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ROSSELLA CORPORENTE

Weighted Integral Inequalities, BMO-Spaces and Applications

TESI DI DOTTORATO DI RICERCA

IL COORDINATORE
Prof. Aldo de Luca

Contents

Introduction

The aim of this work is to study in probabilistic setting some spaces (weighted or not) that are very important in Harmonic Analysis and PDEs. Studying Bounded Mean Oscillation functions (BMO) and their relations with A_p weights we saw that our results can be extended in the theory of probability. In this work we present some of our sharp results in functional spaces that we would like to extend to the BMO-Martingales space.

The theory of weights arises often in varied contexts of mathematical analysis: it has an important role also in the study of the boundary-value problems for linear elliptic equations when "minimal" smoothness is assumed; for example when the boundary of the domain is assumed only to be Lipschitz. Recent references for this topics are [CKL], [K].

The A_p-class of weights was introduced in 1972 by B. Muckenhoupt [M1] in connection with boundness properties of the Hardy-Littlewood maximal operator on weighted Lebesgue spaces. Muckenhoupt weights are important tools in harmonic analysis, partial differential equations and quasiconformal mappings. An important and very useful property of these weights is the "selfimproving" property (see section 1.4), studied also by Gehring in [G] where he introduced another class of weights verifying a reverse Hölder inequality, the G_q-class. Gehring class was singled out in connection with local integrability properties of the gradient of quasiconformal mappings.

There is a number of works devoted to the study of relationships between Gehring and Muckenhoupt classes. In particular Coifman and Fefferman [CoFe] proved that any Gehring class is contained in some Muckenhoupt class, and viceversa.

Chapter 1 (section 1.1) contains properties and relations between A_p and

G_q classes. In section 1.2 we present our first result about a limit case of a well known Theorem of Johnson and Neugebauer [JN1]. Namely, following [C], we prove that if $h : \mathbb{R} \longrightarrow \mathbb{R}$ is an increasing homeomorphism onto such that h, h^{-1} are locally absolutely continuous, then

$$A_\infty(h') = G_1((h^{-1})').$$

This equality represents the best quantitative version of the known result that $h' \in A_\infty$ iff $(h^{-1})' \in A_\infty$. Here we have defined for weights $w : \mathbb{R} \longrightarrow \mathbb{R}^+ \cup \{0\}$ and $v : \mathbb{R} \longrightarrow \mathbb{R}^+ \cup \{0\}$the following constants

$$A_\infty(w) = \sup_Q \left(\fint_Q w dx \right) \left(\exp \fint_Q \log \frac{1}{w} dx \right)$$

$$G_1(v) = \sup_Q \left(\exp \fint_Q \frac{v}{v_Q} \log \frac{v}{v_Q} \right)$$

where $v_Q = \fint_Q v \, dx$, according to classical papers of Hruscev and Fefferman ([H],[CoFe]).

In section 1.3 we report results about the improvement of integrability exponent of A_p and G_q classes (see [DS], [Ko], [Po], [BSW], [V]). At the end of the Chapter we give an application to the solvability of the Neumann Problem for elliptic operators in divergence form with L^p data in the half plane.

Another aim of this thesis is to extend Johnson and Neugebauer Theorem to the more general Young functions. To do this, following Kerman and Torchinsky [KT] and Migliaccio [M], we introduced the A_Φ and G_Ψ weights classes, where Φ, Ψ are Young functions (Chapter 2).

In sections 2.1 and 2.2 we report some preliminaries definitions and properties that we need in the following. In sections 2.3 and 2.4 we show A_Φ and G_Ψ relations. In the last section we present our Theorems about a generalization of Johnson and Neugebauer result (see Theorem 2.16 and 2.17).

In Chapter 3 we show properties about Hardy-Littlewood maximal function and the connection with A_p and G_q classes.

In Chapter 4 we investigated the space of funcions of Bounded Mean Oscillation, BMO, introduced by John and Nirenberg [JN] in 1961. This space has become extremely important in various areas of analysis including harmonic analysis, PDEs and function theory. BMO-Space is also of interest since it may be considered an appropriate substitute for L^∞. Since BMO contains unbounded functions a natural question is “How large can be functions in BMO?”. The answer is given by John and Nirenberg Theorem [JN].

In sections 4.1, 4.2, 4.3 we report the most important properties of this functions space. In section 4.4 we show Korenovskii result that, in the one dimensional case, give the exact constant in John and Nirenberg Theorem and relations between A_p and BMO given by Garnett and Jones in [GJ]. In section 4.5 we present our results about the improvement of a recent Theorem of Gotoh [Go]. We proved that if h is an increasing homeomorphism from $\mathbb{R}$ into itself and if $\omega = h'$ verifies the A_∞ condition:

$$\frac{\int_E \omega \, dx}{\int_I \omega \, dx} \leq K \left(\frac{|E|}{|I|} \right)^\alpha$$

for any interval $I \subset \mathbb{R}$ and for each measurable set $E \subset I$, where $K \geq 1 \geq \alpha > 0$, then

$$\|f \circ h^{-1}\|_* \leq \frac{K}{\alpha} e^{2+\frac{2}{e}} \|f\|_*$$

for any $f \in BMO(\mathbb{R})$ (see [ACS]).

Another aim consisted to give an explicit bound for the distance to L^∞ after composition. We proved in [ACS] that if $h : \mathbb{R} \longrightarrow \mathbb{R}$ is an increasing homeomorphism such that $(h^{-1})'$ belongs to the A_p-class, then for any $f \in BMO(\mathbb{R})$

$$\varepsilon(f \circ h^{-1}) \leq p \, \varepsilon(f).$$

Moreover, there exists an equivalent norm $\| \cdot \|'_*$ on BMO such that

$$dist\,'(f \circ h^{-1}, L^\infty) \leq p \, dist\,'(f, L^\infty).$$

In the last section of the Chapter we introduced the BMO^R-Space and the class A_p^R, where in the definitions cubes are replaced by rectangles in $\mathbb{R}^n$. We had focus attention on the connections between A_2^R-class and BMO^R (see [W1]).

In Chapter 5 we showed how BMO functions spaces can be extended in probability theory. We investigated about BMO-martingales space and its relations with the probabilistic version of A_p-condition.

In section 5.4 we present how results about the distance in BMO to L^∞ can be extended to probability setting. In the last section we give some ideas for applications of BMO-martingales space in Mathematical Finance (see [Ge], [DeS], [DMSSS]).

Chapter 1

Weighted Integral Inequalities

In this Chapter, first we recall basic definitions and some known results about weighted inequalities, in particular we focus attention on A_p and G_q weights, respectively from Muckenhoupt and Gehring (reverse Hölder) weighted inequalities. Then we show in details our results in the dimension one (see [C]). In the last section we show an application to partial differential equations.

1.1 A_p and G_q classes and constants

We begin recalling some definitions.

Definition 1.1. A non negative measurable function w (weight) on the space $\mathbb{R}^n$ satisfies the A_p**-condition**, $1 < p < \infty$ if there exists a constant $A \geq 1$ such that, for any cube $Q \subset \mathbb{R}^n$ with edges parallel to the coordinate axes, one has

$$\fint_Q w dx \left(\fint_Q w^{-\frac{1}{p-1}} dx \right)^{p-1} \leq A \tag{1.1}$$

where $\fint_Q w\ dx = \frac{1}{|Q|} \int_Q w\ dx$ denotes the mean value of w over Q. We call the A_p**-constant** of w as

$$A_p(w) = \sup_Q \fint_Q w dx \left(\fint_Q w^{-\frac{1}{p-1}} dx \right)^{p-1}, \quad 1 < p < \infty \tag{1.2}$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with edges parallel to the coordinate axes.

The A_p-class was introduced in 1972 by B. Muckenhoupt [M1] in connection with boundness properties of the Hardy-Littlewood Maximal Operator M (see Chapter 3 for details) defined on the weighted space $L^p_{loc}(\mathbb{R}^n, wdx)$ by

$$Mf(x) = \sup_{x\in Q} \fint_Q |f(y)|dy. \tag{1.3}$$

In fact, the following Theorem holds:

Theorem 1.1. *[M1] If $1 < p < \infty$, then Mf is bounded on $L^p(w)$ if and only if $w \in A_p$.*

Almost simultaneously (1973) another important class of weights was singled out by F.W. Gehring [G], the G_q-class, $1 < q < \infty$, in connection with local integrability properties of the gradient of quasiconformal mappings.

Definition 1.2. A weight v on the space $\mathbb{R}^n$ satisfies the **G_q-condition** if there exists a constant $G \geq 1$ such that, for all cubes $Q \subset \mathbb{R}^n$ as above, we have

$$\frac{\left(\fint_Q v^q(x)dx\right)^{\frac{1}{q}}}{\fint_Q v(x)dx} \leq G \tag{1.4}$$

and we refer to (1.4) as a "reverse" Hölder inequality. We call the **G_q-constant** of v as

$$G_q(v) = \sup_Q \left[\frac{\left(\fint_Q v^q dx\right)^{\frac{1}{q}}}{\fint_Q v dx}\right]^{q'} \tag{1.5}$$

with $q' = \frac{q}{q-1}$, where the supremum is taken over all cubes $Q \in \mathbb{R}^n$ with sides parallel to the coordinate axes.

Note that the previous definitions hold when $1 < p < \infty$, now we report the specific definitions when $p = 1$ and $p = \infty$.

Definition 1.3. **A_1-class** consists of all weights w such that $A_1(w)$ is finite, where

$$A_1(w) = \sup_Q \frac{\fint_Q w dx}{\operatorname{ess\,inf}_{x\in Q} w(x)}. \tag{1.6}$$

Definition 1.4. A_∞**-class** consists of all weights w such that $A_\infty(w)$ is finite, where

$$A_\infty(w) = \sup_Q \left(\fint_Q w dx \right) \left(\exp \fint_Q \log \frac{1}{w} dx \right). \tag{1.7}$$

The numbers in (1.6) (1.7) are respectively called A_1 and A_∞ **constants** of w.

Note that Definition 1.4 is due to Hruscev in [H] but there is also a characterization that gives an equivalent definition of A_∞-class, namely

Proposition 1.2. *([M2], [CoFe]) A locally integrable weight* $w : \mathbb{R}^n \longrightarrow [0, +\infty)$ *belongs to the* A_∞*-class iff there exist constants* $0 < \alpha \leq 1 \leq K$ *so that*

$$\frac{|F|}{|Q|} \leq K \left(\frac{\int_F w dx}{\int_Q w dx} \right)^\alpha \tag{1.8}$$

for each cube $Q \subset \mathbb{R}^n$ *with sides parallel to the coordinate axes and for each measurable set* $F \subset Q$.

In the same spirit we define G_∞-class and G_1-class.

Definition 1.5. G_∞**-class** consists of all weights v such that $G_\infty(v)$ is finite, where

$$G_\infty(v) = \sup_Q \frac{\operatorname*{ess\,sup}_{x \in Q} v}{\fint_Q v \, dx}. \tag{1.9}$$

Definition 1.6. G_1**-class** consists of all weights v such that $G_1(v)$ is finite, where

$$G_1(v) = \sup_Q \left(\exp \fint_Q \frac{v}{v_Q} \log \frac{v}{v_Q} dx \right) \tag{1.10}$$

with $v_Q = \fint_Q v \, dx$.

The numbers in (1.9) (1.10) are respectively called G_∞ and G_1 **constants** of v.

The following characterization gives an equivalent definition of G_1-class and is a somewhat "dual" definition of (1.8):

Proposition 1.3. *A locally integrable weight* $v : \mathbb{R}^n \longrightarrow [0,\infty)$ *belongs to* G_1*-class iff there exist constants* $0 < \beta \leq 1 \leq H$ *so that*

$$\frac{\int_E w dx}{\int_Q w dx} \leq H \left(\frac{|E|}{|Q|}\right)^{\beta} \tag{1.11}$$

for each cube $Q \subset \mathbb{R}^n$ *with sides parallel to the coordinate axes and for each measurable set* $E \subset Q$*.*

Now we report two results that are very useful to illustrate the properties of A_p and G_q weights.

Theorem 1.4. *[W] A locally integrable weight* w *is in* A_p*,* $p > 1$ *if and only if there exists* $1 < p_1 < p$ *such that for every cube* Q

$$\left(\frac{|F|}{|Q|}\right)^{p_1} \leq A_{p_1}(w)\frac{\int_F w dx}{\int_Q w dx}$$

for every measurable subset F *of* Q*.*

Theorem 1.5. *[M] A locally integrable weight* v *is in* G_q*,* $q > 1$ *if and only if there exists* $q_1 > q$ *such that for every cube* Q

$$\left(\frac{\int_E v dx}{\int_Q v dx}\right)^{q_1'} \leq G_{q_1}(v)\frac{|E|}{|Q|}$$

where $q_1' = \frac{q_1}{q_1 - 1}$*, for every measurable subset* E *of* Q*.*

The following Theorem shows a first relation between Gehring and Muckenhoupt weights, namely when a Muckenhoupt weight verifies a reverse Hölder inequality (Gehring condition).

Theorem 1.6. *Let* $w \in A_p$*,* $1 < p < \infty$*. Then there exist constants* C *and* $\varepsilon > 0$*, depending only on* p *and the* A_p *constant of* w*, such that for any cube* Q*,*

$$\left(\fint_Q w^{1+\varepsilon} dx\right)^{\frac{1}{1+\varepsilon}} \leq C \fint_Q w \, dx$$

The following propositions resume some common properties of A_p and G_q weights (see for instance [GR]).

Proposition 1.7. *[GR] Let w be a non negative measurable function on the space $\mathbb{R}^n$. Then*

1. $A_1 \subset A_p \subset A_q$, *for* $1 \leq p \leq q < \infty$.
2. $G_\infty \subset G_q \subset G_p$, *for* $1 < p \leq q < \infty$.
3. *If* $w \in A_p, 1 < p < \infty$ *, then there exists* $1 < q < p$ *such that* $w \in A_q$.
4. *If* $w \in G_q, 1 < q < \infty$ *, then there exists* $q < p < \infty$ *such that* $w \in G_p$.
5. $A_\infty = \bigcup\limits_{1 \leq p < \infty} A_p = \bigcup\limits_{1 < q \leq \infty} G_q$.
6. *If* $1 < p < \infty$, $w \in A_p$ *if and only if* $w^{1-p'} \in A_{p'}$, *with* $p' = \frac{p}{p-1}$.
7. $w \in A_p$ *for some* p, *if and only if* $w \in G_q$ *for some* q.

Proposition 1.8. *[GR] Let w be a non negative measurable function on the space $\mathbb{R}^n$. Then*

1. $w \in A_p \Longrightarrow w \in A_q$, $q \geq p$ *and* $w^\alpha \in A_p$, $0 \leq \alpha \leq 1$.
2. *if* $w_1, w_2 \in A_p$, *then* $w_1^\alpha w_2^{1-\alpha} \in A_p$, $0 \leq \alpha \leq 1$.
3. *If* $w_o, w_1 \in A_1$ *then* $w_o w_1^{1-p} \in A_p$.
4. $w \in A_p$, $1 < p < \infty$, *iff there exist* $u, v \in A_1$, *so that* $w = uv^{1-p}$.
5. *if* $w \in A_p$, *then* $w^\tau \in A_p$ *for some* $\tau > 1$.

A_∞ and G_q classes are closely related as the following theorem [StWh] shows.

Theorem 1.9. *[StWh] Let $w : \mathbb{R}^n \longrightarrow [0, +\infty)$ be a weight, then*

$$w \in G_q \iff w^q \in A_\infty$$

In 1972 Muckenhoupt proved the following result, also known “backward propagation” of the A_p condition:

Theorem 1.10. *[M1] Let $w : \mathbb{R}^n \longrightarrow [0, \infty)$ be a locally integrable weight. If $w \in A_p$, then $\exists\, \delta > 0$ such that $w \in A_{p-\delta}$.*

Two years later Coifman and Fefferman proved in [CoFe] the following Lemma.

Lemma 1.11. *[CoFe] If $w \in A_p$, then $w \in A_{p-\varepsilon}$, where $\varepsilon \sim A_p(w)^{1-p'}$, $\frac{1}{p} + \frac{1}{p'} = 1$, and exists a constant C such that*

$$A_{p-\varepsilon}(w) \leq C A_p(w).$$

The following Theorem is a well known result due to F.W. Gehring [G] about the improvement of the integrability exponent in a reverse Hölder inequality also known "forward propagation" of G_q condition:

Theorem 1.12. *[G] Let $v : \mathbb{R}^n \longrightarrow [0, \infty)$ be a locally integrable weight. If $v \in G_q$, then $\exists\, \varepsilon > 0$ such that $v \in G_p$ for $p \in [q, q + \varepsilon)$.*

There are some limiting relations between constants defined above, in fact in [SW] was proved the following

Theorem 1.13. *[SW] Let $w : \mathbb{R}^n \longrightarrow [0, \infty)$ be a locally integrable weight. Then*

$$A_\infty(w) = \lim_{p\to\infty} A_p(w) \tag{1.12}$$

and in [MS] the following

Theorem 1.14. *[MS] Let $v : \mathbb{R}^n \longrightarrow [0, \infty)$ be a locally integrable weight. Then*

$$G_1(v) = \lim_{q\to 1} G_q(v). \tag{1.13}$$

The formulas (1.12) and (1.13) give a quantitative version of the equalities

$$A_\infty = \bigcup_{p>1} A_p = \bigcup_{q>1} G_q = G_1.$$

proved by Muckenhoupt in [M2].

A more general class of weights, including both the A_p and G_q classes was proposed by B. Bojarski [Bo] and I. Wik [Wi].

Definition 1.7. Let $u : \mathbb{R}^n \longrightarrow [0, +\infty)$ be a weight, u satisfies the **B_r^s-condition**, $1 < r < s < \infty$ if there exists a constant $B \geq 1$ such that, for any cube $Q \subset \mathbb{R}^n$ with edges parallel to the coordinate axes, one has

$$\frac{\left(\fint_Q u^s \, dx\right)^{1/s}}{\left(\fint_Q u^r \, dx\right)^{1/r}} \leq B \tag{1.14}$$

and we call **B_r^s-constant** of u

$$B_r^s(u) = \sup_Q \frac{\left(\fint_Q u^s \, dx\right)^{1/s}}{\left(\fint_Q u^r \, dx\right)^{1/r}},$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes.

It is immediate to check that

$$B_1^q(u) = G_q(u)^{1/q'}$$

and

$$B^1_{-\frac{1}{p-1}}(u) = A_p(u)$$

hence A_p and G_q classes are included in the system of B_r^s classes.

1.2 A precise relation among A_∞ and G_1 constants in one dimension

In this section we confine ourselves to the case $n = 1$. The definitions introduced before continue to hold if we consider intervals instead cubes. We prove directly the equality

$$G_1((h^{-1})') = A_\infty(h')$$

for an increasing homeomorphism h on the real line, where $G_1(v)$ is the limit as $q \to 1$ of the Gehring constant $G_q(v)$ and $A_\infty(w)$ is the limit as $p \to \infty$ of the Muckenhoupt constant $A_p(w)$, for v, w non negative weights on $\mathbb{R}$.

We begin reporting a result contained in [JN1] that shows us the A_p-regularity of the derivative of a homeomorphism of the real line and the derivative of his inverse.

Theorem 1.15. *[JN1] (Johnson-Neugebauer) Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing homeomorphism onto such that h, h^{-1} are locally absolutely continuous. Then*

$$h' \in A_p \Longleftrightarrow (h^{-1})' \in G_q, \qquad \frac{1}{p} + \frac{1}{q} = 1 \tag{1.15}$$

and

$$A_p(h') = G_q(h^{-1})'. \tag{1.16}$$

Proof. If $(h^{-1})' \in G_q$ then for every interval J,

$$\left(⨍_J [(h^{-1})'(t)]^q dt \right)^{\frac{1}{q}} \leq G_q((h^{-1})')^{\frac{1}{p}} ⨍_J (h^{-1})'(t) dt$$

where $p = \frac{q}{q-1}$ and $h(I) = J$. Let

$$L = ⨍_I h'(x) dx \left(⨍_I [h'(x)]^{1-q} dx \right)^{p-1}.$$

Note that $⨍_I h'(x) dx = \dfrac{|J|}{|I|}$, so we have to estimate only the second average integral in L. By change of variables $t = h(x)$ we have

$$⨍_I [h'(x)]^{1-q} dx = \frac{1}{|I|} \int_J \frac{1}{h'[(h^{-1}(t))]^q} dt = \frac{|J|}{|I|} \left(⨍_J [(h^{-1})'(t)]^q dt \right) \leq$$

$$\leq \frac{|J|}{|I|} G_q((h^{-1})')^{\frac{q}{p}} \left(⨍_J (h^{-1})'(t) dt \right)^q = G_q((h^{-1})')^{\frac{q}{p}} \left(\frac{|J|}{|I|} \right)^{1-q}.$$

Consequently,

$$L \leq \frac{|J|}{|I|} \left(\frac{|J|}{|I|} \right)^{(1-q)(p-1)} G_q((h^{-1})')^{\frac{q(p-1)}{p}} = G_q((h^{-1})')$$

and taking the supremum over all intervals $I \subset \mathbb{R}$, we get

$$A_p(h') \leq G_q((h^{-1})').$$

Conversely, if $h' \in A_p$, we have, by change of variables $t = h(x)$

$$\frac{\left(⨍_J [(h^{-1})'(t)]^q dt \right)^{\frac{1}{q}}}{⨍_J (h^{-1})'(t) dt} = \left(\frac{1}{|J|} \int_I [(h'(x)]^{1-q} dx \right)^{\frac{1}{q}} \frac{|J|}{|I|}$$

Raising both sides to the power p we have

$$\left\{\frac{\left(\fint_J [(h^{-1})'(t)]^q dt\right)^{\frac{1}{q}}}{\fint_J (h^{-1})'(t)dt}\right\}^p \le \frac{|J|}{|I|}\left(\fint_I [(h'(x)]^{1-q}dx\right)^{p-1} \le A_p(h').$$

Now taking the supremum over all interval $J \subset \mathbb{R}$, we get

$$G_q(h^{-1})' \le A_p(h')$$

and the Theorem is proved. □

An immediately consequence of the previous result is the following Lemma.

