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NONLINEAR STABILITY FOR REACTION-DIFFUSION LOTKA-VOLTERRA MODELS

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Contents

Introduction

This thesis concerns with reaction-diffusion equations which model diffusion phenomena of the real world.

In the past several years, reaction-diffusion equations have attracted a great deal of attention from mathematicians and other scientists. In fact typical problems which arises in chemical, biological or physical areas are modelled by these equations.

The aim of this thesis is to use some Liapunov functions for reaction-diffusion models, introduced by Rionero, to obtain the nonlinear stability for the steady state solution (biologically meaningfull) of a generalized Lotka-Volterra model.

The plan of the thesis is as follows.

Chapter 1 is dedicated to recall some fundamental results connected with parabolic equations and to introduce reaction-diffusion equations. These equations represent an important class of evolution equations which arise in many real world phenomena such as fluid dynamics, plasma physics, crystal growth and, last but not list, biological population genetics.

Chapter 2 is a review of general stability theory. After some basic concepts related to the dynamical systems, the Liapunov direct method is recalled and in particular some Liapunov functionals, introduced by Rionero et al. for reaction-diffusion models are recalled. Successively, in order to recall that for P.D.Es. stability is topologically-dependent, we consider the well known example concerning the linear stability of Couette flow of an ideal incompressible fluid.

In Chapter 3, a binary reaction-diffusion system of partial differential equations is considered. In order to link the L^2-stability (instability) of an assigned solution, to the stability (instability) of the zero solution of a suitable linear binary system of ordinary differential equations associated to the problem at hand, a peculiar Rionero-Liapunov function is introduced.

Finally Chapter 4 is devoted to the coexistence problem for a generalized Lotka-Volterra predator-prey model, with Beddington-De Angelis functional response and Robin type boundary conditions. By using the Rionero-Liapunov functionals introduced in Chapter 3, conditions guaranteeing the nonlinear L^2-stability of the biologically meaningfull equilibrium state are furnished.

Chapter 1

Preliminaries and Fundamental Issues on Parabolic Equations

1.1 Introduction

Mathematical equations have always provided a language in which to formulate physical concepts. A mathematical model is an equation, or a set of equations, whose solutions describe the behavior of the related physical phenomena. In general, a mathematical model is a (simplified) description of a phenomenon of the real world expressed in mathematical terms.

Mathematical modelling involves physical observation, selection of the relevant physical variables, formulation of the equations, analysis of the equations, simulation, and, finally, the validation of the model. In this last step information from the simulations and solutions is fed back into the model to test if the model can describe the phenomenon, otherwise some modifications and refinements have been made.

The aim of this thesis concerns with models involving partial differential equations (P.D.Es.) of parabolic type. For the sake of completeness, this chapter is devoted to recall some fundamental concepts related to the parabolic equations (Cfr. [11], [14], [21]) and reaction-diffusion equations (Cfr. [26] [39]).

1.2 Initial boundary value problem for parabolic equations

Let Ω be an open, bounded subset of $\Re^n$, and let us set $\Omega_T = \Omega \times (0, T]$ for some fixed time $T > 0$. The most general form of parabolic equation is the following

$$u_t - Lu = f$$

where $f : \Omega_T \rightarrow \Re$ is an assigned function, $u : \overline{\Omega}_T \rightarrow \Re$ is the unknown, and L denotes, for each time t, a second order partial differential operator, having either the divergence form

$$Lu = \sum_{i,j=1}^{n} \left(a^{ij}(x,t)u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x,t)u_{x_i} + c(x,t)u$$

or the nondivergence form

$$Lu = \sum_{i,j=1}^{n} a^{ij}(x,t)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x,t)u_{x_i} + c(x,t)u$$

for given coefficients a^{ij}, b^i and c $(i, j = 1, ..., n)$.

To model concrete physical processes one has to add to the differential equations some auxiliary conditions, i.e. initial and boundary conditions. For example, if we want to determine the temperature inside a body at an arbitrary time, we must in addition know the temperature distribution in the body at the initial time (initial condition) and the temperature regime on the boundary $\partial\Omega$ of the body Ω (boundary condition). Different kinds of boundary conditions can be added to the parabolic equation, i.e.

- Neumann boundary conditions

$$\nabla u \cdot n = a(x,\, t) \quad x \in \partial\Omega,\, t > 0$$

  where n is the unit outward normal to $\partial\Omega$ and $a(x,\, t)$ is an assigned function;

- Dirichlet boundary conditions

$$u = b(x, t) \quad x \in \partial\Omega, t > 0$$

  where $b(x,\, t)$ is a prescribed function;

- Robin boundary conditions or mixed boundary conditions

$$\alpha(x, t)u + \beta(x, t)\nabla u \cdot n = \gamma(x, t) \quad x \in \partial\Omega, t > 0$$

  where α, β and γ are given functions.

Let us consider the initial boundary value problem

$$\begin{cases} u_t - Lu = f & \Omega_T \\ u = 0 & \Omega \times \{0\} \\ u = g & \partial\Omega \times [0, T] \end{cases} \tag{1.1}$$

where $g : \Omega \to \Re$ is given and

$$\begin{aligned} & a^{ij}, b^i, c \in L^\infty(\Omega_T) \\ & f \in L^2(\Omega_T) \\ & g \in L^2(\Omega) \end{aligned} \tag{1.2}$$

with $a^{ij} = a^{ji}$ $(i, j = 1, ..., n)$.

Let us define the time dependent bilinear form

$$B[u, v; t] := \int_\Omega \left[\sum_{i,j=1}^{n} a^{ij}(\cdot, t) u_{x_i} v_{x_j} + \sum_{i=1}^{n} b^i(\cdot, t) u_{x_i} v + c(\cdot, t) uv \right] dx$$

for $u, v \in H_0^1(\Omega)$ and $0 \leq t \leq T$, where

$$H_0^1 = \left\{ \varphi : \varphi^2 + (\nabla\varphi)^2 \in L^2,\ \varphi = 0 \ \ on \ \ \partial\Omega \right\}$$

and H^{-1} is the dual space on H_0^1.

Let us associate with u a mapping

$$\underline{u} : [0, T] \to H_0^1(\Omega)$$

defined by

$$[\underline{u}(t)](x) := u(x, t) \quad x \in \Omega,\ t \in [0, T]$$

and similarly let us define

$$\underline{f} : [0, T] \to L^2(\Omega)$$

by

$$[\underline{f}(t)](x) := f(x, t) \quad x \in \Omega,\ t \in [0, T].$$

If we fix a function $v \in H_0^1(\Omega)$, on multiplying (1.1) by v and on integrating by parts, it turns out that

$$\langle \underline{u}', v \rangle - B[\underline{u}, v; t] = (\underline{f}, v) \quad 0 \leq t \leq T$$

where $\underline{u}' = \dfrac{d\underline{u}}{dt}$, $(\cdot, \cdot)$ is the inner product in $L^2(\Omega)$, and $\langle \cdot, \cdot \rangle$ is the pair of $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. These considerations motivate the following definition of weak solution.

Definition 1.2.1. *A function*

$$\underline{u} \in L^2\left(0, T; H_0^1(\Omega)\right), \qquad \text{with} \;\; \underline{u}' \in L^2\left(0, T; H^{-1}(\Omega)\right),$$

is a weak solution of the parabolic initial boundary value problem (1.1) *provided*

$$\begin{aligned} &\langle \underline{u}', v\rangle + B[\underline{u}, v; t] = (\underline{f}, v) \\ &\underline{u}(0) = g \end{aligned}$$

for each $v \in H_0^1(\Omega)$ *and* $0 \leq t \leq T$.

The following theorem holds

Theorem 1.2.1. *There exists a weak solution of* (1.1), *and it is unique.*

For the proof see [11].

1.3 The maximum principle

One of the most useful and best known tools employed in the study of partial differential equations is the maximum principle (Cfr. [14], [21]). This principle enables us to obtain informations about solutions of differential equations without any explicit knowledge of the solutions themselves. In particular, the maximum principle is an useful tool to approximate solutions, a subject of great interest for many scientists.
Let us consider parabolic second order equation of the type

$$Lu(t, x) = f(t, x) \tag{1.3}$$

where

$$Lu = \sum_{i,j=1}^{n} \left(a^{ij}(x,t)u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x,t)u_{x_i} + c(x,t)u - \frac{\partial u}{\partial t}(x,t) \tag{1.4}$$

in the $(n+1)$-dimensional domain $\Omega_T = \Omega \times (0, T)$. Let us assume that

(i) L is parabolic in Ω_T, i.e., $\forall (x,t) \in \Omega,\ \forall \xi \in \Re^n\ \sum a_{ij}\xi_i\xi_j > 0$,

(ii) the coefficients of L are continuous functions in Ω,

(iii) $c(x,t) \leq 0$ in Ω_T.

We also assume that u has two continuous x-derivatives and one continuous t-derivative in Ω_T.

Let us remark that, for any point $P_0 \in \Omega_T$, we denote by $S(P_0)$ the set of all points Q in Ω_T which can be connected to P_0 by a simple continuous curve in Ω_T, along which the t-coordinate is nondecreasing from Q to P_0.

Theorem 1.3.1. ***Strong maximum principle*** *Let (i), (ii) and (iii) hold. If $Lu \geq 0$ ($Lu \leq 0$) in Ω_T and if u has in Ω_T a positive maximum (negative minimum) which is attained at a point $P_0(x_0, t_0)$, then $u(P) = u(P_0)$, $\forall P \in S(P_0)$.*

For the proof of this theorem, we need to recall the following lemmas.

Lemma 1.3.1. *Let (i), (ii) and (iii) hold. Assume that either $Lu > 0$ throughout Ω_T or that $Lu \geq 0$ and $c(x,t) < 0$ throughout Ω_T. Then u cannot have a positive maximum in Ω_T.*

Lemma 1.3.2. *Let (i), (ii) and (iii) hold. Let $Lu \geq 0$ in Ω_T and let u have a positive maximum M in Ω_T. Suppose that Ω_T contains a closed solid ellipsoid E:*

$$\sum_{i=1}^{n} \lambda_i (x_i - x_i^*)^2 + \lambda_0 (t - t^*)^2 \leq R^2 \quad (\lambda_j > 0, R > 0)$$

and that $u < M$ in the interior of E and $u(\overline{x}, \overline{t}) = M$ at some point $\overline{P} = (\overline{x}, \overline{t})$ on the boundary ∂E of E. Then $\overline{x} = x^*$, where $x^* = (x_1^*, ... x_n^*)$.

Lemma 1.3.3. *Let (i), (ii) and (iii) hold. If $Lu \geq 0$ in Ω_T and if u has a positive maximum in Ω_T which is attained at a point $P_0 = (x_0, t_0)$, then $u(P) = u(P_0)$ for all $P \in C(P_0)$.*

Lemma 1.3.4. *Let (i), (ii) and (iii) hold. Let R be a rectangle*

$$x_{i0} - a_i \leq x_i \leq x_{i0} + a_i \qquad t_0 - a_0 \leq t \leq t_0 \qquad (i = 1, ...n)$$

contained in Ω_T, and let $Lu \geq 0$ in Ω_T. If u has a positive maximum in R which is attained at the point $P_0 = (x_0, t_0)$, then $u(P) = u(P_0)$ for all $P \in R$.

Now we can give the proof of the maximum principle.

Proof. Suppose that $u(P) \neq u(P_0)$ in $S(P_0)$, then there exists a point $Q \in S(P_0)$ such that $u(Q) < u(P_0)$. Connect Q to P_0 by a simple continuous curve γ lying in $S(P_0)$ such that t-coordinate is nondecreasing from Q to P_0. On γ there exists a point P_1 such that $u(P_1) = u(P_0)$ and $u(\overline{P}) < u(P_1)$ for all points $\overline{P} \in \gamma$ lying between Q and P_1. Denote by γ_0 the subarc of γ lying between Q and P_1. Construct a rectangle

$$x_{i1} - a_i \leq x_i \leq x_{i1} + a_i \qquad t_1 - a \leq t \leq t_1 \qquad (i = 1, ...n)$$

where $P_1 = (x_{11}, ...x_{n1}, t_1)$ and a is sufficiently small so that the rectangle lies in Ω. Applying Lemma 1.3.4 it follows that $u = u(P_1)$ in this rectangle. Hence $u(P) = u(P_1)$ on the segment of γ_0 lying in the rectangle. This however contradicts the definition of P_1.

As a consequence of this theorem, we may obtain an uniqueness result.

Let Lu defined by (1.4), let β be a continuous function on $\partial\Omega \times (0, T]$, and let τ be a direction defined at each point of $\partial\Omega \times (0, T]$ in a continuous manner.

Consider the initial boundary value problem

$$\begin{cases} Lu(x,t) - \dfrac{\partial u}{\partial t} = f(x,t) & \Omega_T \\ \\ u(x,0) = \varphi(x) & \Omega \times \{0\} \\ \\ \dfrac{\partial u(x,t)}{\partial \tau} + \beta(x,t)u(x,t) = \psi(x,t) & \partial\Omega \times (0,T] \\ \end{cases} \tag{1.5}$$

which is said to be regular if τ is never tangent to $\partial\Omega \times (0,T]$.

Definition 1.3.1. *The boundary $\partial\Omega$ of a domain Ω belongs to the class C^m, or $C^{m+\alpha}$, if there exist local representations of $\partial\Omega$, in neighborhoods of each of its points, having the form $x_i = h(x_1, ...x_{i-1}, x_{i+1}, ...x_n)$, where the functions h belong (locally) to C^m, or $C^{m+\alpha}$, respectively.*

Theorem 1.3.2. *Let L be a parabolic operator with continuous coefficients in $\overline{\Omega}_T = \overline{\Omega} \times [0,T]$ and let $\partial\Omega$ belong to $C^{1+\lambda}$ $(0 < \lambda < 1)$. If f is Holder continuous (exponent α) in x, uniformly in Ω_T, if $\varphi(x)$ is continuous in $\overline{\Omega}_T$ and vanished in some Ω_T-neighborhood of the boundary of Ω_T, and if ψ is continuous on $\partial\Omega \times (0,T]$, then there exists a unique solution of the problem* (1.5).

For the proof see [14].

1.4 Reaction diffusion equations

An important class of evolution equations is represented by reaction-diffusion equations, which arise in many fields of application such as heat transfer, combustion, reaction chemistry, fluid dynamics, plasma physics, crystal growth, biological population genetics and neurology (Cfr. [8] and [25]).

