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# On the regularity of solutions to some classes of obstacle problems

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# Chapter 1

# Introduction

The classical obstacle problem consists in finding the equilibrium position of an elastic membrane whose boundary is held fixed, and which is constrained to lie above a given obstacle. The mathematical formulation of the problem is to seek minimizers of the Dirichlet integral

$$\int_\Omega |Du|^2 dx$$

in a domain $\Omega \subset \mathbb{R}^n$, among all functions $u \in u_0 + W_0^{1,2}(\Omega)$ above a prescribed obstacle $\psi$.

The same mathematical problem is motivated by numerous applications: fluid filtration in porous media, elasto-plasticity, optimal control, and financial mathematics. We refer to [53] for an overview of these applications.

The obstacle problem appeared in the mathematical literature in the work of Stampacchia [99] in the special case $\psi = \chi_E$ and related to the capacity of a subset $E \Subset \Omega$ in potential theory. In an earlier independent work, Fichera [50] solved the first unilateral problem, the so-called *Signorini problem* in elastostatics. It consists in finding the elastic equilibrium configuration of an anisotropic non-homogeneous elastic body, resting on a rigid frictionless surface and subject only to its mass forces.

The study of the regularity theory for obstacle problems is a classical and important topic in Partial Differential Equations and Calculus of Variations. It is well known that the solution to the obstacle problem cannot be of class $\mathcal{C}^2$ independent on how regular the obstacle is; this led to the origin of the concept of weak solution and to the theory of *variational inequalities*, after the fundamental work by Lions and Stampacchia [81]. The underlying principle is that the regularity of solutions to obstacle problems is influenced by the one of the obstacle itself. A solution cannot be more regular than the assigned obstacle function, as immediately follows considering the so called coincidence set, i.e. the set where the solution and the obstacle coincide. For linear obstacle problems, solutions are as regular as the obstacle (see [7, 10, 73]). However, this is no longer the case in the nonlinear setting for general integrands without any specific structure. In this situation, extra regularity must be imposed on the obstacle to balance, in some sense, both the nonlinearity and the non-standard growth. Hence along the years, there have been intense research activities for the regularity of the obstacle problem in this direction.

A first important result by Michael and Ziemer [86] establishes Hölder continuity of solutions to the obstacle problem when the obstacle itself is Hölder continuous. Choe [17] proved that if the gradient of the obstacle is Hölder continuous, the same happens for the gradient of solutions. Other results that deserved to be quoted are [18, 55, 80], in the case of a single obstacle problem, and [4] in the case of double obstacle problems. Since then, many regularity results have been obtained in different situations: for instance we quote [42] in the setting of Morrey and Campanato spaces, [8, 43, 44, 91] in the setting of non-standard growth conditions (see also [9, 43, 92] for Calderón-Zygmund case). Moreover we refer to [2, 93] for gradient continuity for nonlinear obstacle problems, [45] for global results up to the boundary, [5] for the parabolic case, [6] for the porous medium problem.

This thesis deals with the regularity theory for solutions to some classes of obstacle problems. In Chapter 2, we recall some notation and preliminary results. In Chapter 3, we prove higher integrability and differentiability properties of solutions to obstacle problems associated to an elliptic variational integral of the type

$$\mathcal{F}(w,\Omega) := \int_\Omega F(x,w,Dw)dx, \tag{1.0.1}$$

where $\Omega$ is a bounded open set of $\mathbb{R}^n$, $n \geq 2$. More precisely, given an obstacle function $\psi : \Omega \to [-\infty, +\infty)$, the functional $\mathcal{F}$ is minimized under the obstacle condition $w \geq \psi$. This problem can be reformulated as a variational inequality, i.e. $u \geq \psi$ is a minimizer of $\mathcal{F}$ if, and only if, $u$ solves the following variational inequality

$$\int_\Omega \langle \mathcal{A}(x,u,Du), D(\varphi - u)\rangle dx \geq \int_\Omega \mathcal{B}(x,u,Du)(\varphi - u)dx,$$

for every $\varphi \geq \psi$, where we set

$$\mathcal{A}(x,v,\xi) := D_\xi F(x,v,\xi) \quad \text{and} \quad \mathcal{B}(x,v,\xi) := -D_v F(x,v,\xi).$$

We assume $p$-ellipticity and standard $p$-growth conditions on the operator $\mathcal{A}(x,v,\xi)$, for an exponent $p \geq 2$.
We present two main results contained in the paper [67]. First, we establish a Calderón-Zygmund type estimate for the gradient of the solution, stating that $Du$ is as integrable as the gradient of the obstacle. Next, this estimate allows us to prove that a higher differentiability in the scale of Besov spaces of the gradient of the obstacle transfers to the gradient of the solution. Here, the main novelty is the treatment of $v$-dependence in the integrand, respectively the coefficients.

The results in Chapter 3 holds for minimizers to functionals as in (1.0.1) where the integrand $F$ has $p$-growth

$$\xi^p \lesssim F(x,v,\xi) \lesssim (1+\xi^2)^{\frac{p}{2}}, \qquad p \in (2,+\infty). \tag{1.0.2}$$

Now, let us take a look at the following functionals, where $1 < p < q$ are fixed numbers:

$$\mathcal{F}_1(w,\Omega) := \int_\Omega \sum_{i=1}^n a_i(x)|D_i w|^{p_i} dx, \qquad 1 \leq a_i(x) \leq L,\ 1 < p := p_1 \leq p_2 \leq ... \leq p_n =: q,$$

$$\mathcal{F}_2(w,\Omega) := \int_\Omega [|Dv|^p + a(x)|Dw|^q]dx, \qquad 0 \leq a(x) \leq L,$$

$$\mathcal{F}_3(w,\Omega) := \int_\Omega |Dw|^{p(x)}dx, \qquad 1 < p \leq p(x) \leq q.$$

None of the above integrands satisfies the growth conditions (1.0.2) for any possible choice of the exponent $p > 1$. But all of them satisfy the more general growth conditions

$$\xi^p \lesssim F(x,\xi) \lesssim (1+\xi^2)^{\frac{q}{2}}, \qquad 1 < p < q. \tag{1.0.3}$$

Functionals satisfying conditions (1.0.3) are called functionals with $(p, q)$-*growth conditions*. The study of regularity of minima of functionals with non-standard growth of $(p, q)$-type was initiated by Marcellini in the seminal papers [82, 83, 85]. When referring to $(p, q)$-growth conditions (1.0.3), we call the quantity $q/p > 1$ *the gap ratio of the integrand* $F$, or simply, *the gap*. A condition that ensures the regularity of minima is that the gap $q/p$ does not differ too much from 1, in other words, the difference between the growth exponents $p$ and $q$ is not too large.

The main point here is that functionals satisfying (1.0.3) exhibit nonuniform ellipticity features, which emerge when looking at the Euler-Lagrange equation $\operatorname{div} D_\xi F(x, Du) = 0$. This lead us to the definition of the *rate of non-uniform ellipticity* quantified by the ratio

$$\mathcal{R}(\xi, B) := \frac{\sup_{x\in B} \text{ of the highest eigenvalue of } D_{\xi\xi}F(x,\xi)}{\inf_{x\in B} \text{ of the lowest eigenvalue of } D_{\xi\xi}F(x,\xi)}$$

on any ball $B \subset \Omega$, that in the nonuniformly elliptic case becomes unbounded as $|\xi| \to \infty$. This is not the case of $p$-Laplacean type functionals, i.e. $F(x, \xi) \approx |\xi|^p$, for which $\mathcal{R}(\xi, B) \equiv 1$. This occurs instead in the case of the double phase functional $\mathcal{F}_2(w, \Omega)$, where it is $\mathcal{R}(\xi, B) \approx 1 + \|a\|_{L^\infty(B)}|\xi|^{q-p}$ on any ball $B$ intersecting the set $\{a(x) = 0\}$. Another example is given by the variable exponent energy $\mathcal{F}_3(w, \Omega)$ and in this case it is $\mathcal{R}(\xi, B) \approx |\xi|^{p_+ - p_-}$, for $|\xi|$ large, where $p_- := \min_{x\in B} p(x)$ and $p_+ := \max_{x\in B} p(x)$.

The rest of the thesis is devoted to the study of regularity properties of solutions to obstacle problems with non-standard growth obtained in [37, 68, 69, 70].

Chapter 4 deals with the higher differentiability of fractional order of the solutions to a class of obstacle problems with $(p, q)$-growth conditions, for $2 \le p < q$. In particular, we consider variational obstacle problems of the form

$$\min\left\{ \int_\Omega f(x, Dw)dx \;:\; w \in W^{1,p}(\Omega),\; w \ge \psi \text{ a.e. in } \Omega \right\}, \tag{1.0.4}$$

where $f$ is a Caratheódory function with radial structure. The novelty of our results consists in the fact that the gradient of the obstacle and the partial map $x \mapsto D_\xi f(x, \xi)$ are only assumed to be differentiable of fractional order in the sense that they belong to a certain Besov space. More precisely, we prove higher fractional differentiability of the solution in the sense of

$$(1 + |Du|^2)^{\frac{p-2}{4}} Du \in B^\alpha_{2,\sigma,\mathrm{loc}}(\Omega),$$

provided the obstacle function satisfies

$$D\psi \in B^\gamma_{2q-p,\sigma,\mathrm{loc}}(\Omega),$$

for $0 < \alpha < \gamma < 1$, and if the gap between $q$ and $p$ is not too large. We show two main theorems, one for the case $\sigma = \infty$ and a second one for the case of a finite $\sigma \le \frac{2n}{n-2\alpha}$, in which the parameters $\gamma = \alpha$ are admissible.

In Chapter 5, we investigate the local boundedness of solutions to obstacle problems of the type

$$\min\left\{ \int_\Omega F(x, w, Dw)dx \;:\; w \in W^{1,p}(\Omega),\; w \ge \psi \text{ a.e. in } \Omega \right\}, \tag{1.0.5}$$

where the integrand $F$ satisfies $(p,q)$-growth conditions, for some exponents $1 < p \leq q$, and the function $\psi \in W^{1,p}(\Omega)$ is the obstacle.

The study of the local boundedness for minimizers of integral functionals is a classic topic in Partial Differential Equations and Calculus of Variations starting from the classical result by De Giorgi [66] and often is the first step in the analysis of the regularity properties of the solutions.

In the last years there has been an intense research activity concerning the local boundedness of minimizers of unconstrained problems, in case of energy densities satisfying non-standard growth conditions. In [27, 28, 30], the authors established the local boundedness of minimizers of integral functionals satisfying the anisotropic growth conditions

$$c_1 \sum_{i=1}^{n} |\xi_i|^{p_i} \leq f(x,\xi) \leq c_2(1+|\xi|^q).$$

In this case, local boundedness is proven under the condition $q < \bar{p}^*$, where $\frac{1}{\bar{p}} = \sum_{i=1}^{n} \frac{1}{p_i}$ and $\bar{p}^* = \frac{n\bar{p}}{n-\bar{p}}$. Recently, in [29] the previous boundedness result has been obtained also in the bordeline case $q = \bar{p}^*$. Moreover, the functionals, there considered, allows the dependence on $u$. Other related boundedness result that deserved to be quoted are [11, 32, 76, 77, 89].

The analogous study for solutions of obstacle problems under non-standard growth has been exploited in [15], where the local boundedness has been established assuming that

$$q < p_n^* = \frac{np}{n-p}, \tag{1.0.6}$$

provided the obstacle is locally bounded. We also point out a particular case of $(p,q)$-growth condition considered in the paper by Chlebicka and De Filippis [16]. There the authors proved the local boundedness for solutions to double-phase obstacle problems, where the model functional is given by

$$\mathcal{F}(w,\Omega) := \int_\Omega [|Dw|^p + a(x)|Dw|^q]dx,$$

for a non-negative function $a \in \mathcal{C}^{0,\alpha}$, with $\alpha \in (0,1]$. For this kind of problems, the local boundedness of the solutions is proven assuming that the obstacle, beside the boundedness, belongs to the Sobolev space $W^{1,K(\cdot)}$, with $K(t) = t^p + a(x)t^q$, and the exponent $p$, $q$ and $\alpha$ satisfy the relations

$$\frac{q}{p} \leq 1 + \frac{\alpha}{n} \qquad \text{and} \qquad 1 < p < q \leq n.$$

A restriction on the closeness between the growth exponents cannot be avoided, since thanks to the well known Marcellini's counterexample [85], we know that if

$$q > \frac{(n-1)p}{n-1-p} = p_{n-1}^* \qquad 1 < p < n-1$$

minimizers of functionals with $(p,q)$-growth can be unbounded even in the unconstrained setting. In a very recent paper [72], Hirsch and Schäffner proved that the sharp bound

$$\frac{1}{q} \geq \frac{1}{p} - \frac{1}{n-1} \tag{1.0.7}$$

is sufficient to the local boundedness for unconstrained minimizers.
We show that (1.0.7) is sufficient also to establish the local boundedness of solutions to the obstacle problems (1.0.5). Here, we improve the previous local boundedness result for obstacle problems contained in [15] in different directions: we admit the dependece of the integrand $F$ not only on $x$ and $Du$, but on $u$ too; the bound (1.0.7) is less restrictive than (1.0.6) and it is essentially sharp for local boundedness, since, in view of a counterexample by Franchi, Serapioni and Serra Cassano [52], the conclusion of Theorem 5.0.1 is false if condition (1.0.7) is replaced by

$$\frac{1}{p} + \frac{1}{q} < \frac{1}{n-1} + \varepsilon$$

for any $\varepsilon > 0$, already for unconstrained minimizers. Furthermore, our result shows that solutions to double-phase obstacle problems are locally bounded under weaker assumptions on the data with respect to the one considered in [16] (see Corollary 5.3.2).

In Chapter 6, we exploit the result in Chapter 5 in order to analyse the higher differentiability of a priori bounded solutions $u \in W^{1,p}(\Omega)$ to the obstacle problem (1.0.4).

As we already mentioned, it is well known that in order to get regularity of minimizers to functionals with non-standard growth conditions, even the boundedness, a restriction between $p$ and $q$ need to be imposed, usually expressed in the form

$$q \leq c(n)p, \qquad \text{with } c(n) \to 1 \text{ as } n \to \infty. \tag{1.0.8}$$

We refer to [62, 85] for counterexamples. Now, it happens that when minimizers are bounded, then we do not have to require a relation between $p$ and $q$ of the type (1.0.8) in order to prove regularity results for the gradient.

When dealing with bounded minimizers, both for unconstrained and constrained problems with $(p, q)$-growth, regularity results for the gradient can be proved under dimension-free conditions on the gap $q/p$ and weaker assumptions on the data of the problem (see for instance [12, 22, 34, 60, 61, 65]). Moreover, in [60, 61, 65], the higher differentiability of integer order of bounded solutions to (1.0.4) is obtained assuming that the map $x \mapsto D_\xi f(x, \xi)$ and the gradient of the obstacle belong to a Sobolev class that is not related to the dimension $n$ but to the ellipticity and the growth exponents of the functional.

In Chapter 5, we prove that, assuming the local boundedness of the obstacle $\psi$, the solution to obstacle problem (1.0.4) is locally bounded under a sharp relation between $p$ and $q$. Here, we study higher fractional differentiability properties of bounded solutions to obstacle problems satisfying $(p, q)$-growth conditions. The novelty of our resutls consists in showing that, even in the fractional setting, the higher differentiability properties of bounded solutions to (1.0.4) hold true assuming that the Besov type regularity on the partial map $x \mapsto D_\xi f(x, \xi)$ is not related to the dimension $n$.

Finally, in Chapter 7, we study the differentiability properties of the gradient of the solutions to obstacle problems driven by the double-phase energy density

$$\mathcal{F}(w, \Omega) := \int_\Omega b(x, w)[|Dw|^p + a(x)|Dw|^q]\, dx, \qquad w \geq \psi \text{ a.e. in } \Omega, \tag{1.0.9}$$

where $0 \leq a(\cdot) \in \mathcal{C}^{0,\alpha}(\Omega)$, $\psi$ is the obstacle function and $0 < b_0 \leq b(\cdot) \leq B_0$ is Hölder continuous in all its arguments. Assuming the condition on the gap $q/p$

$$\frac{q}{p} < 1 + \frac{\alpha}{n},$$

we prove the following implication

$$D\psi \in B^{\gamma}_{2q-p,\infty,\mathrm{loc}}(\Omega),\ 0<\alpha<\gamma<1 \implies V_p(Du),\ \sqrt{a(\cdot)}V_q(Du) \in B^{t}_{2,\infty,\mathrm{loc}}(\Omega), \quad \forall t \in (0,\tilde{\sigma}),$$

where $V_s(z) := |z|^{\frac{s-2}{2}}z$, $s \in \{p,q\}$ and $\tilde{\sigma} \in (0,1)$ is a threshold parameter depending only on the data of the problem.
The main difficulty of this work is the dependence of the integrand on the $w$-variable, since for functional (1.0.9) the Euler-Lagrange equation is not available due to the mere Hölder continuity of $w \mapsto b(x,w)$. In fact, in general, for proving higher differentiability for minima of variational integrals, the availability of the Euler-Lagrange equation is a crucial ingredient already in the case of $p$-Laplacian.

The double-phase energy density given by (1.0.9) is a model case of functions $f$ satisfying the following set of conditions

$$\nu_1|z|^p \le f(x,w,\xi) \le L_1(1+|z|^q)$$

$$\nu_2|z|^{p-2}|\lambda|^2 \le \langle D_{\xi\xi}f(x,w,\xi)\lambda,\lambda\rangle \le L_2(1+|z|^{q-2})|\lambda|^2$$

$$|f(x_1,w_1,\xi)-f(x_2,w_2,\xi)| \le l_1\omega_\delta(|x_1-x_2|+|w_1-w_2|)(1+|z|^q)$$

for all $x, x_1, x_2 \in \Omega$, $w, w_1, w_2 \in \mathbb{R}$ and every $\xi, \lambda \in \mathbb{R}^n$, where $0<\nu_1\le L_1$, $0<\nu_2\le L_2$, $l_1 \ge 1$ are fixed constants and $\omega_\delta : \mathbb{R}^+ \to [0,1]$ is a function defined by $\omega_\delta(\rho)=\min\{\rho^\delta,1\}$, for some $\delta \in (0,1)$.

The regularity theory for obstacle problems driven by quasilinear operators of the $p$-Laplacian type started with the contributions of Duzaar and Fuchs [40], Duzaar [39], Choe and Lewis [21] and Fuchs [54]. It is worth noticing that double phase functionals are a useful tool to study the behaviour of strongly anisotropic materials whose hardening properties are strongly dependent on the point and connected to the exponent ruling the growth of the gradient variable. The coefficient $a(\cdot)$ regulates the mixture between two different materials, with $p$ and $q$ hardening, respectively (see, for instance, [103, 104]). The regularity properties of local minimizers to such functionals recently have been investigated for unconstrained problems. In particular, we quote the works [23] by Colombo and Mingione, [3] by Baroni, Colombo and Mingione and [24] by Coscia, who dealt with the functional defined by

$$\mathcal{F}(w,\Omega) := \int_\Omega b(x,w)[|Dw|^p + a(x)|Dw|^p \log(e+|Dw|)]dx.$$

Furthermore, a higher fractional differentiability [102] and a Lipschitz continuity result [35] have been proved for solutions to double phase elliptic obstacle problems.

# Chapter 2

# Definitions and preliminary tools

Here, we introduce some notation. We will use the symbols $C$ or $c$ to denote general positive constants. Different occurrences from line to line will be still denoted using the same letters. Relevant dependencies on parameters will be emphasized using parentheses or subscripts. We denote by $B(x,r) = B_r(x) = \{y \in \mathbb{R}^n : |y-x| < r\}$ the ball centered at $x$ of radius $r$. We shall omit the dependence on the center and on the radius when no confusion arises. Moreover, we set $S_r := \{x \in \mathbb{R}^n : |x| = r\}$ the sphere of radius $r$ on $\mathbb{R}^n$. For a function $v \in L^1(B)$, the symbol

$$v_B := \fint_B v(x)dx = \frac{1}{|B|}\int_B v(x)dx.$$

will denote the integral mean of the function $v$ over the ball $B$.

In the following, $\Omega$ will denote a bounded open set of $\mathbb{R}^n$, with $n \geq 2$. Define the integral functional

$$\mathcal{F}(u,\Omega) := \int_\Omega f(x,u,Du)dx \tag{2.0.1}$$

where $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function. Given a function $\psi \in W^{1,p}(\Omega)$, $1 \leq p < +\infty$, which is called *obstacle*, we define a closed convex set

$$\mathcal{K}_\psi(\Omega) := \{w \in u_0 + W_0^{1,p}(\Omega) : w \geq \psi \text{ a.e. in } \Omega\}, \tag{2.0.2}$$

where $u_0 \in W^{1,p}(\Omega)$ is a fixed boundary datum. To avoid trivialities, in what follows we shall assume that $\mathcal{K}_\psi(\Omega)$ is not empty.

Let us now give the definition of local minimizers of (2.0.1).

**Definition 2.0.1.** *A function $u \in W^{1,p}(\Omega)$ is a local minimizer of (2.0.1) in $\mathcal{K}_\psi(\Omega)$ if and only if $f(x,u,Du) \in L^1_{loc}(\Omega)$ and the minimality condition*

$$\int_{\mathrm{supp}(u-\varphi)} f(x,u,Du)dx \leq \int_{\mathrm{supp}(u-\varphi)} f(x,\varphi,D\varphi)dx$$

*is satisfied for all $\varphi \in \mathcal{K}_\psi(\Omega)$.*

Let $1 \leq p < +\infty$ and $\mu \in [0,1]$. We define an auxiliary function by

$$V_p(\xi) := (\mu^2 + |\xi|^2)^{\frac{p-2}{4}}\xi$$

for all $\xi \in \mathbb{R}^n$. One can easily check that, for $p \geq 2$, it holds

$$|\xi|^p \leq |V_p(\xi)|^2. \tag{2.0.3}$$

For the auxiliary function $V_p$, we recall the following estimate (see e.g. [66, Lemma 8.3]).

**Lemma 2.0.2.** *Let $1 < p < +\infty$. There exists a constant $c = c(n,p) > 0$ such that*

$$c^{-1}(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \leq \frac{|V_p(\xi) - V_p(\eta)|^2}{|\xi - \eta|^2} \leq c(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}}$$

*for any $\xi, \eta \in \mathbb{R}^n$, $\xi \neq \eta$.*

Now we state a well-known iteration lemma (see [66, Lemma 6.1] for the proof).

**Lemma 2.0.3.** *Let $\Phi : [\frac{R}{2}, R] \to \mathbb{R}$ be a bounded nonnegative function, where $R > 0$. Assume that for all $\frac{R}{2} \leq r < s \leq R$ it holds*

$$\Phi(r) \leq \theta\Phi(s) + A + \frac{B}{(s-r)^2} + \frac{C}{(s-r)^\gamma}$$

*where $\theta \in (0,1)$, $A$, $B$, $C \geq 0$ and $\gamma > 0$ are constants. Then there exists a constant $c = c(\theta, \gamma)$ such that*

$$\Phi\left(\frac{R}{2}\right) \leq c\left(A + \frac{B}{R^2} + \frac{C}{R^\gamma}\right).$$

We recall the following higher integrability result, also known as Gehring's Lemma (see e.g. [66, Theorem 6.6]).

**Lemma 2.0.4.** *Let $f \in L^1(B_R)$, and assume that for every balls $B \subset \tilde{B} \Subset B_R$ we have*

$$\fint_B f(x)dx \leq c\left\{\left(\fint_{\tilde{B}} f(x)^m dx\right)^{\frac{1}{m}} + \fint_{\tilde{B}} g(x)dx\right\}$$

*with $0 < m < 1$. Assume that the function $g$ belongs to $L^s(B_R)$ for some $s > 1$. Then, there exists an exponent $r \in (1,s)$ such that $f \in L^r(B_{R/2})$, and moreover it holds*

$$\fint_{B_{R/2}} f(x)^r dx \leq c\left\{\left(\fint_{B_R} f(x)dx\right)^r + \fint_{B_R} g(x)^r dx\right\}.$$

## 2.1 Difference quotient

We recall some properties of the finite difference quotient operator that will be needed in the sequel.

**Definition 2.1.1.** *Let $F$ be a function defined in an open set $\Omega \subset \mathbb{R}^n$ and let $h \in \mathbb{R}^n$. We call the difference quotient of $F$ with respect to $h$ the function*

$$\tau_h^1 F(x) = \tau_h F(x) := F(x+h) - F(x).$$

*Moreover, for every $r \in \mathbb{N} \setminus \{1\}$ we define the $r$-th finite difference quotient with respect to $h$ as*

$$\tau_h^r F(x) := \tau_h(\tau_h^{r-1} F(x)).$$

The function $\tau_h F$ is defined in the set

$$\tau_h\Omega := \{x \in \Omega : x + h \in \Omega\},$$

and hence in the set

$$\Omega_{|h|} := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > |h|\}.$$

We start with the description of some elementary properties that can be found, for example, in [66].

**Proposition 2.1.2.** *Let $F \in W^{1,p}(\Omega)$, with $p \geq 1$, and let $G : \Omega \to \mathbb{R}$ be a measurable function. Then*
*(i) $\tau_h F \in W^{1,p}(\Omega_{|h|})$ and*

$$D_i(\tau_h F) = \tau_h(D_i F).$$

*(ii) If at least one of the functions $F$ or $G$ has support contained in $\Omega_{|h|}$, then*

$$\int_\Omega F\tau_h G dx = \int_\Omega G\tau_{-h} F dx.$$

*(iii) We have*

$$\tau_h(FG)(x) = F(x+h)\tau_h G(x) + G(x)\tau_h F(x).$$

The next result about the finite difference operator is a kind of integral version of Lagrange Theorem.

**Lemma 2.1.3.** *If $0 < \rho < R$, $|h| < \frac{R-\rho}{2}$, $1 < p < +\infty$ and $F \in W^{1,p}(B_R)$, then*

$$\int_{B_\rho} |\tau_h F(x)|^p dx \leq c(n,p)|h|^p \int_{B_R} |DF(x)|^p dx.$$

*Moreover,*

$$\int_{B_\rho} |F(x+h)|^p dx \leq \int_{B_R} |F(x)|^p dx.$$

We conclude this subsection recalling a fractional version of Sobolev embedding property.

**Lemma 2.1.4.** *Let $F \in L^2(B_R)$. Suppose that there exist $\rho \in (0,R)$, $0 < \alpha < 1$ and $M > 0$ such that*

$$\int_{B_\rho} |\tau_h F(x)|^2 dx \leq M^2 |h|^{2\alpha},$$

*for every $h$ such that $|h| < \frac{R-\rho}{2}$. Then $F \in L^{\frac{2n}{n-2\beta}}(B_\rho)$ for every $\beta \in (0,\alpha)$ and*

$$\|F\|_{L^{\frac{2n}{n-2\beta}}(B_\rho)} \leq c(M + \|F\|_{L^2(B_R)}),$$

*with $c = c(n, R, \rho, \alpha, \beta)$.*

## 2.2 Function spaces

In this section, we list the definitions and the properties of some function spaces.

### 2.2.1 Morrey spaces

We recall the definition of Morrey spaces (see [66]).
Let $1 \le p < +\infty$ and $\lambda \ge 0$. By $L^{p,\lambda}(\Omega)$ we denote the linear space of functions $u \in L^p(\Omega)$ such that, if we set $\Omega(x_0,\rho) := \Omega \cap B(x_0,\rho)$, we get

$$\|u\|_{L^{p,\lambda}(\Omega)} := \left\{ \sup_{x_0\in\Omega, 0<\rho<\mathrm{diam}(\Omega)} \rho^{-\lambda} \int_{\Omega(x_0,\rho)} |u(x)|^p dx \right\}^{\frac{1}{p}} < +\infty.$$

It is easy to see that $\|u\|_{L^{p,\lambda}(\Omega)}$ is a norm with respect to which $L^{p,\lambda}(\Omega)$ is a Banach space. Notice that $\|u\|_{L^{p,0}(\Omega)} = \|u\|_{L^p(\Omega)}$, so that $L^{p,0}(\Omega) = L^p(\Omega)$. More generally, using Hölder's inequality, one proves easily that if $s \ge p$ and $\frac{n-\lambda}{p} \ge \frac{n-\kappa}{s}$ the following holds:

$$\|u\|_{L^{p,\lambda}(\Omega)} \le \mathrm{diam}(\Omega)^{\frac{n-\lambda}{p}-\frac{n-\kappa}{s}} \|u\|_{L^{s,\kappa}(\Omega)},$$

and therefore the immersion

$$L^{s,\kappa}(\Omega) \hookrightarrow L^{p,\lambda}(\Omega) \tag{2.2.1}$$

is continuous.

### 2.2.2 Besov spaces

We give the definition of Besov spaces as done in [71, Section 2.5.12].

**Definition 2.2.1.** *Let $1 \le p < +\infty$ and let $\alpha > 0$ be a positive real number. Denote by $r$ the smallest integer larger than $\alpha$.*
*Let $1 \le s < +\infty$. We say that a function $v : \mathbb{R}^n \to \mathbb{R}$ belongs to the Besov space $B^\alpha_{p,s}(\mathbb{R}^n)$ if, and only if, $v \in L^p(\mathbb{R}^n)$ and*

$$[v]_{B^\alpha_{p,s}(\mathbb{R}^n)} := \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \frac{|\tau_h^r v(x)|^p}{|h|^{\alpha p}} dx \right)^{\frac{s}{p}} \frac{dh}{|h|^n} \right)^{\frac{1}{s}} < +\infty.$$

*Equivalently, we could simply say that $v \in L^p(\mathbb{R}^n)$ and $\frac{\tau_h^r v}{|h|^\alpha} \in L^s\left(\frac{dh}{|h|^n}; L^p(\mathbb{R}^n)\right)$.*
*When $s = +\infty$, the Besov space $B^\alpha_{p,\infty}(\mathbb{R}^n)$ consists of functions $v \in L^p(\mathbb{R}^n)$ such that*

$$[v]_{B^\alpha_{p,\infty}(\mathbb{R}^n)} := \sup_{h\in\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \frac{|\tau_h^r v(x)|^p}{|h|^{\alpha p}} dx \right)^{\frac{1}{p}} < +\infty.$$

*Accordingly, for $1 \le s \le +\infty$, the Besov space $B^\alpha_{p,s}(\mathbb{R}^n)$ is normed with*

$$\|v\|_{B^\alpha_{p,s}(\mathbb{R}^n)} := \|v\|_{L^p(\mathbb{R}^n)} + [v]_{B^\alpha_{p,s}(\mathbb{R}^n)}.$$

Observe that, if $1 \le s < +\infty$, integrating for $h \in B(0,\delta)$ for a fixed $\delta > 0$ then an equivalent norm is obtained, because

$$\left( \int_{\{|h|\ge\delta\}} \left( \int_{\mathbb{R}^n} \frac{|\tau_h^r v(x)|^p}{|h|^{\alpha p}} dx \right)^{\frac{s}{p}} \frac{dh}{|h|^n} \right)^{\frac{1}{s}} \le c(n,\alpha,p,s,\delta)\|v\|_{L^p(\mathbb{R}^n)}.$$

In the case $s = +\infty$, one can simply take supremum over $|h| \le \delta$ and obtain an equivalent norm. By construction, one has $B^\alpha_{p,s}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$. One also has the following version of Sobolev embeddings (a proof can be found at [71, Proposition 7.12]).

**Lemma 2.2.2.** *Suppose that $0 < \alpha < 1$.*
*(a) If $1 < p < \frac{n}{\alpha}$ and $1 \leq s \leq p^*_\alpha = \frac{np}{n-\alpha p}$, then there is a continuous embedding $B^\alpha_{p,s}(\mathbb{R}^n) \subset L^{p^*_\alpha}(\mathbb{R}^n)$.*
*(b) If $p = \frac{n}{\alpha}$ and $1 \leq s \leq +\infty$, then there is a continuous embedding $B^\alpha_{p,s}(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$, where $BMO$ denotes the space of functions with bounded mean oscillations* [66, Chapter 2].

We recall the following inclusions between Besov spaces ([71, Proposition 7.10 and Formula (7.35)]).

**Lemma 2.2.3.** *Suppose that $0 < \beta < \alpha < 1$.*
*(a) If $1 < p < +\infty$ and $1 \leq s \leq t \leq +\infty$, then $B^\alpha_{p,s}(\mathbb{R}^n) \subset B^\alpha_{p,t}(\mathbb{R}^n)$.*
*(b) If $1 < p < +\infty$ and $1 \leq s, t \leq +\infty$, then $B^\alpha_{p,s}(\mathbb{R}^n) \subset B^\beta_{p,t}(\mathbb{R}^n)$.*
*(c) If $1 \leq s \leq +\infty$, then $B^\alpha_{\frac{n}{\alpha},s}(\mathbb{R}^n) \subset B^\beta_{\frac{n}{\beta},s}(\mathbb{R}^n)$.*

Combining Lemmas 2.2.2 and 2.2.3, we get the following Sobolev type embedding theorem for Besov spaces $B^\alpha_{p,\infty}(\mathbb{R}^n)$.

**Lemma 2.2.4.** *Suppose that $0 < \alpha < 1$ and $1 < p < \frac{n}{\alpha}$. There is a continuous embedding $B^\alpha_{p,\infty}(\mathbb{R}^n) \subset L^{p^*_\beta}(\mathbb{R}^n)$, for every $0 < \beta < \alpha$. Moreover, for every function $F \in B^\alpha_{p,\infty}(\mathbb{R}^n)$ the following local estimate*

$$\|F\|_{L^{\frac{np}{n-\beta p}}(B_\varrho)} \leq \frac{c}{(R-\varrho)^\delta}(\|F\|_{L^p(B_R)} + [F]_{B^\alpha_{p,q}(B_R)}) \tag{2.2.2}$$

*holds for every ball $B_\varrho \subset B_R$, with $c = c(n, p, q, \alpha, \beta)$ and $\delta = \delta(n, p, q)$.*

Given a domain $\Omega \subset \mathbb{R}^n$, we say that a function $v : \mathbb{R}^n \to \mathbb{R}$ belongs to the local Besov space $B^\alpha_{p,s,\mathrm{loc}}$ if $\varphi v \in B^\alpha_{p,s}(\mathbb{R}^n)$ whenever $\varphi \in \mathcal{C}^\infty_0(\Omega)$. It is worth noticing that one can prove suitable version of Lemmas 2.2.2 and 2.2.3, by using local Besov spaces.

We have the following lemma.

**Lemma 2.2.5.** *Let $1 \leq p < +\infty$, $1 \leq s \leq +\infty$ and $0 < \alpha < 2$. A function $v \in L^p_{loc}(\Omega)$ belongs to the local Besov space $B^\alpha_{p,s,loc}$, if, and only if,*

$$\left\| \frac{\tau^r_h v}{|h|^\alpha} \right\|_{L^s\left(\frac{dh}{|h|^n};L^p(B)\right)} < +\infty,$$

*for any ball $B \subset 2B \subset \Omega$ with radius $r_B$. Here the measure $\frac{dh}{|h|^n}$ is restricted to the ball $B(0, r_B)$ on the $h$-space.*

*Proof.* For the proof of the claim in the case $0 < \alpha < 1$ we refer to [1, Lemma 7]. Therefore, we concentrate only on the case $\alpha \in [1, 2)$. We first note that, given $h \in \mathbb{R}^n$, for any function $\varphi \in \mathcal{C}^\infty_0(\Omega)$ Proposition 2.1.2 yields

$$\begin{aligned}\frac{\tau^2_h(\varphi v)(x)}{|h|^\alpha} &= \frac{\tau_h(\varphi(x+h)\tau_h v(x) + v(x)\tau_h\varphi(x))}{|h|^\alpha} \\ &= \varphi(x+2h)\frac{\tau^2_h v(x)}{|h|^\alpha} + 2\frac{\tau_h v(x)\tau_h\varphi(x+h)}{|h|^\alpha} + v(x)\frac{\tau^2_h\varphi(x)}{|h|^\alpha}.\end{aligned} \tag{2.2.3}$$

It is clear that

$$\left|\frac{\tau_h v(x)\tau_h\varphi(x+h)}{|h|^\alpha}\right| \le \frac{|\tau_h v(x)|}{|h|^{\alpha-1}}\|D\varphi\|_\infty,$$

and so, since $\alpha-1<1$ and $v\in B^{\alpha-1}_{p,s,\mathrm{loc}}(\Omega)$ from Lemma 2.2.3, one has

$$\frac{\tau_h v(x)\tau_h\varphi(x+h)}{|h|^\alpha}\in L^s\left(\frac{dh}{|h|^n};L^p(\mathbb{R}^n)\right).$$

Moreover, it holds

$$\begin{aligned}\left|\frac{v(x)\tau_h^2\varphi(x)}{|h|^\alpha}\right| &\le |v(x)|\|\tau_h D\varphi\|_\infty|h|^{1-\alpha}\\ &\le |v(x)|\|D^2\varphi\|_\infty|h|^{2-\alpha}\end{aligned}$$

and therefore we have

$$\frac{v(x)\tau_h^2\varphi(x)}{|h|^\alpha}\in L^s\left(\frac{dh}{|h|^n};L^p(\mathbb{R}^n)\right).$$

As a consequence, we have the equivalence

$$\varphi v\in B^\alpha_{p,s}(\mathbb{R}^n)\Longleftrightarrow \varphi(x+2h)\frac{\tau_h^2 v(x)}{|h|^\alpha}\in L^s\left(\frac{dh}{|h|^n};L^p(\mathbb{R}^n)\right).$$

However, it is clear that $\varphi(x+2h)\frac{\tau_h^2 v(x)}{|h|^\alpha}\in L^s\big(\frac{dh}{|h|^n};L^p(\mathbb{R}^n)\big)$ for every $\varphi\in\mathcal{C}^\infty_0(\Omega)$ if, and only if, the same happens for every $\varphi=\chi_B$ and all ball $B\subset 2B\subset\Omega$. This concludes the proof. □

It is known that Besov spaces of fractional order $\alpha\in(0,1)$ can be characterized in pointwise terms. We give the following definition according to [74].

**Definition 2.2.6.** *Given a measurable function $v:\mathbb{R}^n\to\mathbb{R}$, a fractional $\alpha$-Hajlasz gradient for $v$ is a sequence $\{g_k\}_k$ of measurable, non-negative functions $g_k:\mathbb{R}^n\to\mathbb{R}$, together with a null set $N\subset\mathbb{R}^n$, such that the inequality*

$$|v(x)-v(y)|\le(g_k(x)+g_k(y))|x-y|^\alpha$$

*holds whenever $k\in\mathbb{Z}$ and $x,y\in\mathbb{R}^n\setminus N$ are such that $2^{-k}\le|x-y|<2^{-k+1}$. We say that $\{g_k\}_k\in l^s(\mathbb{Z};L^p(\mathbb{R}^n))$ if*

$$\|\{g_k\}_k\|_{l^s(L^p)}=\left(\sum_{k\in\mathbb{Z}}\|g_k\|^s_{L^p(\mathbb{R}^n)}\right)^{\frac{1}{s}}<+\infty.$$

The following result was proved in [74].

**Theorem 2.2.7.** *Let $0<\alpha<1$, $1\le p<+\infty$ and $1\le s\le+\infty$. Let $v\in L^p(\mathbb{R}^n)$. One has $v\in B^\alpha_{p,s}(\mathbb{R}^n)$ if, and only if, there exists a fractional $\alpha$-Hajlasz gradient $\{g_k\}_k\in l^s(\mathbb{Z};L^p(\mathbb{R}^n))$ for $v$. Moreover,*

$$\|v\|_{B^\alpha_{p,s}(\mathbb{R}^n)}\simeq\inf\|\{g_k\}_k\|_{l^s(L^p)},$$

*where the infimum runs over all possible fractional $\alpha$-Hajlasz gradients for $v$.*

### 2.2.3 Fractional Sobolev spaces

We recall here the definition of Sobolev spaces of fractional order (also known as Slobodeskii spaces).

**Definition 2.2.8.** *Let $\Omega \subset \mathbb{R}^n$ be a domain and let $0 < s < 1$ and $1 \le p < +\infty$. We say that a function $v : \Omega \to \mathbb{R}$ belongs to the space $W^{s,p}(\Omega)$ if and only if*

$$[v]^p_{W^{s,p}(\Omega)} := \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^p}{|x-y|^{n+sp}} dxdy < +\infty.$$

*Then we set*

$$\|v\|_{W^{\sigma,p}(\Omega)} := \|v\|_{L^p(\Omega)} + [v]_{W^{s,p}(\Omega)}.$$

*When $s > 1$ is not an integer, we write $s = m + \sigma$, with $m \in \mathbb{N}$ and $\sigma \in (0,1)$, and then we let*

$$W^{s,p}(\Omega) := \{v \in W^{m,p}(\Omega) : D^m v \in W^{\sigma,p}(\Omega)\},$$

*normed with*

$$\|v\|_{W^{\sigma,p}(\Omega)} := \|v\|_{W^{m,p}(\Omega)} + \|D^m v\|_{W^{\sigma,p}(\Omega)}.$$

The next lemma states fundamental embedding properties between Sobolev and Besov spaces (see e.g. [88, 100]).

**Lemma 2.2.9.** *Let $\gamma > 0$ and $1 \le p < +\infty$.*
*(a) If $1 \le s \le +\infty$, then there is a continuous embedding $B^\gamma_{p,s}(\mathbb{R}^n) \subset W^{m,p}(\mathbb{R}^n)$, for every $m \in \mathbb{N}$ such that $m < \gamma$.*
*(b) If $1 \le s \le \min\{p, 2\}$, then there is a continuous embedding $B^\gamma_{p,s}(\mathbb{R}^n) \subset W^{\gamma,p}(\mathbb{R}^n)$.*

## 2.3 Gagliardo-Nirenberg inequality

In this section, we collect some interpolation inequalities in the setting of Besov spaces.

**Lemma 2.3.1.** *Let $v \in W^{1,p}_{loc}(\mathbb{R}^n)$. If $Dv \in B^\gamma_{p,s,loc}(\mathbb{R}^n)$, for some $1 \le s \le +\infty$ and $0 < \gamma < 1$, then $v \in B^{1+\gamma}_{p,s,loc}(\mathbb{R}^n)$. Moreover, the following estimate*

$$[v]_{B^{1+\gamma}_{p,s}(B_\rho)} \le c[Dv]_{B^\gamma_{p,s}(B_R)}$$

*holds for every ball $B_\rho \subset B_R$, with $c = c(n,p)$.*

*Proof.* We give the proof of Lemma 2.3.1 only for $s = +\infty$, since the case $s$ finite can be obtained in a similar way.
Fix $0 < \rho < R$, $|h| < \frac{R-\rho}{2}$ and consider balls $B_\rho \subset B_R$. Since $1 < 1+\gamma < 2$, we have that

$$[v]_{B^{1+\gamma}_{p,\infty}(B_\rho)} = \sup_{h \in \mathbb{R}^n} \left( \int_{B_\rho} \frac{|\tau_h^2 v(x)|^p}{|h|^{(1+\gamma)p}} dx \right)^{\frac{1}{p}}.$$

Now, using $v \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ and Lemma 2.1.3, we obtain

$$\int_{B_\rho} \frac{|\tau_h(\tau_h v(x))|^p}{|h|^{(1+\gamma)p}} dx \le c|h|^p \int_{B_R} \frac{|\tau_h Dv(x)|^p}{|h|^{(1+\gamma)p}} dx$$

$$= c\int_{B_R}\frac{|\tau_h Dv(x)|^p}{|h|^{\gamma p}}dx \le c[Dv]^p_{B^\gamma_{p,\infty}(B_R)},$$

which is finite by the assumption on $Dv$. This completes the proof. □

The following interpolation inequality can be found in [98].

**Lemma 2.3.2.** *Let $1 \le p < +\infty$, $1 \le s \le +\infty$, $\gamma > 0$ and $0 < \theta < 1$. Then, the following interpolation inequality*

$$\|v\|_{B^{\theta\gamma}_{p/\theta,s/\theta}(\mathbb{R}^n)} \le c\|v\|^\theta_{B^\gamma_{p,s}(\mathbb{R}^n)}\|v\|^{1-\theta}_{L^\infty(\mathbb{R}^n)} \tag{2.3.1}$$

*holds for every $v \in B^\gamma_{p,s}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.*

Now, Lemmas 2.3.1 and 2.3.2 lead to the following higher integrability result.

**Proposition 2.3.3.** *Let $v \in W^{1,p}_{loc}(\mathbb{R}^n) \cap L^\infty_{loc}(\mathbb{R}^n)$ and let $Dv \in B^\gamma_{p,\infty,loc}(\mathbb{R}^n)$, for some $1 \le p < +\infty$ and $0 < \gamma < 1$. Then $Dv \in L^{p(1+\beta)}_{loc}(\mathbb{R}^n)$, for every $0 < \beta < \gamma$. Moreover, the following estimate*

$$\int_{B_\rho}|Dv|^{p(1+\beta)}dx \le C\|v\|^{p\beta}_{L^\infty(B_R)}\left([Dv]^p_{B^\gamma_{p,\infty}(B_R)} + \frac{1}{(R-\rho)^{2p}}\|v\|^p_{W^{1,p}(B_R)}\right)$$

*holds for every ball $B_\rho \subset B_R$, with $C = C(n,p,\gamma,\beta)$.*

*Proof.* Thanks to Lemma 2.3.1, we obtain

$$v \in B^{1+\gamma}_{p,\infty}(\mathbb{R}^n) \quad \text{locally.}$$

Then, Lemma 2.3.2 yields

$$v \in B^{\theta(1+\gamma)}_{p/\theta,\infty}(\mathbb{R}^n) \quad \text{locally,} \tag{2.3.2}$$

for every $\theta \in (0,1)$.
Choosing $\theta = \frac{1}{1+\beta}$, for $0 < \beta < \gamma$, we have

$$\theta(1+\gamma) = \frac{1+\gamma}{1+\beta} > 1.$$

Let us consider $0 < \rho < R \le 1$ and fix balls $B_\rho \subset B_R$ and a cut-off function $\eta \in \mathcal{C}^\infty_0(B_{\frac{R+\rho}{2}})$ such that $0 \le \eta \le 1$, $\eta = 1$ on $B_\rho$, $|D\eta| \le \frac{C}{R-\rho}$ and $|D^2\eta| \le \frac{C}{(R-\rho)^2}$. By virtue of Lemma 2.2.9, we have

$$\int_{B_\rho}|Dv|^{p(1+\beta)}dx \le \|\eta v\|^{p(1+\beta)}_{W^{1,p(1+\beta)}(\mathbb{R}^n)} \le c\|\eta v\|^{p(1+\beta)}_{B^{\frac{1+\gamma}{1+\beta}}_{p(1+\beta),\infty}(\mathbb{R}^n)}. \tag{2.3.3}$$

From Lemma 2.3.2, we get

$$\|\eta v\|^{p(1+\beta)}_{B^{\frac{1+\gamma}{1+\beta}}_{p(1+\beta),\infty}(\mathbb{R}^n)} \le c\|v\|^{p\beta}_{L^\infty(B_R)}\|\eta v\|^p_{B^{1+\gamma}_{p,\infty}(\mathbb{R}^n)}. \tag{2.3.4}$$

Using identity (2.2.3) and the properties of $\eta$, we infer

$$\|\eta v\|^p_{B^{1+\gamma}_{p,\infty}(\mathbb{R}^n)} \le C\|v\|^p_{L^p(B_{\frac{R+\rho}{2}})} + C\sup_{|h|\le\frac{R-\rho}{4}}\int_{\mathbb{R}^n}|\eta(x+2h)|^p\frac{|\tau^2_h v|^p}{|h|^{p(1+\gamma)}}dx$$

$$
\begin{aligned}
&+ C \sup_{|h|\le \frac{R-\rho}{4}} \int_{\mathbb{R}^n} |\tau_h \eta(x+h)|^p \frac{|\tau_h v|^p}{|h|^{p(1+\gamma)}} dx + C \sup_{|h|\le \frac{R-\rho}{4}} \int_{B_{\frac{3R+\rho}{4}}} |v|^p \frac{|\tau_h^2 \eta(x)|^p}{|h|^{p(1+\gamma)}} dx \\
\le\; & C\|v\|^p_{L^p(B_{\frac{R+\rho}{2}})} + C \sup_{|h|\le \frac{R-\rho}{4}} \int_{B_{\frac{3R+\rho}{4}}} \frac{|\tau_h^2 v|^p}{|h|^{p(1+\gamma)}} dx \\
&+ C \sup_{|h|\le \frac{R-\rho}{4}} \int_{\mathbb{R}^n} \|D\eta\|^p_{L^\infty(B_{\frac{3R+\rho}{4}})} \frac{|\tau_h v|^p}{|h|^{p\gamma}} dx \\
&+ C \sup_{|h|\le \frac{R-\rho}{4}} |h|^{p(1-\gamma)} \int_{B_{\frac{3R+\rho}{4}}} |v|^p \|D^2\eta\|^p_{L^\infty(B_{\frac{3R+\rho}{4}})} dx \\
\le\; & C\|v\|^p_{L^p(B_{\frac{R+\rho}{2}})} + C[v]^p_{B^{1+\gamma}_{p,\infty}(B_{\frac{3R+\rho}{4}})} \\
&+ \frac{C}{(R-\rho)^p} \sup_{|h|\le \frac{R-\rho}{4}} \int_{B_{\frac{3R+\rho}{4}}} \frac{|\tau_h v|^p}{|h|^{p\gamma}} dx \\
&+ \frac{C}{(R-\rho)^{2p}} \sup_{|h|\le \frac{R-\rho}{4}} |h|^{p(1-\gamma)} \int_{B_{\frac{3R+\rho}{4}}} |v|^p dx.
\end{aligned}
$$

Now, exploiting Lemma 2.1.3 and using the fact that $R-\rho<1$, we obtain

$$
\begin{aligned}
\|\eta v\|^p_{B^{1+\gamma}_{p,\infty}(\mathbb{R}^n)} \le\; & C\|v\|^p_{L^p(B_{\frac{R+\rho}{2}})} + C[v]^p_{B^{1+\gamma}_{p,\infty}(B_{\frac{3R+\rho}{4}})} \\
&+ \frac{C}{(R-\rho)^p} \sup_{|h|\le \frac{R-\rho}{4}} |h|^{p(1-\gamma)} \int_{B_R} |Dv|^p dx \\
&+ \frac{C}{(R-\rho)^{2p}} \int_{B_{\frac{3R+\rho}{4}}} |v|^p dx \\
\le\; & C[v]^p_{B^{1+\gamma}_{p,\infty}(B_{\frac{3R+\rho}{4}})} + \frac{C}{(R-\rho)^{2p}} \|v\|^p_{W^{1,p}(B_R)}.
\end{aligned} \tag{2.3.5}
$$

Combining inequalities (2.3.3), (2.3.4) and (2.3.5) and Lemma 2.3.1, we derive

$$
\begin{aligned}
\int_{B_\rho} |Dv|^{p(1+\beta)} dx &\le C\|v\|^{p\beta}_{L^\infty(B_R)} [v]^p_{B^{1+\gamma}_{p,\infty}(B_{\frac{3R+\rho}{4}})} + \frac{C}{(R-\rho)^{2p}} \|v\|^{p\beta}_{L^\infty(B_R)} \|v\|^p_{W^{1,p}(B_R)} \\
&\le C\|v\|^{p\beta}_{L^\infty(B_R)} [Dv]^p_{B^{\gamma}_{p,\infty}(B_R)} + \frac{C}{(R-\rho)^{2p}} \|v\|^{p\beta}_{L^\infty(B_R)} \|v\|^p_{W^{1,p}(B_R)}
\end{aligned}
$$

i.e. the desired estimate. □

The following proposition is an immediate consequence of the previous result.

**Proposition 2.3.4.** *Let $v \in W^{1,p}_{loc}(\mathbb{R}^n) \cap L^\infty_{loc}(\mathbb{R}^n)$, for some $p \ge 2$, and assume that $V_p(Dv) \in B^{\gamma}_{2,\infty,loc}(\mathbb{R}^n)$, for some $0<\gamma<1$. Then $Dv \in L^{p+2\beta}_{loc}(\mathbb{R}^n)$, for every $0<\beta<\gamma$. Moreover, the following inequality*

$$
\int_{B_\rho} |Dv|^{p+2\beta} dx \le C\|v\|^{2\beta}_{L^\infty(B_R)} \left( [V_p(Dv)]^2_{B^{\gamma}_{2,\infty}(B_R)} + \frac{1}{(R-\rho)^{2p}} \|v\|^p_{W^{1,p}(B_R)} \right)
$$

*holds for every ball $B_\rho \subset B_R$, with $C = C(n,p,\gamma,\beta)$.*

*Proof.* By Lemma 2.0.2, we get

$$|\tau_h Dv(x)|^p \le |\tau_h Dv|^2(\mu^2 + |Dv(x+h)|^2 + |Dv(x)|^2)^{\frac{p-2}{2}} \le c|\tau_h V_p(Dv(x))|^2,$$

where in the first inequality we used the fact that $p \ge 2$. Dividing by $|h|^{2\gamma}$ and integrating over the ball $B_R$ the preceding inequality imply

$$\int_{B_R} \frac{|\tau_h Dv|^p}{|h|^{p\frac{2\gamma}{p}}}dx = \int_{B_R} \frac{|\tau_h Dv|^p}{|h|^{2\gamma}}dx \le c\int_{B_R} \frac{|\tau_h V_p(Dv)|^2}{|h|^{2\gamma}}dx \qquad \text{for every } h. \tag{2.3.6}$$

Now, thanks to the assumption on $V_p(Dv)$, taking the supremum over $h$ leads us to

$$Dv \in B^{\frac{2\gamma}{p}}_{p,\infty}(\mathbb{R}^n) \quad \text{locally.}$$

This together with Lemma 2.3.1 yields

$$v \in B^{1+\frac{2\gamma}{p}}_{p,\infty}(\mathbb{R}^n) \quad \text{locally.}$$

By virtue of Lemma 2.3.3 and (2.3.6), it follows

$$\int_{B_\rho} |Dv|^{p+2\beta}dx \le C\|v\|^{2\beta}_{L^\infty(B_R)}[V_p(Dv)]^2_{B^\gamma_{2,\infty}(B_R)} + \frac{C}{(R-\rho)^{2p}}\|v\|^{2\beta}_{L^\infty(B_R)}\|v\|^p_{W^{1,p}(B_R)}$$

for every $0 < \beta < \gamma$. □

Arguing similarly, but assuming that $V_p(Dv) \in B^\gamma_{2,s,\text{loc}}(\mathbb{R}^n)$, for some $1 \le s < +\infty$, we can prove the following result.

**Proposition 2.3.5.** *Let $v \in W^{1,p}_{loc}(\mathbb{R}^n) \cap L^\infty_{loc}(\mathbb{R}^n)$, for some $p \ge 2$, and assume that $V_p(Dv) \in B^\gamma_{2,s,loc}(\mathbb{R}^n)$, for some $0 < \gamma < 1$ and $1 \le s \le \frac{4}{p+2\gamma}$. Then $Dv \in L^{p+2\gamma}_{loc}(\mathbb{R}^n)$. Moreover, the following inequality*

$$\int_{B_\rho} |Dv|^{p+2\gamma}dx \le C\|v\|^{2\gamma}_{L^\infty(B_R)}\left([V_p(Dv)]^2_{B^\gamma_{2,s}(B_R)} + \frac{1}{(R-\rho)^{2p}}\|v\|^p_{W^{1,p}(B_R)}\right)$$

*holds for every ball $B_\rho \subset B_R$, with $C = C(n, p, \gamma)$.*

# Chapter 3

# Gradient regularity for nonlinear obstacle problems with standard growth

In this chapter we deal with the regularity theory for obstacle problems involving elliptic operators. The results we report are contained in the paper [67].

We are interested in the regularity properties of the function $u : \Omega \to \mathbb{R}$ belonging to $\mathcal{K}_\psi(\Omega)$ and satisfying the variational inequality

$$\int_\Omega \langle \mathcal{A}(x, u, Du), D(\varphi - u)\rangle dx \geq \int_\Omega \mathcal{B}(x, u, Du)(\varphi - u)dx, \qquad \forall \varphi \in \mathcal{K}_\psi(\Omega), \tag{3.0.1}$$

where $\psi \in W^{1,p}(\Omega)$ is the obstacle function and $\mathcal{K}_\psi(\Omega)$ is the admissible class defined in (2.0.2). We suppose that there exist positive constants $\nu, L$, a parameter $\mu \in [0, 1]$, that will allow us to consider in our analysis both the degenerate case and the non-degenerate one, and an exponent $p \geq 2$ such that the following conditions are satisfied:

$$\langle \mathcal{A}(x, u, \xi) - \mathcal{A}(x, u, \eta), \xi - \eta\rangle \geq \nu|\xi - \eta|^2(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \tag{A1}$$

$$|\mathcal{A}(x, u, \xi) - \mathcal{A}(x, u, \eta)| \leq L|\xi - \eta|(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \tag{A2}$$

$$|\mathcal{A}(x, u, \xi) - \mathcal{A}(y, v, \xi)| \leq \omega(|x - y| + |u - v|)(\mu^2 + |\xi|^2)^{\frac{p-1}{2}} \tag{A3}$$

for all $x, y \in \Omega$, $u, v \in \mathbb{R}$, $\xi, \eta \in \mathbb{R}^n$, where $\omega : [0, +\infty) \to [0, 1]$ is given by $\omega(\rho) = \min\{\rho^\alpha, 1\}$, with $0 < \alpha < 1$. As a consequence of assumptions (A2) and (A3), we also have

$$|\mathcal{A}(x, u, \xi)| \leq l(1 + |\xi|^2)^{\frac{p-1}{2}} \tag{A4}$$

for all $x \in \Omega$, $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$ and for a positive constant $l$. Moreover, we suppose that there exist a non negative function $a \in L^{\frac{p}{p-1}}_{\mathrm{loc}}(\Omega)$ and a positive exponent $r < p - 1$ such that

$$|\mathcal{B}(x, v, \xi)| \leq |\xi|^r + |v|^r + a(x) \tag{B1}$$

for a.e. $x \in \Omega$ and every $v \in \mathbb{R}$, $\xi \in \mathbb{R}^n$.

First, we will present higher integrability properties of the gradient of the solution to (3.0.1), having as initial information the integrability properties of the gradient of the obstacle function. Accordingly, we shall provide a local Calderón-Zygmund type estimate for variational inequalities of the form (3.0.1). The result tells that the gradient of the solution is as integrable as the gradient of the obstacle, and this assertion is sharp, as easily follows by considering the regularity of the solution on the coincidence set, i.e. that portion of the domain where the solution and the obstacle coincide.

There have been many research activities on the Calderón-Zygmund theory for obstacle problems. We refer to [5, 9, 17, 80, 90] and references therein, for the regularity of solutions to obstacle problems with constant growth. In particular, Bögelein, Duzaar and Mingione in [5] attained local Calderón-Zygmund type estimates for elliptic and parabolic obstacle problems involving possibly degenerate operators in divergence form of $p$-Laplacian type with irregular obstacles. The result was extended to a global one in [9], where a discontinuous nonlinearity and a nonsmooth domain are involved. In the setting of variable growth exponent, we quote [43, 49] and [8] for local and global Calderón-Zygmund estimates, respectively.

Compared with this works, the novelty of our result consists in obtaining a local estimate for elliptic obstacle problems involving operators with an explicit dependence on the $u$-variable, other than on its gradient $Du$ and on the $x$-variable. More precisely, we are going to prove the following result.

**Theorem 3.0.1.** *Let $1 < q < +\infty$ and assume that $a^{\frac{p}{p-1}}, |D\psi|^p \in L^q_{loc}(\Omega)$. If $q \leq \frac{n}{p}$, assume that $D\psi \in L^{pt,\kappa}_{loc}(\Omega)$, for some $t > 1$ and $n - p < \kappa < n$. Then, under assumptions (A1)–(A4) and (B1), the weak solution $u \in \mathcal{K}_\psi(\Omega)$ of the variational inequality (3.0.1) belongs to $W^{pq}_{loc}(\Omega)$. Moreover, there exists a constant $C = C(n, p, q, \nu, L, l)$ such that the following inequality*

$$\int_{B_r} |Du|^{pq} dx \leq C \int_{B_r} (a^{\frac{pq}{p-1}} + |u|^{pq} + |D\psi|^{pq} + |Du|^p + 1) dx \tag{3.0.2}$$

*holds for every ball $B_r \Subset \Omega$.*

In Theorem 3.0.1 the continuity assumption of the solution $u$ cannot be removed, since the technique used for the proof relies on the fact that the map $x \mapsto \mathcal{A}(x, u(x), \xi)$ is *locally uniformly in VMO* (see Section 3.2.1 for the precise definition). Indeed, as shown by an explicit example in the paper by Di Gironimo, Esposito and Sgambati [38], the composition of a *VMO* function with a $\mathcal{C}^\infty$ function is not a priori *VMO*.