Lemma 1.16. *Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing homeomorphism onto such that h, h^{-1} are locally absolutely continuous. Then*

$$h' \in A_\infty \Longleftrightarrow (h^{-1})' \in A_\infty.$$

Note that (1.16) continue to hold also in the limit case via limiting formulas contained in [MS] and [SW]. A direct proof of this result is contained in [C] where is proved the following Theorem.

Theorem 1.17. *[C] Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing homeomorphism onto such that h, h^{-1} are locally absolutely continuous. Then*

$$A_\infty(h') = G_1((h^{-1})'). \tag{1.17}$$

We begin by proving the following Lemma.

Lemma 1.18. *[C] Let I and J be two intervals such that $h(I) = J$. Set*

$$L_I = \fint_I h'(x)dx \exp \fint_I \log \frac{1}{h'(x)} dx$$

and

$$M_J = \exp\left(\fint_J \frac{(h^{-1}(t))'}{\fint_J ((h^{-1}(t))')dt} \log \frac{(h^{-1}(t))'}{\fint_J ((h^{-1}(t))')dt}\, dt\right)$$

Then we have

$$L_I = M_J. \tag{1.18}$$

Proof. We have

$$\fint_I h'(x)dx = \frac{|J|}{|I|} \tag{1.19}$$

and $\displaystyle\fint_I \log\frac{1}{h'(x)}dx$ with the change of variables $x = h^{-1}(t)$ becomes

$$\frac{1}{|I|}\int_J \log\frac{1}{h'(h^{-1}(t))}\frac{dt}{h'(h^{-1}(t))} = \frac{1}{|I|}\int_J \log[(h^{-1}(t))'](h^{-1}(t))'dt \tag{1.20}$$

so we have

$$L_I = \frac{|J|}{|I|}\exp\left[\frac{|J|}{|I|}\fint_J \log[(h^{-1}(t))'](h^{-1}(t))'dt\right] \tag{1.21}$$

Now we consider that

$$\fint_J \frac{(h^{-1}(t))'}{\fint_J (h^{-1}(t))'dt}\log\frac{(h^{-1}(t))'}{\fint_J (h^{-1}(t))'dt}\,dt = \frac{1}{|J|}\int_J \frac{(h^{-1}(t))'}{\frac{|I|}{|J|}}\log\frac{(h^{-1}(t))'}{\frac{|I|}{|J|}}dt =$$

$$= \frac{1}{|I|}\int_J \log\frac{(h^{-1}(t))'}{\frac{|I|}{|J|}}(h^{-1}(t))'dt = \frac{1}{|I|}\int_J (h^{-1}(t))'[\log(h^{-1}(t))' - \log\frac{|I|}{|J|}]dt =$$

$$= \frac{1}{|I|}\int_J (h^{-1}(t))'\log(h^{-1}(t))'dt - \frac{1}{|I|}\int_J (h^{-1}(t))'\log\frac{|I|}{|J|}dt =$$

$$= \frac{1}{|I|}\int_J (h^{-1}(t))'\log(h^{-1}(t))'dt - \log\frac{|I|}{|J|}.$$

So we have

$$M_J = \exp\left(\frac{|J|}{|I|}\fint_J (h^{-1}(t))'\log(h^{-1}(t))'dt - \log\frac{|I|}{|J|}\right) =$$

$$= \exp\left(\frac{|J|}{|I|}\fint_J (h^{-1}(t))'\log(h^{-1}(t))'dt\right)\cdot\exp\left(-\log\frac{|I|}{|J|}\right)$$

and then

$$M_J = \frac{|J|}{|I|}\exp\left[\frac{|J|}{|I|}\fint_J \log[(h^{-1}(t))'](h^{-1}(t))'dt\right] \tag{1.22}$$

In the end (1.21) is equal to (1.22) and the proof is completed. □

Proof. (of **Theorem 1.10**)

By Lemma 1.18 we have to prove only that

$$\sup_I L_I = \sup_J M_J$$

Fix the interval I_o and set $J_o = h(I_o)$. By Lemma 1.18 we have

$$L_{I_o} = M_{J_o} \leq \sup_J M_J,$$

taking the supremum on the left hand side as I_o varies among all intervals in $\mathbb{R}$ we obtain

$$\sup_I L_I \leq \sup_J M_J$$

By a similar argument we get the reverse inequality and our result is proved.

□

1.3 Improvement of the integrability exponent

In this section we report some results about the so-called "sharp self-improvement of exponents" property of the A_p and G_q classes in one dimension.

Let us begin with some results about the improvement of the integrability exponent of a function that is in G_q. The following Theorem is contained in [DS].

Theorem 1.19. *[DS] Let $v : \mathbb{R} \longrightarrow \mathbb{R}$ be a non increasing and nonnegative function. If $v \in G_q$, then $v \in G_p$ with $p \in [q, \beta)$ and β is the solution of the equation*

$$\varphi(x) = 1 - B^q \, \frac{x-q}{x} \left(\frac{x}{x-1} \right)^q = 0$$

where B is such that

$$\left(-\!\!\!\!\!\!\int_I v^q dx \right)^{\frac{1}{q}} \leq B -\!\!\!\!\!\!\int_I v dx.$$

Then, for $q \leq \sigma < \beta$ we have

$$[G_\sigma(v)]^{\frac{1}{\sigma'}} \leq B^{\frac{1}{q'}} \left[\frac{q}{\sigma \varphi(\sigma)} \right]^{\frac{1}{q}}.$$

The result is sharp.

Note that Theorem 1.19 also shows that the best integrability exponent of all non increasing functions in G_q is equal to the best integrability exponent of a power type function in G_q.

Theorem 1.19 was generalized to all functions by Korenovskii [Ko] who also gave a A_p version of the same result.

Theorem 1.20. *[Ko] Let* $1 < q < \infty$ *and let* $f \in L^q(I)$*. If* $f \in G_q$*, then* $f \in G_p$ *for* $p \in [q, \beta)$ *and* β *verifies the equation*

$$1 - M^q \frac{x-q}{x} (\frac{x}{x-1})^q = 0$$

where M *is such that*

$$\left(\fint_I f^q dx \right)^{\frac{1}{q}} \leq M \fint_I f dx.$$

Theorem 1.21. *[Ko] Let* $w : \mathbb{R} \longrightarrow \mathbb{R}$ *be a nonincreasing and nonnegative function. If* $w \in A_q$ *then* $w \in A_p$ *with* $p \in [q, \eta)$ *where* η *is the solution of the equation*

$$\psi(x) = 1 - \frac{q-x}{q-1} (Ax)^{\frac{1}{q-1}} = 0$$

where A *is such that*

$$\fint_I w dx \left(\fint_I w^{-\frac{1}{p-1}} dx \right)^{p-1} \leq A.$$

Then, for $\eta < \rho \leq q$ *we have*

$$A_\rho(w) \leq A \left[\frac{\rho - 1}{(p-1)\psi(\rho)} \right]^{p-1}.$$

It is worth noting that in the special case $p = q = 2$ we have explicit values of β and η and the above theorems enjoy a simpler presentation.

Corollary 1.22. *If* $G_2(v) = G < \infty$*, then for* $2 \leq r < 1 + \sqrt{\frac{G}{G-1}}$

$$[G_r(v)]^2 \leq \frac{2G(r-1)^2}{r[(r-1)^2 - Gr(r-2)]}.$$

Corollary 1.23. *If* $A_2(\omega) = A < \infty$*, then for* $1 + \sqrt{\frac{A-1}{A}} < s \leq 2$

$$A_s(\omega) \leq \frac{A(s-1)}{1 - As(2-s)}.$$

Various relations occurring among A_p and A_2 constants of weights and their powers are collected in the following

Lemma 1.24. *[S1] Let* $\omega : \mathbb{R} \longrightarrow \mathbb{R}$ *be a weight. For* $p > 1$ *we have*

$$[A_2(\omega^{\frac{1}{p-1}})]^{p-1} \leq A_p(\omega)\, A_p(\omega^{-1}). \tag{1.23}$$

For $1 < p \leq 2$ *we have*

$$A_p(\omega) \leq [A_2(\omega^{\frac{1}{p-1}})]^{p-1}. \tag{1.24}$$

For $q > 1$ we have

$$A_2(\omega) \le A_q(\omega) A_q(\omega^{-1}). \tag{1.25}$$

Proof. For any interval $I \subset \mathbb{R}$, Hölder inequality implies

$$1 \le \fint_I \omega \fint_I \omega^{-1}$$

hence

$$\left[\fint_I \omega^{\frac{1}{p-1}} \fint_I \omega^{-\frac{1}{p-1}}\right]^{p-1} \le$$

$$\le \fint_I \omega \left(\fint_I \omega^{-\frac{1}{p-1}}\right)^{p-1} \cdot \fint_I \omega^{-1} \left(\fint_I \omega^{\frac{1}{p-1}}\right)^{p-1} \le A_p(\omega) A_p(\omega^{-1})$$

taking supremum with respect to all intervals I we obtain (1.23).

Fix an interval I and take p such that $1 < p \le 2$; then we have $1 \le \frac{1}{p-1}$ and Jensen inequality implies

$$\fint_I \omega \le \left(\fint_I \omega^{\frac{1}{p-1}}\right)^{p-1}$$

hence

$$\fint_I \omega \left(\fint_I \omega^{-\frac{1}{p-1}}\right)^{p-1} \le \left[\fint_I \omega^{-\frac{1}{p-1}} \cdot \fint_I \omega^{\frac{1}{p-1}}\right]^{p-1} \le \left[A_2(\omega^{\frac{1}{p-1}})\right]^{p-1}.$$

Taking supremum with respect to all intervals I we obtain (1.24).

If $q > 1$ assume

$$A_q(\omega) A_q(\omega^{-1}) = A < \infty.$$

Since that $A_q(w) = [A_p(w^{-\frac{1}{q-1}})]^{q-1}$ where $p = q/(q-1)$, we have

$$A_p(\omega^{\frac{1}{q-1}}) A_p(\omega^{-\frac{1}{q-1}}) = A^{\frac{1}{q-1}}.$$

Replacing ω with $\omega^{\frac{1}{q-1}}$ in (1.23) we get

$$\left[A_2((\omega^{\frac{1}{q-1}})^{\frac{1}{p-1}})\right]^{p-1} \le A_p(\omega^{\frac{1}{q-1}})\, A_p(\omega^{-\frac{1}{q-1}})$$

But $(q-1)(p-1) = 1$, hence

$$A_2(\omega) \le A$$

that is (1.25). □

To state an exact continuation theorem in B_r^s-classes (see Definition 1.7), in view of the optimal integrability results established in [Po], let us introduce two auxiliary functions on $[0,1]$. Given $B \geq 1$, if $0 < r < s$ let us define

$$\varphi(y) = 1 - B^s(1-y)\left(\frac{s}{s-ry}\right)^{s/r} \qquad y \in [0,1]$$

while, if $r < 0 < s$ we define

$$\chi(y) = 1 - B^{-r}(1-y)\left(\frac{r}{r-sy}\right)^{r/s} \qquad y \in [0,1]$$

Theorem 1.25. *Assume the weight $u : [a,b] \longrightarrow [0,\infty)$ satisfies the condition*

$$B_r^s(u) = B < \infty$$

and let x_0 be the unique solution to the equation

$$\left(\frac{x}{x-s}\right)^{1/s} = B\left(\frac{x}{x-r}\right)^{1/r}.$$

Then we have:

1. *if* $0 < r < s$,
$$B_r^\sigma(u) \leq B\left[\frac{s}{\sigma\varphi(\frac{s}{\sigma})}\right]^{1/s}$$
for $s \leq \sigma < x_0$ (and $\varphi(\frac{s}{x_0}) = 0$);

2. *if* $r < 0 < s$,
$$B_\rho^s(u) \leq B\left[\frac{r}{\rho\chi(\frac{r}{\rho})}\right]^{-1/r}$$
for $x_0 < \rho \leq r$ (and $\chi(\frac{r}{x_0}) = 0$).

The proof follows through suitable calculations from the proof of Theorem 1.3 in [Po].

Up to now we have been dealing with the self-improvement of exponents p and q in A_p and G_q classes respectively or of exponents r and s in B_r^s classes.

Now we consider the problem of the exact G_q-class pertaining to all A_p-weights. This was solved for $p = 1$ in [BSW] and for $p > 1$ has been recently settled by Vasyunin [V], who found the exact range of exponents q so that a weight in the A_p-class belongs to the G_q-class. Let us first state the main result in [BSW].

Theorem 1.26. *Let w belong to the A_1-class with $A_1(w) = A$. Then for every $1 \le q < \frac{A}{A-1}$*

$$[G_q(w)]^{q-1} \le \frac{1}{A^{q-1}(A+q-qA)}. \tag{1.26}$$

The constant on the right hand side as well as the upper bound of q cannot be improved. In fact, the weight $w(t) = \frac{t^{\frac{1}{(A-1)}}}{A}$ is an extremal, which gives equality in (1.26) and lies in L^q if and only if $q < \frac{A}{A-1}$.

In order to state the result from [V], we fix $p > 1$ and $\delta > 1$ and denote by $x = x(p, \delta)$ the positive solution to the equation

$$(1-x)(1-x/p)^{-p} = \frac{1}{\delta}.$$

Then $0 < x \le 1$ and we put

$$p^* = p^*(p, \delta) = \frac{p-x}{x(p-1)}$$

we have the following.

Theorem 1.27. *[V] Suppose that a weight ω belongs to A_p and let $A = A_p(\omega)$. Then ω belongs to G_q for each $1 \le q < p^*(p, A)$. The bound for q is optimal.*

We report a result contained in [S1] that gives a simple proof of previous theorem in a special case.

Theorem 1.28. *[S1] Suppose that a non-decreasing weight $\omega : [a, b] \to [0, \infty)$ belongs to A_2 and $A = A_2(\omega)$. Then for $1 \le q < \sqrt{\frac{A}{A-1}}$, ω^{-1} belongs to G_q and for any $[c, d] \subset [a, b]$*

$$\left(\fint_c^d \omega^{-q}\, dx \right)^{1/q} \le \frac{q}{A - q^2(A-1)} \fint_c^d \omega^{-1}\, dx \tag{1.27}$$

The result is sharp.

Before proving Theorem 1.28 we state an useful Lemma ([Ko],[S2]).

Lemma 1.29. *Let ω be a non-decreasing function in $[a, b]$ and $0 < \alpha < 1$. Then*

$$\left(\int_a^b \omega^{-1/\alpha}\, dx \right)^{\alpha} \le \alpha \int_a^b (x-a)^{\alpha-1} \omega^{-1}\, dx.$$

Proof. *(of Theorem 1.28)* To prove the Theorem we use the same method first adopted in [S2] for reverse Hölder inequalities of G_2 type.
Let us define for $0 < \alpha < 1$

$$\gamma(\alpha) = 1 - A(1 - \alpha^2)$$

and note that $\gamma\left(\sqrt{\frac{A-1}{A}}\right) = 0$, $\gamma(\alpha) > 0$ for $\alpha > \sqrt{\frac{A-1}{A}}$.

Let us prove that, for any $c < d$

$$\int_c^d (x-c)^{\alpha-1}\omega^{-1} \le \frac{(d-c)^{\alpha-1}}{\gamma(\alpha)} \int_c^d \omega^{-1} \tag{1.28}$$

for $\alpha > \sqrt{\frac{A-1}{A}}$.

By Fubini's theorem and our assumption on ω, we have

$$\frac{1}{\alpha-1}\left[(d-c)^{\alpha-1}\int_c^d \omega^{-1} - \int_c^d (x-c)^{\alpha-1}\omega^{-1}\right] =$$

$$= \int_c^d (x-c)^{\alpha-1} \fint_c^x \omega^{-1} \ \le \ A\int_c^d (x-c)^{\alpha-1}\left(\fint_c^x \omega\right)^{-1} \tag{1.29}$$

We invoke now the weighted Hardy's inequality

$$\int_c^d (x-c)^{\alpha-1}\left(\fint_c^x \omega\right)^{-1} \le (1+\alpha)\int_c^d (x-c)^{\alpha-1}\omega^{-1}(x)$$

which enables us to deduce by (1.29) that

$$\frac{1}{\alpha-1}(d-c)^{\alpha-1}\int_c^d \omega^{-1} \le \left[\frac{1}{\alpha-1} + A(1+\alpha)\right]\int_c^d (x-c)^{\alpha-1}\omega^{-1}(x).$$

Hence

$$[1 - A(1-\alpha^2)]\int_c^d (x-c)^{\alpha-1}\omega^{-1}(x) \le (d-c)^{\alpha-1}\int_c^d \omega^{-1}$$

which is (1.28).

Next, we combine (1.28) with Lemma 1.29 obtaining

$$\frac{1}{\alpha}\left(\int_c^d \omega^{-1/\alpha}\,dx\right)^{\alpha} \le \frac{(d-c)^{(\alpha-1)}}{\gamma(\alpha)}\int_c^d \omega^{-1}dx$$

hence

$$\left(\fint_c^d \omega^{-1/\alpha}\,dx\right)^{\alpha} \le \frac{\alpha}{1-A+A\alpha^2}\fint_c^d \omega^{-1}dx$$

and (1.27) follows, for $\sqrt{\frac{A-1}{A}} < \alpha < 1$. □

Another point of view concerns the improvement of power exponents pertaining to A_2 weights. The following Theorem has a lot of applications in BMO-spaces (see Chapter 4). Namely, assume that the weights ω belongs to A_2 and set $A = A_2(\omega)$. Then, it is easy to check that

$$A_2(\omega^\theta) \leq A^\theta \qquad \text{for } \ 0 \leq \theta \leq 1.$$

Passing to exponents $\tau > 1$ is possible, as a consequence of Muckenhoupt's work, as we have already seen in this section. In fact the following Theorem describes the so called optimal "self-improvement of exponents" property of the A_2 class.

Theorem 1.30. *[AS] Assume $A_2(\omega) = A < \infty$, then for $1 \leq \tau < \sqrt{\frac{A}{A-1}}$ we have $\omega^\tau \in A_2$ and*

$$A_2(\omega^\tau)^{\frac{1}{2\tau}} \leq \frac{\tau A}{A - \tau^2(A-1)}. \tag{1.30}$$

The upper bound on τ cannot be improved.

