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Geometric and functional inequalities for PDEs with Dirichlet and Robin boundary conditions

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Contents

Introduction

The task of optimizing the shape of an object to achieve maximum efficiency has interested people for a long time. One of the first evidence can be found in Virgil's *Aeneid*, where it is narrated that Queen Dido negotiated with the king of the tribes of Libya to be given as much land as could be enclosed by a bull's hide. The Queen managed to find a very clever solution, she cut the hide into thin strips, tied them together into a rope, and looped it around a plot of land by the shoreline in such a way as to maximize the enclosed area.

This legend is linked to the *Isoperimetric problem* that consists in finding the curve that encloses the largest area among the planar curves of fixed length. The solution to the Isoperimetric problem has been known for a long time, Queen Dido already knew it, and it states that among the planar curves of given length, the disc, and the disc only, encloses the largest area. For rigorous proof of this statement, one had to wait since the last two centuries, with the proof given in the nineteenth century by Steiner [149] and Edler [69] and the most complete proof given only by De Giorgi [63], almost 65 years ago, starting from a general definition of Perimeter. This leads to the, nowadays classical, *Isoperimetric inequality* that can be stated in any dimension: if $\Omega \subseteq \mathbb{R}^n$, then

$$P(\Omega) \geq n\omega_n^{\frac{1}{n}} |\Omega|^{1-\frac{1}{n}},$$

where ω_n is the $n-$measure of the unit ball in $\mathbb{R}^n$, $P(\Omega)$ is the classical perimeter introduced by De Giorgi, and $|\Omega|$ is the $n-$dimensional volume.

With the advent of computers and advancements in technology, this subject has garnered renewed attention, leading to the emergence of a prominent branch of mathematics known as Shape Optimization. This field of mathematical analysis is aimed at solving the problem of maximizing or, equivalently, minimizing a *Shape functional* $\mathcal{F}$ in a class of admissible sets $\mathcal{A}$,

$$\max\,(\min)\,\mathcal{F}(\Omega), \quad \text{where } \Omega \in \mathcal{A}. \tag{1}$$

From a mathematical perspective, problem (1) gives various intriguing aspects. It raises questions regarding the existence and uniqueness of solutions, the identification of optimality conditions, and the exploration of geometric or qualitative properties of the optimal shape(s), whenever it exists. Answering these questions is not always easy, as it shown by the long time we needed to get a rigorous proof of the Isoperimetric inequality.

Very often, one refers to an Isoperimetric Problem as a Shape Optimization problem where the ball is the solution. The study of such problems started with the conjecture of Saint Venant [64] about the optimal shape of the cross-section of a prism in order to maximize its torsional rigidity, that is the ability to resist torsional stress. This was finally settled by Pólya in 1948, proving that the bar with maximal torsional rigidity has a circular cross-section. From a mathematical point of view, if one denotes by Ω the cross-section of the bar, the torsional rigidity is the L^1-norm of the unique and positive solution in $H_0^1(\Omega)$ to the following problem

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \in \mathbb{R}^n$ is an open set with finite Lebesgue measure. So, if u_Ω is this unique solution, known as *torsion function*, the torsional rigidity, or *torsion*, is defined as

$$T(\Omega) = \int_\Omega u_\Omega \, dx,$$

and the Saint-Venant inequality states that

$$T(\Omega)|\Omega|^{-\frac{n+2}{n}} \leq T(B)|B|^{-\frac{n+2}{n}}, \tag{2}$$

where B is any open ball in $\mathbb{R}^n$.

Another problem that goes back to the end of the 19th Century, is the one of finding the planar membrane, fixed at its boundary, that has the minimal principal frequency. In the famous book of Lord Rayleigh, *The Theory of Sound* [134], the author conjectured that, among all planar sets with fixed area, the disk minimizes the principal frequency. This can be restated in a mathematical framework by saying that the disk minimizes the first Dirichlet-Laplacian eigenvalue, that is the smallest $\lambda > 0$ such that the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

admits a nontrivial solution in $H_0^1(\Omega)$. This conjecture was proved 50 years later by two simultaneous but independent works, one by Faber [72] and one by Krahn [108], and it was completely solved later with the work of Pólya and Szegö [132], leading to the following inequality

$$\lambda_1(\Omega)|\Omega|^{2/n} \geq \lambda_1(B)|B|^{2/n}, \tag{3}$$

being B any open ball in $\mathbb{R}^n$.

Exploring isoperimetric inequalities requires a blend of analysis, geometry, and the theory of partial differential equations. A key tool in addressing isoperimetric problems is the *Schwarz symmetrization*, a technique also referred to as the spherically symmetric and decreasing rearrangement of functions. Roughly speaking, the Schwarz symmetrization of a measurable function consists of rearranging the superlevel set of the function in a spherical shape in order to keep all the L^p norms fixed.

Symmetrization techniques in the context of qualitative properties of solutions to second-order elliptic boundary value problems were introduced by Talenti in [152]. In this seminal paper, the author considered an open, bounded and Lipschitz set $\Omega \subset \mathbb{R}^n$, the ball $\Omega^\sharp$ with the same measure as Ω and the solutions u_D and v_D to the following problems

$$\begin{cases} -\Delta u_D = f & \text{in } \Omega, \\ u_D = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta v_D = f^\sharp & \text{in } \Omega^\sharp, \\ v_D = 0 & \text{on } \partial\Omega^\sharp, \end{cases} \tag{4}$$

where $f \in L^2(\Omega)$ is a positive function and $f^\sharp$ is its Schwarz rearrangement of f (see Definition 1.2.3). In this setting, Talenti proved the following pointwise estimate:

$$u_D^\sharp(x) \leq v_D(x), \qquad \text{for all } x \in \Omega^\sharp. \tag{5}$$

For the sake of completeness, we observe that this result was proved more generally for a uniformly elliptic linear operator in divergence form in place of the Laplace operator Δ.

The result of Talenti has proven to be a very powerful tool, as it allows one to demonstrate both Saint Venant's conjecture (2) and Lord Rayleigh's conjecture (3).

Over the years, this topic has gained more and more interest. A version of this result for nonlinear operators in divergence form is contained in [153], which includes as a special instance the case of the $p-$Laplace operator. Further extensions can be found in [5] for anisotropic elliptic operators, in [7] for the parabolic case, and in [20, 155] for higher-order operators.

Once a comparison result holds, it is natural to ask whether the equality cases can be characterized and, so, if a *rigidity result* is in force. In [6], the rigidity result linked to problem (4) is proved. Indeed, the authors proved that if equality holds in (5), then Ω is a ball, u is radially symmetric and decreasing, and $f = f^\sharp$. Rigidity results for a generic linear, elliptic second-order operator can be found in [74] and [105].

The spherical symmetrization technique heavily depends on the boundary conditions we consider. In particular, for a long time, it was believed that comparison results could not be proved by means of spherical rearrangement argument when dealing with Robin boundary conditions until the recent paper [8]. The authors consider the following problems

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta v = f^\sharp & \text{in } \Omega^\sharp, \\ \dfrac{\partial v}{\partial \nu} + \beta v = 0 & \text{on } \partial\Omega^\sharp, \end{cases}$$

where ν is the outer unit normal to $\partial\,\Omega$, and they prove the following comparison involving Lorentz norms (see Definition 1.2.4) of u and v whenever f is a non-negative function in $L^2(\Omega)$ and β is a positive parameter,

$$\begin{aligned} \|u\|_{L^{k,1}(\Omega)} &\leq \|v\|_{L^{k,1}(\Omega^\sharp)} \quad \forall 0 < k \leq \frac{n}{2n-2}, \\ \|u\|_{L^{2k,2}(\Omega)} &\leq \|v\|_{L^{2k,2}(\Omega^\sharp)} \quad \forall 0 < k \leq \frac{n}{3n-4}. \end{aligned}$$

In particular, in the case $f \equiv 1$, they prove

$$\|u\|_{L^p(\Omega)} \leq \|v\|_{L^p(\Omega^\sharp)}, \quad p = 1, 2,$$

and, if $n = 2$, the pointwise comparison

$$u^\sharp(x) \leq v(x), \qquad \text{for all } x \in \Omega^\sharp. \tag{6}$$

Generalizations of the results contained in [8] can be found for the anisotropic case in [139], for mixed boundary conditions in [4], for the case of the Hermite operator in [55].

The starting point of the present Thesis is the study of symmetrization techniques in the context of nonlinear differential equations with Robin boundary conditions. Indeed, Chapter 2 of this Thesis aims to generalize the results contained in [8] to the nonlinear case, considering the $p-$Laplace operator

$$\Delta_p u := -\text{div}(|\nabla u|^{p-2}\nabla u)$$

instead of the Laplacian. We consider the problem

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + \beta |u|^{p-2}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

and we establish a comparison with the solution to the following symmetrized problem

$$\begin{cases} -\Delta_p v = f^\sharp & \text{in } \Omega^\sharp, \\ |\nabla v|^{p-2}\dfrac{\partial v}{\partial \nu} + \beta |v|^{p-2} v = 0 & \text{on } \partial\Omega^\sharp, \end{cases} \tag{8}$$

where $\Omega^\sharp$ is the ball centered in the origin with the same measure of Ω and $f^\sharp$ is the Schwarz rearrangement of f. In particular, we obtain

Theorem. *Let u and v be the solutions to problem (7) and (8) respectively. Then, we have*

$$\|u\|_{L^{k,1}(\Omega)} \leq \|v\|_{L^{k,1}(\Omega^\sharp)} \;\forall\, 0 < k \leq \frac{n(p-1)}{(n-1)p},$$

$$\|u\|_{L^{pk,p}(\Omega)} \leq \|v\|_{L^{pk,p}(\Omega^\sharp)} \;\forall\, 0 < k \leq \frac{n(p-1)}{(n-2)p+n}.$$

As a consequence of this Theorem, we have that, if $p \geq n$

$$\|u\|_{L^1(\Omega)} \leq \|v\|_{L^1(\Omega^\sharp)} \quad \text{and} \quad \|u\|_{L^p(\Omega)} \leq \|v\|_{L^p(\Omega^\sharp)}.$$

In the particular case when the right-hand side f is constant, we obtain further information

Theorem. *Assume that $f \equiv 1$ and let u and v be the solutions to (7) and (8) respectively.*

i. If $1 < p \leq \dfrac{n}{n-1}$ then

$$u^\sharp(x) \leq v(x) \qquad x \in \Omega^\sharp,$$

ii. if $p > \dfrac{n}{n-1}$ and $0 < k \leq \dfrac{n(p-1)}{n(p-1)-p}$, then

$$\|u\|_{L^{k,1}(\Omega)} \leq \|v\|_{L^{k,1}(\Omega^\sharp)},$$
$$\|u\|_{L^{pk,p}(\Omega)} \leq \|v\|_{L^{pk,p}(\Omega^\sharp)}.$$

As a consequence of this Theorem, in the case $f \equiv 1$ we have that

$$\|u\|_{L^1(\Omega)} \leq \|v\|_{L^1(\Omega^\sharp)} \quad \text{and} \quad \|u\|_{L^p(\Omega)} \leq \|v\|_{L^p(\Omega^\sharp)} \qquad \text{for } p > 1,$$

while we have the point-wise comparison only for $p \leq \dfrac{n}{n-1}$.

We said that Talenti's result enables one to prove the Faber-Krahn inequality (3) regarding the first Dirichlet eigenvalue. On the other hand, if we consider the first Robin eigenvalue, defined by the following Rayleigh quotient

$$\lambda_{p,\beta}(\Omega) = \min_{\substack{\omega \in W^{1,p}(\Omega) \\ \omega \neq 0}} \frac{\displaystyle\int_\Omega |\nabla \omega|^p \, dx + \beta \int_{\partial\Omega} |\omega|^p \, d\mathcal{H}^{n-1}(x)}{\displaystyle\int_\Omega |\omega|^p \, dx},$$

it was proved by Bossel [33] and Daners [62], and then generalized in [45, 47, 48, 50], that among Lipschitz sets of given volume it holds

$$\lambda_{p,\beta}(\Omega) \geq \lambda_{p,\beta}(B),$$

being B a ball with the same measure of Ω.

Our result provides an alternative demonstration for the Bossel-Daners inequality, at least when $p \geq n$. Actually, the Bossel-Daners inequality is more general, since it holds for every $p > 1$, but it is obtained with completely different tools than the ones we used to prove the results in Chapter 2.

Talenti's result also allowed proving Saint-Venant's conjecture in the stronger form of the pointwise inequality (5) between torsion functions. In the case of Robin boundary conditions, one can define the Robin torsion of Ω as the L^1-norm of u, solution to (7) when $f \equiv 1$ and $p = 2$

$$T(\Omega) := \int_\Omega u \, dx,$$

or, equivalently, as the maximum of the following Rayleigh quotient:

$$T(\Omega) = \max_{\substack{\varphi \in H^1(\Omega) \\ \varphi \not\equiv 0}} \frac{\left(\displaystyle\int_\Omega |\varphi(x)| \, dx \right)^2}{\displaystyle\int_\Omega |\nabla \varphi(x)|^2 \, dx + \beta \int_{\partial\Omega} \varphi^2 \, d\mathcal{H}^1}.$$

In [47] the authors proved that the Robin torsional rigidity is maximum on balls among bounded and Lipschitz sets of fixed Lebesgue measure, and the proof of this Saint-Venant type inequality relies on reflection arguments (see also [49]). Theorem 1.2 in [8] enables one to prove the same result with symmetrization techniques; anyway, it is still an open problem to establish if, for $q \in (1, +\infty)$, the ball maximizes the L^q norm of the torsion function among open, bounded and Lipschitz sets (see [49, Open Problem 1]). First pieces of evidence in this direction are provided in [140], where it is proved that the ball is a critical shape for every L^q norm in dimension $n > 2$, as in the case $n = 2$ the Open Problem 1 in [49] is solved in the stronger, and *Talentian*, form (6).

To further investigate in this direction, we ask if it is possible to characterize the equality cases in the comparison for solutions to (7) and (8) in the most general case, with a generic $f \in L^2(\Omega)$ positive, $1 < p < +\infty$, and $n \geq 2$, case in which we have the following

$$\|u\|_{L^{pk,p}(\Omega)} \leq \|v\|_{L^{pk,p}(\Omega^\sharp)}, \qquad \forall\, 0 < k \leq \frac{n(p-1)}{(n-2)p+n}.$$

The result we obtain can be considered the analogous to the Theorem proved by Alvino Lions and Trombetti in [6] for the Dirichlet boundary conditions.

Theorem. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open and Lipschitz set and let $\Omega^\sharp$ be the ball centered at the origin with the same measure as Ω. Let u be the solution to (7) and let v be a solution to (8). If*

$$\|u\|_{L^{pk,p}(\Omega)} = \|v\|_{L^{pk,p}(\Omega^\sharp)}, \quad \text{for some } k \in \left]0, \frac{n(p-1)}{(n-2)p+n}\right]$$

then, there exists $x_0 \in \mathbb{R}^n$ such that

$$\Omega = \Omega^\sharp + x_0, \qquad u(\cdot + x_0) = v(\cdot), \qquad f(\cdot + x_0) = f^\sharp(\cdot).$$

Of course, a similar result holds when $f \equiv 1$, when k is allowed to vary in the wider range

$$0 < k \leq \frac{n(p-1)}{n(p-1)-p}.$$

If one wants to prove a rigidity result of this kind, the first thing to prove is that the set Ω coincides, up to a translation, with the set $\Omega^\sharp$. This can be done by showing that the superlevel sets of u are balls. The main difficulty is to prove that these balls are concentric.

In the case of the Laplace operator with Dirichlet boundary conditions studied in [7, 73], this is proven by applying the steepest descent method introduced in [19], which strongly relies on the continuity of both the solution and of its gradient.

In the problem considered in Chapter 2, we decided to avoid the use of the steepest descent method, exploiting a different technique, as this method strongly relies on the continuity of the function. In the case of the $p-$Laplace equation, the continuity of the solution up to the boundary depends on the regularity of the given datum f. To overcome this regularity issue, we show that u is a solution to a suitable Dirichlet problem and it satisfies the Pólya-Szegő inequality

$$\int_{\Omega^\sharp} \left|\nabla u^\sharp\right|^p dx \leq \int_\Omega |\nabla u|^p \, dx$$

with the equality sign. Then, we can conclude that u is radially symmetric and decreasing, using the classical result contained in [43].

To the best of our knowledge, rigidity results for nonlinear operators with Dirichlet boundary conditions are not present in the literature. In Section 2.2 we obtain, as a corollary of our results, the rigidity for the $p-$Laplace operator with Dirichlet boundary conditions in any dimension (see Corollary 2.2.4).

The latter result gives new information about the Open Problem 1 in [49]. Indeed, the Torsion problem can be obtained by choosing $f \equiv 1$ and $p = 2$ and our result ensures that the ball is the *only* maximizer of the L^2 norm of the Torsion function.

Finally, we devote the end of Chapter 2 to some examples and further considerations. The most significant is linked to the following semilinear problem

$$\begin{cases} -\Delta u = f(u) & \text{in } B, \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial B, \end{cases} \tag{9}$$

being B a ball in $\mathbb{R}^n$. The analogous problem with Dirichlet boundary conditions was studied by Gidas, Ni, and Nirenberg in [88], where they proved that, under suitable assumptions on the function f, the solution u must be radial and decreasing. Later, Lions in [113] was able to remove the regularity assumption on f, by imposing its positivity and using the symmetrization techniques by Talenti. We ask if the theory developed in [8] in the context of Robin boundary conditions can be applied to problem (9), but we obtain a negative answer

Theorem. *Let $n \geq 2$. There exists a positive superharmonic function u that is a solution to* (9) *and that is not radially symmetric.*

So, it seems that a result á la Gidas-Ni-Nirenberg cannot be proved, neither in the superharmonic case, in the context of Robin boundary conditions.

In Chapter 3, we focus our attention on the *eigenvalues* of the $p-$Laplace operator with Robin boundary conditions. In particular, we study the limit, as p goes to infinity, of the first eigenvalue of the $p-$Laplacian, defined as the minimum of the following Rayleigh quotient

$$\lambda_{p,\beta^p} = \inf_{w\in W^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega |\nabla w|^p\,dx + \beta^p \int_{\partial\Omega} |w|^p\,d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega |w|^p\,dx}.$$

We first prove that

$$\lim_{p\to+\infty} (\lambda_{p.\beta^p})^{1/p} = \Lambda_{\infty,\beta} =: \inf_{\substack{w\in W^{1,\infty}(\Omega)\\ \|w\|_{L^\infty(\Omega)}=1}} \max\left\{\|\nabla w\|_{L^\infty(\Omega)}, \beta\|w\|_{L^\infty(\partial\Omega)}\right\},$$

and we give a geometric characterization of this quantity, precisely:

$$\Lambda_{\infty,\beta} = \frac{1}{1/\beta + r(\Omega)},$$

where $r(\Omega)$ denotes the inradius of Ω, i.e. the radius of the largest ball contained in Ω. Thereafter, we prove that $\Lambda_{\infty,\beta}$ is the first eigenvalue of the $\infty-$Laplacian, in the sense that equation

$$\begin{cases} \min\{\, |\nabla u| - \Lambda u, -\Delta_\infty u\,\} = 0 & \text{in } \Omega,\\ -\min\left\{\, |\nabla u| - \beta u, -\dfrac{\partial u}{\partial \nu}\,\right\} = 0 & \text{on } \partial\Omega, \end{cases}$$

that can be seen as the limit $p-$Laplace eigenvalue equation, admits non-trivial solutions only if $\Lambda \geq \Lambda_{\infty,\beta}$.

Similar results, in the case of Dirichlet and Neumann boundary conditions, were obtained in [102, 101, 26, 71, 137]. More specifically, in [102, 101], Juutinen, Lindqvist and Manfredi studied the Dirichlet case as $p \to +\infty$, providing a complete characterization of the limiting solutions in terms of geometric quantities. Indeed, the first eigenvalue of the $p-$Laplace operator $\{\lambda_p^D\}$ happens to satisfy

$$\lim_{p\to\infty} \left(\lambda_p^D\right)^{1/p} = \lambda_\infty^D := \frac{1}{r(\Omega)}.$$

The related eigenfunctions v_p^D also converge (up to a subsequence) to some Lipschitz function v_∞^D. Most importantly, the authors showed that there exists a natural viscosity formulation of the eigenvalue problem for the ∞-Laplacian, for which λ_∞^D and v_∞^D turn out to be the first eigenvalue and first eigenfunction, respectively.

The Neumann case seems to be more subtle. It was investigated in [71, 137] and, similarly to the Dirichlet case, the authors established that the first non-trivial eigenvalues of the $p-$Laplacian $\{\lambda_p^N\}$ satisfy

$$\lim_{p\to\infty} \left(\lambda_p^N\right)^{1/p} = \lambda_\infty^N := \frac{2}{\mathrm{diam}(\Omega)},$$

where $\mathrm{diam}(\Omega)$ is the intrinsic diameter of Ω, i.e. the supremum of the geodetic distance between two points of Ω.

However, while both first eigenvalues and first eigenfunctions converge (as $p \to \infty$) and are solutions to some appropriate eigenvalue problem for the ∞-Laplacian, in [71], the authors are able to prove that they actually converge to the *first* eigenvalue e *first* eigenfunction only in the case of a convex domain Ω. Whether or not the same holds true in the general case, is still an open problem.

Thanks to study of the limiting behaviour of the Robin $p-$Laplace eigenvalue, we can focus our attention on the study of the limit of the $p-$Poisson equation with Robin boundary conditions:

$$\begin{cases} -\Delta_p v = f & \text{in } \Omega \\ |\nabla v|^{p-2}\dfrac{\partial v}{\partial \nu} + \beta^p |v|^{p-2} v = 0 & \text{on } \partial\Omega, \end{cases}$$

when $f \in L^\infty(\Omega)$ is a non-negative function.

We prove that there exists (up to a subsequence) a limiting solution v_∞ as $p \to \infty$ and we establish conditions on f which are equivalent to the uniqueness of v_∞.

The ∞-Poisson problem for Dirichlet boundary conditions has already been studied in [26] by Bhattacharya, DiBenedetto and Manfredi, while, to the best of our knowledge, similar results have not been addressed in the case of Neumann boundary conditions.

Chapter 4 deals with different Shape Optimization problems in the class of convex sets. We focus on different geometrical quantities: the Robin-eigenvalues, the Torsional rigidity, and the Cheeger constant.

In the first part of Chapter 4, we study the different behavior of eigenvalues of the $p-$Laplace operator with Robin boundary conditions in the case $\beta > 0$ and $\beta < 0$. For the sake of completeness, we recall that, in the case of $\beta = 0$, we recover the Neumann boundary condition, for which the first eigenvalue is zero and the associated eigenfunctions are constants.

As we already said, in the case $\beta > 0$, Bossel [33] and Daners [62] proved a Faber-Krahn inequality for the first eigenvalue of the Robin-Laplacian in two and higher dimensional case, respectively. In particular, they proved that among sets of given volume, the one which minimizes the first Robin-eigenvalue is the ball. This Faber-Krahn inequality for fixed volume and the following rescaling property (see [44])

$$\lambda_{p,\beta}(t\Omega) \le \frac{1}{t}\lambda_{p,\beta}(\Omega) \le \lambda_{p,\beta}(\Omega), \quad \forall t > 1,$$

give a Faber-Krahn inequality for fixed perimeters, i.e.

$$\lambda_{p,\beta}(\Omega^\star) \le \lambda_{p,\beta}(\Omega),$$

where $\Omega^\star$ is the ball having the same perimeter as Ω.

Our aim is to give a continuity bound, in terms of the isoperimetric deficit, to the ratio

$$\frac{\lambda_{p,\beta}(\Omega) - \lambda_{p,\beta}(\Omega^\star)}{\lambda_{p,\beta}(\Omega)},$$

indeed, we prove

Theorem. *Let β be a positive parameter. Let Ω be a bounded, open and convex set in $\mathbb{R}^n$ and let $\Omega^\star$ be the ball, centered at the origin, such that $P(\Omega) = P(\Omega^\star) = \rho$. Denote by $\lambda_{p,\beta}(\Omega)$ and $\lambda_{p,\beta}(\Omega^\star)$ the first eigenvalues of the $p-$Laplacian operator with Robin boundary conditions respectively on Ω and $\Omega^\star$, and by v a positive eigenfunction associated to $\lambda_{p,\beta}(\Omega^\star)$, then*

$$\frac{\lambda_{p,\beta}(\Omega) - \lambda_{p,\beta}(\Omega^\star)}{\lambda_{p,\beta}(\Omega)} \le C(n,p,\beta,\rho)\left(1 - \frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{P(\Omega)^{\frac{n}{n-1}}}\right),$$

where ω_n *is the measure of the unitary ball in* $\mathbb{R}^n$*, and* $C(n,p,\beta,\rho) = \dfrac{\|v\|_\infty^p |\Omega^\star|}{\|v\|_p^p}$.

We observe that this result can be seen as a generalization to the Robin case of the result in [36], which holds true in the case of Dirichlet eigenvalues of the $p-$Laplacian. Incidentally, let us mention that the anisotropic case is studied in [65].

On the other hand, when β is a negative parameter, the authors in [46] proved a reverse Faber-Krahn inequality for the first eigenvalue of the Robin-Laplacian among convex sets of given perimeter. In particular, they proved that among convex sets of given perimeter the ball $\Omega^\star$ maximizes the first Robin eigenvalue of the $p-$Laplacian, i.e.

$$\lambda_{p,\beta}(\Omega) \leq \lambda_{p,\beta}(\Omega^\star).$$

For completeness' sake, we quote that in [15], it has already been proved that the disc maximizes the first eigenvalue under a perimeter constraint, among C^2 domains in $\mathbb{R}^2$, while the question remained open in arbitrary dimension.

This question is related to the conjecture of Bareket (see [22]) claiming that the ball maximizes $\lambda_{p,\beta}(\Omega)$ among all Lipschitz sets with given volume. Freitas and Krejčiřík in [77] proved that the conjecture is false, giving a counter-example based on the asymptotic behavior of the eigenvalues on a disc and an annulus of the same area when $\beta \to -\infty$. They also proved that among sets of area equal to 1, the conjecture is true, provided β is close to 0. For more details see [93].

Concerning the quantitative version of the reverse Faber-Krahn, we can find it in [57] for the case $p = 2$ among convex sets of fixed perimeter, where the techniques introduced by Fuglede in [83] are used.

In Chapter 4, we recover the result in [57] obtaining a quantitative version of the reverse Faber-Krahn inequality for all $p \in (1, +\infty)$, using a different approach. Indeed, following the method introduced by Payne and Weinberger in [129], we establish a comparison using the so-called parallel coordinates method. In particular, we firstly prove a lower bound in terms of perimeter and measure of Ω, that is

Theorem. *Let* β *be a negative parameter. Let* Ω *be a bounded, open and convex set in* $\mathbb{R}^n$ *and let* $\Omega^\star$ *be the ball, centered at the origin, such that* $P(\Omega) = P(\Omega^\star) = \rho$*. Denote by* $\lambda_{p,\beta}(\Omega)$ *and* $\lambda_{p,\beta}(\Omega^\star)$ *the first eigenvalues of the* $p-$*Laplacian operator with Robin boundary conditions respectively on* Ω *and* $\Omega^\star$*, and by* v *a positive eigenfunction associated to* $\lambda_{p,\beta}(\Omega^\star)$*, then*

$$\frac{\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega)}{|\lambda_{p,\beta}(\Omega)|} \geq C(n,p,\beta,\rho)\left(1 - \frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{P(\Omega)^{\frac{n}{n-1}}}\right),$$

where ω_n *is the measure of the unitary ball in* $\mathbb{R}^n$*, and* $C(n,p,\beta,\rho) = \dfrac{v_m^p|\Omega^\star|}{\|v\|_p^p}$ *with* $v_m = \min\limits_{\Omega^\star} v$.

Then we prove the quantitative result as in [57].

Theorem. *Let* $n \geq 2$*,* $\rho > 0$ *and* $\beta < 0$*. Then, there exist two positive constants* $C(n,p,\beta,\rho) > 0$ *and* $\delta_0(n,p,\beta,\rho) > 0$*, such that, for all* $\Omega \subset \mathbb{R}^n$ *bounded and convex with* $P(\Omega) = \rho$ *and* $\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega) \leq \delta_0$*, it holds*

$$\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega) \geq C(n,p,\beta,\rho)g(\mathcal{A}_{\mathcal{H}}^\star(\Omega))$$

where $\Omega^\star$ *is a ball with the same perimeter of* Ω*,* $\mathcal{A}_{\mathcal{H}}^\star$ *is the Hausdorff asymmetry defined in* (1.53) *and* g *is defined in* (1.57).

In the second part, we study a generalization and find a quantitative result for Pólya's inequality, that gives an estimate from below of the torsion of a non-empty open, bounded and convex set, in terms of its perimeter and measure.

In [131] the author proved that, among all bounded, open and convex planar sets, the following inequality holds

$$\frac{T(\Omega)P^2(\Omega)}{|\Omega|^3} \geq \frac{1}{3} \tag{10}$$

and equality is asymptotically achieved by a sequence of thinning rectangles. An upper bound of the same functional was given by Makai in [118], where he proved that among all bounded, open and convex planar sets, it holds

$$\frac{T(\Omega)P^2(\Omega)}{|\Omega|^3} \leq \frac{2}{3}, \tag{11}$$

which is asymptotically achieved by a sequence of thinning triangles (for the exact definition of thinning domains see Definition 1.47). These estimates were generalized to the $p-$torsion in [76]. More precisely, the authors proved that, in the class of bounded, open and convex planar sets,

$$\frac{1}{q+1} < \frac{T_p(\Omega)P^q(\Omega)}{|\Omega|^{q+1}} < \frac{2^{q+1}}{(q+2)(q+1)} \qquad q = \frac{p}{p-1},$$

where the lower and the upper bounds hold asymptotically on a sequence of thinning rectangles and on a sequence of thinning isosceles triangles, respectively. In [65], the authors generalized the lower bound in every dimension, proving that, for bounded, open and convex sets $\Omega \subseteq \mathbb{R}^n$, it holds

$$\frac{T_p(\Omega)P^q(\Omega)}{|\Omega|^{q+1}} > \frac{1}{q+1},$$

and they extended such results also to the anisotropic case.

This problem has interested also the authors in [40], where they considered the functional

$$H_k(\Omega) = \frac{P(\Omega)T^k(\Omega)}{|\Omega|^{\alpha_k}} \qquad \alpha_k = 1 + k + \frac{2k-1}{n}$$

and proved that, among bounded, open, and convex sets in $\mathbb{R}^n$, this functional is bounded if and only if $k = 1/2$. More precisely, they proved the following:

$$\frac{1}{\sqrt{3}} \leq H_{\frac{1}{2}}(\Omega) \leq \frac{2^n n^{3n/2}}{\omega_n} \left(\frac{n}{n+2}\right)^{\frac{1}{2}}. \tag{12}$$

We observe that, in the planar case, the lower bound in (12) coincides with the one given in (10), while the upper bound is strictly larger than the one given in (11). It is conjectured that, in the higher dimensional case, the optimal upper bound is

$$H_{\frac{1}{2}}(\Omega) \leq n \left(\frac{2}{(n+1)(n+2)}\right)^{\frac{1}{2}}.$$

In Chapter 4, we first generalize inequality (10) replacing the torsion with the $(f,p)-$torsion. In particular, we consider $\Omega \subset \mathbb{R}^n$, $n \geq 2$, to be a non-empty, bounded, open and convex set and $p \in (1, +\infty)$. The $(f,p)-$torsion is defined as the L^1-norm of the solution to

$$\begin{cases} -\Delta_p u(x) = f(d(x, \partial\Omega)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f : [0, r(\Omega)] \to [0, +\infty[$ is a continuous, non-increasing and not identically zero function, $d(\cdot, \partial\Omega) : \Omega \to [0, +\infty[$ is the distance function from the boundary and $r(\Omega)$ is the inradius of Ω. This class of functions, depending only on the distance, are the so-called web functions, see as a reference [59]. Equivalently, one can define the (f, p)-torsion of Ω as

$$T_{f,p}(\Omega) = \max_{\substack{\varphi \in W_0^{1,p}(\Omega) \\ \varphi \not\equiv 0}} \frac{\left(\displaystyle\int_\Omega f(d(x, \partial\Omega)) |\varphi(x)| \, dx \right)^{\frac{p}{p-1}}}{\left(\displaystyle\int_\Omega |\nabla \varphi(x)|^p \, dx \right)^{\frac{1}{p-1}}}.$$

The first result that we prove, following the method of proof used in [131] with the use of web functions as test functions, is a lower bound for the (f, p)-torsional rigidity.

Theorem. *Let Ω be a non-empty, bounded, open and convex set of $\mathbb{R}^n$, $n \geq 2$, and let $f : [0, r(\Omega)] \to [0, +\infty[$ be a continuous and non-increasing function such that $f \not\equiv 0$. Then, it holds*

$$T_{f,p}(\Omega) \geq c_p \frac{\mu_f^{q+1}(\Omega)}{f(0) P^q(\Omega)},$$

where

$$c_p = \frac{p-1}{2p-1}, \qquad q = \frac{p}{p-1},$$

and

$$\mu_f(\Omega) = \int_\Omega f(x) \, dx.$$

Moreover, the equality sign is asymptotically achieved by a sequence of thinning cylinders.

For the definition of thinning cylinder see Definition 1.4.9. Later, we focus our study on the case $f \equiv 1$ and $n = 2$ and we obtain some quantitative estimates. If we define the following scaling invariant functional

$$\mathcal{F}_p(\Omega) = \frac{T_p(\Omega) P^q(\Omega)}{|\Omega|^{q+1}}, \qquad q = \frac{p}{p-1},$$

the statement of our Theorem, in the case $f \equiv 1$, reads

$$\mathcal{F}_p(\Omega) \geq c_p,$$

and if we consider a sequence of thinning cylinders $\{\Omega_l\}_{l \in \mathbb{N}}$, we have

$$\mathcal{F}_p(\Omega_l) \xrightarrow{l \to 0} c_p.$$

This leads to the following stability issue: if $\mathcal{F}_p(\Omega)$ is close to c_p, can we say that Ω is close in some sense to a thin cylinder? The following result gives us information on the nature of the geometry of Ω: when $\mathcal{F}_p(\Omega) - c_p$ is sufficiently small, the set Ω is a thin domain.

The main novelty consists indeed in the following quantitative results of the Pólya-type estimates proved in [131, 76, 65] by means of suitable deficits. For completeness, we recall some standard references about isoperimetric quantitative results, see for example [84, 86, 39, 38, 87, 126]. The main difference between these results and ours is that the equality in Pólya's estimates is achieved asymptotically for a sequence of thinning cylinders. Hence, the proof of quantitative result must take into account that there is not a minimum, as in the classical isoperimetric stability results. What we prove is the following.

Theorem. *Let Ω be a non-empty, bounded, open and convex set of $\mathbb{R}^n$ and let $f \equiv 1$. Then,*

$$\mathcal{F}_p(\Omega) - c_p \geq K(n,p)\left(\frac{w(\Omega)}{diam(\Omega)}\right)^{n-1},$$

where $K(n,p)$ is a positive constant depending only on p and the dimension of the space n and the quantity $w(\Omega)$ is the minimal width *of Ω, defined in Definition (1.12). In particular, in the case $n=2$, the exponent of the quantity $\dfrac{w(\Omega)}{diam(\Omega)}$ is sharp.*

We prove a second quantitative result in the case $p = n = 2$.

Theorem. *Let Ω be a non-empty, bounded, open and convex set in $\mathbb{R}^2$, let $f \equiv 1$ and let $p = 2$. Then, there exists a positive constant $\tilde{K}$ such that*

$$\mathcal{F}_2(\Omega) - c_2 = \frac{T(\Omega)P^2(\Omega)}{|\Omega|^3} - \frac{1}{3} \geq \tilde{K}\left(\frac{|\Omega \triangle Q|}{|\Omega|}\right)^3,$$

where $\Omega \triangle Q$ denotes the symmetric difference between Ω and a rectangle Q with sides $P(\Omega)/2$ and $w(\Omega)$ containing Ω.

Eventually, in Chapter 4 we deal with the problem of minimizing and maximizing the Cheeger constant of a convex set Ω.

The Cheeger constant of Ω was introduced by Jeff Cheeger in [54] in connection with the first eigenvalue of the Laplacian, and it is defined as

$$h(\Omega) := \inf\left\{\frac{P(E)}{|E|} \,:\, E \,\text{measurable and}\, E \subseteq \Omega,\, |E| > 0\right\}. \tag{13}$$

The minimum in (13) is achieved when Ω has Lipschitz boundary, see as a reference [127], and the set E that realizes this minimum is called a *Cheeger set* of Ω. For the properties of the Cheeger constant and for an introductory survey, see for example [3, 103, 127]. We point out that, in the case of planar convex sets, the authors in [3] prove that the Cheeger set is unique,so it can be denoted by C_Ω, while in [103] the authors give a characterization for the Cheeger constant.

The problem of finding the Cheeger constant of a domain has been widely considered and has several applications. One of the possible interpretations of the Cheeger constant can be found for instance in the context of maximal flow and minimal cut problems (see [150]) and this has applications in the medical images process (see [17]). The Cheeger problem also appears in the study of plate failure under stress (see [104]). For these reasons, it is useful to have estimates of the Cheeger constant in terms of geometric quantities that can be easily computed.

Here, we are interested in describing all possible inequalities involving the Cheeger constant of a given compact, bounded and convex set $\Omega \subset \mathbb{R}^2$ with nonempty interior and two among the following geometrical quantities: the area $|\Omega|$, the perimeter $P(\Omega)$, the inradius $r(\Omega)$, the circumradius $R(\Omega)$, the minimal width $w(\Omega)$ and the diameter $\operatorname{diam}(\Omega)$. So, we aim to study the associated Blaschke–Santaló diagrams of these triplets.

A Blaschke–Santaló diagram is a tool that allows one to visualize all the possible inequalities between three geometric quantities. More precisely, we consider three homogenous shape functionals (J_1, J_2, J_3), that is to say that for every $i \in \{1,2,3\}$ there exists $\alpha_i \in \mathbb{R}$ such that $J_i(t\Omega) = t^{\alpha_i} J_i(\Omega)$ for every $t > 0$, and we want to find a system of inequalities describing the set

$$\{(J_1(\Omega), J_2(\Omega))|\ J_3(\Omega) = 1,\, \Omega \in \mathcal{K}^2\},$$

where we denote by $\mathcal{K}^2$ the class of compact sets in $\mathbb{R}^2$ that are convex and with nonempty interior. This kind of diagram was introduced by Blaschke in [27], in order to investigate all possible relations between the volume, the surface area, and the integral mean curvature in the class of compact convex sets in $\mathbb{R}^3$. Following the idea of Blaschke, Santaló in [141] proposed the study of these diagrams for all the triplets of the following geometrical quantities: area, perimeter, inradius, circumradius, minimal width and diameter; these diagrams were studied under the constraint of convexity and six of them are still not completely solved. We refer to the introduction in [67] for the accurate state of art. Moreover, for classical results about Blaschke–Santaló diagram, we refer for example to [32, 94, 95, 96, 97, 98, 141] and for more recent results we recall [34, 35, 66, 67, 80, 81, 115].

In [78] and [79] the author studied two Blaschke–Santaló diagrams involving the Cheeger constant. More precisely, in [78], it is studied the Blaschke–Santaló diagram between the Cheeger constant, the area and the inradius, and it is proved that, if Ω in $\mathcal{K}^2$, then

$$\frac{1}{r(\Omega)} + \frac{\pi r(\Omega)}{|\Omega|} \leq h(\Omega) \leq \frac{1}{r(\Omega)} + \sqrt{\frac{\pi}{|\Omega|}},$$

where the upper bound is achieved by (and only by) sets that are homothetic to their form body (see Definition 1.4.1), meanwhile the lower one is achieved by (and only by) stadiums. Then, in [79], it is studied the diagram between the Cheeger constant, the area, and the perimeter and it is proved that if $\Omega \in \mathcal{K}^2$, then

$$\frac{P(\Omega) + \sqrt{4\pi|\Omega|}}{2|\Omega|} \leq h(\Omega) \leq \frac{P(\Omega)}{|\Omega|},$$

where the upper bound is achieved by any set that is Cheeger of itself (in particular stadiums), meanwhile the lower one is achieved, for example, by circumscribed polygons. We also recall that in [91] the maximization problem of the Cheeger constant among sets of constant width is studied.

In Chapter 4, we first prove the following

Theorem. *Let $\Omega \in \mathcal{K}^2$, then the minimization and the maximization Shape Optimization problems of the Cheeger constant $h(\Omega)$ admit a solution whenever we fix two geometrical quantity between the following: $|\Omega|$, $P(\Omega)$, $r(\Omega)$, $R(\Omega)$, $w(\Omega)$, $diam(\Omega)$.*

In some cases, we are able to provide a complete description of the relative Blaschke–Santaló diagrams. For the precise definitions of the below-mentioned extremal sets, see Section 1.4. For the explicit bounds, see Propositions 4.3.5, 4.3.6 and 4.3.7 and for the description of the corresponding diagrams we refer to Proposition 4.3.8.

Theorem. *The following results hold*

(i) The maximum and the minimum in $\mathcal{K}^2_{P,r} := \{\Omega \in \mathcal{K}^2 : P(\Omega) = P,\, r(\Omega) = r\}$, where $P \geq 2\pi r$, are achieved respectively by sets that are homothetic to their form body and stadiums.

(ii) The maximum in $\mathcal{K}^2_{d,r} = \{\Omega \in \mathcal{K}^2 : diam(\Omega) = d,\, r(\Omega) = r\}$, where $d \geq 2r$ is achieved by symmetrical two-cup bodies; moreover, there exists $D_0 > 0$ such that if $d \geq rD_0$ the minimum in $\mathcal{K}^2_{d,r}$ is achieved by symmetrical spherical slices, while, if $d < rD_0$, the minimum is achieved by regular smoothed nonagons.

(iii) The maximum and the minimum in $\mathcal{K}^2_{R,r}\{\Omega \in \mathcal{K}^2 : R(\Omega) = R,\, r(\Omega) = r\}$, where $R \geq r$ are achieved respectively by two-cup bodies and symmetrical spherical slices.

For the remaining classes of sets, for which we obtain partial results, we refer to the end of Chapter 4.

Chapter 1

Preliminaries

1.1 Notations

- n is the dimension of the Euclidian space;
- $|\cdot|$ stands both for the Euclidean norm in $\mathbb{R}^n$ and the Lebesgue measure in $\mathbb{R}^n$;
- $x \cdot y$ is the standard Euclidean scalar product for $n \geq 2$;
- $\mathcal{H}^k(\cdot)$, for $k \in [0, n)$, is the $k-$dimensional Hausdorff measure in $\mathbb{R}^n$;
- ω_n is measure of the unit $n-$dimensional ball;
- χ_A is characteristic function of a set A

1.1.1 Basic definitions

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, open set and let $E \subseteq \mathbb{R}^n$ be a measurable set. We recall the definition of the perimeter of E in Ω, that is

$$P(E; \Omega) = \sup \left\{ \int_E \mathrm{div}\varphi \, dx : \ \varphi \in C_c^\infty(\Omega; \mathbb{R}^n), \ ||\varphi||_\infty \leq 1 \right\}.$$

The perimeter of E in $\mathbb{R}^n$ will be denoted by $P(E)$ and, if $P(E) < \infty$, we say that E is a set of finite perimeter. Some references for results relative to the sets of finite perimeter are for example [117, 13]. We observe that a feature of this definition is that in this way the perimeter is not affected by modifications on sets of measure 0. Moreover, if E has Lipschitz boundary, we have that

$$P(E) = \mathcal{H}^{n-1}(E).$$

By their respectively definitions, we have that $P(E)$ and $|E|$ satisfy the following scaling properties, for $t > 0$,

$$P(tE) = t^{n-1} P(E), \qquad |tE| = t^n |E|.$$

We recall the classical isoperimetric inequality and we refer the reader, for example, to [125, 51, 53, 154] and to the original paper by De Giorgi [63].

Theorem 1.1.1 (Isoperimetric Inequality)**.** *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter. Then,*

$$n\omega_n^{\frac{1}{n}}|E|^{\frac{n-1}{n}} \leq P(E), \tag{1.1}$$

where ω_n is the measure of the unit ball in $\mathbb{R}^n$. Equality occurs if and only if E is (equivalent to) a Ball.

If Ω is an open and Lipschitz set, it holds the following coarea formula. Some references for results relative to the sets of finite perimeter and the coarea formula are, for instance, [13, 117].

Theorem 1.1.2 (Coarea formula)**.** *Let $f : \Omega \to \mathbb{R}$ be a Lipschitz function and let $u : \Omega \to \mathbb{R}$ be a measurable function. Then,*

$$\int_\Omega u|\nabla f(x)|dx = \int_{\mathbb{R}} dt \int_{(\Omega\cap f^{-1}(t))} u(y)\, d\mathcal{H}^1(y). \tag{1.2}$$

1.2 Rearrangements of functions

Let us recall some basic notions about rearrangements. We refer for instance to [106, 153] for all the details.

Definition 1.2.1. Let $u\colon \Omega \to \mathbb{R}$ be a measurable function, the *distribution function* of u is defined as

$$\mu\colon [0,+\infty) \to [0,+\infty) \qquad \mu(t) = |\{\, x \in \Omega \,:\, |u(x)| > t \,\}|.$$

From its definition, it is clear that μ is a decreasing function, moreover, one can prove that μ is right continuous and

$$\mu(t^-) = |\{\, x \in \Omega \,:\, |u(x)| \geq t \,\}|.$$

By use of Coarea formula (1.2), one can deduce the following expression for μ

$$\mu(t) = |\{u > t\} \cap \{\nabla u = 0\}| + \int_t^{+\infty} \left(\int_{u=s} \frac{1}{|\nabla u|}\, d\,\mathcal{H}^{n-1} \right) ds, \tag{1.3}$$

as a consequence, for almost all $t \in (0,+\infty)$,

$$\infty > -\mu'(t) \geq \int_{u=t} \frac{1}{|\nabla u|} d\mathcal{H}^{n-1} \tag{1.4}$$

and if μ is absolutely continuous, equality holds in (1.4).

Definition 1.2.2. Let $u\colon \Omega \to \mathbb{R}$ be a measurable function, the *decreasing rearrangement* of u is defined as

$$u^*\colon \mathbb{R} \to \mathbb{R}^+ \qquad u^*(s) = \inf\{\, t > 0 \mid \mu(t) \leq s \,\}.$$

Equivalently, one can say that u^* is the distribution function of μ.

From Definitions 1.2.1 and 1.2.2 one can prove that

$$u^*(\mu(t)) \leq t, \quad \forall t \geq 0,$$

$$\mu(u^*(s)) \leq s \quad \forall s \geq 0.$$

Remark 1.2.1. We observe that the function $u^*(\cdot)$ is the generalized inverse of the function $\mu(\cdot)$.

Definition 1.2.3. Let $u\colon \Omega \to \mathbb{R}$ be a measurable function. The *Schwarz rearrangement* of u is the function $u^\sharp$ whose superlevel sets are balls with the same measure as the superlevel sets of u. The functions $u^\sharp$ and u^* are linked by the relation

$$u^\sharp(x) = u^*(\omega_n |x|^n).$$

It can be easily checked that the functions u, u^* e $u^\sharp$ are equi-distributed, i.e. they have the same distribution function, and it holds

$$\|u\|_{L^p(\Omega)} = \|u^*\|_{L^p(0,|\Omega|)} = \|u^\sharp\|_{L^p(\Omega^\sharp)}, \quad \text{for all } p \geq 1.$$

The following Lemma, whose proof is contained for instance in [43], gives a characterization of the absolutely continuity of μ.

Lemma 1.2.1. *Let $u \in W^{1,p}(\mathbb{R}^n)$, with $p \in (1, +\infty)$. The distribution function μ of u is absolutely continuous if and only if*

$$\Big|\{0 < u < \mathit{esssup}\, u^\sharp\} \cap \{\nabla u^\sharp = 0\}\Big| = 0. \tag{1.5}$$

We also recall the Hardy-Littlewood inequality, an important propriety of the decreasing rearrangement, that holds for every h, g measurable functions

$$\int_\Omega |h(x)g(x)|\, dx \leq \int_{\Omega^\sharp} h^\sharp(x) g^\sharp(x)\, dx = \int_0^{|\Omega|} h^*(s) g^*(s)\, ds.$$

So, by choosing $h(\cdot) = \chi_{\{|u|>t\}}$, one has

$$\int_{|u|>t} |g(x)|\, dx \leq \int_0^{\mu(t)} g^*(s)\, ds.$$

The equality case in the Hardy-Littlewood inequality can be characterized as follows.

Lemma 1.2.2. *Let $h, g \in L^2(\Omega)$ be two positive functions. If*

$$\int_\Omega hg\; dx = \int_{\Omega^\sharp} h^\sharp g^\sharp\; dx,$$

then, for every $\tau \geq 0$ there exists $t \geq 0$ such that we have, up to zero measure set,

$$\{g > \tau\} = \{h > t\}.$$

If we ask that the function u is a *Sobolev function*, i.e. $u \in W^{1,p}(\mathbb{R}^n)$, then $u^\sharp$ is a Sobolev function too and the gradient does not increase under symmetrization, as it is stated in the Pólya-Szegő inequality [132].

Theorem 1.2.3 (Pólya-Szegő)**.** *Let $u \in W^{1,p}(\mathbb{R}^n)$, then $u^\sharp \in W^{1,p}(\mathbb{R}^n)$ and*

$$\|\nabla u^\sharp\|_{L^p(\mathbb{R}^n)} \leq \|\nabla u\|_{L^p(\mathbb{R}^n)}. \tag{1.6}$$

If $u = u^\sharp$ of course the equality holds in the Pólya-Szegő inequality. In the celebrated paper [43], the authors show that the converse is true under the additional assumption (1.8).

Theorem 1.2.4 (Brothers-Ziemer)**.** *Let $u \in W^{1,p}(\mathbb{R}^n)$, let $\mu(t)$ be its distribution function and let*

$$u_M := \begin{cases} \|u\|_\infty & \text{if } u \in L^\infty(\mathbb{R}^n) \\ +\infty & \text{otherwise.} \end{cases}$$

If

$$\int_{\mathbb{R}^n} |\nabla u|^p = \int_{\mathbb{R}^n} |\nabla u^\sharp|^p, \tag{1.7}$$

and

$$\Big|\Big\{ \, |\nabla u^\sharp| = 0 \, \Big\} \cap \Big\{ \, 0 < u^\sharp < u_M \, \Big\}\Big| = 0, \tag{1.8}$$

then there exists a translate of $u^\sharp$ which is almost everywhere equal to u.

Remark 1.2.2. We observe that in [56], it is proved that the condition

$$|\{ \, |\nabla u| = 0 \, \} \cap \{ \, 0 < u < u_M \, \}| = 0 \tag{1.9}$$

implies (1.8). So, if we have (1.7) and (1.9), there exists a translated of $u^\sharp$ which is almost everywhere equal to u.

Remark 1.2.3. We observe that the Pólya -Szegő inequality (1.6) and the relative rigidity result 1.2.4 hold also if we assume $u \in W_0^{1,p}(\Omega)$. Indeed, it is easily proved that for every $u \in W_0^{1,p}(\Omega)$ one has $u^\sharp \in W_0^{1,p}(\Omega^\sharp)$.

1.2.1 Lorentz Norm

Thanks to the distribution function μ, one can define the *Lorentz space* $L^{p,q}(\Omega)$.

Definition 1.2.4. Let $0 < p < +\infty$ and $0 < q \leq +\infty$. The Lorentz space $L^{p,q}(\Omega)$ is the space of those functions such that the quantity:

$$\|g\|_{L^{p,q}} = \begin{cases} p^{\frac{1}{q}} \left(\displaystyle\int_0^\infty t^q \mu(t)^{\frac{q}{p}} \, \frac{dt}{t} \right)^{\frac{1}{q}} & 0 < q < \infty \\ \sup\limits_{t>0} \left(t^p \mu(t) \right) & q = \infty \end{cases}$$

is finite.

Let us observe that for $p = q$ the Lorentz space coincides with the L^p space, as a consequence of the well known *Cavalieri's Principle*

$$\int_\Omega |g|^p = p \int_0^{+\infty} t^{p-1} \mu(t) \, dt.$$

See [155] for more details on Lorentz space.

1.2.2 Gronwall's lemma

The following Lemma will be crucial in Chapter 2.

Lemma 1.2.5 (Gronwall). *Let $\xi(t) : [\tau_0, +\infty[\to \mathbb{R}$ be a continuous and differentiable function satisfying, for some non negative constant C, the following differential inequality*

$$\tau\xi'(\tau) \leq (q-1)\xi(\tau) + C \quad \forall \tau \geq \tau_0 > 0.$$

Then we have

(i) $\xi(\tau) \leq \left(\xi(\tau_0) + \frac{C}{q-1}\right)\left(\frac{\tau}{\tau_0}\right)^{q-1} - \frac{C}{q-1} \quad \forall \tau \geq \tau_0;$

(ii) $\xi'(\tau) \leq \left(\frac{(q-1)\xi(\tau_0) + C}{\tau_0}\right)\left(\frac{\tau}{\tau_0}\right)^{q-2} \quad \forall \tau \geq \tau_0.$

Proof. Dividing both sides of the differential inequality by τ^q, we obtain

$$\left(\frac{\xi'(\tau)}{\tau^{q-1}} - (q-1)\frac{\xi(\tau)}{\tau^q}\right) = \left(\frac{\xi(\tau)}{\tau^{q-1}}\right)' \leq \frac{C}{\tau^q}.$$

Now, we integrate from τ_0 to τ and we obtain

$$\int_{\tau_0}^{\tau} \left(\frac{\xi(t)}{t^{q-1}}\right)' dt \leq \int_{\tau_0}^{\tau} \frac{C}{t^q}\, dt$$
$$\Longrightarrow \xi(\tau) \leq \left(\xi(\tau_0) + \frac{C}{q-1}\right)\left(\frac{\tau}{\tau_0}\right)^{q-1} - \frac{C}{q-1},$$

which gives *(i)*.

In order to obtain *(ii)*, we just take into account *(i)* in the differential inequality. □

1.2.3 Viscosity solutions

We recall the definition of viscosity solutions to a boundary value problem, see [58] for more details.

Definition 1.2.5. Let us consider the following boundary value problem

$$\begin{cases} F(x,u,\nabla u, D^2 u) = 0 & \text{in } \Omega, \\ B(x,u,\nabla u) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.10}$$

where $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}$ and $B : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are two continuous functions.