Next, we study the higher fractional differentiability properties of the gradient of the solution $u \in W^{1,p}(\Omega)$ to the variational inequality (3.0.1), provided the gradient of the obstacle $\psi$ possesses some extra differentiability properties.

Such analysis has been carried out for the first time in the work by Eleuteri and Passarelli di Napoli [46], in the setting of standard growth of $p$-type, with $p \geq 2$. More precisely, the authors consider solutions to variational inequalities of the form

$$\int_\Omega \langle \mathbf{a}(x, Du), D(\varphi - u) \rangle dx \geq 0 \tag{3.0.3}$$

for all $\varphi \in \mathcal{K}_\psi(\Omega)$. It is usually observed that the regularity of the solutions to (3.0.3) is strictly connected to the analysis of the regularity of the solutions to partial differential equations of the form

$$\operatorname{div} \mathbf{a}(x, Du) = \operatorname{div} \mathbf{a}(x, D\psi).$$

It is well known that no extra differentiability properties for the solutions of partial differential equations of the type

$$\operatorname{div} \mathbf{a}(x, Du) = \operatorname{div} \mathcal{G} \tag{3.0.4}$$

can be expected even if $\mathcal{G}$ is smooth, unless some assumption is given on the $x$-dependence of $\mathbf{a}$. On the other hand, recent results concerning the higher differentiability of solutions to (3.0.4) show that the weak differentiability of integer or fractional order of the map $x \mapsto \mathbf{a}(x, \xi)$ is a sufficient condition (see [64, 65, 95, 96, 97] for the case of Sobolev space with integer order and [1, 19] for the fractional one).

In [46] it turns out that the differentiability of $\mathbf{a}$, as function of the $x$-variable, is sufficient also in the context of obstacle problems to prove that the differentiability of the gradient of the obstacle transfers to the gradient of the solution. Moreover, we refer to [58, 59] for higher differentiability properties for obstacle problems with sub-quadratic growth, that is the case $1 < p < 2$.

Here we continue the study of the higher differentiability properties of the solutions to obstacle problems. Our result covers a more general class of variational inequalities with respect to the works quoted above. Indeed, in (3.0.1) we consider operators which allow also the dependence on the $u$-variable. This kind of problem has already been treated in [47] for minimizers of integral functionals, but there the energy densities are not supposed to be differentiable with respect to the $u$-variable. Therefore, minimizers are not solutions to suitable variational inequalities.

We have the following theorem.

**Theorem 3.0.2.** *Assume that assumptions* (A1)–(A4) *and* (B1) *hold for an exponent* $2 \le p < \frac{n}{\beta}$*, where* $0 < \beta < 1$*. If* $p \le \frac{n}{\beta+1}$*, assume that* $D\psi \in L^{pt,\kappa}_{loc}(\Omega)$*, for some* $t > 1$ *and* $n-p < \kappa < n$*. Let* $u \in \mathcal{K}_\psi(\Omega)$ *be the solution to the obstacle problem* (3.0.1)*. We set*

$$\gamma := \begin{cases} \beta & \text{if } p = 2 \\ \min\{\beta, \alpha\theta\} & \text{if } p \neq 2 \end{cases}$$

*where* $\theta$ *is the Hölder exponent of* $u$*. Then the following implication*

$$a \in L^{\frac{np}{(n-\beta p)(p-1)}}(\Omega), D\psi \in B^{\beta}_{p,s}(\Omega) \Longrightarrow (\mu^2 + |Du|^2)^{\frac{p-2}{4}} Du \in B^{\gamma}_{2,s}(\Omega)$$

*holds locally, provided* $s \le p^*_\beta = \frac{np}{n-\beta p}$.

The proof of Theorem 3.0.2 is achieved by means of difference quotient method, that is quite natural when trying to establish higher differentiabilty results and local gradient estimates (see for instance [84, 85]). Here the difficulties come from the set of admissible test functions that have to take into account the presence of the obstacle. In order to overcome this problem, we need to consider difference quotient involving both the solution and the obstacle, so that the function satisfies the constraint of belonging to the admissible class $\mathcal{K}_\psi(\Omega)$.

Observe that, due to the local nature of our regularity results, we are not requiring further properties on the boundary datum $u_0$ in (2.0.2). Existence of solutions to the (3.0.1) can be easily proved through classical results regarding variational inequalities, so we will mainly concentrate on the regularity results.

Finally, we observe that the Morrey regularity of the gradient of the obstacle is only needed to get the local Hölder continuity of the solution (see Theorem 3.1.1). Therefore if we deal with a priori local Hölder continuous minimizers, then the result holds without the hypothesis $D\psi \in L^{pt,\kappa}_{\text{loc}}(\Omega)$.

The structure of this chapter is as follows. After recalling a Hölder continuity result for obstacle problems in Section 3.1, we prove Theorem 3.0.1 in Section 3.2. In particular, we give the definition of *VMO* functions in Section 3.2.1. In Section 3.2.2, we provide comparison estimates by a method of approximation. Then, in Section 3.2.3, we prove our local higher integrability result for the gradient of the solution. Finally, Section 3.3 is devoted to the proof of Theorem 3.0.2.

## 3.1 Preliminary result

We now present a regularity result we need in the sequel. The proof can be found in [42, Theorem 2.10].

**Theorem 3.1.1.** *Let $u \in W^{1,p}(\Omega)$ be the weak solution to the variational inequality (3.0.1), under assumptions (A1)–(A4) and (B1). Suppose that the obstacle $\psi$ fulfils the assumption*

$$D\psi \in L^{\tilde{q},\lambda}_{loc}(\Omega),$$

*where $\tilde{q} = pq$ for some $q > 1$ and $n - p < \lambda < n$. Then, we have $u \in \mathcal{C}^{0,\delta}_{loc}(\Omega)$, with $\delta = 1 - \frac{n-\lambda}{p}$.*

**Remark 3.1.2.** *In Theorem 3.0.1, we do not need to assume for $q > \frac{n}{p}$ a Morrey regularity on the gradient of the obstacle, in order to get the Hölder continuity of the solution. Indeed, thanks to (2.2.1), for every $q > \frac{n}{p}$ there exist $t > 1$ and $n - p < \kappa < n$ such that*

$$D\psi \in L^{pq}_{loc}(\Omega) \hookrightarrow L^{pt,\kappa}_{loc}(\Omega).$$

## 3.2 Local gradient estimates

In this section we will prove Theorem 3.0.1. First, we derive comparison estimates for the solution to (3.0.1). Next, using a Vitali type covering lemma and performing a truncation technique lead us to a local estimate for the gradient of the solution.
Throughout this section, we denote by $u \in \mathcal{K}_\psi(\Omega)$ the weak solution of (3.0.1) and we will assume that hypotheses of Theorem 3.0.1 are in force.

### 3.2.1 VMO coefficients

For convenience of notation, we set

$$\mathbf{A}(x,\xi) := \mathcal{A}(x, u(x), \xi)$$

for every $x \in \Omega$ and every $\xi \in \mathbb{R}^n$. Consider $\mathbf{A}_{B_r(x_0)}(\xi)$ which is the integral average of $\mathbf{A}(\cdot,\xi)$ over $B_r(x_0)$, i.e.

$$\mathbf{A}_{B_r(x_0)}(\xi) := \fint_{B_r(x_0)} \mathbf{A}(x,\xi)dx.$$

One can easily check that $\mathbf{A}_{B_r(x_0)}(\xi)$ also satisfies assumptions (A1), (A2), (A3) and (A4). Setting

$$H(x, B_r(x_0)) := \sup_{\xi\neq 0} \frac{|\mathbf{A}(x,\xi) - \mathbf{A}_{B_r(x_0)}(\xi)|}{(\mu^2 + |\xi|^2)^{\frac{p-1}{2}}}, \tag{3.2.1}$$

we will say that the map $x \mapsto \mathbf{A}(x,\xi)$ is *locally uniformly in VMO* if for each compact set $K \subset \Omega$ we have that

$$\lim_{R\to 0} \sup_{r<R} \sup_{x_0\in K} \fint_{B_r(x_0)} H(x, B_r(x_0))dx = 0. \tag{3.2.2}$$

**Remark 3.2.1.** *We point out that it follows from* (3.2.2) *and the growth condition* (A4) *that for any* $s \geq 1$,

$$\lim_{R \to 0} \sup_{r<R} \sup_{x_0 \in K} \fint_{B_r(x_0)} H(x, B_r(x_0))^s dx = 0.$$

Now, we prove that assumption (A3) and the Hölder continuity of the solution $u$ yield that $\mathbf{A}$ is locally uniformly in *VMO*. The proof can be deduced by that of [1, 19] taking into account that the operator $\mathbf{A}$ implicitly depends on the solution $u$. We report it here for the sake of completeness.

**Lemma 3.2.2.** *Let* $\mathbf{A}$ *be such that* (A3) *holds. If* $q \leq \frac{n}{p}$*, assume that* $D\psi \in L^{pt,\lambda}_{loc}(\Omega)$*, for some* $t > 1$ *and* $n - p < \lambda < n$*. Then,* $\mathbf{A}$ *is locally uniformly in VMO, that is* (3.2.2) *holds.*

*Proof.* First, we note that from Theorem 3.1.1 and Remark 3.1.2 we deduce that $u \in C^{0,\delta}_{\mathrm{loc}}(\Omega)$, for some $\delta \in (0,1)$. Now, let $B \subset \Omega$ be a ball. Assumption (A3) implies

$$\begin{aligned} &\fint_B \sup_{\xi \neq 0} \frac{|\mathbf{A}(x,\xi) - \mathbf{A}_B(\xi)|}{(\mu^2 + |\xi|^2)^{\frac{p-1}{2}}} dx \\ &\qquad \leq \fint_B \sup_{\xi \neq 0} \fint_B \frac{|\mathbf{A}(x,\xi) - \mathbf{A}(y,\xi)|}{(\mu^2 + |\xi|^2)^{\frac{p-1}{2}}} dy dx \\ &\qquad \leq \fint_B \fint_B \omega(|x-y| + |u(x) - u(y)|) dy dx \end{aligned}$$

Recalling that $\omega(\rho) = \min\{\rho^\alpha, 1\}$, for some $\alpha \in (0,1)$, we infer

$$\begin{aligned} &\fint_B \sup_{\xi \neq 0} \frac{|\mathbf{A}(x,\xi) - \mathbf{A}_B(\xi)|}{(\mu^2 + |\xi|^2)^{\frac{p-1}{2}}} dx \\ &\qquad \leq C \fint_B \fint_B (|x-y|^\alpha + |u(x) - u(y)|^\alpha) dy dx \\ &\qquad = C \fint_B \fint_B |x-y|^\alpha dy dx + C \fint_B \fint_B |u(x) - u(y)|^\alpha dy dx \\ &\qquad \leq C(n,\alpha)|B|^{\frac{\alpha}{n}} + C \fint_B \fint_B |x-y|^{\alpha\delta} dy dx \\ &\qquad \leq C(n,\alpha)|B|^{\frac{\alpha}{n}} + C(n,\alpha)|B|^{\frac{\alpha\delta}{n}} \end{aligned}$$

where in the last inequality we used the Hölder continuity of the solution $u$. Thus

$$\fint_B \sup_{\xi \neq 0} \frac{|\mathbf{A}(x,\xi) - \mathbf{A}_B(\xi)|}{(\mu^2 + |\xi|^2)^{\frac{p-1}{2}}} dx \to 0,$$

as $|B| \to 0$. □

### 3.2.2 Comparison estimates

We start with the following comparison principle.

**Lemma 3.2.3.** *Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded open set. Suppose that $f, g \in W^{1,p}(\mathcal{O})$ satisfy*

$$\begin{cases} -\mathrm{div}\boldsymbol{A}(x, Df) \leq -\mathrm{div}\boldsymbol{A}(x, Dg) & \text{in } \mathcal{O} \\ f \leq g & \text{on } \partial\mathcal{O}, \end{cases}$$

*in the weak sense that*

$$\int_{\mathcal{O}} \langle \boldsymbol{A}(x, Df) - \boldsymbol{A}(x, Dg), D\varphi \rangle dx \leq 0 \qquad \forall \varphi \in W_0^{1,p}(\mathcal{O}) \text{ with } \varphi \geq 0 \tag{3.2.3}$$

*and $(f - g)_+ \in W_0^{1,p}(\mathcal{O})$. Then, $f \leq g$ a.e. in $\mathcal{O}$.*

*Proof.* Taking $\varphi = (f - g)_+$ as test function in (3.2.3), we obtain

$$\int_{\mathcal{O} \cap \{f > g\}} \langle \mathbf{A}(x, Df) - \mathbf{A}(x, Dg), Df - Dg \rangle dx \leq 0.$$

From the ellipticity assumption (A1), we find that

$$\int_{\mathcal{O} \cap \{f > g\}} |Df - Dg|^2 (\mu^2 + |Df|^2 + |Dg|^2)^{\frac{p-2}{2}} dx = 0$$

which implies $Df = Dg$ a.e. in $\mathcal{O} \cap \{f > g\}$ and therefore $D((f - g)_+) = 0$ a.e. in $\mathcal{O}$. Since $(f - g)_+ \in W_0^{1,p}(\mathcal{O})$, we have that $f \leq g$ a.e. in $\mathcal{O}$. □

We next discuss higher integrability results for homogeneous problems.

**Lemma 3.2.4.** *Let $v \in W^{1,p}(B_R)$ be a weak solution of*

$$\mathrm{div}\boldsymbol{A}(x, Dv) = 0 \text{ in } B_R, \tag{3.2.4}$$

*where $B_R \subset \Omega$. Then, there exists $\sigma \in (0, 1)$ such that*

$$\fint_{B_{\rho/2}} |Dv|^{p(1+\sigma)} dx \leq c \left\{ \left( \fint_{B_\rho} |Dv|^p dx \right)^{1+\sigma} + 1 \right\}$$

*for every ball $B_\rho \subset B_R$, with a constant $c = c(n, p, \nu, L, l)$.*

*Proof.* We consider a ball $B_\rho \subset B_R$ and we test (3.2.4) by the function $\varphi := \eta^p(v - v_\rho)$, where $\eta \in \mathcal{C}_0^\infty(B_\rho)$ is a cut-off function with $\eta = 1$ on $B_{\rho/2}$, $0 \leq \eta \leq 1$ and $|\nabla \eta| \leq \frac{c}{\rho}$. We obtain

$$\begin{aligned} 0 &= \int_{B_{2r}} \langle \mathbf{A}(x, Dv), D\varphi \rangle dx \\ &= \int_{B_{2r}} \eta^p \langle \mathbf{A}(x, Dv), Dv \rangle dx + p \int_{B_{2r}} \eta^{p-1}(v - v_\rho) \langle \mathbf{A}(x, Dv), D\eta \rangle dx \\ &= I + II. \end{aligned}$$

From the ellipticity assumption (A1), it follows

$$\langle \mathbf{A}(x, \xi), \xi \rangle \geq \frac{\nu}{2} |\xi|^p - \tilde{c}$$

for every $\xi \in \mathbb{R}^n$ and so

$$I \geq \frac{\nu}{2} \int_{B_{2r}} \eta^p |Dv|^p dx - \tilde{c}|B_\rho|.$$

Now, we take care of the term $II$. Using assumption (A4), properties of $\eta$ and Young's inequality, we get

$$\begin{aligned} II \leq & \frac{c}{\rho} \int_{B_\rho} \eta^{p-1} |\mathbf{A}(x,\xi)||v - v_\rho| dx \\ \leq & \frac{c}{\rho} \int_{B_\rho} \eta^{p-1}(1 + |Dv|^{p-1})|v - v_\rho| dx \\ \leq & \frac{c}{\rho} \left( \int_{B_\rho} \eta^p (1 + |Dv|^p) dx \right)^{\frac{p-1}{p}} \left( \int_{B_\rho} |v - v_\rho|^p dx \right)^{\frac{1}{p}} \\ \leq & \frac{\nu}{2} \int_{B_\rho} \eta^p (1 + |Dv|^p) dx + \frac{c}{\rho^p} \int_{B_\rho} |v - v_\rho|^p dx. \end{aligned}$$

Joining the preceding estimates and re-absorbing the terms containing $Dv$ in the right-hand side into the left-hand side yield

$$\int_{B_{\rho/2}} |Dv|^p dx \leq \frac{c}{\rho^p} \int_{B_\rho} |v - v_\rho|^p dx + c|B_\rho|.$$

With an application of Sobolev-Poincarè inequality, we find that

$$\begin{aligned} \fint_{B_{\rho/2}} |Dv|^p dx \leq & \frac{c}{\rho^p} \fint_{B_\rho} |v - v_\rho|^p dx + c \\ \leq & c \left\{ \left( \fint_{B_\rho} |Dv|^{\frac{np}{n+p}} dx \right)^{\frac{n+p}{n}} + 1 \right\}. \end{aligned}$$

Now, the higher integrability of $Dv$ follows by Lemma 2.0.4. □

For the proof of next lemma see [78, 79].

**Lemma 3.2.5.** *Let $v \in W^{1,p}(B_R)$ be a weak solution of*

$$\operatorname{div} \boldsymbol{A}_{B_R}(Dv) = 0 \quad \text{in } B_R$$

*where $B_R \subset \Omega$. Then, $Dv \in L^\infty(B_{R/2})$ and the following estimate holds*

$$\|Dv\|_{L^\infty(B_{R/2})} \leq c \left( \fint_{B_R} |Dv|^p dx + 1 \right)$$

*for a constant $c = c(n, p, \nu, L, l)$.*

Fix $B_R \Subset \Omega$ and $y \in B_R$. Let $B_{4r} = B_{4r}(y) \subset \Omega$ be a ball with $r \leq R_0/4$, where $R_0 > 0$ will be determined later. By $k \in u + W_0^{1,p}(B_{4r})$ we denote the weak solution of

$$\begin{cases} \operatorname{div} \mathbf{A}(x, Dk) = \operatorname{div} \mathbf{A}(x, D\psi) & \text{in } B_{4r} \\ k = u & \text{on } \partial B_{4r}, \end{cases} \tag{3.2.5}$$

and $w \in k + W_0^{1,p}(B_{4r})$ the weak solution of

$$\begin{cases} \operatorname{div}\mathbf{A}(x, Dw) = 0 & \text{in } B_{4r} \\ w = k & \text{on } \partial B_{4r}. \end{cases} \tag{3.2.6}$$

We prove the following

**Lemma 3.2.6.** *Let $\varepsilon \in (0,1)$ and $\lambda \geq 1$. There exists $\delta = \delta(n, p, \nu, L, l, \varepsilon) > 0$ such that if*

$$\fint_{B_{4r}} (|Du|^p + |u|^p) dx \leq \lambda \qquad \text{and} \qquad \fint_{B_{4r}} \Psi^p dx \leq \delta\lambda, \tag{3.2.7}$$

*where $\Psi(x) := a(x)^{\frac{1}{p-1}} + |D\psi(x)| + 1$, then*

$$\fint_{B_{4r}} |Dw|^p dx \leq c\lambda \tag{3.2.8}$$

*for a constant $c = c(n, p, \nu, L, l)$ and*

$$\fint_{B_{4r}} |Du - Dw|^p dx \leq \varepsilon\lambda. \tag{3.2.9}$$

*Proof.* Taking $k - u \in W_0^{1,p}(B_{4r})$ as test function in (3.2.5), we have

$$\fint_{B_{4r}} \langle \mathbf{A}(x, Dk), Dk - Du\rangle dx = \fint_{B_{4r}} \langle \mathbf{A}(x, D\psi), Dk - Du\rangle dx$$

and so

$$\fint_{B_{4r}} \langle \mathbf{A}(x, Dk) - \mathbf{A}(x, Du), Dk - Du\rangle dx = \fint_{B_{4r}} \langle \mathbf{A}(x, D\psi) - \mathbf{A}(x, Du), Dk - Du\rangle dx.$$

Using the equality $\operatorname{div}\mathbf{A}(x, Dk) = \operatorname{div}\mathbf{A}(x, D\psi)$ in the ball $B_{4r}$, the ellipticity assumption (A1) and assumption (A2), we get

$$\begin{aligned} \nu \fint_{B_{4r}} |Dk - Du|^p dx \leq & \fint_{B_{4r}} \langle \mathbf{A}(x, Dk) - \mathbf{A}(x, Du), Dk - Du\rangle dx \\ \leq & \fint_{B_{4r}} \langle \mathbf{A}(x, D\psi) - \mathbf{A}(x, Du), Dk - Du\rangle dx \\ \leq & L \fint_{B_{4r}} (1 + |D\psi|^{p-1} + |Du|^{p-1})|Dk - Du| dx \\ \leq & \frac{\nu}{2} \fint_{B_{4r}} |Dk - Du|^p dx + C(p, \nu, L) \fint_{B_{4r}} (1 + |D\psi|^p + |Du|^p) dx, \end{aligned}$$

where in the last line we used Young's inequality. Re-absorbing the first integral in the right-hand side by the left-hand side we obtain

$$\fint_{B_{4r}} |Dk - Du|^p dx \leq C(p, \nu, L) \fint_{B_{4r}} (1 + |D\psi|^p + |Du|^p) dx$$

which by virtue of assumption (3.2.7) yields

$$\fint_{B_{4r}} |Dk - Du|^p dx \le c\lambda \tag{3.2.10}$$

with a constant $c := c(p, \nu, L, \varepsilon)$. On the other hand, taking $w - k \in W_0^{1,p}(B_{4r})$ as test function in (3.2.6), we have

$$\fint_{B_{4r}} \langle \mathbf{A}(x, Dw), Dw - Dk\rangle dx = 0$$

that leads to

$$\fint_{B_{4r}} \langle \mathbf{A}(x, Dw) - \mathbf{A}(x, Dk), Dw - Dk\rangle dx = -\fint_{B_{4r}} \langle \mathbf{A}(x, Dk), Dw - Dk\rangle dx.$$

By using the ellipticity assumption (A1) in the left-hand side and assumption (A4) in the right-hand side, we get

$$\begin{aligned} \nu \fint_{B_{4r}} |Dw - Dk|^p dx \le & \fint_{B_{4r}} \langle \mathbf{A}(x, Dw) - \mathbf{A}(x, Dk), Dw - Dk\rangle dx \\ \le & l \fint_{B_{4r}} (1 + |Dk|^2)^{\frac{p-1}{2}} |Dw - Dk| dx \\ \le & \frac{\nu}{2} \fint_{B_{4r}} |Dw - Dk|^p dx + C(p, \nu, l) \fint_{B_{4r}} (1 + |Dk|^p) dx \end{aligned}$$

which yields

$$\fint_{B_{4r}} |Dw - Dk|^p dx \le C(p, \nu, l) \fint_{B_{4r}} (1 + |Dk|^p) dx.$$

Thanks to (3.2.7) and (3.2.10), we infer

$$\fint_{B_{4r}} |Dw|^p dx \le c\lambda,$$

for a constant $c := (p, \nu, L, l, \varepsilon)$.

In view of Lemma 3.2.3 and (3.2.5), we see that $k \ge \psi$ a.e. in $B_{4r}$. We extend $k$ by $u$ in $\Omega \setminus B_{4r}$, hence, we have $k \in \mathcal{K}_\psi(\Omega)$. Therefore we can take $\varphi = k$ in (3.0.1), thus obtaining

$$\int_{B_{4r}} \langle \mathbf{A}(x, Du), D(k - u)\rangle dx \ge \int_{B_{4r}} \mathcal{B}(x, u, Du)(k - u) dx. \tag{3.2.11}$$

Taking $k - u \in W_0^{1,p}(B_{4r})$ as test function in (3.2.5), we also get

$$\int_{B_{4r}} \langle \mathbf{A}(x, Dk), D(k - u)\rangle dx = \int_{B_{4r}} \langle \mathbf{A}(x, D\psi), D(k - u)\rangle dx. \tag{3.2.12}$$

We then subtract (3.2.11) from (3.2.12) to find that

$$\fint_{B_{4r}} \langle \mathbf{A}(x, Dk) - \mathbf{A}(x, Du), D(k - u)\rangle dx$$

$$\leq \fint_{B_{4r}} \langle \mathbf{A}(x, D\psi), D(k-u)\rangle dx - \fint_{B_{4r}} \mathcal{B}(x,u,Du)(k-u)dx. \tag{3.2.13}$$

We first estimate the left-hand side of (3.2.13). Since $p \geq 2$, the ellipticity assumption (A1) implies

$$\nu \fint_{B_{4r}} |Dk - Du|^p dx \leq \fint_{B_{4r}} \langle \mathbf{A}(x, Dk) - \mathbf{A}(x, Du), D(k-u)\rangle dx. \tag{3.2.14}$$

Next, we estimate the right-hand side of (3.2.13). By assumptions (A4) and (B1), from Young's inequality and Sobolev-Poincaré inequality, denoting by $c_0$ the Poincarè constant, we have that for any positive numbers $\eta$ and $\theta$,

$$\begin{aligned}
&\fint_{B_{4r}} \langle \mathbf{A}(x, D\psi), D(k-u)\rangle dx - \fint_{B_{4r}} \mathcal{B}(x,u,Du)(k-u)dx \\
&\quad\leq l \fint_{B_{4r}} (1 + |D\psi|^{p-1})|Dk - Du| dx + \fint_{B_{4r}} (|Du|^r + |u|^r + a)|k-u| dx \\
&\quad\leq \eta \fint_{B_{4r}} |Dk - Du|^p dx + C(\eta, l, p) \fint_{B_{4r}} (1 + |D\psi|^p) dx \\
&\qquad + \frac{\eta}{c_0} \fint_{B_{4r}} |k-u|^p dx + C(\eta, c_0, p) \fint_{B_{4r}} (|Du|^{\frac{rp}{p-1}} + |u|^{\frac{rp}{p-1}} + a^{\frac{p}{p-1}}) dx \\
&\quad\leq \eta \fint_{B_{4r}} |Dk - Du|^p dx + C(\eta, l, c_0, p) \fint_{B_{4r}} (1 + |D\psi|^p + a^{\frac{p}{p-1}}) dx \\
&\qquad + \eta \fint_{B_{4r}} |Dk - Du|^p dx + C(\eta, c_0, p) \fint_{B_{4r}} (|Du|^{\frac{rp}{p-1}} + |u|^{\frac{rp}{p-1}}) dx \\
&\quad\leq 2\eta \fint_{B_{4r}} |Dk - Du|^p dx + C(\eta, l, c_0, p) \fint_{B_{4r}} \Psi^p dx \\
&\qquad + \theta \fint_{B_{4r}} (|Du|^p + |u|^p) dx + C(\theta, \eta, c_0, p) \\
&\quad\leq 2\eta \fint_{B_{4r}} |Dk - Du|^p dx + C(\eta, l, c_0, p)\delta\lambda + \theta\lambda,
\end{aligned} \tag{3.2.15}$$

where in the last line we used (3.2.7). Combining (3.2.13), (3.2.14) and (3.2.15), we derive that

$$\fint_{B_{4r}} |Dk - Du|^p dx \leq 2\eta \fint_{B_{4r}} |Dk - Du|^p dx + C(\eta, l, n, p)\delta\lambda + \theta\lambda.$$

Choosing

$$\eta = \frac{1}{4}, \qquad \theta = \frac{\varepsilon}{4} \qquad \text{and} \qquad \delta \leq \frac{\varepsilon}{4C(\eta, l, n, p)},$$

we get

$$\fint_{B_{4r}} |Dk - Du|^p dx \leq \varepsilon\lambda. \tag{3.2.16}$$

On the other hand, by taking $w - k \in W_0^{1,p}(B_{4r})$ as test function in (3.2.5) and (3.2.6), we have

$$\fint_{B_{4r}} \langle \mathbf{A}(x, Dw) - \mathbf{A}(x, Dk), D(w-k)\rangle dx = \fint_{B_{4r}} \langle \mathbf{A}(x, D\psi), D(k-w)\rangle dx.$$

In a similar way of estimating (3.2.16), we can find $\delta = \delta(n, \nu, L, l, p, \varepsilon) > 0$ such that

$$\fint_{B_{4r}} |Dw - Dk|^p dx \leq \varepsilon\lambda.$$

This completes the proof of the lemma. $\square$

Denote by $v \in w + W_0^{1,p}(B_{2r})$ the unique weak solution of

$$\begin{cases} \operatorname{div}\mathbf{A}_{B_{2r}}(Dv) = 0 & \text{in } B_{2r} \\ v = w & \text{on } \partial B_{2r}, \end{cases} \tag{3.2.17}$$

where $w \in k + W_0^{1,p}(B_{4r})$ is the weak solution of (3.2.6).

**Lemma 3.2.7.** *Let $\varepsilon \in (0,1)$ and $\lambda \geq 1$. There exists a radius $R_0 = R_0(n, p, \nu, L, l, \varepsilon) \in (0, R)$ such that if*

$$\fint_{B_{2r}} |Dw|^p dx \leq \lambda \tag{3.2.18}$$

*for some $r \leq R_0/4$, then*

$$\fint_{B_{2r}} |Dv|^p dx \leq c\lambda$$

*for a constant $c = c(n, p, \nu, L, l)$ and*

$$\fint_{B_{2r}} |Dw - Dv|^p dx \leq \varepsilon\lambda.$$

*Proof.* We test (3.2.6) and (3.2.17) by $v - w \in W_0^{1,p}(B_{2r})$ to see that

$$\begin{aligned} I :=& \fint_{B_{2r}} \langle \mathbf{A}_{B_{2r}}(Dv) - \mathbf{A}_{B_{2r}}(Dw), Dv - Dw\rangle dx \\ =& \fint_{B_{2r}} \langle \mathbf{A}(x, Dw) - \mathbf{A}_{B_{2r}}(Dw), Dv - Dw\rangle dx =: J. \end{aligned} \tag{3.2.19}$$

We estimate from below the term $I$ using the ellipticity assumption (A1). We have

$$I \geq \nu \fint_{B_{2r}} |Dv - Dw|^p dx. \tag{3.2.20}$$

Now, we take care of the term $J$. Recalling the definition of the function $H$ in (3.2.1), using Young's inequality and (3.2.18), we infer

$$\begin{aligned} J \leq& \fint_{B_{2r}} |\mathbf{A}(x, Dw) - \mathbf{A}_{B_{2r}}(Dw)||Dv - Dw| dx \\ \leq& \fint_{B_{2r}} H(x, B_{2r})(1 + |Dw|^{p-1})|Dv - Dw| dx \\ \leq& \frac{\nu}{2} \fint_{B_{2r}} |Dv - Dw|^p dx + c(p, \nu) \fint_{B_{2r}} H(x, B_{2r})^{\frac{p}{p-1}}(1 + |Dw|^p) dx. \end{aligned}$$

Then Hölder's inequality and Lemma 3.2.4 applied to the function $w$ imply

$$\begin{aligned} c \fint_{B_{2r}} & H(x, B_{2r})^{\frac{p}{p-1}} (1 + |Dw|^p) dx \\ & \leq c \biggl( \fint_{B_{2r}} H(x, B_{2r})^{\frac{p}{p-1} \frac{1+\sigma}{\sigma}} dx \biggr)^{\frac{\sigma}{1+\sigma}} \biggl( \fint_{B_{2r}} (1 + |Dw|^{p(1+\sigma)}) dx \biggr)^{\frac{1}{1+\sigma}} \\ & \leq c \biggl( \fint_{B_{2r}} H(x, B_{2r})^{\frac{p}{p-1} \frac{1+\sigma}{\sigma}} dx \biggr)^{\frac{\sigma}{1+\sigma}} \biggl( \fint_{B_{2r}} (1 + |Dw|^{p}) dx \biggr) \\ & \leq c\lambda \biggl( \fint_{B_{2r}} H(x, B_{2r})^{\frac{p}{p-1} \frac{1+\sigma}{\sigma}} dx \biggr)^{\frac{\sigma}{1+\sigma}} \end{aligned}$$

for some $\sigma \in (0, 1)$, with $c := c(n, p, \nu, L, l)$. From Remark 3.2.1 it follows that there exists $0 < R_0 < R$ sufficiently small so that

$$\biggl( \fint_{B_{2r}} H(x, B_{2r})^{\frac{p}{p-1} \frac{1+\sigma}{\sigma}} dx \biggr)^{\frac{\sigma}{1+\sigma}} \leq \frac{\varepsilon \nu}{2c}$$

for all $r \leq R_0/4$. Thus, we can conclude that

$$J \leq \frac{\nu}{2} \fint_{B_{2r}} |Dv - Dw|^p dx + \frac{\varepsilon \lambda \nu}{2}. \tag{3.2.21}$$

Inserting (3.2.20) and (3.2.21) in (3.2.19) and re-absorbing the first integral in the right-hand side by the left-hand side, we obtain

$$\fint_{B_{2r}} |Dv - Dw|^p \leq \varepsilon \lambda,$$

which, together with (3.2.18), yields

$$\fint_{B_{2r}} |Dv|^p \leq c\lambda. \qquad \square$$

Combining Lemmas 3.2.5, 3.2.6 and 3.2.7, we finally obtain the following comparison estimates.

**Lemma 3.2.8.** *Let $\varepsilon \in (0, 1)$ and $\lambda \geq 1$. There exist a positive constant $\delta = \delta(n, p, \nu, L, l, \varepsilon)$ and a radius $R_0 = R_0(n, p, \nu, L, l, \varepsilon) \in (0, R)$ such that if*

$$\fint_{B_{4r}} (|Du|^p + |u|^p) dx \leq \lambda \qquad \textit{and} \qquad \fint_{B_{4r}} \Psi^p dx \leq \delta \lambda,$$

*for some ball $B_{4r} \subset \Omega$ with $r \leq R_0/4$, then there exist $w \in W^{1,p}(B_r)$ and $v \in W^{1,\infty}(B_r)$ such that*

$$\fint_{B_r} |Du - Dw|^p dx \leq \varepsilon \lambda, \quad \fint_{B_r} |Dw - Dv|^p dx \leq \varepsilon \lambda \quad \textit{and} \quad \|Dv\|^p_{L^\infty(B_r)} \leq c_1 \lambda,$$

*for a constant $c_1 = c_1(n, p, \nu, L, l, \varepsilon)$, where $B_r \Subset B_{4r}$ are concentric balls.*

### 3.2.3 Proof of Theorem 3.0.1

In this subsection we will derive the gradient local estimate (3.0.2). We start by showing that hypotheses of Lemma 3.2.8 are satisfied.

Let $B_R \Subset \Omega$ and $0 < R_0 \le \min\{R, \operatorname{dist}(\partial\Omega, \partial B_R)\}$ that will be determined later. For the weak solution $u \in W^{1,p}(\Omega)$ of (3.0.1) and for the function $\Psi(x) := a(x)^{\frac{1}{p-1}} + |D\psi(x)| + 1$, we define

$$E_R(|Du|^p, \lambda) := \{x \in B_R : |Du(x)|^p + |u(x)|^p > \lambda\}$$

and

$$E_R(\Psi^p, \lambda) := \{x \in B_R : \Psi(x)^p > \lambda\}$$

for any $\lambda > 0$. Moreover, we set

$$\lambda_0 := \fint_{B_R} \left( |Du|^p + |u|^p + \frac{1}{\delta}\Psi^p \right) dx, \tag{3.2.22}$$

where $\delta > 0$ will be determined later. For any $y \in B_R$, we define a continuous map $G(y, \cdot) : (0, R_0] \to [0, \infty)$ by

$$G(y,r) := \fint_{B_r(y)} \left( |Du|^p + |u|^p + \frac{1}{\delta}\Psi^p \right) dx.$$

Then, for almost every $y \in E_R(|Du|^p, \lambda)$, it follows from the Lebesgue's Theorem that

$$\lim_{r\to 0} G(y,r) = |Du(y)|^p + |u(y)|^p + \frac{1}{\delta}\Psi(y)^p > \lambda. \tag{3.2.23}$$

On the other hand, for any $r \in [\frac{R}{\sigma^{1/n}}, R_0]$, with $\sigma \ge (R/R_0)^n$, we compute

$$G(y,r) \le \frac{|B_R|}{|B_r(y)|} \fint_{B_R} \left( |Du|^p + |u|^p + \frac{1}{\delta}\Psi^p \right) dx = \frac{R^n}{r^n}\lambda_0 \le \sigma\lambda_0 < \lambda \tag{3.2.24}$$

for $\lambda$ chosen sufficiently large such that

$$\lambda > \lambda_1 := \sigma\lambda_0. \tag{3.2.25}$$

Now fix any $\lambda > \lambda_1$. Since $G(y, \cdot)$ is a continuous map, estimates (3.2.23) and (3.2.24) imply that for almost every $y \in E_R(|Du|^p, \lambda)$, there exists a number $r_y \in (0, \frac{R}{\sigma^{1/n}})$ such that

$$G(y, r_y) = \lambda \quad \text{and} \quad G(y,r) < \lambda \quad \text{for all } r \in (r_y, R_0].$$

In view of Vitali covering Lemma, there is a countable family of mutually disjoint balls $\{B_{r_i}(y^i)\}_{i\in\mathbb{N}}$, with $y^i \in E_R(|Du|^p, \lambda)$ and $r_i \in (0, \frac{R}{\sigma^{1/n}})$, and negligible set $N \subset \mathbb{R}^n$ such that

$$E_R(|Du|^p, \lambda) \subset \bigcup_{i\ge 1} B_{5r_i}(y^i) \cup N, \tag{3.2.26}$$

$$\fint_{B_{r_i}(y^i)} \left( |Du|^p + |u|^p + \frac{1}{\delta}\Psi^p \right) dx = \lambda, \tag{3.2.27}$$

$$\fint_{B_r(y^i)} \left( |Du|^p + |u|^p + \frac{1}{\delta}\Psi^p \right) dx < \lambda \quad \text{for each } r \in (r_i, R_0]. \tag{3.2.28}$$

Now let us fix $i \in \mathbb{N}$. From (3.2.27) we have

$$|B_{r_i}(y^i)| = \frac{1}{\lambda}\int_{B_{r_i}(y^i)} (|Du|^p + |u|^p)dx + \frac{1}{\lambda}\int_{B_{r_i}(y^i)} \frac{\Psi^p}{\delta}dx =: \frac{1}{\lambda}\mathbf{I} + \frac{1}{\delta\lambda}\mathbf{II}, \tag{3.2.29}$$

with the obvious meaning of $\mathbf{I}, \mathbf{II}$. For $\mathbf{I}$ we have

$$\begin{aligned}\frac{1}{\lambda}\mathbf{I} =& \frac{1}{\lambda}\bigg(\int_{B_{r_i}(y^i)\cap\{|Du|^p+|u|^p\leq\frac{\lambda}{4}\}} (|Du|^p + |u|^p)dx + \int_{B_{r_i}(y^i)\cap\{|Du|^p+|u|^p>\frac{\lambda}{4}\}} (|Du|^p + |u|^p)dx\bigg)\\ \leq& \frac{|B_{r_i}(y^i)|}{4} + \frac{1}{\lambda}\int_{B_{r_i}(y^i)\cap E_R(|Du|^p,\frac{\lambda}{4})} (|Du|^p + |u|^p)dx.\end{aligned} \tag{3.2.30}$$

In a similar way, we get the following estimate for $\mathbf{II}$

$$\begin{aligned}\frac{1}{\delta\lambda}\mathbf{II} =& \frac{1}{\lambda}\bigg(\int_{B_{r_i}(y^i)\cap\{\Psi^p\leq\frac{\delta\lambda}{4}\}} \Psi^p dx + \int_{B_{r_i}(y^i)\cap\{\Psi^p>\frac{\delta\lambda}{4}\}} \Psi^p dx\bigg)\\ \leq& \frac{|B_{r_i}(y^i)|}{4} + \frac{1}{\delta\lambda}\int_{B_{r_i}(y^i)\cap E_R(\Psi^p,\frac{\delta\lambda}{4})} \Psi^p dx.\end{aligned} \tag{3.2.31}$$

Inserting estimates (3.2.30) and (3.2.31) in (3.2.29) and re-absorbing the term $|B_{r_i}(y_i)|$ from the right into the left we find that

$$|B_{r_i}(y^i)| \leq \frac{2}{\lambda}\bigg(\int_{B_{r_i}(y^i)\cap E_R(|Du|^p,\frac{\lambda}{4})} (|Du|^p + |u|^p)dx + \frac{1}{\delta}\int_{B_{r_i}(y^i)\cap E_R(\Psi^p,\frac{\delta\lambda}{4})} \Psi^p dx\bigg). \tag{3.2.32}$$

Now, choosing $\sigma \geq (20R/R_0)^n$, we have that $B_{20r_i}(y^i) \subset \Omega$, with $5r_i \leq R_0/4$. Hence, from inequality (3.2.28) we derive that

$$\fint_{B_{20r_i}(y^i)} (|Du|^p + |u|^p)dx \leq \lambda \quad \text{and} \quad \fint_{B_{20r_i}(y^i)} \Psi^p dx \leq \delta\lambda, \tag{3.2.33}$$

for every $\lambda > \lambda_1$, where $\lambda_1$ was defined in (3.2.25).

Let $\varepsilon \in (0,1)$. According to Lemma 3.2.8, there exist a positive constant $\delta = \delta(n,p,\nu,L,l,\varepsilon)$ and a sufficiently small radius $R_0 = R_0(n,p,\nu,L,l,\varepsilon)$ such that (3.2.33) implies the existence of two functions $w_i \in W^{1,p}(B_{5r_i}(y^i))$ and $v_i \in W^{1,\infty}(B_{5r_i}(y^i))$ satisfying

$$\fint_{B_{5r_i}(y^i)} |Du - Dw_i|^p dx \leq \varepsilon\lambda, \quad \fint_{B_{5r_i}(y^i)} |Dw_i - Dv_i|^p dx \leq \varepsilon\lambda \tag{3.2.34}$$

and

$$\|Dv_i\|^p_{L^\infty(B_{5r_i}(y^i))} \leq c_1\lambda, \tag{3.2.35}$$

for a positive constant $c_1 = c_1(n,p,\nu,L,l,\varepsilon)$ which is independent of $i$ and $\lambda$.

Now we are in the position to prove Theorem 3.0.1.

*Proof of Theorem 3.0.1.* Let $c_2 = 2 \cdot 4^{p-1}(c_1 + 2) > 1$, where $c_1$ is given in (3.2.35). For each $\lambda \geq \lambda_1$, we recall the covering $\{B_{r_i}(y^i)\}$ satisfying (3.2.26)–(3.2.28). We define

$$\tilde{E}_R(|Du|^p, \lambda) := \{x \in B_R : |Du(x)|^p > \lambda\}.$$

Since we have $\tilde{E}_R(|Du|^p, c_2\lambda) \subset E_R(|Du|^p, c_2\lambda) \subset E_R(|Du|^p, \lambda)$, we can conclude that

$$\int_{\tilde{E}_R(|Du|^p, c_2\lambda)} |Du|^p dx \leq \sum_{i=1}^{\infty} \Big( \int_{\tilde{E}_R(|Du|^p, c_2\lambda) \cap B_{5r_i}(y^i)} |Du|^p dx \Big). \tag{3.2.36}$$

We use the definition of $c_2$ and (3.2.35), to find that for almost every $x \in \tilde{E}_R(|Du|^p, c_2\lambda) \cap B_{5r_i}(y^i)$ it holds

$$\begin{aligned} |Du|^p \leq & 4^{p-1}(|Du - Dw_i|^p + |Dw_i - Dv_i| + |Dv_i|^p) \\ \leq & 4^{p-1}(|Du - Dw_i|^p + |Dw_i - Dv_i|^p + |Dv_i|^p + 2) \\ \leq & 4^{p-1}(|Du - Dw_i|^p + |Dw_i - Dv_i|^p + (c_1 + 2)\lambda) \\ \leq & 4^{p-1}(|Du - Dw_i|^p + |Dw_i - Dv_i|^p) + \frac{|Du|^p}{2}. \end{aligned}$$

Then, from (3.2.34) we have

$$\begin{aligned} \int_{\tilde{E}_R(|Du|^p, c_2\lambda) \cap B_{5r_i}(y^i)} |Du|^p dx \leq & 2 \cdot 4^{p-1} \Big( \int_{B_{5r_i}(y^i)} |Du - Dw_i|^p dx + \int_{B_{5r_i}(y^i)} |Dw_i - Dv_i|^p dx \Big) \\ \leq & 4 \cdot 4^{p-1} 5^n |B_{r_i}(y^i)| \varepsilon \lambda. \end{aligned} \tag{3.2.37}$$

Therefore, combining (3.2.37) and (3.2.32), we have for every $i \in \mathbb{N}$

$$\begin{aligned} & \int_{\tilde{E}_R(|Du|^p, c_2\lambda) \cap B_{5r_i}(y^i)} |Du|^p dx \\ & \leq c_3 \varepsilon \Big( \int_{B_{r_i}(y^i) \cap E_R(|Du|^p, \frac{\lambda}{4})} |Du|^p dx + \frac{1}{\delta} \int_{B_{r_i}(y^i) \cap E_R(\Psi^p, \frac{\delta\lambda}{4})} \Psi^p dx \Big), \end{aligned} \tag{3.2.38}$$

where $c_3 = 8 \cdot 4^{p-1} 5^n$. Since $\{B_{r_i}(y^i)\}$ are mutually disjoints, summing on $i \in \mathbb{N}$, it follows from (3.2.36) and (3.2.38) that

$$\int_{\tilde{E}_R(|Du|^p, c_2\lambda)} |Du|^p dx \leq c_3 \varepsilon \Big( \int_{E_R(|Du|^p, \frac{\lambda}{4})} |Du|^p dx + \frac{1}{\delta} \int_{E_R(\Psi^p, \frac{\delta\lambda}{4})} \Psi^p dx \Big) \tag{3.2.39}$$

for any $\lambda > \lambda_1$.
Now, we introduce the cutting operator

$$T_k(\sigma) := \min\{k, \sigma\}$$

for every $k, \sigma \in \mathbb{R}$ and we define for every $\tilde{\lambda} > 0$

$$F_k(|Du|^p, \tilde{\lambda}) := \{x \in B_R : T_k(|Du(x)|^p + |u(x)|^p) > \tilde{\lambda}\}$$

and

$$\tilde{F}_k(|Du|^p, \tilde{\lambda}) := \{x \in B_R : T_k(|Du(x)|^p) > \tilde{\lambda}\}.$$

Notice that (3.2.39) implies

$$\int_{\tilde{F}_k(|Du|^p, c_2\lambda)} |Du|^p dx \leq c_3\varepsilon\Big(\int_{F_k(|Du|^p, \frac{\lambda}{4})} |Du|^p dx + \frac{1}{\delta}\int_{F_k(\Psi^p, \frac{\delta\lambda}{4})} \Psi^p dx\Big). \tag{3.2.40}$$

Multiplying (3.2.40) for $\lambda^{q-2}$ and integrating with respect to $\lambda$ for $\lambda \in (\lambda_1, +\infty)$, we get

$$\begin{aligned}\int_{\lambda_1}^{+\infty} \lambda^{q-2}\Big(\int_{\tilde{F}_k(|Du|^p, c_2\lambda)} |Du|^p dx\Big) d\lambda \leq & c_2\varepsilon \int_{\lambda_1}^{+\infty} \lambda^{q-2}\Big(\int_{F_k(|Du|^p, \frac{\lambda}{4})} (|Du|^p + |u|^p) dx\Big) d\lambda \\ & + c_2\varepsilon \int_{\lambda_1}^{+\infty} \lambda^{q-2}\Big(\int_{F_k(\Psi^p, \frac{\delta\lambda}{4})} \frac{\Psi^p}{\delta} dx\Big) d\lambda. \end{aligned} \tag{3.2.41}$$

Now we estimate the left hand side of (3.2.41). By using Fubini Theorem, we obtain

$$\begin{aligned}
&\int_{\lambda_1}^{+\infty} \lambda^{q-2}\Big(\int_{\tilde{F}_k(|Du|^p, c_2\lambda)} |Du|^p dx\Big) d\lambda \\
&= \int_{\tilde{F}_k(|Du|^p, c_2\lambda)} |Du|^p \Big(\int_{\lambda_1}^{\frac{T_k(|Du|^p)}{c_2}} \lambda^{q-2} d\lambda\Big) dx \\
&= \frac{1}{q-1}\int_{\tilde{F}_k(|Du|^p, c_2\lambda)} |Du|^p \Big\{\frac{[T_k(|Du|^p)]^{q-1}}{c_2^{q-1}} - \lambda_1^{q-1}\Big\} dx \\
&= \frac{1}{q-1}\int_{B_R} \frac{|Du|^p [T_k(|Du|^p)]^{q-1}}{c_2^{q-1}} dx - \frac{1}{q-1}\int_{\{T_k(|Du|^p) \leq c_2\lambda\}} \frac{|Du|^p [T_k(|Du|^p)]^{q-1}}{c_2^{q-1}} dx \\
&\quad - \frac{1}{q-1}\int_{B_R} \lambda_1^{q-1} |Du|^p dx + \frac{1}{q-1}\int_{\{T_k(|Du|^p) \leq c_2\lambda\}} \lambda_1^{q-1} |Du|^p dx \\
&\geq \frac{1}{q-1}\int_{B_R} \frac{|Du|^p [T_k(|Du|^p)]^{q-1}}{c_2^{q-1}} dx - \frac{1}{q-1}\int_{\{T_k(|Du|^p) \leq c_2\lambda\}} \frac{|Du|^p c_2^{q-1}\lambda_1^{q-1}}{c_2^{q-1}} dx \\
&\quad - \frac{1}{q-1}\int_{B_R} \lambda_1^{q-1} |Du|^p dx + \frac{1}{q-1}\int_{\{T_k(|Du|^p) \leq c_2\lambda\}} \lambda_1^{q-1} |Du|^p dx \\
&= \frac{1}{q-1}\int_{B_R} \frac{|Du|^p [T_k(|Du|^p)]^{q-1}}{c_2^{q-1}} dx - \frac{1}{q-1}\int_{B_R} \lambda_1^{q-1} |Du|^p dx.
\end{aligned} \tag{3.2.42}$$

Similarly, we estimate the integrals in the right hand side of (3.2.41). Thus we get

$$\begin{aligned} & c_2\varepsilon \int_{\lambda_1}^{+\infty} \lambda^{q-2}\Big(\int_{F_k(|Du|^p, \frac{\lambda}{4})} (|Du|^p + |u|^p) dx\Big) d\lambda \\ & \quad \leq c_2\varepsilon \frac{4^{q-1}}{q-1}\int_{B_R} (|Du|^p + |u|^p)[T_k(|Du|^p + |u|^p)]^{q-1} dx \end{aligned} \tag{3.2.43}$$

and

$$c_2\varepsilon \int_{\lambda_1}^{+\infty} \lambda^{q-2}\Big(\int_{F_k(\Psi^p, \frac{\delta\lambda}{4})} \frac{\Psi^p}{\delta} dx\Big) d\lambda \leq c_2\varepsilon \frac{4^{q-1}}{\delta^{q-1}}\int_{B_R} \Psi^{pq} dx. \tag{3.2.44}$$

Consider $s, t \geq 0$. If $s \leq t$, then $T_k(s) \leq T_k(t)$. Moreover, we have

$$(s+t)T_k(s+t) \leq \begin{cases} 2sT_k(2s) & \text{if } t \leq s \\ 2tT_k(2t) & \text{if } s \leq t, \end{cases}$$

that implies

$$(s+t)T_k(s+t) \leq 4(sT_k(s) + tT_k(t)). \tag{3.2.45}$$

By virtue of (3.2.45), inserting (3.2.42), (3.2.43) and (3.2.44) in (3.2.41) we obtain

$$\begin{aligned} \frac{1}{q-1} \int_{B_R} & \frac{|Du|^p [T_k(|Du|^p)]^{q-1}}{c_2^{q-1}} dx \\ \leq & \frac{1}{q-1} \int_{B_R} \lambda_1^{q-1} |Du|^p dx + c_2 \varepsilon \frac{4^{q-1}}{q-1} \int_{B_R} |Du|^p [T_k(|Du|^p)]^{q-1} dx \\ & + c_2 \varepsilon \frac{4^{q-1}}{q-1} \int_{B_R} |u|^p [T_k(|u|^p)]^{q-1} dx + c_2 \varepsilon \frac{4^{q-1}}{\delta^{q-1}} \int_{B_R} \Psi^{pq} dx. \end{aligned}$$

For $\varepsilon \in (0,1)$ sufficiently small, we can re-absorb the term containing $|Du|^p[T_k(|Du|^p)]^{q-1}$ from the right into the left thus getting

$$\int_{B_R} |Du|^p [T_k(|Du|^p)]^{q-1} dx \leq C \int_{B_R} (|Du|^p + |u|^p [T_k(|u|^p)]^{q-1} + \Psi^{pq}) dx,$$

for a constant $C := C(n,p,q,\nu,L,l)$. Eventually, by letting $k \to +\infty$, we derive

$$\int_{B_R} |Du|^{pq} dx \leq C \int_{B_R} (|Du|^p + |u|^{pq} + \Psi^{pq}) dx.$$

This completes the proof of Theorem 3.0.1. □

## 3.3 Higher differentiability

This section is devoted to the proof of Theorem 3.0.2. The main point is the gradient local estimate proved in Theorem 3.0.1 that ensures that the gradient of the solution to (3.0.1) is as integrable as the gradient of the obstacle.

Before proceding with the proof, it is convenient to fix some further notation. We set $p' = \frac{p}{p-1}$ the Hölder conjugate exponent of $p$. For a ball $\mathcal{B} \Subset \Omega$ of radius $R$, we shall denote by $\mathcal{Q}_1 = \mathcal{Q}_1(\mathcal{B})$ and $\mathcal{Q}_2 = \mathcal{Q}_2(\mathcal{B})$ the largest and the smallest cubes, concentric to $\mathcal{B}$ and with sides parallel to the coordinate axes, contained in $\mathcal{B}$ and containing $\mathcal{B}$ respectively. Clearly $|\mathcal{B}| \approx |\mathcal{Q}_1| \approx |\mathcal{Q}_2| \approx R^n$. We also denote the enlarged ball by $\hat{\mathcal{B}} = 4\mathcal{B}$. We set

$$\mathcal{Q}_1 = \mathcal{Q}_1(\mathcal{B}), \quad \hat{\mathcal{Q}}_2 = \mathcal{Q}_2(\hat{\mathcal{B}})$$

so that we have the following chain of inclusions

$$\mathcal{Q}_1 \subset \mathcal{B} \Subset 2\mathcal{B} \Subset \mathcal{Q}_1(\hat{\mathcal{B}}) \subset \hat{\mathcal{B}} \subset \hat{\mathcal{Q}}_2.$$

In the sequel, we shall always take $\mathcal{B}$ such that $\mathcal{Q}_2(\hat{\mathcal{B}}) \Subset \Omega$.

*Proof of Theorem 3.0.2.* Let $u \in W^{1,p}(\Omega)$ be the weak solution to (3.0.1). From Lemma 2.2.2 we have $D\psi \in B^{\beta}_{p,s,\mathrm{loc}}(\Omega) \hookrightarrow L^{\frac{np}{n-\beta p}}_{\mathrm{loc}}(\Omega)$; consequently an application of Theorem 3.0.1 with $q = \frac{n}{n-\beta p}$ assures that $Du \in L^{\frac{np}{n-\beta p}}_{\mathrm{loc}}(\Omega)$.

