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Results in Second-Order Elasticity with Live Loads (obtained with Mathematica)

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(**Abstract**)

We present a generalization of Signorini's method to the case of live loads which allows us to derive approximate solutions to some pure traction-value problems in finite elastostatics. The boundary value problems and the corresponding compatibility conditions are formulated in order to determine the displacement of the system up to the second-order approximation. In particular, we consider the case of homogeneous and isotropic elastic bodies and we solve the following two pure traction-value problems with live loads: (i) a sphere subjected to the action of a uniform pressure field; (ii) a hollow circular cylinder whose inner and outer surfaces are subjected to uniform pressures. Then, starting from these solutions, we suggest experiments to determine the second-order constitutive constants of the elastic body. Expressions of the second-order material constants in terms of displacements and Lamé coefficients are determined. Further we apply the generalized Signorini's perturbation scheme to analyze radial expansion/contraction of an hollow cylinder made of an isotropic functionally graded elastic material, whose material moduli depend upon the radial coordinate only. We study the case of an hollow circular cylinder under uniform internal and external pressure. The displacement of the system and its state of stress are determined up to the second-order approximation. We consider both compressible and incompressible functionally graded elastic bodies.

The quantitative analysis of all the described problems has been carried out with the aid of the software Mathematica by Wolfram Research. The programs which have allowed to find the solutions to these problems are presented and their characteristics are discussed.

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Introduction

1.1 Preliminary Considerations

The nonlinear Theory of Elasticity presents many interesting problems both from mathematical and physical points of view. The main issues are essentially related to the following topics.

1. Equations governing the equilibrium and the motion of an elastic system are not linear. This implies that, except for some cases of incompressible materials, it is not possible to obtain an analytical solution to these equations. Then, it is necessary to resort to procedures which allow us to determine *approximate solutions*, which are extremely useful in practical applications. It is also common to turn to numerical routines which are presently very effective thanks to the extraordinary developments in modern computers.

2. A further complication of the theory for live loads is that boundary conditions can be assigned only as a function of the unknown deformation.

3. Owing to the nonlinearity of the equations, the wave propagation problems are very hard to study and may present blow-up phenomena.

4. The determination of the response of a nonlinear elastic material is not an easy task. For homogeneous and isotropic solid bodies it reduces to finding the form of the specific strain energy. This situation is more complex than the one in the

linear Theory of Elasticity. In the latter case the response of an isotropic material is completely characterized by only two constants, the *Lamé coefficients*, which can be experimentally determined by a simple tension test.

In this thesis we deal with points 1, 2, and 4. The study of wave propagation in nonlinear materials represents the next step of the present work.

1.2 A Brief Historical Survey

In 1930, Signorini [1] suggested a perturbation method to find approximate solutions of boundary-value problems of finite elasticity in the presence of *dead loads* (loads independent of the deformation). This procedure is essentially an application of Poincarè's perturbation method (e.g., see [2] and [3]) to equations of finite elasticity. Furthermore, in [4] and [5] Signorini's method was used to investigate the uniqueness of solutions as well as the position of the classical linear theory with respect to the nonlinear theory. Later, Stoppelli [6-10] proved a local theorem of existence, uniqueness, and analytic dependence on a parameter for the solution to the traction-value problem, when the applied dead loads do not have an *axis of equilibrium*, and the existence and analyticity of solutions when the dead loads have an axis of equilibrium (see also Tolotti [11]). A discussion of Stoppelli's work can be found in [12-14]. In [15-19] Capriz and Podio-Guidugli investigate the compatibility of the linear and nonlinear elasticity theories and show that a very large class of traction-value problems can be solved by perturbation methods of Signorini's type. In particular, in [16], the authors provide a series expansion to construct an approximate solution to the static balance equations of elastic bodies under dead loads and discuss the meaning and implication of the Fredholm-type conditions for the existence of such an expansion. Furthermore, in [18] and [19] traction-value problems of finite elasticity are analyzed in the presence of loads depending on the deformation (*live loads*). Although every realistic load depends on the deformation, the introduction of live loads leads to difficult mathematical problems. A crucial contribution in this framework, has been given by Valent in many papers which are collected in [20]. In this book, Valent proves theorems of existence, uniqueness, and analytic dependence on a parameter for boundary-value problems of place and traction in finite elastostatics with dead loads and some special types of live loads.

1.3 Detailed Summary of the Work

This thesis is divided into six chapters.

The first chapter is devoted to the statement of the equilibrium problem in finite elastostatics. First, the balance equations are written in the Eulerian and the Lagrangian formulation for an arbitrary continuous system and then they are specialized to the case of elastic systems, both in the linear and the nonlinear framework.

The second chapter is concerned with some issues regarding the formulation of the equilibrium problem. We start from the classification of boundary-value problems of finite elastostatics and then discuss difficulties related to the pure traction-value problem. In particular, we analyze the concept of live loads and describe the resulting mathematical complications. Further, we discuss the nature of the global equilibrium conditions, observing that in the pure traction problem and in the presence of live loads they are essentially compatibility conditions for the data and the displacements.

The third chapter deals with the description of the classical Signorini's method for dead loads. In order to obtain conditions under which the perturbation method can be applied, the equilibrium equations, the boundary conditions and the corresponding compatibility conditions are written in a nondimensional form. Reference is made to the existence and the uniqueness results obtained by Stoppelli [6, 7, 9, 10] and Van Buren [21].

In the fourth, starting from the results obtained by Valent in Chapter 6 of [20], we provide a generalization of Signorini's method to the case of live loads. In the framework of second-order elasticity theory we solve two traction-value problems with live loads and design four experiments which allow us to determine the second-order constitutive constants for the given material.

In order to investigate the effects of material inhomogeneity on the response of isotropic elastic bodies, we apply the generalized Signorini's perturbation technique to study the equilibrium boundary value problem of functionally graded materials (FGMs), whose material moduli depend on the radial coordinate only. Functionally graded composite materials have been the subject of intense research in recent years. These materials are special composites commonly made by ceramics and metals. The ceramic offers thermal barrier effects and protects the metal from corrosion and oxidation, while the material is toughened and strengthened by the metallic composition. The compositions and the volume fractions of the constituents are varied gradually, thus giving a non-uniform microstructure in the material which results in continuously graded macroproperties, which are the most distinctive features of FGMs. Chapter 5 is devoted to a brief overview on design and applications of these materials.

In the last chapter the generalized perturbation method obtained in Chapter 4 is used to solve the traction-value problem of an hollow circular cylinder made of a functionally graded material. In particular we analyze the case in which the inner and outer surface of the hollow cylinder are loaded by a uniform pressure. We write the explicit solutions both for compressible and incompressible FGMs.

1.4 Quantitative Analysis and the Software Mathematica

Even though the results described so far have a strict theoretical value, they would not have been obtained without the aid of the software Mathematica by Wolfram Research. As we have already remarked, the non linearity of the equations governing the equilibrium of an elastic material usually prevents us from their explicit integration. Since the applications require a quantitative descriptions of solutions (displacement and stress fields), we have to resort to a perturbation method which allows us to obtain approximate solutions to the examined problems. As we shall see in the sequel, this scheme simplifies at some extent the mathematical difficulties of the equilibrium boundary value problems of finite elasticity. Its distinctive feature is that it transforms a *non linear* system of partial differential equations (PDEs) into a finite set of *linear* PDEs. In spite of this, solving these linear PDEs is not an easy task. Scientific computing based on the software Mathematica has represented a crucial support to overcome this difficulty. Indeed, it has contributed to the development of analytic calculations by symbolic calculus, such as the integrations of differential equations, the solutions of algebraic systems, the representation and the simplification of some quantities of interest in the analyzed problems. Besides its scientific purposes, this research work is based on the idea that mathematical methods and scientific computing should be dealt jointly. At the end of Chapters 4 and 6 we present and discuss the programs which have been written to solve the described problems.

Chapter 1
Equilibrium of an Elastic System

1.1 Introduction

In this chapter *the local equilibrium equations* are written for a continuous system, both in the *Eulerian* and in the *Lagrangian* formulation. These equations are general relations, that is, they are valid for all continuous systems and do not depend on the material of the body. However, it is well known from experience that two bodies having the same geometric characteristics react differently when subjected to the same mechanical loads and thermal conditions. Thus it is necessary to introduce some criteria which allow us to distinguish between the macroscopic behaviors of different material bodies. The mathematical description of different material behaviors is the object of *the theory of constitutive equations*. Even if these equations represent the material constitution of the body, they must fulfill certain general principles, called *constitutive axioms*, which impose restrictions on their forms.

In Section 5 of the present chapter the constitutive equations of a *continuous elastic system* are derived. It is noted there that, although constitutive axioms impose severe restrictions on the form of constitutive equations, they still allow the theory wide margin of arbitrariness, which can be filled only by experimental data. This is due to the fact that the macroscopic behavior of a continuous body is related to its molecular structure. Since this structure does not enter into the continuum description, the constitutive equation of a particular material must be determined experimentally. The experimental determination of the constitutive equations of an elastic material is an extremely complicated task. It can be simplified by exploiting the *symmetry* of the material. After having introduced the concept of *isotropy group of a material*, the constitutive equations of *homogeneous and isotropic* elastic bodies are exhibited. In the last Section these equations are specialized to the case of *linear elastic materials*.

1.2 Finite Deformations

We consider a three-dimensional continuous system S moving in an inertial reference frame $\mathbf{I}$ in which is assigned a Cartesian coordinate system $R \equiv (O, \mathbf{e}_i)$, $i = 1, 2, 3$, where O is the origin and $\mathbf{e}_i$ the unit vectors. The region of space occupied by points of S at a certain time instant t is called the *configuration* of S at the instant t and denoted by $C(t)$. In order to determine the motion of S, it is necessary in the first place to "label" all points of S and then to follow them during the motion, assigning their position in R at every instant.

To this aim a *reference configuration* C_* is introduced, that is a possible configuration of S, and we call material or *Lagrangian* coordinates the coordinates (X_L), $L = 1, 2, 3$, in R of a particle $p \in S$ in the configuration C_*.

The configuration $C\,(t)$ of S at the instant t is the *actual* or the *present configuration* of S and the coordinates (x_i) in R of the particle $p \in S$ in C are referred to as spatial or *Eulerian* coordinates. Any quantity ϕ associated to the motion of S can be expressed in either the Lagrangian or the Eulerian form depending on whether it is intended as a function of the variables (X_L) or (x_i) , in other words depending on whether it is assumed to be defined on C_* or on C.

We call *finite deformation* from C_* to C the vectorial function

$$\mathbf{x} = \mathbf{x}(\mathbf{X}) \tag{1.1}$$

which maps any point $\mathbf{X} \in C_*$ onto the corresponding position in C, or equivalently, the three scalar functions

$$x_i = x_i(X_L) \qquad i, L = 1, 2, 3. \tag{1.2}$$

The functions (1.2) are assumed to be

1. one-to-one, and
2. of class C^1 together with their inverse.

The first assumption assures that the system neither fractures during the motion nor does a crack close; the second one translates into mathematical terms the basic property of the matter that two particles cannot simultaneously occupy the same place (*impenetrability principle*). In particular, we require that at any point $\mathbf{X} \in C_*$ functions (1.2) fulfill the condition

$$J = \det\left(\frac{\partial x_i}{\partial X_L}\right) > 0,$$

in order to guarantee the right-hand orientation of the frame of references. In other words (1.2) are diffeomorphisms preserving the topological properties of the reference configuration. Differentiation of (1.2) yields

$$dx_i = \frac{\partial x_i}{\partial X_L}(\mathbf{X})\, dX_L, \tag{1.3}$$

where we have used Einstein's summation notation in which the summation over two repeated indices is understood. Equation (1.3) defines at any point $\mathbf{X} \in C_*$ a linear transformation which maps an infinitesimal vector $d\mathbf{X}$ coming from $\mathbf{X}$ onto the corresponding infinitesimal vector $d\mathbf{x}$ coming from $\mathbf{x}(\mathbf{X})$. This transformation is called *deformation gradient* at $\mathbf{X}$ and it is represented in the reference frame $\mathbf{R}$ by the matrix

$$\mathbf{F} = (F_{iL}) \equiv \left(\frac{\partial x_i}{\partial X_L}\right). \tag{1.4}$$

It is easy to realize that equation (1.3) contains all the information regarding the deformation of the volume element at $\mathbf{X}$ when passing from C_* to C.

In the sequel we shall make use of the following formulae, which can be deduced from (1.3) and relate the corresponding infinitesimal surface elements $d\boldsymbol{\sigma}_*$ and $d\boldsymbol{\sigma}$, as well as the infinitesimal volume elements dc_* and dc in C_* and C respectively (for their derivation see [23])

$$d\boldsymbol{\sigma} = J(\mathbf{F}^{-1})^T d\boldsymbol{\sigma}_*, \tag{1.5}$$

$$dc = J dc_*. \tag{1.6}$$

From equation (1.5) we can write

$$|d\boldsymbol{\sigma}|^2 = d\sigma_i d\sigma_i = J^2 \left|d\boldsymbol{\sigma}_*\right|^2 (\mathbf{F}^{-1})_{Li} N_{*L} (\mathbf{F}^{-1})_{Ki} N_{*K}. \tag{1.7}$$

If we introduce the *right Cauchy-Green tensor*

$$\mathbf{C} = \mathbf{F}^T\mathbf{F}, \qquad C_{LM} = \frac{\partial x_k}{\partial X_L}\frac{\partial x_k}{\partial X_M},$$

since

$$(\mathbf{F}^{-1})_{Li}(\mathbf{F}^{-1})_{Mi} = (\mathbf{C}^{-1})_{LM},$$

from (1.7) we obtain the following relation which will play an important role in the sequel

$$d\sigma = J\sqrt{\mathbf{N}_*\mathbf{C}^{-1}\mathbf{N}_*}d\sigma_*. \tag{1.8}$$

The deformation of S when passing from C_* to C can be described in a completely equivalent way by resorting to the *displacement field* $\mathbf{u}(\mathbf{X})$ defined by the following relation

$$\mathbf{u}(\mathbf{X}) = \mathbf{x}(\mathbf{X}) - \mathbf{X}. \tag{1.9}$$

Introducing the *displacement gradient* $\mathbf{H}$ by the definition

$$\mathbf{H} = \nabla\mathbf{u}, \qquad H_{iL} = \frac{\partial u_i}{\partial X_L},$$

from (1.9) it follows that

$$\mathbf{H} = \mathbf{F} - \mathbf{1}, \tag{1.10}$$

where $\mathbf{1}$ is the 3×3 identity matrix.

1.3 Mass Conservation Equation

The mass of a continuous system S is assumed to be continuously distributed over the whole region $C(t)$ occupied by S at the instant t. In other words, we postulate the existence of a function $\rho(\mathbf{x}, t)$ of class C^1, called *mass density*, such that, if c_* is a part of S in C_* and c its image through the deformation $\mathbf{x}(\mathbf{X})$, the mass of c_* at time t is given by

$$m(c_*) = \int_c \rho(\mathbf{x}, t)\, dc. \tag{1.11}$$

Like every quantity associated with S, the mass density can be expressed in Lagrangian form. In this case it will be denoted by the symbol ρ_* to highlight that it is a function assigned on C_*, namely $\rho_* = \rho_*(\mathbf{X}, t)$. Hence, the following relation holds

$$m(c_*) = \int_c \rho(\mathbf{x}, t)\, dc = \int_{c_*} \rho_*(\mathbf{X}, t)\, dc_*,$$

from which, having in mind the rule of the variable change in the multiple integrals (equation (1.6)), it follows that

$$\int_{c_*} [\rho(\mathbf{x}(\mathbf{X}, t), t)\, J - \rho_*(\mathbf{X}, t)] dc_* = 0.$$

Since c_* is an arbitrary volume, at any point of c_* in which the integrand function is regular we obtain the following *local Lagrangian formulation* of the *mass conservation*

$$\rho J = \rho_*. \tag{1.12}$$

1.4 Eulerian Formulation of the Equilibrium Equation

In the Mechanics of the Continuous Systems loads acting on the material region $c \subset C$ from its exterior c^e are divided into *mass forces* (or *body forces*), continuously distributed over the region c and *contact forces* (or *surface loads*) acting on the boundary ∂c. The resultant force $\mathbf{R}$ and the resultant moment $\mathbf{M}_0$ with respect to a fixed point *O* are given by

$$\mathbf{R}(c, c^e) = \int_c \rho \mathbf{b} dc + \int_{\partial c} \mathbf{t} d\sigma, \tag{1.13}$$

$$\mathbf{M}_0(c, c^e) = \int_c (\mathbf{x} - \mathbf{x}_0) \times \rho \mathbf{b} dc + \int_{\partial c} (\mathbf{x} - \mathbf{x}_0) \times \mathbf{t} d\sigma, \tag{1.14}$$

where $\mathbf{x}_0$ is the position vector of O, ρ is the mass density of S and the *specific mass force* $\mathbf{b}$ and the *traction* $\mathbf{t}$ are defined on c and ∂c respectively.

The first integral in (1.13) takes into account all forces acting over c from the exterior of S. These actions are assumed to be known "a priori". They may be gravitational, electromagnetic, thermal etc. and are expressed by the assigned specific force field $\mathbf{b}(\mathbf{x}, t)$ acting on the whole volume c and whose values are not influenced by the motion of S.

On the contrary, $\mathbf{t}$ represents a contact force field acting at the boundary ∂c of c; the behavior of these forces is deeply related to the motion of S and therefore they are unknown quantities.

We remark that the previous assumptions, although natural, are extremely restrictive. First of all, the mass forces acting on c can originate from the region outside of c but not necessarily from that outside of S. However, in this case they cannot be assumed as known since they depend on the motion of the system. Further, the assumption regarding the contact forces implies that the action of all bodies contacting ∂c is equivalent to the vector $\mathbf{t}d\sigma$. If one takes into account phenomena related to the inner structure of the medium, then it can be assumed that these forces are better represented by a force $\mathbf{t}d\sigma$ and a couple $\mathbf{m}d\sigma$. This assumption gives rise to a branch of Continuum Mechanics referred to as the *theory of Cosserat (or micropolar) Continua*[1]. Hence in a micropolar body, in addition to body forces, the existence of an independent body couple density $\mathbf{m}$ is postulated. If it is assumed, as we do in the sequel, that $\mathbf{m} = \mathbf{0}$, S is called a *simple continuum*.

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1 For a deteiled description of micropolar continua, see [24-27].

We postulate that the force acting upon the surface element $d\sigma$ is related to the deformation of particles close to $d\sigma$ only, and depends on the orientation of $d\sigma$ only through its outward normal vector $\mathbf{n}$. This is known as *Cauchy's postulate* and can be expressed in mathematical terms as

$$\mathbf{t} = \mathbf{t}\left(\mathbf{x}, t, \mathbf{n}\right). \tag{1.15}$$

This is one of the fundamental assumptions in the Mechanics of Simple Continua.

We assume that at equilibrium for any arbitrary material volume $c \subset C$ of S the following conditions hold

$$\mathbf{R}\left(c, c^e\right) = \mathbf{0}, \qquad \mathbf{M}_0\left(c, c^e\right) = \mathbf{0},$$

which, from (1.13) and (1.14), become

$$\int_{\partial c} \mathbf{t} d\sigma + \int_c \rho \mathbf{b} dc = \mathbf{0}, \tag{1.16}$$

$$\int_{\partial c} \left(\mathbf{x} - \mathbf{x}_0\right) \times \mathbf{t} d\sigma + \int_c \left(\mathbf{x} - \mathbf{x}_0\right) \times \rho \mathbf{b} dc = \mathbf{0}. \tag{1.17}$$

A fundamental theorem due to Cauchy shows that, under proper regularity conditions, $\mathbf{t}$ is a linear function of $\mathbf{n}$, i.e.,

$$\mathbf{t} = \mathbf{T}\left(\mathbf{x}, t\right) \cdot \mathbf{n}, \tag{1.18}$$

where $\mathbf{T}$ is a second order tensorial field independent of $\mathbf{n}$ known as the *Cauchy stress tensor*[2]. Equation (1.18) allows us to write the *local* form of equation (1.16). In fact,

[2] For a detailed treatment of the Cauchy theorem see [23].

applying the Gauss theorem we can write the first integral as

$$\int_{\partial c} \mathbf{t} d\sigma = \int_{\partial c} \mathbf{T}\mathbf{n} d\sigma = \int_{c} \nabla_{\mathbf{x}} \cdot \mathbf{T} dc, \tag{1.19}$$

where $\nabla_{\mathbf{x}} \cdot \mathbf{T}$ is a vector whose components in R are $\dfrac{\partial T_{ij}}{\partial x_j}$. Substituting equation (1.19) into (1.16) we obtain

$$\int_{c} (\rho \mathbf{b} + \nabla_{\mathbf{x}} \cdot \mathbf{T}) dc = \mathbf{0}.$$

Since the integration domain is arbitrary, at all points of c where the integrand functions are regular, the relation written above implies the following local form of the equilibrium of S

$$\rho \mathbf{b} + \nabla_{\mathbf{x}} \cdot \mathbf{T} = \mathbf{0}. \tag{1.20}$$

In order to derive the *local* equation corresponding to equation (1.17), we remark that by virtue of the Cauchy theorem, the second integral can be written as

$$\begin{aligned}\int_{\partial c} \epsilon_{ijk} \left(x_j - x_{0j}\right) t_k = \int_{\partial c} \epsilon_{ijk} \left(x_j - x_{0j}\right) T_{kh} n_h = \\ \int_{c} \epsilon_{ijk} T_{kj} dc + \int_{c} \epsilon_{ijk} \left(x_j - x_{0j}\right) \frac{\partial T_{kh}}{\partial x_h} dc\end{aligned}$$

where ϵ_{ijk} is the *Levi-Civita* symbol. Then, substituting into (1.17) and taking into account (1.20) we obtain

$$\int_{c} \epsilon_{ijk} T_{kj} dc = 0.$$

Since c is an arbitrary material volume, it follows that

$$\epsilon_{ijk} T_{kj} = 0,$$

which implies that the Cauchy stress tensor is symmetric

$$\mathbf{T} = \mathbf{T}^T. \tag{1.21}$$

We conclude that the local equilibrium equation of a continuous body in the Eulerian form is given by equation (1.20) in which $\mathbf{T}$ is a symmetric tensor.

1.5 Lagrangian Formulation of the Equilibrium Equation

In the previous section the fields ρ and $\mathbf{T}$ have been written in Eulerian form. However there are many physical problems of practical interest in which it is more convenient to express these fields as functions of $\mathbf{X}$ in C_*.

From the mass conservation equation in Lagrangian form, $\rho_* = \rho J$, the following identity follows

$$\int_c \rho \mathbf{b} dc = \int_{c_*} \rho_* \mathbf{b} dc_* \tag{1.22}$$

for any region $c_* \subset C_*$, where c is the material volume corresponding to c_* in the actual configuration C.

We define the *first Piola-Kirchhoff stress tensor* $\mathbf{T}_*$ by the condition

$$\int_{\partial c_*} \mathbf{T}_* \cdot \mathbf{N}_* d\sigma_* = \int_{\partial c} \mathbf{T} \cdot \mathbf{n} d\sigma, \tag{1.23}$$

where $\mathbf{N}_*$ is the unit outward normal vector to the surface element $d\sigma_*$ of ∂c_*. Recalling equation (1.5), i. e.,

$$d\sigma n_i = J \left(F^{-1}\right)_{Li} N_{*L} d\sigma_*,$$

equation (1.23) becomes

$$\int_{\partial c_*} (T_{*iL} - T_{ij} J \left(F^{-1}\right)_{Lj}) \mathbf{N}_{*L} d\sigma_* = 0, \qquad \forall \mathbf{c}_* \subset C_*,$$

from which we have

$$\mathbf{T}_* = J\mathbf{T}\left(\mathbf{F}^{-1}\right)^T. \tag{1.24}$$

Equation (1.24) expresses the relation between Cauchy's stress tensor $\mathbf{T}$ and the first Piola-Kirchhoff stress tensor $\mathbf{T}_*$. From (1.22) and (1.23) it follows that the integral equilibrium equation of S can be put in the form

$$\int_{\partial c_*} \mathbf{T}_* \cdot \mathbf{N}_* d\sigma_* + \int_{c_*} \rho_* \mathbf{b} dc_* = \mathbf{0}, \qquad \forall \mathbf{c}_* \subset C_*. \tag{1.25}$$

By employing the same arguments as those used in the previous section we conclude from (1.25) that, if the involved fields are regular, then the local equilibrium equation in Lagrangian form is

$$\nabla_{\mathbf{X}} \cdot \mathbf{T}_* + \rho_* \mathbf{b} = \mathbf{0}, \tag{1.26}$$

where $\nabla_{\mathbf{X}} \cdot \mathbf{T}_*$is a vector whose i-th component in the reference frame R is given by $\dfrac{\partial T_{*iL}}{\partial X_L}$.

Equation (1.26) presents the following two advantages over equation (1.20):

1. the mass density ρ_* is a known function of the point $\mathbf{X}$;
2. the involved fields and the function $\mathbf{x}(\mathbf{X})$ are defined in the known and fixed region C_*.

1.6 Elastic Bodies

The local equilibrium equations are general relations, that is, they are valid for all continuous systems and do not depend on the material of the body. However, it is well known from experience that two bodies having the same geometric characteristics may react differently when subjected to the same mechanical loads and thermal conditions. This means that for a continuous system the knowledge of the equilibrium conditions and of the external forces acting upon the body is not sufficient to determine the deformation field.

The above considerations can be expressed in a more formal way by noting that equation (1.26) constitutes a system of three partial differential equations in the 6 unknown components of $\mathbf{T}_*(\mathbf{X})$[3]. This means that the equilibrium equations do not form a closed set of field equations, so we add additional relations connecting the stress tensor $\mathbf{T}_*$ to the the deformation $\mathbf{x}(\mathbf{X})$ or the displacement $\mathbf{u}(\mathbf{X})$. Thus it is necessary to introduce criteria which allow us to distinguish between macroscopic behaviors of different material bodies. These relations are called *constitutive equations* because they translate in mathematical terms the material constitution of the body. In order to get these equations one can start from the assumption that the macroscopic response of a body depends on its molecular structure. This means that in principle the response functions can be obtained from Statistical Mechanics in terms of the average of microscopic quantities. Such an investigation is very stimulating both from theoretical and practical points of view. In this way, we can improve material properties and create new materials that respond to technological demands. But this approach is not straightforward if applied to complex materials, which are of interest in applications. In Continuum Mechanics the basic assumption of a continuously distributed matter cancels its discrete structure, so constitutive equations are determined experimentally. Even if these equations translate the material constitution of the body, they have to fulfill certain general principles, called *constitutive axioms*, which

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[3] Because $\mathbf{T}^T = \mathbf{T}$, from (1.24) it can be seen that only 6 components of the first Piola-Kirchhoff stress tensor are independent.

impose restrictions on their form. For an exhaustive study of this subject see pages 134-155 of [13].

Here we only remark that constitutive equations are assumed to be *objective*, i.e., we postulate that the material behavior is independent of the observer (*principle of material frame indifference*) and further that in any evolution of the system they satisfy the second law of thermodynamics (*principle of dissipation of Coleman and Noll*)[4].

A material point $\mathbf{X}$ in a continuous system S is *elastic* if the Cauchy stress tensor $\mathbf{T}$ at the point $\mathbf{X}$ of S depends on the deformation which S has experienced in the neighborhood of $\mathbf{X}$ when passing from the reference configuration C_* to the actual configuration C. Having in mind that the deformation at $\mathbf{X} \in C_*$ is completely described by the deformation gradient $\mathbf{F}$, the material point $\mathbf{X}$ is said to be *elastic* if

$$\mathbf{T}(\mathbf{X}, t) = \mathbf{f}\left(\mathbf{F}, \mathbf{X}, t\right). \tag{1.27}$$

It can be shown (see [13]) that equation (1.27) satisfies the principle of dissipation and the material frame indifference principle if the Cauchy stress tensor has the following form

$$\mathbf{T} = \rho \frac{\partial \psi}{\partial \mathbf{F}} \mathbf{F}^T, \tag{1.28}$$

where $\psi = \psi\left(\mathbf{F}\right)$ is called the *specific strain energy*.

An elastic material point $\mathbf{X}$ is *incompressible* if it can undergo only volume preserving deformations,i.e.,

$$\det \mathbf{C} = 1. \tag{1.29}$$

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4 See the fundamental memoir [28].

For an incompressible material equation (1.28) becomes

$$\mathbf{T} = -p\mathbf{1} + \rho \frac{\partial \psi}{\partial \mathbf{F}} \mathbf{F}^T, \tag{1.30}$$

where the *pressure* $p(\mathbf{x}) = \widetilde{p}(\mathbf{X})$ is an unknown function. In other words, the constitutive equation for an incompressible elastic material contains a pressure which is unknown in the problem. This is similar to the situation in the Dynamics of Rigid Bodies. There, the assumption of material rigidity makes all components of stress tensor unknown.

From equation (1.28), having in mind the relation (1.24) and the mass conservation equation in Lagrangian form (1.12), we obtain the following expression for the first Piola-Kirchhoff stress tensor for an unconstrained elastic medium

$$\mathbf{T}_* = \rho_* \frac{\partial \psi}{\partial \mathbf{F}}. \tag{1.31}$$

We conclude that the Lagrangian equilibrium equations for an elastic system are given by (1.26) in which the first Piola-Kirchhoff stress tensor is given by (1.31).

We point out that although constitutive axioms impose severe restrictions on the constitutive equation (1.27), they still allow the theory a wide margin of arbitrariness, which can be filled only by experimental data. The experimental determination of the constitutive equations of an elastic material is an extremely complicated task. From equations (1.28) and (1.31) it can be seen that it consists in the determination of the specific strain energy function $\psi(\mathbf{F})$. On the other hand the symmetry properties of the material can simplify it to some extent. We recall that the *material symmetry group* or simply the *isotropy group* is the group of unimodular transformations (i.e. whose determinant equals 1) of the material coordinates under which the constitutive equations are invariant.[5] In this sense, the symmetry group of an *isotropic* material, i.e., a material whose properties are independent of the particular direction, is the whole group of orthogonal transformations. In mathematical terms, for any orthogonal matrix $\mathbf{Q}$ the function $\mathbf{f}(\mathbf{F})$ must satisfy

$$\mathbf{f}(\mathbf{F})= \mathbf{f}(\mathbf{FQ}).$$

It can be shown (see [23]) that an elastic material point, whose constitutive equations satisfy the objectivity principle and the second principle of thermodynamics, is an isotropic solid if and only if

$$\psi = \psi\left(I, II, III\right), \qquad \mathbf{T} = \varphi_0 \mathbf{1} + \varphi_1 \mathbf{B} + \varphi_2 \mathbf{B}^2, \tag{1.32}$$

where $I, II, III,$ are the three principal invariants of the *left Cauchy-Green tensor* $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ and functions φ_i, $i = 0, 1, 2$, are defined as follows

$$\begin{aligned}
\varphi_0 &= 2\rho III \frac{\partial \psi}{\partial III}, \\
\varphi_1 &= 2\rho \left(\frac{\partial \psi}{\partial I} + I \frac{\partial \psi}{\partial II} \right), \\
\varphi_2 &= -2\rho \frac{\partial \psi}{\partial II}.
\end{aligned} \tag{1.33}$$

A simple application of the *Cayley-Hamilton theorem* allows us to obtain the following form of $(1.32)_2$

$$\mathbf{T} = f_0 \mathbf{1} + f_1 \mathbf{B} + f_2 \mathbf{B}^{-1}, \tag{1.34}$$

[5] This form of the notion of symmetry group was introduced by W. Noll [29].

where f_i, $i = 0, 1, 2$, are related to the φ_i by the following relations:[6]

$$\begin{aligned} f_0 &= \varphi_0 - II\varphi_2, \\ f_1 &= \varphi_1 - I\varphi_2, \\ f_2 &= III\varphi_2. \end{aligned} \tag{1.35}$$

Using (1.33) in (1.35), we obtain

$$\begin{aligned} f_0 &= 2\rho\left(II\frac{\partial\psi}{\partial II} + III\frac{\partial\psi}{\partial III}\right), \\ f_1 &= 2\rho\frac{\partial\psi}{\partial I}, \\ f_2 &= -2\rho III\frac{\partial\psi}{\partial II}. \end{aligned} \tag{1.36}$$

For an incompressible isotropic solid, we have

$$\psi = \psi(I, II), \quad \mathbf{T} = -p\mathbf{1} + 2\rho\frac{\partial\psi}{\partial I}\mathbf{B} - 2\rho\frac{\partial\psi}{\partial II}\mathbf{B}^{-1}. \tag{1.37}$$

For the proof of these results and a detailed description of the symmetry properties of a continuum system we refer the reader to [13] and [23].

---

6 The Cayley-Hamilton theorem states that an $n \times n$ square matrix $\mathbf{A}$ satisfies the following identity

$$(-\mathbf{A})^n + I_1(-\mathbf{A})^{n-1} + ... + I_{n-1}(-\mathbf{A}) + I_n\mathbf{1} = \mathbf{0},$$

where $I_1, I_2, ..., I_n$ are scalar functions depending on the components of $\mathbf{A}$ and are called *principal invariants* of $\mathbf{A}$. For $n = 3$ we have

$$\mathbf{A}^3 - I_A\mathbf{A}^2 + II_A\mathbf{A} - III_A\mathbf{1} = \mathbf{0},$$

where

$$\begin{aligned} I_A &= A_{kk} = tr\mathbf{A}, \\ II_A &= \frac{1}{2}(A_{kk}A_{ll} - A_{kl}A_{lk}) = \frac{1}{2}(tr\mathbf{A})^2 - \frac{1}{2}tr\mathbf{A}^2, \\ III_A &= \det\mathbf{A}. \end{aligned}$$

In particular for the left Cauchy-Green tensor we obtain

$$\mathbf{B}^3 - I\mathbf{B}^2 + II\mathbf{B} - III\mathbf{1} = \mathbf{0}.$$

Multiplying the preceding equation by $\mathbf{B}^{-1}$ and solving for $\mathbf{B}^2$ we get

$$\mathbf{B}^2 = I\mathbf{B} - II\mathbf{I} + III\mathbf{B}^{-1},$$

which, substituted into $(1.32)_2$ gives (1.34).