Viscosity supersolution A lower semi-continuous function u is a viscosity supersolution to (1.10) if, whenever we fix $x_0 \in \overline{\Omega}$, for every $\phi \in C^2(\overline{\Omega})$ such that $u(x_0) = \phi(x_0)$ and x_0 is a strict minimum in Ω for $u - \phi$, then

- if $x_0 \in \Omega$, the following holds

$$F\Big(x_0, \phi(x_0), \nabla\phi(x_0), D^2\phi(x_0)\Big) \geq 0$$

- if $x_0 \in \partial\Omega$, the following holds

$$\max\Big\{\ F\Big(x_0, \phi(x_0), \nabla\phi(x_0), D^2\phi(x_0)\Big), B\left(x_0, \phi(x_0), \nabla\phi(x_0)\right)\Big\} \geq 0$$

Viscosity subsolution An upper semi-continuous function u is a viscosity subsolution to (1.10) if, whenever we fix $x_0 \in \overline{\Omega}$, for every $\phi \in C^2(\overline{\Omega})$ such that $u(x_0) = \phi(x_0)$ and x_0 is a strict maximum in Ω for $u - \phi$, then

- if $x_0 \in \Omega$, the following holds

$$F\Big(x_0, \phi(x_0), \nabla\phi(x_0), D^2\phi(x_0)\Big) \leq 0$$

- if $x_0 \in \partial\Omega$, the following holds

$$\min\Big\{\ F\Big(x_0, \phi(x_0), \nabla\phi(x_0), D^2\phi(x_0)\Big), B\left(x_0, \phi(x_0), \nabla\phi(x_0)\right)\Big\} \leq 0$$

Viscosity solution A continuous function u is a viscosity solution to (1.10) if it is both a super and subsolution.

Remark 1.2.4. The condition $u - \phi$ has a strict maximum or minimum can be relaxed: it is sufficient to ask that $u - \phi$ has a local maximum or minimum in a ball $B_R(x_0)$ for some positive R.

1.3 Properties of Convex sets

Let Ω be an open set of $\mathbb{R}^n$. We say that Ω is a convex set if $\forall t \in [0,1]$ and for any $x, y \in \Omega$,

$$tx + (1-t)y \in \Omega.$$

Since $\partial\Omega$ is locally a graph of a concave function, then Ω an open set with Lipschitz boundary, hence $P(\Omega) = \mathcal{H}^{n-1}(\partial\Omega)$.

In the class

$$\mathcal{K}^n := \{\Omega \mid \Omega \text{ is a bounded and convex set of } \mathbb{R}^n \text{ with nonempty interior }\},$$

we can define the following operations

Definition 1.3.1. Let $\Omega, K \subset \mathbb{R}^n$ two convex bounded sets. We define the *Minkowski sum* $(+)$ and *difference* $(\sim)$ as

$$\Omega + K := \{x + y \,:\, x \in \Omega,\, y \in K\},$$
$$\Omega \sim K := \{x \in \mathbb{R}^2 \,:\, x + K \subseteq \Omega\}.$$

We can introduce a distance on the set $\mathcal{K}^n$, known as the *Hausdorff distance*

Definition 1.3.2. Let $\Omega, K \subset \mathbb{R}^n$ two non-empty compact sets, we define the Hausdorff distance between Ω and K as

$$d_{\mathcal{H}}(\Omega, K) = \inf\left\{\varepsilon > 0 \,:\, \Omega \subset K + B_\varepsilon,\ K \subset \Omega + B_\varepsilon\right\}.$$

Note that, in the case that Ω and K are convex sets, we have that $d_{\mathcal{H}}(\Omega, K) = d_{\mathcal{H}}(\partial\Omega, \partial K)$.

Let $\{\Omega_k\}_{k\in\mathbb{N}}$ be a sequence of non-empty, open, bounded convex subsets of $\mathbb{R}^n$, we say that Ω_k *converges to* Ω *in the Hausdorff sense* and we denote

$$\Omega_k \xrightarrow{\mathcal{H}} \Omega$$

if and only if $d_{\mathcal{H}}(\Omega_k, \Omega) \to 0$ as $k \to \infty$.

We have the following compactness result (see as a reference [143], Theorem 1.8.7).

Theorem 1.3.1. *Let $\{\Omega_k\}_{k\in\mathbb{N}}$ a bounded sequence of compact convex sets with nonempty interior. Then, there exists a subsequence $\{\Omega_{k_n}\}_{k_n\in\mathbb{N}}$ converging to a convex set $\Omega^\star$ in the Hausdorff sense.*

Let us now recall the following definitions:

Definition 1.3.3. Let $\Omega \in \mathcal{K}^n$. The *distance function from the boundary of* Ω is the function $d(\cdot, \partial\Omega) : \Omega \to [0, +\infty[$ defined as

$$d(x, \partial\Omega) = \inf_{y\in\partial\Omega} |x - y|. \tag{1.11}$$

The *inradius* $r(\Omega)$ of Ω is defined as

$$r(\Omega) = \sup_{x\in\Omega} d(x, \partial\Omega).$$

Finally, the *circumradius* $R(\Omega)$ is defined as

$$R(\Omega) = \min_{x\in\Omega} \max_{y\in\partial\Omega} |x - y|.$$

We need now to introduce the support function of a convex set.

Definition 1.3.4. Let Ω be a bounded, open and convex set of $\mathbb{R}^n$. The support function of Ω is defined as

$$h_\Omega(y) = \sup_{x\in\Omega} x \cdot y, \qquad y \in \mathbb{R}^n.$$

Definition 1.3.5. Let Ω a bounded, open and convex set of $\mathbb{R}^n$, the width of Ω in the direction $y \in \mathbb{R}$ is defined as

$$\omega_\Omega(y) = h_\Omega(y) + h_\Omega(-y)$$

and the minimal width of Ω as

$$w(\Omega) = \min\{\omega_\Omega(y) \mid y \in \mathbb{S}^{n-1}\}. \tag{1.12}$$

1.3.1 Quermassintegrals

For the content of this section, we will refer to [143]. Let $K \subset \mathbb{R}^n$ be a non-empty, bounded, convex set, let B be the unitary ball centered at the origin and let $\rho > 0$. We can write the *Steiner formula* for the Minkowski sum $K + \rho B$ as

$$|K + \rho B| = \sum_{i=0}^{n} \binom{n}{i} W_i(K)\rho^i. \tag{1.13}$$

The coefficients $W_i(K)$ are known in the literature as *quermassintegrals* of K. In particular, $W_0(K) = |K|$, $nW_1(K) = P(K)$ and $W_n(K) = \omega_n$. If K has C^2 boundary, the quermassintegrals can be written in terms of principal curvatures of K. More precisely, denoting with H_j the j-th normalized elementary symmetric function of the principal curvature $\kappa_1, \dots, \kappa_{n-1}$ of ∂K, i.e.

$$H_0 = 1, \qquad H_j = \binom{n-1}{j}^{-1} \sum_{1\le i_1<\ldots<i_j\le n-1} \kappa_{i_1}\ldots\kappa_{i_j}, \qquad j = 1, \dots, n-1,$$

then the quermassintegrals can be written as

$$W_i(K) = \frac{1}{n}\int_{\partial K} H_{i-1}\, d\mathcal{H}^{n-1}, \qquad i = 1, \dots, n. \tag{1.14}$$

Moreover, the Steiner formula holds true also for quermassintegrals, that is

$$W_j(K + \rho B) = \sum_{i=0}^{n-j} \binom{n-j}{i} W_{j+i}(K)\rho^i, \qquad j = 0, \dots, n-1.$$

For $j = 1$ we have

$$\begin{aligned} P(K + \rho B_1) &= n\sum_{i=0}^{n-1} \binom{n-1}{i} W_{i+1}(K)\rho^i \\ &= P(K) + n(n-1)W_2(K)\rho + \ldots + nW_n(K)\rho^{n-1}, \end{aligned}$$

from which follows

$$\lim_{\rho\to 0} \frac{P(K + \rho B) - P(K)}{\rho} = n(n-1)W_2(K), \tag{1.15}$$

and if ∂K is of class C^2, formula (1.14) gives

$$\lim_{\rho\to 0}\frac{P(K+\rho B)-P(K)}{\rho}=(n-1)\int_{\partial K} H_1\,d\mathcal{H}^{n-1}.$$

Furthermore, the *Aleksandrov-Fenchel* inequalities hold true

$$\left(\frac{W_j(K)}{\omega_n}\right)^{\frac{1}{n-j}}\geq\left(\frac{W_i(K)}{\omega_n}\right)^{\frac{1}{n-i}},\qquad 0\leq i<j\leq n-1, \tag{1.16}$$

where equality holds if and only if K is a ball. When $i=0$ and $j=1$, formula (1.16) reduces to the classical isoperimetric inequality, i.e.

$$P(K)\geq n\omega_n^{\frac{1}{n}}|K|^{\frac{n-1}{n}}.$$

In the following, we will use (1.16) for $i=1$ and $j=2$, that is

$$W_2(K)\geq n^{-\frac{n-2}{n-1}}\omega_n^{\frac{1}{n-1}}P(K)^{\frac{n-2}{n-1}}. \tag{1.17}$$

1.3.2 Inner parallel sets

When dealing with convex analysis, the superlevel sets of the distance function from $\partial\Omega$ play a key role.

Definition 1.3.6. Let Ω be an open, bounded and convex set, the *inner parallel set* of Ω at distance $t\in[0,r(\Omega)]$ is

$$\Omega_t=\{x\in\Omega\,:\,d(x,\partial\Omega)\geq t\}.$$

Remark 1.3.1. We remark that, by definition, we have that

$$\Omega_t=\Omega\sim tB_1.$$

Moreover, we observe that for any $y\in\mathbb{S}^1$ and for every $\Omega,K\in\mathcal{K}^n$, one has, see e.g. [143],

$$h_{\Omega\sim K}(y)\leq h_\Omega(y)-h_K(y),$$

so, in the case $K=tB_1$, this reads

$$h_{\Omega_t}(y)\leq h_\Omega(y)-t. \tag{1.18}$$

Moreover, as it is observed in [99, Proposition 3.2], one has

$$R(\Omega+K)\leq R(\Omega)+R(K) \tag{1.19}$$

and, if $K=tB_1$, equality holds in (1.19).

We are now in a position to prove the following Lemma. We will state (and hance prove) it only in the case $n=2$, as we will apply it only in the planar case.

Lemma 1.3.2. *Let $\Omega \in \mathcal{K}^2$. We have for every $t \in [0, r(\Omega)]$:*

$$r(\Omega_t) = r(\Omega) - t, \tag{1.20}$$
$$diam(\Omega_t) \le diam(\Omega) - 2t, \tag{1.21}$$
$$w(\Omega_t) \le w(\Omega) - 2t, \tag{1.22}$$
$$R(\Omega_t) \le R(\Omega) - t, \tag{1.23}$$
$$P(\Omega_t) \le P(\Omega) - 2\pi t. \tag{1.24}$$

Proof. The proof of (1.20) can be found in [110, Lemma 1.4].

Let us now prove (1.21). Let $x_t, y_t \in \Omega_t$ be two diametrical points of Ω_t (i.e., such that $|x_t - y_t| = \mathrm{diam}(\Omega_t)$). We denote by $x, y \in \Omega$ the points corresponding to the intersection of the line containing x_t and y_t with the boundary of Ω. We have

$$\mathrm{diam}(\Omega) \ge |x - y| = |x - x_t| + |x_t - y_t| + |y_t - y| = |x - x_t| + \mathrm{diam}(\Omega_t) + |y_t - y| \ge \mathrm{diam}(\Omega_t) + 2t,$$

where the last inequality is a consequence of the fact that $x_t, y_t \in \Omega_t = \{x \in \Omega \mid \mathrm{d}(x, \partial\Omega) \ge t\}$.
The proof of (1.22) follows directly from the definition of width and (1.18).

We prove now (1.23). As observed in Remark 1.3.1, and, in particular, by formula (1.19), for every $\Omega \in \mathcal{K}^2$, we have that $R(\Omega + tB_1) = R(\Omega) + t$. Thus, we have

$$R(\Omega_t) = R(\Omega_t + tB_1) - t \le R(\Omega) - t.$$

The last inequality follows from the inclusion $\Omega_t + tB_1 \subset \Omega$.

Formula (1.24) is the classical Steiner formula and we refer to [143] and [151]. □

Let us denote by μ the distribution function of the distance from the boundary and by $P(t) = P(\Omega_t)$. By coarea formula (1.2), recalling that $|\nabla d| = 1$ almost everywhere, we have

$$\mu(t) = \int_{\{d>t\}} dx = \int_{\{d>t\}} \frac{|\nabla d|}{|\nabla d|}\, dx = \int_t^{r(\Omega)} \frac{1}{|\nabla d|} \int_{\{d=s\}} d\mathcal{H}^{n-1}\, ds = \int_t^{r(\Omega)} P(s)\, ds;$$

hence, the function $\mu(t)$ is absolutely continuous, decreasing and its derivative is $\mu'(t) = -P(t)$ almost everywhere. Moreover, it is possible to prove that the perimeter $P(t)$ is non-increasing and absolutely continuous. Indeed, by the Brunn-Minkowski Theorem ([143, Theorem 7.4.5]) and the concavity of the distance function, the map

$$t \mapsto P(\Omega_t)^{\frac{1}{n-1}},$$

is concave in $[0, r(\Omega)]$, hence absolutely continuous in $(0, r(\Omega))$. Moreover, there exists its right derivative at 0 and it is negative, since $P(\Omega_t)^{\frac{1}{n-1}}$ is strictly monotone decreasing.

Finally, let us consider the case $n = 2$. For Ω non-empty bounded, open and convex set of $\mathbb{R}^2$, the Steiner formulas for the inner parallel sets hold (see [151]):

$$P(\Omega_t) \le P(\Omega) - 2\pi t \qquad \forall t \in [0, r(\Omega)], \tag{1.25}$$

$$|\Omega_t| \ge |\Omega| - P(\Omega)t + \pi t^2 \qquad \forall t \in [0, r(\Omega)], \tag{1.26}$$

equality holding in both (1.25) and (1.26) for the stadii (see [76]).

Remark 1.3.2. Steiner formula (1.13) for *outer* parallel sets hold in any dimension. When considering the *inner* parallel sets, the Stainer formulas (1.25) and (1.26) hold only in dimension $n = 2$.

As a consequence of the Alexandrov-Fenchel inequality (1.16) and the isoperimetric inequality for the quermassintegrals (see [143]), we have

$$-P'(\Omega_t) \geq n(n-1)\omega_n^{\frac{1}{n-1}} \left(\frac{P(\Omega_t)}{n}\right)^{\frac{n-2}{n-1}}, \tag{1.27}$$

that, for $n = 2$, reads

$$-P'(\Omega_t) \geq 2\pi, \tag{1.28}$$

with equality if Ω is a ball or a stadium.

The following result is contained in [36]. We report its proof for the reader's convenience.

Lemma 1.3.3. *Let Ω be a bounded, convex, open set in $\mathbb{R}^n$. Then for almost every $t \in (0, r(\Omega))$*

$$-\frac{d}{dt}P(\Omega_t) \geq n(n-1)W_2(\Omega_t), \tag{1.29}$$

and equality holds if Ω is a ball.

Proof. For every $s \in (0, t)$ it holds

$$\Omega_t + sB_1 \subset \Omega_{t-s},$$

and if Ω is a ball, the two sets coincide. The monotonicity of the perimeter with respect to the inclusion of convex sets and formula (1.15) give, for almost every $t \in (0, r_\Omega)$,

$$\begin{aligned} -\frac{d}{dt}P(\Omega_t) &= \lim_{s\to 0^+} \frac{P(\Omega_{t-s}) - P(\Omega_t)}{s} \\ &\geq \lim_{s\to 0^+} \frac{P(\Omega_t + sB_1) - P(\Omega_t)}{s} = n(n-1)W_2(\Omega_t). \end{aligned}$$

□

Combining the chain rule, the previous lemma and the fact that $|\nabla d(x, \partial\Omega)| = 1$ almost everywhere, we obtain

Lemma 1.3.4. *Let $f\colon [0, +\infty) \to [0, +\infty)$ be a strictly increasing C^1 function with $f(0) = 0$. Set $u(x) = f(d(x, \partial\Omega))$ and*

$$E_t = \{\, x \in \Omega \,:\, u(x) > t \,\} = \Omega_{f^{-1}(t)},$$

then

$$-\frac{d}{dt}P(E_t) \geq (n-1)\frac{W_2(E_t)}{|\nabla u|_{u=t}}. \tag{1.30}$$

1.4 Extremal shapes and their properties

In this Section, we describe special shapes that appear in Chapter 4. We start with planar shapes, then we move to the $n-$dimensional setting.

Firstly, we recall the definition of the form body of convex set Ω, following [143]. A point $x \in \partial\Omega$ is called *regular* if the supporting hyperplane at x is uniquely defined. The set of all regular points of $\partial\Omega$ is denoted by $\mathrm{reg}(\Omega)$. We also let $U(\Omega)$ denote the set of all outward pointing unit normals to $\partial\Omega$ at points of $\mathrm{reg}(\Omega)$.

Definition 1.4.1. The form body $\hat{\Omega}$ of a set $\Omega \in \mathcal{K}^2$ is defined as

$$\hat{\Omega} = \bigcap_{u \in U(\Omega)} \{x \in \mathbb{R}^2 : (x, u) \leq 1\}.$$

In particular, a polygon whose incircle touches all its sides is homothetic to its form body.

Definition 1.4.2. A *stadium* $\mathcal{R}$ is defined as the convex hull of the union of two balls in $\mathbb{R}^2$ with the same radius (see Figure 1.1).

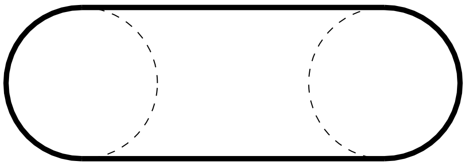

Figure 1.1: Stadium

Definition 1.4.3. The *symmetrical spherical slice* $\mathcal{S}$ of diameter d and width $w < d$ is the convex set obtained by the intersection of a ball of radius $d/2$ and a strip of width w centered at the center of the ball (see Figure 1.2).

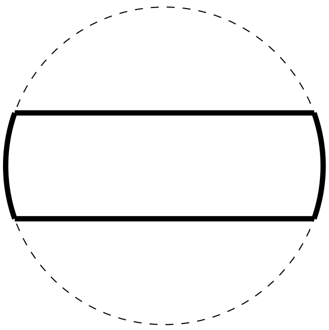

Figure 1.2: Symmetrical spherical slice.

Definition 1.4.4. A *two-cup body* $\mathcal{C}$ is the convex hull of a ball in $\mathbb{R}^2$ with two points that are symmetric with respect to the center of the ball (see Figure 1.3). In particular, a two-cup body is a "homothetic-to-its-form-body" set.

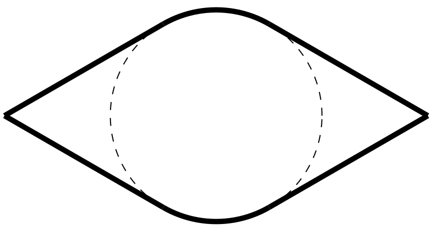

Figure 1.3: Two-cup body.

Definition 1.4.5. A *subequilateral triangle* T_I is an isosceles triangle with the two equal angles greater than $\pi/3$.

The following class of sets (introduced in [159], see also [141]) represents a way to pass in a continuous manner, with respect to the Hausdorff distance, from the equilateral triangle to the Reuleaux triangle, the set enclosed in a curve of constant width constructed by drawing arcs from each polygon vertex of an equilateral triangle between the other two vertices.

Definition 1.4.6. A *Yamanouti set* Y is the convex hull of the set obtained by an equilateral triangle by constructing on any side an arc of circle centered in the opposite vertex and with radius less or equal to the side itself (see Figure 1.4).

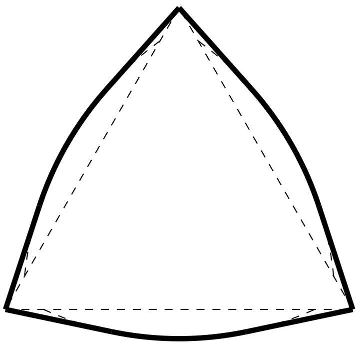

Figure 1.4: Yamanouti set.

In [67] the authors define the smoothed regular nonagon as follows.

Definition 1.4.7. Let $r > 0$ and $2r < d < 2\sqrt{3}r$. The *smoothed regular nonagon* of inradius r and diameter d, that we denote by $\mathcal{N}$, is the convex set enclosed in an equilateral triangle T_E with barycenter in the origin and such that $r(T_E) = r$, obtained following the construction below. Let η_i the normal angles to the sides of T_E and let

$$\tau := (3 + \sqrt{d^2 - 3r^2})/2, \quad \text{and} \quad h := \sqrt{d^2 - \tau^2}.$$

We define now the points A_i, B_i, M_i, for $i = 1, 2, 3$:

$$A_i := r\begin{pmatrix}\cos\eta_i + h\sin\eta_i \\ \sin\eta_i - h\cos\eta_i\end{pmatrix}, \; B_i := r\begin{pmatrix}\cos\eta_i - h\sin\eta_i \\ \sin\eta_i + h\cos\eta_i\end{pmatrix}, \; M_i := r(1-\tau)\begin{pmatrix}\cos\eta_i \\ \sin\eta_i\end{pmatrix}.$$

We obtain $\mathcal{N}$ as follows (see Figure 1.5):

- the points A_i, B_i and M_i, for $i = 1, 2, 3$, belong to $\partial\mathcal{N}$;
- $\widehat{B_1M_3}$ and $\widehat{M_1A_3}$ are diametrically opposed arcs of the same circle of diameter d, the same for the pairs $\widehat{B_2M_1}$ and $\widehat{M_2A_1}$, $\widehat{M_2B_3}$ and $\widehat{M_3A_2}$;
- the boundary contains the segments $\overline{A_iB_i}$, for $i = 1, 2, 3$, and the contact point I_i with the incircle is the middle of the corresponding segment.

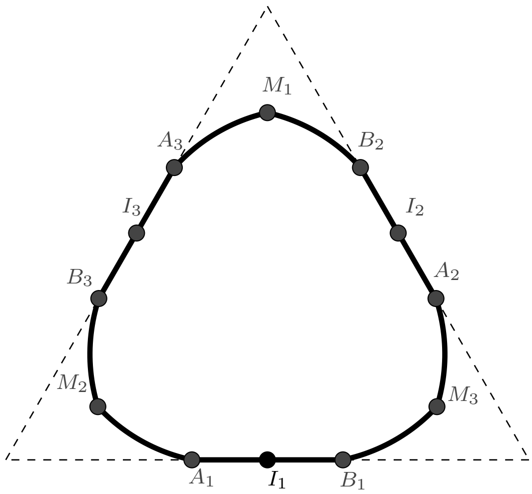

Figure 1.5: Smoothed regular nonagon

Definition 1.4.8. A nonagon of constant width is the convex set obtained following the construction below (see also Figure 1.6). Let Γ and γ be the circumcircle and the incircle of a constant width set: they are concentric and $d = w = R + r$. The nonagon of constant width can be constructed in the following way: an equilateral triangle PQR is inscribed in the circle Γ, and now we take the circular arcs of radius $R+r$ drawn about the three vertex points. These arcs touch γ at the opposite points $\bar{P}, \bar{Q}, \bar{R}$ of P, Q, R, respectively. Furthermore, we construct three circles of radius $(R + r)/2$ that have the sides of the triangle as chords and whose centers lie inside the triangle. The required constant width set has 3-fold symmetry and it is formed by nine arcs of the six constructed circles

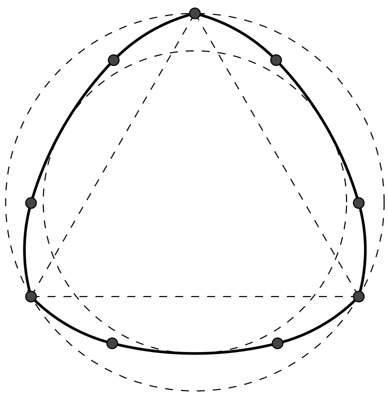

Figure 1.6: Nonagon of constant width

We recall now some of the Blaschke–Santaló sharp inequalities that we will need in the sequel between three of the following geometric quantities: perimeter, area, inradius, circumradius, diameter, and width.

Firstly, let us consider the diagram $(|\cdot|, \mathrm{diam}, r)$. We have the following two theorems.

Theorem 1.4.1 ([96] & [67] Theorem 1)**.** *Let $\Omega \in \mathcal{K}^2$. Then, it holds*

$$|\Omega| \geq r(\Omega)\sqrt{diam^2(\Omega) - 4r^2(\Omega)} + r^2(\Omega)\left(\pi - 2\arccos\left(\frac{2r(\Omega)}{diam(\Omega)}\right)\right), \tag{1.31}$$

where equality holds if and only if Ω *is a two-cup body.*

Theorem 1.4.2 ([67], Theorem 2)**.** *Let* $\Omega \in \mathcal{K}^2$*. Then, it holds*

$$|\Omega| \leq \psi\left(diam\left(\Omega\right), r(\Omega)\right), \tag{1.32}$$

where

$$\psi(d,r) := \begin{cases} \dfrac{3\sqrt{3}r}{2}(\sqrt{d^2-3r^2}-r)+\dfrac{3d^2}{2}\left(\dfrac{\pi}{3}-\arccos\left(\dfrac{\sqrt{3}r}{d}\right)\right), & \text{if } d \leq rD^* \\ r\sqrt{d^2-4r^2}+\dfrac{d^2}{2}\arcsin\left(\dfrac{2r}{d}\right), & \text{if } d \geq rD^* \end{cases} \tag{1.33}$$

and D^* *is the unique number in* $[2, 2\sqrt{3}]$ *for which the two expressions of the function* $\psi(d,1)$ *are equal.*

Moreover, if $diam(\Omega) \leq r(\Omega)D^*$*, we have equality in* (1.32) *if and only if* Ω *is a regular smoothed nonagon, while, if* $diam(\Omega) > r(\Omega)D^*$*, we have equality if and only if* Ω *is a symmetrical spherical slice.*

As far as the diagram $(|\cdot|, w, R)$ is concerned, we recall the following.

Theorem 1.4.3 ([96], Theorem 3)**.** *Let* $\Omega \in \mathcal{K}^2$*. Then, it holds*

$$|\Omega| \leq \chi(w(\Omega), R(\Omega)), \tag{1.34}$$

where

$$\chi(w(\Omega), R(\Omega)) := \frac{w(\Omega)}{2}\sqrt{4R(\Omega)^2 - w(\Omega)^2} + 2R(\Omega)^2 \arcsin\frac{w(\Omega)}{2R(\Omega)}, \tag{1.35}$$

and equality in (1.34) *holds if and only if* Ω *is a symmetrical spherical slice.*

Theorem 1.4.4 ([96], Theorem 6)**.** *Let* $\Omega \in \mathcal{K}^2$*. Then, if* $w(\Omega) \leq \frac{3}{2}R(\Omega)$*, it holds*

$$16|\Omega|^6 \geq R^2(\Omega)w^2(\Omega)\left(16|\Omega|^4 - R^2(\Omega)w^6(\Omega)\right) \tag{1.36}$$

and equality holds if and only if Ω *is a subequilateral triangle.*

We recall the following inequality from the diagram (R, r, w).

Theorem 1.4.5 ([98], Theorem 2)**.** *Let* $\Omega \in \mathcal{K}^2$*. Then, it holds*

$$\left(4r(\Omega) - w(\Omega)\right)\left(w(\Omega) - 2r(\Omega)\right) \leq \frac{2r^3(\Omega)}{R(\Omega)} \tag{1.37}$$

and equality holds if and only if Ω *is an isosceles triangle.*

Theorem 1.4.6 ([141], Section 10)**.** *Let* $\Omega \in \mathcal{K}^2$*. Then, it holds*

$$w(\Omega) \leq R(\Omega) + r(\Omega), \tag{1.38}$$

where equality is achieved by any set of constant width.

The following theorem deals with the $(|\cdot|, r, R)$ diagram.

Theorem 1.4.7 ([96], Theorems 1 and 2)**.** *Let $\Omega \in \mathcal{K}^2$. Then, it holds*

$$|\Omega| \geq 2r(\Omega)\left(\sqrt{R^2(\Omega) - r^2(\Omega)} + r \arcsin \frac{r(\Omega)}{R(\Omega)}\right), \tag{1.39}$$

and equality in (1.39) *holds if and only if Ω is a two-cup body. Moreover, we have*

$$|\Omega| \leq \varphi(R(\Omega), r(\Omega)), \tag{1.40}$$

where

$$\varphi(R(\Omega), r(\Omega)) := 2\left(r\sqrt{R^2(\Omega) - r^2(\Omega)} + R^2(\Omega) \arcsin \frac{r(\Omega)}{R(\Omega)}\right), \tag{1.41}$$

and equality in (1.41) *holds if and only if Ω is a symmetrical spherical slice.*

Now we quote two inequalities concerning the $(|\cdot|, w, r)$ and (P, w, r) diagrams.

Theorem 1.4.8 ([96], Theorem 5)**.** *Let $\Omega \in \mathcal{K}^2$. Then, it holds*

$$(w(\Omega) - 2r(\Omega))^2(4r(\Omega) - w(\Omega))|\Omega|^2 \leq r^4(\Omega)w^3(\Omega), \tag{1.42}$$

$$(w(\Omega) - 2r(\Omega))^2(4r(\Omega) - w(\Omega))P^2(\Omega) \leq 4r^2(\Omega)w^3(\Omega) \tag{1.43}$$

In both inequalities, equality holds if and only if Ω is a subequilateral triangle.

Finally, we recall this result from the (d, w, r) diagram.

Theorem 1.4.9 ([94], Theorem 1-2)**.** *Let $\Omega \in \mathcal{K}^2$. Then, it holds*

$$diam^2(\Omega)(w(\Omega) - 2r(\Omega))^2(4r(\Omega) - w(\Omega)) \leq 4r^4(\Omega)w(\Omega), \tag{1.44}$$

where equality holds if and only if Ω is a subequilateral triangle T_I, and

$$w(\Omega) - r(\Omega) \leq \frac{\sqrt{3}}{3} diam(\Omega), \tag{1.45}$$

and equality holds if Ω is a Yamanouti set.

We turn now to the $n-$dimensional case.

Definition 1.4.9. Let Ω_l be a sequence of non-empty, bounded, open and convex sets of $\mathbb{R}^n$. We say that Ω_l is a sequence of thinning domains if

$$\frac{w(\Omega_l)}{\mathrm{diam}(\Omega_l)} \xrightarrow{l \to 0} 0. \tag{1.46}$$

In particular, if $l > 0$ and C is a bounded, open and convex set of $\mathbb{R}^{n-1}$ with unitary $(n-1)$-dimensional measure, then, if $l \to 0$, the sequence

$$\Omega_l = l^{-\frac{1}{n-1}} C \times \left[-\frac{l}{2}, \frac{l}{2}\right] \tag{1.47}$$

is called a sequence of thinning cylinders. Moreover, in the case $n = 2$, the sequence (1.47) is called sequence of thinning rectangles.

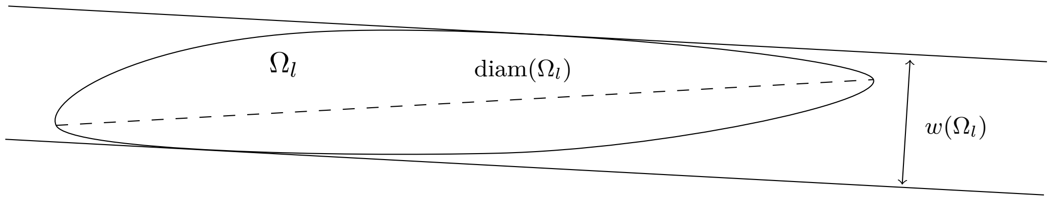

Figure 1.7: Minimal width and diameter of a convex set.

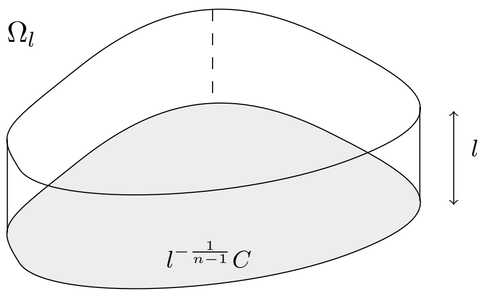

Figure 1.8: Thinning cylinders.

We recall the following estimate, which is proved in [30] in the planar case and is generalized in [37] to all dimensions.

Proposition 1.4.10. *Let Ω be a non-empty bounded, open and convex set of $\mathbb{R}^n$. Then,*

$$\frac{1}{n} \leq \frac{|\Omega|}{P(\Omega)r(\Omega)} < 1. \tag{1.48}$$

The upper bound is sharp on a sequence of thinning cylinders, while the lower bound is sharp, for example, on balls. Moreover, for $n = 2$, any circumscribed polygon, that is a polygon whose incircle touches all the sides, verifies the lower bound with the equality sign.

In the planar case, the following inequalities hold true (see as a reference [145, 144, 142]).

Proposition 1.4.11. *Let Ω be a bounded, open and convex set of $\mathbb{R}^2$. Then,*

$$2 \leq \frac{w(\Omega)}{r(\Omega)} \leq 3. \tag{1.49}$$

The upper bound is achieved by equilateral triangles and the lower bound is achieved by disks. Moreover,

$$(w(\Omega) - 2r(\Omega))\, P(\Omega) \leq \frac{2}{\sqrt{3}} w^2(\Omega), \tag{1.50}$$

with equality holding for equilateral triangles, and

$$|\Omega| \leq r(\Omega)\, (P(\Omega) - \pi r(\Omega))\, , \tag{1.51}$$

with equality holding for the stadii (convex hull of two identical disjoint balls). Eventually,

$$2\,diam(\Omega) < P(\Omega) \leq \pi\, diam(\Omega), \tag{1.52}$$

where the lower bound is asymptotically achieved by a sequence of thinning rectangles and the upper bound by sets of constant width.

1.4.1 Asymmetry index $\mathcal{A}^\star(E)$

In Chapter 3, we will prove a quantitative result, so we need a geometric quantity that gives us information about the shape of Ω. We will consider the Hausdorff asymmetry index, as already done in [57].

For any measurable set E of finite perimeter, we define two isoperimetric deficits

$$\mathcal{D}(E) := P(E) - P(E^\sharp), \quad \mathcal{M}(E) := |E^\star| - |E|,$$

where $E^\sharp$ and $E^\star$ are the balls with the same measure and the same perimeter of E, respectively.

The Hausdorff asymmetry indices are

$$\mathcal{A}^\star_{\mathcal{H}}(E) = \min_{x\in\mathbb{R}^n} \left\{ d_{\mathcal{H}}(E, B_r(x)),\ P(\Omega) = P(B_r(x)) \right\}, \tag{1.53}$$

and

$$\mathcal{A}^\sharp_{\mathcal{H}}(E) = \min_{x\in\mathbb{R}^n} \left\{ d_{\mathcal{H}}(E, B_r(x)),\ |\Omega| = |B_r(x)| \right\}. \tag{1.54}$$

Lemma 2.9 in [87] tells us how these two indices are related one to the other.

Lemma 1.4.12. *Let $n \geq 2$ and let $E \subset \mathbb{R}^n$ be a bounded, open and convex set with $\mathcal{D}(E) \leq \delta$, then*

$$\mathcal{A}^\star_H(E) \leq C(n)\mathcal{A}^\sharp_H(E). \tag{1.55}$$

With these definitions, we can recall the quantitative isoperimetric inequality proved in [83, 85].

Theorem 1.4.13 (Fuglede)**.** *Let $n \geq 2$, and let E be a bounded open and convex set with $|E| = \omega_n$. There exists δ_F, C, depending only on n, such that if $\mathcal{D}(E) \leq \delta_F$ then*

$$\mathcal{D}(E) \geq Cg(\mathcal{A}^\sharp_{\mathcal{H}}(E)), \tag{1.56}$$

where g is defined by

$$g(s) = \begin{cases} s^2 & \text{if } n = 2 \\ f^{-1}(s^2) & \text{if } n = 3 \\ s^{\frac{n+1}{2}} & \text{if } n \geq 4 \end{cases} \tag{1.57}$$

and $f(t) = \sqrt{t\log(\frac{1}{t})}$ for $0 < t < e^{-1}$.

We are interested in a modified version of this theorem, in terms of $\mathcal{M}(E)$, so we have

Lemma 1.4.14. *Let $E \subset \mathbb{R}^n$ be a bounded, open and convex set and let $E^\star$ be the ball satisfying $P(E) = P(E^\star) = \rho$. Then, there exist δ, C, depending only on n and ρ, such that, if*

$$\mathcal{M}(E) = |E^\star| - |E| \leq \delta, \tag{1.58}$$

then

$$\mathcal{M}(E) \geq Cg(\mathcal{A}^\star_{\mathcal{H}}(E)),$$

where g is the function defined in (1.57).

Proof. Let us start by setting $\delta < \dfrac{|E^\star|}{2}$, so

$$|E| > \frac{|E^\star|}{2}. \tag{1.59}$$

Let us divide the proof in two steps. First we assume that $|E| = \omega_n$. By the differentiability of the function $h(t) = t^{\frac{n-1}{n}}$, there exists $\xi \in (|E|, |E^\star|)$ such that

$$|E^\star|^{\frac{n-1}{n}} - |E|^{\frac{n-1}{n}} = h'(\xi)\left(|E^\star| - |E|\right) \leq \frac{n-1}{n\omega_n^{\frac{1}{n}}}\left(|E^\star| - |E|\right). \tag{1.60}$$

Moreover,

$$|E^\star|^{\frac{n-1}{n}} - |E|^{\frac{n-1}{n}} = \frac{P(E)}{n\omega_n^{\frac{1}{n}}} - |E|^{\frac{n-1}{n}} = \frac{P(E)}{n\omega_n^{\frac{1}{n}}} - \frac{P(E^\sharp)}{n\omega_n^{\frac{1}{n}}} = \frac{\mathcal{D}(E)}{n\omega^{\frac{1}{n}}}. \tag{1.61}$$

Hence, by (1.60) and (1.61), we get

$$\mathcal{D}(E) \leq (n-1)\mathcal{M}(E).$$

So, choosing $\delta \leq \dfrac{\delta_F(n)}{n-1}$, where $\delta_F(n)$ is the one provided by Theorem 1.4.13, we can apply the aforementioned Theorem to write

$$P(E) \geq n\omega_n\Big(1 + \gamma(n)g\big(\mathcal{A}^\sharp_{\mathcal{H}}(E)\big)\Big),$$

obtaining

$$\begin{aligned}|E^\star| - |E| &= \frac{P(E)^{\frac{n}{n-1}}}{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}} - |E| \geq \omega_n\left(1 + \gamma(n)g\big(\mathcal{A}^\sharp_{\mathcal{H}}(E)\big)\right)^{\frac{n}{n-1}} - \omega_n \\ &\geq \frac{n\omega_n\gamma(n)}{n-1}g\big(\mathcal{A}^\sharp_{\mathcal{H}}(E)\big),\end{aligned}$$

where in the last step we used Bernoulli's inequality

$$(1+x)^r \geq 1 + rx \qquad x \geq -1,\ r \geq 1.$$

Applying Lemma 1.4.12, we eventually have

$$|\Omega^\star| - |\Omega| \geq C(n)g\big(\mathcal{A}^\star_{\mathcal{H}}(\Omega)\big).$$

Now we treat the general case. Let us rescale the set E as

$$E_1 = \left(\frac{\omega_n}{|E|}\right)^{\frac{1}{n}} E,$$

so we have

$$\begin{aligned}\mathcal{M}(E_1) = |E_1^\star| - |E_1| &= \frac{\omega_n}{|E|}|E^\star| - \omega_n \\ &= \frac{\omega_n}{|E|}\mathcal{M}(E) \leq \frac{2\omega_n}{|E^\star|}\mathcal{M}(E),\end{aligned}$$

where the last inequality follows from (1.59).

In order to apply the previous case to the set E_1, we need

$$\mathcal{M}(E_1) \leq \frac{\delta_F}{n-1},$$

which is guaranteed if we choose

$$\mathcal{M}(E) \leq \frac{|E^\star|}{2(n-1)\omega_n}\,\delta_F = \delta(n,\rho).$$

□

Chapter 2

Comparison and Rigidity results for solutions to the $p-$Laplace equation with Robin Boundary conditions

Let β be a positive parameter and let Ω be a bounded open set of $\mathbb{R}^n$, $n \geq 2$, with Lipschitz boundary. Let $f \in L^{p'}(\Omega)$ be a non-negative function. We consider the following problem

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + \beta |u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

A function $u \in W^{1,p}(\Omega)$ is a weak solution to (2.1) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \varphi \, dx + \beta \int_{\partial\Omega} |u|^{p-2} u \varphi \, d\mathcal{H}^{n-1}(x) = \int_\Omega f \varphi \, dx, \quad \forall \varphi \in W^{1,p}(\Omega). \tag{2.2}$$

We want to compare the solution to the following symmetrized problem

$$\begin{cases} -\Delta_p v = f^\sharp & \text{in } \Omega^\sharp \\ |\nabla v|^{p-2}\dfrac{\partial v}{\partial \nu} + \beta |v|^{p-2}v = 0 & \text{on } \partial\Omega^\sharp, \end{cases} \tag{2.3}$$

where $\Omega^\sharp$ is the ball centered in the origin with the same measure of Ω and $f^\sharp$ is the Schwarz rearrangement of f (see Definition 1.2.3.)

This chapter is devoted to establishing a comparison between the solution to problem (2.1) and (2.3), extending to the non-linear case the results obtained by Alvino, Nitsch and Trombetti in [8], where they establish a comparison between a suitable norm of u and v, respectively solution to

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial\Omega. \end{cases} \qquad \begin{cases} -\Delta v = f^\sharp & \text{in } \Omega^\sharp \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial\Omega^\sharp. \end{cases}$$

They proved that if f is a non-negative function in $L^2(\Omega)$, then

$$\|u\|_{L^{k,1}(\Omega)} \leq \|v\|_{L^{k,1}(\Omega^\sharp)} \quad \forall 0 < k \leq \frac{n}{2n-2}$$
$$\|u\|_{L^{2k,2}(\Omega)} \leq \|v\|_{L^{2k,2}(\Omega^\sharp)} \quad \forall 0 < k \leq \frac{n}{3n-4}.$$

Moreover, the authors in [8] were able to establish a comparison á la Talenti,

$$u^\sharp(x) \le v(x), \quad \forall x \in \Omega^\sharp$$

in the case $f \equiv 1$ and $n = 2$.

The contents of this chapter can be also found in [9, 120, 52].

2.1 Comparison results

Let us start by proving that equation (2.1) admits a unique solution. Let us consider the functional

$$\mathfrak{F}(w) = \frac{1}{p}\int_\Omega |\nabla w|^p \, dx + \frac{\beta}{p}\int_{\partial\Omega} |w|^p \, d\mathcal{H}^{n-1}(x) - \int_\Omega fw \, dx, \tag{2.4}$$

defined on $W^{1,p}(\Omega)$. This functional is well defined and its Euler-Lagrange equation is exactly (2.1), so showing that the functional (2.4) admits a minimum is equivalent to proving that (2.1) has a solution.

Theorem 2.1.1. *Let Ω be an open, bounded and Lipschitz set. Then, the functional (2.4) admits a unique minimum in $W^{1,p}(\Omega)$.*

Proof. Let us show that the functional is bounded from below. Indeed, using the parametric Young inequality, we have

$$\begin{aligned}
\mathfrak{F}(u) &\ge \frac{1}{p}\int_\Omega |\nabla u|^p \, dx + \frac{\beta}{p}\int_{\partial\Omega} |u|^p \, d\mathcal{H}^{n-1}(x) - \frac{\varepsilon^p}{p}\int_\Omega |u|^p \, dx - \frac{1}{p'\varepsilon^{p'}}\int_\Omega |f|^{p'} \, dx \\
&\ge \frac{1}{p}\left(\int_\Omega |\nabla u|^p \, dx + \beta\int_{\partial\Omega} |u|^p \, d\mathcal{H}^{n-1}(x)\right) - \frac{\varepsilon^p}{p}\int_\Omega |u|^p \, dx - \frac{1}{p'\varepsilon^{p'}}\int_\Omega |f|^{p'} \, dx \\
&\ge \frac{\lambda_{p,\beta}(\Omega) - \varepsilon^p}{p}\int_\Omega |u|^p \, dx - \frac{1}{p'\varepsilon^{p'}}\int_\Omega |f|^{p'} \, dx
\end{aligned}$$

In the last inequality, we used the Sobolev inequality with trace term

$$\int_\Omega |\nabla u|^p + \beta\int_{\partial\Omega} |u|^p \ge \lambda_{p,\beta}(\Omega)\int_\Omega |u|^p.$$

The quantity $\lambda_{p,\beta}(\Omega)$ denotes the first eigenvalue of the $p-$Laplacian with Robin boundary conditions, which can be also seen as a trace constant of the set Ω.

If ε is small enough, the quantity

$$\frac{\lambda_{p,\beta}(\Omega) - \varepsilon^p}{p}$$

is non-negative, and then

$$\mathfrak{F}(u) \ge -\frac{1}{p'\varepsilon^{p'}}\int_\Omega |f|^{p'},$$

so

$$m = \inf_{W^{1,p}} \mathfrak{F}(u) > -\infty.$$

Let us show that the minimum is achieved. Let $\{u_i\}$ be a minimizing sequence. We can assume that $\mathfrak{F}(u_i) \leq m+1$, $\forall i$. Using again the Young inequality, we have

$$\begin{aligned} m+1 &\geq \frac{1}{p}\int_\Omega |\nabla u_i|^p\,dx + \frac{\beta}{p}\int_{\partial\Omega} |u_i|^p\,d\mathcal{H}^{n-1}(x) - \int_\Omega f u_i\,dx \\ &\geq \frac{1}{p}\int_\Omega |\nabla u_i|^p\,dx + \frac{\beta}{p}\int_{\partial\Omega} |u_i|^p\,d\mathcal{H}^{n-1}(x) - \frac{\varepsilon^p}{p}\int_\Omega |u_i|^p\,dx - \frac{1}{p'\varepsilon^{p'}}\int_\Omega |f|^{p'}\,dx, \end{aligned}$$

then

$$\begin{aligned} m+1+\frac{1}{p'\varepsilon^{p'}}\int_\Omega |f|^{p'}\,dx &\geq \frac{1}{2p}\left(\int_\Omega |\nabla u_i|^p\,dx + \beta\int_{\partial\Omega} |u_i|^p\,d\mathcal{H}^{n-1}(x)\right) - \frac{\varepsilon^p}{p}\int_\Omega |u_i|^p\,dx \\ &\quad + \frac{1}{2p}\left(\int_\Omega |\nabla u_i|^p\,dx + \beta\int_{\partial\Omega} |u_i|^p\,d\mathcal{H}^{n-1}(x)\right) \\ &\geq \frac{1}{2p}\int_\Omega |\nabla u_i|^p\,dx + \left(\frac{\lambda_{1,\beta}(\Omega) - 2\varepsilon^p}{2p}\right)\int_\Omega |u_i|^p\,dx. \end{aligned}$$

The minimizing sequence $\{u_i\}$ is bounded in $W^{1,p}(\Omega)$, so there exists a subsequence $\{\, u_{i_k} \,\}$ weakly converging in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$ to a function u. Let us show that u is the minimum.

The function t^p is strictly convex for $p>1$, so

$$|u_{i_k}|^p \geq |u|^p + p|u|^{p-2}u(u_{i_k} - u), \tag{2.5}$$

$$|\nabla u_{i_k}|^p \geq |\nabla u|^p + p|\nabla u|^{p-2}\nabla u(\nabla u_{i_k} - \nabla u). \tag{2.6}$$

If we combine (2.5), (2.6) and the definition (2.4) of $\mathfrak{F}(u_{i_k})$, we obtain

$$\begin{aligned} \int_\Omega f u_{i_k}\,dx + \mathfrak{F}(u_{i_k}) &\geq \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \int_\Omega |\nabla u|^{p-2}\nabla u(\nabla u_{i_k} - \nabla u)\,dx \\ &\quad + \frac{\beta}{p}\int_{\partial\Omega} |u|^p\,d\mathcal{H}^{n-1}(x) + \beta\int_{\partial\Omega} |u|^{p-2}u(u_{i_k} - u)\,d\mathcal{H}^{n-1}(x). \end{aligned}$$

Passing to the limit as $k\to\infty$, by the weak convergence of $\{u_{i_k}\}$ the integral over Ω on the right-hand side goes to 0. The integral over $\partial\Omega$ goes to 0 as well. Indeed, the space $W^{1,p}(\Omega)$ is compactly embedded in $L^p(\partial\Omega)$ (for more details, see [121, Section 2.5]), and $u_{i_k} - u \to 0$ in $L^p(\partial\Omega)$. So, we obtain

$$m \geq \mathfrak{F}(u).$$

This ensures us that u is the minimum of the functional.

The uniqueness of the minimum follows from the fact that $\mathfrak{F}(u)$ is the sum of a strictly convex part and a linear part. □

Let us now recall some properties of the solution to (2.1) and (2.3). We observe that the solutions u and v to (2.1) and (2.3) respectively are both $p-$superharmonic and then, by the strong maximum principle in [158], it follows that they achieve their minima on the boundary. If we denote by u_m and v_m the minimum of u and v respectively, thanks to the positiveness of β and the Robin boundary conditions, we have that $u_m \geq 0$ and $v_m \geq 0$. Hence u and v are strictly positive in the interior of Ω. Moreover, we can observe that

$$u_m = \min_\Omega u \leq \min_{\Omega^\sharp} v = v_m, \tag{2.7}$$

indeed, by the weak formulation (2.2), we have

$$\begin{aligned} v_m^{p-1}\mathrm{P}(\Omega^\sharp) &= \int_{\partial\Omega^\sharp} v(x)^{p-1}\, d\mathcal{H}^{n-1}(x) = \frac{1}{\beta}\int_{\Omega^\sharp} f^\sharp\, dx = \frac{1}{\beta}\int_\Omega f\, dx \\ &= \int_{\partial\Omega} u(x)^{p-1}\, d\mathcal{H}^{n-1}(x) \\ &\geq u_m^{p-1}\mathrm{P}(\Omega) \geq u_m^{p-1}\mathrm{P}(\Omega^\sharp). \end{aligned}$$

Remark 2.1.1. Let us observe that in general inequality (2.7) is strict: indeed, from

$$v_m^{p-1}\mathrm{P}(\Omega^\sharp) \geq u_m^{p-1}\mathrm{P}(\Omega)$$

equality in (2.7) implies that Ω is a ball.

Let us fix some notation that will be used throughout this chapter. We will denote by

$$U_t = \{x \in \Omega : u(x) > t\}, \quad \partial U_t^{int} = \partial U_t \cap \Omega, \quad \partial U_t^{ext} = \partial U_t \cap \partial\Omega,$$

and by

$$\mu(t) = |U_t|, \quad P_u(t) = \mathrm{P}(U_t).$$

If v is the solution to (2.3), using the same notations, we set

$$V_t = \left\{x \in \Omega^\sharp : v(x) > t\right\}, \quad \phi(t) = |V_t|, \quad P_v(t) = \mathrm{P}(V_t).$$

Because of the invariance of the $p-$Laplacian and of the Schwarz rearrangement of f by rotation, the solution v to (2.3) is radially symmetric. Since v is radial, positive, and decreasing along the radius, for $0 \leq t \leq v_m$, $V_t = \Omega^\sharp$, while, for $v_m < t < \max_{\Omega^\sharp} v$, V_t is a ball, concentric to $\Omega^\sharp$ and strictly contained in it.

A consequence of (2.7) that will be used in what follows is that

$$\mu(t) \leq \phi(t) = |\Omega| \quad \forall t \leq v_m. \tag{2.8}$$

The following two lemmas will be crucial in the proof of the main Theorems.

Lemma 2.1.2. *Let u be the solution to (2.1) and let v be the solution to (2.3). Then, for almost every $t > 0$, we have*

$$\gamma_n \mu(t)^{\left(1-\frac{1}{n}\right)\frac{p}{p-1}} \leq \left(\int_0^{\mu(t)} f^*(s)\, ds\right)^{\frac{1}{p-1}} \left(-\mu'(t) + \frac{1}{\beta^{\frac{1}{p-1}}}\int_{\partial U_t^{ext}} \frac{1}{u}\, d\mathcal{H}^{n-1}(x)\right) \tag{2.9}$$

and

$$\gamma_n \phi(t)^{\left(1-\frac{1}{n}\right)\frac{p}{p-1}} = \left(\int_0^{\phi(t)} f^*(s)\, ds\right)^{\frac{1}{p-1}} \left(-\phi'(t) + \frac{1}{\beta^{\frac{1}{p-1}}}\int_{\partial V_t^{ext}} \frac{1}{v}\, d\mathcal{H}^{n-1}(x)\right). \tag{2.10}$$

where $\gamma_n = \left(n\omega_n^{1/n}\right)^{\frac{p}{p-1}}$.

Proof. Let $t > 0$ and $h > 0$. In the weak formulation (2.2), we choose the following test function

$$\varphi(x) = \begin{cases} 0 & \text{if } u < t \\ u - t & \text{if } t < u < t + h \\ h & \text{if } u > t + h, \end{cases} \tag{2.11}$$

obtaining

$$\begin{aligned} \int_{U_t \setminus U_{t+h}} |\nabla u|^p \, dx + \beta h \int_{\partial U_{t+h}^{ext}} u^{p-1} \, d\mathcal{H}^{n-1}(x) + \beta \int_{\partial U_t^{ext} \setminus \partial U_{t+h}^{ext}} u^{p-1}(u - t) \, d\mathcal{H}^{n-1}(x) \\ = \int_{U_t \setminus U_{t+h}} f(u - t) \, dx + h \int_{U_{t+h}} f \, dx. \end{aligned} \tag{2.12}$$

Dividing (2.12) by h, using coarea formula (1.2) and letting h go to 0, we have that for a.e. $t > 0$

$$\int_{\partial U_t} g(x) \, d\mathcal{H}^{n-1}(x) = \int_{U_t} f \, dx,$$

where

$$g = \begin{cases} |\nabla u|^{p-1} & \text{if } x \in \partial U_t^{int}, \\ \beta u^{p-1} & \text{if } x \in \partial U_t^{ext}. \end{cases} \tag{2.13}$$

Using the isoperimetric inequality, for a.e. $t \in [0, +\infty)$ we have

$$n\omega_n^{\frac{1}{n}} \mu(t)^{\frac{n-1}{n}} \leq P(U_t) = \int_{\partial U_t} d\mathcal{H}^{n-1}(x) \tag{2.14}$$

$$\leq \left(\int_{\partial U_t} g \, d\mathcal{H}^{n-1}(x) \right)^{\frac{1}{p}} \left(\int_{\partial U_t} \frac{1}{g^{\frac{1}{p-1}}} \, d\mathcal{H}^{n-1}(x) \right)^{1-\frac{1}{p}} \tag{2.15}$$

$$= \left(\int_{\partial U_t} g \, d\mathcal{H}^{n-1}(x) \right)^{\frac{1}{p}} \left(\int_{\partial U_t^{int}} \frac{1}{|\nabla u|} \, d\mathcal{H}^{n-1}(x) + \frac{1}{\beta^{\frac{1}{p-1}}} \int_{\partial U_t^{ext}} \frac{1}{u} \, d\mathcal{H}^{n-1}(x) \right)^{1-\frac{1}{p}} \tag{2.16}$$

$$\leq \left(\int_0^{\mu(t)} f^*(s) \, ds \right)^{\frac{1}{p}} \left(-\mu'(t) + \frac{1}{\beta^{\frac{1}{p-1}}} \int_{\partial U_t^{ext}} \frac{1}{u} \, d\mathcal{H}^{n-1}(x) \right)^{1-\frac{1}{p}}, \tag{2.17}$$

and, so, (2.9) follows. Finally, we notice that, if v is the solution to (2.3), then all the inequalities above are equalities, and, consequently, we have (2.10). □

Lemma 2.1.3. *For all $\tau \geq v_m$, we have*

$$\int_0^\tau t^{p-1} \left(\int_{\partial U_t^{ext}} \frac{1}{u(x)} \, d\mathcal{H}^{n-1}(x) \right) dt \leq \frac{1}{p\beta} \int_0^{|\Omega|} f^*(s) \, ds, \tag{2.18}$$

and

$$\int_0^\tau t^{p-1} \left(\int_{\partial V_t \cap \partial \Omega^\sharp} \frac{1}{v(x)} \, d\mathcal{H}^{n-1}(x) \right) dt = \frac{1}{p\beta} \int_0^{|\Omega|} f^*(s) \, ds. \tag{2.19}$$

Proof. If we integrate the quantity

$$t^{p-1}\left(\int_{\partial U_t^{ext}} \frac{1}{u(x)}\, d\mathcal{H}^{n-1}(x)\right),$$

from 0 to $+\infty$, by Fubini theorem, we obtain

$$\begin{aligned}\int_0^\infty \tau^{p-1}\left(\int_{\partial U_\tau^{ext}} \frac{1}{u(x)}\, d\mathcal{H}^{n-1}(x)\right) d\tau &= \int_{\partial\Omega}\left(\int_0^{u(x)} \frac{\tau^{p-1}}{u(x)}\, d\tau\right) d\mathcal{H}^{n-1}(x)\\ &= \frac{1}{p}\int_{\partial\Omega} u(x)^{p-1}\, d\mathcal{H}^{n-1}(x)\\ &= \frac{1}{p\beta}\int_0^{|\Omega|} f^*(s)\, ds,\end{aligned}$$

where the last equality follows from the fact that u solves (2.1).