Let us fix a ball $B_R$ such that $B_{2R} \Subset \Omega$ and a cut-off function $\eta \in \mathcal{C}^\infty_0(B_R)$, $\eta \equiv 1$ on $B_{R/2}$ such that $|\nabla\eta| \le \frac{c}{R}$. Without loss of generality, we suppose $R \le 1$. Then, for $|h| < \frac{R}{4}$, we consider

$$\varphi_1(x) = u(x) + t\eta^2(x)\tau_h(u-\psi)(x)$$

and

$$\varphi_2(x) = u(x) + t\eta^2(x-h)\tau_{-h}(u-\psi)(x)$$

which belong to the admissible class $\mathcal{K}_\psi(\Omega)$ for all $t \in [0,1)$. Using $\varphi_1$ and $\varphi_2$ as test functions in (3.0.1), we obtain

$$\begin{aligned}\int_\Omega \langle \mathcal{A}(x,u,Du), D(\eta^2\tau_h(u-\psi))\rangle dx + \int_\Omega \langle \mathcal{A}(x,u,Du), D(\eta^2(x-h)\tau_{-h}(u-\psi))\rangle dx \\ \ge \int_\Omega \mathcal{B}(x,u(x),Du(x))[\eta^2(x)\tau_h(u-\psi) + \eta^2(x-h)\tau_{-h}(u-\psi)]dx.\end{aligned}$$

By means of a simple change of variable in the second integral in the left hand side of the previous inequality, we find that

$$\begin{aligned} I :=& \int_\Omega \langle \mathcal{A}(x+h,u(x+h),Du(x+h)) - \mathcal{A}(x,u(x),Du(x)), D(\eta^2\tau_h(u-\psi))\rangle dx \\ \le& -\int_\Omega \mathcal{B}(x,u(x),Du(x))[\eta^2(x)\tau_h(u-\psi) + \eta^2(x-h)\tau_{-h}(u-\psi)]dx \\ =& \int_\Omega \mathcal{B}(x,u(x),Du(x))\tau_{-h}(\eta^2\tau_h(u-\psi))dx =: J.\end{aligned}$$

We can write the integral $I$ as follows

$$\begin{aligned} I =& \int_\Omega \langle \mathcal{A}(x+h,u(x+h),Du(x+h)) - \mathcal{A}(x+h,u(x+h),Du(x)), \eta^2 D\tau_h u\rangle dx \\ &- \int_\Omega \langle \mathcal{A}(x+h,u(x+h),Du(x+h)) - \mathcal{A}(x+h,u(x+h),Du(x)), \eta^2 D\tau_h \psi\rangle dx \\ &+ \int_\Omega \langle \mathcal{A}(x+h,u(x+h),Du(x+h)) - \mathcal{A}(x+h,u(x+h),Du(x)), 2\eta D\eta\tau_h(u-\psi)\rangle dx \\ &+ \int_\Omega \langle \mathcal{A}(x+h,u(x+h),Du(x)) - \mathcal{A}(x,u(x),Du(x)), \eta^2 D\tau_h u\rangle dx \\ &- \int_\Omega \langle \mathcal{A}(x+h,u(x+h),Du(x)) - \mathcal{A}(x,u(x),Du(x)), \eta^2 D\tau_h \psi\rangle dx \\ &+ \int_\Omega \langle \mathcal{A}(x+h,u(x+h),Du(x)) - \mathcal{A}(x,u(x),Du(x)), 2\eta D\eta\tau_h(u-\psi)\rangle dx \\ =:& I_1 + I_2 + I_3 + I_4 + I_5 + I_6.\end{aligned}$$

Thus, we get

$$I_1 \le |I_2| + |I_3| + |I_4| + |I_5| + |I_6| + |J|. \tag{3.3.1}$$

We proceed estimating the various pieces arising up from (3.3.1).
**Estimate for** $I_1$. The ellipticity assumption (A1) implies

$$I_1 \geq \nu \int_\Omega \eta^2 |\tau_h Du|^2 (\mu^2 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} dx. \tag{3.3.2}$$

**Estimate for** $I_2$. From the growth condition (A2), Young's inequality and Hölder's inequality, we get

$$\begin{aligned}
|I_2| \leq & L \int_\Omega \eta^2 |\tau_h Du| (\mu^2 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} |\tau_h D\psi| dx \\
\leq & \varepsilon \int_\Omega \eta^2 |\tau_h Du|^2 (\mu^2 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} dx \\
& + C_\varepsilon(L) \int_\Omega \eta^2 |\tau_h D\psi|^2 (\mu^2 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} dx \\
\leq & \varepsilon \int_\Omega \eta^2 |\tau_h Du|^2 (\mu^2 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} dx \\
& + C_\varepsilon(L) \left( \int_{B_R} |\tau_h D\psi|^p dx \right)^{\frac{2}{p}} \left( \int_{B_{2R}} (1 + |Du|^p) dx \right)^{\frac{p-2}{p}} \\
\leq & \varepsilon \int_\Omega \eta^2 |\tau_h Du|^2 (\mu^2 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} dx \\
& + C_\varepsilon(n,p,\beta,L) \left( \int_{B_R} |\tau_h D\psi|^p dx \right)^{\frac{2}{p}} \left( \int_{B_{2R}} (1 + |Du|^{\frac{np}{n-\beta p}}) dx \right)^{\frac{(p-2)(n-\beta p)}{np}},
\end{aligned} \tag{3.3.3}$$

where we also used the properties of $\eta$.
**Estimate for** $I_3$. Similarly as for the estimate of $I_2$, we obtain

$$\begin{aligned}
|I_3| \leq & 2L \int_\Omega |D\eta| \eta |\tau_h Du| (\mu^2 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} |\tau_h (u - \psi)| dx \\
\leq & \varepsilon \int_\Omega \eta^2 |\tau_h Du|^2 (\mu^2 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} dx \\
& + C_\varepsilon(L) \int_\Omega |D\eta|^2 |\tau_h (u-\psi)|^2 (\mu^2 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} dx \\
\leq & \varepsilon \int_\Omega \eta^2 |\tau_h Du|^2 (\mu^2 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} dx \\
& + \frac{C_\varepsilon(n,p,\beta,L)}{R^2} R^{\beta p} \left( \int_{B_{2R}} (1 + |Du|^{\frac{np}{n-\beta p}}) dx \right)^{\frac{(p-2)(n-\beta p)}{np}} \left( \int_{B_R} |\tau_h (u-\psi)|^{\frac{np}{n-\beta p}} dx \right)^{\frac{2(n-\beta p)}{np}} \\
\leq & \varepsilon \int_\Omega \eta^2 |\tau_h Du|^2 (\mu^2 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} dx \\
& + \frac{C_\varepsilon(n,p,\beta,L)}{R^2} |h|^2 R^{\beta p} \left( \int_{B_{2R}} (1 + |Du|^{\frac{np}{n-\beta p}}) dx \right)^{\frac{(p-2)(n-\beta p)}{np}} \\
& \cdot \left( \int_{B_{2R}} |D(u-\psi)|^{\frac{np}{n-\beta p}} dx \right)^{\frac{2(n-\beta p)}{np}}.
\end{aligned} \tag{3.3.4}$$

**Estimate for** $I_4$. Assumption (A3), Young's inequality and the properties of $\eta$ imply

$$\begin{aligned}|I_4| \leq& \int_\Omega \eta^2 |\tau_h Du| \omega(|\tau_h u| + |h|)(\mu^2 + |Du(x)|^2)^{\frac{p-1}{2}} dx\\ \leq& \varepsilon \int_\Omega \eta^2 |\tau_h Du|^2 (\mu^2 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} dx\\ &+ C_\varepsilon \int_{B_R} \omega(|\tau_h u| + |h|)^2 (1 + |Du|^p) dx.\end{aligned}$$

By Lemma 2.2.2, we know that $D\psi \in L^{\frac{np}{n-\beta}}_{\mathrm{loc}}(\Omega)$. Therefore, Theorem 3.1.1 and Remark 3.1.2 with $q = \frac{n}{n-\beta p}$ yield that the solution $u$ is Hölder continuous. Let $\theta$ be the Hölder exponent of $u$. By virtue of Hölder's inequality, we get

$$\begin{aligned}|I_4| \leq& \varepsilon \int_\Omega \eta^2 |\tau_h Du|^2 (\mu^2 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} dx\\ &+ C_\varepsilon(n,p,\beta)(|h|^{2\alpha\theta} + |h|^{2\alpha}) R^{\beta p} \left( \int_{B_R} (1 + |Du|^{\frac{np}{n-\beta p}}) dx \right)^{\frac{n-\beta p}{n}}. \end{aligned} \tag{3.3.5}$$

**Estimate for** $I_5$. Arguing as we did for the estimate of the integral $I_4$, we get

$$\begin{aligned}|I_5| \leq& \int_\Omega \eta^2 |\tau_h D\psi| \omega(|\tau_h u| + |h|)(\mu^2 + |Du(x)|^2)^{\frac{p-1}{2}} dx\\ \leq& C(|h|^{\alpha\theta} + |h|^\alpha) \left( \int_{B_R} |\tau_h D\psi|^p dx \right)^{\frac{1}{p}} \left( \int_{B_R} (1 + |Du|^p) dx \right)^{\frac{1}{p'}}\\ \leq& C \left( \int_{B_R} |\tau_h D\psi|^p dx \right)^{\frac{2}{p}} + C(|h|^{2\alpha\theta} + |h|^{2\alpha}) \left( \int_{B_R} (1 + |Du|^p) dx \right)^{\frac{2}{p'}}\\ \leq& C \left( \int_{B_R} |\tau_h D\psi|^p dx \right)^{\frac{2}{p}}\\ &+ C(n,p,\beta)(|h|^{2\alpha\theta} + |h|^{2\alpha}) R^{2\beta(p-1)} \left( \int_{B_R} (1 + |Du|^{\frac{np}{n-\beta p}}) dx \right)^{\frac{2(n-\beta p)}{np'}}. \end{aligned} \tag{3.3.6}$$

**Estimate for** $I_6$. Assumption (A3), Young's and Hölder's inequality and Lemma 2.1.3 lead us to

$$\begin{aligned}|I_6| \leq& \int_\Omega \eta |D\eta| |\tau_h(u-\psi)| \omega(|\tau_h u| + |h|)(\mu^2 + |Du(x)|^2)^{\frac{p-1}{2}} dx\\ \leq& C \int_\Omega |D\eta|^2 |\tau_h(u-\psi)|^2 (\mu^2 + |Du|^2)^{\frac{p-2}{2}} dx\\ &+ C \int_\Omega \eta^2 \omega(|\tau_h u| + |h|)^2 (\mu^2 + |Du|^2)^{\frac{p}{2}} dx\\ \leq& \frac{C}{R^2} \left( \int_{B_R} |\tau_h(u-\psi)|^p dx \right)^{\frac{2}{p}} \left( \int_{B_R} (1 + |Du|^p) dx \right)^{\frac{p-2}{p}}\\ &+ C(n,p,\beta)(|h|^{2\alpha\theta} + |h|^{2\alpha}) R^{\beta p} \left( \int_{B_R} (1 + |Du|^{\frac{np}{n-\beta p}}) dx \right)^{\frac{n-\beta p}{n}}\end{aligned}$$

$$\leq \frac{C(n,p,\beta)}{R^2}|h|^2 R^{\beta p}\left(\int_{B_{2R}}|D(u-\psi)|^{\frac{np}{n-\beta p}}dx\right)^{\frac{2(n-\beta p)}{np}}\left(\int_{B_R}(1+|Du|^{\frac{np}{n-\beta p}})dx\right)^{\frac{(p-2)(n-\beta p)}{np}}$$
$$+ C(n,p,\beta)(|h|^{2\alpha\theta}+|h|^{2\alpha})R^{\beta p}\left(\int_{B_R}(1+|Du|^{\frac{np}{n-\beta p}})dx\right)^{\frac{n-\beta p}{n}}, \tag{3.3.7}$$

where we also used the Hölder continuity of the solution.
**Estimate for** $J$. Using assumption (B1), Young's and Hölder's inequality and properties of $\eta$, we get

$$\begin{aligned}
|J| \leq& \int_\Omega |\mathcal{B}(x,u,Du)||\tau_{-h}(\eta^2\tau_h(u-\psi))|dx\\
\leq&\left(\int_{B_R}|B(x,u,Du)|^{p'}dx\right)^{\frac{1}{p'}}\left(\int_\Omega|\tau_{-h}(\eta^2\tau_h(u-\psi))|^p dx\right)^{\frac{1}{p}}\\
\leq& C(n,p)|h|\left(\int_{B_R}(|Du|^{rp'}+|u|^{rp'}+a^{p'})dx\right)^{\frac{1}{p'}}\left(\int_\Omega|D(\eta^2\tau_h(u-\psi))|^p dx\right)^{\frac{1}{p}}\\
\leq& C(n,p)|h|\left(\int_{B_R}(|Du|^{p}+|u|^{p}+a^{p'}+1)dx\right)^{\frac{1}{p'}}\\
&\cdot\left\{\left(\int_\Omega \eta^p|D\eta|^p|\tau_h(u-\psi)|^p dx\right)^{\frac{1}{p}}\right.\\
&\quad+\left(\int_\Omega \eta^{2p}|\tau_h Du|^2(\mu^2+|Du(x+h)|^2+|Du(x)|^2)^{\frac{p-2}{2}}dx\right)^{\frac{1}{p}}\\
&\quad\left.+\left(\int_\Omega \eta^{2p}|\tau_h D\psi|^p dx\right)^{\frac{1}{p}}\right\}\\
\leq& C(n,p,\beta)|h|R^{\beta(p-1)}\left(\int_{B_R}(|Du|^{\frac{np}{n-\beta p}}+|u|^{\frac{np}{n-\beta p}}+a^{\frac{np'}{n-\beta p}}+1)dx\right)^{\frac{n-\beta p}{np'}}\\
&\cdot\left\{\frac{|h|}{R}\left(\int_{B_{2R}}|D(u-\psi)|^p dx\right)^{\frac{1}{p}}\right.\\
&\quad+\left(\int_\Omega \eta^2\eta^{2(p-1)}|\tau_h Du|^2(\mu^2+|Du(x+h)|^2+|Du(x)|^2)^{\frac{p-2}{2}}dx\right)^{\frac{1}{p}}\\
&\quad\left.+\left(\int_{B_R}|\tau_h D\psi|^p dx\right)^{\frac{1}{p}}\right\}\\
\leq& \frac{C(n,p,\beta)}{R}|h|^2 R^{\beta(p-1)}\left(\int_{B_R}(|Du|^{\frac{np}{n-\beta p}}+|u|^{\frac{np}{n-\beta p}}+a^{\frac{np'}{n-\beta p}}+1)dx\right)^{\frac{n-\beta p}{np'}}\\
&\cdot\left(\int_{B_{2R}}|D(u-\psi)|^p dx\right)^{\frac{1}{p}}\\
&+C_\varepsilon(n,p,\beta)|h|^{p'}R^{\beta p}\left(\int_{B_R}(|Du|^{\frac{np}{n-\beta p}}+|u|^{\frac{np}{n-\beta p}}+a^{\frac{np'}{n-\beta p}}+1)dx\right)^{\frac{n-\beta p}{n}}\\
&+\varepsilon\int_\Omega \eta^2|\tau_h Du|^2(\mu^2+|Du(x+h)|^2+|Du(x)|^2)^{\frac{p-2}{2}}dx
\end{aligned}$$

$$
\begin{aligned}
&+ C(n,p,\beta)|h|R^{\beta(p-1)}\left(\int_{B_R}(|Du|^{\frac{np}{n-\beta p}}+|u|^{\frac{np}{n-\beta p}}+a^{\frac{np'}{n-\beta p}}+1)dx\right)^{\frac{n-\beta p}{np'}}\\
&\quad\cdot\left(\int_{B_R}|\tau_h D\psi|^p dx\right)^{\frac{1}{p}}\\
\le&\frac{C(n,p,\beta)}{R}|h|^2R^{\beta p}\left(\int_{B_R}(|Du|^{\frac{np}{n-\beta p}}+|u|^{\frac{np}{n-\beta p}}+a^{\frac{np'}{n-\beta p}}+1)dx\right)^{\frac{n-\beta p}{np'}}\\
&\quad\cdot\left(\int_{B_{2R}}|D(u-\psi)|^{\frac{np}{n-\beta p}}dx\right)^{\frac{n-\beta p}{np}}\\
&+ C_\varepsilon(n,p,\beta)|h|^{p'}R^{\beta p}\left(\int_{B_R}(|Du|^{\frac{np}{n-\beta p}}+|u|^{\frac{np}{n-\beta p}}+a^{\frac{np'}{n-\beta p}}+1)dx\right)^{\frac{n-\beta p}{n}}\\
&+\varepsilon\int_\Omega \eta^2|\tau_h Du|^2(\mu^2+|Du(x+h)|^2+|Du(x)|^2)^{\frac{p-2}{2}}dx\\
&+ C(n,p,\beta)|h|^2R^{2\beta(p-1)}\left(\int_{B_R}(|Du|^{\frac{np}{n-\beta p}}+|u|^{\frac{np}{n-\beta p}}+a^{\frac{np'}{n-\beta p}}+1)dx\right)^{\frac{2(n-\beta p)}{np'}}\\
&+ C(n,p,\beta)\left(\int_{B_R}|\tau_h D\psi|^p dx\right)^{\frac{2}{p}}.
\end{aligned}
\tag{3.3.8}
$$

Inserting estimates (3.3.2), (3.3.3), (3.3.4), (3.3.5), (3.3.6), (3.3.7) and (3.3.8) in (3.3.1), we get

$$
\begin{aligned}
\nu&\int_\Omega \eta^2|\tau_h Du|^2(\mu^2+|Du(x+h)|^2+|Du(x)|^2)^{\frac{p-2}{2}}dx\\
\le&5\varepsilon\int_\Omega \eta^2|\tau_h Du|^2(\mu^2+|Du(x+h)|^2+|Du(x)|^2)^{\frac{p-2}{2}}dx\\
&+C_\varepsilon\left(\int_{B_R}|\tau_h D\psi|^p dx\right)^{\frac{2}{p}}\left(\int_{B_{2R}}(1+|Du|^{\frac{np}{n-\beta p}})dx\right)^{\frac{(p-2)(n-\beta p)}{np}}\\
&+C_\varepsilon|h|^2R^{-2+\beta p}\left(\int_{B_{2R}}(1+|Du|^{\frac{np}{n-\beta p}})dx\right)^{\frac{(p-2)(n-\beta p)}{np}}\left(\int_{B_{2R}}|D(u-\psi)|^{\frac{np}{n-\beta p}}dx\right)^{\frac{2(n-\beta p)}{np}}\\
&+C_\varepsilon(|h|^{2\alpha\theta}+|h|^{2\alpha})R^{\beta p}\left(\int_{B_R}(1+|Du|^{\frac{np}{n-\beta p}})dx\right)^{\frac{n-\beta p}{n}}\\
&+C_\varepsilon\left(\int_{B_R}|\tau_h D\psi|^p dx\right)^{\frac{2}{p}}+C(|h|^{2\alpha\theta}+|h|^{2\alpha})R^{2\beta(p-1)}\left(\int_{B_R}(1+|Du|^{\frac{np}{n-\beta p}})dx\right)^{\frac{2(n-\beta p)}{np'}}\\
&+C_\varepsilon|h|^2R^{-2\beta p}\left(\int_{B_{2R}}|D(u-\psi)|^{\frac{np}{n-\beta p}}dx\right)^{\frac{2(n-\beta p)}{np}}\left(\int_{B_R}(1+|Du|^{\frac{np}{n-\beta p}})dx\right)^{\frac{(p-2)(n-\beta p)}{np}}\\
&+C_\varepsilon(|h|^{2\alpha\theta}+|h|^{2\alpha})R^{\beta p}\left(\int_{B_R}(1+|Du|^{\frac{np}{n-\beta p}})dx\right)^{\frac{n-\beta p}{n}}\\
&+C_\varepsilon|h|^2R^{-1+\beta p}\left(\int_{B_R}(|Du|^{\frac{np}{n-\beta p}}+|u|^{\frac{np}{n-\beta p}}+a^{\frac{np'}{n-\beta p}}+1)dx\right)^{\frac{n-\beta p}{np'}}
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \left(\int_{B_{2R}} |D(u-\psi)|^{\frac{np}{n-\beta p}} dx\right)^{\frac{n-\beta p}{np}}\\
&+ C_\varepsilon |h|^{p'} R^{\beta p}\left(\int_{B_R} (|Du|^{\frac{np}{n-\beta p}} + |u|^{\frac{np}{n-\beta p}} + a^{\frac{np'}{n-\beta p}} + 1) dx\right)^{\frac{n-\beta p}{n}}\\
&+ C_\varepsilon |h|^2 R^{2\beta(p-1)}\left(\int_{B_R} (|Du|^{\frac{np}{n-\beta p}} + |u|^{\frac{np}{n-\beta p}} + a^{\frac{np'}{n-\beta p}} + 1) dx\right)^{\frac{2(n-\beta p)}{np'}}, \qquad (3.3.9)
\end{aligned}
$$

with a constant $C_\varepsilon := C_\varepsilon(n, p, \beta, L)$.
For a better readability, we define

$$
\begin{aligned}
C_1 &:= C_\varepsilon \left\{1 + \left(\int_{B_{2R}} (1 + |Du|^{\frac{np}{n-\beta p}}) dx\right)^{\frac{(p-2)(n-\beta p)}{np}}\right\},\\
C_2 &:= C_\varepsilon \left(\int_{B_{2R}} (1 + |Du|^{\frac{np}{n-\beta p}}) dx\right)^{\frac{(p-2)(n-\beta p)}{np}} \left(\int_{B_{2R}} |D(u-\psi)|^{\frac{np}{n-\beta p}} dx\right)^{\frac{2(n-\beta p)}{np}},\\
C_3 &:= C_\varepsilon \left(\int_{B_R} (1 + |Du|^{\frac{np}{n-\beta p}}) dx\right)^{\frac{n-\beta p}{n}},\\
C_4 &:= C_\varepsilon \left(\int_{B_R} (1 + |Du|^{\frac{np}{n-\beta p}}) dx\right)^{\frac{2(n-\beta p)}{np'}},\\
C_5 &:= C_\varepsilon \left(\int_{B_R} (|Du|^{\frac{np}{n-\beta p}} + |u|^{\frac{np}{n-\beta p}} + a^{\frac{np'}{n-\beta p}} + 1) dx\right)^{\frac{n-\beta p}{np'}} \left(\int_{B_{2R}} |D(u-\psi)|^{\frac{np}{n-\beta p}} dx\right)^{\frac{n-\beta p}{np}},\\
C_6 &:= C_\varepsilon \left(\int_{B_R} (|Du|^{\frac{np}{n-\beta p}} + |u|^{\frac{np}{n-\beta p}} + a^{\frac{np'}{n-\beta p}} + 1) dx\right)^{\frac{n-\beta p}{n}},\\
C_7 &:= C_\varepsilon \left(\int_{B_R} (|Du|^{\frac{np}{n-\beta p}} + |u|^{\frac{np}{n-\beta p}} + a^{\frac{np'}{n-\beta p}} + 1) dx\right)^{\frac{2(n-\beta p)}{np'}},
\end{aligned}
$$

so that the previous estimate can be rewritten as

$$
\begin{aligned}
\nu &\int_\Omega \eta^2 |\tau_h Du|^2 (\mu^2 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} dx\\
&\leq 5\varepsilon \int_\Omega \eta^2 |\tau_h Du|^2 (\mu^2 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} dx\\
&\quad + C_1 \left(\int_{B_R} |\tau_h D\psi|^p dx\right)^{\frac{2}{p}}\\
&\quad + C_2 |h|^2 R^{-2+\beta p} + C_3 (|h|^{2\alpha} R^{\beta p} + |h|^{2\alpha\theta} R^{\beta p})\\
&\quad + C_4 (|h|^{2\alpha} R^{2\beta(p-1)} + |h|^{2\alpha\theta} R^{2\beta(p-1)})\\
&\quad + C_5 |h|^2 R^{-1+\beta p} + C_6 |h|^{p'} R^{\beta p} + C_7 |h|^2 R^{2\beta(p-1)}.
\end{aligned}
$$

Choosing $\varepsilon = \frac{\nu}{10}$ and using Lemma 2.0.2 we get the following estimate

$$
\int_{B_{R/2}} |\tau_h V_p(Du)|^2 dx
$$

$$
\begin{aligned}
&\leq C_1\left(\int_{B_R}|\tau_h D\psi|^p dx\right)^{\frac{2}{p}}\\
&\quad + C_2|h|^2R^{-2+\beta p} + C_3(|h|^{2\alpha}R^{\beta p} + |h|^{2\alpha\theta}R^{\beta p})\\
&\quad + C_4(|h|^{2\alpha}R^{2\beta(p-1)} + |h|^{2\alpha\theta}R^{2\beta(p-1)})\\
&\quad + C_5|h|^2R^{-1+\beta p} + C_6|h|^{p'}R^{\beta p} + C_7|h|^2R^{2\beta(p-1)}. \qquad (3.3.10)
\end{aligned}
$$

for every balls $B_R \subset B_{2R} \Subset \Omega$.
Let us fix arbitrary open subsets $\Omega' \Subset \Omega'' \Subset \Omega$ and choose $x_0 \in \Omega'$. Let $\sigma \in (0,1)$ be chosen later and consider the ball $\mathcal{B} = \mathcal{B}(h) = \mathcal{B}(x_0, |h|^\sigma)$ with $|h|$ sufficiently small, depending on the dimension $n$, the parameter $\sigma$ and the distance between $\Omega'$ and the boundary of $\Omega''$ such that $\hat{\mathcal{Q}}_2 \Subset \Omega''$.
Estimate (3.3.10) applied over the ball $\mathcal{B}$ implies

$$
\begin{aligned}
&\int_{\mathcal{B}}|\tau_h V_p(Du)|^2 dx\\
&\quad\leq C_1\left(\int_{2\mathcal{B}}|\tau_h D\psi|^p dx\right)^{\frac{2}{p}}\\
&\quad\quad + C_2|h|^{2-2\sigma+\sigma\beta p} + C_3(|h|^{2\alpha+\sigma\beta p} + |h|^{2\alpha\theta+\sigma\beta p})\\
&\quad\quad + C_4(|h|^{2\alpha+2\sigma\beta(p-1)} + |h|^{2\alpha\theta+2\sigma\beta(p-1)})\\
&\quad\quad + C_5|h|^{2-\sigma+\sigma\beta p} + C_6|h|^{p'+\sigma\beta p} + C_7|h|^{2+2\sigma\beta(p-1)}. \qquad (3.3.11)
\end{aligned}
$$

Since $\sigma \in (0,1)$ and $|h| \leq 1$, we have

$$|h|^{\sigma\beta p} \leq |h|^{2-\sigma+\sigma\beta p} \leq |h|^{2-2\sigma+\sigma\beta p}$$

and

$$|h|^{2\alpha\theta+2\sigma\beta(p-1)} \leq |h|^{2\alpha\theta+\sigma\beta p} \iff p \geq 2. \qquad (3.3.12)$$

Now, consider the case $p > 2$. We have

$$|h|^{p'+\sigma\beta p} = |h|^{2-2\sigma+\sigma\beta p} \iff \sigma = \frac{p-2}{2(p-1)}$$

and

$$|h|^{p'+\sigma\beta p} = |h|^{p'(1+\beta\frac{p-2}{2})}.$$

Notice that

$$p'\left(1+\beta\frac{p-2}{2}\right) > 2\beta \iff \beta p^2 - 6\beta p + 2p + 4\beta > 0$$

and

$$p \geq 2\ ,\ \beta \in (0,1) \implies \beta p^2 - 6\beta p + 2p + 4\beta > 0.$$

Choosing $\sigma = \frac{p-2}{2(p-1)}$ estimate (3.3.11) becomes

$$\int_{\mathcal{B}}|\tau_h V_p(Du)|^2 dx$$

$$
\begin{aligned}
&\leq C_1\Big(\int_{2\mathcal{B}}|\tau_h D\psi|^p dx\Big)^{\frac{2}{p}}\\
&\quad+2(C_3+C_4)|h|^{2\alpha\theta+\beta(p-2)}\\
&\quad+(C_2+C_5+C_6)|h|^{p'(1+\beta\frac{p-2}{2})}\\
&\quad+C_7|h|^{2+\beta(p-2)}. \qquad (3.3.13)
\end{aligned}
$$

At this point, arguing as in [75, Lemma 4.5] a covering argument allows us to replace the cubes $\mathcal{Q}_1$ and $\hat{\mathcal{Q}}_2$ with the fixed open subsets $\Omega'$ and $\Omega''$, respectively. Indeed for each $|h|\in\mathbb{R}^n$ sufficiently small we can find balls $\mathcal{B}_1=\mathcal{B}_1(x_1,|h|^\sigma),...,\mathcal{B}_K=\mathcal{B}_K(x_K,|h|^\sigma)$, being $K=K(h)\in\mathbb{N}$, such that the corresponding inner cubes $\mathcal{Q}_1(\mathcal{B}_1),...,\mathcal{Q}_1(\mathcal{B}_K)$ are disjoint and satisfy

$$
\Big|\Omega'\setminus\bigcup_{k=1}^K\mathcal{Q}_1(\mathcal{B}_k)\Big|=0.
$$

By our assumption we have that $\mathcal{Q}_2(\hat{\mathcal{B}}_k)\subset\Omega''$, for every $k\leq K$ and each of the dilated outer cubes $\mathcal{Q}_2(\hat{\mathcal{B}}_k)$ intersects at most $(16\sqrt{n})$ of the other cubes $\mathcal{Q}_2(\hat{\mathcal{B}}_j)$, with $j\neq k$. Hence, after summing up (3.3.13) over the inner cubes $\mathcal{Q}_1\in\{\mathcal{Q}_1(\mathcal{B}_1),...,\mathcal{Q}_1(\mathcal{B}_K)\}$, and enlarging the constant by a fixed factor only depending on $n$ and $p$ (in particular independent of $h$), we arrive at

$$
\begin{aligned}
&\int_{\Omega'}|\tau_h V_p(Du)|^2dx\\
&\leq C_1\Big(\int_{\Omega''}|\tau_h D\psi|^p dx\Big)^{\frac{2}{p}}\\
&\quad+2(C_3+C_4)|h|^{2\alpha\theta+\beta(p-2)}\\
&\quad+(C_2+C_5+C_6)|h|^{p'(1+\beta\frac{p-2}{2})}\\
&\quad+C_7|h|^{2+\beta(p-2)}.
\end{aligned}
$$

Then, by (3.3.12), we get

$$
\int_{\Omega'}|\tau_h V_p(Du)|^2dx\leq C\Big(\int_{\Omega''}|\tau_h D\psi|^p dx\Big)^{\frac{2}{p}}+C\big(|h|^{2\alpha\theta+\frac{\beta p(p-2)}{2(p-1)}}+|h|^{p'(1+\beta\frac{p-2}{2})}\big).
$$

Dividing by $|h|^{2\gamma}$, raising to the power $\frac{s}{2}$ and integrating with respect to the measure $\frac{dh}{|h|^n}$ over the ball $B_{\frac{R}{4}}(0)$ both sides of preceding estimates, we obtain

$$
\begin{aligned}
&\int_{B_{\frac{R}{4}}(0)}\Big(\int_{\Omega'}\frac{|\tau_h V_p(Du)|^2}{|h|^{2\gamma}}dx\Big)^{\frac{s}{2}}\frac{dh}{|h|^n}\\
&\quad\leq C\int_{B_{\frac{R}{4}}(0)}\Big(\int_{\Omega''}\frac{|\tau_h D\psi|^p}{|h|^{\gamma p}}dx\Big)^{\frac{s}{p}}\frac{dh}{|h|^n}\\
&\qquad+C\int_{B_{\frac{R}{4}}(0)}|h|^{(\alpha\theta+\frac{\beta p(p-2)}{4(p-1)}-\gamma)s}\frac{dh}{|h|^n}+C\int_{B_{\frac{R}{4}}(0)}|h|^{(\frac{p'}{2}(1+\beta\frac{p-2}{2})-\gamma)s}\frac{dh}{|h|^n}
\end{aligned}
$$

$$=: F_1 + F_2 + F_3,$$

where we denote $\gamma = \min\{\beta, \alpha\theta\}$.
Thanks to the assumption on $\psi$, we have that

$$F_1 = C\|D\psi\|^s_{B^\gamma_{p,s}(\Omega'')} < +\infty.$$

The integrals $F_2$ and $F_3$ can be easily calculated in polar coordinates.

$$F_2 = C\int_0^{R/4} \rho^{(\alpha\theta+\frac{\beta p(p-2)}{4(p-1)}-\gamma)s-1} d\rho < +\infty$$

and

$$F_3 = C\int_0^{R/4} \rho^{(\frac{p'}{2}(1+\beta\frac{p-2}{2})-\gamma)s-1} d\rho < +\infty,$$

since $p > 2$ and $p'\big(1+\beta\frac{p-2}{2}\big) > 2\beta$. Then, $V_p(Du) \in B^\gamma_{2,s}(\Omega)$ locally.
Now, let $p = 2$. Notice that

$$|h|^{2\alpha\theta+2\sigma\beta} = |h|^{2-2\sigma+2\sigma\beta} \Leftrightarrow \sigma = 1-\alpha\theta$$

and

$$2\alpha\theta + 2\beta(1-\alpha\theta) > 2\beta.$$

Choosing $\sigma = 1-\alpha\theta$, estimate (3.3.11) becomes

$$\begin{aligned}
&\int_{\mathcal{B}} |\tau_h V_2(Du)|^2 dx \\
&\quad \le C_1 \int_{2\mathcal{B}} |\tau_h D\psi|^2 dx \\
&\qquad + (C_2 + 2C_3 + 2C_4 + C_5)|h|^{2\alpha\theta+2\beta(1-\alpha\theta)} \\
&\qquad + (C_6 + C_7)|h|^{2+2\beta(1-\alpha\theta)}.
\end{aligned}$$

Arguing as we did for the case $p > 2$, we finally obtain

$$\begin{aligned}
&\int_{B_{\frac{R}{4}}(0)} \left(\int_{\Omega'} \frac{|\tau_h V_2(Du)|^2}{|h|^{2\beta}} dx\right)^{\frac{s}{2}} \frac{dh}{|h|^n} \\
&\quad \le C\int_{B_{\frac{R}{4}}(0)} \left(\int_{\Omega''} \frac{|\tau_h D\psi|^2}{|h|^{2\beta}} dx\right)^{\frac{s}{2}} \frac{dh}{|h|^n} \\
&\qquad + C\int_{B_{\frac{R}{4}}(0)} |h|^{\alpha\theta(1-\beta)s} \frac{dh}{|h|^n} + C\int_{B_{\frac{R}{4}}(0)} |h|^{(1-\beta)s} \frac{dh}{|h|^n} < +\infty,
\end{aligned}$$

that is $V_2(Du) \in B^\gamma_{2,s}(\Omega)$ locally. □

# Chapter 4

# Higher differentiability for obstacle problems with non-standard growth

Here, we provide some higher fractional differentiability results, obtained in the paper [69], for the gradient of the solutions $u \in W^{1,p}(\Omega)$ to obstacle problems with $(p,q)$-growth of the form

$$\min\left\{\int_{\Omega} F(x, Dw)dx \ : \ w \in \mathcal{K}_{\psi}(\Omega)\right\}, \tag{4.0.1}$$

where $\psi \in W^{1,p}(\Omega)$ is the obstacle function and $\mathcal{K}_{\psi}(\Omega)$ is the admissible class defined in (2.0.2). We assume that $F : \Omega \times \mathbb{R}^n \to [0, +\infty)$ is a Carathéodory function and there exists a function $f : \Omega \times [0, +\infty) \to [0, +\infty)$ satisfying the condition

$$F(x, \xi) = f(x, |\xi|) \tag{F1}$$

for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^n$. Moreover, we also assume that there exist positive constants $\tilde{\nu}$, $\tilde{L}$, $\tilde{l}$, exponents $2 \leq p < q < +\infty$ and a parameter $\mu \in [0, 1]$, that will allow us to consider in our analysis both the degenerate and the non-degenerate situation, such that the following assumptions are satisfied:

$$\frac{1}{\tilde{l}}(|\xi|^p - \mu^p) \leq F(x, \xi) \leq \tilde{l}(\mu^2 + |\xi|^2)^{\frac{q}{2}} \tag{F2}$$

$$\langle D_{\xi\xi}F(x, \xi)\lambda, \lambda\rangle \geq \tilde{\nu}(\mu^2 + |\xi|^2)^{\frac{p-2}{2}}|\lambda|^2 \tag{F3}$$

$$|D_{\xi\xi}F(x, \xi)| \leq \tilde{L}(\mu^2 + |\xi|^2)^{\frac{q-2}{2}} \tag{F4}$$

for a.e. $x, y \in \Omega$ and every $\xi \in \mathbb{R}^n$. Recently, in [31] it has been proved that (F3) and (F4) imply the growth conditions (F2).

When dealing with the vectorial case, i.e., when the integrand $F$ is such that $F : \Omega \times \mathbb{R}^{nN} \to \mathbb{R}$ with $N > 1$, everywhere local regularity fails in general (see e.g. [87] and references therein). A way to recover everywhere higher regularity is to restrict to integrands with the special structure defined in assumption (F1), also called *Uhlenbeck structure* after the pioneering work [101]. However, the above considerations do not affect our problem, since we are dealing with the scalar case. Here we are interested in an additional property of the Uhlenbeck structure, which is of fundamental importance in the framework of non-standard growth conditions, both in the scalar and in the vectorial setting. Indeed, functionals with $(p,q)$-growth satisfying (F1) can be approximated from below with a sequence of integrands with $p$-growth (see Lemma 4.2.2 below). This rules out the occurrence of the so-called *Lavrentiev phenomenon* (see [33] for a comprehensive discussion). Indeed, in this framework the first big problem arising is an inequality of the type

$$\inf_{v\in W^{1,p}(\Omega)\cap\{w\geq\psi\}}\int_\Omega F(x,Dv)dx < \inf_{v\in W^{1,q}(\Omega)\cap\{w\geq\psi\}}\int_\Omega F(x,Dv)dx. \tag{4.0.2}$$

This is a clear obstruction to regularity, since (4.0.2) prevents minimizers to belong to $W^{1,q}(\Omega)$. We observe that the Lavrentiev phenomenon cannot happen if $p=q$ or if $F(x,\xi)=F(\xi)$ and the obstacle is sufficiently regular.

We say that function $F$ satisfies assumption (F5) if there exist a non-negative function $k\in L^r_{\mathrm{loc}}(\Omega)$, with $r>\frac{n}{\alpha}$ and $0<\alpha<1$, such that

$$|D_\xi F(x,\xi)-D_\xi F(y,\xi)|\leq |x-y|^\alpha(k(x)+k(y))(\mu^2+|\xi|^2)^{\frac{q-1}{2}} \tag{F5}$$

for a.e. $x,y\in\Omega$ and every $\xi\in\mathbb{R}^n$.
On the other hand, we say that assumption (F6) is satisfied if there exists a sequence of measurable non-negative functions $g_k\in L^r_{\mathrm{loc}}(\Omega)$ such that

$$\sum_{k=1}^{\infty}\|g_k\|^\sigma_{L^r(\Omega)}<\infty,$$

and at the same time

$$|D_\xi F(x,\xi)-D_\xi F(y,\xi)|\leq |x-y|^\alpha(g_k(x)+g_k(y))(\mu^2+|\xi|^2)^{\frac{q-1}{2}} \tag{F6}$$

for a.e. $x,y\in\Omega$ such that $2^{-k}\mathrm{diam}(\Omega)\leq|x-y|<2^{-k+1}\mathrm{diam}(\Omega)$ and for every $\xi\in\mathbb{R}^n$.

It is well known that the regularity of the minima often comes from the fact that they are also solutions to the corresponding Euler-Lagrange equation, in the unconstrained setting, or, in the case of obstacle problems, to the corresponding variational inequality. It is worth observing that in the case of standard growth conditions, i.e. $p=q$, $u\in W^{1,p}(\Omega)$ is the solution to the obstacle problem (4.0.1) if, and only if, $u\in\mathcal{K}_\psi(\Omega)$ solves the variational inequality

$$\int_\Omega\langle\mathcal{A}(x,Du),D(\varphi-u)\rangle dx\geq 0,\qquad \forall\varphi\in\mathcal{K}_\psi(\Omega), \tag{4.0.3}$$

where we set

$$\mathcal{A}(x,\xi):=D_\xi F(x,\xi).$$

On the other hand, in the framework of $(p,q)$-growth conditions, the question of whether minimizers are in fact solutions of the corresponding Euler-Lagrange equation or variational inequality is an issue that requires a careful investigation (see [13, 14, 48]). However, once we have attained more regularity for the solutions to (4.0.1) by means of our higher differentiability result, we will able to reformulate the minimization problem as a variational inequality.

From conditions (F2)–(F4), we deduce the existence of positive constants $\nu, L, l$ such that the following $p$-ellipticity and $q$-growth conditions are satisfied by the map $\mathcal{A}$:

$$|\mathcal{A}(x,\xi)|\leq l(\mu^2+|\xi|^2)^{\frac{q-1}{2}} \tag{C1}$$

$$\langle \mathcal{A}(x,\xi) - \mathcal{A}(x,\eta), \xi - \eta\rangle \geq \nu|\xi-\eta|^2(\mu^2+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}} \tag{C2}$$

$$|\mathcal{A}(x,\xi) - \mathcal{A}(x,\eta)| \leq L|\xi-\eta|(\mu^2+|\xi|^2+|\eta|^2)^{\frac{q-2}{2}} \tag{C3}$$

for a.e. $x, y \in \Omega$ and every $\xi, \eta \in \mathbb{R}^n$.

In [56, 57] Gavioli proved that the weak differentiability of integer order of the partial map $x \mapsto \mathcal{A}(x,\xi)$ is a sufficient condition to prove that an extra differentiability of integer order of the gradient of the obstacle transfers to the gradient of the solutions to obstacle problems with $(p,q)$-growth conditions. Other higher differentiability results that deserved to be quoted are [51], in the setting of variable growth exponents, and [20, 102], in the case of double phase obstacle problems.

In [69], we continue the study of the higher differentiablity properties of solutions to (3.0.1) in case of $(p,q)$-growth conditions. The novelty of our results consists in assuming that both the gradient of the obstacle and the partial map $x \mapsto \mathcal{A}(x,\xi)$ belong to a suitable Besov space. In particular, our aim is to extend the higher differentiability results in [46] (see Theorems 4.1.1 and 4.1.2 in Section 4.1) to the case of functionals with $(p,q)$-growth.

**Theorem 4.0.1.** *Let $F(x,\xi)$ satisfy (F1)–(F5) for exponents $2 \leq p < \frac{n}{\alpha} < r$, $p < q$ such that*

$$\frac{q}{p} < 1 + \frac{\alpha}{n} - \frac{1}{r}. \tag{4.0.4}$$

*Let $u \in \mathcal{K}_\psi(\Omega)$ be the solution to the obstacle problem (4.0.1). Then we have*

$$D\psi \in B^{\gamma}_{2q-p,\infty,loc}(\Omega) \Rightarrow (\mu^2+|Du|^2)^{\frac{p-2}{4}}Du \in B^{\alpha}_{2,\infty,loc}(\Omega),$$

*provided $0 < \alpha < \gamma < 1$.*

It is reasonable to expect that, strengthening the regularity assumptions both on the integrand and the obstacle, we are able to prove better regularity properties than those obtained in Theorem 4.0.1. Indeed, we have

**Theorem 4.0.2.** *Let $F(x,\xi)$ satisfy (F1)–(F4) and (F6) for exponents $2 \leq p < \frac{n}{\alpha} < r$, $p < q$ such that*

$$\frac{q}{p} < 1 + \frac{\min\{\alpha,\gamma\}}{n} - \frac{1}{r}, \tag{4.0.5}$$

*where $0 < \gamma < 1$. Let $u \in \mathcal{K}_\psi(\Omega)$ be the solution to the obstacle problem (4.0.1). Then we have*

$$D\psi \in B^{\gamma}_{2q-p,\sigma,loc}(\Omega) \Rightarrow (\mu^2+|Du|^2)^{\frac{p-2}{4}}Du \in B^{\min\{\alpha,\gamma\}}_{2,\sigma,loc}(\Omega),$$

*provided $\sigma \leq \frac{2n}{n-2\min\{\alpha,\gamma\}}$.*

We notice that the condition (4.0.4) is the natural counterpart in the fractional setting of the gap bound

$$\frac{q}{p} < 1 + \frac{1}{n} - \frac{1}{r}$$

cosidered in [56, 57] in the framework of Sobolev spaces of integer order.

This chapter is structured as follows. In Section 4.1, we recall some regularity results for obstacle problems with $p$-growth. In Section 4.2, we prove the existence of a sequence of functions with standard growth conditions that monotonically converges to our initial problems.

Section 4.3 is devoted to the proof of Theorem 4.0.1. In particular, in Section 4.3.1, we show that the closeness condition (4.0.4) ensures the validity of a priori estimates for approximating problems satisfying $p$-growth conditions. Then, in Section 4.3.2, we complete the proof of Theorem 4.0.1 by a convergence argument. Eventually, in Section 4.4 we prove Theorem 4.0.2, focusing only on the a priori estimate, since the limit procedure works exactly in the same way as for the previous result.

## 4.1 Preliminary results on standard growth conditions

For sake of clarity, we would like to recall the following regularity results (see [46] for the proof), which will be used in order to prove Theorems 4.0.1 and 4.0.2.

**Theorem 4.1.1.** *Assume that $\mathcal{A}(x,\xi)$ satisfies (C1)–(C3) for an exponent $2 \leq p = q < \frac{n}{\alpha}$ and let $u \in \mathcal{K}_\psi(\Omega)$ be the solution to the obstacle problem (4.0.3). If there exists a sequence of measurable non-negative functions $g_k \in L^{\frac{n}{\alpha}}_{loc}(\Omega)$ such that*

$$\sum_{k=1}^{\infty} \|g_k\|^{\sigma}_{L^{\frac{n}{\alpha}}(\Omega)} < \infty,$$

*and at the same time*

$$|\mathcal{A}(x,\xi) - \mathcal{A}(y,\xi)| \leq |x-y|^{\alpha}(g_k(x) + g_k(y))(\mu^2 + |\xi|^2)^{\frac{p-1}{2}},$$

*for a.e. $x, y \in \Omega$ such that $2^{-k} diam(\Omega) \leq |x-y| < 2^{-k+1} diam(\Omega)$ and for every $\xi \in \mathbb{R}^n$, then the following implication*

$$D\psi \in B^{\gamma}_{p,\sigma,loc}(\Omega) \Rightarrow (\mu^2 + |Du|^2)^{\frac{p-2}{4}} Du \in B^{\min\{\alpha,\gamma\}}_{2,\sigma,loc}(\Omega)$$

*holds, provided $\sigma \leq p^*_\gamma = \frac{np}{n-\gamma p}$.*

In the case of a regularity of the type $B^{\alpha}_{p,\infty}$, which is the weakest one in the scale of Besov spaces, both on the coefficients and on the gradient of the obstacle, we have the following.

**Theorem 4.1.2.** *Assume that $\mathcal{A}(x,\xi)$ satisfies (C1)–(C3) for an exponent $2 \leq p = q < \frac{n}{\alpha}$ and let $u \in \mathcal{K}_\psi(\Omega)$ be the solution to the obstacle problem (4.0.3). If there exists a non-negative function $k \in L^{\frac{n}{\alpha}}_{loc}(\Omega)$ such that*

$$|\mathcal{A}(x,\xi) - \mathcal{A}(y,\xi)| \leq |x-y|^{\alpha}(k(x) + k(y))(\mu^2 + |\xi|^2)^{\frac{p-1}{2}},$$

*for a.e. $x, y \in \Omega$ and for every $\xi \in \mathbb{R}^n$, then the following implication*

$$D\psi \in B^{\gamma}_{p,\infty,loc}(\Omega) \Rightarrow (\mu^2 + |Du|^2)^{\frac{p-2}{4}} Du \in B^{\alpha}_{2,\infty,loc}(\Omega)$$

*holds, provided $0 < \alpha < \gamma < 1$.*

## 4.2 Approximation results

The main tool to prove Theorems 4.0.1 and 4.0.2 is Lemma 4.2.2, which allows us to approximate from below the function $F$ with a sequence of functions $(F_j)$ satisfying $p$-growth.

We first recall the following theorem, which follows immediately from assumptions (F2) and (F3) (see [26, 36]).

**Theorem 4.2.1.** *Let $F : \Omega \times \mathbb{R}^n \to [0, +\infty), F = F(x, \xi)$, be a Carathéodory function. Then, assumptions (F2) and (F3) imply that there exist positive constants $c_0(p, q, \tilde{\nu}, \tilde{l})$, $c_1(p, \tilde{\nu})$ and a convex function $g : \Omega \times \mathbb{R}^n \to [-c_0, +\infty)$ such that it holds*

$$F(x, \xi) = c_1(\mu^2 + |\xi|^2)^{\frac{p}{2}} + g(x, \xi),$$

*for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^n$.*

In the next lemma, we adapt a well known approximation result, which can be found in [26], to the case when the map $x \mapsto D_\xi F(x, \xi)$ has a Besov regularity.

**Lemma 4.2.2.** *Let $F : \Omega \times \mathbb{R}^n \to [0, +\infty), F = F(x, \xi)$, be a Carathéodory function, convex with respect to $\xi$, satisfying assumptions (F1)–(F5). Then there exists a sequence $(F_j)$ of Carathéodory functions $F_j : \Omega \times \mathbb{R}^n \to [0, +\infty)$, convex with respect to the last variable, monotonically convergent to $F$, such that*

*(i) for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^n$, $F_j(x, \xi) = \tilde{F}_j(x, |\xi|)$,*

*(ii) for a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^n$ and for every $j$, $F_j(x, \xi) \le F_{j+1}(x, \xi) \le F(x, \xi)$,*

*(iii) for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^n$, we have $\langle D_{\xi\xi} F_j(x, \xi)\lambda, \lambda\rangle \ge \nu_1(\mu^2 + |\xi|^2)^{\frac{p-2}{2}}|\lambda|^2$, with $\nu_1$ depending only on $p$ and $\tilde{\nu}$,*

*(iv) for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^n$, there exist $L_1$, independent of $j$, and $\bar{L}_1$, depending on $j$, such that*

$$1/L_1(|\xi|^p - \mu^2) \le F_j(x, \xi) \le L_1(\mu^2 + |\xi|^q),$$
$$F_j(x, \xi) \le \bar{L}_1(p, q, j)(\mu^2 + |\xi|^p),$$

*(v) there exists a constant $C(p, q, j) > 0$ such that*

$$|D_\xi F_j(x, \xi) - D_\xi F_j(y, \xi)| \le |x - y|^\alpha (k(x) + k(y))(\mu^2 + |\xi|^2)^{\frac{q-1}{2}},$$
$$|D_\xi F_j(x, \xi) - D_\xi F_j(y, \xi)| \le C(p, q, j)|x - y|^\alpha (k(x) + k(y))(\mu^2 + |\xi|^2)^{\frac{p-1}{2}}$$

*for a.e. $x, y \in \Omega$ and for every $\xi \in \mathbb{R}^n$.*

*Proof.* According to Theorem 4.2.1, which holds under hypotheses (F2) and (F3), there exist two positive constants $c_0 = c_0(p, q, \tilde{\nu}, \tilde{l}, \tilde{L})$ and $c_1 = c_1(p, \tilde{\nu})$ and a convex function $g : \Omega \times \mathbb{R}^n \to [-c_0, +\infty)$ such that

$$F(x, \xi) = c_1(\mu^2 + |\xi|^2)^{\frac{p}{2}} + g(x, \xi). \tag{4.2.1}$$

Moreover there exists $\tilde{g} : \Omega \times [0, +\infty) \to [-c_0, +\infty)$ such that $\tilde{g}(x, |\xi|) = g(x, \xi)$ for any $\xi \in \mathbb{R}^n$. Since $n \ge 2$, for a.e. $x \in \Omega$, $t \mapsto \tilde{g}(x, t)$ is convex and increasing. For any $j \in \mathbb{N}$, we might then define $\tilde{g}_j : \Omega \times [0, +\infty) \to [-c_0, +\infty)$ as

$$\tilde{g}_j(x, t) = \tilde{g}(x, t), \qquad \forall (x, t) \in \Omega \times [0, j],$$

$$\tilde{g}_j(x,t) = \tilde{g}(x,j) + D_t\tilde{g}(x,j)(t-j), \qquad \forall (x,t) \in \Omega \times (j,\infty).$$

We notice that, by definition, for a.e. $x \in \Omega$, $t \mapsto \tilde{g}_j(x,t)$ is convex and increasing in $[0,+\infty)$ and $\tilde{g}_j(x,t) \leq \tilde{g}_{j+1}(x,t) \leq \tilde{g}(x,t)$. Combining assumption (F2), the definition of $\tilde{g}_j(x,t)$ and (4.2.1), we infer

$$\begin{aligned} &\tilde{g}_j(x,t) \leq \tilde{l}(\mu^2+t^2)^{\frac{q}{2}}, \\ &\tilde{g}_j(x,t) \leq c(q,\tilde{l},j)(\mu^2+t^2)^{\frac{p}{2}}. \end{aligned} \tag{4.2.2}$$

We now want to show that $D_t\tilde{g}_j$ has a (F5)-type growth. It is easy to see that $D_t\tilde{g}_j(x,t) = D_t\tilde{g}(x,j)$ for $t \geq j$. In particular, assumption (F5) yields

$$|D_t\tilde{g}(x,j) - D_t\tilde{g}(y,j)| \leq |x-y|^\alpha (k(x)+k(y))(\mu^2+j^2)^{\frac{q-1}{2}}.$$

Hence, for a.e. $x \in \Omega$ and every $t > 0$,

$$|D_t\tilde{g}(x,t) - D_t\tilde{g}(y,t)| \leq |x-y|^\alpha (k(x)+k(y))(\mu^2+t^2)^{\frac{q-1}{2}}. \tag{4.2.3}$$

Moreover, for $t \leq j$, according to (4.2.1) and (4.2.3), we obtain

$$\begin{aligned} |D_t\tilde{g}(x,t) - D_t\tilde{g}(y,t)| \leq& |x-y|^\alpha (k(x)+k(y))(\mu^2+t^2)^{\frac{p-1}{2}}(\mu^2+t^2)^{\frac{q-p}{2}} \\ \leq& |x-y|^\alpha (k(x)+k(y))(\mu^2+t^2)^{\frac{p-1}{2}}(\mu^2+j^2)^{\frac{q-p}{2}} \\ \leq& c(p,q,j)|x-y|^\alpha (k(x)+k(y))(\mu^2+t^2)^{\frac{p-1}{2}}. \end{aligned}$$

On the other hand, in the same way, for $t > j$, we get

$$\begin{aligned} |D_t\tilde{g}(x,t) - D_t\tilde{g}(y,t)| \leq& |x-y|^\alpha (k(x)+k(y))(\mu^2+j^2)^{\frac{p-1}{2}}(\mu^2+j^2)^{\frac{q-p}{2}} \\ \leq& |x-y|^\alpha (k(x)+k(y))(\mu^2+t^2)^{\frac{p-1}{2}}(\mu^2+j^2)^{\frac{q-p}{2}} \\ \leq& c(p,q,j)|x-y|^\alpha (k(x)+k(y))(\mu^2+t^2)^{\frac{p-1}{2}}. \end{aligned}$$

Eventually, for any $j$, we define $g_j : \Omega \times \mathbb{R}^n \to [-c_0,+\infty)$ as

$$g_j(x,\xi) = \tilde{g}_j(x,|\xi|).$$

Statements $(i),(ii),(iii),(v)$ directly follow by setting $F_j : \Omega \times \mathbb{R}^n \to [0,+\infty)$

$$F_j(x,\xi) := c_1(\mu^2+|\xi|^2)^{\frac{p}{2}} + g_j(x,\xi).$$

Property $(iv)$ is obtained combining (4.2.1) with (4.2.2) and the definition of $F_j$. □

**Remark 4.2.3.** *It is worth noting that an analogous version of Lemma 4.2.2 can be proved similarly, supposing* (F6) *instead of* (F5)*. In particular, statement* $(v)$ *would change as follows.*

*(v) There exists a constant* $C(p,q,j) > 0$ *such that*

$$\begin{aligned} &|D_\xi F_j(x,\xi) - D_\xi F_j(y,\xi)| \leq |x-y|^\alpha (g_k(x)+g_k(y))(\mu^2+|\xi|^2)^{\frac{q-1}{2}}, \\ &|D_\xi F_j(x,\xi) - D_\xi F_j(y,\xi)| \leq C(p,q,j)|x-y|^\alpha (g_k(x)+g_k(y))(\mu^2+|\xi|^2)^{\frac{p-1}{2}} \end{aligned}$$

*for a.e.* $x,y \in \Omega$ *such that* $2^{-k}diam(\Omega) \leq |x-y| < 2^{-k+1}diam(\Omega)$ *and for every* $\xi \in \mathbb{R}^n$.

## 4.3 Proof of Theorem 4.0.1

In order to prove Theorem 4.0.1, in Section 4.3.1, we derive a suitable a priori estimate for minimizers of obstacle problems with $p$-growth conditions, while in Section 4.3.2, we conclude showing that the a priori estimate is preserved when passing to the limit.

### 4.3.1 A priori estimate

Let us consider

$$\min\left\{\int_\Omega F_j(x,Dw)dx \ : \ w\in\mathcal{K}_\psi(\Omega)\right\}, \tag{4.3.1}$$

where $F_j:\Omega\times\mathbb{R}^n\to[0,+\infty)$, $F_j=F_j(x,\xi)$, was set in Lemma 4.2.2.
Denoting

$$\mathcal{A}_j(x,\xi):=D_\xi F_j(x,\xi),$$

one can easily check that $\mathcal{A}_j$ satisfies assumptions (C1)–(C3) and (F5) and the following conditions:

$$|\mathcal{A}_j(x,\xi)|\le l_1(p,q,j)(\mu^2+|\xi|^2)^{\frac{p-1}{2}} \tag{4.3.2}$$

$$|\mathcal{A}_j(x,\xi)-\mathcal{A}_j(x,\eta)|\le L_1(p,q,j)|\xi-\eta|(\mu^2+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}} \tag{4.3.3}$$

$$|\mathcal{A}_j(x,\xi)-\mathcal{A}_j(y,\xi)|\le\Theta(p,q,j)|x-y|^\alpha(k(x)+k(y))(\mu^2+|\xi|^2)^{\frac{p-1}{2}} \tag{4.3.4}$$

for a.e. $x,y\in\Omega$ and every $\xi,\eta\in\mathbb{R}^n$. It is well known that $u_j\in\mathcal{K}_\psi(\Omega)$ is the minimizer of the problem (4.3.1) if, and only if, the following variational inequality holds

$$\int_\Omega\langle\mathcal{A}_j(x,Du_j),D(\varphi-u_j)\rangle dx\ge 0,\quad\forall\varphi\in\mathcal{K}_\psi(\Omega). \tag{4.3.5}$$

We have the following theorem.

**Theorem 4.3.1.** *Let $\mathcal{A}_j(x,\xi)$ satisfy (C1)–(C3) and (F5) and (4.3.2)– (4.3.4) for exponents $2\le p<\frac{n}{\alpha}<r$, $p<q$ satisfying (4.0.4). Let $u_j\in\mathcal{K}_\psi(\Omega)$ be the solution to the obstacle problem (4.3.5). Suppose that $k\in L^r_{loc}(\Omega)$ and $D\psi\in B^\gamma_{2q-p,\infty,loc}(\Omega)$, for $0<\alpha<\gamma<1$. Then, the following estimate*

$$\int_{B_{R/4}}|\tau_hV_p(Du_j)|^2dx\le C|h|^{2\alpha}\left\{\int_{B_R}(1+|Du_j|^p)dx+\|D\psi\|_{B^\gamma_{2q-p,\infty}(B_R)}+\|k\|_{L^r(B_R)}\right\}^\kappa, \tag{4.3.6}$$

*holds for all balls $B_{R/4}\subset B_R\Subset\Omega$, for positive constants $C=C(n,p,q,r,\alpha,\nu,L,R)$, $\kappa=\kappa(n,p,q,r,\alpha)$, both independent of $j$.*

*Proof.* We start by observing that, since $p<2q-p$, we have

$$D\psi\in B^\gamma_{2q-p,\infty,\mathrm{loc}}(\Omega)\Rightarrow D\psi\in B^\gamma_{p,\infty,\mathrm{loc}}(\Omega),$$

thus an application of Theorem 4.1.2 implies

$$(\mu^2+|Du_j|^2)^{\frac{p-2}{4}}Du_j\in B^\alpha_{2,\infty,\mathrm{loc}}(\Omega),$$

which by Lemma 2.2.4 yields

$$Du_j \in L^{\frac{np}{n-2\beta}}_{\mathrm{loc}}(\Omega),$$

for all $0 < \beta < \alpha$. Therefore, the integral

$$\int_{\Omega'} (1 + |Du_j|)^{\frac{np}{n-2\beta}} dx$$

is finite, for every $\Omega' \Subset \Omega$ and $\beta \in (0, \alpha)$.