1.7 Linear Elastic Bodies

The deformation $\mathbf{x}(\mathbf{X})$ is said to be *infinitesimal* if components of the displacement field (1.9) and those of the displacement gradient (1.10) are *first order quantities*, that is, if their powers or products can be neglected as compared to the quantities themselves. When the deformation $\mathbf{x}(\mathbf{X})$ from C_* to C is infinitesimal and C_* represents a *natural state* (that is $\mathbf{T} = \mathbf{0}$ in C_*), from (1.32) we derive the constitutive equation for an *isotropic linear elastic* material. We note that for an infinitesimal deformation we obtain the following expressions for $\mathbf{B}$ and $\mathbf{B}^2$:

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = (\mathbf{1} + \mathbf{H})\left(\mathbf{1} + \mathbf{H}^T\right) = \mathbf{1} + 2\mathbf{E} + ..., \tag{1.38}$$

$$\mathbf{B}^2 = \mathbf{1} + 4\mathbf{E} + ...,$$

where the symmetric tensor

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T\right) \tag{1.39}$$

is known as the *infinitesimal strain tensor* and plays a fundamental role in the theory of infinitesimal deformations. If we assume that functions φ_i, $i = 0, 1, 2$, can be approximated by their Taylor series expansions in the invariants of $\mathbf{B}$ in the neighborhood of $\mathbf{B} = \mathbf{1}$ (absence of deformation), up to first order terms in $\mathbf{H}$, we obtain

$$\varphi_i = a_i + b_i (I - 3) = a_i + 2b_i I_E, \tag{1.40}$$

where I_E is the first principal invariant of $\mathbf{E}$. The condition $\mathbf{T} = \mathbf{0}$ for $\mathbf{B} = \mathbf{1}$ requires that constants a_i and b_i satisfy the relation

$$a_1 + a_2 + a_3 = 0.$$

Hence, for small deformations of an elastic isotropic material, the stress tensor becomes

$$\mathbf{T} = \lambda I_E \mathbf{1} + 2\mu \mathbf{E}, \tag{1.41}$$

where coefficients λ and μ, which depend in a proper way on constants a_i and b_i, are called *Lamé coefficients* of the elastic material.

From (1.41), it follows

$$\text{tr}\mathbf{T} = (3\lambda + 2\mu)\,\text{tr}\mathbf{E},$$

so that, assuming $(3\lambda + 2\mu) \neq 0$, combining with (1.41) we obtain the following inverse relation

$$\mathbf{E} = \frac{1}{2\mu}\mathbf{T} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}(\text{tr}\mathbf{T})\mathbf{1}. \tag{1.42}$$

In order to provide a physical interpretation to these coefficients, we assume that the elastic body S is loaded by a uniform traction whose intesity is t along the direction parallel to the base unit vector $\mathbf{e}_1$, so that

$$\mathbf{t}_{\mathbf{e}_1} = \mathbf{T}\,\mathbf{e}_1 = t\mathbf{e}_1.$$

The only nonvanishing component of the stress tensor $\mathbf{T}$, relative to the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, is $T_{11} = t$. From (1.42) we get the deformation components

$$\begin{aligned}
E_{11} &= \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)} t, \\
E_{22} &= E_{33} = -\frac{\lambda}{2\mu(3\lambda + 2\mu)} t, \\
E_{12} &= E_{13} = E_{23} = 0.
\end{aligned}$$

The quantities

$$
\begin{aligned}
E_Y &= \frac{t}{E_{11}} = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu},\\
\nu &= -\frac{E_{22}}{E_{33}} = -\frac{E_{33}}{E_{11}} = \frac{\lambda}{2(\lambda + \mu)},
\end{aligned}
$$

are called *Young's modulus* and *Poisson's ratio*, respectively and for a homogeneous body are constants. Young's modulus indicates the ratio between the traction per unit surface area and the linear dilation produced in the same direction; Poisson's ratio is the ratio between the contraction, in the direction orthogonal to $\mathbf{t}$, and the dilation along $\mathbf{t}$.

The Cauchy stress tensor for infinitesimal deformations of an anisotropic elastic material can be written as

$$\mathbf{T} = \mathbb{C}\mathbf{E}, \tag{1.43}$$

where the *elasticity tensor* $\mathbb{C}$ is a fourth order tensor characterized by the following symmetries

$$\mathbb{C}_{ijhk} = \mathbb{C}_{ijkh} = \mathbb{C}_{jikh}.$$

These properties can be proved starting from the symmetry properties of tensors $\mathbf{T}$ and $\mathbf{E}$:

$$T_{ij} = \mathbb{C}_{ijhk}E_{hk} = \mathbb{C}_{ijkh}E_{kh} = \mathbb{C}_{ijkh}E_{hk} = T_{ji} = \mathbb{C}_{jihk}E_{hk}.$$

In particular, for an isotropic linear elastic medium, from (1.41) and (1.43) it follows that the elasticity tensor becomes

$$\mathbb{C}_{ijhk} = \lambda\delta_{ij}\delta_{hk} + \mu(\delta_{ih}\delta_{jk} + \delta_{jh}\delta_{ik}).$$

In the Theory of Elasticity it is useful to consider the *Green-Saint Venant strain tensor*

$$\mathbf{G} = \mathbf{E} + \frac{1}{2}\mathbf{H}^T\mathbf{H}. \tag{1.44}$$

It is easy to see that in an infinitesimal deformation

$$\mathbf{G} \backsimeq \mathbf{E}, \tag{1.45}$$

where the notation $\backsimeq$ means that quantities on the left and the right-hand side of (1.45) differ by an order greater than $|\mathbf{u}|$ and $|\mathbf{H}|$. The following relations between the invariants of $\mathbf{G}$ and $\mathbf{B}$ hold:

$$\begin{aligned} 2I_{\mathbf{G}} &= I - 3, \\ 4II_{\mathbf{G}} &= II - 2I + 3, \\ 8III_{\mathbf{G}} &= III - II + I - 1, \end{aligned} \tag{1.46}$$

and

$$\begin{aligned} I &= 2I_{\mathbf{G}} + 3, \\ II &= 4II_{\mathbf{G}} + 4I_{\mathbf{G}} + 3, \\ III &= 8III_{\mathbf{G}} + 4II_{\mathbf{G}} + 2I_{\mathbf{G}} + 1. \end{aligned} \tag{1.47}$$

Chapter 2
Nonlinear Elastostatics

2.1 Introduction

In this chapter we formulate the boundary-value problems of finite elastostatics. We study in depth the meaning and the difficulties related to the assignment of the traction boundary conditions. In fact, in the case of the pure traction-value problem the boundary condition depends on the unknown deformation. In order to describe this, we discuss two examples. The first example is concerned with an elastic system at equilibrium in an uniform pressure field acting upon its boundary. The second example deals with an elastic system subjected to the action of elastic surface forces.

A further complication is represented by the fact that, for the case of pure traction-value problem, the global equilibrium conditions depend also on the unknown deformation.

2.2 The Local Equilibrium Equations

Let S be an elastic system at equilibrium in a reference configuration C_*. Let us assume that under the action of body forces and surface tractions acting on a part or on the whole boundary ∂S, S reaches a new equilibrium configuration C. The task of an elastostatic problem is to determine the finite deformation $\mathbf{x}\left(\mathbf{X}\right)$, or equivalently the displacement

$\mathbf{u}(\mathbf{X})$, which S undergoes when passing from C_* to C, under the action of the aforesaid forces.

Since the function $\mathbf{x} = \mathbf{x}(\mathbf{X})$ is defined on the reference configuration C_*, it is necessary to resort to the Lagrangian equilibrium equations. Also the traction acting upon the boundary ∂C of the equilibrium configuration C can not be assigned since ∂C appears among the unknowns of the problem. At the equilibrium the following equations hold (see (1.24)):

$$\begin{cases} \nabla_{\mathbf{X}} \cdot \mathbf{T}_* + \rho_* \mathbf{b} = \mathbf{0}, & \forall \mathbf{X} \in C_*, \\ \mathbf{T}_* \cdot \mathbf{N}_* = \mathbf{t}_*, & \forall \mathbf{X} \in \partial C_*', \end{cases} \tag{2.48}$$

where ρ_* is the mass density in the reference configuration, $\mathbf{T}_*$ the first Piola-Kirchhoff stress tensor and $\mathbf{N}_*$ the outward unit vector normal to the part $\partial C_*'$ of ∂C_* on which the surface forces with density $\mathbf{t}_*$ act. (If $\partial C_*' \neq \partial C_*$, then boundary conditions on $\partial C_* - \partial C_*'$ need to be given).

For an elastic material, by virtue of (1.28), we have

$$\mathbf{T} = \rho \frac{\partial \psi}{\partial \mathbf{F}} \mathbf{F}^T, \tag{2.49}$$

from which, remembering the definition (1.24) of the first Piola-Kirchhoff stress tensor, the Lagrangian formulation of the mass equation $\rho J = \rho_*$ and the definition (1.10) of the displacement gradient $\mathbf{H}$, we obtain

$$\mathbf{T}_* = \rho_* \frac{\partial \psi(\mathbf{F})}{\partial \mathbf{F}} = \rho_* \frac{\partial \widetilde{\psi}(\mathbf{H})}{\partial \mathbf{H}}. \tag{2.50}$$

Substitution of (2.50) into $(2.48)_1$ yields the following system of three *non linear* second order partial differential equations in the three scalar unknown functions $x_j(\mathbf{X})$ which are regular in the domain C_*:

$$A_{LMij}\left(\mathbf{X},\mathbf{H}\right)\frac{\partial^2 u_j}{\partial X_L \partial X_M}+r_i\left(\mathbf{X},\mathbf{H}\right)+\rho_* b_i\left(\mathbf{X}\right)=0, \quad i=1,2,3, \tag{2.51}$$

where

$$A_{LMij}=\frac{\partial T_{*iL}}{\partial H_{jM}}=\rho_*\frac{\partial^2\psi}{\partial H_{iL}\partial H_{jM}}, \quad r_i=\left(\frac{\partial T_{*iL}}{\partial X_L}\right). \tag{2.52}$$

In particular, when the body is homogeneous in the reference configuration, the coefficients A_{LMij} do not depend explicitly on $\mathbf{X}$ and $r_i=0$, $\forall i$.

The main goal of the non linear elastostatics is to find the finite deformation $\mathbf{x}(\mathbf{X})$ of S from equations (2.51), together with boundary conditions $(2.48)_2$.

2.3 Some Considerations about Boundary Conditions

In the boundary-value problem (2.48) there is no mention of conditions on the part $\partial C''_*=\partial C-\partial C'_*$ of the boundary of S on which no surfaces forces are applied. Here, we assume that this part is fixed or deformed in a known way by virtue of suitable constraints. That is

$$\mathbf{x}\left(\mathbf{X}\right)=\mathbf{x}_0\left(\mathbf{X}\right), \qquad \forall\mathbf{X}\in\partial C''_*. \tag{2.53}$$

If $\partial C'_*=\phi$, the boundary-value problem (2.48) is a *pure displacement problem*; if $\partial C''_*=\phi$ we deal with a *pure traction-value problem*, and finally the problem is said to be *mixed* if $\partial C''_*\subset\partial C_*$.

The following remarks can be made concerning the boundary-value problems.

1. In a boundary-value problem the data are usually *assigned* functions on the boundary of the domain which is to be determined as a part of the solution. For instance, in the

Dirichlet problem for the *Laplace* equation on the domain Ω, we assign the value of the unknown function on the boundary of Ω. In the *Neumann* problem the value of the normal derivative of the unknown function is given. Finally, in the mixed problem we assign the value of the unknown function on a part of $\partial\Omega$ and the value of the normal derivative on the remaining part of $\partial\Omega$. However, the assignment of the surface traction on the boundary in a non linear elastostatic problem is not as easy as it may appear.

Let us consider, for example, an elastic system S at equilibrium in the absence of body forces (i.e., $\mathbf{b} = \mathbf{0}$) under a uniform pressure p_0 acting on the boundary ∂C of the actual configuration. The corresponding Eulerian boundary-value problem is (see (1.20))

$$\begin{cases} \nabla_{\mathbf{x}} \cdot \mathbf{T} = \mathbf{0}, & \forall \mathbf{x} \in C, \\ \mathbf{T} \cdot \mathbf{N} = -p_0 \mathbf{N}, & \forall \mathbf{x} \in \partial C'. \end{cases} \tag{2.54}$$

This boundary-value problem can be formulated in the Lagrangian form. Having in mind equation (1.5) and the definition of the first Piola-Kirchhoff stress tensor we obtain

$$T_{ij} N_j = T_{ij} J \left(\mathbf{F}^{-1}\right)_{Lj} N_{*L} \frac{d\sigma_*}{d\sigma} = T_{*iL} N_{*L} \frac{d\sigma_*}{d\sigma}.$$

From (1.5) and $(2.54)_2$, it follows that

$$t_i = T_{ij} N_j = -p_0 N_i = -p_0 J \left(\mathbf{F}^{-1}\right)_{Li} N_{*L} \frac{d\sigma_*}{d\sigma}.$$

Comparing the two preceding relations we obtain

$$T_{*iL} N_{*L} = -p_0 J \left(\mathbf{F}^{-1}\right)_{Li} N_{*L}.$$

Thus the Lagrangian formulation of the problem (2.54) becomes

$$\begin{cases} \nabla_{\mathbf{X}} \cdot \mathbf{T}_* = \mathbf{0}, & \forall \mathbf{X} \in C_*, \\ \mathbf{T}_* \cdot \mathbf{N}_* = -p_0 J \left(\mathbf{F}^{-1}\right)^T \mathbf{N}_* \equiv \mathbf{t}_*, & \forall \mathbf{X} \in \partial C'_*. \end{cases} \tag{2.55}$$

It is easy to realize that $\mathbf{t}_*$ is not a known function of the point $\mathbf{X} \in \partial C_*$. In fact it contains the unknown deformation $\mathbf{x}(\mathbf{X})$ through its gradient $\mathbf{F}$. In other words, the function $\mathbf{t}_*(\mathbf{X})$ can not be given.

Similarly, let us consider an elastic system S subjected to the action of the elastic forces $\mathbf{t}(\mathbf{x}) = -kh(\mathbf{x})\,\mathbf{i}$ (where $h(\mathbf{x})$ is the elongation of a linear spring at the point $\mathbf{X}$, k is a constant, and $\mathbf{i}$ is a unit vector) acting on the part $\partial C'$ of the boundary ∂C of its present equilibrium configuration. Because of (1.8) and having in mind that the relation between $\mathbf{t}_*$ and $\mathbf{t}$ is

$$\mathbf{t}_* d\sigma_* = \mathbf{t} d\sigma,$$

we obtain the following relation between $\mathbf{t}_*$ and the actual stress vector $\mathbf{t}$

$$\mathbf{t}_* = J\mathbf{t}\sqrt{\mathbf{N}_* \mathbf{C}^{-1} \mathbf{N}_*}. \tag{2.56}$$

Hence, the data to assign on the part $\partial C'_*$ corresponding to the part $\partial C'$ is

$$\mathbf{t}_*(\mathbf{F}, \mathbf{X}) = \mathbf{t}\frac{d\sigma}{d\sigma_*} = -J\sqrt{\mathbf{N}_* \mathbf{C}^{-1} \mathbf{N}_*}\, kh(\mathbf{x}(\mathbf{X}))\,\mathbf{i}, \tag{2.57}$$

which depends on the unknown deformation and, consequently, can not be assigned. These two examples together with the fact that equations (2.51) are essentially non linear make the boundary-value problems of elastostatics very complicated. Hence, it is natural to attempt to simplify the problems described above by limiting the analysis to those problems in which boundary conditions can be assigned. All loads, which in C_* depend on the deformation are called *live loads*; differently, loads which can be given as known functions of $\mathbf{X} \in \partial C_*$ are called *dead loads*. These dead loads have been widely studied in the literature. However, they are very difficult to realize in practice. As a matter of the fact, from the condition

$$\mathbf{t}_* \left(\mathbf{X}\right) = \frac{d\sigma}{d\sigma_*}\mathbf{t}$$

it follows that the traction $\mathbf{t}$ acting upon $\partial C'$ has to be assigned in such a way that $\mathbf{t}_*$ be independent of the deformation but depend only on $\mathbf{X}$.

2. Equations (2.48) represent *necessary conditions* for the equilibrium of S. Therefore, together with (2.48), we have to consider the *global equilibrium conditions* which express the vanishing of the resultant force and the resultant moment with respect to a point O of all forces acting on S. In order to write them, we denote by $\mathbf{\Phi}$ the reaction due to constraints which are necessary to realize the displacement (2.53). Then the following global equilibrium conditions must hold

$$\begin{gathered} \int_{C_*} \rho_* \mathbf{b} dc_* + \int_{\partial C'_*} \mathbf{t}_* d\sigma_* + \int_{\partial C'_*} \mathbf{\Phi} d\sigma_* = \mathbf{0}, \\ \int_{C_*} \rho_* \mathbf{r} \times \mathbf{b} dc_* + \int_{\partial C'_*} \mathbf{r} \times \mathbf{t}_* d\sigma_* + \int_{\partial C'_*} \mathbf{r} \times \mathbf{\Phi} d\sigma_* = \mathbf{0}. \end{gathered} \tag{2.58}$$

It is plain that for a mixed-value and a displacement-value problem the constraints have to satisfy these conditions. On the contrary, for the traction-value problem conditions (2.58) reduce to the following

$$\begin{gathered} \int_{C_*} \rho_* \mathbf{b} dc_* + \int_{\partial C_*} \mathbf{t}_* d\sigma_* = \mathbf{0}, \\ \int_{C_*} \rho_* \mathbf{r} \times \mathbf{b} dc_* + \int_{\partial C_*} \mathbf{r} \times \mathbf{t}_* d\sigma_* = \mathbf{0}. \end{gathered} \tag{2.59}$$

Since $\mathbf{t}_*$ depends on $\mathbf{x}\left(\mathbf{X}\right)$ and $\mathbf{F}$ on ∂C_* and owing to the presence of $\mathbf{r} = (\mathbf{x}(\mathbf{X}) - \mathbf{x}_0)$ in (2.59), it is not possible to establish if the data satisfy these conditions unless we know the deformation corresponding to the force system $(\mathbf{t}_*, \mathbf{b})$. For this reason, Signorini [1] suggested to regard (2.59) as *compatibility conditions*, i.e., if there exists no deformation satisfying equations (2.48) and (2.53), then the traction boundary-value problem has no solution. In the particular case of dead loads, condition $(2.59)_1$ is an "a priori" restriction upon the traction data, while $(2.59)_2$ still remains a compatibility condition due to the presence of $\mathbf{r}$.

Chapter 3
Signorini's Method for Dead Loads

3.1 A Citation

In the footnote 1 at page 118 of the Non-Linear field Theories of Mechanics by C. Truesdell and W. Noll [33] there is written:"In the period between the two great wars, knowledge of the classical theory of finite elastic strain sank so far that "engineering" paper sprouted here and there with "new" theories, all either pointlessly special or wrong. Only in Italy, due to the teaching and writing of SIGNORINI, was the true theory still widely known. The experts of the older generation, such as Hadamard, HILBERT, and HAMEL, seem to have lost interest".

3.2 Dimensional Analysis of the Equilibrium Equations

In 1930, Signorini [1] suggested a perturbation method to find approximate solutions of boundary-value problems of finite elasticity in the presence of *dead loads*. First the elastic system S is assumed to be in equilibrium, in the absence of forces, in a homogeneous, isotropic and unstressed configuration C_*. Subsequently, under the action of a system of mass forces $\mathbf{b}$ and surface tractions $\mathbf{t}$, the continuum S deforms until it assumes a new equilibrium configuration C. The basic assumption of Signorini's perturbation method is that the response of S to applied loads is not so different from its response if the system behaved like a *linear elastic material*. Consequently, the first step in order to check if the method can be applied is to write equations (2.48) in a nondimensional form. To this aim, we introduce the following *reference quantities*

$$\tilde{T}, \quad l, \quad L, \quad \tilde{b}, \quad \tilde{t}, \quad \tilde{\rho}, \tag{3.60}$$

where $\tilde{T}$ has the dimension of stress and describes the internal state of the system in C_*; l and L are lengths which represent measures for the displacements and the characteristic dimensions of the body, respectively; $\tilde{b}$ and $\tilde{t}$ are reference mass and surface forces, and finally $\tilde{\rho}$ has the dimension of mass density. If we continue to use the same notations for the nondimensional quantities, the pure traction boundary-value problem (2.48) becomes

$$\begin{aligned} \frac{\tilde{T}}{L}\nabla_* \cdot \mathbf{T}_* &= -\tilde{\rho}\rho_*\tilde{b}\mathbf{b}, \\ \tilde{T}\mathbf{T}_* \cdot \mathbf{N}_* &= \tilde{t}\mathbf{t}. \end{aligned} \tag{3.61}$$

On the other hand, if we use the Cauchy stress tensor of linear elasticity as a measure of the state of stress of the body, from the constitutive equation (1.41)

$$\mathbf{T} = \lambda I_E \mathbf{1} + 2\mu\mathbf{E},$$

in which I_E is the trace of the infinitesimal deformation tensor $\mathbf{E}$ and λ and μ are the Lamé coefficients, it follows that

$$\tilde{T} \simeq \Gamma\frac{l}{L} \equiv \alpha\Gamma,$$

where $\Gamma = \max\{\lambda, \mu\}$ and $\alpha \equiv \dfrac{l}{L}$. Substituting into (3.61) the preceding expression of $\tilde{T}$, we obtain the following system

$$\begin{aligned} \nabla_* \cdot \mathbf{T}_* &= -\frac{L\tilde{b}\tilde{\rho}}{\alpha\Gamma}\rho_*\mathbf{b}, \\ \mathbf{T}_* \cdot \mathbf{N}_* &= \frac{\tilde{t}}{\alpha\Gamma}\mathbf{t}, \end{aligned} \tag{3.62}$$

where all quantities are nondimensional. It is now easy to understand that if we want C_* to be the unperturbed state of S , we must have

$$\epsilon \equiv \frac{L\tilde{b}\tilde{\rho}}{\alpha\Gamma} \simeq \frac{\tilde{t}}{\alpha\Gamma} << 1. \tag{3.63}$$

This relation allows us to estimate the order of magnitude of the body forces and surface tractions acting upon S starting from the material of the body (Γ), its dimensions (L) and the relative magnitude of displacements ($\alpha = l/L$).

In terms of reference and nondimensional quantities, the compatibility condition $(2.59)_1$ can be written as

$$\tilde{\rho}\tilde{b}L^3 \int_{C_*} \rho_*\mathbf{b}dc_* + \tilde{t}L^2 \int_{\partial C_*} \mathbf{t}_* d\sigma_* = \mathbf{0},$$

which, dividing by $\alpha\Gamma L^2$ and having in mind the value of the small parameter ϵ given by (3.63) becomes

$$\int_{C_*} \rho_*\epsilon\mathbf{b}dc_* + \int_{\partial C_*} \epsilon\mathbf{t}_* d\sigma_* = \mathbf{0}. \tag{3.64}$$

In the same way, for the second compatibility condition $(2.59)_2$ we have

$$\tilde{\rho}\tilde{b}lL^3 \int_{C_*} \rho_*\mathbf{r} \times \mathbf{b}dc_* + \tilde{t}lL^2 \int_{\partial C_*} \mathbf{r} \times \mathbf{t}_* d\sigma_* = \mathbf{0},$$

and, dividing by $\alpha\Gamma lL^2$, we obtain

$$\int_{C_*} \rho_*\mathbf{r} \times \epsilon\mathbf{b}dc_* + \int_{\partial C_*} \mathbf{r} \times \epsilon\mathbf{t}_* d\sigma_* = \mathbf{0}. \tag{3.65}$$

3.3 Signorini's Method for Dead Loads

We note that the use of an approximation procedure makes no sense unless we have ensured that there exists at least a solution to the equilibrium problem. Consequently, the existence and uniqueness results play a crucial role within the boundary-value problems of finite elasticity. Unfortunately, the known existence and uniqueness theorems for solutions to the partial differential equations are not general enough to include the boundary-value problems we have described above. Van Buren [21] proved an existence and uniqueness theorem for the solution of the mixed problem involving dead loads. This result starts from the Banach-Caccioppoli theorem on the inverse functions and is *local*, i.e., it is valid for finite deformations which are not far from the linear deformations. Furthermore, it requires that an existence and uniqueness theorem hold for the corresponding linear problem. For a mixed boundary-value problem Van Buren showed that if the first Piola-Kirchhoff stress tensor depends analytically on the displacement gradient and the applied body and surface loads are analytical functions of the perturbation parameter ϵ then there exists a unique solution of the problem which is an analytical function of ϵ. Hence, provided that $\epsilon \ll 1$, the perturbation method can be applied if the following main hypotheses are satisfied:

1. The first Piola-Kirchhoff stress tensor $\mathbf{T}_* = J\mathbf{T}\left(\mathbf{F}^{-1}\right)^T$ depends analytically on the displacement gradient $\mathbf{H}$;
2. $\mathbf{b}(\epsilon, \mathbf{X})$ and $\mathbf{t}_*(\epsilon, \mathbf{X})$ are analytical functions of ϵ.

In order to derive a sequence of linear problems to be solved, we write

$$\mathbf{T}_* = \mathbf{A}\left(\mathbf{H}\right) = \sum_{n=1}^{\infty} \mathbf{A}_n\left(\mathbf{H}\right), \quad \mathbf{A}\left(\mathbf{0}\right) = \mathbf{0}, \tag{3.66}$$

where functions $\mathbf{A}_n\left(\mathbf{H}\right)$ are homogeneous polynomials of degree n in $\mathbf{H}$[7], and further

$$\mathbf{b} = \sum_{n=1}^{\infty} \epsilon^n \mathbf{b}_n, \quad \mathbf{t}_* = \sum_{n=1}^{\infty} \epsilon^n \mathbf{t}_{*n}, \tag{3.68}$$

$$\mathbf{u} = \sum_{n=1}^{\infty} \epsilon^n \mathbf{u}_n, \tag{3.69}$$

where series (3.66), (3.68), and (3.69) are absolutely and uniformly convergent in a proper radius of convergence.

Assuming that $\mathbf{H}_n = \nabla \mathbf{u}_n$, from (3.69) it follows that

$$\mathbf{H} = \sum_{n=1}^{\infty} \epsilon^n \mathbf{H}_n. \tag{3.70}$$

Substituting from (3.70) into (3.66), we have

$$\begin{aligned} & T_{*iL} = C_{(1)iLjM}\left(\epsilon H_{1jM} + \epsilon^2 H_{2jM} + ...\right) + \\ & + C_{(2)iLjMhN}\left(\epsilon H_{1jM} + \epsilon^2 H_{2jM} + ...\right)\left(\epsilon H_{1hN} + \epsilon^2 H_{2hN} + ...\right) + ... \\ = \; & \epsilon C_{(1)iLjM} H_{1jM} + \epsilon^2 \left(C_{(1)iLjM} H_{2jM} + C_{(2)iLjMhN} H_{1jM} H_{1hN}\right) + ... \end{aligned}$$

or equivalently

$$\mathbf{T}_* = \sum_{n=1}^{\infty} \epsilon^n \left(\mathbf{C}_{(1)} \mathbf{H}_n + \mathbf{B}_n\left(\mathbf{H}_1, ..., \mathbf{H}_{n-1}\right)\right), \tag{3.71}$$

where $\mathbf{C}_{(1)}$ is a fourth order tensor and $\mathbf{B}_n\left(\mathbf{H}_1, ..., \mathbf{H}_{n-1}\right)$, $n = 2, 3, ...$, are homogeneous polynomial of degree n in variables $\mathbf{H}_1, ..., \mathbf{H}_{n-1}$, while $\mathbf{B}_1 = \mathbf{0}$. From (3.67) it follows

[7] In fact it is sufficient to note that

$$T_{*iL} = A_{iL}\left(\mathbf{H}\right) = C_{(1)iLjM} H_{jM} + C_{(2)iLjMhN} H_{jM} H_{hN} + ... \tag{3.67}$$

also that $\mathbf{C}_{(1)}$ has to be identified with the tensor $\mathbb{C}$ of the linear theory of elasticity (see equation (1.43)), so that (3.71) becomes

$$\mathbf{T}_* = \sum_{n=1}^{\infty} \epsilon^n \left(\mathbb{C} \cdot \mathbf{E}_n + \mathbf{B}_n \left(\mathbf{H}_1, ..., \mathbf{H}_{n-1}\right)\right), \tag{3.72}$$

where $\mathbf{E}_n = \dfrac{\left(\mathbf{H}_n + \mathbf{H}_n^T\right)}{2}$.

Substituting from (3.68) and (3.72) into (2.48) with homogeneous displacement boundary condition, we obtain

$$\begin{cases} \nabla \cdot (\mathbb{C} \cdot \mathbf{E}_n) + \rho_* \hat{\mathbf{b}}_n = \mathbf{0}, \quad on\ C_*, \\ (\mathbb{C} \cdot \mathbf{E}_n) \cdot \mathbf{N}_* = \hat{\mathbf{t}}_{*n}, \quad on\ \partial C'_*, \\ \mathbf{u}_n = \mathbf{0}, \quad on\ \partial C''_*, \quad n = 1, 2, ... \end{cases} \tag{3.73}$$

where the following definitions have been made

$$\begin{cases} \rho_* \hat{\mathbf{b}}_n \equiv \rho_* \mathbf{b}_n + \nabla \cdot \mathbf{B}_n \left(\mathbf{H}_1, ..., \mathbf{H}_{n-1}\right), \\ \hat{\mathbf{t}}_{*n} \equiv \mathbf{t}_{*n} - \mathbf{B}_n \left(\mathbf{H}_1, ..., \mathbf{H}_{n-1}\right) \cdot \mathbf{N}_*. \end{cases} \tag{3.74}$$

When $n = 1$, $\hat{\mathbf{b}}_1 = \mathbf{b}_1$, $\hat{\mathbf{t}}_{*1} = \mathbf{t}_1$ and equations (3.73) coincide with those of the mixed boundary-value problem in the linear theory of elasticity. More generally, if we assume that fields $\mathbf{u}_1, ..., \mathbf{u}_{m-1}$ are solutions to the problems (3.73) for $n = 1, ..., m-1$; then, equations (3.73) written for $n = m$ define a new mixed boundary-value problem *for the same material* and for *the same domain* C_*, but with loads $\hat{\mathbf{b}}_n$, $\hat{\mathbf{t}}_{*n}$ depending in a known way on $\mathbf{u}_1, ..., \mathbf{u}_{m-1}$. In other words, the determination of the $m-th$ term of series (3.69) reduces to the solution of m mixed boundary-value problems for the same body in C_* , but with different loads. The great advantage of using Signorini's method lies in the fact that it allows us to pass from a non linear problem to a set of *linear* problems.

Van Buren's theorem can not be directly extended to the case of the pure traction-value problem for the following two reasons. First, the linear and the nonlinear pure traction boundary-value problems admit at least a solution if the applied loads are balanced , i.e.,

$$\int_{C_*} \rho_* \mathbf{b} dC_* + \int_{\partial C_*} \mathbf{t}_* d\sigma_* = \mathbf{0}, \tag{3.75}$$

$$\int_{C_*} \mathbf{r} \times \mathbf{b}\rho_* dC_* + \int_{\partial C_*} \mathbf{r} \times \mathbf{t}_* d\sigma_* = \mathbf{0}, \qquad \mathbf{r} = \mathbf{r}_* + \mathbf{u}. \tag{3.76}$$

Second, for the linear problem there does not exist a uniqueness theorem since the solution is determined to within an arbitrary infinitesimal rigid displacement (see Theorem 10.4 of [23]). This means that in order to obtain a unique solution to the linear pure traction-value problem, we have to add further conditions to the displacement. For instance, we may require that

$$\mathbf{u}(\mathbf{0}) = \mathbf{0}, \qquad \mathbf{H}(\mathbf{0}) = \mathbf{H}^T(\mathbf{0}), \tag{3.77}$$

which exclude the translation and the infinitesimal rigid rotation respectively. Concerning Signorini's method for the pure traction-value problem with dead loads, Stoppelli [6-10] proved a local existence and uniqueness theorem, and analytic dependence of the solution on a parameter, when the applied dead loads do not have an *axis of equilibrium*, and the existence and analyticity of solutions when the dead loads have an axis of equilibrium (see also Tolotti [11]). As Van Buren's theorem, Stoppelli's theorem is an application of Banach-Caccioppoli theorem on the inverse functions to the pure traction-value problem and is local. A discussion of Stoppelli's papers can be found in [12-14], and [33].

In [4] and [5] Signorini's method was used to investigate the uniqueness of the above solutions as well as the position of the classical linear theory with respect to the nonlinear theory. Later, Capriz and Podio-Guidugli [15-19] investigated the compatibility of the linear and the nonlinear elasticity theories and showed that a very large class of traction-value problems can be solved by perturbation methods of Signorini's type.

Chapter 4
Signorini's Method for Traction-Value Problems with Live Loads

4.1 Introduction

Starting from the fundamental paper [1], the research relative to pure traction-value problems (see [4-17]) has essentially been developed only in the presence of dead loads $\mathbf{b} = \mathbf{b}(\mathbf{X})$ and $\mathbf{t}_* = \mathbf{t}_*(\mathbf{X})$. Nevertheless, it is easy to realize that the physically meaningful loads depend on the deformation. Further the hypothesis of live loads in the pure traction-value problems introduces the following mathematical difficulties (see Section 2.3):

1. The boundary conditions depend on the unknown deformation.

2. The global equilibrium conditions represent compatibility conditions for the data and the displacement and can not be verified a priori.

Only since the 80's the research in finite elastostatics has been devoted to pure traction-value problems with live loads. In fact, in [18], [19], and [30] Signorini's method has been extended to live traction-value problems by Grioli, Capriz and Podio-Guidugli. In particular, in [30] Grioli studied an equilibrium problem for a heavy solid immersed in a homogeneous incompressible fluid and, later, in [31] and [32] Grioli provided a new perturbation procedure for the pure traction boundary-value problems introducing a convenient constitutive parameter. In [19], starting from two problems suggested by Grioli, Capriz and Podio-Guidugli generalized the results of [18] and presented a perturbation method for the pure traction-value problems with live loads for *almost rigid* hyperelastic bodies. An important contribution in the framework of non linear elasticity with live loads has been made by Valent in many papers which are collected in [20]. In this book, Valent extended Stoppelli's theorem to some pure traction-value problems with live loads. In particular, Valent proved some local theorems of existence, uniqueness and analytic dependence on a parameter which allow us to use Signorini's method for pure traction-value problem in which the prescribed surface traction is parallel to the normal to the boundary of the body. More precisely, within this class Valent considered the traction-value problem for loads invariant under translations and rotations, and the case of a heavy body submerged in a homogeneous liquid.

In this chapter, starting from the results of Valent [Chapter 6, 20], we study traction-value problems for a body subjected to a uniform pressure. Besides the introduction, this chapter is divided into five sections. In the second section, we provide a generalization of Signorini's method to the case of live loads. Furthermore, we formulate the boundary-value problems and the corresponding compatibility conditions (global equilibrium conditions) in order to determine the displacement of the system up to the second-order approximation. In the third section, we solve two live traction-value problems for an elastic continuous system of simple geometry (sphere and hollow cylinder) in a uniform pressure field. The obtained solutions allow us to propose four experiments for determining the second-order constitutive constants for the given material (Section 4.4). Finally, in the last two sections we present the programs `Sphere` and `HollowCylinder` which have been used to solve the described problems.

