, 0) = \pi_0(\mathbf{x}), \quad \theta(\mathbf{x}, 0) = \theta_0(\mathbf{x}). \tag{9.7}$$

Accounting for $(9.5)_2$, taking the divergence of $(9.5)_1$, system (9.5) becomes:

$$\begin{cases} \Delta\pi + \widehat{\beta}\dfrac{\partial \pi}{\partial z} = \mathsf{Ra}\dfrac{\partial \theta}{\partial z}, \\ \mathbf{u} = -\nabla\pi + \mathsf{Ra}\theta\mathbf{k} - \widehat{\beta}\pi\mathbf{k}, \\ \dfrac{\partial \theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = w + \Delta\theta. \end{cases} \tag{9.8}$$

In the sequel, we will focus on bi-dimensional perturbations in the plane (x, z) and assume the perturbations fields $\pi, \mathbf{u}, \theta$ to be periodic functions in the horizontal direction x with period $\frac{2\pi}{a_x}$, a_x being the wavenumber. Without loss of generality, in the sequel we will assume that the wavelength is 1, so $\frac{2\pi}{a_x} = 1$ and we will consider the periodicity cell $V = [0, 1] \times [0, 1]$.

9.2 LINEAR ANALYSIS

To perform the linear instability analysis of the basic solution, let us consider the linear version of (9.8):

$$\begin{cases} \Delta\pi + \widehat{\beta}\dfrac{\partial \pi}{\partial z} = \mathsf{Ra}\dfrac{\partial \theta}{\partial z}, \\ \mathbf{u} = -\nabla\pi + \mathsf{Ra}\theta\mathbf{k} - \widehat{\beta}\pi\mathbf{k}, \\ \dfrac{\partial \theta}{\partial t} = w + \Delta\theta, \end{cases} \tag{9.9}$$

together with boundary conditions:

$$w = \theta = 0 \quad \text{and} \quad \frac{\partial \pi}{\partial z} = -\widehat{\beta}\pi \qquad \text{on } z = 0, 1. \tag{9.10}$$

By virtue of the Robin boundary condition (9.10) on the pressure, it is possible to choose:

$$\pi = e^{-\widehat{\beta}z}\Pi(x, z, t). \tag{9.11}$$

Therefore equation $(9.9)_1$ becomes:

$$\Delta\Pi - \widehat{\beta}\frac{\partial \Pi}{\partial z} = \mathsf{Ra}e^{\widehat{\beta}z}\frac{\partial \theta}{\partial z} \tag{9.12}$$

and the Robin boundary conditions $\dfrac{\partial \pi}{\partial z} = -\widehat{\beta}\pi$ becomes the Neumann condition given by:

$$\frac{\partial \Pi}{\partial z} = 0 \qquad z = 0, 1. \tag{9.13}$$

Introducing the stream function Φ such that

$$u = -\frac{\partial \Phi}{\partial z}, \quad w = \frac{\partial \Phi}{\partial x} \tag{9.14}$$

and considering the curl of $(9.9)_2$ projected on the y-axis, one obtains:

$$\Delta \Phi = \mathsf{Ra}\frac{\partial \theta}{\partial x} - \widehat{\beta} e^{-\widehat{\beta} z}\frac{\partial \Pi}{\partial x}. \tag{9.15}$$

Hence, to perform the linear instability analysis of the conduction solution, we consider the following system:

$$\begin{cases} \Delta \Pi - \widehat{\beta}\dfrac{\partial \Pi}{\partial z} = \mathsf{Ra} e^{\widehat{\beta} z}\dfrac{\partial \theta}{\partial z} \\ \Delta \Phi = \mathsf{Ra}\dfrac{\partial \theta}{\partial x} - \widehat{\beta} e^{-\widehat{\beta} z}\dfrac{\partial \Pi}{\partial x} \\ \dfrac{\partial \theta}{\partial t} = \dfrac{\partial \Phi}{\partial x} + \Delta \theta \end{cases} \tag{9.16}$$

to which we add the boundary conditions:

$$\theta = \frac{\partial \Pi}{\partial z} = \Delta \Phi = 0 \qquad \text{on } z = 0, 1. \tag{9.17}$$

9.2.1 *Normal mode solutions*

By virtue of (9.17), since system (9.16) is linear, we assume normal mode solutions:

$$\begin{aligned} \theta(x,z,t) &= \sum_{m,n=0}^{\infty} [A^1_{mn}(t)\cos(2\pi m x)\sin(\pi n z) + A^2_{mn}(t)\sin(2\pi m x)\sin(\pi n z)], \\ \Pi(x,z,t) &= \sum_{m,n=0}^{\infty} [B^1_{mn}(t)\cos(2\pi m x)\cos(\pi n z) + B^2_{mn}(t)\sin(2\pi m x)\cos(\pi n z)], \\ \Delta\Phi(x,z,t) &= \sum_{m,n=0}^{\infty} [C^1_{mn}(t)\cos(2\pi m x)\sin(\pi n z) + C^2_{mn}(t)\sin(2\pi m x)\sin(\pi n z)]. \end{aligned} \tag{9.18}$$

In order to get zero mean value on V, we assume $(m,n) \in \mathbb{N} \times \mathbb{N}_0$. Applying the laplacian operator to $(9.16)_3$ and by virtue of (9.18), one obtains:

$$\begin{cases} \sum_{m,n} [B^1_{mn}\cos(2\pi mx) + B^2_{mn}\sin(2\pi mx)][-\alpha_{mn}\cos(n\pi z) + \widehat{\beta} n\pi \sin(n\pi z)] \\ \qquad = \mathsf{Ra}\sum_{m,n} n\pi[A^1_{mn}\cos(2\pi mx) + A^2_{mn}\sin(2\pi mx)]e^{\widehat{\beta}z}\cos(n\pi z) \\ \sum_{m,n}[C^1_{mn}\cos(2\pi mx) + C^2_{mn}\sin(2\pi mx)]\sin(n\pi z) \\ \qquad = \sum_{m,n=0} 2\pi m \Big\{ \mathsf{Ra}[-A^1_{mn}\sin(2\pi mx) + A^2_{mn}\cos(2\pi mx)]\sin(\pi nz) \\ \qquad -\widehat{\beta} e^{-\widehat{\beta}z}[-B^1_{mn}\sin(2\pi mx) + B^2_{mn}\cos(2\pi mx)]\cos(n\pi z) \Big\} \\ \sum_{m,n} -\alpha_{mn}[(\dot{A}^1_{mn} + \alpha_{mn}A^1_{mn})\cos(2\pi mx) + (\dot{A}^2_{mn} + \alpha_{mn}A^2_{mn})\sin(2\pi mx)]\sin(n\pi z) \\ \qquad = \sum_{m,n} 2\pi m[-C^1_{mn}\sin(2\pi mx) + C^2_{mn}\cos(2\pi mx)]\sin(n\pi z) \end{cases} \tag{9.19}$$

where $\alpha_{mn} = (2\pi m)^2 + (\pi n)^2$ and $\dot{A}^i_{mn} = \dfrac{dA^i_{mn}}{dt}$. From $(9.19)_3$, it immediately follows that

$$\begin{aligned} C^1_{mn} &= \frac{\alpha_{mn}}{2\pi m}(\dot{A}^2_{mn} + \alpha_{mn}A^2_{mn}), \\ C^2_{mn} &= -\frac{\alpha_{mn}}{2\pi m}(\dot{A}^1_{mn} + \alpha_{mn}A^1_{mn}). \end{aligned} \tag{9.20}$$

Let us multiply $(9.19)_1$ by $\cos(k\pi z)$ and integrate with respect to $z \in (0,1)$, therefore we get:

$$\begin{aligned} &\sum_{m,n}[B^1_{mn}\cos(2\pi mx) + B^2_{mn}\sin(2\pi mx)]\Big[-\alpha_{mn}\int_0^1 \cos(n\pi z)\cos(k\pi z)dz \\ &\qquad + \widehat{\beta} n\pi \int_0^1 \sin(n\pi z)\cos(k\pi z)dz\Big] = \\ &\sum_{m,n} \mathsf{Ra}\, n\pi[A^1_{mn}\cos(2\pi mx) + A^2_{mn}\sin(2\pi mx)]\int_0^1 e^{\widehat{\beta}z}\cos(n\pi z)\cos(k\pi z)dz \end{aligned} \tag{9.21}$$

namely:

$$\begin{aligned} &\sum_{m,n}[B^1_{mn}\cos(2\pi mx) + B^2_{mn}\sin(2\pi mx)]\left[-\frac{1}{2}\alpha_{mk}\delta_{nk} + \widehat{\beta}F_{nk}\right] = \\ &\qquad \mathsf{Ra}\sum_{m,n}[A^1_{mn}\cos(2\pi mx) + A^2_{mn}\sin(2\pi mx)]\frac{\widehat{\beta}}{2}\mathcal{L}_{nk}(\widehat{\beta}) \end{aligned} \tag{9.22}$$

with:

$$F_{nk} = \begin{cases} 0 & \text{if } n = k \\ \dfrac{n^2((-1)^{n+k} - 1)}{(k-n)(k+n)} & \text{if } n \neq k \end{cases}$$

$$\mathcal{L}_{nk}(\widehat{\beta}) = n\pi(e^{\widehat{\beta}}(-1)^{k+n} - 1)\left(\frac{1}{\pi^2(k+n)^2 + \widehat{\beta}^2} + \frac{1}{\pi^2(k-n)^2 + \widehat{\beta}^2}\right). \tag{9.23}$$

Setting

$$\mathcal{D}^m_{nk}(\widehat{\beta}) = \delta_{nk} + \widehat{\beta} \begin{cases} \dfrac{-2n}{\alpha_{mk}}\left(\dfrac{1}{n-k} + \dfrac{1}{n+k}\right) & \text{if} \quad n+k \ \text{ odd} \\ 0 & \text{if} \quad n+k \ \text{ even} \end{cases} \tag{9.24}$$

by virtue of linearity, from (9.22) one obtains

$$\sum_{n=0}^{\infty} B^i_{mn}\mathcal{D}^m_{nk}(\widehat{\beta}) = -\frac{\mathsf{Ra}\widehat{\beta}}{\alpha_{mk}} \sum_{n=0}^{\infty} A^i_{mn}\mathcal{L}_{nk}(\widehat{\beta}), \qquad i = 1, 2. \tag{9.25}$$

9.2.2 *The main theorem*

Let us remark that the $N \times N$ matrix $\mathcal{D}^m$ is invertible since it is strictly diagonally dominant for small $\widehat{\beta}$ (see [113]). Moreover, through a fixed point argument, an estimate on the compressibility factor $\widehat{\beta}$ guaranteeing the invertibility of the matrix $\mathcal{D}^m$ for all $N \in \mathbb{N}$ is obtained. The following theorem holds.

Theorem 9.2.1. If

$$\widehat{\beta} < \frac{\pi^2}{2c} \tag{9.26}$$

with $c = \dfrac{1}{8}[2\pi\coth(2\pi) + 1]$, then the matrix $\mathcal{D}^m$ is invertible for all $N \in \mathbb{N}$.

Proof. Let us consider the basis functions:

$$\phi^i_{mn}(x,z) = \begin{cases} \cos(2\pi m x)\cos(n\pi z) & \text{if} \ \ i = 1 \\ \sin(2\pi m x)\cos(n\pi z) & \text{if} \ \ i = 2 \end{cases} \tag{9.27}$$

which are the eigenfunctions of the Laplace operator:

$$\Delta\phi^i_{mn} = -\alpha_{mn}\phi^i_{mn}, \tag{9.28}$$

$\alpha_{mn} = 4\pi^2 m^2 + \pi^2 n^2$ being the eigenvalues. Since:

$$b_{mn} := \|\phi^i_{mn}\|^2 = \begin{cases} \dfrac{1}{2} & n = 0 \\ \dfrac{1}{4} & \text{otherwise} \end{cases} \tag{9.29}$$

defining $\gamma_{mn} = \sqrt{\alpha_{mn} b_{mn}}$, the following normalization can be introduced:

$$\psi^i_{mn} = \frac{\phi^i_{mn}}{\gamma_{mn}}. \tag{9.30}$$

Equation $(9.16)_1$ can be written in terms of (9.30) as:

$$-\sum_{\substack{i=1,2 \\ m,n}} B^i_{mn}(-\Delta\psi^i_{mn}) - \widehat{\beta} \sum_{\substack{i=1,2 \\ m,n}} B^i_{mn} \frac{\partial \psi^i_{mn}}{\partial z} = \sum_{\substack{i=1,2 \\ m,n}} e^{\widehat{\beta} z} \mathsf{Ra} A^i_{mn} \psi^i_{mn}. \tag{9.31}$$

If we multiply (9.31) by ψ^j_{lr} and integrate on V we obtain:

$$-\sum_{\substack{i=1,2 \\ m,n}} B^i_{mn} \left\langle \nabla\psi^i_{mn}, \nabla\psi^j_{lr} \right\rangle - \widehat{\beta} \sum_{\substack{i=1,2 \\ m,n}} B^i_{mn} \left\langle \frac{\partial \psi^i_{mn}}{\partial z}, \psi^j_{lr} \right\rangle = \sum_{\substack{i=1,2 \\ m,n}} F^{i,j}_{mnlr} \tag{9.32}$$

where:

$$F^{i,j}_{mnlr} = e^{\widehat{\beta} z} \mathsf{Ra} A^i_{mn} \left\langle \psi^i_{mn}, \psi^j_{lr} \right\rangle. \tag{9.33}$$

From (9.27) and (9.30), it follows that:

$$\begin{aligned} \left\langle \frac{\partial \psi^i_{mn}}{\partial z}, \psi^j_{lr} \right\rangle &= \frac{1}{\gamma_{mn}\gamma_{lr}} \left\langle \frac{\partial \phi^i_{mn}}{\partial z}, \phi^j_{lr} \right\rangle \\ &= -\frac{\delta_{ij}\delta_{ml}}{\gamma_{mn}\,\gamma_{lr}} \frac{n}{2} \begin{cases} \frac{1}{n+r} + \frac{1}{n-r} & \text{se } n + r \geq 1 \text{ odd} \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{9.34}$$

and equation (9.32) becomes:

$$-B^j_{lr} + \widehat{\beta} \sum_{\substack{i \,|\, m,n \\ n+r \geq 1 \text{ odd}}} B^i_{mn} \frac{n}{2} \frac{\delta_{ij}\delta_{ml}}{\gamma_{mn}\gamma_{lr}} \left(\frac{1}{n+r} + \frac{1}{n-r} \right) - \sum_{i|m,n} F^{i,j}_{mnlr} = 0. \tag{9.35}$$