In the sequel we will profusely use the following inequality:

$$2q - p \le \frac{r(2q-p)}{r-2} \le \frac{np}{n-2\beta}, \tag{4.3.7}$$

for $\beta \in (\frac{\alpha nr}{nr+2(\alpha r-n)}, \alpha)$. The first part of inequality (4.3.7) is trivial, while the second part comes from (4.0.4). Namely,

$$\frac{r(2q-p)}{r-2} \le \frac{np}{n-2\beta} \Leftrightarrow \frac{q}{p} \le \frac{nr-n-\beta r}{r(n-2\beta)}$$

and

$$1 + \frac{\alpha}{n} - \frac{1}{r} < \frac{nr-n-\beta r}{r(n-2\beta)} \Leftrightarrow \beta > \frac{\alpha nr}{nr+2(\alpha r-n)}.$$

Fix $0 < \frac{R}{4} < \rho < s < t < t' < \frac{R}{2}$ such that $B_R \Subset \Omega$ and a cut-off function $\eta \in \mathcal{C}^1_0(B_t)$ with $0 \le \eta \le 1$, $\eta = 1$ on $B_s$ and $|D\eta| \le \frac{C}{t-s}$. Then, for $|h| \le t' - t$, we consider

$$\varphi_1(x) = u_j(x) + t\eta^2(x)\tau_h(u_j - \psi)(x)$$

and

$$\varphi_2(x) = u_j(x) + t\eta^2(x-h)\tau_{-h}(u_j - \psi)(x),$$

which belong to the admissible class $\mathcal{K}_\psi(\Omega)$, for every $t \in [0,1)$. Choosing $\varphi_1$ and $\varphi_2$ as test functions in (4.3.5), we obtain

$$\int_\Omega \langle \mathcal{A}_j(x, Du_j), D(\eta^2\tau_h(u_j - \psi))\rangle dx + \int_\Omega \langle \mathcal{A}_j(x, Du_j), D(\eta^2(x-h)\tau_{-h}(u_j - \psi))\rangle dx \ge 0$$

By means of a simple change of variable in the second integral in the left hand side of the previous inequality, we find that

$$\int_\Omega \langle \mathcal{A}_j(x+h, Du_j(x+h)) - \mathcal{A}_j(x, Du_j(x)), D(\eta^2\tau_h(u_j - \psi))\rangle dx \le 0$$

We can write previous inequality as follows

$$\begin{aligned} 0 \ge &\int_\Omega \langle \mathcal{A}_j(x+h, Du_j(x+h)) - \mathcal{A}_j(x+h, Du_j(x)), \eta^2 D\tau_h u_j\rangle dx \\ &- \int_\Omega \langle \mathcal{A}_j(x+h, Du_j(x+h)) - \mathcal{A}_j(x+h, Du_j(x)), \eta^2 D\tau_h \psi\rangle dx \end{aligned}$$

$$
\begin{aligned}
&+\int_\Omega \langle \mathcal{A}_j(x+h, Du_j(x+h)) - \mathcal{A}_j(x+h, Du_j(x)), 2\eta D\eta\tau_h(u_j-\psi)\rangle dx\\
&+\int_\Omega \langle \mathcal{A}_j(x+h, Du_j(x)) - \mathcal{A}_j(x, Du_j(x)), \eta^2 D\tau_h u_j\rangle dx\\
&-\int_\Omega \langle \mathcal{A}_j(x+h, Du_j(x)) - \mathcal{A}_j(x, Du_j(x)), \eta^2 D\tau_h \psi\rangle dx\\
&+\int_\Omega \langle \mathcal{A}_j(x+h, Du_j(x)) - \mathcal{A}_j(x, Du_j(x)), 2\eta D\eta\tau_h(u_j-\psi)\rangle dx\\
=:&I_1+I_2+I_3+I_4+I_5+I_6, \qquad (4.3.8)
\end{aligned}
$$

that yields

$$
I_1 \le |I_2|+|I_3|+|I_4|+|I_5|+|I_6| \qquad (4.3.9)
$$

The ellipticity assumption (C2) implies

$$
I_1 \ge \nu \int_\Omega \eta^2|\tau_h Du_j|^2(\mu^2+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{p-2}{2}}dx \qquad (4.3.10)
$$

From the growth condition (C3), Young's and Hölder's inequalities and the assumption on $D\psi$, we get

$$
\begin{aligned}
|I_2| \le& L\int_\Omega \eta^2|\tau_h Du_j|(\mu^2+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{q-2}{2}}|\tau_h D\psi| dx\\
\le& \varepsilon\int_\Omega \eta^2|\tau_h Du_j|^2(\mu^2+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{p-2}{2}}dx\\
&+C_\varepsilon(L)\int_\Omega \eta^2|\tau_h D\psi|^2(\mu^2+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{2q-p-2}{2}}dx\\
\le& \varepsilon\int_\Omega \eta^2|\tau_h Du_j|^2(\mu^2+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{p-2}{2}}dx\\
&+C_\varepsilon(L)\left(\int_{B_t}|\tau_h D\psi|^{2q-p}dx\right)^{\frac{2}{2q-p}}\left(\int_{B_{t'}}(1+|Du_j|)^{2q-p}dx\right)^{\frac{2q-p-2}{2q-p}}\\
\le& \varepsilon\int_\Omega \eta^2|\tau_h Du_j|^2(\mu^2+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{p-2}{2}}dx\\
&+C_\varepsilon(L)|h|^{2\gamma}[D\psi]^2_{B^\gamma_{2q-p,\infty}(B_R)}\left(\int_{B_{t'}}(1+|Du_j|)^{2q-p}dx\right)^{\frac{2q-p-2}{2q-p}}\\
\le& \varepsilon\int_\Omega \eta^2|\tau_h Du_j|^2(\mu^2+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{p-2}{2}}dx\\
&+C_\varepsilon(L)|h|^{2\gamma}[D\psi]^{2q-p}_{B^\gamma_{2q-p,\infty}(B_R)}+C_\varepsilon(L,n,p,q)|h|^{2\gamma}\int_{B_{t'}}(1+|Du_j|)^{2q-p}dx.
\end{aligned}
$$

Therefore, from (4.3.7), we infer

$$
|I_2| \le \varepsilon\int_\Omega \eta^2|\tau_h Du_j|^2(\mu^2+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{p-2}{2}}dx
$$

$$+ C_\varepsilon(p,q,r,L)|h|^{2\gamma}[D\psi]^{2q-p}_{B^\gamma_{2q-p,\infty}(B_R)} + C_\varepsilon(p,q,r,L)|h|^{2\gamma}\left(\int_{B_{t'}}(1+|Du_j|)^{\frac{r(2q-p)}{r-2}}dx\right)^{\frac{r-2}{r}}. \tag{4.3.11}$$

Arguing analogously, we get

$$\begin{aligned}
|I_3| \le & 2L\int_\Omega |D\eta|\eta|\tau_h Du_j|(1+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{q-2}{2}}|\tau_h(u_j-\psi)|dx \\
\le & \varepsilon\int_\Omega \eta^2|\tau_h Du_j|^2(\mu^2+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{p-2}{2}}dx \\
& + \frac{C_\varepsilon(L)}{(t-s)^2}\int_{B_t}|\tau_h(u_j-\psi)|^2(\mu^2+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{2q-p-2}{2}}dx \\
\le & \varepsilon\int_\Omega \eta^2|\tau_h Du_j|^2(\mu^2+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{p-2}{2}}dx \\
& + \frac{C_\varepsilon(L)}{(t-s)^2}\left(\int_{B_t}|\tau_h\psi|^{2q-p}dx\right)^{\frac{2}{2q-p}}\left(\int_{B_{t'}}(1+|Du_j|)^{2q-p}dx\right)^{\frac{2q-p-2}{2q-p}} \\
& + \frac{C_\varepsilon(L)}{(t-s)^2}\left(\int_{B_t}|\tau_h u_j|^{2q-p}dx\right)^{\frac{2}{2q-p}}\left(\int_{B_{t'}}(1+|Du_j|)^{2q-p}dx\right)^{\frac{2q-p-2}{2q-p}}.
\end{aligned}$$

Using Young's inequality and Lemma 2.1.3, we obtain

$$\begin{aligned}
|I_3| \le & \varepsilon\int_\Omega \eta^2|\tau_h Du_j|^2(\mu^2+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{p-2}{2}}dx \\
& + \frac{C_\varepsilon(n,p,q,L)}{(t-s)^2}|h|^2\left(\int_{B_R}|D\psi|^{2q-p}dx\right)^{\frac{2}{2q-p}}\left(\int_{B_{t'}}(1+|Du_j|)^{2q-p}dx\right)^{\frac{2q-p-2}{2q-p}} \\
& + \frac{C_\varepsilon(n,p,q,L)}{(t-s)^2}|h|^2\int_{B_{t'}}(1+|Du_j|)^{2q-p}dx \\
\le & \varepsilon\int_\Omega \eta^2|\tau_h Du_j|^2(\mu^2+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{p-2}{2}}dx \\
& + \frac{C_\varepsilon(n,p,q,L)}{(t-s)^2}|h|^2\int_{B_R}|D\psi|^{2q-p}dx \\
& + \frac{C_\varepsilon(n,p,q,L)}{(t-s)^2}|h|^2\int_{B_{t'}}(1+|Du_j|)^{2q-p}dx.
\end{aligned} \tag{4.3.12}$$

Recalling the first inequality of (4.3.7), we can write

$$\begin{aligned}
|I_3| \le & \varepsilon\int_\Omega \eta^2|\tau_h Du_j|^2(\mu^2+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{p-2}{2}}dx \\
& + \frac{C_\varepsilon(n,p,q,L)}{(t-s)^2}|h|^2\int_{B_R}|D\psi|^{2q-p}dx \\
& + \frac{C_\varepsilon(n,p,q,L,R)}{(t-s)^2}|h|^2\left(\int_{B_{t'}}(1+|Du_j|)^{\frac{r(2q-p)}{r-2}}dx\right)^{\frac{r-2}{r}}.
\end{aligned} \tag{4.3.13}$$

In order to estimate the integral $I_4$, we use assumption (F5), and Young's and Hölder's inequalities as follows

$$\begin{aligned}|I_4| \leq& \int_\Omega \eta^2 |\tau_h Du_j||h|^\alpha (k(x+h)+k(x))(1+|Du_j(x)|)^{\frac{q-1}{2}} dx\\ \leq& \varepsilon \int_\Omega \eta^2 |\tau_h Du_j|^2 (\mu^2 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{\frac{p-2}{2}} dx\\ &+ C_\varepsilon |h|^{2\alpha} \int_{B_t} (k(x+h)+k(x))^2 (1+|Du_j|)^{2q-p} dx\\ \leq& \varepsilon \int_\Omega \eta^2 |\tau_h Du_j|^2 (\mu^2 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{\frac{p-2}{2}} dx\\ &+ C_\varepsilon |h|^{2\alpha} \left(\int_{B_R} k^r dx\right)^{\frac{2}{r}} \left(\int_{B_t} (1+|Du_j|)^{\frac{r(2q-p)}{r-2}} dx\right)^{\frac{r-2}{r}}. \end{aligned} \tag{4.3.14}$$

We now take care of $I_5$. Similarly as above, exploiting assumption (F5) and Hölder's inequality, we infer

$$\begin{aligned}|I_5| \leq& \int_\Omega \eta^2 |\tau_h D\psi||h|^\alpha \left(k(x+h)+k(x)\right) \left(1+|Du_j|^2\right)^{\frac{q-1}{2}} dx\\ \leq& |h|^\alpha \left(\int_{B_{t'}} k^r dx\right)^{\frac{1}{r}} \left(\int_{B_t} |\tau_h D\psi|^{\frac{r}{r-1}} (1+|Du_j|)^{\frac{r(q-1)}{r-1}} dx\right)^{\frac{r-1}{r}}\\ \leq& |h|^\alpha \left(\int_{B_R} k^r dx\right)^{\frac{1}{r}} \left(\int_{B_t} |\tau_h D\psi|^{2q-p} dx\right)^{\frac{1}{2q-p}} \left(\int_{B_t} (1+|Du_j|)^{\frac{r(q-1)(2q-p)}{(r-1)(2q-p)-r}} dx\right)^{\frac{(r-1)(2q-p)-r}{r(2q-p)}}. \end{aligned}$$

Now, we observe

$$\frac{r(q-1)(2q-p)}{(r-1)(2q-p)-r} \leq \frac{r(2q-p)}{r-2} \Leftrightarrow p-2+r(q-p) \geq 0, \tag{4.3.15}$$

which is true by assumption, that is $p \geq 2$, $r > \frac{n}{\alpha} > 2$ and $q > p$. Thus, (4.3.15) and Hölder's inequality imply

$$|I_5| \leq C|h|^{\alpha+\gamma} \left(\int_{B_R} k^r dx\right)^{\frac{1}{r}} [D\psi]_{B^\gamma_{2q-p,\infty}(B_R)} \left(\int_{B_t} (1+|Du_j|)^{\frac{r(2q-p)}{r-2}} dx\right)^{\frac{(r-2)(q-1)}{r(2q-p)}}, \tag{4.3.16}$$

for a constant $C := C(n,p,q,r,R)$.

From assumption (F5), the hypothesis $|D\eta| < \frac{C}{t-s}$ and Hölder's inequality, we infer the following estimate for $I_6$.

$$\begin{aligned}|I_6| \leq& \frac{C}{t-s}|h|^\alpha \int_{B_t} |\tau_h \psi|(k(x+h)+k(x))(1+|Du_j|^2)^{\frac{q-1}{2}} dx\\ &+ \frac{C}{t-s}|h|^\alpha \int_{B_t} |\tau_h u_j|(k(x+h)+k(x))(1+|Du_j|^2)^{\frac{q-1}{2}} dx\\ \leq& \frac{C}{t-s}|h|^\alpha \left(\int_{B_R} k^r dx\right)^{\frac{1}{r}} \left(\int_{B_t} |\tau_h \psi|^{2q-p} dx\right)^{\frac{1}{2q-p}} \end{aligned}$$

$$\cdot\left(\int_{B_t}(1+|Du_j|)^{\frac{r(q-1)(2q-p)}{(r-1)(2q-p)-r}}dx\right)^{\frac{(r-1)(2q-p)-r}{r(2q-p)}}$$

$$+\frac{C}{t-s}|h|^{\alpha}\left(\int_{B_{t'}}k^r dx\right)^{\frac{1}{r}}\left(\int_{B_t}|\tau_h u_j|^{\frac{r}{r-1}}(1+|Du_j|)^{\frac{r(q-1)}{r-1}}dx\right)^{\frac{r-1}{r}}.$$

Using once again Hölder's inequality, we have

$$\begin{aligned}|I_6|\leq&\frac{C}{t-s}|h|^{\alpha+1}\left(\int_{B_R}k^r dx\right)^{\frac{1}{r}}\left(\int_{B_{t'}}|D\psi|^{2q-p}dx\right)^{\frac{1}{2q-p}}\\&\cdot\left(\int_{B_t}(1+|Du_j|)^{\frac{r(q-1)(2q-p)}{(r-1)(2q-p)-r}}dx\right)^{\frac{(r-1)(2q-p)-r}{r(2q-p)}}\\&+\frac{C}{t-s}|h|^{\alpha}\left(\int_{B_R}k^r dx\right)^{\frac{1}{r}}\left(\int_{B_t}|\tau_h u_j|^{\frac{rq}{r-1}}dx\right)^{\frac{r-1}{rq}}\left(\int_{B_t}(1+|Du_j|)^{\frac{rq}{r-1}}dx\right)^{\frac{(r-1)(q-1)}{rq}}.\end{aligned}$$

From Lemma 2.1.3 we find that

$$\begin{aligned}|I_6|\leq&\frac{C}{t-s}|h|^{\alpha+1}\left(\int_{B_R}k^r dx\right)^{\frac{1}{r}}\left(\int_{B_R}|D\psi|^{2q-p}dx\right)^{\frac{1}{2q-p}}\\&\cdot\left(\int_{B_t}(1+|Du_j|)^{\frac{r(2q-p)}{r-2}}dx\right)^{\frac{(q-1)(r-2)}{r(2q-p)}}\\&+\frac{C(n,p)}{t-s}|h|^{\alpha+1}\left(\int_{B_R}k^r dx\right)^{\frac{1}{r}}\left(\int_{B_{t'}}(1+|Du_j|)^{\frac{rq}{r-1}}dx\right)^{\frac{r-1}{rq}}.\end{aligned}$$

We remark that

$$\frac{rq}{r-1}\leq\frac{r(2q-p)}{r-2}\Leftrightarrow p+r(q-p)\geq 0, \tag{4.3.17}$$

which is true by assumption, that is $p\geq 2$, $r>\frac{n}{\alpha}>2$ and $q>p$. Hence

$$\begin{aligned}|I_6|\leq&\frac{C}{t-s}|h|^{\alpha+1}\left(\int_{B_R}k^r dx\right)^{\frac{1}{r}}\left(\int_{B_R}|D\psi|^{2q-p}dx\right)^{\frac{1}{2q-p}}\\&\cdot\left(\int_{B_t}(1+|Du_j|)^{\frac{r(2q-p)}{r-2}}dx\right)^{\frac{(q-1)(r-2)}{r(2q-p)}}\\&+\frac{C(n,p,q,r,R)}{t-s}|h|^{\alpha+1}\left(\int_{B_R}k^r dx\right)^{\frac{1}{r}}\left(\int_{B_{t'}}(1+|Du_j|)^{\frac{r(2q-p)}{r-2}}dx\right)^{\frac{q(r-2)}{r(2q-p)}}.\end{aligned}\tag{4.3.18}$$

Inserting estimates (4.3.10), (4.3.11), (4.3.13), (4.3.14), (4.3.16) and (4.3.18) in (4.3.9), we derive

$$\begin{aligned}&\nu\int_{\Omega}\eta^2|\tau_h Du_j|^2(\mu^2+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{p-2}{2}}dx\\&\quad\leq 3\varepsilon\int_{\Omega}\eta^2|\tau_h Du_j|^2(\mu^2+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{p-2}{2}}dx\end{aligned}$$

$$
\begin{aligned}
&+ C_\varepsilon |h|^{2\gamma} [D\psi]^{2q-p}_{B^\gamma_{2q-p,\infty}(B_R)} + C_\varepsilon |h|^{2\gamma} \left( \int_{B_{t'}} (1+|Du_j|)^{\frac{r(2q-p)}{r-2}} dx \right)^{\frac{r-2}{r}} \\
&+ \frac{C_\varepsilon}{(t-s)^2} |h|^2 \int_{B_R} |D\psi|^{2q-p} dx \\
&+ \frac{C_\varepsilon}{(t-s)^2} |h|^2 \left( \int_{B_{t'}} (1+|Du_j|)^{\frac{r(2q-p)}{r-2}} dx \right)^{\frac{r-2}{r}} \\
&+ C_\varepsilon |h|^{2\alpha} \left( \int_{B_R} k^r dx \right)^{\frac{2}{r}} \left( \int_{B_t} (1+|Du_j|)^{\frac{r(2q-p)}{r-2}} dx \right)^{\frac{r-2}{r}} \\
&+ C_\varepsilon |h|^{\alpha+\gamma} \left( \int_{B_R} k^r dx \right)^{\frac{1}{r}} [D\psi]_{B^\gamma_{2q-p,\infty}(B_R)} \left( \int_{B_t} (1+|Du_j|)^{\frac{r(2q-p)}{r-2}} dx \right)^{\frac{(r-2)(q-1)}{r(2q-p)}} \\
&+ \frac{C_\varepsilon}{t-s} |h|^{\alpha+1} \left( \int_{B_R} k^r dx \right)^{\frac{1}{r}} \left( \int_{B_R} |D\psi|^{2q-p} dx \right)^{\frac{1}{2q-p}} \\
&\cdot \left( \int_{B_t} (1+|Du_j|)^{\frac{r(2q-p)}{r-2}} dx \right)^{\frac{(q-1)(r-2)}{r(2q-p)}} \\
&+ \frac{C_\varepsilon}{t-s} |h|^{\alpha+1} \left( \int_{B_R} k^r dx \right)^{\frac{1}{r}} \left( \int_{B_{t'}} (1+|Du_j|)^{\frac{r(2q-p)}{r-2}} dx \right)^{\frac{q(r-2)}{r(2q-p)}}, \qquad (4.3.19)
\end{aligned}
$$

with a constant $C_\varepsilon := C_\varepsilon(n,p,q,r,L,R)$. We now introduce the following interpolation inequality

$$
\|Dw\|_{\frac{r(2q-p)}{r-2}} \le \|Dw\|_p^\delta \|Dw\|^{1-\delta}_{\frac{np}{n-2\beta}}, \qquad (4.3.20)
$$

where $0 < \delta < 1$ is defined through the equality

$$
\frac{r-2}{r(2q-p)} = \frac{\delta}{p} + \frac{(1-\delta)(n-2\beta)}{np} \qquad (4.3.21)
$$

which implies

$$
\delta = \frac{nr(p-q) - np + \beta r(2q-p)}{\beta r(2q-p)}, \quad 1-\delta = \frac{n[r(q-p)+p]}{\beta r(2q-p)}.
$$

Hence we get the following inequalities

$$
\begin{aligned}
\left( \int_{B_{t'}} (1+|Du_j|)^{\frac{r(2q-p)}{r-2}} dx \right)^{\frac{r-2}{r}} \le& \left( \int_{B_{t'}} (1+|Du_j|)^p dx \right)^{\frac{\delta(2q-p)}{p}} \\
&\cdot \left( \int_{B_{t'}} (1+|Du_j|)^{\frac{np}{n-2\beta}} dx \right)^{\frac{(n-2\beta)[r(q-p)+p]}{\beta pr}}, \qquad (4.3.22) \\
\left( \int_{B_t} (1+|Du_j|)^{\frac{r(2q-p)}{r-2}} dx \right)^{\frac{(r-2)(q-1)}{r(2q-p)}} \le& \left( \int_{B_t} (1+|Du_j|)^p dx \right)^{\frac{\delta(q-1)}{p}} \\
&\cdot \left( \int_{B_t} (1+|Du_j|)^{\frac{np}{n-2\beta}} dx \right)^{\frac{(n-2\beta)(q-1)p'}{p}}, \qquad (4.3.23)
\end{aligned}
$$

$$\left(\int_{B_{t'}}(1+|Du_j|)^{\frac{r(2q-p)}{r-2}}dx\right)^{\frac{q(r-2)}{r(2q-p)}} \leq \left(\int_{B_{t'}}(1+|Du_j|)^p dx\right)^{\frac{\delta q}{p}} \cdot \left(\int_{B_{t'}}(1+|Du_j|)^{\frac{np}{n-2\beta}}dx\right)^{\frac{(n-2\beta)q[r(q-p)+p]}{\beta pr(2q-p)}}, \qquad (4.3.24)$$

where $p' = \frac{r(q-p)+p}{\beta r(2q-p)}$.

Inserting (4.3.22), (4.3.23) and (4.3.24) in (4.3.19), and exploiting the bounds

$$\frac{n[r(q-p)+p]}{\beta pr} < 1, \quad \frac{n(q-1)[r(q-p)+p]}{\beta rp(2q-p)} < 1, \quad \frac{nq[r(q-p)+p]}{\beta pr(2q-p)} < 1, \qquad (4.3.25)$$

which hold by assumption (4.0.4) and for $\beta \in (\frac{n[r(q-p)+p]}{pr}, \alpha)$, from Young's inequality, we infer

$$\begin{aligned}
&\nu \int_\Omega \eta^2|\tau_h Du_j|^2(\mu^2+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{p-2}{2}}dx \\
&\leq 3\varepsilon \int_\Omega \eta^2|\tau_h Du_j|^2(\mu^2+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{p-2}{2}}dx \\
&\quad + C_\varepsilon|h|^{2\gamma}[D\psi]^{2q-p}_{B^\gamma_{2q-p,\infty}(B_R)} + C_{\varepsilon,\theta}|h|^{2\gamma}\left(\int_{B_R}(1+|Du_j|)^p dx\right)^{\frac{\delta(2q-p)\tilde{p}}{p}} \\
&\quad + \theta|h|^{2\gamma}\left(\int_{B_{t'}}(1+|Du_j|)^{\frac{np}{n-2\beta}}dx\right)^{\frac{n-2\beta}{n}} \\
&\quad + \frac{C_\varepsilon}{(t-s)^2}|h|^2\int_{B_R}|D\psi|^{2q-p}dx \\
&\quad + \theta|h|^2\left(\int_{B_{t'}}(1+|Du_j|)^{\frac{np}{n-2\beta}}dx\right)^{\frac{n-2\beta}{n}} \\
&\quad + \frac{C_{\varepsilon,\theta}}{(t-s)^{2\tilde{p}}}|h|^2\left(\int_{B_R}(1+|Du_j|)^p dx\right)^{\frac{\tilde{p}\delta(2q-p)}{p}} \\
&\quad + C_{\varepsilon,\theta}|h|^{2\alpha}\left(\int_{B_R}k^r dx\right)^{\frac{2\tilde{p}}{r}} \\
&\quad \cdot \left(\int_{B_R}(1+|Du_j|)^p dx\right)^{\frac{\tilde{p}\delta(2q-p)}{p}} \\
&\quad + \theta|h|^{2\alpha}\left(\int_{B_{t'}}(1+|Du_j|)^{\frac{np}{n-2\beta}}dx\right)^{\frac{n-2\beta}{n}} \\
&\quad + C_\theta|h|^{\alpha+\gamma}\left(\int_{B_R}k^r dx\right)^{\frac{p''}{r}} \\
&\quad \cdot [D\psi]^{p''}_{B^\gamma_{2q-p,\infty}(B_R)} \\
&\quad \cdot \left(\int_{B_R}(1+|Du_j|)^p dx\right)^{\frac{\delta(q-1)(2q-p)p''}{p}}
\end{aligned}$$

$$
\begin{aligned}
&+\theta|h|^{\alpha+\gamma}\left(\int_{B_t}(1+|Du_j|)^{\frac{np}{n-2\beta}}dx\right)^{\frac{n-2\beta}{n}}\\
&+\frac{C_\theta}{(t-s)^{p''}}|h|^{\alpha+1}\left(\int_{B_R}k^r dx\right)^{\frac{p''}{r}}\left(\int_{B_R}|D\psi|^{2q-p}dx\right)^{\frac{p''}{2q-p}}\\
&\cdot\left(\int_{B_R}(1+|Du_j|)^p dx\right)^{\frac{\delta(q-1)p''}{p}}\\
&+\theta|h|^{\alpha+1}\left(\int_{B_t}(1+|Du_j|)^{\frac{np}{n-2\beta}}dx\right)^{\frac{n-2\beta}{n}}\\
&+\frac{C_\theta}{(t-s)^{p^*}}|h|^{\alpha+1}\left(\int_{B_R}k^r dx\right)^{\frac{p^*}{r}}\left(\int_{B_R}(1+|Du_j|)^p dx\right)^{\frac{p^*\delta q}{p}}\\
&+\theta|h|^{\alpha+1}\left(\int_{B_{t'}}(1+|Du_j|)^{\frac{np}{n-2\beta}}dx\right)^{\frac{n-2\beta}{n}},
\end{aligned}
\tag{4.3.26}
$$

for some constant $\theta\in(0,1)$, where we set $\tilde p=\frac{\beta pr}{\beta pr-n[r(q-p)+p]}$, $p''=\frac{\beta rp(2q-p)}{\beta rp(2q-p)-(q-1)n[r(q-p)+p]}$, $p^*=\frac{p}{p-(1-\delta)q}$.
For a better readability we now define

$$
\begin{aligned}
A:=&C_\varepsilon[D\psi]^{2q-p}_{B^\gamma_{2q-p,\infty}(B_R)}+C_{\varepsilon,\theta}\left(\int_{B_R}(1+|Du_j|)^p dx\right)^{\frac{\delta(2q-p)\tilde p}{p}}\\
&+C_{\varepsilon,\theta}\left(\int_{B_R}k^r dx\right)^{\frac{2\tilde p}{r}}\left(\int_{B_R}(1+|Du_j|)^p dx\right)^{\frac{\tilde p\delta(2q-p)}{p}}\\
&+C_\theta\left(\int_{B_R}k^r dx\right)^{\frac{p''}{r}}[D\psi]^{p''}_{B^\gamma_{2q-p,\infty}(B_R)}\left(\int_{B_R}(1+|Du_j|)^p dx\right)^{\frac{\delta(q-1)(2q-p)p''}{p}}\\
B_1:=&C_\varepsilon\int_{B_R}|D\psi|^{2q-p}dx,\\
B_2:=&C_{\varepsilon,\theta}\left(\int_{B_R}(1+|Du_j|)^p dx\right)^{\frac{\tilde p\delta(2q-p)}{p}},\\
B_3:=&C_\theta\left(\int_{B_R}k^r dx\right)^{\frac{p''}{r}}\left(\int_{B_R}|D\psi|^{2q-p}dx\right)^{\frac{p''}{2q-p}}\left(\int_{B_R}(1+|Du_j|)^p dx\right)^{\frac{\delta(q-1)p''}{p}},\\
B_4:=&C_\theta\left(\int_{B_R}k^r dx\right)^{\frac{p^*}{r}}\left(\int_{B_R}(1+|Du_j|)^p dx\right)^{\frac{p^*\delta q}{p}},
\end{aligned}
$$

so that we can rewrite the previous estimate as

$$
\begin{aligned}
\nu&\int_\Omega\eta^2|\tau_h Du_j|^2(\mu^2+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{p-2}{2}}dx\\
&\le 3\varepsilon\int_\Omega\eta^2|\tau_h Du_j|^2(\mu^2+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{p-2}{2}}dx\\
&\quad+\theta(|h|^{2\alpha}+|h|^{\alpha+\gamma}+|h|^{\alpha+1})\left(\int_{B_t}(1+|Du_j|)^{\frac{np}{n-2\beta}}dx\right)^{\frac{n-2\beta}{n}}
\end{aligned}
$$

$$+\theta(|h|^2+|h|^{2\gamma}+|h|^{\alpha+1})\left(\int_{B_{t'}}(1+|Du_j|)^{\frac{np}{n-2\beta}}dx\right)^{\frac{n-2\beta}{n}}$$
$$+(|h|^{2\gamma}+|h|^{2\alpha}+|h|^{\alpha+\gamma})A+|h|^2\frac{B_1}{(t-s)^2}+|h|^2\frac{B_2}{(t-s)^{2\tilde p}}$$
$$+|h|^{\alpha+1}\frac{B_3}{(t-s)^{p''}}+|h|^{\alpha+1}\frac{B_4}{(t-s)^{p^*}}.$$

Choosing $\varepsilon=\frac{\nu}{6}$, we can reabsorb the first integral in the right hand side of the previous estimate by the left hand side, thus getting

$$\int_\Omega \eta^2|\tau_h Du_j|^2(\mu^2+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{p-2}{2}}dx$$
$$\le 3\theta|h|^{2\alpha}\left(\int_{B_t}(1+|Du_j|)^{\frac{np}{n-2\beta}}dx\right)^{\frac{n-2\beta}{n}}+3\theta|h|^{2\alpha}\left(\int_{B_{t'}}(1+|Du_j|)^{\frac{np}{n-2\beta}}dx\right)^{\frac{n-2\beta}{n}}$$
$$+|h|^{2\alpha}A+|h|^2\frac{B_1}{(t-s)^2}+|h|^2\frac{B_2}{(t-s)^{\tilde p}}+|h|^{2\alpha}\frac{B_3}{(t-s)^{p''}}+|h|^{2\alpha}\frac{B_4}{(t-s)^{p^*}},$$

where we used the fact that $\alpha<\gamma$. Using Lemma 2.0.2 in the left hand side of the previous inequality, recalling that $\eta=1$ on $B_s$, we get

$$\int_{B_s}|\tau_h V_p(Du_j)|^2dx\le |h|^{2\alpha}\Bigg\{3\theta\left(\int_{B_t}(1+|Du_j|)^{\frac{np}{n-2\beta}}dx\right)^{\frac{n-2\beta}{n}}+3\theta\left(\int_{B_{t'}}(1+|Du_j|)^{\frac{np}{n-2\beta}}dx\right)^{\frac{n-2\beta}{n}}$$
$$+A+\frac{B_1}{(t-s)^2}+\frac{B_2}{(t-s)^{\tilde p}}+\frac{B_3}{(t-s)^{p''}}+\frac{B_4}{(t-s)^{p^*}}\Bigg\}. \tag{4.3.27}$$

Lemma 2.1.4 and inequality (2.0.3) imply

$$\left(\int_{B_s}|Du_j|^{\frac{np}{n-2\beta}}dx\right)^{\frac{n-2\beta}{n}}\le 3\theta\left(\int_{B_t}(1+|Du_j|)^{\frac{np}{n-2\beta}}dx\right)^{\frac{n-2\beta}{n}}+3\theta\left(\int_{B_{t'}}(1+|Du_j|)^{\frac{np}{n-2\beta}}dx\right)^{\frac{n-2\beta}{n}}$$
$$+A+\frac{B_1}{(t-s)^2}+\frac{B_2}{(t-s)^{2\tilde p}}+\frac{B_3}{(t-s)^{p''}}+\frac{B_4}{(t-s)^{p^*}}, \tag{4.3.28}$$

for some $\beta:=\beta(n,p,q,r)\in(0,\alpha)$.
Setting

$$\Phi(r)=\left(\int_{B_r}|Du_j|^{\frac{np}{n-2\beta}}dx\right)^{\frac{n-2\beta}{n}},$$

we can write inequality (4.3.28) as

$$\Phi(s)\le 3\theta\Phi(t)+3\theta\Phi(t')+A+\frac{B_1}{(t-s)^2}+\frac{B_2}{(t-s)^{2\tilde p}}+\frac{B_3}{(t-s)^{p''}}+\frac{B_4}{(t-s)^{p^*}}. \tag{4.3.29}$$

By virtue of Lemma 2.0.3, choosing $0<\theta<1/3$, we obtain

$$\Phi(\varrho)\le c\Big(3\theta\Phi(t')+A+\frac{B_1}{R^2}+\frac{B_2}{R^{2\tilde p}}+\frac{B_3}{R^{p''}}+\frac{B_4}{R^{p^*}}\Big), \tag{4.3.30}$$

for some constant $c := c(n,p,q,r,\alpha,\theta)$. Then, applying Lemma 2.0.3 again, we get

$$\Phi\left(\frac{R}{4}\right) \le \tilde{c}\left(A + \frac{B_1}{R^2} + \frac{B_2}{R^{2\tilde{p}}} + \frac{B_3}{R^{p''}} + \frac{B_4}{R^{p^*}}\right), \tag{4.3.31}$$

with $\tilde{c} := \tilde{c}(n,p,q,r,\alpha)$.
Now, recalling the definition of $\Phi$, we obtain

$$\left(\int_{B_{R/4}} |Du_j|^{\frac{np}{n-2\beta}} dx\right)^{\frac{n-2\beta}{n}} \le \tilde{c}\left\{\int_{B_R} (1+|Du_j|^p)dx + \|D\psi\|_{B^\gamma_{2q-p,\infty}(B_R)} + \|k\|_{L^r(B_R)}\right\}^\kappa, \tag{4.3.32}$$

thus, from inequalities (4.3.32) and (4.3.27), we deduce the a priori estimate

$$\int_{B_{R/4}} |\tau_h V_p(Du_j)|^2 dx \le C|h|^{2\alpha}\left\{\int_{B_R} (1+|Du_j|^p)dx + \|D\psi\|_{B^\gamma_{2q-p,\infty}(B_R)} + \|k\|_{L^r(B_R)}\right\}^\kappa, \tag{4.3.33}$$

where $C := C(n,p,q,r,\alpha,\nu,L,R)$ and $\kappa := \kappa(n,p,q,r,\alpha)$ are both independent of $j$. □

### 4.3.2 Passage to the limit

Let $u \in \mathcal{K}_\psi(\Omega)$ be the solution to (4.0.1), and let $F_j$ be defined as in Lemma 4.2.2. From Theorem 4.2.1, there exists $c_1 > 0$ such that

$$|\xi|^p \le c_1(1 + F_j(x,\xi)), \quad \forall j \in \mathbb{N}. \tag{4.3.34}$$

Fixed $B_R \Subset \Omega$, let $u_j$ be the solution of the problem

$$\min\left\{\int_{B_R} F_j(x,Dw)dx : w \ge \psi \text{ a.e. in } B_R,\ w \in u + W^{1,p}_0(B_R)\right\}.$$

From (4.3.34), the minimality of $u_j$ implies

$$\begin{aligned}\int_{B_R} |Du_j|^p dx \le& c_1 \int_{B_R} (1 + F_j(x,Du_j))dx \\ \le& c_1 \int_{B_R} (1 + F_j(x,Du))dx \\ \le& c_1 \int_{B_R} (1 + F(x,Du))dx,\end{aligned} \tag{4.3.35}$$

where in the last inequality we used Lemma 4.2.2. Thus, up to subsequences,

$$u_j \rightharpoonup \tilde{u} \text{ in } u + W^{1,p}_0(B_R) \tag{4.3.36}$$

and

$$u_j \to \tilde{u} \text{ in } L^p(B_R). \tag{4.3.37}$$

For any $j$, $F_j$ satisfies the assumptions of Theorem 4.3.1. Combining (4.3.32) and (4.3.35) we get

$$\|Du_j\|_{L^{\frac{np}{n-2\beta}}(B_{R/4})} \leq \tilde{c}\left\{\int_{B_R}(1+F(x,Du))dx + \|D\psi\|_{B^{\gamma}_{2q-p,\infty}(B_R)} + \|k\|_{L^r(B_R)}\right\}^{\tilde{\kappa}}, \tag{4.3.38}$$

thus, by (4.3.36), (4.3.38) and the weak lower semicontinuity pf the norm, we infer

$$\begin{aligned}\|D\tilde{u}\|_{L^{\frac{np}{n-2\beta}}(B_{R/4})} \leq& \liminf_{j\to\infty} \|Du_j\|_{L^{\frac{np}{n-2\beta}}(B_{R/4})} \\ \leq& \tilde{c}\left\{\int_{B_R}(1+F(x,Du))dx + \|D\psi\|_{B^{\gamma}_{2q-p,\infty}(B_R)} + \|k\|_{L^r(B_R)}\right\}^{\tilde{\kappa}}.\end{aligned} \tag{4.3.39}$$

Let $j_0, j \in \mathbb{N}$ such that $j_0 < j$. Then, by Lemma 4.2.2 and the fact that $u_j$ is the minimum for $F_j$, we might write

$$\begin{aligned}\int_{B_R} F_{j_0}(x,Du_j)dx \leq& \int_{B_R} F_j(x,Du_j)dx \\ \leq& \int_{B_R} F_j(x,Du)dx \leq \int_{B_R} F(x,Du)dx.\end{aligned}$$

Now from the weak lower semicontinuity of $F_{j_0}$ and (4.3.36), it holds, for every $j_0 \in \mathbb{N}$, that

$$\int_{B_R} F_{j_0}(x,D\tilde{u})dx \leq \liminf_{j\to+\infty}\int_{B_R} F_{j_0}(x,Du_j)dx \leq \int_{B_R} F(x,Du)dx.$$

Combining these last inequalities, we get

$$\int_{B_R} F(x,D\tilde{u})dx = \lim_{j_0\to+\infty}\int_{B_R} F_{j_0}(x,D\tilde{u})dx \leq \int_{B_R} F(x,Du)dx, \tag{4.3.40}$$

where we also applied the monotone convergence theorem, according to Lemma 4.2.2. Moreover, by the weak convergence (4.3.36), the limit function $\tilde{u}$ still belongs to $\mathcal{K}_\psi(B_R)$, since this set is convex and closed. Thus, we can conclude that

$$\tilde{u} = u \quad \text{a.e. in } B_R \tag{4.3.41}$$

by the strict convexity of $F$, and, recalling estimate (4.3.39),

$$\|Du\|_{L^{\frac{np}{n-2\beta}}(B_{R/4})} \leq \tilde{c}\left\{\int_{B_R}(1+F(x,Du))dx + \|D\psi\|_{B^{\gamma}_{2q-p,\infty}(B_R)} + \|k\|_{L^r(B_R)}\right\}^{\tilde{\kappa}}. \tag{4.3.42}$$

Finally, we can repeat the proof of Theorem 4.3.1 obtaining $V_p(Du) \in B^{\alpha}_{2,\infty,\mathrm{loc}}(\Omega)$.

## 4.4 Proof of Theorem 4.0.2

This section is devoted to the proof of Theorem 4.0.2. We here focus only on the derivation of the a priori estimate. Indeed, the limit procedure is achieved using the same arguments presented in Sections 4.2 (cnfr. Remark 4.2.3) and 4.3.2.

*Proof of Theorem 4.0.2.* We a priori assume that the integral

$$\int_{\Omega'} (1+|Du|)^{\frac{np}{n-2\lambda}} dx$$

is finite, for every $\Omega' \Subset \Omega$, where we set $\lambda := \min\{\alpha, \gamma\}$.

Arguing analogously as in the proof of Theorem 4.3.1, we define the integrals $I_1$–$I_6$ according to (4.3.8) and we are able to derive estimates (4.3.9) and (4.3.10). We need to treat differently the integrals $I_2 - I_6$ in which the new assumptions (F6) on the map $x \mapsto \mathcal{A}(x,\xi)$ comes into the play.

Similarly as we did for (4.3.7) but using this time (4.0.5), we get

$$2q-p \leq \frac{r(2q-p)}{r-2} \leq \frac{np}{n-2\lambda}. \tag{4.4.1}$$

Consider the integral $I_2$, then according to $L^p$ embeddings and Young's inequality,

$$\begin{aligned}|I_2| \leq &\varepsilon \int_\Omega \eta^2|\tau_h Du|^2(\mu^2+|Du(x+h)|^2+|Du(x)|^2)^{\frac{p-2}{2}}dx\\ &+C_\varepsilon(L)\left(\int_{B_t}|\tau_h D\psi|^{2q-p}dx\right)^{\frac{2}{2q-p}}\left(\int_{B_{t'}}(1+|Du|)^{2q-p}dx\right)^{\frac{2q-p-2}{2q-p}}\\ \leq &\varepsilon \int_\Omega \eta^2|\tau_h Du|^2(\mu^2+|Du(x+h)|^2+|Du(x)|^2)^{\frac{p-2}{2}}dx\\ &+C_\varepsilon(L)\left(\int_{B_{R/2}}|\tau_h D\psi|^{2q-p}dx\right)^{\frac{2}{2q-p}}\left(\int_{B_{t'}}(1+|Du|)^{\frac{r(2q-p)}{r-2}}dx\right)^{\frac{(r-2)(2q-p-2)}{r(2q-p)}},\end{aligned} \tag{4.4.2}$$

where in the last inequality we used (4.4.1).

In order to take care of $I_3$, we are able to perform the same computations which led us to (4.3.13), that is

$$\begin{aligned}|I_3| \leq &\varepsilon \int_\Omega \eta^2|\tau_h Du|^2(\mu^2+|Du(x+h)|^2+|Du(x)|^2)^{\frac{p-2}{2}}dx\\ &+\frac{C_\varepsilon(n,p,q,L)}{(t-s)^2}|h|^2\int_{B_R}|D\psi|^{2q-p}dx\\ &+\frac{C_\varepsilon(n,p,q,L,R)}{(t-s)^2}|h|^2\left(\int_{B_{t'}}(1+|Du|)^{\frac{r(2q-p)}{r-2}}dx\right)^{\frac{r-2}{r}}.\end{aligned} \tag{4.4.3}$$

Now, we estimate the integral $I_4$. For $h \in \mathbb{R}^n$ such that $2^{-k}(t'-t) \leq |h| \leq 2^{-k+1}(t'-t)$, $k \in \mathbb{N}$, assumption (F6), Young's and Hölder's inequalities yield that

$$\begin{aligned}|I_4| \leq &\int_\Omega \eta^2|\tau_h Du||h|^\alpha(g_k(x+h)+g_k(x))(1+|Du(x)|)^{\frac{q-1}{2}}dx\\ \leq &\varepsilon \int_\Omega \eta^2|\tau_h Du|^2(\mu^2+|Du(x+h)|^2+|Du(x)|^2)^{\frac{p-2}{2}}dx\\ &+C_\varepsilon|h|^{2\alpha}\int_{B_t}(g_k(x+h)+g_k(x))^2(1+|Du|)^{2q-p}dx\end{aligned}$$

$$\leq \varepsilon \int_\Omega \eta^2 |\tau_h Du|^2 (\mu^2 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} dx$$
$$+ C_\varepsilon |h|^{2\alpha} \left( \int_{B_t} (g_k(x+h) + g_k(x))^r dx \right)^{\frac{2}{r}} \left( \int_{B_t} (1+|Du|)^{\frac{r(2q-p)}{r-2}} dx \right)^{\frac{r-2}{r}}. \tag{4.4.4}$$

Exploiting assumption (F6), Hölder's inequality and (4.3.15), we infer the following estimate for the integral $I_5$

$$|I_5| \leq \int_\Omega \eta^2 |\tau_h D\psi| |h|^\alpha \left( g_k(x+h) + g_k(x) \right) \left(1+|Du|^2\right)^{\frac{q-1}{2}} dx$$
$$\leq |h|^\alpha \left( \int_{B_t} (g_k(x+h) + g_k(x))^r dx \right)^{\frac{1}{r}} \left( \int_{B_t} |\tau_h D\psi|^{2q-p} dx \right)^{\frac{1}{2q-p}}$$
$$\cdot \left( \int_{B_t} (1+|Du|)^{\frac{r(q-1)(2q-p)}{(r-1)(2q-p)-r}} dx \right)^{\frac{(r-1)(2q-p)-r}{r(2q-p)}}$$
$$\leq C(n,p,q,r,R)|h|^\alpha \left( \int_{B_{R/2}} (g_k(x+h) + g_k(x))^r dx \right)^{\frac{1}{r}} \left( \int_{B_{R/2}} |\tau_h D\psi|^{2q-p} dx \right)^{\frac{1}{2q-p}}$$
$$\cdot \left( \int_{B_t} (1+|Du|)^{\frac{r(2q-p)}{r-2}} dx \right)^{\frac{(r-2)(q-1)}{r(2q-p)}}, \tag{4.4.5}$$

where $2^{-k}(t'-t) \leq |h| \leq 2^{-k+1}(t'-t)$, $k \in \mathbb{N}$.

Similarly as above, from assumption (F6), (4.3.15), hypothesis $|D\eta| < \frac{C}{t-s}$ and Hölder's inequality, we can estimate the integral $I_6$ as follows

$$|I_6| \leq \frac{C}{t-s} |h|^\alpha \int_{B_t} |\tau_h \psi| (g_k(x+h) + g_k(x))(1+|Du|^2)^{\frac{q-1}{2}} dx$$
$$+ \frac{C}{t-s} |h|^\alpha \int_{B_t} |\tau_h u| (g_k(x+h) + g_k(x))(1+|Du|^2)^{\frac{q-1}{2}} dx$$
$$\leq \frac{C}{t-s} |h|^\alpha \left( \int_{B_t} (g_k(x+h) + g_k(x))^r dx \right)^{\frac{1}{r}} \left( \int_{B_t} |\tau_h \psi|^{2q-p} dx \right)^{\frac{1}{2q-p}}$$
$$\cdot \left( \int_{B_t} (1+|Du|)^{\frac{r(q-1)(2q-p)}{(r-1)(2q-p)-r}} dx \right)^{\frac{(r-1)(2q-p)-r}{r(2q-p)}}$$
$$+ \frac{C}{t-s} |h|^\alpha \left( \int_{B_t} (g_k(x+h) + g_k(x))^r dx \right)^{\frac{1}{r}} \left( \int_{B_t} |\tau_h u|^{\frac{rq}{r-1}} dx \right)^{\frac{r-1}{rq}}$$
$$\cdot \left( \int_{B_t} (1+|Du|)^{\frac{rq}{r-1}} dx \right)^{\frac{(r-1)(q-1)}{rq}}$$
$$\leq \frac{C(n,p,q,r,R)}{t-s} |h|^{\alpha+1} \left( \int_{B_{R/2}} (g_k(x+h) + g_k(x))^r dx \right)^{\frac{1}{r}} \left( \int_{B_R} |D\psi|^{2q-p} dx \right)^{\frac{1}{2q-p}}$$
$$\cdot \left( \int_{B_t} (1+|Du|)^{\frac{r(2q-p)}{r-2}} dx \right)^{\frac{(r-2)(q-1)}{r(2q-p)}}$$
$$+ \frac{C(n,q,r)}{t-s} |h|^{\alpha+1} \left( \int_{B_{R/2}} (g_k(x+h) + g_k(x))^r dx \right)^{\frac{1}{r}} \left( \int_{B_{t'}} (1+|Du|)^{\frac{rq}{r-1}} dx \right)^{\frac{r-1}{r}}. \tag{4.4.6}$$

Inserting estimates (4.3.10), (4.4.2), (4.4.3), (4.4.4), (4.4.5) and (4.4.6) in (4.3.9), we infer

$$\begin{aligned}
\nu \int_\Omega & \eta^2 |\tau_h Du|^2 (\mu^2 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} dx \\
\leq & 3\varepsilon \int_\Omega \eta^2 |\tau_h Du|^2 (\mu^2 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} dx \\
& + C_\varepsilon \left( \int_{B_{R/2}} |\tau_h D\psi|^{2q-p} dx \right)^{\frac{2}{2q-p}} \left( \int_{B_{t'}} (1+|Du|)^{\frac{r(2q-p)}{r-2}} dx \right)^{\frac{(r-2)(2q-p-2)}{r(2q-p)}} \\
& + \frac{C_\varepsilon}{(t-s)^2} |h|^2 \int_{B_R} |D\psi|^{2q-p} dx \\
& + \frac{C_\varepsilon}{(t-s)^2} |h|^2 \left( \int_{B_{t'}} (1+|Du|)^{\frac{r(2q-p)}{r-2}} dx \right)^{\frac{r-2}{r}} \\
& + C_\varepsilon |h|^{2\alpha} \left( \int_{B_t} (g_k(x+h) + g_k(x))^r dx \right)^{\frac{2}{r}} \left( \int_{B_t} (1+|Du|)^{\frac{r(2q-p)}{r-2}} dx \right)^{\frac{r-2}{r}} \\
& + C_\varepsilon |h|^{\alpha} \left( \int_{B_{R/2}} (g_k(x+h) + g_k(x))^r dx \right)^{\frac{1}{r}} \left( \int_{B_{R/2}} |\tau_h D\psi|^{2q-p} dx \right)^{\frac{1}{2q-p}} \\
& \cdot \left( \int_{B_t} (1+|Du|)^{\frac{r(2q-p)}{r-2}} dx \right)^{\frac{(r-2)(q-1)}{r(2q-p)}} \\
& + \frac{C_\varepsilon}{t-s} |h|^{\alpha+1} \left( \int_{B_{R/2}} (g_k(x+h) + g_k(x))^r dx \right)^{\frac{1}{r}} \left( \int_{B_R} |D\psi|^{2q-p} dx \right)^{\frac{1}{2q-p}} \\
& \cdot \left( \int_{B_t} (1+|Du|)^{\frac{r(2q-p)}{r-2}} dx \right)^{\frac{(r-2)(q-1)}{r(2q-p)}} \\
& + \frac{C_\varepsilon}{t-s} |h|^{\alpha+1} \left( \int_{B_{R/2}} (g_k(x+h) + g_k(x))^r dx \right)^{\frac{1}{r}} \left( \int_{B_{t'}} (1+|Du|)^{\frac{rq}{r-1}} dx \right)^{\frac{r-1}{r}}, \qquad (4.4.7)
\end{aligned}$$

with a constant $C_\varepsilon := C_\varepsilon(n,p,q,r,L,R)$. Thanks to (4.4.1), we have the following interpolation inequality

$$\|Dw\|_{\frac{r(2q-p)}{r-2}} \leq \|Dw\|_p^\delta \|Dw\|_{\frac{np}{n-2\lambda}}^{1-\delta}, \qquad (4.4.8)$$

where $0 < \delta < 1$ is defined through the equality

$$\frac{r-2}{r(2q-p)} = \frac{\delta}{p} + \frac{(1-\delta)(n-2\lambda)}{np}. \qquad (4.4.9)$$

Hence, using the interpolation inequality (4.4.8), from estimate (4.4.7), we infer

$$\begin{aligned}
\nu \int_\Omega & \eta^2 |\tau_h Du|^2 (\mu^2 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} dx \\
\leq & 3\varepsilon \int_\Omega \eta^2 |\tau_h Du|^2 (\mu^2 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} dx
\end{aligned}$$

$$
\begin{aligned}
&+ C_\varepsilon \left( \int_{B_{R/2}} |\tau_h D\psi|^{2q-p} dx \right)^{\frac{2}{2q-p}} \left( \int_{B_R} (1+|Du|)^p dx \right)^{\frac{\delta(2q-p-2)}{p}} \\
&\cdot \left( \int_{B_{t'}} (1+|Du|)^{\frac{np}{n-2\lambda}} dx \right)^{\frac{(1-\delta)(n-2\lambda)(2q-p-2)}{np}} \\
&+ \frac{C_\varepsilon}{(t-s)^2} |h|^2 \int_{B_R} |D\psi|^{2q-p} dx \\
&+ \frac{C_\varepsilon}{(t-s)^2} |h|^2 \left( \int_{B_R} (1+|Du|)^p dx \right)^{\frac{\delta(2q-p)}{p}} \\
&\cdot \left( \int_{B_{t'}} (1+|Du|)^{\frac{np}{n-2\lambda}} dx \right)^{\frac{(1-\delta)(n-2\lambda)(2q-p)}{np}} \\
&+ C_\varepsilon |h|^{2\alpha} \left( \int_{B_{R/2}} (g_k(x+h)+g_k(x))^r dx \right)^{\frac{2}{r}} \left( \int_{B_R} (1+|Du|)^p dx \right)^{\frac{\delta(2q-p)}{p}} \\
&\cdot \left( \int_{B_t} (1+|Du|)^{\frac{np}{n-2\lambda}} dx \right)^{\frac{(1-\delta)(n-2\lambda)(2q-p)}{np}} \\
&+ C_\varepsilon |h|^{\alpha} \left( \int_{B_{R/2}} (g_k(x+h)+g_k(x))^r dx \right)^{\frac{1}{r}} \left( \int_{B_{R/2}} |\tau_h D\psi|^{2q-p} dx \right)^{\frac{1}{2q-p}} \\
&\cdot \left( \int_{B_R} (1+|Du|)^p dx \right)^{\frac{\delta(q-1)}{p}} \left( \int_{B_t} (1+|Du|)^{\frac{np}{n-2\lambda}} dx \right)^{\frac{(1-\delta)(n-2\lambda)(q-1)}{np}} \\
&+ \frac{C_\varepsilon}{t-s} |h|^{\alpha+1} \left( \int_{B_{R/2}} (g_k(x+h)+g_k(x))^r dx \right)^{\frac{1}{r}} \left( \int_{B_R} |D\psi|^{2q-p} dx \right)^{\frac{1}{2q-p}} \\
&\cdot \left( \int_{B_R} (1+|Du|)^p dx \right)^{\frac{\delta(q-1)}{p}} \left( \int_{B_t} (1+|Du|)^{\frac{np}{n-2\lambda}} dx \right)^{\frac{(1-\delta)(n-2\lambda)(q-1)}{np}} \\
&+ \frac{C_\varepsilon}{t-s} |h|^{\alpha+1} \left( \int_{B_{R/2}} (g_k(x+h)+g_k(x))^r dx \right)^{\frac{1}{r}} \left( \int_{B_R} (1+|Du|)^p dx \right)^{\frac{\delta q}{p}} \\
&\cdot \left( \int_{B_{t'}} (1+|Du|)^{\frac{np}{n-2\lambda}} dx \right)^{\frac{(1-\delta)(n-2\lambda)q}{np}} .
\end{aligned} \tag{4.4.10}
$$

Choosing $\varepsilon = \frac{\nu}{6}$ yields

$$
\begin{aligned}
&\int_\Omega \eta^2 |\tau_h Du|^2 (\mu^2 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} dx \\
\le & C \left( \int_{B_{R/2}} |\tau_h D\psi|^{2q-p} dx \right)^{\frac{2}{2q-p}} \left( \int_{B_R} (1+|Du|)^p dx \right)^{\frac{\delta(2q-p-2)}{p}} \\
&\cdot \left( \int_{B_{t'}} (1+|Du|)^{\frac{np}{n-2\lambda}} dx \right)^{\frac{(1-\delta)(n-2\lambda)(2q-p-2)}{np}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{C}{(t-s)^2}|h|^2\int_{B_R}|D\psi|^{2q-p}dx\\
&+\frac{C}{(t-s)^2}|h|^2\left(\int_{B_R}(1+|Du|)^p dx\right)^{\frac{\delta(2q-p)}{p}}\\
&\cdot\left(\int_{B_{t'}}(1+|Du|)^{\frac{np}{n-2\lambda}}dx\right)^{\frac{(1-\delta)(n-2\lambda)(2q-p)}{np}}\\
&+C|h|^{2\alpha}\left(\int_{B_{R/2}}(g_k(x+h)+g_k(x))^r dx\right)^{\frac{2}{r}}\left(\int_{B_R}(1+|Du|)^p dx\right)^{\frac{\delta(2q-p)}{p}}\\
&\cdot\left(\int_{B_t}(1+|Du|)^{\frac{np}{n-2\lambda}}dx\right)^{\frac{(1-\delta)(n-2\lambda)(2q-p)}{np}}\\
&+|h|^{\alpha}\left(\int_{B_{R/2}}(g_k(x+h)+g_k(x))^r dx\right)^{\frac{1}{r}}\left(\int_{B_{R/2}}|\tau_h D\psi|^{2q-p}dx\right)^{\frac{1}{2q-p}}\\
&\cdot\left(\int_{B_R}(1+|Du|)^p dx\right)^{\frac{\delta(q-1)}{p}}\left(\int_{B_t}(1+|Du|)^{\frac{np}{n-2\lambda}}dx\right)^{\frac{(1-\delta)(n-2\lambda)(q-1)}{np}}\\
&+\frac{C}{t-s}|h|^{\alpha+1}\left(\int_{B_{R/2}}(g_k(x+h)+g_k(x))^r dx\right)^{\frac{1}{r}}\left(\int_{B_R}|D\psi|^{2q-p}dx\right)^{\frac{1}{2q-p}}\\
&\cdot\left(\int_{B_R}(1+|Du|)^p dx\right)^{\frac{\delta(q-1)}{p}}\left(\int_{B_t}(1+|Du|)^{\frac{np}{n-2\lambda}}dx\right)^{\frac{(1-\delta)(n-2\lambda)(q-1)}{np}}\\
&+\frac{C}{t-s}|h|^{\alpha+1}\left(\int_{B_{R/2}}(g_k(x+h)+g_k(x))^r dx\right)^{\frac{1}{r}}\left(\int_{B_R}(1+|Du|)^p dx\right)^{\frac{\delta q}{p}}\\
&\cdot\left(\int_{B_{t'}}(1+|Du|)^{\frac{np}{n-2\lambda}}dx\right)^{\frac{(1-\delta)(n-2\lambda)q}{np}}.
\end{aligned}
\tag{4.4.11}
$$

for a positive constant $C:=C(n,p,q,r,\nu,L,R)$.