Proof. Let us recall that the exact continuation of Muckenhoupt condition A_2 in one dimension ([Ko], [S1], [V]) reads as follows: for $1 + \sqrt{\frac{A-1}{A}} < s \leq 2$

$$A_s(\omega) \leq \frac{A}{\psi(s)} \tag{1.31}$$

with

$$\psi(s) = \frac{1}{s-1}[1 - As(2-s)]. \tag{1.32}$$

In particular, we deduce for any interval $I \subset \mathbb{R}$

$$⨍_I \omega^{-\frac{1}{s-1}} \leq \left[\frac{1}{⨍_I \omega} \cdot \frac{A}{\psi(s)} \right]^{1/(s-1)} \tag{1.33}$$

and also, taking into account that $A = A_2(\omega) = A_2(\omega^{-1})$ we deduce that

$$⨍_I \omega^{\frac{1}{s-1}} \leq \left[\frac{1}{⨍_I \omega^{-1}} \cdot \frac{A}{\psi(s)} \right]^{1/(s-1)}. \tag{1.34}$$

Multiplying (1.33) and (1.34) and using the Hölder inequality in the form

$$1 \leq \fint_I \omega \fint_I \omega^{-1},$$

we obtain

$$\fint_I \omega^{\frac{1}{s-1}} \fint_I \omega^{-\frac{1}{s-1}} \leq \left[\frac{A}{\psi(s)}\right]^{2/(s-1)}.$$

Hence, for $1 + \sqrt{\frac{A-1}{A}} < s \leq 2$ we have

$$A_2(\omega^{\frac{1}{s-1}}) \leq \left[\frac{A}{\psi(s)}\right]^{2/(s-1)}.$$

If we set $\tau = \frac{1}{s-1}$ we obtain immediately, for the range $1 < \tau < \sqrt{\frac{A}{A-1}}$,

$$[A_2(\omega^\tau)]^{1/2\tau} \leq \frac{A}{\varphi(\tau)}$$

where $\varphi(\tau) = \tau\left[1 - A(1 - \frac{1}{\tau^2})\right]$ which coincides with (1.30).
The optimality is seen by mean of power functions. Namely, choose $\omega(x) = |x|^r$ with $0 < r < 1$, then we have

$$A_2(|x|^r) = \frac{1}{1 - r^2}$$

and $A_2(|x|^{r\tau}) = \dfrac{1}{1 - \tau^2 r^2} < \infty$ if and only if $1 < \tau < \sqrt{\frac{A}{A-1}} = \dfrac{1}{r}$. □

1.4 Application: The Neumann Problem

In this section we report an application of Theorem 1.15 to the solvability of Neumann Problem for divergence form elliptic operators and L^p data in the half plane ([K]).

Let us start with the definition of Sobolev space.

Definition 1.8. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $1 \leq p \leq \infty$, the Sobolev space $W^{1,p}(\Omega)$ is the set of functions $u \in L^p(\Omega) : \exists g_1, g_2, ..., g_N \in L^p(\Omega)$ such that

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} = -\int_\Omega g_i \varphi, \qquad \forall \varphi \in C_c^\infty(\Omega), \qquad \forall i = 1, ..., N.$$

Let $A(x) = (a_{ij}(x))_{i,j=1}^n$ be a real, symmetric, $n \times n$ matrix, with $A(x) \in L^\infty(\mathbb{R}^n)$, and A uniformly elliptic,

$$\lambda|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq \lambda^{-1}|\xi|^2$$

for all $\xi \in \mathbb{R}^n \backslash \{0\}$, where $\lambda > 0$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let

$$W^{1,2}(\Omega) = \left\{ u \in L^2(\Omega) : \int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 < \infty \right\}$$

and

$$W^{1,2}_{loc}(\Omega) = \left\{ u \in L^2_{loc}(\Omega) : \varphi u \in W^{1,2}(\Omega) \quad \forall \varphi \in C^\infty_o(\Omega) \right\}.$$

Let B be a bounded Lipschitz domain and given $\mu \in W^{2,-1/2}(\partial B) = (W^{2,1/2}(\partial B))^*$, with $\langle 1, \mu\rangle$=0, we say that $u \in W^{2,1}(B)$ is the variational solution to the Neumann problem

$$(N) \begin{cases} Lu = 0, \text{ in } B \\ A\nabla u \cdot \vec{N}|_{\partial B} = \mu, \end{cases} \tag{1.35}$$

if, given any $\varphi \in W^2_1(B)$, $\int_B \varphi = 0$, we have

$$\int_B A\nabla u \cdot \nabla\varphi = \langle Tr(\varphi), \mu\rangle.$$

Note that the Lax-Milgran lemma shows that there exists a unique (modulo constants) solution of the problem (1.35).

Now we report the definition of non-tangential maximal function given by Kenig in [K].

Definition 1.9. [K] If $Q \in \partial B$, $\Gamma(Q) \subseteq B$ is a truncated cone with vertex at Q, for $u \in L^2_{loc}(B)$, the **non-tangential maximal function** $\widetilde{N}$ is

$$\widetilde{N}(u) = \sup_{X\in\Gamma} \left(\fint_{B\left(X, \frac{\delta(X)}{2}\right)} |u(z)|^2 \, dz \right)^{1/2}.$$

Definition 1.10. [K] We say that the Neumann problem 1.35 for L with data in $L^p(\partial B, d\sigma)$ is solvable (abbreviated N_p holds) if, whenever $f \in L^2(\partial B, d\sigma) \cap L^p(\partial B, d\sigma)$, and $\int_{\partial B} f d\sigma = 0$, the solution to 1.35 with $\mu = f$, verifies

$$\|\widetilde{N}(\nabla u)\|_{L^p(\partial B, d\sigma)} \leq C\|f\|_{L^p(\partial B, d\sigma)}$$

where $\widetilde{N}$ is the non-tangential maximal function of Definition 1.9.

Recall that an orientation preserving homeomorphism $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\Phi(x,t) = (\alpha(x,t), \beta(x,t))$ is called K-quasiconformal if $\Phi \in W^{1,2}_{loc}(\mathbb{R}^2)$, and

$$\left(\frac{\partial\alpha}{\partial x}\right)^2 + \left(\frac{\partial\alpha}{\partial t}\right)^2 + \left(\frac{\partial\beta}{\partial x}\right)^2 + \left(\frac{\partial\beta}{\partial t}\right)^2 \leq K\left[\frac{\partial\alpha}{\partial x}\frac{\partial\beta}{\partial t} - \frac{\partial\alpha}{\partial t}\frac{\partial\beta}{\partial x}\right],$$

i.e. Φ maps infinitesimal discs onto infinitesimal ellipses, with uniformly bounded eccentricity.

The connection with the subject is that quasiconformal mappings in the plane preserve the class of solutions to divergence form elliptic operators with bounded measurable coefficients. In fact if Φ in addition is a homeomorphism from $\mathbb{R}^2_+ \longrightarrow \mathbb{R}^2_+$, and $\triangle v = 0$ in $\mathbb{R}^2_+$, then $u = v \circ \Phi$ verifies $Lu = 0$ in $\mathbb{R}^2_+$, where $L = \text{div } A\nabla$, and $A(x,t) = (D\Phi^t)^{-1}|D\Phi|(D\Phi)^{-1}$, where $D\Phi$ is the matrix of partial derivatives of Φ, and $|D\Phi|$ its determinant. We quote an important result due to Beurling-Ahlfors [BA] that is very useful in the following,

Theorem 1.31. *[BA]*

(i) *Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be a homeomorphism. Then, there exists $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ K-quasiconformal such that $h = \Phi|_{\mathbb{R}}$ if and only if dh is a doubling measure, i.e.*

$$\int_{2I} dh \leq C \int_I dh$$

for all intervals $I \subset \mathbb{R}$, where $2I$ is the interval concentric with I, of double length.

(ii) *There exist $h : \mathbb{R} \longrightarrow \mathbb{R}$ a homeomorphism with dh doubling, such that dh is purely singular with respect to Lebesgue measure (i.e. the support of dh has Lebesgue measure 0), and hence the two measures are mutually singular.*

Now we show how Theorem 1.15 can be applied to the solvability of Neumann problems for divergence elliptic operators with L^p data. Let us consider the following Neumann problem

$$\begin{cases} \Delta v = 0, \text{ in } \mathbb{R}^2_+ \\ \frac{\partial v}{\partial t}|_{\mathbb{R}} = f. \end{cases} \tag{1.36}$$

Let us consider a quasiconformal mapping $\Phi : \mathbb{R}^2_+ \longrightarrow \mathbb{R}^2_+$ with $\Phi(x,0) = h(x)$ where $h : \mathbb{R} \longrightarrow \mathbb{R}$ is a homeomorphism and let us consider the pull-black Laplacian matrix: $A(x,t) = (D\Phi^t)^{-1}|D\Phi|(D\Phi)^{-1}$. If we compose a solution of (1.36) with Φ we have that $u = v \circ \Phi$ is the solution of the following Neumann problem

$$(N_p) \begin{cases} \operatorname{div} A\nabla u = 0, \text{ in } \mathbb{R}^2_+ \\ A\nabla u \cdot \vec{N}|_{\mathbb{R}} = (f \circ h)h'. \end{cases} \tag{1.37}$$

In fact using the variational formulation of the Neumann problem and assuming that $dh = h'dx$, we can see that the Neumann data for $u = v \circ \Phi$ are $(f \circ h)h'$. For this to belong to L^p, we need that

$$\int |f \circ h|^p |h'|^p < \infty$$

which is equivalent to, if $y = h(x)$, $dy = h'(x)dx$, the fact that $f \in L^p(wdx)$, where $w(y) = |h'(h^{-1}(y))|^{p-1}$.

Now if (N_p) was solvable for L, then all derivatives of u, restricted to $\mathbb{R}$ would be in L^p. This implies, in particular, that, if in (N_p) $f \in L^p(wdx)$, $\frac{\partial v}{\partial y} = H(f) \in L^p(wdx)$, where $H(f)$ is the classical Hilbert transform

$$Hf(y) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|x-y|>\varepsilon} \frac{f(x)}{x-y} dy,$$

the bound

$$\|Hf\|_{L^p(w)} \leq C \|f\|_{L^p(w)} \tag{1.38}$$

would hold. But it is well known that (1.38) holds if and only if $w \in A_p$ (see [HMW]). But $w \in A_p$ is equivalent to $(h^{-1})' \in A_q$, with $q = \frac{p}{p-1}$. Now by Theorem 1.15 we know that this equivalent to $h' \in G_p$. So we have that if (N_p) holds, then $h' \in G_p$.

In general it is possible construct h such that dh is doubling, h is absolutely continuous but $h' \notin G_p$ and so (N_p) does not always hold.

Remark 1.1. *Note that by geometric properties of quasiconformal mappings, the condition $h' \in G_p$ is also sufficient to solve Neumann problems for such divergence elliptic operators in the half plane.*

Remark 1.2. *Another relevant fact is that the condition* $h' \in G_p$ *is necessary and sufficient for the solvability of Dirichlet problem in* $L^{p'}$ *for such an operator, where* p' *is the conjugate exponent of* p*. It can, in fact, be shown that the Neumann problem in* L^p *is solvable if and only if the Dirichlet problem is solvable in* $L^{p'}$, $\frac{1}{p} + \frac{1}{p'} = 1$ *for operators in* $\mathbb{R}^2_+$ *which arise as the pullback of the Laplacian under quasiconformal change of variable.*

Chapter 2

Weighted Integral Inequalities in Orlicz Spaces

In this chapter we extend some results contained in the Chapter 1 to Orlicz Spaces. In particularly we prove in the context of Orlicz classes a result of Johnson and Neugebauer contained in [JN1] (see Theorem 1.15).

2.1 Preliminaries

Let us fix notations and recall some definitions.

A *Young function* is a convex function $\Phi : [0,\infty) \longrightarrow [0,\infty)$ such that Φ is increasing on $[0,\infty)$, satisfying

$$\lim_{t\to 0}\frac{\Phi(t)}{t} = 0, \qquad \lim_{t\to\infty}\frac{\Phi(t)}{t} = \infty.$$

Φ has a derivative φ which is nondecreasing and nonnegative, $\varphi(0+) = 0$ and $\varphi(\infty) = \infty$, so that

$$\Phi(t) = \int_0^t \varphi(x)dx$$

and we can take φ to be right-continuous. The Young function complementary to Φ is given by

$$\Psi(t) = \sup_s\{st - \Phi(s)\} = \int_0^t \psi(x)dx$$

where $\psi(x) = \inf\{s : \varphi(s) \geq x\}$. These functions verify the Young's inequality

$$ab \leq \Phi(a) + \Psi(b) \qquad \forall\ a, b > 0.$$

More in general, if a Young function Φ is not necessarily convex we have an Orlicz function. The Orlicz space, $L^\Phi(\Omega)$ consists of all measurable functions f on Ω such that

$$\int_\Omega \Phi\left(\frac{|f|}{\lambda}\right) dx < \infty \quad \text{for some } \lambda > 0.$$

$L^\Phi(\Omega)$ is a complete linear metric space with respect the following distance function:

$$dist_{\Phi(f,g)} = \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(\frac{|f-g|}{\lambda}\right) dx \leq \lambda\right\}.$$

If Φ is a Young function, $L^\Phi(\Omega)$ can be equipped with the Luxemburg norm

$$\|f\|_{L^\Phi} = \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(\frac{|f|}{\lambda}\right) dx \leq 1\right\},$$

and becomes a Banach space. If we put $\Phi(t) = t^p$, $0 < p < \infty$ then the space $L^\Phi(\Omega)$ coincides with the usual Lebesgue space $L^p(\Omega)$. Note that $L^p(\Omega)$ is a Banach space only when $p \geq 1$.

In the same way we can define the weighted Orlicz classes. The weighted Orlicz class is the set of all functions f for which

$$\int_\Omega \Phi\left(\frac{|f|}{\lambda}\right) w dx < \infty \quad \text{for some } \lambda > 0,$$

where w is a non negative measurable function on the space $\mathbb{R}^n$. As before $L^\Phi_w(\Omega)$ denotes the weighted Orlicz space.

In the following we are going to report some common important properties and results about Young functions.

Definition 2.1. Let Φ be a Young function. Φ satisfies the Δ_2-condition ($\Phi \in \Delta_2$) if there is $c > 0$ such that

$$\Phi(2t) \leq c\Phi(t), \qquad \forall t \geq 0. \tag{2.1}$$

Note that if $\forall \Phi, \Psi \in L^\Phi(\Omega)$ complementary functions we have that both verify Δ_2- condition, $L^\Phi(\Omega)$ is a reflexive Orlicz space.

Definition 2.2. Let $\Phi \in \Delta_2$ be a Young function and let us define

$$h_\Phi(\lambda) = \sup_{t>0} \frac{\Phi(\lambda t)}{\Phi(t)}, \quad \lambda > 0 \tag{2.2}$$

the numbers

$$\underline{\alpha}(\Phi) = \lim_{\lambda\to 0^+} \frac{\log h_\Phi(\lambda)}{\log \lambda} = \sup_{0<\lambda<1} \frac{\log h_\Phi(\lambda)}{\log \lambda} \tag{2.3}$$

and

$$\overline{\alpha}(\Phi) = \lim_{\lambda\to \infty} \frac{\log h_\Phi(\lambda)}{\log \lambda} = \inf_{1<\lambda<\infty} \frac{\log h_\Phi(\lambda)}{\log \lambda} \tag{2.4}$$

are called the lower index of Φ and the upper index of Φ, respectively. Sometimes these indices are called the fundamental indices of Φ.

The numbers $\underline{\alpha}(\Phi)$ and $\overline{\alpha}(\Phi)$ are reciprocals of the Boyd indices (see [B]). In the same way we can define the fundamental indices of the complementary function Ψ, $\underline{\alpha}(\Psi)$ and $\bar{\alpha}(\Psi)$. We have the following properties:

$$1 \leq \underline{\alpha}(\Phi) \leq \overline{\alpha}(\Phi), \text{ and } \underline{\alpha}(\Phi) > 1 \Longleftrightarrow \Psi \in \Delta_2. \tag{2.5}$$

$$1 \leq \underline{\alpha}(\Psi) \leq \overline{\alpha}(\Psi), \text{ and } \underline{\alpha}(\Psi) > 1 \Longleftrightarrow \Phi \in \Delta_2. \tag{2.6}$$

and, moreover, the couples $\underline{\alpha}(\Psi)$, $\overline{\alpha}(\Phi)$, and $\overline{\alpha}(\Psi)$, $\underline{\alpha}(\Phi)$ behave similarly as conjugate exponents of power functions, namely we have

$$\underline{\alpha}(\Psi) = \frac{\overline{\alpha}(\Phi)}{\overline{\alpha}(\Phi) - 1} \tag{2.7}$$

and

$$\overline{\alpha}(\Psi) = \frac{\underline{\alpha}(\Phi)}{\underline{\alpha}(\Phi) - 1}. \tag{2.8}$$

The following Theorem gives a simple formula to compute fundamental indices.

Theorem 2.1. *[F2] If there exist*

$$r_0 = \lim_{t\to 0} \frac{t\Phi'(t)}{\Phi(t)} \quad \textit{and} \quad r_\infty = \lim_{t\to\infty} \frac{t\Phi'(t)}{\Phi(t)},$$

then

$$\underline{\alpha}(\Phi) = \min\{r_0, r_\infty\} \quad \textit{and} \quad \overline{\alpha}(\Phi) = \max\{r_0, r_\infty\}.$$

By previous Theorem we give some examples of Young functions and fundamental indices.

Example 2.1. The Young function $\Phi(t) = |t|^p \log^\alpha(a + |t|)$, with $1 < p < \infty$ and $\alpha \geq 0$ has the following fundamental indices:

$$\begin{cases} \underline{\alpha}(\Phi) = \overline{\alpha}(\Phi) = p, \quad \text{if } a > 1, \\ \underline{\alpha}(\Phi) = p, \ \overline{\alpha}(\Phi) = p + \alpha, \ \text{if } a = 1. \end{cases}$$

Example 2.2. The Young function $\Phi(t) = t^2 - \frac{\log(1+t^2)}{2}$, $t \geq 0$ has lower and upper index equal to 2.

The following Proposition is very useful in practice because give a list of results about fundamental indices.

Proposition 2.2. *[FK] Let Φ and Ψ be complementary Young functions verifying Δ_2-condition. Let $e_r = |t|^r$, $r > 0$. Then*

1. $\underline{\alpha}(\Phi^{-1}) = \dfrac{1}{\overline{\alpha}(\Phi)}$; $\qquad \overline{\alpha}(\Phi^{-1}) = \dfrac{1}{\underline{\alpha}(\Phi)}$
2. $\underline{\alpha}(\Phi \circ \Psi) \geq \underline{\alpha}(\Phi)\underline{\alpha}(\Psi)$; $\qquad \overline{\alpha}(\Phi \circ \Psi) \leq \overline{\alpha}(\Phi)\overline{\alpha}(\Psi)$
3. $\underline{\alpha}(e_r \circ \Phi) = \underline{\alpha}(\Phi)r$; $\qquad \overline{\alpha}(e_r \circ \Phi) = \overline{\alpha}(\Phi)r$
4. $\underline{\alpha}(\Phi \circ e_r) = \underline{\alpha}(\Phi)r$; $\qquad \overline{\alpha}(\Phi \circ e_r) = \overline{\alpha}(\Phi)r$
5. $\underline{\alpha}(\Phi\Psi) \geq \underline{\alpha}(\Phi) + \underline{\alpha}(\Psi)$; $\qquad \overline{\alpha}(\Phi\Psi) \leq \overline{\alpha}(\Phi) + \overline{\alpha}(\Psi)$
6. $\underline{\alpha}(\Phi e_r) = \underline{\alpha}(\Phi) + r$; $\qquad \overline{\alpha}(\Phi e_r) = \overline{\alpha}(\Phi) + r$

2.2 Characterizations of indices by growth exponents and integral means

At first we report a result about the equivalence between a growth condition and Δ_2-condition.

Theorem 2.3. *[KR] Let Φ be a Young function, then*

$$\Phi \in \Delta_2 \Longleftrightarrow p\Phi(t) \leq t\Phi'(t) \leq q\Phi(t) \quad \forall t > 0 \tag{2.9}$$

with $1 < p \leq q$.

Moreover we have this connections with fundamental indices:

Lemma 2.4. *[FK] Let Φ be a Young function satisfying the growth condition $p\Phi(t) \leq t\Phi'(t) \leq q\Phi(t), \ \forall t > 0$, with $1 < p \leq q$, then we have*

$$p \leq \underline{\alpha}(\Phi) \leq \overline{\alpha}(\Phi) \leq q. \tag{2.10}$$

where $\underline{\alpha}(\Phi)$ and $\overline{\alpha}(\Phi)$ are fundamental indices of Φ (see (2.3) and (2.4)).

Theorem 2.5. *[KR] Let Φ and Ψ be complementary Young functions and suppose that their derivatives are continuous, then*

$$p\Psi(t) \leq t\Psi'(t) \leq q\Psi(t) \Longleftrightarrow \frac{q}{q-1}\Phi(t) \leq t\Phi'(t) \leq \frac{p}{p-1}\Phi(t), \quad \forall\, t > 0 \tag{2.11}$$

with $1 < p \leq q$.

Now we give an example of growth exponents of a Young function.