Now, let us introduce the following continuous functions:

$$\mathcal{P} : B \in \mathbb{R}^N \longmapsto -B^j_{lr} + \widehat{\beta} \sum_{\substack{i|m,n \\ n+r\geq 1 \text{ odd}}} B^i_{mn} \frac{n}{2} \frac{\delta_{ij}\delta_{ml}}{\gamma_{mn}\gamma_{lr}} \left(\frac{1}{n+r} + \frac{1}{n-r} \right) - \sum_{i|m,n} F^{i,j}_{mnlr} \in \mathbb{R}^N$$

$$\mathcal{G} : B \in \mathbb{R}^N \longmapsto \widehat{\beta} \sum_{\substack{i|m,n \\ n+r\geq 1 \text{ odd}}} B^i_{mn} \frac{n}{2} \frac{\delta_{ij}\delta_{ml}}{\gamma_{mn}\gamma_{lr}} \left(\frac{1}{n+r} + \frac{1}{n-r} \right) - \sum_{i|m,n} F^{i,j}_{mnlr} \in \mathbb{R}^N \tag{9.36}$$

so the algebraic system (9.35) - equivalent to system (9.25) - can be written as $\mathcal{P}(B) = 0$. Let us observe that the invertibility of $\mathcal{D}^m$ is equivalent to prove that system (9.35) admits a non-trivial solution. Moreover, B is a solution of (9.35) if and only if B is a fixed point of $\mathcal{G}$:

$$\mathcal{P}(B) = 0 \iff \mathcal{G}(B) = B. \tag{9.37}$$

The existence of a fixed point for $\mathcal{G}$ is guaranteed by the Leray–Schauder theorem, provided that:

$$\{B \in \mathbb{R}^N \mid B = \lambda\mathcal{G}(B),\ 0 \leq \lambda \leq 1\} \subset B_R(0), \tag{9.38}$$

$B_R(0)$ being a ball of radius $R > 0$ centred in 0, hence:

$$\{B \in \mathbb{R}^N \mid B = \lambda\mathcal{G}(B),\ 0 \leq \lambda \leq 1\}^{\complement} \supset \mathscr{B} := \{B \in \mathbb{R}^N \mid B = \lambda\mathcal{G}(B),\ \lambda > 1\}. \tag{9.39}$$

If $B \in \mathscr{B}$, then $\lambda B + (1-\lambda)B = \lambda\mathcal{G}(B)$, i.e. $(1-\lambda)B = \lambda\mathcal{P}(B)$, therefore:

$$\frac{1-\lambda}{\lambda}|B|^2 = \mathcal{P}(B) \cdot B \tag{9.40}$$

$|\cdot|$ being the standard euclidean norm. Therefore, from (9.38) and (9.40) we can state that the proof of the existence of a fixed pointy for $\mathcal{G}$ is equivalent to prove that:

$$\begin{aligned} \mathcal{P}(B) \cdot B = &- \sum_{l,r} (B^j_{lr})^2 + \frac{\widehat{\beta}}{2} \sum_{\substack{i|m,n,r \\ n+r\geq 1 \text{ odd}}} B^i_{mn} B^i_{mr} \frac{1}{\gamma_{mn}\gamma_{mr}} \left(\frac{n}{n+r} + \frac{n}{n-r} \right) \\ &- \sum_{i|m,n,l,r} F^{i,j}_{mnlr} B^i_{lr} \end{aligned} \tag{9.41}$$

is negative for $|B| > R$. For notational convenience let us set:

$$\widetilde{B}^i_{mnr} = \frac{B^i_{mn}}{\gamma_{mr}} \quad \text{and} \quad \widetilde{B}^i_{mrn} = \frac{B^i_{mr}}{\gamma_{mn}} \tag{9.42}$$

and hence, from (9.41) we have:

$$\frac{1}{2}\sum_{\substack{i|m,n,r\\ n+r\geq 1 \text{ odd}}} \widetilde{B}^i_{mnr}\widetilde{B}^i_{mrn}\left(\frac{n}{n+r}+\frac{n}{n-r}\right) =: I+J. \tag{9.43}$$

Therefore:

$$\begin{aligned} I &= \sum_{\substack{i|m,n,r\\ n+r\geq 1 \text{ odd}}} \widetilde{B}^i_{mnr}\widetilde{B}^i_{mrn}\frac{n}{n+r} \\ &= \sum_{\substack{i|m\\ n \text{ even}, r \text{ odd}}} \widetilde{B}^i_{mnr}\widetilde{B}^i_{mrn}\frac{n}{n+r} + \sum_{\substack{i|m\\ n \text{ odd}, r \text{ even}}} \widetilde{B}^i_{mnr}\widetilde{B}^i_{mrn}\frac{n}{n+r} \\ &= \sum_{\substack{i|m\\ n \text{ even}, r \text{ odd}}} \widetilde{B}^i_{mnr}\widetilde{B}^i_{mrn}\frac{n}{n+r} + \sum_{\substack{i|m\\ n \text{ even}, r \text{ odd}}} \widetilde{B}^i_{mnr}\widetilde{B}^i_{mrn}\frac{r}{n+r} \\ &= \sum_{\substack{i|m\\ n \text{ even}, r \text{ odd}}} \widetilde{B}^i_{mnr}\widetilde{B}^i_{mrn} \end{aligned} \tag{9.44}$$

and similarly with J

$$\begin{aligned} J &= \sum_{\substack{i|m,n,r\\ n+r\geq 1 \text{ odd}}} \widetilde{B}^i_{mnr}\widetilde{B}^i_{mrn}\frac{n}{n-r} \\ &= \sum_{\substack{i|m\\ n \text{ even}, r \text{ odd}}} \widetilde{B}^i_{mnr}\widetilde{B}^i_{mrn}\frac{n}{n-r} + \sum_{\substack{i|m\\ n \text{ odd}, r \text{ even}}} \widetilde{B}^i_{mnr}\widetilde{B}^i_{mrn}\frac{n}{n-r} \\ &= \sum_{\substack{i|m\\ n \text{ even}, r \text{ odd}}} \widetilde{B}^i_{mnr}\widetilde{B}^i_{mrn}\frac{n}{n-r} + \sum_{\substack{i|m\\ n \text{ even}, r \text{ odd}}} \widetilde{B}^i_{mnr}\widetilde{B}^i_{mrn}\frac{r}{n-r} \\ &= \sum_{\substack{i|m\\ n \text{ even}, r \text{ odd}}} \widetilde{B}^i_{mnr}\widetilde{B}^i_{mrn}. \end{aligned} \tag{9.45}$$

By virtue of (9.44) and (9.45), Cauchy-Schwarz and Young inequalities and since $\gamma_{mn}^2 \geq \frac{4\pi^2+n^2\pi^2}{4}$, from (9.43) it follows:

$$
\begin{aligned}
&\sum_{\substack{i|m \\ n \text{ even}, r \text{ odd}}} \widetilde{B}^i_{mnr}\widetilde{B}^i_{mrn} = \sum_{\substack{i \\ n \text{ even}, r \text{ odd}}} \widetilde{B}^i_{\cdot,nr} \cdot \widetilde{B}^i_{\cdot,rn} \leq \sum_{\substack{i \\ n \text{ even}, r \text{ odd}}} |\widetilde{B}^i_{\cdot,nr}|\,|\widetilde{B}^i_{\cdot,rn}| \\
&\leq \frac{1}{2}\left[\sum_{\substack{i \\ n \text{ even}, r \text{ odd}}} |\widetilde{B}^i_{\cdot,nr}|^2 + \sum_{\substack{i \\ n \text{ even}, r \text{ odd}}} |\widetilde{B}^i_{\cdot,rn}|^2\right] \\
&= \frac{1}{2}\left[\sum_{\substack{i\,|\,m \\ n \text{ even}, r \text{ odd}}} \frac{|B^i_{mn}|^2}{\gamma^2_{mr}} + \sum_{\substack{i\,|\,m \\ n \text{ even}, r \text{ odd}}} \frac{|B^i_{mr}|^2}{\gamma^2_{mn}}\right] \\
&\leq 2\left[\sum_{\substack{i\,|\,m \\ n \text{ even}}} |B^i_{mn}|^2 \sum_{r \text{ odd}} \frac{1}{4\pi^2+r^2\pi^2} + \sum_{\substack{i\,|\,m \\ r \text{ odd}}} |B^i_{mr}|^2 \sum_{n \text{ even}} \frac{1}{4\pi^2+n^2\pi^2}\right] \\
&\leq \frac{2}{\pi^2}\left[\sum_{\substack{i\,|\,m \\ n \text{ even}}} |B^i_{mn}|^2 + \sum_{\substack{i\,|\,m \\ r \text{ odd}}} |B^i_{mr}|^2\right] \sum_n \frac{1}{4+n^2} \\
&= \frac{2}{\pi^2}|B|^2 \sum_n \frac{1}{4+n^2} = \frac{2}{\pi^2}|B|^2 c
\end{aligned}
\tag{9.46}
$$

where $c = \dfrac{1}{8}[2\pi\coth(2\pi)+1]$. Finally, from (9.41) and (9.46) one gets:

$$
\mathcal{P}(B) \cdot B \leq -|B|^2 + \widehat{\beta} c \frac{2}{\pi^2}|B|^2 + K|B| \tag{9.47}
$$

with $K = |F|$. Therefore, for $|B| > R := K/(1 - \widehat{\beta} c 2\pi^{-2})$ and if $\widehat{\beta} < \frac{\pi^2}{2c}$ it follows $\mathcal{P}(B) \cdot B < 0$. □

Solving system (9.25), we get component-wise the same relation for the coefficients B^1_{mj} and B^2_{mj}, i.e. for $i = 1, 2$:

$$
B^i_{mj} = -\frac{\mathsf{Ra}\widehat{\beta}}{\alpha_{mk}} \sum_{n,k=0}^{\infty} A^i_{mn} \mathcal{L}_{nk}(\widehat{\beta}) [\mathcal{D}^m(\widehat{\beta})]^{-1}_{kj}. \tag{9.48}
$$

Now, let us substitute (9.20) and (9.48) in $(9.19)_2$, obtaining:

$$\sum_{m,j=0}^{\infty}[\frac{\alpha_{mj}}{2\pi m}(\dot{A}^2_{mj}+\alpha_{mj}A^2_{mj})\cos(2\pi mx)-\frac{\alpha_{mj}}{2\pi m}(\dot{A}^1_{mj}+\alpha_{mj}A^1_{mj})\sin(2\pi mx)]\sin(j\pi z)=$$
$$\sum_{m,j=0}^{\infty}2\pi m\Big\{\mathsf{Ra}[-A^1_{mj}\sin(2\pi mx)+A^2_{mj}\cos(2\pi mx)]\sin(\pi jz)$$
$$-\widehat{\beta}e^{-\widehat{\beta}z}\Big[\frac{\mathsf{Ra}\widehat{\beta}}{\alpha_{mj}}\sum_{n,k=0}^{\infty}A^1_{mn}\mathcal{L}_{nk}(\widehat{\beta})[\mathcal{D}^m(\widehat{\beta})]^{-1}_{kj}\sin(2\pi mx)$$
$$-\frac{\mathsf{Ra}\widehat{\beta}}{\alpha_{mj}}\sum_{n,k=0}^{\infty}A^2_{mn}\mathcal{L}_{nk}(\widehat{\beta})[\mathcal{D}^m(\widehat{\beta})]^{-1}_{kj}\cos(2\pi mx)\Big]\cos(j\pi z)\Big\}. \tag{9.49}$$

Let us multiply (9.49) by $\sin(h\pi z)$ and integrate with respect to $z\in(0,1)$, therefore we get:

$$\sum_{m,j=0}^{\infty}\Big[\frac{\alpha_{mj}}{2\pi m}(\dot{A}^2_{mj}+\alpha_{mj}A^2_{mj})\cos(2\pi mx)-\frac{\alpha_{mj}}{2\pi m}(\dot{A}^1_{mj}+\alpha_{mj}A^1_{mj})\sin(2\pi mx)\Big]$$
$$\int_0^1\sin(j\pi z)\sin(h\pi z)dz=$$
$$\sum_{m,j=0}^{\infty}2\pi m\Big\{\mathsf{Ra}[-A^1_{mj}\sin(2\pi mx)+A^2_{mj}\cos(2\pi mx)]\int_0^1\sin(\pi jz)\sin(h\pi z)dz \tag{9.50}$$
$$-\widehat{\beta}\Big[\frac{\mathsf{Ra}\widehat{\beta}}{\alpha_{mj}}\sum_{n,k=0}^{\infty}A^1_{mn}\mathcal{L}_{nk}(\widehat{\beta})[\mathcal{D}^m(\widehat{\beta})]^{-1}_{kj}\sin(2\pi mx)$$
$$-\frac{\mathsf{Ra}\widehat{\beta}}{\alpha_{mj}}\sum_{n,k=0}^{\infty}A^2_{mn}\mathcal{L}_{nk}(\widehat{\beta})[\mathcal{D}^m(\widehat{\beta})]^{-1}_{kj}\cos(2\pi mx)\Big]\int_0^1 e^{-\widehat{\beta}z}\cos(j\pi z)\sin(h\pi z)dz\Big\}$$