For example to model the dispersive behaviour of populations (of cells or animals) or concentrations (of chemicals) one often uses a continuum approach employing density functions to describe the distribution of basic particles.
Let $u(x,t) : \Omega \times \Re^+ \to \Re$, where $\Omega \subset \Re^n$, be the particle density function or concentration. Let $Q(x,t,...)$ be the net creation rate of particles at $x \in \Omega$ at time t (for instance the birth rate per unit volume minus the death rate per unit volume). Let $J(x,t,...)$ be the flux density, i.e. for any unit vector $n \in \Re^n$, the scalar product $J \cdot n$ is the net rate at which particles cross a unit area in a plane perpendicular to n (positive in n direction).
For any regular subset $S \subset \Omega$

$$\int_S u dx$$

denotes the population mass in S. We assume that the rate of change of this mass is due to particle creation or degradation inside S, and to the inflow and outflow of particles through the boundary ∂S, i.e.

$$\frac{d}{dt}\int_S u dx = -\int_{\partial S} J \cdot n d\sigma + \int_S Q dx \tag{1.6}$$

where n denotes the outward normal to ∂S. Applying the divergence theorem, equation (1.6) becomes

$$\int_S u_t dx = \int_S \left[-\nabla \cdot J + Q\right] dx. \tag{1.7}$$

But S is arbitrary in Ω so the local balance or conservation equation follows

$$u_t = -\nabla \cdot J + Q. \tag{1.8}$$

For a given model we must specify Q and J. For example, we may follow the theory of diffusion founded by the physiologist Fick. According to Fick's law the flux J is proportional to the gradient of the density, i.e.

$$J = -D\nabla u. \tag{1.9}$$

Under the assumption that D is a positive constant, (1.8) becomes the following reaction diffusion equation

$$u_t = D\Delta u + Q(x, t, ...)\,.$$

There are many more formulations for the flux terms in diffusive process, Okubo in [26] and Gurtin and Mac Camy in [17], provide a good account of such processes applied in biology.

Generally if $u \in I\!R^n$, (1.8), by virtue of (1.9) becomes

$$u_t = \nabla \cdot (D\nabla u) + F(u) \quad x \in \Omega \subset \Re^n,\, t > 0 \tag{1.10}$$

where $u \in \Re^n$, D is the diffusion matrix and $F(u)$ is a nonlinear smooth function of u which represent the reaction term.

If D is a diagonal matrix, i.e. there is no cross-diffusion among the species

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ 0 & 0 & \cdots & d_n \end{pmatrix}$$

with $d_i \geq 0$, $1 \leq i \leq n$, then (1.10) reduces to

$$u_t = D\Delta u + F(u)\,, \qquad u \in I\!R^n\,. \tag{1.11}$$

1.5 Reaction diffusion equations of two biological populations

An ecological model represented by equation (1.11) is the following

$$\begin{cases} u_t = \alpha\Delta u + uM(u,v) & (x,t) \in \Omega \times \Re^+ \\ v_t = \beta\Delta v + vN(u,v) & (x,t) \in \Omega \times \Re^+ \end{cases} \tag{1.12}$$

which describes the classical two species interactions, when diffusion and spatial dependence are taken into account.

Here Ω is a bounded region in $\Re^n$, $\alpha, \beta \geq 0$ are constants; $u > 0$ and $v > 0$ are scalar functions of (x,t) which represent population densities; M and N are their respective growth rates, that we assume to be smooth. Together with (1.12), we have the initial conditions

$$\begin{cases} u(x,0) = u_0(x) & x \in \Omega \\ v(x,0) = v_0(x) & x \in \Omega \end{cases} \tag{1.13}$$

and the homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad on \quad \partial\Omega \times \Re^+. \tag{1.14}$$

It is assumed that both u_0 and v_0 are bounded nonnegative smooth functions. The boundary conditions (1.14) are to be interpreted as "no flux" conditions, i.e. there is no migration of either species across $\partial\Omega$. Ω is here considered as the habitat of u and v.

We now consider the three classical ecological interactions that are determined by the signs of the partial derivatives $\dfrac{\partial M}{\partial v} = M_v$ and $\dfrac{\partial N}{\partial u} = N_u$.

- In the predator-prey interaction, the derivatives are of opposite sign

  $$M_v < 0, \quad N_u > 0$$

  where u denotes the prey density, and v the predator density.

- Competition refers to the case in which both derivatives are negative

$$M_v < 0, \quad N_u < 0.$$

- Symbiosis refers to the case in which both derivatives are positive

$$M_v > 0, \quad N_u > 0.$$

There are always present, in any environment, specific resource limitations which place a definite upper bound on the growth rates. Such limits to the growth are intimately connected with pointwise bounds on u and v; i.e., they imply the existence of bounded invariant regions.

Chapter 2

Stability. Liapunov Direct Method

2.1 Introduction

In modelling a real world phenomenon, in general it may happens that the mathematical model considered contains some errors. These arise in the measurements of the data (initial data, boundary data, forces, geometry of the domain in which the phenomenon takes place, parameters contained in the evolution equation,...) and in errors in formulating the model. The question arises therefore of how these errors may influence the solution. This is the concept of continuous dependence and, more generally, of stability.

Qualitative theory of solutions of differential equations originates in the developments due to Poincaré and Liapunov. The basic idea of the so called Liapunov second method is to generalize the statement that if the potential energy of a physical system is a minimum (maximum) at an equilibrium point, than the equilibrium point is stable (unstable). In this method, the potential energy function is replaced by a more general kind of function, the Liapunov function, and stability properties of an equilibrium are deduced from the properties of its time derivative along motions of the dynamical system being investigated.

Liapunov's work is applicable to all kinds of evolutionary systems, including some for which the concept of energy has no meaning, so the interpretation of Liapunov function is extended to a notion of generalized energy function. Another interpretation of Liapunov function is that of a generalized distance function, when it is viewed as representing a measure of the distance of a trajectory or motion at a time istant t from an invariant set, usually an equilibrium.

In this chapter, following the books [13]-[22], we introduce the Liapunov direct method and we give some examples of Liapunov functions for reaction-diffusion equations (Cfr.also [1]).

2.2 Dynamical Systems

Let us consider the following initial value problem

$$\begin{cases} u_t = F(u) \\ \\ u(x,\,0) = u_0(x) \qquad \text{in } \Omega \\ \\ A(u,\,\nabla u) = \hat{u} \qquad \text{on } \partial\Omega \times [0,T] \end{cases} \tag{2.1}$$

where $u = (u_1, ..., u_n) \in I\!R^n,\ (n \geq 1),$, $u_0 \in C(I\!R)$ and $\hat{u}$ are prescribed real functions. Let $u(u_0, t)$, with $u(u_0, 0) = u_0$ be a global solution of the problem. Then u is a dynamical system according to the following definition.

Definition 2.2.1. *A dynamical system on a metric space X is a mapping*

$$v : (v_0, t) \in X \times \Re \to v(v_0, t) \in X$$

such that

$$\begin{cases} v(v_0, 0) = v_0 \\ v(v_0, t+\tau) = v(v(v_0, t), \tau) \quad \forall t, \tau \in \Re^+ \end{cases}$$

where $v_0 \in X$.

For a dynamical system v, the function

$$v(v_0, \cdot) : t \in \Re \to v(v_0, t) \in X\,, \qquad v_0 \in X$$

is called a motion associated to the initial data v_0, and is denoted by $v(v_0, t)$ or by $v(t)$.

Definition 2.2.2. *The motion $v(v_0, t)$ is steady and v_0 is an equilibrium, or a critical point, if*

$$v(t) = v_0\,, \quad \forall t \in \Re\,.$$

Definition 2.2.3. *If $\exists \tau : v(t+\tau) = v(t)$, $\forall t \in \Re$, the motion v is periodic in time with period τ.*

Definition 2.2.4. *A semigroup on a metric space X is a one parameter family $\{S(t)\}_{t \geq 0}$ of operators, $S(t) : X \to X$ such that, for all $t, \tau \in \Re^+, x \in X$, one has*

$$i) S(0) = I$$

$$ii) S(t+\tau) = S(t)S(\tau)\,.$$

The equivalence between a semigroup $\{S(t)\}_{t \geq 0}$ and a dynamical system v is immediately seen by setting

$$v(v_0, t) = S(t)v_0 \qquad v_0 \in X, t \in \Re^+\,.$$

Other basic properties may, in general, be needed in the study of a dynamical system. We recall here the following

iii) $v(\cdot, t) : X \to X$ is continuous $\forall t \geq 0$

iv) $v(v_0, \cdot) : \Re^+ \to X$ is continuous $\forall v_0 \in X$

v) $v(v_0, \cdot) : \Re^+ \to X$ is injective.

Let v be a dynamical system on a metric spac (X, d) and let us consider the open ball $S(x, r)$ centered at x and having radius $r > 0$. Essentially, the idea of continuous dependence of a particular motion $v(v_0, \cdot)$ is that any other motion $v(v_1, \cdot)$, starting at the same initial instant from a position v_1 sufficiently close to v_0, will remain as closed as desired to the basic motion for all finite time $T > 0$.

In a mathematically rigorous way, it means what it follows.

Definition 2.2.5. *A motion $v(v_0, \cdot)$ depends continuously on the initial data iff* $\forall T > 0, \forall \epsilon > 0$

$$\exists \delta(\epsilon, T) : v_1 \in S(v_0, \delta) \Rightarrow v(v_1, t) \in S(v(v_0, t), \epsilon), \quad \forall t \in [0, T]$$

2.3 Liapunov stability

The Liapunov stability of a basic motion $v(v_0, \cdot)$ of a dynamical system v extends the requirement of continuous dependence to the infinite interval of time $(0, +\infty)$.

Definition 2.3.1. *A motion $v(v_0, \cdot)$ is Liapunov stable (with respect to perturbations to the initial data) iff*

$$\forall \epsilon > 0, \exists \delta(\epsilon) > 0 : v_1 \in S(v_0, \delta) \Rightarrow v(v_1, t) \in S(v(v_0, t), \epsilon), \forall t \in \Re^+ .$$

A motion is unstable if it is not stable.

Definition 2.3.2. *A motion $v(v_0,\cdot)$ is said to be an attractor on a set Y if*

$$v_1 \in Y \Rightarrow \lim_{t\to\infty} d[v(v_0,t), v(v_1,t)] = 0\,.$$

The biggest set Y on which it holds, is called the basin of attraction of $v(v_0,\cdot)$.

Definition 2.3.3. *The motion $v(v_0,\cdot)$ is asymptotically stable if it is stable and if there exists $\delta_1 > 0$ such that $v(v_0,\cdot)$ is attractive on $S(v_0,\delta_1)$. In particular, $v(v_0,\cdot)$ is exponentially stable if there exist $\delta_1 > 0$, $\lambda(\delta_1) > 0$ and $M(\delta_1) > 0$ such that*

$$v_1 \in S(v_0,\delta_1) \Rightarrow d[v(v_1,t), v(v_0,t)] \le Me^{-\lambda t} d(v_1,v_0), \forall t \in \Re^+\,.$$

Exponential stability is the strongest stability property which corresponds not only to (uniform) asymptotic stability, but gives also quantitative description of the behaviour of solutions.
It is always possible to express the stability of a given basic motion $v(v_0,t)$ through the stability of the zero solution of the perturbed dynamical system

$$u : (u_0,t) \in X \times \Re^+ \to v(v_0+u_0,t) - v(v_0,t)\,.$$

If the dynamical system v is linear, i.e. $v(\cdot,t)$ is a linear operator of X on X, $\forall t \in \Re^+$, then the stability of every motion is determined by the stability of zero solution. When v is nonlinear, the stability of the trivial solution does not determine the stability of every motion.

2.4 Topology dependent stability

Partial differential equations (P.D.Es.) are (generally) embedded in a normed linear infinite dimensional space. Then it follows that a solution of P.D.Es.

could be stable with respect one choice of metric and unstable with respect to another choice. This means that for P.D.Es. stability depends on the topology.

In order to show that for P.D.Es. stability is topology-dependent, let us consider the well-known example concerning the linear stability of Couette flow of an ideal incompressible fluid.

Let us recall that the motion of an incompressible homogeneous viscous fluid occurring in a fixed region $\Omega \subseteq \Re^3$ is described by the Navier- Stokes equations

$$\begin{cases} v_t + v \cdot \nabla v = -\nabla p + \nu \Delta v + F \quad \Omega \times \Re^+ \\ \nabla \cdot v = 0 \quad \Omega \times \Re^+ \end{cases} \tag{2.2}$$

where $v(x,t)$ is the velocity field, $p(x,t)$ is the pressure field, $\nu > 0$ is the kinematic viscosity and $F(x,t)$ is the body force acting on the fluid.

The equations of motion of a perfect incompressible fluid can be obtained from (2.2) on setting $\nu = 0$. When F is a conservative force ($F = -\nabla U$), one has

$$\begin{cases} v_t + v \cdot \nabla v = -\nabla (p + U) \quad \Omega \times \Re^+ \\ \nabla \cdot v = 0 \quad \Omega \times \Re^+ \end{cases} \tag{2.3}$$

with the initial and the boundary condition given by

$$\begin{cases} v(x,0) = v_0(x) \quad \Omega \\ v \cdot n = 0 \quad \partial\Omega \times \Re^+ \end{cases} \tag{2.4}$$

where n denotes the outward unit normal to $\partial\Omega$. Because of the vectorial identities

$$\begin{cases} a \cdot \nabla a = (\nabla \times a) \times a + \dfrac{1}{2} \nabla a^2 \\ \\ \nabla \times (a \times b) = b \cdot \nabla a - a \cdot \nabla b + (\nabla \cdot b) a - (\nabla \cdot a) b \,, \end{cases}$$

one obtains on taking curl of both sides of $(2.3)_1$

$$\Theta_t + v \cdot \nabla \Theta = \Theta \cdot \nabla v \tag{2.5}$$

where $\Theta = \nabla \times v$ is the vorticity vector.