Using Lemma 2.0.2 in the left hand side of the previous estimate, recalling that $\eta=1$ on $B_s$ and dividing both sides by $|h|^{2\lambda}$, we get

$$
\begin{aligned}
&\int_{B_s}\frac{|\tau_h V_p(Du)|^2}{|h|^{2\lambda}}dx\\
\le{}&C\left(\int_{B_{R/2}}\frac{|\tau_h D\psi|^{2q-p}}{|h|^{\lambda(2q-p)}}dx\right)^{\frac{2}{2q-p}}\left(\int_{B_R}(1+|Du|)^p dx\right)^{\frac{\delta(2q-p-2)}{p}}\\
&\cdot\left(\int_{B_{t'}}(1+|Du|)^{\frac{np}{n-2\lambda}}dx\right)^{\frac{(1-\delta)(n-2\lambda)(2q-p-2)}{np}}\\
&+\frac{C}{(t-s)^2}|h|^{2(1-\lambda)}\left\{\int_{B_R}|D\psi|^{2q-p}dx+\left(\int_{B_R}(1+|Du|)^p dx\right)^{\frac{\delta(2q-p)}{p}}\right.\\
&\left.\cdot\left(\int_{B_{t'}}(1+|Du|)^{\frac{np}{n-2\lambda}}dx\right)^{\frac{(1-\delta)(n-2\lambda)(2q-p)}{np}}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+ C|h|^{2(\alpha-\lambda)}\left(\int_{B_{R/2}} (g_k(x+h)+g_k(x))^r dx\right)^{\frac{2}{r}} \left(\int_{B_R} (1+|Du|)^p dx\right)^{\frac{\delta(2q-p)}{p}} \\
&\cdot \left(\int_{B_t} (1+|Du|)^{\frac{np}{n-2\lambda}} dx\right)^{\frac{(1-\delta)(n-2\lambda)(2q-p)}{np}} \\
&+ C|h|^{\alpha-\lambda}\left(\int_{B_{R/2}} (g_k(x+h)+g_k(x))^r dx\right)^{\frac{1}{r}} \left(\int_{B_{R/2}} \frac{|\tau_h D\psi|^{2q-p}}{|h|^{\lambda(2q-p)}} dx\right)^{\frac{1}{2q-p}} \\
&\cdot \left(\int_{B_R} (1+|Du|)^p dx\right)^{\frac{\delta(q-1)}{p}} \left(\int_{B_t} (1+|Du|)^{\frac{np}{n-2\lambda}} dx\right)^{\frac{(1-\delta)(n-2\lambda)(q-1)}{np}} \\
&+ \frac{C}{t-s}|h|^{\alpha+1-2\lambda}\left(\int_{B_{R/2}} (g_k(x+h)+g_k(x))^r dx\right)^{\frac{1}{r}} \\
&\cdot \left\{\left(\int_{B_R} (1+|Du|)^p dx\right)^{\frac{\delta(q-1)}{p}} \left(\int_{B_t} (1+|Du|)^{\frac{np}{n-2\lambda}} dx\right)^{\frac{(1-\delta)(n-2\lambda)(q-1)}{np}}\right. \\
&\left. + \left(\int_{B_R} (1+|Du|)^p dx\right)^{\frac{\delta q}{p}} \left(\int_{B_{t'}} (1+|Du|)^{\frac{np}{n-2\lambda}} dx\right)^{\frac{(1-\delta)(n-2\lambda)q}{np}}\right\}. \qquad (4.4.12)
\end{aligned}
$$

We need now to take the $L^\sigma$ norm with the measure $\frac{dh}{|h|^n}$ restricted to the ball $B(0,R/4)$ on the $h$-space of the $L^2$ norm of the difference quotient of order $\lambda$ of the function $V_p(Du)$. Since the functions $g_k$ are defined for $2^{-k}(t'-t) \le |h| \le 2^{-k+1}(t'-t)$ we interpret the ball $B(0,t'-t)$ as

$$
B(0,t'-t) = \bigcup_{k=1}^{\infty} B(0,2^{-k+1}(t'-t)) \setminus B(0,2^{-k}(t'-t)) =: \bigcup_{k=1}^{\infty} E_k.
$$

We obtain the following estimate

$$
\begin{aligned}
&\int_{B_{t'-t}(0)} \left(\int_{B_s} \frac{|\tau_h V_p(Du)|^2}{|h|^{2\lambda}} dx\right)^{\frac{\sigma}{2}} \frac{dh}{|h|^n} \\
\le & C\left(\int_{B_R} (1+|Du|)^p dx\right)^{\frac{\delta(2q-p-2)\sigma}{2p}} \left(\int_{B_{t'}} (1+|Du|)^{\frac{np}{n-2\lambda}} dx\right)^{\frac{(1-\delta)(n-2\lambda)(2q-p-2)\sigma}{2np}} \\
&\cdot \int_{B_{R/4}(0)} \left(\int_{B_{R/2}} \frac{|\tau_h D\psi|^{2q-p}}{|h|^{\lambda(2q-p)}} dx\right)^{\frac{\sigma}{2q-p}} \frac{dh}{|h|^n} \\
&+ \frac{C}{(t-s)^\sigma}\left(\int_{B_R} (1+|Du|)^p dx\right)^{\frac{\delta(2q-p)\sigma}{2p}} \left(\int_{B_{t'}} (1+|Du|)^{\frac{np}{n-2\lambda}} dx\right)^{\frac{(1-\delta)(n-2\lambda)(2q-p)\sigma}{2np}} \\
&\cdot \int_{B_{R/4}(0)} |h|^{(1-\lambda)\sigma} \frac{dh}{|h|^n} \\
&+ C\left(\int_{B_R} (1+|Du|)^p dx\right)^{\frac{\delta(2q-p)\sigma}{2p}} \left(\int_{B_t} (1+|Du|)^{\frac{np}{n-2\lambda}} dx\right)^{\frac{(1-\delta)(n-2\lambda)(2q-p)\sigma}{2np}} \\
&\cdot \sum_{k=1}^{\infty} \int_{E_k} |h|^{(\alpha-\lambda)\sigma} \left(\int_{B_{R/2}} (g_k(x+h)+g_k(x))^r dx\right)^{\frac{\sigma}{r}} \frac{dh}{|h|^n}
\end{aligned}
$$

$$
\begin{aligned}
&+ C\sum_{k=1}^{\infty}\int_{E_k}|h|^{(\alpha-\lambda)\frac{\sigma}{2}}\left(\int_{B_{R/2}}(g_k(x+h)+g_k(x))^r dx\right)^{\frac{\sigma}{2r}}\left(\int_{B_{R/2}}\frac{|\tau_h D\psi|^{2q-p}}{|h|^{\lambda(2q-p)}}dx\right)^{\frac{\sigma}{2(2q-p)}}\frac{dh}{|h|^n}\\
&\cdot\left(\int_{B_R}(1+|Du|)^p dx\right)^{\frac{\delta(q-1)\sigma}{2p}}\left(\int_{B_t}(1+|Du|)^{\frac{np}{n-2\lambda}}dx\right)^{\frac{(1-\delta)(n-2\lambda)(q-1)\sigma}{2np}}\\
&+\frac{C}{(t-s)^{\sigma/2}}\sum_{k=1}^{\infty}\int_{E_k}|h|^{(\alpha+1-2\lambda)\frac{\sigma}{2}}\left(\int_{B_{R/2}}(g_k(x+h)+g_k(x))^r dx\right)^{\frac{\sigma}{2r}}\frac{dh}{|h|^n}\\
&\cdot\left\{\left(\int_{B_R}(1+|Du|)^p dx\right)^{\frac{\delta(q-1)\sigma}{2p}}\left(\int_{B_t}(1+|Du|)^{\frac{np}{n-2\lambda}}dx\right)^{\frac{(1-\delta)(n-2\lambda)(q-1)\sigma}{2np}}\right.\\
&\left.+\left(\int_{B_R}(1+|Du|)^p dx\right)^{\frac{\delta q\sigma}{2p}}\left(\int_{B_{t'}}(1+|Du|)^{\frac{np}{n-2\lambda}}dx\right)^{\frac{(1-\delta)(n-2\lambda)q\sigma}{2np}}\right\}. \qquad (4.4.13)
\end{aligned}
$$

Note that, since $\lambda \le \gamma$, the integral

$$J_1 := \int_{B_{t'-t}(0)}\left(\int_{B_{R/2}}\frac{|\tau_h D\psi|^{2q-p}}{|h|^{\lambda(2q-p)}}dx\right)^{\frac{\sigma}{2q-p}}\frac{dh}{|h|^n}$$

is controlled by the norm in the Besov space $B^{\gamma}_{2q-p,\sigma}$ on $B_{R/2}$ of the gradient of the obstacle which is finite by assumptions. The integral

$$J_2 := \int_{B_{t'-t}(0)}|h|^{(1-\lambda)\sigma}\frac{dh}{|h|^n}$$

can be calculated in polar coordinates as follows

$$J_2 = C(n)\int_0^{t'-t}\varrho^{(1-\lambda)\sigma-1}d\varrho \le C(n)\int_0^{R/4}\varrho^{(1-\lambda)\sigma-1}d\varrho = C(n,\alpha,\gamma,\sigma,R),$$

since $\lambda \in (0,1)$.
Now, we take care of the integral

$$J_3 := \sum_{k=1}^{\infty}\int_{E_k}|h|^{(\alpha-\lambda)\sigma}\left(\int_{B_{R/2}}(g_k(x+h)+g_k(x))^r dx\right)^{\frac{\sigma}{r}}\frac{dh}{|h|^n}.$$

Recalling that $|h| \le 1$ and $\alpha \ge \lambda$, we have

$$J_3 \le \sum_{k=1}^{\infty}\int_{E_k}\left(\int_{B_{R/2}}(g_k(x+h)+g_k(x))^r dx\right)^{\frac{\sigma}{r}}\frac{dh}{|h|^n}.$$

We write the right hand sinde of the previous estimate in polar coordinates, so $h \in E_k$ if, and only if, $h = m\xi$ for some $2^{-k+1}(t'-t) \le m < 2^{-k}(t'-t)$ and some $\xi$ in the unit sphere $\mathbb{S}^{n-1}$ on $\mathbb{R}^n$. We denote by $dS(\xi)$ the surface measure on $\mathbb{S}^{n-1}$. We infer

$$J_3 \le \sum_{k=1}^{\infty}\int_{m_{k-1}}^{m_k}\int_{\mathbb{S}^{n-1}}\left(\int_{B_{R/2}}(g_k(x+h)+g_k(x))^r dx\right)^{\frac{\sigma}{r}}dS(\xi)\frac{dm}{m}$$

$$=\sum_{k=1}^{\infty}\int_{m_{k-1}}^{m_k}\int_{\mathbb{S}^{n-1}}\|(\tau_{m\xi}g_k+g_k)\|^{\sigma}_{L^r(B_{R/2})}dS(\xi)\frac{dm}{m},$$

where we set $m_k=2^{-k}(t'-t)$. We note that for each $\xi\in\mathbb{S}^{n-1}$ and $m_{k-1}\le m\le m_k$

$$\begin{aligned}\|(\tau_{m\xi}g_k+g_k)\|_{L^r(B_{R/2})}\le&\|g_k\|_{L^r(B_{R/2-m_k\xi})}+\|g_k\|_{L^r(B_{R/2})}\\ \le&2\|g_k\|_{L^r(B_{3R/4})},\end{aligned}$$

hence

$$J_3\le C(n)\|\{g_k\}_k\|^{\sigma}_{l^{\sigma}(L^r(B_R))},$$

which is finite by assumption (F6).
Recalling that $|h|\le 1$, $\alpha\ge\lambda$ and using the Young's inequality with exponent 2, we deduce the following estimate

$$\begin{aligned}&\sum_{k=1}^{\infty}\int_{E_k}|h|^{(\alpha-\lambda)\frac{\sigma}{2}}\left(\int_{B_{R/2}}(g_k(x+h)+g_k(x))^r dx\right)^{\frac{\sigma}{2r}}\left(\int_{B_{R/2}}\frac{|\tau_h D\psi|^{2q-p}}{|h|^{\lambda(2q-p)}}dx\right)^{\frac{\sigma}{2(2q-p)}}\frac{dh}{|h|^n}\\ \le&C\sum_{k=1}^{\infty}\int_{E_k}\left(\int_{B_{R/2}}(g_k(x+h)+g_k(x))^r dx\right)^{\frac{\sigma}{r}}\frac{dh}{|h|^n}+C\int_{B_{t'-t}(0)}\left(\int_{B_{R/2}}\frac{|\tau_h D\psi|^{2q-p}}{|h|^{\lambda(2q-p)}}dx\right)^{\frac{\sigma}{2q-p}}\frac{dh}{|h|^n}\end{aligned}$$

where the two integrals in the right hand side can be estimated as the integrals $J_1$ and $J_3$. Similarly, we obtain

$$\begin{aligned}&\sum_{k=1}^{\infty}\int_{E_k}|h|^{(\alpha+1-2\lambda)\frac{\sigma}{2}}\left(\int_{B_{R/2}}(g_k(x+h)+g_k(x))^r dx\right)^{\frac{\sigma}{2r}}\frac{dh}{|h|^n}\\ \le&\int_{B_{t'-t}(0)}|h|^{(\alpha+1-2\lambda)\sigma}\frac{dh}{|h|^n}+\sum_{k=1}^{\infty}\int_{E_k}\left(\int_{B_{R/2}}(g_k(x+h)+g_k(x))^r dx\right)^{\frac{\sigma}{r}}\frac{dh}{|h|^n}.\end{aligned}$$

The latter term can be estimated as the integral $J_3$; the first integral can be calculated as we did for $J_2$ thus obtaining

$$\int_{B_{t'-t}(0)}|h|^{(\alpha+1-2\lambda)\sigma}\frac{dh}{|h|^n}\le C(n)\int_0^{R/4}\varrho^{(1-\lambda)\sigma-1}d\varrho\le C(n,\alpha,\gamma,\sigma,R),$$

since $0<\lambda\le\alpha<1$.

Estimate (4.4.13) can be written as follows

$$\begin{aligned}&\int_{B_{t'-t}(0)}\left(\int_{B_s}\frac{|\tau_h V_p(Du)|^2}{|h|^{2\lambda}}dx\right)^{\frac{\sigma}{2}}\frac{dh}{|h|^n}\\ \le&C\left(\int_{B_R}(1+|Du|)^p dx\right)^{\frac{\delta(2q-p-2)\sigma}{2p}}\left(\int_{B_{t'}}(1+|Du|)^{\frac{np}{n-2\lambda}}dx\right)^{\frac{(1-\delta)(n-2\lambda)(2q-p-2)\sigma}{2np}}\\ &+\frac{C}{(t-s)^{\sigma}}\left(\int_{B_R}(1+|Du|)^p dx\right)^{\frac{\delta(2q-p)\sigma}{2p}}\left(\int_{B_{t'}}(1+|Du|)^{\frac{np}{n-2\lambda}}dx\right)^{\frac{(1-\delta)(n-2\lambda)(2q-p)\sigma}{2np}}\end{aligned}$$

$$
\begin{aligned}
&+C\left(\int_{B_R}(1+|Du|)^p dx\right)^{\frac{\delta(2q-p)\sigma}{2p}}\left(\int_{B_t}(1+|Du|)^{\frac{np}{n-2\lambda}}dx\right)^{\frac{(1-\delta)(n-2\lambda)(2q-p)\sigma}{2np}}\\
&+C\left(\int_{B_R}(1+|Du|)^p dx\right)^{\frac{\delta(q-1)\sigma}{2p}}\left(\int_{B_t}(1+|Du|)^{\frac{np}{n-2\lambda}}dx\right)^{\frac{(1-\delta)(n-2\lambda)(q-1)\sigma}{2np}}\\
&+\frac{C}{(t-s)^{\sigma/2}}\left(\int_{B_R}(1+|Du|)^p dx\right)^{\frac{\delta(q-1)\sigma}{2p}}\left(\int_{B_t}(1+|Du|)^{\frac{np}{n-2\lambda}}dx\right)^{\frac{(1-\delta)(n-2\lambda)(q-1)\sigma}{2np}}\\
&+\frac{C}{(t-s)^{\sigma/2}}\left(\int_{B_R}(1+|Du|)^p dx\right)^{\frac{\delta q\sigma}{2p}}\left(\int_{B_{t'}}(1+|Du|)^{\frac{np}{n-2\lambda}}dx\right)^{\frac{(1-\delta)(n-2\lambda)q\sigma}{2np}}\\
=&H_1+H_2+H_3+H_4+H_5+H_6,
\end{aligned}
\tag{4.4.14}
$$

for a constant $C := C(n,p,q,r,\sigma,\alpha,\gamma,\nu,L,R,\|D\psi\|_{B^{\gamma}_{2q-p,\sigma}(B_{R/2})},\|\{g_k\}_k\|_{l^{\sigma}(L^r(B_R))})$.
We proceed estimating the various pieces arising up from (4.4.14).

By assumption (4.0.5), we have that

$$
\frac{(1-\delta)(2q-p-2)}{p}<1 \quad \text{and} \quad \frac{(1-\delta)(2q-p)}{p}<1.
$$

Thus, using the Young's inequality, we deduce the following estimate

$$
\begin{aligned}
H_1+H_2+H_3\le &C_\theta\left(\int_{B_R}(1+|Du|)^p dx\right)^{\sigma'}+C_\theta\left(\int_{B_R}(1+|Du|)^p dx\right)^{\sigma''}\\
&+\frac{C_\theta}{(t-s)^{\frac{\sigma p}{p-(1-\delta)(2q-p)}}}\left(\int_{B_R}(1+|Du|)^p dx\right)^{\sigma''}\\
&+2\theta\left(\int_{B_{t'}}(1+|Du|)^{\frac{np}{n-2\lambda}}dx\right)^{\frac{(n-2\lambda)\sigma}{2n}}\\
&+\theta\left(\int_{B_t}(1+|Du|)^{\frac{np}{n-2\lambda}}dx\right)^{\frac{(n-2\lambda)\sigma}{2n}},
\end{aligned}
\tag{4.4.15}
$$

for $0<\theta<1$, where we set $\sigma'=\frac{\delta(2q-p-2)\sigma}{2[p-(1-\delta)(2q-p-2)]}$, $\sigma''=\frac{\delta(2q-p)\sigma}{2[p-(1-\delta)(2q-p)]}$.
According to the second inequality of (4.3.25) with $\beta$ replaced by $\lambda$, the use of Young's inequality yields

$$
\begin{aligned}
H_4+H_5\le &C_\theta\left(\int_{B_R}(1+|Du|)^p dx\right)^{\sigma'''}+\frac{C_\theta}{(t-s)^{\frac{\sigma p}{2[p-(1-\delta)(q-1)]}}}\left(\int_{B_R}(1+|Du|)^p dx\right)^{\sigma'''}\\
&+2\theta\left(\int_{B_t}(1+|Du|)^{\frac{np}{n-2\lambda}}dx\right)^{\frac{(n-2\lambda)\sigma}{2n}},
\end{aligned}
\tag{4.4.16}
$$

where we set $\sigma''=\frac{p(q-1)\sigma}{2[p-(1-\delta)(q-1)]}$.
Similarly, recalling the third inequality of (4.3.25) with $\beta$ replaced by $\lambda$, we deduce that

$$
H_6\le\frac{C_\theta}{(t-s)^{\frac{\sigma p}{2[p-(1-\delta)q]}}}\left(\int_{B_R}(1+|Du|)^p dx\right)^{\tilde{\sigma}}+\theta\left(\int_{B_{t'}}(1+|Du|)^{\frac{np}{n-2\lambda}}dx\right)^{\frac{(n-2\lambda)\sigma}{2n}},
\tag{4.4.17}
$$

where we set $\tilde{\sigma} = \frac{pq\sigma}{2[p-(1-\delta)q]}$.

To simplify the presentation, we now define

$$A := C_\theta \Big( \int_{B_R} (1+|Du|)^p dx \Big)^{\sigma'} + C_\theta \Big( \int_{B_R} (1+|Du|)^p dx \Big)^{\sigma''} + C_\theta \Big( \int_{B_R} (1+|Du|)^p dx \Big)^{\sigma'''},$$

$$B_1 := C_\theta \Big( \int_{B_R} (1+|Du|)^p dx \Big)^{\sigma''}, \quad B_2 := C_\theta \Big( \int_{B_R} (1+|Du|)^p dx \Big)^{\sigma'''},$$

$$B_3 := C_\theta \Big( \int_{B_R} (1+|Du|)^p dx \Big)^{\tilde{\sigma}}, \quad \pi_1 := \frac{\sigma p}{p-(1-\delta)(2q-p)},$$

$$\pi_2 := \frac{\sigma p}{2[p-(1-\delta)(q-1)]}, \quad \pi_3 := \frac{\sigma p}{2[p-(1-\delta)q]},$$

so that, inserting estimates (4.4.15), (4.4.16) and (4.4.17) in (4.4.14), we obtain

$$\begin{aligned} &\int_{B_{t'-t}(0)} \Big( \int_{B_s} \frac{|\tau_h V_p(Du)|^2}{|h|^{2\lambda}} dx \Big)^{\frac{\sigma}{2}} \frac{dh}{|h|^n} \\ &\le 3\theta \Big( \int_{B_t} (1+|Du|)^{\frac{np}{n-2\lambda}} dx \Big)^{\frac{(n-2\lambda)\sigma}{2n}} \\ &\quad + 3\theta \Big( \int_{B_{t'}} (1+|Du|)^{\frac{np}{n-2\lambda}} dx \Big)^{\frac{(n-2\lambda)\sigma}{2n}} \\ &\quad + A + \frac{B_1}{(t-s)^{\pi_1}} + \frac{B_2}{(t-s)^{\pi_2}} + \frac{B_3}{(t-s)^{\pi_3}}. \end{aligned}$$

Lemma 2.2.2 $(a)$ and inequality (2.0.3) imply

$$\begin{aligned} &\Big( \int_{B_s} |Du|^{\frac{np}{n-2\lambda}} dx \Big)^{\frac{(n-2\lambda)\sigma}{2n}} \\ &\le 3\theta \Big( \int_{B_t} (1+|Du|)^{\frac{np}{n-2\lambda}} dx \Big)^{\frac{(n-2\lambda)\sigma}{2n}} \\ &\quad + 3\theta \Big( \int_{B_{t'}} (1+|Du|)^{\frac{np}{n-2\lambda}} dx \Big)^{\frac{(n-2\lambda)\sigma}{2n}} \\ &\quad + A + \frac{B_1}{(t-s)^{\pi_1}} + \frac{B_2}{(t-s)^{\pi_2}} + \frac{B_3}{(t-s)^{\pi_3}}. \end{aligned}$$

Arguing as in the proof of Theorem 4.3.1, we finally obtain

$$\Big( \int_{B_{R/4}} |Du|^{\frac{np}{n-2\lambda}} dx \Big)^{\frac{(n-2\beta)\sigma}{2n}} \le \tilde{c} \Big\{ \int_{B_R} (1+|Du|^p) dx + \|D\psi\|_{B^\gamma_{2q-p,\sigma}(B_{R/2})} + \|\{g_k\}_k\|_{l^\sigma(L^r(B_R))} \Big\}^\kappa,$$

which implies

$$\int_{B_{t'-t}(0)} \Big( \int_{B_{R/4}} \frac{|\tau_h V_p(Du)|^2}{|h|^{2\lambda}} dx \Big)^{\frac{\sigma}{2}} \frac{dh}{|h|^n} dx$$

$$\leq C\Big\{\int_{B_R}(1+|Du|^p)dx + \|D\psi\|_{B^\gamma_{2q-p,\sigma}(B_{R/2})} + \|\{g_k\}_k\|_{l^\sigma(L^r(B_R))}\Big\}^\kappa,$$

for every $t'-t \leq R/4$. Therefore, we get

$$\int_{B_{R/4}(0)}\left(\int_{B_{R/4}}\frac{|\tau_h V_p(Du)|^2}{|h|^{2\lambda}}dx\right)^{\frac{\sigma}{2}}\frac{dh}{|h|^n}dx$$
$$\leq C\Big\{\int_{B_R}(1+|Du|^p)dx + \|D\psi\|_{B^\gamma_{2q-p,\sigma}(B_{R/2})} + \|\{g_k\}_k\|_{l^\sigma(L^r(B_R))}\Big\}^\kappa,$$

for some constants $C := C(n,p,q,r,\sigma,\alpha,\gamma,\nu,L,R)$ and $\kappa := \kappa(n,p,q,r,\sigma,\alpha,\gamma)$. □

# Chapter 5

# Local boundedness for obstacle problems with non-standard growth conditions

Here we deal with the local boundedness of the solutions $u \in W^{1,p}(\Omega)$ to variational obstacle problems of the type

$$\min\left\{\int_\Omega F(x,w,Dw)dx \ : \ w \in \mathcal{K}_\psi(\Omega)\right\}, \tag{5.0.1}$$

where the function $\psi \in W^{1,p}(\Omega)$ is the obstacle and the set $\mathcal{K}_\psi(\Omega)$ was defined in (2.0.2).

We assume that the energy density $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function such that

$$(v,\xi) \mapsto F(x,v,\xi) \text{ is convex}, \tag{5.0.2}$$

$$c_1|\xi|^p \le F(x,v,\xi) \le c_2(1+|s|^\gamma+|\xi|^q), \tag{5.0.3}$$

for almost all $x \in \Omega$ and all $v \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, where $1 < p \le q$, $0 \le \gamma$ and $c_1, c_2 > 0$ are fixed constants.

More precisely, we are to going to prove the following theorem contained in [37].

**Theorem 5.0.1.** *Let $u \in W^{1,p}(\Omega)$ be the solution to (5.0.1) under assumptions (5.0.2) and (5.0.3), for exponents $1 < p \le q$, $0 \le \gamma$ verifying (1.0.7) and*

$$\gamma \le p_n^* := \begin{cases} \frac{np}{n-p} & \text{if } p < n \\ \text{any finite exponent in } [q,\infty) & \text{if } p \ge n. \end{cases} \tag{5.0.4}$$

*If $\psi \in L^\infty_{loc}(\Omega)$, then $u \in L^\infty_{loc}(\Omega)$ and the following estimate*

$$\sup_{B_{R_0/2}} |u| \le C(\sup_{B_{R_0}} |\psi| + \|u\|_{W^{1,p}(B_{R_0})})^\pi$$

*holds for every ball $B_{R_0} \Subset \Omega$, for constants $C = C(n,p,q,R_0)$ and $\pi = \pi(n,p,q)$.*

The proof of Theorem 5.0.1 is achieved following the strategy first proposed in [15], i.e. using the well known De Giorgi method that consists in deriving a suitable Caccioppoli inequality on the superlevel sets of the solution to (5.0.1). In order to do so, one has to use test functions obtained truncating the solution. Here, the difficulties come from the set of admissible test

functions that must belong to the admissible class $\mathcal{K}_\psi(\Omega)$ and this is where the local boundedness of the obstacle $\psi$ comes into play. We also remark that the crucial tool in order to achieve the result under the sharp bound on the gap between the exponents is the Sobolev inequality on the spheres as done in [72].
This chapter is organized as follows. In Section 5.1 we collect some results that will be needed in the sequel. in Section 5.2 we derive a Caccioppoli inequality for the minimizer of (5.0.1). Finally, Section 5.3 is devoted to the proof of Theorem 5.0.1.

## 5.1 Preliminary Results

A key ingredient in the proof of Theorem 5.0.1 is the following lemma, that can be found in [72, Lemma 2.1].

**Lemma 5.1.1.** *Let $n \geq 2$. For any $0 < \rho < \sigma < \infty$, $v \in L^1(B_\sigma)$ and $s > 1$, we set*

$$I(\rho,\sigma,v) := \inf\left\{ \int_{B_\sigma} |v||D\eta|^s dx : \eta \in \mathcal{C}^1_0(B_\sigma), \eta \geq 0, \eta = 1 \text{ in } B_\rho \right\}.$$

*Then for every $\delta \in (0,1]$*

$$I(\rho,\sigma,v) \leq (\sigma-\rho)^{s-1+\frac{1}{\delta}} \left( \int_\rho^\sigma \left( \int_{\partial B_r} |v| d\mathcal{H}^{n-1} \right)^\delta dr \right)^{\frac{1}{\delta}}.$$

Next Lemma, whose proof can be found in [66, Lemma 7.1], allows us to iterate the Caccioppoli type estimate and it is crucial to establish the local boundedness result.

**Lemma 5.1.2.** *Let $\alpha > 0$ and let $(J_i)$ be a sequence of real positive numbers, such that*

$$J_{i+1} \leq A\lambda^i J_i^{1+\alpha},$$

*with $A > 0$ and $\lambda > 1$. If $J_0 \leq A^{-\frac{1}{\alpha}} \lambda^{-\frac{1}{\alpha^2}}$, then*

$$J_i \leq \lambda^{-\frac{i}{\alpha}} J_0 \quad \text{and} \quad \lim_{i \to +\infty} J_i = 0.$$

We conclude this section recalling the Sobolev inequality on spheres (see e.g. [63, Chapter 16]).

**Lemma 5.1.3.** *Let $v \in W^{1,m}(S_1, d\mathcal{H}^{n-1})$ with $m \in [1, n-1)$. Then there exists $c = c(n,m)$ such that*

$$\left( \int_{S_1} |v|^{m^*} d\mathcal{H}^{n-1} \right)^{\frac{1}{m^*}} \leq c \left( \int_{S_1} (|Dv|^m + |v|^m) d\mathcal{H}^{n-1} \right)^{\frac{1}{m}},$$

*where $\frac{1}{m^*} = \frac{1}{m} - \frac{1}{n-1}$.*

## 5.2 Caccioppoli Inequality

If $u \in W^{1,p}(\Omega)$, $k \in \mathbb{R}$ and $B_R \subset \Omega$ is a ball, we set

$$A_{k,R} := \{x \in B_R : u(x) > k\}.$$

The main result of this section is the following Caccioppoli type inequality.

**Theorem 5.2.1.** *Let $u \in W^{1,p}(\Omega)$ be the solution to (5.0.1) under assumptions (5.0.2) and (5.0.3), for exponents $1 < p \leq q$ verifying (1.0.7) and $0 \leq \gamma$. Assume that $\psi \in L^\infty_{loc}(\Omega)$. Then the following inequality*

$$\int_{B_\rho} |D(u-k)_+|^p dx \leq \frac{C}{(R-\rho)^\mu} \|(u-k)_+\|^q_{W^{1,p}(B_R)} |A_{k,R}|^{q(\frac{1}{q_*}-\frac{1}{p})} + Ck^\gamma |A_{k,R}| \tag{5.2.1}$$

*holds for all balls $B_\rho \subset B_R \subset B_{R_0} \Subset \Omega$ and for every $k \geq \max\{\sup_{B_{R_0}} \psi, 1\}$, where*

$$\frac{1}{q_*} := \min\left\{\frac{1}{q} + \frac{1}{n-1}, 1\right\}, \tag{5.2.2}$$

*$\mu := q - 1 + \frac{q}{q_*}$ and with $C = C(n, q, R_0)$.*

*Proof.* Let us fix $B_{R_0}(x_0) \Subset \Omega$. Let $\frac{R_0}{2} \leq \rho \leq s < t \leq R \leq R_0$ and let $\eta \in \mathcal{C}_0^\infty(B_t)$ be a cut-off function such that $0 \leq \eta \leq 1$, $\eta = 1$ in $B_s$, $|D\eta| \leq \frac{2}{t-s}$. By virtue of the assumption $\psi \in L^\infty_{\text{loc}}(\Omega)$, we may fix $k \geq \max\{\sup_{B_{R_0}} \psi, 1\}$ and define $u_k := (u-k)_+ = \max\{u-k, 0\}$. Note that $\varphi = u - \eta^\sigma u_k$, with $\sigma > q$, belongs to $\mathcal{K}_\psi(\Omega)$. Indeed,

$$\varphi = \begin{cases} u \geq \psi & \text{if } u \leq k \\ u - \eta^\sigma(u-k) = (1-\eta^\sigma)(u-k) + k \geq k \geq \psi & \text{if } u \geq k \end{cases}. \tag{5.2.3}$$

By the minimality of $u$ and the convexity assumption (5.0.2), we get

$$\begin{aligned} &\int_{A_{k,s}} F(x, u, Du)dx \leq \int_{A_{k,t}} F(x, u+\varphi, Du + D\varphi)dx \\ &= \int_{A_{k,t}} F\left(x, (1-\eta^\sigma)u + \eta^\sigma k, (1-\eta^\sigma)Du - \sigma\eta^\sigma \frac{D\eta}{\eta}(u-k)\right)dx \\ &\leq \int_{A_{k,t}} \left[(1-\eta^\sigma)F(x,u,Du) + \eta^\sigma F\left(x, k, -\sigma\frac{D\eta}{\eta}(u-k)\right)\right]dx. \end{aligned}$$

Taking into account that $\text{supp}(1-\eta^\sigma) \subset A_{k,t} \setminus A_{k,s}$, $\sigma > q$, $t \leq R$ and the growth assumption (5.0.3), we obtain

$$\begin{aligned} &\int_{A_{k,s}} F(x,u,Du)dx \\ &\leq \int_{A_{k,t}\setminus A_{k,s}} (1-\eta^\sigma)F(x,u,Du)dx + c_2 \int_{A_{k,t}} \eta^\sigma \left(1 + k^q + \sigma^q \frac{|D\eta|^q}{\eta^q}(u-k)^q\right)dx \\ &\leq \int_{A_{k,t}\setminus A_{k,s}} F(x,u,Du)dx + Ck^q|A_{k,R}| + C\int_{A_{k,t}} |D\eta|^q (u-k)^q dx. \end{aligned} \tag{5.2.4}$$

Exploiting Lemma 5.1.3 after the use of Hölder's inequality in case $q \in (1, \frac{n-1}{n-2})$, we infer

$$\left(\int_{S_1} |v|^q d\mathcal{H}^{n-1}\right)^{\frac{1}{q}} \le c(n,q)\left(\int_{S_1} (|Dv|^{q_*} + |v|^{q_*}) d\mathcal{H}^{n-1}\right)^{\frac{1}{q_*}}, \tag{5.2.5}$$

for every $v \in W^{1,q_*}(S_1, d\mathcal{H}^{n-1})$, where $q_* \ge 1$ is given by

$$\frac{1}{q_*} := \min\left\{\frac{1}{q} + \frac{1}{n-1}, 1\right\}.$$

We set

$$\mathcal{A}(s,t) := \{\eta \in \mathcal{C}_0^\infty(B_t) : 0 \le \eta \le 1,\ \eta = 1 \text{ in } B_s\}.$$

Combining (5.2.5) applied to $v(y) = u_k^q(x_0 + ry)$ with $r \in (s,t)$ and Lemma 5.1.1 with $\delta = \frac{q_*}{q}$, we get

$$\begin{aligned}
&\inf_{\mathcal{A}(s,t)} \int_{B_t} |D\eta|^q u_k^q dx \\
&\le (t-s)^{-(q-1+\frac{q}{q_*})}\left(\int_s^t r^{(n-1)\frac{q_*}{q}}\left(\int_{S_1} u_k^q(x_0+ry) d\mathcal{H}^{n-1}(y)\right)^{\frac{q_*}{q}} dr\right)^{\frac{q}{q_*}} \\
&\le c(n,q)(t-s)^{-(q-1+\frac{q}{q_*})}\left(\int_s^t r^{(n-1)\frac{q_*}{q}} \int_{S_1} (r^{q_*}|Du_k(x_0+ry)|^{q_*}\right. \\
&\qquad\qquad \left. + u_k^{q_*}(x_0+ry)) d\mathcal{H}^{n-1}(y) dr\right)^{\frac{q}{q_*}} \\
&= c(n,q)(t-s)^{-(q-1+\frac{q}{q_*})}\left(\int_s^t r^{(n-1)q_*(\frac{1}{q}+\frac{1}{n-1}-\frac{q_*}{q})} \int_{S_r} (|Du_k(x_0+y)|^{q_*}\right. \\
&\qquad\qquad \left. + r^{-q_*} u_k^{q_*}(x_0+y)) d\mathcal{H}^{n-1}(y) dr\right)^{\frac{q}{q_*}}.
\end{aligned}$$

Using the fact that $\frac{R_0}{2} \le s < t \le R_0$, we deduce

$$\begin{aligned}
&\inf_{\mathcal{A}(s,t)} \int_{B_t} |D\eta|^q u_k^q dx \\
&\le C(n,q,R_0)(t-s)^{-(q-1+\frac{q}{q_*})}\left(\int_s^t \int_{S_r} (|Du_k(x_0+y)|^{q_*} + u_k^{q_*}(x_0+y)) d\mathcal{H}^{n-1}(y) dr\right)^{\frac{q}{q_*}}.
\end{aligned} \tag{5.2.6}$$

Now, notice that inequality (1.0.7) implies $q_* \le p$. Thus, using Hölder's inequality in estimate (5.2.6), we obtain

$$\begin{aligned}
\inf_{\mathcal{A}(s,t)} \int_{B_t} |D\eta|^q u_k^q dx \le& \frac{C(n,q,R_0)}{(t-s)^\mu} \|u_k\|^q_{W^{1,q_*}(B_t \setminus B_s)} \\
\le& \frac{C(n,q,R_0)}{(t-s)^\mu} \|u_k\|^q_{W^{1,p}(B_t \setminus B_s)} |A_{k,t}|^{q(\frac{1}{q_*}-\frac{1}{p})} \\
\le& \frac{C(n,q,R_0)}{(t-s)^\mu} \|u_k\|^q_{W^{1,p}(B_R)} |A_{k,R}|^{q(\frac{1}{q_*}-\frac{1}{p})},
\end{aligned} \tag{5.2.7}$$

where $\mu = q - 1 + \frac{q}{q_*}$.
Since estimate (5.2.4) holds for every $\eta \in \mathcal{A}(s,t)$, by (5.2.7), we get

$$\begin{aligned}&\int_{A_{k,s}} F(x,u,Du)dx \\ &\leq \int_{A_{k,t}\setminus A_{k,s}} F(x,u,Du)dx + Ck^{\gamma}|A_{k,R}| \\ &\quad + \frac{C(n,q,R_0)}{(t-s)^{\mu}}\|u_k\|^q_{W^{1,p}(B_R)}|A_{k,R}|^{q(\frac{1}{q_*}-\frac{1}{p})}.\end{aligned}$$

Adding the integral $\int_{A_{k,s}} F(x,u,Du)dx$ to both sides of the previous estimate, by Lemma 2.0.3 we get

$$\begin{aligned}&\int_{A_{k,\rho}} F(x,u,Du)dx \\ &\leq C(n,q,R_0)\left\{k^{\gamma}|A_{k,R}| + \frac{1}{(R-\rho)^{\mu}}\|u_k\|^q_{W^{1,p}(B_R)}|A_{k,R}|^{q(\frac{1}{q_*}-\frac{1}{p})}\right\}.\end{aligned} \tag{5.2.8}$$

Eventually, inequality (5.2.8) and the growth condition at (5.0.3) yield

$$\begin{aligned}&\int_{B_{\rho}} |D(u-k)_+|^p dx \\ &\leq C(n,q,R_0)\left\{k^{\gamma}|A_{k,R}| + \frac{1}{(R-\rho)^{\mu}}\|u_k\|^q_{W^{1,p}(B_R)}|A_{k,R}|^{q(\frac{1}{q_*}-\frac{1}{p})}\right\},\end{aligned}$$

i.e. the conclusion. □

## 5.3 Proof of the Main Result

Before giving the proof of Theorem 5.0.1, we need to introduce some notations.
For a fixed ball $B_{R_0} \Subset \Omega$, define two sequences by setting

$$\rho_i := \frac{R_0}{2}\left(1 + \frac{1}{2^i}\right),$$

$$k_i := 2d\left(1 - \frac{1}{2^{i+1}}\right),$$

where $d \geq \max\{\sup_{B_{R_0}} \psi, 1\}$ will be determined later. Note that

$$\frac{R_0}{2} \leq \rho_i \leq R_0, \quad \rho_i \searrow \frac{R_0}{2}$$

and

$$d \leq k_i \leq 2d, \quad k_i \nearrow 2d.$$

The use of estimate (5.2.1) on the concentric balls $B_{\rho_{i+1}} \subset B_{\rho_i}$ translates into

$$\|D(u-k_{i+1})_+\|^p_{L^p(B_{\rho_{i+1}})}$$

$$\leq \frac{C}{(\rho_i-\rho_{i+1})^{\mu}}\|(u-k_{i+1})_+\|^q_{W^{1,p}(B_{\rho_i})}|A_{k_{i+1},\rho_i}|^{q(\frac{1}{q*}-\frac{1}{p})}+Ck^{\gamma}_{i+1}|A_{k_{i+1},\rho_i}|, \tag{5.3.1}$$

for every $i \in \mathbb{N}$.
Moreover, define the sequence $(J_i)$ setting

$$J_i := \|(u-k_i)_+\|^p_{W^{1,p}(B_{\rho_i})}.$$

We begin proving an inequality that will be crucial for the proof of Theorem 5.0.1.

**Lemma 5.3.1.** *Let $u \in W^{1,p}(\Omega)$ be the solution to (5.0.1) under assumptions (5.0.2) and (5.0.3), for exponents $1 < p \leq q$, $0 \leq \gamma$ verifying (1.0.7), (5.0.4) respectively. Then, there exists a constant $\tilde{C} > 0$ such that for every $i \in \mathbb{N}$*

$$J_{i+1} \leq \frac{\tilde{C}}{d^{\theta}}\lambda^i J_i^{1+\alpha},$$

*for some $\lambda > 1$ and $\alpha > 0$, where* $\frac{1}{d^{\theta}} := \max\left\{\frac{1}{d^{p_n^*-p}}, \frac{1}{d^{q(\frac{1}{q*}-\frac{1}{p})p_n^*}}, \frac{1}{d^{p_n^*-\gamma}}\right\}.$

*Proof.* We observe that $k_{i+1}-k_i < u-k_i$ on $A_{k_{i+1},\rho_i}$ for every $i \in \mathbb{N}$. By Sobolev inequality we get

$$\begin{aligned}
|A_{k_{i+1},\rho_i}| &\leq \int_{A_{k_{i+1},\rho_i}}\left(\frac{u-k_i}{k_{i+1}-k_i}\right)^{p_n^*}dx \leq \frac{\|(u-k_i)_+\|^{p_n^*}_{L^{p_n^*}(B_{\rho_i})}}{(k_{i+1}-k_i)^{p_n^*}} \\
&\leq C(n,p)\frac{J_i^{\frac{p_n^*}{p}}}{(k_{i+1}-k_i)^{p_n^*}},
\end{aligned} \tag{5.3.2}$$

where $p_n^*$ was introduced in (5.0.4).
Furthermore, we have

$$\begin{aligned}
&\|(u-k_{i+1})_+\|^p_{W^{1,p}(B_{\rho_i})} \\
&= \int_{B_{\rho_i}}(u-k_{i+1})^p_+dx + \int_{B_{\rho_i}}|D(u-k_{i+1})_+|^p dx \\
&= \int_{A_{k_{i+1},\rho_i}}(u-k_{i+1})^p dx + \int_{A_{k_{i+1},\rho_i}}|D(u-k_{i+1})|^p dx \\
&\leq \int_{A_{k_i,\rho_i}}(u-k_i)^p dx + \int_{A_{k_i,\rho_i}}|D(u-k_i)|^p dx = J_i.
\end{aligned} \tag{5.3.3}$$

Inserting estimates (5.3.2) and (5.3.3) in (5.3.1), we obtain

$$\|D(u-k_{i+1})_+\|^p_{L^p(B_{\rho_{i+1}})} \leq \frac{C}{(\rho_i-\rho_{i+1})^{\mu}}\frac{J_i^{\frac{q}{p}\left(1+p_n^*(\frac{1}{q*}-\frac{1}{p})\right)}}{(k_{i+1}-k_i)^{q(\frac{1}{q*}-\frac{1}{p})p_n^*}} + C\frac{k^{\gamma}_{i+1}J_i^{\frac{p_n^*}{p}}}{(k_{i+1}-k_i)^{p_n^*}}. \tag{5.3.4}$$

By using Hölder's and Sobolev inequalities and estimate (5.3.2), it follows

$$\|(u-k_{i+1})_+\|^p_{L^p(B_{\rho_{i+1}})} \leq \|(u-k_i)_+\|^p_{L^{p_n^*}(B_{\rho_i})}|A_{k_{i+1},\rho_i}|^{1-\frac{p}{p_n^*}} \leq C\frac{J_i^{\frac{p_n^*}{p}}}{(k_{i+1}-k_i)^{p_n^*-p}}. \tag{5.3.5}$$

Summing (5.3.4) and (5.3.5), we obtain

$$J_{i+1} \le C\Bigg[\frac{J_i^{\frac{p_n^*}{p}}}{(k_{i+1}-k_i)^{p_n^*-p}} + \frac{1}{(\rho_i-\rho_{i+1})^\mu}\frac{J_i^{\frac{q}{p}\left(1+p_n^*(\frac{1}{q_*}-\frac{1}{p})\right)}}{(k_{i+1}-k_i)^{q(\frac{1}{q_*}-\frac{1}{p})p_n^*}} + \frac{k_{i+1}^\gamma J_i^{\frac{p_n^*}{p}}}{(k_{i+1}-k_i)^{p_n^*}}\Bigg], \tag{5.3.6}$$

for a constant $C := C(n,p,q,R_0)$.
Recalling the definition of $k_i$, $\rho_i$ and using the fact that $k_i \le 2d$, we can write inequality (5.3.6) as follows

$$\begin{aligned} &J_{i+1} \\ &\le C\Bigg[\frac{J_i^{\frac{p_n^*}{p}}}{d^{p_n^*-p}}2^{(i+2)(p_n^*-p)} + \frac{2^{(i+2)\left(\mu+q(\frac{1}{q_*}-\frac{1}{p})p_n^*\right)}}{R_0^\mu}\frac{J_i^{\frac{q}{p}\left(1+p_n^*(\frac{1}{q_*}-\frac{1}{p})\right)}}{d^{q(\frac{1}{q_*}-\frac{1}{p})p_n^*}} + \frac{2^{(i+2)p_n^*}}{d^{p_n^*-\gamma}}J_i^{\frac{p_n^*}{p}}\Bigg] \\ &\le \frac{C}{R_0^\mu d^\theta}2^{i\tau}\Big(J_i^{\frac{p_n^*}{p}} + J_i^{\frac{q}{p}\left(1+p_n^*(\frac{1}{q_*}-\frac{1}{p})\right)} + J_i^{\frac{p_n^*}{p}}\Big), \end{aligned} \tag{5.3.7}$$

where

$$\frac{1}{d^\theta} := \max\left\{\frac{1}{d^{p_n^*-p}}, \frac{1}{d^{q(\frac{1}{q_*}-\frac{1}{p})p_n^*}}, \frac{1}{d^{p_n^*-\gamma}}\right\},$$

$$\tau := \max\left\{p_n^*-p, \mu+q\left(\frac{1}{q_*}-\frac{1}{p}\right)p_n^*, p_n^*\right\}.$$

Notice that by the definition of $J_i$ and $d$, we easily derive

$$J_i \le \|(u-\sup_{B_{R_0}}\psi)_+\|^p_{W^{1,p}(B_{R_0})} \quad \forall i \in \mathbb{N}.$$

Setting

$$\sigma := \min\left\{\frac{p_n^*}{p}, \frac{q}{p}\left(1+p_n^*\left(\frac{1}{q_*}-\frac{1}{p}\right)\right)\right\} > 1 \tag{5.3.8}$$

and

$$\beta := \max\left\{\frac{p_n^*}{p}-\sigma, \frac{q}{p}\left(1+p_n^*\left(\frac{1}{q_*}-\frac{1}{p}\right)\right)-\sigma\right\} \ge 0,$$

estimate (5.3.7) can be written as

$$J_{i+1} \le \frac{C}{R_0^\mu d^\theta}2^{i\tau}\big(1+\|(u-\sup_{B_{R_0}}\psi)_+\|^p_{W^{1,p}(B_{R_0})}\big)^\beta J_i^\sigma = \frac{\tilde{C}}{d^\theta}2^{i\tau}J_i^\sigma. \qquad \square$$

Now, we are in position to give the proof of our main result.

*Proof of Theorem 5.0.1.* Notice that $u \ge \psi$ and $\psi \in L^\infty_{\mathrm{loc}}(\Omega)$ imply that $u$ is bounded from below. Hence, we only need to show that $u$ is bounded from above.
In order to show that the minimizer $u$ is bounded in the ball $B_{R_0/2}$, we prove that the sequence $(J_i)$ satisfies the assumptions of Lemma 5.1.2. Indeed, Lemma 5.1.2 yields

$$\lim_{i\to+\infty} J_i = 0. \tag{5.3.9}$$

On the other hand, we have

$$\lim_{i\to+\infty} J_i = \|(u-2d)_+\|^p_{W^{1,p}(B_{R_0/2})}. \tag{5.3.10}$$

Thus, (5.3.9) and (5.3.10) allow us to conclude that

$$\sup_{B_{R_0/2}} u \le 2d.$$

Denote

$$\varepsilon := \frac{1}{q_*} - \frac{1}{p}.$$

If $\gamma < p_n^*$ and $\varepsilon > 0$, by Lemma 5.3.1 the sequence $(J_i)$ satisfies assumptions of Lemma 5.1.2 with

$$A = \frac{\tilde{C}}{d^\theta}, \quad \lambda = 2^\tau \quad \text{and} \quad \alpha = \sigma - 1,$$

where $\sigma$ was defined in (5.3.8).
We also have that

$$J_0 \le A^{-\frac{1}{\alpha}}\lambda^{-\frac{1}{\alpha^2}} \quad \text{with} \quad d \ge \tilde{C}^{\frac{1}{\theta}}\lambda^{\frac{1}{\alpha\theta}}\|u\|^{p\frac{\alpha}{\theta}}_{W^{1,p}(B_{R_0})}.$$

Therefore, (5.3.9) holds.
If $\gamma < p_n^*$ and $\varepsilon = 0$, Lemma 5.3.1 implies that the sequence $(J_i)$ satisfies Lemma 5.1.2 with

$$A = \tilde{C}, \quad \lambda = 2^\tau \quad \text{and} \quad \alpha = \sigma - 1.$$

Therefore, by Lemma 5.1.2 we have that (5.3.9) holds if

$$J_0 \le A^{-\frac{1}{\alpha}}\lambda^{-\frac{1}{\alpha^2}}. \tag{5.3.11}$$

By definition, $J_0 = \|(u-d)_+\|^p_{W^{1,p}(B_{R_0})}$. We choose $d > 0$ large enough so that (5.3.11) holds; this is possible since $u \in W^{1,p}(B_{R_0})$ and

$$J_0 = \|(u-d)_+\|^p_{W^{1,p}(A_{d,R_0})} \le \|u\|^p_{W^{1,p}(A_{d,R_0})} \longrightarrow 0 \quad \text{as } d \to +\infty.$$

With this choice of $d$, we get (5.3.9).
Eventually, the case $\gamma = p_n^*$ and $\varepsilon \ge 0$ can be treated as the previous one. □

**Corollary 5.3.2.** *Let us consider functionals $F$ satisfying the double-side bound*

$$\nu H(x,z) \le F(x,s,z) \le LH(x,z), \tag{5.3.12}$$

*for constants $0 < \nu \le L$, where*

$$H(x,z) := |z|^p + a(x)|z|^q,$$

*with $1 < p < q$ and $0 \le a(x) \in L^\infty(\Omega)$. Then, under the closeness condition (1.0.7) on the exponents $p, q$, solutions to (5.0.1) with $F$ verifying (5.3.12) are locally bounded provided the obstacle is locally bounded.*

*Proof.* The arguments are essentially the same of Theorem 5.0.1. Indeed, an analogous version of Theorem 5.2.1 and Lemma 5.3.1 can be proved similarly with exponent $\gamma = 0$.
Let $u \in W^{1,p}(\Omega)$ be the solution to (5.0.1) with energy density satisfying the growth condition (5.3.12). Using the same notation introduced in the proof of Theorem 5.2.1, we have

$$\begin{aligned}&\int_{A_{k,s}} F(x,u,Du)dx \leq \int_{A_{k,t}} F(x,u+\varphi,Du+D\varphi)dx \\ &\leq L\int_{A_{k,t}} H(x,Du+D\varphi)dx \\ &= L\int_{A_{k,t}} H\Big(x,(1-\eta^\sigma)Du-\sigma\eta^\sigma\frac{D\eta}{\eta}(u-k)\Big)dx. \end{aligned} \tag{5.3.13}$$

By the very definition of $H(x,z)$, we have that for a.e. $x \in \Omega$ and every $z_1, z_2 \in \mathbb{R}^n$

$$H(x,z_1+z_2) \leq 2^q(H(x,z_1)+H(x,z_2)). \tag{5.3.14}$$

Using (5.3.14), the fact that $\eta \leq 1$ and the boundedness of the function $a$, we estimate inequality (5.3.13) as follows

$$\begin{aligned}&\int_{A_{k,s}} F(x,u,Du)dx \\ &\leq C(q,L)\int_{A_{k,t}} H(x,(1-\eta^\sigma)Du)dx \\ &\quad + C(q,L)\int_{A_{k,t}} H\Big(x,-\sigma\eta^\sigma\frac{D\eta}{\eta}(u-k)\Big)dx \\ &\leq C(q,L)\int_{A_{k,t}} (1-\eta^\sigma)^p H(x,Du)dx \\ &\quad + C(q,L,\|a\|_\infty)\int_{A_{k,t}} (1+|D\eta|^q(u-k)^q)dx.\end{aligned}$$

Taking into account that $\mathrm{supp}(1-\eta^\sigma) \subset A_{k,t}\setminus A_{k,s}$, the growth assumption (5.3.12) and $t \leq R$, we obtain

$$\begin{aligned}&\int_{A_{k,s}} H(x,Du)dx \leq C\int_{A_{k,s}} F(x,u,Du)dx \\ &\leq C\int_{A_{k,t}\setminus A_{k,s}} H(x,Du)dx + C|A_{k,R}| + C\int_{A_{k,t}} |D\eta|^q(u-k)^q dx.\end{aligned}$$

Recalling the definition of $H(x,Du)$, the previous estimate yields the Caccioppoli inequality (5.2.1) with $\gamma = 0$.
It is worth noting that assumption (1.0.7) improves the bound on the gap $q/p$ established in [16] for obtaining the local boundedness of solutions to double phase obstacle problems. □

# Chapter 6

# Regularity for bounded solutions to obstacle problems with non-standard growth

In this chapter we study the higher fractional differentiability properties of the gradient of bounded solutions $u \in W^{1,p}(\Omega)$ to obstacle problems of the form

$$\min\left\{\int_\Omega F(x, Dw)dx \ : \ w \in \mathcal{K}_\psi(\Omega)\right\}, \tag{6.0.1}$$

where $\psi \in W^{1,p}(\Omega)$ is the obstacle function and $\mathcal{K}_\psi(\Omega)$ was defined in (2.0.2). The results we report have been proved in [68].

In what follows, we assume that the energy density $F : \Omega \times \mathbb{R}^n \to [0, +\infty)$ is a Carathéodory map satisfying assumptions (F1)–(F4), for some exponents $2 \leq p < q$. We say that function $F$ satisfies assumption (F7) if there exists a non-negative function $g \in L_{\mathrm{loc}}^{\frac{p+2\beta}{p+\beta-q}}(\Omega)$, with $0 < \beta < \alpha < 1$, such that

$$|D_\xi F(x, \xi) - D_\xi F(y, \xi)| \leq |x - y|^\alpha (g(x) + g(y))(\mu^2 + |\xi|^2)^{\frac{q-1}{2}} \tag{F7}$$

for a.e. $x, y \in \Omega$ and every $\xi \in \mathbb{R}^n$.
On the other hand, we say that assumption (F8) is satisfied if there exists a sequence of measurable non-negative functions $g_k \in L_{\mathrm{loc}}^{\frac{p+2\alpha}{p+\alpha-q}}(\Omega)$, with $0 < \alpha < 1$, such that

$$\sum_{k=1}^{\infty} \|g_k\|^\sigma_{L^{\frac{p+2\alpha}{p+\alpha-q}}(\Omega)} < \infty,$$

for some $\sigma \geq 1$, and at the same time

$$|D_\xi F(x, \xi) - D_\xi F(y, \xi)| \leq |x - y|^\alpha (g_k(x) + g_k(y))(\mu^2 + |\xi|^2)^{\frac{q-1}{2}} \tag{F8}$$

for a.e. $x, y \in \Omega$ such that $2^{-k}\mathrm{diam}(\Omega) \leq |x - y| < 2^{-k+1}\mathrm{diam}(\Omega)$ and for every $\xi \in \mathbb{R}^n$.

More precisely, we shall prove the following theorems.

**Theorem 6.0.1.** *Let $F(x, \xi)$ satisfy (F1)–(F4) and (F7) for exponents $2 \leq p < q$ such that*

$$q < p + \beta \tag{6.0.2}$$

*and*

$$\frac{1}{q} \geq \frac{1}{p} - \frac{1}{n-1}. \tag{6.0.3}$$

*Let $u \in \mathcal{K}_\psi(\Omega)$ be the solution to the obstacle problem (6.0.1). If $\psi \in L^\infty_{loc}(\Omega)$, then we have*

$$D\psi \in B^\alpha_{\frac{p+2\beta}{p+1+\beta-q},\infty,loc}(\Omega) \Rightarrow (\mu^2 + |Du|^2)^{\frac{p-2}{4}} Du \in B^\alpha_{2,\infty,loc}(\Omega),$$

*provided* $0 < \beta < \alpha < 1$.

On the other hand, a Besov regularity of the type $B^\alpha_{r,\sigma}$, with $\sigma$ finite, is stronger than the one of the type $B^\alpha_{p,\infty}$. In this case, we prove the higher fractional differentiability properties for bounded minimizers under weaker assumptions both on the coefficients of $D_\xi F(x,\xi)$ and on the gradient of the obstacle and on the bound for the gap $q/p$. The main difference is that a stronger embedding theorem between Sobolev and Besov spaces holds (see Proposition 2.3.5).

**Theorem 6.0.2.** *Let $F(x,\xi)$ satisfy (F1)–(F4) and (F8) for exponents $2 \leq p < q$ verifying (6.0.3) and*

$$q < p + \alpha. \tag{6.0.4}$$

*Let $u \in \mathcal{K}_\psi(\Omega)$ be the solution to the obstacle problem (6.0.1). If $\psi \in L^\infty_{loc}(\Omega)$, then we have*

$$D\psi \in B^\alpha_{\frac{p+2\alpha}{p+1+\alpha-q},\sigma,loc}(\Omega) \Rightarrow (\mu^2 + |Du|^2)^{\frac{p-2}{4}} Du \in B^\alpha_{2,\sigma,loc}(\Omega),$$

*for a constant $\sigma \geq 1$ such that $\sigma(\alpha + \frac{p}{2}) \leq 2$.*

Note that Theorems 6.0.1 and 6.0.2 improve the results contained in Chapter 4. Here, the local boundedness allows us to use an interpolation inequality (see Lemma 2.3.2) that gives the higher local integrability $L^{p+2\alpha}$ of the gradient of the solutions. Such higher integrability is the key tool in order to weaken the assumptions on the partial map $x \mapsto D_\xi F(x,\xi)$ and on $D\psi$ with respect to the ones in Theorems 4.0.1 and 4.0.2. Indeed, for $p < n - 2\alpha$ and $q < p + \alpha - \frac{\alpha(p+2\alpha)}{n}$, we have $L^{\frac{n}{\alpha}} \subset L^{\frac{p+2\alpha}{p+\alpha-q}}$, and, moreover, under our assumption on the gap, i.e. $q < p + \alpha$, $B^\alpha_{2q-p,\sigma} \subset B^\alpha_{\frac{p+2\alpha}{p+1+\alpha-q},\sigma}$, for every $\sigma \geq 1$.

We observe that the bound (6.0.3) is only needed to get the local boundedness of the solution (see Theorem 5.0.1). Therefore, if we deal with a priori bounded minimizers, then the result holds without the hypotesis (6.0.3).

The structure of this chapter is the following. In Section 6.1, we prove Theorem 6.0.1. In particular, we derive the a priori estimates in Section 6.1.1, and, in Section 6.1.2, we pass to the limit in the approximating problems. Eventually, in Section 6.2, we give the proof of Theorem 6.0.2, focusing only on the a priori estimate since the approximation procedure works exactly in same way as in the proof of Theorem 6.0.1.

## 6.1 Proof of Theorem 6.0.1

This section is devoted to the proof of Theorem 6.0.1. In particular, in Section 6.1.1, we derive the a priori estimates for regular minimizers of obstacle problems (6.0.1), while in Section 6.1.2, we conclude through an approximation argument.

### 6.1.1 A priori estimate

We have the following theorem.

**Theorem 6.1.1.** *Let $F(x, \xi)$ satisfy (F1)–(F4) and (F7) for exponents $2 \leq p < q$ such that (6.0.2) and (6.0.3) hold. Let $u \in \mathcal{K}_\psi(\Omega)$ be the solution to the obstacle problem (6.0.1). Suppose that*

$$g \in L_{loc}^{\frac{p+2\beta}{p+\beta-q}}(\Omega), \ \psi \in L^\infty_{loc}(\Omega) \quad \textit{and} \quad D\psi \in B^\alpha_{\frac{p+2\beta}{p+1+\beta-q},\infty,loc}(\Omega),$$

*for $0 < \beta < \alpha < 1$. If we a priori assume that*

$$V_p(Du) \in B^\alpha_{2,\infty,loc}(\Omega),$$

*then the following estimates*

$$\begin{aligned}\int_{B_{R/4}} |Du|^{p+2\beta} dx \leq & C(\|\psi\|_{L^\infty(B_R)} + \|u\|_{W^{1,p}(B_R)})^\pi \\ & \cdot \left( \int_{B_R} (g^{\frac{p+2\beta}{p+\beta-q}} + 1) dx + \|D\psi\|_{B^\alpha_{\frac{p+2\beta}{p+1+\beta-q},\infty}(B_R)} \right)^\pi \end{aligned} \tag{6.1.1}$$

*and*

$$\begin{aligned}\int_{B_{R/4}} |\tau_h V_p(Du)|^2 dx \leq & C|h|^{2\alpha}(\|\psi\|_{L^\infty(B_R)} + \|u\|_{W^{1,p}(B_R)})^\pi \\ & \cdot \left( \int_{B_R} (g^{\frac{p+2\beta}{p+\beta-q}} + 1) dx + \|D\psi\|_{B^\alpha_{\frac{p+2\beta}{p+1+\beta-q},\infty}(B_R)} \right)^\pi \end{aligned} \tag{6.1.2}$$

*hold for all balls $B_{R/4} \subset B_R \Subset \Omega$, for positive constants $C = C(n, p, q, \beta, \nu, L, R)$ and $\pi = \pi(n, p, q, \beta)$.*

*Proof.* By virtue of assumption (6.0.3) and Theorem 5.0.1, $u \in L^\infty_{\text{loc}}(\Omega)$. Hence, using Proposition 2.3.4, we deduce that

$$Du \in L^{p+2\beta}_{\text{loc}}(\Omega). \tag{6.1.3}$$

Notice that $Du \in L^{p+2\beta}_{\text{loc}}(\Omega)$ implies that the $u$ satisfies the variational inequality (3.0.1) for every $\varphi \in W^{1,q}(\Omega)$ such that $\varphi \geq \psi$. Indeed, let $\varphi \in W^{1,q}(\Omega)$, $\varphi \geq \psi$, then the function $u + \varepsilon(\varphi - u)$ belongs to the admissible class, for every $\varepsilon \in (0, 1)$, since

$$u + \varepsilon(\varphi - u) = \varepsilon\varphi + (1 - \varepsilon)u \geq \psi.$$

Hence, by minimality of $u$, we get

$$\int_\Omega F(x, Du) dx \leq \int_\Omega F(x, Du + \varepsilon D(\varphi - u)) dx,$$

which leads to

$$\int_\Omega [F(x, Du + \varepsilon D(\varphi - u)) - F(x, Du)] dx \geq 0.$$

From Lagrange's theorem, there exists $\theta \in (0,1)$ such that

$$\int_\Omega \langle \mathcal{A}(x, Du + \varepsilon\theta D(\varphi - u)), \varepsilon D(\varphi - u)\rangle dx \geq 0,$$

where we set $\mathcal{A}(x,\xi) := D_\xi F(x,\xi)$. Since $\varepsilon > 0$, we get

$$\int_\Omega \langle \mathcal{A}(x, Du + \varepsilon\theta D(\varphi - u)), D(\varphi - u)\rangle dx \geq 0. \tag{6.1.4}$$

Since assumptions (F2)–(F4) are in force, then the operator $\mathcal{A}$ satisfies conditions (C1)–(C3). Now, from assumption (C1), we obtain

$$\begin{aligned} &|\langle \mathcal{A}(x, Du + \varepsilon\theta D(\varphi - u)), D(\varphi - u)\rangle| \\ &\quad \leq |\mathcal{A}(x, Du + \varepsilon\theta D(\varphi - u))||D(\varphi - u)| \\ &\quad \leq C(1 + |Du + \varepsilon\theta D(\varphi - u)|^{q-1})|D(\varphi - u)| \\ &\quad \leq C(1 + |Du|^q + |D\varphi|^q), \end{aligned}$$

where we also used that $\varepsilon, \theta \in (0,1)$.
On the other hand, by virtue of assumption (6.0.2) and (6.1.3), we have

$$1 + |Du|^q + |D\varphi|^q \in L^1_{\mathrm{loc}}(\Omega).$$

Therefore, by applying the Dominated convergence Theorem, we can pass to the limit for $\varepsilon \to 0^+$ in (6.1.4), getting the inequality

$$\int_\Omega \langle \mathcal{A}(x, Du), D(\varphi - u)\rangle dx \geq 0, \tag{6.1.5}$$

for every $\varphi \in W^{1,q}(\Omega)$ such that $\varphi \geq \psi$.