Analogously

$$\int_0^\infty \tau^{p-1}\left(\int_{\partial V_\tau\cap\partial\Omega^\sharp} \frac{1}{v(x)}\, d\mathcal{H}^{n-1}(x)\right) d\tau = \frac{1}{p\beta}\int_0^{|\Omega|} f^*(s)\, ds.$$

Since u is positive, we obtain, $\forall t\geq 0$,

$$\int_0^t \tau^{p-1}\left(\int_{\partial U_\tau^{ext}} \frac{1}{u(x)}\, d\mathcal{H}^{n-1}(x)\right) d\tau \leq \frac{1}{p\beta}\int_0^{|\Omega|} f^*(s)\, ds,$$

on the other hand, since $\partial V_t\cap\partial\Omega^\sharp$ is empty for $t\geq v_m$, we have

$$\int_0^t \tau^{p-1}\left(\int_{\partial V_\tau\cap\partial\Omega^\sharp} \frac{1}{v(x)}\, d\mathcal{H}^{n-1}(x)\right) d\tau = \frac{1}{p\beta}\int_0^{|\Omega|} f^*(s)\, ds.$$

and the proof of lemma 2.1.3 is complete. □

Remark 2.1.2. It can be observed that, since $\partial V_t\cap\partial\Omega^\sharp$ is empty for $t\geq v_m$ and $\phi(t)=|\Omega|$ for $t\leq v_m$, for all $\delta>0$ and for all t, we have

$$\begin{aligned}&\int_0^t \tau^{p-1}\phi(\tau)^\delta\left(\int_{\partial V_\tau\cap\partial\Omega^\sharp} \frac{1}{v(x)}\, d\mathcal{H}^{n-1}(x)\right) d\tau =\\ &\int_0^{v_m} \tau^{p-1}\phi(\tau)^\delta\left(\int_{\partial V_\tau\cap\partial\Omega^\sharp} \frac{1}{v(x)}\, d\mathcal{H}^{n-1}(x)\right) d\tau =\\ &\int_0^{+\infty} \tau^{p-1}\phi(\tau)^\delta\left(\int_{\partial V_\tau\cap\partial\Omega^\sharp} \frac{1}{v(x)}\, d\mathcal{H}^{n-1}(x)\right) d\tau = \frac{|\Omega|^\delta}{p\beta}\int_0^{|\Omega|} f^*(s)\, ds.\end{aligned}$$

Now we are in position to state and prove the main Theorems.

Theorem 2.1.4. *Let u and v be the solutions to problem (2.1) and (2.3) respectively. Then we have*

$$\|u\|_{L^{k,1}(\Omega)} \leq \|v\|_{L^{k,1}(\Omega^\sharp)}, \qquad \forall\, 0<k\leq \frac{n(p-1)}{p(n-1)}, \tag{2.20}$$

$$\|u\|_{L^{pk,p}(\Omega)} \leq \|v\|_{L^{pk,p}(\Omega^\sharp)}, \qquad \forall\, 0<k\leq \frac{n(p-1)}{(n-2)p+n}. \tag{2.21}$$

Proof. Let $0 < k \le \dfrac{n(p-1)}{p(n-1)}$, so $\delta = \dfrac{1}{k} - \dfrac{p(n-1)}{n(p-1)}$ is positive.

Multiplying (2.1.2) by $t^{p-1}\mu(t)^\delta$ and integrating from 0 to $\tau \ge v_m$, by the previous Lemma, we obtain

$$\begin{aligned}\int_0^\tau \gamma_n t^{p-1}\mu(t)^{\frac{1}{k}}\,dt \le \int_0^\tau (-\mu'(t))\, t^{p-1}\mu(t)^\delta \left(\int_0^{\mu(t)} f^*(s)\,ds\right)^{\frac{1}{p-1}} dt \\ + \frac{|\Omega|^\delta}{p\beta^{\frac{p}{p-1}}}\left(\int_0^{|\Omega|} f^*(s)\,ds\right)^{\frac{p}{p-1}}.\end{aligned} \tag{2.22}$$

Setting $F(l) = \int_0^l \omega^\delta \left(\int_0^\omega f^*(s)\,ds\right)^{\frac{1}{p-1}} d\omega$, we can integrate by parts both sides of the last inequality, getting

$$\begin{aligned}\tau^{p-1}\left(\left(\int_0^\tau \gamma_n\mu(t)^{\frac{1}{k}}\,dt\right) + F(\mu(\tau))\right) \le (p-1)\int_0^\tau t^{p-2}\left(\left(\int_0^t \gamma_n\mu(s)^{\frac{1}{k}}\,ds\right) + F(\mu(t))\right) dt \\ + \frac{|\Omega|^\delta}{p\beta^{\frac{p}{p-1}}}\left(\int_0^{|\Omega|} f^*(s)\,ds\right)^{\frac{p}{p-1}}.\end{aligned}$$

Setting $\xi(\tau) = \int_0^\tau t^{p-2}\left(\int_0^t \gamma_n\mu(s)^{\frac{1}{k}}\,ds + F(\mu(t))\right) dt$ and $C = \dfrac{|\Omega|^\delta}{p\beta^{\frac{p}{p-1}}}\left(\int_0^{|\Omega|} f^*(s)\,ds\right)^{\frac{p}{p-1}}$, we are in the hypothesis of Lemma 1.2.5 (Gronwall), namely

$$\tau\xi'(\tau) \le (p-1)\xi(\tau) + C,$$

so, choosing $\tau_0 = v_m$, we have

$$\tau^{p-2}\left(\int_0^\tau \gamma_n\mu(s)^{\frac{1}{k}}\,ds + F(\mu(\tau))\right) \le \left(\frac{(p-1)\xi(v_m) + C}{v_m}\right)\left(\frac{\tau}{v_m}\right)^{p-2},$$

where

$$\xi(v_m) = \int_0^{v_m} t^{p-2}\left(\int_0^t \gamma_n\mu(s)^{\frac{1}{k}}\,ds + F(\mu(t))\right) dt.$$

The previous inequality becomes an equality if we replace $\mu(t)$ with $\phi(t)$. Since $\mu(t) \le \phi(t) = |\Omega|, \quad \forall t \le v_m$, and $F(l)$ is monotone, we obtain

$$\int_0^{v_m} t^{p-2}\left(\int_0^t \gamma_n\mu(s)^{\frac{1}{k}}\,ds + F(\mu(t))\right) dt \le \int_0^{v_m} t^{p-2}\left(\int_0^t \gamma_n\phi(s)^{\frac{1}{k}}\,ds + F(\phi(t))\right) dt,$$

hence

$$\int_0^\tau \gamma_n\mu(s)^{\frac{1}{k}}\,ds + F(\mu(\tau)) \le \int_0^\tau \gamma_n\phi(s)^{\frac{1}{k}}\,ds + F(\phi(\tau)).$$

Passing to the limit as $\tau \to \infty$, we get

$$\int_0^\infty \mu(t)^{\frac{1}{k}}\,dt \le \int_0^\infty \phi(t)^{\frac{1}{k}}\,dt,$$

and hence

$$\|u\|_{L^{k,1}(\Omega)} \le \|v\|_{L^{k,1}(\Omega^\sharp)}, \qquad \forall\, 0 < k \le \frac{n(p-1)}{p(n-1)}.$$

To prove the inequality (2.21), it is enough to show that

$$\int_0^\infty t^{p-1}\mu(t)^{\frac{1}{k}}\,dt \le \int_0^\infty t^{p-1}\phi(t)^{\frac{1}{k}}\,dt. \tag{2.23}$$

Let us consider equation (2.22), let us integrate by parts the first term on the right-hand side from 0 to τ and then let us pass to the limit as $\tau \to \infty$, we have

$$\int_0^\infty \gamma_n t^{p-1}\mu(t)^{\frac{1}{k}}\,dt \le (p-1)\int_0^\infty t^{p-2}F(\mu(t))\,dt + \frac{|\Omega|^\delta}{p\beta^{1+\frac{1}{p-1}}}\left(\int_0^{|\Omega|} f^*(s)\,ds\right)^{\frac{p}{p-1}}.$$

Therefore, if we show that

$$\int_0^\infty t^{p-2}F(\mu(t))\,dt \le \int_0^\infty t^{p-2}F(\phi(t))\,dt, \tag{2.24}$$

we obtain (2.23). To this aim, we multiply (2.1.2) by $t^{p-1}F(\mu(t))\mu(t)^{-\frac{(n-1)p}{n(p-1)}}$ and integrate. First, we observe that, by the choice $k \le \dfrac{n(p-1)}{(n-2)p+n}$, it follows that the function $h(l) = F(l)l^{-\frac{(n-1)p}{n(p-1)}}$ is non decreasing. Hence, we obtain

$$\int_0^\tau \gamma_n t^{p-1}F(\mu(t))\,dt \le \int_0^\tau \Big(-\mu'(t)\Big)t^{p-1}\mu(t)^{-\frac{(n-1)p}{n(p-1)}}F(\mu(t))\left(\int_0^{\mu(t)} f^*(s)\,ds\right)^{\frac{1}{p-1}}dt$$
$$+ F(|\Omega|)\frac{|\Omega|^{-\frac{(n-1)p}{n(p-1)}}}{p\beta^{\frac{p}{p-1}}}\left(\int_0^{|\Omega|} f^*(s)\,ds\right)^{\frac{p}{p-1}}.$$

If we integrate by parts both sides of the last expression and set

$$C = F(|\Omega|)\frac{|\Omega|^{-\frac{p(n-1)}{n(p-1)}}}{p\beta^{\frac{p}{p-1}}}\left(\int_0^{|\Omega|} f^*(s)\,ds\right)^{\frac{p}{p-1}},$$

we obtain

$$\tau\int_0^\tau \gamma_n t^{p-2}F(\mu(t))\,dt + \tau H_\mu(\tau) \le \int_0^\tau\int_0^t r^{p-2}F(\mu(r))\,drdt + \int_0^\tau H_\mu(t)\,dt + C, \tag{2.25}$$

where

$$H_\mu(\tau) = -\int_\tau^{+\infty} t^{p-2}\mu(t)^{-\frac{p(n-1)}{n(p-1)}}F(\mu(t))\Big(\int_0^{\mu(t)} f^*(s)\,ds\Big)^{\frac{1}{p-1}}d\mu(t).$$

Setting

$$\xi(\tau) = \int_0^\tau\int_0^t \gamma_n r^{p-2}F(\mu(r))\,dr + \int_0^t H_\mu(t)\,dt,$$

then (2.25) becomes

$$\tau\xi'(\tau) \le \xi(\tau) + C.$$

So lemma 1.2.5, with $\tau_0 = v_m$, gives

$$\int_0^\tau \gamma_n t^{p-2} F(\mu(t))\,dt + H_\mu(\tau) \le \left(\frac{(p-1)\displaystyle\int_0^{v_m} t^{p-2} F(\mu(t)\,dt + H_\mu(v_m) + C}{v_m} \right) \left(\frac{\tau}{v_m}\right)^{p-2}.$$

Of course, the inequality holds as equality if we replace $\mu(t)$ with $\phi(t)$, so we get, keeping in mind (2.8),

$$\int_0^\tau \gamma_n t^{p-2} F(\mu(t)\,dt + H_\mu(\tau) \le \int_0^\tau \gamma_n F(\phi(t))\,dt + H_\phi(\tau).$$

Letting $\tau \to \infty$, one has

$$\int_0^\infty t^{p-2} F(\mu(t))dt \le \int_0^\infty t^{p-2} F(\phi(t))dt,$$

as $H_\mu(\tau), H_\phi(\tau) \to 0$ as $\tau \to \infty$. This proves (2.24), and hence (2.21).

The fact that both H_μ and H_ϕ go to 0 as τ goes to infinity can be easily deduced distinguishing the cases.

- If $p \ge 2$

$$\begin{aligned} t^{p-2}\mu(t) &= \int_{u>t} t^{p-2}\,dx \le \int_{u>t} u^{p-2}\,dx \le \|u\|_{L^p}^{p-2} \mu(t)^{\frac{2}{p}}, \\ \Rightarrow |H_\mu(\tau)| &= \int_\tau^{+\infty} t^{p-2} F(\mu(t))\mu(t)^{-\frac{p(n-1)}{n(p-1)}} \Big(\int_0^{\mu(t)} f^*(s)\,ds\Big)(-\mu'(t))\,dt \\ &\le \Big(\int_0^{|\Omega|} f^*(s)\,ds\Big) \|u\|_{L^p}^{p-2} \int_\tau^{+\infty} F(\mu(t))\mu(t)^{\frac{2}{p}-\frac{p(n-1)}{n(p-1)}-1}(-\mu'(t))\,dt \xrightarrow{\tau\to+\infty} 0. \end{aligned}$$

- If $p < 2$

$$\begin{aligned} |H_\mu(\tau)| &= \int_\tau^{+\infty} t^{p-2} F(\mu(t))\mu(t)^{-\frac{p(n-1)}{n(p-1)}} \Big(\int_0^{\mu(t)} f^*(s)\,s\Big)(-\mu'(t))\,dt \\ &\le \tau^{p-2} \int_\tau^{+\infty} F(\mu(t))\mu(t)^{-\frac{p(n-1)}{n(p-1)}} \Big(\int_0^{\mu(t)} f^*(s)\,s\Big)(-\mu'(t))\,dt \xrightarrow{\tau\to+\infty} 0. \end{aligned}$$

  and analogously for H_ϕ, which concludes the proof. □

We observe that from Theorem 2.1.4, we have that, if $p \ge n$, we can choose $k = 1$ in (2.20) and (2.21), obtaining

$$\|u\|_{L^1(\Omega)} \le \|v\|_{L^1(\Omega^\sharp)} \quad \text{and} \quad \|u\|_{L^p(\Omega)} \le \|v\|_{L^p(\Omega^\sharp)}.$$

If we suppose that the right-hand side f in (2.1) is constantly equal to 1, we achieve more information.

Theorem 2.1.5. *Assume that $f \equiv 1$ and let u and v be the solutions to (2.1) and (2.3) respectively.*

1. *If $1 \le p \le \dfrac{n}{n-1}$ then*

$$u^\sharp(x) \le v(x) \qquad x \in \Omega^\sharp, \tag{2.26}$$

2. *if* $p > \dfrac{n}{n-1}$ *and* $0 < k \leq \dfrac{n(p-1)}{n(p-1)-p}$, *then*

$$\|u\|_{L^{k,1}(\Omega)} \leq \|v\|_{L^{k,1}(\Omega^\sharp)}, \tag{2.27}$$

$$\|u\|_{L^{pk,p}(\Omega)} \leq \|v\|_{L^{pk,p}(\Omega^\sharp)}. \tag{2.28}$$

Proof. Firstly, we observe that $\displaystyle\int_0^{\mu(t)} f^*(s)\,ds = \mu(t)$, so (2.1.2) becomes

$$\gamma_n \mu(t)^{\left(1-\frac{1}{n}-\frac{1}{p}\right)\frac{p}{p-1}} \leq -\mu'(t) + \frac{1}{\beta^{\frac{1}{p-1}}} \int_{\partial U_t^{\text{ext}}} \frac{1}{u}\, d\mathcal{H}^{n-1}(x). \tag{2.29}$$

Let us multiply both sides of (2.29) by $t^{p-1}\mu(t)^\delta$, where $\delta = -\left(1-\frac{1}{n}-\frac{1}{p}\right)\frac{p}{p-1}$. We point out that $\delta \geq 0$ for $p \leq \frac{n}{n-1}$. Hence, integrating from 0 to $\tau \geq v_m$, we have

$$\begin{aligned}\int_0^\tau \gamma_n t^{p-1} &\leq \int_0^\tau t^{p-1}\mu(t)^\delta(-\mu'(t))\,dt + \frac{1}{\beta^{\frac{1}{p-1}}}\int_0^\tau t^{p-1}\mu(t)^\delta \int_{\partial U_t^{\text{ext}}} \frac{1}{u}\,d\mathcal{H}^{n-1}(x)\\ &\leq \int_0^\tau t^{p-1}\mu(t)^\delta(-\mu'(t))\,dt + \frac{|\Omega|^{\delta+1}}{p\beta^{\frac{p}{p-1}}}.\end{aligned} \tag{2.30}$$

Taking into account Remark 2.1.2, if we replace $\mu(t)$ with $\phi(t)$ the previous inequality holds as equality. Hence, we get

$$\int_0^\tau t^{p-1}\mu(t)^\delta(-\mu'(t))\,dt \geq \int_0^\tau t^{p-1}\phi(t)^\delta(-\phi'(t))\,dt.$$

Then, integration by parts gives

$$-\tau^{p-1}\frac{\mu(\tau)^{\delta+1}}{\delta+1} + (p-1)\int_0^\tau t^{p-2}\frac{\mu(t)^{\delta+1}}{\delta+1}\,dt \geq -\tau^{p-1}\frac{\phi(\tau)^{\delta+1}}{\delta+1} + (p-1)\int_0^\tau t^{p-2}\frac{\phi(t)^{\delta+1}}{\delta+1}\,dt.$$

Finally, using Gronwall's Lemma 1.2.5 with the function $\xi(\tau) = \displaystyle\int_0^\tau s^{p-2}\left(\frac{\mu(s)^{\delta+1}-\phi(s)^{\delta+1}}{\delta+1}\right) ds$ we obtain

$$\tau^{p-2}\left(\frac{\mu^{\delta+1}(\tau)-\phi^{\delta+1}(\tau)}{\delta+1}\right) \leq (p-1)\frac{\tau^{p-2}}{v_m^{p-2}}\int_0^{v_m} s^{p-2}\left(\frac{\mu^{\delta+1}(s)-\phi^{\delta+1}(s)}{\delta+1}\right) ds.$$

Equation (2.8) ensures us that the quantity on the right-hand side is non-positive, so

$$\mu(\tau) \leq \phi(\tau) \quad \forall \tau \geq v_m,$$

and, remembering that,

$$\mu(\tau) \leq \phi(\tau) = |\Omega| \quad \forall \tau \leq v_m,$$

we get the point-wise inequality (2.26).

Now, we want to show (2.27). Proving

$$\|u\|_{L^{k,1}(\Omega)} \leq \|v\|_{L^{k,1}(\Omega^\sharp)},$$

is equivalent to show

$$\int_0^{+\infty} \mu(t)^{\frac{1}{k}}\, dt \leq \int_0^{+\infty} \phi(t)^{\frac{1}{k}}\, dt. \tag{2.31}$$

Let us multiply (2.29) by $t^{p-1}\mu(t)^{\frac{1}{k}-\left(1-\frac{1}{n}-\frac{1}{p}\right)\frac{p}{p-1}}$ and integrate from 0 to $\tau \geq v_m$. Then, using Lemma 2.1.3 and Remark 2.1.2, we obtain

$$\int_0^{\tau} \gamma_n t^{p-1}\mu(t)^{\frac{1}{k}}\, dt \leq \int_0^{\tau} t^{p-1}\mu(t)^{\frac{1}{k}-\left(1-\frac{1}{n}-\frac{1}{p}\right)\frac{p}{p-1}}(-\mu'(t))\, dt + \frac{|\Omega|^{\frac{1}{k}-\left(1-\frac{1}{n}-\frac{1}{p}\right)\frac{p}{p-1}+1}}{p\beta^{\frac{p}{p-1}}} \tag{2.32}$$

and equality holds if we replace μ with ϕ. In order to simplify the notation, we set

$$\eta = \frac{1}{k} - \left(1-\frac{1}{n}-\frac{1}{p}\right)\frac{p}{p-1}, \qquad C = \frac{|\Omega|^{\eta+1}}{p\beta^{\frac{p}{p-1}}}.$$

We point out that (2.32) follows by (2.30) if $\eta \geq 0$, namely

$$0 < k \leq \frac{n(p-1)}{n(p-1)-p}.$$

With these notations and the fact that μ is a non-increasing function, we have from (2.32) that

$$\int_0^{\tau} \gamma_n t^{p-1}\mu(t)^{\frac{1}{k}}\, dt \leq \int_0^{\tau} -t^{p-1}\mu(t)^{\eta}\, d\mu(t) + C. \tag{2.33}$$

Let us set $G(\ell) = \int_0^{\ell} w^{\eta}\, dw = \frac{\ell^{\eta+1}}{\eta+1}$, let us integrate by parts both sides of (2.33) in order to obtain

$$\begin{aligned} &\gamma_n \tau^{p-1}\int_0^{\tau} \mu(t)^{\frac{1}{k}}\, dt + \tau^{p-1}G(\mu(\tau)) \\ &\leq (p-1)\left[\int_0^{\tau} \gamma_n t^{p-2}\int_0^{t} \mu(r)^{\frac{1}{k}}\, dr dt + \int_0^{\tau} t^{p-2}G(\mu(t))\, dt\right] + C. \end{aligned} \tag{2.34}$$

Setting

$$\xi(\tau) = \int_0^{\tau}\left(\gamma_n t^{p-2}\int_0^{t} \mu(r)^{\frac{1}{k}}\, dr\right) dt + \int_0^{\tau} t^{p-2}G(\mu(t))\, dt,$$

(2.34) reads as follows

$$\tau\xi'(\tau) \leq (p-1)\xi(\tau) + C.$$

Hence, using Gronwall's Lemma 1.2.5 with $\tau_0 = v_m$, we get

$$\gamma_n \tau^{p-2}\int_0^{\tau} \mu(t)^{\frac{1}{k}}\, dt + \tau^{p-2}G(\mu(\tau)) \leq \left(\frac{(p-1)\xi(v_m)+C}{v_m}\right)\left(\frac{\tau}{v_m}\right)^{p-2},$$

where

$$\xi(v_m) = \int_0^{v_m} \gamma_n t^{p-2}\int_0^{t} \mu(r)^{\frac{1}{k}}\, dr\, dt + \int_0^{v_m} t^{p-2}G(\mu(t))\, dt.$$

Again, if we replace μ with ϕ, the previous inequality holds as an equality, and $\xi(v_m)$ is less or equal then the same quantity obtained by replacing μ with ϕ, as (2.7) holds. By (2.8), we have

$$\tau^{p-2}\left(\gamma_n\int_0^\tau \mu(t)^{\frac{1}{k}}\,dt+G(\mu(\tau))\right)\leq \tau^{p-2}\left(\gamma_n\int_0^\tau \phi(t)^{\frac{1}{k}}\,dt+G(\phi(\tau))\right).$$

Passing to the limit as $\tau\to+\infty$, we get

$$\int_0^{+\infty}\mu(t)^{\frac{1}{k}}\,dt\leq\int_0^{+\infty}\phi(t)^{\frac{1}{k}}\,dt,$$

namely (2.31) and so (2.27).

To conclude the proof, we have to show that (2.28) holds, that is

$$\|u\|_{L^{pk,p}(\Omega)}\leq\|v\|_{L^{pk,p}(\Omega^\sharp)}\qquad\forall\, 0<k\leq\frac{n(p-1)}{n(p-1)-p},$$

or, equivalently,

$$\int_0^{+\infty}t^{p-1}\mu(t)^{\frac{1}{k}}\,dt\leq\int_0^{+\infty}t^{p-1}\phi(t)^{\frac{1}{k}}\,dt.$$

We consider (2.33), pass to the limit as $\tau\to+\infty$ and integrate by parts the first term on the right-hand side

$$\int_0^{+\infty}\gamma_n t^{p-1}\mu(t)^{\frac{1}{k}}\,dt\leq(p-1)\int_0^{+\infty}t^{p-2}G(\mu(t))\,dt+C.$$

Hence it is enough to show that

$$\int_0^{+\infty}t^{p-2}G(\mu(t))\,dt\leq\int_0^{+\infty}t^{p-2}G(\phi(t))\,dt.$$

To this aim, we multiply (2.29) by $t^{p-1}G(\mu(t))\mu(t)^{-\left(1-\frac{1}{n}-\frac{1}{p}\right)\frac{p}{p-1}}$ and integrate from 0 to $\tau\geq v_m$

$$\begin{aligned}\int_0^\tau\gamma_n t^{p-1}G(\mu(t))\,dt\leq&\int_0^\tau t^{p-1}G(\mu(t))\mu(t)^{-\left(1-\frac{1}{n}-\frac{1}{p}\right)\frac{p}{p-1}}\,d\mu(t)\\&+\frac{1}{\beta^{\frac{1}{p-1}}}\int_0^\tau t^{p-1}G(\mu(t))\mu(t)^{-\left(1-\frac{1}{n}-\frac{1}{p}\right)\frac{p}{p-1}}\left(\int_{\partial U_t^{ext}}\frac{1}{u}\,d\mathcal{H}^{n-1}(x)\right)dt.\end{aligned}$$

Since $k\leq\dfrac{n(p-1)}{n(p-1)-p}$, using Lemma 2.1.3 and the fact that the function $G(\ell)\ell^{-\left(1-\frac{1}{n}-\frac{1}{p}\right)\frac{p}{p-1}}$ is non-decreasing, we obtain

$$\int_0^\tau\gamma_n t^{p-1}G(\mu(t))\,dt\leq\int_0^\tau t^{p-1}G(\mu(t))\mu(t)^{-\left(1-\frac{1}{n}-\frac{1}{p}\right)\frac{p}{p-1}}\,d\mu(t)+C,\tag{2.35}$$

with

$$C=\frac{1}{p\beta^{\frac{p}{p-1}}}G(|\Omega|)|\Omega|^{-\left(1-\frac{1}{n}-\frac{1}{p}\right)\frac{p}{p-1}+1}.$$

If we replace μ with ϕ the previous inequality holds as an equality, as pointed out in Remark (2.1.2). Now, let us integrate by parts both sides of (2.35), obtaining

$$\tau\int_0^\tau\gamma_n t^{p-2}G(\mu(t))\,dt+\tau H(\tau)\leq\int_0^\tau\int_0^t\gamma_n t^{p-2}G(\mu(r))\,drdt+\int_0^\tau H_\mu(t)\,dt+C,\tag{2.36}$$

where

$$H_\mu(\tau) = -\int_\tau^{+\infty} t^{p-2} G(\mu(t))\mu(t)^{-\left(1-\frac{1}{n}-\frac{1}{p}\right)\frac{p}{p-1}}\, d\mu(t).$$

Setting

$$\xi(\tau) = \int_0^\tau \int_0^t \gamma_n t^{p-2} G(\mu(r))\, drdt + \int_0^\tau H_\mu(t)\, dt,$$

equation (2.36) reads as follows

$$\tau\xi'(\tau) \le \xi(\tau) + C.$$

Again, using Gronwall's Lemma 1.2.5, we get

$$\int_0^\tau \gamma_n t^{p-2} G(\mu(t))\, dt + H_\mu(\tau) \le \left(\frac{(p-1)\xi(v_m)+C}{v_m}\right)\left(\frac{\tau}{v_m}\right)^{p-2},$$

with

$$\xi(v_m) = \int_0^{v_m} \int_0^t \gamma_n t^{p-2} G(\mu(r))\, drdt + \int_0^{v_m} H_\mu(t)\, dt.$$

We recall that for ϕ the previous inequalities hold as equalities and that G is non-decreasing, so $\xi(v_m)$ is less or equal to the same quantity obtained by replacing μ with ϕ. Hence, we obtain

$$\int_0^\tau \gamma_n t^{p-2} G(\mu(t))\, dt + H_\mu(\tau) \le \int_0^\tau \gamma_n t^{p-2} G(\phi(t))\, dt + H_\phi(\tau)$$

and passing to the limit as $\tau \to +\infty$, and we finally get

$$\int_0^{+\infty} t^{p-2} G(\mu(t))\, dt \le \int_0^{+\infty} t^{p-2} G(\phi(t))\, dt,$$

indeed, as in the proof of Theorem 2.1.4, $H_\mu(\tau)$ and $H_\phi(\tau)$ go to 0 as $\tau \to \infty$. That concludes the proof. □

As a consequence of Theorems 2.1.4 and 2.1.5, we obtain

Corollary 2.1.6. *Let u and v be the solutions to (2.1) and (2.3) respectively. Then, if $p \ge n$, we have*

$$\|u\|_{L^1(\Omega)} \le \|v\|_{L^1(\Omega^\sharp)} \quad \text{and} \quad \|u\|_{L^p(\Omega)} \le \|v\|_{L^p(\Omega^\sharp)}.$$

Moreover in the case $f \equiv 1$, Theorem 2.1.5 gives

$$\|u\|_{L^1(\Omega)} \le \|v\|_{L^1(\Omega^\sharp)} \quad \text{and} \quad \|u\|_{L^p(\Omega)} \le \|v\|_{L^p(\Omega^\sharp)}, \qquad \forall p > 1,$$

and the point-wise comparison for $p \le \dfrac{n}{n-1}$.

Proof. If $p \ge n$ the upper bounds of k, in both cases (2.20) e (2.21), are greater than 1 and so we can choose $k = 1$. The assertion follows from the fact that

$$\|\cdot\|_{L^{p,p}(\Omega)} = \|\cdot\|_{L^p(\Omega)}.$$

Analogously if $f \equiv 1$. □

2.1.1 The Faber–Krahn inequality

Using tools from the previous section, we are able to give a new proof of the Faber-Krahn inequality with Robin boundary conditions in the case $p \geq n$. This topic was already studied in the papers by Bucur, Giacomini, Daners and Trebeschi, [45], [47] , [48] and [50] where the authors proved the Faber-Krahn inequality for the eigenvalues of the $p-$Laplacian with Robin boundary conditions, for every $p > 1$. Although they are more general, the results in [45] are obtained with completely different tools than the ones studied here.

We recall that the first eigenvalue of $p-$Laplace operator with Robin boundary conditions is obtained as the minimum of the Rayleigh quotients,

$$\lambda_{p,\beta}(\Omega) = \min_{\substack{\omega \in W^{1,p}(\Omega) \\ \omega \neq 0}} \frac{\displaystyle\int_\Omega |\nabla \omega|^p \, dx + \beta \int_{\partial\Omega} |\omega|^p \, d\mathcal{H}^{n-1}(x)}{\displaystyle\int_\Omega |\omega|^p \, dx}. \tag{2.37}$$

Of course, if u achieves the minimum of Rayleigh quotients, so does $|u|$. From this, we have that u in non-negative. Furthermore, as a consequence of Harnack inequality, if $u_{\lambda_1} \geq 0$ in Ω, $u_{\lambda_1} > 0$ in Ω.

Moreover, one can prove that the first eigenvalue is simple. Indeed, as shown in [111], if Ω is smooth enough and u and v are two eigenfunctions referred to the first eigenvalue, we can choose as test function $\varphi_1 = \dfrac{u^p - v^p}{u^{p-1}}$ in

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi_1 \, dx + \beta \int_{\partial\Omega} u^{p-1} \varphi_1 \, d\mathcal{H}^{n-1}(x) = \int_\Omega \lambda_1 u^{p-1} \varphi_1 \, dx,$$

and $\varphi_2 = \dfrac{v^p - u^p}{v^{p-1}}$ in

$$\int_\Omega |\nabla v|^{p-2} \nabla v \nabla \varphi_2 \, dx + \beta \int_{\partial\Omega} v^{p-1} \varphi_2 \, d\mathcal{H}^{n-1}(x) = \int_\Omega \lambda_1 v^{p-1} \varphi_2 \, dx.$$

Summing the two equations, we have

$$\begin{aligned}
0 = &\int_\Omega \left\{ 1 + (p-1) \left(\frac{v}{u} \right)^p \right\} |\nabla u|^p + \left\{ 1 + (p-1) \left(\frac{u}{v} \right)^p \right\} |\nabla v|^p \\
&- \int_\Omega p \left(\frac{v}{u} \right)^{p-1} |\nabla u|^{p-2} \nabla u \nabla v + p \left(\frac{u}{v} \right)^{p-1} |\nabla v|^{p-2} \nabla v \nabla u \\
&= \int_\Omega (u^p - v^p) \left(|\nabla \log u|^p - |\nabla \log v|^p \right) \\
&- \int_\Omega p v^p |\nabla \log u|^{p-2} |\nabla \log u| \left(\nabla \log v - \nabla \log u \right) \\
&- \int_\Omega p u^p |\nabla \log v|^{p-2} |\nabla \log v| \left(\nabla \log u - \nabla \log v \right).
\end{aligned}$$

Now, we can use the following inequalities, see e.g. [111], which hold true for each w_1 and

$w_2 \in \mathbb{R}^n$,

$$
\begin{aligned}
|w_2|^p \geq |w_1|^p + p|w_1|^{p-2}w_1 \cdot (w_2 - w_1) + \frac{|w_2 - w_1|^p}{2^{p-1}-1}, \quad \text{if} \quad p \geq 2, \\
|w_2|^p \geq |w_1|^p + p|w_1|^{p-2}w_1 \cdot (w_2 - w_1) + C(p)\frac{|w_2 - w_1|^2}{(|w_1| + |w_2|)^{2-p}}, \quad \text{if} \quad 1 < p < 2.
\end{aligned}
\tag{2.38}
$$

If, for instance, we consider the case $p \geq 2$, we choose $w_2 = \nabla \log u$ and $w_1 = \nabla \log v$, we obtain

$$
\frac{1}{2^{p-1}-1}\int_\Omega \left(\frac{1}{v^p} + \frac{1}{u^p}\right) |v\nabla u - u\nabla v|^p = 0.
$$

Hence, we obtain that $v\nabla u = u\nabla v$ a.e. in Ω, and so there exists a constant K for which $u = Kv$. This means that λ_1 is simple.

The following corollary of Theorem 2.1.4 holds true, that is Faber-Krahn inequality.

Corollary 2.1.7. *If $p \geq n$, we have*

$$
\lambda_{p,\beta}(\Omega) \geq \lambda_{p,\beta}(\Omega^\sharp).
$$

Proof. Let u an eigenfunction of the $p-$Laplacian with Robin boundary conditions, then it solves

$$
\begin{cases}
-\Delta_p u = \lambda_{p,\beta}(\Omega)\, |u|^{p-2} u & \text{in } \Omega \\
|\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + \beta |u|^{p-2}u = 0 & \text{on } \partial\Omega.
\end{cases}
$$

Now, let z be a solution to the following problem

$$
\begin{cases}
-\Delta_p z = \lambda_{p,\beta}(\Omega)\, |u^\sharp|^{p-2} u^\sharp & \text{in } \Omega^\sharp \\
|\nabla z|^{p-2}\dfrac{\partial z}{\partial \nu} + \beta |z|^{p-2}z = 0 & \text{on } \partial\Omega^\sharp.
\end{cases}
$$

In that case, as $p \geq n$, corollary 2.1.6 gives

$$
\int_\Omega |u|^p \, dx = \int_{\Omega^\sharp} \left|u^\sharp\right|^p dx \leq \int_{\Omega^\sharp} |z|^p \, dx,
$$

and hence, by Hölder inequality

$$
\int_{\Omega^\sharp} (u^\sharp)^{p-2} u^\sharp z \, dx \leq \left(\int_{\Omega^\sharp} \left|u^\sharp\right|^p dx\right)^{\frac{p-1}{p}} \left(\int_{\Omega^\sharp} z^p \, dx\right)^{\frac{1}{p}} \leq \int_{\Omega^\sharp} z^p \, dx.
$$

Therefore, we can write the eigenvalue $\lambda_{p,\beta}(\Omega)$ in the following way, obtaining

$$
\begin{aligned}
\lambda_{p,\beta}(\Omega) &= \frac{\displaystyle\int_{\Omega^\sharp} |\nabla z|^p \, dx + \beta \int_{\partial\Omega^\sharp} z^p \, d\mathcal{H}^{n-1}(x)}{\displaystyle\int_{\Omega^\sharp} (u^\sharp)^{p-2} u^\sharp z \, dx} \\
&\geq \frac{\displaystyle\int_{\Omega^\sharp} |\nabla z|^p \, dx + \beta \int_{\partial\Omega^\sharp} z^p \, d\mathcal{H}^{n-1}(x)}{\displaystyle\int_{\Omega^\sharp} z^p \, dx} \geq \lambda_{p,\beta}(\Omega^\sharp).
\end{aligned}
$$

□

2.2 Rigidity results

In the present section, we focus our study on the rigidity for the $p-$Laplace operator. In this case, the comparison results are obtained in [9] and were exploited in Section 2.1.

In Theorems 2.1.4 and 2.1.5, we establish a comparison result between suitable Lorentz norms of the solutions u and v to problems (2.1) and (2.3), that are

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + \beta |u|^{p-2}u = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta_p v = f^\sharp & \text{in } \Omega^\sharp \\ |\nabla v|^{p-2}\dfrac{\partial v}{\partial \nu} + \beta |v|^{p-2}v = 0 & \text{on } \partial\Omega^\sharp. \end{cases}$$

respectively. In particular, we proved that

$$\|u\|_{L^{pk,p}(\Omega)} \le \|v\|_{L^{pk,p}(\Omega^\sharp)}, \qquad \forall\, 0 < k \le \frac{n(p-1)}{(n-2)p+n}, \tag{2.39}$$

and in the case $f \equiv 1$,

$$u^\sharp(x) \le v(x), \quad 1 \le p \le \frac{n}{n-1}$$

and

$$\|u\|_{L^{pk,p}(\Omega)} \le \|v\|_{L^{pk,p}(\Omega^\sharp)}, \qquad \forall\, 0 < k \le \frac{n(p-1)}{(n-2)p+n}, \quad \forall p > 1. \tag{2.40}$$

We now aim to characterize the equality case in (2.39) and (2.40), answering to the open problem contained in [119]. For simplicity, we state the main Theorem only in the case $f \in L^{p'}(\Omega)$ positive, since in the case $f \equiv 1$ the proof is analogous, as we observe in Remark 2.2.1. The main goal of the Section is to prove the following

Theorem 2.2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open and Lipschitz set and let $\Omega^\sharp$ be the ball centered at the origin with the same measure as Ω. Let u be the solution to (2.1) and let v be a solution to (2.3). If*

$$\|u\|_{L^{pk,p}(\Omega)} = \|v\|_{L^{pk,p}(\Omega^\sharp)}, \quad \text{for some } k \in \left]0, \frac{n(p-1)}{(n-2)p+n}\right] \tag{2.41}$$

then, there exists $x_0 \in \mathbb{R}^n$ such that

$$\Omega = \Omega^\sharp + x_0, \qquad u(\cdot + x_0) = v(\cdot), \qquad f(\cdot + x_0) = f^\sharp(\cdot).$$

The idea of the proof of Theorem 2.2.1 is the following. First of all, we prove that hypothesis (2.41) implies that the superlevel sets of u are balls. The main difficulty is to prove that these balls are concentric.

Differently from the case of the Laplace operator with Dirichlet boundary conditions studied in [7, 73], we can't apply directly the steepest descent method introduced in [19], because it strongly relays on the continuity of both the solution and of its gradient. In the case of the $p-$Laplace equation, the continuity of the solution up to the boundary depends on the regularity of the given datum f. To overcome this regularity issue we show that u is a solution to a suitable Dirichlet problem and it satisfies the Pólya-Szegő inequality with equality sign. Then, we can conclude that u is radially symmetric and decreasing, using the classical result contained in [43]. We make use of Lemma 2.2.3, where the rigidity of the Poisson problem for the $p-$Laplace operator with Dirichlet boundary condition is proved under the assumption $f \in L^{p'}(\Omega)$ and positive.

In order to prove the main Theorem 2.2.1, we divide the proof into the following steps. First of all, we prove that, under the assumptions of Theorem 2.2.1, equality holds in (2.9), i.e.

$$\gamma_n \mu(t)^{\left(1-\frac{1}{n}\right)\frac{p}{p-1}} = \left(\int_0^{\mu(t)} f^*(s)\,ds\right)^{\frac{1}{p-1}} \left(-\mu'(t) + \frac{1}{\beta^{\frac{1}{p-1}}} \int_{\partial U_t^{\text{ext}}} \frac{1}{u}\, d\mathcal{H}^{n-1}(x)\right);$$

this is the content of Proposition 2.2.2. Then, in Proposition 2.2.5, we prove that equality in (2.9) implies the fact that Ω is a ball and u and f are radial functions. In order to prove this last step, we need the key Lemma 2.2.3.

Proposition 2.2.2. *Let u be the solution to* (2.1) *and let v be the solution to* (2.3)*. If there exists k*

$$k \in \left]0, \frac{n(p-1)}{(n-2)p+n}\right] \quad \textit{such that} \quad \|u\|_{L^{pk,p}(\Omega)} = \|v\|_{L^{pk,p}(\Omega^\sharp)},$$

then equality holds in (2.9) *for almost every t.*

Proof. Since we are assuming that $\|u\|_{L^{pk,p}(\Omega)} = \|v\|_{L^{pk,p}(\Omega^\sharp)}$, we have that

$$\int_0^{+\infty} t^{p-1}\mu(t)^{\frac{1}{k}}\,dt = \int_0^{+\infty} t^{p-1}\phi(t)^{\frac{1}{k}}\,dt. \tag{2.42}$$

Let us multiply (2.9) by $t^{p-1}\mu(t)^\alpha$, where $\alpha = \dfrac{1}{k} - \left(1-\dfrac{1}{n}\right)\dfrac{p}{p-1}$, and let us integrate from 0 to $+\infty$:

$$\begin{aligned} &\gamma_n \int_0^{+\infty} t^{p-1}\mu^{\frac{1}{k}}(t)\,dt \\ &\le \int_0^{+\infty} \left(\int_0^{\mu(t)} f^*(s)\,ds\right)^{\frac{1}{p-1}} \left(-\mu'(t) + \frac{1}{\beta^{\frac{1}{p-1}}}\int_{\partial U_t^{ext}} \frac{1}{u}\,d\mathcal{H}^{n-1}(x)\right) t^{p-1}\mu(t)^\alpha\,dt \\ &\le \int_0^{+\infty} t^{p-1}\mu(t)^\alpha \left(\int_0^{\mu(t)} f^*(s)\,ds\right)^{\frac{1}{p-1}} (-\mu'(t))\,dt + \frac{|\Omega|^\alpha}{p\beta^{\frac{p}{p-1}}}\left(\int_0^{|\Omega|} f^*(s)\,ds\right)^{\frac{p}{p-1}}, \end{aligned} \tag{2.43}$$

where in the last inequality we have used $\mu(t) \le |\Omega|$ and (2.18) in Lemma 2.1.3. For v, it holds

$$\begin{aligned} &\gamma_n \int_0^{+\infty} t^{p-1}\phi^{\frac{1}{k}}(t)\,dt \\ &= \int_0^{+\infty} \left(\int_0^{\phi(t)} f^*(s)\,ds\right)^{\frac{1}{p-1}} \left(-\phi'(t) + \frac{1}{\beta^{\frac{1}{p-1}}}\int_{\partial V_t^{ext}} \frac{1}{u}\,d\mathcal{H}^{n-1}(x)\right) t^{p-1}\phi(t)^\alpha\,dt \\ &= \int_0^{+\infty} t^{p-1}\phi(t)^\alpha \left(\int_0^{\phi(t)} f^*(s)\,ds\right)^{\frac{1}{p-1}} (-\phi'(t))\,dt + \frac{|\Omega|^\alpha}{p\beta^{\frac{p}{p-1}}}\left(\int_0^{|\Omega|} f^*(s)\,ds\right)^{\frac{p}{p-1}}. \end{aligned} \tag{2.44}$$

We observe that the left-hand-side of (2.43) and the left-hand-side of (2.44) are equal from (2.42). So, it follows

$$\int_0^{+\infty} t^{p-1}\phi(t)^\alpha \left(\int_0^{\phi(t)} f^*(s)\,ds\right)^{\frac{1}{p-1}} (-\phi'(t))\,dt \le \int_0^{+\infty} t^{p-1}\mu(t)^\alpha \left(\int_0^{\mu(t)} f^*(s)\,ds\right)^{\frac{1}{p-1}} (-\mu'(t))\,dt. \tag{2.45}$$

Setting $F(l) = \displaystyle\int_0^l \omega^\delta \left(\int_0^\omega f^*(s)\,ds\right)^{\frac{1}{p-1}} d\omega$, and integrating (2.45) by parts, we get

$$\int_0^\infty t^{p-2}F(\phi(t))\,dt \le \int_0^\infty t^{p-2}F(\mu(t))\,dt,$$

being $\mu(t) = \phi(t) = 0$ for $t > v_M$. In the proof of Theorem 2.1.4, it is proved that

$$\int_0^\infty t^{p-2}F(\mu(t))\,dt \le \int_0^\infty t^{p-2}F(\phi(t))\,dt. \tag{2.46}$$

and we recall here the proof for the reader's convenience. In order to do that, we multiply (2.9) by $t^{p-1}F(\mu(t))\mu(t)^{-\frac{(n-1)p}{n(p-1)}}$ and we integrate between 0 and $\tau > v_m$. First, we observe that, by the hypothesis $k \le \dfrac{n(p-1)}{(n-2)p+n}$, it follows that the function $h(l) = F(l)l^{-\frac{(n-1)p}{n(p-1)}}$ is non decreasing. Hence, we obtain

$$\int_0^\tau \gamma_n t^{p-1}F(\mu(t))\,dt \le \int_0^\tau \Big(-\mu'(t)\Big)t^{p-1}\mu(t)^{-\frac{(n-1)p}{n(p-1)}}F(\mu(t))\left(\int_0^{\mu(t)} f^*(s)\,ds\right)^{\frac{1}{p-1}} dt$$
$$+ F(|\Omega|)\frac{|\Omega|^{-\frac{p(n-1)}{n(p-1)}}}{p\beta^{\frac{p}{p-1}}}\left(\int_0^{|\Omega|} f^*(s)\,ds\right)^{\frac{p}{p-1}}.$$

If we integrate by parts both sides of the last expression and we set

$$C = F(|\Omega|)\frac{|\Omega|^{-\frac{p(n-1)}{n(p-1)}}}{p\beta^{\frac{p}{p-1}}}\left(\int_0^{|\Omega|} f^*(s)\,ds\right)^{\frac{p}{p-1}},$$

we obtain

$$\tau\int_0^\tau \gamma_n t^{p-2}F(\mu(t))\,dt + \tau H_\mu(\tau) \le \int_0^\tau\int_0^t r^{p-2}F(\mu(r))\,dr dt + \int_0^\tau H_\mu(t)\,dt + C,$$

where

$$H_\mu(\tau) = -\int_\tau^{+\infty} t^{p-2}\mu(t)^{-\frac{p(n-1)}{n(p-1)}}F(\mu(t))\Big(\int_0^{\mu(t)} f^*(s)\,ds\Big)^{\frac{1}{p-1}} d\mu(t).$$

Setting now

$$\xi(\tau) = \int_0^\tau\int_0^t \gamma_n r^{p-2}F(\mu(r))\,dr + \int_0^t H_\mu(t)\,dt,$$

inequality (2.25) becomes

$$\tau\xi'(\tau) \le \xi(\tau) + C.$$

So, Lemma 1.2.5, with $\tau_0 = v_m$ and $q = 2$, gives

$$\int_0^\tau \gamma_n t^{p-2} F(\mu(t))\, dt + H_\mu(\tau) \leq \left(\frac{\displaystyle\int_0^{v_m} t^{p-2} F(\mu(t)\, dt + H_\mu(v_m) + C}{v_m} \right).$$

Of course, the inequality holds as equality if we replace $\mu(t)$ with $\phi(t)$, so we get:

$$\int_0^\tau \gamma_n t^{p-2} F(\mu(t))\, dt + H_\mu(\tau) \leq \int_0^\tau \gamma_n F(\phi(t))\, dt + H_\phi(\tau),$$

keeping in mind that $\mu(t) \leq \phi(t) = |\Omega|$ for $t \leq v_m$. Now, letting $\tau \to \infty$, one has

$$\int_0^\infty t^{p-2} F(\mu(t)) dt \leq \int_0^\infty t^{p-2} F(\phi(t)) dt,$$

since $H_\mu(\tau), H_\phi(\tau) \to 0$.

So, we get equality in (2.43) and, consequently, in (2.9) for almost every t, indeed

$$\begin{aligned}
&\gamma_n \int_0^{+\infty} t^{p-1} \mu^{\frac{1}{k}}(t)\, dt \\
&\leq \int_0^{+\infty} \left(\int_0^{\mu(t)} f^*(s)\, ds \right)^{\frac{1}{p-1}} \left(-\mu'(t) + \frac{1}{\beta^{\frac{1}{p-1}}} \int_{\partial U_t^{ext}} \frac{1}{u}\, d\mathcal{H}^{n-1}(x) \right) t^{p-1} \mu(t)^\alpha\, dt \\
&\leq \int_0^{+\infty} t^{p-2} F(\mu(t))\, dt + \frac{|\Omega|^\alpha}{p\beta^{\frac{p}{p-1}}} \left(\int_0^{|\Omega|} f^*(s)\, ds \right)^{\frac{p}{p-1}} \\
&= \int_0^{+\infty} t^{p-2} F(\phi(t))\, dt + \frac{|\Omega|^\alpha}{p\beta^{\frac{p}{p-1}}} \left(\int_0^{|\Omega|} f^*(s)\, ds \right)^{\frac{p}{p-1}} \\
&= \int_0^{+\infty} \left(\int_0^{\phi(t)} f^*(s)\, ds \right)^{\frac{1}{p-1}} \left(-\phi'(t) + \frac{1}{\beta^{\frac{1}{p-1}}} \int_{\partial V_t^{ext}} \frac{1}{v}\, d\mathcal{H}^{n-1}(x) \right) t^{p-1} \phi(t)^\alpha\, dt \\
&= \gamma_n \int_0^{+\infty} t^{p-1} \phi^{\frac{1}{k}}(t)\, dt = \gamma_n \int_0^{+\infty} t^{p-1} \mu^{\frac{1}{k}}(t)\, dt.
\end{aligned}$$

□

In the following Lemma we prove that a solution to a Dirichlet problem, such that its distribution function satisfies the differential equation (2.48), is necessarily defined on a ball and it has to be radial and decreasing.

Lemma 2.2.3. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and Lipschitz set. Let $f \in L^{p'}(\Omega)$ be a positive function, let w be a weak solution to*

$$\begin{cases} -\Delta_p w = f & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.47}$$

and let σ be the distribution function of w. If σ satisfies the following condition

$$\gamma_n \sigma(t)^{\left(1-\frac{1}{n}\right)\frac{p}{p-1}} = \left(\int_0^{\sigma(t)} f^*(s)\,ds\right)^{\frac{1}{p-1}} (-\sigma'(t)), \quad \textit{for a.e. } t \in [0, w_M] \tag{2.48}$$

then, there exists x_0 such that

$$\Omega = \Omega^\sharp + x_0, \quad w(\cdot + x_0) = w^\sharp(\cdot), \quad f(\cdot + x_0) = f^\sharp(\cdot).$$

Proof. First of all, we recall that w is a weak solution to (2.47) if and only if

$$\int_\Omega |\nabla w|^{p-2} \nabla w \nabla \varphi \, dx = \int_\Omega f\varphi \, dx, \qquad \forall \varphi \in W_0^{1,p}(\Omega). \tag{2.49}$$

Arguing as in the proof of (2.9) in Lemma 2.1.2, choosing the same test function φ, defined in (2.11),

$$\varphi(x) = \begin{cases} 0 & \text{if } w < t \\ w - t & \text{if } t < w < t + h \\ h & \text{if } w > t + h, \end{cases}$$

one obtains

$$\int_{\partial W_t} |\nabla w|^{p-1} \, d\mathcal{H}^{n-1}(x) = \int_{W_t} f(x)\,dx \leq \int_0^{\sigma(t)} f^\star(s)\,ds, \tag{2.50}$$

where $W_t = \{x \in \Omega \ : w(x) > t\}$. Let us observe that the function φ is a $W_0^{1,p}(\Omega)$ function as the level sets of w do not intersect the boundary of Ω.

If we apply the isoperimetric inequality to the superlevel set W_t, the Hölder inequality and the Hardy-Littlewood inequality, we get, for almost every t,

$$n\omega_n^{\frac{1}{n}} \sigma(t)^{\frac{n-1}{n}} \leq P(W_t) = \int_{\partial W_t} d\mathcal{H}^{n-1}(x) \tag{2.51}$$

$$\leq \left(\int_{\partial W_t} |\nabla w|^{p-1} \, d\mathcal{H}^{n-1}(x)\right)^{\frac{1}{p}} \left(\int_{\partial W_t} \frac{1}{|\nabla w|} \, d\mathcal{H}^{n-1}(x)\right)^{1-\frac{1}{p}} \tag{2.52}$$

$$\leq \left(\int_0^{\sigma(t)} f^*(s)\,ds\right)^{\frac{1}{p}} (-\sigma'(t))^{1-\frac{1}{p}}. \tag{2.53}$$

So, hypothesis (2.48) ensures us that equality holds in the isoperimetric inequality (2.51), in the Hölder inequality (2.52) and in the Hardy-Littlewood inequality (2.53).
We now divide the proof into three steps.
Step 1. Let us prove that the superlevel set $\{ w > t \}$ is a ball for all $t \in [0, w_M)$. Equality in (2.51) implies that, for almost every t, W_t is a ball. On the other hand, for all $t \in [0, w_M)$, there exists a sequence $\{ t_k \}$ such that

1. $t_k \to t$;
2. $t_k > t_{k+1}$;

3. $\{w > t_k\}$ is a ball for all k.