The Euler equations (2.5) for vorticity become simpler in the context of two dimensional motion $v = (v_1, v_2, 0)$, with $v_i = v_i(x_1, x_2, t)$. In fact introducing the stream function $\Psi(x_1, x_2, t)$ and setting

$$v = \nabla^\perp \Psi, \quad \nabla^\perp = \left(\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1}, 0 \right),$$

one immediately obtains

$$\nabla \cdot v = 0, \quad \Theta = -\Delta \Psi e_3, \quad \Theta \cdot \nabla v = 0$$

with e_i unit vector along x_i axes, and (2.5) becomes

$$\frac{\partial \Delta \Psi}{\partial t} = \frac{\partial(\Psi, \Delta\Psi)}{\partial(x_1, x_2)}. \tag{2.6}$$

From $(2.4)_2$ it turns out that Ψ must be a constant on $\partial\Omega$ and therefore, because Ψ is defined modulo a constant, one can append to (2.6) the initial and boundary conditions

$$\begin{cases} \Psi(x_1, x_2, 0) = \varphi(x_1, x_2) & \text{in } \Omega \\ \\ \Psi = c & \text{on } \partial\Omega, \end{cases} \tag{2.7}$$

where φ is prescribed function and c is an arbitrary constant.

Let φ be a steady solution to (2.6), $(2.7)_2$ and let us consider the stability of the basic motion $\varphi_* = \nabla^\perp \varphi$ with respect to planar perturbations $u = \nabla^\perp \Phi(x_1, x_2, t)$. One immediately obtains the following initial boundary value problem

$$\begin{cases} \dfrac{\partial \Delta \Phi}{\partial t} = \dfrac{\partial(\Phi, \Delta\Phi + \Delta\varphi)}{\partial(x_1, x_2)} + \dfrac{\partial(\varphi, \Delta\Phi)}{\partial(x_1, x_2)} \\ \\ \Phi(x_1, x_2, 0) = \Phi_0(x_1, x_2) & \text{in } \Omega \\ \\ \Phi = 0 & \text{on } \partial\Omega \end{cases} \tag{2.8}$$

where Φ_0 is prescribed. Linearizing with respect to Φ, $(2.8)_1$ gives

$$\frac{\partial \Delta\Phi}{\partial t} = \frac{\partial(\Phi, \Delta\varphi)}{\partial(x_1, x_2)} + \frac{\partial(\varphi, \Delta\Phi)}{\partial(x_1, x_2)}\,. \tag{2.9}$$

As basic motion v_*, let us consider the Couette flow in the flat pipe

$$\Omega = \{(x_1, x_2) : x_1 \in \Re, |x_2| \leq 1\}$$

given by

$$v_* = \nabla^\perp \varphi, \quad \varphi = \frac{1}{2}x_2^2\,.$$

Then (2.8) gives

$$\begin{cases} \dfrac{\partial \Delta\Phi}{\partial t} + x_2 \dfrac{\partial(\Delta\Phi)}{\partial x_1} = 0 \\ \\ \Phi(x_1, x_2, 0) = \Phi_0(x_1, x_2) \\ \Phi(x_1, \pm 1, t) = 0 \quad t \in \Re^+ \end{cases}$$

and hence

$$\begin{cases} \dfrac{\partial \omega}{\partial t} + x_2 \dfrac{\partial \omega}{\partial x_1} = 0 \\ \\ \omega(x_1, x_2, 0) = \omega_0(x_1, x_2) \end{cases}$$

where $\omega = -\Delta\Phi$ and $\omega_0 = -\Delta\Phi_0$. Assuming $\omega_0 \in C^1(\Omega)$, immediately one has

$$\omega(x_1, x_2, t) = \omega_0(x_1 - x_2 t, x_2)\,.$$

Since

$$\sup_\Omega |\omega(x_1, x_2, t)| = \sup_\Omega |\omega_0(x_1, x_2)| \quad \forall t \geq 0 \tag{2.10}$$

$$\begin{aligned} \sup_\Omega |\omega_{x_2}| &= \sup_\Omega |-t\omega_{0x_1}(x_1 - x_2 t, x_2) + \omega_{0x_2}(x_1 - x_2 t, x_1)| \\ &\geq t \sup_\Omega |\omega_{0x_1}(x, y)| - \sup_\Omega |\omega_{0x_2}(x, y)| \quad \forall t \end{aligned}$$

one has stability with respect the norm (2.10) but instability with respect the norm $\sup\limits_\Omega |\omega| + \sup\limits_\Omega |\omega_{x_2}|$.

2.5 General estimates based on the first order inequalities

The strategy of qualitative analysis is to obtain estimates and properties of the state vector u of a phenomenon F without solving explicity the P.D.Es. modelling F. In this strategy a central role is played by the inequalities that one is able to obtain from the P.D.E. at hand.

In the present section we introduce a general estimate (Gronwall's Lemma) for U satisfying the first order inequality

$$\dot{U} \leq f(t)U + g(t) \qquad t \geq t_0 \tag{2.11}$$

where f and g are known functions of t.

Subsequently we will concentrate on the simple case $\dot{U} \leq 0$ in order to show, through the Liapunov direct method, how one can obtain much important information on the behaviour of the state vector u and hence on the phenomenon at hand.

The following is the Gronwall's Lemma in differential form.

Lemma 2.5.1. *Let* (2.11) *hold and let* U, $\dot{U}$, f *and* g *belong to* $L^1_{loc}(]t_0, \infty[)$, *i.e. are locally integrable. Then the following estimates holds*

$$U(t) \leq U(t_0) exp\left[\int_{t_0}^{t} f(\tau)d\tau\right] + \int_{t_0}^{t} g(\tau) exp\left[\int_{t_0}^{t} f(s)ds\right] d\tau \quad t \geq t_0\,. \tag{2.12}$$

Proof. Setting

$$\omega(t) = exp\left[-\int_{t_0}^{t} f(\tau)d\tau\right]$$

it follows that

$$\frac{d}{dt}(\omega U) \leq \omega g$$

and hence, integrating on (t_1, t_2) with $t_2 \geq t_1 \geq t_0$, one has

$$\omega(t_2)U(t_2) \leq \omega(t_1)U(t_1) + \int_{t_0}^{t} \omega(\tau)g(\tau)d\tau\,. \tag{2.13}$$

The usual Gronwall estimate (2.12) then immediately follows for $t_1 = t_0$, $t_2 = t$.

Let us remark now that if f, g, U are positive then $\omega(t) \leq 1$, $\forall t \geq t_0$, and therefore (2.13), for $t_1 = t_0$, $t_2 = t + \xi$, $\xi \geq 0$, implies that

$$U(t+\xi) \leq \left[U(t_1) + \int_{t_1}^{t+\xi} g(\tau)d\tau\right] exp\left[\int_{t_1}^{t+\xi} f(\tau)d\tau\right] \tag{2.14}$$

and the following Gronwall's Lemma in integral form holds.

Lemma 2.5.2. *Let the assumptions of Lemma 2.4.1 hold and let U, f and g be nonnegative. If there exist three positive constants α, β, γ such that ($\forall t \geq t_0$)*

$$\int_{t}^{t+\delta} f(\tau)d\tau \leq \beta \qquad \int_{t}^{t+\delta} g(\tau)d\tau \leq \alpha \tag{2.15}$$

then, for $\xi \in]0, \delta[$ and $t \geq t_0$, the following estimate holds

$$U(t+\xi) \leq \left[\frac{1}{\xi}\int_{t}^{t+\xi} U(\tau)d\tau + \alpha\right] e^{\beta}\,. \tag{2.16}$$

Proof. Inequality (2.16) is an immediate consequence of (2.14). In fact for $t_1 \in [t, t+\xi]$, (2.14) gives

$$U(t+\xi) \leq [U(t_1) + \alpha]e^{\beta}$$

and hence, integrating with respect to t_1 on $(t, t+\xi)$, (2.16) follows.

Let us emphasize the importance of (2.16) considering the case

$$f = \lambda = const.(> 0) \quad g = t_0 = 0\,.$$

Then (2.12) gives

$$U(t) \leq U(0)e^{\lambda t}$$

i.e., a bound for U gorwing (exponentially) with t.

Another important inequality for the qualitative analysis of a P.D.Es. model is the following Poincaré inequality [13].

Theorem 2.5.1. *Let Ω be a bounded, connected, open subset of $\Re^n$, with a C' boundary $\partial\Omega$. Assume $1 \leq p \leq \infty$. Then there exists a constant γ, depending only on n, p and Ω, such that*

$$\|u - (u)_\Omega\|_{L^p(\Omega)} \leq \gamma \|\nabla u\|_{L^p(\Omega)} \qquad \forall u \in W^{1,p}(\Omega) \tag{2.17}$$

where $(u)_\Omega = \int_\Omega u d\Omega$ is the average of u over Ω.

2.6 Liapunov functions

In 1893 A.M. Liapunov – in order to establish conditions ensuring stability of solutions of ordinary differential equations (O.D.Es.) – introduced a method which is called the direct or second method. This method – based on knowing the sign of the time derivative, along the solutions, of an auxiliary function, but without any recourse to them – has been recognized to be very general and powerful, and has been used for over 65 years in the qualitative thory of O.D.Es. . The first generalization of the Liapunov direct method to P.D.Es. and, in general, to evolution equations other that O.D.Es., appeared only in the years 1957-59. Our aim is to introduce the fundamental ideas and problems of the Liapunov direct method in the light of its applications to phenomena which are modelled (essentially) by P.D.Es. .

Definition 2.6.1. *Let v be a dynamical system on a metric space X. A functional $V : X \to I\!R$ is a Liapunov function on a subset $I \subset X$ if V is continuous on I, and a nonincreasing function of time along the solutions having the initial data on I.*

In order to ensure that $V[v(x,\cdot)]$ is a nonincreasing function of time, in the sequel we assume that V is differentiable with respect to time and that the derivative is non-positive. However, it is standard in literature to ensure that V is a non-incresing by requiring that the generalized time derivative

$$\overset{\bullet}{V} := \lim_{t\to 0^+} \inf \frac{1}{t}\{V[v(x,t)] - V(x)\}, \qquad x \in I \tag{2.18}$$

(coinciding with the ordinary derivative when V is differentiable) is non-positive.

In the sequel – for some $\alpha \leq \infty$ – we denote by $\Sigma_\alpha \subset X$ a subset of the set

$$\Sigma(X,\alpha) := \{x \in X \ : \ V(x) \leq \alpha\}$$

and by $\Sigma_{(\alpha,\beta)}$ the intersection $\Sigma_\alpha \cap \Sigma(X,\beta)$ for $\beta \leq \alpha$. The following Theorem holds

Theorem 2.6.1. *Let v be a dynamical system on a metric space X and let V be a Liapunov function on Σ_α, having a non-positive time derivative. Then*

i) $\Sigma_{(\alpha,\beta)}$ and $\overline{\Sigma}_{(\alpha,\beta)}$, $\forall\beta \leq \alpha$, are positive invariant.

ii) $V[v(x,\cdot)]$, $\overline{\Sigma}_\alpha$, is a non-increasing function of time.

iii) $V[v(x,\cdot)]$ is differentiable a.e. with

$$V[v(x,t)] \leq V(x) + \int_0^t \overset{\bullet}{V}[v(x,\tau)]d\tau \qquad (x,t) \in \Sigma_\alpha \times \Re^+ \tag{2.19}$$

Theorem 2.6.1 shows that the Liapunov functions can be used to determine some positive invariant sets. This role is important because if a bounded (or precompact) set $S \cap \Omega$ can be shown to be positive invariant, then the positive orbit $\gamma(x)$, given by $x \in S \to \gamma(x) \in S$, is bounded (or precompact). We notice that the Theorem 2.6.1 continues to hold under weaker conditions on V. In fact, instead of the continuity of V, it is enough to require its lower semicontinuity, i.e. to require that the set $\{x \in \Omega, V(x) \leq \alpha\}$ is closed $\forall x \in \Omega, \forall \alpha \in \Re$. However the continuity of V is needed in the Liapunov direct method.

2.7 The Liapunov direct method

The stability of a given motion can be expressed, for a normed linear space X, through the stability of the zero solution of the perturbed dynamical system. For this reason, one can introduce the direct method for investigating the stability of an equilibrium position only. Assuming, for the sake of simplicity, that X is a normed linear space, and denoting by F_r, with $r = const. > 0$, the set of the function $\varphi : [0, r) \to \Re^+$ continuous, strictly increasing and satisfying $\varphi(0) = 0$, then the Liapunov direct method can be summarized by the following two Theorems.

Theorem 2.7.1. *Let u be a dynamical system on X and let O be an equilibrium point. If V is a Liapunov function on the open ball $S(O, r)$, for some $r > 0$, such that:*
i) $V(O) = 0$,
ii) $\exists f \in F_r : V(u) \geq f(\|u\|), \ \forall u \in S(O, r)$,
then O is stable. If, in addition,
iii) $\exists g \in F_r : \overset{\bullet}{V}(u) \leq -g(\|u\|), \ \forall u \in S(O, r)$,
then O is asymptotically stable.

Proof. Let us assume $\epsilon < r$ and introduce

$$\alpha = \inf_{\|u\|=\epsilon} V(u) \geq f(\epsilon) > 0\,, \quad (\epsilon \neq 0)\,.$$

In view of i), ii) and Theorem 2.6.1, it follows that $S(O,\epsilon)$ contains a positive invariant component Σ_α of $\Sigma(X,\alpha)$. The stability is then immediately obtained observing that, by virtue of i) and of the V continuity, there exists $\delta(\epsilon) > 0$ such that $S(O,\delta) \subset \Sigma_\alpha$ and therefore

$$u_0 \in S(O,\delta) \Rightarrow \gamma(u_0) \subset \Sigma_\alpha \subset S(O,\epsilon).$$

Turning now to the asymptotic stability, by (2.19), ii) and iii), it follows that

$$0 \leq f[\|u(u_0,t)\|] \leq V[u(u_0,t)] \leq V(u_0) - \int_0^t g(\|u(u_0,\tau)\|)d\tau \qquad (2.20)$$

$\forall u_0 \in S(0,\delta)$. Because $V[u(u_0,t)] : \Re^+ \to \Re^+$ is a bounded nonincreasing function, then there exists a $\beta \in \Re^+$ such that

$$0 \leq \inf_{t\in\Re^+} V[u(u_0,t)] = \beta \leq V(u_0) \leq \alpha\,.$$

But $\beta > 0$ implies $\gamma(u_0) \cap \Sigma(X,\beta) = \emptyset$ and, by the V continuity the existence of $r^* > 0$ such that $\forall\ t \in \Re^+$

- $\gamma(u_0) \cap S(O,r^*) = \emptyset$
- $\|u(u_0,t)\| > r^*$
- $g(r^*) \leq g(\|u(u_0,t)\|)\,.$

Consequently (2.20) gives

$$0 < V[u(u_0,t)] \leq V(u_0) - \int_0^t g(r^*)d\tau \leq V(u_0) - tg(r^*) < 0, \quad t > \frac{V(u_0)}{g(r^*)}$$

which is impossible. Therefore $\beta = 0$ and the asymptotic stability then follows.