Fix $0 < \frac{R}{4} < \rho < s < t < t' < \frac{R}{2}$ such that $B_R \Subset \Omega$ and a cut-off function $\eta \in \mathcal{C}^1_0(B_t)$ such that $0 \leq \eta \leq 1$, $\eta = 1$ on $B_s$, $|D\eta| \leq \frac{C}{t-s}$. Now, for $|h| \leq t' - t$, we consider functions

$$\varphi_1(x) = u(x) + t\eta^2(x)\tau_h(u - \psi)(x)$$

and

$$\varphi_2(x) = u(x) + t\eta^2(x - h)\tau_{-h}(u - \psi)(x),$$

which belong to the admissible class $\mathcal{K}_\psi(\Omega)$, for every $t \in [0,1)$. Choosing $\varphi_1$ and $\varphi_2$ as test functions in (6.1.5), we obtain

$$\int_\Omega \langle \mathcal{A}(x, Du), D(\eta^2\tau_h(u - \psi))\rangle dx + \int_\Omega \langle \mathcal{A}(x, Du), D(\eta^2(x - h)\tau_{-h}(u - \psi))\rangle dx \geq 0.$$

By means of a simple change of variable in the second integral in the left hand side of the previous inequality, we infer

$$\int_\Omega \langle \mathcal{A}(x + h, Du(x + h)) - \mathcal{A}(x, Du(x)), D(\eta^2\tau_h(u - \psi))\rangle dx \leq 0.$$

We can write previous inequality as follows

$$
\begin{aligned}
0 \geq & \int_\Omega \langle \mathcal{A}(x+h, Du(x+h)) - \mathcal{A}(x+h, Du(x)), \eta^2 D\tau_h u\rangle dx \\
& - \int_\Omega \langle \mathcal{A}(x+h, Du(x+h)) - \mathcal{A}(x+h, Du(x)), \eta^2 D\tau_h \psi\rangle dx \\
& + \int_\Omega \langle \mathcal{A}(x+h, Du(x+h)) - \mathcal{A}(x+h, Du(x)), 2\eta D\eta \tau_h (u-\psi)\rangle dx \\
& + \int_\Omega \langle \mathcal{A}(x+h, Du(x)) - \mathcal{A}(x, Du(x)), \eta^2 D\tau_h u\rangle dx \\
& - \int_\Omega \langle \mathcal{A}(x+h, Du(x)) - \mathcal{A}(x, Du(x)), \eta^2 D\tau_h \psi\rangle dx \\
& + \int_\Omega \langle \mathcal{A}(x+h, Du(x)) - \mathcal{A}(x, Du(x)), 2\eta D\eta \tau_h (u-\psi)\rangle dx \\
=: & I_1 + I_2 + I_3 + I_4 + I_5 + I_6,
\end{aligned}
\tag{6.1.6}
$$

that yields

$$
I_1 \leq |I_2| + |I_3| + |I_4| + |I_5| + |I_6|. \tag{6.1.7}
$$

The ellipticity assumption (C2) and Lemma 2.0.2 imply

$$
\begin{aligned}
I_1 \geq & \nu \int_\Omega \eta^2 |\tau_h Du|^2 (\mu^2 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} dx \\
\geq & C(n,p,\nu) \int_\Omega \eta^2 |\tau_h V_p(Du)|^2 dx.
\end{aligned}
\tag{6.1.8}
$$

From the growth condition (C3), Young's and Hölder's inequalities, Lemma 2.0.2 and assumption on $D\psi$, we get

$$
\begin{aligned}
|I_2| \leq & L \int_\Omega \eta^2 |\tau_h Du| (\mu^2 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{q-2}{2}} |\tau_h D\psi| dx \\
\leq & \varepsilon \int_\Omega \eta^2 |\tau_h Du|^2 (\mu^2 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} dx \\
& + C_\varepsilon(L) \int_\Omega \eta^2 |\tau_h D\psi|^2 (\mu^2 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{2q-p-2}{2}} dx \\
\leq & \varepsilon \int_\Omega \eta^2 |\tau_h V_p(Du)|^2 dx \\
& + C_\varepsilon(L) \left( \int_{B_t} |\tau_h D\psi|^{\frac{p+2\beta}{p+1+\beta-q}} dx \right)^{\frac{2(p+1+\beta-q)}{p+2\beta}} \left( \int_{B_{t'}} (1+|Du|)^{p+2\beta} dx \right)^{\frac{2q-p-2}{p+2\beta}} \\
\leq & \varepsilon \int_\Omega \eta^2 |\tau_h V_p(Du)|^2 dx \\
& + C_\varepsilon(L) |h|^{2\alpha} [D\psi]^2_{B^\alpha_{\frac{p+2\beta}{p+1+\beta-q},\infty}(B_R)} \left( \int_{B_{t'}} (1+|Du|)^{p+2\beta} dx \right)^{\frac{2q-p-2}{p+2\beta}}.
\end{aligned}
\tag{6.1.9}
$$

Arguing analogously, we get

$$
\begin{aligned}
|I_3| \le& 2L\int_\Omega |D\eta|\eta|\tau_h Du|(\mu^2+|Du(x+h)|^2+|Du(x)|^2)^{\frac{q-2}{2}}|\tau_h(u-\psi)|dx\\
\le&\varepsilon\int_\Omega \eta^2|\tau_h Du|^2(\mu^2+|Du(x+h)|^2+|Du(x)|^2)^{\frac{p-2}{2}}dx\\
&+\frac{C_\varepsilon(L)}{(t-s)^2}\int_{B_t}|\tau_h(u-\psi)|^2(\mu^2+|Du(x+h)|^2+|Du(x)|^2)^{\frac{2q-p-2}{2}}dx\\
\le&\varepsilon\int_\Omega \eta^2|\tau_h V_p(Du)|^2dx\\
&+\frac{C_\varepsilon(L)}{(t-s)^2}\left(\int_{B_{t'}}|\tau_h(u-\psi)|^{\frac{p+2\beta}{p+1+\beta-q}}dx\right)^{\frac{2(p+1+\beta-q)}{p+2\beta}}\left(\int_{B_{t'}}(1+|Du|)^{p+2\beta}dx\right)^{\frac{2q-p-2}{p+2\beta}}.
\end{aligned}
$$

Using Lemma 2.1.3, we obtain

$$
\begin{aligned}
|I_3| \le&\varepsilon\int_\Omega \eta^2|\tau_h V_p(Du)|^2dx\\
&+\frac{C_\varepsilon(n,p,q,\beta,L)}{(t-s)^2}|h|^2\left(\int_{B_t}|D(u-\psi)|^{\frac{p+2\beta}{p+1+\beta-q}}dx\right)^{\frac{2(p+1+\beta-q)}{p+2\beta}}\left(\int_{B_{t'}}(1+|Du|)^{p+2\beta}dx\right)^{\frac{2q-p-2}{p+2\beta}}.
\end{aligned}
\tag{6.1.10}
$$

In order to estimate the integral $I_4$, we use assumption (F7), Young's and Hölder's inequalities and Lemma 2.0.2 as follows

$$
\begin{aligned}
|I_4| \le&\int_\Omega \eta^2|\tau_h Du||h|^\alpha(g(x+h)+g(x))(1+|Du|^2)^{\frac{q-1}{2}}dx\\
\le&\varepsilon\int_\Omega \eta^2|\tau_h Du|^2(\mu^2+|Du(x+h)|^2+|Du(x)|^2)^{\frac{p-2}{2}}dx\\
&+C_\varepsilon|h|^{2\alpha}\int_{B_t}(g(x+h)+g(x))^2(1+|Du|)^{2q-p}dx\\
\le&\varepsilon\int_\Omega \eta^2|\tau_h V_p(Du)|^2dx\\
&+C_\varepsilon|h|^{2\alpha}\left(\int_{B_R}g^{\frac{p+2\beta}{p+\beta-q}}dx\right)^{\frac{2(p+\beta-q)}{p+2\beta}}\left(\int_{B_t}(1+|Du|)^{p+2\beta}dx\right)^{\frac{2q-p}{p+2\beta}}.
\end{aligned}
\tag{6.1.11}
$$

We now take care of $I_5$. Similarly as above, exploiting assumption (F7) and Hölder's inequality, we infer

$$
\begin{aligned}
|I_5| \le&\int_\Omega \eta^2|\tau_h D\psi||h|^\alpha\left(g(x+h)+g(x)\right)\left(1+|Du|^2\right)^{\frac{q-1}{2}}dx\\
\le&|h|^\alpha\left(\int_{B_{t'}}g^{\frac{p+2\beta}{p+\beta-q}}dx\right)^{\frac{p+\beta-q}{p+2\beta}}\left(\int_{B_t}|\tau_h D\psi|^{\frac{p+2\beta}{q+\beta}}(1+|Du|)^{\frac{(q-1)(p+2\beta)}{q+\beta}}dx\right)^{\frac{q+\beta}{p+2\beta}}\\
\le&|h|^\alpha\left(\int_{B_{t'}}g^{\frac{p+2\beta}{p+\beta-q}}dx\right)^{\frac{p+\beta-q}{p+2\beta}}\left(\int_{B_t}|\tau_h D\psi|^{\frac{p+2\beta}{p+1+\beta-q}}dx\right)^{\frac{p+1+\beta-q}{p+2\beta}}
\end{aligned}
$$

$$\cdot\left(\int_{B_t}(1+|Du|)^{\frac{(q-1)(p+2\beta)}{2q-p-1}}dx\right)^{\frac{2q-p-1}{p+2\beta}}.$$

Now, we observe

$$\frac{(q-1)(p+2\beta)}{2q-p-1}<p+2\beta\Longleftrightarrow p<q. \tag{6.1.12}$$

Hence, (6.1.12) together with Hölder's inequality yields

$$|I_5|\le C|h|^{2\alpha}\left(\int_{B_R}g^{\frac{p+2\beta}{p+\beta-q}}dx\right)^{\frac{p+\beta-q}{p+2\beta}}[D\psi]_{B^{\alpha}_{\frac{p+2\beta}{p+1+\beta-q},\infty}(B_R)}\left(\int_{B_t}(1+|Du|)^{p+2\beta}dx\right)^{\frac{q-1}{p+2\beta}}, \tag{6.1.13}$$

with a constant $C:=C(n,p,q,\beta,R)$. From assumption (F7), hypothesis $|D\eta|<\frac{C}{t-s}$ and Hölder's inequality, we infer the following estimate for $I_6$.

$$\begin{aligned}|I_6|\le&\frac{C}{t-s}|h|^{\alpha}\int_{B_t}|\tau_h(u-\psi)|(g(x+h)+g(x))(1+|Du|^2)^{\frac{q-1}{2}}dx\\ \le&\frac{C}{t-s}|h|^{\alpha}\left(\int_{B_{t'}}g^{\frac{p+2\beta}{p+\beta-q}}dx\right)^{\frac{p+\beta-q}{p+2\beta}}\left(\int_{B_t}|\tau_h(u-\psi)|^{\frac{p+2\beta}{p+1+\beta-q}}dx\right)^{\frac{p+1+\beta-q}{p+2\beta}}\\ &\cdot\left(\int_{B_t}(1+|Du|)^{\frac{(q-1)(p+2\beta)}{2q-p-1}}dx\right)^{\frac{2q-p-1}{p+2\beta}}.\end{aligned}$$

Using once again Hölder's inequality, inequality (6.1.12) and Lemma 2.1.3, we have

$$\begin{aligned}|I_6|\le&\frac{C}{t-s}|h|^{\alpha+1}\left(\int_{B_R}g^{\frac{p+2\beta}{p+\beta-q}}dx\right)^{\frac{p+\beta-q}{p+2\beta}}\left(\int_{B_{t'}}|D(u-\psi)|^{\frac{p+2\beta}{p+1+\beta-q}}dx\right)^{\frac{p+1+\beta-q}{p+2\beta}}\\ &\cdot\left(\int_{B_t}(1+|Du|)^{p+2\beta}dx\right)^{\frac{q-1}{p+2\beta}},\end{aligned} \tag{6.1.14}$$

for a constant $C:=C(n,p,q,\beta,R)$.
Inserting estimates (6.1.8), (6.1.9), (6.1.10), (6.1.11), (6.1.13) and (6.1.14) in (6.1.7), we infer

$$\begin{aligned}&C(n,p,\nu)\int_{\Omega}\eta^2|\tau_hV_p(Du)|^2dx\\ \le&3\varepsilon\int_{\Omega}\eta^2|\tau_hV_p(Du)|^2dx\\ &+C_\varepsilon|h|^{2\alpha}[D\psi]^2_{B^{\alpha}_{\frac{p+2\beta}{p+1+\beta-q},\infty}(B_R)}\left(\int_{B_{t'}}(1+|Du|)^{p+2\beta}dx\right)^{\frac{2q-p-2}{p+2\beta}}\\ &+\frac{C_\varepsilon}{(t-s)^2}|h|^2\left(\int_{B_t}|D(u-\psi)|^{\frac{p+2\beta}{p+1+\beta-q}}dx\right)^{\frac{2(p+1+\beta-q)}{p+2\beta}}\left(\int_{B_{t'}}(1+|Du|)^{p+2\beta}dx\right)^{\frac{2q-p-2}{p+2\beta}}\\ &+C_\varepsilon|h|^{2\alpha}\left(\int_{B_R}g^{\frac{p+2\beta}{p+\beta-q}}dx\right)^{\frac{2(p+\beta-q)}{p+2\beta}}\left(\int_{B_t}(1+|Du|)^{p+2\beta}dx\right)^{\frac{2q-p}{p+2\beta}}\end{aligned}$$

$$
+ C_\varepsilon |h|^{2\alpha} \left( \int_{B_R} g^{\frac{p+2\beta}{p+\beta-q}} dx \right)^{\frac{p+\beta-q}{p+2\beta}} [D\psi]_{B^{\alpha}_{\frac{p+2\beta}{p+1+\beta-q},\infty}(B_R)} \left( \int_{B_t} (1+|Du|)^{p+2\beta} dx \right)^{\frac{q-1}{p+2\beta}}
$$
$$
+ \frac{C_\varepsilon}{t-s} |h|^{\alpha+1} \left( \int_{B_R} g^{\frac{p+2\beta}{p+\beta-q}} dx \right)^{\frac{p+\beta-q}{p+2\beta}} \left( \int_{B_{t'}} |D(u-\psi)|^{\frac{p+2\beta}{p+1+\beta-q}} dx \right)^{\frac{p+1+\beta-q}{p+2\beta}}
$$
$$
\cdot \left( \int_{B_t} (1+|Du|)^{p+2\beta} dx \right)^{\frac{q-1}{p+2\beta}}, \tag{6.1.15}
$$

with $C_\varepsilon := C_\varepsilon(n,p,q,\beta,L,R)$. Using Lemma 2.0.2, $\eta \le 1$ and $u \in W^{1,p}(\Omega)$, we obtain

$$
\int_\Omega \eta^2 |\tau_h V_p(Du)|^2 dx \le c(n,p) \int_\Omega |\tau_h Du|^2 (\mu^2 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} dx < \infty.
$$

Choosing $\varepsilon = \frac{C(n,p,\nu)}{6}$, we can reabsorb the first integral in the right hand side of the previous estimate by the left hand side, thus getting

$$
\begin{aligned}
&\int_\Omega \eta^2 |\tau_h V_p(Du)|^2 dx \\
\le& C|h|^{2\alpha} M_R^2 \left( \int_{B_{t'}} (1+|Du|)^{p+2\beta} dx \right)^{\frac{2q-p-2}{p+2\beta}} \\
&+ \frac{C}{(t-s)^2} |h|^2 \left( \int_{B_t} |D(u-\psi)|^{\frac{p+2\beta}{p+1+\beta-q}} dx \right)^{\frac{2(p+1+\beta-q)}{p+2\beta}} \left( \int_{B_{t'}} (1+|Du|)^{p+2\beta} dx \right)^{\frac{2q-p-2}{p+2\beta}} \\
&+ C|h|^{2\alpha} M_R^2 \left( \int_{B_t} (1+|Du|)^{p+2\beta} dx \right)^{\frac{2q-p}{p+2\beta}} \\
&+ C|h|^{2\alpha} M_R \left( \int_{B_t} (1+|Du|)^{p+2\beta} dx \right)^{\frac{q-1}{p+2\beta}} \\
&+ \frac{C}{t-s} |h|^{\alpha+1} M_R \left( \int_{B_{t'}} |D(u-\psi)|^{\frac{p+2\beta}{p+1+\beta-q}} dx \right)^{\frac{p+1+\beta-q}{p+2\beta}} \left( \int_{B_t} (1+|Du|)^{p+2\beta} dx \right)^{\frac{q-1}{p+2\beta}},
\end{aligned} \tag{6.1.16}
$$

for a constant $C := C(n,p,q,\beta,\nu,L,R)$, where we set $M_R := \|g\|_{L^{\frac{p+2\beta}{p+\beta-q}}(B_R)} + \|D\psi\|_{B^{\alpha}_{\frac{p+2\beta}{p+1+\beta-q},\infty}(B_R)}$.
From Young's inequality, we infer

$$
\begin{aligned}
&\int_\Omega \eta^2 |\tau_h V_p(Du)|^2 dx \\
\le& C_\theta |h|^{2\alpha} + \theta |h|^{2\alpha} \int_{B_{t'}} (1+|Du|)^{p+2\beta} dx \\
&+ \frac{C_\theta}{(t-s)^{\tilde{p}}} |h|^2 \int_{B_t} |D(u-\psi)|^{\frac{p+2\beta}{p+1+\beta-q}} dx + \theta |h|^2 \int_{B_{t'}} (1+|Du|)^{p+2\beta} dx \\
&+ C_\theta |h|^{2\alpha} + \theta |h|^{2\alpha} \int_{B_t} (1+|Du|)^{p+2\beta} dx
\end{aligned}
$$

$$
\begin{aligned}
&+ C_\theta|h|^{2\alpha} + \theta|h|^{2\alpha}\int_{B_t}(1+|Du|)^{p+2\beta}dx \\
&+ \frac{C_\theta}{(t-s)^{p^*}}|h|^{\alpha+1}\left(\int_{B_{t'}}|D(u-\psi)|^{\frac{p+2\beta}{p+1+\beta-q}}dx\right)^{\frac{\bar{p}(p+1+\beta-q)}{p+2\beta}} \\
&+ \theta|h|^{\alpha+1}\int_{B_t}(1+|Du|)^{p+2\beta}dx,
\end{aligned}
\tag{6.1.17}
$$

with $C_\theta := C_\theta(n,p,q,\beta,\nu,L,R,M_R)$, where $\tilde{p} := \frac{p+2\beta}{p+\beta-q}$ and $\bar{p} := \frac{p+2\beta}{p+2\beta-q+1}$.
Using Young's inequality, we estimate the third and the second last integral appearing in the right hand side of estimate (6.1.17) as follows

$$
\begin{aligned}
&\frac{C_\theta}{(t-s)^{\tilde{p}}}|h|^2\int_{B_t}|D(u-\psi)|^{\frac{p+2\beta}{p+1+\beta-q}}dx \\
&\le \frac{C_\theta}{(t-s)^{\tilde{p}}}|h|^2\int_{B_t}|Du|^{\frac{p+2\beta}{p+1+\beta-q}}dx + \frac{C_\theta}{(t-s)^{\tilde{p}}}|h|^2\int_{B_t}|D\psi|^{\frac{p+2\beta}{p+1+\beta-q}}dx \\
&\le \theta|h|^2\int_{B_t}|Du|^{p+2\beta}dx + \frac{C_\theta}{(t-s)^{p'}}|h|^2|B_R| + \frac{C_\theta}{(t-s)^{\tilde{p}}}|h|^2,
\end{aligned}
\tag{6.1.18}
$$

and analogously

$$
\begin{aligned}
&\frac{C_\theta}{(t-s)^{\bar{p}}}|h|^{\alpha+1}\left(\int_{B_{t'}}|D(u-\psi)|^{\frac{p+2\beta}{p+1+\beta-q}}dx\right)^{\frac{\bar{p}(p+1+\beta-q)}{p+2\beta}} \\
&\le C_\theta|h|^{\alpha+1} + \frac{C_\theta}{(t-s)^{p''}}|h|^{\alpha+1}\int_{B_{t'}}|D(u-\psi)|^{\frac{p+2\beta}{p+1+\beta-q}}dx \\
&\le C_\theta|h|^{\alpha+1} + \theta|h|^2\int_{B_t}|Du|^{p+2\beta}dx + \frac{C_\theta}{(t-s)^{\tilde{p}}}|h|^2|B_R| + \frac{C_\theta}{(t-s)^{p''}}|h|^2,
\end{aligned}
\tag{6.1.19}
$$

where $p' := \frac{p+1+\beta-q}{p+\beta-q}$, $p'' := \frac{p+2\beta}{p+1+\beta-q}$.
Inserting (6.1.18) and (6.1.19) in (6.1.17), we get

$$
\begin{aligned}
&\int_\Omega \eta^2|\tau_h V_p(Du)|^2dx \\
&\le C_\theta|h|^{2\alpha} + \theta|h|^{2\alpha}\int_{B_{t'}}(1+|Du|)^{p+2\beta}dx \\
&\quad + \theta|h|^2\int_{B_t}|Du|^{p+2\beta}dx + \frac{C_\theta}{(t-s)^{p'}}|h|^2|B_R| + \frac{C_\theta}{(t-s)^{\tilde{p}}}|h|^2 \\
&\quad + \theta|h|^2\int_{B_{t'}}(1+|Du|)^{p+2\beta}dx \\
&\quad + C_\theta|h|^{2\alpha} + \theta|h|^{2\alpha}\int_{B_t}(1+|Du|)^{p+2\beta}dx \\
&\quad + C_\theta|h|^{2\alpha} + \theta|h|^{2\alpha}\int_{B_t}(1+|Du|)^{p+2\beta}dx
\end{aligned}
$$

$$+ C_\theta|h|^{\alpha+1} + \theta|h|^2 \int_{B_t} |Du|^{p+2\beta} dx + \frac{C_\theta}{(t-s)^{\tilde{p}}}|h|^2|B_R|$$
$$+ \frac{C_\theta}{(t-s)^{p''}}|h|^2 + \theta|h|^{\alpha+1} \int_{B_t} (1+|Du|)^{p+2\beta} dx. \tag{6.1.20}$$

We can rewrite the previous estimate as

$$\begin{aligned}
&\int_\Omega \eta^2|\tau_h V_p(Du)|^2 dx \\
\le& 5\theta|h|^{2\alpha} \int_{B_t} (1+|Du|^{p+2\beta}) dx + 2\theta|h|^{2\alpha} \int_{B_{t'}} (1+|Du|^{p+2\beta}) dx \\
&+ C_\theta|h|^{2\alpha}\left(1 + \frac{1}{(t-s)^{p'}} + \frac{1}{(t-s)^{\tilde{p}}} + \frac{1}{(t-s)^{p''}}\right),
\end{aligned}$$

for a constant $C_\theta := C_\theta(n,p,q,\beta,\nu,L,R,M_R)$.
Dividing both sides of the previous estimate by $|h|^{2\alpha}$, recalling that $\eta = 1$ in $B_s$ and passing to the limit as $t' \to t^+$, we get

$$\begin{aligned}
&\int_{B_s} \frac{|\tau_h V_p(Du)|^2}{|h|^{2\alpha}} dx \\
\le& 7\theta \int_{B_t} (1+|Du|^{p+2\beta}) dx \\
&+ C_\theta\left(1 + \frac{1}{(t-s)^{p'}} + \frac{1}{(t-s)^{\tilde{p}}} + \frac{1}{(t-s)^{p''}}\right),
\end{aligned} \tag{6.1.21}$$

for every $h \in \mathbb{R}^n$.
Since $u \in L^\infty_{\text{loc}}(\Omega)$ and $V_p(Du) \in B^\alpha_{2,\infty,\text{loc}}(\Omega)$, by virtue of Proposition 2.3.4 and Theorem 5.0.1, we infer the following inequality

$$\begin{aligned}
\int_{B_\rho} |Du|^{p+2\beta} dx \le& C\|u\|^{2\beta}_{L^\infty(B_s)} \sup_h \int_{B_s} \frac{|\tau_h V_p(Du)|^2}{|h|^{2\alpha}} dx \\
&+ \frac{C}{(s-\rho)^{2p}}\|u\|^{2\beta}_{L^\infty(B_s)}\|u\|^p_{W^{1,p}(B_s)} \\
\le& C(\|\psi\|_{L^\infty(B_R)} + \|u\|_{W^{1,p}(B_R)})^\pi \sup_h \int_{B_s} \frac{|\tau_h V_p(Du)|^2}{|h|^{2\alpha}} dx \\
&+ \frac{C}{(s-\rho)^{2p}} C(\|\psi\|_{L^\infty(B_R)} + \|u\|_{W^{1,p}(B_R)})^\pi,
\end{aligned} \tag{6.1.22}$$

for a constant $\pi := \pi(n,p,q)$.
Taking the supremum over $h$ in the left hand side of (6.1.21) and using estimate (6.1.22), we obtain

$$\begin{aligned}
\int_{B_\rho} |Du|^{p+2\beta} dx \le& C(\|\psi\|_{L^\infty(B_R)} + \|u\|_{W^{1,p}(B_R)})^\pi \theta \int_{B_t} (1+|Du|^{p+2\beta}) dx \\
&+ C_\theta(\|\psi\|_{L^\infty(B_R)} + \|u\|_{W^{1,p}(B_R)})^\pi \left[1 + \frac{1}{(t-s)^{p'}} + \frac{1}{(t-s)^{\tilde{p}}} + \frac{1}{(t-s)^{p''}}\right]
\end{aligned}$$

$$+\frac{C}{(s-\rho)^{2p}}(\|\psi\|_{L^\infty(B_R)}+\|u\|_{W^{1,p}(B_R)})^\pi, \tag{6.1.23}$$

for every $0<\frac{R}{4}<\rho<s<t<\frac{R}{2}$.
Now, choosing $s$ such that $s-\rho=t-s$, i.e. $s=\frac{t+\rho}{2}$, it follows

$$\begin{aligned}\int_{B_\rho}|Du|^{p+2\beta}dx \le& C(\|\psi\|_{L^\infty(B_R)}+\|u\|_{W^{1,p}(B_R)})^\pi\theta\int_{B_t}(1+|Du|^{p+2\beta})dx\\ &+C_\theta(\|\psi\|_{L^\infty(B_R)}+\|u\|_{W^{1,p}(B_R)})^\pi\left[1+\frac{1}{(t-\rho)^{p'}}+\frac{1}{(t-\rho)^{\tilde p}}+\frac{1}{(t-\rho)^{p''}}\right]\\ &+\frac{C}{(t-\rho)^{2p}}(\|\psi\|_{L^\infty(B_R)}+\|u\|_{W^{1,p}(B_R)})^\pi. \end{aligned}\tag{6.1.24}$$

Setting

$$\Phi(r)=\int_{B_r}|Du|^{p+2\beta}dx,$$

we can write inequality (6.1.24) as

$$\begin{aligned}\Phi(\rho)\le& C(\|\psi\|_{L^\infty(B_R)}+\|u\|_{W^{1,p}(B_R)})^\pi\theta\Phi(t)\\ &+C_\theta(\|\psi\|_{L^\infty(B_R)}+\|u\|_{W^{1,p}(B_R)})^\pi\left[1+\frac{1}{(t-\rho)^{p'}}+\frac{1}{(t-\rho)^{\tilde p}}+\frac{1}{(t-\rho)^{p''}}\right]\\ &+\frac{C}{(t-\rho)^{2p}}(\|\psi\|_{L^\infty(B_R)}+\|u\|_{W^{1,p}(B_R)})^\pi.\end{aligned}$$

By virtue of Lemma 2.0.3, choosing $\theta$ such that $C(\|\psi\|_{L^\infty(B_R)}+\|u\|_{W^{1,p}(B_R)})^\pi\theta=\frac{1}{2}$, we obtain

$$\begin{aligned}\Phi\left(\frac{R}{4}\right)\le& C(\|\psi\|_{L^\infty(B_R)}+\|u\|_{W^{1,p}(B_R)})^\pi\left[1+\frac{1}{R^{p'}}+\frac{1}{R^{\tilde p}}+\frac{1}{R^{p''}}\right]\\ &+\frac{C}{R^{2p}}(\|\psi\|_{L^\infty(B_R)}+\|u\|_{W^{1,p}(B_R)})^\pi,\end{aligned}$$

with $C:=C(n,p,q,\beta,\nu,L,R,M_R)$.
Now, recalling the definition of $\Phi$, we obtain

$$\begin{aligned}\int_{B_{R/4}}|Du|^{p+2\beta}dx \le& C(\|\psi\|_{L^\infty(B_R)}+\|u\|_{W^{1,p}(B_R)})^\pi\\ &\cdot\left(\int_{B_R}(k^{\frac{p+2\beta}{p+\beta-q}}+1)dx+\|D\psi\|_{B^\alpha_{\frac{p+2\beta}{p+1+\beta-q},\infty}(B_R)}\right)^\pi,\end{aligned}\tag{6.1.25}$$

thus, inserting (6.1.25) in (6.1.21), we deduce the a priori estimate

$$\begin{aligned}\int_{B_{R/4}}|\tau_hV_p(Du)|^2dx \le& C|h|^{2\alpha}(\|\psi\|_{L^\infty(B_R)}+\|u\|_{W^{1,p}(B_R)})^\pi\\ &\cdot\left(\int_{B_R}(g^{\frac{p+2\beta}{p+\beta-q}}+1)dx+\|D\psi\|_{B^\alpha_{\frac{p+2\beta}{p+1+\beta-q},\infty}(B_R)}\right)^\pi,\end{aligned}$$

for constants $C:=C(n,p,q,\beta,\nu,L,R)$ and $\pi:=\pi(n,p,q,\beta)$. □

Now, we are able to establish the following higher differentiability result for obstacle problems with $p$-growth.

**Theorem 6.1.2.** *Assume that* $\mathcal{A}(x,\xi)$ *satisfies* (C1)–(C3) *for an exponent* $2 \le p = q$ *and let* $u \in \mathcal{K}_\psi(\Omega)$ *be the solution to the obstacle problem* (6.0.1). *If there exist a non-negative function* $g \in L^\infty_{loc}(\Omega)$ *and an exponent* $0 < \alpha < 1$ *such that*

$$|\mathcal{A}(x,\xi) - \mathcal{A}(y,\xi)| \le |x-y|^\alpha (g(x)+g(y))(\mu^2+|\xi|^2)^{\frac{p-1}{2}},$$

*for a.e.* $x, y \in \Omega$ *and for every* $\xi \in \mathbb{R}^n$, *then the following implication*

$$\psi \in L^\infty_{loc}(\Omega), D\psi \in B^\alpha_{\frac{p+2\beta}{1+\beta},\infty,loc}(\Omega) \Rightarrow (\mu^2+|Du|^2)^{\frac{p-2}{4}} Du \in B^\alpha_{2,\infty,loc}(\Omega),$$

*holds, provided* $0 < \beta < \alpha$.

*Proof.* Using Proposition 2.3.3, we infer $D\psi \in L^{p+2\beta}_{\text{loc}}(\Omega)$. Hence, [8, Theorem 2.6] yields $Du \in L^{p+2\beta}_{\text{loc}}(\Omega)$. Arguing as in the proof of Theorem 6.1.1, we derive estimate (6.1.21) in the case $p = q$. This completes the proof. □

### 6.1.2 Passage to the limit

*Proof of Theorem 6.0.1.* Let $u \in \mathcal{K}_\psi(\Omega)$ be the solution to (6.0.1), and let $F_j$ be defined as in Lemma 4.2.2. Fixed $B_R \Subset \Omega$, let $u_j$ be the solution of the problem

$$\min\left\{\int_{B_R} F_j(x,Dw)dx : w \ge \psi \text{ a.e. in } B_R,\ w \in u + W^{1,p}_0(B_R)\right\}. \tag{6.1.26}$$

Setting

$$\mathcal{A}_j(x,\xi) := D_\xi F_j(x,\xi),$$

one can easily check that $\mathcal{A}_j$ satisfies (C1)–(C3) and (F7) and the following assumptions:

$$|\mathcal{A}_j(x,\xi)| \le l_1(j)(\mu^2+|\xi|^2)^{\frac{p-1}{2}} \tag{6.1.27}$$

$$|\mathcal{A}_j(x,\xi) - \mathcal{A}_j(x,\eta)| \le L_1(j)|\xi-\eta|(\mu^2+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}} \tag{6.1.28}$$

$$|\mathcal{A}_j(x,\xi) - \mathcal{A}_j(y,\xi)| \le \Theta(j)|x-y|^\alpha(g(x)+g(y))(\mu^2+|\xi|^2)^{\frac{p-1}{2}} \tag{6.1.29}$$

for a.e. $x, y \in \Omega$, for every $\xi, \eta \in \mathbb{R}^n$ and $j \in \mathbb{N}$. It is well known that $u_j \in \mathcal{K}_\psi(B_R)$ is the minimizer of problem (6.1.26) if, and only if, the following variational inequality holds

$$\int_\Omega \langle \mathcal{A}_j(x,Du_j), D(\varphi - u_j)\rangle dx \ge 0, \quad \forall \varphi \in \mathcal{K}_\psi(B_R). \tag{6.1.30}$$

Let $\Omega' \Subset \Omega$ be an open set. Fix a non-negative smooth kernel $\phi \in C^\infty_0(B_1(0))$ such that $\int_{B_1(0)} \phi = 1$ and consider the corresponding family of mollifiers $\{\phi_m\}_{m\in\mathbb{N}}$. Setting

$$g_m = g * \phi_m$$

and

$$\mathcal{A}_{jm}(x,\xi) = \int_{B_1(0)} \phi(y)\mathcal{A}_j(x+my,\xi)dy, \tag{6.1.31}$$

an easy computation shows that $\mathcal{A}_{jm}$ satisfies assumptions (C1)–(C3) and (6.1.27), (6.1.28) and the conditions:

$$|\mathcal{A}_{jm}(x,\xi) - \mathcal{A}_{jm}(y,\xi)| \le |x-y|^{\alpha}(g_m(x)+g_m(y))(\mu^2+|\xi|^2)^{\frac{q-1}{2}}$$

$$|\mathcal{A}_{jm}(x,\xi) - \mathcal{A}_{jm}(y,\xi)| \le \Theta(j)|x-y|^{\alpha}(g_m(x)+g_m(y))(\mu^2+|\xi|^2)^{\frac{p-1}{2}}$$

for a.e. $x, y \in \Omega$, for every $\xi, \eta \in \mathbb{R}^n$ and every $j, m \in \mathbb{N}$.

**Step 1.** Fixed $j \in \mathbb{N}$, let $\mathcal{A}_{jm}$ be defined as in (6.1.31) and let $u_{jm} \in u_j + W_0^{1,p}(B_R)$ be the solution to the variational inequality

$$\int_{B_R} \langle \mathcal{A}_{jm}(x, Du_{jm}), D(\varphi - u_{jm})\rangle dx \ge 0, \quad \forall \varphi \in \mathcal{K}_{\psi}(B_R). \tag{6.1.32}$$

By the ellipticity assumption (C2), we have

$$\begin{aligned}
&\nu \int_{B_R} (\mu^2 + |Du_j|^2 + |Du_{jm}|^2)^{\frac{p-2}{2}} |Du_{jm} - Du_j|^2 dx \\
&\le \int_{B_R} \langle \mathcal{A}_{jm}(x, Du_{jm}) - \mathcal{A}_{jm}(x, Du_j), Du_{jm} - Du_j\rangle dx \\
&= \int_{B_R} \langle \mathcal{A}_{jm}(x, Du_{jm}), Du_{jm} - Du_j\rangle dx \\
&\quad - \int_{B_R} \langle \mathcal{A}_{jm}(x, Du_j), Du_{jm} - Du_j\rangle dx \\
&= \int_{B_R} \langle \mathcal{A}_{jm}(x, Du_{jm}), Du_{jm} - Du_j\rangle dx \\
&\quad - \int_{B_R} \langle \mathcal{A}_{j}(x, Du_j), Du_{jm} - Du_j\rangle dx \\
&\quad + \int_{B_R} \langle \mathcal{A}_{j}(x, Du_j) - \mathcal{A}_{jm}(x, Du_j), Du_{jm} - Du_j\rangle dx
\end{aligned} \tag{6.1.33}$$

Since $u_j$ and $u_{jm}$ are solutions to (6.1.30) and (6.1.32) respectively, we notice that

$$\int_{B_R} \langle \mathcal{A}_{jm}(x, Du_{jm}), Du_{jm} - Du_j\rangle dx - \int_{B_R} \langle \mathcal{A}_{j}(x, Du_j), Du_{jm} - Du_j\rangle dx \le 0 \tag{6.1.34}$$

Combining (6.1.33) and (6.1.34), we get

$$\begin{aligned}
&\nu \int_{B_R} (\mu^2 + |Du_j|^2 + |Du_{jm}|^2)^{\frac{p-2}{2}} |Du_{jm} - Du_j|^2 dx \\
&\le \int_{B_R} \langle \mathcal{A}_{j}(x, Du_j) - \mathcal{A}_{jm}(x, Du_j), Du_{jm} - Du_j\rangle dx \\
&\le \left(\int_{B_R} |\mathcal{A}_{j}(x, Du_j) - \mathcal{A}_{jm}(x, Du_j)|^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \left(\int_{B_R} |Du_{jm} - Du_j|^p dx\right)^{\frac{1}{p}},
\end{aligned} \tag{6.1.35}$$

where in the last inequality we used Hölder's inequality.
Since $p \geq 2$, from (6.1.35), we obtain

$$\int_{B_R} |Du_{jm} - Du_j|^p dx \leq C \int_{B_R} |\mathcal{A}_j(x, Du_j) - \mathcal{A}_{jm}(x, Du_j)|^{\frac{p}{p-1}} dx. \tag{6.1.36}$$

Since $\mathcal{A}_{jm}(x, Du_j)$ satisfies

$$|\mathcal{A}_{jm}(x, Du_j)| \leq l_1(j)(\mu^2 + |Du_j|^2)^{\frac{p-1}{2}},$$

and

$$\mathcal{A}_{jm}(x, Du_j) \to \mathcal{A}_j(x, Du_j) \quad \text{a.e. in } \Omega$$

as $m \to +\infty$, applying the Dominated convergence Theorem, we have

$$\mathcal{A}_{jm}(x, Du_j) \to_{m\to\infty} \mathcal{A}_j(x, Du_j) \quad \text{strongly in } L^{\frac{p}{p-1}}(\Omega).$$

Therefore, passing to the limit for $m \to \infty$ in (6.1.36), we deduce that

$$u_{jm} \to u_j \text{ in } W^{1,p}(B_R). \tag{6.1.37}$$

Moreover, since $g \in L^{\frac{p+2\beta}{p+\beta-q}}_{\text{loc}}(\Omega)$, we have

$$g_m \to g \text{ in } L^{\frac{p+2\beta}{p+\beta-q}}_{\text{loc}}(\Omega). \tag{6.1.38}$$

By virtue of Theorem 6.1.2, $V_p(Du_{jm}) \in B^{\alpha}_{2,\infty,\text{loc}}(B_R)$. Hence, from Theorem 6.1.1, $u_{jm}$ satisfies the a priori estimate

$$\begin{aligned}\int_{B_{R/4}} |Du_{jm}|^{p+2\beta} dx \leq & C(\|\psi\|_{L^\infty(B_R)} + \|u_{jm}\|_{W^{1,p}(B_R)})^\pi \\ & \cdot \left( \int_{B_R} (g_m^{\frac{p+2\beta}{p+\beta-q}} + 1) dx + \|D\psi\|_{B^{\alpha}_{\frac{p+2\beta}{p+1+\beta-q},\infty}(B_R)} \right)^\pi,\end{aligned} \tag{6.1.39}$$

for constants $C := C(n, p, q, \beta, \nu, L, R)$ and $\pi := \pi(n, p, q, \beta)$, both independent of $j$ and $m$.
Finally, by the weak lower semicontinuity of the norm, (6.1.37) and (6.1.38), we get

$$\begin{aligned}\int_{B_{R/4}} |Du_j|^{p+2\beta} dx \leq & \liminf_{m\to\infty} \int_{B_{R/4}} |Du_{jm}|^{p+2\beta} dx \\ \leq & C(\|\psi\|_{L^\infty(B_R)} + \|u_j\|_{W^{1,p}(B_R)})^\pi \\ & \cdot \left( \int_{B_R} (g^{\frac{p+2\beta}{p+\beta-q}} + 1) dx + \|D\psi\|_{B^{\alpha}_{\frac{p+2\beta}{p+1+\beta-q},\infty}(B_R)} \right)^\pi.\end{aligned} \tag{6.1.40}$$

**Step 2.** From Theorem 4.2.1, there exists $c_1 > 0$ such that

$$|\xi|^p \leq c_1(1 + F_j(x, \xi)), \quad \forall j \in \mathbb{N}.$$

The previous estimate and the minimality of $u_j$ imply

$$\int_{B_R} |Du_j|^p dx \leq c_1 \int_{B_R} (1 + F_j(x, Du_j)) dx$$

$$\begin{aligned}
&\le c_1 \int_{B_R} (1 + F_j(x, Du))dx \\
&\le c_1 \int_{B_R} (1 + F(x, Du))dx, \qquad (6.1.41)
\end{aligned}$$

where in the last inequality we used Lemma 4.2.2. Thus, up to subsequences,

$$u_j \rightharpoonup \tilde{u} \text{ in } u + W_0^{1,p}(B_R). \qquad (6.1.42)$$

Now, fix $j_0 \in \mathbb{N}$. Then, by Lemma 4.2.2 and the fact that $u_j$ is the minimum for $F_j$, for every $j > j_0$, we might write

$$\begin{aligned}
\int_{B_R} F_{j_0}(x, Du_j)dx &\le \int_{B_R} F_j(x, Du_j)dx \\
&\le \int_{B_R} F_j(x, Du)dx \le \int_{B_R} F(x, Du)dx.
\end{aligned}$$

From weak lower semicontinuity of $F_{j_0}$ and (6.1.42), it holds,

$$\int_{B_R} F_{j_0}(x, D\tilde{u})dx \le \liminf_{j\to+\infty} \int_{B_R} F_{j_0}(x, Du_j)dx \le \int_{B_R} F(x, Du)dx.$$

Combining these last inequalities, we get

$$\int_{B_R} F(x, D\tilde{u})dx = \lim_{j_0\to+\infty} \int_{B_R} F_{j_0}(x, D\tilde{u})dx \le \int_{B_R} F(x, Du)dx,$$

where we also applied the monotone convergence theorem, according to Lemma 4.2.2. Moreover, by the weak convergence (6.1.42), the limit function $\tilde{u}$ still belongs to $\mathcal{K}_\psi(B_R)$, since this set is convex and closed. Thus, by strict convexity of $F$, we have that $\tilde{u} = u$ a.e. in $B_R$.

Now, from estimates (6.1.40) and (6.1.41), it follows

$$\begin{aligned}
\int_{B_{R/4}} |Du_j|^{p+2\beta}dx \le& C(\|\psi\|_{L^\infty(B_R)} + \|u_j\|_{W^{1,p}(B_R)})^\pi \\
&\cdot \left(\int_{B_R} (g^{\frac{p+2\beta}{p+\beta-q}} + 1)dx + \|D\psi\|_{B^\alpha_{\frac{p+2\beta}{p+1+\beta-q},\infty}(B_R)}\right)^\pi \\
\le& C\left(\|\psi\|_{L^\infty(B_R)} + \int_{B_R} (1 + |u|^p + F(x, Du))dx\right)^\pi \\
&\cdot \left(\int_{B_R} (g^{\frac{p+2\beta}{p+\beta-q}} + 1)dx + \|D\psi\|_{B^\alpha_{\frac{p+2\beta}{p+1+\beta-q},\infty}(B_R)}\right)^\pi,
\end{aligned}$$

for constants $C := C(n, p, q, \beta, \nu, L, R)$ and $\pi := \pi(n, p, q, \beta)$, both independent of $j$. Hence, from (6.1.42) and weak lower semicontinuity, it follows

$$\begin{aligned}
\int_{B_{R/4}} |Du|^{p+2\beta}dx \le& \liminf_{j\to\infty} \int_{B_{R/4}} |Du_j|^{p+2\beta}dx \\
\le& C\left(\|\psi\|_{L^\infty(B_R)} + \int_{B_R} (1 + |u|^p + F(x, Du))dx\right)^\pi \\
&\cdot \left(\int_{B_R} (g^{\frac{p+2\beta}{p+\beta-q}} + 1)dx + \|D\psi\|_{B^\alpha_{\frac{p+2\beta}{p+1+\beta-q},\infty}(B_R)}\right)^\pi.
\end{aligned}$$

Eventually, proceeding as in the proof of Theorem 6.1.1, we get $V_p(Du) \in B^\alpha_{2,\infty,\mathrm{loc}}(\Omega)$. □

## 6.2 Proof of Theorem 6.0.2

*Proof of Theorem 6.0.2.* We derive only the a priori estimates, since the approximation procedure is achieved using the same arguments presented in Section 6.1.2.
We a priori assume that $V_p(Du) \in B^{\alpha}_{2,\sigma,\mathrm{loc}}(\Omega)$. By virtue of assumption (6.0.3) and Theorem 5.0.1, $u \in L^{\infty}_{\mathrm{loc}}(\Omega)$. Hence, using Proposition 2.3.5, we deduce that

$$Du \in L^{p+2\alpha}_{\mathrm{loc}}(\Omega).$$

Arguing analogously as in the proof of Theorem 6.1.1, we define the integrals $I_1$-$I_6$ according to (6.1.6) and we are able to derive estimates (6.1.7) and (6.1.8) for the integral $I_1$. We need to treat differently the integrals $I_2$-$I_6$ in which the new assumptions on the coefficients of the map $\mathcal{A}(x,\xi)$ and on the gradient of the obstacle come into the play.

Consider the integral $I_2$. From the growth condition (C3), Young's and Hölder's inequalities, Lemma 2.0.2 and the assumption $D\psi \in B^{\alpha}_{\frac{p+2\alpha}{p+1+\alpha-q},\sigma,\mathrm{loc}}(\Omega)$, we get

$$\begin{aligned}|I_2| \leq& \varepsilon \int_{\Omega} \eta^2 |\tau_h V_p(Du)|^2 dx \\ &+ C_\varepsilon(L)\left(\int_{B_t} |\tau_h D\psi|^{\frac{p+2\alpha}{p+1+\alpha-q}} dx\right)^{\frac{2(p+1+\alpha-q)}{p+2\alpha}} \left(\int_{B_{t'}} (1+|Du|)^{p+2\alpha} dx\right)^{\frac{2q-p-2}{p+2\alpha}}. \end{aligned} \tag{6.2.1}$$

Performing the same computations which led us to (6.1.10) with $\beta = \alpha$, we get the following estimate for $I_3$

$$\begin{aligned}|I_3| \leq& \varepsilon \int_{\Omega} \eta^2 |\tau_h V_p(Du)|^2 dx \\ &+ \frac{C_\varepsilon(n,p,q,\alpha,L)}{(t-s)^2}|h|^2\left(\int_{B_t} |D(u-\psi)|^{\frac{p+2\alpha}{p+1+\alpha-q}} dx\right)^{\frac{2(p+1+\alpha-q)}{p+2\alpha}} \left(\int_{B_{t'}} (1+|Du|)^{p+2\alpha} dx\right)^{\frac{2q-p-2}{p+2\alpha}}. \end{aligned} \tag{6.2.2}$$

Now, we take care of the integral $I_4$. For $h \in \mathbb{R}^n$ such that $2^{-k}(t'-t) \leq |h| \leq 2^{-k+1}(t'-t)$, $k \in \mathbb{N}$, applying lemma 2.0.2, assumption (F7), Young's and Hölder's inequalities give that

$$\begin{aligned}|I_4| \leq& \int_{\Omega} \eta^2 |\tau_h Du||h|^{\alpha}(g_k(x+h)+g_k(x))(1+|Du(x)|^2)^{\frac{q-1}{2}} dx \\ \leq& \varepsilon \int_{\Omega} \eta^2 |\tau_h Du|^2 (\mu^2 + |Du(x+h)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} dx \\ &+ C_\varepsilon |h|^{2\alpha} \int_{B_t} (g_k(x+h)+g_k(x))^2 (1+|Du(x)|)^{2q-p} dx \\ \leq& \varepsilon \int_{\Omega} \eta^2 |\tau_h V_p(Du)|^2 dx \\ &+ C_\varepsilon |h|^{2\alpha}\left(\int_{B_t} (g_k(x+h)+g_k(x))^{\frac{p+2\alpha}{p+\alpha-q}} dx\right)^{\frac{2(p+\alpha-q)}{p+2\alpha}} \left(\int_{B_t} (1+|Du|)^{p+2\alpha} dx\right)^{\frac{2q-p}{p+2\alpha}}. \end{aligned} \tag{6.2.3}$$

Exploiting assumption (F7) and Hölder's inequality, we infer

$$|I_5| \leq \int_{\Omega} \eta^2 |\tau_h D\psi||h|^{\alpha}\left(g_k(x+h)+g_k(x)\right)\left(1+|Du|^2\right)^{\frac{q-1}{2}} dx$$

$$
\begin{aligned}
&\leq |h|^{\alpha}\left(\int_{B_t}(g_k(x+h)+g_k(x))^{\frac{p+2\alpha}{p+\alpha-q}}dx\right)^{\frac{p+\alpha-q}{p+2\alpha}}\left(\int_{B_t}|\tau_h D\psi|^{\frac{p+2\alpha}{q+\alpha}}(1+|Du|)^{\frac{(q-1)(p+2\alpha)}{q+\alpha}}dx\right)^{\frac{q+\alpha}{p+2\alpha}}\\
&\leq |h|^{\alpha}\left(\int_{B_t}(g_k(x+h)+g_k(x))^{\frac{p+2\alpha}{p+\alpha-q}}dx\right)^{\frac{p+\alpha-q}{p+2\alpha}}\left(\int_{B_t}|\tau_h D\psi|^{\frac{p+2\alpha}{p+1+\alpha-q}}dx\right)^{\frac{p+1+\alpha-q}{p+2\alpha}}\\
&\quad\cdot\left(\int_{B_t}(1+|Du|)^{\frac{(q-1)(p+2\alpha)}{2q-p-1}}dx\right)^{\frac{2q-p-1}{p+2\alpha}}.
\end{aligned}
$$

Now, we observe

$$
\frac{(q-1)(p+2\alpha)}{2q-p-1}<p+2\alpha \Longleftrightarrow p<q. \tag{6.2.4}
$$

Hence

$$
\begin{aligned}
|I_5| \leq &C|h|^{\alpha}\left(\int_{B_t}(g_k(x+h)+g_k(x))^{\frac{p+2\alpha}{p+\alpha-q}}dx\right)^{\frac{p+\alpha-q}{p+2\alpha}}\left(\int_{B_t}|\tau_h D\psi|^{\frac{p+2\alpha}{p+1+\alpha-q}}dx\right)^{\frac{p+1+\alpha-q}{p+2\alpha}}\\
&\cdot\left(\int_{B_t}(1+|Du|)^{p+2\alpha}dx\right)^{\frac{q-1}{p+2\alpha}},
\end{aligned} \tag{6.2.5}
$$

for a constant $C:=C(n,p,q,\alpha,R)$, where $h\in\mathbb{R}^n$ is such that $2^{-k}(t'-t)\leq|h|\leq 2^{-k+1}(t'-t)$, $k\in\mathbb{N}$.

Similary to above, from assumption (F7), hypothesis $|D\eta|<\frac{C}{t-s}$ and Hölder's inequality, we are able to estimate the integral $I_6$ as follows

$$
\begin{aligned}
|I_6| \leq &\frac{C}{t-s}|h|^{\alpha}\int_{B_t}|\tau_h(u-\psi)|(g_k(x+h)+g_k(x))(1+|Du|^2)^{\frac{q-1}{2}}dx\\
\leq &\frac{C}{t-s}|h|^{\alpha}\left(\int_{B_t}(g_k(x+h)+g_k(x))^{\frac{p+2\alpha}{p+\alpha-q}}dx\right)^{\frac{p+\alpha-q}{p+2\alpha}}\left(\int_{B_t}|\tau_h(u-\psi)|^{\frac{p+2\alpha}{p+1+\alpha-q}}dx\right)^{\frac{p+1+\alpha-q}{p+2\alpha}}\\
&\cdot\left(\int_{B_t}(1+|Du|)^{\frac{(q-1)(p+2\alpha)}{2q-p-1}}dx\right)^{\frac{2q-p-1}{p+2\alpha}}.
\end{aligned}
$$

Using once again Hölder's inequality, inequality (6.2.4) and Lemma 2.1.3, we have

$$
\begin{aligned}
|I_6| \leq &\frac{C}{t-s}|h|^{\alpha+1}\left(\int_{B_t}(g_k(x+h)+g_k(x))^{\frac{p+2\alpha}{p+\alpha-q}}dx\right)^{\frac{p+\alpha-q}{p+2\alpha}}\left(\int_{B_{t'}}|D(u-\psi)|^{\frac{p+2\alpha}{p+1+\alpha-q}}dx\right)^{\frac{p+1+\alpha-q}{p+2\alpha}}\\
&\cdot\left(\int_{B_t}(1+|Du|)^{p+2\alpha}dx\right)^{\frac{q-1}{p+2\alpha}},
\end{aligned} \tag{6.2.6}
$$

with $C:=C(n,p,q,\alpha,L)$.