Since $\{ w > t \} = \cup_k \{ w > t_k \}$ can be written as an increasing union of balls, $\{w > t\}$ is a ball for all t and, in particular, $\Omega = \{w > 0\}$ is a ball too and we obtain that $\Omega = x_0 + \Omega^\sharp$.
From now on, we can assume without loss of generality that $x_0 = 0$.
Step 2. Let us prove that the superlevel sets are concentric balls.
Equality in (2.52) implies also equality in Hölder inequality, i.e.

$$\int_{\partial W_t} d\mathcal{H}^{n-1}(x) = \left(\int_{\partial W_t} |\nabla w|^{p-1}\, d\mathcal{H}^{n-1}(x)\right)^{\frac{1}{p}} \left(\int_{\partial W_t} \frac{1}{|\nabla w|}\, d\mathcal{H}^{n-1}(x)\right)^{1-\frac{1}{p}}.$$

This means that, for almost every t, $|\nabla w|$ is constant $\mathcal{H}^{n-1}-$almost everywhere on ∂W_t , and we denote by C_t the ($\mathcal{H}^{n-1}-$a.e.) constant value of $|\nabla w|$ on ∂W_t. We claim that $C_t \neq 0$ for almost every t. Indeed, (2.50) and the positivity of f ensure us that

$$P(W_t)C_t^{p-1} = \int_{\partial W_t} |\nabla w|^{p-1}\, d\mathcal{H}^{n-1}(x) = \int_{W_t} f(x)\, dx > 0.$$

Integrating (2.48), we obtain $w^\sharp(x) = z(x)$, for all $x \in \Omega^\sharp$, where z is the solution to

$$\begin{cases} -\Delta_p z = f^\sharp & \text{in } \Omega^\sharp \\ z = 0 & \text{on } \partial\Omega^\sharp, \end{cases} \tag{2.54}$$

and it has the following explicit form:

$$z(x) = \int_{\omega_n |x|^n}^{|\Omega|} \frac{1}{\gamma_n} \left(\int_0^s f^\star(r)\, dr\right)^{1/(p-1)} \frac{1}{s^{(1-1/n)(p/(p-1))}}\, ds,$$

so it easily follows that

$$\left|\left\{ |\nabla w^\sharp| = 0 \right\} \cap \left\{ 0 < w^\sharp < w_M \right\}\right| = 0. \tag{2.55}$$

Using (1.5) in Lemma 1.2.1, we have that (2.55) implies the absolute continuity of σ.
Now, we denote by $C_t^\sharp$ the ($\mathcal{H}^{n-1}-$a.e.) constant value of $\left|\nabla w^\sharp\right|$ on $\partial W_t^\sharp$. We recall that it holds

$$-\sigma'(t) = \int_{\partial W_t^\sharp} \frac{1}{|\nabla w^\sharp|} = \frac{P(\partial W_t^\sharp)}{C_t^\sharp}.$$

and, by the absolute continuity of σ, we have

$$-\sigma'(t) = \int_{\partial W_t} \frac{1}{|\nabla w|} = \frac{P(\partial W_t)}{C_t}.$$

Since w and $w^\sharp$ are equi-distributed, we have,

$$\frac{P(\partial W_t)}{C_t} = \frac{P(\partial W_t^\sharp)}{C_t^\sharp}$$

Moreover, since $P(\partial W_t) = P(\partial W_t^\sharp)$, we have that $C_t = C_t^\sharp$. So, by the coarea formula, we get

$$\int_\Omega |\nabla w|^p\,dx = \int_0^{+\infty}\int_{\partial W_t} |\nabla w|^{p-1}\,d\mathcal{H}^{n-1}(x) = \int_0^{+\infty} C_t^{p-1} P(W_t)\,dt\,d\mathcal{H}^{n-1}(x)$$
$$= \int_0^{+\infty} \left(C_t^\sharp\right)^{p-1} P(W_t)\,dt\,d\mathcal{H}^{n-1}(x) = \int_0^{+\infty}\int_{\partial W_t^\sharp} |\nabla w^\sharp|^{p-1}\,d\mathcal{H}^{n-1}(x) = \int_{\Omega^\sharp} |\nabla w^\sharp|^p\,dx.$$

By Lemma 1.2.4, we conclude that $w = w^\sharp$.

Step 3. Let us prove that f is radial and decreasing.

Equality in (2.53) reads, for almost every t,

$$\int_{W_t} f(x)\,dx = \int_0^{\sigma(t)} f^*(s)\,ds.$$

Moreover, for all $\tau \in [0, w_M)$, there exists a sequence $\{\tau_k\}$ such that

1. $\tau_k \to \tau$;
2. $\tau_k > \tau_{k+1}$;
3. $\displaystyle\int_{W_{\tau_k}} f(x)\,dx = \int_0^{\sigma(\tau_k)} f^*(s)\,ds,$

and, by the continuity of $\sigma(\cdot)$, we have

$$\int_0^{\sigma(\tau)} f^*(s)\,ds = \lim_k \int_0^{\sigma(\tau_k)} f^*(s) = \lim_k \int_{W_{\tau_k}} f(x)\,dx = \int_{W_\tau} f(x)\,dx.$$

By Lemma 1.2.2, we have that for all τ, there exists α_τ such that

$$\{w > \tau\} = \{f > \alpha_\tau\}.$$

Consequently, we have that also f is radial and decreasing, so $f = f^\sharp$. □

As a direct consequence of Lemma 2.2.3, we obtain the rigidity for the $p-$Laplace operator with Dirichlet boundary conditions.

Corollary 2.2.4. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and Lipschitz set. Let $f \in L^{p'}(\Omega)$ be a positive function and let w and z be weak solutions respectively to*

$$\begin{cases} -\Delta_p w = f & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta_p z = f^\sharp & \text{in } \Omega^\sharp \\ z = 0 & \text{on } \partial\Omega^\sharp. \end{cases}$$

If $w^\sharp(x) = z(x)$, for all $x \in \Omega^\sharp$, then there exists $x_0 \in \mathbb{R}^n$ such that

$$\Omega = \Omega^\sharp + x_0, \qquad w(\cdot + x_0) = z(\cdot), \qquad f(\cdot + x_0) = f^\sharp(\cdot).$$

Proof. From the proof of Lemma 2.2.3, it follows that the distribution function of w, denoted by σ, satisfies

$$n\omega_n^{\frac{1}{n}}\sigma(t)^{\frac{n-1}{n}} \leq \left(\int_0^{\sigma(t)} f^*(s)\,ds\right)^{\frac{1}{p}} (-\sigma'(t))^{1-\frac{1}{p}}. \tag{2.56}$$

Now, we integrate (2.56) from 0 to t, obtaining

$$w^*(t) = \int_{\sigma(t)}^{|\Omega|} \frac{1}{\gamma_n}\left(\int_0^s f^\star(r)\,dr\right)^{1/(p-1)} \frac{1}{s^{(1-1/n)(p/(p-1))}}\,ds = z^*(t).$$

So, if $w^\sharp = z$, we have $w^* = z^*$, and consequently we obtain equality in (2.56) for almost every $t \in [0, w_M]$. We can conclude by applying Lemma 2.2.3.

□

To conclude the proof of Theorem 2.2.1, we need the following

Proposition 2.2.5. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and Lipschitz set and let $\Omega^\sharp$ be the ball with the same measure as Ω. Let u be the solution to (2.1) and let μ be its distribution function. If equality holds in (2.9), then there exists $x_0 \in \mathbb{R}^n$ such that*

$$\Omega = \Omega^\sharp + x_0, \qquad u(\cdot + x_0) = v(\cdot), \qquad f(\cdot + x_0) = f^\sharp(\cdot).$$

Proof. Firstly, we claim that the superlevel sets $\{\, u > t \,\}$ are balls for every $t \in [0, u_M)$. Equality in (2.9) implies the equality in (2.14), i.e.

$$n\omega_n^{\frac{1}{n}}\mu(t)^{\frac{n-1}{n}} = P(U_t), \quad \text{for a. e. } t \in [0, u_M]$$

that means that almost every superlevel set is a ball. Arguing as in Step 1 of Lemma 2.2.3, we can conclude that every superlevel set is a ball, so, $\Omega = \{u > u_m\}$ is a ball and we obtain that $\Omega = x_0 + \Omega^\sharp$.

Let us observe that for every $t, s \in [u_m, u_M]$ with $t < s$, as both U_t and U_s are balls, we have that $\partial U_t \cap \partial U_s$ contains at most one point. In particular, the function $w = u - u_m$ is a weak solution to the Dirichlet problem (2.47) in Ω.

We claim that $\sigma(t) = |\{\, w > t \,\}|$ satisfies (2.48). Since $\{\, w > t \,\} = \{\, u > t + u_m \,\}$, we have $\sigma(t) = \mu(t + u_m)$ for all $t \in [0, u_M - u_m]$. Moreover, we have

$$\int_{\partial U_t} \frac{1}{u}\,d\mathcal{H}^{n-1}(x) = 0, \quad \forall t > u_m$$

So, using the fact that we have equality in (2.9) by hypothesis, we get

$$\begin{aligned}
\gamma_n\sigma(t)^{\left(1-\frac{1}{n}\right)\frac{p}{p-1}} &= \gamma_n\mu(t+u_m)^{\left(1-\frac{1}{n}\right)\frac{p}{p-1}} \\
&= \left(\int_0^{\mu(t+u_m)} f^*(s)\,ds\right)^{\frac{1}{p-1}} \left(-\mu'(t+u_m) + \frac{1}{\beta^{\frac{1}{p-1}}}\int_{\partial U^{\text{ext}}_{t+u_m}} \frac{1}{u}\,d\mathcal{H}^{n-1}(x)\right) \\
&= \left(\int_0^{\sigma(t)} f^*(s)\,ds\right)^{\frac{1}{p-1}} (-\sigma'(t)),
\end{aligned}$$

for all $t \in (0, u_M - u_m)$. So, we can conclude by Lemma 2.2.3. □

We can finish now with the proof of the main Theorem.

Proof of Theorem 2.2.1. From Proposition 2.2.2, we have that the hypothesis of Theorem 2.2.1

$$\|u\|_{L^{pk,p}(\Omega)} = \|v\|_{L^{pk,p}(\Omega^\sharp)}, \quad \text{for some } k \in \left]0, \frac{n(p-1)}{(n-2)p+n}\right]$$

implies the following equality for almost every $t \in (0, u_M)$

$$\gamma_n \mu(t)^{\left(1-\frac{1}{n}\right)\frac{p}{p-1}} = \left(\int_0^{\mu(t)} f^*(s)\, ds\right)^{\frac{1}{p-1}} \left(-\mu'(t) + \frac{1}{\beta^{\frac{1}{p-1}}} \int_{\partial U_t^{\text{ext}}} \frac{1}{u}\, d\mathcal{H}^{n-1}(x)\right),$$

where $\mu(t)$ is the distribution function of u.

Now, we are in position to apply Proposition 2.2.5, and, so, there exists $x_0 \in \mathbb{R}^n$ such that

$$\Omega = \Omega^\sharp + x_0, \qquad u(\cdot + x_0) = v(\cdot), \qquad f(\cdot + x_0) = f^\sharp(\cdot).$$

□

Remark 2.2.1. In Theorem 2.1.5, we also prove that in the case $f \equiv 1$, it holds

$$\|u\|_{L^{pk,p}(\Omega)} \le \|v\|_{L^{pk,p}(\Omega^\sharp)}, \qquad \text{if } 0 < k \le \frac{n(p-1)}{n(p-1)-p}.$$

We stress that the proof of Theorem 2.2.1 can be adapted to the case $f \equiv 1$, regardless of the fact that now the admissible k varies in a wider range.

2.3 Further remarks

We extended the results obtained for the Laplacian in [8] to the $p-$Laplacian. Many problems remain open, such as

Open Problem In the assumptions of Theorem 2.1.5, does the point-wise comparison hold also for $p > \frac{n}{n-1}$?

We have already observed in the corollary 2.1.6 that if $p \ge n$ we have an estimate on the L^p norms of u and v. Can we generalize this estimate also for $q \ne p$? We know for sure that for $q = \infty$ this can't be done, as it can be seen in the following example.

Example 2.3.1. Let $\Omega \subseteq \mathbb{R}^n$ be the union of two disjoint balls, B_1 and B_r with radii 1 and r respectively. We choose $\beta < \left(\frac{n-1}{p-1}\right)^{p-1}$ with $p \ne n$, and we fix $f = 1$ on B_1 and $f = 0$ on B_r. Both u and v can be explicitly computed. We have $\|u\|_\infty - \|v\|_\infty = Cr^n + o(r^n)$, where C is a positive constant.

Proof. We want an explicit expression of u and v respectively. Starting from u, it is a solution to

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + \beta |u|^{p-2} u = 0 & \text{on } \partial\Omega. \end{cases}$$

with $f|_{B_1} = 1$ and $f|_{B_r} = 0$.

It's clear that $u|_{B_r} = 0$ and $u(x) = u(|x|)$ it's radial on B_1.

So the equation (2.1) becomes

$$s^{n-1}\Delta_p u(s) = \frac{d}{ds}\left(s^{n-1}|u'(s)|^{p-2}u'(s)\right),$$

and then

$$\frac{d}{ds}\left(s^{n-1}|u'(s)|^{p-2}u'(s)\right) = s^{n-1}\Delta_p u(s) = -s^{n-1}$$
$$s^{n-1}|u'(s)|^{p-2}u'(s) = -\frac{s^n}{n} + c.$$

We set $c = 0$, in order to have a C^1-solution.

$$|u'(s)|^{p-2}u'(s) = -\frac{s}{n} \implies u'(s) = -\frac{s^{\frac{1}{p-1}}}{n^{\frac{1}{p-1}}}, \qquad \alpha = \frac{1}{p-1}.$$

If we integrate, we obtain

$$u(s) = -\frac{p-1}{n^\alpha p}s^{\frac{p}{p-1}} + A.$$

The Robin boundary conditions become

$$|u'(1)|^{p-2}u'(1) + \beta u(1)^{p-1} = 0 \quad (u \geq 0),$$

now we can compute the value of A

$$-\frac{1}{n} + \beta\left(-\frac{p-1}{n^\alpha p} + A\right)^{p-1} = 0 \implies A = \frac{1}{(n\beta)^\alpha} + \frac{p-1}{n^\alpha p}.$$

So

$$u(s) = \frac{p-1}{n^\alpha p}\left(1 - s^{\frac{p}{p-1}}\right) + \frac{1}{(n\beta)^\alpha}.$$

As u is decreasing, we have

$$\|u\|_\infty = u(0) = \frac{p-1}{n^\alpha p} + \frac{1}{(n\beta)^\alpha}.$$

Now, let us compute $v(s)$. We will do this firstly for $s \in (0,1)$, then for $s \in (1,\overline{r})$ where $\overline{r} = (1+r^n)^{\frac{1}{n}}$ is determined by the condition $|\Omega| = |\Omega^\sharp|$.

Let $s < 1$

$$\frac{d}{ds}\left(s^{n-1}|v'(s)|^{p-2}v'(s)\right) = -s^{n-1},$$

$$|v'(s)|^{p-2}v'(s) = -\frac{s}{n} \implies v'(s) = -\frac{s^{\frac{1}{p-1}}}{n^\alpha},$$

$$v(s) = -\frac{p-1}{n^\alpha p}s^{\frac{p}{p-1}} + B.$$

Now we can't determine B as before, as v is not identically 0 in the annulus $B_{\overline{r}} \backslash B_1$.

Let $s > 1$ and $p \neq n$

$$\frac{d}{ds}\left(s^{n-1}|v'(s)|^{p-2}v'(s)\right) = 0$$

$$|v'(s)|^{p-2}v'(s) = \frac{C}{s^{n-1}}$$

by imposing the continuity of the derivative for $s = 1$, we obtain that $C = -1/n$

$$v'(s) = -\frac{s^{-\frac{n-1}{p-1}}}{n^{\alpha}},$$

$$v(s) = -\frac{p-1}{n^{\alpha}(p-n)}s^{\frac{p-n}{p-1}} + D,$$

and by Robin conditions

$$|v'(\overline{r})|^{p-2}v'(\overline{r}) + \beta v(\overline{r})^{p-1} = 0,$$

$$-\frac{\overline{r}^{-n-1}}{n} + \beta\left(-\frac{p-1}{n^{\alpha}(p-n)}\overline{r}^{\frac{p-n}{p-1}} + D\right)^{(p-1)} = 0,$$

$$D = \frac{1}{(n\beta)^{\alpha}}\overline{r}^{-\frac{n-1}{p-1}} + \frac{p-1}{n^{\alpha}(p-n)}\overline{r}^{\frac{p-n}{p-1}}.$$

By imposing the continuity of v for $s = 1$, we have

$$B = \frac{p-1}{n^{\alpha}p} + \frac{\overline{r}^{-\frac{n-1}{p-1}}}{(n\beta)^{\alpha}} + \frac{p-1}{n^{\alpha}(p-n)}\left(\overline{r}^{\frac{p-n}{p-1}} - 1\right),$$

that is to say

$$v(s) = \begin{cases} u(s) + \frac{1}{(n\beta)^{\alpha}}\left(\overline{r}^{-\frac{n-1}{p-1}} - 1\right) + \frac{p-1}{n^{\alpha}(p-n)}\left(\overline{r}^{\frac{p-n}{p-1}} - 1\right) & \text{if } s < 1 \\ \frac{1}{(n\beta)^{\alpha}}\overline{r}^{-\frac{n-1}{p-1}} + \frac{p-1}{n^{\alpha}(p-n)}\left(\overline{r}^{\frac{p-n}{p-1}} - s^{\frac{p-n}{p-1}}\right) & \text{if } 1 < s < \overline{r}. \end{cases}$$

For convenience's sake, we set $h = \frac{1}{(n\beta)^{\alpha}}\left(\overline{r}^{-\frac{n-1}{p-1}} - 1\right) + \frac{p-1}{n^{\alpha}(p-n)}\left(\overline{r}^{\frac{p-n}{p-1}} - 1\right)$.

So we have

$$\|v\|_{L^{\infty}(\Omega^{\sharp})} = \|v\|_{L^{\infty}(B_1)} = \|u\|_{L^{\infty}(\Omega)} + h = u(0) + h.$$

By using Taylor expansion of the function $(1 + r^n)^{\delta}$ we get

$$h = \left(-\frac{1}{(n\beta)^{\alpha}}\frac{n-1}{n(p-1)} + \frac{1}{n^{\alpha+1}}\right)r^n + o(r^n),$$

so, if we choose $\beta < \left(\frac{n-1}{p-1}\right)^{p-1}$, we get

$$\|v\|_{L^{\infty}(\Omega^{\sharp})} = \|u\|_{L^{\infty}(\Omega)} - Cr^n + o(r^n) \text{ where } C > 0.$$

Next example 2.3.2 is a counterexample to the corollary 2.1.6 in the case $n > p$.

Example 2.3.2. Let $\Omega \subseteq \mathbb{R}^n$, $p < n$ be the union of two disjoint balls B_1 and B_r with radii 1 and r respectively. We choose $\beta \leq \left(\frac{n-p}{p(p-1)}\right)^{p-1}$ and we fix $f = 1$ on B_1 and $f = 0$ on B_r. Both u and v can be explicitly computed. We have $\|u\|_p^p - \|v\|_p^p = Cr^n + o(r^n)$, where C is a positive constant.

Proof. Let us consider the Taylor expansion of $(1+y)^p$, we get

$$\|v\|^p_{L^p(B_1)} = \int_{B_1} (u+h)^p = \|u\|^p_{L^p(B_1)} + p\|u\|^{p-1}_{L^{p-1}(B_1)} h + o(r^n).$$

Moreover,

$$\|v\|^p_{L^p(B_{\overline{r}}\setminus B_1)} = \frac{\omega_n}{(n\beta)^{\alpha p}} r^n + o(r^n),$$

as if $1 < s < \overline{r}$

$$\frac{1}{(n\beta)^\alpha} \overline{r}^{-\frac{n-1}{p-1}} \leq v(s) \leq \frac{1}{(n\beta)^\alpha} \overline{r}^{-\frac{n-1}{p-1}} + \frac{p-1}{n^\alpha (p-n)} \left(\overline{r}^{\frac{p-n}{p-1}} - 1 \right),$$

thus

$$v(s) = \frac{1}{(n\beta)^\alpha} + O(r^n),$$

and by integration we obtain the value of the norm in $L^p(B_{\overline{r}} \setminus B_1)$.

So

$$\|v\|^p_{L^p(\Omega^\sharp)} = \|v\|^p_{L^p(B_1)} + \|v\|^p_{L^p(B_{\overline{r}}\setminus B_1)} = \|u\|^p_{L^p(\Omega)} + p\|u\|^{p-1}_{L^{p-1}(B_1)} h + \frac{\omega_n}{(n\beta)^{\alpha p}} r^n + o(r^n),$$

and recalling that

$$h = \left(-\frac{1}{(n\beta)^\alpha} \frac{n-1}{n(p-1)} + \frac{1}{n^{\alpha+1}} \right) r^n + o(r^n),$$

we get

$$\|v\|^p_{L^p(\Omega^\sharp)} = \|u\|^p_{L^p(\Omega)} + \left[p\|u\|^{p-1}_{L^{p-1}(B_1)} \left(-\frac{1}{(n\beta)^\alpha} \frac{n-1}{n(p-1)} + \frac{1}{n^{\alpha+1}} \right) + \frac{\omega_n}{(n\beta)^{\alpha p}} \right] r^n + o(r^n).$$

We have to understand whether

$$p\|u\|^{p-1}_{L^{p-1}(B_1)} \left(-\frac{1}{(n\beta)^\alpha} \frac{n-1}{n(p-1)} + \frac{1}{n^{\alpha+1}} \right) + \frac{\omega_n}{(n\beta)^{\alpha p}} < 0. \tag{2.57}$$

If we choose $\beta < \left(\frac{n-1}{p-1}\right)^{p-1}$ we have $-\frac{1}{(n\beta)^\alpha} \frac{n-1}{n(p-1)} + \frac{1}{n^{\alpha+1}} < 0$. In order to have (2.57), we need

$$\|u\|^{p-1}_{L^{p-1}(B_1)} > \frac{\omega_n}{(n\beta)^{\alpha p}} \left[\frac{n(p-1)n^\alpha \beta^\alpha}{p(n-1) - p\beta^\alpha (p-1)} \right],$$

$$\|u\|^{p-1}_{L^{p-1}(B_1)} > \frac{\omega_n}{(n\beta)^{\alpha (p-1)}} \left[\frac{n(p-1)}{p(n-1) - p\beta^\alpha (p-1)} \right].$$

If we show that

$$\left[\frac{n(p-1)}{p(n-1) - p\beta^\alpha (p-1)} \right] \leq 1, \tag{2.58}$$

then

$$u(s) > \frac{1}{(n\beta)^\alpha} \implies \|u\|^{p-1}_{L^{p-1}(B_1)} > \frac{\omega_n}{(n\beta)^{\alpha(p-1)}}.$$

We just have to verify (2.58)

$$\begin{aligned}
\left[\frac{n(p-1)}{p(n-1)-p\beta^\alpha(p-1)}\right] \leq 1 &\iff n(p-1) \leq p(n-1) - p\beta^\alpha(p-1) \\
&\iff p-n \leq -p(p-1)\beta^\alpha < 0 \quad \text{(if and only if } p<n\text{!)} \\
&\iff \beta \leq \left(\frac{n-p}{p(p-1)}\right)^{p-1}
\end{aligned}$$

2.4 A remark on solutions to semilinear equations

One of the applications of symmetrization techniques introduced by Talenti in [152] can be found in a paper by Lions [113]. In this paper, the author considers the following semilinear problem

$$\begin{cases} -\Delta u = f(u) & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{2.59}$$

where B is a ball of $\mathbb{R}^2$ of radius R, that was already studied by Gidas, Ni, and Nirenberg in the celebrated paper [88, Theorem 1]. They proved the following result in $\mathbb{R}^n$.

Theorem 2.4.1 (Gidas-Ni-Nirenberg)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f = f_1 + f_2$, where f_1 is a locally Lipschitz function and f_2 is non-decreasing. Then, any positive solution $u \in C^2(\overline{B})$ to the problem* (2.59)*, where B is a ball of $\mathbb{R}^n$ of radius R, has to be radial and*

$$\frac{\partial u}{\partial r} < 0, \quad \text{for } 0 < r < R.$$

In order to prove this result the authors make use of the method of moving planes, first introduced by Aleksandrov in [2], and then applied by Serrin in [147] in the context of PDEs.

In the case $n = 2$, Lions in [113] gives an alternative proof of Theorem 2.4.1, that allows considering weaker smoothness assumptions on f and u, under the additional hypothesis $f \geq 0$. The technique used in [113] relies on the Schwartz symmetrization, the isoperimetric inequality and the Pohozaev identity. His techniques were applied and generalized in [107] to the context of the $n-$Laplacian, and, eventually, in [146] to the solutions to $p-$Laplace equation for any p, in any dimension $n \geq 2$.

We briefly recall the outline of Lion's proof of the radial symmetry of any positive solution to (2.59).

Let u be a positive solution to (2.59) and let us denote by $\mu(t)$ the distribution function of u. By exploiting the proof of Lemma 2.2.3, one can infer

$$4\pi\mu(t) \leq (-\mu'(t)) \int_{U_t} f(u)\, dx. \tag{2.60}$$

If we multiply by $f(t)$ and integrate between 0 and $M > \max_B u$, we get

$$4\pi \int_B F(u)\, dx = 4\pi \int_0^M f(t)\mu(t)\, dt \leq \int_0^M \left(\int_{U_t} f(u)\, dx \right) f(t)(-\mu'(t))\, dt \leq \frac{1}{2} \left(\int_B f(u)\, dx \right)^2$$

where $F(t) = \displaystyle\int_0^t f(s)\, ds$, and in the last inequality we have used that

$$t \longrightarrow \int_{U_t} f(u)\, dx$$

is non-increasing. On the other hand, the following well-known identity holds, known as the *Pohozaev Identity* (see [130])

$$\int_B F(u)\, dx = \frac{R}{4} \int_{\partial B} \left(\frac{\partial u}{\partial \nu} \right)^2 d\mathcal{H}^1(x),$$

that implies

$$\pi R \int_{\partial B} \left(\frac{\partial u}{\partial \nu}\right)^2 d\mathcal{H}^1(x) = 4\pi \int_B F(u)\,dx \le \frac{1}{2}\left(\int_B f(u)\,dx\right)^2.$$

Eventually, if we apply the Holder inequality to the right-hand side

$$\left(\int_B f(u)\,dx\right)^2 = \left(\int_{\partial B} \frac{\partial u}{\partial \nu} d\mathcal{H}^1(x)\right)^2 \le \left(\int_{\partial B} \left(\frac{\partial u}{\partial \nu}\right)^2 d\mathcal{H}^1(x)\right) 2\pi R,$$

everything reduces to equality and, in particular, equality holds in (2.60). One can conclude by applying Lemma (2.2.3) to deduce the radial symmetry of u.

The proof by Lion allows us to obtain the result of Gidas, Ni and Nirenberg under weaker assumptions. Anyway, the result contained in [88] is a milestone in proving symmetry results: in the linear case see, for instance, [24, 23, 75, 135] and, for the symmetry results in the case of the $p-$Laplace operator, we refer to [21, 60, 61, 41]. In all the afore-mentioned papers, it is possible to prove the radiality of positive solution to (2.59) either under the regularity hypothesis on f stated in Theorem 2.4.1 (using the moving plane method as in [88]) or under the assumption $f \ge 0$ (with symmetrization techniques as in [113, 146, 107]). For a sign changing f, the Lipschitz continuity property cannot be relaxed to Hölder continuity, as remarked in [88, Section 2.3]. Indeed, in this case, the authors find a solution to (2.59) that is not radially symmetric, see also [42] for the $p-$Laplacian case.

One could wonder if a similar analysis can be done also for positive solutions to the Robin problem

$$\begin{cases} -\Delta u = f(u) & \text{in } B \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial B \end{cases} \tag{2.61}$$

whenever β is a positive parameter. Up to our knowledge, in the literature, there are few results dealing with symmetry properties of the solutions to differential equations with Robin boundary conditions. For instance, in [25], the authors consider the following problem

$$\begin{cases} -\varepsilon^2 \Delta u = f(u) - u & \text{in } B \\ \varepsilon \dfrac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial B \end{cases} \tag{2.62}$$

where $\varepsilon > 0$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, of the form $f = f_1 - f_2$ for $t \ge 0$, with $f_1, f_2 \ge 0$, satisfying some structural growth conditions (for the precise details see [25, Section 1]). Under these assumptions, it is possible to prove the existence of a positive solution to (2.62) by making use of the Palais-Smale condition and the mountain pass Lemma. The authors show that there exists a $\beta_* > 1$, such that, for $\beta > \beta_*$, the least energy solution has the maximum in the center of the ball, while, for $\beta \le \beta_*$ and $\varepsilon \to 0$, the least energy solution has the maximum near the boundary, and, consequently, it cannot be radially symmetric. For completeness sake, we recall that the case $\beta = 0$, i.e. the Neumann problem, and the case $\beta = +\infty$, i.e the Dirichlet problem, have been studied, for instance, respectively, in [122, 123] and [124]. Moreover, these results hold in the more general case of $\Omega \subset \mathbb{R}^n$ bounded and smooth, as well as the one proved in [25] for the Robin boundary conditions.

As we proved inequality (2.9) for solutions to (2.61) and a Pohozaev type identity for solutions to (2.61) holds, see for instance [68, Proposition 2.8]

$$2\int_B F(u)\,dx = R\int_{\partial B}\left(\frac{\partial u}{\partial \nu}\right)^2 - \frac{1}{2}|\nabla u|^2 + F(u)\,d\mathcal{H}^1(x),$$

one can try to adapt Lion's proof in the Robin case. anyway, in Theorem 2.4.2, we show that in the one-dimensional case, the symmetry result for the solution to (2.61) holds under the standard hypotheses of Gidas, Ni, Nirenberg and the additional hypothesis $f \geq 0$. On the other hand, this is not the case for $n \geq 2$, as pointed out in Corollary 2.4.4:

> *In dimension $n \geq 2$, there exists a positive superharmonic function φ that is a solution to (2.61) and that is not radially symmetric.*

Let us start by analyzing the one-dimensional case, in which the symmetry holds.

Theorem 2.4.2. *Let $R > 0$ and let $I =]-R, R[$ be the open ball of radius R. Let $u \in C^2([-R, R])$ be a solution to*

$$\begin{cases} -u'' = f(u) & \text{in } I, \\ \dfrac{\partial u}{\partial \nu}(x) + \beta u(x) = 0 & \text{in } x = \pm R, \end{cases}$$

where $\beta > 0$. Let us assume that f satisfies the following assumptions:

(i). $f \geq 0$ in $\mathbb{R}$, f is not identically zero in $u(I)$,

(ii). $f = f_1 + f_2$, where f_1 is locally Lipschitz in $\mathbb{R}$ and f_2 is non decreasing.

Then, $u(x) = u(-x)$ for all $x \in [-R, R]$. Moreover,

$$u'(x) < 0, \quad x \in [0, R].$$

Proof. We divide the proof in two steps. In the first step, we prove that the function u is strictly positive and, in the second one, we prove that we can apply the result contained in Theorem 2.4.1.

Step 1. We start by proving that $u > 0$ in $[-R, R]$. Since $u'' \leq 0$, u' is non increasing in $]-R, R[$, so the minimum of u on $[-R, R]$ is achieved either in $-R$ or in R. Let us denote by x_m the minimum point of u in $[-R, R]$. From the Robin boundary conditions, we have that

$$-\beta u(x_m) = \frac{\partial u}{\partial \nu}(x_m) \leq 0,$$

and, as a consequence, $u \geq 0$ in $[-R, R]$.

Now we want to prove that $u > 0$ in $[-R, R]$. By contradiction, we assume that $u(x_m) = 0$. If $x_m = -R$, the Robin boundary conditions imply

$$0 = \beta u(-R) = -\frac{\partial u}{\partial \nu}(-R) = u'(-R) \geq u'(x), \quad \forall x \in I,$$

where we have observed that u' is non increasing. This implies that also u is a non increasing function, so $-R$ should be both a minimum and a maximum. This is not possible, since u should be constant and this contradicts the hypothesis $f \not\equiv 0$ in $u(I)$. Therefore, we have that $x_m = R$ and, arguing as before, we have

$$0 = -\beta u(R) = \frac{\partial u}{\partial \nu}(R) = u'(R) \leq u'(x), \quad \forall x \in I,$$

So u is a non-decreasing function, the point R is both a minimum and a maximum and we get a contradiction as before.

Step 2. We prove now that $u(R) = u(-R)$. Let us assume by contradiction that $u(R) \neq u(-R)$. Without loss of generality, we can suppose $u(R) < u(-R)$. As a consequence of Step 1, the function u is strictly increasing in a neighborhood of the point $-R$, so, by the continuity of u, there exists $y \in I$ such that $u(y) = u(-R)$. Therefore, the following quantity is well defined

$$\lambda := \inf\{t \in I \,:\, u(t) = u(-R)\}$$

and $\lambda > -R$. Moreover, the continuity of u also implies $u(\lambda) = u(-R)$ and $u(x) > u(-R)$ in $(-R, \lambda)$.

We define now the function $v := u - u(-R)$, that is a positive solution to

$$\begin{cases} -\Delta v = \tilde{f}(v) & \text{in } (-R, \lambda), \\ v = 0 & \text{in } x = -R, \ x = \lambda, \end{cases}$$

where $\tilde{f}(v) = f(v + u(R))$. So we can use Theorem 2.4.1 in the interval $(-R, \lambda)$. We have that u is symmetric with respect to the line $x = 2^{-1}(\lambda - R)$ and, as a consequence, we get

$$\frac{du}{d\nu}(-R) = -\frac{du}{dx}(-R) = \frac{du}{dx}(\lambda) < 0. \tag{2.63}$$

Using (2.63) and the fact that u' is non increasing, we obtain

$$\beta u(R) = -\frac{du}{d\nu}(R) = -\frac{du}{dx}(R) \geq -\frac{du}{dx}(\lambda) = -\frac{du}{d\nu}(-R) = \beta u(-R),$$

and, therefore,

$$u(R) \geq u(-R),$$

that is a contradiction.

From Step 1 and 2, we have that the funtion $v = u - u(R)$ is a positive solution to

$$\begin{cases} -\Delta v = \tilde{f}(v) & \text{in } I, \\ v = 0 & \text{in } x = \pm R. \end{cases}$$

So we can conclude by using Theorem 2.4.1.

□

We do not know if the hypotheses $i) - ii)$ on the function f are the optimal ones to obtain the symmetry result. Nevertheless, we will show in Remark 2.4.1 that the assumption $f \geq 0$ cannot be removed.

Theorem 2.4.3. *Let $B_R \subset \mathbb{R}^n$, $n \geq 2$, be the ball centered at the origin with radius R, let β be a positive constant and let $x_0 \neq 0$ in B_R. Then, there exists a the positive function $\varphi \in C^\infty(\overline{B_R})$ that is a non radial function in B_R (i.e. $\varphi(x) \neq \varphi(|x|)$) and it is a solution to*

$$\begin{cases} -\Delta \varphi = f(\varphi) & \text{in } B_R, \\ \dfrac{\partial \varphi}{\partial \nu} + \beta \varphi = 0 & \text{on } \partial B_R, \end{cases} \tag{2.64}$$

where

$$f(t) = c_1 t \left[c_2 t^{\frac{1}{\beta R}} + c_3 t^{\frac{2}{\beta R}} \right], \tag{2.65}$$

with $c_1, c_3 > 0$, $c_2 \in \mathbb{R}$ *defined as follows:*

$$c_1 = 2\beta R, \quad c_2 = -2(\beta R + 1) + n, \quad c_3 = 2(\beta R + 1)\alpha^2, \quad \alpha^2 = R^2 - |x_0|^2.$$

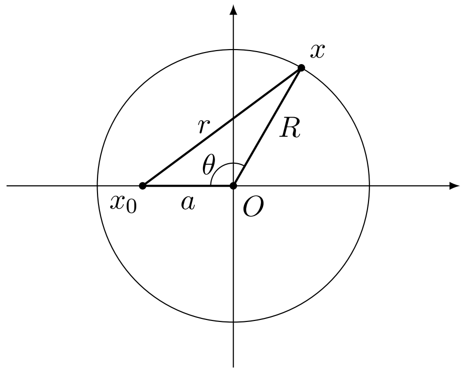

Figure 2.1: Construction of the function φ

Proof. We define the following quantities (see Figure 2.1):

- $a := |x_0|$,
- $\alpha^2 := R^2 - a^2 > 0$.

We show the existence of a positive function $\varphi(x) = \varphi(|x - x_0|) = \varphi(r)$ such that

$$\frac{\partial \varphi}{\partial \nu} + \beta \varphi = 0 \qquad \text{on } \partial B_R \tag{2.66}$$

where ν is the unit outer normal to ∂B_R. Let us fix $x \in \partial B_R$. Being

$$\nabla(\varphi(x)) = \varphi'(r)\, \frac{x - x_0}{r}, \qquad \nu(x) = \frac{x}{R},$$

the Robin boundary conditions (2.66) becomes

$$\varphi'(r)\, \frac{x - x_0}{r} \cdot \frac{x}{R} + \beta \varphi(r) = 0. \tag{2.67}$$

Denoting, now, by θ the angle between the vectors x_0 and x, we have the following relation

$$\cos(\theta) = \frac{R^2 + a^2 - r^2}{2aR}. \tag{2.68}$$

So, from (2.67) and (2.68), recalling that $\alpha^2 = R^2 - a^2$, we have

$$\frac{\varphi'(r)}{rR} \left(R^2 - Ra\cos(\theta) \right) + \beta\varphi(r) = \frac{\varphi'(r)}{2rR} \left(r^2 + \alpha^2 \right) + \beta\varphi(r) = 0,$$

and, therefore,

$$\frac{\varphi'(r)}{\varphi(r)} = -(2\beta R)\,\frac{r}{r^2+\alpha^2}. \tag{2.69}$$

Integrating (2.69), we get

$$\varphi(r) = \frac{c}{(r^2+\alpha^2)^{\beta R}}. \tag{2.70}$$

If we choose $c = 1$, we have

$$\varphi'(r) = -\frac{2\beta Rr}{(r^2+\alpha^2)^{\beta R+1}},$$
$$\varphi''(r) = -(-\beta R-1)\frac{4\beta Rr^2}{(r^2+\alpha^2)^{\beta R+2}} - \frac{2\beta R}{(r^2+\alpha^2)^{\beta R+1}},$$

and, consequently,

$$\begin{aligned}
-\Delta\varphi &= -\varphi''(r) - \frac{n-1}{r}\varphi'(r) = \\
&= \frac{2\beta R}{(r^2+\alpha^2)^{\beta R+1}}\left[n - 2(\beta R+1) + (\beta R+1)\frac{2\alpha^2}{r^2+\alpha^2}\right] \\
&= 2\beta R\varphi(r)^{\frac{1}{\beta R}+1}\left[n - 2(\beta R+1) + 2(\beta R+1)\alpha^2\varphi(r)^{\frac{1}{\beta R}}\right] \\
&= f(\varphi(r)),
\end{aligned}$$

where f is the function defined in (2.65). So, we have proved the desired claim, since we have found a non-radial function of the form $\varphi(x) = \varphi(|x-x_0|)$, defined in (2.70), that satisfies (2.64). □

As a consequence of Theorem 2.4.3, we obtain the following Corollary.

Corollary 2.4.4. *Let $n \geq 2$. There exists a positive superharmonic function φ that is a solution to* (2.61) *and that is not radially symmetric.*

Proof. In the case $n = 2$, the right-hand side of (2.65) becomes

$$f(t) = 4\beta Rt\left(-\beta Rt^{\frac{1}{\beta R}} + \alpha^2(1+\beta R)t^{\frac{2}{\beta R}}\right).$$

We notice that

$$f(t) \geq 0, \qquad \text{if } t \geq \left(\frac{\beta R}{\alpha^2(1+\beta R)}\right)^{\beta R}, \tag{2.71}$$

so the function $f\circ\varphi$ is positive if φ satisfies (2.71) for all $x \in B_R$, and this follows by imposing the following geometric constraint

$$\beta \leq \frac{R-a}{R(R+a)}.$$

If $n \geq 3$, we can choose the constant $c_2 \geq 0$, by imposing the condition

$$\beta \leq \frac{n-2}{2R} \tag{2.72}$$

and, under these assumptions, we have that $f(t) \geq 0$ for $t \geq 0$.

Therefore, we can see that, by imposing the geometrical constraints (2.71) and (2.72) respectively for $n = 2$ and $n \geq 3$, the function φ defined in (2.70) is an example of positive superharmonic function, that is non radial and that satisfies (2.64). □

We conclude with a remark for the one dimensional case.

Remark 2.4.1. The function φ defined in Theorem 2.4.3, in the case $n = 1$, satisfies the problem

$$\begin{cases} -\varphi'' = f(\varphi) & \text{in } (-R, +R), \\ \dfrac{\partial \varphi}{\partial \nu} + \beta\varphi = 0 & \text{in } x = \pm R. \end{cases}$$

We note that $\varphi \in C^{\infty}([-R, R])$ and f is a locally Lipschitz function, but, f does not satisfy the hypothesis $i)$, that is the positiveness. Indeed, by straightforward computations, we obtain that $f \circ \varphi$ is a sign changing function in $\varphi([-R, +R])$ for every $\beta > 0$ and $R > a > 0$.

Chapter 3

On the eigenvalues of the $\infty-$Laplace operator with Robin boundary conditions

The first eigenvalue of the $p-$Laplace operator is defined as the least real number λ for which the equation

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u = 0 & \text{in } \Omega \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial \nu} + \beta^p |u|^{p-2} u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

admits nontrivial weak solutions in a given bounded domain Ω. In this case, a weak solution is a function u_p in $W^{1,p}(\Omega)$ satisfying

$$\int_\Omega |\nabla u_p|^{p-2} \nabla u_p \nabla \varphi \, dx + \beta^p \int_{\partial\Omega} |u_p|^{p-2} u_p \varphi \, d\mathcal{H}^{n-1}(x) = \lambda \int_\Omega |u_p|^{p-2} u_p \varphi \, dx, \quad \forall \varphi \in W^{1,p}(\Omega),$$

The first eigenvalue can be also characterized as the minimum of the Rayleigh quotient

$$\lambda_{p,\beta}(\Omega) = \min_{\substack{\omega \in W^{1,p}(\Omega) \\ \omega \neq 0}} \frac{\displaystyle\int_\Omega |\nabla \omega|^p \, dx + \beta^p \int_{\partial\Omega} |\omega|^p \, d\mathcal{H}^{n-1}(x)}{\displaystyle\int_\Omega |\omega|^p \, dx}, \tag{3.2}$$

where $1 < p < \infty$, and in the case case $p = 2$ one obtains the principal frequency of a membrane Ω and the associated first eigenfunction u describes the shape of the membrane when it vibrates emitting its gravest sound.

In the first part of the chapter, we will focus on the behavior, when $p \to +\infty$, of the first $p-$Laplacian eigenvalues with Robin boundary conditions and the limit of the associated eigenfunctions. In the second part, as an application, we will study the $p-$Poisson equation when the datum f belongs to $L^\infty(\Omega)$.

The content of this chapter can be also found in [11].

In this Chapter, Ω is a bounded Lipschitz set in $\mathbb{R}^n$ and the boundary parameter β is positive and erased to the power p. As the set Ω is fixed, we will often denote $\lambda_{p,\beta^p}(\Omega)$ simply by λ_{p,β^p}.

3.1 The ∞-eigenvalue problem

Our aim is to show that the sequence of the $p-$eigenvalue satisfies

$$\lim_{p\to+\infty}(\lambda_{p,\beta^p})^{1/p}=\Lambda_{\infty,\beta}=:\inf_{\substack{w\in W^{1,\infty}(\Omega)\\ \|w\|_{L^\infty(\Omega)}=1}}\max\left\{\|\nabla w\|_{L^\infty(\Omega)},\beta\|w\|_{L^\infty(\partial\Omega)}\right\},\tag{3.3}$$

First of all, let us recall the following Lemma, whose proof can be found in [136].

Lemma 3.1.1. *Given $f,g\in W^{1,\infty}(\Omega)$, then*

$$\lim_{p\to\infty}\left(\int_\Omega |f|^p+\int_\Omega |g|^p\right)^{1/p}=\max\left\{\|f\|_\infty,\|g\|_\infty\right\}.$$

Lemma 3.1.1 brings to the following estimate

$$\limsup_{p\to\infty}\left(\lambda_{p,\beta^p}\right)^{1/p}\le\Lambda_{\infty,\beta},\tag{3.4}$$

so we only need to prove the liminf inequality.

Lemma 3.1.2. *Let $\{\lambda_{p,\beta^p}\}_{p>1}$ be the sequence of the first eigenvalues of the $p-$Laplacian operator with Robin boundary condition. Then,*

$$\lim_{p\to\infty}\left(\lambda_{p,\beta^p}\right)^{\frac{1}{p}}=\Lambda_{\infty,\beta},$$

where $\Lambda_{\infty,\beta}$ is defined in (3.3).

Moreover, if $\{u_p\}_{p>1}$ is the sequence of eigenfunctions associated to $\{\lambda_{p,\beta^p}\}_{p>1}$, then there exists a function $u_\infty\in W^{1,\infty}(\Omega)$ such that, up to a subsequence,

$$\begin{aligned}&u_p\to u_\infty &&\text{uniformly in }\Omega\\ &\nabla u_p\to\nabla u_\infty &&\text{weakly in }L^q(\Omega),\forall q.\end{aligned}$$

Proof. As a consequence of (3.4), the sequence $\{u_p\}_{p>1}$ of eigenfunctions associated to λ_{p,β^p} is uniformly bounded in $W^{1,q}(\Omega)$: indeed, if $q<p$, by Hölder inequality,

$$\|\nabla u_p\|_{L^q(\Omega)}\le\|\nabla u_p\|_{L^p(\Omega)}|\Omega|^{\frac1q-\frac1p}\le\lambda_{p,\beta^p}^{1/p}|\Omega|^{\frac1q-\frac1p}\le C,\tag{3.5}$$

$$\|u_p\|_{L^q(\Omega)}\le\|u_p\|_{L^p(\Omega)}|\Omega|^{\frac1q-\frac1p}\le|\Omega|^{\frac1q-\frac1p}\le C,\tag{3.6}$$

where the constant C is independent of p.

By a classical argument of diagonalization, see for instance [26], we can extract a subsequence u_{p_j} such that

$$\begin{gathered}u_{p_j}\to u_\infty\ \text{uniformly}\implies\|u_{p_j}\|_{L^{p_j}}\to\|u_\infty\|_{L^\infty},\\ \nabla u_{p_j}\to\nabla u_\infty\ \text{weakly in}\ L^q(\Omega),\ \forall q>1.\end{gathered}$$

Moreover, from (3.5) and (3.6), the following inequality holds

$$\frac{\|\nabla u_\infty\|_{L^q(\Omega)}}{\|u_\infty\|_{L^q(\Omega)}} \le \liminf_{p\to\infty} \frac{\|\nabla u_p\|_{L^q(\Omega)}}{\|u_p\|_{L^q(\Omega)}} \le \liminf_{p\to\infty} \frac{\|\nabla u_p\|_{L^p(\Omega)}}{\|u_p\|_{L^q(\Omega)}} |\Omega|^{\frac{1}{q}-\frac{1}{p}}$$
$$\le \frac{|\Omega|^{\frac{1}{q}}}{\|u_\infty\|_{L^q(\Omega)}} \liminf_{p\to\infty} \left(\lambda_{p,\beta^p}\right)^{\frac{1}{p}}.$$

Letting $q \to \infty$ we obtain

$$\|\nabla u_\infty\|_{L^\infty(\Omega)} \le \liminf_{p\to\infty} \left(\lambda_{p,\beta^p}\right)^{\frac{1}{p}}.$$

Similarly

$$\beta\|u_p\|_{L^q(\partial\Omega)} \le \beta\| \; u_p\|_{L^p(\partial\Omega)} |\partial\Omega|^{\frac{1}{q}-\frac{1}{p}} \le \lambda_{p,\beta^p}^{1/p} |\partial\Omega|^{\frac{1}{q}-\frac{1}{p}} \le C,$$

gives us

$$\beta\|u_\infty\|_{L^\infty(\partial\Omega)} \le \liminf_{p\to\infty} \left(\lambda_{p,\beta^p}\right)^{\frac{1}{p}},$$

hence

$$\Lambda_{\infty,\beta} \le \liminf_{p\to\infty} \lambda_{p,\beta^p}^{1/p}.$$

□

Now we want to show that the limit u_∞ solves an eigenvalue problem in the viscosity sense. For this purpose, we need the following proposition

Proposition 3.1.3. *A continuous weak solution u to (3.1) is a viscosity solution to (3.1).*

Proof. The proof is similar to the one in [102, 71] for the $p-$Laplacian with other boundary conditions.

We only write explicitly the proof that u_p satisfies the boundary conditions in the viscosity sense.

Let u_p be a continuous weak solution to (3.1), let $x_0 \in \partial\Omega$, and let us consider a function ϕ such that $\phi(x_0) = u_p(x_0)$ and such that $u_p - \phi$ has a strict minimum at x_0. Then

$$\max\Big\{ -|\nabla\phi(x_0)|^{p-2}\Delta\phi(x_0) - (p-2)|\nabla\phi(x_0)|^{p-4}\Delta_\infty\phi(x_0) - \lambda_{p,\beta^p}|\phi(x_0)|^{p-2}\phi(x_0),$$
$$|\nabla\phi(x_0)|^{p-2}\frac{\partial\phi(x_0)}{\partial\nu} + \beta^p|\phi(x_0)|^{p-2}\phi(x_0)\Big\} \ge 0. \tag{3.7}$$

Assume by contradiction that both terms are negative. If we choose r sufficiently small, in $\overline{\Omega}\cap B_r(x_0)$, we have

$$-|\nabla\phi(x)|^{p-2}\Delta\phi(x) - (p-2)|\nabla\phi(x)|^{p-4}\Delta_\infty\phi(x) - \lambda_{p,\beta^p}|\phi(x)|^{p-2}\phi(x) < 0$$

and, in $\partial\Omega \cap B_r(x_0)$,

$$|\nabla\psi(x)|^{p-2}\frac{\partial\psi(x)}{\partial\nu} + \beta^p|\psi(x)|^{p-2}\psi(x) < 0, \quad \text{where } \psi = \phi + \frac{m}{2}.$$

Then

$$\int_{\{\,\psi>u\,\}\cap B_r(x_0)} |\nabla\psi|^{p-2}\nabla\psi\nabla(\psi-u)\,dx$$
$$< \lambda_{p,\beta^p}\int_{\{\,\psi>u\,\}\cap B_r(x_0)} |\phi|^{p-2}\phi(\psi-u)\,dx - \beta^p\int_{\partial\Omega\cap B_r(x_0)\cap\{\,\psi>u\,\}} |\psi|^{p-2}\psi(\psi-u)\,d\mathcal{H}^{n-1}(x),$$

using the definition of weak solution, we have

$$\begin{aligned} C(N,p)\int_{\{\,\psi>u\,\}\cap B_r(x_0)} &|\nabla\psi-\nabla u|^p\,dx \\ &\le \int_{\{\,\psi>u\,\}\cap B_r(x_0)} \Big\langle|\nabla\psi|^{p-2}\nabla\psi-|\nabla u|^{p-2}\nabla u,\nabla(\psi-u)\Big\rangle\,dx \\ &< \lambda_{p,\beta^p}\int_{\{\,\psi>u\,\}\cap B_r(x_0)}\Big(|\phi|^{p-2}\phi-|u|^{p-2}u\Big)(\psi-u)\,dx \\ &-\beta^p\int_{\partial\Omega\cap B_r(x_0)\cap\{\,\psi>u\,\}}\Big(|\psi|^{p-2}\psi-|u|^{p-2}u\Big)(\psi-u)\,d\mathcal{H}^{n-1}(x)<0 \end{aligned}$$

which gives a contradiction. □

Now we can prove the following

Theorem 3.1.4. *Let u_∞ be the function given in Theorem 2.2.1. Then u_∞ is a viscosity solution to*

$$\begin{cases} \min\{\,|\nabla u|-\Lambda_{\infty,\beta}u,-\Delta_\infty u\,\}=0 & \text{in } \Omega,\\ -\min\left\{\,|\nabla u|-\beta u,-\dfrac{\partial u}{\partial\nu}\,\right\}=0 & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

Proof. We divide the proof in two steps.

Step 1 u_∞ is a viscosity supersolution.
Let $x_0\in\Omega$ and let $\phi\in C^2(\Omega)$ be such that $u_\infty-\phi$ has a strict minimum in x_0. We want to show

$$\min\{\,|\nabla\phi(x_0)|-\Lambda_{\infty,\beta}\phi(x_0),-\Delta_\infty\phi(x_0)\,\}\ge 0$$

Notice that $u_p-\phi$ has a minimum in x_p and $x_p\to x_0$. If we set $\phi_p(x)=\phi(x)+c_p$ with $c_p=u_p(x_p)-\phi(x_p)\to 0$ when p goes to infinity, we have that $u_p(x_p)=\phi_p(x_p)$ and $u_p-\phi_p$ has a minimum in x_p, so Proposition 3.1.3 implies

$$-|\nabla\phi_p(x_p)|^{p-2}\Delta\phi_p(x_p)-(p-2)|\nabla\phi_p(x_p)|^{p-4}\Delta_\infty\phi(x_p)-\lambda_{p,\beta^p}|\phi_p(x_p)|^{p-2}\phi_p(x_p)\ge 0. \tag{3.9}$$

Now dividing by $(p-2)|\nabla\phi_p(x_p)|^{p-4}$, we obtain

$$-\Delta_\infty\phi_p(x_p)-\frac{|\nabla\phi_p(x_p)|^2\Delta\phi_p(x_p)}{p-2}\ge\frac{|\nabla\phi_p(x_p)|^4}{(p-2)\phi_p(x_p)}\left(\frac{\lambda_{p,\beta^p}^{1/p}\phi_p(x_p)}{|\nabla\phi_p(x_p)|}\right)^p \tag{3.10}$$

This gives us $|\nabla\phi(x_0)|-\Lambda_{\infty,\beta}\phi(x_0)\ge 0$ since, otherwise, the right-hand side of (3.10) would go to infinity, in contradiction with the fact that $\phi\in C^2(\Omega)$. Moreover $-\Delta_\infty\phi(x_0)\ge 0$, just taking the limit.

Then, $\min\{\,|\nabla\phi(x_0)|-\Lambda_{\infty,\beta}\phi(x_0),-\Delta_\infty\phi(x_0)\,\}\ge 0$ and u_∞ is a viscosity supersolution.