Theorem 2.7.2. *Let u be a dynamical system on $X \times \Re^+$, and let O be an equilibrium point. If V is a Liapunov function on the open set $A_r = S(O,r) \cap \Sigma(X,0)$, for some $r > 0$, and* *i)* $V(O) = 0$,

ii) $\exists g \in F_r : \overset{\bullet}{V}(u) \leq -g[-V(u)], \ \ u \in A_r$,

iii) $A_\epsilon \neq \emptyset, \quad \forall \epsilon > 0$,

then O is unstable.

Proof. Because of i) and the V continuity, there exists $0 < \epsilon < r$ such that

$$u \in S(O, \epsilon) \Rightarrow V(u) > -1\,.$$

The point O cannot be stable for otherwise one could find a $\delta(\epsilon) > 0$ such that

$$u_0 \in S(O, \delta) \Rightarrow \gamma(u_0) \in S(O, \epsilon)$$

and hence, by ii), $u_0 \in A_\delta \Rightarrow \gamma(u_0) \in A_\epsilon$ and $V[u(u_0, t)] \leq V(u_0) < 0$, $\forall t \in \Re^+$. Consequently $g[-V(u)] \geq g[-V(u_0)]$ on $\gamma(u_0)$ and (2.19) gives

$$-1 < V[u(u_0, t)] \leq V(u_0) - \int_0^t g[-V(u_0)]d\tau \leq V(u_0) - tg[-V(u_0)] < -1$$

for $t > \dfrac{V(u_0)}{g[-V(u_0)]}$, which is impossible. Therefore O is unstable.

2.8 Liapunov functions for some reaction-diffusion systems

In the present section, on considering the initial boundary value problem

$$\begin{cases} u_t = u_{xx} + F(x, t, u, u_x, u_{xx}) & x \in (0, 1), t > 0 \\ u(0, t) = u(1, t) = 0 & \forall t \geq 0 \\ u(x, 0) = u_0(x) & x \in [0, 1]\ , \end{cases} \tag{2.21}$$

under the assumption that

$$u = 0 \Rightarrow F = 0,$$

we will recall some suitable Liapunov functionals for different type of reaction term.

i) *The diffusion equation (i.e. $F = 0$).* In this case, the following functionals

$$\begin{cases} U = \dfrac{1}{2}\left\|u(t)\right\|^2 = \dfrac{1}{2}\displaystyle\int_0^1 u^2(x,t)dx \\ V = \dfrac{1}{2}\left\|u_x(t)\right\|^2 = \dfrac{1}{2}\displaystyle\int_0^1 u_x^2(x,t)dx \\ W = U + V \end{cases} \tag{2.22}$$

are Liapunov functionals. In fact, along the solutions of (2.21), by virtue of the Poincaré inequality, it turns out that

$$\begin{gathered} \dot{U} = \int_0^1 uu_t dx = \int_0^1 uu_{xx} dx = -\int_0^1 u_x^2 dx \le -\frac{\gamma}{2}U \\ \dot{V} = \int_0^1 u_x u_{xt} dx = -\int_0^1 u_{xx}^2 dx \le -\frac{\gamma}{2}V \\ \dot{W} = -\int_0^1 (u_x^2 + u_{xx}^2 dx \le -\frac{\gamma}{2}W, \end{gathered}$$

with $\gamma = const. > 0$. Hence the solution $u \equiv 0$ is asymptotically exponentially stable according to

$$\begin{cases} U \le U_0 e^{-\gamma t} \\ V \le V_0 e^{-\gamma t} \\ W \le W_0 e^{-\gamma t}\,. \end{cases}$$

ii) *A nonhomogeneous linear case.* If

$$F = (g(x)u)_x$$

with $g \in C^1([0,1])$, on choosing

$$V = \frac{1}{2}\int_0^1 e^{\varphi(x)} u^2 dx, \quad \varphi(x) = \int_0^x g(\xi)d\xi$$

along the solutions of (2.21), by virtue of

$$\inf_{x\in[0,1]} \varphi \equiv \alpha \le \varphi(x) \le \beta \equiv \sup_{x\in[0,1]} \varphi$$

it turns out that

$$\dot{V} = \int_0^1 e^{\varphi} u u_t dx = \int_0^1 e^{\varphi} u[u_{xx} + (gu)_x] dx = -\int_0^1 e^{\varphi}(gu + u_x)^2 dx$$

$$= -\int_0^1 e^{-\varphi}[(e^{\varphi}u)_x]^2 dx \leq -e^{-\beta} \int_0^1 [(e^{\varphi}u)_x]^2 dx$$

$$\leq -\frac{e^{-\beta}\gamma}{2} \int_0^1 e^{2\varphi} u^2 dx \leq -\frac{e^{\alpha-\beta}\gamma}{2} \int_0^1 e^{\varphi} u^2 dx \leq cV$$

where $c = -\dfrac{e^{\alpha-\beta}\gamma}{2}$. Hence one recover the asymptotic exponential stability of $u \equiv 0$ with respect to the V-norm.

iii) *The nonlinear case* $F = g(u_x)u_{xx}$. In this case, equation $(2.21)_1$ becomes

$$u_t = [1 + g(u_x)]u_{xx}$$

which is a diffusion equation with the nonconstant diffusion coefficient given by

$$k = 1 + g(u_x).$$

According to the physical meaning of k, it is natural to require that

$$1 + g(u_x) \geq 0.$$

Choosing the function $(2.22)_2$ along the solution of (2.21), it turns out that

$$\dot{V} = \int_0^1 (1 + g)u_{xx}^2 dx.$$

Assuming

$$\exists \epsilon = const. > 0 : g(\xi) > \epsilon - 1 \qquad \forall \xi \in \Re$$

it follows that

$$\dot{V} \leq -\epsilon \int_0^1 u_{xx}^2 dx \leq -\epsilon\gamma V$$

and hence the asymptotic stability of the zero solution is recovered.

iv) *Diffusion with variable conductivity.* Let us consider the initial boundary value problem

$$\begin{cases} u_t = \dfrac{\partial}{\partial x}[(k+g(u))u_x] & x \in (0,1), t>0 \\ u(0,t) = u(1,t) = 0 & \forall t \geq 0 \\ u(x,0) = u_0(x) & x \in [0,1] \end{cases} \tag{2.23}$$

with $u = u(x,t)$, $k = const. > 0$ and $g \in C^1(\Re)$. This problem arises in the heat diffusion phenomena, when thermal conductivity depends on the temperature. This happens, for instance, in the "cold ice" of glaciers. $(2.23)_1$, more generally, can be written

$$u_t = \Delta\varphi(u) \tag{2.24}$$

with $\varphi(u) = \int_0^1 [k+g(s)]ds$.

Equation (2.24) models many other phenomena like diffusion of biological populations, diffusion of fluids through porous media and heat diffusion in the Stefan problem. Setting

$$F(u) = \int_0^u g(\xi)d\xi$$

it follows that

$$u = 0 \Rightarrow F = 0$$

and $(2.23)_1$ becomes

$$u_t = u_{xx} + \frac{\partial^2}{\partial x^2}F(u).$$

Choosing the function $(2.22)_1$ as Liapunov function, it turns out that

$$\dot{U} = -k\int_0^1 u_x^2 dx - \int_0^1 u_x \frac{\partial F}{\partial x}dx = -\int_0^1 (k+g)u_x^2 dx.$$

Hence, assuming

$$\exists \epsilon = const. > 0 : k + g(\xi) \geq \epsilon \quad \forall \xi \in \Re$$

it follows that

$$\dot{U} = -\epsilon \int_0^1 u_x^2 dx \leq -\epsilon\gamma U, \quad \gamma = const, > 0$$

i.e. the asymptotic exponential stability of the zero solution of (2.21).

v) *The case* $u_t = \Delta F(u)$. Let $\Omega \subset \Re^3$ be a sufficiently smooth bounded domain. Let us consider

$$\begin{cases} u_t = \Delta F(u) & \Omega \times \Re^+ \\ u(x,0) = u_0(x) & \Omega \times \{0\} \\ u(x,t) = u_1(x) & \partial\Omega \times \Re^+ \end{cases} \tag{2.25}$$

where $F \in C^2(\Omega)$, $u_0 \in C(\overline{\Omega})$ and $u_1 \in C(\partial\Omega)$ are assigned functions. Some interesting results, relating to the asymptotic behaviour of solutions of (2.25), have been obtained in [12], [28]. For the sake of completness here we recall the following ones. Let us consider the steady boundary value problem

$$\begin{cases} \Delta F(U) = 0 & \Omega \times \Re^+ \\ U = u_1(x) & \partial\Omega \times \Re^+ \end{cases} \tag{2.26}$$

and let us put $u = U + v$, it follows that

$$\begin{cases} v_t = \Delta L & \Omega \times \Re^+ \\ v(x,0) = v_0(x) & \Omega \times \{0\} \\ v(x,t) = 0 & \partial\Omega \times \Re^+ . \end{cases}$$

The approach is based on the introduction of the following peculiar Liapunov functional

$$V(t) = \int_\Omega G(U,v) d\Omega$$

with

$$G(U,v) = \int_0^v L(U,\overline{v}) d\overline{v}. \tag{2.27}$$

For $u \in \Re$, let

$$F'(u) \geq m \tag{2.28}$$

where m is a positive constant.

Lemma 2.8.1. *Supposing that $v \in \Re$, then*

1. $G(U,0) = \left[\dfrac{\partial G}{\partial v}\right]_{v=0} = 0$

2. $\dfrac{\partial^2 G}{\partial v^2} > 0$

3. $G(U,v) \geq \dfrac{1}{2}mv^2$

4. $\left(\dfrac{\partial G}{\partial v}\right)^2 = L^2(U,v) \geq 2mG(U,v).$

The following theorem holds.

Theorem 2.8.1. *Let* (2.28) *hold and let* (2.26) *be solvable. Then U is asymptotically exponentially stable in the L^2-norm and is the asymptotic state of any solution of the initial boundary value problem* (2.25) *in this norm.*

Let us introduce a second type of Liapunov functional as follows

$$V_n(t) = \int_\Omega G_n(U,v)d\Omega$$

Let us now consider the following Lemma

Lemma 2.8.2. *Let $v \in \Re$ and let* (2.27) *hold. On setting*

$$G_n(U,v) = \int_0^v L^{2n+1}(U,\overline{v}d\overline{v}, \quad G_0 = G$$

(n being a positive integer or zero), it follows that

1. $0 \leq G_n < vL^{2n+1}$

2. $G_n \geq \dfrac{m^{2n+1}}{2(n+1)} v^{2(n+1)}$

3. $L^{2n+2} \geq m G_n$.

Introducing a second type of Liapunov function, in [12], Flavin and Rionero obtain the following two generalizations of Theorem 3.5.1 .

Theorem 2.8.2. *Let* (2.28) *hold and let the boundary value problem* (2.26) *be solvable. Then* U *is asymptotically exponentially stable in the* L^{2n}*-norm,* $n \in \Re^+$*, and is the asymptotic state of each solution of* (2.25) *in the* L^{2n}*-norm as* $t \to \infty$*.*

Theorem 2.8.3. *Let* (2.28) *hold and let the boundary value problem* (2.26) *be solvable.*
a) Then the steady state U *is stable in the* L^∞*-norm.*
b) If $F' < M$*, with* M *positive constant, then the steady state* U *is exponentially asymptotically stable in the* L^∞*-norm, and it is the asymptotic state in this norm.*

We notice that the Poincaré inequality holds also on noncompact domains, bounded at least in one direction. Therefore Theorems 2.8.1 - 2.8.3 continue to hold also for these domains, at least with respect to perturbations spatially periodic in the directions in which the domains are unbounded.

2.9 Linear stability

Let H be a Hilbert space endowed with a scalar product $\langle \cdot, \cdot \rangle$ and associated norm $|\cdot|$. We consider in H the following perturbed dynamical system $u(u_0, t)$

$$\begin{cases} u_t + Lu + Nu = 0 \\ \\ u(x, 0) = u_0(x), \end{cases} \tag{2.29}$$

where $u(u_0, t)$ has been obtained on perturbing at the initial time the basic motion $v(v_0, t)$. In (2.29) we assume that L is a linear operator (possibly unbounded), and N is a nonlinear operator with $N(0) = 0$ in order to ensure that (2.29) admits the null solution.

On linearizing system (2.29), one studies the linear stability of the basic motion $v(v_0, t)$ or equivalently the linear stability of the null solution of equation $(2.29)_1$.

Hence, the linear stability is governed by the system

$$\begin{cases} u_t + Lu = 0 \\ \\ u(x, 0) = u_0(x)\,. \end{cases} \tag{2.30}$$

Let us assume that

(i) L is an autonomous, densely defined and closed operator such that for $\lambda \in C$, $(L - \lambda I)^{-1}$ is compact (where I is the identity operator in H), i.e. L is an operator with compact resolvent. In such hypotheses the following theorem holds true.

Theorem 2.9.1. *The spectrum of the operator L consists entirely of an at most denumerable number of eigenvalues $\{\sigma_n\}_{n \in N}$ with finite (both algebraic and geometric) multiplicites and, moreover, such eigenvalues can cluster only at infinity.*

Generally the linear operator L is not symmetric. For this reason, on looking for solutions of (2.29) of the type $u = \phi(x)e^{-\sigma t}$, the eigenvalues σ satisfying the equation

$$L\phi = \sigma\phi \tag{2.31}$$

are not necessarily real eigenvalues, but they can be ordered in the following way

$$re(\sigma_1) \leq re(\sigma_2) \leq ... \leq re(\sigma_n) \leq ... .$$

On indicating Σ the set of the eigenvalues that are solutions of the equation (2.31), in the linear stability theory one introduces the following definition

Definition 2.9.1. *The null solution of system (2.29) is linearly stable if*

$$re(\sigma_1) > 0 . \tag{2.32}$$

The null solution of system (2.29) is linearly unstable if it is not stable.