Example 2.3. The Young function

$$\Phi(t) = \begin{cases} t^2 & t \in [0,1] \\ e^{2(t-1)} & t \in [1,2] \\ \frac{e^2}{16}t^4 & t \in [2,+\infty[\end{cases}$$

verifies the following growth condition

$$2\Phi(t) \leq t\Phi'(t) \leq 4\Phi(t), \ \forall t > 0.$$

In many applications (calculus of variations, interpolation etc) it is useful to assume that a Young function is, in a certain sense between two powers t^p and t^q, namely we have

Lemma 2.6. *[CF] If Φ is a Young function verifying a growth condition*

$$p\Phi(t) \leq t\Phi'(t) \leq q\Phi(t), \quad \forall\, t > 0$$

with $1 < p \leq q$, then $\exists C > 0$ such that

$$\Phi(\lambda t) \leq C \max\{\lambda^p, \lambda^q\}\Phi(t), \quad \forall\, \lambda, t > 0. \tag{2.12}$$

This inequality, in turn gives also

$$\Phi(\lambda t) \geq C^{-1} \min\{\lambda^p, \lambda^q\}\Phi(t), \quad \forall\, \lambda, t > 0. \tag{2.13}$$

Now we report a theorem due to Fiorenza [F] where is proved that the Jensen mean $\Psi^{-1}\left(\fint_Q \Psi(w)dx\right)$ lies between the L_p- norm and the L_q-norm.

Theorem 2.7. *[F] Let $w \in L^1_{loc}(\mathbb{R}^n)$ be a nonnegative weight and let Ψ a Young function verifying the condition*

$$p\Psi(t) \leq t\Psi'(t) \leq q\Psi(t), \quad \forall\, t > 0$$

then

$$\frac{1}{C}\left(\fint_Q w^p\right)^{\frac{1}{p}} \leq \Psi^{-1}\left(\fint_Q \Psi(w)dx\right) \leq C\left(\fint_Q w^q\right)^{\frac{1}{q}} \tag{2.14}$$

where $C = (\frac{q}{p})^{\frac{1}{p}}$.

Furthermore we have the following proposition that slightly improve the previous theorem.

Proposition 2.8. *[FK] The inequality* (2.14) *holds with every $0 < p < \underline{\alpha}(\Psi)$ and every $\overline{\alpha}(\Psi) < q < \infty$.*

2.3 A_Φ and G_Ψ classes and constants

In this section we report the extension of definitions of A_p and G_q condition to more general Young functions and also some known results about them.

In [KT], Kerman and Torchinsky extended the definition of A_p-condition to Orlicz spaces, we have

Definition 2.3. Let $w \in L^1_{loc}(\mathbb{R}^n)$ be a nonnegative weight and let Φ, Ψ be complementary Young functions verifying Δ_2-condition, we say that w satisfies A_Φ**-condition** if $\exists\, A \geq 1$ such that

$$\forall \varepsilon > 0, \quad \left(\fint_Q \varepsilon w dx\right)\varphi\left(\fint_Q \varphi^{-1}\left(\frac{1}{\varepsilon w}\right)dx\right) \leq A \tag{2.15}$$

where $\varphi(t) = \Phi'(t)$. Moreover, we define A_Φ**-constant** as

$$A_\Phi(w) = \sup_{\varepsilon>0}\sup_{Q}\left[\fint_Q \varepsilon w dx\, \varphi\left(\fint_Q \varphi^{-1}\left(\frac{1}{\varepsilon w}\right)dx\right)\right] \tag{2.16}$$

for any cube $Q \subset \mathbb{R}^n$.

Furthermore we can extend the definition of G_q-condition to Orlicz spaces, we have

Definition 2.4. Let $v \in L^1_{loc}(\mathbb{R}^n)$ be a nonnegative weight and let Φ, Ψ be complementary Young functions verifying Δ_2-condition, we say that v satisfies G_Ψ**-condition** if $\exists B \geq 1$ such that

$$\forall \varepsilon > 0, \quad \frac{\Psi^{-1}\left(\fint_Q \Psi\left(\frac{v}{\varepsilon}\right) dx\right)}{\fint_Q \frac{v}{\varepsilon} dx} \leq B. \tag{2.17}$$

Moreover, we define G_Ψ**-constant** as

$$G_\Psi(v) = \sup_{\varepsilon>0} \sup_Q \Phi\left[\frac{\Psi^{-1}\left(\fint_Q \Psi\left(\frac{v}{\varepsilon}\right) dx\right)}{\fint_Q \frac{v}{\varepsilon} dx}\right] \tag{2.18}$$

for any cube $Q \subset \mathbb{R}^n$.

Note that if we put $\Phi(t) = \dfrac{t^p}{p}$ and $\Psi(t) = \dfrac{t^q}{q}$, with $\frac{1}{p} + \frac{1}{q} = 1$, we get A_p and G_q conditions.

Now we are going to report properties about connections between A_Φ and A_p classes and between G_Ψ and G_q classes.

In [KT], Kerman and Torchinsky proved the following

Theorem 2.9. *[KT] Let $w \in L^1_{loc}(\mathbb{R}^n)$ be a nonnegative weight and let Φ, Ψ be complementary Young functions verifying Δ_2-condition. The following conditions are equivalent:*

i) $w(x) \in A_\Phi$

ii) $w(x) \in A_p - $ *class, where* $p = \overline{\alpha}(\Phi)$.

In [M], Migliaccio proved the following

Theorem 2.10. *[M] Let $w \in L^1_{loc}(\mathbb{R}^n)$ be a nonnegative weight and let Φ, Ψ be complementary Young functions verifying Δ_2-condition. We have*

$$w \in G_\Psi \Longrightarrow \textit{for } q \leq \underline{\alpha}(\Psi),\ w \in G_q. \tag{2.19}$$

2.4 The RH_Ψ condition

In the paper [HSV] the authors introduced another definition that extend G_q-condition in the context Orlicz spaces. Furthermore, they proved a result like Theorem 2.10 but they proved a necessary and sufficient condition between G_q and RH_Ψ.

Definition 2.5. Let $v \in L^1_{loc}(\mathbb{R}^n)$ be a nonnegative weight and let Φ, Ψ be complementary Young functions verifying Δ_2-condition, we say that v satisfies RH_Ψ**-condition** if $\exists C > 0$ such that

$$\int_Q \Psi\left(\frac{\Psi^{-1}(\frac{\varepsilon}{|Q|})}{C\fint_Q v ds} v(x)\right)\frac{dx}{\varepsilon} \leq 1 \tag{2.20}$$

for any cube $Q \subset \mathbb{R}^n$ and $\varepsilon > 0$.

Note that if $\Psi(t) = t^q$, $q > 1$ we have that RH_Ψ-condition coincides with G_q-condition. We also remark that the parameter ε is necessary to make the class RH_Ψ invariant under dilatations, in the sense that if $v(x) \in RH_\Psi$ then $v(\lambda x) \in RH_\Psi$ with the constant C independent of $\lambda > 0$.

Theorem 2.11. *[HSV] Let $v \in L^1_{loc}(\mathbb{R}^n)$ be a nonnegative weight and let Φ, Ψ be complementary Young functions verifying Δ_2-condition. The following conditions are equivalent:*

i) $v \in RH_\Psi$

ii) *For any $b \geq 0$, $\Psi(bv) \in A_\infty$ with a uniform constant.*

iii) $v^q \in A_\infty$

iv) *$v \in G_q$, where $q = \overline{\alpha}(\Psi)$.*

The proof of the previous Theorem is based on the following important property of RH_Ψ classes.

Proposition 2.12. *Let Φ, Ψ be complementary Young functions verifying Δ_2-condition. If $v \in RH_\Psi$, then $\exists\, r > 1$ such that $v \in RH_{\Psi^r}$.*

2.5 The One Dimensional Case

In this section we prove an extension of Theorem 1.15 to more general Young functions that preserve the same property of monotonicity and convexity of power functions. We begin to prove some auxiliary Lemmas and then main Theorems.

Lemma 2.13. *If φ is a Young function such that*

$$\exists\ \epsilon \in]0,1[\ :\ \varphi^{\epsilon} \text{ is concave}, \tag{2.21}$$

then the function Γ_ρ defined by

$$\Gamma_\rho : t \in [0,+\infty[\to \Gamma_\rho(t) = \varphi^{-1}(t^\rho)$$

is convex for all $\rho \geq 1/\epsilon$.

Proof. Let us fix $\rho > 1/\epsilon$. It is

$$\Gamma'_\rho(t) = \frac{d}{dt}(\varphi^{-1}(t^\rho)) = (\varphi^{-1})'(t^\rho) \cdot \rho t^{\rho-1} = \frac{1}{\varphi'(\varphi^{-1}(t^\rho))} \cdot \rho t^{\rho-1} \quad \forall t > 0$$

so that, setting $t = \varphi(s)^{1/\rho}$, it is

$$\Gamma'_\rho(\varphi(s)^{1/\rho}) = \frac{1}{\varphi'(s)} \cdot \rho[\varphi(s)^{1/\rho}]^{\rho-1} = \frac{\rho}{\varphi'(s)}\varphi(s)^{1/\rho'} \qquad \forall s > 0.$$

Since φ is strictly increasing, the assertion is proven if we show that the function on the right hand side is increasing. But this follows observing that, since $0 < 1/\rho \leq \epsilon$, the function $\varphi^{1/\rho}$ is concave, and therefore its derivative

$$(\varphi^{1/\rho})' = \frac{1}{\rho}\varphi^{\frac{1}{\rho}-1}\varphi' = \frac{1}{\rho}\varphi^{-\frac{1}{\rho'}}\varphi'$$

is decreasing. □

The following example shows that there exist Young functions satisfying the Δ_2-condition along with their complementary function, such that (2.21) does not hold.

Example 2.4. It is sufficient to consider the function

$$\varphi(t) = \begin{cases} t^2 & t \in [0,1] \\ e^{2(t-1)} & t \in [1,2] \\ \frac{e^2}{16}t^4 & t \in [2,+\infty[\end{cases}$$

It is straighforward to check that $2\varphi(t) \leq t\varphi'(t) \leq 4\varphi(t)$ $\forall t > 0$. Moreover, (2.21) does not hold because

$$\varphi^{\epsilon}(t) = (2\epsilon)^2 e^{2\epsilon(t-1)} > 0 \qquad \forall t \in [1,2].$$

Lemma 2.14. *In the same hypothesis of Lemma 2.13, $\forall \rho \geq \frac{1}{\varepsilon}$, we have*

$$\left[\fint_I f^{\frac{1}{\rho}} ds \right]^{\rho} \leq \varphi\left(\fint_I \varphi^{-1}(f) ds \right), \quad \forall f : \varphi^{-1}(f) \in L^1(\mathbb{R}) \tag{2.22}$$

for all intervals $I \subset \mathbb{R}$.

Proof. If we set in (2.22) $f^{\frac{1}{\rho}} = g$, then the thesis of Lemma becomes:

$$\left[\fint_I g ds \right]^{\rho} \leq \varphi\left(\fint_I \varphi^{-1}(g^{\rho}) ds \right)$$

and equivalently

$$\fint_I g ds \leq \varphi^{\frac{1}{\rho}}\left(\fint_I \varphi^{-1}(g^{\rho}) ds \right).$$

If we set in previous inequality $\varphi^{-1}(t^{\rho}) = \lambda(t) = s$, then we have to show that

$$\fint_I g \, ds \leq \lambda^{-1}\left(\fint_I \lambda(g) ds \right)$$

and this is true iff λ is convex, by Jensen inequality. The fact that λ is convex comes from Lemma 2.13, with $\lambda = \Gamma$. □

Lemma 2.15. *Let Φ, Ψ be complementary Young functions verifying Δ_2-condition. Suppose that $\exists\ \sigma : 0 < \sigma < \sigma_o$ such that $q(\Psi) = \sup \frac{t\Psi'(t)}{\Psi(t)} < (\rho+1)' + \sigma$, then*

$$\Psi^{-1}\left(\fint_I \Psi(f) ds \right) \leq \left(\fint_I f^{(\rho+1)'+\sigma} ds \right)^{\frac{1}{(\rho+1)'+\sigma}}, \quad \forall f \in L^1_{loc}(\mathbb{R})$$

for all intervals $I \subset \mathbb{R}$.

Proof. From Theorem 2.7 we know that

$$\Psi^{-1}\left(\fint_I \Psi(f) ds \right) \leq \left(\fint_I f^{q(\Psi)} ds \right)^{\frac{1}{q(\Psi)}}$$

since $q(\Psi) < (\rho+1)' + \sigma$, the Lemma is proved. □

Now we are able to prove a first important Theorem.

Theorem 2.16. *Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing homeomorphism onto such that h, h^{-1} are locally absolutely continuous. Let Φ, Ψ be complementary Young functions verifying the Δ_2-condition, then*

$$(h^{-1})' \in G_\Psi \Longrightarrow h' \in A_\Phi$$

Proof. To prove this implication we use some known results presented before. If $(h^{-1})' \in G_\Psi$ from Theorem 2.10 we know that $(h^{-1})' \in G_q$, where $q = \underline{\alpha}(\Psi)$. By Theorem 1.15 we know that $(h^{-1})' \in G_q \Longleftrightarrow h' \in A_p$, with $p = \frac{q}{q-1} = \overline{\alpha}(\Phi)$. Moreover, Kerman and Torchinski in [KT](see Theorem 2.9) proved that $h' \in A_p \Longleftrightarrow h' \in A_\Phi$. So we have $(h^{-1})' \in G_\Psi \Longrightarrow h' \in A_\Phi$. $\square$

Let us set

$$\varphi_{p,\alpha}(s) = \frac{s^p}{\log^\alpha(e+s)},\ \alpha > 0,\ p > 1$$

$$\Phi_{p,\alpha}(t) = \int_0^t \varphi_{p,\alpha}(s)\,ds$$

$$\Psi_{p,\alpha}(t) = \text{complementary function of } \Phi_{p,\alpha}(t).$$

Now we are able to prove the main Theorem of the section.

Theorem 2.17. *Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing homeomorphism onto such that h, h^{-1} are locally absolutely continuous, then*

$$\forall M > 1\ \exists \alpha > 0 : h' \in A_{\Phi_{p,\alpha}},\ A_{\Phi_{p,\alpha}}(h') \le M \Longrightarrow (h^{-1})' \in G_{\Psi_{p,\alpha}}.$$

Proof. We know, by definition, that

(2.23)

$$A_{\Phi_{p,\alpha}}(h') \le M \Longleftrightarrow \forall \varepsilon > 0, \left(\fint_I \varepsilon h' ds\right) \varphi_{p,\alpha}\left(\fint_I \varphi_{p,\alpha}^{-1}\left(\frac{1}{\varepsilon h'}\right) ds\right) \le M, \forall I \subset \mathbb{R}.$$

By Lemma 2.14 applied to $f = \frac{1}{h'}$, $\rho = p$ (note that $\epsilon = \frac{1}{p}$ is such that $\varphi_{p,\alpha}^\epsilon$ is concave) we have

$$\left[\fint_I \left(\frac{1}{h'}\right)^{\frac{1}{p}} ds\right]^p \le \varphi_{p,\alpha}\left(\fint_I \varphi_{p,\alpha}^{-1}\left(\frac{1}{h'}\right) ds\right),\ \forall I \subset \mathbb{R}.$$

which together with (2.23) gives

$$\left(\fint_I h' ds\right) \varphi_{p,\alpha}\left(\fint_I \varphi_{p,\alpha}^{-1}\left(\frac{1}{h'}\right) ds\right) \le M$$

i.e.

$$h' \in A_{p+1}, \qquad A_{p+1}(h') \leq M.$$

By Theorem 1.15 we have

$$h' \in A_{p+1} \Longleftrightarrow (h^{-1})' \in G_{(p+1)'}$$

and

$$A_{p+1}(h') = G_{(p+1)'}((h^{-1})') \leq M.$$

Therefore, by Theorem 1.19,

$$\exists \beta = \beta(p, M) : (h^{-1})' \in G_q, \ \forall q \in [(p+1)', (p+1)' + \beta[.$$

Fix $\sigma \in]0, \beta[$ and define $\tau = \tau(p, M) > 0$ by

$$(p+1)' + \sigma = (p+1-\tau)'$$

and choose $\alpha > 0$ sufficiently small, so that

$$(p-\tau)\varphi_{p,\alpha}(t) \leq t\varphi'_{p,\alpha}(t) \quad \forall t. \tag{2.24}$$

Inequality (2.24) comes from the following

$$\inf_t \frac{t\varphi'_{p,\alpha}(t)}{\varphi_{p,\alpha}(t)} = p - \alpha \sup_t \frac{t}{(e+t)\log(e+t)} \underset{\alpha \to 0}{\longrightarrow} p.$$

Now integrating (2.24)

$$(p+1-\tau)\Phi_{p,\alpha}(t) \leq t\Phi'_{p,\alpha}(t) \ \ \forall t$$

$$t\Psi'_{p,\alpha}(t) \leq (p+1-\tau)'\Psi_{p,\alpha}(t) = [(p+1)' + \sigma]\Psi_{p,\alpha}(t) \ \ \forall t$$

Finally, using Lemma 2.15 applied to $f = \frac{(h^{-1})'}{\varepsilon}$, we get

$$\Psi^{-1}_{p,\alpha}\left(⨍_I \Psi_{p,\alpha}\left(\frac{(h^{-1})'}{\varepsilon}\right) ds \right) \leq \left(⨍_I \left(\frac{(h^{-1})'}{\varepsilon}\right)^{(p+1)'+\sigma} ds \right)^{\frac{1}{(p+1)'+\sigma}}, \ \forall \varepsilon > 0 \ \forall I \subset \mathbb{R}$$

for all $0 < \sigma < \beta$, and therefore by Theorem 1.19 we get that for some $\overline{M} > 0$ it is

$$\left\{ ⨍_I [(h^{-1})']^{(p+1)'+\sigma} ds \right\}^{\frac{1}{(p+1)'+\sigma}} \leq \overline{M} ⨍_I (h^{-1})' ds$$

dividing by ε

$$\left\{ \fint_I \left[\frac{(h^{-1})'}{\varepsilon} \right]^{(p+1)'+\sigma} ds \right\}^{\frac{1}{(p+1)'+\sigma}} \leq \overline{M} \fint_I \frac{(h^{-1})'}{\varepsilon} ds$$

and so

$$\Psi_{p,\alpha}^{-1} \left(\fint_I \Psi_{p,\alpha} \left(\frac{(h^{-1})'}{\varepsilon} \right) ds \right) \leq \overline{M} \fint_I \frac{(h^{-1})'}{\varepsilon} ds$$

i.e.

$$(h^{-1})' \in G_{\Psi_{p,\alpha}}.$$

$\square$

Corollary 2.18. $\forall M > 1 \; \exists \; \overline{\alpha} > 0 \; : \; h' \in A_{\Phi_{p,\alpha}}, \; A_{\Phi_{p,\alpha}}(h') \leq M \Longrightarrow (h^{-1})' \in G_{\Psi_{p,\alpha}}, \; \forall \alpha \in [0, \overline{\alpha}]$.

Remark 2.1. *Let Φ be such that $\underline{\alpha}(\Phi) = \overline{\alpha}(\Phi) = p$. Then for any $\tau > 0$ there exists Φ_1 equivalent to Φ such that*

$$p - \tau \leq \frac{s\Phi_1'(s)}{\Phi_1(s)}.$$

If $\Phi_1^{\frac{1}{p}}$ is concave, then the argument of Theorem 2.17 applies, and we can assert that

$$h' \in A_\Phi \Longrightarrow (h^{-1})' \in G_\Psi$$

getting the converse of Theorem 2.16.

Remark 2.2. *Choosing $\alpha = 0$ in Corollary 2.18 we can see that our result generalizes Theorem 1.15 ([JN]).*

Chapter 3

The Hardy-Littlewood Maximal Function

In this chapter we will introduce the Hardy-Littlewood maximal function and some properties related to it. We focus attention on weighted norm inequalities for Hardy-Littlewood type maximal operators. The importance of the maximal operator stems from the fact that it controls many operators arising naturally in analysis.

3.1 Definitions

We begin recalling some definitions.

The concept of the maximal function can be traced back to G.H. Hardy and J.E. Littlewood [HL] and has been under study since then.

Definition 3.1. Let $B_r = B(0, r)$ be the Euclidean ball of radius r centered at the origin. The **Hardy-Littlewood maximal function** of a locally integrable function f on $\mathbb{R}^n$ is defined by

$$Mf(x) = \sup_{r>0} \fint_{B_r} |f(x-y)|dy. \tag{3.1}$$

Note that the maximal operator M is sublinear and homogeneous, that is, $M(f+g) \leq Mf + Mg$ and $M(\lambda f) = \lambda(Mf)$, $\forall \lambda \geq 0$.

Another definition of Hardy-Littlewood maximal function is based on cubes in place of balls, namely

Definition 3.2. If Q_r is the cube $[-r, r]^n$, we define the **centered Hardy-Littlewood maximal function** as

$$M_c f(x) = \sup_{r>0} \frac{1}{(2r)^n} \int_{Q_r} |f(x-y)| dy. \tag{3.2}$$

Note that when $n = 1$, M and M_c coincide. If $n > 1$ then there exist constants c_n and C_n, depending only on n, such that

$$c_n M_c f(x) \leq M f(x) \leq C_n M_c f(x) \tag{3.3}$$

Because of inequality (3.3), the two operators M and M_c are essentially interchangeable, and we will use whichever is more appropriate, depending on the circumstances. In fact, we can also define a more general maximal function:

Definition 3.3. Let f be a locally integrable function on $\mathbb{R}^n$. The **non-centered Hardy-Littlewood maximal function** is defined by:

$$M^* f(x) = \sup_{x \in Q} \fint_Q |f(y)| dy. \tag{3.4}$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to coordinate axes and containing x.