Let us fix $x_0\in\partial\Omega$, $\phi\in C^2(\overline{\Omega})$ such that $u-\phi$ has a strict minimum in x_0, our aim is to prove that

$$\max\left\{\min\{\,|\nabla\phi(x_0)|-\Lambda_{\infty,\beta}\phi(x_0),-\Delta_\infty\phi(x_0)\,\},-\min\left\{\,|\nabla\phi(x_0)|-\beta\phi(x_0),-\frac{\partial\phi}{\partial\nu}(x_0)\,\right\}\right\}\ge 0$$

If for infinitely many $x_p\in\Omega$ (3.9) holds true, then we get

$$\min\{\,|\nabla\phi(x_0)|-\Lambda_{\infty,\beta}\phi(x_0),-\Delta_\infty\phi(x_0)\,\}\ge 0.$$

If for infinitely many p, $x_p \in \partial\Omega$ the following holds true

$$|\nabla \phi_p(x_p)|^{p-2}\frac{\partial \phi_p(x_p)}{\partial \nu} + \beta^p|\phi_p(x_p)|^{p-2}\phi_p(x_p) \geq 0,$$

then

$$|\nabla \phi_p(x_p)|^{p-2}\left(-\frac{\partial \phi_p(x_p)}{\partial \nu}\right) \leq \beta^p|\phi_p(x_p)|^{p-2}\phi_p(x_p).$$

Only two cases can occur:

- $-\dfrac{\partial \phi}{\partial \nu}(x_0) \leq 0$;
- $-\dfrac{\partial \phi}{\partial \nu}(x_0) > 0$, then letting p to infinity in the following

$$\left(|\nabla \phi_p(x_p)|^{p-2}\left(-\frac{\partial \phi_p(x_p)}{\partial \nu}\right)\right)^{1/p} \leq \left(\beta^p|\phi_p(x_p)|^{p-2}\phi_p(x_p)\right)^{1/p}$$

  we get $|\nabla\phi(x_0)| \leq \beta\phi(x_0)$.

That is

$$-\min\left\{\ |\nabla\phi(x_0)| - \beta\phi(x_0), -\frac{\partial \phi}{\partial \nu}(x_0)\ \right\} \geq 0.$$

Step 2 u_∞ is a viscosity subsolution.
Let us fix $x_0 \in \Omega$, $\phi \in C^2(\Omega)$ such that $u_\infty - \phi$ has a strict maximum. We want to prove that

$$\min\left\{|\nabla\phi(x_0)| - \Lambda_{\infty,\beta}\phi(x_0), -\Delta_\infty\phi(x_0)\right\} \leq 0,$$

so it is enough to prove that only one of the two terms in the bracket is non positive.

For instance, assume that $-\Delta_\infty\phi(x_0) > 0$, we can argue as in (2.26), but now, all the inequality involving the second order differential operator are reversed and we get

$$\lambda_{p,\beta^p}\phi_p^{p-1}(x_p) \geq (p-2)|\nabla\phi_p(x_p)|^{p-4}\left[-\frac{|\nabla\phi_p(x_p)|^2\Delta\phi_p(x_p)}{p-2} - \Delta_\infty\phi_p(x_p)\right].$$

As $-\Delta_\infty\phi(x_0) > 0$, the term in the big square bracket is non-negative, we can erase everything to the power $1/p$, obtaining

$$\Lambda_{\infty,\beta}\phi(x_0) \geq |\nabla\phi(x_0)|,$$

which shows that u_∞ is a viscosity subsolution to (3.8).

Similar arguments to step 1 give us the boundary conditions for viscosity subsolution. □

We are also able to give a geometric characterization of $\Lambda_{\infty,\beta}$.

Lemma 3.1.5. *Let $\Lambda_{\infty,\beta}$ be the quantity defined in (3.3) and let $r(\Omega)$ be the inradius of Ω. Then*

$$\Lambda_{\infty,\beta} = \min_{x_0\in\Omega}\frac{1}{\frac{1}{\beta} + d(x_0,\partial\Omega)} = \frac{1}{\frac{1}{\beta} + r(\Omega)}.$$

Proof. The function $\frac{1}{\beta} + d(x, \partial\Omega) \in W^{1,\infty}(\Omega)$, moreover

$$\|\nabla\left(1/\beta + d(x, \partial\Omega)\right)\|_{L^\infty(\Omega)} = 1 \quad \text{and} \quad \beta\|1/\beta + d(x, \partial\Omega)\|_{L^\infty(\partial\Omega)} = 1.$$

Then

$$\Lambda_{\infty,\beta} \leq \min_{x_0\in\Omega} \frac{1}{\frac{1}{\beta} + d(x_0, \partial\Omega)}.$$

In order to prove the reverse inequality, we consider $w \in W^{1,\infty}(\Omega)$ such that $\|w\|_{L^\infty(\Omega)} = 1$.

The following facts can occur

Case 1 If $\beta\|w\|_{L^\infty(\partial\Omega)} \leq \|\nabla w\|_{L^\infty(\Omega)}$, then

$$\max\left\{\|\nabla w\|_{L^\infty(\Omega)}, \beta\|w\|_{L^\infty(\partial\Omega)}\right\} = \|\nabla w\|_{L^\infty(\Omega)}.$$

We choose $x \in \Omega$ and y equal to the point on the boundary such that $|x - y| = d(x, \partial\Omega)$. So, we have

$$\begin{aligned}
|w(x)| &\leq |w(x) - w(y)| + |w(y)| \\
&\leq \|\nabla w\|_{L^\infty(\Omega)}|x - y| + \|w\|_{L^\infty(\partial\Omega)} \\
&\leq \|\nabla w\|_{L^\infty(\Omega)} d(x, \partial\Omega) + \frac{1}{\beta}\|\nabla w\|_{L^\infty(\Omega)} \\
&= \|\nabla w\|_{L^\infty(\Omega)} \left(\frac{1}{\beta} + d(x, \partial\Omega)\right) \\
&\leq \|\nabla w\|_{L^\infty(\Omega)}\|1/\beta + d(x, \partial\Omega)\|_{L^\infty(\Omega)},
\end{aligned}$$

and then

$$\frac{\|\nabla w\|_{L^\infty(\Omega)}}{\|w\|_{L^\infty(\Omega)}} \geq \frac{1}{\|1/\beta + d(x, \partial\Omega)\|_{L^\infty(\Omega)}}$$

Case 2 If $\beta\|w\|_{L^\infty(\partial\Omega)} > \|\nabla w\|_{L^\infty(\Omega)}$, then

$$\max\left\{\|\nabla w\|_{L^\infty(\Omega)}, \beta\|w\|_{L^\infty(\partial\Omega)}\right\} = \beta\|w\|_{L^\infty(\partial\Omega)}.$$

With the same choice of x and y, we have

$$\begin{aligned}
|w(x)| &\leq |w(x) - w(y)| + |w(y)| \\
&\leq \|\nabla w\|_{L^\infty(\Omega)}|x - y| + \|w\|_{L^\infty(\partial\Omega)} \\
&\leq \beta\|w\|_{L^\infty(\partial\Omega)} d(x, \partial\Omega) + \|w\|_{L^\infty(\partial\Omega)} \\
&= \beta\|w\|_{L^\infty(\partial\Omega)} \left(d(x, \partial\Omega) + \frac{1}{\beta}\right) \\
&\leq \beta\|w\|_{L^\infty(\partial\Omega)}\|1/\beta + d(x, \partial\Omega)\|_{L^\infty(\Omega)}.
\end{aligned}$$

Hence,

$$\frac{\beta\|w\|_{L^\infty(\partial\Omega)}}{\|w\|_{L^\infty(\Omega)}} \geq \frac{1}{\|1/\beta + d(x, \partial\Omega)\|_{L^\infty(\Omega)}}.$$

Finally we get $\forall w \in W^{1,\infty}(\Omega) : \|w\|_{L^\infty(\Omega)} = 1$,

$$\max\left\{\|\nabla w\|_{L^\infty(\Omega)}, \beta\|w\|_{L^\infty(\partial\Omega)}\right\} \geq \frac{1}{\|1/\beta + d(x,\partial\Omega)\|_{L^\infty(\Omega)}}$$

and then the desired inequality

$$\Lambda_{\infty,\beta} \geq \min_{x_0\in\Omega} \frac{1}{\frac{1}{\beta} + d(x_0,\partial\Omega)}.$$

□

Theorem 3.1.6. *Let $\Lambda_{\infty,\beta}$ be the quantity defined in (3.3). Then*

$$\Lambda_{\infty,\beta}(\Omega) \geq \Lambda_{\infty,\beta}(\Omega^\sharp),$$

where $\Omega^\sharp$ is the ball centered at the origin with the same measure of Ω.

Proof. The Faber-Krahn inequality for the first eigenvalue of the $p-$Laplacian with Robin boundary condition (for instance see [45]) states that

$$\lambda_{p,\beta^p}(\Omega) \geq \lambda_{p,\beta^p}(\Omega^\sharp).$$

Letting p go to infinity, we have

$$\Lambda_{\infty,\beta}(\Omega) \geq \Lambda_{\infty,\beta}(\Omega^\sharp).$$

This can follow also from the geometric characterization in Lemma 3.1.5

$$\Lambda_{\infty,\beta} = \frac{1}{\frac{1}{\beta} + r(\Omega)}$$

as the ball maximizes the inradius among sets of given volume. □

Remark 3.1.1. One can easily prove that the function $\frac{1}{\beta} + d(x,\partial\Omega)$ is an eigenfunction if the domain $\Omega = B_R(x_0)$. This is not true if Ω is a square: see for instance [102].

3.1.1 The first Robin $\infty-$eigenvalue

Now we want to show that $\Lambda_{\infty,\beta}$ is the first eigenvalue of (3.8), that is the smallest Λ such that

$$\begin{cases} \min\{\,|\nabla u| - \Lambda u, -\Delta_\infty u\,\} = 0 & \text{in } \Omega, \\ -\min\left\{\,|\nabla u| - \beta u, -\dfrac{\partial u}{\partial \nu}\,\right\} = 0 & \text{on } \partial\Omega \end{cases}$$

admits a non-trivial solution.

Theorem 3.1.7. *Let Ω be a bounded and open set of class C^2 in $\mathbb{R}^n$. If for some Λ, problem (3.8) admits a non-trivial eigenfunction u, then $\Lambda \geq \Lambda_{\infty,\beta}$.*

Proof. Let Λ be an eigenvalue to (3.8), let u be a corresponding eigenfunction. We normalize it in this way

$$\max_{x\in\Omega} u(x) = \frac{1}{\Lambda}.$$

Then u is viscosity subsolution to

$$\min\{\, |\nabla u| - 1, -\Delta_\infty u \,\} = 0 \text{ in } \Omega.$$

For every $\varepsilon > 0$ and $\gamma > 0$, let us consider the function

$$g_{\varepsilon,\gamma} = \frac{1}{\beta} + (1+\varepsilon)d(x,\partial\Omega) - \gamma d(x,\partial\Omega)^2.$$

It is well known (see [89]) that in a small tubular neighbourhood Γ_μ of $\partial\Omega$, the boundary $\partial\Omega$ and the distance function $d(x,\partial\Omega)$ share the same regularity: so both $d(x,\partial\Omega)$ and $g_{\varepsilon,\gamma}$ are $C^2(\Gamma_\mu)$.

Moreover, by direct computation, one can check that if

$$\gamma < \frac{\varepsilon}{2r(\Omega)},$$

then $g_{\varepsilon,\gamma}$ is a viscosity supersolution to

$$\min\{\, |\nabla g_{\varepsilon,\gamma}| - 1, -\Delta_\infty g_{\varepsilon,\gamma} \,\} = 0 \text{ in } \Omega.$$

Hence, Theorem 2.1 in [100] ensures that

$$m_\varepsilon = \inf_{x\in\Omega}(g_{\varepsilon,\gamma}(x) - u(x)) = \inf_{x\in\partial\Omega}(g_{\varepsilon,\gamma}(x) - u(x)).$$

Assume by contradiction that $m_\varepsilon < -\frac{\varepsilon}{\beta}$, and set $v = g_{\varepsilon,\gamma} - m_\varepsilon$. We observe that $v \geq u$ in Ω and $v(x_0) = u(x_0)$, where x_0 is the point which achieves the minimum on the boundary, so we can use it as test function in the definition of viscosity subsolution for u.

Assuming $\gamma < \frac{\varepsilon}{2r(\Omega)}$, we obtain

$$\begin{aligned}
\nabla v(x) &= [1+\varepsilon - 2\gamma d(x,\partial\Omega)]\, \nabla d(x,\partial\Omega),\\
|\nabla v(x_0)| &= 1+\varepsilon - 2\gamma d(x_0,\partial\Omega) > 1,\\
-\frac{\partial v}{\partial \nu}(x_0) &= -\,[1+\varepsilon - 2\gamma d(x_0,\partial\Omega)]\, \nabla d(x_0,\partial\Omega)\cdot \nu > 0,\\
-\Delta_\infty v(x_0) &= 2\gamma\, [1+\varepsilon - 2\gamma d(x_0,\partial\Omega)]^2\, |\nabla d(x_0,\partial\Omega)|^4 > 0.
\end{aligned}$$

The fact that $m_\varepsilon < -\frac{\varepsilon}{\beta}$ implies

$$|\nabla v(x_0)| - \beta v(x_0) = \varepsilon + \beta m_\varepsilon < 0.$$

Therefore

$$-\min\left\{ |\nabla v| - \beta v, -\frac{\partial v}{\partial \nu} \right\} > 0 \quad \text{and} \quad \min\{|\nabla v| - 1, -\Delta_\infty v\} > 0,$$

against the fact that

$$\min\left\{ \min\{|\nabla v| - 1, -\Delta_\infty v\}, -\min\left\{ |\nabla v| - \beta v, -\frac{\partial v}{\partial \nu} \right\} \right\} \leq 0.$$

So we have

$$g_{\varepsilon,\gamma}(x) - u(x) \geq m_\varepsilon \geq -\frac{\varepsilon}{\beta},$$

letting ε and γ go to zero, it follows

$$\frac{1}{\beta} + d(x, \partial\Omega) \geq u(x), \qquad \forall x \in \Omega.$$

Hence

$$\frac{1}{\Lambda_{\infty,\beta}} = \max_{x\in\Omega} \left(\frac{1}{\beta} + d(x, \partial\Omega)\right) \geq \max_{x\in\Omega} u(x) = \frac{1}{\Lambda},$$

which concludes the proof. □

3.2 The $p-$Poisson equation

The study of the limiting behavior of the $p-$Laplacian eigenvalues can be used to study the $p-$Poisson equation with Robin boundary conditions

$$\begin{cases} -\Delta_p v = f & \text{in } \Omega \\ |\nabla v|^{p-2}\dfrac{\partial v}{\partial \nu} + \beta^p |v|^{p-2} v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.11}$$

when $p \to \infty$, whenever f is a function in $L^\infty(\Omega)$ and $\beta > 0$. We recall that a function v_p is a weak solution to (3.11) if it satisfies

$$\int_\Omega |\nabla v_p|^{p-2}\nabla v_p \nabla\varphi\, dx + \beta^p \int_{\partial\Omega} |v_p|^{p-2} v_p \varphi\, d\mathcal{H}^{n-1}(x) = \int_\Omega f\varphi\, dx, \quad \forall \varphi \in W^{1,p}(\Omega). \tag{3.12}$$

As we prove in Chapter 2, the solution to this equation is the unique minimum of the functional

$$J_p(\varphi) = \frac{1}{p}\int_\Omega |\nabla\varphi|^p\, dx + \frac{\beta^p}{p}\int_{\partial\Omega} |\varphi|^p\, d\mathcal{H}^{n-1}(x) - \int_\Omega f\varphi\, dx. \tag{3.13}$$

If we let *formally* p go to ∞ in (3.13), we obtain the functional

$$\varphi \longrightarrow \min \int_\Omega -f\varphi\, dx \quad \varphi \in W^{1,\infty}(\Omega). \tag{3.14}$$

The limit procedure imposes two extra constraints to (3.14), namely

$$\|\nabla\varphi\|_\infty \leq 1, \quad \beta\|\varphi\|_{L^\infty(\partial\Omega)} \leq 1.$$

The following proposition holds true.

Proposition 3.2.1. *Let v_p be the solution to (3.11). Then there exists a subsequence $\left\{ v_{p_j} \right\}_j$ such that*

$$v_{p_j} \to v_\infty \ \textit{uniformly}, \qquad \nabla v_{p_j} \to \nabla v_\infty \ \textit{weakly in } L^m(\Omega), \forall m > 1.$$

Moreover

$$\|\nabla v_\infty\|_\infty \leq 1 \qquad \beta\|v_\infty\|_{L^\infty(\partial\Omega)} \leq 1.$$

Proof. Choosing $\varphi = v_p$ in (3.12), we have

$$\int_\Omega |\nabla v_p|^p + \beta^p \int_{\partial\Omega} v_p^p = \int_\Omega f v_p,$$

and Young inequality gives

$$\int_\Omega |\nabla v_p|^p + \beta^p \int_{\partial\Omega} v_p^p - \frac{1}{\varepsilon_p^p p} \int_\Omega v_p^p \le \frac{\varepsilon_p^{p'}}{p'} \int_\Omega f^{p'}.$$

Taking into account (3.2), we get

$$\left(1 - \frac{1}{p\lambda_{p,\beta^p}\varepsilon_p^p}\right)\left[\int_\Omega |\nabla v_p|^p + \beta^p \int_{\partial\Omega} v_p^p\right] \le$$
$$\le \int_\Omega |\nabla v_p|^p + \beta^p \int_{\partial\Omega} v_p^p - \frac{1}{\varepsilon_p^p p} \int_\Omega v_p^p \le \frac{\varepsilon_p^{p'}}{p'} \int_\Omega f^{p'}.$$

Choosing ε_p such that $1 - \dfrac{1}{p\lambda_{p,\beta^p}\varepsilon_p^p} = \dfrac{1}{2}$, we have

$$\int_\Omega |\nabla v_p|^p + \beta^p \int_{\partial\Omega} v_p^p \le 2\frac{\varepsilon_p^{p'}}{p'} \int_\Omega f^{p'} \le C \int_\Omega f^{p'} \tag{3.15}$$

where the constant C is independent of p, thanks to Lemma 3.1.2.

Hence

$$\left(\int_\Omega |\nabla v_p|^p\right)^{1/p} \le \left(C \int_\Omega f^{p'}\right)^{1/p} \le \left(C|\Omega|\,\|f\|_\infty^{p'}\right)^{1/p},$$
$$\left(\beta^p \int_{\partial\Omega} v_p^p\right)^{1/p} \le \left(C \int_\Omega f^{p'}\right)^{1/p} \le \left(C|\Omega|\,\|f\|_\infty^{p'}\right)^{1/p}.$$

Analogously

$$\left(\int_\Omega v_p^p\right)^{1/p} \le C\left(\int_\Omega f^{p'}\right)^{1/p}. \tag{3.16}$$

If $p > m$, Hölder inequality gives

$$\begin{aligned}
\left(\int_\Omega |\nabla v_p|^m\right)^{1/m} &\le \left(\int_\Omega |\nabla v_p|^p\right)^{1/p} |\Omega|^{1/m-1/p} \le \left(C\|f\|_\infty^{p'}\right)^{1/p} |\Omega|^{1/m},\\
\beta\left(\int_{\partial\Omega} v_p^m\right)^{1/m} &\le \left(\beta^p \int_{\partial\Omega} v_p^p\right)^{1/p} |\partial\Omega|^{1/m-1/p} \le \left(C\|f\|_\infty^{p'}\right)^{1/p} |\partial\Omega|^{1/m},\\
\left(\int_\Omega v_p^m\right)^{1/m} &\le \left(\int_\Omega v_p^p\right)^{1/p} |\Omega|^{1/m-1/p} \le C\left(\|f\|_\infty^{p'}\right)^{1/p} |\Omega|^{1/m},
\end{aligned} \tag{3.17}$$

Then there exists v_{p_j} such that

$$v_{p_j} \to v_\infty \text{ uniformly}, \qquad \nabla v_{p_j} \to \nabla v_\infty \text{ weakly in } L^m(\Omega),\ \forall m > 1,$$

moreover

$$\|\nabla v_\infty\|_m \le \liminf_{j\to\infty} \left\|\nabla v_{p_j}\right\|_m \le \lim_{j\to\infty} \left(C\|f\|_\infty^{p'_j}\right)^{1/p_j} |\Omega|^{1/m} = |\Omega|^{1/m},$$

$$\beta\|v_\infty\|_{L^m(\partial\Omega)} = \beta \lim_{j\to\infty} \left\|v_{p_j}\right\|_{L^m(\partial\Omega)} \le \lim_{j\to\infty} \left(C\|f\|_\infty^{p'_j}\right)^{1/p_j} |\partial\Omega|^{1/m} = |\partial\Omega|^{1/m},$$

and then

$$\|\nabla v_\infty\|_{L^\infty(\Omega)} \le 1, \qquad \beta\|v_\infty\|_{L^\infty(\partial\Omega)} \le 1.$$

□

We are also able to link the so obtained function v_∞ with the functional (3.14), indeed

Theorem 3.2.2. *The functional*

$$J_\infty(\varphi) = -\int_\Omega f\varphi \qquad \varphi \in W^{1,\infty}(\Omega) \tag{3.18}$$

admits at least one minimum $\overline{\varphi}$ satisfying $\|\nabla\overline{\varphi}\|_{L^\infty(\Omega)} \le 1$ and $\beta\|\overline{\varphi}\|_{L^\infty(\partial\Omega)} \le 1$.

Moreover, if v_∞ is a limit of a subsequence of $\{v_p\}$, then v_∞ is a minimizer of (3.18).

Proof. Let v_∞ a limit of a subsequence of $\{v_p\}$ and let us assume it is not a minimum of J_∞. This implies that there exists $\varphi \in W^{1,\infty}(\Omega)$ such that

$$-\int_\Omega f\varphi < -\int_\Omega f v_\infty.$$

We want to show that there exists a function ϕ and an exponent p, such that $J_p(\phi) < J_p(v_p)$, which contradicts the minimality of v_p.

First of all, we recall that exists a sequence $v_{p_i} \rightharpoonup v_\infty$ in $W^{1,m}(\Omega)$ $\forall m$. Then

$$\int_\Omega f v_{p_i} \to \int_\Omega f v_\infty$$

and so there exists $\bar{i}$ such that we still have

$$-\int_\Omega f\varphi < -\int_\Omega f v_{p_i} \qquad \forall i > \bar{i}. \tag{3.19}$$

Two cases can occur:

case 1 $\exists i > \bar{i}$

$$\int_\Omega |\nabla\varphi|^{p_i} + \beta^{p_i}\int_{\partial\Omega} |\varphi|^{p_i} \le \int_\Omega |\nabla v_{p_i}|^{p_i} + \beta^{p_i}\int_{\partial\Omega} v_{p_i}^{p_i}$$

Then

$$\begin{aligned} J_{p_i}(\varphi) &= \frac{1}{p}\int_\Omega |\nabla\varphi|^{p_i} + \frac{\beta^{p_i}}{p}\int_{\partial\Omega} |\varphi|^{p_i} - \int_\Omega f\varphi \\ &< \frac{1}{p}\int_\Omega |\nabla v_{p_i}|^{p_i} + \frac{\beta^{p_i}}{p}\int_{\partial\Omega} v_{p_i}^{p_i} - \int_\Omega f v_{p_i} = J_{p_i}(v_{p_i}), \end{aligned}$$

which is a contradiction.

case 2 $\forall i > \bar{i}$

$$\int_\Omega |\nabla\varphi|^{p_i} + \beta^{p_i}\int_{\partial\Omega}|\varphi|^{p_i} > \int_\Omega |\nabla v_{p_i}|^{p_i} + \beta^{p_i}\int_{\partial\Omega} v_{p_i}^{p_i}.$$

Considering $\phi = \alpha\varphi$ with $\alpha \in (0,1)$:

$$-\int_\Omega f\phi = -\alpha\int_\Omega f\varphi < -\int_\Omega f v_{p_i} < 0 \qquad \forall i > \bar{i},$$

we have

$$\int_\Omega |\nabla\phi|^{p_i} + \beta^{p_i}\int_{\partial\Omega}|\phi|^{p_i} = \alpha^{p_i}\left[\int_\Omega |\nabla\varphi|^{p_i} + \beta^{p_i}\int_{\partial\Omega}|\varphi|^{p_i}\right].$$

Moreover

$$M \le \int_\Omega f v_{p_i} = \int_\Omega |\nabla v_{p_i}|^{p_i} + \beta^{p_i}\int_{\partial\Omega} v_{p_i}^{p_i},$$

$$\int_\Omega |\nabla\varphi|^{p_i} + \beta^{p_i}\int_{\partial\Omega}|\varphi|^{p_i} \le |\Omega|\|\nabla\varphi\|^{p_i}_{L^\infty(\Omega)} + \beta^{p_i}|\partial\Omega|\|\varphi\|^{p_i}_{L^\infty(\partial\Omega)} \le |\Omega| + |\partial\Omega|.$$

We now choose p_i:

$$0 \xleftarrow{i\to\infty} \alpha^{p_i} \le \frac{M}{|\Omega|+|\partial\Omega|} \le \frac{\int_\Omega |\nabla v_{p_i}|^{p_i} + \beta^{p_i}\int_{\partial\Omega} v_{p_i}^{p_i}}{\int_\Omega |\nabla\varphi|^{p_i} + \beta^{p_i}\int_{\partial\Omega}|\varphi|^{p_i}}$$

obtaining

$$\int_\Omega |\nabla\phi|^{p_i} + \beta^{p_i}\int_{\partial\Omega}|\phi|^{p_i} \le \int_\Omega |\nabla v_{p_i}|^{p_i} + \beta^{p_i}\int_{\partial\Omega} v_{p_i}^{p_i}.$$

□

Proposition 3.2.3. *Let v_p be the solution to (3.11) and let v_∞ be any limit of a subsequence of $\{v_p\}_{p>1}$. Then*

$$v_\infty(x) \le \frac{1}{\beta} + d(x,\partial\Omega). \tag{3.20}$$

Proof. We notice that

$$|v_\infty(x) - v_\infty(y)| \le |x-y|$$

as we have proven that $\|\nabla v_\infty\|_\infty \le 1$. This holds true for every x, y in Ω. In particular, we can choose y equal to the point on the boundary which realizes $|x-y| = d(x,\partial\Omega)$. So, we have

$$v_\infty(x) \le v_\infty(y) + d(x,\partial\Omega) \le \frac{1}{\beta} + d(x,\partial\Omega),$$

as v_∞ also satisfies $\beta\|v_\infty\|_{L^\infty(\partial\Omega)} \le 1$. □

Remark 3.2.1. We explicitly observe that the estimate $\varphi(x) \le \frac{1}{\beta} + d(x,\partial\Omega)$ holds for every admissible function φ.

Proposition 3.2.4. *Assume $f > 0$ in Ω. Then the sequence of solution to (3.11) converges strongly in $W^{1,m}(\Omega)$, for all $m > 1$, to*

$$\bar{v}_\infty(x) = \frac{1}{\beta} + d(x,\partial\Omega).$$

Proof. Let v_∞ be any limit of a subsequence $\left\{ v_{p_j} \right\} \subset \{ v_p \}$. Theorem 3.2.2 gives that v_∞ is a minimum of the functional J_∞, in the class $\{\varphi \in W^{1,\infty}(\Omega) : \|\nabla\varphi\|_\infty \leq 1, \beta\|\varphi\|_{L^\infty(\partial\Omega)} \leq 1\}$.

The function $\frac{1}{\beta} + d(x, \partial\Omega)$ is a competitor and then

$$\int_\Omega f\left(v_\infty - \frac{1}{\beta} - d(x,\partial\Omega)\right) \geq 0. \tag{3.21}$$

Then (3.20) gives $v_\infty(x) = \frac{1}{\beta} + d(x, \partial\Omega)$.

Since every subsequence of $\{ v_p \}$ has a subsequence converging to $\frac{1}{\beta} + d(x,\partial\Omega)$ weakly in $W^{1,m}(\Omega)$, the whole sequence $\{ v_p \}$ converges to $\frac{1}{\beta} + d(x,\partial\Omega)$ weakly in $W^{1,m}(\Omega)$, and in particular, in $C^\alpha(\overline{\Omega})$ and its gradient weakly in $L^m(\Omega)$.

It remains to prove the strong convergence in $W^{1,m}(\Omega)$.

Clarkson's inequality implies for $p, q > m$

$$\int_\Omega \frac{|\nabla v_p + \nabla v_q|^m}{2^m} + \int_\Omega \frac{|\nabla v_p - \nabla v_q|^m}{2^m} \leq \frac{1}{2}\int_\Omega |\nabla v_p|^m + \frac{1}{2}\int_\Omega |\nabla v_q|^m$$

From (3.17) we deduce

$$\lim_{p\to\infty} \int_\Omega |\nabla v_p|^m \leq |\Omega|,$$

then semicontinuity of L^m-norm gives

$$\limsup_{p,q} \int_\Omega \frac{|\nabla v_p + \nabla v_q|^m}{2^m} \leq |\Omega| = \int_\Omega |\nabla d(x,\partial\Omega)|^m \leq \liminf_{p,q} \int_\Omega \frac{|\nabla v_p + \nabla v_q|^m}{2^m}$$

and then

$$\limsup_{p,q} \int_\Omega \frac{|\nabla v_p - \nabla v_q|^m}{2^m} = 0.$$

□

Remark 3.2.2. If Supp$f \subset \Omega$, then we can deduce that $v_\infty(x) = \frac{1}{\beta} + d(x,\partial\Omega)$ for all $x \in$ Supp$f$, while in $\Omega \setminus$ Supp$f$ inequality (3.20) can be strict.

Definition 3.2.1. We denote by $\mathcal{R}$ the set of discontinuity of the function $\nabla d(x, \partial\Omega)$. This set consists of points $x \in \Omega$ for which $d(x,\partial\Omega)$ is achieved by more than one point y on the boundary.

Then it holds true the following

Theorem 3.2.5. *Let f be a non-negative function in Ω, then function $\overline{v}_\infty(x) = \dfrac{1}{\beta} + d(x,\partial\Omega)$ is the unique extremal function of* (3.18) *if and only if $\mathcal{R} \subset$ Suppf.*

Proof. Suppose that $\mathcal{R} \subset$ Supp$f$ and let w be a minimum of (3.18)in the class $\{\varphi \in W^{1,\infty}(\Omega) : \|\nabla\varphi\|_\infty \leq 1, \beta\|\varphi\|_{L^\infty(\partial\Omega)} \leq 1\}$. By Remark 3.2.1 we have

$$w(x) \leq \frac{1}{\beta} + d(x,\partial\Omega) \qquad \forall x \in \Omega,$$

and arguing as in Remark 3.2.2 we have

$$w(x) = \frac{1}{\beta} + d(x,\partial\Omega) \qquad \forall x \in \overline{\text{supp} f}.$$

Assume by contradiction that there exists $x \in (\overline{\text{Supp}f})^c$ such that

$$w(x) < \frac{1}{\beta} + d(x, \partial\Omega),$$

setting $\eta = \nabla d(x, \partial\Omega)$, we choose the smallest t such that $y = x + t\eta$ belongs to $\partial(\text{Supp}f)$ (Lemma 3.2.6 will justify this choice), thus

$$\begin{aligned}\|\nabla w\|_{L^\infty(\Omega)}|y - x| &\geq w(y) - w(x)\\ &> d(y, \partial\Omega) - d(x, \partial\Omega) = \nabla d(\xi, \partial\Omega) \cdot (y - x) = |y - x|\end{aligned}$$

where the last equality follows from Lemma 3.2.6.

So we have $\|\nabla w\|_{L^\infty(\Omega)} > 1$ that is a contradiction.

Assume now that $w(x) = \dfrac{1}{\beta} + d(x, \partial\Omega)$ is the unique extremal of (3.18), and assume by contradiction that $\mathcal{R} \not\subset \text{Supp}f$. Then, thanks to the Aronsson theorem (see [18]), we can construct a function φ different from w which coincides with w in $\text{Supp}f$ and such that it is admissible for (3.14). Then φ is a minimum too.
This contradicts the fact that w is the unique minimum of (3.14), so $\mathcal{R} \subset \text{Supp}f$. □

We have to prove Lemma 3.2.6 to complete the proof.

Lemma 3.2.6. *Let $x \in \Omega \setminus \mathcal{R}$ and let us set $\eta = \nabla(d(x, \partial\Omega))$. Let us consider $y_t = x + t\eta$, then there exists T such that $y_T \in \mathcal{R}$ and $y_t \notin \mathcal{R}$ for all $t < T$. Moreover,*

$$\nabla d(x + t\eta, \partial\Omega) = \eta \qquad \forall t \in [0, T).$$

Proof. Let us consider the following Cauchy problem

$$\begin{cases}\dot\gamma(t) = \nabla d(\gamma(t), \partial\Omega),\\ \gamma(0) = x\end{cases}$$

in a maximal interval $[0, T)$. We have that

- $L(\gamma) = \displaystyle\int_0^T |\dot\gamma(t)|\, ds = T$;

- $\dfrac{d}{dt} d(\gamma(t), \partial\Omega) = \nabla d(\gamma(t), \partial\Omega)\dot\gamma(t) = 1$, then

$$T = \int_0^T \frac{d}{dt} d(\gamma(t), \partial\Omega)\, dt = d(\gamma(T), \partial\Omega) - d(x, \partial\Omega).$$

These considerations give us the following:

- $T < \infty$, otherwise d is unbounded, and this is a contradiction as Ω is bounded;

- $\gamma(T) \in \mathcal{R}$, otherwise one can extend the solution for $t > T$, in contradiction with the fact that $[0, T)$ is the maximal interval.

In the end, if $y = \gamma(T)$, we have

$$d(y, \partial\Omega) = d(x, \partial\Omega) + T = d(x, \partial\Omega) + L(\gamma),$$

$L(\gamma) \geq |y - x|$, and they are equal if and only if γ is a segment.

If $L(\gamma) > |y - x|$ then

$$d(y, \partial\Omega) = d(x, \partial\Omega) + L(\gamma) > d(x, \partial\Omega) + |y - x| \geq |y - z|,$$

with $z \in \partial\Omega$ such that $d(x, \partial\Omega) = |x - z|$, and this is a contradiction, because $d(y, \partial\Omega)$ is the infimum. Then $L(\gamma) = |y - x|$ and, remembering the fact $\dot{\gamma}(t) = \nabla d(\gamma(t), \partial\Omega)$, whose norm is 1, then γ is a segment and

$$\nabla d(\gamma(t), \partial\Omega) = \eta \qquad \forall t \in [0, T).$$

This conclude the proof. □

3.2.1 The limit PDE

We have proved that any limit v_∞ of subsequence $\{ v_p \}$ is a minimum of the functional (3.18) defined in $W^{1,\infty}(\Omega)$. Now we want to understand if such limits are solutions to a certain PDE, which, in some sense, is the Euler-Lagrange equation of the functional (3.18).

Proposition 3.2.7. *Let $f \in L^\infty(\Omega) \cap C(\overline{\Omega})$ be a non-negative function. Then any limit v_∞ of a subsequence $\{ v_p \}$ satisfies*

$$|\nabla v_\infty| \leq 1 \qquad\qquad \textit{in the viscosity sense.} \tag{3.22}$$

Proof. The proof follows the techniques contained in [26, 102, 160, 114].

Let $x_0 \in \Omega$, let $\varphi \in C^2(\Omega)$ be such that $v_\infty - \varphi$ has a local maximum at x_0

$$(v_\infty - \varphi)(x_0) \geq (v_\infty - \varphi)(x),\ \forall x \in B_R(x_0),$$

then

$$|\nabla\varphi(x_0)| \leq 1. \tag{3.23}$$

Indeed, let

$$C = \sup_{q>1} \max \left\{ \|v_q\|_{L^\infty(B_R(x_0))}; \|\varphi\|_{L^\infty(B_R(x_0))} \right\},$$

we consider the following sequence

$$f_q(x) = v_q(x) - \varphi(x) - k|x - x_0|^a, \quad k = \frac{4C}{R^a}, \quad a > 2.$$

So $f_q(x) \leq -3C$ for $x \in \partial B_R(x_0)$, and $f_q(x_0) \geq -2C$. Then $f_q(x)$ attains its maximum at some point x_q in the interior of $B_R(x_0)$. and it holds

$$\nabla v_q(x_q) = \nabla\varphi(x_q) + ka(x_q - x_0)|x_q - x_0|^{a-2}.$$

Moreover, the sequence $\{ x_q \}$ must converge to x_0.

Assume by contradiction that $|\nabla\varphi(x_0)| > 1$, so there exists $\delta \in (0, 1)$ such that $|\nabla\varphi(x_0)| \geq 1 + \delta$. Choosing $\overline{q}$ large enough, we have

$$|\nabla v_q(x_q)| > 1 + \frac{\delta}{2} - ka|x_q - x_0|^{a-1} \geq 1 + \frac{\delta}{4}, \quad \forall q \geq \bar{q}.$$

By Lemma 1.1 of Part III in [26],

$$|\nabla v_q(x)| \leq \left(\frac{\gamma}{R^n}\right)^{\frac{1}{q}} \left(\int_{B_{\frac{R}{2}}(x_0)} (1 + |\nabla v_q|)^q \, dx\right)^{\frac{1}{q}}, \qquad \forall x \in B_{\frac{R}{2}}(x_0)$$

where γ is a constant independent of q.

For q sufficiently large, this contradict $|\nabla v_q(x_q)| > 1 + \frac{\delta}{4}$. □

Proposition 3.2.8. *Let $f \in L^\infty(\Omega) \cap C(\Omega)$ be a non-negative function, then a continuous weak solution to (3.11) is a viscosity solution.*

Proof. The proof follows the same techniques of Proposition 3.1.3. □

Theorem 3.2.9. *Let $f \in L^\infty(\Omega) \cap C(\Omega)$ be a non-negative function. Then any v_∞ satisfies*

$$|\nabla v_\infty| = 1 \qquad \text{on } \{ f > 0 \} \qquad \text{in the viscosity sense} \tag{3.24}$$

$$-\Delta_\infty v_\infty = 0 \qquad \text{on } (\overline{\{ f > 0 \}})^c \qquad \text{in the viscosity sense} \tag{3.25}$$

Proof. We first prove (3.24). Let $x_0 \in \Omega \cap \{ f > 0 \}$ and let $\varphi \in C^2(\Omega)$ such that $v_\infty - \varphi$ has a strict minimum in x_0. We want to show

$$|\nabla\varphi(x_0)| \geq 1$$

Let us denote by x_p the minimum of $v_p - \varphi$, we have that $x_p \to x_0$, then $x_p \in B_R(x_0) \subset \{ f > 0 \}$ for p large enough. Setting $\varphi_p(x) = \varphi(x) + c_p$, then $c_p = v_p(x_p) - \varphi(x_p) \to 0$ when p goes to infinity. We notice that $v_p(x_p) = \varphi_p(x_p)$ and $v_p - \varphi_p$ has a minimum in x_p, so by Proposition 3.2.8,

$$-|\nabla\varphi_p(x_p)|^{p-2}\Delta\varphi_p(x_p) - (p-2)|\nabla\varphi_p(x_p)|^{p-4}\Delta_\infty\varphi(x_p) \geq f(x_p) > 0.$$

Dividing by $(p-2)|\nabla\varphi_p(x_p)|^{p-4}$, we obtain

$$-\Delta_\infty\varphi_p(x_p) - \frac{|\nabla\varphi_p(x_p)|^2\Delta\varphi_p(x_p)}{p-2} \geq \frac{f(x_p)}{(p-2)|\nabla\varphi_p(x_p)|^{p-4}} \tag{3.26}$$

This gives us $|\nabla\varphi(x_0)| \geq 1$, otherwise the right-hand side would go to infinity, in contradiction with the fact that $\varphi \in C^2(\Omega)$.

We stress that, if we let $p \to \infty$ in (3.26), we get $-\Delta_\infty\varphi(x_0) \geq 0$, so v_∞ is a supersolution to $-\Delta_\infty u = 0$.

Now, we only have to prove that v_∞ is a viscosity solution to $\Delta_\infty\varphi = 0$ in $(\overline{\{ f > 0 \}})^c$.

If we fix $x_0 \in (\overline{\{ f > 0 \}})^c$, and $\varphi \in C^2(\Omega)$ such that $v_\infty - \varphi$ has a strict maximum in x_0, we can choose R such that $B_R(x_0) \subset (\overline{\{ f > 0 \}})^c$. The function $v_p - \varphi$ has a maximum $x_p \to x_0$, and $x_p \in B_R(x_0)$ for p large enough.

The definition of viscosity subsolution implies

$$-|\nabla\varphi_p(x_p)|^{p-2}\Delta\varphi_p(x_p) - (p-2)|\nabla\varphi_p(x_p)|^{p-4}\Delta_\infty\varphi(x_p) \leq f(x_p) = 0.$$

Without loss of generality, we may assume $|\nabla\varphi(x_0)| \neq 0$. Dividing both side of the last equation by $(p-2)|\nabla\varphi_p(x_p)|^{p-4}$, we obtain

$$-\Delta_\infty\varphi_p(x_p) \leq \frac{|\nabla\varphi_p(x_p)|^2\Delta\varphi_p(x_p)}{p-2}.$$

Letting $p \to \infty$, we get

$$-\Delta_\infty\varphi(x_0) \leq 0.$$

Analogously if $\varphi \in C^2(\Omega)$ is such that $v_\infty - \varphi$ has a minimum at x_0, a symmetric argument shows that $-\Delta_\infty\varphi(x_0) \geq 0$. □

We conclude with some illustrative examples.

Example 3.2.3. We consider the case $\Omega = B_1(0)$ and $f = 1$. The uniqueness of the solution and the invariance under rotation of the $p-$Laplacian ensure that the solution v is radially symmetric and

$$r^{n-1}\Delta_p v_p = \frac{d}{dr}\left(r^{n-1}\left|v_p'\right|^{p-2}v_p'\right).$$

Setting $\alpha = 1/(p-1)$, we have

$$v_p(x) = -\frac{p-1}{n^\alpha p}|x|^{\frac{p}{p-1}} + \frac{1}{(n\beta^p)^\alpha} + \frac{p-1}{n^\alpha p}$$

Then

$$v_\infty(x) = -|x| + \frac{1}{\beta} + 1 = \frac{1}{\beta} + d(x, \partial\Omega)$$

Example 3.2.4. We fix $0 < \varepsilon < 1$ and we consider

$$f = \begin{cases} 1 & \text{if } x \in B_\varepsilon(0) \\ 0 & \text{if } x \in B_1(0) \setminus B_\varepsilon(0). \end{cases}$$

In this case, v_p is radially symmetric, and

$$v_p = \begin{cases} \frac{p-1}{n^\alpha p}\left(\varepsilon^{\frac{p}{p-1}} - |x|^{\frac{p}{p-1}}\right) + \frac{\varepsilon^{n\alpha}(p-1)}{n^\alpha(p-n)}\left(1 - \varepsilon^{\frac{p-n}{p-1}}\right) + \frac{\varepsilon^{n\alpha}}{(n\beta^p)^\alpha} & \text{if } x \in B_\varepsilon(0) \\ \frac{\varepsilon^{n\alpha}(p-1)}{n^\alpha(p-n)}\left(1 - |x|^{\frac{p-n}{p-1}}\right) + \frac{\varepsilon^{n\alpha}}{(n\beta^p)^\alpha} & \text{if } x \in B_1(0) \setminus B_\varepsilon(0) \end{cases}$$

Letting p go to infinity, we obtain

$$v_\infty = \frac{1}{\beta} + d(x, \partial\Omega).$$

Chapter 4

Sharp inequalities for some geometric functionals in the class of convex sets

In this Chapter, we will consider three geometric functionals of a convex set Ω: the first Robin eigenvalue of the $p-$Laplacian, the $p-$Torsional rigidity, and the Cheeger constant. We aim to prove sharp bounds of these quantities in terms of other geometrical quantities of a convex set Ω. In some cases, we also aim to analyze the stability of such inequalities

Firstly, we focus on the eigenvalues of the $p-$Laplace operator with Robin boundary conditions, allowing the Robin parameter β to vary in $\mathbb{R}$. We prove an upper bound for this quantity in terms of the isoperimetric ratio when the boundary parameter is positive, while we prove a quantitative version of the reverse Faber-Krahn type inequality for the first Robin eigenvalue of the p-Laplacian whene β is negative, among convex sets with prescribed perimeter.

Then, we consider the $(f,p)-$torsional rigidity for the Poisson problem with Dirichlet boundary conditions, denoted by $T_{f,p}(\Omega)$. We prove a Pólya type lower bound for $T_{f,p}(\Omega)$ in any dimension; then, we consider the planar case and we provide two quantitative estimates in the case $f \equiv 1$.

Eventually, we focus our attention on the Cheeger constant

$$h(\Omega) = \inf_{E \subseteq \Omega} \left\{ \frac{P(E)}{|E|}, |E| > 0 \right\}$$

as we are interested in finding sharp bounds for the Cheeger constant via different geometrical quantities, such as the area $|\cdot|$, the perimeter P, the inradius r, the circumradius R, the minimal width w and the diameter diam. In particular, we completely solve the Blasche-Santaló diagrams describing all the possible inequalities involving the Cheeger constant, the perimeter, and the inradius, then, the Cheeger constant, the diameter and the inradius, and, finally, the Cheeger constant, the circumradius, and the inradius.

The content of this chapter can be also found in [10, 12, 82]

4.1 Estimates for Robin $p-$Laplacian eigenvalues with prescribed perimeter

Let us consider the eigenvalue problem for the $p-$Laplacian

$$\begin{cases} -\Delta_p u = \lambda_{p,\beta}(\Omega)|u|^{p-2}u & \text{in } \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + \beta|u|^{p-2}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

In this section, Ω is a bounded, open and *convex* set in $\mathbb{R}^n$ and $\beta \in \mathbb{R}$. We want to analyze the different behavior of the eigenvalues with respect to the sign of the boundary parameter β.

It is clear from the definition of the eigenvalue (3.2) that for fixed $1 < p < +\infty$ and $\Omega \subset \mathbb{R}^n$, the map

$$\beta \in \mathbb{R} \mapsto \lambda_{p,\beta}(\Omega)$$

is increasing and it holds (see for instance [116])

$$\lambda_{p,N} < \lambda_{p,\beta} < \lambda_{p,D}, \qquad \forall \beta \in (0, +\infty),$$

and

$$\lim_{\beta\to 0^+} \lambda_{p,\beta} = \lambda_{p,N} = 0, \qquad \lim_{\beta\to+\infty} \lambda_{p,\beta} = \lambda_{p,D},$$

where $\lambda_{p,N}$ and $\lambda_{p,D}$ are the first eigenvalues of the Neumann $p-$Laplacian and Dirichlet $p-$Laplacian respectively.

Now we consider the case of $\Omega = B_R$, we will denote by $v_{p,\beta}$ the solution to (4.1). We aim to prove that, if we fix the value of the eigenfunction in the origin, the map

$$\beta \to v_{p,\beta}, \quad \text{such that } v_{p,\beta(0)} = C,$$

is decreasing, in the sense that

$$\text{if } \beta_1 < \beta_2 \Rightarrow v_{p,\beta_1}(x) \geq v_{p,\beta_2}(x), \qquad \forall x \in B_R.$$

With this aim, let us recall that the first eigenvalue of $p-$Laplacian is simple and that the corresponding eigenfunction is radially symmetric, i.e. there exists $h\colon [0,R] \to \mathbb{R}$ such that $v_{p,\beta}(x) = h(|x|)$. In the following, we will denote with $v_{p,\beta}$ both the eigenfunction and the function h.

Lemma 4.1.1. *Let $0 < r < R$, $1 < p < +\infty$, $\beta \in \mathbb{R}$. Let us denote by $\lambda_{p,\beta}$ the first eigenvalue of the $p-$Laplacian on the ball B_R defined in (3.2) with boundary parameter β and let $v_{p,\beta}$ be the corresponding eigenfunction. Then, $v_{p,\beta}|_{B_r}$ is the first eigenfunction of the $p-$Laplacian on the ball B_r with boundary parameter*

$$\gamma = -\frac{|v'_{p,\beta}|^{p-2}(r)v'_{p,\beta}(r)}{v^{p-1}_{p,\beta}(r)}.$$

Proof. Let us suppose $p = 2$, the general case is analogous. For sake of simplicity, we denote by $\lambda_\beta := \lambda_{2,\beta}$ and $v_\beta := v_{2,\beta}$.

By the radiality of v_β, we can infer that it is a solution to

$$\begin{cases} -\Delta v_\beta = \lambda_\beta v_\beta & \text{in } B_r \\ \dfrac{\partial v_\beta}{\partial \nu} + \gamma v_\beta = 0 & \text{on } \partial B_r. \end{cases}$$

Let us suppose by contradiction that λ_β is not the first eigenvalue of the Robin Laplacian with boundary parameter γ. So we can choose

$$\lambda_\gamma < \lambda_\beta, \tag{4.2}$$

the first Robin eigenvalue with boundary parameter γ and w_γ the associated eigenfunction, that is

$$\lambda_\gamma = \frac{\displaystyle\int_{B_r} |\nabla w_\gamma|^2\,dx + \gamma \int_{\partial B_r} w_\gamma^2\,d\mathcal{H}^{n-1}(x)}{\displaystyle\int_{B_r} w_\gamma^2\,dx} = \min_{\psi\in W^{1,2}(B_r)} \frac{\displaystyle\int_{B_r} |\nabla \psi|^2\,dx + \gamma \int_{\partial B_r} \psi^2\,d\mathcal{H}^{n-1}(x)}{\displaystyle\int_{B_r} \psi^2\,dx}. \tag{4.3}$$

We know that w_γ is unique up to a multiplicative constant, so we can choose the constant such that $v_\beta = w_\gamma$ on ∂B_r.

Let us consider the function

$$f(x) = \begin{cases} w_\gamma(x) & \text{if } x \in B_r \\ v_\beta(x) & \text{if } x \in B_R \setminus B_r, \end{cases}$$

we can use it as a test function in the definition of λ_β. In particular, we have

$$\begin{aligned} \lambda_\beta &\leq \frac{\displaystyle\int_{B_R} |\nabla f|^2 + \beta \int_{\partial B_r} f^2\,d\mathcal{H}^{n-1}(x)}{\displaystyle\int_{B_R} f^2\,dx} \\ &= \frac{\displaystyle\int_{B_r} |\nabla w_\gamma|^2 + \int_{B_R\setminus B_r} |\nabla v_\beta|^2 + \beta \int_{\partial B_R} v_\beta^2\,d\mathcal{H}^{n-1}(x)}{\displaystyle\int_{B_r} w_\gamma^2\,dx + \int_{B_R\setminus B_r} v_\beta^2\,dx}. \end{aligned}$$

If we add and subtract $\gamma \displaystyle\int_{\partial B_r} w_\gamma^2\,d\mathcal{H}^{n-1}(x)$ and $\displaystyle\int_{B_r} |\nabla v_\beta|^2\,dx$, and we recall that v_β achieves (3.2) on B_R with value λ_β and w_γ achieves (3.2) on B_r with value λ_γ, we have

$$\lambda_\beta \leq \frac{\lambda_\gamma \displaystyle\int_{B_r} w_\gamma^2\,dx + \lambda_\beta \int_{B_R} v_\beta^2\,dx - \gamma \int_{\partial B_r} w_\gamma^2\,d\mathcal{H}^{n-1}(x) - \int_{B_r} |\nabla v_\beta|^2\,dx}{\displaystyle\int_{B_r} w_\gamma^2\,dx + \int_{B_R\setminus B_r} v_\beta^2\,dx}.$$

Since $v_\beta = w_\gamma$ on ∂B_r and v_β is an eigenfunction on B_r with eigenvalue λ_β, we get

$$
\begin{aligned}
\lambda_\beta &\leq \frac{\lambda_\gamma \displaystyle\int_{B_r} w_\gamma^2\,dx + \lambda_\beta \int_{B_R} v_\beta^2\,dx - \lambda_\beta \int_{B_r} v_\beta^2\,dx}{\displaystyle\int_{B_r} w_\gamma^2\,dx + \int_{B_R\setminus B_r} v_\beta^2\,dx} \\
&= \frac{\lambda_\gamma \displaystyle\int_{B_r} w_\gamma^2\,dx + \lambda_\beta \int_{B_R\setminus B_r} v_\beta^2\,dx}{\displaystyle\int_{B_r} w_\gamma^2\,dx + \int_{B_R\setminus B_r} v_\beta^2\,dx} \\
&< \lambda_\beta \frac{\displaystyle\int_{B_r} w^2\,dx + \int_{B_R\setminus B_r} w^2\,dx}{\displaystyle\int_{B_r} w^2\,dx + \int_{B_R\setminus B_r} w^2\,dx} = \lambda_\beta,
\end{aligned}
$$

where in the last formula we use (4.2), and this is an absurd.

□

Proposition 4.1.2. *Let $R > 0$, $1 < p < +\infty$ and $\beta_1 < \beta_2$. Let us denote by λ_{p,β_1} and λ_{p,β_2} the eigenvalues defined in (3.2) and let v_{p,β_1} and v_{p,β_2} be the corresponding eigenfunctions normalized such that $v_{p,\beta_1}(0) = v_{p,\beta_2}(0) > 0$, then*

$$v_{p,\beta_1}(x) \geq v_{p,\beta_2}(x) \qquad \forall x \in B_R. \tag{4.4}$$

Proof. Let us suppose $p = 2$, the general case is analogous. For sake of simplicity, we denote by $\lambda_{\beta_i} := \lambda_{2,\beta_i}$ and $v_{\beta_i} := v_{2,\beta_i}$.