4.2 A Generalization of Signorini's Method to Live Loads

4.2.1 Dimensional Analysis

For the sake of clarity, here we reproduce briefly the same arguments as used earlier to describe the dimensional analysis for dead loads. We assume that the elastic system S is in equilibrium in the absence of forces in a homogeneous, isotropic and unstressed configuration C_*. Then, by virtue of the action of a system of mass forces $\mathbf{b}$ and surface tractions $\mathbf{t}$, S assumes a new equilibrium configuration C. We assume that the response of S is close to the linear elastic response. Consequently, the first step in order to check if Signorini's method can be applied is to write equations (2.48) in a nondimensional form. In order to do that, we introduce the following reference quantities (see section 3.1)

$$\tilde{T}, \quad l, \quad L, \quad \tilde{b}, \quad \tilde{t}, \quad \tilde{\rho}, \tag{4.78}$$

and continue to use the same notations for the nondimensional quantities, the pure traction boundary-value problem (2.48) and compatibility conditions (2.59) (see equations (3.64) and (3.65)). We write them as

$$\begin{cases} \nabla_* \cdot \mathbf{T}_* = -\epsilon \mathbf{b} & \text{in } C_*, \\ \mathbf{T}_* \cdot \mathbf{N}_* = \epsilon \mathbf{t}_* & \text{on } \partial C_*, \end{cases} \tag{4.79}$$

$$\begin{cases} \displaystyle\int_{C_*} \rho_* \epsilon \mathbf{b} dc_* + \int_{\partial C_*} \epsilon \mathbf{t}_* d\sigma_* = \mathbf{0}, \\ \displaystyle\int_{C_*} \rho_* \mathbf{r} \times \epsilon \mathbf{b} dc_* + \int_{\partial C_*} \mathbf{r} \times \epsilon \mathbf{t}_* d\sigma_* = \mathbf{0}, \end{cases} \tag{4.80}$$

where, adopting the Cauchy stress tensor $\mathbf{T}$ of the linear theory to describe the stress state, the perturbation parameter ϵ is given by

$$\epsilon \equiv \frac{L}{\alpha\Gamma}\tilde{\rho}\tilde{b} \simeq \frac{\tilde{t}}{\alpha\Gamma} \tag{4.81}$$

in which $\Gamma = \max\{\lambda, \mu\}$, λ and μ are the Lamé coefficients, and $\alpha = l/L$.

Provided that $\epsilon \ll 1$, the perturbation method can be applied if the following hypotheses are satisfied:

1. The first Piola-Kirchhoff stress tensor $\mathbf{T}_*$ depends analytically on the displacement gradient $\mathbf{H}$;

2. $\mathbf{b}(\epsilon, \mathbf{X}, \mathbf{u}, \mathbf{H})$ and $\mathbf{t}_*(\epsilon, \mathbf{X}, \mathbf{u}, \mathbf{H})$ are analytical functions of ϵ.

Note that in this case the body force and surface tractions depend on the displacement field and its gradient.

Then, if local theorems of existence and uniqueness for the problem (4.79) hold and the solution is an analytic function of ϵ which satisfies the compatibility conditions (4.80) together with the assigned loads, problem (4.79)-(4.80) reduces to solving *a set of linear pure traction boundary-value problems with dead loads*.

4.2.2 Signorini's Method for Live Traction-Value Problem

In view of the applications of Section 4.3 and 4.4, the aforesaid perturbation method will be used to solve *up to the second order* of approximation the following problem

$$\begin{cases} \nabla_* \cdot \mathbf{T}_* = \mathbf{0} \quad \text{in } C_*, \\ \mathbf{T}_* \cdot \mathbf{N}_* = \epsilon \mathbf{t}_* \quad \text{on } \partial C_*, \end{cases} \tag{4.82}$$

$$\int_{\partial C_*} \epsilon \mathbf{t}_* d\sigma_* = \mathbf{0}, \qquad \int_{\partial C_*} \mathbf{r} \times \epsilon \mathbf{t}_* d\sigma_* = \mathbf{0}, \tag{4.83}$$

where

$$\mathbf{t}_* = J\mathbf{t}\sqrt{\mathbf{N}_* \cdot \mathbf{C}^{-1}\mathbf{N}_*}, \tag{4.84}$$

(see equation (2.56)).

The first Piola-Kirchhoff stress tensor $\mathbf{T}_*$ of an elastic, isotropic and homogeneous body up to second order in $\mathbf{H}$ is given by (see [13])

$$\begin{aligned} \mathbf{T}_* &= \lambda I_{\mathbf{E}}\mathbf{I} + 2\mu\mathbf{E} + \left[\frac{\lambda}{2}\left(I_{\mathbf{H}\mathbf{H}^T} + 2I_{\mathbf{E}}^2\right) + \beta_1 I_{\mathbf{E}}^2 + \beta_2 II_{\mathbf{E}}\right]\mathbf{I} \\ &\quad + \beta_3 I_{\mathbf{E}}\mathbf{E} + \beta_4\mathbf{E}^2 - \lambda I_{\mathbf{E}}\mathbf{H}^T - \mu\left(\mathbf{H}^T\right)^2, \end{aligned} \tag{4.85}$$

where $\mathbf{E} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T\right)$ is the *infinitesimal strain tensor*, β_i, $i = 1, \cdots, 4$, are the *second order constitutive constants*, and $I_{\mathbf{E}}$ and $II_{\mathbf{E}}$ denote the first and the second principal invariants of the tensor $\mathbf{E}$, respectively.

The series expansion of the displacement $\mathbf{u}$ up to second-order terms in the small parameter ϵ is given by

$$\mathbf{u} = \epsilon\mathbf{u}_1 + \epsilon^2\mathbf{u}_2. \tag{4.86}$$

From (4.86) we can easily deduce the following relations:

$$
\begin{aligned}
\mathbf{H} &= \epsilon\mathbf{H}_1 + \epsilon^2\mathbf{H}_2, \quad \mathbf{E} = \epsilon\mathbf{E}_1 + \epsilon^2\mathbf{E}_2, \quad I_{\mathbf{E}} = \epsilon I_{\mathbf{E}_1} + \epsilon^2 I_{\mathbf{E}_2}, \\
I_{\mathbf{H}\mathbf{H}^T} &= \epsilon^2 I_{\mathbf{H}_1\mathbf{H}_1^T}, \quad I_{\mathbf{E}}^2 = \epsilon^2 I_{\mathbf{E}_1}^2, \quad II_{\mathbf{E}} = \epsilon^2 II_{\mathbf{E}_1}, \quad I_{\mathbf{E}}\mathbf{E} = \epsilon^2 I_{\mathbf{E}_1}\mathbf{E}_1, \qquad (4.87) \\
\mathbf{E}^2 &= \epsilon^2\mathbf{E}_1^2, \quad I_{\mathbf{E}}\mathbf{H}^T = \epsilon^2 I_{\mathbf{E}_1}\mathbf{H}_1^T, \quad (\mathbf{H}^T)^2 = \epsilon^2(\mathbf{H}_1^T)^2.
\end{aligned}
$$

Substituting the second-order expansions (4.87) into (4.85) we obtain

$$
\mathbf{T}_* = \epsilon\mathbf{T}_{*1} + \epsilon^2\left(\mathbf{T}_{*2} + \mathbf{B}_{*1}\right), \qquad (4.88)
$$

where

$$
\mathbf{T}_{*i} = \lambda I_{\mathbf{E}_i}\mathbf{I} + 2\mu\mathbf{E}_i \quad i = 1, 2, \qquad (4.89)
$$

$$
\mathbf{B}_{*1} = \left[\frac{\lambda}{2}\left(I_{\mathbf{H}_1\mathbf{H}_1^T} + 2I_{\mathbf{E}_1}^2\right) + \beta_1 I_{\mathbf{E}_1}^2 + \beta_2 II_{\mathbf{E}_1}\right]\mathbf{I} + \beta_3 I_{\mathbf{E}_1}\mathbf{E}_1 \qquad (4.90)
$$

$$
+\beta_4\mathbf{E}_1^2 - \lambda I_{\mathbf{E}_1}\mathbf{H}_1^T - \mu\left(\mathbf{H}_1^T\right)^2.
$$

We now derive the form of the traction (4.84) up to second order terms. First, from (4.86) we get the following expression for $\mathbf{t} = \mathbf{t}\left(\mathbf{X}, \mathbf{u}\left(\epsilon\right), \mathbf{H}\left(\epsilon\right)\right)$

$$
\epsilon\mathbf{t} = \epsilon\mathbf{t}_1\left(\mathbf{X}\right) + \epsilon^2\left[\left(\nabla_{\mathbf{u}}\mathbf{t}\right)_{\mathbf{0}}\mathbf{u}_1 + \left(\nabla_{\mathbf{H}}\mathbf{t}\right)_{\mathbf{0}}\mathbf{H}_1\right]. \qquad (4.91)
$$

Furthermore, for a matrix $\mathbf{A}$ written as

$$
\mathbf{A} = \mathbf{1} + \mathbf{S},
$$

$$
\begin{aligned}
\det\mathbf{A} &= 1 + I_{\mathbf{S}} + II_{\mathbf{S}} + III_{\mathbf{S}}, \qquad (4.92) \\
\mathbf{A}^{-1} &= \mathbf{1} - \mathbf{S} + \mathbf{S}^2 + 0(\mathbf{S}^2). \qquad (4.93)
\end{aligned}
$$

Therefore, for

$$\mathbf{F} = \mathbf{1} + \mathbf{H},$$

from (4.92) and (4.93), to within an error of third order in the components of $\mathbf{H}$, we get

$$J = \det \mathbf{F} \simeq 1 + I_{\mathbf{H}} + II_{\mathbf{H}} = 1 + I_{\mathbf{H}} + \frac{1}{2}\left[I_{\mathbf{H}}^2 - I_{\mathbf{H}^2}\right], \tag{4.94}$$

$$\mathbf{F}^{-1} = \mathbf{1} - \mathbf{H} + \mathbf{H}^2 + o\left(\mathbf{H}^2\right), \tag{4.95}$$

$$\begin{aligned} \mathbf{C}^{-1} &= \left(\mathbf{F}^T\mathbf{F}\right)^{-1} = \mathbf{F}^{-1}\left(\mathbf{F}^T\right)^{-1} \simeq \left(\mathbf{1} - \mathbf{H} + \mathbf{H}^2\right)\left(\mathbf{1} - \mathbf{H}^T + \left(\mathbf{H}^T\right)^2\right) \\ &\simeq \mathbf{1} - \left(\mathbf{H} + \mathbf{H}^T\right) + \left(\mathbf{H}^T\right)^2 + \mathbf{H}\mathbf{H}^T + \mathbf{H}^2. \end{aligned} \tag{4.96}$$

From $(4.87)_1$ it follows that

$$\mathbf{H}^2 \simeq \epsilon^2 \mathbf{H}_1^2 \tag{4.97}$$

$$I_{\mathbf{H}} = \epsilon I_{\mathbf{H}_1} + \epsilon^2 I_{\mathbf{H}_2}, \qquad I_{\mathbf{H}^2} = \epsilon^2 I_{\mathbf{H}_1^2}. \tag{4.98}$$

Substituting relations (4.98) into (4.94) we obtain

$$\begin{aligned} J &\simeq 1 + \epsilon I_{\mathbf{H}_1} + \epsilon^2 I_{\mathbf{H}_2} + \frac{1}{2}\left[\left(\epsilon I_{\mathbf{H}_1} + \epsilon^2 I_{\mathbf{H}_2}\right)^2 - \epsilon^2 I_{\mathbf{H}_1^2}\right] \\ &\simeq 1 + \epsilon I_{\mathbf{H}_1} + \epsilon^2 I_{\mathbf{H}_2} + \frac{1}{2}\epsilon^2\left(I_{\mathbf{H}_1}^2 - I_{\mathbf{H}_1^2}\right) \end{aligned}$$

and, since

$$II_{\mathbf{H}_1} = \frac{1}{2}\left(I_{\mathbf{H}_1}^2 - I_{\mathbf{H}_1^2}\right),$$

we finally come to the following second-order expression for J

$$J = 1 + \epsilon I_{\mathbf{H}_1} + \epsilon^2\left[I_{\mathbf{H}_2} + II_{\mathbf{H}_1}\right]. \tag{4.99}$$

By using equations $(4.87)_1$, $(4.87)_{10}$, and (4.97) we get

$$
\begin{aligned}
\mathbf{C}^{-1} &= \mathbf{1} - \left(\epsilon\mathbf{H}_1 + \epsilon^2\mathbf{H}_2 + \epsilon\mathbf{H}_1^T + \epsilon^2\mathbf{H}_2^T\right) + \left(\epsilon\mathbf{H}_1^T + \epsilon^2\mathbf{H}_2^T\right)^2 + \\
&\quad \left(\epsilon\mathbf{H}_1 + \epsilon^2\mathbf{H}_2\right)\left(\epsilon\mathbf{H}_1^T + \epsilon^2\mathbf{H}_2^T\right) + \left(\epsilon\mathbf{H}_1 + \epsilon^2\mathbf{H}_2\right)^2 \\
&\simeq \mathbf{1} - \left(\epsilon\mathbf{H}_1 + \epsilon^2\mathbf{H}_2 + \epsilon\mathbf{H}_1^T + \epsilon^2\mathbf{H}_2^T\right) + \left(\epsilon\mathbf{H}_1^T\right)^2 + \epsilon^2\mathbf{H}_1\mathbf{H}_1^T + \epsilon^2\mathbf{H}_1^2 \\
&\simeq \mathbf{1} - \epsilon\left(\mathbf{H}_1 + \mathbf{H}_1^T\right) + \epsilon^2\left(\mathbf{H}_1^2 + \left(\mathbf{H}_1^T\right)^2 + \mathbf{H}_1\mathbf{H}_1^T - \mathbf{H}_2 - \mathbf{H}_2^T\right),
\end{aligned}
$$

and remembering that

$$
2\mathbf{E}_i = \left(\mathbf{H}_i + \mathbf{H}_i^T\right), \qquad i = 1, 2,
$$

we have

$$
\mathbf{C}^{-1} \simeq \mathbf{1} - 2\epsilon\mathbf{E}_1 - \epsilon^2\left(2\mathbf{E}_2 - \mathbf{H}_1^2 - \mathbf{H}_1\mathbf{H}_1^T - \left(\mathbf{H}_1^T\right)^2\right). \tag{4.100}
$$

From (4.100) it follows that

$$
\mathbf{N}_* \cdot \mathbf{C}^{-1}\mathbf{N}_* = 1 - 2\epsilon\mathbf{N}_* \cdot \mathbf{E}_1\mathbf{N}_* - \epsilon^2\mathbf{N}_* \cdot \left(2\mathbf{E}_2 - \mathbf{H}_1^2 - \mathbf{H}_1\mathbf{H}_1^T - \left(\mathbf{H}_1^T\right)^2\right)\mathbf{N}_*.
$$

Using the Taylor series expansion

$$
\sqrt{1 - 2a\epsilon - b\epsilon^2} = 1 - \epsilon a - \left(\frac{a^2 + b}{2}\right)\epsilon^2 + o(\epsilon^2),
$$

and introducing the following constants

$$
\begin{aligned}
a &= \mathbf{N}_* \cdot \mathbf{E}_1\mathbf{N}_* \\
b &= \mathbf{N}_* \cdot \left(2\mathbf{E}_2 - \mathbf{H}_1^2 - \mathbf{H}_1\mathbf{H}_1^T - \left(\mathbf{H}_1^T\right)^2\right)\mathbf{N}_*,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\sqrt{\mathbf{N}_* \cdot \mathbf{C}^{-1}\mathbf{N}_*} &= \sqrt{1 - 2a\epsilon - \epsilon^2 b} \\
&\simeq 1 - \epsilon a - \left(\frac{a^2 + b}{2}\right)\epsilon^2.
\end{aligned} \tag{4.101}
$$

Thus from equations (4.91), (4.99), and (4.101) the relation (4.84) can be written as

$$\begin{aligned}\epsilon \mathbf{t}_* &= J\epsilon \mathbf{t}\sqrt{\mathbf{N}_* \cdot \mathbf{C}^{-1}\mathbf{N}_*} \simeq \left(1 + \epsilon I_{\mathbf{H}_1} + \epsilon^2 \left[I_{\mathbf{H}_2} + II_{\mathbf{H}_1}\right]\right) \\ &\quad \left(\epsilon \mathbf{t}_1(\mathbf{X}) + \epsilon^2 \mathbf{f}\right)\left[1 - \epsilon a - \left(\frac{a^2 + b}{2}\right)\epsilon^2\right],\end{aligned}$$

where

$$\mathbf{f} = (\nabla_{\mathbf{u}}\mathbf{t})_{\mathbf{0}}\,\mathbf{u}_1 + (\nabla_{\mathbf{H}}\mathbf{t})_{\mathbf{0}}\,\mathbf{H}_1.$$

Retaining only the terms up to second order in ϵ we obtain

$$\begin{aligned}\epsilon \mathbf{t}_* &= \left(\epsilon \mathbf{t}_1 + \epsilon^2 \mathbf{f} + \epsilon^2 I_{\mathbf{H}_1}\mathbf{t}_1\right)\left[1 - \epsilon a - \left(\frac{a^2 + b}{2}\right)\epsilon^2\right] \\ &\simeq \epsilon \mathbf{t}_1 - \epsilon^2 \mathbf{t}_1 a + \epsilon^2 \mathbf{f} + \epsilon^2 I_{\mathbf{H}_1}\mathbf{t}_1 \\ &= \epsilon \mathbf{t}_1 + \epsilon^2\left[I_{\mathbf{H}_1}\mathbf{t}_1 - \mathbf{t}_1 \mathbf{N}_* \cdot \mathbf{E}_1 \mathbf{N}_* + (\nabla_{\mathbf{u}}\mathbf{t})_{\mathbf{0}}\,\mathbf{u}_1 + (\nabla_{\mathbf{H}}\mathbf{t})_{\mathbf{0}}\,\mathbf{H}_1\right].\end{aligned}$$

Thus, the the traction (4.84) assumes the following form up to second order terms

$$\epsilon \mathbf{t}_* = \epsilon \mathbf{t}_{*1} + \epsilon^2 \mathbf{t}_{*2}, \tag{4.102}$$

where

$$\begin{aligned}\mathbf{t}_{*1} &= \mathbf{t}_1, \\ \mathbf{t}_{*2} &= I_{\mathbf{H}_1}\mathbf{t}_1 - \mathbf{t}_1 \mathbf{N}_* \cdot \mathbf{E}_1 \mathbf{N}_* + (\nabla_{\mathbf{u}}\mathbf{t})_{\mathbf{0}}\,\mathbf{u}_1 + (\nabla_{\mathbf{H}}\mathbf{t})_{\mathbf{0}}\,\mathbf{H}_1.\end{aligned} \tag{4.103}$$

Finally, because of equations (4.88) and (4.102), problem (4.82) reduces to solving the following two *linear boundary-value problems with dead loads*

$$\begin{cases} \nabla_* \cdot \mathbf{T}_{*1} = \mathbf{0} \quad \text{in } C_*, \\ \mathbf{T}_{*1} \cdot \mathbf{N}_* = \mathbf{t}_{*1} \quad \text{on } \partial C_*, \end{cases} \tag{4.104}$$

$$\begin{cases} \nabla_* \cdot (\mathbf{T}_{*2} + \mathbf{B}_{*1}) = \mathbf{0} \quad \text{in } C_*, \\ (\mathbf{T}_{*2} + \mathbf{B}_{*1}) \cdot \mathbf{N}_* = \mathbf{t}_{*2} \quad \text{on } \partial C_*, \end{cases} \tag{4.105}$$

respectively with the following compatibility conditions

$$\begin{aligned}\int\limits_{\partial C_*} \mathbf{t}_{*1} d\sigma_* &= \mathbf{0}, \\ \int\limits_{\partial C_*} (\mathbf{X} + \mathbf{u}_1) \times \mathbf{t}_{*1} d\sigma_* &= \mathbf{0},\end{aligned} \tag{4.106}$$

$$\begin{aligned}\int\limits_{\partial C_*} \mathbf{t}_{*2} d\sigma_* &= \mathbf{0}, \\ \int\limits_{\partial C_*} \left[(\mathbf{X} + \mathbf{u}_1) \times \mathbf{t}_{*2} + \mathbf{u}_2 \times \mathbf{t}_{*1}\right] d\sigma_* &= \mathbf{0}.\end{aligned} \tag{4.107}$$

Here moments are taken with respect to the origin of the coordinate system.

In the next section, we solve the problem (4.82)-(4.83) only when S is subjected to a *uniform pressure*. Furthermore, since the uniform pressure fields are live loads which are invariant under translations, the theorems of existence, uniqueness, and continuous dependence on a parameter, proved in [20], hold for the problem (4.82)-(4.83).

Equations (4.106) are compatibility conditions imposing restrictions both on the applied forces and on the corresponding displacement which is solution to the problem (4.104). In other words, in (4.106) the following relations

$$\int\limits_{\partial C_*} \mathbf{t}_{*1} d\sigma_* = \mathbf{0}, \quad \int\limits_{\partial C_*} \mathbf{X} \times \mathbf{t}_{*1} d\sigma_* = \mathbf{0},$$

representing a restriction on the applied forces, have to be verified a priori. However, the condition which guarantees the physical meaning of the displacement $\mathbf{u}_1$

$$\int\limits_{\partial C_*} \mathbf{u}_1 \times \mathbf{t}_{*1} d\sigma_* = \mathbf{0},$$

can only be verified a posteriori.

The inspection of the compatibility conditions (4.107) can only be made a posteriori.

4.3 Two Live Traction-Value Problems

In this section problems (4.104)-(4.107) are solved for the case of a continuum with spherical and cylindrical geometry. Then, starting from the obtained solutions, some experimental procedures are suggested to determine the constitutive constants β_i of the body.

4.3.1 The First Traction-Value Problem: Sphere

Let S be a sphere of radius R made of an elastic, homogeneous, and isotropic material. Suppose S be at equilibrium in a current configuration C under the action of a uniform pressure field

$$\mathbf{t} = -p_0 \mathbf{N}, \tag{4.108}$$

where p_0 is a positive constant and $\mathbf{N}$ is the unit outward normal vector to the boundary ∂C of S (see Figure 4.1).

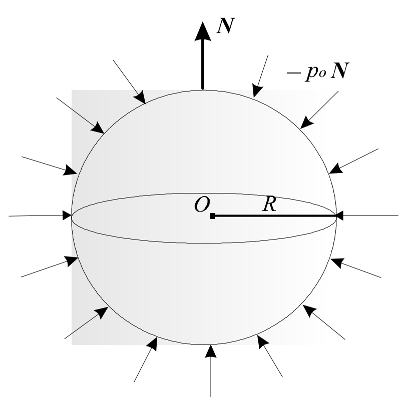

Fig. 4.1. A sphere loaded by a uniform pressure.

In order to write the Lagrangian equilibrium equations in a nondimensional form, it is useful to introduce the following characteristic quantities (see (4.78))

$$\begin{aligned} \tilde{t} &= \tilde{p}, \quad l = \tilde{u}(R), \\ \tilde{T} &= \max\{\lambda, \mu\} \equiv \Gamma, \quad L = R, \end{aligned} \tag{4.109}$$

where $\tilde{p}$ is a pressure which induces a linear response and $\tilde{u}(R)$ is the displacement of the boundary of S in the linear approximation.

From (4.81) and (4.109), the perturbation parameter ϵ is given by

$$\epsilon = \frac{\tilde{p}}{\alpha\Gamma}. \tag{4.110}$$

Owing to spherical symmetry, we search for the solution of pure traction-value problem (4.79)-(4.80) in the following form

$$\mathbf{u}(r) = \left[\epsilon u_1(r) + \epsilon^2 u_2(r)\right] \mathbf{a}_r, \tag{4.111}$$

where $\mathbf{a}_r$ is the radial unit vector of the physical basis associated with the spherical coordinates $\{r, \varphi, \theta\}$.

The first-order boundary-value problem (4.104) is

$$\begin{cases} \nabla_* \cdot \mathbf{T}_{*1} = \mathbf{0} & \text{in } C_*, \\ \mathbf{T}_{*1} \cdot \mathbf{N}_* = \mathbf{t}_{*1} & \text{on } \partial C_*. \end{cases} \tag{4.112}$$

In order to write the problem (4.112) in spherical coordinates we start by writing the metric tensor (g_{ij}) associated with the natural (or holonomic) basis $(\mathbf{e}_i)$.

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2\sin^2\theta & 0 \\ 0 & 0 & r^2 \end{pmatrix}, (g^{ij}) = (g_{ij})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2\sin^2\theta} & 0 \\ 0 & 0 & \frac{1}{r^2} \end{pmatrix}. \tag{4.113}$$

First, we recall that the relation between the unit vectors of the natural basis $(\mathbf{e}_i)$ and the unit vectors $(\mathbf{a}_i)$ of the physical basis is[8]

$$\mathbf{a}_i = \frac{1}{\sqrt{g_{ii}}}\mathbf{e}_i \quad \text{(no sum over } i\text{)}, \tag{4.114}$$

while the relation between the reciprocal basis is

$$\mathbf{a}^i = \sqrt{g_{ii}}\mathbf{e}^i \quad \text{(no sum over } i\text{)}. \tag{4.115}$$

The covariant components of the first-order displacement gradient $\mathbf{H}_1$ in the basis $\mathbf{e}^i \otimes \mathbf{e}^j$ are given by

$$(\mathbf{H}_1)_{ij} = \frac{\partial u_i}{\partial x_j} - \Gamma^h_{ij} u_h \tag{4.116}$$

where the *Christoffel symbols* Γ^h_{ij} are related to the metric coefficients g_{ij} and g^{ij} by the following formulae

$$\Gamma^l_{jh} = \frac{1}{2} g^{li}(g_{ij,h} + g_{hi,j} - g_{jh,i}). \tag{4.117}$$

From (4.113) it follows that the non-zero Christoffel symbols are

$$\Gamma^3_{31} = \Gamma^3_{13} = \Gamma^2_{21} = \Gamma^2_{12} = \frac{1}{r}, \quad \Gamma^1_{22} = -r\sin^2\theta, \quad \Gamma^1_{33} = -r, \tag{4.118}$$

$$\Gamma^2_{23} = \Gamma^2_{32} = \cot\theta, \quad \Gamma^3_{22} = -\sin\theta\cos\theta. \tag{4.119}$$

It is now an easy task to compute from (4.116), (4.118), and (4.119) the covariant components of the first-order displacement gradient $\mathbf{H}_1$ in the basis $\mathbf{e}^i \otimes \mathbf{e}^j$

$$
\begin{aligned}
(\mathbf{H}_1)_{11} &= u_1', \quad (\mathbf{H}_1)_{22} = r\sin^2\theta\, u_1, \quad (\mathbf{H}_1)_{33} = r\, u_1, \\
(\mathbf{H}_1)_{ij} &= 0, \quad i \neq j,
\end{aligned}
$$

[8] Note that from (4.113) and (4.114) it follows that

$$\mathbf{a}_1 = \mathbf{e}_1, \quad \mathbf{a}_2 = \frac{\mathbf{e}_2}{r\sin\theta}, \quad \mathbf{a}_3 = \frac{\mathbf{e}_3}{r}.$$

where $u_1' = \dfrac{du_1}{dr}$. The non-zero contravariant components of $\mathbf{H}_1$ are given by the relation

$$(\mathbf{H}_1)^{ij} = g^{ik} g^{jh} (\mathbf{H}_1)_{kh}$$

and, from (4.113), they assume the following values

$$(\mathbf{H}_1)^{11} = u_1', \quad (\mathbf{H}_1)^{22} = \frac{u_1}{r^3 \sin^2 \theta}, \quad (\mathbf{H}_1)^{33} = \frac{u_1}{r^3}.$$

Hence, we can write

$$\mathbf{H}_1 = u_1' \mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{u_1}{r^3 \sin^2 \theta} \mathbf{e}_2 \otimes \mathbf{e}_2 + \frac{u_1}{r^3} \mathbf{e}_3 \otimes \mathbf{e}_3$$

and, from (4.114), we finally obtain

$$\mathbf{H}_1 = u_1' \mathbf{a}_1 \otimes \mathbf{a}_1 + \frac{u_1}{r} \mathbf{a}_2 \otimes \mathbf{a}_2 + \frac{u_1}{r} \mathbf{a}_3 \otimes \mathbf{a}_3,$$

or in matrix notation

$$\mathbf{H}_1 = \begin{pmatrix} u_1' & 0 & 0 \\ 0 & \dfrac{u_1}{r} & 0 \\ 0 & 0 & \dfrac{u_1}{r} \end{pmatrix}. \tag{4.120}$$

From (4.120) we have

$$I_{\mathbf{H}_1} = u_1' + 2\frac{u_1}{r}, \quad \mathbf{E}_1 = \mathbf{H}_1,$$

and from (4.89) we obtain

$$\mathbf{T}_{*1} = \begin{pmatrix} (\lambda + 2\mu) u_1' + 2\lambda \dfrac{u_1}{r} & 0 & 0 \\ 0 & \lambda u_1' + 2(\lambda + \mu) \dfrac{u_1}{r} & 0 \\ 0 & 0 & \lambda u_1' + 2(\lambda + \mu) \dfrac{u_1}{r} \end{pmatrix}. \tag{4.121}$$

We recall that the physical components of the divergence of a symmetric tensor field $\mathbf{T}$ in spherical coordinates are given by (see [33])

$$
\begin{aligned}
(\nabla\cdot\mathbf{T})_r &= \frac{\partial T_{rr}}{\partial r}+\frac{1}{r}\frac{\partial T_{r\theta}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial T_{r\varphi}}{\partial\varphi}+\\
&\quad \frac{1}{r}(2T_{rr}-T_{\theta\theta}-T_{\varphi\varphi}+\cot\theta T_{r\theta}),\\
(\nabla\cdot\mathbf{T})_\theta &= \frac{\partial T_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial T_{\theta\varphi}}{\partial\varphi}+\\
&\quad \frac{1}{r}[3T_{r\theta}+\cot\theta(T_{\theta\theta}-T_{\varphi\varphi})],\\
(\nabla\cdot\mathbf{T})_\varphi &= \frac{\partial T_{r\varphi}}{\partial r}+\frac{1}{r}\frac{\partial T_{\theta\varphi}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial T_{\varphi\varphi}}{\partial\varphi}+\\
&\quad \frac{1}{r}(3T_{r\varphi}+2\cot\theta T_{\theta\varphi}).
\end{aligned}
\tag{4.122}
$$

It can be easily seen from (4.121) that the relations (4.122) assume the following form

$$
\begin{aligned}
(\nabla\cdot\mathbf{T}_{*1})_r &= \frac{d(\mathbf{T}_{*1})_{rr}}{dr}+\frac{1}{r}[2(\mathbf{T}_{*1})_{rr}-(\mathbf{T}_{*1})_{\theta\theta}-(\mathbf{T}_{*1})_{\varphi\varphi}]\\
&= (\lambda+2\mu)\left(u_1''+\frac{2}{r}u_1'-\frac{2}{r^2}u_1\right),\\
(\nabla\cdot\mathbf{T}_{*1})_\theta &= 0,\quad (\nabla\cdot\mathbf{T}_{*1})_\varphi=0,
\end{aligned}
\tag{4.123}
$$

and, having in mind that $\mathbf{N}_*=(1,0,0)^T$,

$$
\mathbf{T}_{*1}\cdot\mathbf{N}_*=(\lambda+2\mu)\,u_1'+\frac{2\lambda}{r}u_1. \tag{4.124}
$$

Furthermore, from (4.108) and $(4.103)_1$ it follows that

$$
\mathbf{t}_{*1}=-p_0\mathbf{N}_*. \tag{4.125}
$$

Hence, from (4.123) and (4.124), and (4.125), the boundary-value problem (4.112) becomes

$$
\begin{cases}
r^2u_1''+2ru_1'-2u_1=0,\\
\left[(\lambda+2\mu)\,u_1'+\dfrac{2\lambda}{r}u_1\right]_{r=R}=-p_0.
\end{cases}
\tag{4.126}
$$

The general integral of the differential equation $(4.126)_1$ is

$$u_1(r) = \bar{A}r + \frac{\bar{B}}{r^2}. \tag{4.127}$$

Substituting from (4.127) into the boundary condition $(4.126)_2$ and due to the spherical symmetry of the problem we must have

$$u(0) = 0,$$

by routine calculations we get

$$\bar{A} = -\frac{p_0}{3\lambda + 2\mu}, \quad \bar{B} = 0.$$

Thus, the solution of system (4.126) is

$$u_1(r) = -\frac{p_0}{3\lambda + 2\mu} r. \tag{4.128}$$

On the other hand, $\mathbf{t}_{*1} = -p_0 \mathbf{N}_*$, so that from (4.128) we easily verify that the first-order compatibility conditions (4.106) are satisfied.

The second-order boundary-value problem is

$$\begin{cases} \nabla_* \cdot (\mathbf{T}_{*2} + \mathbf{B}_{*1}) = \mathbf{0} & \text{in } C_*, \\ (\mathbf{T}_{*2} + \mathbf{B}_{*1}) \cdot \mathbf{N}_* = \mathbf{t}_{*2} & \text{on } \partial C_*. \end{cases} \tag{4.129}$$

From the first-order displacement (4.128) we obtain

$$\begin{aligned} \mathbf{H}_1 &= \mathbf{E}_1 = \bar{A}\mathbf{1}, \quad \mathbf{E}_1^2 = (\mathbf{H}_1^T)^2 = \bar{A}^2\mathbf{1}, \\ I_{\mathbf{H}_1} &= I_{\mathbf{E}_1} = 3\bar{A}, \quad I_{\mathbf{H}_1\mathbf{H}_1^T} = 3\bar{A}^2, \quad II_{\mathbf{E}_1} = 3\bar{A}^2. \end{aligned} \tag{4.130}$$

Substitution of expressions (4.130) into (4.90) and $(4.103)_2$ yields

$$\begin{aligned} \mathbf{B}_{*1} &= \Lambda \bar{A}^2 \mathbf{1}, \\ \mathbf{t}_{*2} &= -2p_0 \bar{A} \mathbf{N}_*, \end{aligned} \tag{4.131}$$

where

$$\Lambda = \frac{15}{2}\lambda - \mu + 9\beta_1 + 3\beta_2 + 3\beta_3 + \beta_4.$$

Since Λ and $\bar{A}$ are constants, equation $(4.131)_1$ implies that

$$\nabla_* \cdot \mathbf{B}_{*1} = \mathbf{0}. \tag{4.132}$$

Using the same arguments as those used to derive the expression of the divergence of the first-order stress tensor we obtain

$$\begin{aligned}
(\nabla \cdot \mathbf{T}_{*2})_r &= \frac{\partial (\mathbf{T}_{*2})_{rr}}{\partial r} + \frac{1}{r}[2(\mathbf{T}_{*2})_{rr} - (\mathbf{T}_{*2})_{\theta\theta} - (\mathbf{T}_{*2})_{\varphi\varphi}] \\
&= (\lambda + 2\mu)\left(u_2'' + \frac{2}{r}u_2' - \frac{2}{r^2}u_2\right), \\
(\nabla \cdot \mathbf{T}_{*2})_\theta &= 0, \quad (\nabla \cdot \mathbf{T}_{*2})_\varphi = 0.
\end{aligned} \tag{4.133}$$

Furthermore

$$\mathbf{T}_{*2} \cdot \mathbf{N}_* = (\lambda + 2\mu)\, u_2' + \frac{2\lambda}{r} u_2. \tag{4.134}$$

Therefore, from (4.133), (4.132) and (4.131), the second-order boundary-value problem (4.105) becomes

$$\left\{ \begin{array}{l} r^2 u_2'' + 2r u_2' - 2u_2 = 0, \\ \left[(\lambda + 2\mu)\, u_2' + \dfrac{2\lambda}{r} u_2\right]_{r=R} + \Lambda \bar{A}^2 = -2p_0 \bar{A}, \end{array} \right. \tag{4.135}$$

and it admits the following solution

$$u_2\,(r) = \left(2\bar{A}^2 + \frac{\Lambda \bar{A}^3}{p_0}\right) r. \tag{4.136}$$

From $(4.131)_2$ and (4.136), it is easy to verify that the second-order compatibility conditions (4.107) are also satisfied.