Hence:

$$\sum_{m=0}^{\infty}\Big[\frac{\alpha_{mh}}{2\pi m}(\dot{A}^2_{mh}+\alpha_{mh}A^2_{mh})\cos(2\pi mx)-\frac{\alpha_{mh}}{2\pi m}(\dot{A}^1_{mh}+\alpha_{mh}A^1_{mh})\sin(2\pi mx)\Big]$$
$$=\sum_{m=0}^{\infty}2\pi m\mathsf{Ra}[-A^1_{mh}\sin(2\pi mx)+A^2_{mh}\cos(2\pi mx)]$$
$$-\sum_{m=0}^{\infty}\widehat{\beta}^2\mathsf{Ra}2\pi m\sum_{j,n,k=0}^{\infty}\frac{1}{\alpha_{mj}}A^1_{mn}\mathcal{L}_{nk}(\widehat{\beta})[\mathcal{D}^m(\widehat{\beta})]^{-1}_{kj}\mathcal{N}_{jh}(\widehat{\beta})\sin(2\pi mx) \tag{9.51}$$
$$+\sum_{m=0}^{\infty}\widehat{\beta}^2\mathsf{Ra}2\pi m\sum_{j,n,k=0}^{\infty}\frac{1}{\alpha_{mj}}A^2_{mn}\mathcal{L}_{nk}(\widehat{\beta})[\mathcal{D}^m(\widehat{\beta})]^{-1}_{kj}\mathcal{N}_{jh}(\widehat{\beta})\cos(2\pi mx)$$

with

$$\mathcal{N}_{jh}(\widehat{\beta})=\pi(1-e^{-\widehat{\beta}}(-1)^{h+j})\left(\frac{h+j}{\pi^2(h+j)^2+\widehat{\beta}^2}+\frac{h-j}{\pi^2(h-j)^2+\widehat{\beta}^2}\right). \tag{9.52}$$

By the linear independence of the sinus and cosinus functions with respect to the variable x, we get, for $i = 1, 2$:

$$\begin{aligned} \frac{\alpha_{mh}}{2\pi m}(\dot{A}^i_{mh} + \alpha_{mh} A^i_{mh}) &= 2\pi m \mathsf{Ra} A^i_{mh} \\ &+ \widehat{\beta}^2 \mathsf{Ra} 2\pi m \sum_{j,n,k=0}^{\infty} \frac{1}{\alpha_{mj}} A^i_{mn} \mathcal{L}_{nk}(\widehat{\beta})[\mathcal{D}^m(\widehat{\beta})]^{-1}_{kj} \mathcal{N}_{jh}(\widehat{\beta}). \end{aligned} \tag{9.53}$$

Equations (9.53) are first order ODEs with respect to time t. To get a unique solution, system (9.53) decouples and let A^i_{mh} be the only non-vanishing coefficient, which satisfies the following first-order ordinary differential equation:

$$\begin{aligned} \dot{A}^i_{mh} + \alpha_{mh} A^i_{mh} &= \frac{4\pi^2 m^2}{\alpha_{mh}} \mathsf{Ra}\ A^i_{mh} \\ &+ \widehat{\beta}^2 \mathsf{Ra} \frac{4\pi^2 m^2}{\alpha_{mh}} A^i_{mh} \sum_{j,k=0}^{\infty} \frac{1}{\alpha_{mj}} \mathcal{L}_{hk}(\widehat{\beta})[\mathcal{D}^m(\widehat{\beta})]^{-1}_{kj} \mathcal{N}_{jh}(\widehat{\beta}), \end{aligned} \tag{9.54}$$

together with the initial conditions on A^i_{mh} that can be derived from $(9.7)_3$ and $(9.18)_1$. Setting

$$\mathcal{G}_{mh}(\widehat{\beta}) = \frac{4\pi^2 m^2}{\alpha_{mh}} \sum_{j,k=0}^{\infty} \frac{1}{\alpha_{mj}} \mathcal{L}_{hk}(\widehat{\beta})[\mathcal{D}^m(\widehat{\beta})]^{-1}_{kj} \mathcal{N}_{jh}(\widehat{\beta}), \tag{9.55}$$

(9.54) is equivalent to

$$\dot{A}^i_{mh} + A^i_{mh}\left[\alpha_{mh} - \mathsf{Ra}\frac{4\pi^2 m^2}{\alpha_{mh}} - \widehat{\beta}^2 \mathsf{Ra}\ \mathcal{G}_{mh}(\widehat{\beta})\right] = 0, \tag{9.56}$$

whose solution can be easily computed to be:

$$A^i_{mh}(t) = \gamma e^{\left(-\alpha_{mh} + \mathsf{Ra}\frac{4\pi^2 m^2}{\alpha_{mh}} + \widehat{\beta}^2 \mathsf{Ra}\ \mathcal{G}_{mh}(\widehat{\beta})\right)t}, \tag{9.57}$$

γ being a constant depending on the initial conditions. We obtain that the perturbation fields (9.18) have an exponential dependence on time t, so let us define the generalized eigenvalue σ_{mh}:

$$\sigma_{mh} = -\alpha_{mh} + \mathsf{Ra}\frac{4\pi^2 m^2}{\alpha_{mh}} + \widehat{\beta}^2 \mathsf{Ra}\ \mathcal{G}_{mh}(\widehat{\beta}). \tag{9.58}$$

Accounting for (9.58), it arises that the strong principle of exchange of stabilities holds and convection can occur only via stationary motions.

9.2.3 *Critical Rayleigh-Darcy number*

The marginal instability threshold is given by setting $\sigma_{mn} = 0$, i.e.

$$\mathsf{Ra}\Big(\frac{4\pi^2 m^2}{\alpha_{mh}} + \widehat{\beta}^2 \mathcal{G}_{mh}(\widehat{\beta})\Big) - \alpha_{mh} = 0, \tag{9.59}$$

and the critical Rayleigh-Darcy number for the onset of convection is given by:

$$\mathsf{Ra}_L = \inf_{m,h} \frac{\alpha_{mh}^2}{4\pi^2 m^2 + \widehat{\beta}^2 \alpha_{mh} \mathcal{G}_{mh}(\widehat{\beta})}, \tag{9.60}$$

for which, using user-written MatLab codes, we numerically obtain that function $\mathcal{G}_{mh}$ is positive.

In order to compare the obtained results with those ones related to Darcy-Bénard problem when the fluid density constitutive law is given by (1.10), starting from (9.58), let us consider the limit case $\widehat{\beta} \to 0$ and (9.58) becomes

$$\sigma_{mh} = -\alpha_{mh} + \mathsf{Ra}\frac{4\pi^2 m^2}{\alpha_{mh}}. \tag{9.61}$$

Requiring the eigenvalue σ_{mh} to be positive, we get

$$-\alpha_{mh} + \mathsf{Ra}\frac{4\pi^2 m^2}{\alpha_{mh}} > 0, \tag{9.62}$$

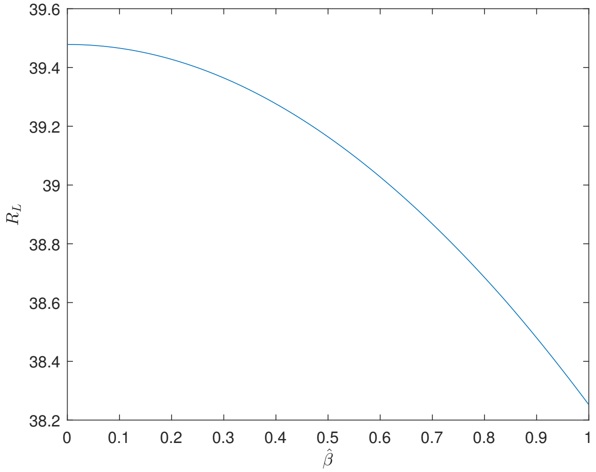
and one immediately recovers the critical Rayleigh-Darcy number for the onset of convection according to the classical Oberbeck-Boussinesq approximation

$$\mathsf{Ra}_c = \min_{m,h} \frac{[(2\pi m)^2 + (h\pi)^2]^2}{4\pi^2 m^2} = 4\pi^2. \tag{9.63}$$

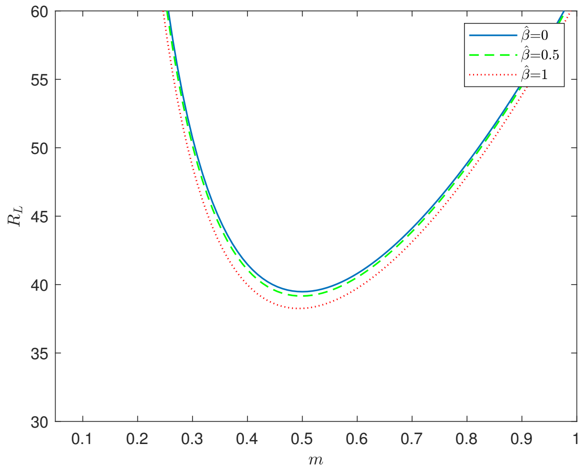
Therefore, comparing (9.60) and (9.63), we get

$$\mathsf{Ra}_L < \mathsf{Ra}_c (= 4\pi^2) \qquad \forall\, \widehat{\beta} > 0. \tag{9.64}$$

In order to analyse the influence of the dimensionless compressibility factor $\widehat{\beta}$ on the onset of convection, we numerically solved (9.60) for quoted values of $\widehat{\beta}$, under restriction (9.26) found in Theorem 9.2.1. In particular, in Figures 9.1(a) – 9.1(b) we choose $\widehat{\beta}$ in a right neighbourhood of 0 to emphasize the behaviour of Ra as a function of $\widehat{\beta}$. From Figures 9.1(a) – 9.1(b) we can conclude that the dimensionless compressibility factor $\widehat{\beta}$ has a *destabilizing* effect on the onset of convective flows: the behaviour of the critical Rayleigh-Darcy number with respect to $\widehat{\beta}$ is decreasing. This result is coherent with the results for the classical Bénard problem found in [97], where a clear fluid is considered.



(a)



(b)

Figure 9.1: (a): Critical Rayleigh-Darcy number Ra_L as function of the compressibility factor $\hat{\beta}$. (b): Neutral curves for quoted values of the compressibility factor $\hat{\beta}$.

9.3 CONCLUSIONS

To the best of our knowledge, in this chapter the Darcy-Bénard problem for an extended-quasi-thermal-incompressible fluid was studied for the first time. We determined the instability threshold for the onset of convection via linear instability analysis of the conduction solution: through a closed algebraic form, we showed that the critical Rayleigh-Darcy number depends on the dimensionless compressibility factor $\widehat{\beta}$ and we rigorously demonstrated that $\widehat{\beta}$ has a destabilizing effect. Moreover, in the limit case $\widehat{\beta} \to 0$ (i.e. according to the classical Oberbeck-Boussinesq approximation), the critical threshold for the Darcy-Bénard problem $4\pi^2$ is recovered. In conclusion, the authors consider the next step in these studies would be to develop the nonlinear stability analysis of the conduction solution (9.3), in order to compare the linear instability threshold Ra_L to the nonlinear stability one.

Part IV

NUMERICAL TECHNIQUES

10 THE CHEBYSHEV-τ METHOD

10.1 INTRODUCTION

Spectral methods belong to the class of discretization schemes for differential equations and are based on the choice of *trial functions* - also called expansion or approximating functions - and *test functions* - also called weight functions. As described in [30], trial functions are used as the basis functions for a truncated series expansion of the solution. Using a truncated expansion instead of the exact solution generates an error, called *residual*. Test functions are used to minimize the residual with respect to a norm or an orthogonality condition (specific to the chosen test function).

The method employed (Galerkin, collocation, or tau) and the choice of the trial functions are features that characterise particular spectral methods. Moreover, the choice of trial functions is what actually distinguishes spectral methods from finite-element and finite-difference methods. In this chapter we will employ a tau method choosing the Chebyshev polynomials as trial functions. For this reason the spectral method we will describe is called Chebyshev-τ method [40]. This method takes advantage of the orthogonality of Chebyshev polynomials with respect to a suitable scalar product to discretize the differential eigenvalue problem. Moreover, Chebyshev polynomials are particularly efficient for discretizing a differential problem because they exhibit some very nice recursive properties, as we will see in this survey.

For many years, spectral methods have been successfully used in many hydrodynamic stability problems and many authors have pointed out the advantages and disadvantages of spectral methods applied to stability problems, see [40, 26, 60, 127, 62, 130]. Particular attention has been devoted to the Chebyshev-τ method as it finds eigenvalues and the eigenfunctions of a differential problem very efficiently. Despite the method being able to determine as many eigenvalues as required, in many stability problems one may only be interested in determining the dominant eigenvalue, which generates the most destabilizing mode.

This chapter is intended to show how to implement the Chebyshev-τ method. Besides a comprehensive description of the theoretical basis for the method, we will provide detailed procedures for discretizing the differential problem. Moreover, we will describe the algorithms we

employed to solve some classical convection problems.
The chapter, based on the contents of a recent survey [3] written in collaboration with Dott. J. A. Gianfrani and Dott. G. Massa, is organised as follows. in Section 10.2 two classical thermal convection problems are described and the mathematical models for are presented. Section 10.3 is devoted to introducing Chebyshev polynomials, their key properties and theoretical aspects of the Chebyshev-τ method. In Section 10.4 algorithms to implement the method are shown in detail and the applications to classical hydrodynamics stability problems are developed in Section 10.5.