2.10 Connection between linear and nonlinear stability

Now we are interested to the connections between the linear and nonlinear stability of an assigned motion. To this end, let us decompose the linear operator L as follows

$$L = L_1 + L_2 \tag{2.33}$$

with: i)$D(L_2) \supset D(L_1) = D(L)$ being $D(\cdot)$ the domain of the associated operator; ii) L_1 symmetric operator with compact resolvent; iii) L_2 skew-symmetric and bounded operator in H^*, being H^* a compact space embedded in H. Hence being L_1 a symetric operator, the eigenvalues $\{\lambda_i\}_{i\in I\!N}$ they can be ordered as follows:

$$\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \leq ...$$

Let us reconsider the problem (2.29), under the hypothesis that the nonlinear operator $N(u)$ is such that $(N(u),\, u) \geq 0$. In such hypothesis, on mulytiplying both the sides of the equation $(2.29)_1$ by u, we obtain the following inequality

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \langle Lu,\, u\rangle \leq 0$$

from which, by virtue of (2.33), one obtains

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \frac{L_1[\phi,\, \phi]}{\|u\|^2}\,\|u\|^2 \leq 0 \tag{2.34}$$

where $L_1[\phi,\, \phi]$ is the bilinear form associated to the operator L_1. Let us suppose that the bilinear form associated to L_1 is defined and bounded on H^* and let us set ϕ the eigenfunction associated to the eigenvalue λ_1. Then

$$\lambda_1 = \min_{\phi\in H_*}\frac{L_1[\phi,\phi]}{\|\phi\|^2}\,. \tag{2.35}$$

Hence from (2.34), by virtue of (2.35), one obtains

$$\|u(t)\|^2 \leq \|u_0\|^2\, e^{-2\lambda_1 t}\,. \tag{2.36}$$

Hence, if one supose that $\lambda_1 > 0$ (linear stability), the null solution of (2.29) is unconditional nonlinearly stable with respect to the H-norm. From the above considerations, it follows that, while the linear stability problem is linked to the study of the eigenvalues associated to the operator L, if $\langle N(u),\, u\rangle \geq 0$, the nonlinear stability involves the study of the eigenvalues of the symmetric part L_1 of L only. Hence, if for example $L_2 = 0$, since the two eigenvalue problems coincide, the linear stability and nonlinear stability thresholds coincide.

Chapter 3

On the Stability of the Solution of a Nonlinear Binary Reaction Diffusion System of P.D.Es.

3.1 Preliminaries

In this chapter, our aim is to recall some fundamental results due to Rionero (cfr. [30], [31]) concerning the nonlinear stability of a biologically meaningfull solution of a binary reaction-diffusion model.

Let $\Omega \subset \Re^3$ be a bounded smooth domain. The nonlinear stability analysis of an equilibrium state in Ω of two substances, for example, diffusing in Ω, can be traced back to the nonlinear stability analysis of the zero solutions of a dimensionless binary system of P.D.Es. like

$$\begin{cases} \overline{u}_t = a_1(x)\overline{u} - a_2(x)\overline{v} + \gamma_1 \Delta \overline{u} + f(\overline{u}, \overline{v}) \\ \\ \overline{v}_t = a_3(x)\overline{u} + a_4(x)\overline{v} + \gamma_2 \Delta \overline{v} + g(\overline{u}, \overline{v}) \\ \end{cases} \tag{3.1}$$

where $a_i : x \in \Omega \to a_i(x) \in \Re$, $a_i \in C(\Omega)$ $(1 \le i \le 4)$; γ_i $(i = 1, 2)$ positive constants; $\overline{u}, \overline{v}$ perturbations (of finite amplitude) to the equilibrium concentrations of the substances and f, g nonlinear smooth functions of $\overline{u}$ and $\overline{v}$ verifying the conditions

$$f(0,0) = g(0,0) = 0.$$

To (3.1) we append the Dirichlet boundary conditons

$$\overline{u} = \overline{v} = 0 \qquad on \quad \partial\Omega \times \Re^+ \tag{3.2}$$

or the Neumann boundary conditions (n being the unit outward normal to $\partial\Omega$)

$$\frac{d\overline{u}}{dn} = \frac{d\overline{v}}{dn} = 0 \qquad on \quad \partial\Omega \times \Re^+ \tag{3.3}$$

with the additional conditions

$$\int_\Omega \overline{u} d\Omega = \int_\Omega \overline{v} d\Omega = 0 \qquad \forall t \in \Re^+, \tag{3.4}$$

in the case (3.3).

We denote by

$< \cdot, \cdot >$ the scalar product in $L^2(\Omega)$;

$< \cdot, \cdot >|_{\overline{\Omega}}$ the scalar product in $L^2(\overline{\Omega})$, with $\overline{\Omega} \subset \Omega$;

$\|\cdot\|$ the $L^2(\Omega)$-norm;

$\|\cdot\|_{\overline{\Omega}}$ the $L^2(\overline{\Omega})$-norm, with $\overline{\Omega} \subset \Omega$;

$H_0^1(\Omega)$ the Sobolev space such that

$$\varphi \in H_0^1(\Omega) \to \left\{\varphi^2 + (\nabla\varphi)^2 \in L^2(\Omega),\ \varphi = 0\ on\ \partial\Omega\right\};$$

$H_*^1(\Omega)$ the Sobolev space such that

$$\varphi \in H_*^1(\Omega) \to \left\{\varphi^2 + (\nabla\varphi)^2 \in L^2(\Omega),\ \frac{d\varphi}{dn} = 0\ on\ \partial\Omega,\ \int_\Omega \varphi d\Omega = 0\right\}$$

and study the stability of ($\overline{u}_* = \overline{v}_* = 0$) in the $L^2(\Omega)$-norm with respect to the perturbations $(\overline{u}, \overline{v})$ belonging, $\forall t \in \Re^+$, to $[H_0^1(\Omega)]^2$ in the case (**??**) and to $[H_*^1(\Omega)]^2$ in the case (3.3) – (3.4).

We also assume that

$$|< \overline{u}, f >| + |< \overline{v}, g >| = o\left(\|\overline{u}\|^2 + \|\overline{v}\|^2\right) \tag{3.5}$$

which is equivalent to require

$$|< \overline{u}, f >| + |< \overline{v}, g >| \leq C\left(\|\overline{u}\| + \|\overline{v}\|\right)\left(\|\nabla\overline{u}\|^2 + \|\nabla\overline{v}\|^2\right)$$

with C positive constant.

To (3.1) we associate the binary linear system of O.D.Es.

$$\begin{cases} \dfrac{d\xi}{dt} = b_1(x)\xi - b_2(x)\eta \\ \\ \dfrac{d\eta}{dt} = b_3(x)\xi + b_4(x)\eta \end{cases} \tag{3.6}$$

with

$$\begin{cases} b_1 = a_1(x) - \gamma_1\overline{\alpha} \qquad b_2 = a_2(x) \\ \\ b_4 = a_4(x) - \gamma_2\overline{\alpha} \qquad b_3 = a_3(x) \\ \\ \end{cases} \tag{3.7}$$

$\overline{\alpha}$ being the positive constant appearing in the Poincaré - Wirtinger inequality

$$\|\nabla\phi\|^2 \geq \overline{\alpha}\,\|\phi\|^2 \tag{3.8}$$

holding both in the spaces $H_0^1(\Omega)$ and $H_*^1(\Omega)$.

As it is well known, $\overline{\alpha} = \overline{\alpha}(\Omega) > 0$ is the lowest eigenvalue λ of

$$\Delta\phi + \lambda\phi = 0$$

respectively in $H_0^1(\Omega)$ and $H_*^1(\Omega)$ (i.e. the principal eigenvalue of $-\Delta$).

Let us notice that the eigenvalues of (3.6) are given by

$$\lambda = \frac{I \pm \sqrt{I^2 - 4A}}{2} \tag{3.9}$$

with

$$\begin{cases} I = b_1 + b_4 \\ \\ A = b_1 b_4 + b_2 b_3 \\ \end{cases}$$

hence the conditions

$$\begin{cases} I < 0 \\ A > 0 \end{cases} \tag{3.10}$$

guarantee the stability of ($\xi_* = \eta_* = 0$), while the instability is guaranteed by

$$\begin{cases} I > 0 \\ A < 0 \end{cases} \tag{3.11}$$

or by

$$A < 0. \tag{3.12}$$

Our aim is to show that the stability (instability) of the critical point ($\xi_* = \eta_* = 0$) of (3.6) implies the stability (instability) of the critical point ($\overline{u}_* = \overline{v}_* = 0$) of (3.1).

3.2 Some peculiar Liapunov functionals

Setting

$$\overline{u} = \alpha u, \qquad \overline{v} = \beta v$$

with α and β suitable constants to be chosen appropriately later, (3.1) can be written as follows

$$\begin{cases} u_t = a_1 u - \dfrac{\beta}{\alpha} a_2 v + \gamma_1 \Delta u + \overline{f} \\ \\ v_t = \dfrac{\alpha}{\beta} a_3 u + a_4 v + \gamma_2 \Delta v + \overline{g} \end{cases} \tag{3.13}$$

with

$$\overline{f} = \frac{1}{\alpha} f \Big|_{(u=\alpha\overline{u}, v=\beta\overline{v})} \qquad \overline{g} = \frac{1}{\beta} g \Big|_{(u=\alpha\overline{u}, v=\beta\overline{v})}.$$

Setting

$$\begin{cases} f^* = \gamma_1(\Delta u + \overline{\alpha} u) \\ \\ g^* = \gamma_2(\Delta v + \overline{\alpha} v) \\ \end{cases}$$

by virtue of (3.7) it follows that

$$\begin{cases} u_t = b_1 u - \dfrac{\beta}{\alpha} b_2 v + f^* + \overline{f} \\ \\ v_t = \dfrac{\alpha}{\beta} b_3 u + b_4 v + g^* + \overline{g} \end{cases} \tag{3.14}$$

under the boundary conditions

$$u = v = 0 \quad \textit{on } \partial\Omega, \quad \forall t \geq 0.$$

Let us consider the following Liapunov functional

$$V = \frac{1}{2} \left[A \left(\|u\|^2 + \|v\|^2 \right) + \left\| b_1 v - \frac{\alpha}{\beta} b_3 u \right\|^2 + \left\| \frac{\beta}{\alpha} b_2 v + b_4 u \right\|^2 \right] \tag{3.15}$$

which is very peculiar for the problem at hand. Infact, along the solutions of (3.14) it turns out that

$$\frac{dV}{dt} = AI\left(\|u\|^2 + \|v\|^2\right) + \Psi^* + \Psi$$

with

$$\begin{cases} \Psi^* = < \alpha_1 u - \alpha_3 v, f^* > + < \alpha_2 v - \alpha_3 u, g^* > \\ \\ \Psi = < \alpha_1 u - \alpha_3 v, \overline{f} > + < \alpha_2 v - \alpha_3 u, \overline{g} > \\ \\ \alpha_1 = A + \dfrac{\alpha^2}{\beta^2} b_3^2 + b_4^2, \quad \alpha_2 = A + b_1^2 + \dfrac{\beta^2}{\alpha^2} b_2^2, \quad \alpha_3 = \dfrac{\alpha}{\beta} b_1 b_3 - \dfrac{\beta}{\alpha} b_2 b_4 \end{cases}$$

and hence the eigenvalues given by (3.9) influence $\frac{dV}{dt}$ in a simple direct way through the product AI.

We notice that, setting

$$f_1^* = -\frac{\beta}{\alpha} b_2 v + f^*, \qquad g_1^* = -\frac{\alpha}{\beta} b_3 u + g^*$$

(3.14) becomes

$$\begin{cases} u_t = b_1 u + f_1^* + \overline{f} \\ \\ v_t = b_4 v + g_1^* + \overline{g}\,. \end{cases} \tag{3.16}$$

Introducing for (3.16) the functional $\widehat{V}$ analogous to (3.15)

$$\widehat{V} = \frac{1}{2}\left[b_1 b_4 \left(\|u\|^2 + \|v\|^2\right) + b_1^2 \|v\|^2 + b_4^2 \|u\|^2\right] \tag{3.17}$$

it follows that, along (3.14)

$$\frac{d\widehat{V}}{dt} = b_1 b_4 (b_1 + b_4) \left(\|u\|^2 + \|v\|^2\right) + \widehat{\Psi}^* + \widehat{\Psi}$$

with

$$\begin{cases} \widehat{\Psi}^* =< \widehat{\alpha}_1 u, f_1^* > + < \widehat{\alpha}_2 v, g_1^* > \\ \\ \widehat{\Psi} =< \widehat{\alpha}_1 u, \overline{f} > + < \widehat{\alpha}_2 v, \overline{g} > \\ \\ \widehat{\alpha}_1 = b_4(b_1+b_4), \qquad \widehat{\alpha}_2 = b_1(b_1+b_4), \qquad \widehat{\alpha}_3 = 0\,. \end{cases}$$

Remark 3.2.1. *Let us notice that, by virtue of* (3.15), $A > 0$ *implies that* V *is a positive definite functional of* (u, v). *Further,* V *denotes a norm equivalent to the* $L^2(\Omega)$*-norm in the sense that there exist two positive constants* k_1, k_2 *such that*

$$k_1(\|u\|^2 + \|v\|^2) \le V \le k_2(\|u\|^2 + \|v\|^2). \tag{3.18}$$

infact, on choosing

$$k_1 = \frac{A}{2}, \qquad k_2 = \max\left\{A, e\left(b_1^2 + \frac{\alpha^2}{\beta^2}\right), e\left(\frac{\beta^2}{\alpha^2}b_2^2 + b_4^2\right)\right\}$$

by virtue of (3.15), (3.18) *immediately follows.*
By virtue of (3.17), $b_1 b_4 > 0$ *implies that* $\widehat{V}$ *is a positive definite functional of* (u, v). *Further it turns out that*

$$k_3(\|u\|^2 + \|v\|^2) \le \widehat{V} \le k_4(\|u\|^2 + \|v\|^2)$$

with

$$k_3 = \frac{1}{2}b_1 b_4, \qquad k_4 = (b_1 + b_4)^2.$$

If we append to (3.1) the Neumann boundary conditions (3.3), if the eigenvalues $\lambda_1 < \lambda_2$ given by (3.9) are real numbers and $b_3 \neq 0$, we may choose $\alpha = \beta$ and introduce the functional

$$W = \frac{1}{2}\left[\|(b_1 - \lambda_1)v - b_3 u\|^2 + \mu\,\|(b_1 - \lambda_2)v - b_3 u\|^2\right]$$

with μ positive parameter to be choose suitably. Then along the solution of (3.14)-(3.3) it follows that

$$\frac{dW}{dt} \leq \delta W + F^* + \mu G^*$$

with

$$\begin{cases} \delta = 2\max(\lambda_1, \mu\lambda_2) \\ \\ F^* =< F, U >, \quad G^* =< G, V > \\ \\ U = (b_1 - \lambda_1)v - b_3 u, \quad V = (b_1 - \lambda_2)v - b_3 u\,. \\ \end{cases}$$

3.3 Nonlinear stability

The following theorem holds.