In conclusion, the pure second-order traction-value problem has the solution

$$\mathbf{u} = \left[\bar{A}\epsilon + \left(2\bar{A}^2 + \frac{\Lambda \bar{A}^3}{p_0}\right)\epsilon^2\right] r\, \mathbf{a}_r. \tag{4.137}$$

It is influenced by the second-order elastic constants through the presence of Λ in (4.137). Even if all the second-order elastic constants vanish, the second-order displacement field is non-zero.

4.3.2 The Second Traction-Value Problem: Hollow Cylinder

We consider an infinite hollow cylinder S made of an elastic, homogeneous and isotropic material. Let R_i and R_e be the internal and the external radii, respectively. Let S be at equilibrium in the current configuration C under the action of the uniform pressure field

$$\mathbf{t} = \begin{cases} -p_i \mathbf{N}_i & \text{on } \partial C_i, \\ -p_e \mathbf{N}_e & \text{on } \partial C_e, \end{cases} \tag{4.138}$$

where p_i and p_e are positive constants, $\mathbf{N}_i$ and $\mathbf{N}_e$ are the unit outward normal vectors to ∂C_i and ∂C_e, respectively (see Figure 4.2).

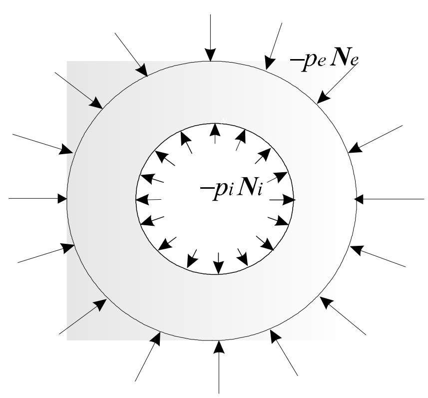

Fig. 4.2. Schematic sketch of a circular hollow cylinder subjected to pressure on the inner and outer surfaces.

Adopting the same arguments as in the previous section, assuming that $L = R_e, \tilde{t} = \hat{p}$ and $l = \tilde{u}\,(R_e)$, where $\tilde{u}\,(r)$ is an infinitesimal displacement, the parameter ϵ can be written as

$$\epsilon = \frac{\hat{p}}{\alpha\Gamma}.$$

Adopting the cylindrical coordinates $\{r, \theta, z\}$, the following first-order pure traction-value problem has to be solved

$$\begin{cases} \nabla_* \cdot \mathbf{T}_{*1} = \mathbf{0} \quad \text{in } C_*, \\ \mathbf{T}_{*1} \cdot \mathbf{N}_{*i} = \mathbf{t}_{*1}^{(i)} \quad \text{on } \partial C_{*i}, \\ \mathbf{T}_{*1} \cdot \mathbf{N}_{*e} = \mathbf{t}_{*1}^{(e)} \quad \text{on } \partial C_{*e}, \end{cases} \tag{4.139}$$

for the unknown displacement $\mathbf{u}_1(r) = u_1(r)\,\mathbf{a}_r$, where $\mathbf{a}_r$ is the radial unit vector of the physical basis $\{\mathbf{a}_r, \mathbf{a}_\theta, \mathbf{a}_z\}$ associated with the cylindrical coordinates. In order to write the problem (4.139) in cylindrical coordinates we start by writing the metric tensor (g_{ij}) associated with the natural basis $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$[9]

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, (g^{ij}) = (g_{ij})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.140}$$

From (4.140) and (4.117) it follows that the non-zero Christoffel symbols are

$$\Gamma^1_{22} = -r, \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}. \tag{4.141}$$

It is now easy to see from (4.116) and (4.141) that the non-zero covariant components of the first-order displacement gradient $\mathbf{H}_1$ in the basis $\mathbf{e}^i \otimes \mathbf{e}^j$ are

$$(\mathbf{H}_1)_{11} = u_1', \quad (\mathbf{H}_1)_{22} = r u_1.$$

The corresponding contravariant components of $\mathbf{H}_1$ are given by the relation

$$(\mathbf{H}_1)^{ij} = g^{ik} g^{jh} (\mathbf{H}_1)_{kh}$$

---

[9] Note that from (4.140) and (4.114) it follows that

$$\mathbf{a}_r = \mathbf{e}_r, \quad \mathbf{a}_\theta = \frac{\mathbf{e}_\theta}{r}, \quad \mathbf{a}_z = \mathbf{e}_z.$$

and, from (4.140), they take the following values

$$(\mathbf{H}_1)^{11} = u_1', \quad (\mathbf{H}_1)^{22} = \frac{u_1}{r^3}.$$

Hence, we can write

$$\mathbf{H}_1 = u_1'\mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{u_1}{r^3}\mathbf{e}_2 \otimes \mathbf{e}_2,$$

and, from (4.114), we obtain

$$\mathbf{H}_1 = u_1'\mathbf{a}_1 \otimes \mathbf{a}_1 + \frac{u_1}{r}\mathbf{a}_2 \otimes \mathbf{a}_2,$$

or in matrix notation

$$\mathbf{H}_1 = \begin{pmatrix} u_1' & 0 & 0 \\ 0 & \dfrac{u_1}{r} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.142}$$

From (4.142) we have

$$I_{\mathbf{H}_1} = u_1' + \frac{u_1}{r}, \quad \mathbf{E}_1 = \mathbf{H}_1,$$

and from (4.89) we get

$$\mathbf{T}_{*1} = \begin{pmatrix} (\lambda+2\mu)u_1' + \lambda\dfrac{u_1}{r} & 0 & 0 \\ 0 & \lambda u_1' + (\lambda+2\mu)\dfrac{u_1}{r} & 0 \\ 0 & 0 & \lambda\left(u_1' + \dfrac{u_1}{r}\right) \end{pmatrix}. \tag{4.143}$$

We recall that the physical components of the divergence of a symmetric tensor field $\mathbf{T}$ in cylindrical coordinates are given by (see [33])

$$\begin{aligned} (\nabla\cdot\mathbf{T})_r &= \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial\theta} + \frac{\partial T_{rz}}{\partial z} + \frac{T_{rr}-T_{\theta\theta}}{r}, \\ (\nabla\cdot\mathbf{T})_\theta &= \frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial\theta} + \frac{\partial T_{\theta z}}{\partial z} + \frac{2}{r}T_{r\theta}, \\ (\nabla\cdot\mathbf{T})_z &= \frac{\partial T_{rz}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta z}}{\partial\theta} + \frac{\partial T_{zz}}{\partial z} + \frac{1}{r}T_{rz}. \end{aligned} \tag{4.144}$$

It can be easily seen from (4.143) that quantities in (4.144) assume the following values

$$
\begin{aligned}
(\nabla\cdot\mathbf{T}_{*1})_r &= \frac{\partial(\mathbf{T}_{*1})_{rr}}{\partial r}+\frac{(\mathbf{T}_{*1})_{rr}-(\mathbf{T}_{*1})_{\theta\theta}}{r}\\
&= (\lambda+2\mu)\left(u_1''+\frac{u_1'}{r}-\frac{u_1}{r^2}\right),\\
(\nabla\cdot\mathbf{T}_{*1})_\theta &= 0,\quad (\nabla\cdot\mathbf{T}_{*1})_z=0,
\end{aligned}
\tag{4.145}
$$

and, since

$$
\mathbf{N}_{*i}=(-1,0,0)^T,\quad \mathbf{N}_{*e}=(1,0,0)^T,
\tag{4.146}
$$

we have

$$
\begin{aligned}
\mathbf{T}_{*1}\cdot\mathbf{N}_{*i} &= -\left[(\lambda+2\mu)\,u_1'+\frac{\lambda}{r}u_1\right],\\
\mathbf{T}_{*1}\cdot\mathbf{N}_{*e} &= (\lambda+2\mu)\,u_1'+\frac{\lambda}{r}u_1.
\end{aligned}
\tag{4.147}
$$

Furthermore from (4.138) and $(4.103)_1$ we have

$$
\mathbf{t}_{*1}^{(i)}=-p_i\mathbf{N}_{*i},\quad \mathbf{t}_{*1}^{(e)}=-p_e\mathbf{N}_{*e}.
\tag{4.148}
$$

Hence, from (4.145), (4.147) and (4.148), the boundary-value problem (4.139) becomes

$$
\begin{cases}
r^2u_1''+ru_1'-u_1=0,\\
\left[(\lambda+2\mu)\,u_1'+\dfrac{\lambda}{r}u_1\right]_{r=R_i}=-p_i,\\
\left[(\lambda+2\mu)\,u_1'+\dfrac{\lambda}{r}u_1\right]_{r=R_e}=-p_e.
\end{cases}
\tag{4.149}
$$

The general solution of (4.149) is

$$
u_1\,(r)=Ar+\frac{B}{r}.
\tag{4.150}
$$

Substituting (4.150) into boundary conditions $(4.149)_{2,3}$, we obtain following values of constants A and B

$$A = \frac{R_e^2 p_e - R_i^2 p_i}{2\,(\lambda+\mu)\,(R_i^2 - R_e^2)}, \quad B = \frac{R_e^2 R_i^2\,(p_e - p_i)}{2\mu\,(R_i^2 - R_e^2)}. \tag{4.151}$$

The second-order boundary-value problem is

$$\begin{cases} \nabla_* \cdot (\mathbf{T}_{*2} + \mathbf{B}_{*1}) = \mathbf{0} \quad \text{in } C_*, \\ (\mathbf{T}_{*2} + \mathbf{B}_{*1}) \cdot \mathbf{N}_{*i} = \mathbf{t}_{*2}^{(i)} \quad \text{on } \partial C_{*i}, \\ (\mathbf{T}_{*2} + \mathbf{B}_{*1}) \cdot \mathbf{N}_{*e} = \mathbf{t}_{*2}^{(e)} \quad \text{on } \partial C_{*e}, \end{cases} \tag{4.152}$$

Adopting the same arguments that brought us to the expression of the first-order stress tensor (4.143), we obtain

$$\mathbf{T}_{*2} = \begin{pmatrix} (\lambda+2\mu)u_2' + \lambda\dfrac{u_2}{r} & 0 & 0 \\ 0 & \lambda u_2' + (\lambda+2\mu)\dfrac{u_2}{r} & 0 \\ 0 & 0 & \lambda\left(u_2' + \dfrac{u_2}{r}\right) \end{pmatrix}, \tag{4.153}$$

from which we get

$$\begin{aligned} (\nabla \cdot \mathbf{T}_{*2})_r &= \frac{\partial (\mathbf{T}_{*2})_{rr}}{\partial r} + \frac{(\mathbf{T}_{*2})_{rr} - (\mathbf{T}_{*2})_{\theta\theta}}{r} \\ &= (\lambda+2\mu)\left(u_2'' + \frac{u_2'}{r} - \frac{u_2}{r^2}\right), \\ (\nabla \cdot \mathbf{T}_{*2})_\theta &= 0, \quad (\nabla \cdot \mathbf{T}_{*2})_z = 0. \end{aligned} \tag{4.154}$$

From the first-order displacement (4.150) we obtain

$$I_{\mathbf{H}_1} = I_{\mathbf{E}_1} = 2A, \quad I_{\mathbf{H}_1\mathbf{H}_1^T} = 2\left(A^2 + \frac{B^2}{r^4}\right), \quad II_{\mathbf{E}_1} = A^4 - \frac{B^2}{r^4}. \tag{4.155}$$

Substituting relations (4.155) into (4.90) we obtain

$$
\begin{aligned}
(\mathbf{B}_{*1})_{11} &= A^2(3\lambda-\mu)+\frac{2AB(\lambda+\mu)}{r^2}+\frac{B^2(\lambda-\mu)}{r^4}\\
&\quad+4A^2\beta_1+\left(A^2-\frac{B^2}{r^4}\right)\beta_2+\left(2A^2-\frac{2AB}{r^2}\right)\beta_3+\left(A-\frac{B}{r^2}\right)^2\beta_4,\\
(\mathbf{B}_{*1})_{22} &= A^2(3\lambda-\mu)-\frac{2AB(\lambda+\mu)}{r^2}+\frac{B^2(\lambda-\mu)}{r^4}\\
&\quad+4A^2\beta_1+\left(A^2-\frac{B^2}{r^4}\right)\beta_2+\left(2A^2+\frac{2AB}{r^2}\right)\beta_3+\left(A+\frac{B}{r^2}\right)^2\beta_4,\\
(\mathbf{B}_{*1})_{33} &= 5A^2\lambda+\frac{B^2\lambda}{r^4}+4A^2\beta_1+\left(A^2-\frac{B^2}{r^4}\right)\beta_2,\\
(\mathbf{B}_{*1})_{ij} &= 0,\quad i\neq j.
\end{aligned}
\tag{4.156}
$$

It is now easy to compute from (4.144) components of the divergence of $\mathbf{B}_{*1}$

$$
\begin{aligned}
(\nabla\cdot\mathbf{B}_{*1})_r &= \frac{\partial(\mathbf{B}_{*1})_{rr}}{\partial r}+\frac{(\mathbf{B}_{*1})_{rr}-(\mathbf{B}_{*1})_{\theta\theta}}{r}\\
&= \frac{4B^2(\mu-\lambda+\beta_2-\beta_4)}{r^5},\\
(\nabla\cdot\mathbf{B}_{*1})_\theta &= 0,\quad (\nabla\cdot\mathbf{B}_{*1})_z=0.
\end{aligned}
\tag{4.157}
$$

In order to write boundary conditions $(4.152)_{2,3}$ we start by noting that (4.153), (4.156), and (4.146) give

$$
(\mathbf{T}_{*2}+\mathbf{B}_{*1})\cdot\mathbf{N}_{*i} = -\left[(\lambda+2\mu)u_2'+\frac{\lambda}{r}u_2+(\mathbf{B}_{*1})_{11}\right]\mathbf{a}_r, \tag{4.158}
$$

$$
(\mathbf{T}_{*2}+\mathbf{B}_{*1})\cdot\mathbf{N}_{*e} = \left[(\lambda+2\mu)u_2'+\frac{\lambda}{r}u_2+(\mathbf{B}_{*1})_{11}\right]\mathbf{a}_r. \tag{4.159}
$$

Further, from (4.138) and $(4.103)_2$ it follows that

$$
\begin{aligned}
\mathbf{t}_{*2}^{(i)} &= p_i\left(\mathbf{H}_1^T-I_{\mathbf{H}_1}\mathbf{1}\right)\mathbf{N}_{*i},\\
\mathbf{t}_{*2}^{(e)} &= p_e\left(\mathbf{H}_1^T-I_{\mathbf{H}_1}\mathbf{1}\right)\mathbf{N}_{*e},
\end{aligned}
$$

which, recalling (4.155) and (4.146), become

$$\mathbf{t}_{*2}^{(i)} = p_i \left(A + \frac{B}{r^2} \right) \mathbf{a}_r, \quad \mathbf{t}_{*2}^{(e)} = -p_e \left(A + \frac{B}{r^2} \right) \mathbf{a}_r. \tag{4.160}$$

Hence, from (4.154), (4.157), (4.158), (4.159), and (4.160) the second-order boundary-value problem (4.152) becomes

$$\begin{cases} r^2 u_2'' + r u_2' - u_2 = -\dfrac{4B^2 \left(\mu - \lambda + \beta_2 - \beta_4\right)}{\left(\lambda + 2\mu\right) r^3}, \\ \left[-\left(\lambda + 2\mu\right) u_2' - \dfrac{\lambda}{r} u_2 - \left(\mathbf{B}_{*1}\right)_{11} \right]_{r=R_i} = D_i p_i, \\ \left[\left(\lambda + 2\mu\right) u_2' + \dfrac{\lambda}{r} u_2 + \left(\mathbf{B}_{*1}\right)_{11} \right]_{r=R_e} = -D_e p_e, \end{cases} \tag{4.161}$$

where

$$D_i = A + \frac{B}{R_i^2}, \quad D_e = A + \frac{B}{R_e^2}. \tag{4.162}$$

It will be useful in the sequel to write $\left(\mathbf{B}_{*1}\right)_{11}$ as follows (see (4.156))

$$\left(\mathbf{B}_{*1}\right)_{11} = \Lambda_0 + \sum_{i=1}^{4} \Lambda_i \beta_i, \tag{4.163}$$

where

$$\begin{aligned} &\Lambda_0 = A^2 \left(3\lambda - \mu\right) + \frac{2AB\left(\lambda + \mu\right)}{r^2} + \frac{B^2 \left(\lambda - \mu\right)}{r^4}, \\ &\Lambda_1 = 4A^2, \quad \Lambda_2 = A^2 - \frac{B^2}{r^4}, \quad \Lambda_3 = 2A^2 - \frac{2AB}{r^2}, \\ &\Lambda_4 = A^2 + \frac{B^2}{r^4} - \frac{2AB}{r^2} = \Lambda_3 - \Lambda_2. \end{aligned} \tag{4.164}$$

The problem (4.161) has the following solution

$$u_2\left(r\right) = rC_1 + \frac{C_2}{r} + \frac{C_3}{r^3}, \tag{4.165}$$

where

$$C_3 = \frac{B^2 \left(\lambda - \mu - \beta_2 + \beta_4\right)}{2\left(\lambda + 2\mu\right)}$$

and the integration constants C_1 and C_2 have to be determined from the boundary conditions $(4.161)_{2,3}$.

In anticipation of the developments in the next section, it is useful to write constants $C_h, h = 1,2,3$, in such a way as to highlight their dependence on the second-order material moduli. In particular, it is possible to write them as

$$C_h = A_{h0} + \sum_{j=1}^{4} A_{hj}\beta_j, \quad h = 1,2,3, \tag{4.166}$$

where for $h = 1,2$,

$$\begin{cases} A_{h0} = \hat{g}_{h0} - (\lambda-\mu)f_h - h_h, \quad A_{h1} = \hat{g}_{h1}, \quad A_{h2} = \hat{g}_{h2} + f_h, \\ A_{h3} = \hat{g}_{h3}, \quad A_{h4} = \hat{g}_{h4} - f_h, \end{cases} \tag{4.167}$$

and

$$A_{30} = \frac{B^2\,(\lambda-\mu)}{2\,(\lambda+2\mu)}, \quad A_{31} = A_{33} = 0, \quad A_{32} = -\frac{B^2}{2\,(\lambda+2\mu)}, \quad A_{34} = -A_{32}, \tag{4.168}$$

where we have introduced the notations

$$\hat{g}_{1j} = \frac{R_i^2\Lambda_j(R_i) - R_e^2\Lambda_j(R_e)}{2(\lambda+\mu)(R_e^2-R_i^2)}, \quad \hat{g}_{2j} = \frac{R_i^2R_e^2(\Lambda_j(R_i)-\Lambda_j(R_e))}{2\mu\,(R_e^2-R_i^2)}, \quad j = 0,\cdots,4, \tag{4.169}$$

$$f_1 = \frac{B^2(\lambda+3\mu)}{2(\lambda+\mu)(\lambda+2\mu)R_e^2R_i^2}, \quad h_1 = \frac{-A(R_i^2p_i - R_e^2p_e) + B(p_e-p_i)}{2(\lambda+\mu)(R_e^2-R_i^2)}, \tag{4.170}$$

$$f_2 = \frac{B^2(\lambda+3\mu)(R_i^2+R_e^2)}{2\mu(\lambda+2\mu)R_e^2R_i^2}, \quad h_2 = \frac{AR_i^2R_e^2(p_e-p_i) + B\,(R_i^2p_e - R_e^2p_i)}{2\mu(R_e^2-R_i^2)}.$$

Since $\Lambda_1 = 4A^2$, (4.167) gives $A_{21} = 0$.

Finally, it can be seen that, owing to the symmetry of the problem and the loads acting on S, the compatibility conditions (4.106) and (4.107) are evidently satisfied.

For a solid cylinder with pressure applied on the outer surface, we must have

$$B = 0, \quad C_2 = 0, \quad C_3 = 0.$$

Thus the second-order traction boundary-value problem has the solution

$$\mathbf{u}(\mathbf{r}) = (\epsilon A + \epsilon^2 C_1) r \mathbf{a}_1.$$

4.4 The Experimental Procedures

The analyses presented in the previous two sections suggest experiments to determine the second-order constitutive constants β_i. Consider a homogeneous and isotropic elastic material S, whose Lamé coefficients are λ and μ. Let S_s and S_c be two specimens of S of spherical and cylindrical geometry, respectively, and assume that geometrical characteristics and forces acting upon S_s and S_c be those described in the previous subsections (see Figures 1 and 2). Then, the displacement fields for S_s and S_c are given by (4.137), (4.150), and (4.165), which can be written in the following dimensional form

$$\mathbf{u}_s = \left[\bar{A} + 2\bar{A}^2 + \frac{\Lambda \bar{A}^3}{p_0} \right] r \, \mathbf{a}_r, \tag{4.171}$$

$$\mathbf{u}_c = \left[(A + C_1) \, r + \frac{(B + C_2)}{r} + \frac{C_3}{r^3} \right] \mathbf{a}_r. \tag{4.172}$$

We propose the following experiments:

1. For a sphere S_s subjected to a uniform pressure $\mathbf{t} = -p_0 \mathbf{N}$, we experimentally measure the displacement $u_s(R)$ of the external surface. Then, (4.171) provides one equation in the unknowns β_i:

$$\left[\bar{A} + \left(2\bar{A}^2 + \frac{\Lambda \bar{A}^3}{p_0} \right) \right] R = u_s(R) \,. \tag{4.173}$$

2. Consider a very long hollow cylinder S_c and assume that $p_i = 0$ and $p_e = \pi_1$. Then, if we denote by $u_{c1}(R_e)$ the experimentally measured displacement of the outer surface, from (4.172) we get the following equation in the unknowns β_i:

$$\left(A^{(1)} + C_1^{(1)}\right) R_e + \frac{\left(B^{(1)} + C_2^{(1)}\right)}{R_e} + \frac{C_3^{(1)}}{R_e^3} = u_{c1}(R_e), \tag{4.174}$$

where $A^{(1)} = A|_{(p_i=0, p_e=\pi_1)}$, $B^{(1)} = B|_{(p_i=0, p_e=\pi_1)}$, and $C_i^{(1)} = C_i|_{(p_i=0, p_e=\pi_1)}$, for $i = 1, 2, 3$.

3. Let the hollow cylinder S_c be subjected to the pressures $p_i = \pi_2$ and $p_e = 0$. Then, instead of (4.174), we obtain

$$\left(A^{(2)} + C_1^{(2)}\right) R_e + \frac{\left(B^{(2)} + C_2^{(2)}\right)}{R_e} + \frac{C_3^{(2)}}{R_e^3} = u_{c2}(R_e), \tag{4.175}$$

where $A^{(2)} = A|_{(p_i=\pi_2, p_e=0)}$, $B^{(2)} = B|_{(p_i=\pi_2, p_e=0)}$, and $C_i^{(2)} = C_i|_{(p_i=\pi_2, p_e=0)}$, for $i = 1, 2, 3$.

4. Finally, let $u_{c3}(R_e)$ be the displacement corresponding to the pressures $p_i = p_e = \pi_3$. Then

$$\left(A^{(3)} + C_1^{(3)}\right) R_e + \frac{\left(B^{(3)} + C_2^{(3)}\right)}{R_e} + \frac{C_3^{(3)}}{R_e^3} = u_{c3}(R_e), \tag{4.176}$$

where $A^{(3)} = A|_{(p_i=\pi_3, p_e=\pi_3)}$, $B^{(3)} = B|_{(p_i=\pi_3, p_e=\pi_3)}$, and $C_i^{(3)} = C_i|_{(p_i=\pi_3, p_e=\pi_3)}$, for $i = 1, 2, 3$.

It is now easy to verify that (4.173)-(4.176) provide an algebraic system of four equations in the unknowns $\beta_1, \cdots, \beta_4$ which has a unique solution. In fact, the determinant of the coefficient matrix $\mathcal{A}$ given by

$$\det \mathcal{A} = \frac{R R_i^4 R_e^5 p_0^2 \pi_1^2 \pi_2^2 \pi_3^2}{64\mu^4 \left(\lambda+\mu\right)^5 \left(3\lambda+2\mu\right)^3 \left(R_e^2-R_i^2\right)^3},$$

is non-zero.

In particular, the solution of the system (4.173)-(4.176) is

$$\begin{aligned}
\beta_1 = {} & 2\frac{\mu^2\left(\lambda+\mu\right)\left(R_e^4-R_i^4\right)}{R_i^2 R_e^3 \pi_1^2} u_{c1}\left(R_e\right) - 4\frac{\mu^2\left(\lambda+\mu\right)\left(R_i^2-R_e^2\right)}{R_i^2 R_e \pi_2^2} u_{c2}\left(R_e\right) - \\
& 2\left[\frac{\left(\lambda+\mu\right)^3}{R_e\pi_3^2} + 2\frac{\mu^2\left(\lambda+\mu\right)}{R_e\pi_3^2} + \frac{\mu^2\left(\lambda+\mu\right)R_e}{R_i^2\pi_3^2}\right] u_{c3}\left(R_e\right) + \\
& \frac{3}{2}\mu + \frac{\left(\lambda+\mu\right)R_i^2}{2R_e^2} + 2\frac{\mu\left(\lambda+2\mu\right)}{\pi_2} - \frac{\left(\lambda+\mu\right)^2}{\pi_3} + 2\mu^2\left(\frac{1}{\pi_1}-\frac{1}{\pi_3}\right) - \\
& \frac{\mu^2 R_e^2}{R_i^2\pi_3} + \frac{\lambda\mu}{\pi_1} + \frac{\mu\left(\lambda+\mu\right)R_i^2}{R_e^2\pi_1} + \frac{\mu^2 R_e^2}{R_i^2\pi_1},
\end{aligned} \tag{4.177}$$

$$\begin{aligned}
\beta_2 = {} & -2\left[\frac{\mu^2\left(\lambda+\mu\right)\left(R_e^4-R_i^4\right)}{R_i^2 R_e^3\pi_1^2} + 2\frac{\mu^2\left(\lambda+\mu\right)\left(R_e^2-R_i^2\right)}{R_i^2 R_e\pi_1^2}\right] u_{c1}\left(R_e\right) + \\
& 8\frac{\mu^2\left(\lambda+\mu\right)\left(R_e^2-R_i^2\right)}{R_i^2 R_e\pi_2^2} u_{c2}\left(R_e\right) + \\
& 2\left[9\frac{\left(\lambda+\mu\right)^3}{R_e\pi_3^2} + 4\frac{\mu^2\left(\lambda+\mu\right)}{R_e\pi_3^2} + 3\frac{\mu^2\left(\lambda+\mu\right)R_e}{R_i^2\pi_3^2}\right] u_{c3}\left(R_e\right) - \frac{\left(3\lambda+2\mu\right)^3}{R p_0^2} u_s\left(R\right) + \\
& \frac{\lambda}{2} - \frac{3}{2}\mu - \frac{\left(\lambda+\mu\right)R_i^2}{2R_e^2} - \frac{\left(3\lambda+2\mu\right)^2}{p_0} - 4\frac{\mu\left(\lambda+2\mu\right)}{\pi_2} + 9\frac{\left(\lambda+\mu\right)^2}{\pi_3} - \\
& 4\mu^2\left(\frac{1}{\pi_1}-\frac{1}{\pi_3}\right) + 3\frac{\mu^2 R_e^2}{R_i^2\pi_3} - 3\frac{\lambda\mu}{\pi_1} - \frac{\mu\left(\lambda+\mu\right)R_i^2}{R_e^2\pi_1} - 3\frac{\mu^2 R_e^2}{R_i^2\pi_1},
\end{aligned} \tag{4.178}$$

$$\beta_3 = 2\left[-3\frac{\mu^2(\lambda+\mu)(R_e^4-R_i^4)}{R_i^2R_e^3\pi_1^2} + 4\frac{\mu^2(\lambda+\mu)(R_e^2-R_i^2)}{R_i^2R_e\pi_1^2}\right]u_{c1}(R_e) +$$
$$4\frac{\mu^2(\lambda+\mu)(R_e^2-R_i^2)}{R_i^2R_e\pi_2^2}u_{c2}(R_e) +$$
$$2\left[-9\frac{(\lambda+\mu)^3}{R_e\pi_3^2} + 2\frac{\mu^2(\lambda+\mu)}{R_e\pi_3^2} - \frac{\mu^2(\lambda+\mu)R_e}{R_i^2\pi_3^2}\right]u_{c3}(R_e) +$$
$$\frac{(3\lambda+2\mu)^3}{Rp_0^2}u_s(R) - \frac{3\lambda+5\mu}{2} - 3\frac{(\lambda+\mu)R_i^2}{2R_e^2} + \frac{(3\lambda+2\mu)^2}{p_0} - 2\frac{\mu(\lambda+2\mu)}{\pi_2} -$$
$$9\frac{(\lambda+\mu)^2}{\pi_3} - 2\mu^2\left(\frac{1}{\pi_1} - \frac{1}{\pi_3}\right) - \frac{\mu^2R_e^2}{R_i^2\pi_3} + \frac{\lambda\mu}{\pi_1} - 3\frac{\mu(\lambda+\mu)R_i^2}{R_e^2\pi_1} + \frac{\mu^2R_e^2}{R_i^2\pi_1}, \tag{4.179}$$

$$\beta_4 = 6\left(\frac{\mu^2(\lambda+\mu)(R_e^4-R_i^4)}{R_i^2R_e^3\pi_1^2} - 2\frac{\mu^2(\lambda+\mu)(R_e^2-R_i^2)}{R_i^2R_e\pi_1^2}\right)u_{c1}(R_e) +$$
$$6\left[3\frac{(\lambda+\mu)^3}{R_e\pi_3^2} + \frac{\mu^2(\lambda+\mu)R_e}{R_i^2\pi_3^2}\right]u_{c3}(R_e) - \frac{(3\lambda+2\mu)^3}{Rp_0^2}u_s(R) + \frac{3\lambda+7\mu}{2} +$$
$$3\frac{(\lambda+\mu)R_i^2}{2R_e^2} - \frac{(3\lambda+2\mu)^2}{p_0} + 9\frac{(\lambda+\mu)^2}{\pi_3} + 3\frac{\mu^2R_e^2}{R_i^2\pi_3} - 3\frac{\lambda\mu}{\pi_1} + 3\frac{\mu(\lambda+\mu)R_i^2}{R_e^2\pi_1} - 3\frac{\mu^2R_e^2}{R_i^2\pi_1}. \tag{4.180}$$

It must be noted that expressions (4.177)-(4.180) of the second-order constitutive constants just appear to be complicated. Indeed, once Lamé coefficients of the material S together with its geometry and the forces acting on it are known, (4.177)-(4.180) only depend on the displacements $u_{c1}(R_e), u_{c2}(R_e), u_{c3}(R_e)$, and $u_s(R)$ which are measured in the experiments. Thus the four second-order elastic constants can be evaluated. Note that loads applied must be such that $\epsilon << 1$, otherwise the response of S to applied loads may not be governed by the second-order elasticity theory used to derive equations (4.177)-(4.180).

4.5 The Program **`Sphere`**

Aim of the Program

The program `Sphere` finds the displacement of a homogeneous, isotropic, compressible, second-order elastic sphere subjected to a live uniform pressure field on its surface (see Section 4.3.1 for a detailed statement of the problem).

Description of the Algorithm

The program is based on the theoretical apparatus built in Section 4.2. First-order displacement gradient, strain tensor, and Piola-Kirchhoff tensor are computed in a symbolic way in order to formulate and solve the first-order equilibrium boundary value problem. Then, starting from the knowledge of first-order displacement and stresses the second-order equilibrium boundary value problem is obtained and solved.

Command Line of the Program Sphere

```
Sphere[N, fext]
```

Input Data

`N` = unit vector field normal to the spherical surface;

`fext` = hydrostatic pressure applied to the spherical surface.

Output Data

`Metric tensor` is the metric tensor (g_{ij}) in spherical coordinates;

`Inverse Metric Tensor` is the inverse metric tensor $(g_{ij})^{-1}$;

`Non-zero Christoffel Symbols` are the Christoffel symbols Γ_{ij}^{k} different from zero;

`First-order Displacement Gradient` $\mathbf{H}_1$;

`First-order Strain Tensor` $\mathbf{E}_1$;

`First-order First Piola-Kirchhoff Stress Tensor` $\mathbf{T}_{*1}$;

`First-order Equilibrium Boundary Value Problem`;

`First-order Displacement` $u_1(r)$;

`First Invariant of Displacement Gradient` $I_{\mathbf{H}_1}$;

`First Invariant of Strain Tensor` $I_{\mathbf{E}_1}$;

`First Invariant of` $\mathbf{H}_1\mathbf{H}_1^T$;

`Second Invariant of Strain Tensor` $\mathbf{E}_1$;

`Second-order Displacement Gradient` $\mathbf{H}_2$;

`Second-order Strain Tensor` $\mathbf{E}_2$;

`Second-order Stress Tensor` $\mathbf{T}_{*2}$;

`Non-zero Components of Tensor` $\mathbf{B}_{*1}$;

`Second-order Components of the applied loads` $\mathbf{t}_{*2}$;

`Second-order Equilibrium Boundary Value Problem`;

`Second-order Displacement` $u_2(r)$.