Since both v_{β_1} and v_{β_2} are radial, we can write the Laplacian in polar coordinates, that is

$$r^{n-1}\Delta u(r) = \left(r^{n-1}u'(r)\right)' \qquad r \in [0, R].$$

Therefore function u_{β_i}, for $i = 1, 2$, satisfies

$$v'_{\beta_i}(r) = -\frac{1}{r^{n-1}}\left[\int_0^r s^{n-1}\lambda_{\beta_i} v_{\beta_i}\,ds\right]. \tag{4.5}$$

Since $\lambda_{\beta_1} < \lambda_{\beta_2}$ and $v_{\beta_1}(0) = v_{\beta_2}(0)$, by continuity there exists $\sigma > 0$ such that

$$\lambda_{\beta_1} v_{\beta_1}(s) < \lambda_{\beta_2} v_{\beta_2}(s), \qquad \forall s \in (0, \sigma),$$

and by (4.5)

$$v'_{\beta_1}(s) > v'_{\beta_2}(s) \qquad s \in (0, \sigma).$$

By classical ODE comparison result, we obtain

$$v_{\beta_1}(s) > v_{\beta_2}(s) \qquad s \in (0, \sigma). \tag{4.6}$$

Let us define

$$A = \{r > \sigma \,:\, v_{\beta_1}(r) = v_{\beta_2}(r)\},$$

we want to prove that A is empty, so by (4.6) we get the claim. Let us suppose by contradiction that $A \neq \varnothing$, hence there exists

$$t = \inf A.$$

By continuity $v_{\beta_1}(t) = v_{\beta_2}(t)$, and this, combined with (4.6), leads to

$$v'_{\beta_1}(t) < v'_{\beta_2}(t). \tag{4.7}$$

Let us set

$$\gamma_i = -\frac{v'_{\beta_i}(t)}{v_{\beta_i}(t)},$$

by (4.7) we have $\gamma_1 > \gamma_2$.
By Lemma 4.1.1, v_{β_1} and v_{β_2} are the first eigenfunctions of the Robin Laplacian in B_t with eigenvalue λ_{γ_1} and λ_{γ_2} with parameter γ_1 and γ_2 respectively. Therefore by monotonicity of eigenvalues with respect to the boundary parameter, we get

$$\lambda_{\beta_1} = \lambda_{\gamma_1} > \lambda_{\gamma_2} = \lambda_{\beta_2}$$

that is absurd. □

The interest in the eigenfunction of the ball is motivated by the Faber-Krahn inequality

$$\lambda_{p,\beta}(\Omega^\sharp) \leq \lambda_{p,\beta}(\Omega), \tag{4.8}$$

that asserts that among the sets of given volume the ball minimizes the first Robin eigenvalue with positive boundary parameter. In what follows, we will study the Robin eigenvalues among convex sets of given perimeter, in this setting it is possible to prove that

$$\lambda_{p,\beta}(\Omega^\star) \leq \lambda_{p,\beta}(\Omega), \tag{4.9}$$

where $\Omega^\star$ is the ball having the same perimeter as Ω. Inequality (4.9) can be easily deduced by (4.8) and the following rescaling property (see [44])

$$\lambda_{p,\beta}(t\Omega) \leq \frac{1}{t}\lambda_{p,\beta}(\Omega) \leq \lambda_{p,\beta}(\Omega), \quad \forall t > 1. \tag{4.10}$$

Our aim is to give a continuity bound, in terms of the isoperimetric deficit, to the ratio

$$\frac{\lambda_{p,\beta}(\Omega) - \lambda_{p,\beta}(\Omega^\star)}{\lambda_{p,\beta}(\Omega)},$$

indeed, we prove

Theorem 4.1.3. *Let β be a positive parameter. Let Ω be a bounded, open and convex set in $\mathbb{R}^n$ and let $\Omega^\star$ be the ball, centered at the origin, such that $P(\Omega) = P(\Omega^\star) = \rho$. Denote by $\lambda_{p,\beta}(\Omega)$ and $\lambda_{p,\beta}(\Omega^\star)$ the first eigenvalues of the $p-$Laplacian operator with Robin boundary conditions respectively on Ω and $\Omega^\star$, and by v a positive eigenfunction associated to $\lambda_{p,\beta}(\Omega^\star)$, then*

$$\frac{\lambda_{p,\beta}(\Omega) - \lambda_{p,\beta}(\Omega^\star)}{\lambda_{p,\beta}(\Omega)} \leq C(n,p,\beta,\rho)\left(1 - \frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{P(\Omega)^{\frac{n}{n-1}}}\right), \tag{4.11}$$

where ω_n is the measure of the unitary ball in $\mathbb{R}^n$, and $C(n,p,\beta,\rho) = \dfrac{\|v\|_\infty^p|\Omega^\star|}{\|v\|_p^p}$.

Proof. The quantity

$$\frac{\lambda_{p,\beta}(\Omega)-\lambda_{p,\beta}(\Omega^\star)}{\lambda_{p,\beta}(\Omega)},$$

is bounded from above by 1, so inequality (4.11) is trivial when

$$\left(1-\frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{P(\Omega)^{\frac{n}{n-1}}}\right)\geq\frac{1}{C(n,p,\beta,\rho)}=\frac{\|v\|_\infty^p|\Omega^\star|}{\|v\|_p^p}.$$

We can assume

$$\left(1-\frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{P(\Omega)^{\frac{n}{n-1}}}\right)<\frac{1}{C(n,p,\beta,\rho)}. \tag{4.12}$$

Let v be the solution to

$$\begin{cases}-\Delta_p v=\lambda_{p,\beta}(\Omega^\star)|v|^{p-2}v & \text{in } \Omega^\star\\ |\nabla v|^{p-2}\dfrac{\partial v}{\partial\nu}+\beta|v|^{p-2}v=0 & \text{on } \partial\Omega^\star,\end{cases} \tag{4.13}$$

as we have already remarked, v is positive and radially symmetric. So the function

$$g(t)=|\nabla v|_{v=t}$$

is well defined for all $t\in(v_m,v_M)$, where $v_m=\min_{\Omega^\star}v$ and $v_M=\max_{\Omega^\star}v$.

Let us define $u(x)=G(d(x,\partial\Omega))$, $x\in\Omega$, where

$$G^{-1}(t)=\int_{v_m}^t\frac{1}{g(s)}\,ds. \qquad v_m<t<v_M.$$

By construction, $u\in W^{1,p}(\Omega)$ and

$$\min_\Omega u=G(0)=v_m,$$
$$\|u\|_\infty\leq v_M,$$
$$|\nabla u|_{u=t}=\left|G'(d(x))\right|_{u=t}=g(t)=|\nabla v|_{v=t}.$$

Let

$$E_t=\{x\in\Omega:\ u(x)>t\},\ \ B_t=\{x\in\Omega^\star:\ v(x)>t\}.$$

By Lemma 1.3.4 and formula (1.17) we have

$$-\frac{d}{dt}P(E_t)\geq(n-1)\frac{W_2(E_t)}{g(t)}\geq(n-1)n^{-\frac{n-2}{n-1}}\omega_n^{\frac{1}{n-1}}\frac{(P(E_t))^{\frac{n-2}{n-1}}}{g(t)},$$

while for v it holds

$$-\frac{d}{dt}P(B_t)=(n-1)\frac{W_2(B_t)}{g(t)}=(n-1)n^{-\frac{n-2}{n-1}}\omega_n^{\frac{1}{n-1}}\frac{(P(B_t))^{\frac{n-2}{n-1}}}{g(t)},$$

and $P(E_0)=P(B_0)$. Then, by classical comparison theorems for ODEs, it holds

$$P(E_t)\leq P(B_t),\quad 0\leq t\leq\|u\|_\infty. \tag{4.14}$$

Denoting by $\mu(t) = |E_t|$ and by $\nu(t) = |B_t|$, the coarea formula (1.2) ensures us that

$$\begin{aligned}-\mu'(t) &= \int_{u=t} \frac{1}{|\nabla u|}\, d\mathcal{H}^{n-1}(x) = \frac{P(E_t)}{g(t)} \leq \frac{P(B_t)}{g(t)} \\ &= \int_{v=t} \frac{1}{|\nabla v|}\, d\mathcal{H}^{n-1}(x) = -\nu'(t),\end{aligned}$$

for almost every $t \in (0, \|u\|_\infty)$. The first equality holds true since $|\nabla u| \neq 0$ in $\{ v_m < u < \|u\|_\infty \}$, so the function μ is absolutely continuous as remarked in Section 1.2 (see also [43]). So the function $\nu - \mu$ is decreasing in $[0, \|u\|_\infty]$, and

$$\begin{aligned}\int_\Omega u^p\, dx &= \int_0^{\|u\|_\infty} pt^{p-1}\mu(t)\, dt = \int_0^{v_M} pt^{p-1}\mu(t)\, dt \\ &= \int_0^{v_M} pt^{p-1}\nu(t)\, dt - \int_0^{v_M} pt^{p-1}(\nu(t)-\mu(t))\, dt \geq \int_{\Omega^\star} v^p\, dx - v_M^p(|\Omega^\star| - |\Omega|).\end{aligned}$$

Moreover, by (4.14), we get

$$\int_{u=t} |\nabla u|^{p-1}\, d\mathcal{H}^{n-1}(x) = g(t)^{p-1}P(E_t) \leq g(t)^{p-1}P(B_t) = \int_{v=t} |\nabla v|^{p-1}\, d\mathcal{H}^{n-1}(x),$$

so, if we integrate over Ω,

$$\int_\Omega |\nabla u|^p\, dx \leq \int_0^{\|u\|_\infty} \int_{v=t} |\nabla v|^{p-1}\, d\mathcal{H}^{n-1}(x)\, dt \leq \int_{\Omega^\star} |\nabla v|^p.$$

We observe that by construction both u and v are constant on $\partial\Omega$, so

$$\beta \int_{\partial\Omega} u^p\, d\mathcal{H}^{n-1}(x) = \beta v_m^p P(\Omega) = \beta \int_{\partial\Omega^\star} v^p\, d\mathcal{H}^{n-1}(x).$$

We eventually get

$$\begin{aligned}\lambda_{p,\beta}(\Omega) &\leq \frac{\displaystyle\int_\Omega |\nabla u|^p\, dx + \beta\int_{\partial\Omega} u^p\, d\mathcal{H}^{n-1}(x)}{\displaystyle\int_\Omega u^p\, dx} \leq \frac{\displaystyle\int_{\Omega^\star} |\nabla v|^p\, dx + \beta\int_{\partial\Omega^\star} v^p\, d\mathcal{H}^{n-1}(x)}{\displaystyle\int_{\Omega^\star} v^p\, dx - v_M^p(|\Omega^\star| - |\Omega|)} \\ &= \lambda_{p,\beta}(\Omega^\star)\, \frac{1}{1 - C(n,p,\beta,\rho)\left(1 - \dfrac{|\Omega|}{|\Omega^\star|}\right)}.\end{aligned}$$

The claim follows from (4.12) as the quantity

$$1 - C(n,p,\beta,\rho)\left(1 - \frac{|\Omega|}{|\Omega^\star|}\right),$$

is non-negative. □

Remark 4.1.1. The constant $C(n,p,\beta,\rho) = \dfrac{\|v\|_\infty^p |\Omega^\star|}{\|v\|_p^p}$ depends on the perimeter of the set Ω and on β. It is possible to give a uniform bound to the constant in (4.11) from above with a constant independent of the parameter β and the perimeter, indeed thanks to Proposition 4.1.2 if we fix the value $v_{p,\beta}(0)$, (4.4) implies that

$$\beta \mapsto \|v_{p,\beta}\|_{L^p}$$

is non-increasing, while the map

$$\beta \mapsto C(n,p,\beta,\rho) = \frac{v_{p,\beta}(0)|\Omega^\star|}{\|v_{p,\beta}\|_p^p}$$

is non-decreasing, for all $\beta \in \mathbb{R}$.

So, if we denote by $v_{p,\infty}$ the first Dirichlet eigenfunction normalized in such a way $v_{p,\beta}(0) = v_{p,\infty}(0)$, we have

$$C(n,p,\beta,\rho) \le \frac{v_{p,\infty}(0)|\Omega^\star|}{\|v_{p,\infty}\|_{L^p}^p} =: C(n,p)$$

that is independent of the perimeter thanks to the rescaling properties of the Dirichlet $p-$Laplacian eigenfunction.

When β is a negative parameter, the authors in [46] proved a reverse Faber-Krahn inequality for the first eigenvalue of the Robin-Laplacian among convex sets of given perimeter. In particular, they proved that among convex sets of given perimeter the ball $\Omega^\star$ maximizes the first Robin eigenvalue of the $p-$Laplacian, i.e.

$$\lambda_{p,\beta}(\Omega) \le \lambda_{p,\beta}(\Omega^\star). \tag{4.15}$$

Once again, we aim to give a quantitative estimate, in terms of the isoperimetric deficit, for the ratio

$$\frac{\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega)}{|\lambda_{p,\beta}(\Omega)|},$$

Theorem 4.1.4. *Let β be a negative parameter. Let Ω be a bounded, open and convex set in $\mathbb{R}^n$ and let $\Omega^\star$ be the ball, centered at the origin, such that $P(\Omega) = P(\Omega^\star) = \rho$. Denote by $\lambda_{p,\beta}(\Omega)$ and $\lambda_{p,\beta}(\Omega^\star)$ the first eigenvalues of the $p-$Laplacian operator with Robin boundary conditions respectively on Ω and $\Omega^\star$, and by v a positive eigenfunction associated to $\lambda_{p,\beta}(\Omega^\star)$, then*

$$\frac{\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega)}{|\lambda_{p,\beta}(\Omega)|} \ge C(n,p,\beta,\rho)\left(1 - \frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{P(\Omega)^{\frac{n}{n-1}}}\right), \tag{4.16}$$

where ω_n is the measure of the unitary ball in $\mathbb{R}^n$, and $C(n,p,\beta,\rho) = \dfrac{v_m^p|\Omega^\star|}{\|v\|_p^p}$ with $v_m = \min_{\Omega^\star} v$.

Proof. Let v be a positive eigenfunction associated to $\lambda_{p,\beta}(\Omega^\star)$, then v is a p sub-harmonic function as $\lambda_{p,\beta}(\Omega^\star) < 0$. We denote by $v_m = v(0) = \min_{\Omega^\star} v$ and by $v_M = \max_{\Omega^\star} v$.

Let us consider the function $\tilde{v} = v_M - v$, that is a positive function with zero trace, and

$$g(t) = |\nabla \tilde{v}|_{\tilde{v}=t}, \quad 0 < t < v_M - v_m.$$

We set $\tilde{u}(x) = G(d(x,\partial\Omega))$, $x \in \Omega$, where $G^{-1}(t) = \displaystyle\int_0^t \frac{1}{g(s)}\,ds$ with $0 < t < v_M - v_m$. By construction, $\tilde{u} \in W_0^{1,p}(\Omega)$. Now we can set $u = v_M - \tilde{u}$, and we have:

$$u_M = \max_{\Omega} u = v_M$$

$$u_m = \min_{\Omega} u = v_M - \max_{\Omega} \tilde{u} \ge v_M - \max_{\Omega^\star} \tilde{v} = \min_{\Omega^\star} v = v_m$$

$$|\nabla u|_{u=t} = |\nabla \tilde{u}|_{\tilde{u}=v_M-t} = |\nabla \tilde{v}|_{\tilde{v}=v_M-t} |\nabla v|_{v=t} = g(t) \quad u_m < t < u_M.$$

Let

$$\widetilde{E}_t = \{x \in \Omega \,:\, \tilde{u}(x) > t\}, \qquad \widetilde{B}_t = \{x \in \Omega \,:\, \tilde{v}(x) > t\},$$
$$E_t = \{x \in \Omega \,:\, u(x) > t\} = \Omega \setminus \overline{\widetilde{E}}_{v_M - t}, \quad B_t = \{x \in \Omega \,:\, v(x) > t\} = \Omega \setminus \overline{\widetilde{B}}_{v_M - t}.$$

By Lemma 1.3.4 and formula (1.17) we have

$$-\frac{d}{dt} P(\widetilde{E}_t) \geq (n-1) \frac{W_2(\widetilde{E}_t)}{g(t)} \geq (n-1) n^{-\frac{n-2}{n-1}} \omega_n^{\frac{1}{n-1}} \frac{(P(\widetilde{E}_t))^{\frac{n-2}{n-1}}}{g(t)},$$

while for $\tilde{v}$ it holds

$$-\frac{d}{dt} P(\widetilde{B}_t) = (n-1) \frac{W_2(\widetilde{B}_t)}{g(t)} = (n-1) n^{-\frac{n-2}{n-1}} \omega_n^{\frac{1}{n-1}} \frac{(P(\widetilde{B}_t))^{\frac{n-2}{n-1}}}{g(t)},$$

and $P(E_0) = P(B_0)$. Then, by classical comparison theorems for ODEs,

$$P(\widetilde{E}_t) \leq P(\widetilde{B}_t), \quad 0 \leq t \leq v_M - v_m.$$

Hence

$$P(E_t) = P(\widetilde{E}_{v_M - t}) \leq P(\widetilde{B}_{v_M - t}) = P(B_t), \quad v_m \leq t \leq v_M. \tag{4.17}$$

Denoting by $\tilde{\mu}(t) = |\widetilde{E}_t|$ and $\tilde{\nu}(t) = |\widetilde{B}_t|$, the coarea formula (1.2) ensures us that

$$\begin{aligned}
-\tilde{\mu}'(t) &= \int_{\tilde{u}=t} \frac{1}{|\nabla \tilde{u}|} \, d\mathcal{H}^{n-1}(x) = \frac{P(\widetilde{E}_t)}{g(t)} \leq \frac{P(\widetilde{B}_t)}{g(t)} \\
&= \int_{\tilde{v}=t} \frac{1}{|\nabla \tilde{v}|} \, d\mathcal{H}^{n-1}(x) = -\tilde{\nu}'(t), \qquad 0 \leq t < v_M - v_m.
\end{aligned}$$

Moreover, setting $\mu(t) = |E_t| = |\Omega| - \tilde{\mu}(v_M - t)$ and by $\nu(t) = |B_t| = |\Omega^\star| - \tilde{\nu}(v_M - t)$, we have $-\mu'(t) \leq -\nu'(t)$ in $[v_m, v_M]$. So the function $\nu - \mu$ is decreasing in $[v_m, v_M]$, and

$$\begin{aligned}
\int_\Omega u^p \, dx &= \int_0^{u_M} p t^{p-1} \mu(t) \, dt = \int_0^{v_M} p t^{p-1} \nu(t) \, dt - \int_0^{v_M} p t^{p-1} (\nu(t) - \mu(t)) \, dt \\
&= \int_0^{v_M} p t^{p-1} \nu(t) \, dt - \int_0^{v_m} p t^{p-1} (\nu(t) - \mu(t)) \, dt - \int_{v_m}^{v_M} p t^{p-1} (\nu(t) - \mu(t)) \, dt \\
&\leq \int_{\Omega^\star} v^p \, dx - v_m^p (|\Omega^\star| - |\Omega|) = \int_{\Omega^\star} v^p \, dx \left[1 - \frac{v_m^p}{\|v\|_p^p} (|\Omega^\star| - |\Omega|) \right].
\end{aligned}$$

By (4.17), we get

$$\int_{\tilde{u}=t} |\nabla \tilde{u}|^{p-1} \, d\mathcal{H}^{n-1}(x) = g(t)^{p-1} P(\widetilde{E}_t) \leq g(t)^{p-1} P(\widetilde{B}_t) = \int_{\tilde{v}=t} |\nabla \tilde{v}|^{p-1} \, d\mathcal{H}^{n-1}(x),$$

so, if we integrate over Ω,

$$\begin{aligned}
\int_\Omega |\nabla u|^p \, dx = \int_\Omega |\nabla \tilde{u}|^p \, dx &= \int_0^{\|\tilde{u}\|_\infty} \int_{\tilde{u}=t} |\nabla \tilde{u}|^{p-1} \, d\mathcal{H}^{n-1}(x) \, dt \\
&\leq \int_0^{\|\tilde{u}\|_\infty} \int_{\tilde{v}=t} |\nabla \tilde{v}|^{p-1} \, d\mathcal{H}^{n-1}(x) \, dt = \int_{\Omega^\star} |\nabla \tilde{v}|^p = \int_{\Omega^\star} |\nabla v|^p.
\end{aligned}$$

We also observe that by construction both u and v are constant on $\partial\Omega$, so

$$\beta \int_{\partial\Omega} u^p \, d\mathcal{H}^{n-1}(x) = \beta u_M^p P(\Omega) = \beta v_M^p P(\Omega) = \beta \int_{\partial\Omega^\star} v^p \, d\mathcal{H}^{n-1}(x).$$

We eventually get

$$\begin{aligned}\lambda_{p,\beta}(\Omega) &\leq \frac{\displaystyle\int_\Omega |\nabla u|^p \, dx + \beta \int_{\partial\Omega} u^p \, d\mathcal{H}^{n-1}(x)}{\displaystyle\int_\Omega u^p \, dx} \leq \frac{\displaystyle\int_{\Omega^\star} |\nabla v|^p \, dx + \beta \int_{\partial\Omega^\star} v^p \, d\mathcal{H}^{n-1}(x)}{\displaystyle\int_{\Omega^\star} v^p \, dx \Big[1 - \frac{v_m^p}{\|v\|_p^p}(|\Omega^\star| - |\Omega|)\Big]} \\ &= \lambda_{p,\beta}(\Omega^\star) \, \frac{1}{1 - \dfrac{v_m^p}{\|v\|_p^p}\,(|\Omega^\star| - |\Omega|)}.\end{aligned}$$

Hence, by direct calculation

$$\frac{\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega)}{|\lambda_{p,\beta}(\Omega)|} \geq \frac{v_m^p}{\|v\|_p^p}(|\Omega^\star| - |\Omega|). \tag{4.18}$$

□

Remark 4.1.2. Unlike the constant in Theorem 4.1.3, the constant $C(n, p, \beta, \rho) = \dfrac{v_m^p |\Omega^\star|}{\|v\|_p^p}$ cannot be bounded from below with a constant independent of the perimeter and of β. Indeed, for example, if $n = p = 2$ and $P(\Omega) = 2\pi$, we have that

$$v_\beta(x) = I_0\left(\sqrt{-\lambda_\beta(B_1)}|x|\right),$$

where I_0 is the modified Bessel function.

We recall that for z sufficiently large (see [1, Section 9.7]), we have

$$I_0(z) \sim \frac{e^z}{\sqrt{2\pi z}}\left[1 + \frac{1}{8z} + \frac{9}{2!(8z)^2} + \dots\right]$$

therefore

$$\|v_\beta(x)\|_{L^2} \sim \frac{e^{-\beta}}{\beta^2} \xrightarrow{\beta \to -\infty} +\infty$$

and

$$C(2, 2, \beta, 2\pi) = \frac{(v_\beta)_m^2 |B_1|}{\|v_\beta\|_2^2} \sim \frac{\beta^2}{e^{-\beta}} \xrightarrow{\beta \to -\infty} 0.$$

Remark 4.1.3. We want to highlight that the constant C depends actually just on n, p and $\rho^{\frac{p-1}{n-1}}\beta$. Indeed, for all $\Omega \subset \mathbb{R}^n$ bounded and convex set with $P(\Omega) = \rho$, we can consider

$$\Omega_1 = \left(\frac{n\omega_n}{\rho}\right)^{\frac{1}{n-1}} \Omega \qquad \Omega_1^\star = \left(\frac{n\omega_n}{\rho}\right)^{\frac{1}{n-1}} \Omega^\star \qquad t := \left(\frac{\rho}{n\omega_n}\right)^{\frac{1}{n-1}}$$

so $P(\Omega_1) = P(\Omega_1^\star) = n\omega_n$ and we have

$$\begin{aligned}\frac{\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega)}{-\lambda_{p,\beta}(\Omega)} &= \frac{\lambda_{p,\beta}(t\Omega_1^\star) - \lambda_{p,\beta}(t\Omega_1)}{-\lambda_{p,\beta}(t\Omega_1)}\\ &= \frac{\lambda_{p,t^{p-1}\beta}(\Omega_1^\star) - \lambda_{p,t^{p-1}\beta}(\Omega_1)}{\lambda_{p,t^{p-1}\beta}(\Omega_1)}\\ &\geq C(n,p,n\omega_n,t^{p-1}\beta)\left(1 - \frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{P(\Omega)^{\frac{n}{n-1}}}\right)\\ &= C\left(n,p,n\omega_n,\left(\frac{\rho}{n\omega_n}\right)^{\frac{p-1}{n-1}}\beta\right)\left(1 - \frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{P(\Omega)^{\frac{n}{n-1}}}\right)\\ &= C(n,p,\rho^{\frac{p-1}{n-1}}\beta)\left(1 - \frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{P(\Omega)^{\frac{n}{n-1}}}\right)\end{aligned}$$

Now we are in position to prove the quantitative result as in [57].

Theorem 4.1.5. *Let $n \geq 2$, $\rho > 0$ and $\beta < 0$. Then, there exist two positive constants $C(n,p,\beta,\rho) > 0$ and $\delta_0(n,p,\beta,\rho) > 0$, such that, for all $\Omega \subset \mathbb{R}^n$ bounded and convex with $P(\Omega) = \rho$ and $\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega) \leq \delta_0$, it holds*

$$\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega) \geq C(n,p,\beta,\rho) g(\mathcal{A}_{\mathcal{H}}^\star(\Omega)) \tag{4.19}$$

where $\Omega^\star$ is a ball with the same perimeter of Ω, $\mathcal{A}_{\mathcal{H}}^\star$ is the Hausdorff asymmetry defined in (1.53) and g is defined in (1.57).

Proof of Theorem 4.1.5. If we choose $w = 1$ in the definition of $\lambda_{p,\beta}(\Omega)$ (3.2), we get

$$|\lambda_{p,\beta}(\Omega)| \geq |\beta|\frac{P(\Omega)}{|\Omega|},$$

and if we combine it with (4.18) we have

$$\begin{aligned}\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega) &\geq |\lambda_{p,\beta}(\Omega)|\frac{v_m^p}{\|v\|_p^p}(|\Omega^\star| - |\Omega|)\\ &\geq |\beta|\frac{P(\Omega)}{|\Omega|}\frac{v_m^p}{\|v\|_p^p}(|\Omega^\star| - |\Omega|)\\ &\geq |\beta|\frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}}{\rho^{\frac{1}{n-1}}}\frac{v_m^p}{\|v\|_p^p}(|\Omega^\star| - |\Omega|).\end{aligned} \tag{4.20}$$

Now, if we suppose that $\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega) \leq \delta_0$, by (4.20) we have

$$|\Omega^\star| - |\Omega| \leq K(n,\rho,p,\beta)\delta_0.$$

So by Lemma 1.4.14 we conclude

$$\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega) \geq C(n,\rho,p,\beta)\, g(\mathcal{A}_{\mathcal{H}}^\star(\Omega)),$$

that is exactly (4.19). □

Remark 4.1.4. Our result, Theorem 4.1.5, applies only when

$$\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega) \le \delta_0.$$

It is possible to get rid of this constraint, obtaining a weaker result.

In order to obtain it, we need the quantitative version of the isoperimetric inequality proved in [84]

$$P(\Omega) \ge P(\Omega^\sharp)(1+\gamma(n)\alpha(\Omega)^2) \qquad \text{where } \alpha(\Omega) = \min\left\{ \frac{|\Omega \triangle B_r|}{|\Omega|} \;\middle|\; |B_r| = |\Omega| \right\}, \tag{4.21}$$

and the following result [70, Lemma 4.2]: there exists a constant $C(n)$ such that if C, W are open and convex sets such that $|C| = |W|$ and $|C \triangle W| < \frac{|C|}{2}$, it holds

$$d_{\mathcal{H}}(C,W) \le C(n)\big[\operatorname{diam}(C) + \operatorname{diam}(W)\big]\left(\frac{|C\triangle W|}{|C|}\right)^{\frac{1}{n}}. \tag{4.22}$$

Moreover, we have to recall that

$$|\Omega^\star| = \frac{P(\Omega)^{\frac{n}{n-1}}}{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}} \qquad \text{and} \qquad P(\Omega^\sharp) = n\omega_n^{\frac{1}{n}}|\Omega|^{\frac{n-1}{n}},$$

so we have

$$\begin{aligned} 1 - \frac{|\Omega|}{|\Omega^\star|} &= 1 - \frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{P(\Omega)} \\ &\ge 1 - \frac{1}{(1+\gamma(n)\alpha^2(\Omega))^{\frac{n}{n-1}}} \\ &\ge 1 - \frac{1}{(1+\gamma(n)\alpha^2(\Omega))} \\ &= \frac{\gamma(n)\alpha^2(\Omega)}{1+\gamma(n)\alpha^2(\Omega)}, \end{aligned}$$

where we used Bernoulli's inequality

$$(1+x)^r \ge 1 + rx \qquad \forall x \ge -1,\, \forall r \ge 0.$$

Since $0 < \alpha^2(\Omega) < 4$ we have

$$1 - \frac{|\Omega|}{|\Omega^\star|} \ge \frac{\gamma(n)}{1+4\gamma(n)}\,\alpha^2(\Omega) = C(n)\alpha^2(\Omega).$$

If $\Omega^\sharp$ is a ball that realizes the minimum in (4.21), then using (4.22) we obtain

$$C(n)\alpha^2(\Omega) \ge C(n)\,\frac{d_{\mathcal{H}}(\Omega,\Omega^\sharp)^{2n}}{\big[\operatorname{diam}(\Omega) + \operatorname{diam}(\Omega^\sharp)\big]^{2n}},$$

and so

$$\frac{\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega)}{|\lambda_{p,\beta}(\Omega)|} \ge C(n,p,\beta\rho)C(n)\frac{d_{\mathcal{H}}(\Omega,\Omega^\sharp)^{2n}}{\big[\operatorname{diam}(\Omega) + \operatorname{diam}(\Omega^\sharp)\big]^{2n}}.$$

4.2 Sharp and quantitative estimates for the $p-$Torsion

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a non-empty, bounded, open, and convex set and let $p \in (1, +\infty)$. We consider the Poisson equation for the $p-$Laplace operator with Dirichlet boundary condition:

$$\begin{cases} -\Delta_p u(x) = f(d(x, \partial\Omega)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.23}$$

where $f : [0, r(\Omega)] \to [0, +\infty[$ is a continuous, non-increasing and not identically zero function and $d(\cdot, \partial\Omega) : \Omega \to [0, +\infty[$ This class of functions, depending only on the distance, are the so-called *web functions*, see as a reference [59]. A function $u \in W_0^{1,p}(\Omega)$ is a weak solution to (4.23) if and only if

$$\int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \nabla \varphi(x)\, dx = \int_\Omega f(d(x, \partial\Omega)) \varphi(x)\, dx \quad \forall \varphi \in W_0^{1,p}(\Omega).$$

The (f, p)-torsional rigidity of Ω, that we denote by $T_{f,p}(\Omega)$, is defined as

$$T_{f,p}(\Omega) = \max_{\substack{\varphi \in W_0^{1,p}(\Omega) \\ \varphi \not\equiv 0}} \frac{\left(\displaystyle\int_\Omega f(d(x, \partial\Omega)) |\varphi(x)|\, dx \right)^{\frac{p}{p-1}}}{\left(\displaystyle\int_\Omega |\nabla \varphi(x)|^p\, dx \right)^{\frac{1}{p-1}}} \tag{4.24}$$

and, if $u_p \in W_0^{1,p}(\Omega)$ is the unique solution to (4.23), we have

$$T_{f,p}(\Omega) = \int_\Omega f u_p\, dx.$$

For the sake of simplicity, when $f \equiv 1$ in Ω, we set $T_p(\Omega) := T_{1,p}(\Omega)$ and, if we are also in the case $p = 2$, we set $T(\Omega) := T_{1,2}(\Omega)$. We recall that the quantities $T(\Omega)$ and $T_p(\Omega)$ are usually called, respectively, torsional rigidity and $p-$torsional rigidity and so, by analogy, we have chosen the above terminology for $T_{f,p}(\Omega)$.

Theorem 4.2.1. *Let Ω be a non-empty, bounded, open and convex set of $\mathbb{R}^n$, $n \geq 2$, and let $f : [0, r(\Omega)] \to [0, +\infty[$ be a continuous and non-increasing function such that $f \not\equiv 0$. Then,*

$$T_{f,p}(\Omega) \geq c_p \frac{\mu_f^{q+1}(\Omega)}{f(0) P^q(\Omega)}, \tag{4.25}$$

where

$$c_p = \frac{p-1}{2p-1}, \qquad q = \frac{p}{p-1},$$

and

$$\mu_f(\Omega) = \int_\Omega f(d(x, \partial\Omega))\, dx.$$

Moreover, the equality sign is asymptotically achieved by a sequence of thinning cylinders.

We split it in two parts: firstly we prove inequality (4.25) and, then, we prove its sharpness.

Step 1: proof of inequality (4.25) in Theorem 4.2.1

Proof. Let us choose in the variational characterization (4.24) $\varphi(x) = g(d(x))$ as a test function, where g is a positive and non-decreasing function in $W^{1,p}([0, r(\Omega)])$ such that $g(0) = 0$. Then, by coarea formula (1.2),

$$\int_\Omega f(d(x,\partial\Omega))\varphi(x)\,dx = \int_0^{r(\Omega)} f(t)g(t)P(t)\,dt \tag{4.26}$$

and

$$\int_\Omega |\nabla\varphi(x)|^p\,dx = \int_0^{r(\Omega)} g'^p(t)P(t)\,dt. \tag{4.27}$$

By (4.24), (4.26) and (4.27) we have

$$T_{f,p}(\Omega) \geq \frac{\left(\displaystyle\int_0^{r(\Omega)} f(t)g(t)P(t)\,dt\right)^{\frac{p}{p-1}}}{\left(\displaystyle\int_0^{r(\Omega)} g'^p(t)P(t)\,dt\right)^{\frac{1}{p-1}}}. \tag{4.28}$$

Now, if we define the following measure

$$\mu_f(E) = \int_E f(d(x,\partial\Omega))\,dx,$$

we have

$$\mu_f(t) := \mu_f(\Omega_t) = \int_{\Omega_t} f(d(x,\partial\Omega))\,dx = \int_t^{r(\Omega)} f(s)P(s)\,ds, \tag{4.29}$$

where $P(s) = P(\Omega_s)$. Since $f(s)P(s)$ is a decreasing function, we get

$$\mu_f(t) \leq (r(\Omega) - t)f(t)P(t). \tag{4.30}$$

From (4.29), we have

$$-\mu_f'(t) = f(t)P(t) \qquad \text{a.e. } t \in [0, r(\Omega)]. \tag{4.31}$$

Using (4.26), (4.31) and integrating by parts, we obtain

$$\int_0^{r(\Omega)} f(t)g(t)P(t)\,dt = -\int_0^{r(\Omega)} g(t)\mu_f'(t)\,dt = \int_0^{r(\Omega)} g'(t)\mu_f(t)\,dt.$$

Consequently, (4.28) becomes

$$T_{f,p}(\Omega) \geq \frac{\left(\displaystyle\int_0^{r(\Omega)} g'(t)\mu_f(t)\,dt\right)^{\frac{p}{p-1}}}{\left(\displaystyle\int_0^{r(\Omega)} g'^p(t)P(t)\,dt\right)^{\frac{1}{p-1}}}.$$

We can choose

$$g(t) = \int_0^t \left(\frac{\mu_f(s)}{P(s)}\right)^{1/(p-1)} ds$$

and we observe that $g \in W^{1,p}([0, r(\Omega)])$, since, using (4.30), we have

$$g(t) \le \int_0^{r(\Omega)} (r(\Omega) - s)^{\frac{1}{p-1}} f(s)^{\frac{1}{p-1}}\, ds \le \|f\|_{L^\infty}^{\frac{1}{p-1}} r(\Omega)^{\frac{p}{p-1}} \in L^p([0, r(\Omega)]),$$
$$g'(t) \le \|f\|_{L^\infty}^{\frac{1}{p-1}} r(\Omega)^{\frac{1}{p-1}} \in L^p([0, r(\Omega)]).$$

So, we have

$$T_{f,p}(\Omega) \ge \int_0^{r(\Omega)} \frac{\mu_f^{\frac{p}{p-1}}(t)}{P^{\frac{1}{p-1}}(t)}\, dt = -\frac{p-1}{2p-1} \int_0^{r(\Omega)} \frac{(\mu_f^{\frac{2p-1}{p-1}}(t))'}{f(t) P^{\frac{p}{p-1}}(t)}\, dt. \tag{4.32}$$

Let us set $c_p = (p-1)/(2p-1)$. Since $f(s)$ is a non-negative and non-increasing function, integrating by parts in (4.32), we get

$$\begin{aligned} T_{f,p}(\Omega) &\ge -c_p \int_0^{r(\Omega)} \frac{(\mu_f^{\frac{2p-1}{p-1}}(t))'}{f(t) P^{\frac{p}{p-1}}(t)}\, dt = -c_p \frac{\mu_f^{\frac{2p-1}{p-1}}(t)}{f(t) P^{\frac{p}{p-1}}(t)}\Bigg|_0^{r(\Omega)} + \\ &\quad - c_p \int_0^{r(\Omega)} \frac{\mu_f^{\frac{2p-1}{p-1}}(t)}{f^2(t) P^{\frac{2p}{p-1}}(t)} \left(f'(t) P^{\frac{p}{p-1}}(t) + \frac{p}{p-1} f(t) P^{\frac{1}{p-1}}(t) P'(t) \right) dt \\ &\ge c_p \frac{\mu_f^{\frac{2p-1}{p-1}}(\Omega)}{f(0) P^{\frac{p}{p-1}}(\Omega)} + \frac{c_p}{P^{\frac{p}{p-1}}(\Omega)} \int_0^{r(\Omega)} \frac{\mu_f^{\frac{2p-1}{p-1}}(t)}{f^2(t)} (-f'(t))\, dt, \end{aligned} \tag{4.33}$$

where in the last inequality we use (4.30) and the fact that $P'(t) \le 0$. Now, since $f(s)$ is non-increasing, we obtain the desired estimate

$$T_{f,p}(\Omega) \ge c_p \frac{\mu_f^{\frac{2p-1}{p-1}}(\Omega)}{f(0) P^{\frac{p}{p-1}}(\Omega)}. \tag{4.34}$$

□

Step 2: proof of the sharpness of (4.25)

Proof. We prove that inequality (4.25) is sharp and that the optimum is asymptotically achieved by the sequence of thinning cylinders Ω_l with unitary measure, as defined in (1.47), that is

$$\Omega_l = l^{-\frac{1}{n-1}} C \times \left(-\frac{l}{2}, \frac{l}{2} \right)$$

where $C \subseteq \mathbb{R}^{n-1}$ is a bounded, open and convex set with unitary $(n-1)-$measure. It is easy to verify that, for $n \ge 3$,

$$\begin{aligned} P(\Omega_l) &= 2\mathcal{H}^{n-1}(l^{-\frac{1}{n-1}} C) + l \mathcal{H}^{n-2}(\partial(l^{-\frac{1}{n-1}} C)) \\ &= 2l^{-1} + l^{\frac{1}{n-1}} \mathcal{H}^{n-2}(\partial C), \end{aligned} \tag{4.35}$$

and we observe that, in the case $n = 2$, we have that $\mathcal{H}^{n-2}(\partial C) = 2$.

Let u be the solution to the following $p-$torsion problem

$$\begin{cases} -\Delta_p u = 1 & \text{in } \Omega_l \\ u = 0 & \text{on } \partial\Omega_l, \end{cases}$$

such that

$$\int_{\Omega_l} u\, dx = T_p(\Omega_l),$$

and let us consider the following function, depending only on the last component x_n of $x \in \mathbb{R}^n$,

$$v(x) = \frac{p-1}{p}\left[\left(\frac{l}{2}\right)^{\frac{p}{p-1}} - |x_n|^{\frac{p}{p-1}}\right],$$

satisfying

$$\begin{cases} -\Delta_p v = 1 & \text{in } \Omega_l \\ v \geq 0 & \text{on } \partial\Omega_l. \end{cases}$$

The comparison principle, see [112], ensures that $u \leq v$ in Ω_l and, as a consequence,

$$\begin{aligned} T_p(\Omega_l) &= \int_{\Omega_l} u\, dx \leq \int_{\Omega_l} v\, dx = \\ &= \frac{p-1}{p}\int_{l^{-\frac{1}{n-1}}C}\int_{-\frac{l}{2}}^{\frac{l}{2}}\left[\left(\frac{l}{2}\right)^{\frac{p}{p-1}} - |x_n|^{\frac{p}{p-1}}\right] dx_n\, d\mathcal{H}^{n-1} \\ &= 2\,\frac{p-1}{p} l^{-1}\int_0^{\frac{l}{2}}\left[\left(\frac{l}{2}\right)^{\frac{p}{p-1}} - x_n^{\frac{p}{p-1}}\right] dx_n \\ &= 2\,\frac{p-1}{p}\left[1 - \frac{p-1}{2p-1}\right] l^{-1}\left(\frac{l}{2}\right)^{\frac{2p-1}{p-1}} = 2c_p l^{-1}\left(\frac{l}{2}\right)^{\frac{2p-1}{p-1}}. \end{aligned} \tag{4.36}$$

By (4.36) and (4.35), we have

$$T_p(\Omega_l)P^{\frac{p}{p-1}}(\Omega_l) \leq 2c_p l^{-1}\left(\frac{l}{2}\right)^{\frac{2p-1}{p-1}}\left(2l^{-1} + l^{\frac{1}{n-1}}\mathcal{H}^{n-2}(\partial C)\right)^{\frac{p}{p-1}} = c_p\left(1 + \frac{l^{\frac{n}{n-1}}}{2}\mathcal{H}^{n-2}(\partial C)\right)^{\frac{p}{p-1}}.$$

Now, since $f(x) \leq f(0)$, we have that, for every bounded, open and convex set Ω,

$$T_{f,p}(\Omega) \leq f^{\frac{p}{p-1}}(0)T_p(\Omega). \tag{4.37}$$

It follows that

$$\begin{aligned} T_{f,p}(\Omega_l)P^{\frac{p}{p-1}}(\Omega_l) &\leq f^{\frac{p}{p-1}}(0)T_p(\Omega_l)P^{\frac{p}{p-1}}(\Omega_l) \\ &\leq c_p f^{\frac{p}{p-1}}(0)\left(1 + \frac{l^{\frac{n}{n-1}}}{2}\mathcal{H}^{n-2}(\partial C)\right)^{\frac{p}{p-1}}. \end{aligned} \tag{4.38}$$

Moreover, we observe that, if f never vanishes, we can use its monotonicity property to bound μ_f from below in the following way:

$$\mu_f(\Omega) = \int_\Omega f(d(x, \partial\Omega))\, dx \geq f(r(\Omega))|\Omega|,$$

obtaining

$$T_{f,p}(\Omega) \geq c_p \frac{f^{\frac{2p-1}{p-1}}(r(\Omega))|\Omega|^{\frac{2p-1}{p-1}}}{f(0)P^{\frac{p}{p-1}}(\Omega)}. \tag{4.39}$$

Joining (4.39) and (4.38), we obtain

$$c_p \frac{f^{\frac{2p-1}{p-1}}(r(\Omega_l))}{f(0)} \leq T_{f,p}(\Omega_l)P^{\frac{p}{p-1}}(\Omega_l) \leq c_p f^{\frac{p}{p-1}}(0)\left(1+\frac{l^{\frac{n}{n-1}}}{2}\mathcal{H}^{n-2}(\partial C)\right)^{\frac{p}{p-1}}.$$

Eventually, passing to the limit when $l \to 0$, observing that $\lim_{l\to 0} r(\Omega_l) = 0$ and that f is continuous, we have

$$T_{f,p}(\Omega_l)P^{\frac{p}{p-1}}(\Omega_l) \longrightarrow c_p f^{\frac{p}{p-1}}(0).$$

□

Remark 4.2.1. If we assume that $f : [0, r(\Omega)] \to [0, +\infty[$ is a function in $L^\infty([0, r(\Omega)])$, then, using the variational characterization (4.24) and the result (4.40) proved in [65],

$$\frac{T_p(\Omega)P^q(\Omega)}{|\Omega|^{q+1}} > \frac{1}{q+1}, \tag{4.40}$$

we have

$$T_{f,p}(\Omega) \geq \left(\inf_{t\in[0,r(\Omega)]} f(t)\right)^{\frac{p}{p-1}} T_p(\Omega) \geq \left(\inf_{t\in[0,r(\Omega)]} f(t)\right)^{\frac{p}{p-1}} c_p \frac{|\Omega|^{\frac{2p-1}{p-1}}}{P(\Omega)^{\frac{p}{p-1}}} \tag{4.41}$$

and the sharpness of (4.41) can be proved in an analogous way as in (4.25).

4.2.1 The quantitative results

The functional

$$\mathcal{F}_p(\Omega) = \frac{T_p(\Omega)P^q(\Omega)}{|\Omega|^{q+1}}, \qquad q = \frac{p}{p-1},$$

is scaling invariant, since for every $t > 0$

$$|t\Omega| = t^n|\Omega|, \qquad P(t\Omega) = t^{n-1}P(\Omega)$$

and

$$T_p(t\Omega) = t^{n+q}T_p(\Omega).$$

We can rewrite inequality (4.25), in the case $f \equiv 1$, as follows

$$\mathcal{F}_p(\Omega) \geq c_p. \tag{4.42}$$

In Theorem 4.2.2 we add a reminder therm to inequality (4.42), that encodes how much the set is "thin".

Theorem 4.2.2. *Let Ω be a non-empty, bounded, open and convex set of $\mathbb{R}^n$ and let $f \equiv 1$. Then,*

$$\mathcal{F}_p(\Omega) - c_p \geq K(n,p)\left(\frac{w(\Omega)}{diam(\Omega)}\right)^{n-1}, \tag{4.43}$$

where $K(n,p)$ is a positive constant depending only on p and the dimension of the space n. In particular, in the case $n=2$, the exponent of the quantity $\dfrac{w(\Omega)}{diam(\Omega)}$ is sharp.

Proof. Let us start by proving (4.43) in the case $n=2$. If $f \equiv 1$, (4.33) becomes

$$T_p(\Omega) \geq c_p \frac{|\Omega|^{\frac{2p-1}{p-1}}}{P^{\frac{p}{p-1}}(\Omega)} + c_p \frac{p}{p-1}\int_0^{R_\Omega} \left(\frac{\mu(t)}{P(t)}\right)^{\frac{2p-1}{p-1}} (-P'(t))\, dt. \tag{4.44}$$

Joining (1.48), (1.49), (1.28) and (4.44), we have that

$$\begin{aligned}
\frac{T_p(\Omega)P^{\frac{p}{p-1}}(\Omega)}{|\Omega|^{\frac{2p-1}{p-1}}} - c_p &> c_p \frac{p}{p-1}\frac{P^{\frac{p}{p-1}}(\Omega)}{|\Omega|^{\frac{2p-1}{p-1}}} \int_0^{r(\Omega)} \left(\frac{\mu(t)}{P(t)}\right)^{\frac{2p-1}{p-1}} (-P'(t))\, dt \\
&\geq \frac{\pi}{2^{\frac{p}{p-1}}}\frac{p}{2p-1}\frac{P^{\frac{p}{p-1}}(\Omega)}{|\Omega|^{\frac{2p-1}{p-1}}}\int_0^{r(\Omega)} (r(\Omega)-t)^{\frac{2p-1}{p-1}}\ dt \\
&\geq \frac{\pi}{2^{\frac{p}{p-1}}}\frac{(p-1)p}{(3p-2)(2p-1)}\frac{r(\Omega)}{P(\Omega)}\left(\frac{r(\Omega)P(\Omega)}{|\Omega|}\right)^{\frac{2p-1}{p-1}} \\
&\geq \frac{\pi}{2^{\frac{p}{p-1}}}\frac{(p-1)p}{(3p-2)(2p-1)}\frac{r(\Omega)}{P(\Omega)}.
\end{aligned}$$

Hence, by applying (1.52) and (1.49) we get

$$\mathcal{F}_p(\Omega) - c_p \geq K(2,p)\frac{w(\Omega)}{\mathrm{diam}(\Omega)}, \tag{4.45}$$

where

$$K(2,p) = \frac{(p-1)p}{2^{\frac{p}{p-1}}3(3p-2)(2p-1)}. \tag{4.46}$$

We now prove that the exponent of $\dfrac{w(\Omega)}{\mathrm{diam}(\Omega)}$ in (4.45) is sharp. In order to do that, we need to find a sequence $\{\Omega_l\}_{l\in\mathbb{N}}$ of convex sets with fixed measure such that

$$M\frac{w(\Omega_l)}{\mathrm{diam}(\Omega_l)} \geq \mathcal{F}_p(\Omega_l) - c_p,$$

for some positive constant M. Let $0<l<1$, we consider the following rectangle

$$\Omega_l = \left(-\frac{1}{2l}, \frac{1}{2l}\right) \times \left(-\frac{l}{2}, \frac{l}{2}\right)$$

and we notice that its inradius and area are $r(\Omega_l) = \frac{l}{2}$ and $|\Omega_l| = 1$. Let u be the unique solution to

$$\begin{cases} -\Delta_p u = 1 & \text{in } \Omega_l \\ u = 0 & \text{on } \partial\Omega_l \end{cases}$$

and let us consider the following function

$$v(y) = \frac{p-1}{p}\left[\left(\frac{l}{2}\right)^{\frac{p}{p-1}} - |y|^{\frac{p}{p-1}}\right],$$

which solves

$$\begin{cases} -\Delta v = 1 & \text{in } \Omega_l \\ v \geq 0 & \text{on } \partial\Omega_l. \end{cases}$$

The comparison principle gives $u \leq v$ in Ω_l and

$$T_p(\Omega_l) = \int_{\Omega_l} u_p \, dx \leq \int_{\Omega_l} v \, dx.$$

Arguing as in (4.36), we have

$$\int_{\Omega_l} v \, dx = c_p \left(\frac{l}{2}\right)^{\frac{p}{p-1}}.$$

On the other hand, the perimeter of the rectangle is given by

$$P(\Omega_l) = \frac{2}{l}\left(1 + l^2\right)$$

and its Taylor expansion with respect to $l > 0$ is

$$P^{\frac{p}{p-1}}(\Omega_l) = \left(\frac{2}{l}\right)^{\frac{p}{p-1}}\left(1 + l^2\right)^{\frac{p}{p-1}} = \left(\frac{2}{l}\right)^{\frac{p}{p-1}}\left(1 + \frac{p}{p-1}l^2 + o(l^2)\right).$$

Using (1.49) and (1.52), we get

$$\begin{aligned} T_p(\Omega_l)P^{\frac{p}{p-1}}(\Omega_l) - c_p &\leq c_p \left(\frac{l}{2}\right)^{\frac{p}{p-1}}\left(\frac{2}{l}\right)^{\frac{p}{p-1}}\left(1 + \frac{p}{p-1}l^2 + o(l^2)\right) - c_p \\ &\leq 2c_p \frac{p}{p-1} l^2 \leq 16 c_p \frac{p}{p-1}\frac{r(\Omega_l)}{P(\Omega_l)} \\ &\leq 4c_p \frac{p}{p-1}\frac{w(\Omega_l)}{\operatorname{diam}(\Omega_l)} \end{aligned}$$

and this concludes the proof in dimension $n = 2$.

Let us now prove (4.43) in the case $n > 2$. If we choose $f \equiv 1$, (4.33) becomes

$$T_p(\Omega) \geq c_p \frac{|\Omega|^{\frac{2p-1}{p-1}}}{P^{\frac{p}{p-1}}(\Omega)} + c_p \frac{p}{p-1}\int_0^{r(\Omega)} \left(\frac{\mu(t)}{P(t)}\right)^{\frac{2p-1}{p-1}} (-P'(t))\, dt. \tag{4.47}$$

Hence, combining (1.27) and (4.47), we have

$$\frac{T_p(\Omega)P^{\frac{p}{p-1}}(\Omega)}{|\Omega|^{\frac{2p-1}{p-1}}} - c_p \geq k(n,p)\frac{P^{\frac{p}{p-1}}(\Omega)}{|\Omega|^{\frac{2p-1}{p-1}}}\int_0^{r(\Omega)} \left(\frac{\mu(t)}{P(t)}\right)^{\frac{2p-1}{p-1}} P(t)^{\frac{n-2}{n-1}}\, dt. \tag{4.48}$$

Moreover, the perimeter is monotone with respect to inclusion among convex sets, so we obtain that

$$P(t) \geq n\omega_n (r(\Omega) - t)^{n-1}, \tag{4.49}$$

and so, using (4.49) in (4.48), we get

$$\begin{aligned}\frac{T_p(\Omega)P^{\frac{p}{p-1}}(\Omega)}{|\Omega|^{\frac{2p-1}{p-1}}} - c_p &\geq k(n,p)\frac{P^{\frac{p}{p-1}}(\Omega)}{|\Omega|^{\frac{2p-1}{p-1}}}\int_0^{r(\Omega)} (r(\Omega)-t)^{\frac{2p-1}{p-1}+n-2}\,dt \\ &= k(n,p)\left(\frac{Rr(\Omega)P(\Omega)}{|\Omega|}\right)^{\frac{2p-1}{p-1}}\frac{r(\Omega)^{n-1}}{P(\Omega)}.\end{aligned} \tag{4.50}$$

If we combine (4.50) with (1.48), with the following estimate (that can be found in [30]):

$$r(\Omega) \geq \begin{cases} w(\Omega)\dfrac{\sqrt{n+2}}{2n+2} & n \text{ even} \\ \\ w(\Omega)\dfrac{1}{2\sqrt{n}} & n \text{ odd,} \end{cases}$$

and with

$$P(\Omega) \leq n\omega_n \left(\frac{n}{2n+2}\right)^{\frac{n-1}{2}} \operatorname{diam}(\Omega)^{n-1},$$

we finally get

$$\frac{T_p(\Omega)P^{\frac{p}{p-1}}(\Omega)}{|\Omega|^{\frac{2p-1}{p-1}}} - c_p \geq K(n,p)\left(\frac{w(\Omega)}{\operatorname{diam}(\Omega)}\right)^{n-1}.$$

□

Remark 4.2.2. As far as the sharpness of (4.43) in the case $n > 2$, we conjecture that the sharp exponent is 1 as in the planar case. Indeed, the minimizing sequence $\{\Omega_l\}$ satisfies

$$T_p(\Omega_l)P^{\frac{p}{p-1}}(\Omega_l) - c_p \approx C\frac{w(\Omega_l)}{\operatorname{diam}(\Omega_l)}.$$

Remark 4.2.3. As already remarked in the Introduction, inequality (4.43) gives information on the set Ω. Indeed, if

$$\mathcal{F}_p(\Omega) - c_p$$

is small, then the ratio between $w(\Omega)$ and $\operatorname{diam}(\Omega)$ has to be necessarily small, i.e. Ω must be a thin domain. Moreover, inequality (4.43) tells us also that the infimum of $\mathcal{F}_p(\Omega)$ is not achieved among bounded, open and convex sets. Assuming by contradiction that there exists a bounded, open and convex set $\tilde{\Omega}$ such that

$$\mathcal{F}_p(\tilde{\Omega}) = c_p,$$

we have that

$$\frac{w(\tilde{\Omega})}{\operatorname{diam}(\tilde{\Omega})} < \varepsilon \qquad \forall \varepsilon > 0,$$

which is impossible.

Theorem 4.2.2 only tells us that any minimizing sequence of $\mathcal{F}_p(\cdot)$ is a sequence of thinning domains. On the other hand, Theorem 4.2.3 gives us more precise information on the geometry of such minimizing sequence in the planar case.

Theorem 4.2.3. *Let Ω be a non-empty, bounded, open and convex set in $\mathbb{R}^2$, let $f \equiv 1$ and let $p = 2$. Then, there exists a positive constant $\tilde{K}$ such that*

$$\mathcal{F}_2(\Omega) - c_2 = \frac{T(\Omega)P^2(\Omega)}{|\Omega|^3} - \frac{1}{3} \geq \tilde{K} \left(\frac{|\Omega \triangle Q|}{|\Omega|} \right)^3, \tag{4.51}$$

where $\Omega \triangle Q$ denotes the symmetric difference between Ω and a rectangle Q with sides $P(\Omega)/2$ and $w(\Omega)$ containing Ω.

Proof. Let Ω be a non-empty, bounded, open and convex set in $\mathbb{R}^2$ and let us consider a rectangle Q with sides $P(\Omega)/2$ and $w(\Omega)$ containing Ω. Such a rectangle exists, since it is enough to choose the shorter side of Q parallel to the direction of $w(\Omega)$ and to recall the lower bound in (1.52) (see Figure 4.1).

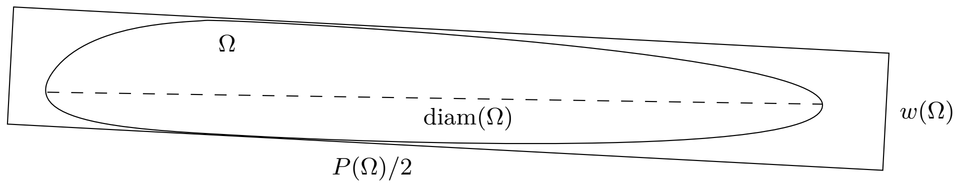

Figure 4.1: Rectangle with sides $P(\Omega)/2$ and $w(\Omega)$ containing Ω.