Again, M^* is pointwise equivalent to M, so in the following we call M the Hardy-Littlewood maximal operator.

Now we give a generalization of the maximal function. Let μ be a positive Borel measure on $\mathbb{R}^n$, finite on compact sets and satisfying the following doubling condition:

$$\mu(2Q) \leq C\mu(Q) \tag{3.5}$$

for every cube Q, with $C > 0$ indipendent of Q. We say that μ is a doubling measure.

Definition 3.4. Let μ as above, $d\mu = w(x)dx$, and let $f \in L^1_{loc}(\mathbb{R}^n)$. The **weighted Hardy-Littlewood maximal function** is defined by:

$$M_w f(x) = \sup_{x \in Q} \frac{1}{w(Q)} \int_Q |f(y)| w(y) dy. \tag{3.6}$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to coordinate axes containing x.

3.2 Weak-type inequalities

We begin recalling the definition of weak-type inequality.

Definition 3.5. Let (X,μ) and (Y,ν) be measure spaces, and let T be an operator from $L^p(X,\mu)$ into the space of measurable functions from Y to $\mathbb{C}$. We say that T is **weak** (p,q), $q<\infty$, if

$$\nu(\{y\in Y : |Tf(y)|>\lambda\}) \leq \left(\frac{C\|f\|_p}{\lambda}\right)^q,$$

and we say that it is weak (p,∞) if it is a bounded operator from $L^p(X,\mu)$ to $L^\infty(Y,\nu)$.

We say that T is strong (p,q) if it is bounded from $L^p(X,\mu)$ to $L^q(Y,\nu)$. If T is strong (p,q) then it is weak (p,q), in fact if we let $E_\lambda=\{y\in Y : |Tf(y)|>\lambda\}$, then

$$\nu(E_\lambda)=\int_{E_\lambda} d\nu \leq \int_{E_\lambda}\left|\frac{Tf(x)}{\lambda}\right|^q d\nu \leq \frac{\|Tf\|_q^q}{\lambda^q} \leq \left(\frac{C\|f\|_p}{\lambda}\right)^q.$$

When $(X,\mu)=(Y,\nu)$ and T is the identity, the weak (p,p) inequality is the classical Chebyshev inequality.

Definition 3.6. Let (X,μ) be a measure space and let $f : X \longrightarrow \mathbb{C}$ be a measurable function. We call the function $a_f : (0,\infty) \longrightarrow [0,\infty]$, given by

$$a_f(x)=\mu(\{x\in X : |f(x)|>\lambda\}),$$

the **distribution function** of f associated with μ.

Proposition 3.1. *Let $\phi : [0,\infty) \longrightarrow [0,\infty)$ be differentiable, increasing and such that $\phi(0)=0$. Then*

$$\int_X \phi(|f(x)|)d\mu = \int_0^\infty \phi'(\lambda)a_f(\lambda)d\lambda.$$

To prove the previous equality it is enough to observe that the left-hand side is equivalent to

$$\int_X\int_0^{|f(x)|}\phi'(\lambda)d\lambda d\mu$$

and change the order of integration. If, in particular, $\phi(\lambda)=\lambda^p$ then

$$\|f\|_p^p = p\int_0^\infty \lambda^{p-1}a_f(\lambda)d\lambda. \tag{3.7}$$

Since weak inequalities measure the size of the distribution function, representation (3.7) of the L^p norm is ideal for proving the following interpolation theorem, which will let us deduce L^p boundness from weak inequalities. It applies to a larger class of operators than linear ones (note that maximal operators are not linear): an operator T from a vector space of measurable functions to measurable functions is sublinear if

$$|T(f_0 + f_1)| \leq |Tf_0| + |Tf_1|,$$

and

$$|T(\lambda f)| = |\lambda||Tf|, \quad \forall \lambda \in \mathbb{C}.$$

Theorem 3.2. ***(Marcinkiewicz Interpolation)*** *Let (X, μ) and (Y, ν) be measure spaces, $1 \leq p_0 < p_1 \leq \infty$, and let T be a sublinear operator from $L^{p_0}(X, \mu) + L^{p_1}(X, \mu)$ to the measurable functions on Y, that is weak (p_0, p_0) and weak (p_1, p_1). Then T is strong (p, p) for $p_0 < p < p_1$.*

Theorem 3.3. *The operator M is weak $(1, 1)$ and strong (p, p), $1 < p \leq \infty$.*

It is immediate from the definition that

$$\|Mf\|_\infty \leq \|f\|_\infty,$$

so by the Theorem 3.2, to prove Theorem 3.3, it will be enough to prove that M is weak $(1, 1)$. Here we give a proof when $n = 1$ and to do this we need the following one-dimensional covering lemma.

Lemma 3.4. *Let $\{I_\alpha\}_{\alpha \in A}$ be a collection of intervals in $\mathbb{R}$ and let K be a compact set contained in their union. Then there exists a finite subcollection $\{T_j\}$ such that*

$$K \subset \bigcup_j I_j \qquad \text{and} \qquad \sum_j \chi_{I_j}(x) \leq 2, \quad x \in \mathbb{R}.$$

Proof. (of Theorem 3.3, $n = 1$) Let $E_\lambda = \{x \in \mathbb{R} : Mf(x) > \lambda\}$. If $x \in E_\lambda$ then there exists an interval I_x centered at x such that

$$\frac{1}{|I_x|} \int_{I_x} |f| > \lambda.$$

Let $K \subset E_\lambda$ be compact. Then $K \subset \bigcup I_x$, so by lemma 3.4 there exists a finite collection $\{I_j\}$ of intervals such that $K \subset \bigcup_j I_j$ and $\sum_j \chi_{I_j} \leq 2$. Hence,

$$|K| \leq \sum_j |I_j| \leq \sum_j \frac{1}{\lambda} \int_{I_j} |f| \leq \frac{1}{\lambda} \int_{\mathbb{R}} \sum_j \chi_{I_j} |f| \leq \frac{2}{\lambda} \|f\|_1.$$

Since the previous inequality holds for every compact $K \subset E_\lambda$, the weak $(1,1)$ inequality for M follows immediately. □

Note that Lemma 3.4 is not valid in dimensions greater than 1. Theorem 3.3 can be proved in $\mathbb{R}^n$ using dyadic maximal function but we don't investigate it here.

3.3 Weighted norm inequalities for maximal operators

A very interesting question in harmonic analysis is what type of weights w have the property that an operator T is bounded in $L^p(w)$, $1 < p < \infty$ where T is bounded in $L^p(\mathbb{R}^n)$. An operator T such that is the Hardy-Littlewood maximal operator, M, in fact we have that M is bounded in $L^p(\mathbb{R}^n)$.

Theorem 3.5. *For every p, with $1 < p \leq \infty$, there is a constant C_p such that, for every $f \in L^p(\mathbb{R}^n)$, we have*

$$\|Mf\|_{L^p} \leq C_p \|f\|_{L^p}.$$

Note that M is not bounded in $L^1(\mathbb{R}^n)$, in fact for $f \geq 0$, Mf is not in L^1 unless $f(x) = 0$ for a.e. x, since $Mf(x) \geq C|x|^{-n}$ for large x, with $C > 0$ if $f \neq 0$.

We have the following very important Theorem about local integrability of maximal operator.

Theorem 3.6. *(Hardy-Littlewood Maximal Theorem) Let f be an integrable function supported in a cube $Q \subset \mathbb{R}^n$. Then $Mf \in L^1(Q)$ if and only if $f \log f \in L^1(Q)$.*

The analogous of Theorem 3.5 for weighted maximal operators is the following

Theorem 3.7. *Let μ a doubling measure in $\mathbb{R}^n$ such that $d\mu = w(x)dx$, then for every p, with $1 < p < \infty$, there is a constant $C_p > 0$ such that for every $f \in L^p(w)$, we have*

$$\left(\int_{\mathbb{R}^n} (M_w f(x))^p w(x)dx\right)^{\frac{1}{p}} \leq C_p \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)dx\right)^{\frac{1}{p}}.$$

Also the Theorem 3.6 can be extended to M_w for a doubling measure μ such that $d\mu = w(x)dx$.

One of the first important Theorem about weights for which the maximal operator is bounded in $L^p(\mathbb{R}^n)$ is due to Muckenhoupt [M1], he proved the following:

Theorem 3.8. *[M1] Let $1 < p < \infty$, then M is a bounded operator in $L^p(w)$ if and only if $w \in A_p$.*

The analogous of Theorem 3.8 in Orlicz Spaces was proved by Kerman and Torchinsky in [KT].

Theorem 3.9. *[KT] Let $w \in L^1_{loc}(\mathbb{R}^n)$ be a nonnegative weight and let Φ, Ψ be complementary Young functions verifying Δ_2-condition. The following conditions are equivalent:*

i) $w(x) \in A_\Phi$

ii) $\int_{\mathbb{R}^n} \Phi((Mf)(x))w(x)dx \leq C \int_{\mathbb{R}^n} \Phi(|f(x)|)w(x)dx$,

*where in **ii)** C is independent of f.*

A generalization of Theorem 3.8 was done by Jawerth [J] in 1986, we have the following

Theorem 3.10. *[J] Let $1 < p < \infty$, and let w be a weight and set $\sigma = w^{1-p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Then*

$$\begin{cases} M : L^p(w) \longrightarrow L^p(w) \\ M : L^{p'}(\sigma) \longrightarrow L^{p'}(\sigma) \end{cases}$$

if and only if

$$\begin{cases} w \in A_p \\ M_w : L^{p'}(w) \longrightarrow L^{p'}(w) \\ M_\sigma : L^p(\sigma) \longrightarrow L^p(\sigma). \end{cases}$$

It is fundamental the fact that M_w is bounded in $L^p(w)$ for every $1 < p < \infty$, if the weight w is doubling (see Theorem 3.7). In particular M_w is bounded if w is a A_∞ weight. The following Theorem due to Perez is a generalization of the previous results.

Theorem 3.11. *[P] The following statements are equivalent.*

1. *For every $1 < p < \infty$, and whenever $w \in A_p$*

$$M : L^p(w) \longrightarrow L^p(w)$$

2. *For every $1 < p < \infty$, and whenever $w \in A_\infty$*

$$M_w : L^p(w) \longrightarrow L^p(w).$$

We have also a weak-type of Theorem 3.8.

Theorem 3.12. *For $1 \leq p < \infty$, the weak (p,p) inequality*

$$w(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}) \leq \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p w(x) dx$$

holds if and only if $w \in A_p$.

Some years later Buckley in [Bu] proved a result which shows how the operator norms specifically depend from the A_p-constant of w.

Theorem 3.13. *[Bu] If $w \in A_p$, then*

$$\|Mf\|^p_{L^p(w)} \leq C(p) A_p(w)^{p'} \|f\|^p_{L^p(w)}$$

where p' is the conjugate exponent of p. The power $A_p(w)^{p'}$ is the best possible.

Before proving the Theorem 3.13 we need some preliminary Lemmas.

Lemma 3.14. *If $f \in L^p(w)$ and $f_{Q_k} \geq \alpha > 0$ for each of the disjoint cubes $\{Q_k\}$, then*

$$\sum_k w(Q_k) \leq A_p(w) \left(\frac{\|f\|_{L^p(w)}}{\alpha} \right)^p.$$

Proof. (of Theorem 3.13) First, we show that for $1 \leq p < \infty$,

$$w(\{Mf > \alpha\}) \leq CA_p(w) \left(\frac{\|f\|_{L^p(w)}}{\alpha} \right)^p. \tag{3.8}$$

Without loss of generality, we assume that $f(x) \geq 0$ and that $\|f\|_{L^p(w)} = 1$. Suppose that $Mf(x) > \alpha > 0$ so that $f_{Q_k} \geq \alpha$ for some cube Q_k centered at x. Let $E_r = \{x : |x| < r, Mf(x) > \alpha\}$. The Besicovich covering Lemma [Be] tells us that E_r can be covered by the union of N_n collections of disjoint cubes, on each of which the mean value of f is at least α. Choose the collection $\{Q_k\}$, whose union has maximal w-measure. Thus,

$$w(E_r) \leq N_n w\left(\bigcup_k Q_k \right) \leq \frac{CA_p(w)}{\alpha^p},$$

by Lemma 3.14. Letting $r \longrightarrow \infty$, we get (3.8).

Suppose now that $p > 1$, if $w \in A_p$ then $w \in A_{p-\varepsilon}$ by Theorem 1.10, with comparable norm, where $\varepsilon \sim A_p(w)^{1-p'}$, see Lemma 1.11, and trivially $w \in A_{p+\varepsilon}$, with norm no larger than $A_p(w)$. Applying the Marcinkiewicz Interpolation Theorem to the corresponding weak-type results at $p - \varepsilon$ and $p + \varepsilon$, we get the strong type result we require with the indicated bound for the operator norm.

To see that the power $A_p(w)^{p'}$ is best possible, we give an example for $\mathbb{R}$ (a similar example works in $\mathbb{R}^n$ for any n). Let $w(x) = |x|^{(p-1)(1-\delta)}$, so that $A_p(w) \sim \frac{1}{\delta^{p-1}}$. Now, $f(x) = |x|^{-1+\delta}\chi_{[0,1]} \in L^p(w)$. It is easy to see that $Mf \geq \frac{f}{\delta}$ and so

$$\frac{\|Mf\|^p_{L^p(w)}}{\|f\|^p_{L^p(w)}} \geq C\delta^{-p} \sim A_p w^{p'}.$$

□

Buckley result can be rewritten also as follows:

$$\|Mf\|_{L^p(w)} \leq C'(p) A_p(w)^{\frac{p'}{p}} \|f\|_{L^p(w)}.$$

Note that $A_p(w)^{\frac{p'}{p}}$ cannot be replaced by $\varphi(A_p(w)^{\frac{p'}{p}})$ for any function φ : $\mathbb{R}_+ \longrightarrow \mathbb{R}_+$ that grows slower than $\frac{p'}{p}$-th power. This can be easily seen by using power functions and power weights. Taking $w \equiv 1$ we see that the constants $C(p)$ must blow up as $p \to 1$.

Buckley [Bu] also showed that the Hilbert transform is bounded on $L^p(w)$ with an operator norm which is at most a multiple of $A_p(w)^\alpha$, where $\max\{1, \frac{p'}{p}\} \leq \alpha \leq p'$. In particular, for $p = 2$ he showed that the dependence on $A_2(w)$ was at least linear, and at most quadratic.

Recently there has been renewed interest in computing the exact dependence of the operator norms from $A_p(w)$. Sharp linear dependence on $A_2(w)$ was obtained by Hukovic, Treil and Volberg [Hu], [HTV] for the dyadic square function on $L^2(w)$ and for the martingale transform. Analogous results were recently obtained for the Beurling transform by Petermichl and Volberg [PV], and latter by Dragičević and Volberg [DV]. Petermichl and Pott [PP] showed that $\alpha \leq \frac{3}{2}$ for the Hilbert transform. Petermichl [Pe] improved this estimate to $\alpha = 1$ when $p \geq 2$.

All of the previous results can be summarized in the following Theorem contained in [DGPP].

Theorem 3.15. *[DGPP] Let T be any of the Hilbert transform, the Beurling transform, the martingale transform, or the dyadic square function. Then for any $1 < p < \infty$ there exist positive constants $C(p)$ such that for all weights w in A_p we have*

$$\|T\|_{L^p(w)} \leq C(p) A_p^\alpha(w),$$

where $\alpha = \max\{1, \frac{p'}{p}\}$. The exponent α in this estimate is sharp for the Hilbert, Beurling and martingale transforms for all $1 < p < \infty$. For the dyadic square function the exponent is sharp for $1 < p \leq 2$.

In 2006 Theorem 3.15 was extended to Riesz transforms by Petermichl [Pe2].

Now we report some others characterizations of weighted integral inequalities for the Maximal Operator by mean of Gehring condition (see definition 1.2).

Lemma 3.16. *[P] Let $1 < p < \infty$. The following statements are equivalent.*

1. *There is a constant $c > 0$, indipendent of B, such that for every nonnegative locally integrable function f*

$$\frac{1}{w(Q)} \int_Q f(y) w(y) dy \leq c \left(\frac{1}{|Q|} \int_Q f(y)^p dy \right)^{\frac{1}{p}} \tag{3.9}$$

2. $w \in G_{p'}$*, with* $\frac{1}{p} + \frac{1}{p'} = 1$*.*

Proof. If we put $f = w^{\frac{p'}{p}}$ in (3.9) we get that $w \in G_{p'}$. Conversely (3.9) follows from 2. by Hölder's inequality:

$$\frac{1}{w(Q)} \int_Q f(y)w(y) \le \frac{1}{w(Q)} \left(\int_Q f(y)^p dy \right)^{\frac{1}{p}} \left(\int_Q w(y)^{p'} dy \right)^{\frac{1}{p'}} =$$

$$= \frac{|Q|}{w(Q)} \left(\fint_Q f(y)^p dy \right)^{\frac{1}{p}} \left(\fint_Q w(y)^{p'} dy \right)^{\frac{1}{p'}} \le c \left(\fint_Q f(y)^p dy \right)^{\frac{1}{p}}.$$

□

Theorem 3.17. *[P] Let* $1 < p < \infty$*. Then*

$$\begin{cases} M_w : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n) \\ M_w : L^{p'}(w^{p'}) \longrightarrow L^{p'}(w^{p'}) \end{cases}$$

if and only if

$$\begin{cases} w \in G_{p'} \\ M : L^{p'}(\mathbb{R}^n) \longrightarrow L^{p'}(\mathbb{R}^n) \\ M_{w^{p'}} : L^p(w^{p'}) \longrightarrow L^p(w^{p'}). \end{cases}$$

Proof. We prove in particular that $w \in G_{p'} \Longrightarrow M_w : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)$. Suppose that $w \in G_{p'}$, by Gehring Theorem we know that $w \in G_{(p-\epsilon)'}$, for some $\epsilon > 0$. Then by Lemma 3.16 we have

$$\frac{1}{w(Q)} \int_Q f(y)w(y)dy \le c \left(\frac{1}{|Q|} \int_Q f(y)^{p-\epsilon} dy \right)^{\frac{1}{p-\epsilon}}$$

and so

$$\int_{\mathbb{R}^n} M_w f(y)^p dy \le \int_{\mathbb{R}^n} (M(f^{p-\epsilon}(y)))^{\frac{p}{p-\epsilon}} dy \le c \int_{\mathbb{R}^n} f(y)^p dy$$

and M_w is bounded in $L^p(\mathbb{R}^n)$.

□

Chapter 4

BMO-Space

In this chapter we will examine some properties related to the space of functions of Bounded Mean Oscillation, BMO, introduced by John and Nirenberg [JN] in 1961. This space has become extremely important in various areas of analysis including harmonic analysis, PDEs and function theory. BMO-Spaces are also of interest since, in the scale of Lebesgue Spaces, they may be considered an appropriate substitute for L^{∞}. Appropriate in the sense that are spaces preserved by a wide class of important operators such as the Hardy-Littlewood maximal function, the Hilbert transform and which can be used as an end point in interpolating L^p spaces.

4.1 Definitions and notations

We begin with some notations. If $\Omega \subset \mathbb{R}^n$ is any measurable set of finite positive measure $|\Omega|$ and f is an integrable function, let us recall that $f_\Omega = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega f dx = \fint_\Omega f dx$ indicates the integral mean of f over Ω.

Definition 4.1. If $f \in L^1_{loc}(\mathbb{R}^n)$, the sharp maximal function $f^\sharp$ of f is defined by

$$f^\sharp(x) = \sup_{x\in Q} \fint_Q |f(y) - f_Q| \, dy,$$

where the supremum is taken over all cubes Q containing x .

The sharp function $f^\sharp$ measures locally, at the point x, the average oscillation of f from its mean value over cubes containing x.

The sharp maximal operator $f \longrightarrow f^\sharp$ is an analogue of the Hardy-Littlewood maximal operator M, but it has certain advantages over it. Of course, $f^\sharp(x) \leq 2Mf(x)$. Note that in the definition of $f^\sharp(x)$ one can take only those cubes Q containing x in its interior.

If f is such that $f^\sharp$ is bounded, we say that f is a function of bounded mean oscillation, and we denote by the initials BMO the space formed by these functions.