4.6 The Program `HollowCylinder`

Aim of the Program

The program `HollowCylinder` finds the displacement of a homogeneous, isotropic, compressible, second-order elastic, infinite hollow cylinder corresponding to a live uniform pressure field applied to the inner and outer surface (see Section 4.3.2 for a complete description of the problem in exam).

Description of the Algorithm

The program has been written starting from the theoretical considerations made in Section 4.2. It computes in a symbolic way all the relevant quantities to formulate and solve the first-order equilibrium boundary value problem. Then, first-order deformation and stresses are used to write the second-order equilibrium boundary value problem and to obtain the corresponding solution.

Command Line of the Program HollowCylinder

```
HollowCylinder[Nint, Next, fint, fext]
```

Input Data

`Nint` = unit vector field normal to the inner surface of the hollow cylinder;

`Next` = unit vector field normal to the outer surface of the hollow cylinder;

`fint` = hydrostatic pressure applied to the inner surface of the hollow cylinder;

`fext` = hydrostatic pressure applied to the outer surface of the hollow cylinder.

Output Data

`Metric tensor` is the metric tensor (g_{ij}) in cylindrical coordinates;

`Inverse Metric Tensor` is the inverse metric tensor $(g_{ij})^{-1}$;

`Non-zero Christoffel Symbols` are the Christoffel symbols Γ_{ij}^{k} different from zero;

`First-order Displacement Gradient` $\mathbf{H}_1$;

`First-order Strain Tensor` $\mathbf{E}_1$;

`First-order First Piola-Kirchhoff Stress Tensor` $\mathbf{T}_{*1}$;

`First-order Equilibrium Boundary Value Problem`;

`First-order Displacement` $u_1(r)$;

`First Invariant of Displacement Gradient` $I_{\mathbf{H}_1}$;

`First Invariant of Strain Tensor` $I_{\mathbf{E}_1}$;

`First Invariant of` $\mathbf{H}_1\mathbf{H}_1^T$;

`Second Invariant of Strain Tensor` $\mathbf{E}_1$;

`Second-order Displacement Gradient` $\mathbf{H}_2$;

`Second-order Strain Tensor` $\mathbf{E}_2$;

`Second-order Stress Tensor` $\mathbf{T}_{*2}$;

`Non-zero Components of Tensor` $\mathbf{B}_{*1}$;

`Second-order Components of the applied loads` $\mathbf{t}_{*2}$;

`Second-order Equilibrium Boundary Value Problem`;

`Second-order Displacement` $u_2(r)$;

`Expressions of Constants` C_i'`s` are the integration constants appearing in the second-order displacement.

Chapter 5
Functionally Graded Materials

5.1 Functionally Graded Materials. An Overview.

Composite materials are made by combining different materials in order to achieve specific properties otherwise not available. However this simple approach may present many limitations. These are essentially due to the presence of a discontinuity of the material structure and/or composition at the interface between the different constituents, which may originate undesirable effects such as residual stresses and lack of adhesion.

In order to overcome these limitations, the concept of Functionally Graded Materials has been recently introduced. In a Functionally Graded Material (**FGM**) the composition and the structure *gradually* change over the volume, resulting in corresponding smooth variations in the properties of the material. The basic structural unit of an FGM is referred to as an *element* [34] or a *material ingredient* [35, 36].

In the simplest case an FGM is made by two different material ingredients which change gradually from one to the other. A nitrided steel, for instance, could be also regarded as an FGM. The most familiar FGM is compositionally graded from a refractory ceramic to a metal. It can achieve incompatible properties such as the heat, wear, and oxidation resistance of ceramics with the high toughness, high strength, machinability, and bonding capability of metals, avoiding severe internal thermal stress. Modern FGMs are constructed for complex requirements, such as the heat shield of a space vehicle entering the earth atmosphere or implants for humans. The gradual transition between the heat or corrosion resistant outer layer (often made of a ceramic material) and the tough metallic base material increases in most cases the life time of the component.

Even if the gradation of the material ingredients does not extend to the whole volume but is located at a specific part of the material such as the surface, an interface or a joint, the material can be still considered an FGM.

The first general idea of structures whose properties vary continuously was introduced for composites and polymeric materials in 1972 ([37, 38]). Various models were suggested for gradients in compositions and concentration along with possible applications for the resulting graded structures. However there was no investigation about how to design, fabricate and study the response of such a structure. In 1985, a new technique was introduced to build a composite material with continuously varying structure (see [39]). The designers recognized that this continuos control of a property could be applied to impart desired properties to any material. At this point the idea of FGMs was introduced to design such materials [34].

5.2 Applications

FGMs can be usefully applied to many fields. Table 5.1 shows a variety of real and potential applications of FGMs in transport system, cutting tools, machine parts, semiconductors, optics and biosystem. Potential applications cover all those cases in which a combination of incompatible functions is required.

Aerospace	space vehicle components, space plane body
Engineering	Cutting tool, shaft, roller, turbine blade
Nuclear Energy	Nuclear reactor components, first wall of fusion reactor
Biomaterials	Implant, Artificial skin, drug delivery system
Optics	Optical fibers, lens
Chemical Plants	Heat exchanger, heat pipe, reaction vessel
Electronics	graded band semiconductor, sensor
Energy Conversion	Thermoelectric generator, fuel cell, solar cell
Commodities	Sport goods, car body, window glass

Table 5.1. Potentially applications for FGMs

In the present section we describe real applications of FGMs. In particular we deal with space vehicle components, cutting tools, machine parts and applications to biomaterials for artificial joints. For an exhaustive treatment of this subject we refer the interested reader to the Chapter 7 of [40].

5.2.1 The Problem of Reentry: Vehicle Protection

Space vehicles flying at hypersonic speed are subjected to extremely high temperatures due to friction between the vehicle surface and the atmosphere. At the present time there exist two type of space vehicles: vehicles which are launched vertically into space by a rocket propulsion system [10]; and fully reusable spacecraft which are based on a horizontal takeoff either from a ground-based runway or from horizontally flying carrier[11].

In the first case, during takeoff, after a sufficient acceleration, the space vehicle separates from the rocket system. During reentry at a velocity greater than $11\,\mathrm{km/\,s}$ the space vehicle is exposed to a rapid heating at altitudes between 120 and $50\ \mathrm{km}$. The leading

[10] This category include the U.S. space shuttle and the capsules used for the Apollo missions.

[11] These spacecrafts have been planned during the late 1980s by the U.S. National Aerospace Plane, the Japanese Single Stage to Orbit, and the German Sanger program.

edge, where the protection shield is located, experiences a maximum temperature which is above $2500\,^\circ\mathrm{C}$. Since the protection shield is exposed to the high reentry temperature only for a few minutes and it is used only one time, it is made by a composite ablative material, and there is no matter if its deterioration makes it unusable for an eventual future mission.[12]

Horizontally launched space vehicles fly in the atmosphere at hypersonic velocities for a longer time than vertically launched vehicles. This means that these latter vehicles are exposed to extreme heating during the takeoff. This is a crucial issue. As a matter of fact, protection system must guarantee that all structural components remain unaltered and work properly during the mission. Initially, the properties of FGMs processed by a particular technique called chemical vapor deposition (see [41-45][13]) were investigated in order to design and develop thermic shields for horizontally launched space vehicles.

In [47] a comparison test between classical C/C composites and FGMs was made in order to establish the different responses to extreme heating exposure. Models of the components of a nose cone made of an hemispherical C/C composites whose diameter was $50\,\mathrm{mm}$ were coated with an ungraded $100\,\mu\mathrm{m}$ thick protective layer of SiC (Silicon carbide). Similar C/C composite models were coated by $100\,\mu\mathrm{m}$ thick Si/C FGM. All the coated nose models were exposed for 1 minute at $1900\,^\circ\mathrm{C}$ to a supersonic gas flow (at Mach 3) containing an amount of oxygen approximately equal to a standard atmosphere. The nose cones with the Si/C FGM layer showed no discernible change in structure even

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[12] The reentry velocity of the U.S. shuttle at an altitude of $120\,\mathrm{km}$ is about $8\,\mathrm{km/\,s}$ and the maximum temperature is about $1500\,^\circ\mathrm{C}$ for few minutes. The structural components exposed to the maximum temperature such as the nose cone or the leading edges are made by non metallic carbon/carbon composites (C/C). Nickel and Titanium alloys are usually used for other parts which are subjected to lower temperatures. The only problem here is represented by weight penalties.

[13] Comprensive reviews on the processing of FGMs can be found in [34, 35, 40, 46]

after 10 cycles. On the contrary, the classical C/C composites deteriorated after the first cycle.

5.2.2 Cutting Tools

FGMs are also used in cutting tools for high speed cutting. One is a graded tungsten carbide/cobalt (WC/Co) throwaway chip [48]. It is designed with a decreasing Co concentration from the surface to the interior, which causes the hardness at the cutting tool's surface to be higher than its interior. A comparison between graded and ungraded cutting tools has been made in [48]. This gradient in hardness result in both considerably higher damage resistance and higher wear resistance than a cutting tool with homogeneous composition. Tough FGM cutting tools based on this model were commercialized in 1996.

5.2.3 Machine parts

The major application of FGMs for machine parts is for joints, largely metal-ceramic joints for gas and steam turbines [49, 50, 51]. The advantage of using an FGM joint is chiefly for thermal stress relaxation and improving the strength and toughness of the joints. Because the rupture strength of graded joints is $3-8$ times higher than for directly bounded joints, they are expected to provide longer service life at elevated temperature.

5.2.4 Biomedical Applications

FGMs have been recently used to obtain biomaterials for artificial joints. Several of the organs of animal such as skin, blood vessels, and bones are composed of multilayers that have different properties. These layers constitute a functionally graded material. Therefore, incompatibility and separation at the interface never occur under normal physiological conditions. This implies that the ideal technique in biomedical applications would be to mimic such natural bonding in order to obtain a fixation to bone that will be stable for many years.

Prostheses coated with a porous graded material have been used for fixation. In this case, the prosthesis is expected to become mechanically fixed to the bone due to ingrowth into the pores of the porous metal coating. Further several calcium phosphate ceramics that can bond to a bone physicochemically, such as bioactive ceramic, have been studied and their clinical applications has been rapidly adopted. The interested reader can refer to [52-61].

Chapter 6
Second-Order effects for FG Elastic Materials

6.1 Introduction

The investigation of the mechanical and thermomechanical behavior of FGMs has created a new field of study in Materials Engineering and Applied Mathematics. For example, analyzing the equilibrium problem or the crack propagation in a non homogeneous material that has a gradually changing composition and structure represents a challenging problem. Local structures and properties and the external thermal and mechanical loads are correlated with FGM's geometry. The profile of stress or strain is determined by this type of correlation, which strongly affects an FGM's thermomechanical stability. The role of stress is essential for both the structural and the functional applications to FGMs.

The mechanical and mathematical modelling of FGMs is currently an active research area. When a continuum mechanics approach is appropriate, the material has to be modeled as a nonhomogeneous body with continuously varying properties. Fracture Mechanics of FGMs using this viewpoint has been discussed in [62], [63], and [64]. Nonhomogeneous theory of elasticity has been used to address some other problems like thick plate theory [65], torsion [66, 67], elastic vibrations [68, 69], and the analysis of Saint-Venant end effects [70 - 73].

In [74] Horgan and Chan have investigated the effects of material inhomogeneity in another fundamental boundary-value problem of linear inhomogeneous isotropic elastostatics, that is the pressurized hollow cylinder (or disk problem). They have analyzed the case of compressible elastic bodies. The solution to the analogous problem for homogeneous isotropic materials (i.e. the classic Lamé problem) have been discussed in many books on linear elasticity (see e.g. Sokolnikoff [75] and Love [76]). However little is known about the corresponding problem for inhomogeneous materials. In the classical textbooks on linear elasticity of Southwell [77] and Timoshenko and Goodier [78], special inhomogeneous configurations have been studied, for example, compound tubes composed of two different homogeneous isotropic materials. Some aspects of the Lamé problem for inhomogeneous materials with continuously varying properties have also been considered in [80 - 83].

A commonly used model is to assume that Poisson's ratio is constant while Young's modulus has a power-law dependence on the radial coordinate[14].

Froli [82] has considered the more complicated case in which both the young modulus and Poisson ratio depend on the spatial variable. When a power-law dependence is taken for both these quantities, a perturbation technique is used to obtain solutions.

The idea of finding optimal values for the coefficients in order to minimize the hoop stress at the inner surface is also considered in [82].

Further, there is a mathematical analogy between the problem of internally pressurized hollow plates of variable thickness (see Vocke [83]) and the problem of pressurized hollow cylinder. In fact, a power-law variation in the plate thickness leads to consideration of an Euler ordinary differential equation of the type we shall consider in a next section.

[14] Poisson's ratio and Young's modulus have been defined in Section 1.7.

In this Chapter we apply the generalized Signorini's perturbation method, presented in Section 4.2, to analyze radial expansion/contraction of hollow cylinders made of elastic and isotropic FGMs, whose material moduli depend upon the radial coordinate only. In particular we consider a very long hollow cylinder whose inner and outer surfaces are loaded by a uniform live pressure. As we have done for the case of homogeneous elastic materials, we formulate the equilibrium boundary value problem in order to determine the displacement of the system and its state of stress up to second-order of approximation. This case has never been investigated before. We analyze both compressible and incompressible hollow cylinders. In the case of compressible material the first-order approximation solution is the same as the solution of the nonhomogeneous linear problem solved in [74], [80 -83]. The second-order solution is completely new.

In most of earlier work the composite materials have assumed to be compressible. However, with the increasing use of rubberlike materials in structural components, there is increasing interest in analyzing deformations of FG incompressible materials. We also note that biological materials are often modeled as incompressible. For this reason, we have discussed the case of incompressible FG hollow cylinders too.

The purpose of this research is to investigate the effects of material inhomogeneity on the nonlinear response of isotropic hollow cylinders or disks under uniform internal and external pressure. This work is motivated by the recent research activity in FGMs as well. It must be noted that mathematical problems arising from nonhomogeneous theory of elasticity are more complicated than those concerning homogeneous theory, even in the linear case. Thus it is easy to realize that when we pass to second-order elasticity we have to face very tough mathematical problems and the expressions of the solutions we find are much more complicated than the classical ones. Further, these solutions, as their analogous in the second-order homogeneous case, are expressed in terms of the material moduli. This implies that we shall be able to appreciate the nonlinear effects only when a proper experimental work will be carried out on FGMs. However the analytical solutions presented here provide checks on the accuracy of eventual numerical schemes and allow for widely applicable parametric studies. For instance, they might be of interest for the design of a set of experiments to determine the second-order material moduli of compressible and incompressible FGMs.

Besides this introduction, the present chapter is divided into eleven sections. In the second section we obtain the constitutive equations for an isotropic FG incompressible elastic material. The first Piola-Kirchhoff stress tensor is written up to the second order of approximation. Next, in Section 6.3 and 6.4, we formulate the first and second-order equilibrium equations, boundary conditions, and incompressibility conditions and we find the corresponding solutions. These latter are then specialized to the case of homogeneous elastic materials. It comes out that displacement, pressure, and stress fields present a singularity in two special cases. Section 6.5 and Section 6.6 are devoted to the computation of displacements and pressure fields in these two cases. A particular dependence of elastic moduli upon the radial coordinate is studied in Section 6.7. In many practical applications it is important to study the state of stress of a pressurized cylindrical cavity. This problem has been addressed in Section 6.8. In the ninth section we consider a compressible elastic FGM. The first Piola-Kirchhoff stress tensor (4.85) is properly written for a nonhomogeneous material and the displacement field is obtained up to the second-order of approximation. In Section 6.10, we show that in the *homogeneous limit* the found displacement field reduce to the solution (4.165) - (4.170). Finally, in the last section, we present the features of the program `FGHollowCylinder` whose use has been crucial for the resolution of the equilibrium problem of a functionally graded, incompressible hollow cylinder.

6.2 Second-Order Effects for Incompressible FG Elastic Bodies

The constitutive equation of an incompressible, homogeneous, isotropic, elastic material may be written as

$$\mathbf{T} = -\tilde{p}\mathbf{1} + f_1\mathbf{B} + f_2\mathbf{B}^{-1}, \tag{6.181}$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ is the *left Cauchy-Green tensor*, $\mathbf{F}$ is the *deformation gradient*, f_1 and f_2 are functions of the two invariants of $\mathbf{B}$, I and II, and $\tilde{p} = P(\mathbf{x}) = p(\mathbf{X})$ is an undetermined function.

To the second order in the displacement gradient $\mathbf{H} = \mathbf{F} - \mathbf{1}$ we have the following expansions

$$\begin{gathered}
\mathbf{B} = \mathbf{1} + 2\mathbf{E} + \mathbf{H}\mathbf{H}^T, \\
\mathbf{B}^{-1} = \mathbf{1} - 2\mathbf{E} - \mathbf{H}\mathbf{H}^T + 4\mathbf{E}^2 + ..., \\
f_1 = a_{11} + a_{12}(I-3) + a_{13}(II-3) + a_{14}(I-3)^2..., \\
f_2 = a_{21} + a_{22}(I-3) + a_{23}(II-3) + a_{24}(I-3)^2...,
\end{gathered} \tag{6.182}$$

where

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T\right).$$

From $(6.182)_1$ it results

$$I \equiv Tr(\mathbf{B}) = 3 + 2I_{\mathbf{E}} + I_{\mathbf{HH}^T}. \tag{6.183}$$

Hence, up to second order terms in $\mathbf{H}$, we have

$$II \equiv \frac{1}{2}(I_{\mathbf{B}}^2 - I_{\mathbf{B}^2}) = 3 + 4I_{\mathbf{E}} + 2I_{\mathbf{HH}^T} + 4II_{\mathbf{E}}. \tag{6.184}$$

Substituting (6.182) into (6.181) and taking into account relations (6.183) and (6.184), we obtain the following expression for the second - order Cauchy stress tensor

$$\mathbf{T} = -p\mathbf{1} + 2\alpha_1\mathbf{E} + \alpha_5 I_{\mathbf{E}}\mathbf{E} + \alpha_1\mathbf{HH}^T + \alpha_6\mathbf{E}^2, \tag{6.185}$$

where the three material constants α_1, α_5, and α_6 can be expressed in terms of some of the eight constants a_{ij}, $i = 1, 2$, $j = 1, 2, 3, 4$, by means of the following relations

$$\begin{aligned} \alpha_1 &= a_{11} - a_{21}, \\ \alpha_5 &= 4a_{12} + 8a_{13} - 4a_{22} - 8a_{23}, \quad \alpha_6 = 4a_{21}, \end{aligned}$$

and the pressure p is given by

$$p = \tilde{p} - a_{11} - a_{21} - \left[\alpha_2 I_{\mathbf{E}} + \frac{\alpha_2}{2} I_{\mathbf{HH}^T} + \alpha_3 II_{\mathbf{E}} + \alpha_4 I_{\mathbf{E}}^2\right],$$

where

$$\begin{aligned} \alpha_2 &= 2a_{12} + 4a_{13} + 2a_{22} + 4a_{23}, \\ \alpha_3 &= 4(a_{13} + a_{23}), \quad \alpha_4 = 4(a_{14} + a_{24}). \end{aligned}$$

We recall that the *first Piola-Kirchhoff stress tensor* is defined as follows

$$\mathbf{T}_* = J\mathbf{T}(\mathbf{F}^{-1})^T, \tag{6.186}$$

where $J = \det \mathbf{F}$.

For an incompressible material it results

$$J = 1. \tag{6.187}$$

Hence, having in mind that up to second-order terms in $\mathbf{H}$ we have

$$(\mathbf{F}^{-1})^T = \mathbf{1} - \mathbf{H}^T + (\mathbf{H}^2)^T,$$

from (6.186) and the incompressibility restriction (6.187) we obtain

$$\mathbf{T}_* = -p[\mathbf{1} - \mathbf{H}^T + (\mathbf{H}^2)^T] + 2\alpha_1\left(\mathbf{E} - \mathbf{E}\mathbf{H}^T + \frac{\mathbf{H}\mathbf{H}^T}{2}\right) + \alpha_5 I_{\mathbf{E}}\mathbf{E} + \alpha_6\mathbf{E}^2. \tag{6.188}$$

We assume that the pressure p, as well as the displacement field $\mathbf{u}$, has a second order expansion in the perturbation parameter ϵ

$$\begin{aligned} p &= \epsilon p_1 + \epsilon^2 p_2, \\ \mathbf{u} &= \epsilon \mathbf{u}_1 + \epsilon^2 \mathbf{u}_2. \end{aligned} \tag{6.189}$$

From (6.189) we easily come to the following relations

$$\begin{aligned} \mathbf{H} &= \epsilon\mathbf{H}_1 + \epsilon^2\mathbf{H}_2, \quad \mathbf{E} = \epsilon\mathbf{E}_1 + \epsilon^2\mathbf{E}_2, \quad I_{\mathbf{E}} = \epsilon I_{\mathbf{E}_1} + \epsilon^2 I_{\mathbf{E}_2}, \\ I_{\mathbf{E}}\mathbf{E} &= \epsilon^2 I_{\mathbf{E}_1}\mathbf{E}_1, \quad \mathbf{E}^2 = \epsilon^2\mathbf{E}_1^2, \quad (\mathbf{H}^T)^2 = \epsilon^2(\mathbf{H}_1^T)^2. \end{aligned} \tag{6.190}$$

Substituting the second-order expansions (6.190) into (6.188) we obtain

$$\mathbf{T}_* = \epsilon\mathbf{T}_{*1} + \epsilon^2(\mathbf{T}_{*2} + \mathbf{B}_{*1}), \tag{6.191}$$

where

$$\mathbf{T}_{*i} = -p_i\mathbf{1} + 2\alpha_1\mathbf{E}_i, \quad i = 1, 2, \tag{6.192}$$

$$\mathbf{B}_{*1} = -\alpha_1\left(2\mathbf{E}_1 - \mathbf{H}_1\right)\mathbf{H}_1^T + p_1\mathbf{H}_1^T + \alpha_5 I_{\mathbf{E}_1}\mathbf{E}_1 + \alpha_6\mathbf{E}_1^2. \tag{6.193}$$

If we interpret α_1 as the classic *shear modulus* μ, as we shall do, then

$$\mathbf{T}_{*i} = -p_i \mathbf{1} + 2\mu \mathbf{E}_i, \quad i = 1, 2, \tag{6.194}$$

$$\mathbf{B}_{*1} = -\mu \left(2\mathbf{E}_1 - \mathbf{H}_1\right) \mathbf{H}_1^T + p_1 \mathbf{H}_1^T + \alpha_5 I_{\mathbf{E}_1} \mathbf{E}_1 + \alpha_6 \mathbf{E}_1^2. \tag{6.195}$$

For a matrix $\mathbf{A}$ written as

$$\mathbf{A} = \mathbf{1} + \mathbf{S},$$

$$\det \mathbf{A} = 1 + I_{\mathbf{S}} + II_{\mathbf{S}} + III_{\mathbf{S}}. \tag{6.196}$$

Therefore, for

$$\mathbf{F} = \mathbf{1} + \mathbf{H},$$

from (6.196), to within an error of third-order in the components of $\mathbf{H}$, we get

$$J = \det \mathbf{F} \simeq 1 + I_{\mathbf{H}} + II_{\mathbf{H}} = 1 + I_{\mathbf{H}} + \frac{1}{2}\left[I_{\mathbf{H}}^2 - I_{\mathbf{H}^2}\right]. \tag{6.197}$$

From (6.190) it follows that

$$I_{\mathbf{H}} = \epsilon I_{\mathbf{H}_1} + \epsilon^2 I_{\mathbf{H}_2}, \quad I_{\mathbf{H}^2} = \epsilon^2 I_{\mathbf{H}_1^2}. \tag{6.198}$$

Substituting (6.198) into (6.197) we obtain

$$J = 1 + \epsilon I_{\mathbf{H}_1} + \epsilon^2 \left[I_{\mathbf{H}_2} + II_{\mathbf{H}_1}\right], \tag{6.199}$$

which yields the following incompressibility restrictions at the first and second-order of approximation respectively

$$\begin{aligned} I_{\mathbf{H}_1} &= 0, \\ I_{\mathbf{H}_2} + II_{\mathbf{H}_1} &= 0. \end{aligned} \tag{6.200}$$

Having in mind that $I_{\mathbf{H}_1} = I_{\mathbf{E}_1}$, $(6.200)_1$ implies that

$$I_{\mathbf{E}_1} = 0. \tag{6.201}$$

In view of (6.201) equations (6.194) and (6.195) become

$$\mathbf{T}_{*i} = -p_i \mathbf{1} + 2\mu \mathbf{E}_i, \quad i = 1, 2, \tag{6.202}$$

$$\mathbf{B}_{*1} = -\mu \left(2\mathbf{E}_1 - \mathbf{H}_1\right) \mathbf{H}_1^T + p_1 \mathbf{H}_1^T + \alpha_6 \mathbf{E}_1^2. \tag{6.203}$$

For a functionally graded material we assume that the material moduli μ and α_6 depend on the radial coordinate in the following way

$$\mu = b_1 r^n, \quad \alpha_6 = b_2 r^n, \tag{6.204}$$

where b_1 and b_2 are constants and n is an arbitrary dimensionless constant.

6.3 Hollow Cylinder

We consider a very long hollow cylinder S made of an incompressible, elastic, homogeneous and isotropic material. Let R_i and R_e be the internal and the external radii, respectively. Let S be at equilibrium in the current configuration C under the action of the uniform pressure field

$$\mathbf{t} = \begin{cases} -p_i \mathbf{N}_i & \text{on } \partial C_i, \\ -p_e \mathbf{N}_e & \text{on } \partial C_e, \end{cases} \tag{6.205}$$

where p_i and p_e are positive constants, $\mathbf{N}_i$ and $\mathbf{N}_e$ are the unit outward normal vectors to ∂C_i and ∂C_e, respectively.

Owing to cylindrical symmetry, we search for the solution of pure traction-value problem in the following form

$$\begin{aligned} \mathbf{u}(r) &= \left[\epsilon u_1(r) + \epsilon^2 u_2(r)\right]\mathbf{a}_r, \\ p(r) &= \epsilon p_1(r) + \epsilon^2 p_2(r), \end{aligned} \tag{6.206}$$

where $\mathbf{a}_r$ is the radial unit vector of the physical basis associated with the cylindrical coordinates $\{r, \varphi, z\}$.

The first-order boundary-value problem and the first-order incompressibility condition are

$$\begin{cases} \nabla_* \cdot \mathbf{T}_{*1} = \mathbf{0} \quad \text{in } C_*, \\ \mathbf{T}_{*1} \cdot \mathbf{N}_{*i} = \mathbf{t}_{*1}^{(i)} \quad \text{on } \partial C_{*i}, \\ \mathbf{T}_{*1} \cdot \mathbf{N}_{*e} = \mathbf{t}_{*1}^{(e)} \quad \text{on } \partial C_{*e}, \end{cases} \tag{6.207}$$

$$I_{\mathbf{H}_1} = 0, \tag{6.208}$$

for the unknown first-order displacement field $\mathbf{u}_1(r) = u_1(r)\,\mathbf{a}_r$, and first-order pressure $p_1(r)$.

The first-order displacement gradient assumes the following form

$$\mathbf{H}_1 = \mathbf{E}_1 = \begin{pmatrix} u_1' & 0 & 0 \\ 0 & \dfrac{u_1}{r} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.209}$$

which implies that

$$I_{\mathbf{H}_1} = u_1' + \frac{u_1}{r}, \quad \mathbf{E}_1 = \mathbf{H}_1. \tag{6.210}$$

The first-order incompressibility condition reduces to the following first-order differential equation

$$u_1' + \frac{u_1}{r} = 0, \tag{6.211}$$

whose solution is

$$u_1(r) = \frac{B_1}{r}, \tag{6.212}$$

in which B_1 is a constant.

Once we know the form of the first-order displacement field, we derive the pressure p_1 from the equilibrium equation $(6.207)_1$. From $(6.202)_1$, (6.209), $(6.210)_2$, and (6.212), it follows that

$$\begin{aligned}
(\mathbf{T}_{*1})_{11} &= -2B_1 b_1 r^{-2+n} - p_1, \\
(\mathbf{T}_{*1})_{22} &= 2B_1 b_1 r^{-2+n} - p_1, \\
(\mathbf{T}_{*1})_{33} &= -p_1, \quad (\mathbf{T}_{*1})_{ij} = 0, \quad i \neq j.
\end{aligned} \tag{6.213}$$

Substitution for $\mathbf{T}$ from (6.213) into $(6.207)_1$ gives

$$2nr^{-3+n} B_1 b_1 + p_1'(r) = 0.$$

Thus

$$p_1(r) = B_2 - \frac{2nB_1 b_1}{n-2} r^{-2+n}, \tag{6.214}$$

where B_2 is a constant of integration which must be determined from the boundary conditions. Since

$$\mathbf{N}_{*i} = (-1, 0, 0)^T, \quad \mathbf{N}_{*e} = (1, 0, 0)^T, \tag{6.215}$$

and

$$\mathbf{t}_{*1}^{(i)} = -p_i \mathbf{N}_{*i}, \quad \mathbf{t}_{*1}^{(e)} = -p_e \mathbf{N}_{*e}, \tag{6.216}$$

boundary conditions $(6.207)_2$ and $(6.207)_3$ can be written as

$$-\frac{4b_1 R_i^{-2+n}}{n-2} B_1 + B_2 = p_i, \quad \frac{4b_1 R_i^{-2+n}}{n-2} B_1 - B_2 = -p_e,$$

from which, if we assume $n \neq 2$, it results

$$B_1 = \frac{R_i^2 R_e^2 (p_i - p_e)(n-2)}{4b_1(R_e^n R_i^2 - R_e^2 R_i^n)}, \qquad B_2 = \frac{p_i R_i^2 R_e^n - p_e R_e^2 R_i^n}{(R_e^n R_i^2 - R_e^2 R_i^n)}. \tag{6.217}$$

It is easy to realize from the expressions of the first-order applied loads (6.216) and the symmetry of the displacement (6.206) that the first-order compatibility conditions (4.106) are verified.

The second-order boundary-value problem and the second-order incompressibility condition become

$$\begin{cases} \nabla_* \cdot (\mathbf{T}_{*2} + \mathbf{B}_{*1}) = \mathbf{0} \quad \text{in } C_*, \\ (\mathbf{T}_{*2} + \mathbf{B}_{*1}) \cdot \mathbf{N}_{*i} = \mathbf{t}_{*2}^{(i)} \quad \text{on } \partial C_{*i}, \\ (\mathbf{T}_{*2} + \mathbf{B}_{*1}) \cdot \mathbf{N}_{*e} = \mathbf{t}_{*2}^{(e)} \quad \text{on } \partial C_{*e}, \end{cases} \tag{6.218}$$

$$I_{\mathbf{H}_2} + II_{\mathbf{H}_1} = I_{\mathbf{H}_2} - \frac{1}{2} I_{\mathbf{H}_1^2} = 0. \tag{6.219}$$

The second-order displacement gradient assumes the following form

$$\mathbf{H}_2 = \begin{pmatrix} u_2' & 0 & 0 \\ 0 & \dfrac{u_2}{r} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.220}$$

while, from (6.212)

$$II_{\mathbf{H}_1} = -\frac{B_1^2}{r^4}. \tag{6.221}$$

Owing to (6.220) and (6.221) equation (6.219) becomes

$$u_2' + \frac{u_2}{r} - \frac{B_1^2}{r^4} = 0,$$

and has the following solution

$$u_2(r) = \frac{B_3}{r} - \frac{B_1^2}{2r^3}, \tag{6.222}$$

where B_3 is a constant of integration.

The second-order pressure $p_2(r)$ can be derived from the second-order equilibrium equation $(6.218)_1$. From (6.222) it follows

$$
\begin{aligned}
(\mathbf{H}_2)_{11} &= -\frac{B_3}{r^2} + \frac{3B_1^2}{2r^4}, \\
(\mathbf{H}_2)_{22} &= \frac{B_3}{r^2} - \frac{B_1^2}{2r^4}, \quad (\mathbf{H}_2)_{33} = 0, \\
(\mathbf{H}_2)_{ij} &= 0, \quad i \neq j, \quad \mathbf{E}_2 = \mathbf{H}_2,
\end{aligned}
\tag{6.223}
$$

and

$$
\begin{aligned}
(\mathbf{T}_{*2})_{11} &= -2b_1 B_3 r^{-2+n} + 3b_1 B_1^2 r^{-4+n} - p_2, \\
(\mathbf{T}_{*2})_{22} &= 2b_1 B_3 r^{-2+n} - b_1 B_1^2 r^{-4+n} - p_2, \quad (\mathbf{T}_{*2})_{33} = -p_2, \\
(\mathbf{T}_{*2})_{ij} &= 0, \quad i \neq j.
\end{aligned}
\tag{6.224}
$$

From (6.209), (6.210), (6.212),and (6.214) we obtain the following components of $\mathbf{B}_{*1}$

$$
\begin{aligned}
(\mathbf{B}_{*1})_{11} &= \frac{B_1^2\left[b_1(n+2) - b_2(2-n)\right]}{(n-2)} r^{-4+n} - \frac{B_1 B_2}{r^2}, \\
(\mathbf{B}_{*1})_{22} &= \frac{B_1^2\left[b_1(2-3n) + b_2(n-2)\right]}{(n-2)} r^{-4+n} + \frac{B_1 B_2}{r^2}, \\
(\mathbf{B}_{*1})_{33} &= 0, \quad (\mathbf{B}_{*1})_{ij} = 0, \quad i \neq j,
\end{aligned}
\tag{6.225}
$$

in which n must different from 2.

Owing to (6.224) and (6.225), the second-order equilibrium equation $(6.218)_1$ becomes

$$
-p_2' + B_1^2\left[4b_1(n-1) + b_2(n-4)\right] r^{n-5} - 2nB_3 b_1 r^{n-3} = 0,
$$

from which we obtain

$$
p_2(r) = B_4 + \frac{B_1^2\left[4b_1(n-1) + b_2(n-4)\right]}{(n-4)} r^{n-4} - \frac{2nB_3 b_1}{(n-2)} r^{n-2},
\tag{6.226}
$$

where B_4 is a constant of integration and $n \neq 2, 4$.