Now, let $\sigma > 0$ be such that

$$\frac{1}{4^3 \cdot 6} - \frac{\pi^2}{2^3 \cdot 3^3} \frac{\sigma^2}{K^2(2)} \geq 0; \tag{4.52}$$

$$\frac{1}{3^3 \cdot 6} - \frac{\pi}{48} \frac{\sigma}{K(2))} - \frac{\pi^2}{2^5 \cdot 3} \frac{\sigma^2}{K^2(2)} \geq 0; \tag{4.53}$$

$$\frac{\pi}{4} - \frac{\pi}{2\sqrt{3}} \frac{\sigma}{K(2)} \geq \frac{4}{3\sqrt{3}}, \tag{4.54}$$

where $K(2) := K(2,2)$ is the constant defined in (4.46). If

$$\frac{T(\Omega)P^2(\Omega)}{|\Omega|^3} - \frac{1}{3} \geq \sigma,$$

then, by (1.49) and (1.48), we have

$$\frac{|Q \triangle \Omega|}{|\Omega|} = \left(\frac{P(\Omega)w(\Omega)}{2|\Omega|} - 1 \right) \leq \left(\frac{3}{2} \frac{P(\Omega)r(\Omega)}{|\Omega|} - 1 \right) \leq 2.$$

So, it follows that

$$\frac{T(\Omega)P^2(\Omega)}{|\Omega|^3} - \frac{1}{3} \geq \frac{\sigma}{2^3} 2^3 \geq \frac{\sigma}{2^3} \left(\frac{|Q \triangle \Omega|}{|\Omega|} \right)^3.$$

On the other hand, let us assume that

$$\frac{T(\Omega)P^2(\Omega)}{|\Omega|^3} - \frac{1}{3} < \sigma. \tag{4.55}$$

By Theorem 4.2.2, we have that

$$\frac{w(\Omega)}{\operatorname{diam}(\Omega)} \leq \frac{1}{K(2)} \left[\frac{T(\Omega)P^2(\Omega)}{|\Omega|^3} - \frac{1}{3} \right] < \frac{\sigma}{K(2)}, \tag{4.56}$$

and we observe that, by the choice of σ made in (4.52)-(4.54), a ball cannot satisfy (4.55).

Now, arguing as in (4.32) with $f \equiv 1$ and $p = 2$, we know that

$$T(\Omega) \geq \int_0^{r(\Omega)} \frac{\mu^2(t)}{P(t)}\, dt. \tag{4.57}$$

We set $\rho = \dfrac{P^2(\Omega)}{4\pi} - |\Omega|$ and $p_R = P(\Omega) - 2\pi r(\Omega)$ and we observe that they are both strictly positive by the isoperimetric inequality and the monotonicity of the perimeter, respectively. Using inequalities (1.25) and (1.26) in (4.57), we have that

$$\begin{aligned} T(\Omega)P^2(\Omega) &\geq P^2(\Omega) \int_0^{r(\Omega)} \frac{(|\Omega| - P(\Omega)t + \pi t^2)^2}{P(\Omega) - 2\pi t}\, dt \\ &= P^2(\Omega) \int_0^{r(\Omega)} \frac{1}{P(\Omega) - 2\pi t} \left(\frac{(P(\Omega) - 2\pi t)^2}{4\pi} - \left(\frac{P^2(\Omega)}{4\pi} - |\Omega| \right) \right)^2 dt \\ &= P^2(\Omega) \int_0^{r(\Omega)} \left(\frac{(P(\Omega) - 2\pi t)^3}{(4\pi)^2} - \frac{\rho}{2\pi} (P(\Omega) - 2\pi t) + \frac{\rho^2}{P(\Omega) - 2\pi t} \right) dt \\ &= \frac{P^2(\Omega)}{2\pi} \left(\frac{P^4(\Omega) - p_R^4}{4(4\pi)^2} - \frac{\rho}{4\pi} \left(P^2(\Omega) - p_R^2 \right) - \rho^2 \log \left(1 - \frac{2\pi r(\Omega)}{P(\Omega)} \right) \right), \end{aligned} \tag{4.58}$$

and, using Newton's formula and the Taylor series for the logarithm, we get

$$\begin{aligned} &P^2(\Omega) - p_R^2 = 4\pi r(\Omega)P(\Omega) - 4\pi^2 r^2(\Omega); \\ &P^4(\Omega) - p_R^4 = 8\pi r(\Omega)P^3(\Omega) - 24\pi^2 r^2(\Omega)P^2(\Omega) + 32\pi^3 r^3(\Omega)P(\Omega) - 16\pi^4 r^4(\Omega); \\ &-\log\left(1 - \frac{2\pi r(\Omega)}{P(\Omega)}\right) = \sum_{i=1}^{\infty} \frac{1}{i} \left(\frac{2\pi r(\Omega)}{P(\Omega)} \right)^i \geq \frac{2\pi r(\Omega)}{P(\Omega)} + \frac{2\pi^2 r^2(\Omega)}{P^2(\Omega)} + \frac{8}{3} \frac{\pi^3 r^3(\Omega)}{P^3(\Omega)} + \frac{4\pi^4 r^4(\Omega)}{P^4(\Omega)}. \end{aligned} \tag{4.59}$$

By (4.59) and (4.58), dividing by $|\Omega|^3$ and subtracting $1/3$, we have

$$\begin{aligned} \frac{T(\Omega)P^2(\Omega)}{|\Omega|^3} - \frac{1}{3} \geq &\frac{1}{3} \left(\frac{P(\Omega)r(\Omega)}{|\Omega|} - 1 \right)^3 + \pi \frac{r^2(\Omega)}{|\Omega|^2} \left(|\Omega| - \frac{2}{3} P(\Omega)r(\Omega) \right) \\ &+ \frac{4}{3}\pi^2 \frac{r^3(\Omega)}{P(\Omega)|\Omega|^2} \left(|\Omega| - \frac{3}{4} P(\Omega)r(\Omega) \right). \end{aligned} \tag{4.60}$$

As an intermediate step we want to prove the following inequality:

$$\begin{aligned} &\frac{1}{3} \left(\frac{P(\Omega)r(\Omega)}{|\Omega|} - 1 \right)^3 + \pi \frac{r^2(\Omega)}{|\Omega|^2} \left(|\Omega| - \frac{2}{3} P(\Omega)r(\Omega) \right) \\ &+ \frac{4}{3}\pi^2 \frac{r^3(\Omega)}{P(\Omega)|\Omega|^2} \left(|\Omega| - \frac{3}{4} P(\Omega)r(\Omega) \right) \geq \frac{1}{6} \left(\frac{P(\Omega)r(\Omega)}{|\Omega|} - 1 \right)^3, \end{aligned} \tag{4.61}$$

that, combined with (4.60), implies

$$\frac{T(\Omega)P^2(\Omega)}{|\Omega|^3}-\frac{1}{3}\geq\frac{1}{6}\left(\frac{P(\Omega)r(\Omega)}{|\Omega|}-1\right)^3, \tag{4.62}$$

where we choose the constant 1/6 as an arbitrary constant less then 1/3. In particular, (4.61) is equivalent to

$$\begin{aligned}\frac{1}{6}(P(\Omega)r(\Omega)-|\Omega|)^3+\pi r^2(\Omega)|\Omega|\left(|\Omega|-\frac{2}{3}P(\Omega)r(\Omega)\right)+\\ +\frac{4}{3}\pi^2\frac{r(\Omega)^3}{P(\Omega)}|\Omega|\left(|\Omega|-\frac{3}{4}P(\Omega)r(\Omega)\right)\geq 0.\end{aligned} \tag{4.63}$$

In order to prove (4.63), we distinguish three cases:

1) if $|\Omega|\geq\frac{3}{4}P(\Omega)r(\Omega)$, then (4.63) is trivial since the left-hand side is the sum of positive quantities;

2) if $\frac{2}{3}P(\Omega)r(\Omega)\leq|\Omega|<\frac{3}{4}P(\Omega)r(\Omega)$, using (1.49), (1.52), (4.52) and (4.56), we have

$$\begin{aligned}\frac{1}{6}(P(\Omega)r(\Omega)-|\Omega|)^3&+\pi r^2(\Omega)|\Omega|\left(|\Omega|-\frac{2}{3}P(\Omega)r(\Omega)\right)+\frac{4}{3}\pi^2\frac{r^3(\Omega)}{P(\Omega)}|\Omega|\left(|\Omega|-\frac{3}{4}P(\Omega)r(\Omega)\right)\\ &\geq P^3(\Omega)r^3(\Omega)\left(\frac{1}{4^3\cdot 6}-\frac{2\pi^2}{3^3}\frac{r^2(\Omega)}{P^2(\Omega)}\right)\\ &\geq P^3(\Omega)r^3(\Omega)\left(\frac{1}{4^3\cdot 6}-\frac{\pi^2}{2^3\cdot 3^3}\frac{w^2(\Omega)}{\mathrm{diam}^2(\Omega)}\right)\\ &\geq P^3(\Omega)R^3_\Omega\left(\frac{1}{4^3\cdot 6}-\frac{\pi^2}{2^3\cdot 3^3}\frac{\sigma^2}{K^2(2)}\right)\geq 0.\end{aligned} \tag{4.64}$$

3) if $\frac{1}{2}P(\Omega)r(\Omega)\leq|\Omega|<\frac{2}{3}P(\Omega)r(\Omega)$, arguing as before, we have

$$\begin{aligned}\frac{1}{6}(P(\Omega)r(\Omega)-|\Omega|)^3&+\pi r^2(\Omega)|\Omega|\left(|\Omega|-\frac{2}{3}P(\Omega)r(\Omega)\right)+\frac{4}{3}\pi^2\frac{r^3(\Omega)}{P(\Omega)}|\Omega|\left(|\Omega|-\frac{3}{4}P(\Omega)r(\Omega)\right)\\ &\geq P^3(\Omega)r^3(\Omega)\left(\frac{1}{3^3\cdot 6}-\frac{\pi}{48}\frac{w(\Omega)}{\mathrm{diam}(\Omega)}-\frac{\pi^2}{2^5\cdot 3}\frac{w^2(\Omega)}{\mathrm{diam}^2(\Omega)}\right)\\ &\geq P^3(\Omega)r^3(\Omega)\left(\frac{1}{3^3\cdot 6}-\frac{\pi}{48}\frac{\sigma}{K(2)}-\frac{\pi^2}{2^5\cdot 3}\frac{\sigma^2}{K^2(2)}\right)\geq 0.\end{aligned} \tag{4.65}$$

So, we have proved the intermediate step (4.62). Now, by combining (4.62) and (1.50), we deduce

$$\frac{T(\Omega)P^2(\Omega)}{|\Omega|^3}-\frac{1}{3}\geq\frac{1}{6}\left[\frac{P(\Omega)r(\Omega)}{|\Omega|}-1\right]^3\geq\frac{1}{6}\left[\frac{P(\Omega)w(\Omega)}{2|\Omega|}-1-\frac{1}{\sqrt{3}}\frac{w^2(\Omega)}{|\Omega|}\right]^3. \tag{4.66}$$

Using (1.51), (1.50), (4.56) and (4.54), we have

$$\begin{aligned}\frac{P(\Omega)w(\Omega)}{2|\Omega|}-1 \geq \frac{P(\Omega)r(\Omega)}{|\Omega|}-1 \geq \pi\frac{r^2(\Omega)}{|\Omega|} &\geq \frac{\pi}{|\Omega|}\left(\frac{w(\Omega)}{2}-\frac{w^2(\Omega)}{\sqrt{3}P(\Omega)}\right)^2\\ &= \frac{w^2(\Omega)}{|\Omega|}\left(\frac{\pi}{4}-\frac{\pi}{\sqrt{3}}\frac{w(\Omega)}{P(\Omega)}+\frac{\pi}{3}\frac{w^2(\Omega)}{P^2(\Omega)}\right)\\ &\geq \frac{w^2(\Omega)}{|\Omega|}\left(\frac{\pi}{4}-\frac{\pi}{2\sqrt{3}}\frac{w(\Omega)}{\operatorname{diam}(\Omega)}\right)\\ &\geq \frac{w^2(\Omega)}{|\Omega|}\left(\frac{\pi}{4}-\frac{\pi}{2\sqrt{3}}\frac{\sigma}{K(2)}\right)\\ &\geq \frac{4}{3\sqrt{3}}\frac{w^2(\Omega)}{|\Omega|}\end{aligned} \tag{4.67}$$

Finally, by combining (4.66) and (4.67), we get the conclusion

$$\frac{T(\Omega)P^2(\Omega)}{|\Omega|^3}-\frac{1}{3} \geq \frac{1}{6}\left[\frac{P(\Omega)r(\Omega)}{|\Omega|}-1\right]^3 \geq \tilde{K}\left[\frac{|Q \triangle \Omega|}{|\Omega|}\right]^3. \tag{4.68}$$

□

The next remark shows that a sequence of thinning triangles is not sharp for (4.43) in the case $n = 2$ and this is the reason why we need Theorem 4.2.3 to obtain more precise information.

Remark 4.2.4. Let us consider a sequence of isosceles triangles $\mathcal{T}_l$ of base L and height l such that $|\mathcal{T}_l| = 1$.

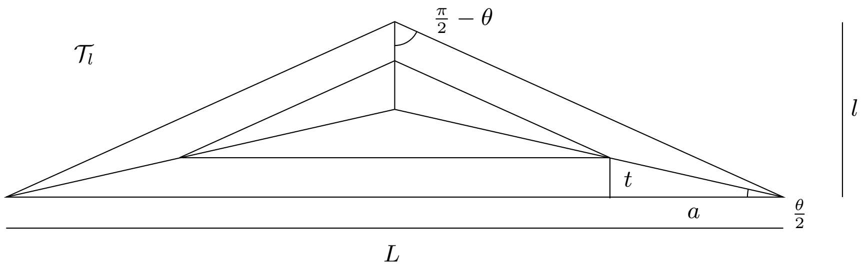

Figure 4.2: Isosceles triangle $\mathcal{T}_l$ of base L and height l.

If we compute (4.62) on the sequence $\mathcal{T}_l$ and we use (1.48), we get, for every l,

$$\frac{T(\mathcal{T}_l)P^2(\mathcal{T}_l)}{|\mathcal{T}_l|^3}-\frac{1}{3} \geq \frac{1}{6}\left(\frac{P(\mathcal{T}_l)r(\mathcal{T}_l)}{|\mathcal{T}_l|}-1\right)^3=\frac{1}{6} \tag{4.69}$$

and, so, the quantity on the left-hand side of (4.69) is bounded away from zero.

Remark 4.2.5. We point out that

$$\frac{P(\Omega)r(\Omega)}{|\Omega|}-1 \geq K\frac{|Q \triangle \Omega|}{|\Omega|},$$

in (4.68) is a quantitative version of the inequality in the right-hand side of (1.48).

4.3 Sharp inequalities involving the Cheeger constant in the planar case

Let Ω be a bounded subset of $\mathbb{R}^2$. The Cheeger constant of Ω, introduced by Jeff Cheeger in [54], is defined as

$$h(\Omega) := \inf \left\{ \frac{P(E)}{|E|} \; : \; E \text{ measurable and } E \subseteq \Omega, \; |E| > 0 \right\}. \tag{4.70}$$

The minimum in (4.70) is achieved when Ω has Lipschitz boundary, see as a reference [127]. For the properties of the Cheeger constant and for an introductory survey, see for example [3, 103, 127]. In particular, in the case of planar convex sets, the authors in [3] prove that the Cheeger set is unique, and the set E that realizes this minimum is called *Cheeger set* of Ω and it is denoted by C_Ω, while in [103] the authors give a characterization for the Cheeger constant.

Theorem 4.3.1. *Let Ω be a convex, planar set. There exists a unique value $t = t^* > 0$ such that $|\Omega_t| = \pi t^2$. Then*

$$h(\Omega) = \frac{1}{t^*}$$

and the Cheeger set of Ω is

$$C_\Omega = \Omega_{t^*} + t^* B_1$$

with B_1 denoting the unit disk.

We are interested in describing all possible inequalities involving the Cheeger constant of a given compact, bounded and convex set $\Omega \subset \mathbb{R}^2$ and two among the following geometrical quantities: the area $|\Omega|$, the perimeter $P(\Omega)$, the inradius $r(\Omega)$, the circumradius $R(\Omega)$, the minimal width $w(\Omega)$ and the diameter $\operatorname{diam}(\Omega)$. So, we aim to study the associated Blaschke–Santaló diagrams of these triplets.

A Blaschke–Santaló diagram is a tool that allows one to visualize all the possible inequalities between three geometric quantities. More precisely, if we consider three shape functionals (J_1, J_2, J_3), we want to find a system of inequalities describing the set

$$\{(J_1(\Omega), J_2(\Omega)) | \; J_3(\Omega) = 1, \; \Omega \in \mathcal{K}^2\},$$

where we denote by $\mathcal{K}^2$ the class of non-empty sets in $\mathbb{R}^2$ that are open, bounded and convex.

We aim to continue what started in [78] and [79], where the author studied the Blaschke–Santaló diagram involving the Cheeger constant. More precisely, in [78], it is studied the Blaschke–Santaló diagram between the Cheeger constant, the area and the inradius, and it is proved that, if

$$\Omega \in \mathcal{K}^2 := \{\Omega \subset \mathbb{R}^2 : \Omega \; \text{ is bounded, compact and convex}\},$$

then

$$\frac{1}{r(\Omega)} + \frac{\pi r(\Omega)}{|\Omega|} \le h(\Omega) \le \frac{1}{r(\Omega)} + \sqrt{\frac{\pi}{|\Omega|}}, \tag{4.71}$$

where the upper bound in (4.71) is achieved by (and only by) sets which are homothetic to their form body (see Definition 1.4.1), for instance circumscribed sets, meanwhile the lower one is achieved by (and only by) stadiums. Then, in [79], it is studied the diagram between the Cheeger constant, the area and the perimeter and it is proved that if $\Omega \in \mathcal{K}^2$, then

$$\frac{P(\Omega) + \sqrt{4\pi|\Omega|}}{2|\Omega|} \le h(\Omega) \le \frac{P(\Omega)}{|\Omega|}, \tag{4.72}$$

where the upper bound is achieved by any set which is Cheeger of itself (in particular stadiums), meanwhile the lower one is achieved, for example, by circumscribed polygons.

Now we state the new results. In order to do that, we need to define the following classes of admissible sets (we refer to [145, Table 2.1] for the associated constraints):

1. $\mathcal{K}^2_{P,r} = \{\Omega \in \mathcal{K}^2 : P(\Omega) = P,\, r(\Omega) = r\}$, where $P \geq 2\pi r$;
2. $\mathcal{K}^2_{d,r} = \{\Omega \in \mathcal{K}^2 : d(\Omega) = d,\, r(\Omega) = r\}$, where $d \geq 2r$;
3. $\mathcal{K}^2_{R,r} = \{\Omega \in \mathcal{K}^2 : R(\Omega) = R,\, r(\Omega) = r\}$, where $R \geq r$;
4. $\mathcal{K}^2_{\omega,d} = \{\Omega \in \mathcal{K}^2 : w(\Omega) = \omega,\, \mathrm{diam}(\Omega) = d\}$, where $\omega \leq d$;
5. $\mathcal{K}^2_{\omega,R} = \{\Omega \in \mathcal{K}^2 : w(\Omega) = \omega,\, R(\Omega) = R\}$, where $2R \geq \omega$;
6. $\mathcal{K}^2_{\omega,P} = \{\Omega \in \mathcal{K}^2 : w(\Omega) = \omega,\, P(\Omega) = P\}$, where $P \geq \pi\omega$;
7. $\mathcal{K}^2_{\omega,A} = \{\Omega \in \mathcal{K}^2 : w(\Omega) = \omega,\, |\Omega| = A\}$, where $\sqrt{3}A \geq \omega^2$;
8. $\mathcal{K}^2_{R,d} = \{\Omega \in \mathcal{K}^2 : R(\Omega) = R,\, \mathrm{diam}(\Omega) = d\}$, where $\sqrt{3}R \leq d < 2R$;
9. $\mathcal{K}^2_{\omega,r} = \{\Omega \in \mathcal{K}^2 : w(\Omega) = \omega,\, r(\Omega) = r\}$, where $2r < w \leq 3r$;
10. $\mathcal{K}^2_{R,A} = \{\Omega \in \mathcal{K}^2 : R(\Omega) = R,\, |\Omega| = A\}$, where $A \leq \pi R^2$;
11. $\mathcal{K}^2_{P,R} = \{\Omega \in \mathcal{K}^2 : P(\Omega) = P,\, R(\Omega) = R\}$, where $4R < P \leq 2\pi R$;
12. $\mathcal{K}^2_{P,d} = \{\Omega \in \mathcal{K}^2 : P(\Omega) = P,\, \mathrm{diam}(\Omega) = d\}$, where $2d < P \leq \pi d$;
13. $\mathcal{K}^2_{d,A} = \{\Omega \in \mathcal{K}^2 : \mathrm{diam}(\Omega) = d,\, |\Omega| = A\}$, where $\pi d^2 \geq 4A$.

Firstly, let us state the following existence result.

Theorem 4.3.2. *Let $\Omega \in \mathcal{K}^2$, then the minimization and the maximization shape optimization problems of the Cheeger constant $h(\Omega)$ admit a solution in the classes of sets defined in* $(1)-(13)$.

Proof. Let us consider the minimization problem of the Cheeger constant in the classes of sets $(1)-(13)$; the maximization problem can be dealt similarly.
For all of these classes of sets, in order to prove the existence of the solution, we consider a minimizing sequence $(\Omega_k)_{k\in\mathbb{N}}$ and we prove that it satisfies the hypothesis of the Blaschke Selection Theorem (see Theorem 1.8.7 in [143]), that is to say its boundedness up to translations. Concerning the class of sets involving a diameter or a circumradius constraint, it is clear that, up to a translation, the minimizing sequence is contained in a sufficiently big ball. So it remains to study the problem in $\mathcal{K}^2_{P,r}$, $\mathcal{K}^2_{\omega,A}$, $\mathcal{K}^2_{r,\omega}$ and $\mathcal{K}^2_{\omega,P}$. Concerning $\mathcal{K}^2_{P,r}$ and $\mathcal{K}^2_{\omega,P}$, we know from [145] that $P = P(\Omega_k) > 2\mathrm{diam}(\Omega_k)$, for every k, so the sequence of the diameters $\mathrm{diam}(\Omega_k)$ is equibounded and, consequently, there exists a sufficiently big ball containing the sequence $(\Omega_k)_k$. As far as $\mathcal{K}^2_{r,\omega}$ is concerned, it is possible to prove the boundness of the minimizing sequence whenever $w(\Omega_k) > 2r(\Omega_k)$, indeed it holds (see [145])

$$\mathrm{diam}(\Omega_k) \leq \frac{w^2(\Omega_k)}{2(w(\Omega_k) - 2r(\Omega_k))}.$$

For the last class $\mathcal{K}^2_{\omega,A}$, from [145], we know that, if $2w(\Omega_k) \leq \sqrt{3}\mathrm{diam}(\Omega_k)$, then

$$2|\Omega_k| \geq w(\Omega_k)\mathrm{diam}(\Omega_k),$$

and, also in this case, the boundedness follows.

So, for every class of sets considered, the Blaschke selection theorem ensures us that, up to a subsequence, Ω_k converges in the Hausdorff sense to a set Ω^*; it remains only to prove that this set belongs to the relative class of admissible sets. We observe that all the considered constraints are stable for the Hausdorff convergence. In particular, in [67] it is proved the stability of the inradius, in [92] the stability of the diameter and in [143] the stability of area, perimeter and width.

It only remains to show that the circumradius is continuous with respect to the Hausdorff distance in the class of admissible sets having a circumradius constraint. Since $R(\Omega_k) = R$, $\forall k \in \mathbb{N}$, we have $\Omega_k \subseteq B_R$. Using the stability of the Hausdorff convergence for the inclusion (see [92], Prop 2.2.17), we have that $\Omega^* \subseteq B_R$, and consequently $R(\Omega^*) \leq R$. By contradiction, let us suppose that $R(\Omega^*) < R$, so there exists $\overline{R} > 0$ such that $R(\Omega^*) < \overline{R} < R$ and so $\Omega^* \subseteq B_{\overline{R}}$. Therefore, by the Hausdorff convergence, for sufficiently large k, $\Omega_k \subseteq B_{\overline{R}}$, but this would imply $R \leq \overline{R}$, that is absurd.

In order to conclude, we observe that in all the above cases, the set Ω^* cannot be a segment, in the sense that the minimizing sequence cannot degenerate loosing one dimension. If we are working in a class of sets involving an inradius or width constraint, then, it is clear that, thanks to the continuity of the inradius and width under Hausdorff convergence, there exists, up to a translation, a sufficiently small ball contained in the minimizing sequence. Moreover, in the case $\mathcal{K}^2_{d,A}$ and $\mathcal{K}^2_{R,A}$, the non-degeneration is ensured by the continuity of the area under Hausdorff distance and the equiboundedness of the diameter. On the other hand, if we consider the minimization problem in $\mathcal{K}^2_{R,d}$, $\mathcal{K}^2_{P,d}$ and $\mathcal{K}^2_{P,R}$, the inradius can be bounded from below by a positive quantity. In [141, Section 9] it is proved that

$$r(\Omega_k) \geq \frac{\mathrm{diam}^2(\Omega_k)\sqrt{4R^2(\Omega_k) - \mathrm{diam}^2(\Omega_k)}}{2R(\Omega_k)\left(2R(\Omega_k) + \sqrt{4R^2(\Omega_k) - \mathrm{diam}^2(\Omega_k)}\right)};$$

in [90, Section 3] it is proved that

$$r(\Omega_k) \geq \frac{P(\Omega_k)}{4} - \frac{\mathrm{diam}(\Omega_k)}{2}$$

which also yields to

$$r(\Omega_k) \geq \frac{P(\Omega_k)}{4} - R(\Omega_k),$$

because $\mathrm{diam}(\Omega_k) \leq 2R(\Omega_k)$.

Recalling now that the Cheeger constant is continuous with respect to the Hausdorff convergence when the sets do not degenerate to a segment (see [128, Proposition 3.1]), the existence part of the theorem is proved. □

The following Lemma will play a key role in the proof of the following theorems.

Lemma 4.3.3. *Let $\Omega \in \mathcal{K}^2$. We assume that there exists a continuous function $g^\Omega : [0, r(\Omega)] \to \mathbb{R}$ such that*

$$\forall t \in [0, r(\Omega)], \qquad |\Omega_t| \leq g^\Omega(t), \qquad \textit{(resp. } |\Omega_t| \geq g^\Omega(t)\textit{)}, \tag{4.73}$$

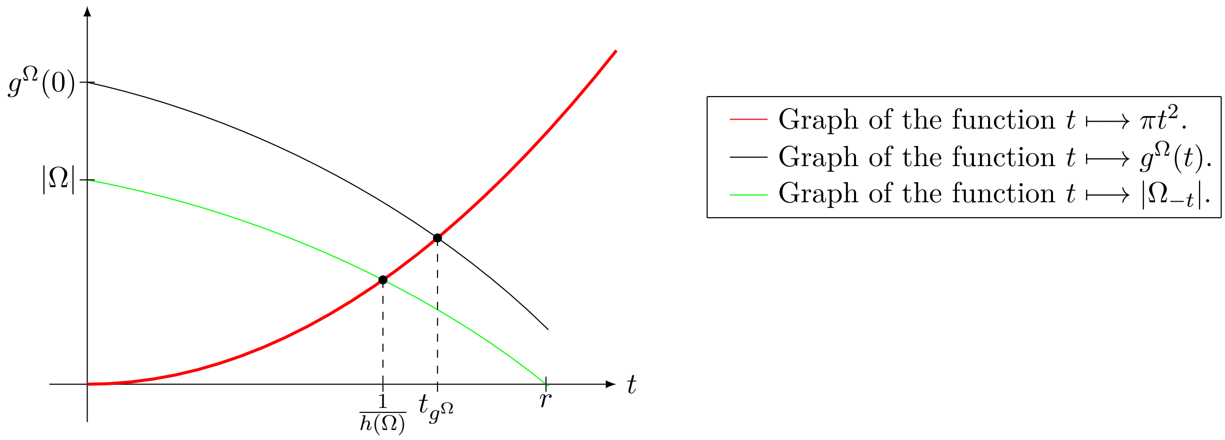

Figure 4.3: Idea of the proof of Lemma 4.3.3

and

$$G_\Omega := \Big\{ t \in [0, r(\Omega)] : g^\Omega(t) = \pi t^2 \Big\} \neq \emptyset. \tag{4.74}$$

We have

$$h(\Omega) \geq \frac{1}{t_{g^\Omega}} \qquad \textit{(resp. } h(\Omega) \leq \frac{1}{t_{g^\Omega}}\textit{)}, \tag{4.75}$$

where t_{g^Ω} is the smallest (resp. largest) solution to the equation $g^\Omega(t) = \pi t^2$ on $[0, r(\Omega)]$.

Proof. From Theorem 4.3.1, we know that there exists a unique $t = t_\Omega > 0$ such that $|\Omega_t| = \pi t^2$ and $h(\Omega) = 1/t_\Omega$. So, it is clear that, if there exists a function $g(t)$ such that (4.73) and (4.74) hold, the smallest (resp. largest) element $t_{g^\Omega} \in G_\Omega$ must satisfy (4.75) (see Figure 4.3). □

In the following Theorem, we consider the triplets of functionals for which we are able to provide the complete description of the relative Blaschke–Santaló diagrams. For the precise definitions of the below-mentioned extremal sets, see Section 1.4 and for the explicit bounds, see Propositions 4.3.5, 4.3.6 and 4.3.7 and for the description of the corresponding diagrams we refer to Proposition 4.3.8.

Theorem 4.3.4. *The following results hold*

(i) The maximum and the minimum of the Cheeger constant h in $\mathcal{K}^2_{P,r}$ are achieved respectively by sets that are homothetic to their form body and stadiums.

(ii) The maximum of the Cheeger constant h in $\mathcal{K}^2_{d,r}$ is achieved by symmetrical two-cup bodies; moreover, there exists $D_0 > 0$ such that if $d \geq rD_0$ the minimum of the Cheeger constant h in $\mathcal{K}^2_{d,r}$ is achieved by symmetrical spherical slices, while, if $d < rD_0$, the minimum of the Cheeger constant h is achieved by regular smoothed nonagons.

(iii) The maximum and the minimum of the Cheeger constant h in $\mathcal{K}^2_{R,r}$ are achieved respectively by two-cup bodies and symmetrical spherical slices.

The triplet (P, h, r)**.**

Proposition 4.3.5. *Let* $\Omega \in \mathcal{K}^2$*. Then, it holds*

$$\frac{1}{r(\Omega)} + \frac{\pi}{P(\Omega) - \pi r(\Omega)} \leq h(\Omega) \leq \frac{1}{r(\Omega)} + \sqrt{\frac{2\pi}{P(\Omega)r(\Omega)}}, \tag{4.76}$$

where equalities are achieved by sets that are homothetic to their form body in the upper bound and by the stadiums in the lower bound.

Proof. We combine the classical convex geometric inequalities (see [29, 31, 145])

$$\frac{P(\Omega)r(\Omega)}{2} \leq |\Omega| \leq r(\Omega)(P(\Omega) - \pi r(\Omega)), \tag{4.77}$$

with the estimates (4.71) to obtain the optimal inequalities (4.76).

The upper bound in (4.71) is an equality for circumscribed sets, since both the upper bound in (4.71) and the lower bound in (4.77) are equalities for circumscribed sets. The lower bound is achieved by stadiums, since both the lower bound in (4.71) and the upper bound in (4.77) are achieved with equality sign on this class of sets. □

The triplet (d, h, r)**.** In the following, we will denote by $\mathcal{S}_{r,d}$ the symmetrical spherical slice of inradius r and diameter d and we will denote by $\mathcal{N}_{r,d}$ the regular smoothed nonagon of same inradius and diameter, see Definitions 1.4.3 and 1.4.7.

Proposition 4.3.6. *Let* $\Omega \in \mathcal{K}^2$*. Then, it holds*

$$h(\Omega) \leq \frac{1}{r(\Omega)} + \sqrt{\frac{\pi}{r(\Omega)\sqrt{diam^2(\Omega) - 4r^2(\Omega)} + r^2(\Omega)\left(\pi - 2\arccos\left(\frac{2r(\Omega)}{diam(\Omega)}\right)\right)}}, \tag{4.78}$$

where equality is achieved if and only if Ω *is a symmetric two-cup body. Moreover, we have*

$$h(\Omega) \geq \frac{1}{t_{g_1^\Omega}} \tag{4.79}$$

where $t_{g_1^\Omega}$ *is the smallest solution to the equation*

$$g_1^\Omega(t) := \psi(diam(\Omega) - 2t, r(\Omega) - t) = \pi t^2 \tag{4.80}$$

on the interval $[0, r(\Omega)]$ *and the function* ψ *is defined in* (1.33)*. Moreover, there exists* D_0 *such that the problem*

$$\min\{h(\Omega) \mid \Omega \in \mathcal{K}^2_{d,r}\}$$

is solved by the smoothed nonagon $\mathcal{N}_{r,d}$ *if* $d < rD_0$ *and by the slice* $\mathcal{S}_{r,d}$ *if* $d \geq rD_0$*.*

Proof. To prove (4.78), one has just to combine the upper bound in (4.71), which is an equality for sets that are homothetic to their form bodies, and (1.31), which is an equality for and only for symmetric two-cup bodies, that are particular sets homothetic to their form bodies.

Let us now prove (4.79). As in Theorem 1.4.2 (see [67]), for any constant parameter $r > 0$, we consider the function

$$\Psi_r(x)=\begin{cases} f_r(x):=\dfrac{3\sqrt{3}r}{2}(\sqrt{x^2-3r^2}-r)+\dfrac{3x^2}{2}\left(\dfrac{\pi}{3}-\arccos\left(\dfrac{\sqrt{3}r}{x}\right)\right), & \text{if } x\leq rD^*\\ g_r(x):=r\sqrt{x^2-4r^2}+\dfrac{x^2}{2}\arcsin\left(\dfrac{2r}{x}\right), & \text{if } x\geq rD^*\end{cases}$$

defined for $x\geq 2r$. The function Ψ_r is strictly increasing. Indeed, we have for every $x<rD^*$

$$f_r'(x)=\frac{3\sqrt{3}rx}{2\sqrt{x^2-3r^2}}+3x\left(\frac{\pi}{3}-\arccos\left(\frac{\sqrt{3}r}{x}\right)\right)-\frac{3\sqrt{3}r}{2\sqrt{1-\frac{3r^2}{x^2}}}=3x\left(\frac{\pi}{3}-\arccos\left(\frac{\sqrt{3}r}{x}\right)\right),$$

and for every $x>rD^*$,

$$g_r'(x)=-\frac{r}{\sqrt{1-\frac{4r^2}{x^2}}}+\frac{rx}{\sqrt{x^2-4r^2}}+x\arcsin\left(\frac{2r}{x}\right)=x\arcsin\left(\frac{2r}{x}\right)>0.$$

Thus, the function f_r is increasing on $[2r,2\sqrt{3}r]$ and is decreasing on $[2\sqrt{3}r,+\infty)$ and the function g_r is increasing on $[2r,+\infty)$. Moreover, we have by Theorem 1.4.2 that $D^*\leq 2\sqrt{3}$. So, the function f_r is increasing on the sub-interval $[2r,rD^*]$ and, since $f_r(rD^*)=g_r(rD^*)=\Psi_r(D^*)$, the continuous function Ψ_r is increasing on $[2r,+\infty)$.

Let $t\in[0,r(\Omega)]$, by applying the result of Theorem 1.4.2 on the convex set Ω_t, we have

$$|\,\Omega_t\,|\leq\Psi_{r(\Omega_t)}(\mathrm{diam}(\Omega_t))=\Psi_{r(\Omega)-t}(\mathrm{diam}(\Omega_t))\leq\Psi_{r(\Omega)-t}(\mathrm{diam}(\Omega)-2t)=:g_1^\Omega(t),$$

where we use the monotonicity of the function $\Psi_{r(\Omega)-t}$ and the estimates (1.20) and (1.21).

Now, using Lemma 4.3.3, we have the following bound for the Cheeger constant

$$h(\Omega)\geq\frac{1}{t_{g_1^\Omega}},\tag{4.81}$$

where $t_{g_1^\Omega}$ is the smallest solution to the equation $g_1^\Omega(t)=\pi t^2$ on the interval $[0,r(\Omega)]$.

It remains to prove that for every $r>0$ and $d\geq 2r$, there exists a convex set of inradius r and diameter d such that (4.81) is an equality. If $d=2r$ then Ω is a ball and thus the equality is trivial. Let us now consider the following two cases:

- If $d\geq rD^*$, we have, for every $t\in[0,r)$,

$$|(\mathcal{S}_{r,d})_t|=|\mathcal{S}_{r-t,d-2t}|=\Psi_{r-t}(d-2t),\tag{4.82}$$

  where the first equality is a consequence of the equality $(\mathcal{S}_{r,d})_t=\mathcal{S}_{r-t,d-2t}$ and the second one is a consequence of [67, Theorem 2] and of the following estimate

$$\mathrm{diam}((\mathcal{S}_{r,d})_t)=d-2t>rD^*-2t>rD^*-tD^*=(r-t)D^*=r((\mathcal{S}_{r,d})_t)D^*,$$

  where we used $D^*\approx 2,3888>2$ (see [67, Theorem 2]). Thus, we have by (4.82)

$$h(\mathcal{S}_{r,d})=\frac{1}{t_{g_1^\Omega}},$$

  with

$$r(\mathcal{S}_{r,d})=r\ \text{ and }\ \mathrm{diam}(\mathcal{S}_{r,d})=d.$$

- If $d \in (2r, rD^*]$, we consider $t^* := \frac{D^* r - d}{D^* - 2}$, that is the value for which the graphs of the functions $t \longmapsto |(\mathcal{N}_{r,d})_t|$ and $t \longmapsto |(\mathcal{S}_{r,d})_t|$ intersect each other, see Figure 4.4. We note that $(\mathcal{N}_{r,d})_t = \mathcal{N}_{r-t,d-2t}$ for every $t \in [0, t^*]$.

  We introduce the quantity $D_0 \in (2, D^*)$ as the (unique) value in the interval $(2, D^*)$ for which the graph of the (decreasing) function $x \longmapsto \frac{D^* - x}{D^* - 2}$ intersects the graph of the (increasing[1]) one $x \longmapsto \frac{1}{h(\mathcal{N}_{1,x})}$. As shown in Figure 4.4, we have the following cases:

  - If $\frac{d}{r} < D_0$, i.e. $t^* > \frac{1}{h(\mathcal{N}_{r,d})}$, we have $h(\mathcal{N}_{r,d}) = \frac{1}{t_{g_1^\Omega}}$, which means that in this case the smoothed nonagon $\mathcal{N}_{r,d}$ provides the equality in (4.79).
  - If $\frac{d}{r} > D_0$, i.e. $t^* < \frac{1}{h(\mathcal{N}_{r,d})}$, we have $h(\mathcal{S}_{r,d}) = \frac{1}{t_{g_1^\Omega}}$, which means that in this case the slice $\mathcal{S}_{r,d}$ provides the equality in (4.79).
  - If $\frac{d}{r} = D_0$, i.e. $t^* = \frac{1}{h(\mathcal{N}_{r,d})}$, we have $h(\mathcal{N}_{r,d}) = h(\mathcal{S}_{r,d}) = \frac{1}{t_{g_1^\Omega}}$, which means that in this case both the smoothed nonagon $\mathcal{N}_{r,d}$ and the slice $\mathcal{S}_{r,d}$ provide the equality in (4.79).

So, the proof is concluded. □

Remark 4.3.1. We note that the symmetrical slices and the smoothed nonagons are not the only sets solving the shape optimization problem $\min\{h(\Omega) \mid \Omega \in \mathcal{K}^2_{d,r}\}$. Indeed, if for example we consider a spherical slice $\mathcal{S}$ and denote by $C_{\mathcal{S}}$ its Cheeger set, we have $h(\mathcal{S}) = h(C_{\mathcal{S}})$ and, by the explicit characterization of the Cheeger sets given in Theorem 4.3.1, we have

$$r(C_{\mathcal{S}}) = r\left(\mathcal{S}_{\frac{1}{h(\mathcal{S})}} + \frac{1}{h(\mathcal{S})}B_1\right) = r\left(\mathcal{S}_{\frac{1}{h(\mathcal{S})}}\right) + \frac{1}{h(\mathcal{S})} = r(\mathcal{S}) - \frac{1}{h(\mathcal{S})} + \frac{1}{h(\mathcal{S})} = r(\mathcal{S})$$

and

$$\mathrm{diam}(C_{\mathcal{S}}) = d(\mathcal{S}),$$

meanwhile, $\mathcal{S} \neq C_{\mathcal{S}}$, which proves the non-uniqueness of the solution of the minimization problem $\min\{h(\Omega) \mid \Omega \in \mathcal{K}^2_{d,r}\}$.

Remark 4.3.2. We give the following explicit lower bound. In [145], it is proved that

$$|\,\Omega\,| < 2\mathrm{diam}(\Omega) r(\Omega).$$

By applying the strategy of Lemma 4.3.3, we obtain that

$$h(\Omega) \geq \frac{4-\pi}{\mathrm{diam}(\Omega) + 2r(\Omega) - \sqrt{(\mathrm{diam}(\Omega) + 2r(\Omega))^2 - 2(4-\pi)\mathrm{diam}(\Omega) r(\Omega)}}.$$

[1]The function $x \longmapsto \frac{1}{h(\mathcal{N}_{1,x})}$ is increasing because $x \longmapsto \mathcal{N}_{1,x}$ is increasing for inclusion, meanwhile the Cheeger constant is decreasing with respect by inclusions.

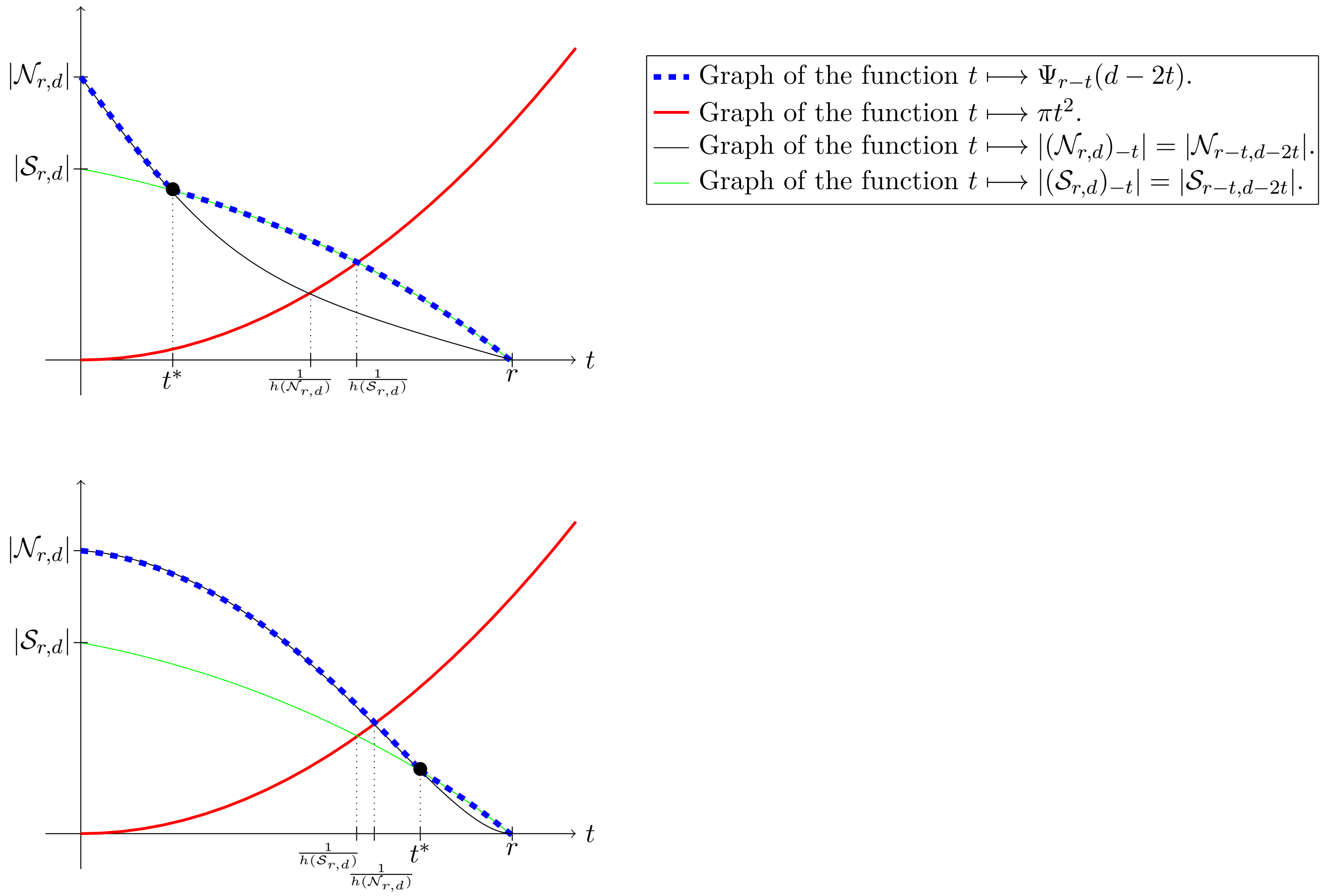

Figure 4.4: Different cases of equality in the inequality (4.79).

The triplet (R, h, r)**.**

Proposition 4.3.7. *Let* $\Omega \in \mathcal{K}^2$*. Then, it holds*

$$h(\Omega) \geq \frac{1}{t_{g_2^\Omega}}, \tag{4.83}$$

where $t_{g_2^\Omega}$ *is the smallest solution of the equation*

$$g_2^\Omega(t) := 2\left((r-t)\sqrt{(R(\Omega)-t)^2 - (r(\Omega)-t)^2} + (R(\Omega)-t)^2 \arcsin\left(\frac{r(\Omega)-t}{R(\Omega)-t}\right)\right) = \pi t^2. \tag{4.84}$$

The equality in (4.83) *is achieved if and only if* Ω *is a symmetrical spherical slice. Moreover, it holds*

$$h(\Omega) \leq \frac{1}{r(\Omega)} + \sqrt{\frac{\pi}{2r(\Omega)\left(\sqrt{R(\Omega)^2 - r(\Omega)^2} + r(\Omega)\arcsin\left(\frac{r(\Omega)}{R(\Omega)}\right)\right)}}, \tag{4.85}$$

where the equality in (4.85) *is achieved by two-cup bodies.*

Proof. In order to prove (4.83), we apply the result of Lemma 4.3.3. Let us introduce the function

$$\varphi : (R, r) \longmapsto 2\left(r\sqrt{R^2 - r^2} + R^2 \arcsin\frac{r}{R}\right),$$

which is increasing with respect to the first variable, indeed

$$\frac{\partial \varphi}{\partial R}(R, r) = 2R \arcsin\left(\frac{r}{R}\right) > 0.$$

By applying (1.40) (where the equality holds only for symmetrical spherical slices), we have, for every $t \in [0, r(\Omega)]$,

$$|\,\Omega_t\,| \leq \varphi(R(\Omega_t), r(\Omega_t)) = \varphi(R(\Omega_t), r(\Omega) - t) \leq \varphi(R(\Omega) - t, r(\Omega) - t) =: g_2^\Omega(t),$$

where the last inequality is a consequence of the monotonicity of the function $R \longmapsto \varphi(R, r)$ and of the fact that $R(\Omega_t) \leq R(\Omega) - t$ (see Lemma 1.3.2). Finally, we conclude by applying the result of Lemma 4.3.3, noticing that condition (4.74) can be easily checked.

In order to prove (4.85), we combine the upper bound in (4.71) and the inequality (1.39). As far as the sharpness of (4.85) is concerned, we observe that (4.71) is sharp on sets that are homothetic to their form body and (1.39) is attained by symmetric two-cup bodies, that are sets that are homothetic to their form body. □

Remark 4.3.3. We can give the following explicit lower bound

$$h(\Omega) \geq \frac{4-\pi}{2(R(\Omega) + r(\Omega)) - \sqrt{4(R(\Omega) + r(\Omega))^2 - 4(4-\pi)R(\Omega)r(\Omega)}},$$

which can be obtained starting from

$$|\Omega| \leq 4R(\Omega)r(\Omega),$$

see [90], and the strategy from Lemma 4.3.3.

4.3.1 Explicit description of the Blaschke-Santaló diagrams

We denote by $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ the Blaschke Santaló diagram respectively corresponding to the triplets (P,h,r), (d,h,r) and (R,h,r). We have defined the quantities $t_{g_1^\Omega}$ and $t_{g_2^\Omega}$ respectively as the smallest solution on $[0,r(\Omega)]$ of equations (4.80) and (4.84). We observe that $t_{g_1^\Omega}$ depends on $r(\Omega)$ and $\text{diam}(\Omega)$, and $t_{g_2^\Omega}$ depends on $r(\Omega)$ and $R(\Omega)$.

In the following Proposition, we are keeping the inradius $r(\Omega)=1$ and consider different values of the remaining variables: diameter for the diagram $\mathcal{D}_2$ and circumradius for $\mathcal{D}_3$. We then use the notation:

$$t_{g_1^\Omega} = t_{g_1}(x) \;\; \text{when } \text{diam}(\Omega) = x \text{ and } \quad t_{g_2^\Omega} = t_{g_2}(x) \;\; \text{when } R(\Omega) = x$$

Proposition 4.3.8. *We obtain the following description of the Blaschke-Santaló diagrams*

$$\mathcal{D}_1 := \left\{(P(\Omega),h(\Omega)) : \Omega\in\mathcal{K}^2,\, r(\Omega)=1\right\} = \left\{(x,y) : 1+\frac{\pi}{x-\pi} \le y \le 1+\sqrt{\frac{2\pi}{x}}\right\}; \tag{4.86}$$

$$\begin{aligned}\mathcal{D}_2 &:= \left\{(diam(\Omega),h(\Omega)) : \Omega\in\mathcal{K}^2,\, r(\Omega)=1\right\}\\ &= \left\{(x,y) : \frac{1}{t_{g_1}(x)} \le y \le 1+\sqrt{\frac{\pi}{\sqrt{x^2-4}+\left(\pi-2\arccos\frac{2}{x}\right)}}\right\};\end{aligned} \tag{4.87}$$

$$\begin{aligned}\mathcal{D}_3 &:= \left\{(R(\Omega),h(\Omega)) : \Omega\in\mathcal{K}^2,\, r(\Omega)=1\right\}\\ &= \left\{(x,y) : \frac{1}{t_{g_2}(x)} \le y \le 1+\sqrt{\frac{\pi}{2\left(\sqrt{x^2-1}+\arcsin\frac{1}{x}\right)}}\right\}.\end{aligned} \tag{4.88}$$

Proof. Let us prove that (4.86) holds. The bounds proved in Proposition 4.3.5 ensure us that

$$D_1 \subseteq \left\{(x,y) : 1+\frac{\pi}{x-\pi} \le y \le 1+\sqrt{\frac{2\pi}{x}}\right\} =: \mathcal{P}\;;$$

it only remains to prove the converse inclusion.

First of all, we observe that the boundary of $\mathcal{P}$ is included in $\mathcal{D}_1$. We explicitly construct a family of convex sets which fills the lower boundary of $\mathcal{P}$. We consider the family of stadiums $\{\,R_l\,\}_{l\ge 0}$ obtained as the convex hull of two balls of radius 1 and centered in $(0,0)$ and $(0,l)$. Indeed one has:

$$P(R_l)=2\pi+2l,\quad r(R_l)=1,\quad h(R_l)=\frac{P(R_l)}{|R_l|}=\frac{2\pi+2l}{\pi+2l}=1+\frac{\pi}{P(R_l)-\pi}.$$

We now construct a family of convex sets filling the upper bound of the boundary of $\mathcal{P}$. We consider the family of two cup bodies $\{\,C_k\,\}_{k\ge 1}$ obtained as the convex hull of the ball B_1 centered at the origin and the points $(-k,0)$ and $(k,0)$. One has (see [96]):

$$P(C_k)=2\left(\sqrt{4k^2-4}+2\arcsin\frac{1}{k}\right),\quad r(C_k)=1,\quad |C_k|=\left(\sqrt{4k^2-4}+2\arcsin\frac{1}{k}\right)$$

and, as it is shown in [78],

$$h(C_k)=\frac{1}{r(C_k)}+\sqrt{\frac{\pi}{|C_k|}}=1+\sqrt{\frac{2\pi}{P(C_k)}}.$$

In order to conclude, we show that we can always construct a continuous path connecting the upper and the lower part of the diagram, covering, in this way, all the area between the two boundaries.

By contradiction, let us assume that there exists a region $\mathcal{A} \subset \mathcal{P} \setminus \mathcal{D}_1$, and let $(x_0, y_0) \in A$. There exist $R_{l_0} \in \{ R_l \}$ and $C_{k_0} \in \{ C_k \}$ such that $P(R_{l_0}) = x_0$ and $P(C_{k_0}) = x_0$. Moreover, let us notice that

$$h(R_{l_0}) \leq y_0 \leq h(C_{k_0}).$$

Now, we define, via the Minkowski sum, the set

$$K_t = tR_{l_0} + (1-t)C_{k_0}.$$

By linearity of $r(\cdot), P(\cdot)$ and $h(\cdot)$ with respect to the Minkowski sum, we have, for every $t \in [0,1]$

$$r(K_t) = 1, \quad P(K_t) = x_0, \quad h(K_t) = th(R_{l_0}) + (1-t)h(C_{k_0}),$$

so there exists $\bar{t} \in (0,1)$ such that $h(K_{\bar{t}}) = y_0$ and this is absurd.

The proofs of (4.87) and (4.88) can be dealt with in an analogous way.

As far as the (d,h,r) diagram is concerned, once one proves that the boundary of the two sets coincides, one has only to prove that the diagram is vertically convex to show (4.87). Indeed, up to rotation, we can assume that the two sets have collinear diameters, and, in this particular case, it holds that the diameter is linear with respect to the Minkowski sum so we can conclude.

As far as the (R,h,r) diagram is concerned, once one proves that the boundary of the two sets coincides, one can conclude by proving that the diagram is vertically convex. In general, it is not true that the circumradius is linear with respect to Minkowski sum, see Remark 1.3.1. On the other hand, to prove that the diagram (R,h,r) is vertically convex, it is enough to prove that the circumradius is linear for a convex combination of a spherical slice and a two-cup body. Indeed, for this kind of sets, the circumradius satisfies the following:

$$R(\Omega) = \frac{\operatorname{diam}(\Omega)}{2}.$$

So by (1.19) and the fact that $2R(\Omega) \geq \operatorname{diam}(\Omega)$, we have

$$R(t\,\Omega_1 + (1-t)\,\Omega_2) \leq tR(\Omega_1) + (1-t)R(\Omega_2) = \frac{1}{2}\left(\operatorname{diam} t(\Omega_1) + (1-t)\operatorname{diam}(\Omega_2)\right) =$$
$$\frac{1}{2}\left(\operatorname{diam}(t\,\Omega_1 + (1-t)\,\Omega_2)\right) \leq R(t\,\Omega_1 + (1-t)\,\Omega_2),$$

and the conclusion follows as in the case of (4.87). □

Remark 4.3.4. Proving Blaschke–Santaló sharp bounds is not equivalent to completely characterizing the diagram. Indeed, once we managed to identify the boundary of the diagram, it remains to show that the diagram is simply connected, which can be a difficult task, see for example [16, Conjecture 5], [115, Open problem 2] and [157, Problem 3].