Definition 4.2. A real valued locally integrable function f on $\mathbb{R}^n$ has bounded mean oscillation, $f \in BMO(\mathbb{R}^n)$ if

$$\sup_Q ⨍_Q |f - f_Q|\, dx = \|f\|_* < \infty \tag{4.1}$$

where the supremum runs over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes. And also

$$BMO(\mathbb{R}^n) = \{f \in L^1_{loc}\mathbb{R}^n : f^\sharp \in L^\infty\}.$$

Endowed with the norm given in (4.1), BMO becomes a Banach space provided we identify functions which differ a.e. by constant; clearly, $\|f\|_* = 0$ for $f(x) = c$ a.e. in $\mathbb{R}^n$.

Remark 4.1. *Note that $L^\infty(\mathbb{R}^n)$ is contained in BMO($\mathbb{R}^n$) and we have*

$$\|f\|_* \leq 2\|f\|_\infty$$

Moreover BMO contains unbounded functions, in fact the function $\log|x|$ *on* $\mathbb{R}$, *is in BMO but it is not bounded, so $L^\infty(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$.*

Now we sketch the proof of the fact that $\log|x|$ is in $BMO(\mathbb{R})$.

Let $I = (a, b) \subset \mathbb{R}$. We show for an appropriate choice of C_I,

$$⨍_I |\log|x| - C_I|\, dx \leq 1, \tag{4.2}$$

which in turn implies that $\|\log|\cdot|\|_* \leq 2$.

To prove (4.2) we consider three cases:

i) $0 < a < b$

ii) $-b < a < b$

iii) the rest

In the case **i)**, we pick $C_I = \log b$ and note that

$$\int_I |\log|x| - \log b|\, dx = \int_{(a,b)} (\log b - \log x)\, dx =$$

$$= \int_{(a,b)} \log b\, dx - \int_{(a,b)} \log x\, dx = (b-a) - a(\log b - \log a).$$

Therefore,

$$⨍_I |\log|x| - \log b|\, dx = 1 - a\frac{\log b - \log a}{b-a},$$

and (4.2) follows since $0 < a < b$.

In the case **ii)** we may restrict ourselves to $-b < a < 0 < b$. Again pick $C_I = \log b$ and note that

$$\int_I |\log|x| - \log b|\, dx = \int_{(a,-a)} |\log|x| - \log b|\, dx + \int_{(-a,b)} (\log b - \log x)\, dx = W + K.$$

From the above computation we have

$$K = (b+a) + a(\log b - \log(-a)).$$

To compute W we observe that the integrand is an even function, so

$$W = 2\lim_{\varepsilon\to 0^+} \int_{(\varepsilon,-a)} (\log b - \log x)\, dx = 2(-a\log b + a\log(-a) - a).$$

Thus

$$W + K = (b-a) + a(\log b - \log(-a))$$

and so

$$⨍_I |\log|x| - \log b|\, dx = 1 - (-a)\frac{(\log b - \log(-a))}{b+a}\frac{b+a}{b-a}.$$

Since $-b < a < 0 < b$ also in this case (4.2) follows.

The remaining cases can be reduced to either i) or ii) since we are dealing with an even function.

Now we give an example of function that does not belong to BMO.

Example 4.1. Let us show that the function $g(x) = \text{sign (x)} \log\frac{1}{|x|}$ does not belong to $BMO([-1,1])$. Indeed, for $0 < h < 1$ and $I \equiv [-h,h]$ we have $g_I = 0$ and

$$⨍_I |g(y) - g_I|\, dy = \frac{1}{2h}\int_{-h}^{h} \left|\log\frac{1}{|x|}\right| dx = \frac{1}{h}\int_0^h \log\frac{1}{x} dx = 1 + \log\frac{1}{h} \overset{h\to 0}{\longrightarrow} \infty$$

This example shows that if the absolute value of a function belongs to the BMO-class, this does not imply that the function itself is a BMO-function.

We shall give a result which provide many example of BMO functions.

Theorem 4.1. *[GR] If w is an A_1 weight (see definition 1.3), then* $\log w \in BMO$ *with a norm depending only on the* $A_1(w)$.

4.2 Estimates of rearrangements of the BMO-functions

The aim of the present section is to show that the non-increasing rearrangement f^* of a BMO-function f is also a BMO-function. The importance of the equimeasurable rearrangements of functions comes from the fact that in certain cases they preserve the properties of the original functions and in the same time have a simpler form. Let us give the definitions.

Definition 4.3. The **non-increasing rearrangement** of the function f is a non-increasing function f^* such that it is equimeasurable with $|f|$, i.e., for all $y > 0$ they have the same distribution function (see Definition 3.6)

$$a_{f^*}(y) = |\{x \in [0, |E|] : f^*(x) > y\}| = |\{t \in E : f^*(t) > y\}| = a_f(y)$$

for any measurable set $E \subset \mathbb{R}^n$.

This property does not define the non-increasing rearrangement uniquely: it can take different values at points of discontinuity (the set of such points is at most countable). For definiteness let us assume in addition that the function f^* is continuous from the left on $(0, |E|]$. The relation between the distribution function and the **non-increasing rearrangement** is given by the following equality:

$$f^*(x) = \inf\{y > 0 : a_f(y) < x\}, \quad 0 < x < |E|.$$

This formula shows that in a certain sense the non-increasing rearrangement is the inverse function to the distribution function.

An equivalent definition of the non-increasing rearrangement can be written in the following way:

$$f^*(x) = \sup_{D \subset E,\, |D| = x} \inf_{y \in D} |f(y)|, \qquad 0 < x < |E|$$

Sometimes instead of the non-increasing rearrangement it is more convenient to use the **non-decreasing rearrangement**. For the function f, measurable on the set $E \subset R^n$, the non-decreasing rearrangement is defined via the following equality:

$$f_*(x) = \inf_{D \subset E,\ |D|=x} \sup_{y \in D} |f(y)|, \qquad 0 < x < |E|.$$

The function f_* is non-negative, it is equimeasurable with $|f|$ on E and it is non-decreasing on $[0, |E|)$. The connection between the non-increasing and non-decreasing rearrangements is given by the equality

$$f_*(x) = f^*(|E| - x)$$

which holds true at every point of continuity, i.e. almost everywhere on $(0, |E|)$.

The equimeasurability of functions f^*, f_* and $|f|$ implies that

$$\int_0^{|E|} \varphi(f^*(u))\, du = \int_0^{|E|} \varphi(f_*(u))\, du = \int_E \varphi(|f(x)|)\, dx$$

The most important properties of the equimeasurable rearrangements f^* and f_* follow directly from their definition and consist in the identities:

$$\sup_{D \subset E,\ |D|=x} \int_D |f(y)|\, dy = \int_0^x f^*(u)\, du, \qquad 0 < x < |E|$$

$$\inf_{D \subset E,\ |D|=x} \int_D |f(y)|\, dy = \int_0^x f_*(u)\, du, \qquad 0 < x < |E|.$$

Often it is useful to consider the following functions

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(u)\, du, \quad f_{**}(t) = \frac{1}{t} \int_0^t f_*(u)\, du, \ \ t > 0.$$

Theorem 4.2. *[BDS] Let $f \in BMO(\mathbb{R}^n)$, then*

$$f^{**}(x) - f^*(x) \leq 2^{n+4} \|f\|_*, \qquad 0 < x < \infty.$$

In particular, from Theorem 4.2 it follows that the rearrangement operator is bounded in BMO.

The following Theorem shows that the non-increasing rearrangement f^* of a BMO-function f is also a BMO-function.

Theorem 4.3. *($(n = 1)$, [GRo]; $(n \geq 1)$, [BDS]) Let $f \in BMO(\mathbb{R}^n)$. Then $f^* \in BMO([0, \infty))$ and*

$$\|f^*\|_* \leq C\|f\|_* \,,$$

where the constant C depends only on the dimension n of the space (one can take $C = 2^{n+5}$).

4.3 The John-Nirenberg inequality

We know that BMO functions are not necessarily bounded, so a natural question is "how large they can be?". Let us consider again the function $\log|x|$. Fix $(0, b) = I \subset \mathbb{R}$ and consider those $x \in I$ where $\log|x|$ is large, i.e., consider the set

$$E_\lambda = \{x \in I : |\log|x| - C_I| > \lambda\}, \qquad \lambda > 0,$$

where $C_I = (\log|\cdot|)_I$. We are interested in E_λ for large values of λ. We can write E_λ as the sum of two sets:

$$E_\lambda = \{x \in I : x > e^{\lambda + C_I}\} \cup \{x \in I : x < e^{-\lambda + C_I}\}.$$

If λ is large the first set is empty and so for λ big enough we get:

$$|E_\lambda| \leq |\{x \in I : x < e^{-\lambda + C_I}\}| = e^{-\lambda} e^{C_I}.$$

Now by Jensen inequality

$$e^{C_I} \leq \fint_I e^{\log x}\, dx = \frac{|I|}{2}$$

and consequently

$$|E_\lambda| \leq \frac{|I|}{2} e^{-\lambda}.$$

The remarkable fact is that a similar estimate holds for arbitrary $f \in BMO$ and $I \subset \mathbb{R}$. More precisely we have

Theorem 4.4. *(**John-Nirenberg**, [JN]) There exist constants C_1, C_2, depending only on the dimension n, such that for every $f \in BMO(\mathbb{R}^n)$ and every cube $Q \subset \mathbb{R}^n$*

$$(4.3) \qquad |\{x \in Q : |f(x) - f_Q| > \lambda\}| \leq C_1\, |Q|\, e^{-\left(\frac{C_2 \lambda}{\|f\|_*}\right)}, \qquad \lambda > 0.$$

In the previous important Theorem the authors showed that the distribution function, corresponding to a function of bounded mean oscillation, is exponentially decreasing.

Remark 4.2. *In terms of equimeasurable rearrangements inequality 4.3 can be rewritten in the following form:*

$$(f - f_Q)^*(x) = \frac{\|f\|_*}{C_2} \log \frac{C_1|Q|}{x}, \qquad 0 < x \leq |Q|. \tag{4.4}$$

So, if $f \in BMO$, then its equimeasurable rearrangement do not grow faster than the logarithmic function as the argument tends to zero.

Remark 4.3. *In a certain sense the John-Nirenberg theorem is invertible. Namely, if f is a locally summable on $\mathbb{R}^n$ function such that for any cube $Q \subset \mathbb{R}^n$*

$$|\{x \in Q : |f(x) - f_Q| > \lambda\}| \leq C_1|Q|\, e^{-C_2\lambda}, \qquad \lambda > 0 \tag{4.5}$$

where the constants C_1 and C_2 do not depend on Q, then we want to prove that $f \in BMO(\mathbb{R}^n)$.

Indeed, let us rewrite (4.5) *in the form*

$$(f - f_Q)^*(x) \leq \frac{1}{C_2} \log \frac{C_1|Q|}{x}, \qquad 0 < x \leq |Q|.$$

Then

$$\frac{1}{|Q|} \int_Q |f(x) - f_Q|\, dx = \frac{1}{|Q|} \int_0^{|Q|} (f - f_Q)^*(y) dy \leq \frac{1}{C_2} \frac{1}{|Q|} \int_0^{|Q|} \log \frac{C_1|Q|}{y}\, dy =$$

$$= \frac{1}{C_2} \int_0^1 \log \frac{C_1}{u}\, du = \frac{1}{C_2}(1 + \log C_1)$$

Taking the supremum over all cubes $Q \subset \mathbb{R}^n$, we obtain

$$\|f\|_* \leq \frac{1}{C_2}(1 + \log C_1).$$

The John-Nirenberg theorem implies the following

Corollary 4.5. *If $f \in BMO(\mathbb{R}^n)$, then $f \in L^p_{loc}(\mathbb{R}^n)$, for any $p < \infty$.*

Proof. It is enough to prove that $f - f_Q \in L^p(Q)$ for any cube $Q \in \mathbb{R}^n$. The John-Nirenberg inequality in the form 4.4 yields

$$\int_Q |f - f_Q|^p dx = \int_0^{|Q|} (f - f_Q)^*(t)\, dt \leq$$

$$\leq \left(\frac{\|f\|_*}{C_2}\right)^p \int_0^{|Q|} \log^p\left(\frac{C_1|Q|}{t}\right) dt =$$

$$= \left(\frac{\|f\|_*}{C_2}\right)^p |Q| C_1 \int_0^{\frac{1}{C_1}} \log^p\left(\frac{1}{u}\right) du < \infty.$$

□

Corollary 4.6. *Let $f \in L^1_{loc}(\mathbb{R}^n)$ verify (4.3), then for $\lambda > \frac{\|f\|_*}{C_2}$ and for any cube Q,*

$$⨍_Q e^{\frac{|f(x)-f_Q|}{\lambda}} dx \leq \frac{C_1}{(C_2 \frac{\lambda}{\|f\|_*}) - 1}.$$

In [GJ] Garnett and Jones gave upper and lower bounds for the distance

$$dist_{BMO}(f, L^\infty) = \inf_{g \in L^\infty} \|f - g\|_*$$

by mean of the quantity

$$\varepsilon(f) = \inf\{\lambda > 0 : \sup_Q ⨍_Q e^{\frac{|f-f_Q|}{\lambda}}\, dx < \infty\}. \tag{4.6}$$

Theorem 4.7. *[GJ] If $f \in L^1_{loc}(\mathbb{R}^n)$ then*

$$k_1\, \varepsilon(f) \leq dist_{BMO}(f, L^\infty) \leq k_2\, \varepsilon(f) \tag{4.7}$$

where k_1, k_2 are constants depending only on the dimension.

Let $p(f) = \inf\{p > 1 : e^f, e^{-f} \in A_p\}$, $\forall f \in L^1_{loc}(\mathbb{R}^n)$, the following Lemma gives an important relation between A_p and BMO functions.

Lemma 4.8. *[GJ] If $f \in L^1_{loc}(\mathbb{R}^n)$ and $p(f) \neq \infty$, then $f \in BMO(\mathbb{R}^n)$ and*

$$p(f) - 1 = \varepsilon(f).$$

We now digress for a second to mention a parallel result to Theorem 4.7 in which BMO is replaced by EXP, the space of exponentially integrable functions.

Let Ω be a measurable set with finite measure $|\Omega|$. We denote by EXP=EXP(Ω) the set of functions $g : \Omega \longrightarrow \mathbb{R}$ such that there exists $\lambda > 0$ for which

$$\fint_{\Omega} e^{|g|/\lambda}\, dx < \infty$$

equipped with the norm

$$\|g\|_{EXP(\Omega)} = \inf\left\{\lambda > 0 : \fint_{\Omega} e^{|g|/\lambda}\, dx \leq 2\right\}. \tag{4.8}$$

Theorem 4.9. *([FLS], [CS]) For every $g \in EXP(\Omega)$ we have*

$$dist_{EXP}(g, L^{\infty}) = \inf\left\{\lambda > 0 : \fint_{\Omega} e^{|g|/\lambda}\, dx < \infty\right\}$$

where the distance is evalued with respect to norm 4.8.

4.4 The one dimensional case

In the first part of this section we report John-Nirenberg inequality in the one dimensional case giving optimal constants. In the second part we show the connection between A_p-class and BMO.

In [Kor] Korenovskii improves the John-Nirenberg Theorem getting the exact exponent in the inequality 4.3 for the one-dimensional case.

Theorem 4.10. *[Kor] Let $f \in BMO(\mathbb{R})$. Then for any interval I and for any $\lambda > 0$*

$$\frac{1}{|I|}|\{x \in I \ : \ |f(x) - f_I| > \lambda\}| \ \leq \ e^{1+\frac{2}{e}}\, exp\left(\frac{-2\lambda}{e\|f\|_*}\right).$$

The constant $(2/e)$ in the exponent cannot be increased.

It is well known that a locally integrable function f belongs to $BMO(\mathbb{R})$ if and only if there exists $\lambda > 0$ such that

$$s(f, \lambda) = \sup_{I} \fint_{I} e^{\frac{|f - f_I|}{\lambda}} dx < \infty \tag{4.9}$$

The following Proposition, whose proof is contained in [T], illustrates the connection among A_2 and BMO.

Proposition 4.11. *[T] A locally integrable function f belongs to $BMO(\mathbb{R})$ if and only if there exists $\lambda > 0$ such that $A_2\left(e^{\frac{f}{\lambda}}\right) < \infty$. Actually, for any $\lambda > 0$ the following inequalities hold:*

$$\frac{1}{2}s(f,\lambda) \leq A_2(e^{\frac{f}{\lambda}}) \leq s(f,\lambda)^2 \tag{4.10}$$

Now we are able to prove the following Proposition (exercise in [Ga]).

Proposition 4.12. *If $f \in BMO(\mathbb{R})$ and $s(f,\lambda) < \infty$, then there exists $\epsilon > 0$ such that*

$$s\left(f, \frac{\lambda}{1+\epsilon}\right) < \infty.$$

Proof. If $f \in BMO$ then there exists $\lambda > 0$ such that $A_2(e^{\frac{f}{\lambda}}) < \infty$ (Proposition 4.11). So if we put, in Theorem 1.30, $w = e^{\frac{f}{\lambda}}$ and $\tau = 1+\epsilon$ we have

$$w^{\tau} = e^{(1+\epsilon)\frac{f}{\lambda}} \in A_2$$

and

$$A_2\left(e^{(1+\epsilon)\dfrac{f}{\lambda}}\right)^{\dfrac{1}{2(1+\epsilon)}} \leq \frac{(1+\epsilon)A}{A-(1+\epsilon)^2(A-1)}$$

Now by (4.10) we have

$$\frac{1}{2}s\left(f, \frac{\lambda}{1+\epsilon}\right) \leq A_2\left(e^{\dfrac{(1+\epsilon)f}{\lambda}}\right) \leq s\left(f, \frac{\lambda}{1+\epsilon}\right)^2$$

and then

$$s\left(f, \frac{\lambda}{1+\epsilon}\right) < \infty$$

that completes the proof. □

Now let us consider the following set

$$I_f = \{\lambda > 0 : A_2(e^{\frac{f}{\lambda}}) < \infty\} \tag{4.11}$$

and describe its properties.

Proposition 4.13. *[AS] The function f belongs to BMO if and only if I_f is a non empty set. If we define*

$$\varepsilon(f) = \inf I_f$$

then

$$I_f = (\varepsilon(f), \infty) \tag{4.12}$$

and

$$\varepsilon(f) \le \frac{e}{2} \|f\|_*. \tag{4.13}$$

Proof. Condition $A_2(e^{f/\lambda}) < \infty$ is equivalent to

$$s(f, \lambda) = \sup_I ⨍_I e^{\frac{|f - f_I|}{\lambda}}\, dx\ < \infty \tag{4.14}$$

where the supremum is taken with respect to all intervals $I \subset \mathbb{R}$. Actually, for $\lambda > 0$ the following inequalities hold (see (4.10))

$$\frac{1}{2}\, s(f, \lambda) \le A_2(e^{f/\lambda}) \le s(f, \lambda)^2. \tag{4.15}$$

Then it is obvious that

$$\lambda_0 \in I_f, \quad \lambda_1 > \lambda_0 \implies \lambda_1 \in I_f.$$

Moreover, due to the Theorem 1.30 the set I_f does not contain its infimum $\varepsilon(f)$. This means that (4.12) holds true. To establish (4.13) we repeat a standard argument ([GR]) invoking Theorem 4.10

$$⨍_I e^{|f - f_I|/\lambda} = \int_0^\infty \frac{e^{t/\lambda}}{\lambda} |\{x \in I : |f(x) - f_I| > t\}|\, dt \le$$

$$\le \int_0^\infty \frac{e^{t/\lambda}}{\lambda}\, e^{(1\,+\,2/e)}\, e^{-(2/e\|f\|_*)\, t}\, |I|\, dt =$$

$$= |I|\, \frac{e^{(1\,+\,2/e)}}{\lambda} \int_0^\infty e^{(\frac{1}{\lambda}\,-\,2/e\|f\|_*)\, t}\, dt = |I|\, \frac{e^{(1\,+\,2/e)}}{\lambda} \left(\frac{2}{e\|f\|_*} - \frac{1}{\lambda} \right)^{-1}$$

if $\lambda > \frac{e}{2}\|f\|_*$. □

Corollary 4.14. *[AS] For any $f \in BMO(\mathbb{R})$*

$$\varepsilon(f) \le \frac{e}{2}\, dist(f, L^\infty). \tag{4.16}$$

Proof. It is easy to check that for $g \in L^\infty$

$$\varepsilon(f) = \varepsilon(f - g).$$

Then, using (4.13) we obtain

$$\varepsilon(f) \le \frac{e}{2}\|f - g\|_*$$

for any $g \in L^\infty$. This immediately implies (4.16). $\square$

More precisely, we obtain another representation for $\varepsilon(f)$.