10.2 CLASSICAL HYDRODYNAMIC STABILITY PROBLEMS

The phenomenon of the onset of convection in a clear, non-isothermal Newtonian fluid is known as Rayleigh-Bénard convection and the related problem is known as the Bénard problem (see [32, chap. II] and [125, chap. 3]). One can show that the following equations govern the linear instability and the nonlinear stability of the conduction solution (see [32, pag. 26] and [125, pag. 54]):

$$\begin{cases} (D^2 - a^2)^2 W - \mathsf{R} a^2 \Theta = 0, \\ (D^2 - a^2)\Theta + \mathsf{R} W = 0, \end{cases} \tag{10.1}$$

where, from now on, $z \in (0,1)$, $D^2 = \frac{d^2}{dz^2}$ and functions $W = W(z)$ e $\Theta = \Theta(z)$ are obtained by separating off the z dependence from the temperature perturbation and the vertical component of the velocity perturbation, respectively. Finally R^2 is the Rayleigh number and, for a fixed fluid, is given by (see [125, pag. 51]):

$$\mathsf{R}^2 = \frac{\alpha g \beta d^4}{\kappa \nu}, \tag{10.2}$$

where g is the modulus of gravitational acceleration, $\beta = \frac{T_L - T_U}{d} > 0$ is the adverse temperature gradient, d is the layer thickness, κ is the thermal diffusivity, α and ν are the coefficient of thermal expansion and the kinematic viscosity of the fluid, respectively. System (10.1) is of order six and so we require six boundary conditions. The following four conditions are frequently enforced:

$$W(0) = W(1) = \Theta(0) = \Theta(1) = 0. \tag{10.3}$$

The final two boundary conditions depend on what type of surfaces bound the fluid layer. The three cases usually considered are (see [32, pag. 21] for a derivation of the following conditions):

i) *free-free*, bottom and top surface free:

$$D^2W(0) = D^2W(1) = 0, \tag{10.4}$$

ii) *rigid-rigid*, bottom and top surface fixed:

$$DW(0) = DW(1) = 0, \tag{10.5}$$

iii) *rigid-free*, bottom surface fixed, top surface free:

$$DW(0) = D^2W(1) = 0. \tag{10.6}$$

Notice that for every fixed wavenumber, system (10.1) forms a generalized eigenvalue problem for the Rayleigh number. Indeed, the system can be rewritten in matrix form as follows:

$$\begin{pmatrix} (D^2-a^2)^2 & 0 \\ 0 & D^2-a^2 \end{pmatrix} \begin{pmatrix} W \\ \Theta \end{pmatrix} = \mathsf{R} \begin{pmatrix} 0 & a^2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} W \\ \Theta \end{pmatrix} \tag{10.7}$$

that is, more briefly and with evident meaning of the symbols:

$$A\mathbf{x} = \mathsf{R}B\mathbf{x}. \tag{10.8}$$

Let R_c be the smallest positive eigenvalue satisfying system (10.8). It can be shown that $\mathsf{R} < \mathsf{R}_c$ is a necessary and sufficient condition for the stability of the conduction solution [32], hence R_c is called the critical Rayleigh number for the onset of convective motions. R_c is determined by solving the following minimum problem:

$$\mathsf{R}_c^{\mathsf{B}} = \min_{a^2 \in \mathbb{R}^+} \{\mathsf{R}^2(a^2) \,|\, \mathsf{R} \text{ verifies (10.7)}\}. \tag{10.9}$$

It turns out that for the problem under examination the critical Rayleigh number for the onset of instabilities in the case of two free boundaries (10.4) is analytically determinable (see [32, pag. 36]), and it is actually:

$$\mathsf{R}_c^{\mathsf{B}} = \frac{27\pi^4}{4} \simeq 657.5114, \tag{10.10}$$

in correspondence with the critical wavenumber:

$$a_c^2 = \frac{\pi^2}{2} \simeq 4.9348. \tag{10.11}$$

The phenomenon of the onset of convection in a porous medium saturated by an incompressible, non-isothermal Newtonian fluid is known as Rayleigh-Bénard convection in porous media and the related problem is known as the Bénard problem for the Darcy equation or the Horton-Rogers-Lapwood problem (see [125, chap. 4] and [90, chap. 6]). One can show that the following equations govern the linear instability and the nonlinear stability of the conduction solution (see [125, pag. 68]):

$$\begin{cases} (D^2 - a^2)W + \mathsf{R}a^2\Theta = 0, \\ (D^2 - a^2)\Theta + \mathsf{R}W = 0, \end{cases} \tag{10.12}$$

where R^2 is the Rayleigh number and, for given fluid and porous medium, is defined as ([125, pag. 64]):

$$\mathsf{R}^2 = \frac{\alpha g \beta k d^2}{\kappa \nu}, \tag{10.13}$$

where k is the permeability of the porous medium. System (10.12) is of order four, so we require four boundary conditions, namely

$$W(0) = W(1) = \Theta(0) = \Theta(1) = 0. \tag{10.14}$$

For every fixed wavenumber, system (10.12) forms a generalized eigenvalue problem for the Rayleigh number. Indeed, the system can be rewritten in matrix form as follows:

$$\begin{pmatrix} D^2 - a^2 & 0 \\ 0 & D^2 - a^2 \end{pmatrix} \begin{pmatrix} W \\ \Theta \end{pmatrix} = \mathsf{R} \begin{pmatrix} 0 & -a^2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} W \\ \Theta \end{pmatrix} \tag{10.15}$$

that is:

$$A\mathbf{x} = \mathsf{R}B\mathbf{x}. \tag{10.16}$$

Let R_c be the smallest positive eigenvalue satisfying system (10.16). Condition $\mathsf{R} < \mathsf{R}_c$ is a necessary and sufficient condition for the stability of the conductive solution [90]. The critical Rayleigh number R_c for the onset of convection is determined by solving the following minimum problem:

$$\mathsf{R}_c^{\mathsf{HRL}} = \min_{a^2 \in \mathbb{R}^+} \{\mathsf{R}^2(a^2) \,|\, \mathsf{R} \text{ verifies } (10.15)\}. \tag{10.17}$$

It turns out that for the problem under examination the critical Rayleigh number for the onset of instabilities is analytically determinable (see [125, pag. 69] and [90, pag. 245]), and it is actually:

$$\mathsf{R}_c^{\mathsf{HRL}} = 4\pi^2 \simeq 39.4784, \tag{10.18}$$

in correspondence with the critical wavenumber:

$$a_c^2 = \pi^2 \simeq 9.8696. \tag{10.19}$$

10.3 THE CHEBYSHEV-τ METHOD

10.3.1 *Remark about general τ method*

When dealing with ordinary differential equations, one can look for a solution as a formal power series expansion:

$$y(z) = b_0 + b_1 z + b_2 z^2 + \dots \tag{10.20}$$

Substituting this solution in the differential equation, it is possible to determine the coefficients with a recurrence relation and boundary conditions allow to uniquely determine the coefficients. However, when dealing with infinite series, convergence problems can occur. It is indeed possible that the series diverges for every point outside $z = 0$. In order to overcome this problem the expansion can be truncated:

$$y(z) = b_0 + b_1 z + b_2 z^2 + \dots + b_n z^n, \tag{10.21}$$

for an arbitrary fixed n. The system becomes overdetermined since we have $n+1$ coefficients to be determined at our disposal, but the number of equations is larger than $n+1$. In order to remove the overdetermination, which arises from the reduction of order of polynomial after derivation, an extra term $\tau p(z)$ (with p a generic polynomial) can be added on the right-hand side of the differential equation. For example in the case of $n+1$ coefficients and $n+3$ equations we will need the error terms:

$$\tau_1 p(z) + \tau_2 q(z), \tag{10.22}$$

with p and q polynomials. The coefficient τ can be considered as an error term introduced in order to gain the solvability of the system. Since the coefficients were determined by successive recurrence, the only ones that cannot be determined are the last ones.

Example

Let us consider the following CAUCHY problem:

$$\begin{cases} 3(1-z)y' + y = 0, \\ y(0) = 1. \end{cases} \tag{10.23}$$

One can look for a solution of the following form:

$$y(z) = a_0 + a_1 z + a_2 z^2. \tag{10.24}$$

Now substituting (10.24) in (10.23) it turns out:

$$-5a_2 z^2 + 2(3a_2 - a_1)z + 3a_1 + a_0 = 0. \tag{10.25}$$

From (10.25), by recurrence one obtains $a_0 = a_1 = a_2 = 0$. The solution will be $y(z) = 0$, which does not satisfy the initial condition $(10.23)_2$. Hence, one needs to introduce a truncation term on the right-hand side of $(10.23)_1$, which leads to the following equation for the coefficients:

$$-5a_2 z^2 + 2(3a_2 - a_1)z + 3a_1 + a_0 = \tau p(z). \tag{10.26}$$

In this case, it is convenient to choose $p(z) = \tau z^2$ because the term that leads to overdetermination is $-5a_2 z^2$. The auxiliary equation for coefficients becomes:

$$-5a_2 z^2 + 2(3a_2 - a_1)z + 3a_1 + a_0 = \tau z^2. \tag{10.27}$$

Therefore:

$$y_p(z) = 1 - \tfrac{1}{3}z - \tfrac{1}{9}z^2, \qquad \tau_p = \tfrac{5}{9}. \tag{10.28}$$

If we chose as truncation term $\tau(2z^2 - 1)$, then:

$$y_c(z) = 1 - \tfrac{6}{13}z - \tfrac{2}{13}z^2, \qquad \tau_c = \tfrac{5}{13}. \tag{10.29}$$

The chosen polynomial $p(z) = 2z^2 - 1$ will turn out to be a Chebyshev polynomial of the first kind. This choice leads to a smaller truncation error, i.e. $\tau_c < \tau_p$. Consequently, in a more general case, it could be convenient to employ Chebyshev polynomials both for the expression of the truncation term and for the expansion of the solution of a specific differential problem. Hence, the method we will describe is a special τ method: the Chebyshev-τ method. For a more detailed discussion of the subject see [76, pag. 464]

10.3.2 *Preliminary*

Let us rewrite system (10.12) as follows:

$$\begin{cases} W'' = a^2 W - \mathsf{R} a^2 \Theta, \\ \Theta'' = \mathsf{R} W + a^2 \Theta, \end{cases} \tag{10.30}$$

with $' := D = \frac{\mathrm{d}}{\mathrm{d}z}$. System (10.30) is a special case of the following more general system of equations:

$$\begin{cases} W'' = \alpha_1 W' + \alpha_2 W + \alpha_3 \Theta' + \alpha_4 \Theta, \\ \Theta'' = \beta_1 W' + \beta_2 W + \beta_3 \Theta' + \beta_4 \Theta, \end{cases} \tag{10.31}$$

where α_j, β_i, with $i = 1, 2, 3, 4$, are known coefficients (which may possibly be functions of z) and $z \in (0, 1)$. Furthermore, one or more of those coefficients contains an eigenvalue R. Boundary conditions are the following:

$$W(0) = W(1) = \Theta(0) = \Theta(1) = 0. \tag{10.32}$$

We begin to describe Chebyshev-τ method in the context of system (10.31). Having this in mind let us define two linear operators L_1 and L_2 as follows:

$$\begin{aligned} L_1(W, \Theta) &= W'' - \alpha_1 W' - \alpha_2 W - \alpha_3 \Theta' - \alpha_4 \Theta, \\ L_2(W, \Theta) &= \Theta'' - \beta_1 W' - \beta_2 W - \beta_3 \Theta' - \beta_4 \Theta. \end{aligned} \tag{10.33}$$

System (10.31) turns out to be equivalent to:

$$\begin{cases} L_1(W, \Theta) = 0, \\ L_2(W, \Theta) = 0, \end{cases} \tag{10.34}$$

in the domain $(-1, 1)$ with boundary conditions:

$$W(-1) = W(1) = \Theta(-1) = \Theta(1) = 0. \tag{10.35}$$

The domains of the functions appearing in system (10.31) have been transformed from $[0, 1]$ to $[-1, 1]$, the latter being the natural domain to work with Chebyshev polynomials. Before proceeding, we need to recall some fundamental properties of these polynomials.

10.3.3 *Chebyshev polynomials: basic properties*

The *Chebyshev polynomials of the first kind of order n* are defined as follows:

$$T_n(z) = \cos\left[n \cos^{-1}(z)\right], \quad z \in [-1, 1], \quad n = 0, 1, \ldots \tag{10.36}$$

From the previous definition, setting $z = \cos\theta$, it turns out that:

$$T_n(z) = \cos(n\theta), \quad \theta \in [0, \pi], \quad n = 0, 1, \ldots \tag{10.37}$$

The *Chebyshev polynomials of the second kind of order n* are defined, setting $z = \cos\theta$, as follows:

$$U_n(z) = \frac{\sin\big((n+1)\theta\big)}{\sin\theta}, \quad \theta \in [0,\pi], \quad n = 0, 1, \ldots \tag{10.38}$$

Polynomials T_n and U_n, with $n \in \mathbb{N}_0$, satisfy the following remarkable properties.

i) The following recurrence relations hold (see [83, pag. 14-16]):

$$\begin{aligned} T_{n+1}(z) &= 2zT_n(z) - T_{n-1}(z), \quad n = 1, 2, \ldots \\ U_{n+1}(z) &= 2zU_n(z) - U_{n-1}(z), \quad n = 1, 2, \ldots \end{aligned} \tag{10.39}$$

where we pose $T_0(z) = 1$, $T_1(z) = z$ and $U_0(z) = 1$, $U_1(z) = 2z$. The first five Chebyshev polynomials of the first and second kind are explicitly reported:

$$\begin{aligned} &T_0(z) = 1 && U_0(z) = 1 \\ &T_1(z) = z && U_1(z) = 2z \\ &T_2(z) = 2z^2 - 1 && U_2(z) = 4z^2 - 1 \\ &T_3(z) = 4z^3 - 3z && U_3(z) = 8z^3 - 4z \\ &T_4(z) = 8z^4 - 8z^2 + 1 && U_4(z) = 16z^4 - 12z^2 + 1 \end{aligned} \tag{10.40}$$

ii) The following relation between Chebyshev polynomials of the first and second kind holds:

$$U_n(z) - U_{n-2}(z) = 2T_n(z), \qquad n = 2, 3, \ldots \tag{10.41}$$

iii) Polynomial $T_n(z)$ has n zeros in the interval $[-1, 1]$, given by (see [83, pag. 31-32]):

$$z_k = \cos\left(\tfrac{2k-1}{2n}\pi\right) \qquad k = 1, 2, \ldots, n \tag{10.42}$$

and $n+1$ extrema given by (see [83, pag. 33]):

$$\widehat{z}_k = \cos\left(\tfrac{k\pi}{n}\right) \qquad k = 0, 1, \ldots, n \tag{10.43}$$

and in these points we have $T_n(\widehat{z}_k) = \cos\left[n\cos^{-1}(\widehat{z}_k)\right] = \cos(k\pi) = (-1)^k$.