Constants B_3 and B_4 can be determined from boundary conditions $(6.218)_2$ and $(6.218)_3$. First, we note that from $(4.103)_2$, (6.209), (6.210), and (6.216) we have

$$\mathbf{t}_{*2}^{(i)} = \frac{B_1 p_i}{R_i^2}\mathbf{a}_r, \quad \mathbf{t}_{*2}^{(e)} = -\frac{B_1 p_e}{R_e^2}\mathbf{a}_r. \tag{6.227}$$

Then, from (6.224), (6.225) and (6.227) boundary conditions $(6.218)_2$ and $(6.218)_3$ can be written as

$$eq_p = 0, \quad p = \{i, e\},$$

where, assuming, as we have done above, that $n \neq 2, 4$,

$$\begin{aligned} eq_p &= -4(n-4)b_1 R_p^{2+n} B_3 + (8 - 6n + n^2) R_p^4 B_4 + \\ &\quad B_1 \left[\left(8 - 6n + n^2\right) R_p^2 (B_2 - p_i) + 8(n-1) b_1 B_1 R_p^n \right], \end{aligned}$$

from which, having in mind expressions (6.217) of constants B_1 and B_2,we get

$$\begin{aligned} B_3 &= \frac{3(n-2)^3 (p_e - p_i)^2 R_e^2 R_i^2 (R_e^4 R_i^n - R_e^n R_i^4)}{16(n-4) b_1^2 (R_e^2 R_i^n - R_i^2 R_e^n)^3}, \\ B_4 &= \frac{3(-2+n)^2 (p_e - p_i)^2 R_e^{2+n} R_i^{2+n} (R_e^2 - R_i^2)}{4(n-4) b_1 (R_e^2 R_i^n - R_i^2 R_e^n)^3}. \end{aligned} \tag{6.228}$$

We note that from (6.227) and (6.206) the second-order compatibility conditions (4.107) are satisfied.

We conclude this section by writing the explicit form of the first and second-order solutions to the equilibrium problem.

$$u_1(r) = \frac{(n-2) R_i^2 R_e^2 (p_i - p_e)}{4 b_1 (R_e^n R_i^2 - R_i^n R_e^2)} \frac{1}{r}, \tag{6.229}$$

$$p_1(r) = \frac{p_i R_e^n R_i^2 - p_e R_i^n R_e^2}{(R_e^n R_i^2 - R_i^n R_e^2)} - \frac{n(p_i - p_e) R_i^2 R_e^2}{2(R_e^n R_i^2 - R_i^n R_e^2)} r^{n-2}, \tag{6.230}$$

$$
\begin{aligned}
u_2(r) &= \frac{3(n-2)^3(p_e-p_i)^2R_e^2R_i^2(R_e^4R_i^n-R_e^nR_i^4)}{16(n-4)b_1^2(R_e^2R_i^n-R_i^2R_e^n)^3}\frac{1}{r}- \\
&\quad \frac{(n-2)^2(p_e-p_i)^2R_i^4R_e^4}{32b_1^2(R_e^2R_i^n-R_i^2R_e^n)^2}\frac{1}{r^3},
\end{aligned} \tag{6.231}
$$

$$
\begin{aligned}
p_2(r) &= \frac{3(-2+n)^2(p_e-p_i)^2R_e^{2+n}R_i^{2+n}(R_e^2-R_i^2)}{4(n-4)b_1(R_e^2R_i^n-R_i^2R_e^n)^3}- \\
&\quad \frac{3(n-2)^2n(p_e-p_i)^2R_e^2R_i^2(R_e^4R_i^n-R_i^4R_e^n)}{8(4-n)b_1\,(R_e^2R_i^n-R_i^2R_e^n)^3}r^{n-2}+ \\
&\quad \frac{(n-2)^2\left[4b_1\,(n-1)+b_2\,(n-4)\right](p_e-p_i)^2R_i^4R_e^4}{16(n-4)b_1^2(R_e^2R_i^n-R_i^2R_e^n)^2}r^{n-4}.
\end{aligned} \tag{6.232}
$$

We point out again that the above expressions for the first and second order displacement and pressure fields hold when $n \neq 2, 4$. This implies that we need to work out these solutions again for the cases $n = 2$ and $n = 4$, as we shall do in the next two sections.

If the material is homogeneous, then we have $n = 0$, and solutions (6.229)-(6.232) reduce to the following ones

$$
\begin{aligned}
u_1(r) &= \frac{R_i^2R_e^2(p_e-p_i)}{2b_1(R_i^2-R_e^2)}\frac{1}{r}, \\
p_1(r) &= \frac{p_iR_i^2-p_eR_e^2}{(R_i^2-R_e^2)},
\end{aligned} \tag{6.233}
$$

$$
\begin{aligned}
u_2(r) &= \frac{3(p_e-p_i)^2R_e^2R_i^2(R_e^2+R_i^2)}{8b_1^2(R_e^2-R_i^2)^2}\frac{1}{r}-\frac{(p_e-p_i)^2R_i^4R_e^4}{8b_1^2(R_e^2-R_i^2)^2}\frac{1}{r^3}, \\
p_2(r) &= -\frac{3(p_e-p_i)^2R_e^2R_i^2}{4b_1(R_e^2-R_i^2)^2}+\frac{(p_e-p_i)^2R_i^4R_e^4\,(b_1+b_2)}{4b_1^2(R_e^2R_i^n-R_i^2R_e^n)^2}\frac{1}{r^4}
\end{aligned} \tag{6.234}
$$

We note that in the homogeneous case the second-order displacement field u_2 does not depend on the second-order material modulus α_6, while the second-order pressure p_2 depends on it. This implies that for the case of an hollow cylinder subjected to a uniform pressure, the second-order effects on the displacement can be evaluated starting from its geometry, loads acting upon it and the shear modulus μ.

6.4 Radial and Hoop Stress

In this section we analyze the state of stress induced in S by the applied tensions. In particular, we give the explicit forms and their variations through the thickness of the hollow cylinder of the radial, tangential and axial components of the first Piola-Kirchhoff stress tensor. It is clear that it is desirable to have a state of stress independent of the radial coordinate and whose values are not too high. We remember that in literature the tangential component of the stress tensor is often referred to as hoop stress[15]. Substitution from equations (6.214) and (6.217) into (6.213) gives the following expressions for the first-order stresses

$$\begin{aligned}
T_{rr}^{(1)} &= \frac{p_e(R_i^{n-2} - r^{n-2}) + p_i(r^{n-2} - R_e^{n-2})}{(R_e^{n-2} - R_i^{n-2})}, \\
T_{\theta\theta}^{(1)} &= \frac{p_e\left[(1-n)\,r^{n-2} + R_i^{n-2}\right] + p_i\ \left[(n-1)\,r^{n-2} - R_e^{n-2}\right]}{(R_e^{n-2} - R_i^{n-2})}, \\
T_{zz}^{(1)} &= \frac{-p_e\left(\dfrac{n}{2}r^{n-2} - R_i^{n-2}\right) + p_i\left(\dfrac{n}{2}r^{n-2} - R_e^{n-2}\right)}{(R_e^{n-2} - R_i^{n-2})}.
\end{aligned} \tag{6.235}$$

It is clear from equation $(6.235)_2$ that for $n = 1$, the hoop stress is uniform throughout the cylinder thickness, and

$$T_{\theta\theta}^{(1)} = -\frac{p_e R_e - p_i\ R_i}{(R_e - R_i)}.$$

Thus $T_{\theta\theta}^{(1)} = 0$ throughout the cylinder thickness for $p_e R_e = p_i\ R_i$.

---

15 In order to lighten the notation, we shall make use of the following symbology

$$\begin{aligned}
(\mathbf{T}_{*1})_{11} &= T_{rr}^{(1)}, \quad ,(\mathbf{T}_{*1})_{22} = T_{\theta\theta}^{(1)}, \quad (\mathbf{T}_{*1})_{22} = T_{zz}^{(1)}, \\
(\mathbf{T}_{*2} + \mathbf{B}_{*1})_{11} &= T_{rr}^{(2)}, \quad (\mathbf{T}_{*2} + \mathbf{B}_{*1})_{22} = T_{\theta\theta}^{(2)}, \quad (\mathbf{T}_{*2} + \mathbf{B}_{*1})_{33} = T_{zz}^{(2)}.
\end{aligned}$$

For a solid circular cylinder subjected to pressure p_e on the outer surface, the constant B_1 into equation (6.212) must vanish in order for the displacement to be finite at the center. Equation (6.212) and (6.214) imply that $u_1 = 0$ and $p_1 = B_2$. From the boundary condition

$$\mathbf{T}_{*1} \cdot \mathbf{N}_{*e} = \mathbf{t}_{*1}^{(e)},$$

having in mind that, if $u_1 = 0$, $\mathbf{T}_{*1} = -p_1 \mathbf{1}$, we obtain $B_2 = p_e$ and

$$T_{rr}^{(1)} = T_{\theta\theta}^{(1)} = T_{zz}^{(1)} = -p_e.$$

The radial displacement of every point is zero, the cylinder does not deform, and the state of stress at any point is that of hydrostatic pressure.

We now consider the case when the hollow cylinder is subjected to internal pressure only, i.e., $p_e = 0$. Thus

$$\begin{aligned}
T_{rr}^{(1)} &= \frac{\left(r^{n-2} - R_e^{n-2}\right)}{\left(R_e^{n-2} - R_i^{n-2}\right)} p_i, \\
T_{\theta\theta}^{(1)} &= \frac{\left[(n-1)\, r^{n-2} - R_e^{n-2}\right]}{\left(R_e^{n-2} - R_i^{n-2}\right)} p_i, \\
T_{zz}^{(1)} &= \frac{\left(\dfrac{n}{2} r^{n-2} - R_e^{n-2}\right)}{\left(R_e^{n-2} - R_i^{n-2}\right)} p_i.
\end{aligned} \tag{6.236}$$

For $n = 0$, i.e., a cylinder made of a homogeneous material, we recover the classical expressions for stresses given in a linear elasticity book even when the cylinder material is incompressible; e.g. see [84]. Note that most linear elasticity books consider the cylinder material to be compressible.

For $n = 1$

$$T_{\theta\theta}^{(1)} = \frac{R_i}{R_e - R_i} p_i$$

and is uniform throughout the cylinder.

We now investigate the sign of $T_{rr}^{(1)}$ and $T_{\theta\theta}^{(1)}$ for $n \neq 1, 2, 4$ and limit our analysis to the case $n \epsilon Z$. We start by noting that the function

$$f(r) = r^{n-2}, \qquad R_i \leq r \leq R_e, \tag{6.237}$$

is always positive and, since $f'(r) = (n-2) * r^{n-3}$, it is monotonically increasing when $n > 2$, and monotonically decreasing when $n < 2$. Its absolute minimum and maximum are summarized in the following table together with the sign of the difference $R_e^{n-2} - R_i^{n-2}$

	Max	min	$R_e^{n-2} - R_i^{n-2}$
$\mathbf{n > 2}$	(R_e, R_e^{n-2})	(R_i, R_i^{n-2})	positive
$\mathbf{n < 2}$	(R_i, R_i^{n-2})	(R_e, R_e^{n-2})	negative

We first study the case $n > 2$.

From $(6.236)_1$ and the values of the table it can be easily seen that for $n > 2$ the radial stress is everywhere compressive, i.e. $T_{rr}^{(1)} < 0$ throughout the thickness of the hollow cylinder.

From $(6.236)_2$ we derive that

$$T_{\theta\theta} \geq 0 \iff (1-n)r^{n-2} + R_e^{n-2} \leq 0,$$

which implies that

$$r^{n-2} \geq \frac{R_e^{n-2}}{n-1}. \tag{6.238}$$

We define

$$\alpha(n) = (n-1)^{\frac{1}{n-2}},$$

and note that $\alpha(3) = 2$, and, since

$$\lim_{n->\infty} \frac{\ln(n-2)}{n-2} = 0,$$

then

$$\lim_{n->\infty}(n-1)^{\frac{1}{n-2}} = e^{\lim\limits_{n->\infty}\frac{\ln(n-2)}{n-2}} = 1.$$

This means that

$$1 < \alpha(n) \leq 2. \tag{6.239}$$

Hence from $(6.236)_2$, (6.238) and (6.239) we can conclude that, if

$$R_i < \frac{R_e}{\alpha(n)} < R_e \Longrightarrow \frac{R_i}{R_e} < \frac{1}{\alpha(n)}, \tag{6.240}$$

then

$$\begin{aligned} T_{\theta\theta} &\geq 0 \quad \text{when} \quad \frac{R_e}{\alpha(n)} \leq r < R_e, \\ T_{\theta\theta} &\leq 0 \quad \text{when} \quad R_i < r \leq \frac{R_e}{\alpha(n)}. \end{aligned} \tag{6.241}$$

If

$$\frac{R_e}{\alpha(n)} \leq R_i \Longrightarrow \frac{1}{\alpha(n)} \leq \frac{R_i}{R_e} < 1, \tag{6.242}$$

then

$$T_{\theta\theta} \geq 0 \quad \text{for all} \;\; R_i \leq r \leq R_e. \tag{6.243}$$

Now we study the case $n < 2$.

From $(6.236)_1$ and the table written above it results that the radial stress is again compressive. Further from $(6.236)_1$ we derive that

$$T_{\theta\theta} \geq 0 \iff (1-n)r^{n-2} + R_e^{n-2} \geq 0,$$

which implies that

$$r^{n-2} \geq \frac{R_e^{n-2}}{n-1}. \tag{6.244}$$

Since the quantity $\dfrac{R_e^{n-2}}{n-1}$ is negative when $n < 2$, then

$$T_{\theta\theta} \geq 0 \quad \text{for all } \ R_i \leq r \leq R_e. \tag{6.245}$$

We note that

$$\frac{d}{dr} T_{\theta\theta} = -\frac{p_i (1-n)\,(n-2)}{R_e^{n-2} - R_i^{n-2}} r^{n-3},$$

which is positive for all r when $n > 2$ and negative for all r when $n < 2$.

From the analysis made above we can conclude that the inhomogeneity in the shear modulus μ strongly affects the through-the-thickness variation of hoop stress.

Further, if we write $(6.236)_2$ as

$$T_{\theta\theta} = -p_i \, \frac{(1-n)\left(\dfrac{r}{R_e}\right)^{n-2} + 1}{1 - \left(\dfrac{R_i}{R_e}\right)^{n-2}},$$

it is easy to recognize that if $r \neq R_e$

$$\lim_{n\to+\infty} T_{\theta\theta} = -p_i, \quad \lim_{n\to-\infty} T_{\theta\theta} = +\infty.$$

If $r = R_e$, then

$$\lim_{n\to+\infty} T_{\theta\theta} = +\infty, \quad \lim_{n\to-\infty} T_{\theta\theta} = 0.$$

Substituting equations (6.228) and (6.232) into (6.224), we obtain the following expressions for second-order radial and hoop stresses

$$\begin{aligned}
T_{rr}^{(2)} &= c_{10} + c_{11} r^{-2} + c_{12} r^{-2+n} + c_{13} r^{-4+n}, \\
T_{\theta\theta}^{(2)} &= c_{20} + c_{21} r^{-2} + c_{22} r^{-2+n} + c_{23} r^{-4+n}, \\
T_{zz}^{(2)} &= c_{30} + c_{31} r^{-2+n} + c_{32} r^{-4+n},
\end{aligned} \tag{6.246}$$

where constants c_{ij} are given by

$$\begin{aligned}
c_{10} &= \frac{3(n-2)^2(p_e-p_i)^2 R_i^{2+n} R_e^{2+n}(R_e^2-R_i^2)}{4(n-4)b_1(R_e^n R_i^2 - R_e^2 R_i^n)^3}, \\
c_{11} &= -\frac{(n-2)(p_e-p_i)R_i^2 R_e^2(-p_i R_e^n R_i^2 + p_e R_e^2 R_i^n)}{4b_1(R_e^n R_i^2 - R_e^2 R_i^n)^2}, \\
c_{12} &= \frac{3(n-2)^2(p_e-p_i)^2 R_i^2 R_e^2(R_e^4 R_i^n - R_e^n R_i^4)}{4(n-4)b_1(R_e^n R_i^2 - R_e^2 R_i^n)^3}, \\
c_{13} &= -\frac{(n-2)(n-1)(p_e-p_i)^2 R_i^4 R_e^4}{2(n-4)b_1(R_e^n R_i^2 - R_e^2 R_i^n)^2},
\end{aligned} \tag{6.247}$$

$$\begin{aligned}
c_{20} &= \frac{3(n-2)^2(p_e-p_i)^2 R_i^{2+n} R_e^{2+n}(R_e^2-R_i^2)}{4(n-4)b_1(R_e^n R_i^2 - R_e^2 R_i^n)^3} = c_{10}, \\
c_{21} &= \frac{(n-2)(p_e-p_i)R_i^2 R_e^2(-p_i R_e^n R_i^2 + p_e R_e^2 R_i^n)}{4b_1(R_e^n R_i^2 - R_e^2 R_i^n)^2}, \\
c_{22} &= \frac{3(n-2)^2(n-1)(p_e-p_i)^2 R_i^2 R_e^2(R_e^4 R_i^n - R_e^n R_i^4)}{4(n-4)b_1(R_e^n R_i^2 - R_e^2 R_i^n)^3}, \\
c_{23} &= -\frac{(n-2)(n^2-4n+3)(p_e-p_i)^2 R_i^4 R_e^4}{2(n-4)b_1(R_e^n R_i^2 - R_e^2 R_i^n)^2},
\end{aligned} \tag{6.248}$$

and

$$\begin{aligned}
c_{30} &= c_{20} = c_{10}, \\
c_{31} &= -\frac{3(n-2)^2\, n\, (p_e-p_i)^2 R_i^2 R_e^2(R_e^4 R_i^n - R_e^n R_i^4)}{8(n-4)b_1(R_e^n R_i^2 - R_e^2 R_i^n)^3}, \\
c_{32} &= -\frac{(n-2)^2\, n\, (p_e-p_i)^2 R_i^4 R_e^4\,[4b_1\,(n-1) + b_2\,(n-4)]}{16(n-4)\, b_1^2\, (R_e^n R_i^2 - R_e^2 R_i^n)^2}.
\end{aligned} \tag{6.249}$$

We note that the second-order elastic constant b_2 appears only in the pressure field but not in the expressions for the radial and hoop stress. Whereas the first-order stresses are independent of the elastic moduli, b_1 appears in the expressions for the second-order stresses. Even though stresses (except for the axial stress) and the radial displacement do not explicitly depend upon b_2, their values for the first-order and second-order elastic materials are different.

For $n = 1$, from $(6.248)_{3,4}$ it follows that $c_{22} = c_{23} = 0$. Thus, from $(6.246)_2$, it is easily seen that the second-order hoop stress is constant throughout the thickness of the cylinder if the constant c_{21}, evaluated for $n = 1$, is equal to zero, i.e. if it results

$$p_i R_i = p_e R_e. \tag{6.250}$$

If the relation written above holds then we have

$$T_{\theta\theta}^{(2)} = -\frac{(p_e - p_i)^2 (R_e^2 - R_i^2)}{4 b_1 (R_i - R_e)^3}.$$

For a homogeneous second-order isotropic, incompressible elastic material the first-order state of stress is obtained putting $n = 0$ into (6.235) and is described by the quantities written below

$$\begin{aligned}
T_{rr}^{(1)} &= \frac{p_i (r^2 - R_e^2) R_i^2 + p_e (R_i^2 - r^2) R_e^2}{r^2 (R_e^2 - R_i^2)}, \\
T_{\theta\theta}^{(1)} &= \frac{p_i \ (r^2 + R_e^2)\, R_i^2 - p_e R_e^2\, (r^2 + R_i^2)}{r^2 (R_e^2 - R_i^2)}, \\
T_{zz}^{(1)} &= \frac{p_e R_e^2 - p_i R_i^2}{(R_i^2 - R_e^2)}.
\end{aligned} \tag{6.251}$$

In the same way the second-order stresses for a homogeneous body can be written from (6.246), (6.247), (6.248) and (6.249)

$$\begin{aligned}
T_{rr}^{(2)} &= d_{10} + d_{11} r^{-2} + d_{12} r^{-4}, \\
T_{\theta\theta}^{(2)} &= d_{20} + d_{21} r^{-2} + d_{22} r^{-4}, \\
T_{zz}^{(2)} &= d_{30} + d_{32} r^{-4},
\end{aligned}$$

where

$$\begin{aligned}
d_{10} &= d_{20} = d_{30} = \frac{3(p_e - p_i)^2 R_i^2 R_e^2}{4b_1(R_e^2 - R_i^2)^2}, \qquad d_{12} = \frac{(p_e - p_i)^2 R_i^4 R_e^4}{4b_1(R_e^2 - R_i^2)^2}, \\
d_{11} &= -\frac{R_i^2 R_e^2 \left[-4\, p_e p_i (R_e^2 + R_i^2) + p_i^2\ (3\, R_e^2 + R_i^2) + p_e^2\, (R_e^2 + 3R_i^2)\right]}{4b_1(R_e^2 - R_i^2)^2}, \\
d_{21} &= -d_{11}, \qquad d_{22} = -\frac{3(p_e - p_i)^2 R_i^4 R_e^4}{4b_1(R_e^2 - R_i^2)^2}, \qquad d_{32} = -\frac{(b_1 + b_2)(p_e - p_i)^2 R_i^4 R_e^4}{4b_1^2(R_e^2 - R_i^2)^2}
\end{aligned}$$

We conclude this section giving an experimental procedure to measure the second-order constant b_2. Recalling that the axial force, F_{ax}, acting on a cross-section of the cylinder is given by

$$F_{ax} = 2\,\pi \int_{R_i}^{R_e} T_{zz}\, r\, dr,$$

and that the axial stress depends upon b_2, an accurate measure of this force can be used to estimate the value of the second-order elastic constant.

6.5 The Case $n = 2$

From equations (6.225), (6.231), and (6.232) it can be easily seen that the case when $n = 2$ must be treated separately. Hence we assume that the material moduli μ and α_6 depend on the radial coordinate r as follows

$$\mu = b_1 r^2, \qquad \alpha_6 = b_2 r^2, \tag{6.252}$$

where b_1 and b_2 are constants[16].

[16] We note that only for the sake of simplicity we have denoted the material constants b_1 and b_2 using the same notation adopted in (6.204). However we must take into account that the nature of these constants depend on the constitutive equations (6.204), that is on the choice of n. For instance, once we pass to dimensional quantities, the physical dimensions of b_1 and b_2 depend on n.

First we note that the first-order incompressibility condition (6.208) remains unchanged, so that the first-order displacement field can be still written as

$$u_1(r) = \frac{\bar{B}_1}{r}, \tag{6.253}$$

where $\bar{B}_1$ is a constant.

By the same arguments we have used in the previous section, we obtain the pressure field p_1 from the equilibrium equation $(6.207)_1$. From (6.202), (6.209), and (6.253), it follows that

$$\begin{aligned}
(\mathbf{T}_{*1})_{11} &= -2\,\bar{B}_1 b_1 - p_1, \\
(\mathbf{T}_{*1})_{22} &= 2\,\bar{B}_1 b_1 - p_1, \\
(\mathbf{T}_{*1})_{33} &= -p_1, \quad (\mathbf{T}_{*1})_{ij} = 0, \quad i \neq j.
\end{aligned} \tag{6.254}$$

Substitution for $\mathbf{T}$ from (6.254) into $(6.207)_1$ gives

$$\frac{4\,\bar{B}_1 b_1}{r} + p_1'(r) = 0,$$

whose solution is

$$p_1(r) = \bar{B}_2 - 4\,\bar{B}_1 b_1 \ln r, \tag{6.255}$$

where $\bar{B}_2$ is a constant of integration.

From (6.254), boundary conditions $(6.207)_2$ and $(6.207)_3$ become

$$2(1 - 2\ln R_i) b_1 \bar{B}_1 + \bar{B}_2 = p_i, \quad 2(2\ln R_e - 1) b_1 \bar{B}_1 - \bar{B}_2 = -p_e,$$

from which we can derive the values of the constants $\bar{B}_1$ and $\bar{B}_2$

$$\bar{B}_1 = -\frac{(p_e - p_i)}{4 b_1 \ln(R_e/R_i)}, \qquad \bar{B}_2 = -\frac{p_e\,[2\ln(R_i) - 1] - p_i\,[2\ln(R_e) - 1]}{2\ln(R_e/R_i)}. \tag{6.256}$$

The second-order incompressibility condition (6.219) yields the following second-order displacement field

$$u_2(r) = \frac{\bar{B}_3}{r} - \frac{\bar{B}_1^2}{2r^3}, \tag{6.257}$$

in which the constant $\bar{B}_3$ must be determined from boundary conditions.

The second-order pressure $p_2(r)$ can be derived from the second-order equilibrium equation $(6.218)_1$. We note that the expression of the second-order displacement gradient $\mathbf{H}_2$ has the same shape we found in the previous section (see (6.223)) while the second-order Piola-Kirchhoff stress tensor $\mathbf{T}_{*2}$ becomes

$$\begin{aligned}
(\mathbf{T}_{*2})_{11} &= -2b_1\bar{B}_3 - p_2 + \frac{3b_1\bar{B}_1^2}{r^2}, \\
(\mathbf{T}_{*2})_{22} &= 2b_1\bar{B}_3 - p_2 - \frac{b_1\bar{B}_1^2}{r^2}, \quad (\mathbf{T}_{*2})_{33} = -p_2, \\
(\mathbf{T}_{*2})_{ij} &= 0, \quad i \neq j,
\end{aligned} \tag{6.258}$$

which coincides with (6.224) for $n = 2$. Substituting equations (6.209), (6.212), and (6.255) into (6.203), we obtain the following components of $\mathbf{B}_{*1}$

$$\begin{aligned}
(\mathbf{B}_{*1})_{11} &= \frac{\bar{B}_1\left\{-\bar{B}_2 + \bar{B}_1\left[(4\ln r - 1)\,b_1 + b_2\right]\right\}}{r^2}, \\
(\mathbf{B}_{*1})_{22} &= \frac{\bar{B}_1\left\{\bar{B}_2 + \bar{B}_1\left[(-4\ln r - 1)\,b_1 + b_2\right]\right\}}{r^2}, \\
(\mathbf{B}_{*1})_{33} &= 0, \quad (\mathbf{B}_{*1})_{ij} = 0, \quad i \neq j.
\end{aligned} \tag{6.259}$$

From (6.258) and (6.259), the second-order equilibrium equation $(6.218)_1$ becomes

$$-p_2' - \frac{4\,\bar{B}_3\,b_1}{r} + \frac{2\,\bar{B}_1^2(2b_1 - b_2)}{r^3} = 0,$$

from which we obtain

$$p_2(r) = \bar{B}_4 - \frac{\bar{B}_1^2(2b_1 - b_2)}{r^2} - 4\,\bar{B}_3 b_1 \ln r, \tag{6.260}$$

where $\bar{B}_4$ is a constant of integration.

From (6.258), (6.259) and (6.227) boundary conditions $(6.218)_2$ and $(6.218)_3$ can be written as

$$
\begin{aligned}
2R_i^2 b_1 \left[1 - 2\ln(R_i)\right] \ \bar{B}_3 + R_i^2 \ \bar{B}_4 &= 4\left[1 + \ln(R_i)\right] \bar{B}_1^2 b_1 + \bar{B}_1(p_i - \bar{B}_2) \\
2R_e^2 b_1 \left[1 - 2\ln(R_e)\right] \ \bar{B}_3 + R_e^2 \ \bar{B}_4 &= 4\left[1 + \ln(R_e)\right] \bar{B}_1^2 b_1 + \bar{B}_1(p_e - \bar{B}_2)
\end{aligned}
$$

from which, having in mind expressions (6.217) of constants B_1 and B_2,we get

$$
\begin{aligned}
\bar{B}_3 &= \frac{3(p_e - p_i)^2 \left(R_e^2 - R_i^2\right)}{32 \, b_1^2 \, R_e^2 \, R_i^2 \left[\ln(R_e/R_i)\right]^3}, \\
\bar{B}_4 &= \frac{3(p_e - p_i)^2 \ \left[(2\ln(R_e) - 1)R_e^2 - (2\ln(R_i) - 1)R_i^2\right]}{16 \, b_1 \, R_e^2 \, R_i^2 \left[\ln(R_e/R_i)\right]^3}.
\end{aligned}
\tag{6.261}
$$

Hence, from (6.256), the first-order displacement and pressure fields (6.253), (6.255) assume the following form

$$
u_1(r) = -\frac{(p_e - p_i)}{4b_1 \left[\ln(R_e/R_i)\right]} \frac{1}{r}, \tag{6.262}
$$

$$
p_1(r) = -\frac{p_e \left[2\ln(R_i) - 1\right] - p_i \left[2\ln(R_e) - 1\right]}{2 \ \ln(R_e/R_i)} + \frac{(p_e - p_i)}{\ln(R_e/R_i)} \ln r. \tag{6.263}
$$

In the same way, substituting (6.261) into (6.257) and (6.260) we get

$$
u_2(r) = \frac{(p_e - p_i)^2}{32 \, b_1^2 \left[\ln(R_e/R_i)\right]^2 r} \left[\frac{3\left(R_e^2 - R_i^2\right)}{R_e^2 \, R_i^2 \ln(R_e/R_i)} - \frac{1}{r^2}\right], \tag{6.264}
$$

$$
\begin{aligned}
p_2(r) = {} & \frac{3(p_e - p_i)^2 \ \left[(2\ln(R_e) - 1)R_e^2 - (2\ln(R_i) - 1)R_i^2\right]}{16 \, b_1 \, R_e^2 \, R_i^2 \left[\ln(R_e) - \ln(R_i)\right]^3} - \\
& \frac{(p_e - p_i)^2 (2b_1 - b_2)}{16 b_1^2 \left[\ln(R_i) - \ln(R_e)\right]^2} \frac{1}{r^2} - \\
& \frac{3(p_e - p_i)^2 \left(R_e^2 - R_i^2\right)}{8 \, b_1 \, R_e^2 \, R_i^2 \left[\ln(R_e) - \ln(R_i)\right]^3} \ln r.
\end{aligned}
\tag{6.265}
$$

Substitution from equations (6.263) and (6.256) into (6.254) gives the following expressions for the first-order stresses

$$
\begin{aligned}
T_{rr}^{(1)} &= \frac{p_i \ln(r/R_e) - p_e \ \ln(r/R_i)}{\ln(R_e/R_i)}, \\
T_{\theta\theta}^{(1)} &= \frac{p_i \ [1+\ln(r/R_e)] - p_e\,[1+\ln(r/R_i)]}{\ln(R_e/R_i)}, \\
T_{zz}^{(1)} &= \frac{p_i \ [1+2 \ \ln(r/R_e)] - p_e\,[1+2 \ \ln(r/R_i)]}{2 \ \ln(R_e/R_i)}.
\end{aligned}
\tag{6.266}
$$

Having in mind equations (6.265) and (6.261), from (6.258) and (6.259) we derive the following expressions for the second-order stresses

$$
\begin{aligned}
T_{rr}^{(2)} &= e_{10}(r) + e_{11}(r)\, r^{-2}, \\
T_{\theta\theta}^{(2)} &= e_{20}(r) + e_{21}(r)\, r^{-2}, \\
T_{zz}^{(2)} &= e_{30}(r) + e_{31}(r)\, r^{-2},
\end{aligned}
\tag{6.267}
$$

where

$$
\begin{aligned}
e_{10} &= \frac{3\,(p_e-p_i)^2 \ [\ln(r/R_e)\,R_e^2 - \ln(r/R_i)\,R_i^2]}{8\,b_1\,[\ln(R_e/R_i)]^3 \ R_e^2\,R_i^2}, \\
e_{11} &= \frac{(p_e-p_i) \ [(3+2\ \ln(r/R_i))\,p_e - (3+2\ \ln(r/R_e))\,p_i]}{8\,b_1\,[\ln(R_e/R_i)]^2}, \\
e_{20} &= \frac{3\,(p_e-p_i)^2 \ [(1+\ln(r/R_e))\,R_e^2 - (1+\ln(r/R_i))\,R_i^2]}{8\,b_1\,[\ln(R_e/R_i)]^3 \ R_e^2\,R_i^2}, \\
e_{21} &= -\frac{(p_e-p_i) \ [(1+2\ \ln(r/R_i))\,p_e - (1+2\ \ln(r/R_e))\,p_i]}{8\,b_1\,[\ln(R_e/R_i)]^2}, \\
e_{30} &= \frac{3\,(p_e-p_i)^2 \ [(1+2\ \ln(r/R_e))\,R_e^2 - (1+2\ \ln(r/R_i))\,R_i^2]}{16\,b_1\,[\ln(R_e/R_i)]^3 \ R_e^2\,R_i^2}, \\
e_{31} &= \frac{(2\,b_1-b_2)\,(p_e-p_i)^2}{16\,b_1^2 \ [\ln(R_e/R_i)]^2}.
\end{aligned}
\tag{6.268}
$$

As for the problem studied in Section 6.3, it can be seen from (6.265) that only the second-order pressure depends upon b_2. The stress distribution in this case has a more complicated variation through the thickness as compared to that when $n \neq \{2, 4\}$

6.6 The Case $n = 4$

Equations (6.231), and (6.232) present another singularity for $n = 4$. This implies that the case $n = 4$ must be treated separately as well. The material moduli μ and α_6 have to depend on the radial coordinate r in the following way

$$\mu = b_1 r^4, \qquad \alpha_6 = b_2 r^4. \tag{6.269}$$

The first-order displacement field is still given by

$$u_1(r) = \frac{\hat{B}_1}{r}, \tag{6.270}$$

where $\hat{B}_1$ is a constant.

From (6.202), (6.209), and (6.270), it follows that

$$\begin{aligned}
(\mathbf{T}_{*1})_{11} &= -2\,\hat{B}_1 b_1 r^2 - p_1, \\
(\mathbf{T}_{*1})_{22} &= 2\,\hat{B}_1 b_1 r^2 - p_1, \\
(\mathbf{T}_{*1})_{33} &= -p_1, \quad (\mathbf{T}_{*1})_{ij} = 0, \quad i \neq j,
\end{aligned} \tag{6.271}$$

and equation $(6.207)_1$ becomes

$$8\,r\,\hat{B}_1 b_1 + p_1'(r) = 0,$$

whose solution is

$$p_1(r) = \hat{B}_2 - 4\,\hat{B}_1 b_1 r^2, \tag{6.272}$$

where $\hat{B}_2$ is a constant of integration.

From (6.271), boundary conditions $(6.207)_2$ and $(6.207)_3$ become

$$-2b_1 R_i^2 \hat{B}_1 + \hat{B}_2 = p_i, \quad 2b_1 R_e^2 \hat{B}_1 - \hat{B}_2 = -p_e,$$

from which we can derive the values of the constants $\bar{B}_1$ and $\bar{B}_2$

$$\hat{B}_1 = \frac{(p_e - p_i)}{2b_1\left(R_i^2 - R_e^2\right)}, \qquad \hat{B}_2 = \frac{p_e R_i^2 - p_i R_e^2}{\left(R_i^2 - R_e^2\right)}. \tag{6.273}$$

The second-order displacement field is

$$u_2(r) = \frac{\hat{B}_3}{r} - \frac{\hat{B}_1^2}{2r^3}, \tag{6.274}$$

in which the constant $\hat{B}_3$ must be determined from boundary conditions.