Theorem 3.3.1. *Let* (3.5) *and* (3.10) *hold. Then* ($\overline{u}_* = \overline{v}_* = 0$) *is nonlinearly asymptotically stable with respect to the* $L^2(\Omega)$*-norm.*

Proof. For any constant $\overline{\epsilon}$ such that

$$0 < \overline{\epsilon} < \inf\left(\frac{|I|}{2\overline{\alpha}}, \frac{A}{\overline{\alpha}|I|}\right),$$

setting

$$\overline{b}_i = b_i + \overline{\alpha\epsilon}, \quad (i = 1, 4) \tag{3.19}$$

it easily turns out that

$$\begin{cases} \overline{I} = \overline{b}_1 + \overline{b}_4 < 0 \\ \\ \overline{A} = \overline{b}_1\overline{b}_4 + \overline{b}_2\overline{b}_3 > 0\,. \\ \end{cases}$$

By virtue of (3.13) and (3.19), we obtain

$$\begin{cases} u_t = \bar{b}_1 u - \dfrac{\beta}{\alpha} b_2 v + \overline{f}^* + \overline{f} \\ \\ v_t = \dfrac{\alpha}{\beta} b_3 u + \bar{b}_4 v + \overline{g}^* + \overline{g} \end{cases} \tag{3.20}$$

with

$$\begin{cases} \overline{f}^* = \overline{\gamma}_1(\Delta u + \overline{\alpha} u) + \overline{\epsilon}\Delta u \\ \\ \overline{g}^* = \overline{\gamma}_2(\Delta v + \overline{\alpha} v) + \overline{\epsilon}\Delta v \, . \\ \end{cases} \tag{3.21}$$

Then, using the substitution

$$\begin{pmatrix} \bar{b}_1 & \bar{b}_4 & \overline{f}^* & \overline{g}^* \\ b_1 & b_4 & f^* & g^* \end{pmatrix}$$

we obtain that along the solutions of (3.20) it turns out that

$$\frac{d\overline{V}}{dt} = \overline{AI}\left(\|u\|^2 + \|v\|^2\right) + \overline{\Psi}^* + \overline{\Psi}$$

with

$$\overline{V} = \frac{1}{2}\left[\overline{A}\left(\|u\|^2 + \|v\|^2\right) + \left\|\bar{b}_1 v - \frac{\alpha}{\beta} b_3 u\right\|^2 + \left\|\frac{\beta}{\alpha} b_2 v - \bar{b}_4 u\right\|^2\right]$$

and

$$\begin{cases} \overline{\Psi}^* =< \overline{\alpha}_1 u - \overline{\alpha}_3 v, \overline{f}^* > + < \overline{\alpha}_2 v - \overline{\alpha}_3 u, \overline{g}^* > \\ \\ \overline{\Psi} =< \overline{\alpha}_1 u - \overline{\alpha}_3 v, \overline{f} > + < \overline{\alpha}_2 v - \overline{\alpha}_3 u, \overline{g} > \\ \\ \overline{\alpha}_1 = \overline{A} + \dfrac{\alpha^2}{\beta^2} b_3^2 + \bar{b}_4^2, \quad \overline{\alpha}_2 = \overline{A} + \bar{b}_1^2 + \dfrac{\beta^2}{\alpha^2} b_2^2, \quad \overline{\alpha}_3 = \dfrac{\alpha}{\beta} \bar{b}_1 b_3 - \dfrac{\beta}{\alpha} b_2 \bar{b}_4 \end{cases} . \tag{3.22}$$

Choosing

$$\alpha = \sqrt{\frac{b_2\bar{b}_4}{\bar{b}_1 b_3}}, \qquad \beta = 1$$

it follows that $\overline{\alpha}_3 = 0$, and by virtue of (3.22), we obtain

$$\overline{\Psi}^* = \overline{\alpha}_1 < u, \overline{f}^* > + \overline{\alpha}_2 < v, \overline{g}^* > =$$

$$\overline{\alpha}_1\overline{\gamma}_1 \left(- \|\nabla u\|^2 + \overline{\alpha}\, \|u\|^2\right) + \overline{\alpha}_2\overline{\gamma}_2 \left(- \|\nabla v\|^2 + \overline{\alpha}\, \|v\|^2\right) - \overline{\epsilon} \left(\overline{\alpha}_1 \|\nabla u\|^2 + \overline{\alpha}_2 \|\nabla v\|^2\right)$$

i.e.

$$\overline{\Psi}^* \leq -k_* \left(\|\nabla u\|^2 + \|\nabla v\|^2\right)$$

with

$$0 < k_* = \overline{\epsilon} \inf \left\{\overline{\alpha}_1, \overline{\alpha}_2\right\} .$$

On the other hand, from (3.5) it follows

$$\overline{\Psi} \leq \frac{\overline{\alpha}_1}{\alpha^2} < \overline{u}, f > + \overline{\alpha}_2 < \overline{v}, g >$$

$$\leq k \left(\frac{\overline{\alpha}_1}{\alpha^2} + \overline{\alpha}_2\right) \left(\alpha^2 + 1\right)^\epsilon \left(\|\overline{u}\|^2 + \|\overline{v}\|^2\right)^\epsilon \left(\|\nabla u\|^2 + \|\nabla v\|^2\right).$$

Therefore we obtain

$$\frac{d\overline{V}}{dt} \leq -\frac{\overline{A}|\overline{I}|}{\overline{k}_2}\overline{V} - \left(k_* - \frac{\widetilde{k}}{\overline{k}_1^\epsilon}\overline{V}^\epsilon\right) \left(\|\nabla u\|^2 + \|\nabla v\|^2\right)$$

with

$$\overline{k}_1 = \frac{\overline{A}}{2}, \qquad \overline{k}_2 = \max\left\{\overline{A}, 2(\overline{b}_1^2 + \alpha^2 b_3^2), 2\left(\frac{1}{\alpha^2} b_2^2 + \overline{b}_4^2\right)\right\}.$$

By recursive arguments, one obtains that

$$\overline{V}_0^\epsilon < \frac{k_*\overline{k}_1^\epsilon}{\widetilde{k}}$$

implies

$$\frac{d\overline{V}}{dt} \leq 0 \qquad \forall t \geq 0$$

and in view of the Poincaré inequality, setting

$$0 < \delta = \frac{1}{\overline{k}_2}\left[\overline{A}|\overline{I}| + \overline{\alpha}\left(k_* - \frac{\widetilde{k}}{\overline{k}_1^{\epsilon}}V_0^{\epsilon}\right)\right]$$

it easily follows

$$\frac{d\overline{V}}{dt} \leq -\delta\overline{V}$$

i.e.

$$\overline{V} \leq \overline{V}_0 e^{-\delta t}.$$

3.4 Instability

The following theorem holds.

Theorem 3.4.1. *Let* (3.5) *and* (3.11) *or* (3.5) *and* (3.12) *hold . Then* ($\overline{u}_* = \overline{v}_* = 0$) *is unstable with respect to the* $L^2(\Omega)$*-norm.*

Proof. By definition, the instability is guaranteed by the existence of at least one destabilizing admissible perturbation. The optimum is when the destabilizing perturbations are dynamically admissible.

In view of (3.14) with $\alpha = \beta = 1$, the L^2-energy system

$$\begin{cases} \dfrac{1}{2}\dfrac{d}{dt}\|u\|^2 = < u, b_1 u - b_2 v > + < u, f^* + f > \\ \\ \dfrac{1}{2}\dfrac{d}{dt}\|v\|^2 = < v, b_3 u + b_4 v > + < v, g^* + g > \end{cases} \tag{3.23}$$

easily follows.

Let us look for solutions of (3.23) having the multiplicative form

$$u = p = X(t)\varphi \qquad v = q = Y(t)\varphi \tag{3.24}$$

with φ principal eigenfunction of $-\Delta$ in $H_0^1(\Omega)$.

Then (3.24) imply

$$\begin{cases} \Delta p + \overline{\alpha} p = \Delta q + \overline{\alpha} q = f^*(p) = g^*(q) = 0 \\ \\ \|\nabla p\|^2 = \overline{\alpha}\, \|p\|^2\,, \qquad \|\nabla q\|^2 = \overline{\alpha}\, \|q\|^2 \end{cases}$$

and any non zero solution of

$$\begin{cases} \dfrac{dX}{dt} = b_1 X - b_2 Y + F(X,Y) \\ \\ \dfrac{dY}{dt} = b_3 X + b_4 Y + G(X,Y) \end{cases} \tag{3.25}$$

with

$$F(X,Y) = \frac{1}{\|\varphi\|^2} < \varphi, f(\varphi X, \varphi Y) >, \qquad G(X,Y) = \frac{1}{\|\varphi\|^2} < \varphi, g(\varphi X, \varphi Y) >$$

nonlinear smooth functions of X, Y such that

$$F(0,0) = G(0,0) = 0\,,$$

is a solution of (3.23).

The global existence of the multiplicative solutions (3.24) of (3.23) is guaranteed by the global existence of the solutions of the binary system of O.D.Es. (3.25), and the instability of the null solution $X^* = Y^* = 0$ of (3.25) implies the instability of the null solution ($\overline{u}_* = \overline{v}_* = 0$) of (3.1).

The linear version of (3.25) coincide with (3.6) hence its eigenvalues are given by (3.9). Both in the cases (3.11) and (3.12), at least one of the eigenvalues is real positive or complex with positive real part. Although in these circumstances it is well known that, in the case at hands, the null solution of (3.25) is nonlinearly unstable, for the sake of completeness, we present here a simple direct proof of it.

In the case (3.11) the appropriate Liapunov functional for the instability is

$$W = \frac{1}{2}\left[A(X^2+Y^2) + (b_1Y - b_3X)^2 + (b_2Y + b_4X)^2\right].$$

Along (3.25) it follows that

$$\frac{dW}{dt} = AI(X^2+Y^2) + \Psi_2$$

with

$$\begin{cases} \Psi_2 = FF_1 + GG_1 \\ \\ F_1 = (A + b_3^2 + b_4^2)X - (b_1b_3 - b_2b_4)Y \\ \\ G_1 = (A + b_1^2 + b_2^2)Y - (b_1b_3 - b_2b_4)X\,. \end{cases}$$

But it easily follows that exists a positive constant k_4 such that

$$\Psi_2 \leq k_4(X^2+Y^2)^{1+\epsilon}$$

and hence

$$\frac{dW}{dt} \geq AI(X^2+Y^2) - k_4(X^2+Y^2)^{1+\epsilon}.$$

Therefore in the sphere S_r of radius $r \leq \left(\frac{AI}{k_4}\right)^{1+\epsilon}$ centered at $(X = Y = 0)$, W is positive definite and $\dfrac{dW}{dt} > 0$. Then the instability is guaranteed by the Liapunov instability theorem.

In the case (3.12), $(X = Y = 0)$ is a saddle point and, via the transformation

$$X_1 = -b_4X + (b_1 - \lambda_1)Y, \qquad Y_1 = -b_4X + (b_1 - \lambda_2)Y,$$

(3.25) can be reduced into

$$\begin{cases} \dfrac{dX}{dt} = \lambda_1 X_1 + F_1 \\ \\ \dfrac{dY}{dt} = \lambda_2 Y_1 + G_1 \end{cases} \tag{3.26}$$

with

$$F_1 = F[X(X_1, Y_1), Y(X_1, Y_1)], \qquad G_1 = G[X(X_1, Y_1), Y(X_1, Y_1)] \tag{3.27}$$

and

$$\begin{cases} X(X_1, Y_1) = \dfrac{1}{b_4(\lambda_1 - \lambda_2)}[(b_1 - \lambda_1)Y_1 + (\lambda_2 - b_1)X_1] \\ \\ Y(X_1, Y_1) = \dfrac{Y_1 - X_1}{\lambda_1 - \lambda_2}\,. \\ \end{cases}$$

In view of $\lambda_1\lambda_2 < 0$, without loss of generality, one can assume $\lambda_1 < 0$. The appropriate Liapunov functional in this case is

$$E = \frac{1}{2}(X_1^2 - Y_1^2)$$

and along the solutions of (3.26) it follows that

$$\frac{dE}{dt} = \lambda_1 X_1^2 + |\lambda_2| Y_1^2 + X_1 F_1 + Y_1 G_1.$$

Setting $\delta = \min\{\lambda_1, |\lambda_2|\}$, (3.27) imply

$$|X_1 F_1 + Y_1 G_1| \leq a(X_1^2 + Y_1^2)^{1+\epsilon}$$

with a positive constant, it turns out that

$$\frac{dE}{dt} > \delta(X_1^2 + Y_1^2) - a(X_1^2 + Y_1^2)^{1+\epsilon}.$$

Therefore in the sphere S_r of radius $r \leq \left(\frac{\delta}{a}\right)^{1/\epsilon}$, centered at $(X_1, Y_1 = 0)$ it turns out that $\frac{dE}{dt} > 0$. By virtue of

$$Y_1 = 0 \Rightarrow E > 0$$

also in the case (3.12) the instability is guaranteed by the Liapunov instability theorem.

We observe that the classical energy method of nonlinear L^2-stability generally does not allow to obtain conditions guaranteing instability.

Remark 3.4.1. *Theorems 2.12.1 and 2.12.2 continue to hold also in the case of Neumann boundary conditions*

$$\frac{du}{dn} = \frac{dv}{dn} = 0$$

(n being the outward normal to $\partial\Omega$) in the class of the perturbations such that

$$\int_\Omega u d\Omega = \int_\Omega v d\Omega = 0.$$

3.5 Stabilizing-destabilizing effect of diffusivity

Immediate consequences of Theorems 2.12.1 and 2.12.2 are the following ones.