iv) The following fundamental product relation holds (see [83, pag. 43]), for every $r, s \in \mathbb{N}_0$:

$$T_r(z)T_s(z) = \tfrac{1}{2}\left[T_{r+s}(z) + T_{|r-s|}(z)\right]. \tag{10.44}$$

v) The following integral relation between Chebyshev polynomials of the first and second kind holds:

$$\int U_n(z)dz = \tfrac{1}{n+1}T_{n+1}(z) + c. \tag{10.45}$$

vi) The first derivative of $T_n(z)$ can be written in terms of the polynomials itself:

$$T_n'(z) = 2n \sideset{}{'}\sum_{\substack{k=0 \\ n-k \text{ odd}}}^{n-1} T_k(z), \tag{10.46}$$

where, from now on the prime at the summation symbol means that, if it appears, the term contributed by $k = 0$ is to be halved. A similar relation for the second derivative of $T_n(z)$ hold (see [83, pag. 50]):

$$T_n''(z) = n \sideset{}{'}\sum_{\substack{k=0 \\ n-k \text{ even}}}^{n-2} (n^2 - k^2)T_k(z). \tag{10.47}$$

Relations (10.46) and (10.47) are special cases of following general formula for $p = 1, 2, \ldots$ (see [84, pag. 597])

$$\frac{\mathrm{d}^p}{\mathrm{d}x^p}T_n(z) = 2^p n \sideset{}{'}\sum_{\substack{k=0 \\ n-p-k \text{ even}}}^{n-p} \binom{\frac{n+p-k}{2} - 1}{\frac{n-p-k}{2}} \frac{\left(\frac{n+p+k}{2} - 1\right)!}{\left(\frac{n-p+k}{2}\right)!} T_k(z). \tag{10.48}$$

vii) Polynomials $T_n(z)$ can be written in terms of powers of z in the following way (see [83, pag. 35-36]):

$$T_n(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} c_{n,k} z^{n-2k}, \tag{10.49}$$

where $\lfloor \cdot \rfloor$ denotes the largest integer not greater than the number it embraces, with:

$$c_{n,k} = (-1)^k 2^{n-2k-1} \frac{n}{n-k}\binom{n-k}{k}. \tag{10.50}$$

For example if $n = 4$:

$$T_4(z) = \sum_{k=0}^{2} c_{4,k} z^{4-2k} = c_{4,0} z^4 + c_{4,1} z^2 + c_{4,2} = 8z^4 - 8z^2 + 1.$$

Powers of z can be written in terms of polynomials $T_n(z)$ in the following way*:

$$z^n = 2^{1-n} \sum_{k=0}^{\lfloor n/2 \rfloor}{}' \binom{n}{k} T_{n-2k}(z). \tag{10.51}$$

viii) The following orthogonality relation holds, for every $r, s \in \mathbb{N}_0$:

$$\langle T_r(z), T_s(z) \rangle := \int_{-1}^{1} T_r(z) T_s(z) \frac{1}{\sqrt{1-z^2}} dz = \begin{cases} \pi & \text{if} \quad r = s = 0, \\ \frac{\pi}{2} & \text{if} \quad r = s \neq 0, \\ 0 & \text{if} \quad r \neq s. \end{cases} \tag{10.52}$$

Moreover, we have that:

$$\| T_n(z) \|^2 := \int_{-1}^{1} T_n^2(z) \frac{1}{\sqrt{1-z^2}} dz = \begin{cases} \pi & \text{if} \quad n = 0, \\ \frac{\pi}{2} & \text{if} \quad n \neq 0. \end{cases} \tag{10.53}$$

Notice that property *viii*) shows that the set $\{T_n(z)\}_{n \geq 0}$ forms an orthogonal system of $L^2(-1,1)$ with respect to the weight function $w(z) = \frac{1}{\sqrt{1-z^2}}$.

10.3.4 *Theoretical aspects of the method*

We will now describe the procedure for determining eigenvalues and eigenfunctions of systems (10.34) with associated boundary conditions (10.35). We defined operators L_1 and L_2 in order to rewrite (10.31) in the equivalent form (10.34). The key idea is to write W and Θ in truncated Chebyshev series:

$$W(z) = \sum_{k=0}^{N+2} w_k T_k(z) \qquad \text{and} \qquad \Theta(z) = \sum_{k=0}^{N+2} \theta_k T_k(z). \tag{10.54}$$

* For details see [83, pag. 34]. Note that in [84, pag. 596] the following equivalent formula is reported:

$$z^n = 2^{1-n} \sum_{\substack{k=0 \\ n-k \text{ even}}}^{n}{}' \binom{n}{\frac{n-k}{2}} T_k(z)$$

Obviously, the exact solution of the differential equation is an infinite series, i.e. $N \to +\infty$. Following the idea introduced in section 10.3.1 in order to solve (10.34) within the approximation (10.54) we solve:

$$\begin{cases} L_1(W,\Theta) = \tau_1 T_{N+1}(z) + \tau_2 T_{N+2}(z), \\ L_2(W,\Theta) = \tau_1' T_{N+1}(z) + \tau_2' T_{N+2}(z), \end{cases} \tag{10.55}$$

where parameters τ_i and τ_i', $i = 1,2$, are effectively error indicators for the truncation in (10.54). In order to determine coefficients w_k and θ_k in (10.54) we use the orthogonality relation (10.52) of the Chebyshev polynomials in the space $L^2(-1,1)$:

$$\begin{cases} \langle L_1(W,\Theta), T_i \rangle = 0, \\ \langle L_2(W,\Theta), T_i \rangle = 0. \end{cases} \qquad i = 0,\ldots,N \tag{10.56}$$

In such a way, $2(N+1)$ equations are obtained, with two additional conditions:

$$\langle L_1(W,\Theta), T_{N+j} \rangle = \tau_j \|T_{N+j}\|^2 \quad \text{and} \quad \langle L_2(W,\Theta), T_{N+j} \rangle = \tau_j' \|T_{N+j}\|^2. \tag{10.57}$$

with $j = 1,2$. The four equations (10.57) just determined give us the coefficients τ_i and τ_i', with $i = 1,2$, which turn out to be the measures of the truncation error.

Let us observe that from definition (10.36) it follows that:

$$\begin{aligned} T_n(1) &= \cos\left[n \cos^{-1}(1)\right] = \cos(0) = 1 \\ T_n(0) &= \cos\left[n \cos^{-1}(0)\right] = \cos\left(n\tfrac{\pi}{2}\right) = \begin{cases} 0 & n = 2k+1 \\ 1 & n = 4k \\ -1 & n = 4k+2 \end{cases} \quad k = 0,1,\ldots \\ T_n(-1) &= \cos\left[n \cos^{-1}(-1)\right] = \cos(n\pi) = (-1)^n \end{aligned} \tag{10.58}$$

so, in particular $T_n(\pm 1) = (\pm 1)^n$. Therefore, given boundary conditions (10.35), we obtain from (10.54) that:

$$\begin{aligned} W(-1) &= \sum_{k=0}^{N+2} w_k T_k(-1) = \sum_{k=0}^{N+2} (-1)^k w_k = 0, \\ W(1) &= \sum_{k=0}^{N+2} w_k T_k(1) = \sum_{k=0}^{N+2} w_k = 0, \\ \Theta(-1) &= \sum_{k=0}^{N+2} \theta_k T_k(-1) = \sum_{k=0}^{N+2} (-1)^k \theta_k = 0, \\ \Theta(1) &= \sum_{k=0}^{N+2} w_k T_k(1) = \sum_{k=0}^{N+2} \theta_k = 0. \end{aligned} \tag{10.59}$$

The $2(N+1)$ equations (10.56) jointly with the 4 equations (10.59) give a system of $2(N+3)$ equations in $2(N+3)$ unknowns w_k, θ_k, with $k = 0, \ldots, N+2$. Since it will be useful in the sequel, let us consider boundary conditions:

$$\begin{aligned} W(0) &= \sum_{k=0}^{N+2} w_k T_k(0) = \sum_{k=0}^{N+2} \cos\left(k\tfrac{\pi}{2}\right) w_k = 0, \\ \Theta(0) &= \sum_{k=0}^{N+2} \theta_k T_k(0) = \sum_{k=0}^{N+2} \cos\left(k\tfrac{\pi}{2}\right) \theta_k = 0. \end{aligned} \tag{10.60}$$

Having in mind (10.56), we proceed in the following way: $\forall i = 0, \ldots, N$

$$\begin{cases} \langle L_1(W,\Theta), T_i\rangle = \langle W'' - \alpha_1 W' - \alpha_2 W - \alpha_3 \Theta' - \alpha_4 \Theta, T_i\rangle = 0, \\ \langle L_2(W,\Theta), T_i\rangle = \langle \Theta'' - \beta_1 W' - \beta_2 W - \beta_3 \Theta' - \beta_4 \Theta, T_i\rangle = 0, \end{cases} \tag{10.61}$$

so, due to the linearity of the scalar product:

$$\begin{cases} \langle W'', T_i\rangle - \alpha_1 \langle W', T_i\rangle - \alpha_2 \langle W, T_i\rangle - \alpha_3 \langle \Theta', T_i\rangle - \alpha_4 \langle \Theta, T_i\rangle = 0, \\ \langle \Theta'', T_i\rangle - \beta_1 \langle W', T_i\rangle - \beta_2 \langle W, T_i\rangle - \beta_3 \langle \Theta', T_i\rangle - \beta_4 \langle \Theta, T_i\rangle = 0, \end{cases} \tag{10.62}$$

therefore, because of (10.54) it follows:

$$\begin{cases} \left\langle \sum_{k=0}^{N+2} w_k T_k'', T_i \right\rangle - \alpha_1 \left\langle \sum_{k=0}^{N+2} w_k T_k', T_i \right\rangle - \alpha_2 \left\langle \sum_{k=0}^{N+2} w_k T_k, T_i \right\rangle \\ \qquad\qquad -\alpha_3 \left\langle \sum_{k=0}^{N+2} \theta_k T_k', T_i \right\rangle - \alpha_4 \left\langle \sum_{k=0}^{N+2} \theta_k T_k, T_i \right\rangle = 0, \\ \left\langle \sum_{k=0}^{N+2} \theta_k T_k'', T_i \right\rangle - \beta_1 \left\langle \sum_{k=0}^{N+2} w_k T_k', T_i \right\rangle - \beta_2 \left\langle \sum_{k=0}^{N+2} w_k T_k, T_i \right\rangle \\ \qquad\qquad -\beta_3 \left\langle \sum_{k=0}^{N+2} \theta_k T_k', T_i \right\rangle - \beta_4 \left\langle \sum_{k=0}^{N+2} \theta_k T_k, T_i \right\rangle = 0. \end{cases} \tag{10.63}$$

Now setting $\gamma_i = \|T_i(z)\|^2$, from (10.52) it immediately turns out that:

$$\left\langle \sum_{k=0}^{N+2} w_k T_k, T_i \right\rangle = w_i \langle T_i, T_i \rangle = \gamma_i w_i, \tag{10.64}$$

and moreover:

$$\left\langle \sum_{k=0}^{N+2} w_k T_k', T_i \right\rangle = \frac{2}{c_i} \gamma_i \sum_{\substack{s=i+1 \\ s+i \text{ odd}}}^{N+2} s w_s, \tag{10.65}$$

with $c_0 = 2$ and $c_i = 1$ for $i = 1, 2, \ldots, N$ and finally:

$$\left\langle \sum_{k=0}^{N+2} w_k T_k'', T_i \right\rangle = \frac{1}{c_i} \gamma_i \sum_{\substack{s=i+2 \\ s+i \text{ even}}}^{N+2} s(s^2 - i^2) w_s, \tag{10.66}$$

always with $c_0 = 2$ and $c_i = 1$ for $i = 1, 2, \ldots, N$. Analogous conclusions are determined for terms θ_k. In view of the above, we can write system (10.63) in the following way:

$$\begin{cases} \sum_{\substack{s=i+2 \\ s+i \text{ even}}}^{N+2} \frac{1}{c_i} s(s^2 - i^2) w_s - \alpha_1 \sum_{\substack{s=i+1 \\ s+i \text{ odd}}}^{N+2} \frac{2}{c_i} s w_s - \alpha_2 \delta_{is} w_s - \alpha_3 \sum_{\substack{s=i+1 \\ s+i \text{ odd}}}^{N+2} \frac{2}{c_i} s \theta_s - \alpha_4 \delta_{is} \theta_s = 0, \\ -\beta_1 \sum_{\substack{s=i+1 \\ s+i \text{ odd}}}^{N+2} \frac{2}{c_i} s w_s - \beta_2 \delta_{is} w_s + \sum_{\substack{s=i+2 \\ s+i \text{ even}}}^{N+2} \frac{1}{c_i} s(s^2 - i^2) \theta_s - \beta_3 \sum_{\substack{s=i+1 \\ s+i \text{ odd}}}^{N+2} \frac{2}{c_i} s \theta_s - \beta_4 \delta_{is} \theta_s = 0, \end{cases} \tag{10.67}$$

with $i = 0, \ldots, N$. The $2(N+1)$ equations (10.67) jointly with the 4 equations (10.59) give us $2(N+3)$ equations in $2(N+3)$ unknowns, which are in vector form:

$$\mathbf{x} = (\boldsymbol{w}, \boldsymbol{\theta}) = \left(w_0, \ldots, w_{N+2}, \theta_0, \ldots, \theta_{N+2} \right). \tag{10.68}$$

System (10.67) in simple matrix form is:

$$B\mathbf{x} = \mathbf{0}, \tag{10.69}$$

with $i = 2(N+1)$ rows and $s = 2(N+3)$ columns, that we can write in blocks:

$$B\mathbf{x} = \begin{pmatrix} D^2 - \alpha_1 D - \alpha_2 I & -\alpha_3 D - \alpha_4 I \\ \\ -\beta_1 D - \beta_2 I & D^2 - \beta_3 D - \beta_4 I \end{pmatrix} \begin{pmatrix} \boldsymbol{w} \\ \\ \boldsymbol{\theta} \end{pmatrix}. \tag{10.70}$$