The second-order Piola-Kirchhoff stress tensor $\mathbf{T}_{*2}$ becomes

$$\begin{aligned}
(\mathbf{T}_{*2})_{11} &= 3\,\hat{B}_1^2 b_1 - 2b_1\hat{B}_3 r^2 - p_2, \\
(\mathbf{T}_{*2})_{22} &= -b_1\hat{B}_1^2 + 2b_1\hat{B}_3 r^2 - p_2, \qquad (\mathbf{T}_{*2})_{33} = -p_2, \\
(\mathbf{T}_{*2})_{ij} &= 0, \quad i \neq j,
\end{aligned} \tag{6.275}$$

which coincides with (6.224) for $n = 4$. Substituting equations (6.209), (6.212), and (6.272) into (6.203), we obtain the following components of $\mathbf{B}_{*1}$

$$\begin{aligned}
(\mathbf{B}_{*1})_{11} &= \frac{\hat{B}_1\left[-\hat{B}_2 + r^2\hat{B}_1\left(3b_1 + b_2\right)\right]}{r^2}, \\
(\mathbf{B}_{*1})_{22} &= \frac{\hat{B}_1\left[\hat{B}_2 + r^2\hat{B}_1\left(-5b_1 + b_2\right)\right]}{r^2}, \\
(\mathbf{B}_{*1})_{33} &= 0, \quad (\mathbf{B}_{*1})_{ij} = 0, \quad i \neq j.
\end{aligned} \tag{6.276}$$

From (6.275) and (6.276), the second-order equilibrium equation $(6.218)_1$ becomes

$$-p_2' - 8\,\hat{B}_3\,b_1 r + \frac{12\,\hat{B}_1^2 b_1}{r} = 0,$$

from which we obtain

$$p_2(r) = \hat{B}_4 - 4\,\hat{B}_3\,b_1 r^2 + 12\,\hat{B}_1^2 b_1 \ln r, \tag{6.277}$$

where $\hat{B}_4$ is a constant of integration.

Substituting equations (6.275), (6.276) and (6.227) into boundary conditions $(6.218)_2$ and $(6.218)_3$, and having in mind expressions (6.273) of constants $\hat{B}_1$ and $\hat{B}_2$, we finally obtain

$$\begin{aligned} \hat{B}_3 &= \frac{3(p_e - p_i)^2\,[\ln(R_e) - \ln(R_i)]}{2\,b_1^2\,\left(R_e^2 - \,R_i^2\right)^3}, \qquad (6.278) \\ \hat{B}_4 &= \frac{(p_e - p_i)^2\,\{b_2(R_i^2 - R_e^2) + 4b_1\,[(3\ln(R_i) - 1)R_e^2 - (3\ln(R_e) - 1)R_i^2]\}}{4\,b_1^2\,\left(R_i^2 - R_e^2\,\right)^3}. \end{aligned}$$

Hence, from (6.273), the first-order displacement and pressure fields (6.270), (6.272) assume the following form

$$u_1(r) = \frac{(p_e - p_i)}{2b_1\,(R_i^2 - R_e^2)}\frac{1}{r}, \tag{6.279}$$

$$p_1(r) = \frac{p_e R_i^2 - p_i R_e^2}{(R_i^2 - R_e^2)} - \frac{2(p_e - p_i)}{(R_i^2 - R_e^2)} r^2. \tag{6.280}$$

In the same way, substituting (6.278) into (6.274) and (6.277) we get

$$\begin{aligned} u_2(r) \;=\; & \frac{3(p_e - p_i)^2\,[\ln(R_e) - \ln(R_i)]}{2\,b_1^2\,\left(R_e^2 - \,R_i^2\right)^3}\frac{1}{r} - \qquad (6.281) \\ & \frac{(p_e - p_i)^2}{8b_1^2\,(R_i^2 - R_e^2)^2}\frac{1}{r^3}, \end{aligned}$$

$$
\begin{aligned}
p_2(r) \quad = \quad & \frac{(p_e - p_i)^2 \left\{ b_2 (R_i^2 - R_e^2) + 4 b_1 \left[(3 \ln(R_i) - 1) R_e^2 - (3 \ln(R_e) - 1) R_i^2 \right] \right\}}{4\, b_1^2 \, \left(R_i^2 - R_e^2 \right)^3} - \\
& \frac{6 (p_e - p_i)^2 \left[\ln(R_e) - \ln(R_i) \right]}{b_1 \, \left(R_e^2 - \, R_i^2 \right)^3} r^2 + \frac{3 (p_e - p_i)^2}{b_1 \left(R_i^2 - R_e^2 \right)^2} \ln r. \qquad (6.282)
\end{aligned}
$$

If we substitute equations (6.273) and (6.273) into (6.271), we obtain the following expressions for the first-order stresses

$$
\begin{aligned}
T_{rr}^{(1)} &= \frac{p_i \, (r^2 - R_e^2) - p_e (r^2 - R_i^2)}{R_e^2 - R_i^2}, \\
T_{\theta\theta}^{(1)} &= \frac{p_i \, (3\, r^2 - R_e^2) - p_e (3\, r^2 - R_i^2)}{R_e^2 - R_i^2}, \qquad (6.283) \\
T_{\theta\theta}^{(1)} &= \frac{p_i \, (2\, r^2 - R_e^2) - p_e (2\, r^2 - R_i^2)}{R_e^2 - R_i^2}.
\end{aligned}
$$

Substitution of equations (6.282) and (6.278) into (6.275) and (6.276) yields the second-order stresses

$$
\begin{aligned}
T_{rr}^{(2)} &= l_{10} \, (r) + l_{11} \, r^{-2} + l_{12} r^2, \\
T_{\theta\theta}^{(2)} &= l_{20} \, (r) + l_{21} \, r^{-2} + l_{22} r^2, \qquad (6.284) \\
T_{zz}^{(2)} &= l_{30} (r) + l_{32} r^2,
\end{aligned}
$$

where

$$
\begin{aligned}
& l_{10}(r) = - \frac{(p_e - p_i)^2 \, \left[(6 \, \ln (r/R_i) - 1) \, R_e^2 - (6 \, \ln (r/R_e) - 1) \, R_i^2 \right]}{2 \, b_1 \, (R_e^2 - R_i^2)^3}, \\
& l_{11} = - \frac{p_i^2 \, R_e^2 + p_e^2 R_i^2 - p_e \, p_i \, (R_e^2 - R_i^2)}{2 \, b_1 \, (R_e^2 - R_i^2)^2}, \qquad (6.285) \\
& l_{12} = \frac{3 \left[\ln (R_e / R_i) \right] (p_e - p_i)^2}{b_1 \, (R_e^2 - R_i^2)^3},
\end{aligned}
$$

$$
\begin{aligned}
& l_{20}\,(r) = -\frac{(p_e - p_i)^2\ \left[(6\ \ln{(r/R_i)} + 5)\, R_e^2 - (6\ \ln{(r/R_e)} + 5)\, R_i^2\right]}{2\, b_1\, (R_e^2 - R_i^2)^3}, \qquad (6.286)\\
& l_{21} = -l_{11}, \qquad l_{22} = \frac{9\,[\ln{(R_e/R_i)}]\,(p_e - p_i)^2}{b_1\,(R_e^2 - R_i^2)^3},\\
& l_{30}(r) = -\frac{(p_e - p_i)^2\,[b_2\,(R_e^2 - R_i^2) + 4 b_1\,(1 + 3\ln{(r/R_i)})\, R_e^2 - (1 + 3\ln{(r/R_e)})\, R_i^2]}{4\, b_1^2\,(R_e^2 - R_i^2)^3},\\
& l_{32} = \frac{6\,[\ln{(R_e/R_i)}]\,(p_e - p_i)^2}{b_1\,(R_e^2 - R_i^2)^3}.
\end{aligned}
$$

6.7 Affine Variations of Elastic Moduli

In this section we present analytical solutions of the pressurized cylinder problem for an affine variation of elastic molduli μ and α_6 upon the non-dimensional radial coordinate r, i.e. when

$$
\mu\,(r) = b_1\,(1 + n\, r)\,, \qquad \alpha_6\,(r) = b_2\,(1 + n\, r) \tag{6.287}
$$

Adopting the same scheme presented in the previous sections, we obtain the following form for the first-order displacement and pressure fields

$$
\begin{aligned}
u_1(r) &= \frac{(p_i - p_e)\ R_e^2\, R_i^2}{2\, r\, b_1\,(R_e - R_i)\,[R_i + R_e\,(1 + 2\, n\, R_i)]}, \qquad (6.288)\\
p_1(r) &= \frac{p_e R_e^2\,(1 + 2\, n\, R_i) - p_i R_i^2\,(1 + 2\, n\, R_e)}{(R_e - R_i)\,[R_i + R_e\,(1 + 2\, n\, R_i)]} + \frac{n\ (p_i - p_e)\ R_e^2\, R_i^2}{r\ (R_e - R_i)\,[R_i + R_e\,(1 + 2\, n\, R_i)]}.
\end{aligned}
$$

Further, at the second-order of approximation, it results

$$
\begin{aligned}
u_2(r) &= \frac{(p_i - p_e)\ R_e^4\ R_i^4}{8\ r^3\ b_1^2 \left(R_e - R_i\right)^2 \left[R_i + R_e\left(1 + 2\ n\ R_i\right)\right]^2} + \\
&\frac{(p_i - p_e)^2\ R_e^2\ R_i^2 \left[3\ R_i^3 + \left(R_e^3 + R_e^2 R_i + R_e R_i^2\right)\left(3 + 4\ n\ R_i\right)\right]}{8\ r\ b_1^2 \left(R_e - R_i\right)^2 \left[R_i + R_e\left(1 + 2\ n\ R_i\right)\right]^3}, \qquad (6.289)\\
p_2(r) &= -\frac{(p_i - p_e)^2 R_e^2 R_i^2 \left[3R_i\left(1 + 2nR_i\right) + 2nR_e^2\left(3 + 4nR_i\right) + R_e\left(8n^2R_i^2 + 10nR_i + 3\right)\right]}{4b_1\left(R_e - R_i\right)^2\left[R_i + R_e\left(1 + 2nR_i\right)\right]^3} + \\
&\frac{n\left(p_i - p_e\right)^2\ R_e^2\ R_i^2 \left[3\ R_i^3 + \left(R_e^3 + R_e^2 R_i + R_e R_i^2\right)\left(3 + 4\ n\ R_i\right)\right]}{4\ r\ b_1 \left(R_e - R_i\right)^2 \left[R_i + R_e\left(1 + 2\ n\ R_i\right)\right]^3} + \\
&\frac{n\ b_2 \left(p_i - p_e\right)^2 R_e^4 R_i^4}{4\ r^3\ b_1^2 \left(R_e - R_i\right)^2 \left[R_i + R_e\left(1 + 2\ n\ R_i\right)\right]^2} + \\
&\frac{\left(b_1 + b_2\right)\left(p_e - p_i\right)^2 R_e^4 R_i^4}{4\ r^4\ b_1^2 \left(R_e - R_i\right)^2 \left[R_i + R_e\left(1 + 2\ n\ R_i\right)\right]^2}
\end{aligned}
$$

The first-order radial and hoop stresses are given by

$$
\begin{aligned}
T_{rr}^{(1)} &= \frac{p_i\left(r - R_e\right)\left[r + \left(1 + 2\ n\ r\right)R_e\right]R_i^2 - p_e\left(r - R_i\right)\left[r + \left(1 + 2\ n\ r\right)R_i\right]R_e^2}{r^2\left(R_e - R_i\right)\left[R_i + R_e\left(1 + 2\ n\ R_i\right)\right]}, \\
T_{\theta\theta}^{(1)} &= \frac{p_i\left(r^2 + 2\ n\ r^2\ R_e + R_e^2\right)R_i^2 - p_e\left(r^2 + 2\ n\ r^2\ R_i + R_i^2\right)R_e^2}{r^2\left(R_e - R_i\right)\left[R_i + R_e\left(1 + 2\ n\ R_i\right)\right]}, \qquad (6.290)
\end{aligned}
$$

while these quantities at the second-order of approximation assume the following expressions

$$
\begin{aligned}
T_{rr}^{(2)} &= g_{10} + g_{11}r^{-1} + g_{12}r^{-2} + g_{13}r^{-4}, \qquad (6.291)\\
T_{\theta\theta}^{(2)} &= g_{20} + g_{21}r^{-1} + g_{22}r^{-2} + g_{23}r^{-4},
\end{aligned}
$$

where

$$
\begin{aligned}
g_{10} &= \frac{(p_e - p_i)^2 R_e^2 R_i^2 \left[3\, R_i (1 + 2nR_i) + 2nR_e^2 (3 + 4nR_i) + R_e \left(8n^2 R_i^2 + 10nR_i + 3\right)\right]}{4b_1 (R_e - R_i)^2 \left[R_i + R_e (1 + 2nR_i)\right]^3}, \\
g_{11} &= -\frac{n (p_e - p_i)^2 R_e^2 R_i^2 \left[3R_i^3 + \left(R_e^3 + R_e^2 R_i + R_e R_i^2\right)(3 + 4nR_i)\right]}{2b_1 (R_e - R_i)^2 \left[R_i + R_e (1 + 2nR_i)\right]^3}, \\
g_{12} &= (p_e - p_i) R_e^2 R_i^2 \left\{ \frac{p_e \left[-R_e^2 R_i - 3R_i^3 - R_e R_i^2 (3 + 4nR_i) + R_e^3 \left(8n^2 R_i^2 + 4nR_i - 1\right)\right]}{4b_1 (R_e - R_i)^2 \left[R_i + R_e (1 + 2nR_i)\right]^3} + \right. \\
&\left. \frac{p_i \left[R_i^3 + R_e^3 (3 + 4nR_i) + R_e \left(R_i^2 - 4nR_i^3\right) + R_e^2 \left(3R_i - 8n^2 R_i^3\right)\right]}{4b_1 (R_e - R_i)^2 \left[R_i + R_e (1 + 2nR_i)\right]^3} \right\}, \\
g_{13} &= \frac{(p_e - p_i)^2 R_e^4 R_i^4}{4b_1 (R_e - R_i)^2 \left[R_i + R_e (1 + 2nR_i)\right]^2},
\end{aligned}
\tag{6.292}
$$

and

$$
g_{20} = g_{10}, \quad g_{21} = 0, \quad g_{22} = -g_{12}, \quad g_{23} = -3\, g_{13}. \tag{6.293}
$$

6.8 Pressurized Cylindrical Hole in an Infinite Space

In the present section we analyze the response of an elastic body with a cylindrical hole subjected to a uniform pressure on the inner surface. It is clear that, in order to study this problem, we have to assume that

$$
p_e = 0, \qquad R_e \to \infty. \tag{6.294}
$$

Before passing to explicit results, we want to underline that the analysis of the state of stress of an elastic solid with a cylindrical cavity under applied tractions is a topic of considerable interest in both practical and theoretical mechanics. As a matter of fact this study can be usefully applied to foundation drilling, oil wells, in situ geotechnical testing, structural and mechanical designs, and borehole technology.

6.8.1 Power law Variation of the Elastic Moduli

For elastic moduli given by equation (6.204), the first-order radial and hoop stress for a pressurized cylindrical hole for $n \neq 2, 4$ are obtained from $(6.235)_{1,2}$ letting $p_e = 0$ and taking the limit as R_e goes to infinity. For the radial stress it results

$$T_{rr}^{(1)} = \begin{cases} -p_i, & n > 2, \\ -p_i \left(\dfrac{R_i}{r}\right)^{2-n}, & n < 2, \end{cases} \tag{6.295}$$

while for the hoop stress we have

$$T_{\theta\theta}^{(1)} = \begin{cases} -p_i, & n > 2, \\ p_i\,(1-n) \left(\dfrac{R_i}{r}\right)^{2-n}, & n < 2. \end{cases} \tag{6.296}$$

From (6.295) and (6.296) it follows that in the case of a homogeneous material ($n = 0$) the first-order radial and hoop stress reduce to

$$T_{rr}^{(1)} = -p_i \left(\frac{R_i}{r}\right)^2, \qquad T_{\theta\theta}^{(1)} = p_i \left(\frac{R_i}{r}\right)^2.$$

Further, we note that the first-order hoop stress vanishes for $n = 1$.

For the second-order radial stress from (6.247) we obtain the following results

$$\begin{aligned}
\lim_{R_e \to +\infty} c_{10} &= 0, \qquad \lim_{R_e \to +\infty} c_{11} = 0, \qquad \forall\, n, \\
\lim_{R_e \to +\infty} c_{12} &= \begin{cases} 0, & n > 2, \\ \dfrac{3\,(n-2)^2\; p_i^2\; R_i^{2-2n}}{4(n-4)b_1}, & n < 2, \end{cases} \\
\lim_{R_e \to +\infty} c_{13} &= \begin{cases} 0, & n > 2, \\ -\dfrac{(n-1)\,(n-2)\; p_i^2\; R_i^{4-2n}}{2(n-4)b_1}, & n < 2, \end{cases}
\end{aligned}$$

which imply that

$$T_{rr}^{(2)} = \begin{cases} 0, & n > 2, \\ \dfrac{p_i^2 R_i^{2-2n}(n-2)r^{-2+n}}{2(n-4)b_1}\left[\dfrac{3\,(n-2)}{2} - \dfrac{(n-1)\;R_i^2}{r^2}\right], & n < 2. \end{cases} \tag{6.297}$$

Finally, since from (6.248) it is

$$\begin{aligned} \lim_{R_e\to+\infty} c_{20} &= 0, \qquad \lim_{R_e\to+\infty} c_{21} = 0, \qquad \forall\, n, \\ \lim_{R_e\to+\infty} c_{22} &= \begin{cases} 0, & n > 2, \\ \dfrac{3\,(n-2)^2\;(n-1)\,p_i^2\;R_i^{2-2n}}{4(n-4)b_1}, & n < 2, \end{cases} \\ \lim_{R_e\to+\infty} c_{23} &= \begin{cases} 0, & n > 2, \\ -\dfrac{(n^2-4n+3)\,(n-2)\;p_i^2\;R_i^{4-2n}}{2(n-4)b_1}, & n < 2, \end{cases} \end{aligned}$$

we get the following expression for the second-order hoop stress

$$T_{\theta\theta}^{(2)} = \begin{cases} 0, & n > 2, \\ \dfrac{p_i^2 R_i^{2-2n}(n-2)(n-1)r^{-2+n}}{2(n-4)b_1}\left[\dfrac{3\,(n-2)}{2} - \dfrac{(n-3)\;R_i^2}{r^2}\right], & n < 2, \end{cases} \tag{6.298}$$

which vanishes when $n = 1$. Thus, letting

$$\begin{aligned} T_{\theta\theta} &= \epsilon T_{\theta\theta}^{(1)} + \epsilon^2 T_{\theta\theta}^{(2)} \\ Trr &= \epsilon T_{rr}^{(1)} + \epsilon^2 T_{rr}^{(2)}, \end{aligned}$$

we reach the following conclusions

$$\begin{aligned} &T_{\theta\theta} = 0, && n = 1, \\ &T_{\theta\theta} = T_{rr} = -p_i, && n > 2, \end{aligned}$$

while the values for $n < 2$ can be obtained combining $(6.295)_2$, $(6.296)_2$, $(6.297)_2$, and $(6.298)_2$.

For a homogeneous cylindrical cavity at the second-order of approximation we have

$$\begin{aligned} T_{rr}^{(2)} &= \frac{p_i^2 R_i^2}{4b_1}\left(\frac{-3r^2 + R_i^2}{r^4}\right), \\ T_{\theta\theta}^{(2)} &= -\frac{3p_i^2 R_i^2}{4b_1}\left(\frac{R_i^2 - r^2}{r^4}\right). \end{aligned}$$

We conclude this section noting that for $n = 2$ and $n = 4$, from (6.266)-(6.268) and (6.283)-(6.286), in the limiting case (6.294) it results

$$T_{rr}^{(1)} = T_{\theta\theta}^{(1)} = -p_i,$$

while the corresponding second-order quantities vanish, i.e.

$$T_{rr}^{(2)} = T_{\theta\theta}^{(2)} = 0.$$

Thus we finally have

$$T_{rr} = T_{\theta\theta} = -p_i.$$

6.8.2 Affine Variation of Elastic Moduli

If the elastic moduli have the form given in (6.287), the first-order radial and hoop stress can be obtained in the pressurized hole limiting case (6.294) from (6.290):

$$\begin{aligned} T_{rr}^{(1)} &= -\frac{(1 + 2\,n\,r)\,p_i\,R_i^2}{(1 + 2\,n\,R_i)\,r^2}, \\ T_{\theta\theta}^{(1)} &= \frac{p_i\,R_i^2}{(1 + 2\,n\,R_i)\,r^2}. \end{aligned}$$

Further, if we note that taking the limit as R_e goes to infinity into constants (6.292) and (6.293), we get

$$\begin{aligned}
\lim_{R_e \to +\infty} g_{10} &= 0, \\
\lim_{R_e \to +\infty} g_{11} &= -\frac{n\, p_i^2\, R_i^2\, (3 + 4\, n\, R_i)}{2\, b_1\, (1 + 2\, n\, R_i)^3}, \\
\lim_{R_e \to +\infty} g_{12} &= -\frac{p_i^2\, R_i^2\, (3 + 4\, n\, R_i)}{4\, b_1\, (1 + 2\, n\, R_i)^3}, \\
\lim_{R_e \to +\infty} g_{13} &= \frac{p_i^2\, R_i^4}{4\, b_1\, (1 + 2\, n\, R_i)^2},
\end{aligned}$$

and

$$\begin{aligned}
\lim_{R_e \to +\infty} g_{20} &= 0, \\
\lim_{R_e \to +\infty} g_{22} &= \frac{p_i^2\, R_i^2\, (3 + 4\, n\, R_i)}{4\, b_1\, (1 + 2\, n\, R_i)^3}, \\
\lim_{R_e \to +\infty} g_{23} &= -\frac{3\, p_i^2\, R_i^4}{4\, b_1\, (1 + 2\, n\, R_i)^2},
\end{aligned}$$

the second-order radial and hoop stresses (6.291) assume the following expressions

$$\begin{aligned}
T_{rr}^{(2)} &= \frac{p_i^2\, R_i^2}{2\, b_1\, (1 + 2\, n\, R_i)^2} \frac{1}{r} \left[\frac{(3 + 4\, n\, R_i)}{(1 + 2\, n\, R_i)} \left(-.n - \frac{1}{2\, r} \right) + \frac{R_i^2}{2r^3} \right], \\
T_{\theta\theta}^{(2)} &= \frac{p_i^2 R_i^2}{4\, b_1\, (1 + 2\, n\, R_i)^2} \frac{1}{r^2} \left[\frac{(3 + 4\, n\, R_i)}{(1 + 2\, n\, R_i)} - \frac{3\, R_i^2}{r^2} \right].
\end{aligned}$$

6.9 Second-Order Effects for an FG Isotropic Compressible Hollow Cylinder

In the present section we solve the problem of the pressurized hollow cylinder presented above for a functionally graded, isotropic, compressible material. In this case the second order expansion of the first Piola-Kirchhoff stress tensor assumes the following form (see

(4.88)-(4.90))

$$\mathbf{T}_* = \epsilon\mathbf{T}_{*1} + \epsilon^2\left(\mathbf{T}_{*2} + \mathbf{B}_{*1}\right), \tag{6.299}$$

where

$$\mathbf{T}_{*i} = a_1(r)I_{\mathbf{E}_i}\mathbf{1} + 2a_2(r)\mathbf{E}_i \quad i = 1, 2, \tag{6.300}$$

$$\begin{aligned}\mathbf{B}_{*1} = &\left[\tfrac{a_1(r)}{2}\left(I_{\mathbf{H}_1\mathbf{H}_1^T} + 2I_{\mathbf{E}_1}^2\right) + a_3(r)I_{\mathbf{E}_1}^2 + a_4(r)II_{\mathbf{E}_1}\right]\mathbf{1} + a_5(r)I_{\mathbf{E}_1}\mathbf{E}_1 \\ &+a_6(r)\mathbf{E}_1^2 - a_1(r)I_{\mathbf{E}_1}\mathbf{H}_1^T - a_2(r)\left(\mathbf{H}_1^T\right)^2.\end{aligned} \tag{6.301}$$

Here we assume, like in [74], that the material moduli depend on the radial coordinate in the following way

$$a_i(r) = a_i r^n, \tag{6.302}$$

where a_i are constants and n is a dimensionless arbitrary constant.

We consider a very long hollow cylinder S made of a compressible, elastic, functionally graded, and isotropic material. Let R_i and R_e be the internal and the external radii, respectively. Let S be at equilibrium in the current configuration C under the action of the uniform pressure field

$$\mathbf{t} = \begin{cases} -p_i\mathbf{N}_i & \text{on } \partial C_i, \\ -p_e\mathbf{N}_e & \text{on } \partial C_e, \end{cases} \tag{6.303}$$

where p_i and p_e are positive constants, $\mathbf{N}_i$ and $\mathbf{N}_e$ are the unit outward normal vectors to ∂C_i and ∂C_e, respectively.

As we have already done, owing to cylindrical symmetry, we search for the solution of pure traction-value problem in the following form

$$\mathbf{u}(r) = \left[\epsilon u_1(r) + \epsilon^2 u_2(r)\right]\mathbf{a}_r, \tag{6.304}$$

where $\mathbf{a}_r$ is the radial unit vector of the physical basis associated with the cylindrical coordinates $\{r, \varphi, z\}$.

The first-order boundary-value problem (4.104) can be written as

$$\begin{cases} \nabla_* \cdot \mathbf{T}_{*1} = \mathbf{0} \quad \text{in } C_*, \\ \mathbf{T}_{*1} \cdot \mathbf{N}_{*i} = \mathbf{t}_{*1}^{(i)} \quad \text{on } \partial C_{*i}, \\ \mathbf{T}_{*1} \cdot \mathbf{N}_{*e} = \mathbf{t}_{*1}^{(e)} \quad \text{on } \partial C_{*e}, \end{cases} \tag{6.305}$$

where from (4.138) and $(4.103)_1$ the applied loads are given by

$$\mathbf{t}_{*1}^{(i)} = -p_i \mathbf{N}_{*i}, \quad \mathbf{t}_{*1}^{(e)} = -p_e \mathbf{N}_{*e}. \tag{6.306}$$

Equations (6.305) lead to the following problem

$$\begin{cases} u_1'' + \dfrac{(n+1)}{r} u_1'(r) + (n\nu - 1)\dfrac{u_1}{r^2} = 0, \\ \left[(a_1 + 2a_2) r^n u_1' + r^{-1+n} a_1 u_1\right]_{r=R_i} = -p_i, \\ \left[(a_1 + 2a_2) r^n u_1' + r^{-1+n} a_1 u_1\right]_{r=R_e} = -p_e, \end{cases} \tag{6.307}$$

where

$$\nu = \frac{a_1}{a_1 + 2a_2}. \tag{6.308}$$

The solution to equation (6.307) is

$$u_1(r) = A_1 r^{-\frac{(n+k)}{2}} + A_2 r^{\frac{(-n+k)}{2}}, \tag{6.309}$$

in which

$$k = \sqrt{n^2 - 4n\nu + 4}, \tag{6.310}$$

and A_1 and A_2 are constants which have to determined from boundary conditions $(6.307)_{2,3}$. It results

$$\begin{aligned} A_1 &= \frac{2\left[p_i R_i^{\frac{1}{2}(2+k-n)} R_e^k - p_e R_i^k R_e^{\frac{1}{2}(2+k-n)}\right]}{\left[(-2+k+n)\, a_1 + 2\,(k+n)\, a_2\right]\left(R_e^k - R_i^k\right)}, \\ A_2 &= \frac{2\left[p_i R_i^{\frac{(2+k)}{2}} R_e^{\frac{n}{2}} - p_e R_i^{\frac{n}{2}} R_e^{\frac{(2+k)}{2}}\right]}{\left[(2+k-n)\, a_1 + 2\,(k-n)\, a_2\right]\left(R_e^k - R_i^k\right)}. \end{aligned} \tag{6.311}$$

The resulting stresses are

$$(\mathbf{T}_{*1})_{rr} = \frac{r^{\frac{1}{2}(-2-k+n)} R_e^{-\frac{n}{2}} R_i^{-\frac{n}{2}} \left[p_i R_i^{\frac{2+k}{2}} R_e^{\frac{n}{2}} \left(r^k - R_e^k \right) + p_e R_i^{\frac{n}{2}} R_e^{\frac{2+k}{2}} \left(-r^k + R_i^k \right) \right]}{R_e^k - R_i^k}, \tag{6.312}$$

$$\begin{aligned}(\mathbf{T}_{*1})_{\varphi\varphi} \;=\; & \frac{r^{\frac{1}{2}(-2-k+n)} R_e^{-\frac{n}{2}} R_i^{-\frac{n}{2}}}{R_e^k - R_i^k} \left[\frac{\left(p_i R_i^{\frac{2+k}{2}} R_e^{\frac{n}{2}} \left(r^k - R_e^k \right) - p_e R_i^{\frac{n}{2}} R_e^{\frac{2+k}{2}} \right) r^k (2 + \nu k - n\nu)}{(k - n + 2\nu)} + \right. \\ & \left. \frac{R_i^{\frac{k}{2}} R_e^{\frac{k}{2}} \left(p_e R_i^{\frac{n+k}{2}} R_e - p_i R_i R_e^{\frac{n+k}{2}} \right) (-2 + \nu k + n\nu)}{(k + n - 2\nu)} \right],\end{aligned} \tag{6.313}$$

$$\begin{aligned}(\mathbf{T}_{*1})_{zz} \;=\; & \frac{r^{\frac{1}{2}(-2-k+n)}}{R_e^k - R_i^k} \nu \left[\frac{\left(p_i R_i^{\frac{2+k-n}{2}} - p_e R_e^{\frac{2+k-n}{2}} \right) r^k (2 + k - n)}{(k - n + 2\nu)} + \right. \\ & \left. \frac{\left(p_e R_i^k R_e^{\frac{2+k-n}{2}} - p_i R_e^k R_i^{\frac{2+k-n}{2}} \right) (-2 + k + n)}{(k + n - 2\nu)} \right]\end{aligned} \tag{6.314}$$

which are in accordance with those found in [74], [80 - 83].