Theorem 3.5.1. *Let* (3.5), (3.10) *and*

$$\begin{cases} I_0 = a_1 + a_4 > 0 \\ \\ A_0 = a_1 a_4 + b_2 b_3 > 0 \\ \end{cases}$$

or

$$A_0 = a_1 a_4 + b_2 b_3 < 0$$

hold. Then $(\overline{u}_* = \overline{v}_* = 0)$, *unstable in absence of diffusivity, is stabilized by diffusivity.*

Theorem 3.5.2. *Let* (3.5), (3.12) *and*

$$\begin{cases} I_0 = a_1 + a_4 < 0 \\ \\ A_0 = a_1 a_4 + b_2 b_3 > 0 \\ \end{cases} \tag{3.28}$$

hold. Then $(\overline{u}_* = \overline{v}_* = 0)$, *stable in absence of diffusivity, is destabilized by diffusivity.*

It remains only to show the consistency of the assumptions (3.28).

From

$$A = \gamma_1\gamma_2\overline{\alpha}^2 - (\gamma_1 a_4 + \gamma_2 a_1)\overline{\alpha} + A_0 < 0 \tag{3.29}$$

it follows that the consistency of (3.28) requires

$$\begin{cases} \gamma_1 \neq \gamma_2 \\ a_1 a_4 < 0\,. \end{cases}$$

Let

$$a_1 < 0 \tag{3.30}$$

then (3.29) becomes

$$\gamma_1 > \frac{1}{a_4}(|a_1| + \gamma_1\overline{\alpha})\gamma_2 + \frac{A_0}{a_4\overline{\alpha}}$$

and the consistency of (3.28) is guaranteed by (3.5), (3.30) and

$$\begin{cases} \gamma_1 > \dfrac{(1+\delta)A_0}{a_4\overline{\alpha}} \\ \\ \gamma_2 < \dfrac{\delta A_0}{(|a_1| + \gamma_1\overline{\alpha})\overline{\alpha}} \end{cases}$$

with $\delta = const. > 0$. Analogously if

$$a_4 < 0 \tag{3.31}$$

the consistency is guaranteed by (3.5), (3.31) and

$$\begin{cases} \gamma_2 \geq \dfrac{(1+\delta)A_0}{a_1\overline{\alpha}} \\ \\ \gamma_< \dfrac{\delta A_0}{(|a_4| + \gamma_2\overline{\alpha})\overline{\alpha}}\,. \end{cases}$$

Remark 3.5.1. *The stabilizing-destabilizing effect of diffusivity on the linear stability is well known* $\{$*see [13],[28]*$\}$*. When* Ω *is a torus and the perturbations verify the plan-form equations, the nonlinear stabilizing-destabilizing effect of diffusivity has been considered in [31].*

Remark 3.5.2. *Theorems 2.12.1 and 2.12.2 allow to obtain the coincidence between the conditions of linear stability (via normal modes) and the conditions of nonlinear stability with respect to the* L^2*-norm.*

Let now $\Im$ denote the identity operator. The scalar

$$\Xi(u, v) =< u, \Im u > + < v, \Im v >$$

is usually interpreted as energy of the perturbation (u, v) to the basic state. Generalizing this point of view, the scalar

$$Q =< u, Fu > + < v, Gv > \tag{3.32}$$

with F and G operators acting on u and v respectively, can be interpreted as energy dissipated or generated by the operators F and G, according to $Q < 0$ or $Q \geq 0$ respectively. In the case of the operators

$$F = \gamma_1 \Delta, \qquad G = \gamma_2 \Delta$$

appearing in (3.1), in view of

$$\begin{cases} < f, \Delta f >=< f, \nabla \cdot \nabla f >= - \|\nabla f\|^2 \\ \forall f \in H_0^1(\Omega), \qquad \forall f \in H_*^1(\Omega) \end{cases}$$

Q is given by

$$Q = -\gamma_1 \|\nabla u\|^2 - \gamma_2 \|\nabla v\|^2$$

and hence the energy is dissipated. By virtue of (3.8),

$$|Q_{max}| = \overline{\alpha}(\gamma_1 \|\nabla u\|^2 + \gamma_2 \|\nabla v\|^2)$$

with

$$\begin{cases} \Delta\overline{u} + \overline{\alpha u} = 0 \\ \\ \Delta\overline{v} + \overline{\alpha v} = 0 \\ \end{cases}$$

denotes the lowest energy dissipated by (3.32). The guideline of this chapter has been to show that the conditions guaranteeing the stability (instability) with respect to the perturbations dissipating the lowest energy, guarantee the stability (instability) with repect to any other perturbation.

Chapter 4

Nonlinear stability for reaction-diffusion Lotka-Volterra model with Beddington-DeAngelis functional response

4.1 Introduction

This chapther is devoted to the coexistence problem for Lotka-Volterra predator-prey model, with Beddington-De Angelis functional response and Robin type boundary conditions.

By using the Rionero-Liapunov functionals introduced in section 3.2, conditions guaranteeing the nonlinear L^2-stability of the biologically meaningful equilibrium state are furnished.

4.2 The model

Denoting by U and V the prey and predator densities respectively, the Lotka-Volterra predator-prey model equations with Beddington-DeAngelis functional response $F(U,V)$ are:

$$\begin{cases} \dfrac{\partial U}{\partial t} = d_1 \Delta U + U(1-U) - \dfrac{aUV}{1+bU+cV} & (x,t) \in \Omega \times \Re^+ \\ \\ \dfrac{\partial V}{\partial t} = d_2 \Delta V - dV + \dfrac{eUV}{1+bU+cV} & (x,t) \in \Omega \times \Re^+ \\ \end{cases} \tag{4.1}$$

where $\Omega \subset \Re^3$ is a bounded smooth domain and a, b, c, d, d_1, d_2 are positive constants.

The dynamics of (4.1), under the boundary conditions ($\beta_i = \text{const.},\ i = 1,2$)

$$\begin{cases} \beta_1 U + (1-\beta_1)\dfrac{dU}{dn} = 0 & \text{on } \partial\Omega \times \Re^+ \\ \\ \beta_2 V + (1-\beta_2)\dfrac{dV}{dn} = 0 & \text{on } \partial\Omega \times \Re^+ \end{cases} \tag{4.2}$$

n being the outward normal to $\partial\Omega$ and $\beta_1, \beta_2 \in [0,1]$, has been deeply studied recently in [5] {Cfr. also [6]}. The analysis is mostly based on the well known Liapunov functional

$$\Xi(t) = \int_\Omega W(U(x,t),\, V(x,t))dx \tag{4.3}$$

with

$$\begin{cases} W = W_1(U) + [a(1+bU^*)/e(1+cV^*)]W_2(V) \\ \\ W_1 = U - U^* - U^* \ln(U/U^*) \\ \\ W_2 = V - V^* - V^* \ln(V/V^*)\,. \end{cases} \tag{4.4}$$

The positive equilibrium state $S = (U^*,\, V^*)$, existing for

$$e > (b+1)d\,, \tag{4.5}$$

is given by

$$\begin{cases} U^* = \dfrac{-[a(e-bd)-ce] + \sqrt{[a(e-bd)-ce]^2 + 4acde}}{2ce} < 1 \\ \\ V^* = -\dfrac{1}{c} + \dfrac{e-bd}{cd}\, U^* > 0\,. \end{cases} \tag{4.6}$$

In particular in [5] the stability of S has been studied in the case in which both the species cannot live Ω, i.e.:

$$\frac{dU}{dn} = \frac{dV}{dn} = 0 \qquad \text{on } \partial\Omega \times \Re^+\,. \tag{4.7}$$

Here we reconsider the stability of S, but under more general boundary conditions. In fact we consider the case in which each specie cannot live Ω only through a part of $\partial\Omega$. Precisely we consider the mixed boundary conditions (Robin type conditions)

$$\begin{cases} \dfrac{dU}{dn} = 0 \;\; on \;\; \Sigma_1 \times \Re^+\,, \qquad U = U^* \;\; on \;\; \Sigma_1^* \times \Re^+ \\ \\ \dfrac{dV}{dn} = 0 \;\; on \;\; \Sigma_2 \times \Re^+\,, \qquad V = V^* \;\; on \;\; \Sigma_2^* \times \Re^+\,, \end{cases} \tag{4.8}$$

with $\partial\Omega = \Sigma_i \cup \Sigma_i^*\,,\ \Sigma_i \cap \Sigma_i^* = \emptyset$, and $\Sigma_i^* \neq \emptyset$ $(i = 1, 2)$.

4.3 Preliminaries

Let (u, v) denote the perturbation to S. It easily follows that

$$\begin{cases} u_t = \left(1 + \dfrac{ah_1}{b} - 2U^*\right) u + \dfrac{ah_2}{c} v - u^2 + d_1 \Delta u - aH(u, v) \\ \\ v_t = -\dfrac{eh_1}{b} u + \left(-\dfrac{eh_2}{c} - d\right) v + d_2 \Delta v + eH(u, v) \end{cases} \tag{4.9}$$

under the boundary conditions

$$\begin{cases} \dfrac{du}{dn} = 0 \quad on \quad \Sigma_1 \times \Re^+ \qquad u = 0 \quad on \quad \Sigma_1^* \times \Re^+ \\ \\ \dfrac{dv}{dn} = 0 \quad on \quad \Sigma_2 \times \Re^+ \qquad v = 0 \quad on \quad \Sigma_2^* \times \Re^+ \,, \end{cases} \tag{4.10}$$

with

$$\begin{cases} H(u, v) = \dfrac{h_1 u^2 + h_2 v^2 + h_3 uv}{\phi^* + bu + cv} \\ \\ h_1 = -\dfrac{bV^*(1 + cV^*)}{\phi^{*2}} \,, \quad h_2 = -\dfrac{cU^*(1 + bU^*)}{\phi^{*2}} \\ \\ h_3 = \dfrac{\phi^* + 2bcU^*V^*}{\phi^{*2}} \,, \quad \phi^* = 1 + bU^* + cV^* \,. \end{cases} \tag{4.11}$$

Denoting by $H_i(\Omega) \subset W^{1,2}(\Omega)$ $(i = 1, 2)$ the functional spaces defined by

$$H_i(\Omega) = \left\{ \varphi : \varphi^2 + |\nabla \varphi|^2 \in L(\Omega), \frac{d\varphi}{dn} = 0 \text{ on } \Sigma_i \times \Re^+, \, \varphi = 0 \text{ on } \Sigma_i^* \times \Re^+ \right\} \tag{4.12}$$

we study the stability of S with respect to the perturbations

$$(u, v) \in H_1(\Omega) \times H_2(\Omega) \,,$$

biologically meaningful, i.e. such that

$$\begin{cases} U = U^* + u > 0 \\ \\ V = V^* + v > 0 \,. \end{cases} \tag{4.13}$$

It is easily verified that system (4.9) can be written

$$\begin{cases} u_t = b_1 u + b_2 v + d_1 \Delta u + f(u,v) \\ \\ v_t = b_3 u + b_4 v + d_2 \Delta v + g(u,v) \end{cases} \tag{4.14}$$

with

$$\begin{cases} b_1 = 1 - 2U^* - \dfrac{aV^*(1+cV^*)}{{\phi^*}^2} \\ \\ b_2 = -\dfrac{aU^*(1+bU^*)}{{\phi^*}^2} (< 0) \\ \\ b_3 = \dfrac{eV^*(1+cV^*)}{{\phi^*}^2} (> 0) \\ \\ b_4 = -d + \dfrac{eU^*(1+bU^*)}{{\phi^*}^2} = -\dfrac{cdV^*}{\phi^*} (< 0) \\ \\ f(u,v) = -u^2 - aH(u,v)\,, \quad g(u,v) = eH(u,v)\,. \end{cases} \tag{4.15}$$

Remark 4.3.1. *We observe that*

i) the positive quadrant $U \geq 0$, $V \geq 0$ is invariant $\{$Cfr. [5] $-$ [6] and [39]$\}$;

ii) the global existence of (u,v) with $(u_0, v_0) \in H_1(\Omega) \times H_2(\Omega)$ can be proved as in [5] $-$ [6];

iii) the infimum

$$\bar{\alpha}_i(\Omega) = \inf_{H_i(\Omega)} \frac{\|\nabla\varphi\|^2}{\|\varphi\|^2}\,, \tag{4.16}$$

exists and is a real positive number $\{$Cfr. [6] and [39]$\}$;

iv) denoting by Ω^* a smooth domain such that $\Omega \subset \Omega^*/\partial\Omega^*$, it turns out that $\bar{\alpha}_1(\Omega) \geq \bar{k}$, $\bar{k}$ being the lowest eigenvalue of

$$\Delta\varphi + \lambda\varphi = 0\,, \qquad \varphi \in W_0^{1,2}(\Omega^*)\,.$$

In particular for $\Omega^* = \left[\delta_1^{(1)},\, \delta_1^{(2)}\right] \times \left[\delta_2^{(1)},\, \delta_2^{(2)}\right] \times \left[\delta_3^{(1)},\, \delta_3^{(2)}\right]$, *it turns out that* {*Cfr. Appendix*}

$$\bar{\alpha}_i(\Omega) \geq \bar{k} = \sum_{j=1}^{3} \frac{1}{\left[\delta_j^{(2)} - \delta_j^{(1)}\right]^2}\,. \tag{4.17}$$

4.4 Nonlinear stability

Following the methodology introduced in [29] − [37], we observe that (4.14) can be written

$$\begin{cases} u_t = b_1^* u + b_2 v + f^*(u) + f(u,v) \\ \\ v_t = b_3 u + b_4^* v + g^*(v) + g(u,v) \end{cases} \tag{4.18}$$

with

$$\begin{cases} 0 < \varepsilon = \text{const.} < 1\,, & \bar{\alpha} = \min\{\bar{\alpha}_1,\, \bar{\alpha}_2\}(> 0) \\ \\ b_i^* = b_i - \varepsilon d_i \bar{\alpha}\,, & i = 1,4 \\ \\ f^*(u) = d_1(\Delta u + \varepsilon\bar{\alpha}u)\,, & g^*(v) = d_2(\Delta v + \varepsilon\bar{\alpha}v)\,. \end{cases} \tag{4.19}$$

Denoting by $\gamma_i\,(i = 1, 2)$ two positive scalings to be chosen suitably later, and setting

$$u = \gamma_1 \bar{u}\,, \quad v = \gamma_2 \bar{v}\,, \quad \mu = \frac{\gamma_1}{\gamma_2} \tag{4.20}$$

in view of (4.18), it turns out that

$$\begin{cases} \bar{u}_t = b_1^* \bar{u} + \dfrac{1}{\mu} b_2 \bar{v} + f^*(\bar{u}) + \bar{f}(\bar{u},\bar{v}) \\ \\ \bar{v}_t = \mu b_3 \bar{u} + b_4^* \bar{v} + g^*(\bar{v}) + \bar{g}(\bar{u},\bar{v}) \end{cases} \tag{4.21}$$

with

$$\begin{cases} \overline{f}(\overline{u},\overline{v}) = \dfrac{1}{\gamma_1} f(u,v)\Big|_{u=\gamma_1\overline{u}, v=\gamma_2\overline{v}} \\ \\ \overline{g}(\overline{u},\overline{v}) = \dfrac{1}{\gamma_2} g(u,v)\Big|_{u=\gamma_1\overline{u}, v=\gamma_2\overline{v}} . \end{cases} \tag{4.22}$$

Our aim is to show that the nonlinear stability of S in the $L^2(\Omega)$-norm can be reduced to the stability of the zero solution of

$$\begin{cases} \dfrac{d\xi}{dt} = b_1^*\xi + \dfrac{1}{\mu} b_2 \eta \\ \\ \dfrac{d\eta}{dt} = \mu b_3 \xi + b_4^* \eta \end{cases} \tag{4.23}$$

i.e. to

$$\begin{cases} I^* = b_1^* + b_4^* = \lambda_1 + \lambda_2 (< 0) \\ \\ A^* = b_1^* b_4^* - b_2 b_3 = \lambda_1 \lambda_2 (> 0) \end{cases} \tag{4.24}$$

$\lambda_i \, (i = 1, 2)$ being the eigenvalues of $\begin{pmatrix} b_1^* & b_2 \\ b_3 & b_4^* \end{pmatrix}$.