By virtue of (10.64), I is an identity-like matrix, while by virtue of (10.65) and (10.66), matrices D and D^2 ($i = N+1$ rows and $s = N+3$ columns) are respectively given by:

$$\begin{aligned} &D_{0,2j-1} = 2j-1, \qquad j \geq 1, \\ &D_{i,i+2j-1} = 2(i+2j-1), \qquad i,j \geq 1, \end{aligned} \tag{10.71}$$

$$D = \begin{pmatrix} 0 & 1 & 0 & 3 & 0 & 5 & 0 & \cdots \\ 0 & 0 & 4 & 0 & 8 & 0 & 12 & \cdots \\ 0 & 0 & 0 & 6 & 0 & 10 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 8 & 0 & 12 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{10.72}$$

and

$$\begin{aligned} &D^2_{0,2j} = \tfrac{1}{2}(2j)^3, \qquad j \geq 1, \\ &D^2_{i,i+2j} = (i+2j)\big((i+2j)^2 - i^2\big) = (i+2j)4j(i+j), \qquad i,j \geq 1, \end{aligned} \tag{10.73}$$

$$D^2 = \begin{pmatrix} 0 & 0 & 4 & 0 & 32 & 0 & 108 & \cdots \\ 0 & 0 & 0 & 24 & 0 & 120 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 48 & 0 & 192 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 80 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{10.74}$$

Notice that $D^2 = D \cdot D$ can be written as a matrix product of D times itself. In order to close system (5.55), we add boundary conditions (9.38) in the following order:

$$\begin{cases} \sum_{\substack{s=i+2 \\ s+i \text{ even}}}^{N+2} \frac{1}{c_i} s(s^2-i^2) w_s - \alpha_1 \sum_{\substack{s=i+1 \\ s+i \text{ odd}}}^{N+2} \frac{2}{c_i} s w_s - \alpha_2 \delta_{is} w_s - \alpha_3 \sum_{\substack{s=i+1 \\ s+i \text{ odd}}}^{N+2} \frac{2}{c_i} s\theta_s - \alpha_4 \delta_{is}\theta_s = 0, \\ \sum_{s=0}^{N+2} w_s = 0, \\ \sum_{s=0}^{N+2} \theta_s = 0, \\ -\beta_1 \sum_{\substack{s=i+1 \\ s+i \text{ odd}}}^{N+2} \frac{2}{c_i} s w_s - \beta_2 \delta_{is} w_s + \sum_{\substack{s=i+2 \\ s+i \text{ even}}}^{N+2} \frac{1}{c_i} s(s^2 - i^2)\theta_s - \beta_3 \sum_{\substack{s=i+1 \\ s+i \text{ odd}}}^{N+2} \frac{2}{c_i} s\theta_s - \beta_4 \delta_{is}\theta_s = 0, \\ \sum_{s=0}^{N+2} (-1)^s w_s = 0, \\ \sum_{s=0}^{N+2} (-1)^s \theta_s = 0. \end{cases} \tag{10.75}$$

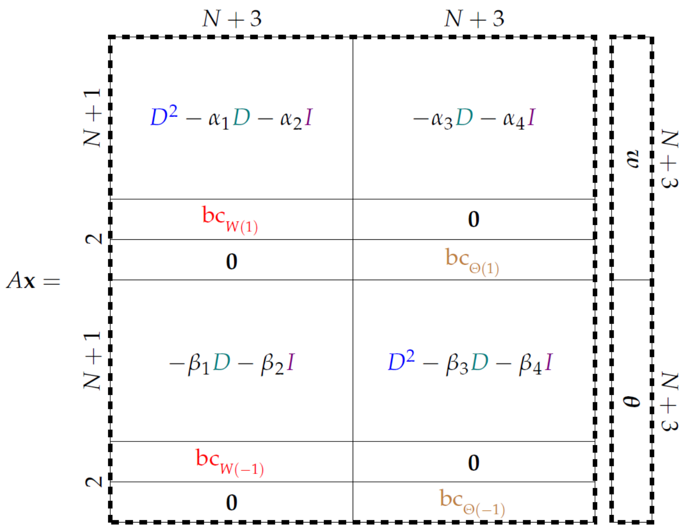
Ultimately the problem can be written in the following matrix form:

$$A\mathbf{x} = \mathbf{0}, \tag{10.76}$$

with A is a numerical squared matrix of order $2(N+3)$ given in blocks by:

$$A\mathbf{x} = \begin{pmatrix} D^2 - \alpha_1 D - \alpha_2 I & -\alpha_3 D - \alpha_4 I \\ \mathrm{bc}_{W(1)} & \mathbf{0} \\ \mathbf{0} & \mathrm{bc}_{\Theta(1)} \\ -\beta_1 D - \beta_2 I & D^2 - \beta_3 D - \beta_4 I \\ \mathrm{bc}_{W(-1)} & \mathbf{0} \\ \mathbf{0} & \mathrm{bc}_{\Theta(-1)} \end{pmatrix} \begin{pmatrix} \boldsymbol{w} \\ \boldsymbol{\theta} \end{pmatrix}. \tag{10.77}$$

We can represent the previous blocks symbolically as follows:



Nonconstant coefficients

Let us now consider the case where one of the coefficients α_2, α_4 and β_2, β_4 is a function of z. Assuming $\alpha_2 = \alpha_2(z)$, the system (10.61) becomes:

$$\begin{cases} \langle L_1(W,\Theta), T_i\rangle = \langle W'' - \alpha_1 W' - \alpha_2(z) W - \alpha_3 \Theta' - \alpha_4 \Theta, T_i\rangle = 0, \\ \langle L_2(W,\Theta), T_i\rangle = \langle \Theta'' - \beta_1 W' - \beta_2 W - \beta_3 \Theta' - \beta_4 \Theta, T_i\rangle = 0, \end{cases} \tag{10.78}$$

with $i = 0, \ldots, N$. As first instance, let α_2 be a linear function of z:

$$\alpha_2(z) = a_1 + a_2 z, \tag{10.79}$$

and without loss of generality let us pose $a_1 = 0$ and $a_2 = 1$. Recalling (10.40) we have $z = T_1(z)$, therefore thanks to the product relation (10.44) it follows that:

$$\begin{aligned} \langle \alpha_2(z) W, T_i\rangle &= \langle zW, T_i\rangle = \left\langle T_1 \sum_{k=0}^{N+2} w_k T_k, T_i \right\rangle = \left\langle \sum_{k=0}^{N+2} w_k T_1 T_k, T_i \right\rangle \\ &= \left\langle \sum_{k=0}^{N+2} w_k \tfrac{1}{2} \left[T_{k+1} + T_{|k-1|} \right], T_i \right\rangle \\ &= \tfrac{1}{2} \left[\left\langle \sum_{k=0}^{N+2} w_k T_{k+1}, T_i \right\rangle + \left\langle \sum_{k=0}^{N+2} w_k T_{|k-1|}, T_i \right\rangle \right]. \end{aligned} \tag{10.80}$$

Observe that for $i = 0$:

$$\frac{1}{2}\left[\left\langle \sum_{k=0}^{N+2} w_k T_{k+1}, T_0 \right\rangle + \left\langle \sum_{k=0}^{N+2} w_k T_{|k-1|}, T_0 \right\rangle\right] = \frac{1}{2}\left[0 + w_1 \langle T_0, T_0\rangle\right] = \gamma_i \frac{1}{2} w_1, \tag{10.81}$$

while for $i = 1$:

$$\frac{1}{2}\left[\left\langle \sum_{k=0}^{N+2} w_k T_{k+1}, T_1 \right\rangle + \left\langle \sum_{k=0}^{N+2} w_k T_{|k-1|}, T_1 \right\rangle\right] = \frac{1}{2}\left[w_0 \langle T_1, T_1\rangle + w_0 \langle T_1, T_1\rangle + w_2 \langle T_1, T_1\rangle\right] = \gamma_i \left[w_0 + \frac{1}{2} w_2\right], \tag{10.82}$$

finally for $i = 2, 3, \ldots, N$:

$$\frac{1}{2}\left[\left\langle \sum_{k=0}^{N+2} w_k T_{k+1}, T_i \right\rangle + \left\langle \sum_{k=0}^{N+2} w_k T_{|k-1|}, T_i \right\rangle\right] = \frac{1}{2}\left[w_{i-1} \langle T_i, T_i\rangle + w_{i+1} \langle T_i, T_i\rangle\right] = \gamma_i \left[\frac{1}{2} w_{i-1} + \frac{1}{2} w_{i+1}\right]. \tag{10.83}$$

Therefore it turns out, for $i = 0, 1, \ldots, N$, that:

$$\frac{1}{2}\left[\left\langle \sum_{k=0}^{N+2} w_k T_{k+1}, T_i \right\rangle + \left\langle \sum_{k=0}^{N+2} w_k T_{|k-1|}, T_i \right\rangle\right] = \gamma_i \left[\frac{c_i}{2} w_{|i-1|} + \frac{1}{2} w_{i+1}\right], \tag{10.84}$$

with $c_0 = 0$, $c_1 = 2$ and $c_i = 1$ for $i = 2, 3, \ldots, N$. Let us rewrite the system (10.78) as follows:

$$\begin{cases} \displaystyle\sum_{\substack{s=i+2 \\ s+i \text{ even}}}^{N+2} \frac{1}{c_i} s(s^2 - i^2) w_s - \alpha_1 \sum_{\substack{s=i+1 \\ s+i \text{ odd}}}^{N+2} \frac{2}{c_i} s w_s - a_2 \left[\frac{c_i}{2} \delta_{i,|s-1|} w_{|s-1|} + \frac{1}{2} \delta_{i,s+1} w_{s+1}\right] \\ \qquad\qquad \displaystyle - \alpha_3 \sum_{\substack{s=i+1 \\ s+i \text{ odd}}}^{N+2} \frac{2}{c_i} s \theta_s - \alpha_4 \delta_{is} \theta_s = 0, \\ \displaystyle -\beta_1 \sum_{\substack{s=i+1 \\ s+i \text{ odd}}}^{N+2} \frac{2}{c_i} s w_s - \beta_2 \delta_{is} w_s + \sum_{\substack{s=i+2 \\ s+i \text{ even}}}^{N+2} \frac{1}{c_i} s(s^2 - i^2) \theta_s \\ \qquad\qquad \displaystyle - \beta_3 \sum_{\substack{s=i+1 \\ s+i \text{ odd}}}^{N+2} \frac{2}{c_i} s \theta_s - \beta_4 \delta_{is} \theta_s = 0, \end{cases} \tag{10.85}$$

or in matrix form:

$$B\mathbf{x} = \mathbf{0}, \tag{10.86}$$

with B a matrix of $i = 2(N+1)$ rows and $s = 2(N+3)$ columns. We can write the system (10.85) in the following block matrix form:

$$B\mathbf{x} = \begin{pmatrix} D^2 - \alpha_1 D - a_2 Z & -\alpha_3 D - \alpha_4 I \\ \\ -\beta_1 D - \beta_2 I & D^2 - \beta_3 D - \beta_4 I \end{pmatrix} \begin{pmatrix} \boldsymbol{w} \\ \\ \boldsymbol{\theta} \end{pmatrix}, \tag{10.87}$$

where Z is a matrix of $i = N+1$ rows and $s = N+3$ columns given by:

$$\begin{aligned} &Z_{1,0} = 1, \\ &Z_{i,i+1} = 1/2, \qquad i \geq 0, \\ &Z_{i+1,i} = 1/2, \qquad i \geq 1, \end{aligned} \tag{10.88}$$

$$Z = \begin{pmatrix} 0 & 1/2 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1/2 & 0 & 0 & 0 & \cdots \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 & \cdots \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 & \cdots \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{10.89}$$

By appropriately adding the boundary conditions, matrix forms analogous to (10.76) and (10.77) can be obtained.

Let us now consider a more generic function $\alpha_2(z)$ defined in $(0,1)$ and $z_0 \in (0,1)$ such that the derivatives $\alpha_2^{(1)}(z_0), \alpha_2^{(2)}(z_0), \ldots, \alpha_2^{(M)}(z_0)$ exist. We consider how such a function can be written in terms of Chebyshev polynomials. The idea we developed involves the TAYLOR expansion of $\alpha_2(z)$ in order to take advantage of (10.51).

For the sake of simplicity, let us assume that $\alpha_2(z)$ may be written as a second order polynomial with a good approximation:

$$\alpha_2(z) = \alpha_2(z_0) + \alpha_2^{(1)}(z_0)\,(z - z_0) + \frac{\alpha_2^{(2)}(z_0)}{2}\,(z - z_0)^2. \tag{10.90}$$

Let us define

$$c_0 = \alpha_2(z_0), \quad c_1 = \alpha_2^{(1)}(z_0), \quad c_2 = \alpha_2^{(2)}(z_0), \tag{10.91}$$

so that (10.90) becomes

$$\alpha_2(z) = c_0 - c_1 z_0 + \tfrac{c_2 z_0^2}{2} + (c_1 - c_2 z_0)\, z + \tfrac{c_2}{2} z^2. \tag{10.92}$$

From (10.40), we may determine the expression of $\alpha_2(z)$ in terms of Chebyshev polynomials:

$$\alpha_2(z) = \left(c_0 - c_1 z_0 + \tfrac{c_2 z_0^2}{2} + \tfrac{c_2}{4}\right) T_0 + (c_1 - c_2 z_0)\, T_1 + \tfrac{c_2}{4} T_2. \tag{10.93}$$

Let us define the following coefficients

$$\begin{aligned} b_0 &= c_0 - c_1 z_0 + \tfrac{c_2 z_0^2}{2} + \tfrac{c_2}{4}, \\ b_1 &= c_1 - c_2 z_0, \\ b_2 &= \tfrac{c_2}{4}, \end{aligned} \tag{10.94}$$

for a simpler expression of (10.93).