The second-order boundary-value problem is

$$\begin{cases} \nabla_* \cdot (\mathbf{T}_{*2} + \mathbf{B}_{*1}) = \mathbf{0} \quad \text{in } C_*, \\ (\mathbf{T}_{*2} + \mathbf{B}_{*1}) \cdot \mathbf{N}_{*i} = \mathbf{t}_{*2}^{(i)} \quad \text{on } \partial C_{*i}, \\ (\mathbf{T}_{*2} + \mathbf{B}_{*1}) \cdot \mathbf{N}_{*e} = \mathbf{t}_{*2}^{(e)} \quad \text{on } \partial C_{*e}, \end{cases} \tag{6.315}$$

where

$$\mathbf{t}_{*2}^{(i)} \;=\; -p_i \mathbf{N}_{*i} \left(I_{\mathbf{H}_1} - \mathbf{N}_{*i} \cdot \mathbf{E}_1 \mathbf{N}_{*i} \right), \tag{6.316}$$

$$\mathbf{t}_{*2}^{(e)} \;=\; -p_e \mathbf{N}_{*e} \left(I_{\mathbf{H}_1} - \mathbf{N}_{*i} \cdot \mathbf{E}_1 \mathbf{N}_{*i} \right).$$

From (6.300), (6.301), and (6.309) the second-order equilibrium equation $(6.315)_1$ becomes

$$u_2'' + \frac{(n+1)}{r} u_2'(r) + (n\nu - 1) \frac{u_2}{r^2} = f(r), \tag{6.317}$$

where

$$f(r) = \frac{1}{8(a_1 + 2a_2)} \left(f_1 r^{-3-k-n} + f_2 r^{-3-n} + f_3 r^{-3+k-n} \right), \tag{6.318}$$

and f_i are given by

$$f_p = \sum_{j=1}^{6} f_{pj} a_j, \qquad p = 1, 2, 3. \tag{6.319}$$

in which

$$\begin{aligned}
f_{11} &= \left[k^3 + 2k^2 n + k\,(2+n)^2 + 4\left(4 - 2n + n^2\right)\right] A_1^2, \\
f_{12} &= -2\left[4 + k^3 + n^2 + kn\,(2+n) + k^2(1+2n)\right] A_1^2, \\
f_{13} &= 2(2+k)(-2+k+n)^2 A_1^2, \\
f_{14} &= -4(2+k)(k+n) A_1^2, \\
f_{15} &= 2\left[k^3 + (n-2)^2 + k^2(-1+2n) + k(n^2-4)\right] A_1^2, \\
f_{16} &= 2\left[4 + k^3 + n^2 + kn\,(2+n) + k^2(1+2n)\right] A_1^2,
\end{aligned} \tag{6.320}$$

$$\begin{aligned}
f_{21} &= -8\left(-4 + k^2 + 2n - n^2\right) A_1 A_2, \\
f_{22} &= 4\left(-4 + k^2 - n^2\right) A_1 A_2, \\
f_{23} &= -8\left[k^2 - (n-2)^2\right] A_1 A_2, \\
f_{24} &= -16 n A_1 A_2, \\
f_{25} &= -4\left[k^2 - (n-2)^2\right] A_1 A_2, \\
f_{26} &= -4\left(-4 + k^2 - n^2\right) A_1 A_2,
\end{aligned} \tag{6.321}$$

and

$$
\begin{aligned}
f_{31} &= -\left[k^3 - 2k^2 n + k\left(2+n\right)^2 - 4\left(4 - 2n + n^2\right)\right] A_2^2, \\
f_{32} &= 2\left[-4 + k^3 - n^2 + kn\left(2+n\right) - k^2(1+2n)\right] A_2^2, \\
f_{33} &= -2(-2+k)(2+k-n)^2 A_2^2, \\
f_{34} &= -4(-2+k)(k-n)^2 A_2^2, \\
f_{35} &= -2\left[k^3 + k^2(1-2n) - (n-2)^2 + k(n^2-4)\right] A_2^2, \\
f_{36} &= -2\left[-4 + k^3 - n^2 + kn\left(2+n\right) - k^2(1+2n)\right] A_2^2.
\end{aligned}
\tag{6.322}
$$

The solution to equation (6.312) is given by

$$
u_2 = A_3 r^{-\frac{(n+k)}{2}} + A_4 r^{\frac{(-n+k)}{2}} + y(r), \tag{6.323}
$$

where A_3 and A_4 are constants,

$$
y(r) = C_1 r^{-1-k-n} + C_2 r^{-1-n} + C_3 r^{-1+k-n}, \tag{6.324}
$$

and

$$
C_q = \sum_{j=1}^{6} c_{qj} a_j, \qquad q = 1, 2, 3, \tag{6.325}
$$

in which

$$
\begin{aligned}
c_{1j} &= \frac{f_{1j}}{\bar{c}_1}, \qquad \bar{c}_1 = 2(2+k+n)(2+3k+n)(a_1+2a_2), \\
c_{2j} &= \frac{f_{2j}}{\bar{c}_2}, \qquad \bar{c}_2 = -2(-2+k-n)(2+k+n)(a_1+2a_2), \\
c_{3j} &= \frac{f_{2j}}{\bar{c}_3}, \qquad \bar{c}_3 = 2(-2+k-n)(-2+3k-n)(a_1+2a_2).
\end{aligned}
\tag{6.326}
$$

Constants A_3 and A_4 must be determined from boundary conditions $(6.315)_{2,3}$, which from (6.300), (**??**), and (6.316) can be written as

$$\begin{cases} \left[(a_1+2a_2)r^n u_2' + r^{-1+n}a_1u_2 + (\mathbf{B}_{*1})_{11}\right]_{r=R_i} = -D_i p_i, \\ \left[(a_1+2a_2)r^n u_2' + r^{-1+n}a_1u_2 + (\mathbf{B}_{*1})_{11}\right]_{r=R_e} = -D_e p_e, \end{cases} \tag{6.327}$$

where

$$D_i = R_i^{\frac{1}{2}(-2-k-n)}\left(A_1 + R_i^k A_2\right), \qquad D_e = R_e^{\frac{1}{2}(-2-k-n)}\left(A_1 + R_e^k A_2\right) \tag{6.328}$$

and

$$(\mathbf{B}_{*1})_{11} = \sum_{j=1}^{6} \Lambda_j(r) * a_j. \tag{6.329}$$

in which

$$\begin{aligned}
\Lambda_1(r) &= \frac{1}{8}\left[12+k^2+2k(n-2)-4n+n^2\right]A_1^2\,r^{-2-k} + \\
&\quad \frac{1}{8}\left[12+k^2-2k(n-2)-4n+n^2\right]A_2^2\,r^{-2+k} + \\
&\quad \frac{(12-k^2-4n+n^2)}{4r^2}A_1A_2, \\
\Lambda_2(r) &= -\frac{1}{4}\left[(k+n)^2A_1^2\,r^{-2-k} + (k-n)^2A_2^2\,r^{-2+k}\right] + \frac{(k^2-n^2)}{2r^2}A_1A_2, \\
\Lambda_3(r) &= \frac{1}{4}\left[(-2+k+n)^2A_1^2\,r^{-2-k} + (2+k-n)^2A_2^2\,r^{-2+k}\right] + \\
&\quad \frac{[(n-2)^2-k^2]}{2r^2}A_1A_2, \\
\Lambda_4(r) &= -\frac{1}{2}\left[(k+n)A_1^2\,r^{-2-k} - (k-n)A_2^2\,r^{-2+k}\right] - \frac{nA_1A_2}{r^2}, \\
\Lambda_5(r) &= \frac{1}{4}\left[k^2+2k(n-1)+n(n-2)\right]A_1^2\,r^{-2-k} + \\
&\quad \frac{1}{4}\left[k^2-2k(n-1)+n(n-2)\right]A_2^2\,r^{-2+k} + \\
&\quad \frac{[-k^2+(n-2)n]A_1A_2}{2r^2}, \\
\Lambda_6(r) &= \frac{1}{4}\left[(k+n)^2A_1^2\,r^{-2-k} + (k-n)^2A_2^2\,r^{-2+k}\right] + \frac{(n^2-k^2)}{2r^2}A_1A_2.
\end{aligned} \tag{6.330}$$

It results

$$
\begin{aligned}
A_3 &= \frac{\sum_{j=1}^{6} F_{3j}a_j + \sum_{j=1}^{2} G_{3j}A_j}{\left[(-2+k+n)\,a_1 + 2(k+n)a_2\right]\left(R_e^k - R_i^k\right)}, \\
A_4 &= \frac{\sum_{j=1}^{6} F_{4j}a_j + \sum_{j=1}^{2} G_{4j}A_j}{\left[(2+k-n)\,a_1 + 2(k-n)a_2\right]\left(R_e^k - R_i^k\right)},
\end{aligned} \tag{6.331}
$$

where

$$
\begin{aligned}
F_{31} &= 2\left(R_e^k R_i^{\frac{1}{2}(2+k-n)}\Lambda_1(R_i) - R_e^{\frac{1}{2}(2+k-n)}R_i^k\Lambda_1(R_e)\right) + \sum_{j=1}^{3} b_jC_j, \\
F_{32} &= 2\left(R_e^k R_i^{\frac{1}{2}(2+k-n)}\Lambda_2(R_i) - R_e^{\frac{1}{2}(2+k-n)}R_i^k\Lambda_2(R_e)\right) + \sum_{j=1}^{3} d_jC_j,
\end{aligned} \tag{6.332}
$$

in which

$$
\begin{aligned}
b_1 &= -2(k+n)R_e^{\frac{1}{2}(-2-k-n)}R_i^{\frac{1}{2}(-2-k-n)}\left[R_e^{\frac{1}{2}(2+3k+n)} - R_i^{\frac{1}{2}(2+3k+n)}\right], \\
b_2 &= 2n\left[R_e^{\frac{1}{2}(-2+k-n)}R_i^k - R_e^k R_i^{\frac{1}{2}(-2+k-n)}\right], \\
b_3 &= 2(k+n)R_e^{-1+k-\frac{n}{2}}R_i^{-1+k-\frac{n}{2}}\left[R_e^{1+\frac{n}{2}}R_i^{\frac{k}{2}} - R_e^{\frac{k}{2}}R_i^{1+\frac{n}{2}}\right], \\
d_1 &= -4(1+k+n)R_e^{\frac{1}{2}(-2-k-n)}R_i^{\frac{1}{2}(-2-k-n)}\left[R_e^{\frac{1}{2}(2+3k+n)} - R_i^{\frac{1}{2}(2+3k+n)}\right], \\
d_2 &= -4(1+n)R_e^{\frac{1}{2}(-2+k-n)}R_i^{\frac{1}{2}(-2+k-n)}\left[R_e^{\frac{1}{2}(2+k+n)} - R_i^{\frac{1}{2}(2+k+n)}\right], \\
d_3 &= 4(-1+k-n)R_e^{-1+k-\frac{n}{2}}R_i^{-1+k-\frac{n}{2}}\left[R_e^{1+\frac{n}{2}}R_i^{\frac{k}{2}} - R_e^{\frac{k}{2}}R_i^{1+\frac{n}{2}}\right],
\end{aligned} \tag{6.333}
$$

$$
\begin{aligned}
F_{41} &= 2\left(R_i^{\frac{1}{2}(2+k-n)}\Lambda_1(R_i) - R_e^{\frac{1}{2}(2+k-n)}\Lambda_1(R_e)\right) + \sum_{j=1}^{3} \bar{b}_jC_j \\
F_{42} &= 2\left(R_i^{\frac{1}{2}(2+k-n)}\Lambda_2(R_i) - R_e^{\frac{1}{2}(2+k-n)}\Lambda_2(R_e)\right) + \sum_{j=1}^{3} \bar{d}_jC_j,
\end{aligned} \tag{6.334}
$$

in which

$$
\begin{aligned}
\bar{b}_1 &= -2(k+n)R_e^{\frac{1}{2}(-2-k-n)}R_i^{\frac{1}{2}(-2-k-n)}\left[R_e^{\frac{1}{2}(2+k+n)} - R_i^{\frac{1}{2}(2+k+n)}\right], \\
\bar{b}_2 &= 2n\left[R_e^{\frac{1}{2}(-2+k-n)} - R_i^{\frac{1}{2}(-2+k-n)}\right], \\
\bar{b}_3 &= 2(k+n)R_e^{-1-\frac{n}{2}}R_i^{-1-\frac{n}{2}}\left[R_e^{1+\frac{n}{2}}R_i^{\frac{3k}{2}} - R_e^{\frac{3k}{2}}R_i^{1+\frac{n}{2}}\right], \qquad (6.335) \\
\bar{d}_1 &= -4(1+k+n)R_e^{\frac{1}{2}(-2-k-n)}R_i^{\frac{1}{2}(-2-k-n)}\left[R_e^{\frac{1}{2}(2+k+n)} - R_i^{\frac{1}{2}(2+k+n)}\right], \\
\bar{d}_2 &= -4(1+n)R_e^{-1-\frac{n}{2}}R_i^{-1-\frac{n}{2}}\left[R_e^{1+\frac{n}{2}}R_i^{\frac{k}{2}} - R_e^{\frac{k}{2}}R_i^{1+\frac{n}{2}}\right], \\
\bar{d}_3 &= 4(-1+k-n)R_e^{-1-\frac{n}{2}}R_i^{-1-\frac{n}{2}}\left[R_e^{1+\frac{n}{2}}R_i^{\frac{3k}{2}} - R_e^{\frac{3k}{2}}R_i^{1+\frac{n}{2}}\right],
\end{aligned}
$$

$$
\begin{aligned}
F_{3l} &= 2\left[R_e^k R_i^{\frac{1}{2}(2+k-n)}\Lambda_l(R_i) - R_e^{\frac{1}{2}(2+k-n)}R_i^k\Lambda_l(R_e)\right], \qquad (6.336) \\
F_{4l} &= 2\left[R_i^{\frac{1}{2}(2+k-n)}\Lambda_l(R_i) - R_e^{\frac{1}{2}(2+k-n)}\Lambda_l(R_e)\right], \quad l = 3, ...6,
\end{aligned}
$$

and

$$
\begin{aligned}
G_{31} &= 2(p_i R_e^k R_i^{-n} - p_e R_e^{-n} R_i^k), \qquad (6.337) \\
G_{32} &= 2R_e^k R_i^k (p_i R_i^{-n} - p_e R_e^{-n}),
\end{aligned}
$$

$$
\begin{aligned}
G_{41} &= 2(p_i R_i^{-n} - p_e R_e^{-n}), \qquad (6.338) \\
G_{42} &= 2(p_i R_i^{k-n} - p_e R_e^{k-n}).
\end{aligned}
$$

It is possible to verify from the expressions of the applied first and second-order loads (6.306) and (6.316) and the symmetry of the displacement (6.304) that the compatibility conditions (4.106) and (4.107) are satisfied.

6.10 Homogeneous Isotropic Solutions

The *homogeneous* isotropic displacements are recovered from (6.309) and (6.323) on letting $n \to 0$ (so that, owing to (6.310), $k \to 2$). Moreover, functions (6.302) reduce to classic material moduli of homogeneous elastic bodies, i.e.

$$\begin{aligned} a_1 &= \lambda, \qquad a_2 = \mu, \\ a_{i+2} &= \beta_i, \qquad i = 1, ...4, \end{aligned}$$

where λ and μ are the *Lamé coefficients* and constants $\beta_i's$ are the *second-order elasticities* of the material. Thus, from (6.309) and (6.311) the first-order displacement $u_1(r)$ becomes

$$u_1(r) = \frac{\bar{A}_1}{r} + \bar{A}_2 r, \tag{6.339}$$

where

$$\bar{A}_1 = \frac{R_i^2 R_e^2 (p_e - p_i)}{2a_2(R_i^2 - R_e^2)}, \qquad \bar{A}_2 = \frac{p_e R_e^2 - p_i R_i^2}{2(a_1 + a_2)(R_i^2 - R_e^2)}, \tag{6.340}$$

and coincides with the first-order solution found in [**?**].

In order to find the expression of the homogeneous second-order displacement $u_2(r)$, we start from noting that as $n \to 0$, from (6.325) and (6.326), we obtain

$$\begin{aligned} \bar{C}_1 &= \frac{(a_1 - a_2 - a_4 + a_6)}{2(a_1 + 2a_2)} \bar{A}_1^2, \\ \bar{C}_2 &= \frac{[a_1^2 - 2a_2(2a_3 + a_4 + a_5) + a_1(-3a_2 - a_4 + a_6)]}{2(a_1 + a_2)(a_1 + 2a_2)} \bar{A}_1 \bar{A}_2, \\ \bar{C}_3 &= -\frac{[a_1^2 + 2a_2(-a_2 + a_5 + a_6) + a_1(-5a_2 + 4a_3 + a_4 + 4a_5 + 3a_6)]}{2(a_1 + a_2)(a_1 + 2a_2)} \bar{A}_2^2, \end{aligned} \tag{6.341}$$

in which we have used the notation

$$\bar{C}_i = \lim_{n \to 0} C_i.$$

In fact, as $n \to 0$, from (6.321), (6.322), and $(6.326)_{2,3}$ it can be seen that while

$$f_{2j} = f_{3j} = 0, \qquad j = 1, ...6, \qquad \bar{c}_2 = \bar{c}_3 = 0,$$

their ratios

$$\frac{f_{2j}}{\bar{c}_2}, \qquad \frac{f_{3j}}{\bar{c}_3}, \qquad j = 1, ...6,$$

tend to a finite value.

Therefore, from (6.323) and (6.324) it follows that the homogeneous second order displacement assumes the following form

$$u_2 = D_1 r + \frac{D_2}{r} + \frac{\bar{C}_1}{r^3},$$

where

$$D_1 = \bar{A}_4 + \bar{C}_3 \qquad D_2 = \bar{A}_3 + \bar{C}_2.$$

and $\bar{A}_3$, $\bar{A}_4$ are the values of constants A_3 and A_4 in the limit as $n \to 0$. From (6.331)-(6.338) and (6.341), it can be seen that constants D_1 and D_2 can be written as

$$D_q = D_{q0} + \sum_{j=1}^{4} D_{qj} a_{j+2}, \quad q = 1, 2, \tag{6.342}$$

where

$$\begin{cases} D_{q0} = \hat{g}_{q0} - (a_1 - a_2) f_q - h_q, \quad D_{q1} = \hat{g}_{q1}, \quad D_{q2} = \hat{g}_{q2} + f_q, \\ D_{q3} = \hat{g}_{q3}, \quad D_{q4} = \hat{g}_{q4} - f_q \end{cases} \tag{6.343}$$

in which we have introduced the notations

$$\hat{g}_{1j} = \frac{R_i^2 \Gamma_j(R_i) - R_e^2 \Gamma_j(R_e)}{2(a_1 + a_2)(R_e^2 - R_i^2)}, \quad \hat{g}_{2j} = \frac{R_i^2 R_e^2 \left[\Gamma_j(R_i) - \Gamma_j(R_e)\right]}{2a_2 \left(R_e^2 - R_i^2\right)}, \quad j = 0, \cdots, 4, \tag{6.344}$$

$$
\begin{aligned}
f_1 &= \frac{\bar{A}_1^2(a_1+3a_2)}{2(a_1+a_2)(a_1+2a_2)R_e^2R_i^2}, \quad h_1 = \frac{-\bar{A}_2(R_i^2p_i - R_e^2p_e) + \bar{A}_1(p_e - p_i)}{2(a_1+a_2)(R_e^2 - R_i^2)}, \\
f_2 &= \frac{\bar{A}_1^2(a_1+3a_2)(R_i^2+R_e^2)}{2a_2(a_1+2a_2)R_e^2R_i^2}, \quad h_2 = \frac{\bar{A}_2R_i^2R_e^2(p_e-p_i) + \bar{A}_1\left(R_i^2p_e - R_e^2p_i\right)}{2a_2(R_e^2-R_i^2)},
\end{aligned}
$$

and

$$
\begin{aligned}
\Gamma_0 &= \bar{A}_2^2\left(3a_1 - a_2\right) + \frac{2\bar{A}_1\bar{A}_2\left(a_1+a_2\right)}{r^2} + \frac{\bar{A}_1^2\left(a_1-a_2\right)}{r^4}, \\
\Gamma_1 &= 4\bar{A}_2^2, \quad \Gamma_2 = \bar{A}_2^2 - \frac{\bar{A}_1^2}{r^4}, \quad \Gamma_3 = 2\bar{A}_2^2 - \frac{2\bar{A}_1\bar{A}_2}{r^2}, \\
\Gamma_4 &= \bar{A}_2^2 + \frac{\bar{A}_1^2}{r^4} - \frac{2\bar{A}_1\bar{A}_2}{r^2},
\end{aligned}
$$

which coincides with the second-order solution found in.(4.165) - (4.170).

6.11 The Program `FGHollowCylinder`

Aim of the Program

The program `FGHollowCylinder` provides analytical expressions for displacements and stresses induced in a functionally graded hollow cylinder by uniform pressures applied to its inner and outer surfaces. The cylinder material is isotropic second-order elastic with all material moduli having similar variation in the radial direction (see Section 6.3 for an exhaustive description of the problem). The results obtained using the program `FGHollowCylinder` allow a complete analysis of the first and second-order state of deformation and stress of the cylinder. Further they can be specialized to the case of the pressurized cylindrical cavity in an infinite space and can be used to design an experimental procedure to measure the second order elastic constant b_2.

Description of the Algorithm

Theoretical bases of the program are provided in Sections 4.2 and 6.2. The program allows to obtain a symbolic form of first and second-order quantities which characterize the response of the cylinder to the applied loads. In particular first and second-order displacements are obtained together with radial, hoop, and axial stresses. The algorithm is structured in such a way to obtain the cylinder response for different material constitutions. In other words it is possible to vary in input the dependence of the elastic moduli upon the radial coordinate and the grade n of the nonhomogeneous material (see equations (6.204) and (6.287)). This means that the program can be usefully used to make a comparison between two different nonhomogeneous material responses and between a homogeneous and a nonhomogeneous material response.

Command Line of the Program FGHollowCylinder

```
FGHollowCylinder[Nint, Next, fint, fext, matmod, grad]
```

Input Data

`Nint` = unit vector field normal to the inner surface of the hollow cylinder;

`Next` = unit vector field normal to the outer surface of the hollow cylinder;

`fint` = hydrostatic pressure applied to the inner surface of the hollow cylinder;

`fext` = hydrostatic pressure applied to the outer surface of the hollow cylinder;

`matmod` = constitutive relations giving the dependence of the elastic moduli upon the radial coordinate r;

`grad` = grade of the functionally graded material.

Output Data

`Metric tensor` is the metric tensor (g_{ij}) in cylindrical coordinates;

`Inverse Metric Tensor` is the inverse metric tensor $(g_{ij})^{-1}$;

`Non-zero Christoffel Symbols` are the Christoffel symbols Γ^k_{ij} different from zero;

`First-order Displacement Gradient` $\mathbf{H}_1$;

First-order Strain Tensor $\mathbf{E}_1$;

First-order Incompressibility Condition;

First-order Displacement $u_1(r)$;

First-order First Piola-Kirchhoff Stress Tensor $\mathbf{T}_{*1}$;

First-order Equilibrium Boundary Value Problem;

First-order pressure $p_1(r)$;

First-order Radial Stress $T_{rr}^{(1)}$;

First-order Hoop Stress $T_{\theta\theta}^{(1)}$;

First-order Axial Stress $T_{zz}^{(1)}$;

Second-order Displacement Gradient $\mathbf{H}_2$;

Second-order Strain Tensor $\mathbf{E}_2$;

First Invariant of the Second-order Displacement Gradient $\mathbf{H}_2$;

Second Invariant of the Fisrt-Order Deformation Gradient $\mathbf{H}_1$;

Second-order Incompressibility Condition;

Second-order Displacement $u_2(r)$;

Second-order Stress Tensor $\mathbf{T}_{*2}$;

Non-zero Components of Tensor $\mathbf{B}_{*1}$;

Second-order Components of the applied loads $\mathbf{t}_{*2}$;

Second-order Equilibrium Boundary Value Problem;

Second-order Pressure $p_2(r)$;

`First-order Fields`: is a synthetic list of first-order relevant quantities (displacement and pressure fields);

`Second-order Fields`: second-order displacement and pressure fields;

`Second-order Radial Stress` $T^{(2)}_{rr}$;

`Second-order Hoop Stress` $T^{(2)}_{\theta\theta}$;

`Second-order Axial Stress` $T^{(2)}_{zz}$.

References

[1] Signorini, A., Sulle deformazioni termoelastiche finite. Proc. 3rd Int. Congr. Appl. Mech. **2**, 80–89, 1930.

[2] Nayfeh, A. H., Introduction to Perturbation Techniques, John Wiley & Sons, New York, 1981.

[3] Marasco, A., Romano, A., Scientific Computing with Mathematica: Mathematical Problems for Ordinary Differential Equations, Birkhauser, Boston, (2001).

[4] Signorini, A., Trasformazioni termoelastiche finite. Memoria II. Ann. Mat. Pura Appl., IV **30**, 1-72, 1949.

[5] Signorini, A., Un semplice esempio di incompatibilità tra la elastostatica classica e la teoria delle deformazioni elastiche finite. Rend. Accad. Naz. Lincei (VIII) **3**, 276-281, 1950.

[6] Stoppelli, F., Un teorema di esistenza e unicità relativo alle equazioni dell' elastostatica isoterma per deformazioni finite. Ricerche Mat. **3**, 247-267, 1954.

[7] Stoppelli, F., Sulla sviluppabilità in serie di potenze di un parametro delle soluzioni delle equazioni dell' elastostatica isoterma. Ricerche Mat. **4**, 58-73, 1955.

[8] Stoppelli, F., Su un sistema di equazioni integro-differenziali interessante l' elastostatica. Ricerche Mat. **6**, 11-26, 1957.

[9] Stoppelli, F., Sull' esistenza di soluzioni delle equazioni dell' elastostatica isoterma nel caso di sollecitazioni dotate di assi di equilibrio I. Ricerche Mat. **6**, 241-287, 1957.

[10] Stoppelli, F., Sull' esistenza di soluzioni delle equazioni dell' elastostatica isoterma nel caso di sollecitazioni dotate di assi di equilibrio II, III. Ricerche Mat. **7**, 71-101, 138-152, 1958.

[11] Tolotti, C., Orientamenti principali di un corpo elastico rispetto alla sua sollecitazione totale. Mem. Accad. Ital. Serv. VII, **13**, 1139-1162, 1942.

[12] Grioli, G., Mathematical Theory of Elastic Equilibrium (Recent Results). Ergebnisse der Angewandte Mathematische, Vol. 7. Springer-Verlag, Berlin, 1962.

[13] Wang, C., Truesdell, C., Introduction to Rational Elasticity. Noordhoff, Groningen, 1973.

[14] Romano, A., Continuum Mechanics Using Mathematica vol. II, Birchauser Boston, to appear.

[15] Capriz, G. & Podio-Guidugli, P., On Signorini's perturbation method in finite elasticity. Arch. Rational Mech. Anal. **57**, 1-30, 1974.

[16] Capriz, G. & Podio-Guidugli, P., The role of Fredholm conditions in Signorini's perturbation method. Arch. Rational Mech. Anal. **70**, 261-288, 1979.

[17] Capriz, G. & Podio-Guidugli, P., Questions of Uniqueness in Finite Elasticity. Trends in Applications of Pure Mathematics to Mechanics, Vol. III, 76-95, Pitman, Boston, 1981.

[18] Capriz, G. & Podio-Guidugli, P., Duality and Stability questions for the linearized traction problem with live loads in elasticity. In; Stability in the Mechanics of Continua (F. H. Schroeder, ed.). Springer-Verlag, Berlin, 1982.

[19] Capriz, G. & Podio-Guidugli, P., A generalization of Signorini's perturbation method suggested by two problems of Grioli. Rend. Sem. Mat. Univ. Padova **68**, 49-162, 1982.

[20] Valent, T., Boundary Value Problems of Finite Elasticity. Springer Tracts in Natural Philosophy. Volume 31, 1988.

[21] Van Buren, W., On the existence and uniqueness of solutions to boundary value problems in finite elasticity. Thesis. Carnagie-Mellon University; Westinghouse Research Laboratories, Report 68-107-MEKMARI, 1968.

[22] Iaccarino, G. L., Marasco A., Romano, A., Signorini's Method for Live Loads and 2-nd Order Effects, Int. J. Engrg. Sci 44, issue 5-6, 312-324, 2006.

[23] Romano, A., Lancellotta, R., Marasco, A., Continuum Mechanics using Mathematica, Birkhauser, 2006.

[24] Truesdell, C., and Toupin, R. A., The classical field theory, Encyclopedia of Physics, 3, No 1, Springer – Verlag, Berlin, 1960.

[25] Eringen, A. C., Foundations of micropolar thermoelasticity, Springer - Verlag, 1970.

[26] Eringen, A. C., Continuum Physics, Volume IV-Polar and Nonlocal Fields Theories, Academic Press, 1976.

[27] Eringen, A. C., Microcontinuum Field Theories: Foundations and Solids, Springer-Verlag, 1999.

[28] Coleman, B. D. & Noll, W., The Thermodynamics of Elastic Material with Heat Conduction and Viscosity, Arch. Rational Mech. Anal. 13, 167-178, 1963.

[29] Noll, W., A Mathematical Theory of the Mechanical Behavior of Continuous Media, Arch. Rational Mech. Anal. 2, 197-226, 1958.

[30] Grioli, G., Statica delle Deformazioni Finite. Problemi di Linearizzazione. Atti del Convegno in memoria di R. Calapso. Ed. V. Veschi, Roma, 1982.

[31] Grioli, G., On the Stress in Rigid Bodies. Meccanica **18**, 3–7, 1983.

[32] Grioli, G., Mathematical Problems in Elastic Equilibrium with Finite Deformations. Applicable Analysis, **15**, 171–186, 1983.

[33] Truesdell, C. and Noll, W., The Nonlinear Field Theories of Mechanics, Handbuch der Physik, Vol. III/3, Springer-Verlag, Berlin Heidelberg New York, 1965.

[34] Hirai, T., Functional Gradient Material, in Processing of Ceramics, Part 2, (ed. R.J. Brook), Material Science and Technology, **17B**, VCH Publishers, Weinheim, 293-341, 1996.

[35] Mortensen, A. and Suresh, S., Functionally Graded Metals and Metal-Ceramic Composites: Part I "Processing", International Materials Reviews, 40(6), 239-265, 1995.

[36] Mortensen, A. and Suresh, S., Fundamentals of Functionally Graded Materials, IOM Communications Ltd., London, 1998.

[37] Bever, M.B. and Duwez, P.E., Gradients in Composites Materials, Mater. Sci. Eng., **10**, 1-8,1972.

[38] Shen, M. and Bever M.B., Gradients in Polymeric Materials, J. Mater. Sci., **7**, 741-746, 1972.

[39] Niino, M. et al., fabrication of a high pressure thrust chamber by the CIP forming method, AIAA Paper No. 84-1227, 1984.

[40] Miyamoto, Y., Kaysser, W.A., Rabin, B.H., Kawasaki, A., Ford, R.G., Functionally Graded Materials: Design, Processing and Applications, Kluwer Academic Publishers, Boston, 1999.

[41] Hirai, T., CVD-Processing, MRS-Bulletin, 20(1), 45-47, 1995

[42] Sasaki, M. et al., Design of SiC/C functionally graded material and its preparation by chemical vapor deposition, J. Ceram. Soc. Jpn. Inter. Ed., 97, 530-534, 1989.

[43] Kowbel, W., Graded-composited ZrC-BN coating for the thermal protection of carbon-carbon composites, in Proceedings Third International Symposium of Ceramic Materials and Components for Engines, (ed. V.J. Tennery), the American Ceramic Society, Westerville, 290-308, 1988.

[44] Kowbel, W., The Mechanism of oxidation protection of C/C composites coated with graded-codeposited carbides, in Proc. of the Second Int'l. Symp. on FGM'92, (eds J.B. Holt, M. Koizumi, T. Hirai, and Z.A. Munir), American Ceramic Society Transactions 34, The American Ceramic Society, Westerville, 237-244, 1993.

[45] Kurihara, K., Sasaki, K., and Kawarada, M., Adhesion improvement of diamond films, in Proc. of The First Int'l. Symp. on FGM'90, Sendai, (eds M. Yamanouchi, M. Koizumi, T. Hirai, and I. Shiota) Functionally Gradient Materials Forum, The Society of Non-Traditional Technology, Tokyo, 65-69, 1990.

[46] Neubrand, A., and Rodel, J., Gradient Materials, An Overview of a Novel Concept, Zeitschrift fur Metallkunde, 88 (5), 358-371, 1997.

[47] Sohda, Y. et al., Carbon/carbon Composites Coated with SiC/C functionally gradient compositions, in Ceramic Transactions, 34, Proc. of The Second Int'l Symp. on FGM'92, (eds J.B. Holt, M. Koizumi, T. Hirai, and Z.A. Munir), American Ceramic Society, Westerville, OH, 125-132, 1993.

[48] Tobioka, M. et al., ACE COAT AC 15 aluminum oxide coated cutting tool for highly efficient machining, in Technical Report "Sumitomodenki", 135, 190-196, 1989.

[49] Heikinheimo, L., Siren, M., and Gasik, M., Al_2O_3 to Ni-superalloy diffusion bonded FG-joints for high temperature applications, in Proc. of The Fourth Int'l. Symp. on FGM'96, Tsukuba, Japan, (Eds. I. Shiota and Y. Miyamoto), Elsevier Science Publ., 313.318, 1997.

[50] Salmi, J., Graded Materials, 2^{nd} Seminar TEKES Technology Program of Material Applications, Espoo, Finland, 1996.

[51] Nuutinen, S. and Heikinheimo, L., The coating-braze combinations in ceramic-matal joints (FG-joints), The State Research Center of Finland (VTT), Report VALB215, Espoo, Finland, 1997.

[52] Bobyn, J.D., et al., The optimum pore size for the fixation of porous-surfaced metal implants by the ingrowth bone, Clin. Orthop., 150, 263-270, 1980.

[53] Cameron, H.U., Macnab, I., and Pilliar, R.M., A porous metal system for joint replacement surgery, Int. J. Artificial Organs 1, 104-109, 1978.

[54] Ducheyne, P. et al., Effect of hydroxyapatite impregnation on skeletal bonding of porous coated implants, J. Biomed. Mater. Res., 14, 225-237, 1980.

[55] Galante, H., et al., Sintered fiber metal composites as a basis for attachment of implants to bone, J. Bone Joint Surg., 53A, 101-114, 1971.

[56] Hench, L.L., et al., Bonding mechanism at the interface of ceramic prosthetic materials, J. Biomed., Mater. Res. Symp., 2, 117-141, 1971.

[57] Oonishi, H., et al., Comparisons of Biological Fixation to the Bone of Titanium coated with hydroxyapatite and with hydroxyapatite reinforced with alumina, Orthop. Ceramic Implants, Japan, 6, 73-80, 1986.

[58] Oonishi, H., et al., the effects of hydroxyapatite coating on bone growth into porous titanium alloy implants, J. Bone Joint Surg., 718, 213-216, 1989.

[59] Oonishi, H., Mechanical and chemical Bonding of Atificial Joints, Clin. Mater., 5, 217-233, 1990.

[60] Pilliar, R.M., Powder Metal Made Orthopaedic implants with porous surface for fixation by tissue ingrowth, Clin. Orthop., 176, 42-51, 1983.

[61] Oonishi, H., Orthopedic Applications of hydroxyapatite, Biomaterials, 12, March, 171-178, 1991.

[62] Erdogan, F., Fracture Mechanics of Functionally Graded Materials, Composites Engineering **5**, 753-770, 1995.

[63] Pindera, M.-J., Aboudi, J., Glaeser, A.M., and Arnold, S.M. (eds), Use of Composites in Multi-Phased and Functionally Graded Materials. Special Issue of Composites Part B: Enginering **28** B, 1997.

[64] Jin, Z.-H. and Batra, R.C., Some Basic Fracture Mechanics Concepts in Functionally Graded Materials, J. Mech. Phys. Solids **44**, 1221-1235, 1996.

[65] Abid Mian, M. and Spencer, A.J.M., Exact Solutions for Functionally Graded Laminated Elastic Materials, J. Mech. Phys. Solids **46**, 2283-2295, 1996.

[66] Rooney, F.T. and Ferrari, M., Torsion and Flexure of inhomogeneous elements, Composites Engineering **5**, 901-911, 1995.

[67] Horgan, C.O. and Chan, A. M., Torsion of Functionally Graded isotropic Linearly Elastic Bars, J. Elasticity **52**, 181-199, 1999.

[68] Loy, C.T., Lam, K.Y., and Reddy, J.N., Vibration of Functionally Graded Cylindrical shells, Internat. J. Mech. Sci. **41**, 309-324, 1999.

[69] Horgan, C.O. and Chan, A.M., Vibration of Inhomogeneous Strings, Rods and Membranes, J. Sound Vibration **225**, 503-513, 1999.

[70] Horgan, C.O. and Payne, L.E., On the Asymptotic Behavior of Solutions of Linear Second-Order Boundary Value Problems on a Semi-infinite Strip, Arch. rational Mech. Anal., **124**, 227-303, 1993.

[71] Horgan, C.O. and miller K.L., Anti-plane Shear Deformations for Homogeneous and Inhomogeneous Anisotropic Linearly Elastic Solids, J. Appl. Mech. **61**, 23-29, 1994.

[72] Scalpato, M.R., and Horgan, C.O., Saint-Venant decay rates for an isotropic Inhomogeneous Linearly Elastic Solid in Anti-Plane Shear, J. Elasticity **48**, 145-166, 1997.

[73] Chan, A.M., and Horgan, C.O., end Effects in Anti-Plane Shear for an Inhomogeneous Isotropic Linearly Elastic Semi-infinite Strip, J. Elasticity **51**, 227-242, 1998.

[74] Horgan, C. O., Chan, A. M., The Pressurized Hollow Cylinder or Disk Problem for Functionally Graded Isotropic Linearly Elastic Materials, Journal Of Elasticity, 55, 43-59, 1999.

[75] Sokolnikoff, I.S., Mathematical theory of Elasticity, 2nd Ed., McGraw-Hill, New York, 1956.

[76] Love, A.E.H., A Treatise on the Mathematical theory of Elasticity, 4th Ed., Dover, New York, 1944.

[78] Southwell, R.V., An Introduction to the Theory of Elasticity, Oxford Univ. Press, Oxford,1936.

[79] Timoshenko, S. and Goodier, Theory of Elasticity, 3rd Ed. McGraw-Hill, New York, 1970.

[80] Lekhnitskii, S. G. , Theory of Elasticity of an Anisotropic Body, Mir, Moscow, 1981.

[81] Lekhnitskii, S. G. , Anisotropic Plates, Gordon and Breach, New York, 1968.

[82] Shaffer, B. W. , Orthotropic Annular Disks in Plane Stress, Journal of Applied Mechanics, 34, 1027-1029, 1967.

[83] Froli, M. , Distribuzione Ottimale dei Moduli Elastici in Corone Circolari non Omogenee, Atti dell' Istituto di Scienza delle Costruzioni dell'Università di Pisa, 18 (251), 13-25, 1987.

[84] Vocke, W., Linear Elasticitat, Profilstab und Profilebene, VEB Fachbuchverlag, Leipzig, 1966.

[85] Batra, R.C., Elements of Continuum Mechanics, AIAA Press, Vienna, 2005.

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