Theorem 4.15. *[AS] For any $f \in BMO$*

$$\varepsilon(f) = \inf\left\{\lambda \sqrt{\frac{A_2(e^{f/\lambda}) - 1}{A_2(e^{f/\lambda})}} \ : \ \lambda \in I_f\right\}. \tag{4.17}$$

Proof. By Theorem 1.30 we deduce that, if $A_2(e^f) = A < \infty$, then

$$\varepsilon(f) \le \sqrt{\frac{A-1}{A}}. \tag{4.18}$$

In fact, for $\omega = e^f$ and $\lambda > \sqrt{\frac{A-1}{A}}$, we deduce $A_2(\omega^{\frac{1}{\lambda}}) < \infty$. Hence the inclusion

$$\left(\sqrt{\frac{A-1}{A}}, \infty\right) \subset I_f$$

holds and this implies (4.18).
Moreover, by applying this observation with f/λ in place of f and using the following property of the functional $\varepsilon(f)$:

$$\varepsilon(\mu f) = \mu\, \varepsilon(f) \qquad\qquad for \quad \mu > 0,$$

we deduce, for $\lambda \in I_f$

$$\frac{1}{\lambda}\varepsilon(f) \le \sqrt{\frac{A_2(e^{f/\lambda}) - 1}{A_2(e^{f/\lambda})}},$$

hence

$$\varepsilon(f) \le \inf\left\{\lambda\sqrt{\frac{A_2(e^{f/\lambda}) - 1}{A_2(e^{f/\lambda})}} : \quad \lambda \in I_f\right\}.$$

To get the inequality (4.17) it is sufficient to observe that

$$\inf\left\{\lambda\sqrt{\frac{A_2(e^{f/\lambda}) - 1}{A_2(e^{f/\lambda})}} : \quad \lambda \in I_f\right\} \le \inf\{\lambda : \lambda \in I_f\} = \varepsilon(f).$$

$\square$

Corollary 4.16. *[AS] For any $f \in BMO$, we have*

$$0 \le \varepsilon(f) \le 1; \tag{4.19}$$

moreover

$$\varepsilon(f) < 1 \tag{4.20}$$

if and only if

$$A_2(e^f) < \infty.$$

Proof. Let us introduce, as in [GJ] and in [T], for $f \in BMO$

$$p(f) = \inf\{p > 1 : A_p(e^{\pm f}) < \infty\},$$

then by Lemma 4.8 one has

$$p(f) = \varepsilon(f) + 1 \le 2.$$

Hence (4.19) holds true.
From (4.18) we deduce that if $A_2(e^f) < \infty$, then

$$\varepsilon(f) \le \sqrt{\frac{A-1}{A}} < 1.$$

Conversely, if $\varepsilon(f) < 1$, there exists $\lambda_0 < 1$ such that

$$A_2(e^{f/\lambda_0}) < \infty.$$

In view of (4.14), (4.15) we obtain

$$s(f, \lambda_0) < \infty$$

and therefore $s(f, 1) < \infty$, which in turns implies $A_2(e^f) < \infty$. □

4.5 Explicit bounds for the norm of composition operators acting on $BMO(\mathbb{R})$

In this section we improve a recent result of Gotoh [Go] who establishes a precise relation among constants in the P. W. Jones [Jo] Theorem about homeomorphisms of the line preserving BMO. We give also an explicit bound for the distance to L^∞ after composition (see [ACS]).

Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing homeomorphism. In the recent paper [Go], the relation between the norm of the operator

$$U : f \in BMO \longrightarrow f \circ h^{-1} \in BMO$$

and the A_∞-constants α, K of $\omega = h'$ according to Proposition 1.2 , was determined.

Theorem 4.17. *[Go] Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing homeomorphism, if h' verifies*

$$\frac{|I|}{|J|} \leq K \left(\frac{\int_I h' \, dx}{\int_J h' \, dx} \right)^\alpha$$

for any interval $J \subset \mathbb{R}$ and for each measurable set $I \subset J$, where $K \geq 1 \geq \alpha > 0$, then

$$\|f \circ h^{-1}\|_* \leq C \frac{K}{\alpha} \tag{4.21}$$

where $C > 0$ is some universal constant.

The following Theorem gives an important relation between A_∞ and $BMO(\mathbb{R})$ that we need in the following.

Theorem 4.18. *[Jo] The following conditions are equivalent*

i) *There exists $c \geq 1$ such that*

$$\|f \circ h^{-1}\|_* \leq c \|f\|_*$$

for any $f \in BMO(\mathbb{R})$;

ii) $h' \in A_\infty$;

iii) $(h^{-1})' \in A_\infty$.

In the following Theorem we identify the constant C in (4.21).

Theorem 4.19. *[ACS] Let h be an increasing homeomorphism from $\mathbb{R}$ into itself and assume that $\omega = h'$ verifies the A_∞ condition:*

$$\frac{\int_E \omega \, dx}{\int_I \omega \, dx} \leq K \left(\frac{|E|}{|I|} \right)^\alpha \tag{4.22}$$

for any interval $I \subset \mathbb{R}$ *and for each measurable set* $E \subset I$*, where* $K \geq 1 \geq \alpha > 0$*. Then*

$$\|f \circ h^{-1}\|_* \leq \frac{K}{\alpha} e^{2+\frac{2}{e}} \|f\|_* \tag{4.23}$$

for any $f \in BMO(\mathbb{R})$*.*

Proof. Following [Go], we fix the interval I and set $I' = h(I)$. It is worth noting that assumption (4.22) for $\omega = h'$ reads as

$$\frac{|h(E)|}{|h(I)|} \leq K \left(\frac{|E|}{|I|} \right)^{\alpha} \tag{4.24}$$

for E measurable, $E \subset I$. Fix $f \in BMO$ and set $g = f \circ h^{-1}$. By the John-Nirenberg Theorem, see Theorem 4.10, if we define for $t > 0$

$$E_t = \{x \in I : |f(x) - f_I| > t\}$$

we have

$$\frac{|E_t|}{|I|} \leq e^{1+\frac{2}{e}} \cdot e^{-\frac{2t}{e\|f\|_*}}. \tag{4.25}$$

On the other hand, let I' be an interval of $\mathbb{R}$, if we set

$$\mu(t) = |\{y \in I' : |g(y) - f_I| > t\}|$$

we have, by (4.24) and (4.25),

$$\mu(t) = |h(E_t)| \leq |h(I)| \cdot K \left(e^{1+\frac{2}{e}} \cdot e^{-\frac{2t}{e\|f\|_*}} \right)^{\alpha}. \tag{4.26}$$

By well known inequalities and identities from measure theory:

$$⨍_{I'} |g - g_{I'}| \leq 2 ⨍_{I'} |g - f_I| = \frac{2}{|I'|} \int_0^\infty \mu(t)\, dt \tag{4.27}$$

and by the simple calculations induced by (4.26)

$$\int_0^\infty \mu(t)\, dt \leq |I'| \cdot K e^{(1+\frac{2}{e})\alpha} \frac{e}{2\alpha} \|f\|_*$$

we arrive at the estimate

$$⨍_{I'} |g - g_{I'}| \leq \frac{K}{\alpha} e^{(2+\frac{2}{e})} \|f\|_*.$$

Taking supremum with respect to the intervals, we obtain (4.23). □

Now our aim is to give an explicit bound for the distance to L^∞ after composition. Let us begin with the following Lemma which is in the same spirit as Theorem 2.7 in [JN1].

Lemma 4.20. *[ACS] Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be a homeomorphism such that $(h^{-1})' \in A_p$, $1 < p < \infty$. Let ω be a weight on $\mathbb{R}$ and set $A_2(\omega) = A$; then, for $0 \le \sigma < \frac{1}{p}\sqrt{\frac{A}{A-1}}$ we have*

$$A_2(\omega^\sigma \circ h^{-1})^{\frac{1}{2}} \le \left[A_p(h^{-1})'\right]^{\frac{1}{p}} \left[\frac{\sigma p A}{A - \sigma^2 p^2 (A-1)}\right]^\sigma . \tag{4.28}$$

The inequality is sharp.

Proof. We will use Theorem 1.30 which describes the so called optimal "self-improvement of exponents" property of the A_2 class. Let σ to be determined later and set

$$L = \fint_I (\omega \circ h^{-1}(x))^\sigma \, dx \fint_I \frac{1}{(\omega \circ h^{-1}(x))^\sigma} dx.$$

We make the change of variables $t = h^{-1}(x)$, $h^{-1}(I) = J$ in the first integral:

$$\frac{1}{|I|}\int_I \omega^\sigma \circ h^{-1}(x)\, dx \ = \ \frac{1}{|I|}\int_J \frac{\omega^\sigma(t)}{(h^{-1})'(h(t))}\, dt \le$$

by Hölder's inequality

$$\le \left(\frac{1}{|I|}\int_J \omega^{\sigma p}(t)dt\right)^{\frac{1}{p}} \left(\frac{1}{|I|}\int_J \frac{1}{[(h^{-1})'(h(t))]^{p'}}\right)^{\frac{1}{p'}}.$$

We change back to the x variable into the last integral, obtaining

$$\frac{1}{|I|}\int_J \frac{1}{[(h^{-1})'(h(t))]^{p'}}\, dt = \frac{1}{|I|}\int_I [(h^{-1})'(x)]^{1-p'} dx$$

hence, taking into account that $\dfrac{|I|}{|J|} = \fint_I (h^{-1})'$,

$$\frac{1}{|I|}\int_I \omega^\sigma \circ h^{-1}(x)\, dx \le \left[\frac{|J|}{|I|}\right]^{\frac{1}{p}} \left(\fint_J \omega^{\sigma p}(t)dt\right)^{\frac{1}{p}} \left(\fint_I (h^{-1})'^{(1-p')}(x)\, dx\right)^{\frac{1}{p'}} \le$$

$$\le \left(\fint_J \omega^{\sigma p}(t)dt\right)^{\frac{1}{p}} \left[A_p(h^{-1})'\right]^{\frac{1}{p}}.$$

Similarly, the second factor in L can be majorized as follows

$$\fint_I (\omega^\sigma \circ h^{-1}(x))^{-\sigma}\, dx \le \left(\fint_J \omega^{-\sigma p}(t)dt\right)^{\frac{1}{p}} \left[A_p(h^{-1})'\right]^{\frac{1}{p}}$$

and hence

$$L \le \left[\fint_J \omega^{\sigma p} \fint_J \omega^{-\sigma p} \right]^{\frac{1}{p}} \left[A_p(h^{-1})' \right]^{\frac{2}{p}}.$$

Taking supremum with respect to J, we obtain

$$L \le [A_2(\omega^{\sigma p})]^{\frac{1}{p}} \left[A_p((h^{-1})') \right]^{\frac{2}{p}}$$

and, finally, taking supremum with respect to I on L

$$A_2(\omega^{\sigma} \circ h^{-1}) \le [A_2(\omega^{\sigma p})]^{\frac{1}{p}} \left[A_p((h^{-1})') \right]^{\frac{2}{p}}.$$

We now choose σ. From Theorem 1.30 it follows that, if $\tau = \sigma p < \sqrt{\frac{A}{A-1}}$, then $A_2(\omega^{\sigma p}) < \infty$. Then, we choose $\sigma < \frac{1}{p}\sqrt{\frac{A}{A-1}}$ and (1.30) gives

$$[A_2(\omega^{\sigma p})]^{\frac{1}{p}} \le \left[\frac{\sigma p A}{A - \sigma^2 p^2 (A-1)} \right]^{2\sigma}.$$

It remains to show that the inequality (4.28) is sharp. This is a consequence of the choice $h(t) = t$ which reduces (4.28) to the form

$$A_2(\omega^{\sigma p})^{1/\sigma p} \le \frac{\sigma p A}{A - \sigma^2 p^2 (A-1)}.$$

which agrees with the sharp implication in Theorem 1.30. □

Let us now consider the functional $\varepsilon(f) = \inf I_f$ where $f \in BMO$ and I_f is defined by (4.11). From Proposition 4.13 and Theorem 4.7 we know that f belongs to BMO if and only if I_f is not empty and that $\varepsilon(f)$ is equivalent to the distance functional

$$dist(f, L^{\infty}) = \inf_{g \in L^{\infty}} \|f - g\|_*.$$

Let us prove the following:

Theorem 4.21. *[ACS] Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing homeomorphism such that $(h^{-1})'$ belongs to the A_p-class. Then for any $f \in BMO(\mathbb{R})$*

$$\varepsilon(f \circ h^{-1}) \le p\, \varepsilon(f). \tag{4.29}$$

Moreover, there exists an equivalent norm $\|\cdot\|'_*$ on BMO such that

$$dist\,'(f \circ h^{-1}, L^{\infty}) \le p\, dist\,'(f, L^{\infty}). \tag{4.30}$$

Proof. Fix $\lambda \in I_f$ and set $A_\lambda = A_2(e^{f/\lambda})$. Let us prove that

$$\varepsilon(f \circ h^{-1}) \leq \lambda\, p \sqrt{\frac{A_\lambda - 1}{A_\lambda}}. \tag{4.31}$$

By previous lemma, with $\omega = e^{f/\lambda}$ we deduce that for $0 \leq \sigma < \frac{1}{p}\sqrt{\frac{A_\lambda}{A_\lambda - 1}}$ one has

$$A_2\left(e^{\dfrac{\sigma f \circ h^{-1}}{\lambda}}\right) < \infty.$$

In other words, for $\mu > \lambda\ p\sqrt{\frac{A_\lambda - 1}{A_\lambda}}$, μ belongs to the set $I_{f \circ h^{-1}}$ and this immediately implies (4.31).

Let us recall that actually (see Theorem 4.15)

$$\varepsilon(f) = \inf\{\lambda\sqrt{\frac{A_\lambda - 1}{A_\lambda}} : \lambda \in I_f\}.$$

Then by (4.31) we get (4.29).

Let us note that if h is an increasing homeomorphism such that $(h^{-1})' \in A_1$ and also $h' \in A_1$, then inequality (4.29) reduces to the optimal identity

$$\varepsilon(f \circ h^{-1}) = \varepsilon(f)$$

for any $f \in BMO$. In this sense our result is sharp. In fact we benefit of the coupled inequality to (4.29)

$$\varepsilon(g \circ h) \leq p\, \varepsilon(g)$$

for any $g \in BMO$ and for any $p > 1$. Passing to the limit in both inequalities we obtain the stated identity.

Now let us observe that, since $(h^{-1})'$ belongs to A_p, in particular it belongs to A_∞ and then by Theorem 4.18 there exists $c > 0$ such that

$$\|f \circ h^{-1}\|_* \leq c\, \|f\|_* \tag{4.32}$$

for any $f \in BMO$. Now it is a routine matter to see that

$$dist(f \circ h^{-1}, L^\infty) \leq c\, dist(f, L^\infty) \tag{4.33}$$

with the same constant c than in (4.32), for any $f \in BMO$. To this end, we note that for any $f, g \in BMO$ (4.32) implies that

$$\|f \circ h^{-1} - g \circ h^{-1}\|_* \leq c\, \|f - g\|_*. \tag{4.34}$$

If we restrict ourselves to $g \in L^\infty$ by (4.34) we deduce

$$\text{(4.35)} \qquad dist(f \circ h^{-1}, L^\infty) \le \|f \circ h^{-1} - g \circ h^{-1}\|_*.$$

In view of (4.34),(4.35) we conclude with (4.33). By mean of Theorem 4.7 and Theorem 4.10 we deduce the inequality

$$\varepsilon(f \circ h^{-1}) \le c\, k_2\, \frac{e}{2}\, \varepsilon(f)$$

which is largely less precise than (4.29).

To prove (4.30) remember ([Ga], p. 258) that, if H denotes the Hilbert transform:

$$Hg(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{g(y)}{x-y}\, dy,$$

and $\varphi \in BMO$, then $\varphi = f + Hg + \alpha$ with $f \in L^\infty$, $g \in L^\infty$ and α constant, and

$$\text{(4.36)} \qquad \|\varphi\|'_* = \inf\{\|f\|_\infty + \|g\|_\infty : \varphi = f + Hg + \alpha\}$$

defines a norm on BMO equivalent to $\|\varphi\|_*$. Now if we set

$$dist\,'(\varphi, L^\infty) = \inf_{\psi \in L^\infty} \|\varphi - \psi\|'_*$$

the identity

$$\text{(4.37)} \qquad dist\,'(\varphi, L^\infty) = \frac{\pi}{2} \varepsilon(\varphi)$$

holds for any $\varphi \in BMO$ ([Ga], Corollary 6.6). If we equipe BMO with the norm (4.36) in view of (4.29), (4.30) holds. □

4.6 BMO^R-Space and the class A_p^R

In this section we will introduce the BMO^R-Space and the class A_p^R, where in the definitions cubes are replaced by rectangles. We will focus attention on the connections between A_2^R-class and BMO^R.

Let us begin with some definitions.

Definition 4.4. A non negative measurable function w on the space $\mathbb{R}^n$ satisfies the A_p^R**-condition**, $1 < p < \infty$ if there exists a constant $A \geq 1$ such that, for any rectangle $R \subset \mathbb{R}^n$ with sides parallel to the coordinate axes, one has

$$\fint_R w dx \left(\fint_R w^{-\frac{1}{p-1}} dx \right)^{p-1} \leq A \tag{4.38}$$

We call the A_p^R**-constant** of w as

$$A_p(w) = \sup_R \fint_R w dx \left(\fint_R w^{-\frac{1}{p-1}} dx \right)^{p-1}, \quad 1 < p < \infty \tag{4.39}$$

where the supremum is taken over all rectangles $R \subset \mathbb{R}^n$ with sides parallel to the coordinate axes.

Definition 4.5. Let $f \in L^1_{loc}(\mathbb{R}^n)$, then $f \in BMO^R(\mathbb{R}^n)$ if

$$\sup_R \fint_R |f - f_R| \, dx = \|f\|_{*,R} < \infty \tag{4.40}$$

where the supremum runs over all rectangles $R \subset \mathbb{R}^n$ with sides parallel to the coordinate axes.

Note that in the multidimensional case the problem of finding the upper bound of C_2 constant in the John-Nirenberg inequality 4.3 is still open. If, instead, we consider the space BMO^R, the maximal value of the constant C_2 in John-Nirenberg inequality is equal to $\dfrac{2}{e}$, as in the one-dimensional case. Namely,

Theorem 4.22. *[Kor] Let $f \in BMO^R(\mathbb{R}^n)$. Then for every rectangle $R \subset \mathbb{R}^n$*

$$|\{x \in Q : |f(x) - f_R| > \lambda\}| \leq e^{1+\frac{2}{e}} \, |R| \, e^{-\left(\frac{\frac{2}{e}\lambda}{\|f\|_{*,R}} \right)}, \qquad \lambda > 0. \tag{4.41}$$

The following proposition gives the exact estimate of the equimeasurable rearrangements of functions satisfying the inverse Jensen inequality respect to rectangles of $\mathbb{R}^n$.

Proposition 4.23. *[Kor] Let ϕ be the class of all positive convex downwards functions φ on $(0,+\infty)$ and let R_0 a fixed rectangle of $\mathbb{R}^n$. Let f be a non-negative function on R_0 satisfying the inverse Jensen inequality*

$$\fint_{R_0} \varphi(f(x))dx \leq A\,\varphi\left(\fint_{R_0} f(x)dx\right) \tag{4.42}$$

with $A > 1$. Then for any interval $I \subset [0, \mu(R_o)]$

$$\fint_I \varphi(f_*(t))dt \leq A\,\varphi\left(\fint_I (f_*(t))dt\right) \tag{4.43}$$

$$\fint_I \varphi(f^*(t))dt \leq A\,\varphi\left(\fint_I (f^*(t))dt\right) \tag{4.44}$$

with the same constant $A > 1$ as in condition (4.42)*.*

Since for $\varphi(u) = u^{-\frac{1}{p-1}}$ $(p > 1)$ the inverse Jensen inequality becomes the Muckenhoupt condition also for the equimeasurable rearrangements of f, we have:

$$A_p^R[f^*] \leq A \qquad A_p^R[f_*] \leq A \tag{4.45}$$

Note that Bojarski, Sbordone and Wik ([BSW]) proved that inequalities (4.45) with $p = 1$ are not true for cubes in $\mathbb{R}^n$, but are true in the one dimensional case.