We now proceed to write the scalar product $\langle \alpha_2(z) W, T_i \rangle$ appearing in (10.78):

$$\begin{aligned} \langle \alpha_2(z) W, T_i \rangle &= \left\langle (b_0 T_0 + b_1 T_1 + b_2 T_2) \sum_{k=0}^{N+2} w_k T_k, T_i \right\rangle \\ &= \left\langle \sum_{k=0}^{N+2} b_0 w_k T_0 T_k, T_i \right\rangle + \left\langle \sum_{k=0}^{N+2} b_1 w_k T_1 T_k, T_i \right\rangle + \left\langle \sum_{k=0}^{N+2} b_2 w_k T_2 T_k, T_i \right\rangle \\ &= \left\langle \sum_{k=0}^{N+2} b_0 w_k T_k, T_i \right\rangle + \left\langle \sum_{k=0}^{N+2} b_1 w_k \tfrac{1}{2} \left[T_{k+1} + T_{|k-1|} \right], T_i \right\rangle \\ &+ \left\langle \sum_{k=0}^{N+2} b_2 w_k \tfrac{1}{2} \left[T_{k+2} + T_{|k-2|} \right], T_i \right\rangle. \end{aligned} \tag{10.95}$$

Now, for $i = 0$:

$$\langle \alpha_2(z) W, T_0 \rangle = b_0 w_0 \gamma_i + \tfrac{1}{2} b_1 w_1 \gamma_i + \tfrac{1}{2} b_2 w_2 \gamma_i, \tag{10.96}$$

for $i = 1$:

$$\langle \alpha_2(z) W, T_1 \rangle = b_0 w_1 \gamma_i + b_1 w_0 \gamma_i + b_1 w_2 \tfrac{1}{2} \gamma_i + \tfrac{1}{2} b_2 w_1 \gamma_i + \tfrac{1}{2} b_2 w_3 \gamma_i, \tag{10.97}$$

for $i = 2$:

$$\langle \alpha_2(z) W, T_2 \rangle = b_0 w_2 \gamma_i + \tfrac{1}{2} b_1 w_1 \gamma_i + \tfrac{1}{2} b_1 w_3 \gamma_i + b_2 w_0 \gamma_i + \tfrac{1}{2} b_2 w_4 \gamma_i, \tag{10.98}$$

finally for $i = 3, 4, \ldots, N$

$$\langle \alpha_2(z) W, T_i \rangle = \gamma_i [b_0 \delta_{k,i} w_k + \tfrac{1}{2} b_1 \delta_{k+1,i} w_{k+1} + \tfrac{1}{2} b_1 \delta_{k-1,i} w_{k-1} + \tfrac{1}{2} b_2 \delta_{k+2,i} w_{k+2} + \tfrac{1}{2} b_2 \delta_{k-2,i} w_{k-2}]. \quad (10.99)$$

Therefore, for $i = 0, 1, 2, \ldots, N$

$$\langle \alpha_2(z) W, T_i \rangle = \sum_{\substack{s=i-2 \\ s \geq 0}}^{i+2} \tfrac{c_s}{2} \gamma_i b_{|s-i|} w_s + \delta_{s=i,1} \tfrac{1}{2} \gamma_i b_{s+i} w_s, \quad (10.100)$$

where $c_s = 2$ for $s = 0, i$; otherwise $c_s = 1$.

The system (10.78) becomes

$$\begin{cases} \displaystyle\sum_{\substack{s=i+2 \\ s+i \text{ even}}}^{N+2} \tfrac{1}{c_i} s(s^2 - i^2) w_s - \alpha_1 \sum_{\substack{s=i+1 \\ s+i \text{ odd}}}^{N+2} \tfrac{2}{c_i} s w_s - \sum_{\substack{s=i-2 \\ s \geq 0}}^{i+2} \frac{c_s}{2} b_{|s-i|} w_s + \delta_{s=i,1} \frac{1}{2} b_{s+i} w_s \\ \qquad \displaystyle -\alpha_3 \sum_{\substack{s=i+1 \\ s+i \text{ odd}}}^{N+2} \tfrac{2}{c_i} s \theta_s - \alpha_4 \delta_{is} \theta_s = 0, \\ \displaystyle -\beta_1 \sum_{\substack{s=i+1 \\ s+i \text{ odd}}}^{N+2} \tfrac{2}{c_i} s w_s - \beta_2 \delta_{is} w_s + \sum_{\substack{s=i+2 \\ s+i \text{ even}}}^{N+2} \tfrac{1}{c_i} s(s^2 - i^2) \theta_s \\ \qquad \displaystyle -\beta_3 \sum_{\substack{s=i+1 \\ s+i \text{ odd}}}^{N+2} \tfrac{2}{c_i} s \theta_s - \beta_4 \delta_{is} \theta_s = 0. \end{cases} \quad (10.101)$$

that is, in matrix form:

$$B\mathbf{x} = \mathbf{0}, \quad (10.102)$$

with B a matrix of $i = 2(N+1)$ rows and $s = 2(N+3)$ columns where

$$B = \begin{pmatrix} D^2 - \alpha_1 D - F & -\alpha_3 D - \alpha_4 I \\ -\beta_1 D - \beta_2 I & D^2 - \beta_3 D - \beta_4 I \end{pmatrix}, \quad (10.103)$$

and F is the matrix representation of $\alpha_2(z)$ of $i = N+1$ rows and $s = N+3$ columns:

$$F = \begin{pmatrix} b_0 & b_1/2 & b_2/2 & 0 & 0 & 0 & 0 & 0 & \cdots \\ b_1 & b_0 + b_2/2 & b_1/2 & b_2/2 & 0 & 0 & 0 & 0 & \cdots \\ b_2 & b_1/2 & b_0 & b_1/2 & b_2/2 & 0 & 0 & 0 & \cdots \\ 0 & b_2/2 & b_1/2 & b_0 & b_1/2 & b_2/2 & 0 & 0 & \cdots \\ 0 & 0 & b_2/2 & b_1/2 & b_0 & b_1/2 & b_2/2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \quad (10.104)$$

As the reader may have noticed, when taking advantage of (10.40) in (10.92), the term z^2 gives an additional contribution in terms of T_0, unlike the term in z which is simply equal to T_1. Looking at (10.40), z^3 will provide an additional term in T_1; z^4 in T_2 and T_0 and so on. Once this idea had been understood, we may proceed to write the code we report in the next section.

10.3.5 *Generalized eigenvalue problems*

A vector $\mathbf{x}$ is said to be an *eigenvector* of the matrix A if there is a non zero real number λ, called *eigenvalue*, such that:

$$A\mathbf{x} = \lambda \mathbf{x}. \quad (10.105)$$

Problem (10.105) is better known as the *eigenvalue problem*. A vector $\mathbf{x}$ is said to be an *eigenvector* of matrices A and B if there exists a non zero real number λ, called *eigenvalue*, such that:

$$A\mathbf{x} = \lambda B\mathbf{x}. \quad (10.106)$$

Problem (10.106) is better known as the *generalized eigenvalue problem*, and this is precisely the kind of problem we are going to deal with. Let us now go back to the general system (10.31):

$$\begin{cases} D^2W = \alpha_1 DW + \alpha_2 W + \alpha_3 D\Theta + \alpha_4 \Theta, \\ D^2\Theta = \beta_1 DW + \beta_2 W + \beta_3 D\Theta + \beta_4 \Theta, \end{cases} \quad (10.107)$$

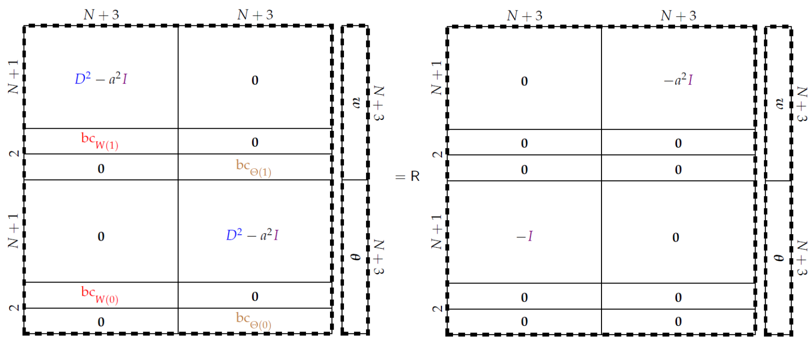
under the following boundary conditions:

$$W(0) = W(1) = \Theta(0) = \Theta(1) = 0. \quad (10.108)$$

In case $\alpha_1 = \alpha_3 = \beta_1 = \beta_3 = 0$, $\alpha_2 = \beta_4 = a^2$, $\alpha_4 = \alpha_4(\mathsf{R}) = -\mathsf{R}a^2$ e $\beta_2 = \beta_2(\mathsf{R}) = -\mathsf{R}$, the system (10.107) becomes:

$$\begin{cases} D^2W = a^2W - \mathsf{R}a^2\Theta, \\ D^2\Theta = -\mathsf{R}W + a^2\Theta, \end{cases} \tag{10.109}$$

and coincides with system (10.12) which is a generalized eigenvalue problem as we remarked before in (10.15) and (10.16). In view of the Chebyshev polynomial theory we have just introduced, we have the following symbolic representation of the problem $A\mathbf{x} = \mathsf{R}B\mathbf{x}$:



At first glance, because of the high order of the derivatives, it seems that problem (10.1) does not fall within the general case (10.107) for which the theory of Chebyshev decomposition was introduced. Let us reconsider system (10.1)

$$\begin{cases} (D^2 - a^2)^2W - \mathsf{R}a^2\Theta = 0, \\ (D^2 - a^2)\Theta + \mathsf{R}W = 0, \end{cases} \tag{10.110}$$

with boundary conditions:

$$W(0) = W(1) = \Theta(0) = \Theta(1) = 0. \tag{10.111}$$

If we introduce the following auxiliary variable:

$$Y = (D^2 - a^2)W, \tag{10.112}$$

system (10.110) can be transformed as follows:

$$\begin{cases} (D^2 - a^2)W - Y = 0, \\ (D^2 - a^2)Y - \mathsf{R}a^2\Theta = 0, \\ (D^2 - a^2)\Theta + \mathsf{R}W = 0, \end{cases} \tag{10.113}$$

therefore the order of derivation is consistent with the theory. Moreover, we need to add new boundary values for the auxiliary variable (10.112), which, because of (10.111), are:

$$\begin{aligned} Y(0) &= (D^2 - a^2)W(0) = D^2W(0) - a^2W(0) = D^2W(0), \\ Y(1) &= (D^2 - a^2)W(1) = D^2W(1) - a^2W(1) = D^2W(1). \end{aligned} \tag{10.114}$$

We now need to specify the type of surface bounding the fluid layer. In particular, if we assume *free-free* conditions (10.4), i.e. $D^2W(0) = D^2W(1) = 0$, it turns out that:

$$Y(0) = Y(1) = 0. \tag{10.115}$$

Ultimately we are dealing with the following generalized eigenvalue problem:

$$\begin{pmatrix} D^2 - a^2 & -1 & 0 \\ 0 & D^2 - a^2 & 0 \\ 0 & 0 & D^2 - a^2 \end{pmatrix} \begin{pmatrix} W \\ Y \\ \Theta \end{pmatrix} = \mathsf{R} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a^2 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} W \\ Y \\ \Theta \end{pmatrix} \tag{10.116}$$

together with boundary conditions:

$$W(0) = W(1) = Y(0) = Y(1) = \Theta(0) = \Theta(1) = 0. \tag{10.117}$$

10.4 NUMERICAL SOLUTIONS: COMPUTATIONAL ASPECTS OF THE METHOD

We anticipate that the present section concerning the numerical implementation of the first in-depth theoretical part will require some basic knowledge of MatLab. In any case, the presented codes will be explained as clearly as possible and applied to the previously introduced hydrodynamic stability problems. Moreover, every presented and cited code is available in the GitHub repository [12].

10.4.1 *Construction of D, D^2 and Z matrices and boundary condition vector*

In order to build the matrix D, we have simply to follow what we computed in (10.65) and (10.71), so only two nested **for** loops are needed, see function **1**:

1

```
function[matrixout] = chebyder(rows,columns)
matrix=zeros(rows,columns);
ck = 2;
for i = 1:rows
    for j = i+1:2:columns
        matrix(i,j)=2*(j-1)/ck;
    end
    ck = 1;
end
matrixout = matrix;
```

Analogously, building the matrix D^2 simply consists in following what we computed in (10.79) and (10.73) therefore only two nested **for** loops are needed, see function **2**:

2

```
function[matrixout] = chebyder2(rows,columns)
matrix = zeros(rows,columns);
ck = 2;
for i = 1:rows
    for j = i+2:2:columns
        matrix(i,j) = (j-1)*((j-1)^2-(i-1)^2)/ck;
    end
    ck = 1;
end
matrixout = matrix;
```

Matrix Z can be built following (10.88), so two consecutive loops **for** are needed, hence function **3**:

3

```
function[matrixout] = chebyZ(rows,columns)
matrix = zeros(rows,columns);
for i = 1:rows
    matrix(i,i+1) = 0.5;
end
for i = 1:columns
```

```
    matrix(i+1,i) = 0.5;
end
matrix(2,1) = 1;
matrixout = matrix(1:rows,1:columns);
```

About boundary conditions in $z = 0$ and $z = 1$, we can proceed recalling what we introduced in (10.59) and (10.60) and writing the conditional constructs shown in function **4**:

4

```
function[vector] = chebybval(columns,eval)
vector = zeros(1,columns);
if eval == 0
    for i = 1:4:columns
        vector(i) = 1;
    end
    for i = 2:2:columns
        vector(i) = 0;
    end
    for i = 3:4:columns
        vector(i) = -1;
    end
else if eval == 1
    for i = 1:columns
        vector(i) = 1;
    end
end
```

Functions `chebyder` **1**, `chebyder2` **2**, `chebyZ` **3** and `chebybval` **4** are the four fundamental building blocks of the method.