Inserting estimates (6.1.8), (6.2.1), (6.2.2), (6.2.3), (6.2.5) and (6.2.6) in (6.1.7) and reabsorbing the integral $\int_\Omega \eta^2|\tau_h V_p(Du)|^2dx$ in the right hand side by the left hand side, we infer

$$
\int_\Omega \eta^2|\tau_h V_p(Du)|^2dx
$$

$$
\begin{aligned}
\leq & C\left(\int_{B_R} |D\psi|^{\frac{p+2\alpha}{p+1+\alpha-q}} dx\right)^{\frac{2(p+1+\alpha-q)}{p+2\alpha}} \left(\int_{B_{t'}} (1+|Du|)^{p+2\alpha} dx\right)^{\frac{2q-p-2}{p+2\alpha}} \\
& + \frac{C}{(t-s)^2}|h|^2 \left(\int_{B_t} |D(u-\psi)|^{\frac{p+2\alpha}{p+1+\alpha-q}} dx\right)^{\frac{2(p+1+\alpha-q)}{p+2\alpha}} \left(\int_{B_{t'}} (1+|Du|)^{p+2\alpha} dx\right)^{\frac{2q-p-2}{p+2\alpha}} \\
& + C|h|^{2\alpha}\left(\int_{B_{R/2}} (g_k(x+h)+g_k(x))^{\frac{p+2\alpha}{p+\alpha-q}} dx\right)^{\frac{2(p+\alpha-q)}{p+2\alpha}} \left(\int_{B_t} (1+|Du|)^{p+2\alpha} dx\right)^{\frac{2q-p}{p+2\alpha}} \\
& + |h|^{\alpha}\left(\int_{B_{R/2}} (g_k(x+h)+g_k(x))^{\frac{p+2\alpha}{p+\alpha-q}} dx\right)^{\frac{p+\alpha-q}{p+2\alpha}} \\
& \cdot \left(\int_{B_R} |\tau_h D\psi|^{\frac{p+2\alpha}{p+1+\alpha-q}} dx\right)^{\frac{p+1+\alpha-q}{p+2\alpha}} \left(\int_{B_t} (1+|Du|)^{p+2\alpha} dx\right)^{\frac{q-1}{p+2\alpha}} \\
& + \frac{C}{t-s}|h|^{\alpha+1}\left(\int_{B_{R/2}} (g_k(x+h)+g_k(x))^{\frac{p+2\alpha}{p+\alpha-q}} dx\right)^{\frac{p+\alpha-q}{p+2\alpha}} \left(\int_{B_{t'}} |D(u-\psi)|^{\frac{p+2\alpha}{p+1+\alpha-q}} dx\right)^{\frac{p+1+\alpha-q}{p+2\alpha}} \\
& \cdot \left(\int_{B_t} (1+|Du|)^{p+2\alpha} dx\right)^{\frac{q-1}{p+2\alpha}}, \qquad (6.2.7)
\end{aligned}
$$

for a positive constant $C := C(n, p, q, \alpha, \nu, L, R)$, where $2^{-k}(t'-t) \leq |h| \leq 2^{-k+1}(t'-t)$, $k \in \mathbb{N}$. Recalling that $\eta = 1$ on $B_s$ and dividing both sides by $|h|^{2\alpha}$, we get

$$
\begin{aligned}
& \int_{B_s} \frac{|\tau_h V_p(Du)|^2}{|h|^{2\alpha}} dx \\
\leq & C\left(\int_{B_R} \frac{|D\psi|^{\frac{p+2\alpha}{p+1+\alpha-q}}}{|h|^{\frac{\alpha(p+2\alpha)}{p+1+\alpha-q}}} dx\right)^{\frac{2(p+1+\alpha-q)}{p+2\alpha}} \left(\int_{B_{t'}} (1+|Du|)^{p+2\alpha} dx\right)^{\frac{2q-p-2}{p+2\alpha}} \\
& + \frac{C}{(t-s)^2}|h|^{2(1-\alpha)} \left(\int_{B_t} |D(u-\psi)|^{\frac{p+2\alpha}{p+1+\alpha-q}} dx\right)^{\frac{2(p+1+\alpha-q)}{p+2\alpha}} \left(\int_{B_{t'}} (1+|Du|)^{p+2\alpha} dx\right)^{\frac{2q-p-2}{p+2\alpha}} \\
& + C\left(\int_{B_{R/2}} (g_k(x+h)+g_k(x))^{\frac{p+2\alpha}{p+\alpha-q}} dx\right)^{\frac{2(p+\alpha-q)}{p+2\alpha}} \left(\int_{B_t} (1+|Du|)^{p+2\alpha} dx\right)^{\frac{2q-p}{p+2\alpha}} \\
& + \left(\int_{B_{R/2}} (g_k(x+h)+g_k(x))^{\frac{p+2\alpha}{p+\alpha-q}} dx\right)^{\frac{p+\alpha-q}{p+2\alpha}} \\
& \cdot \left(\int_{B_R} \frac{|\tau_h D\psi|^{\frac{p+2\alpha}{p+1+\alpha-q}}}{|h|^{\frac{\alpha(p+2\alpha)}{p+1+\alpha-q}}} dx\right)^{\frac{p+1+\alpha-q}{p+2\alpha}} \left(\int_{B_t} (1+|Du|)^{p+2\alpha} dx\right)^{\frac{q-1}{p+2\alpha}} \\
& + \frac{C}{t-s}|h|^{1-\alpha}\left(\int_{B_{R/2}} (g_k(x+h)+g_k(x))^{\frac{p+2\alpha}{p+\alpha-q}} dx\right)^{\frac{p+\alpha-q}{p+2\alpha}} \left(\int_{B_{t'}} |D(u-\psi)|^{\frac{p+2\alpha}{p+1+\alpha-q}} dx\right)^{\frac{p+1+\alpha-q}{p+2\alpha}} \\
& \cdot \left(\int_{B_t} (1+|Du|)^{p+2\alpha} dx\right)^{\frac{q-1}{p+2\alpha}}. \qquad (6.2.8)
\end{aligned}
$$

We need now to take the $L^\sigma$ norm with the measure $\frac{dh}{|h|^n}$ restricted to the ball $B(0, t'-t)$ on the $h$-space of the $L^2$ norm of the difference quotient of order $\alpha$ of the function $V_p(Du)$. Since the functions $g_k$ are defined for $2^{-k}(t'-t) \le |h| \le 2^{-k+1}(t'-t)$ we interpret the ball $B(0,t'-t)$ as

$$B(0,t'-t) = \bigcup_{k=1}^{\infty} B(0, 2^{-k+1}(t'-t)) \setminus B(0, 2^{-k}(t'-t)) =: \bigcup_{k=1}^{\infty} E_k.$$

We obtain the following estimate

$$\begin{aligned}
&\int_{B_{t'-t}(0)} \left( \int_{B_s} \frac{|\tau_h V_p(Du)|^2}{|h|^{2\alpha}} dx \right)^{\frac{\sigma}{2}} \frac{dh}{|h|^n} \\
\le & C \left( \int_{B_{t'}} (1+|Du|)^{p+2\alpha} dx \right)^{\frac{\sigma(2q-p-2)}{2(p+2\alpha)}} \int_{B_{t'-t}(0)} \left( \int_{B_R} \frac{|D\psi|^{\frac{p+2\alpha}{p+1+\alpha-q}}}{|h|^{\frac{\alpha(p+2\alpha)}{p+1+\alpha-q}}} dx \right)^{\frac{\sigma(p+1+\alpha-q)}{p+2\alpha}} \frac{dh}{|h|^n} \\
& + \frac{C}{(t-s)^2} \left( \int_{B_t} |D(u-\psi)|^{\frac{p+2\alpha}{p+1+\alpha-q}} dx \right)^{\frac{\sigma(p+1+\alpha-q)}{p+2\alpha}} \left( \int_{B_{t'}} (1+|Du|)^{p+2\alpha} dx \right)^{\frac{(2q-p-2)\sigma}{2(p+2\alpha)}} \int_{B_{t'-t}(0)} |h|^{\sigma(1-\alpha)} \frac{dh}{|h|^n} \\
& + C \left( \int_{B_t} (1+|Du|)^{p+2\alpha} dx \right)^{\frac{(2q-p)\sigma}{2(p+2\alpha)}} \sum_{k=1}^{\infty} \int_{E_k} \left( \int_{B_{R/2}} (g_k(x+h)+g_k(x))^{\frac{p+2\alpha}{p+\alpha-q}} dx \right)^{\frac{\sigma(p+\alpha-q)}{p+2\alpha}} \frac{dh}{|h|^n} \\
& + \sum_{k=1}^{\infty} \int_{E_k} \left( \int_{B_{R/2}} (g_k(x+h)+g_k(x))^{\frac{p+2\alpha}{p+\alpha-q}} dx \right)^{\frac{(p+\alpha-q)\sigma}{2(p+2\alpha)}} \left( \int_{B_R} \frac{|\tau_h D\psi|^{\frac{p+2\alpha}{p+1+\alpha-q}}}{|h|^{\frac{\alpha(p+2\alpha)}{p+1+\alpha-q}}} dx \right)^{\frac{(p+1+\alpha-q)\sigma}{2(p+2\alpha)}} \frac{dh}{|h|^n} \\
& \cdot \left( \int_{B_t} (1+|Du|)^{p+2\alpha} dx \right)^{\frac{(q-1)\sigma}{2(p+2\alpha)}} \\
& + \frac{C}{t-s} \sum_{k=1}^{\infty} \int_{E_k} |h|^{\frac{\sigma(1-\alpha)}{2}} \left( \int_{B_{R/2}} (g_k(x+h)+g_k(x))^{\frac{p+2\alpha}{p+\alpha-q}} dx \right)^{\frac{(p+\alpha-q)\sigma}{2(p+2\alpha)}} \\
& \cdot \left( \int_{B_{t'}} |D(u-\psi)|^{\frac{p+2\alpha}{p+1+\alpha-q}} dx \right)^{\frac{(p+1+\alpha-q)\sigma}{2(p+2\alpha)}} \left( \int_{B_t} (1+|Du|)^{p+2\alpha} dx \right)^{\frac{(q-1)\sigma}{2(p+2\alpha)}} .
\end{aligned} \tag{6.2.9}$$

Note that, since $\alpha \le \gamma$, the integral

$$J_1 := \int_{B_{t'-t}(0)} \left( \int_{B_R} \frac{|\tau_h D\psi|^{\frac{p+2\alpha}{p+1+\alpha-q}}}{|h|^{\frac{\alpha(p+2\alpha)}{p+1+\alpha-q}}} dx \right)^{\frac{\sigma(p+1+\alpha-q)}{p+2\alpha}} \frac{dh}{|h|^n}$$

is controlled by the norm in the Besov space $B^{\alpha}_{\frac{p+2\alpha}{p+1+\alpha-q},\sigma}$ on $B_R$ of the gradient of the obstacle which is finite by assumptions. The integral

$$J_2 := \int_{B_{t'-t}(0)} |h|^{\sigma(1-\alpha)} \frac{dh}{|h|^n}$$

can be calculated in polar coordinates as follows

$$J_2 = C(n) \int_0^{t'-t} \varrho^{\sigma(1-\alpha)-1} d\varrho \le C(n) \int_0^{R/4} \varrho^{\sigma(1-\alpha)-1} d\varrho = C(n,\alpha,\sigma,R),$$

since $t'-t \leq \frac{R}{4}$ and $\alpha \in (0,1)$.
Now, we take care of the integral

$$J_3 := \sum_{k=1}^{\infty} \int_{E_k} \left( \int_{B_{R/2}} (g_k(x+h) + g_k(x))^{\frac{p+2\alpha}{p+\alpha-q}} dx \right)^{\frac{\sigma(p+\alpha-q)}{p+2\alpha}} \frac{dh}{|h|^n}.$$

We write the right hand sinde of the previous estimate in polar coordinates, so $h \in E_k$ if, and only if, $h = m\xi$ for some $2^{-k+1}(t'-t) \leq m < 2^{-k}(t'-t)$ and some $\xi$ in the unit sphere $\mathbb{S}^{n-1}$ on $\mathbb{R}^n$. We denote by $dS(\xi)$ the surface measure on $\mathbb{S}^{n-1}$. We infer

$$\begin{aligned} J_3 \leq & \sum_{k=1}^{\infty} \int_{m_{k-1}}^{m_k} \int_{\mathbb{S}^{n-1}} \left( \int_{B_{R/2}} (g_k(x+h) + g_k(x))^{\frac{p+2\alpha}{p+\alpha-q}} dx \right)^{\frac{\sigma(p+\alpha-q)}{p+2\alpha}} dS(\xi) \frac{dm}{m} \\ = & \sum_{k=1}^{\infty} \int_{m_{k-1}}^{m_k} \int_{\mathbb{S}^{n-1}} \|(\tau_{m\xi} g_k + g_k)\|^{\sigma}_{L^{\frac{p+2\alpha}{p+\alpha-q}}(B_{R/2})} dS(\xi) \frac{dm}{m}, \end{aligned}$$

where we set $m_k = 2^{-k}(t'-t)$. We note that for each $\xi \in \mathbb{S}^{n-1}$ and $m_{k-1} \leq m \leq m_k$

$$\begin{aligned} \|(\tau_{m\xi} g_k + g_k)\|_{L^{\frac{p+2\alpha}{p+\alpha-q}}(B_{R/2})} \leq & \|g_k\|_{L^{\frac{p+2\alpha}{p+\alpha-q}}(B_{R/2-m_k\xi})} + \|g_k\|_{L^{\frac{p+2\alpha}{p+\alpha-q}}(B_{R/2})} \\ \leq & 2\|g_k\|_{L^{\frac{p+2\alpha}{p+\alpha-q}}(B_{R/2+R/4})}, \end{aligned}$$

where in the last inequality we used that $t'-t \leq \frac{R}{4}$. Hence

$$J_3 \leq C(n) \|\{g_k\}_k\|^{\sigma}_{l^{\sigma}(L^{\frac{p+2\alpha}{p+\alpha-q}}(B_R))},$$

which is finite by assumption (F8).
Using the Young's inequality with exponent 2, we deduce the following estimate

$$\begin{aligned} & \sum_{k=1}^{\infty} \int_{E_k} \left( \int_{B_{R/2}} (g_k(x+h) + g_k(x))^{\frac{p+2\alpha}{p+\alpha-q}} dx \right)^{\frac{\sigma(p+\alpha-q)}{2(p+2\alpha)}} \left( \int_{B_{R/2}} \frac{|\tau_h D\psi|^{\frac{p+2\alpha}{p+1+\alpha-q}}}{|h|^{\frac{\alpha(p+2\alpha)}{p+1+\alpha-q}}} dx \right)^{\frac{\sigma(p+1+\alpha-q)}{2(p+2\alpha)}} \frac{dh}{|h|^n} \\ \leq & C \sum_{k=1}^{\infty} \int_{E_k} \left( \int_{B_{R/2}} (g_k(x+h) + g_k(x))^{\frac{p+2\alpha}{p+\alpha-q}} dx \right)^{\frac{\sigma(p+\alpha-q)}{p+2\alpha}} \frac{dh}{|h|^n} \\ & + C \int_{B_{t'-t}(0)} \left( \int_{B_{R/2}} \frac{|\tau_h D\psi|^{\frac{p+2\alpha}{p+1+\alpha-q}}}{|h|^{\frac{\alpha(p+2\alpha)}{p+1+\alpha-q}}} dx \right)^{\frac{\sigma(p+1+\alpha-q)}{p+2\alpha}} \frac{dh}{|h|^n} \end{aligned}$$

where the two integrals in the right hand side can be estimated as the integrals $J_1$ and $J_3$.
Similarly, we obtain

$$\begin{aligned} & \sum_{k=1}^{\infty} \int_{E_k} |h|^{(1-\alpha)\frac{\sigma}{2}} \left( \int_{B_{R/2}} (g_k(x+h) + g_k(x))^{\frac{p+2\alpha}{p+\alpha-q}} dx \right)^{\frac{\sigma(p+\alpha-q)}{2(p+2\alpha)}} \frac{dh}{|h|^n} \\ \leq & \int_{B_{t'-t}(0)} |h|^{(1-\alpha)\sigma} \frac{dh}{|h|^n} + \sum_{k=1}^{\infty} \int_{E_k} \left( \int_{B_{R/2}} (g_k(x+h) + g_k(x))^{\frac{p+2\alpha}{p+\alpha-q}} dx \right)^{\frac{\sigma(p\alpha-q)}{p+2\alpha}} \frac{dh}{|h|^n}. \end{aligned}$$

The first term and the latter one can be estimated as the integral $J_2$ and $J_3$, respectively.

Estimate (6.2.9) can be written in the following way

$$\begin{aligned}
&\int_{B_{t'-t}(0)}\left(\int_{B_s}\frac{|\tau_h V_p(Du)|^2}{|h|^{2\alpha}}dx\right)^{\frac{\sigma}{2}}\frac{dh}{|h|^n}\\
\leq& C\left(\int_{B_{t'}}(1+|Du|)^{p+2\alpha}dx\right)^{\frac{\sigma(2q-p-2)}{2(p+2\alpha)}}\\
&+\frac{C}{(t-s)^2}\left(\int_{B_t}|D(u-\psi)|^{\frac{p+2\alpha}{p+1+\alpha-q}}dx\right)^{\frac{\sigma(p+1+\alpha-q)}{p+2\alpha}}\left(\int_{B_{t'}}(1+|Du|)^{p+2\alpha}dx\right)^{\frac{(2q-p-2)\sigma}{2(p+2\alpha)}}\\
&+C\left(\int_{B_t}(1+|Du|)^{p+2\alpha}dx\right)^{\frac{(2q-p)\sigma}{2(p+2\alpha)}}+C\left(\int_{B_t}(1+|Du|)^{p+2\alpha}dx\right)^{\frac{(q-1)\sigma}{2(p+2\alpha)}}\\
&+\frac{C}{t-s}\left(\int_{B_{t'}}|D(u-\psi)|^{\frac{p+2\alpha}{p+1+\alpha-q}}dx\right)^{\frac{(p+1+\alpha-q)\sigma}{2(p+2\alpha)}}\left(\int_{B_t}(1+|Du|)^{p+2\alpha}dx\right)^{\frac{(q-1)\sigma}{2(p+2\alpha)}}, \quad (6.2.10)
\end{aligned}$$

for a constant $C := C(n,p,q,\sigma,\alpha,\nu,L,R,\|D\psi\|_{B^{\gamma}_{\frac{p+2\alpha}{p+1+\alpha-q},\sigma}(B_R)},\|\{g_k\}_k\|_{l^{\sigma}(L^{\frac{p+2\alpha}{p+\alpha-q}}(B_R))})$.

From Young's inequality, we infer

$$\begin{aligned}
&\int_{B_{t'-t}(0)}\left(\int_{B_s}\frac{|\tau_h V_p(Du)|^2}{|h|^{2\alpha}}dx\right)^{\frac{\sigma}{2}}\frac{dh}{|h|^n}\\
\leq&\theta\left(\int_{B_{t'}}(1+|Du|)^{p+2\alpha}dx\right)^{\frac{\sigma}{2}}+C_\theta\\
&+\frac{C_\theta}{(t-s)^{p''}}\left(\int_{B_t}|D(u-\psi)|^{\frac{p+2\alpha}{p+1+\alpha-q}}dx\right)^{\frac{\sigma}{2}}+\theta\left(\int_{B_{t'}}(1+|Du|)^{p+2\alpha}dx\right)^{\frac{\sigma}{2}}\\
&+\theta\left(\int_{B_t}(1+|Du|)^{p+2\alpha}dx\right)^{\frac{\sigma}{2}}+\theta\left(\int_{B_t}(1+|Du|)^{p+2\alpha}dx\right)^{\frac{\sigma}{2}}\\
&+\frac{C_\theta}{(t-s)^{\bar{p}}}\left(\int_{B_{t'}}|D(u-\psi)|^{\frac{p+2\alpha}{p+1+\alpha-q}}dx\right)^{\frac{(p+1+\alpha-q)\sigma}{2(p+1+2\alpha-q)}}+\theta\left(\int_{B_t}(1+|Du|)^{p+2\alpha}dx\right)^{\frac{\sigma}{2}},
\end{aligned} \quad (6.2.11)$$

for $0<\theta<1$, where we denote $p'' := \frac{p+2\alpha}{p+1+\alpha-q}$ and $\bar{p} := \frac{p+2\alpha}{p+1+2\alpha-q}$.

We estimate the second and the penultimate integral appearing in the right hand side of estimate as follows

$$\begin{aligned}
&\frac{C_\theta}{(t-s)^{p''}}\left(\int_{B_t}|D(u-\psi)|^{\frac{p+2\alpha}{p+1+\alpha-q}}dx\right)^{\frac{\sigma}{2}}\\
\leq&\frac{C_\theta}{(t-s)^{p''}}\left(\int_{B_t}|Du|^{\frac{p+2\alpha}{p+1+\alpha-q}}dx\right)^{\frac{\sigma}{2}}+\frac{C_\theta}{(t-s)^{p''}}\left(\int_{B_t}|D\psi|^{\frac{p+2\alpha}{p+1+\alpha-q}}dx\right)^{\frac{\sigma}{2}}\\
\leq&\theta\left(\int_{B_t}|Du|^{p+2\alpha}dx\right)^{\frac{\sigma}{2}}+\frac{C_\theta(L)}{(t-s)^{\frac{\tilde{p}\sigma}{2}}}|B_R|^{\frac{\sigma}{2}}+\frac{C_\theta}{(t-s)^{p''}}\left(\int_{B_R}|D\psi|^{\frac{p+2\alpha}{p+1+\alpha-q}}dx\right)^{\frac{\sigma}{2}}
\end{aligned} \quad (6.2.12)$$

and similarly

$$\begin{aligned}
&\frac{C_\theta}{(t-s)^{\bar{p}}}\left(\int_{B_{t'}}|D(u-\psi)|^{\frac{p+2\alpha}{p+1+\alpha-q}}dx\right)^{\frac{\sigma(p+1+\alpha-q)}{2(p+1+2\alpha-q)}}\\
&\leq \frac{C_\theta}{(t-s)^{p'''}}+C_\theta\Big(\int_{B_{t'}}|D(u-\psi)|^{\frac{p+2\alpha}{p+1+\alpha-q}}dx\Big)^{\frac{\sigma}{2}}\\
&\leq \frac{C_\theta}{(t-s)^{p'''}}+\theta\Big(\int_{B_{t'}}|Du|^{p+2\alpha}dx\Big)^{\frac{\sigma}{2}}+|B_R|^{\frac{\sigma}{2}}+C_\theta\Big(\int_{B_R}|D\psi|^{\frac{p+2\alpha}{p+1+\alpha-q}}dx\Big)^{\frac{\sigma}{2}} \qquad (6.2.13)
\end{aligned}$$

where we set $\tilde{p}:=\frac{p+2\alpha}{p+\alpha-q}$ and $p''':=\frac{p+2\alpha}{\alpha}$.
Inserting estimates (6.2.12) and (6.2.13) in (6.2.11), we obtain

$$\begin{aligned}
&\int_{B_{t'-t}(0)}\Big(\int_{B_s}\frac{|\tau_h V_p(Du)|^2}{|h|^{2\alpha}}dx\Big)^{\frac{\sigma}{2}}\frac{dh}{|h|^n}\\
&\leq 4\theta\Big(\int_{B_t}(1+|Du|)^{p+2\alpha}dx\Big)^{\frac{\sigma}{2}}+3\theta\Big(\int_{B_{t'}}(1+|Du|)^{p+2\alpha}dx\Big)^{\frac{\sigma}{2}}\\
&\quad+C_\theta+\frac{C_\theta(L)}{(t-s)^{\frac{\tilde{p}\sigma}{2}}}|B_R|^{\frac{\sigma}{2}}+\frac{C_\theta}{(t-s)^{p''}}\Big(\int_{B_R}|D\psi|^{\frac{p+2\alpha}{p+1+\alpha-q}}dx\Big)^{\frac{\sigma}{2}}\\
&\quad+\frac{C_\theta}{(t-s)^{p'''}}+|B_R|^{\frac{\sigma}{2}}+C_\theta\Big(\int_{B_R}|D\psi|^{\frac{p+2\alpha}{p+1+\alpha-q}}dx\Big)^{\frac{\sigma}{2}}. \qquad (6.2.14)
\end{aligned}$$

Now, by virtue of Proposition 2.3.5, we infer the following inequality

$$\begin{aligned}
\Big(\int_{B_\rho}|Du|^{p+2\alpha}dx\Big)^{\frac{\sigma}{2}} \leq\, & C\|u\|^{\sigma\gamma}_{L^\infty(B_s)}\int_{B_{t'-t}(0)}\Big(\int_{B_s}\frac{|\tau_h V_p(Du)|^2}{|h|^{2\alpha}}dx\Big)^{\frac{\sigma}{2}}\frac{dh}{|h|^n}\\
&+\frac{C}{(s-\rho)^{\sigma p}}\|u\|^{\sigma\gamma}_{L^\infty(B_s)}\|u\|^{\frac{\sigma p}{2}}_{W^{1,p}(B_s)}. \qquad (6.2.15)
\end{aligned}$$

Combining inequalities (6.2.14) and (6.2.15) and arguing as in the proof of Theorem 6.1.1, we obtain

$$\begin{aligned}
\Big(\int_{B_{R/4}}|Du|^{p+2\alpha}dx\Big)^{\frac{\sigma}{2}} \leq\, & C(\|\psi\|_{L^\infty(B_R)}+\|u\|_{W^{1,p}(B_R)})^{\pi}\\
&\cdot\Big(\|\{g_k\}_k\|^{\sigma}_{l^\sigma(L^{\frac{p+2\alpha}{p+\alpha-q}}(B_R))}+\|D\psi\|_{B^\gamma_{\frac{p+2\alpha}{p+1+\alpha-q},\infty}(B_R)}+1\Big)^{\pi}, \qquad (6.2.16)
\end{aligned}$$

which yields

$$\begin{aligned}
\int_{B_{t'-t}(0)}\Big(\int_{B_{R/4}}\frac{|\tau_h V_p(Du)|^2}{|h|^{2\alpha}}dx\Big)^{\frac{\sigma}{2}}\frac{dh}{|h|^n} \leq\, & C(\|\psi\|_{L^\infty(B_R)}+\|u\|_{W^{1,p}(B_R)})^{\pi}\\
&\cdot\Big(\|\{g_k\}_k\|^{\sigma}_{l^\sigma(L^{\frac{p+2\alpha}{p+\alpha-q}}(B_R))}+\|D\psi\|_{B^\gamma_{\frac{p+2\alpha}{p+1+\alpha-q},\infty}(B_R)}+1\Big)^{\pi},
\end{aligned}$$

for every $t' - t \leq \frac{R}{4}$. Hence, we eventually get

$$\int_{B_{R/4}(0)} \left( \int_{B_{R/4}} \frac{|\tau_h V_p(Du)|^2}{|h|^{2\alpha}} dx \right)^{\frac{\sigma}{2}} \frac{dh}{|h|^n} \leq C(\|\psi\|_{L^\infty(B_R)} + \|u\|_{W^{1,p}(B_R)})^\pi \cdot \left( \|\{g_k\}_k\|^\sigma_{l^\sigma(L^{\frac{p+2\alpha}{p+\alpha-q}}(B_R))} + \|D\psi\|_{B^\gamma_{\frac{p+2\alpha}{p+1+\alpha-q},\infty}(B_R)} + 1 \right)^\pi,$$

for constants $C := C(n, p, q, \sigma, \alpha, \nu, L, R)$ and $\pi := \pi(n, p, q, \alpha, \sigma)$. □

# Chapter 7

# Higher differentiability for double phase obstacle problems

In this chapter we shall consider higher fractional differentiability properties of the gradient of the solutions $u \in W^{1,p}(\Omega)$ to variational obstacle problems of the form

$$\min\left\{\int_\Omega F(x,w,Dw)dx \;:\; w \in \mathcal{K}_\psi(\Omega)\right\}, \tag{7.0.1}$$

where the energy density $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$F(x,w,\xi) := b(x,w)H(x,\xi), \tag{7.0.2}$$

being

$$H(x,\xi) := |\xi|^p + a(x)|\xi|^q, \tag{7.0.3}$$

for some exponents $2 \leq p < q$, where $\psi \in W^{1,p}(\Omega)$ is the obstacle function and the class $\mathcal{K}_\psi(\Omega)$ was set in (2.0.2). The results we present are contained in the work [70].
We assume that the coefficients $a(x)$ and $b(x,w)$ satisfy the following assumptions:
**Assumption 1:**

(i) $a : \Omega \to [0,+\infty)$ is a bounded and measurable function such that

$$|a(x) - a(y)| \leq \omega_a(|x-y|),$$

for all $x, y \in \Omega$, where $\omega_a : \mathbb{R}^+ \to [0,1]$ is defined by $\omega_a(\rho) = \min\{\rho^\alpha, 1\}$, for some $\alpha \in (0,1)$;

(ii) the function $b : \Omega \times \mathbb{R} \to (0,+\infty)$ is a bounded Carathéodory function, i.e. there exist $0 < \nu \leq L$ such that

$$0 < \nu \leq b(x,w) \leq L < +\infty.$$

**Assumption 2:**

(i) there exists a function $\omega_b : \mathbb{R}^+ \to [0,1]$ defined by $\omega_b(\rho) = \min\{\rho^\beta, 1\}$, for some $\beta \in (0,1)$, such that

$$|b(x,u) - b(y,v)| \leq \omega_b(|x-y| + |u-v|),$$

for all $x, y \in \Omega$ and every $u, v \in \mathbb{R}$.

We point out that the choice of stating Assumption 1 and 2 separately is due to the fact that they are needed independently.

The main difficulty of this the regularity result contained in [70] is the dependence of our double phase functional both on the $x$-variable and the $w$-variable, where the map $w \mapsto b(x,w)H(x,z)$ is non-differentiable. In order to deal with this issue, we follow the strategy proposed in [75] and later used in [47]. Namely, we introduce the so-called "freezed" functional defined in (7.1.1) and the solution to the corresponding obstacle problem (see (7.1.2)) for which we prove a higher differentiability result in the scale of Besov spaces following the argument in [69]. The idea is to compare the solution $u$ to the original obstacle problem (7.0.1) and the solution $v$ to the "freezed" one (7.1.2). More precisely, we estimate the fractional difference quotients of $u$ and $v$, in an integral sense, gaining a Besov regularity for $u$. In order to do so, we also have to derive some ad hoc higher integrability results, both at the interior and up to boundary, that is for the solution $u$ of the original obstacle problem (7.0.1) and the solution $v$ to the freezed one (7.1.2) respectively. The first one is obtained adapting the argument in [41], while the second one generalizes the result by Cupini, Fusco and Petti in [25]. Eventually, we use a boot-strapping argument to get the maximal higher fractional differentiability.

The main result of this chapter is the following

**Theorem 7.0.1.** *Let $u \in W^{1,p}(\Omega)$ be the solution to the obstacle problem (7.0.1), with $F$ defined by (7.0.2), under Assumptions 1 and 2, for exponents $2 \le p < \frac{n}{\alpha}$, $p < q$ verifying*

$$\frac{q}{p} < 1 + \frac{\alpha}{n}.$$

*If $D\psi \in B^{\gamma}_{2q-p,\infty,loc}(\Omega)$, for some $0 < \alpha < \gamma < 1$, then there exists a threshold parameter $\tilde{\sigma} = \tilde{\sigma}(p,q,n,\alpha,\beta) \in (0,1)$ such that*

$$V_p(Du),\ \sqrt{a(x)}V_q(Du) \in B^{t}_{2,\infty,loc}(\Omega), \quad \forall t \in (0,\tilde{\sigma}).$$

This chapter is organized as follows. In Section 7.1, we show that the solution to the freezed obstacle problem (7.1.2) satisfies a variational inequality and moreover we present interior and up to the boundary higher integrability properties, which will be crucial for the comparison argument, as already mentioned. In Section 7.2, we prove the higher fractional differentiability of the solution to the freezed obstacle problem (7.1.2). We remark that the procedure used in order to do so requires the assumption $p \ge 2$. The comparison argument is presented in Section 7.3. Finally, in Section 7.4, we show that a suitable fractional differentiability property on the gradient of the obstacle transfers to a higher fractional differentiability for the gradient of the minimizer, so that we are eventually able to prove Theorem 7.0.1.

We point out that, in order to prove the higher integrability of the solution to the original obstacle problem (see Theorem 7.1.2) and the higher differentiability of the solution to the freezed obstacle problem in Section 7.2, Assumption 1 *(ii)* is the only one needed on the function $b(x,w)$. On the other hand, in order to prove the comparison lemma (see Lemma 7.3.3), we require Assumption 2 on the coefficient $b(x,w)$.

## 7.1 Higher integrability

The results contained in this section will be crucial for the comparison argument presented in Section 7.3.

Let $u$ be the solution to the obstacle problem (7.0.1) and fix a ball $B = B_{\frac{R}{2}}(x_0) \Subset \Omega$, for a given radius $R > 0$ and $x_0 \in \Omega$. Let us consider the so-called "freezed" functional

$$\int_B \tilde{F}(x, Dw)dx := \int_B b(x_0, u_B)H(x, Dw)dx, \tag{7.1.1}$$

where $H$ was defined in (7.0.3), and let $v \in u + W_0^{1,p}(B)$ be the solution to

$$\min\left\{\int_B \tilde{F}(x, Dw)dx \ : \ w \in \mathcal{K}_\psi(\Omega), \ w = u \text{ on } \partial B\right\}. \tag{7.1.2}$$

Now, we show that a local minimizer of functional (7.1.1) satisfies a variational inequality. More precisely, we have

**Proposition 7.1.1.** *A function $v \in u + W_0^{1,p}(B)$ is the solution to (7.1.2) if and only if it satisfies the following variational inequality*

$$\int_B \langle D_\xi H(x, Dv), D(\varphi - v)\rangle dx \geq 0, \tag{7.1.3}$$

*for every $\varphi \in u + W_0^{1,p}(B) \cap \mathcal{K}_\psi(\Omega)$ such that $H(x, D\varphi) \in L^1(B)$.*

*Proof.* It is clear that the solution $v$ to (7.1.3) must also be the minimizer of (7.1.2). Conversely, let $g = v + \varepsilon(\varphi - v)$ for $\varepsilon \in (0, 1)$. Then, $g$ belongs to the obstacle class. Indeed,

$$g = v + \varepsilon(\varphi - v) = \varepsilon\varphi + (1 - \varepsilon)v \geq \psi.$$

We notice that $H(x, D(v + \varepsilon(\varphi - v))) \in L^1(B)$. Moreover,

$$\int_B H(x, Dv)dx \leq \int_B H(x, Dv + \varepsilon D(\varphi - v))dx,$$

which leads to

$$\int_B H(x, Dv + \varepsilon D(\varphi - v))dx - \int_B H(x, Dv)dx \geq 0.$$

From Lagrange's theorem, for $\theta \in (0, 1)$ it holds

$$\int_B \langle D_\xi H(x, Dv + \varepsilon\theta D(\varphi - v)), \varepsilon D(\varphi - v)\rangle dx \geq 0.$$

Dividing for $\varepsilon > 0$ both sides of the preceding inequality, we have

$$\int_B \langle D_\xi H(x, Dv + \varepsilon\theta D(\varphi - v)), D(\varphi - v)\rangle dx \geq 0. \tag{7.1.4}$$

By a standard calculation, it immediately follows that

$$|\langle D_\xi H(x, \xi), \lambda\rangle| \leq C(p, q)\left(H(x, \xi) + H(x, \lambda)\right),$$

for every $x \in \Omega$ and every $\xi, \lambda \in \mathbb{R}^n$. Therefore,

$$|\langle D_\xi H(x, Dv + \varepsilon\theta D(\varphi - v)), D(\varphi - v)\rangle|$$

$$
\begin{aligned}
&\le C\left(H(x,Dv+\varepsilon\theta D(\varphi-v))+H(x,D(\varphi-v))\right)\\
&\le C\left(H(x,Dv)+H(x,\varepsilon\theta D(\varphi-v))+H(x,D\varphi)+H(x,Dv)\right)\\
&\le C\left(H(x,Dv)+(\varepsilon\theta)^pH(x,D\varphi)+(\varepsilon\theta)^pH(x,Dv)+H(x,D\varphi)\right), \qquad (7.1.5)
\end{aligned}
$$

where in the last passage we also used the direct property $H(x,\varepsilon\theta\xi)\le(\varepsilon\theta)^pH(x,\xi)$. For $\varepsilon\to 0$, the second and the third term on the right hand side of (7.1.5) go to zero. Hence, the right hand side tends to $C\left(H(x,Dv)+H(x,D\varphi)\right)$ in $L^1$. Then, we can pass to the limit for $\varepsilon\to 0$ in (7.1.4) applying the Dominated convergence theorem, which concludes the proof. □

If $u$ is the solution to (7.0.1), then we are able to establish for $u$ a higher integrability result.

**Theorem 7.1.2.** *Let $u$ be the solution to the obstacle problem (7.0.1) where the integrand satisfies Assumption 1, for exponents $2\le p<q$ verifying*

$$\frac{q}{p}\le 1+\frac{\alpha}{n}.$$

*If the function $\psi$ is such that $H(x,D\psi)\in L^{m_1}_{loc}(\Omega)$, for some $m_1>1$, then there exist an exponent $m_1>m_2>1$ and a positive constant $C$ such that it holds*

$$\left(\fint_{B_{\frac{R}{2}}}(H(x,Du))^{m_2}dx\right)^{\frac{1}{m_2}}\le C\left[\fint_{B_R}H(x,Du)dx+\left(\fint_{B_R}(H(x,D\psi))^{m_1}dx\right)^{\frac{1}{m_1}}\right].$$

*for all balls $B_{\frac{R}{2}}\subset B_R\Subset\Omega$.*

*Proof.* Let $\frac{R}{2}\le t<s\le R\le 1$ and let $\eta\in C_0^\infty(B_R)$ be a cut-off function such that $0\le\eta\le 1, \eta\equiv 1$ on $B_t, \eta\equiv 0$ outside $B_s, |D\eta|\le\frac{2}{s-t}$. We set $\varphi=\eta(x)(u(x)-u_{B_R})-\eta(x)(\psi(x)-\psi_{B_R})$ and $g=u-\varphi\in\mathcal{K}_\psi(\Omega)$. We observe that $g=u$ on $\partial B_s$ and $g=\psi-\psi_{B_R}+u_{B_R}$ on $B_t$, therefore $Dg=D\psi$ on $B_t$. Using Assumption 1 $(ii)$ and the fact that $u$ is a local minimizer, we have

$$
\begin{aligned}
&\int_{B_t}H(x,Du)dx\\
&\quad\le C\int_{B_t}F(x,u,Du)dx\\
&\quad\le C\int_{B_s}F(x,g,Dg)dx\\
&\quad\le C\int_{B_s}|Dg|^p+a(x)|Dg|^qdx\\
&\quad\le C\int_{B_s}[|D\eta|(\psi-\psi_{B_R})+\eta|D\psi|+|D\eta|(u-u_{B_R})+(1-\eta)|Du|]^p\\
&\qquad\qquad+a(x)\left[|D\eta|(\psi-\psi_{B_R})+\eta|D\psi|+|D\eta|(u-u_{B_R})+(1-\eta)|Du|\right]^q dx\\
&\quad\le C\int_{B_s}(1-\eta)^p\left(|Du|^p+a(x)|Du|^q\right)dx\\
&\qquad+C\int_{B_s}\left[\left|\frac{u-u_{B_R}}{s-t}\right|^p+a(x)\left|\frac{u-u_{B_R}}{s-t}\right|^q\right]dx
\end{aligned}
$$

$$
\begin{aligned}
&+ C\int_{B_s}\left[\left|\frac{\psi-\psi_{B_R}}{s-t}\right|^p + a(x)\left|\frac{\psi-\psi_{B_R}}{s-t}\right|^q\right]dx\\
&+ C\int_{B_s}\left(|D\psi|^p + a(x)|D\psi|^q\right)dx\\
\le{}& C\int_{B_s\setminus B_t} H(x,Du)dx\\
&+\frac{C}{|s-t|^p}\int_{B_R}|u-u_{B_R}|^p dx + \frac{C}{|s-t|^q}\int_{B_R} a(x)|u-u_{B_R}|^q dx\\
&+\frac{C}{|s-t|^p}\int_{B_R}|\psi-\psi_{B_R}|^p dx + \frac{C}{|s-t|^q}\int_{B_R} a(x)|\psi-\psi_{B_R}|^q dx\\
&+ C\int_{B_R} H(x,D\psi)dx.
\end{aligned}
$$

Adding the quantity $C\int_{B_t} H(x,Du(x))dx$ to both sides of the previous estimate, by Lemma 2.0.3 we get

$$
\begin{aligned}
\int_{B_{\frac{R}{2}}} H(x,Du)dx \le C\Bigg[&\frac{1}{R^p}\int_{B_R}|u-u_{B_R}|^p dx + \frac{1}{R^q}\int_{B_R} a(x)|u-u_{B_R}|^q dx\\
&+\frac{1}{R^p}\int_{B_R}|\psi-\psi_{B_R}|^p dx + \frac{1}{R^q}\int_{B_R} a(x)|\psi-\psi_{B_R}|^q dx\\
&+\int_{B_R} H(x,D\psi)dx\Bigg].
\end{aligned}
$$

Setting $\tilde{H}(x,u(x)) := |u(x)|^p + a(x)|u(x)|^q$ and $\tilde{H}(x,\psi(x)) := |\psi(x)|^p + a(x)|\psi(x)|^q$, we can write the previous inequality as

$$
\begin{aligned}
&\fint_{B_{\frac{R}{2}}} H(x,Du)dx\\
\le{}& \fint_{B_R}\tilde{H}\left(x,\frac{u-u_{B_R}}{R}\right)dx + \fint_{B_R}\tilde{H}\left(x,\frac{\psi-\psi_{B_R}}{R}\right)dx + \fint_{B_R} H(x,D\psi)dx. \qquad (7.1.6)
\end{aligned}
$$

According to [94, Theorem 2.13] and Hölder's inequality, it holds

$$
\begin{aligned}
\fint_{B_R}\tilde{H}\left(x,\frac{u-u_{B_R}}{R}\right)dx &\le \left(\fint_{B_R}\left(\tilde{H}\left(x,\frac{u-u_{B_R}}{R}\right)\right)^{d_1}dx\right)^{\frac{1}{d_1}}\\
&\le \left(\fint_{B_R}\left(H(x,Du)\right)^{d_2}dx\right)^{\frac{1}{d_2}}, \qquad (7.1.7)
\end{aligned}
$$

where $d_2 < 1 < d_1$ depend on $n, p, q, \alpha$. Analogously,

$$
\begin{aligned}
\fint_{B_R}\tilde{H}\left(x,\frac{\psi-\psi_{B_R}}{R}\right)dx &\le \left(\fint_{B_R}\left(\tilde{H}\left(x,\frac{\psi-\psi_{B_R}}{R}\right)\right)^{d_1}dx\right)^{\frac{1}{d_1}}\\
&\le \left(\fint_{B_R}\left(H(x,D\psi)\right)^{d_2}dx\right)^{\frac{1}{d_2}}. \qquad (7.1.8)
\end{aligned}
$$

Inserting (7.1.7) and (7.1.8) in (7.1.6) and exploiting Hölder's inequality, we infer

$$\fint_{B_{\frac{R}{2}}} H(x,Du)dx \le C\left[\left(\fint_{B_R}(H(x,Du))^{d_2}\right)^{\frac{1}{d_2}} + \fint_{B_R} H(x,D\psi)dx\right]. \tag{7.1.9}$$

Since $H(x,D\psi(x)) \in L^{m_1}$, for $m_1 > 1$, from Lemma 2.0.4 it follows that there exists $m_1 > m_2 > 1$ such that $H(x,Du(x)) \in L^{m_2}$. Then, holding to $d_2 < 1$, we might write

$$\begin{aligned}\fint_{B_{\frac{R}{2}}}(H(x,Du))^{m_2}dx &\le C\left[\left(\fint_{B_R} H(x,Du)dx\right)^{m_2} + \fint_{B_R}(H(x,D\psi))^{m_2}dx\right]\\ &\le C\left[\left(\fint_{B_R} H(x,Du)dx\right)^{m_2} + \left(\fint_{B_R}(H(x,D\psi)^{m_1}dx\right)^{\frac{m_2}{m_1}}\right].\end{aligned}$$

Hence,

$$\left(\fint_{B_{\frac{R}{2}}}(H(x,Du))^{m_2}dx\right)^{\frac{1}{m_2}} \le C\left[\fint_{B_R} H(x,Du)dx + \left(\fint_{B_R}(H(x,D\psi)^{m_1}dx\right)^{\frac{1}{m_1}}\right]. \qquad \square$$

The higher integrability of the minimizer $u$ stated in Theorem 7.1.2 allows us to prove the following higher integrability up to the boundary result for the solution to the freezed obstacle problem (7.1.2).

**Theorem 7.1.3.** *Let $v \in u + W_0^{1,p}(B_{\frac{R}{2}})$ be the solution to the obstacle problem (7.1.2) where the integrand $\tilde{F}$ satisfies Assumption 1, for exponents $2 \le p < q$ verifying*

$$\frac{q}{p} \le 1 + \frac{\alpha}{n}.$$

*If the function $\psi$ is such that $H(x,D\psi) \in L^{m_1}_{loc}(\Omega)$, for some $m_1 > 1$, then $H(x,Du) \in L^{m_2}_{loc}(\Omega)$, for some $m_1 > m_2 > 1$, and there exist a constant $C$ and an exponent $m_3$, with $m_1 > m_2 > m_3 > 1$, such that $H(x,Dv) \in L^{m_3}_{loc}(\Omega)$ and*

$$\left(\fint_{B_{\frac{R}{2}}}(H(x,Dv))^{m_3}\,dx\right)^{\frac{1}{m_3}} \le C\left[\left(\fint_{B_R}(H(x,Du))^{m_2}dx\right)^{\frac{1}{m_2}} + \left(\fint_{B_R}(H(x,D\psi))^{m_2}dx\right)^{\frac{1}{m_2}}\right].$$

*Proof.* We start setting

$$w(x) := \begin{cases} v(x) & \text{if } x \in B_{\frac{R}{2}} \\ u(x) & \text{if } x \in B_R \setminus B_{\frac{R}{2}}. \end{cases} \tag{7.1.10}$$

We first consider $B_\rho(x_1) \subset B_{\frac{R}{2}}$. In this case the Caccioppoli inequality (7.1.9) holds, namely

$$\fint_{B_{\frac{\rho}{2}}} H(x,Dv)dx \le C\left[\left(\fint_{B_\rho}(H(x,Dv))^{d_2}dx\right)^{\frac{1}{d_2}} + \fint_{B_\rho} H(x,D\psi)dx\right]. \tag{7.1.11}$$

Let us now focus on the case $B_\rho(x_1) \subset B_R$, with $x_1 \in \partial B_{\frac{R}{2}}$. Fix $\frac{\rho}{2} \le t < s \le \rho$ and a cut-off function $\eta$ between $B_s(x_1)$ and $B_t(x_1)$, with $|D\eta| \le \dfrac{2}{t-s}$. Let us set $g(x) := (1-\eta(x))v(x) + \eta(x)u(x)$. It is straightforward that $g \in u + W_0^{1,p}$ and $g(x) \ge \psi(x)$. Since $v$ is a minimizer, according to the definition of $H$ and Assumption 1 (ii), we have

$$\begin{aligned}\int_{B_t \cap B_{\frac{R}{2}}} H(x, Dv)dx &\le C \int_{B_t \cap B_{\frac{R}{2}}} \tilde{F}(x, Dv)dx \\ &\le C \int_{B_s \cap B_{\frac{R}{2}}} \tilde{F}(x, Dg)dx.\end{aligned}$$

Therefore, from the definitions of $g$ and $\eta$, we get

$$\begin{aligned}&\int_{B_t \cap B_{\frac{R}{2}}} H(x, Dv)dx \\ &\quad \le C \Bigg[ \int_{B_s \cap B_{\frac{R}{2}}} \left( \frac{1}{(t-s)^p} |u-v|^p + a(x) \frac{1}{(t-s)^q} |u-v|^q \right) dx \\ &\qquad + \int_{(B_s \setminus B_t) \cap B_{\frac{R}{2}}} H(x, Dv)dx \\ &\qquad + \int_{B_s} H(x, Du)dx \Bigg].\end{aligned}$$

As before, adding the quantity $C \int_{B_t \cap B_{\frac{R}{2}}} H(x, Dv)dx$ to both sides of the previous inequality, by Lemma 2.0.3 we get

$$\begin{aligned}&⨍_{B_{\frac{\rho}{2}} \cap B_{\frac{R}{2}}} H(x, Dv)dx \\ &\quad \le C \Bigg[ ⨍_{B_\rho \cap B_{\frac{R}{2}}} \left( \frac{1}{\rho^p} |u-v|^p + a(x) \frac{1}{\rho^q} |u-v|^q \right) dx \\ &\qquad + ⨍_{B_\rho} H(x, Du)dx \Bigg].\end{aligned} \tag{7.1.12}$$

We set

$$\tilde{H}\left(x, \frac{u-v}{\rho}\right) := \frac{1}{\rho^p}|u-v|^p + a(x)\frac{1}{\rho^q}|u-v|^q.$$

Adapting the argument in [23, Remark 2] and [94, Theorem 2.13] and exploiting Hölder's inequality, we have

$$⨍_{B_\rho \cap B_{\frac{R}{2}}} \tilde{H}\left(x, \frac{u-v}{\rho}\right) dx \le \left( ⨍_{B_\rho \cap B_{\frac{R}{2}}} \left( \tilde{H}\left(x, \frac{u-v}{\rho}\right)\right)^{d_1} dx \right)^{\frac{1}{d_1}}$$

$$\leq \left( \fint_{B_\rho \cap B_{\frac{R}{2}}} (H(x, Du - Dv))^{d_2}\, dx \right)^{\frac{1}{d_2}}, \tag{7.1.13}$$

where $d_2 < 1 < d_1$ depend on $n, p, q, \alpha$. Inserting (7.1.13) in (7.1.12), it yields

$$\begin{aligned} \fint_{B_{\frac{\rho}{2}} \cap B_{\frac{R}{2}}} H(x, Dv)dx \leq & C \left[ \left( \fint_{B_\rho} (H(x, Du))^{d_2}\, dx \right)^{\frac{1}{d_2}} \right. \\ & + \left( \fint_{B_\rho \cap B_{\frac{R}{2}}} (H(x, Dv))^{d_2}\, dx \right)^{\frac{1}{d_2}} \\ & \left. + \fint_{B_\rho} H(x, Du)dx \right] \\ \leq & C \left[ \left( \fint_{B_\rho \cap B_{\frac{R}{2}}} (H(x, Dv))^{d_2}\, dx \right)^{\frac{1}{d_2}} + \fint_{B_\rho} H(x, Du)dx \right]. \end{aligned}$$

Therefore, from the definition of $w$ in (7.1.10), we infer

$$\begin{aligned} \fint_{B_{\frac{\rho}{2}}} H(x, Dw)dx \leq & C \left[ \left( \fint_{B_\rho} (H(x, Dw))^{d_2}\, dx \right)^{\frac{1}{d_2}} + \fint_{B_\rho} H(x, Du)dx \right. \\ & \left. + \fint_{B_\rho} H(x, D\psi)dx \right]. \end{aligned} \tag{7.1.14}$$

Hence, by (7.1.11) it follows that (7.1.14) holds not only if $B_\rho(x_1) \subset B_{\frac{R}{2}}$ or $B_\rho(x_1) \cap B_{\frac{R}{2}} \neq \emptyset$, but also when $B_\rho(x_1) \subset B_R$ and $x_1 \in \partial B_{\frac{R}{2}}$.

We now take care of the case $B_\rho(x_1) \cap \partial B_{\frac{R}{2}} \neq \emptyset$ and $B_{4\rho} \subset B_R$. We fix $x_2 \in B_\rho(x_1) \cap \partial B_{\frac{R}{2}}$.

$$\begin{aligned} & \fint_{B_{\frac{\rho}{2}}(x_1)} H(x, Dw)dx \\ & \leq 3^N \fint_{B_{\frac{3\rho}{2}}(x_2)} H(x, Dw)dx \\ & \leq C \left[ \left( \fint_{B_{3\rho(x_2)}} (H(x, Dw))^{d_2} dx \right)^{\frac{1}{d_2}} + \fint_{B_{3\rho}(x_2)} H(x, Du)dx + \fint_{B_{3\rho}(x_2)} H(x, D\psi)dx \right] \\ & \leq C \left[ \left( \fint_{B_{4\rho(x_1)}} (H(x, Dw))^{d_2} dx \right)^{\frac{1}{d_2}} + \fint_{B_{4\rho}(x_1)} H(x, Du)dx + \fint_{B_{4\rho}(x_1)} H(x, D\psi)dx \right]. \end{aligned}$$

Since this estimate holds for every $B_{\frac{\rho}{2}}$ such that $B_{4\rho} \subset B_R$, by a covering argument it follows that inequality (7.1.14) holds for every $B_{\frac{\rho}{2}}$ such that $B_\rho \subset B_R$. Now, since $H(x, D\psi) \in L^{m_1}$, $m_1 > 1$, Theorem 7.1.2 yields that there exists $m_2$, with $1 < m_2 < m_1$, such that

$H(x,Du) \in L^{m_2}$. Therefore, according to Lemma 2.0.4, there exists $m_3$, with $1 < m_3 < m_2 < m_1$, such that,

$$\left( \fint_{B_{\frac{\rho}{2}}(x_1)} (H(x,Dw))^{m_3}\, dx \right)^{\frac{1}{m_3}} \leq C \left[ \fint_{B_\rho(x_1)} H(x,Dw)dx + \left( \fint_{B_\rho(x_1)} (H(x,Du))^{m_2} dx \right)^{\frac{1}{m_2}} + \left( \int_{B_\rho(x_1)} (H(x,D\psi))^{m_2} dx \right)^{\frac{1}{m_2}} \right].$$

In particular, for $\rho = R$ and $x_1 = x_0$, recalling the definition of $w$ we have

$$\begin{aligned} &\left( \fint_{B_{\frac{R}{2}}} (H(x,Dv))^{m_3}\, dx \right)^{\frac{1}{m_3}} \\ \leq &C \left[ \fint_{B_{\frac{R}{2}}} H(x,Dv)dx + \fint_{B_R \setminus B_{\frac{R}{2}}} H(x,Du)dx + \left( \fint_{B_R} (H(x,Du))^{m_2} dx \right)^{\frac{1}{m_2}} + \left( \fint_{B_R} (H(x,D\psi))^{m_2} dx \right)^{\frac{1}{m_2}} \right] \\ \leq &C \left[ \fint_{B_{\frac{R}{2}}} H(x,Dv)dx + \left( \fint_{B_R} (H(x,Du))^{m_2} dx \right)^{\frac{1}{m_2}} + \left( \fint_{B_R} (H(x,D\psi))^{m_2} dx \right)^{\frac{1}{m_2}} \right]. \end{aligned}$$

Since $v$ is a minimizer and recalling that $m_2 > 1$, it holds

$$\left( \fint_{B_{\frac{R}{2}}} (H(x,Dv))^{m_3}\, dx \right)^{\frac{1}{m_3}} \leq C \left[ \left( \fint_{B_R} (H(x,Du))^{m_2} dx \right)^{\frac{1}{m_2}} + \left( \fint_{B_R} (H(x,D\psi))^{m_2} dx \right)^{\frac{1}{m_2}} \right],$$

i.e. the conclusion. □

**Remark 7.1.4.** *We point out that Theorems 7.1.2 and 7.1.3 hold true also under the more general hypothesis $q > p > 1$. However, they are stated for $q > p \geq 2$ for later purpose in Section 7.4.*

## 7.2 Higher differentiability for comparison maps

The higher differentiability of the solution $v$ to (7.1.2) has been already establish in Chapter 4 under weaker assumptions on the coefficients of the energy density. Indeed, Lemma 7.2.1

below can be seen as a special case of Theorem 4.0.1 where the coefficients are assumed to be bounded. Here, we only give the proof of the a priori bounds, in order to establish precise estimates on the difference quotient that will be crucial for the comparison argument. On the other hand, the approximation procedure is achieved using the same arguments in Section 4.3.2, therefore it will not be presented.
Before stating the result, it is worth noticing that Assumption 1 implies that there exist positive constants $\tilde{l}, \tilde{\nu}, \tilde{L}$ such that the following conditions are satisfied:

$$|D_\xi \tilde{F}(x,\xi)| \le \tilde{l}(|\xi|^{p-1} + a(x)|\xi|^{q-1}) \tag{H1}$$

$$\langle D_\xi \tilde{F}(x,\xi) - D_\xi \tilde{F}(x,\eta), \xi - \eta\rangle \ge \tilde{\nu}(|\xi-\eta|^2(|\xi|^2+|\eta|^2)^{\frac{p-2}{2}} + a(x)|\xi-\eta|^2(|\xi|^2+|\eta|^2)^{\frac{q-2}{2}}) \tag{H2}$$

$$|D_\xi \tilde{F}(x,\xi) - D_\xi \tilde{F}(x,\eta)| \le \tilde{L}(|\xi-\eta|(|\xi|^2+|\eta|^2)^{\frac{p-2}{2}} + a(x)|\xi-\eta|(|\xi|^2+|\eta|^2)^{\frac{q-2}{2}}) \tag{H3}$$

$$|D_\xi \tilde{F}(x,\xi) - D_\xi \tilde{F}(y,\xi)| \le |x-y|^\alpha |\xi|^{q-1} \tag{H4}$$

for every $x, y \in \Omega$ and every $\xi, \eta \in \mathbb{R}^n$.

The following lemma holds.

**Lemma 7.2.1.** *Let $v \in u + W_0^{1,p}(B)$ be the solution to (7.1.2) under Assumption 1, for exponents $2 \le p < \frac{n}{\alpha}$, $p < q$ satisfying*

$$\frac{q}{p} < 1 + \frac{\alpha}{n}. \tag{7.2.1}$$

*If*

$$D\psi \in B^{\gamma}_{2q-p,\infty}(B),$$

*for $0 < \alpha < \gamma < 1$, then*

$$V_p(Dv) \in B^{\alpha}_{2,\infty,loc}(B)$$

*and the following estimate*

$$\begin{aligned}\int_{B_{r/4}} & (|\tau_h V_p(Dv)|^2 + a(x+h)|\tau_h V_q(Dv)|^2)dx \\ & \le C|h|^{2\alpha}\left\{\frac{1}{r^{2\tilde{p}}}\left(\int_{B_r}(1+|Dv|^p+|D\psi|^{2q-p})dx\right)^{\kappa} + [D\psi]^{2q-p}_{B^{\gamma}_{2q-p,\infty}(B_r)}\right\}\end{aligned} \tag{7.2.2}$$

*holds for all balls $B_{r/4} \subset B_r \Subset B$, with $C = C(n,p,q,\alpha,\|a\|_\infty)$, $\tilde{p} = \tilde{p}(n,p,q,\alpha) > 1$ and $\kappa = \kappa(n,p,q,\alpha) < \tilde{p}$.*

*Proof.* We a priori assume that $Dv \in L^{\frac{np}{n-2\mu}}_{\text{loc}}(B)$, for all $\frac{\alpha n}{n+2\alpha} < \mu < \alpha$.
In the sequel we will profusely use the following inequality:

$$2q - p \le \frac{np}{n-2\mu}, \tag{7.2.3}$$

for $\mu \in [\frac{\alpha n}{n+2\alpha}, \alpha)$. Indeed,

$$2q - p \le \frac{np}{n-2\mu} \Leftrightarrow \frac{q}{p} \le \frac{n-\mu}{n-2\mu}$$

and

$$1 + \frac{\alpha}{n} \le \frac{n-\mu}{n-2\mu} \Leftrightarrow \mu \ge \frac{\alpha n}{n+2\alpha}.$$

Fix $0 < \frac{r}{4} < \rho < s < t < t' < \frac{r}{2}$ such that $B_r \Subset B$ and a cut-off function $\eta \in \mathcal{C}^1_0(B_t)$ such that $0 \leq \eta \leq 1$, $\eta = 1$ on $B_s$, $|D\eta| \leq \frac{C}{t-s}$. Due to the local nature of our results, there is no loss of generality in supposing $r \leq 1$, that we will do from now on.
Now, for $|h| < \frac{r}{4}$, we consider functions

$$\varphi_1(x) = u(x) + t\eta^2(x)\tau_h(u-\psi)(x)$$

and

$$\varphi_2(x) = u(x) + t\eta^2(x-h)\tau_{-h}(u-\psi)(x),$$

which belong to the admissible class $\mathcal{K}_\psi(\Omega)$, for every $t \in [0,1)$. Choosing $\varphi_1$ and $\varphi_2$ as test functions in (7.1.3) and arguing analogously as in the proof of Theorem 4.3.1, we obtain the following estimate

$$\begin{aligned}
0 \geq & \int_\Omega \langle D_\xi H(x+h, Dv(x+h)) - D_\xi H(x+h, Dv(x)), \eta^2 D\tau_h v\rangle dx \\
& - \int_\Omega \langle D_\xi H(x+h, Dv(x+h)) - D_\xi H(x+h, Dv(x)), \eta^2 D\tau_h \psi\rangle dx \\
& + \int_\Omega \langle D_\xi H(x+h, Dv(x+h)) - D_\xi H(x+h, Dv(x)), 2\eta D\eta \tau_h(v-\psi)\rangle dx \\
& + \int_\Omega \langle D_\xi H(x+h, Dv(x)) - D_\xi H(x, Dv(x)), \eta^2 D\tau_h v\rangle dx \\
& - \int_\Omega \langle D_\xi H(x+h, Dv(x)) - D_\xi H(x, Dv(x)), \eta^2 D\tau_h \psi\rangle dx \\
& + \int_\Omega \langle D_\xi H(x+h, Dv(x)) - D_\xi H(x, Dv(x)), 2\eta D\eta \tau_h(v-\psi)\rangle dx \\
=: & I_1 + I_2 + I_3 + I_4 + I_5 + I_6,
\end{aligned}$$

that yields

$$I_1 \leq |I_2| + |I_3| + |I_4| + |I_5| + |I_6|. \tag{7.2.4}$$

The ellipticity assumption (H2) and the properties of $a(x)$ imply

$$\begin{aligned}
I_1 \geq & \tilde{\nu} \int_\Omega \eta^2 |\tau_h Dv|^2 (|Dv(x+h)|^2 + |Dv(x)|^2)^{\frac{p-2}{2}} dx \\
& + \tilde{\nu} \int_\Omega \eta^2 a(x+h) |\tau_h Dv|^2 (|Dv(x+h)|^2 + |Dv(x)|^2)^{\frac{q-2}{2}} dx \\
\geq & \tilde{\nu} \int_\Omega \eta^2 (|\tau_h V_p(Dv)|^2 + a(x+h)|\tau_h V_q(Dv)|^2) dx.
\end{aligned} \tag{7.2.5}$$

From the growth condition (H3), the boundedness of $a(x)$ and Young's inequality, we get

$$\begin{aligned}
|I_2| \leq & \tilde{L} \int_\Omega \eta^2 |\tau_h Dv| (|Dv(x+h)|^2 + |Dv(x)|^2)^{\frac{p-2}{2}} |\tau_h D\psi| dx \\
& + \tilde{L} \int_\Omega \eta^2 a(x+h) |\tau_h Dv| (|Dv(x+h)|^2 + |Dv(x)|^2)^{\frac{q-2}{2}} |\tau_h D\psi| dx \\
\leq & \tilde{L} \int_\Omega \eta^2 |\tau_h Dv| (|Dv(x+h)|^2 + |Dv(x)|^2)^{\frac{p-2}{2}} |\tau_h D\psi| dx
\end{aligned}$$

$$+\tilde{L}\|a\|_\infty \int_\Omega \eta^2|\tau_h Dv|(|Dv(x+h)|^2+|Dv(x)|^2)^{\frac{q-2}{2}}|\tau_h D\psi|dx$$
$$\leq \varepsilon \int_\Omega \eta^2|\tau_h Dv|^2(|Dv(x+h)|^2+|Dv(x)|^2)^{\frac{p-2}{2}}dx$$
$$+C_\varepsilon(\tilde{L},\|a\|_\infty)\int_\Omega \eta^2|\tau_h D\psi|^2(1+|Dv(x+h)|^2+|Dv(x)|^2)^{\frac{2q-p-2}{2}}dx.$$

The calculations performed in Theorem 4.3.1 and Lemma 2.0.2 lead us to the following estimate for the integral $I_2$

$$\begin{aligned}|I_2| \leq& \varepsilon \int_\Omega \eta^2|\tau_h V_p(Dv)|^2dx\\ &+C_\varepsilon(\tilde{L},\|a\|_\infty)|h|^{2\gamma}[D\psi]^{2q-p}_{B^\gamma_{2q-p,\infty}(B_r)}\\ &+C_\varepsilon(p,q,\tilde{L},\|a\|_\infty)|h|^{2\gamma}\int_{B_{t'}}(1+|Dv|)^{2q-p}dx.\end{aligned} \tag{7.2.6}$$

Now, we consider the integral $I_3$. From assumption (H3), hypothesis $|D\eta|\leq \frac{C}{t-s}$ and Young's inequality, we get

$$\begin{aligned}|I_3| \leq& 2\tilde{L}\int_\Omega |D\eta|\eta|\tau_h Dv|(|Dv(x+h)|^2+|Dv(x)|^2)^{\frac{p-2}{2}}|\tau_h(v-\psi)|dx\\ &+2\tilde{L}\|a\|_\infty\int_\Omega |D\eta|\eta|\tau_h Dv|(1+|Dv(x+h)|^2+|Dv(x)|^2)^{\frac{q-2}{2}}|\tau_h(v-\psi)|dx\\ \leq& \varepsilon\int_\Omega \eta^2|\tau_h Dv|^2(|Dv(x+h)|^2+|Dv(x)|^2)^{\frac{p-2}{2}}dx\\ &+\frac{C_\varepsilon(L,\|a\|_\infty)}{(t-s)^2}\int_{B_t}|\tau_h(v-\psi)|^2(|Dv(x+h)|^2+|Dv(x)|^2)^{\frac{2q-p-2}{2}}dx,\end{aligned}$$

where we also used the boundedness of function $a(x)$. Arguing analogously as in the proof of Theorem 6.0.1, we can estimate the integral $I_3$ as follows

$$\begin{aligned}|I_3| \leq& \varepsilon\int_\Omega \eta^2|\tau_h V_p(Dv)|^2dx\\ &+\frac{C_\varepsilon(n,p,q,\tilde{L},\|a\|_\infty)}{(t-s)^2}|h|^2\int_{B_r}|D\psi|^{2q-p}dx\\ &+\frac{C_\varepsilon(n,p,q,\tilde{L},\|a\|_\infty)}{(t-s)^2}|h|^2\int_{B_{t'}}(1+|Dv|)^{2q-p}dx.\end{aligned} \tag{7.2.7}$$

In order to estimate the integral $I_4$, we use assumption (H4), Young's inequality and Lemma 2.0.2 as follows

$$\begin{aligned}|I_4| \leq& \int_\Omega \eta^2|\tau_h Dv||h|^\alpha|Dv|^{q-1}dx\\ \leq& \varepsilon\int_\Omega \eta^2|\tau_h Dv|^2(|Dv(x+h)|^2+|Dv(x)|^2)^{\frac{p-2}{2}}dx\\ &+C_\varepsilon|h|^{2\alpha}\int_{B_t}|Dv|^{2q-p}dx\end{aligned}$$

$$\leq \varepsilon \int_\Omega \eta^2 |\tau_h V_p(Dv)|^2 dx$$
$$+ C_\varepsilon |h|^{2\alpha} \int_{B_t} |Dv|^{2q-p} dx. \tag{7.2.8}$$

We now take care of $I_5$. Similarly as above, exploiting assumption (H4) and Hölder's inequality, we infer

$$|I_5| \leq \int_\Omega \eta^2 |\tau_h D\psi||h|^\alpha |Dv|^{q-1} dx$$
$$\leq |h|^\alpha \left( \int_{B_t} |\tau_h D\psi|^{2q-p} dx \right)^{\frac{1}{2q-p}} \left( \int_{B_t} |Dv|^{\frac{(q-1)(2q-p)}{2q-p-1}} dx \right)^{\frac{2q-p-1}{2q-p}}.$$

Now, we observe

$$\frac{(q-1)(2q-p)}{2q-p-1} < 2q-p \Leftrightarrow p < q. \tag{7.2.9}$$

Thus, Hölder's inequality yields

$$|I_5| \leq C(n,p,q)|h|^{\alpha+\gamma} [D\psi]_{B^\gamma_{2q-p,\infty}(B_r)} \left( \int_{B_t} |Dv|^{2q-p} dx \right)^{\frac{q-1}{2q-p}}$$
$$\leq C(n,p,q)|h|^{\alpha+\gamma} [D\psi]^q_{B^\gamma_{2q-p,\infty}(B_r)}$$
$$+ C(n,p,q)|h|^{\alpha+\gamma} \left( \int_{B_t} |Dv|^{2q-p} dx \right)^{\frac{q}{2q-p}}. \tag{7.2.10}$$

From assumption (H4), hypothesis $|D\eta| \leq \frac{C}{t-s}$ and Hölder's inequality, we infer the following estimate for $I_6$.

$$|I_6| \leq \frac{C}{t-s} |h|^\alpha \int_{B_t} |\tau_h \psi||Dv|^{q-1} dx$$
$$+ \frac{C}{t-s} |h|^\alpha \int_{B_t} |\tau_h v||Dv|^{q-1} dx$$
$$\leq \frac{C}{t-s} |h|^\alpha \left( \int_{B_t} |\tau_h \psi|^{2q-p} dx \right)^{\frac{1}{2q-p}} \left( \int_{B_t} |Dv|^{\frac{(q-1)(2q-p)}{2q-p-1}} dx \right)^{\frac{2q-p-1}{2q-p}}$$
$$+ \frac{C}{t-s} |h|^\alpha \left( \int_{B_t} |\tau_h v|^{2q-p} dx \right)^{\frac{1}{2q-p}} \left( \int_{B_t} |Dv|^{\frac{(q-1)(2q-p)}{2q-p-1}} dx \right)^{\frac{2q-p-1}{2q-p}}.$$

Using Lemma 2.1.3, (7.2.9) and Hölder's and Young's inequality, we have

$$|I_6| \leq \frac{C(n,p,q)}{t-s} |h|^{\alpha+1} \left( \int_{B_{t'}} |D\psi|^{2q-p} dx \right)^{\frac{1}{2q-p}} \left( \int_{B_t} |Dv|^{\frac{(q-1)(2q-p)}{2q-p-1}} dx \right)^{\frac{2q-p-1}{2q-p}}$$
$$+ \frac{C(n,p,q)}{t-s} |h|^{\alpha+1} \left( \int_{B_{t'}} |Dv|^{2q-p} dx \right)^{\frac{1}{2q-p}} \left( \int_{B_t} |Dv|^{\frac{(q-1)(2q-p)}{2q-p-1}} dx \right)^{\frac{2q-p-1}{2q-p}}$$

$$
\begin{aligned}
&\leq \frac{C(n,p,q)}{t-s}|h|^{\alpha+1}\left(\int_{B_r}|D\psi|^{2q-p}dx\right)^{\frac{1}{2q-p}}\left(\int_{B_t}|Dv|^{2q-p}dx\right)^{\frac{q-1}{2q-p}} \\
&\quad+\frac{C(n,p,q)}{t-s}|h|^{\alpha+1}\left(\int_{B_{t'}}|Dv|^{2q-p}dx\right)^{\frac{q}{2q-p}} \\
&\leq \frac{C(n,p,q)}{t-s}|h|^{\alpha+1}\left(\int_{B_r}|D\psi|^{2q-p}dx\right)^{\frac{q}{2q-p}} \\
&\quad+\frac{C(n,p,q)}{t-s}|h|^{\alpha+1}\left(\int_{B_{t'}}|Dv|^{2q-p}dx\right)^{\frac{q}{2q-p}}. \qquad (7.2.11)
\end{aligned}
$$

Inserting estimates (7.2.5), (7.2.6), (7.2.7), (7.2.8), (7.2.10) and (7.2.11) in (7.2.4), we infer

$$
\begin{aligned}
&\nu\int_\Omega \eta^2(|\tau_h V_p(Dv)|^2+a(x+h)|\tau_h V_p(Dv)|^2)dx \\
&\leq 3\varepsilon\int_\Omega \eta^2|\tau_h V_p(Dv)|^2dx \\
&\quad+C_\varepsilon(\tilde{L},p,q,\|a\|_\infty)|h|^{2\gamma}[D\psi]^{2q-p}_{B^{\gamma}_{2q-p,\infty}(B_r)} \\
&\quad+C_\varepsilon(\tilde{L},p,q,\|a\|_\infty)|h|^{2\gamma}\int_{B_{t'}}(1+|Dv|)^{2q-p}dx \\
&\quad+\frac{C_\varepsilon(\tilde{L},n,p,q,\|a\|_\infty)}{(t-s)^2}|h|^2\int_{B_r}|D\psi|^{2q-p}dx \\
&\quad+\frac{C_\varepsilon(\tilde{L},n,p,q,\|a\|_\infty)}{(t-s)^2}|h|^2\int_{B_{t'}}(1+|Dv|)^{2q-p}dx \\
&\quad+C_\varepsilon|h|^{2\alpha}\int_{B_t}|Dv|^{2q-p}dx \\
&\quad+C(q)|h|^{\alpha+\gamma}[D\psi]^q_{B^{\gamma}_{2q-p,\infty}(B_r)}+C(q)|h|^{\alpha+\gamma}\left(\int_{B_t}|Dv|^{2q-p}dx\right)^{\frac{q}{2q-p}} \\
&\quad+\frac{C(n,p,q)}{t-s}|h|^{\alpha+1}\left(\int_{B_r}|D\psi|^{2q-p}dx\right)^{\frac{q}{2q-p}} \\
&\quad+\frac{C(n,p,q)}{t-s}|h|^{\alpha+1}\left(\int_{B_{t'}}|Dv|^{2q-p}dx\right)^{\frac{q}{2q-p}}. \qquad (7.2.12)
\end{aligned}
$$

We now introduce the following interpolation inequality

$$
\|Dw\|_{2q-p}\leq \|Dw\|_p^{\delta}\|Dw\|^{1-\delta}_{\frac{np}{n-2\mu}},
$$

where $0<\delta<1$ is defined through the condition

$$
\frac{1}{(2q-p)}=\frac{\delta}{p}+\frac{(1-\delta)(n-2\mu)}{np}
$$

which implies

$$
\delta=\frac{n(p-q)+\mu(2q-p)}{\mu(2q-p)},\quad 1-\delta=\frac{n(q-p)}{\mu(2q-p)}.
$$

Hence we get the following inequalities

$$\int_{B_{t'}}(1+|Dv|)^{2q-p}dx \leq \left(\int_{B_{t'}}(1+|Dv|)^p dx\right)^{\frac{\delta(2q-p)}{p}}\left(\int_{B_{t'}}(1+|Dv|)^{\frac{np}{n-2\mu}}dx\right)^{\frac{(n-2\mu)(q-p)}{\mu p}}, \tag{7.2.13}$$

$$\left(\int_{B_{t'}}|Dv|^{2q-p}dx\right)^{\frac{q}{2q-p}} \leq \left(\int_{B_{t'}}|Dv|^p dx\right)^{\frac{\delta q}{p}}\cdot\left(\int_{B_{t'}}|Dv|^{\frac{np}{n-2\mu}}dx\right)^{\frac{(n-2\mu)qp'}{p}}, \tag{7.2.14}$$

where $p' = \frac{q-p}{\mu(2q-p)}$.