However, when two quantities in the Blaschke–Santaló triplets are linear and continuous with respect to the Minkowski sum, we can fill the diagram, similarly as in the proof of Proposition 4.3.8. This is the case for all triplets analyzed in the present section. We refer to [81] for a proof of the simple connectedness of a Blaschke–Santaló diagram where the linearity assumption does not hold for two functionals.

Regarding the classes of sets $\mathcal{K}^2_{\omega,d}$ $\mathcal{K}^2_{R,\omega}$, $\mathcal{K}^2_{\omega,P}$ and $\mathcal{K}^2_{A,\omega}$, we can identify part of the boundary of the Blasche- Santaló diagram. See Propositions 4.3.9, 4.3.10, 4.3.11 and 4.3.12 for the explicit bounds.

In the class $\mathcal{K}^2_{d,R}$ we have solved the maximization problem, while in the class $\mathcal{K}^2_{\omega,R}$ the minimization one; see Propositions 4.3.13 and 4.3.10 for the explicit bounds.

The triplet (ω, h, d).

Proposition 4.3.9. *Let $\Omega \in \mathcal{K}^2$. Then, it holds*

$$h(\Omega) \geq \frac{1}{t_{g_3^\Omega}} \tag{4.89}$$

where $t_{g_3^\Omega}$ is the smallest solution to

$$g_3^\Omega(t) := \frac{w(\Omega)-2t}{2}\sqrt{(diam(\Omega)-2t)^2-(w(\Omega)-2t)^2} + \frac{(diam(\Omega)-2t)^2}{2}\arcsin\left(\frac{w(\Omega)-2t}{diam(\Omega)-2t}\right) = \pi t^2.$$

The equality in (4.89) is achieved by symmetrical spherical slices. Moreover,

- *if $w(\Omega) \leq \frac{\sqrt{3}}{2}diam(\Omega)$, it holds*
$$h(\Omega) \leq h(T_I), \tag{4.90}$$
where T_I is the subequilateral triangle such that $w(T_I) = w(\Omega)$ and $diam(T_I) = diam(\Omega)$. The equality in (4.90) is achieved by the isosceles triangle T_I;

- *and if $\frac{\sqrt{3}}{2}diam(\Omega) \leq w(\Omega) \leq diam(\Omega)$, we have*
$$h(\Omega) \leq \frac{\sqrt{3}}{\sqrt{3}w(\Omega)-diam(\Omega)} + \sqrt{\frac{2\pi}{\pi w(\Omega)^2 - \sqrt{3}diam(\Omega)^2 + 6w(\Omega)\left(\tan\left(\arccos\left(\frac{w(\Omega)}{diam(\Omega)}\right)\right) - \arccos\left(\frac{w(\Omega)}{diam(\Omega)}\right)\right)}}. \tag{4.91}$$

The equality in (4.91) is achieved by equilateral triangles.

Proof. Let us start by proving the lower bound (4.89), by using the strategy given in Lemma 4.3.3. For every $\Omega \in \mathcal{K}^2$, it holds (see [109] and also [145])

$$|\Omega| \leq \frac{w(\Omega)}{2}\sqrt{\mathrm{diam}^2(\Omega) - w^2(\Omega)} + \frac{\mathrm{diam}^2(\Omega)}{2}\arcsin\left(\frac{w(\Omega)}{\mathrm{diam}(\Omega)}\right)$$

with equality if and only if Ω is a symmetrical spherical slice. If we denote by

$$f(d,w) = \frac{w}{2}\sqrt{d^2-w^2} + \frac{d^2}{2}\arcsin\left(\frac{w}{d}\right),$$

we have

$$\frac{\partial f}{\partial d}(d,w) = d\arcsin\left(\frac{w}{d}\right) > 0,$$

$$\frac{\partial f}{\partial w}(d,w) = \sqrt{d^2 - w^2} > 0.$$

Thus, using Lemma 4.3.3, we have

$$|\Omega_t| \leq f(\mathrm{diam}(\Omega_t), w(\Omega_t)) \leq f(\mathrm{diam}(\Omega) - 2t, w(\Omega) - 2t)$$

and

$$h(\Omega) \geq \frac{1}{t_{g_3^\Omega}},$$

where $t_{g_3^\Omega}$ is the smallest solution to

$$g_3^\Omega(t) := f(\mathrm{diam}(\Omega) - 2t, w(\Omega) - 2t) = \pi t^2.$$

We note that condition (4.74) can be easily checked. In order to prove the upper bound (4.90), we consider the following minimization problem for the area in the class of convex sets with given diameter and width, studied in [148] and [145]:

(i) if $2w(\Omega) \leq \sqrt{3}\mathrm{diam}(\Omega)$, then

$$2|\Omega| \geq w(\Omega)\mathrm{diam}(\Omega), \tag{4.92}$$

where equality is achieved by triangles;

(ii) if $\sqrt{3}\mathrm{diam}(\Omega) \leq 2w(\Omega) \leq 2\mathrm{diam}(\Omega)$, then

$$\begin{aligned} 2|\Omega| \geq& \pi w^2(\Omega) - \sqrt{3}\mathrm{diam}^2(\Omega) + \\ & 6w^2(\Omega)\left(\tan\left(\arccos\left(\frac{w(\Omega)}{\mathrm{diam}(\Omega)}\right)\right) - \arccos\left(\frac{w(\Omega)}{\mathrm{diam}(\Omega)}\right)\right) \\ =& |T_Y|, \end{aligned} \tag{4.93}$$

where the equality is achieved by the Yamanouti set T_Y such that $w(T_Y) = w(\Omega)$ and $\mathrm{diam}(T_Y) = \mathrm{diam}(\Omega)$.

Moreover, if we consider the minimization problem of the inradius in the class of convex set with given diameter and width, we have from Theorem 1.4.9 (see (1.44) and (1.45)):

$$r(\Omega) \geq \begin{cases} r(T_I), & \text{if } 2w(\Omega) \leq \sqrt{3}\mathrm{diam}(\Omega) \\ r(T_Y) & \text{if } \sqrt{3}\mathrm{diam}(\Omega) \leq 2w(\Omega) \leq 2\mathrm{diam}(\Omega). \end{cases} \tag{4.94}$$

So combining (4.71) with the estimates (4.92), (4.93) and (4.94), we obtain (4.89)

$$h(\Omega) \leq \begin{cases} \frac{1}{r(T_I)} + \sqrt{\frac{\pi}{|T_I|}} = h(T_I), & \text{if } 2w(\Omega) \leq \sqrt{3}\mathrm{diam}(\Omega) \\ \frac{1}{r(T_Y)} + \sqrt{\frac{\pi}{|T_Y|}} & \text{if } \sqrt{3}\mathrm{diam}(\Omega) \leq 2w(\Omega) \leq 2\mathrm{diam}(\Omega). \end{cases}$$

The explicit formula given in the inequality (4.91) is obtained by using (4.93) and

$$r(T_Y) = \omega(T_Y) - \frac{\mathrm{diam}(T_Y)}{\sqrt{3}},$$

see [94, Theorem 2]. □

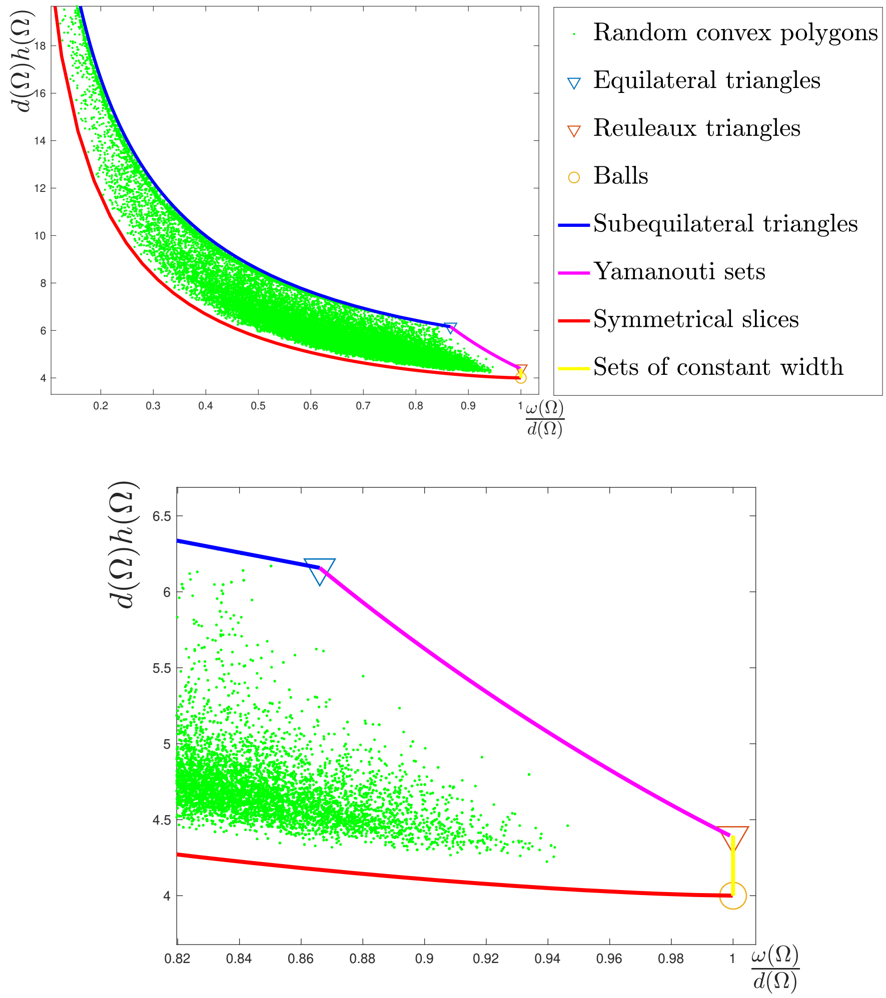

Figure 4.5: Blaschke–Santaló diagram of the triplet (ω, h, d).

We are also able to give an explicit, but not sharp, lower bound of the Cheeger constant in terms of the width and the diameter.

Remark 4.3.5. Let $\Omega \in \mathcal{K}^2$. Then, it holds

$$h(\Omega) > \frac{1}{w(\Omega)} + \frac{1}{\mathrm{diam}(\Omega)} + \sqrt{\left(\frac{1}{w(\Omega)} + \frac{1}{\mathrm{diam}(\Omega)}\right)^2 - \frac{4-\pi}{w(\Omega)\mathrm{diam}(\Omega)}}, \tag{4.95}$$

where equality is asymptotically achieved by a sequence of thin collapsing rectangles. In order to prove (4.95), it is enough to consider the inequality

$$|\Omega| \leq w(\Omega)\mathrm{diam}(\Omega)$$

and to use the strategy of Lemma 4.3.3.

Conjecture 4.3.6. As far as the diagram (ω, h, d) is concerned, we conjecture that, if $\sqrt{3}/2\mathrm{diam}(\Omega) \leq w(\Omega) \leq \mathrm{diam}(\Omega)$, then for all $\Omega \in \mathcal{K}^2$

$$h(\Omega) \leq h(Y)$$

where Y is the Yamanouti set with $w(Y) = w(\Omega)$ and $\mathrm{diam}(Y) = \mathrm{diam}(\Omega)$ (see Figure 4.5 and 1.4).

The triplet (ω, h, R).

Proposition 4.3.10. *Let $\Omega \in \mathcal{K}^2$. Then, it holds*

$$h(\Omega) \geq \frac{1}{t_{g_4^\Omega}}, \tag{4.96}$$

where $t_{g_4^\Omega}$ is the smallest solution of the equation

$$g_4^\Omega(t) := \frac{(w(\Omega)-2t)}{2}\sqrt{4\left(R(\Omega)-t\right)^2-\left(w(\Omega)-2t\right)^2}+2(R(\Omega)-t)^2\arcsin\left(\frac{w(\Omega)-2t}{2(R(\Omega)-t)}\right)=\pi t^2$$

on $[0, r(\Omega)]$.

The equality in (4.96) is achieved by symmetrical spherical slices. Moreover, it holds

$$h(\Omega) \leq h(T_I), \qquad \text{if } w(\Omega) \leq \frac{3}{2}R(\Omega), \tag{4.97}$$

where T_I is the subequilateral triangle such that $R(T_I) = R(\Omega)$ and $\omega(T_I) = w(\Omega)$. The equality in (4.97) is realized by the subequilateral triangle T_I.

Proof. Let us start by proving the lower bound (4.96), by using the strategy given in Lemma 4.3.3.

Let us recall the function defined in (1.35):

$$\chi : (\omega, R) \longmapsto \frac{\omega}{2}\sqrt{4R^2-\omega^2}+2R^2\arcsin\frac{\omega}{2R}.$$

We have, for every $R, \omega > 0$,

$$\frac{\partial\chi}{\partial R}(\omega, R) = 4R\arcsin\frac{\omega}{2R} \geq 0 \quad \text{and} \quad \frac{\partial\chi}{\partial\omega}(\omega, R) = \sqrt{4R^2-\omega^2} \geq 0.$$

Thus, using Theorem 1.4.3, we have, for every $t \in [0, r(\Omega))$,

$$|\Omega_t| \leq \chi(w(\Omega_t), R(\Omega_t)) \leq \chi(w(\Omega)-2t, R(\Omega)-t) := g_4^\Omega(t),$$

where we use (1.22) and (1.23). By Lemma 4.3.3, we have

$$h(\Omega) \geq \frac{1}{t_{g_4^\Omega}},$$

where $t_{g_4^\Omega}$ is the smallest solution to the equation $g_4^\Omega(t) = \pi t^2$ on the interval $[0, r(\Omega)]$. We observe that condition (4.74) can be easily checked.

Let us now prove the upper bound (4.97). We recall the inequality (4.71):

$$h(\Omega) \leq \frac{1}{r(\Omega)} + \sqrt{\frac{\pi}{|\Omega|}}, \tag{4.98}$$

where equality is achieved for instance by circumscribed sets, in particular, by triangles. In order to prove (4.97), we consider the following facts:

(i) In (1.36), it is proved that, if $\Omega \in \mathcal{K}^2$,

$$|\Omega| \geq |T_I| \quad \text{if } w(\Omega) \leq \frac{3}{2}R(\Omega), \tag{4.99}$$

where T_I is the subequilateral triangle with given width and circumradius and nothing is known if $w(\Omega) \geq 3/2R(\Omega)$;

(ii) In Theorem 1.4.5, see (1.37), and Theorem 1.4.6, see (1.38), it is proved that if

$$r(\Omega) \geq \begin{cases} r(T_I) & \text{if } w(\Omega) \leq \frac{3}{2}R(\Omega) \\ w(\Omega) - R(\Omega) & \text{if } w(\Omega) \geq \frac{3}{2}R(\Omega), \end{cases} \tag{4.100}$$

where, if $w(\Omega) \geq \frac{3}{2}R(\Omega)$, equality is achieved by any set obtained by an equilateral triangle of circumradius $R(\Omega)$ by replacing the edges by three equal circular arcs, in particular each arc of circle is centered on the height relative to the same side.

So, if $w(\Omega) \leq \frac{3}{2}R(\Omega)$, combining (4.98) with (4.99) and (4.100), we get the thesis. □

In the following Remark, we give some non-sharp bounds, that are explicit.

Remark 4.3.7. We can prove that

$$h(\Omega) \leq \frac{3}{w(\Omega)} + \sqrt{\frac{\pi}{\sqrt{3}R(\Omega)w(\Omega)}}. \tag{4.101}$$

We recall the following inequalities, proved in [145]:

$$|\Omega| \geq \sqrt{3}R(\Omega)w(\Omega) \quad \text{and} \quad w(\Omega) \leq 3r(\Omega). \tag{4.102}$$

By combining these estimates and the upper bound in (4.71), we have

$$h(\Omega) \leq \frac{3}{w(\Omega)} + \sqrt{\frac{\pi}{\sqrt{3}R(\Omega)w(\Omega)}}.$$

Since the equality in (4.71) is achieved by circumscribed sets, while the equalities in (4.102) are achieved by equilateral triangles, that are particular circumscribed sets, we have the equality in (4.101) for equilateral triangles.

Moreover, another non-sharp upper bound can be obtained by using the strategy from Lemma 4.3.3, starting from

$$|\Omega| < 2R(\Omega)w(\Omega),$$

which is asymptotically achieved by a sequence of rectangles with circumradius that goes to infinity (see [90]). We get $|\Omega_t| \leq 2R(\Omega_t)w(\Omega_t) \leq 2(R(\Omega) - t)(w(\Omega) - 2t)$ and, consequently,

$$h(\Omega) \geq \frac{4-\pi}{(2R(\Omega) + w(\Omega)) - \sqrt{(2R(\Omega) + w(\Omega))^2 - (4-\pi)(2R(\Omega)w(\Omega))}}.$$

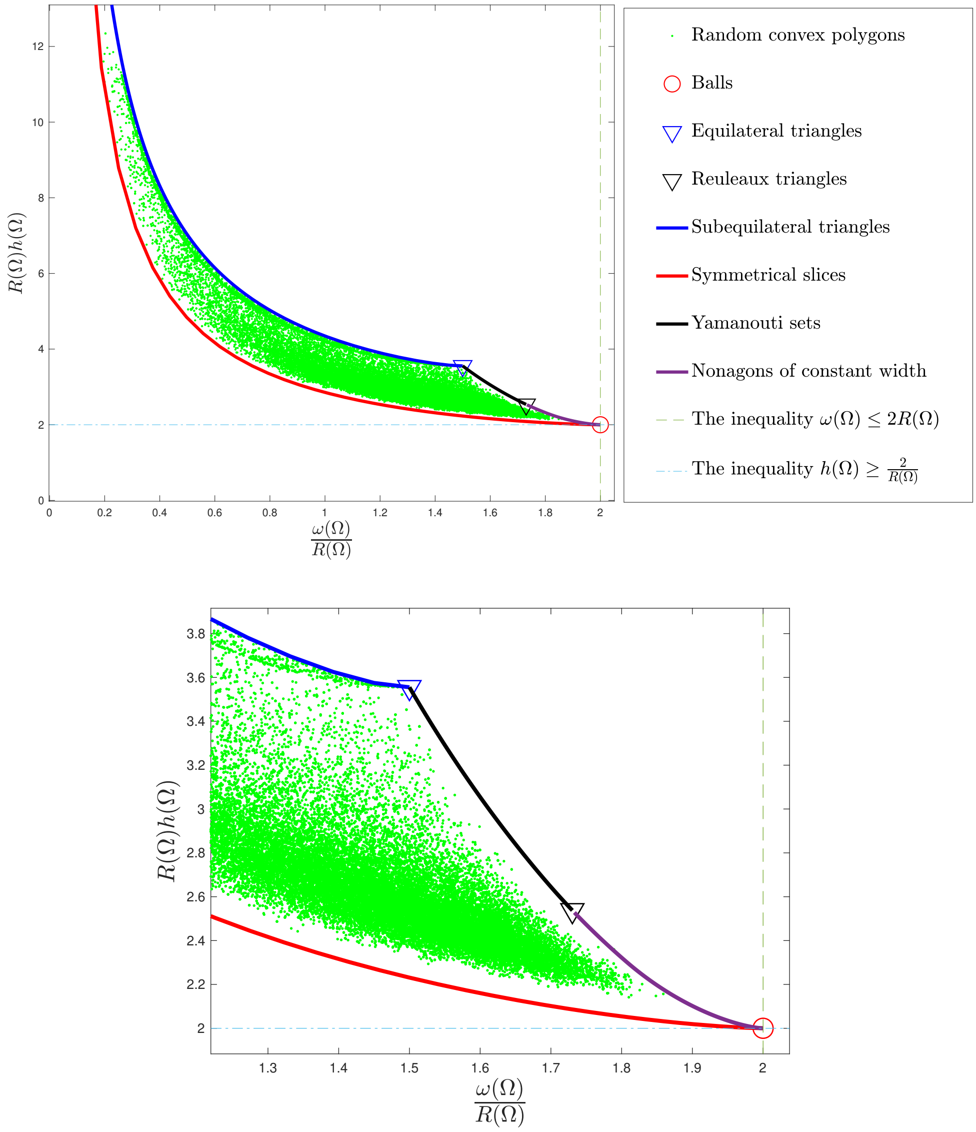

Figure 4.6: Blaschke–Santaló diagram of the triplet (ω, h, R).

Conjecture 4.3.8. As far as the diagram (ω, h, R) is concerned, we conjecture that, if $w(\Omega) \geq \frac{3}{2}R(\Omega)$, then, for all $\Omega \in \mathcal{K}^2$,

$$h(\Omega) \leq h(\bar{T}),$$

where $\bar{T}$ is a Yamanouti set if

$$\frac{3}{2} \leq \frac{w(\Omega)}{R(\Omega)} \leq K$$

and a *nonagon of constant width* (see Figure 1.6) if

$$K \leq \frac{w(\Omega)}{R(\Omega)} \leq 2,$$

being K the value that one obtains if computes the ratio

$$\frac{w(\Omega)}{R(\Omega)}$$

on a Relaoux triangle. (see Figure (4.6) for the Blaschke-Santaló diagram).

The triplet (ω, h, P)**.**

Proposition 4.3.11. *Let $\Omega \in \mathcal{K}^2$. Then, it holds*

$$h(\Omega) \geq \frac{2}{w(\Omega)} + \frac{2\pi}{2P(\Omega) - \pi w(\Omega)}, \tag{4.103}$$

where equality is achieved by stadiums. Moreover, if $P(\Omega) \geq 2\sqrt{3}w(\Omega)$, it holds

$$h(\Omega) \leq h(T_I), \tag{4.104}$$

where T_I is the subequilateral triangle such that $P(T_I) = P(\Omega)$ and $w(T_I) = w(\Omega)$. The equality in (4.104) is achieved by the subequilateral triangle T_I.

Proof. The inequality (4.103) is a consequence of (4.105) and the inequality

$$|\,\Omega\,| \leq \frac{w(\Omega)}{2}\left(P(\Omega) - \frac{\pi w(\Omega)}{2}\right),$$

which is an equality for stadiums (see for example [145] and the references therein).

Let us now assume that $P(\Omega) \geq 2\sqrt{3}w(\Omega)$. In order to prove (4.104), we recall inequality (1.43):

$$P(\Omega) \leq \sqrt{\frac{4r^2(\Omega)w^3(\Omega)}{(w(\Omega) - 2r(\Omega))^2(4r(\Omega) - w(\Omega))}} =: f_{w(\Omega)}(r(\Omega))$$

By direct computations, we prove that the continuous function

$$f_{w(\Omega)} : r \longmapsto \sqrt{\frac{4r^2w^3(\Omega)}{(w(\Omega) - 2r)^2(4r - w(\Omega))}}$$

is strictly increasing on $\left[\frac{w(\Omega)}{3}, \frac{w(\Omega)}{2}\right)$. Let us denote by $g_{w(\Omega)}$ the inverse function of $f_{w(\Omega)}$, which is also continuous and strictly increasing. We have

$$r(\Omega) \geq g_{w(\Omega)}(P(\Omega)) = r(T_I),$$

where T_I is any subequilateral triangle such that $w(T_I) = w(\Omega)$ and $P(T_I) = P(\Omega)$. Moreover, since $P(\Omega) \geq 2\sqrt{3}w(\Omega)$, we have by the results contained in [159],

$$|\,\Omega\,| \geq |T_I|,$$

(see also [145] as a reference). Finally, we obtain

$$h(\Omega) \leq \frac{1}{r(\Omega)} + \sqrt{\frac{\pi}{|\,\Omega\,|}} \leq \frac{1}{r(T_I)} + \sqrt{\frac{\pi}{|T_I|}} = h(T_I).$$

□

Conjecture 4.3.9. As far as the diagram (ω, h, P) is concerned, we conjecture that, if $\pi w(\Omega) \leq P(\Omega) \leq 2\sqrt{3}w(\Omega)$, then, for all $\Omega \in \mathcal{K}^2$,

$$h(\Omega) \leq h(Y),$$

where Y is the Yamanouti set with $P(Y) = P(\Omega)$ and $w(Y) = w(\Omega)$. (see Figure (4.7)).

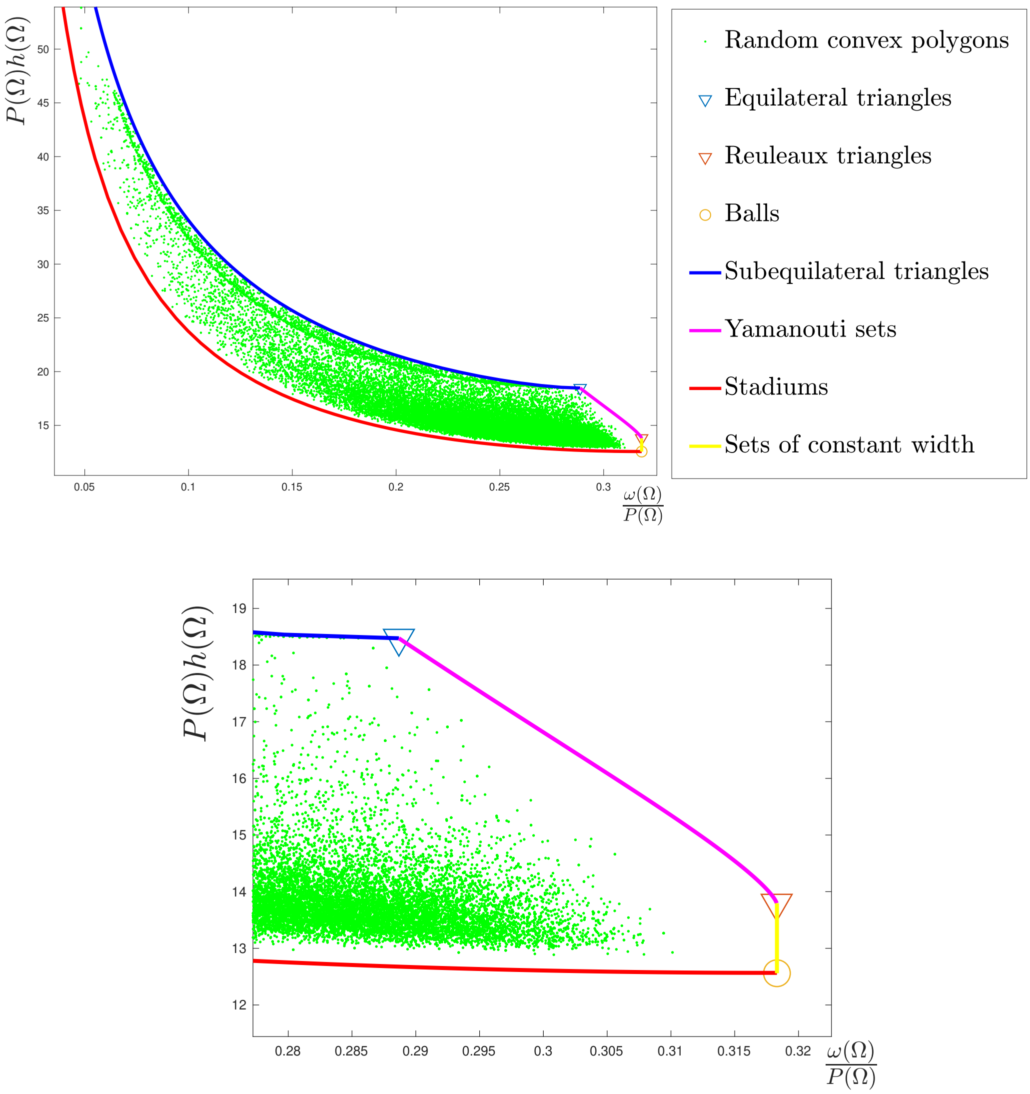

Figure 4.7: Blaschke–Santaló diagram of the triplet (ω, h, P).

The triplet $(\omega, h, |\cdot|)$**.**

Proposition 4.3.12. *Let* $\Omega \in \mathcal{K}^2$*. Then, it holds*

$$h(\Omega) \geq \frac{2}{w(\Omega)} + \frac{\pi w(\Omega)}{2|\Omega|}, \tag{4.105}$$

where equality is achieved by stadiums. Moreover, it holds

$$h(\Omega) \leq h(T_I), \tag{4.106}$$

where T_I *is a subequilateral triangle such that* $|\,\Omega\,| = |T_I|$ *and* $w(\Omega) = w(T_I)$*. The equality in* (4.106) *is achieved by the subequilateral triangle* T_I *(see Figure 4.8).*

Proof. Let us prove the inequality (4.105). We recall the lower bound in (4.71)

$$h(\Omega) \geq \frac{1}{r(\Omega)} + \frac{\pi r(\Omega)}{2|\,\Omega\,|},$$

which is an equality if and only if Ω is a stadium. Since the function $r \longmapsto \frac{1}{r} + \frac{\pi r}{2|\,\Omega\,|}$ is strictly decreasing and $r(\Omega) \leq \frac{w(\Omega)}{2}$ (where equality holds for stadiums), we obtain (4.105).

Let us now prove inequality (4.106). We start by recalling inequality (1.42):

$$|\,\Omega\,| \leq \sqrt{\frac{r^4(\Omega)w^3(\Omega)}{(w(\Omega)-2r(\Omega))^2(4r(\Omega)-w(\Omega))}} =: f_{w(\Omega)}(r(\Omega)).$$

By direct computations, we prove that the continuous function

$$f_{w(\Omega)} : r \longmapsto \sqrt{\frac{r^4 w(\Omega)^3}{(w(\Omega)-2r)^2(4r-w(\Omega))}}$$

is strictly increasing on $\left[\frac{w(\Omega)}{3}, \frac{w(\Omega)}{2}\right)$. Let us denote by $g_{w(\Omega)}$ the inverse function of $f_{w(\Omega)}$, which is also continuous and strictly increasing. We have

$$r(\Omega) \geq g_{w(\Omega)}(|\,\Omega\,|) = r(T_I),$$

where T_I is any subequilateral triangle such that $w(T_I) = w(\Omega)$ and $|T_I| = |\,\Omega\,|$. Thus, we have

$$h(\Omega) \leq \frac{1}{r(\Omega)} + \sqrt{\frac{\pi}{|\,\Omega\,|}} \leq \frac{1}{r(T_I)} + \sqrt{\frac{\pi}{|T_I|}} = h(T_I),$$

with equality if and only if $\Omega = T_I$. □

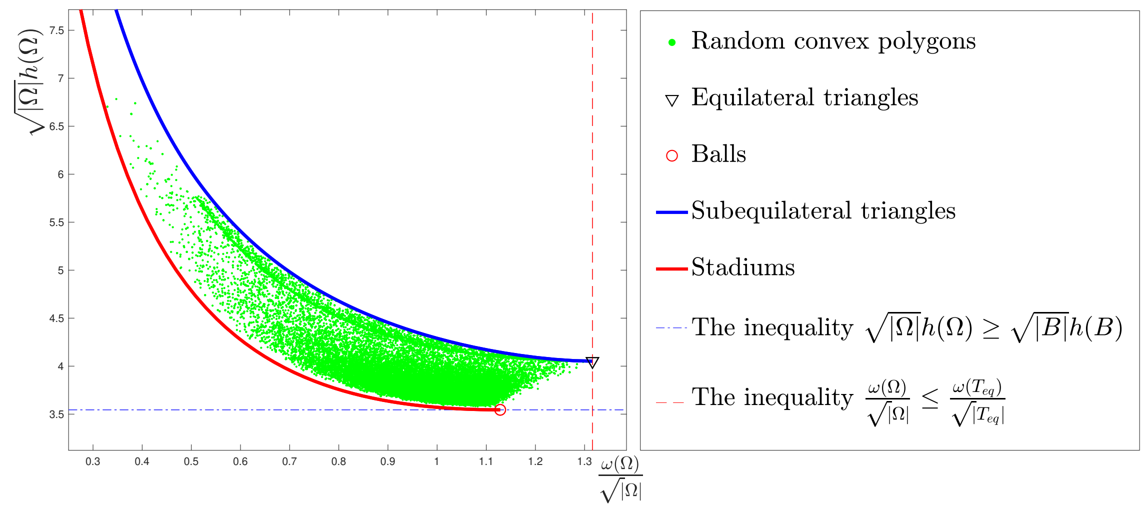

Figure 4.8: Blaschke–Santaló diagram of the triplet $(\omega, h, |\cdot|)$.

Remark 4.3.10. One may use classical convex geometry inequalities to obtain simpler bounds than the implicit one given in (4.106). Indeed, if we combine the inequalities in (4.71) and the following classical ones

$$\frac{2}{w(\Omega)} \leq \frac{1}{r(\Omega)} \leq \frac{2}{w(\Omega)} + \frac{w(\Omega)}{\sqrt{3}|\,\Omega\,|}$$

where the lower bound is realized in particular by stadiums, and the upper bound by equilateral triangles (see for example [145] and the references therein), we obtain the following inequalities

$$h(\Omega) \leq \frac{2}{w(\Omega)} + \frac{w(\Omega)}{\sqrt{3}|\,\Omega\,|} + \sqrt{\frac{\pi}{|\,\Omega\,|}} \tag{4.107}$$

and

$$h(\Omega) \leq \frac{2}{w(\Omega) - \frac{w^3(\Omega)}{4|\,\Omega\,|}} + \sqrt{\frac{\pi}{|\,\Omega\,|}}. \tag{4.108}$$

The bound (4.107) is achieved by equilateral triangles and (4.108) is asymptotically achieved by a sequence of thin subequilateral triangles.

The triplet (R, h, d).

Proposition 4.3.13. *Let $\Omega \in \mathcal{K}^2$. Then, it holds*

$$h(\Omega) \leq \frac{2R(\Omega)\left(2R(\Omega) + \sqrt{4R^2(\Omega) - diam^2(\Omega)}\right)}{diam^2(\Omega)\sqrt{4R^2(\Omega) - diam^2(\Omega)}} + \sqrt{\frac{4\pi R^2(\Omega)}{diam^3(\Omega)\sqrt{4R^2(\Omega) - diam^2(\Omega)}}}, \tag{4.109}$$

where equality is achieved by subequilateral triangles.

Proof. The inequality (4.109) is obtained by combining the upper bound in (4.71), which is an equality for circumscribed sets (in particular subequilateral triangles), and

$$r(\Omega) \geq \frac{\mathrm{diam}^2(\Omega)\sqrt{4R^2(\Omega) - \mathrm{diam}^2(\Omega)}}{2R(\Omega)\left(2R(\Omega) + \sqrt{4R^2(\Omega) - \mathrm{diam}^2(\Omega)}\right)}, \qquad |\,\Omega\,| \geq \frac{\mathrm{diam}^3(\Omega)\sqrt{4R^2(\Omega) - \mathrm{diam}^2(\Omega)}}{4R^2(\Omega)},$$

respectively proved in [141] and [95], where the equalities hold only for subequilateral triangles. □

Conjecture 4.3.11. As far as the diagram (R, h, d) is concerned, we conjecture that

$$h(\Omega) \geq h(N),$$

where N is the nonagon of constant width in Figure 1.6 (see Figure 4.9).

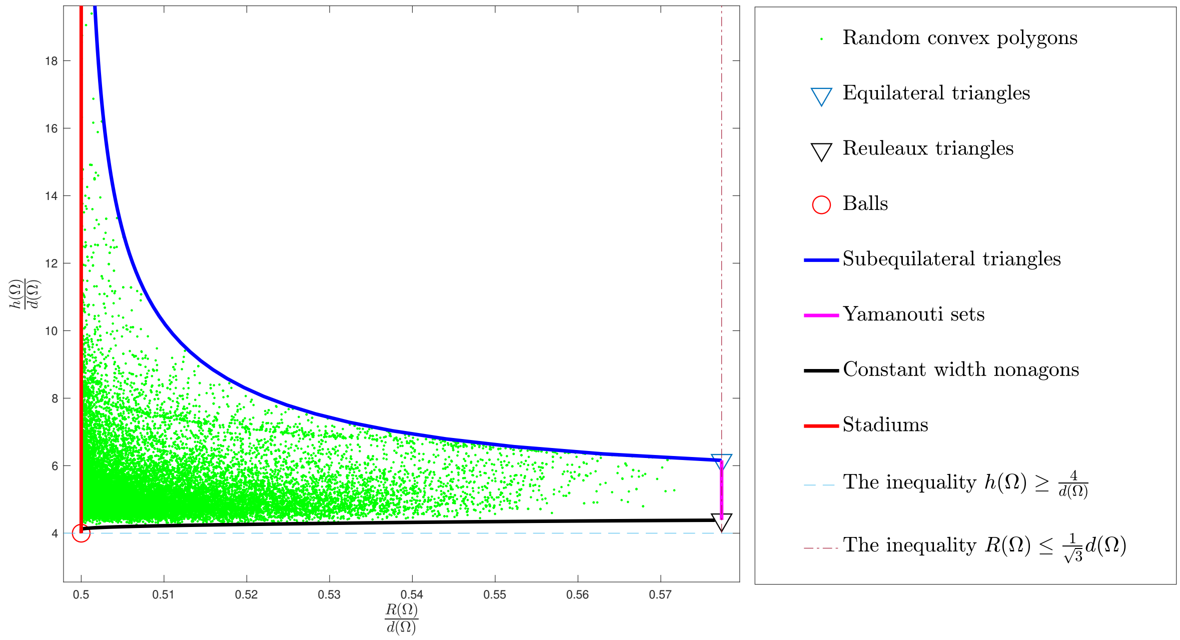

Figure 4.9: Blaschke–Santaló diagram of the triplet (R, h, d).

The triplet (ω, h, r)**.**

Proposition 4.3.14. *Let $\Omega \in \mathcal{K}^2$. Then, it holds*

$$h(\Omega) \geq \frac{1}{r(\Omega)} + \frac{1}{r(\Omega)} \sqrt{\pi \left(1 - \frac{2r(\Omega)}{w(\Omega)}\right) \sqrt{\frac{4r(\Omega)}{w(\Omega)}} - 1}, \tag{4.110}$$

where the equality is achieved by subequilateral triangles (see Figure 4.10).

Proof. The proof of (4.110) is inspired by the proof of [96, Theorem 5]. It is known that the incircle of a set Ω meets the boundary of Ω either in two diametrically opposite points, or in three points that form the vertices of a triangle, see [31]. In the first case, we have $w(\Omega) = 2r(\Omega)$, thus the inequality (4.110) is equivalent to $h(\Omega) \geq \frac{1}{r(\Omega)}$, that easily follows from the definition of $h(\Omega)$ and from the inequality $|E| < r(E)P(E)$ (see [145]):

$$h(\Omega) = \inf_{\substack{E \text{ is measurable} \\ E\subset\Omega,\ |E|>0}} \frac{P(E)}{|E|} \geq \inf_{\substack{E \text{ is measurable} \\ E\subset\Omega,\ |E|>0}} \frac{1}{r(E)} \geq \frac{1}{r(\Omega)}.$$

In the second case, let us denote by T a triangle formed by three lines of support common to Ω and the incircle. We have $r(\Omega) = r(T)$ and, by $\Omega \subset T$ and the monotonicity with respect to the inclusion, we get

$$h(\Omega) \geq h(T) \tag{4.111}$$

and

$$w(\Omega) \leq w(T). \tag{4.112}$$

So, inequality (4.110) is equivalent to the following one

$$\frac{1}{r(\Omega)h(\Omega)} f\left(\frac{r(\Omega)}{w(\Omega)}\right) \leq 1,$$

where $f : x \in \left[\frac{1}{3}, \frac{1}{2}\right] \longmapsto 1 + \sqrt{\pi\,(1-2x)\,\sqrt{4x-1}}$. We observe that the function

$$g : x \in \left[\frac{1}{3}, \frac{1}{2}\right] \longmapsto (1-2x)\,\sqrt{4x-1}$$

is decreasing. Indeed,

$$g'(x) = \frac{4(1-3x)}{\sqrt{4x-1}} \leq 0.$$

Thus, f is also decreasing on $\left[\frac{1}{3}, \frac{1}{2}\right]$. Then, since $\frac{r(\Omega)}{w(\Omega)} \geq \frac{r(T)}{w(T)}$ by (4.112), we have

$$f\left(\frac{r(\Omega)}{w(\Omega)}\right) \leq f\left(\frac{r(T)}{w(T)}\right).$$

Moreover, we get, by (4.111),

$$\frac{1}{r(\Omega)h(\Omega)} \leq \frac{1}{r(T)h(T)}.$$

Thus, we obtain

$$\frac{1}{r(\Omega)h(\Omega)} f\left(\frac{r(\Omega)}{w(\Omega)}\right) \leq \frac{1}{r(T)h(T)} f\left(\frac{r(T)}{w(T)}\right) = \frac{1}{1 + r(T)\sqrt{\frac{\pi}{|T|}}} f\left(\frac{r(T)}{w(T)}\right),$$

where we used the equality $h(T) = \frac{1}{r(T)} + \sqrt{\frac{\pi}{|T|}}$, which holds because T is a triangle, see [78]. Now, we use the inequality

$$|T| \leq \frac{r(T)^2}{\left(1 - \frac{2r(T)}{w(T)}\right)\sqrt{\frac{4r(T)}{w(T)} - 1}},$$

which is an equality if and only if T is a subequilateral triangle, see [96, Theorem 5]. So, we have

$$\frac{1}{r(\Omega)h(\Omega)} f\left(\frac{r(\Omega)}{w(\Omega)}\right) \leq \frac{1}{1 + r(T)\sqrt{\frac{\pi}{|T|}}} f\left(\frac{r(T)}{w(T)}\right) \leq 1,$$

which ends the proof. □

Remark 4.3.12. It is possible to give a non-sharp upper bound

$$h(\Omega) \leq \frac{1}{r(\Omega)} + \frac{\sqrt{\pi\sqrt{3}}}{w(\Omega)}$$

In order to prove it, we combine the upper bound in (4.71) and $w^2(\Omega) \leq \sqrt{3}|\,\Omega\,|$, see [145].

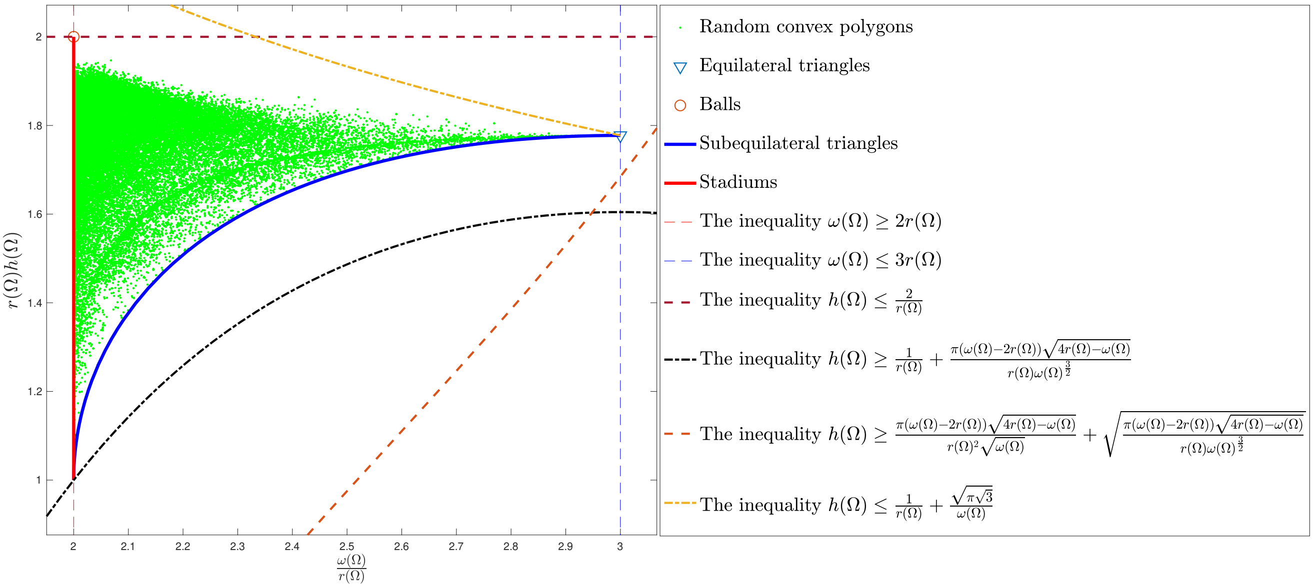

Figure 4.10: Blaschke–Santaló diagram of the triplet (ω, h, r).

4.3.2 Numerical results and Blaschke–Santaló diagrams

In this Section, we introduce numerical tools, that we have used to obtain more information on the diagrams and state the conjectures.

We want to provide a numerical approximation of the diagrams studied in the previous section. To do so, a natural idea is to generate a large number of convex sets (more precisely convex polygons) and for each, we compute the involved functionals. Nevertheless, the task of (properly) generating random convex polygons is quite challenging and interesting on its own. The main difficulty is that one wants to design an efficient and fast algorithm that allows obtaining a uniform distribution of the generated random convex polygons. For clarity, let us discuss two different (naive) approaches:

- One easy way to generate random convex polygons is by rejection sampling. We generate a random set of points in a square; if they form a convex polygon, we return it, otherwise, we try again. Unfortunately, the probability of a set of n points uniformly generated inside a given square to be in convex position is equal to

$$p_n = \left(\frac{\binom{2n-2}{n-1}}{n!} \right)^2,$$

see [156]. Thus, the random variable X_n corresponding to the expected number of iterations needed to obtain a convex distribution follows a geometric law of parameter p_n, which means that its expectation is given by

$$\mathbb{E}(X_n) = \frac{1}{p_n} = \left(\frac{n!}{\binom{2n-2}{n-1}} \right)^2.$$

For example, if $N = 20$, the expected number of iterations is approximately equal to 2.10^9, and, since one iteration is performed in an average of 0.7 seconds, this means that the algorithm will need about 50 years to provide one convex polygon with 20 sides;

- another natural approach is to generate random points and take their convex hull. This method is quite fast, as one can compute the convex hull of N points in a $\mathcal{O}(N\log(N))$ time (see [14] for example), but it is not quite relevant since it yields to a biased distribution.

In order to avoid the issues stated above, we use an algorithm presented in [138], that is based on the work of P. Valtr [156], where the author computes the probability of a set of n points uniformly generated inside a given square to be in convex position. The author remarks (in Section 4) that the proof yields a fast and non-biased method to generate random convex sets inside a given square. We also refer to [138] for a nice description of the steps of the method and a beautiful animation where one can follow each step; one can also find an implementation of Valtr's algorithm in Java that we decided to translate in Matlab. To obtain the Blaschke-Santaló diagram, we generate 10^5 random convex polygons of unit area and number of sides between 3 and 30, for which we compute the involved functionals. We then obtain clouds of dots that provide approximations of the diagrams. This approach has been used in several works, we refer for example to [16], [78] and [81].

Let us give few details on the numerical computation of the functionals involved in the paper:

- The **Cheeger constant** is computed by using a code implemented by Beniamin Bogosel in [28] based on the characterization of the Cheeger sets of planar convex sets given in [103] and the toolbox Clipper, a very good implementation of polygon offset computation by Agnus Johnson.

- The **inradius** is also computed by using the tootbox Clipper and the fact that $r(\Omega)$ is the solution to the equation $|\,\Omega_{-t}\,| = 0$.

- The **diameter** is computed via a fast method of computation, which consists of finding all antipodal pairs of points and looking for the diametrical one between them. This is classically known as Shamos algorithm [133].

- The **area** is computed by using Matlab's function `"polyarea"`.

- The **minimum width** of a polygon Ω of vertices $\{A_1,\cdots,A_N\}$ is computed by using the following formula
$$w(\Omega) = \min_{i\in[1,N]}\max_{j\in[1,N]} \text{dist}(A_j,(A_iA_{i+1})),$$
where $\text{dist}(A_j,(A_iA_{i+1}))$ corresponds to the distance between the point A_j and the line (A_iA_{i+1}) (with the convention $A_{N+1} := A_1$).

- The **circumradius** of a convex set Ω can be written as follows
$$R(\Omega) = \min_{c\in\Omega}\max_{x\in\Omega}|c-x|.$$
It is computed by using Matlab's routine `"fminmax"`.

4.3.3 Summary tables with the results

In this first table, we summarize the results relative to the diagrams that are completely solved.

Param.	Inequality	Extremal sets	Ref.
P, h, A	$h \leq \dfrac{P}{A}$ $h \geq \dfrac{P+\sqrt{4\pi A}}{2A}$	Cheeger to itself circumscribed sets	[79]
r, h, A	$h \leq \dfrac{1}{r} + \sqrt{\dfrac{\pi}{A}}$ $h \geq \dfrac{1}{r} + \dfrac{\pi r}{A}$	circumscribed sets stadiums	[78]
P, h, r	$h \leq \dfrac{1}{r} + \sqrt{\dfrac{2\pi}{Pr}}$ $h \geq \dfrac{1}{r} + \dfrac{\pi}{P-\pi r}$	circumscribed sets stadiums	Prop. 4.3.5
d, h, r	$h \leq \dfrac{1}{r} + \sqrt{\dfrac{\pi}{r\sqrt{d^2-4r^2}+r^2\left(\pi - 2\arccos\left(\frac{2r}{d}\right)\right)}}$ $h \geq \dfrac{1}{t_{g_2^\Omega}}$ (ii)	two-cup bodies spherical slices	Prop. 4.3.7
R, h, r	$h \leq \dfrac{1}{r} + \sqrt{\dfrac{\pi}{2r\left(\sqrt{R^2-r^2}+r\arcsin\left(\frac{r}{R}\right)\right)}}$ $h \geq \dfrac{1}{t_{g_2^\Omega}}$ (ii)	two-cup bodies spherical slices	Prop. 4.3.7

(i) $t_{g_1^\Omega}$ is the smallest solution on $[0, r(\Omega)]$ to

$$g_1^\Omega(t) := \psi(\mathrm{diam}(\Omega) - 2t, r(\Omega) - t) = \pi t^2,$$

where

$$\psi(d,r) := \begin{cases} \dfrac{3\sqrt{3}r}{2}(\sqrt{d^2-3r^2}-r) + \dfrac{3d^2}{2}\left(\dfrac{\pi}{3} - \arccos\left(\dfrac{\sqrt{3}r}{d}\right)\right), & \text{if } d \leq rD^* \\ r\sqrt{d^2-4r^2} + \dfrac{d^2}{2}\arcsin\left(\dfrac{2r}{d}\right), & \text{if } d \geq rD^*. \end{cases}$$

and D^* is the unique number in $[2, 2\sqrt{3}]$ for which the two expression of the function $\psi(d,r)$ are equal.
(ii) $t_{g_2^\Omega}$ is the smallest solution on $[0, r(\Omega)]$ to

$$g_2^\Omega(t) := 2\left((r-t)\sqrt{(R(\Omega)-t)^2-(r(\Omega)-t)^2} + (R(\Omega)-t)^2\arcsin\left(\frac{r(\Omega)-t}{R(\Omega)-t}\right)\right) = \pi t^2.$$

In this second table, we summarize the results of the partially solved Blaschke–Santaló diagrams.

Param.	Condition	Inequality	Extremal sets
ω, h, d	$\omega \leq \frac{3d}{2}$ $\frac{\sqrt{3}d}{2} \leq \omega \leq d$	$h \leq \frac{1}{r(\omega,d)} + \sqrt{\frac{2\pi}{\omega d}}$ *(iii)* $h \leq \frac{\sqrt{3}}{\sqrt{3}\omega - d} + \sqrt{\frac{2\pi}{\pi\omega^2 - \sqrt{3}d^2 + 6\omega(\tan(\arccos(\frac{\omega}{d})) - \arccos(\frac{\omega}{d}))}}$ $h(\Omega) \geq \frac{1}{t_{g_3^\Omega}}$ *(iv)*	subeq. tr. eq. tr. sph. slices
ω, h, R	$\omega \leq \frac{3}{2}R$	$h \leq \frac{1}{r(\omega,R)} + \sqrt{\frac{\pi}{A(\omega,R)}}$ *(v)* $h \geq \frac{1}{t_{g_4^\Omega}}$ *(vi)*	subeq. tr. sph. slices
ω, h, P	$P \geq 2\sqrt{3}\omega$	$h \leq \frac{1}{r(\omega,P)} + \sqrt{\frac{\pi}{A(\omega,P)}}$ *(vii)* $h \geq \frac{2}{\omega} + \frac{2\pi}{2P - \pi\omega}$	subeq tr. stadiums
ω, h, A		$h \leq \frac{1}{r(\omega,A)} + \sqrt{\frac{\pi}{A}}$ *(viii)* $h \geq \frac{2}{\omega} + \frac{\pi\omega}{2A}$	subeq. tr. stadiums
R, h, d		$h \leq \frac{2R(2R+\sqrt{4R^2-d^2})}{d^2\sqrt{4R^2-d^2}} + \sqrt{\frac{4\pi R^2}{d^3\sqrt{4R^2-d^2}}}$	subeq. tr.
ω, h, r		$h \geq \frac{1}{r} + \frac{1}{r}\sqrt{\pi\left(1-\frac{2r}{\omega}\right)\sqrt{\frac{4r}{\omega}-1}}$	subeq. tr.

(iii) $r(\omega, d)$ is given by

$$d^2\,(\omega - 2r(\omega,d))^2\,(4r(\omega,d) - \omega) = 4r^4(\omega,d)\,\omega.$$

(iv) $t_{g_3^\Omega}$ is the smallest solution to

$$g_3^\Omega(t) := f(d(\Omega) - 2t, \omega(\Omega) - 2t) = \pi t^2,$$

where

$$f(d,w) = \frac{w}{2}\sqrt{d^2 - w^2} + \frac{d^2}{2}\arcsin\left(\frac{w}{d}\right)$$

(v) $r(\omega, R)$ is given by

$$(4r(\omega,R) - \omega)\,(\omega - 2r(\omega,R)) = \frac{2r^3(\omega,R)}{R}$$

and $A(\omega, R)$ is given by

$$16A(\omega,R)^6 = R^2\omega^2\left(16A(\omega,R)^4 - R^2\omega^6\right).$$

(vi) $t_{g_4^\Omega}$ is the smallest solution on $[0, r(\Omega)]$ to

$$g_4^\Omega(t) := \chi(\omega(\Omega) - 2t, R(\Omega) - t) = \pi t^2,$$

where

$$\chi(\omega, R) := \frac{\omega}{2}\sqrt{4R^2 - \omega^2} + 2R^2 \arcsin\frac{\omega}{2R}.$$

(vii) $r(\omega, P)$ is given by

$$(\omega - 2r(\omega, P))^2(4r(\omega, P) - \omega)P^2 = 4r(\omega, P)^2\omega^3$$

and $A(\omega, P)$ is the middle root of the equation

$$128PA(\omega, P)^3 - 16\omega(5P^2 + \omega^2)A(\omega, P)^2 + 16\omega^2P^3A(\omega, P) - \omega^3P^4 = 0$$

(viii) $r(\omega, A)$ is given by

$$(\omega - 2r(\omega, A))^2(4r(\omega, A) - \omega)A^2 = r^4(\omega, A)\,\omega^3.$$

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