Theorem 4.4.1. *Let*

$$b_1 < d_1 \bar{\alpha} \, . \tag{4.25}$$

Then $S = (U^*, V^*)$ is nonlinearly asymptotically exponentially stable with respect to the $L^2(\Omega)$-norm.

Proof. Let $\varepsilon > 0$ be such that $b_1 < \varepsilon d_1 \bar{\alpha}$. Then (4.24) hold. Denoting by $\| \cdot \|$ and $< \cdot, \cdot >$ respectively the norm and the scalar product in $L^2(\Omega)$, let us introduce the Rionero-Liapunov functional $\{[29] - [36]\}$

$$E = \frac{1}{2} \left[A^* (\|\bar{u}\|^2 + \|\bar{v}\|^2) + \|b_1^* \bar{v} - \mu b_3 \bar{u}\|^2 + \|\frac{b_2}{\mu} \bar{v} - b_4^* \bar{u}\|^2 \right] \tag{4.26}$$

which time derivative along the solution of (4.21) is given by

$$\frac{dE}{dt} = A^* I^*(\|\bar{u}\|^2 + \|\bar{v}\|^2) + \Psi + \Psi^* \tag{4.27}$$

with

$$\begin{cases} \Psi =< \alpha_1 \bar{u} - \alpha_3 \bar{v},\, \bar{f} > + < \alpha_2 \bar{v} - \alpha_3 \bar{u},\, \bar{g} > \\ \\ \Psi^* =< \alpha_1 \bar{u} - \alpha_3 \bar{v},\, f^* > + < \alpha_2 \bar{v} - \alpha_3 \bar{u},\, g^* > \\ \\ \alpha_1 = A^* + \mu^2 b_3^2 + (b_4^*)^2, \quad \alpha_2 = A^* + (b_1^*)^2 + \dfrac{1}{\mu^2} b_2^2 \\ \\ \alpha_3 = \mu b_1^* b_3 + \dfrac{1}{\mu} b_2 b_4^* . \end{cases} \tag{4.28}$$

By virtue of (4.15), (4.19) and (4.25) it turns out that

$$\begin{cases} b_1^* b_3 < 0 \\ \\ b_2 b_4^* > 0\,, \end{cases} \tag{4.29}$$

hence choosing

$$\mu = \bar{\mu} = \sqrt{\left| \frac{b_2 b_4^*}{b_1^* b_3} \right|} \tag{4.30}$$

it follows that

$$\alpha_3 = 0\,. \tag{4.31}$$

In view of

$$\begin{cases} < \bar{u}, f^* >= d_1[-\|\nabla \bar{u}\|^2 + \varepsilon \bar{\alpha} \|\bar{u}\|^2] < -d_1(1-\varepsilon)\|\nabla \bar{u}\|^2 \\ \\ < \bar{v}, g^* >= d_2[-\|\nabla \bar{v}\|^2 + \varepsilon \bar{\alpha} \|\bar{v}\|^2] < -d_2(1-\varepsilon)\|\nabla \bar{v}\|^2 \end{cases} \tag{4.32}$$

one obtains

$$\frac{dE}{dt} \le A^* I^*(\|\bar{u}\|^2 + \|\bar{v}\|^2) - (1-\varepsilon)[d_1\|\nabla \bar{u}\|^2 + d_2\|\nabla \bar{v}\|^2] + \Psi \tag{4.33}$$

with

$$\Psi = \alpha_1 < \bar{u},\, \bar{f} > + \alpha_2 < \bar{v},\, \bar{g} > \; . \tag{4.34}$$

In view of (4.20) and (4.30) it follows that

$$\begin{cases} \gamma_1 = \bar{\mu}\, \gamma_2 \\ \\ |\bar{f}(\bar{u},\, \bar{v})| \leq \gamma_2 \left(c_1 \bar{u}^2 + c_2 \bar{v}^2 + c_3 |\bar{u}\, \bar{v}| \right) \\ \\ |\bar{g}(\bar{u},\, \bar{v})| \leq \gamma_2 \left(c_4 \bar{u}^2 + c_5 \bar{v}^2 + c_6 |\bar{u}\, \bar{v}| \right) \end{cases} \tag{4.35}$$

with c_i $(i = 1, ..., 6)$ positive constants depending only on $\bar{\mu}$. In fact (4.13) imply

$$\phi^* + b\gamma_1 \bar{u} + c\gamma_2 \bar{v} > 1 \tag{4.36}$$

and hence

$$\begin{aligned} |\bar{f}(\bar{u},\, \bar{v})| &= \left| -\gamma_1 \bar{u}^2 - a \frac{h_1 \gamma_1 \bar{u}^2 + h_2 \gamma_2^2 / \gamma_1 \bar{v}^2 + h_3 \gamma_2 \bar{u}\bar{v}}{\phi^* + b\gamma_1 \bar{u} + c\gamma_2 \bar{v}} \right| = \\ &= \left| -\bar{\mu}\gamma_2 \bar{u}^2 - a \frac{h_1 \bar{\mu}\gamma_2 \bar{u}^2 + h_2 \gamma_2 / \bar{\mu} \bar{v}^2 + h_3 \gamma_2 \bar{u}\bar{v}}{\phi^* + b\gamma_1 \bar{u} + c\gamma_2 \bar{v}} \right| \leq \\ &\leq \gamma_2 [c_1 \bar{u}^2 + c_2 \bar{v}^2 + c_3 |\bar{u}\, \bar{v}|] \end{aligned} \tag{4.37}$$

with

$$c_1 = \bar{\mu}(1 + a|h_1|), \quad c_2 = a|h_2|/\bar{\mu}, \quad c_3 = a|h_3|\, . \tag{4.38}$$

Analogously, it is can easily proved that $(4.35)_2$ holds true, with

$$c_4 = e|h_1|\bar{\mu}^2, \quad c_5 = e|h_2|, \quad c_6 = e|h_3|/\bar{\mu}\, . \tag{4.39}$$

Then, by virtue of (4.34)-(4.39), it follows that

$$|\Psi| \leq \gamma_2 [\eta_1 < |\bar{u}|^3 > + \eta_2 < |\bar{v}|^3 > + \eta_3 < |\bar{u}|\, |\bar{v}|^2 > + \eta_4 < |\bar{u}|^2\, |\bar{v}| >] \tag{4.40}$$

with η_i $(i = 1, .., 4)$ positive constants given by

$$\begin{cases} \eta_1 = |\alpha_1| c_1 \,, \quad \eta_2 = |\alpha_2| c_5 \\ \\ \eta_3 = |\alpha_1| c_2 + |\alpha_2| c_6 \\ \\ \eta_4 = |\alpha_1| c_3 + |\alpha_2| c_4 \end{cases} \tag{4.41}$$

and the Hölder inequality implies

$$|\Psi| \leq \gamma_2 (\|\bar{u}\|^2 + \|\bar{v}\|^2)^{1/2} [[(\eta_1 + \eta_4) \|\bar{u}\|_4^2 + (\eta_2 + \eta_3) \|\bar{v}\|_4^2] \,. \tag{4.42}$$

From (4.46), in view of the embedding inequality {[40]}:

$$\|f\|_4^2 \leq k(\Omega) \|\nabla f\|_2^2 \,, \qquad k(\Omega) = \text{positive constant} \tag{4.43}$$

it turns out that

$$|\Psi| \leq \gamma_2 \, k \, M \, (\|\nabla \bar{u}\|^2 + \|\nabla \bar{v}\|^2)(\|\bar{u}\|^2 + \|\bar{v}\|^2)^{1/2} \,, \tag{4.44}$$

with

$$M = \max\{\eta_1 + \eta_4, \, \eta_2 + \eta_3\} > 0 \,. \tag{4.45}$$

By virtue of $A^* > 0$, it easily follows that E is positive definite and that exist two positive constants k_i $(i = 1, 2)$ such that

$$k_1 (\|\bar{u}\|^2 + \|\bar{v}\|^2) \leq E \leq k_2 (\|\bar{u}\|^2 + \|\bar{v}\|^2) \,. \tag{4.46}$$

Therefore

$$(\|\bar{u}\|^2 + \|\bar{v}\|^2)^{1/2} \leq \frac{2E^{1/2}}{k_1^{1/2}} \tag{4.47}$$

and (4.33), (4.44), (4.46)-(4.47) imply

$$\frac{dE}{dt} \leq -k_3 E - (k_4 - \gamma_2 k_5 E^{1/2})(\|\nabla \bar{u}\|^2 + \|\nabla \bar{v}\|^2) \tag{4.48}$$

with

$$\begin{cases} k_3 = \dfrac{A^*|I^*|}{k_1}\,, \quad k_4 = (1-\varepsilon)\inf(d_1,\, d_2) \\[2em] k_5 = \dfrac{2kM}{k_1^{1/2}}\,. \end{cases} \tag{4.49}$$

By recursive argument

$$0 < \gamma_2 < \frac{k_4}{k_5\, E_0^{1/2}} \tag{4.50}$$

implies

$$\frac{dE}{dt} \le -k_3 E \qquad \forall\, t \ge 0 \tag{4.51}$$

i.e.

$$E(t) \le E_0\, e^{-k_3 t}\,. \tag{4.52}$$

Theorem 4.4.2. *Let*

$$bU^*(1-U^*) < U^* + d_1 \phi^* \bar{\alpha}\,. \tag{4.53}$$

Then $S = (U^*,\, V^*)$ is asymptotically nonlinearly stable according to (4.52)*.*

Proof. S is a steady state, hence

$$\begin{cases} U^*(1-U^*) - a\dfrac{U^*V^*}{\phi^*} = 0 \\[2em] \phi^* = 1 + bU^* + cV^* \end{cases} \tag{4.54}$$

i.e.

$$\begin{cases} \dfrac{aV^*}{\phi^*} = 1 - U^* \\[2em] 1 + cV^* = \phi^* - bU^*\,. \end{cases} \tag{4.55}$$

By virtue of (4.55), it turns out that

$$b_1 = 1 - 2U^* - (1 - U^*)\frac{\phi^* - bU^*}{\phi^*} =$$

$$= 1 - 2U^* - (1 - U^*)\left(1 - \frac{bU^*}{\phi^*}\right) = \tag{4.56}$$

$$= 1 - 2U^* - 1 + U^* + \frac{bU^*}{\phi^*}(1 - U^*) = -U^* + \frac{bU^*}{\phi^*}(1 - U^*)$$

i.e.

$$b_1 = \frac{U^*}{\phi^*}\left[-\phi^* + b(1 - U^*)\right] . \tag{4.57}$$

From (4.25) and (4.57), (4.53) immediately follows.

Remark 4.4.1. *We observe that*

i) in the case of the boundary conditions (4.7) *one has* $\bar{\alpha} = 0$, *and* (4.53) *reduces to*

$$b(1 - U^*) < 1\,, \tag{4.58}$$

which is the stability condition given in [5]*;*

ii) (4.53) *is only a sufficient condition for guaranteeing* (4.24).

Appendix

Let Ω^* be a smooth bounded domain such that $\Omega \subset \Omega^*/\partial\Omega^*$ and

$$\bar{k} = \inf_{\varphi \in W_0^{1,2}(\Omega^*)} \frac{\|\nabla\varphi\|^2}{\|\varphi\|^2} . \tag{4.59}$$

Then

$$\|\varphi\|^2 \leq \bar{k}\|\nabla\varphi\|^2 \qquad \forall\, \varphi \in W_i^{1,2}(\Omega, \beta_i) . \tag{4.60}$$

In fact, setting

$$\varphi = \begin{cases} \Psi\,, & \mathbf{x} \in \Omega \\ \\ 0\,, & \mathbf{x} \in \Omega^*/\Omega \end{cases} \tag{4.61}$$

$\Psi \in W_i^{1,2}(\Omega, \beta_i) \to \varphi \in W_0^{1,2}(\Omega)$ hence (4.59) $\Rightarrow$ (4.60). In particular for $\Omega^* = \left[\delta_1^{(1)},\, \delta_1^{(2)}\right] \times \left[\delta_2^{(1)},\, \delta_2^{(2)}\right] \times \left[\delta_3^{(1)},\, \delta_3^{(2)}\right]$ it follows that

$$\bar{k} \geq \sum_{j=1}^{3} \frac{1}{\left[\delta_j^{(2)} - \delta_j^{(1)}\right]^2}\,. \tag{4.62}$$

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