10.5 APPLICATION TO THE CLASSICAL PROBLEMS

HORTON-ROGERS-LAPWOOD *problem*

As we said in section 10.3.5, in order to solve the HORTON-ROGERS-LAPWOOD problem, we need to work on the generalized eigenvalue problem $A\mathbf{x} = \mathrm{R}B\mathbf{x}$ which is symbolically represented in page 212. The next-to-last step is to build matrices A and B, see **5**:

5

```
function[A,B] = fullmatrices_HRL(N,x)
                              ▷ N is the number of Chebyshev polynomials;
                                      ▷ x is a fixed wavenumber (x=a^2);
                            ▷ construction of matrices D^2 and D^2 - a^2 I:
M = chebyder2(N,N);
X = M-x*eye(N);
                          ▷ construction of bc vectors in z = 0 e z = 1:
v_0 = chebybval(N,0);
v_1 = chebybval(N,1);
                                              ▷ matrices initialization:
A = zeros(2*N,2*N);
B = zeros(2*N,2*N);
                                           ▷ construction of matrix A:
for i = 1:N-2
    for j = 1:N
        A(i,j) = X(i,j);
    end
end
for i = N+1:2*N-2
    for j = N+1:2*N
        A(i,j) = X(i-N,j-N);
    end
end
                               ▷ complete the matrix A blocks with bc:
for j = 1:N
    A(N-1,j) = v_0(j);
    A(N,j) = v_1(j);
end
for j = N+1:2*N
    A(2*N-1,j) = v_0(j-N);
    A(2*N,j) = v_1(j-N);
end
                                           ▷ construction of matrix B:
I=eye(N-2,N-2);
for i=1:N-2
    for j=N+1:2*N-2
        B(i,j) = -x*I(i,j-N);
    end
```

```
end
for i = N+1:2*N-2
    for j = 1:N-2
        B(i,j) = -I(i-N,j);
    end
end
```

The real last step is actually the resolution of the generalized eigenvalue problem and the determination of the critical Rayleigh number, as done in **6**:

6

```
                    ▷ choose the number of Chebyshev polynomials
                                    ▷ and the discretization step:
N = 40;
step = 0.01;
                                                          ▷ x = a^2
x = 0.7:step:40;
        ▷ determination of the numerical curve of marginal stability:
for i = 1:length(x)
    [A,B] = fullmatrices_HRL(N,x(i));
    R = eig(A,B,'qz');
    R_pos_r = zeros(length(R),1);
                         ▷ selection of real positive eigenvalues:
    for j = 1:length(R)
        if R(j)>0 & abs(imag(R(j)))<10^(-10)
            R_pos_r(j) = R(j);
        end
    end
                             ▷ ordering of positive eigenvalues:
    R_pos_r_ord = sort(R_pos_r);
          ▷ selection of the first positive eigenvalue and its square:
    flag = 0;
    for j = 1:length(R_pos_r_ord)
        if R_pos_r_ord(j)>0 & flag == 0
            R_x(i) = R_pos_r_ord(j);
            R_x_sqr(i) = R_x(i)^2;
            flag = 1;
        end
```

```
        end
    end
                    ▷ determination of the critical Rayleigh number:
    [M,i] = min(R_x_sqr);
    wave_c = double(x(i))
    R_c = R_x_sqr(i)
```

Recalling what we analytically found in (10.18) and (10.19), setting `step` $= 0.0001$ and `x` $= 9.7 :$ `step` $: 10$, the code gives us as output the following critical numbers:

$$a^2_{c,\text{Chebyshev-}\tau} = 9.8696 \quad \text{and} \quad \mathsf{R}^{\mathsf{HRL}}_{c,\text{Chebyshev-}\tau} = 39.4784. \qquad (10.118)$$

In Figure 10.1 the numerically computed neutral curve (i.e. the instability threshold for the onset of steady convection) is depicted. In Figure 10.2, the solid blue line depicts the neutral curve determined analytically, while the dashed black line depicts the numerically computed neutral curve. The two lines overlap, therefore the numerical approach computes the exact solution (10.17).

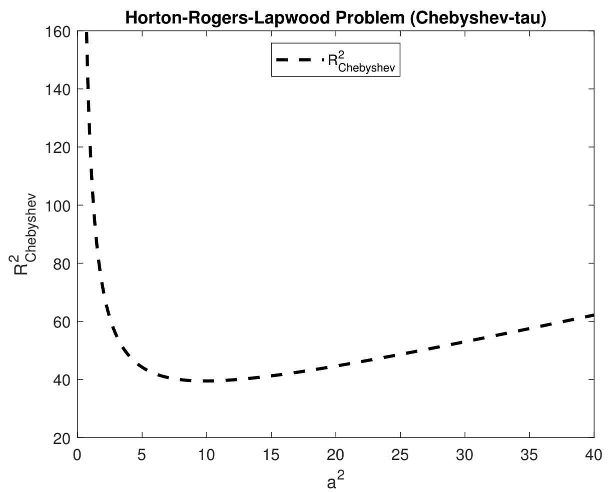


Figure 10.1: Plot of the marginal stability curve for the Horton-Rogers-Lapwood problem.

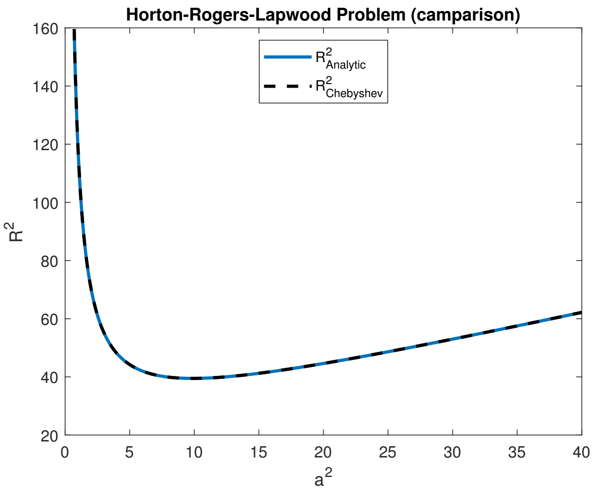


Figure 10.2: Comparison between the numerical and analytical curves for the HORTON-ROGERS-LAPWOOD problem.

BÉNARD *problem*

For the BÉNARD problem, following the procedure we introduced in section 10.3.5 (page 212), in order to build matrices A and B we may proceed as in function `fullmatrices_B`, available at [4]. The last step is the resolution of the generalized eigenvalue problem and the determination of the critical Rayleigh number. To this aim, it is sufficient to replace `fullmatrices_HRL` with `fullmatrices_B` in code **6** (line 9). Recalling what we analytically found in (10.10) and (10.11), the resulting code will produce as output the following critical numbers and the neutral curve depicted in Figures 10.3:

$$a^2_{c,\text{Chebyshev-}\tau} = 4.9348 \qquad \text{and} \qquad \mathrm{R}^{\mathrm{B}_{\textit{free-free}}}_{c,\text{Chebyshev-}\tau} = 657.5114. \tag{10.119}$$

In Figure 10.4 the solid blue line depicts the neutral curve determined analytically, while the dashed black line depicts the numerically computed neutral curve. The overlapping between these curves shows once again that the CHEBYSHEV-τ method manages to find the exact critical Rayleigh number defined by (10.9).

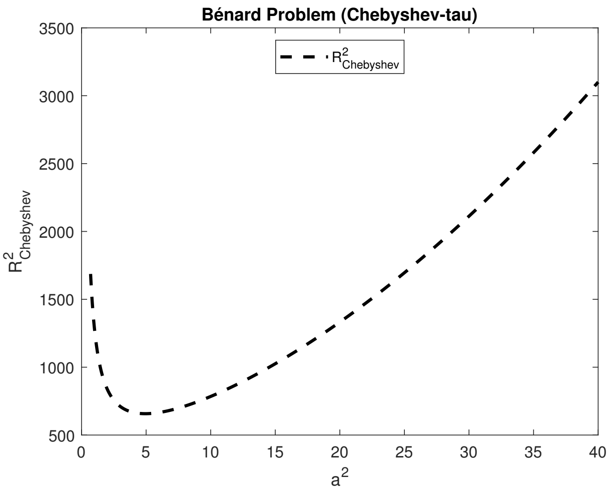


Figure 10.3: Plot of the marginal stability curve for the Bénard problem.

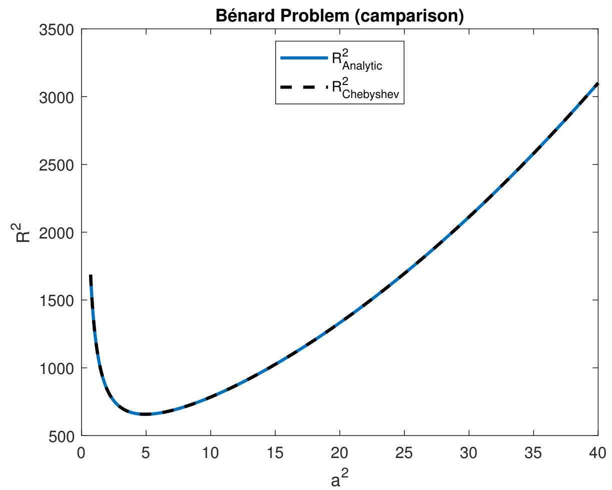


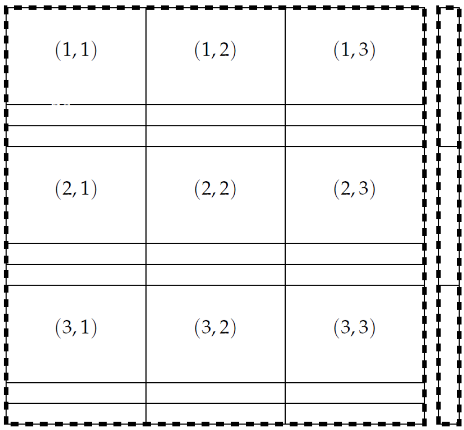
Figure 10.4: Comparison between the numerical and analytical curves for the Bénard problem.

10.5.1 *Methodological remark*

Given a certain generalized eigenvalue problem to solve with the Chebyshev-τ method, in order to avoid confusion, it is definitely convenient to sketch a symbolic representation of the system as we did in section 10.3.5 (page 212). We can therefore identify the blocks that form matrices A and B. Given a certain block of order (p,q) in the nested for loops, code 7 follows:

7

```
                                              ▷ p-q block
for i = (p-1)*N+1:p*N-2
    for j = (q-1)*N+1:q*N
        A(i,j) = X(i-(p-1)*N,j-(q-1)*N);
    end
end
                                              ▷ p-q bc
for j = (q-1)*N+1:q*N
    A(p*N-1,j) = bv_0(j-(q-1)*N);
    A(p*N,j) = bv_1(j-(q-1)*N);
end
```



The codes reported in the chapters are in the GitHub repository [12], and other codes used in the thesis can be found in [5, 6, 10, 9, 8, 11, 7].

CONCLUSION

The present doctoral thesis is focused on investigating some hydrodynamic stability problems. In particular the onset of convective motions in presence of quadratic density

$$\rho(T) = \rho_0[1 - \alpha(T - T_0)^2],$$

and in presence of pressure-dependent density

$$\rho(p, T) = \rho_0[1 - \alpha(T - T_0) + \beta(p - p_0)].$$

The related results are exposed in the central part of the thesis, parts II and III respectively.
The undertaken qualitative investigations aim to find the critical value for the dimensionless parameter, the Rayleigh number, that captures the physics of the problem. To approach the stability analysis of the basic steady solution, we performed a linear analysis obtaining information about the fate of small-amplitude disturbances. In this regard, we found the critical thresholds of the Rayleigh number for the onset of convection. Moreover, in order to find sufficient conditions for the stability of the steady solution, we analysed the full nonlinear system. To this aim, the identification of a suitable weighted Lyapunov functional leads us to determine the nonlinear stability results. For the problems exposed in the second part of the thesis, a subcritical instability region exists, where we cannot say anything about the future behaviour of the perturbations. Moreover, in order to establish the well-posedness of the Darcy-Bénard model for slightly-compressible fluids in porous media, we rigorously proved the existence, uniqueness and regularity of the solution of the problem at stake. Afterwards we performed the linear analysis determining instability results. The previous results were reached also taking advantage of numerical schemes. In this regard, we implemented and employed the Chebyshev-τ method to solve the differential eigenvalue problem in the linear analysis as well as the Euler-Lagrange system arising from the nonlinear analysis.

Throughout the doctoral program we investigated the compressibility effect on porous convection and we tried to cover as much as possible a good variety of physical set-ups where penetrative convection in porous

media may be modelled. In the following, we briefly summarize the main results achieved in the present thesis.
Concerning the density inversion phenomenon, the onset of porous penetrative convection in a horizontal layer uniformly heated from below is investigated. In particular, the critical pressure value for the thermodynamic consistency of the model is determined, global nonlinear stability results are determined and, for all the problems involving the Veronis' law, we proved the stabilizing effect of the upper plane temperature. Regarding the layer's inclination we proved that the transverse perturbations destabilize only up to a certain critical angles, that the most destabilizing are the longitudinal ones and, looking at the limit case of an horizontal layer, we recovered the results found in the previous study. As regards bi-disperse porous convection, the effect of double porosity structure on the onset of penetrative convection is investigated and the results are coherent, in the limit of a single porosity material, with the ones found in the previous investigation. Relating to the LTNE hypothesis accounting for a Darcy-Brinkman model, we found the stabilizing effect of the Darcy's number and in the limit case of LTE we found the expected coherent results. Concerning the EQTI fluids, after a rigorous proof of the well-posedness of the model under investigation, we were able to carry on a linear analysis of the conduction solution finding, in closed form, the critical linear Rayleigh number, proving that the compressibility factor has a destabilizing effect on the onset of convection and recovering the classical result in the limit case when the compressibility factor goes to zero.

In conclusion, we want to outline some future perspectives which take shape spontaneously from the contents of this thesis. An interesting problem to address would be to set back in the framework of inclined porous penetrative convection and asking what happens if the layer become vertical. This is a classical problem know as *Gill's problem*, [55, 119, 22]. In addition, a very interesting foundational question, would be to investigate the thermodynamic consistency of the density law which appears in double-diffusive convection problems [65, 81, 123] (convection problems in which a salt/chemical component C is dissolved at bottom of the layer), namely

$$\rho(p, T, C) = \rho_0[1 - \alpha(T - T_0) + \beta(p - p_0) + \gamma(C - C_0)],$$

where γ is Chemical expansion coefficient and C_0 a reference concentration, with particular regard to the effect of the parameter γ on the onset of convection.

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[†] To Jueun, with love.