Inserting (7.2.13) and (7.2.14) in (7.2.12), and exploiting the bounds

$$\frac{n(q-p)}{\mu p} < 1, \quad \frac{nq(q-p)}{\mu p(2q-p)} < 1,$$

which hold by assumption (7.2.1) and for $\mu \in (\frac{n(q-p)}{p}, \alpha)$, from Young's inequality, we infer

$$\begin{aligned}
&\nu\int_\Omega \eta^2(|\tau_h V_p(Dv)|^2 + a(x+h)|\tau_h V_p(Dv)|^2)dx\\
\leq& 3\varepsilon\int_\Omega \eta^2|\tau_h V_p(Dv)|^2 dx\\
&+ C_\varepsilon(\tilde{L}, p, q, \|a\|_\infty)|h|^{2\gamma}[D\psi]^{2q-p}_{B^\gamma_{2q-p,\infty}(B_r)}\\
&+ C_{\varepsilon,\theta}(\tilde{L}, n, p, q, \|a\|_\infty)|h|^{2\gamma}\left(\int_{B_r}(1+|Dv|)^p dx\right)^{\frac{\delta(2q-p)\tilde{p}}{p}}\\
&+ \theta|h|^{2\gamma}\left(\int_{B_{t'}}(1+|Dv|)^{\frac{np}{n-2\mu}}dx\right)^{\frac{n-2\mu}{n}}\\
&+ \frac{C_\varepsilon(\tilde{L}, n, p, q, \|a\|_\infty)}{(t-s)^2}|h|^2\int_{B_r}|D\psi|^{2q-p}dx\\
&+ \theta|h|^2\left(\int_{B_{t'}}(1+|Dv|)^{\frac{np}{n-2\mu}}dx\right)^{\frac{n-2\mu}{n}} + \frac{C_{\varepsilon,\theta}(\tilde{L}, n, p, q, \|a\|_\infty)}{(t-s)^{2\tilde{p}}}|h|^2\left(\int_{B_r}(1+|Dv|)^p dx\right)^{\frac{\tilde{p}\delta(2q-p)}{p}}\\
&+ C_{\varepsilon,\theta}|h|^{2\alpha}\left(\int_{B_r}|Dv|^p dx\right)^{\frac{\tilde{p}\delta(2q-p)}{p}} + \theta|h|^{2\alpha}\left(\int_{B_t}|Dv|^{\frac{np}{n-2\mu}}dx\right)^{\frac{n-2\mu}{n}}\\
&+ C_\theta(q)|h|^{\alpha+\gamma}[D\psi]^q_{B^\gamma_{2q-p,\infty}(B_r)} + C_\theta(n,p,q)|h|^{\alpha+\gamma}\left(\int_{B_r}|Dv|^p dx\right)^{\frac{p^*\delta q}{p}}\\
&+ \theta|h|^{\alpha+\gamma}\left(\int_{B_t}|Dv|^{\frac{np}{n-2\mu}}dx\right)^{\frac{n-2\mu}{n}}\\
&+ \frac{C_\theta(n,p,q)}{t-s}|h|^{\alpha+1}\left(\int_{B_r}|D\psi|^{2q-p}dx\right)^{\frac{q}{2q-p}}\\
&+ \frac{C_\theta(n,p,q)}{(t-s)^{p^*}}|h|^{\alpha+1}\left(\int_{B_r}|Dv|^p dx\right)^{\frac{p^*\delta q}{p}} + \theta|h|^{\alpha+1}\left(\int_{B_{t'}}|Dv|^{\frac{np}{n-2\mu}}dx\right)^{\frac{n-2\mu}{n}}.
\end{aligned} \tag{7.2.15}$$

for some constant $\theta \in (0,1)$, where we set $\tilde{p} = \frac{\mu p}{\mu p - n(q-p)}$, $p^* = \frac{\mu p(2q-p)}{\mu p(2q-p) - n(q-p)q}$.

For a better readability we now define

$$\begin{aligned}
A :=& C_\varepsilon(\tilde{L}, p, q, \|a\|_\infty)[D\psi]^{2q-p}_{B^\gamma_{2q-p,\infty}(B_r)} + C_{\varepsilon,\theta}(\tilde{L}, n, p, q, \|a\|_\infty)\left(\int_{B_r}(1+|Dv|)^p dx\right)^{\frac{\delta(2q-p)\tilde{p}}{p}} \\
&+ C_{\varepsilon,\theta}\left(\int_{B_r}|Dv|^p dx\right)^{\frac{\tilde{p}\delta(2q-p)}{p}} \\
&+ C_\theta(q)|h|^{\alpha+\gamma}[D\psi]^q_{B^\gamma_{2q-p,\infty}(B_r)} + C_\theta(n,p,q)|h|^{\alpha+\gamma}\left(\int_{B_r}|Dv|^p dx\right)^{\frac{p^*\delta q}{p}} \\
B_1 :=& C_\varepsilon(\tilde{L}, n, p, q, \|a\|_\infty)\int_{B_r}|D\psi|^{2q-p}dx, \\
B_2 :=& C_{\varepsilon,\theta}(\tilde{L}, n, p, q, \|a\|_\infty)\left(\int_{B_r}(1+|Dv|)^p dx\right)^{\frac{\tilde{p}\delta(2q-p)}{p}}, \\
B_3 :=& C_\theta(n,p,q)\left(\int_{B_r}|D\psi|^{2q-p}dx\right)^{\frac{q}{2q-p}}, \\
B_4 :=& C_\theta(n,p,q)\left(\int_{B_r}|Dv|^p dx\right)^{\frac{p^*\delta q}{p}},
\end{aligned}$$

so that we can rewrite the previous estimate as

$$\begin{aligned}
&\nu\int_\Omega \eta^2(|\tau_h V_p(Dv)|^2 + a(x+h)|\tau_h V_p(Dv)|^2)dx \\
\leq& 3\varepsilon\int_\Omega \eta^2|\tau_h V_p(Dv)|^2 dx \\
&+ \theta(|h|^{2\alpha} + |h|^{\alpha+\gamma})\left(\int_{B_t}(1+|Dv|)^{\frac{np}{n-2\mu}}dx\right)^{\frac{n-2\mu}{n}} \\
&+ \theta(|h|^2 + |h|^{2\gamma} + |h|^{\alpha+1})\left(\int_{B_{t'}}(1+|Dv|)^{\frac{np}{n-2\mu}}dx\right)^{\frac{n-2\mu}{n}} \\
&+ (|h|^{2\gamma} + |h|^{2\alpha} + |h|^{\alpha+\gamma})A + |h|^2\frac{B_1}{(t-s)^2} + |h|^2\frac{B_2}{(t-s)^{2\tilde{p}}} \\
&+ |h|^{\alpha+1}\frac{B_3}{(t-s)^{p''}} + |h|^{\alpha+1}\frac{B_4}{(t-s)^{p^*}}.
\end{aligned}$$

Choosing $\varepsilon = \frac{\nu}{6}$, we can reabsorb the first integral in the right hand side of the previous estimate by the left hand side, thus getting

$$\begin{aligned}
&\int_\Omega \eta^2(|\tau_h V_p(Dv)|^2 + a(x+h)|\tau_h V_q(Dv)|^2)dx \\
\leq& 3\theta|h|^{2\alpha}\left(\int_{B_t}(1+|Dv|)^{\frac{np}{n-2\mu}}dx\right)^{\frac{n-2\mu}{n}} + 3\theta|h|^{2\alpha}\left(\int_{B_{t'}}(1+|Dv|)^{\frac{np}{n-2\mu}}dx\right)^{\frac{n-2\mu}{n}} \\
&+ |h|^{2\alpha}A + |h|^2\frac{B_1}{(t-s)^2} + |h|^2\frac{B_2}{(t-s)^{2\tilde{p}}} + |h|^{2\alpha}\frac{B_3}{t-s} + |h|^{2\alpha}\frac{B_4}{(t-s)^{p^*}}, \qquad (7.2.16)
\end{aligned}$$

where we used the fact that $\alpha < \gamma$.

Since the right hand side of (7.2.16) depends on the integrability of $Dv$, in order to exploit inequality (7.2.2), we need to derive an a priori estimate for the gradient of the minimzer $v$. First, we bound (7.2.16) from below as follows

$$\begin{aligned}
&\int_{B_s} |\tau_h V_p(Dv)|^2 dx \\
\leq &\int_{B_s} (|\tau_h V_p(Dv)|^2 + a(x+h)|\tau_h V_q(Dv)|^2) dx \\
\leq &|h|^{2\alpha}\Bigg\{ 2\theta \left( \int_{B_t} (1+|Dv|)^{\frac{np}{n-2\mu}} dx \right)^{\frac{n-2\mu}{n}} + 3\theta \left( \int_{B_{t'}} (1+|Dv|)^{\frac{np}{n-2\mu}} dx \right)^{\frac{n-2\mu}{n}} \\
&+ A + \frac{B_1}{(t-s)^2} + \frac{B_2}{(t-s)^{2\tilde{p}}} + \frac{B_3}{t-s} + \frac{B_4}{(t-s)^{p^*}} \Bigg\},
\end{aligned} \tag{7.2.17}$$

where we also used that $\eta = 1$ on $B_s$. Then, Lemma 2.2.4 and equality (2.0.3) imply

$$\begin{aligned}
\left( \int_{B_s} |Dv|^{\frac{np}{n-2\mu}} dx \right)^{\frac{n-2\mu}{n}} \leq &2\theta \left( \int_{B_t} (1+|Dv|)^{\frac{np}{n-2\mu}} dx \right)^{\frac{n-2\mu}{n}} + 3\theta \left( \int_{B_{t'}} (1+|Dv|)^{\frac{np}{n-2\mu}} dx \right)^{\frac{n-2\mu}{n}} \\
&+ A + \frac{B_1}{(t-s)^2} + \frac{B_2}{(t-s)^{2\tilde{p}}} + \frac{B_3}{t-s} + \frac{B_4}{(t-s)^{p^*}},
\end{aligned} \tag{7.2.18}$$

for all $\mu \in (\frac{n(q-p)}{p}, \alpha)$.
Now, applaying the iteration Lemma 2.0.3 twice, we obtain

$$\left( \int_{B_{r/4}} |Dv|^{\frac{np}{n-2\mu}} dx \right)^{\frac{n-2\mu}{n}} \leq C|h|^{2\alpha} \left\{ \frac{1}{r^{2\tilde{p}}} \left( \int_{B_r} (1+|Dv|^p + |D\psi|^{2q-p}) dx \right)^{\kappa} + [D\psi]^{2q-p}_{B^{\gamma}_{2q-p,\infty}(B_r)} \right\}, \tag{7.2.19}$$

thus, using Lemma 2.2.4, from inequalities (7.2.19) and (7.2.17), we deduce the a priori estimate

$$\begin{aligned}
&\int_{B_{r/4}} (|\tau_h V_p(Dv)|^2 + a(x+h)|\tau_h V_q(Dv)|^2) dx \\
\leq &C|h|^{2\alpha} \left\{ \frac{1}{r^{2\tilde{p}}} \left( \int_{B_r} (1+|Dv|^p + |D\psi|^{2q-p}) dx \right)^{\kappa} + [D\psi]^{2q-p}_{B^{\gamma}_{2q-p,\infty}(B_r)} \right\},
\end{aligned} \tag{7.2.20}$$

for constants $C := C(n,p,q,\alpha,\|a\|_\infty)$ and $\kappa := \frac{\delta(2q-p)\tilde{p}}{p} < \tilde{p}$. □

According to the previous result, we state the following remarks, which will be crucial for the proof of Theorem 7.4.1.

**Remark 7.2.2.** *From Proposition 2.1.2, it follows that*

$$|\tau_h(\sqrt{a(x)} V_q(Dv))|^2 \leq C a(x+h)|\tau_h V_q(Dv)|^2 + C|V_q(Dv)|^2 |\tau_h a(x)|. \tag{7.2.21}$$

*Combining (7.2.20) and (7.2.21), we obtain*

$$\int_{B_{r/4}} |\tau_h(\sqrt{a(x)} V_q(Dv))|^2 dx$$

$$\leq \int_{B_{r/4}} (|\tau_h V_p(Dv)|^2 + |\tau_h(\sqrt{a(x)}V_q(Dv))|^2 dx$$
$$\leq C \int_{B_{r/4}} (|\tau_h V_p(Dv)|^2 + a(x+h)|\tau_h V_q(Dv)|^2 + |V_q(Dv)|^2|\tau_h a(x)|)dx$$
$$\leq C|h|^{2\alpha}\left\{\frac{1}{r^{2\tilde{p}}}\left(\int_{B_r}(1+|Dv|^p+|D\psi|^{2q-p})dx\right)^{\kappa} + [D\psi]^{2q-p}_{B^{\gamma}_{2q-p,\infty}(B_r)}\right\} + C|h|^{\alpha}\|Dv\|^q_{L^q(B_r)}$$
$$\leq C|h|^{\alpha}\left\{\frac{1}{r^{2\tilde{p}}}\left(\int_{B_r}(1+|Dv|^p+|D\psi|^{2q-p})dx\right)^{\kappa} + \|Dv\|^q_{L^q(B_r)} + [D\psi]^{2q-p}_{B^{\gamma}_{2q-p,\infty}(B_r)}\right\}, \quad (7.2.22)$$

*which is finite by Theorem 7.2.1. Therefore,*

$$\sqrt{a(x)}V_q(Dv) \in B^{\frac{\alpha}{2}}_{2,\infty,loc}(B).$$

*Lemma (2.2.4) yields*

$$a(x)|Dv|^q \in L^{\frac{n}{n-2\beta}}_{loc}(B), \quad \forall \beta < \frac{\alpha}{2}.$$

**Remark 7.2.3.** *Choosing $\mu < \alpha$ s.t. $q = \frac{np}{n-2\mu}$, estimates (7.2.19) and (7.2.22) yield*

$$\int_{B_{r/4}} (|\tau_h V_p(Dv)|^2 + |\tau_h(\sqrt{a(x)}V_q(Dv))|^2 dx$$
$$\leq C|h|^{\alpha}\left\{\frac{1}{r^{2\tilde{p}}}\left(\int_{B_r}(1+|Dv|^p+|D\psi|^{2q-p})dx\right)^{\kappa} + [D\psi]^{2q-p}_{B^{\gamma}_{2q-p,\infty}(B_r)}\right\}^{\frac{q}{p}}$$
$$\leq C|h|^{\alpha}\left\{\frac{1}{r^{2\tilde{p}_1}}\left(\int_{B_r}(1+|Dv|^p+|D\psi|^{2q-p})dx\right)^{\kappa_1} + [D\psi]^{q_1}_{B^{\gamma}_{2q-p,\infty}(B_r)} + 1\right\}, \quad (7.2.23)$$

*where $\frac{\tilde{p}q}{p} = \tilde{p}_1 > 1$, $\frac{\kappa q}{p} = \kappa_1 < \tilde{p}_1$ and $\frac{(2q-p)q}{p} = q_1 < \tilde{p}_1$, with $\tilde{p}$ and $\kappa$ introduced in (7.2.15) and (7.2.20) respectively.*

## 7.3 Comparison

In this section we prove a comparison lemma (see Lemma 7.3.3 below), where we estimate the distance between the solution $u$ to the problem (7.0.1) and the solution $v$ to the problem (7.1.2). In order to do so, we first need the following lemma.

**Lemma 7.3.1.** *Let $\tilde{F} : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be the function defined in (7.1.1) under Assumption 1. Then there exists a positive constant $c = c(r, n, \nu)$ such that the following inequality holds for every $x \in \Omega$ and every $z_1, z_2 \in \mathbb{R}^n$*

$$c(|V_p(z_1) - V_p(z_2)|^2 + a(x)|V_q(z_1) - V_q(z_2)|^2) \leq \tilde{F}(x, z_1) - \tilde{F}(x, z_2) - \langle D_\xi \tilde{F}(x, z_2), z_1 - z_2\rangle. \quad (7.3.1)$$

*Proof.* We start proving that for every $r \geq 2$ there exists a constant $c = c(r, n)$ such that

$$c(r, n)|V_r(z_1) - V_r(z_2)|^2 \leq g_r(z_1) - g_r(z_2) - \langle D_\xi g_r(z_2), z_1 - z_2\rangle,$$

where we denote $g_r(z) := |z|^r$.

Let us consider the function $G_r : [0,1] \to \mathbb{R}$ defined by $G_r(t) := g_r(tz_1 + (1-t)z_2)$. Since $G_r \in \mathcal{C}^2([0,1])$, by using Taylor's formula with integral remainder, we obtain

$$G_r(1) = G_r(0) + G_r'(0) + \int_0^1 (1-s)G_r''(s)ds. \tag{7.3.2}$$

Since

$$\begin{aligned} G_r'(t) =& \langle D_\xi g_r(tz_1 + (1-t)z_2), z_1 - z_2\rangle, \\ G_r''(t) =& \langle D_{\xi\xi} g_r(tz_1 + (1-t)z_2)(z_1 - z_2), z_1 - z_2\rangle, \end{aligned}$$

from (7.3.2) we get

$$\begin{aligned} & g_r(z_1) - g_r(z_2) - \langle D_\xi g_r(z_2), z_1 - z_2\rangle \\ &\quad = \int_0^1 (1-s)\langle D_{\xi\xi} g_r(sz_1 + (1-s)z_2)(z_1 - z_2), z_1 - z_2\rangle ds \\ &\quad \geq c(r)|z_1 - z_2|^2 \int_0^1 (1-s)|sz_1 + (1-s)z_2|^{r-2} ds. \end{aligned} \tag{7.3.3}$$

Now, we want to estimate from below $|sz_1 + (1-s)z_2|^{r-2}$. If $|z_1| \leq |z_2|$ and $s \in [3/4, 1]$, then $-1/4 \leq s - 1 \leq 0$ and

$$|sz_1 + (1-s)z_2| \geq s|z_1| + (s-1)|z_2| \geq \frac{3}{4}|z_1| - \frac{1}{4}|z_2| \geq \frac{1}{4}(|z_1| + |z_2|),$$

while, if $|z_2| > |z_2|$ and $s \in [0, 1/4]$, then $3/4 \leq 1 - s \leq 1$ and

$$|sz_1 + (1-s)z_2| \geq -s|z_1| + (1-s)|z_2| \geq -\frac{1}{4}|z_1| + \frac{3}{4}|z_2| \geq \frac{1}{4}(|z_1| + |z_2|).$$

Therefore

$$|sz_1 + (1-s)z_2|^{r-2} \geq 4^{2-r}(|z_1| + |z_2|)^{r-2} \tag{7.3.4}$$

holds on a suitable subinterval of $[0,1]$. Eventually, inserting (7.3.4) in (7.3.3) we obtain

$$\begin{aligned} g_r(z_1) - g_r(z_2) - \langle D_\xi g_r(z_2), z_1 - z_2\rangle \geq& c(r)(|z_1| + |z_2|)^{r-2}|z_1 - z_2|^2 \\ \geq& c(r,n)|V_r(z_1) - V_r(z_2)|^2, \end{aligned}$$

where in the last inequality we used Lemma 2.0.2.

At this point, using the bound from below on $b$ in Assumption 1 and estimate (7.3.1) we deduce

$$\begin{aligned} & \tilde{F}(x, z_1) - \tilde{F}(x, z_2) - \langle D_\xi \tilde{F}(x, z_2), z_1 - z_2\rangle \\ &\quad = b(x_0, u_B)[g_p(z_1) - g_p(z_2) - \langle D_\xi g_p(z_2), z_1 - z_2\rangle \\ &\qquad + a(x)(g_q(z_1) - g_q(z_2) - \langle D_\xi g_q(z_2), z_1 - z_2\rangle)] \\ &\quad \geq c(r, n, \nu)(|V_p(z_1) - V_p(z_2)|^2 + a(x)|V_q(z_1) - V_q(z_2)|^2), \end{aligned}$$

which is the desired estimate. □

**Remark 7.3.2.** *In the proof of Lemma 7.3.3 we will take advantage of the higher integrability results established in Section 7.1, in particular in the case* $\frac{q}{p} < 1 + \frac{\alpha}{n}$.

*Indeed, the assumption* $D\psi \in B^{\gamma}_{2q-p,\infty,loc}(\Omega)$ *and Lemma 2.2.4 imply that* $D\psi \in L^{\frac{n(2q-p)}{n-\mu(2q-p)}}$, *for every* $0 < \mu < \gamma$. *Therefore,* $H(x, D\psi)$ *belong to some* $L^m$, *with* $m > 1$.

**Lemma 7.3.3.** *Let* $u$ *be the solution to* (7.0.1) *and* $v \in u + W^{1,p}_0(B)$ *be the solution to* (7.1.2), *under Assumptions 1 and 2, for exponents* $2 \le p < q$ *verifying*

$$\frac{q}{p} < 1 + \frac{\alpha}{n}.$$

*If*

$$D\psi \in B^{\gamma}_{2q-p,\infty,loc}(\Omega),$$

*for* $0 < \alpha < \gamma < 1$, *then*

$$\begin{aligned}\int_B |V_p(Du) - V_p(Dv)|^2 + a(x)|V_q(Du) - V_q(Dv)|^2 dx \\ \le CR^{\sigma} \int_{2B} (1 + (H(x, Du))^m + (H(x, D\psi))^m) dx,\end{aligned} \tag{7.3.5}$$

*with* $\sigma = \min\{\beta, m - 1\}$, *where* $\beta$ *is the exponent appearing in the Assumption 2 and where* $m$ *is the minimum of the two higher integrability exponents of Theorems 7.1.2 and 7.1.3.*

*Proof.* Assumption 1, the definition of $\tilde{F}$ and the minimality of $v$ imply

$$\int_B H(x, Dv)dx \le C \int_B \tilde{F}(x, Dv)dx \le \int_B \tilde{F}(x, Du)dx \le C \int_B H(x, Du)dx, \tag{7.3.6}$$

on the other hand, Theorem 7.1.3 yields

$$\int_B (H(x, Dv))^m dx \le \int_B [(H(x, Du))^m + (H(x, D\psi))^m]dx, \tag{7.3.7}$$

for some $m > 1$. From inequality (7.3.1) we get

$$\begin{aligned}\int_B |V_p(Du) - V_p(Dv)|^2 + a(x)|V_q(Du) - V_q(Dv)|^2 dx \\ \le C \int_B \tilde{F}(x, Du) - \tilde{F}(x, Dv) - \langle D_\xi \tilde{F}(x, Dv), Du - Dv\rangle dx,\end{aligned}$$

moreover, recalling inequality (7.1.3), i.e.

$$\int_B \langle D_\xi H(x, Dv), Du - Dv\rangle dx \ge 0,$$

and that $b(x_0, u_B) \ge \nu > 0$, we deduce

$$\int_B |V_p(Du) - V_p(Dv)|^2 + a(x)|V_q(Du) - V_q(Dv)|^2 dx \le C \int_B \tilde{F}(x, Du) - \tilde{F}(x, Dv)dx.$$

Hence, we can write the previous estimate as follows

$$
\begin{aligned}
\int_B &|V_p(Du)-V_p(Dv)|^2+a(x)|V_q(Du)-V_q(Dv)|^2dx\\
&\le C\int_B \tilde{F}(x,Du)-\tilde{F}(x,Dv)dx\\
&= C\int_B [b(x_0,u_B)H(x,Du)-b(x_0,u_B)H(x,Dv)]dx\\
&= C\int_B [b(x_0,u_B)H(x,Du)-b(x,u_B)H(x,Du)]dx\\
&\quad + C\int_B [b(x,u_B)H(x,Du)-b(x,u)H(x,Du)]dx\\
&\quad + C\int_B [b(x,u)H(x,Du)-b(x,v)H(x,Dv)]dx\\
&\quad + C\int_B [b(x,v)H(x,Dv)-b(x,v_B)H(x,Dv)]dx\\
&\quad + C\int_B [b(x,v_B)H(x,Dv)-b(x,u_B)H(x,Dv)]dx\\
&\quad + C\int_B [b(x,u_B)H(x,Dv)-b(x_0,u_B)H(x,Dv)]dx\\
&= C[I_1+I_2+I_3+I_4+I_5+I_6]. \qquad (7.3.8)
\end{aligned}
$$

We proceed estimating the various pieces arising up from (7.3.8).

By Assumption 2 and estimate (7.3.6), we get

$$
\begin{aligned}
I_1+I_6 &\le \int_B \omega_b(|x-x_0|)H(x,Du)dx+\int_B \omega_b(|x-x_0|)H(x,Dv)dx\\
&\le \int_B |x-x_0|^\beta (H(x,Du)+H(x,Dv))dx\\
&\le CR^\beta \int_B H(x,Du)dx\\
&\le CR^\beta \int_B [1+(H(x,Du))^m]dx. \qquad (7.3.9)
\end{aligned}
$$

Now, we take care of the integral $I_2$. From Assumption 2, Young's and Poincaré's inequalities, we infer

$$
\begin{aligned}
I_2 &\le \int_B \omega_b(|u-u_B|)H(x,Du)dx\\
&= \int_B \frac{1}{R^{\frac{\sigma}{1+\sigma}}}\omega_b(|u-u_B|)R^{\frac{\sigma}{1+\sigma}}H(x,Du)dx\\
&\le C\int_B \frac{1}{R}\omega_b(|u-u_B|)^{\frac{1+\sigma}{\sigma}}+R^\sigma(H(x,Du))^{1+\sigma}dx\\
&\le CR^\sigma \int_B \frac{1}{R^{1+\sigma}}|u-u_B|^{1+\sigma}+(H(x,Du))^{1+\sigma}dx\\
&\le CR^\sigma \int_B |Du|^{1+\sigma}+(H(x,Du))^{1+\sigma}dx
\end{aligned}
$$

$$\le CR^{\sigma}\int_B (1+|Du|^{p(1+\sigma)}) + (H(x,Du))^{1+\sigma}dx$$
$$\le CR^{\sigma}\int_B [1+(H(x,Du))^m]dx, \tag{7.3.10}$$

where $\sigma := \min\{\beta, m-1\}$.

The minimality of $u$ yields that

$$I_3 \le 0. \tag{7.3.11}$$

Arguing analogously as for the integral $I_2$, we obtain

$$\begin{aligned} I_4 \le& \int_B \omega_b(|v-v_B|)H(x,Dv)dx \\ \le& CR^{\sigma}\int_B \frac{1}{R^{1+\sigma}}|v-v_B|^{1+\sigma} + (H(x,Dv))^{1+\sigma}dx \\ \le& CR^{\sigma}\int_B |Dv|^{1+\sigma} + (H(x,Dv))^{1+\sigma}dx \\ \le& CR^{\sigma}\int_B [1+(H(x,Dv))^m]dx \\ \le& CR^{\sigma}\int_B [1+(H(x,Du))^m + (H(x,D\psi))^m]dx, \end{aligned} \tag{7.3.12}$$

where in the last inequality we used (7.3.7).

Since $u = v$ on $\partial B$, using Poicaré inequality for the function $u - v$, we infer the following estimate for $I_5$.

$$\begin{aligned} I_5 \le& \int_B \omega_b(|u_B - v_B|)H(x,Dv)dx \\ \le& CR^{\sigma}\int_B \frac{1}{R^{1+\sigma}}\omega_b(|u_B-v_B|)^{\frac{1+\sigma}{\sigma}} + (H(x,Du))^{1+\sigma}dx \\ \le& CR^{\sigma}\int_B \frac{1}{R^{1+\sigma}}|u-v|^{1+\sigma} + (H(x,Dv))^{1+\sigma}dx \\ \le& CR^{\sigma}\int_B |Du|^{1+\sigma} + |Dv|^{1+\sigma} + (H(x,Dv))^{1+\sigma}dx \\ \le& CR^{\sigma}\int_B [1+(H(x,Du))^m + (H(x,Dv))^m]dx \\ \le& CR^{\sigma}\int_B [1+(H(x,Du))^m + (H(x,D\psi))^m]dx, \end{aligned} \tag{7.3.13}$$

where in the inequality we used estimate (7.3.6). Finally, inserting estimates (7.3.9)–(7.3.13) in (7.3.8), we get the desired estimate. □

## 7.4 Main result

In order to prove Theorem 7.0.1 we follow the strategy first proposed in [75].
Before proving our main result, in Section 7.4.1, we fix some further notation and derive a

preliminary regularity theorem for solutions to (7.0.1).
As done in Section 3.3, for a ball $\mathcal{B} \Subset \Omega$ of radius $R$, we will denote by $\mathcal{Q}_1 = \mathcal{Q}_1(\mathcal{B})$ and $\mathcal{Q}_2 = \mathcal{Q}_2(\mathcal{B})$ the largest and the smallest cubes, concentric to $\mathcal{B}$ and with sides parallel to the coordinate axes, contained in $\mathcal{B}$ and containing $\mathcal{B}$ respectively. We also denote the enlarged ball by $\hat{\mathcal{B}} = 4\mathcal{B}$ and we set $\mathcal{Q}_1 = \mathcal{Q}_1(\mathcal{B})$, $\hat{\mathcal{Q}}_2 = \mathcal{Q}_2(\hat{\mathcal{B}})$. In what follows, we shall always take $\mathcal{B}$ such that $\mathcal{Q}_2(\hat{\mathcal{B}}) \Subset \Omega$.

Our next result shows that a fractional differentiability property on the gradient of the obstacle transfers to a higher fractional differentiability for the gradient of the minimizer.

**Theorem 7.4.1.** *Let $u$ be the solution to (7.0.1) under Assumptions 1 and 2, for exponents $2 \leq p < \frac{n}{\alpha}$, $p < q$ verifying*

$$\frac{q}{p} < 1 + \frac{\alpha}{n}.$$

*Then the following implication*

$$D\psi \in B^{\gamma}_{2q-p,\infty,loc}(\Omega) \Rightarrow V_p(Du), \sqrt{a(x)}V_q(Du) \in B^{\sigma_\alpha}_{2,\infty,loc}(\Omega)$$

*holds provided $0 < \alpha < \gamma < 1$, with $\sigma_\alpha = \sigma_\alpha(p, q, n, \alpha, \beta, m)$, where $\beta$ is the exponent appearing in the Assumption 2 and where $m$ is the minimum of the two higher integrability exponents of Theorems 7.1.2 and 7.1.3.*

*Proof.* Let us fix arbitrary open subsets $\Omega' \Subset \Omega'' \Subset \Omega$ and choose $x_0 \in \Omega'$. We recall the definition of $\tilde{p}_1$ from (7.2.23). Let $\delta \in \left(0, \frac{\alpha}{2\tilde{p}_1}\right)$ be chosen later and consider the ball $\mathcal{B} = \mathcal{B}(x_0, |h|^\delta)$ with $|h|$ sufficiently small, depending on the dimension $n$, the parameter $\delta$ and the distance between $\Omega'$ and the boundary of $\Omega''$ such that $\hat{\mathcal{Q}}_2 \Subset \Omega''$. Furthermore, let $v \in u + W^{1,p}_0(B)$ be the solution to (7.1.2) with $B = \hat{\mathcal{B}}$.

We estimate the difference quotient for $V_p(Du)$ and $\sqrt{a(x)}V_q(Du)$ as follows

$$\begin{aligned}
&\int_{\mathcal{B}} |\tau_h V_p(Du)|^2 + |\tau_h(\sqrt{a(x)}V_q(Du))|^2 dx \\
&\quad = \int_{\mathcal{B}} |V_p(Du(x+h)) - V_p(Du(x))|^2 dx \\
&\qquad + \int_{\mathcal{B}} |\sqrt{a(x+h)}V_q(Du(x+h)) - \sqrt{a(x)}V_q(Du(x))|^2 dx \\
&\quad \leq C \int_{\mathcal{B}} |V_p(Du(x+h)) - V_p(Dv(x+h))|^2 dx \\
&\qquad + C \int_{\mathcal{B}} |V_p(Dv(x+h)) - V_p(Dv(x))|^2 dx \\
&\qquad + C \int_{\mathcal{B}} |V_p(Dv(x)) - V_p(Du(x))|^2 dx \\
&\qquad + C \int_{\mathcal{B}} |\sqrt{a(x+h)}V_q(Du(x+h)) - \sqrt{a(x+h)}V_q(Dv(x+h))|^2 dx \\
&\qquad + C \int_{\mathcal{B}} |\sqrt{a(x+h)}V_q(Dv(x+h)) - \sqrt{a(x)}V_q(Dv(x))|^2 dx \\
&\qquad + C \int_{\mathcal{B}} |\sqrt{a(x)}V_q(Dv(x)) - \sqrt{a(x)}V_q(Du(x))|^2 dx. \qquad (7.4.1)
\end{aligned}$$

Notice that if $x \in \mathcal{B}$, then $x+h \in \hat{\mathcal{B}}$, for $|h| \leq 1$. Thus, we get

$$\begin{aligned}
&\int_{\mathcal{B}} |V_p(Du(x+h)) - V_p(Dv(x+h))|^2 dx \\
&\qquad + \int_{\mathcal{B}} |\sqrt{a(x+h)} V_q(Du(x+h)) - \sqrt{a(x+h)} V_q(Dv(x+h))|^2 dx \\
&\quad \leq \int_{\hat{\mathcal{B}}} |V_p(Du) - V_p(Dv)|^2 + a(x)|V_q(Du) - V_q(Dv)|^2 dx. \qquad (7.4.2)
\end{aligned}$$

Inserting inequality (7.4.2) in (7.4.1), we obtain

$$\begin{aligned}
&\int_{\mathcal{B}} |\tau_h V_p(Du)|^2 + |\tau_h(\sqrt{a(x)} V_q(Du))|^2 dx \\
&\quad \leq C \int_{\hat{\mathcal{B}}} |V_p(Du) - V_p(Dv)|^2 + a(x)|V_q(Du) - V_q(Dv)|^2 dx \\
&\qquad + C \int_{\mathcal{B}} |\tau_h V_p(Dv)|^2 + |\tau_h(\sqrt{a(x)} V_q(Dv))|^2 dx \\
&\quad =: J_1 + J_2. \qquad (7.4.3)
\end{aligned}$$

From estimate (7.3.5) applied over the ball $\hat{\mathcal{B}}$, we infer

$$J_1 \leq C|h|^{\sigma\delta} \int_{\hat{\mathcal{Q}}_2} (1 + (H(x,Du))^m + (H(x,D\psi))^m) dx, \qquad (7.4.4)$$

where we used that the radius of $\hat{\mathcal{B}}$ is proportional to $|h|^\delta$. Now estimate (7.2.23) (see Remark 7.2.3) applied over the ball $\mathcal{B}$ yields

$$J_2 \leq C|h|^{\alpha - 2\delta\tilde{p}_1} \left( \int_{\hat{\mathcal{B}}} (1 + |Dv|^p + |D\psi|^{2q-p}) dx \right)^{\kappa_1} + C|h|^\alpha [D\psi]^{q_1}_{B^\gamma_{2q-p,\infty}(\hat{\mathcal{B}})} + C|h|^\alpha, \qquad (7.4.5)$$

recalling that the radius of $\mathcal{B}$ is $|h|^\delta$. Inserting (7.4.4) and (7.4.5) in (7.4.3), we get

$$\begin{aligned}
&\int_{\mathcal{B}} |\tau_h V_p(Du)|^2 + |\tau_h(\sqrt{a(x)} V_q(Du))|^2 dx \\
&\quad \leq C|h|^{\sigma\delta} \int_{\hat{\mathcal{Q}}_2} (1 + (H(x,Du))^m + (H(x,D\psi))^m) dx \\
&\qquad + C|h|^{\alpha - 2\delta\tilde{p}_1} \left( \int_{\hat{\mathcal{B}}} (1 + |Dv|^p + |D\psi|^{2q-p}) dx \right)^{\kappa_1} + C|h|^\alpha [D\psi]^{q_1}_{B^\gamma_{2q-p,\infty}(\hat{\mathcal{B}})} + C|h|^\alpha \\
&\quad \leq C|h|^{\sigma\delta} \int_{\hat{\mathcal{Q}}_2} (1 + (H(x,Du))^m + (H(x,D\psi))^m) dx \\
&\qquad + C|h|^{\alpha - 2\delta\tilde{p}_1} \left( \int_{\hat{\mathcal{Q}}_2} (1 + H(x,Du)) dx \right)^{\kappa_1} + C|h|^\alpha [D\psi]^{q_1}_{B^\gamma_{2q-p,\infty}(\hat{\mathcal{Q}}_2)} \\
&\qquad + C|h|^{\alpha - 2\delta\tilde{p}_1} \left( \int_{\hat{\mathcal{Q}}_2} |D\psi|^{2q-p} dx \right)^{\kappa_1} + C|h|^\alpha, \qquad (7.4.6)
\end{aligned}$$

where in the last inequality we used (7.3.6).

Now we choose $\delta$ in order to minimize the right hand side of the previous estimate. It is easy to check that the best possible estimate is given by the choice

$$\delta = \frac{\alpha}{\sigma + 2\tilde{p}_1} \in \left(0, \frac{\alpha}{2\tilde{p}_1}\right).$$

With such a choice of $\delta$ estimate (7.4.6) becomes

$$\begin{aligned}&\int_{\mathcal{B}} |\tau_h V_p(Du)|^2 + |\tau_h(\sqrt{a(x)}V_q(Du))|^2 dx \\ \leq& C|h|^{\frac{\alpha\sigma}{\sigma+2\tilde{p}_1}} \left\{ \int_{\hat{\mathcal{Q}}_2} (1 + (H(x,Du))^m + (H(x,D\psi))^m) dx + \|D\psi\|_{B^{\gamma}_{2q-p,\infty}(\hat{\mathcal{Q}}_2)} + 1 \right\}^{\kappa_2}, \qquad (7.4.7)\end{aligned}$$

where $\kappa_2 := \kappa_2(n,p,q,\alpha)$.

At this stage, using the same covering argument presented in the proof of Theorem 3.0.2, for each $|h| \in \mathbb{R}^n$ sufficiently small we can find balls $\mathcal{B}_1 = \mathcal{B}_1(x_1, |h|^\sigma), ..., \mathcal{B}_K = \mathcal{B}_K(x_K, |h|^\sigma)$, being $K = K(h) \in \mathbb{N}$, such that the corresponding inner cubes $\mathcal{Q}_1(\mathcal{B}_1), ..., \mathcal{Q}_1(\mathcal{B}_K)$ are disjoint and satisfy

$$\left| \Omega' \setminus \bigcup_{k=1}^{K} \mathcal{Q}_1(\mathcal{B}_k) \right| = 0.$$

By our assumption we have that $\mathcal{Q}_2(\hat{\mathcal{B}}_k) \subset \Omega''$, for every $k \leq K$ and each of the dilated outer cubes $\mathcal{Q}_2(\hat{\mathcal{B}}_k)$ intersects at most $(16\sqrt{n})$ of the other cubes $\mathcal{Q}_2(\hat{\mathcal{B}}_j)$, with $j \neq k$. Hence, after summing up (7.4.7) over the inner cubes $\mathcal{Q}_1 \in \{\mathcal{Q}_1(\mathcal{B}_1), ..., \mathcal{Q}_1(\mathcal{B}_K)\}$, and enlarging the constant by a fixed factor only depending on $n$ and $p$ (in particular independent of $h$), we arrive at

$$\begin{aligned}&\int_{\Omega'} |\tau_h V_p(Du)|^2 + |\tau_h(\sqrt{a(x)}V_q(Du))|^2 dx \\ \leq& C|h|^{\frac{\alpha\sigma}{\sigma+2\tilde{p}_1}} \left\{ \int_{\Omega''} (1 + (H(x,Du))^m + (H(x,D\psi))^m) dx + \|D\psi\|_{B^{\gamma}_{2q-p,\infty}(\Omega'')} + 1 \right\}^{\kappa_2}. \qquad (7.4.8)\end{aligned}$$

Since the right hand side of the previous estimate is finite by our assumptions, it follows by arbitrariness of $\Omega'$ that

$$V_p(Du),\ \sqrt{a(x)}V_q(Du) \in B^{\frac{\alpha\sigma}{2(\sigma+2\tilde{p}_1)}}_{2,\infty}(\Omega) \quad \text{locally}.$$

Setting

$$\sigma_\alpha := \frac{\alpha\sigma}{2(\sigma + 2\tilde{p}_1)}, \qquad (7.4.9)$$

it follows the conclusion. □

### 7.4.1 Proof of Theorem 7.0.1

We are now able to give the proof of the main result of this chapter.

*Proof of Theorem 7.0.1.* Let us consider the function

$$A(t) = \frac{\alpha\sigma}{2[2(\tilde{p}_1 - \kappa_1 t) + \sigma]}, \qquad \forall t \in \left(0, \frac{\sigma + 2\tilde{p}_1 - \sqrt{(\sigma + 2\tilde{p}_1)^2 - 4\kappa_1\alpha\sigma}}{4\kappa_1}\right) =: (0, \tilde{\sigma}), \quad (7.4.10)$$

where $\tilde{p}_1, \kappa_1$ are defined in (7.2.23), $\sigma$ is defined in Lemma 7.3.3 and $\alpha$ is the exponent appearing in Assumption 1.

It is easy to see that $t \mapsto A(t)$ is increasing and that

$$t < A(t) < \tilde{\sigma}, \tag{7.4.11}$$
$$A(\tilde{\sigma}) = \tilde{\sigma}. \tag{7.4.12}$$

It is worth noticing that

$$\sigma_\alpha < \tilde{\sigma} < \frac{\alpha\sigma}{2\tilde{p}_1}, \tag{7.4.13}$$

where $\sigma_\alpha$ was introduced in (7.4.9). Indeed, owing to (7.4.9), the first part of inequality (7.4.13) holds if, and only if,

$$(\sigma + 2\tilde{p}_1)\sqrt{(\sigma + 2\tilde{p}_1)^2 - 4\kappa_1\alpha\sigma} < (\sigma + 2\tilde{p}_1)^2 - 2\kappa_1\alpha\sigma.$$

The last inequality is satisfied if, and only if,

$$(\sigma + 2\tilde{p}_1)^4 - 4\alpha\kappa_1\sigma(\sigma + 2\tilde{p}_1)^2 < (\sigma + 2\tilde{p}_1)^4 + 4\kappa_1^2\alpha^2\sigma^2 - 4\kappa_1\alpha\sigma(\sigma + 2\tilde{p}_1)^2,$$

that is equivalent to

$$0 < 4\kappa_1^2\alpha^2\sigma^2.$$

On the other hand, the second part of inequality (7.4.13) is valid if, and only if,

$$\tilde{p}_1(\sigma + 2\tilde{p}_1) - 2\kappa_1\alpha\sigma < \tilde{p}_1\sqrt{(\sigma + 2\tilde{p}_1)^2 - 4\kappa_1\alpha\sigma},$$

or, equivalently,

$$\tilde{p}_1^2(\sigma + 2\tilde{p}_1)^2 + 4\kappa_1^2\alpha^2\sigma^2 - 4\tilde{p}_1\kappa_1\alpha\sigma(\sigma + 2\tilde{p}_1) < \tilde{p}_1^2(\sigma + 2\tilde{p}_1)^2 - 4\tilde{p}_1^2\kappa_1\alpha\sigma.$$

The previous inequality can be written as

$$\kappa_1\alpha\sigma - \tilde{p}_1\sigma < \tilde{p}_1^2,$$

that holds true since $1 < \kappa_1 < \tilde{p}_1$ and $\alpha, \sigma \in (0, 1)$.

Let us now fix

$$\theta_0 \in \left(0, \frac{\alpha\sigma}{2(\sigma + 2\tilde{p}_1)}\right)$$

and denote

$$\theta_j = A(\theta_{j-1}), \quad \forall j \in \mathbb{N},\ j \geq 1.$$

Hence, the sequence $(\theta_j)_j$ is increasing and

$$\lim_j \theta_j = \tilde{\sigma}. \tag{7.4.14}$$

Now we define the sequence $(\iota_j)_j$ inductively as follows:

$$\begin{aligned}
\iota_0 &= \frac{\theta_0}{2} + \frac{\alpha\sigma}{4(\sigma + 2\tilde{p}_1)} < \frac{\alpha\sigma}{2(\sigma + 2\tilde{p}_1)}, \\
\iota_j &= \frac{\theta_j + A(\iota_{j-1})}{2}.
\end{aligned}$$

Using the fact that $A$ is increasing and (7.4.11), (7.4.12), we obtain

$$\theta_j < \iota_j < \tilde{\sigma}, \qquad \forall j \in \mathbb{N}, \tag{7.4.15}$$

and therefore, from (7.4.14), it follows that

$$\lim_j \iota_j = \tilde{\sigma}. \tag{7.4.16}$$

Arguing by induction, we shall prove that

$$V_p(Du),\ \sqrt{a(x)}V_q(Du) \in B^{\iota_j}_{2,\infty,\mathrm{loc}}(\Omega) \quad \forall j \in \mathbb{N}.$$

The case $j = 0$ follows from Theorem 7.4.1 and our choice of $\iota_0$. Now, let us prove the implication

$$V_p(Du),\ \sqrt{a(x)}V_q(Du) \in B^{\iota_{j-1}}_{2,\infty,\mathrm{loc}}(\Omega) \Rightarrow V_p(Du),\ \sqrt{a(x)}V_q(Du) \in B^{\iota_j}_{2,\infty,\mathrm{loc}}(\Omega). \tag{7.4.17}$$

By virtue of Lemma 2.2.4, the assumptions $V_p(Du), \sqrt{a(x)}V_q(Du) \in B^{\iota_{j-1}}_{2,\infty,\mathrm{loc}}(\Omega)$ imply

$$V_p(Du),\ \sqrt{a(x)}V_q(Du) \in L^{\frac{2n}{n-2\lambda}}(\hat{\mathcal{Q}}_2),$$

for every $0 < \lambda < \iota_{j-1}$ and so, recalling equality (2.0.3), we have that

$$|Du|^p,\ a(x)|Du|^q \in L^{\frac{n}{n-2\lambda}}(\hat{\mathcal{Q}}_2).$$

In particular, it follows

$$H(x, Du) \in L^{\frac{n}{n-2\lambda}}(\hat{\mathcal{Q}}_2),$$

for every $0 < \lambda < \iota_{j-1}$. Moreover, the assumption $D\psi \in B^{\gamma}_{2q-p,\infty,\mathrm{loc}}(\Omega)$ and Lemma 2.2.4 imply that $D\psi \in L^{\frac{n(2q-p)}{n-\pi(2q-p)}}(\hat{\mathcal{Q}}_2)$, for every $0 < \pi < \gamma$. Therefore, using Hölder's inequality in estimate (7.4.6) we infer

$$\begin{aligned}
&\int_{\mathcal{B}} |\tau_h V_p(Du)|^2 + |\tau_h(\sqrt{a(x)}V_q(Du))|^2 dx \\
&\quad \le C|h|^{\sigma\delta}\int_{\hat{\mathcal{Q}}_2}(1 + (H(x, Du))^m + (H(x, D\psi))^m)dx \\
&\qquad + C|h|^{\alpha-2\delta\tilde{p}_1+2\delta\kappa_1\lambda}\left(\int_{\hat{\mathcal{Q}}_2}(1 + H(x, Du))^{\frac{n}{n-2\lambda}}dx\right)^{\frac{(n-2\lambda)\kappa_1}{n}} + C|h|^{\alpha}[D\psi]^{q_1}_{B^{\gamma}_{2q-p,\infty}(\hat{\mathcal{Q}}_2)} \\
&\qquad + C|h|^{\alpha-2\delta\tilde{p}_1+(2q-p)\delta\kappa_1\pi}\left(\int_{\hat{\mathcal{Q}}_2}|D\psi|^{\frac{n(2q-p)}{n-\pi(2q-p)}}dx\right)^{\frac{(n-\pi(2q-p))\kappa_1}{n}} + C|h|^{\alpha} \\
&\quad \le C|h|^{\sigma\delta}\int_{\hat{\mathcal{Q}}_2}(1 + (H(x, Du))^m + (H(x, D\psi))^m)dx \\
&\qquad + C|h|^{\alpha-2\delta\tilde{p}_1+2\delta\kappa_1\lambda}\left(\int_{\hat{\mathcal{Q}}_2}(1 + H(x, Du))^{\frac{n}{n-2\lambda}}dx\right)^{\frac{(n-2\lambda)\kappa_1}{n}} + C|h|^{\alpha}[D\psi]^{q_1}_{B^{\gamma}_{2q-p,\infty}(\hat{\mathcal{Q}}_2)} \\
&\qquad + C|h|^{\alpha-2\delta\tilde{p}_1+2\delta\kappa_1\lambda}\left(\int_{\hat{\mathcal{Q}}_2}|D\psi|^{\frac{n(2q-p)}{n-\pi(2q-p)}}dx\right)^{\frac{(n-\pi(2q-p))\kappa_1}{n}} + C|h|^{\alpha},
\end{aligned} \tag{7.4.18}$$

for some $\pi \geq \frac{2\lambda}{2q-p}$, where we used the fact that the radius of the cube $\hat{\mathcal{Q}}_2$ is proportional to $|h|^\delta$. Therefore, choosing $\delta$ in order to maximize the right hand side of (7.4.18), namely

$$\delta = \frac{\alpha}{\sigma + 2(\tilde{p}_1 - k_1\lambda)},$$

we have

$$\begin{aligned}
&\int_{\mathcal{B}} |\tau_h V_p(Du)|^2 + |\tau_h(\sqrt{a(x)}V_q(Du))|^2 dx \\
&\leq C|h|^{\frac{\alpha\sigma}{\sigma+2(\tilde{p}_1-k_1\lambda)}} \left\{ \int_{\hat{\mathcal{Q}}_2} (1 + (H(x,Du))^m + (H(x,D\psi))^m) dx \right. \\
&\quad \left. + \int_{\hat{\mathcal{Q}}_2} (1 + H(x,Du))^{\frac{n}{n-2\lambda}} dx + \|D\psi\|_{B^\gamma_{2q-p,\infty}(\hat{\mathcal{Q}}_2)} + 1 \right\}^{\kappa^*},
\end{aligned} \tag{7.4.19}$$

where $\kappa^* := \kappa^*(n,p,q,\mu,\lambda)$. Thus, again through a covering argument, we deduce that

$$V_p(Du),\ \sqrt{a(x)}V_q(Du) \in B^{\frac{\alpha\sigma}{2[\sigma+2(\tilde{p}_1-k_1\lambda)]}}_{2,\infty,\mathrm{loc}}(\Omega) = B^{A(\lambda)}_{2,\infty,\mathrm{loc}}(\Omega), \quad \forall \lambda < \iota_{j-1}.$$

We have just proved the following implication

$$V_p(Du),\ \sqrt{a(x)}V_q(Du) \in B^{\iota_{j-1}}_{2,\infty,\mathrm{loc}}(\Omega) \Rightarrow V_p(Du),\ \sqrt{a(x)}V_q(Du) \in B^{t}_{2,\infty,\mathrm{loc}}(\Omega), \tag{7.4.20}$$

for all $t < A(\iota_{j-1})$.

Since $A$ is increasing, it follows from (7.4.15) that $\theta_j < A(\iota_{j-1})$. Moreover, the definition of $\iota_j$ implies $\iota_j < A(\iota_{j-1})$. Therefore, (7.4.17) follows from (7.4.20). Besides, from (7.4.15) and (7.4.16), we infer

$$V_p(Du),\ \sqrt{a(x)}V_q(Du) \in B^{t}_{2,\infty,\mathrm{loc}}(\Omega), \quad \forall t \in (0, \tilde{\sigma}).$$

It is worth noting that the exponent $\tilde{\sigma}$ defined in (7.4.10) is bigger than $\sigma_\alpha$. Therefore, Theorem 7.0.1 improves the higher fractional differentiability result established in Theorem 7.4.1. □

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