Theorem 4.24. *[Kor] Assume that the non-negative function f satisfies*

$$A_2^R(f) = A < \infty \tag{4.46}$$

uniformly over all rectangles $R \subset R_o$, where $R_o \subset \mathbb{R}^n$ is a fixed rectangle and $A > 1$. Then for every $s \in \left(-\sqrt{\frac{A}{A-1}}, -1\right) \cup \left(1, \sqrt{\frac{A}{A-1}}\right)$ there exist positive constants B' and B'' depending only by A and s such that

$$\frac{1}{B'}\left(\fint_R f^{-1}(x)dx\right)^{-1} \leq \left(\fint_R f^s(x)dx\right)^{\frac{1}{s}} \leq B''\left(\fint_R f(x)dx\right) \tag{4.47}$$

The proof of Theorem 4.24 requires two Lemmas about monotone functions of one variable:

Lemma 4.25. *[Kor] Let h be a non-increasing function of one variable in [a,b] such that*

$$(4.48)\qquad \fint_a^t h(x)dx \le A\left(\fint_a^t h^{-1}(x)dx\right)^{-1} \qquad a \le t \le b$$

namely $A_2(h) = A < \infty$*, then for any* $s \in \left(1, \sqrt{\frac{A}{A-1}}\right)$ *there exists a constant* $B'' > 0$ *such that*

$$(4.49)\qquad \left(\fint_a^b h^s(x)dx\right)^{\frac{1}{s}} \le B''\left(\fint_a^b h(x)dx\right)$$

Lemma 4.26. *[Kor] Let h be a non-decreasing function of one variable in [a,b] such that*

$$(4.50)\qquad \fint_a^t h(x)dx \le A\left(\fint_a^t h^{-1}(x)dx\right)^{-1} \qquad a \le t \le b$$

namely $A_2(h) = A < \infty$*, then for any* $s \in \left(-\sqrt{\frac{A}{A-1}}, -1\right)$ *there exists a constant* $B' > 0$ *such that*

$$(4.51)\qquad \left(\fint_a^b h^s(x)dx\right)^{\frac{1}{s}} \ge \frac{1}{B'}\left(\fint_a^b h^{-1}(x)dx\right)^{-1}$$

Now we are able to prove Theorem 4.24.

Proof. (of Theorem 4.24)

From properties of equimeasurable rearrangements we have

$$(4.52)\qquad \int_0^{\mu(R)} (f_*(x))^p dx = \int_0^{\mu(R)} (f^*(x))^p dx = \int_R f^p(x)dx$$

for any real p. Fix some segment $R \subset R_o$, then by (4.45) the function $h \equiv f_*$ satisfies the condition of Lemma 4.25 with $[a,b] = [0, \mu(R)]$. Hence if $s \in (1, \sqrt{\frac{A}{A-1}})$ we have:

$$\left(\fint_0^{\mu(R)} ((f_*(x))^s dx\right)^{\frac{1}{s}} \le B''\left(\fint_0^{\mu(R)} f_*(x)dx\right)$$

and from (4.52)

$$\left(\fint_R f^s(x)dx\right)^{\frac{1}{s}} \le B''\left(\fint_R f(x)dx\right)$$

that is the right inequality in (4.47).

Similarly by (4.45) the function $h \equiv f^*$ satisfies the condition of Lemma 4.26 with $[a, b] = [0, \mu(R)]$. Hence if $s \in (-\sqrt{\frac{A}{A-1}}, -1)$ we have:

$$\left(\fint_0^{\mu(R)} (f^*)^s(x)dx \right)^{\frac{1}{s}} \geq \frac{1}{B'} \left(\fint_0^{\mu(R)} (f^*)^{-1}(x)dx \right)^{-1}$$

and from (4.52)

$$\frac{1}{B'} \left(\fint_R f^{-1}(x)dx \right)^{-1} \leq \left(\fint_R f^s(x)dx \right)^{\frac{1}{s}}$$

that is the left inequality in (4.47). □

Remark 4.4. *Rewriting* (4.47) *in this way*

$$\frac{1}{B'' \left(\fint_R f(x)dx \right)} \leq \left(\fint_R f^s(x)dx \right)^{-\frac{1}{s}} \leq B' \left(\fint_R f^{-1}(x)dx \right)$$

and multiplying by $\fint_R f(x)dx$ *we have*

$$\frac{1}{B''} \leq \left(\fint_R f^s(x)dx \right)^{-\frac{1}{s}} \fint_R f(x)dx \leq B' \left(\fint_R f^{-1}(x)dx \right) \left(\fint_R f(x)dx \right)$$

Now if we put $s = \frac{1}{1-\tau}$ *we have*

$$\frac{1}{B''} \leq \left(\fint_R f^{\frac{1}{1-\tau}}(x)dx \right)^{\tau-1} \fint_R f(x)dx \leq B' \left(\fint_R f^{-1}(x)dx \right) \left(\fint_R f(x)dx \right)$$

taking the supremum over all rectangles R *we have:*

$$\frac{1}{B''} \leq A_\tau^R(f) \leq B'A$$

for any $\tau \in \left(0, 1 - \sqrt{\frac{A-1}{A}}\right) \cup \left(1 + \sqrt{\frac{A-1}{A}}, 2\right)$.

We can observe that the constant B' *is the same of Lemma 4.26 and from Corollary 1.23 we have*

$$A_\tau^R(f) \leq A \frac{(\tau - 1)}{\frac{1}{A} - \tau(2 - \tau)} \tag{4.53}$$

for any $\tau \in \left(1 + \sqrt{\frac{A-1}{A}}, 2\right)$.

Now we are able to prove the analogous of Theorem 1.30 for A_2^R-class:

Theorem 4.27. *Assume $A_2^R(w) = A < \infty$, then for $1 < \gamma < \sqrt{\frac{A}{A-1}}$ we have $w^\gamma \in A_2^R$ and*

$$A_2^R(w^\gamma)^{\frac{1}{2\gamma}} \leq \frac{\gamma A}{A - \gamma^2(A-1)}. \tag{4.54}$$

Proof. If we put $\psi(\tau) = \frac{\frac{1}{A} - \tau(2-\tau)}{\tau - 1}$ in (4.53) we have

$$A_\tau^R \leq \frac{A}{\psi(\tau)}$$

for any $\tau \in \left(1 + \sqrt{\frac{A-1}{A}}, 2\right)$. In particular we deduce for any rectangle $R \subset \mathbb{R}^n$

$$\fint_R w^{-\frac{1}{\tau-1}} \leq \left[\frac{1}{\fint_R w} \cdot \frac{A}{\psi(\tau)} \right]^{\frac{1}{\tau-1}} \tag{4.55}$$

and also, taking into account that $A = A_2^R(w) = A_2^R(w^{-1})$, we deduce that

$$\fint_R w^{\frac{1}{\tau-1}} \leq \left[\frac{1}{\fint_R w^{-1}} \frac{A}{\psi(\tau)} \right]^{\frac{1}{\tau-1}} \tag{4.56}$$

Multiplying (4.55) and (4.56) and using the Hölder inequality in the form

$$1 \leq \fint_R \omega \fint_R \omega^{-1},$$

we get

$$\fint_R \omega^{\frac{1}{\tau-1}} \fint_R \omega^{-\frac{1}{\tau-1}} \leq \left[\frac{A}{\psi(\tau)} \right]^{2/(\tau-1)}.$$

Hence, for $1 + \sqrt{\frac{A-1}{A}} < \tau \leq 2$ we have

$$A_2^R(\omega^{\frac{1}{\tau-1}}) \leq \left[\frac{A}{\psi(\tau)} \right]^{2/(\tau-1)}.$$

If we set $\gamma = \frac{1}{\tau-1}$ we obtain immediately, for the range $1 < \gamma < \sqrt{\frac{A}{A-1}}$,

$$[A_2^R(\omega^\gamma)]^{1/2\gamma} \leq \frac{A}{\varphi(\gamma)}$$

where $\varphi(\gamma) = \gamma \left[1 - A(1 - \frac{1}{\gamma^2}) \right]$ which coincides with (4.54).
The optimality is seen by mean of power functions. Namely, choose $\omega(x) = |x|^r$ with $0 < r < 1$, then we have

$$A_2^R(|x|^r) = \frac{1}{1 - r^2}$$

and $A_2^R(|x|^{r\gamma}) = \frac{1}{1-\gamma^2 r^2} < \infty$ if and only if $1 < \gamma < \sqrt{\frac{A}{A-1}} = \frac{1}{r}$. □

Theorem 4.28. *Let $\lambda > 0$ and $f \in BMO^R(\mathbb{R}^n)$ satisfy*

$$A_2^R\left(e^{\frac{f}{\lambda}}\right) = A < \infty.$$

Then for $0 < \epsilon < \sqrt{\frac{A}{A-1}} - 1$ we have

$$A_2^R\left(e^{(1+\epsilon)\frac{f}{\lambda}}\right) < \infty$$

The bound for ϵ is sharp.

Proof. If we put $w = e^{\frac{f}{\lambda}}$ in Theorem 4.27 and $\gamma = 1 + \varepsilon$ we have

$$w^\gamma = e^{(1+\varepsilon)\frac{f}{\lambda}} \in A_2^R$$

and

$$A_2^R(e^{(1+\varepsilon)\frac{f}{\lambda}})^{\frac{1}{2(1+\varepsilon)}} \leq \frac{(1+\varepsilon)A}{A-(1+\varepsilon)^2(A-1)}$$

for $0 < \epsilon < \sqrt{\frac{A}{A-1}} - 1$. Since $A > 1$ we have

$$A_2^R\left(e^{(1+\epsilon)\frac{f}{\lambda}}\right) < \infty$$

that completes the proof. □

Another important fact related to A_2^R weights class is due to B.D. Wick ([W1]) that proves the following

Lemma 4.29. *[W1] A weight $w : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is in A_2^R if and only if $w \in A_2$ uniformly in each variable separately. Moreover, we have the relationship*

$$\max\{\sup_x A_2(w(x,\cdot)), \sup_y A_2(w(\cdot,y))\} \leq A_2^R(w) \leq$$

$$\leq C^2(\max\{\sup_x A_2(w(x,\cdot)), \sup_y A_2(w(\cdot,y))\})^2,$$

and the constant C depends only on the dimensions of the space being considered.

Wick also proved that there is an equivalent norm in the space BMO^R, namely we have the following

Lemma 4.30. *[W1] Let $f \in BMO^R(\mathbb{R}^2)$, then*

$$\frac{1}{4}\|f\|_{*,R} \leq \max\{\sup_x \|f(x,\cdot)\|_*, \sup_y \|f(\cdot,y)\|_*\} \leq \|f\|_{*,R}.$$

The following Theorem shows that there is also in the rectangles case the same log-exp relationship between A_p classes and BMO spaces like in the one dimensional case (see Proposition 4.11), namely

Theorem 4.31. *[W1] Let* $w \in A_2^R$. *Then* $\log w \in BMO^R$. *Conversely, if* $v \in BMO^R$*, then for* $|\alpha| < \frac{C}{\|v\|_{*,R}}$ *we have* $e^{\alpha v} \in A_2^R$*, where* C *is an absolute dimensional constant.*

Chapter 5

BMO-Martingales and Applications

In this chapter we will show how we can extend BMO functions spaces to probability theory.

After preliminaries basic definitions and notations we focus attention on the space of BMO-martingales and its connections with probabilistic A_p-condition. In the forth section we speak about the distance in BMO to L^∞ and the connection with the classical functions case. In the last section we report some applications in mathematical finance.

5.1 Preliminaries

At first we recall some concepts from general probability theory.

Definition 5.1. If Ω is a given set, then a **σ-algebra** $\mathcal{F}$ on Ω is a family $\mathcal{F}$ on Ω with the following properties:

(i) $\emptyset \in \mathcal{F}$

(ii) $F \in \mathcal{F} \Longrightarrow F^C \in \mathcal{F}$, where $F^C = \Omega \backslash F$

(iii) $A_1, A_2, ... \in \mathcal{F} \Longrightarrow A := \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is called a measurable space.

Definition 5.2. A **probability measure** P on a measurable space $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \longrightarrow [0,1]$ such that

(a) $P(\emptyset) = 0,\ P(\Omega) = 1$

(b) if $A_1, A_2, ... \in \mathcal{F}$ and $\{A_i\}_{i=1}^{\infty}$ is disjoint then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$.

The triple $(\Omega, \mathcal{F}, P)$ is called **probability space**.

The subsets F of Ω which belong to $\mathcal{F}$ are called $\mathcal{F}$-measurable sets. In a probability context these sets are called events and we use the interpretation

$$P(F) = \text{"the probability that the event F occurs"}.$$

In particular, if $P(F) = 1$ we say that "F occurs with probability 1", or "almost surely (a.s.)".

Definition 5.3. If $(\Omega, \mathcal{F}, P)$ is a given probability space, then the function $Y : \Omega \longrightarrow \mathbb{R}^n$ is called **$\mathcal{F}$-measurable** if

$$Y^{-1}(U) := \{w \in \Omega; Y(w) \in U\} \in \mathcal{F}$$

for every open sets $U \in \mathbb{R}^n$.

Definition 5.4. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. A **random variable** X is an $\mathcal{F}$-measurable function $X : \Omega \longrightarrow \mathbb{R}^n$. Every random variable induces a probability measure μ_X on $\mathbb{R}^n$, defined by

$$\mu_X(B) = P(X^{-1}(B)),\ \forall B \in \mathcal{B}$$

where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}^n$ and μ_X is called the **distribution** of X. If $\int_\Omega |X(w)|dP(w) < \infty$, then the number

$$E[X] := \int_\Omega X(w)dP(w) = \int_{\mathbb{R}^n} x d\mu_X(x)$$

is called **the expectation** of X. More generally, if $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is Borel measurable and $\int_\Omega |f(X(w))|dP(w) < \infty$ then we have

$$E[f(X)] := \int_\Omega f(X(w))dP(w) = \int_{\mathbb{R}^n} f(x)d\mu_X(x).$$

Definition 5.5. Two subsets $A, B \in \mathcal{F}$ are called **independent** if

$$P(A \cap B) = P(A) \cdot P(B).$$

If two random variables $X, Y : \Omega \longrightarrow \mathbb{R}^n$ are independent, then

$$E[XY] = E[X]E[Y],$$

provided that $E[X] < \infty$ and $E[Y] < \infty$.

Definition 5.6. Let $T \subset \mathbb{R}_+$, a **stochastic process** is a parametrized collection of random variables

$$\{X_t\}_{t \in T}$$

defined on a probability space $(\Omega, \mathcal{F}, P)$ and assuming values in $\mathbb{R}^n$.

The parameter space T is usually the halfline $[0, \infty)$. It may be useful for the intuition to think of t as "time" and each w as an individual "particle" or "experiment".

Definition 5.7. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. A **filtration** $\mathcal{F}_t$ is a family of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$ such that

(i) $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$,

(ii) $\mathcal{F}_t = \bigcap_{u>t} \mathcal{F}_u$ for all $t \geq 0$.

Definition 5.8. A real valued stochastic process $M = (M_t, \mathcal{F}_t)$ is called a **martingale** (resp. **supermatingale**, **submartingale**)if

(i) each M_t is $\mathcal{F}_t$-measurable, i.e., M is adapted to the filtration $\mathcal{F}_t$,

(ii) $E[|M_t|] < \infty$ for all t,

(iii) $E[M_s|\mathcal{F}_t] = M_t$ for all $s \geq t$

(resp. $E[M_s|\mathcal{F}_t] \leq M_t$, $E[M_s|\mathcal{F}_t] \geq M_t$).

There are two very important classes of stochastic processes, one is martingales the other is Markov processes, and there is the most important (continuous) stochastic process Brownian motion which belongs to both classes.

The two-dimensional Brownian motion was observed in 1828 by Robert Brown as diffusion of pollen in water. Later the one-dimensional Brownian motion was used by Louis Bachelier around 1900 in modeling of financial markets and in 1905 by Albert Einstein. A first rigorous proof of its (mathematical) existence was given by Norbert Wiener in 1921. Later on, various different proofs of its existence were given.

Definition 5.9. Let $(W_t, \mathcal{F}_t)_{t\in T}$ be an $\mathbb{R}$-valued continuous stochastic process on $(\Omega, \mathcal{F}, P)$. Then $(W_t, \mathcal{F}_t)_{t\in T}$ is called a **standard Brownian motion** if

1. $W_0 = 0$ a.s.
2. $W_t - W_s \sim \mathcal{N}(0, t-s)$, where $\mathcal{N}$ is the normal standard distribution
3. $W_t - W_s$ independent of $\mathcal{F}_s$

Definition 5.10. A random variable $\tau : \Omega \longrightarrow T \cup \{+\infty\}$ is a **stopping time** if $\forall t \in T, \{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$.

Definition 5.11. An adapted process $M = (M_t, \mathcal{F}_t)$ is said to be a **local martingale** if there exists a sequence of increasing stopping times τ_n with $\lim\limits_{n\to\infty} \tau_n = \infty$ a.s. such that $(M_{t\wedge\tau_n} I_{\{\tau_n>0\}}, \mathcal{F}_t)$ is a martingale for each n. Such a sequence (τ_n) of stopping times is called **fundamental sequence**.

Now we suppose that any local martingale adapted to this filtration is continuous. Note that the following properties are equivalent (see [ESY]):

1. any local martingale is continuous,
2. any stopping time is predictable,
3. for every stopping time τ and every $\mathcal{F}_t$-measurable random variable U, there exists a continuous local martingale M with $M_\tau = U$ a.s..

In the following we assume that $M_0 = 0$. Let us denote by $\langle M\rangle$ the continuous increasing process such that $M^2 - \langle M\rangle$ is also a local martingale.

In plain words, the martingale property means, that the process, given the present time s has no tendency in future times $t \geq s$, that is the average over all future possible states of X_t gives just the present state X_s. In difference to this, the Markov process has no memory, that is the average of X_t knowing the past is the same as the average of X_t knowing the present.

Definition 5.12. An adapted process $M = (M_t, \mathcal{F}_t)$ is said to be a **semimartingale** if X_t can be written as $M_t + A_t$ where M is local martingale and A is a stochastic process that is locally of bounded variation.

The next formula plays an extremely important role in stochastic calculus.

Theorem 5.1. *(Itô's formula) Let $X = M + A$ be a continuous semimartingale, and let f be a real valued function on $\mathbb{R}$ which is twice differentiable. Then*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d\langle M_s\rangle. \tag{5.1}$$

Note that the second term on the right hand side is the stochastic integral.

The Itô's formula shows that the class of semimartingales is invariant under composition with C^2-function.

5.2 BMO-Martingales

In this section we will speak about the BMO-Martingales space. We recall that the space of BMO functions was introduced by John and Nirenberg in 1961 [JN] and they gave the first important result on BMO functions (see Chapter 4). In 1971 Fefferman [Fe] characterize the space of BMO-functions as the dual of the Hardy space H_1. On the other hand, in 1972 Getoor and Sharpe [GS] introduced the concept of a conformal martingale and by using conformal martingales they established the duality of H_1 and BMO in probabilistic setting.

Let us begin with some definitions.

Definition 5.13. Let M be a continuous local martingale, then we can define the **exponential local martingale** $\varepsilon(M)_t$ as

$$\varepsilon(M)_t = \exp\left(M_t - \frac{1}{2}\langle M\rangle_t\right) \quad (0 \leq t < \infty)$$

with $\varepsilon(M)_0 = 1$.

A noteworthy fact is that, supposing that the exponential local martingale $\varepsilon(M)$ is uniformly integrable, it is not necessarily a true martingale. In fact, generally, we have $E[\varepsilon(M)_t] \leq 1$ for every t, because $\varepsilon(M)$ is a positive supermartingale with $\varepsilon(M)_0 = 1$. Therefore, it is a martingale if and only if $E[\varepsilon(M)_t] = 1$ for every t.

Example 5.1. Let $B = (B_t, \mathcal{F}_t)$ be a one-dimensional Brownian motion starting at 0. For each $t > 0$ we have

$$E[\varepsilon(B)_t] = \int_{-\infty}^{\infty} \exp\left(x - \frac{t}{2}\right) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right)\, dx =$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-t)^2}{2t}\right)\, dx = 1$$

and hence $\varepsilon(B)$ is a true martingale. However, since $\varepsilon(B)_\infty = 0$ a.s., it is not a uniformly integrable martingale.

Definition 5.14. Let $M = (M_t, \mathcal{F}_t)$ be a uniformly integrable martingale with $M_0 = 0$ and set

$$\|M\|_{BMO} = \sup_{\tau} \|E[|M_\infty - M_\tau||\mathcal{F}_\tau]\|_\infty$$

where the supremum is taken over all stopping times τ. The space of BMO-**martingales** is the class of all uniformly integrable martingale such that

$$\|M\|_{BMO} < \infty,$$

and $\|\cdot\|_{BMO}$ is a norm in this space.

More in particular we can also define the space BMO_p.

Definition 5.15. Let $M = (M_t, \mathcal{F}_t)$ be a uniformly integrable martingale with $M_0 = 0$ and for $1 \leq p < \infty$ we set

$$\|M\|_{BMO_p} = \sup_{\tau} \|E[|M_\infty - M_\tau|^p|\mathcal{F}_\tau]^{\frac{1}{p}}\|_\infty$$

where the supremum is taken over all stopping times τ. The space of BMO_p-**martingales** is the class of all uniformly integrable martingale such that

$$\|M\|_{BMO_p} < \infty,$$

and $\|\cdot\|_{BMO_p}$ is a norm in this space.

From the Hölder inequality it follows at once that for $p < q$, $BMO_q \subset BMO_p$.

Now let L^∞ be the class of all bounded martingales and let H^∞ be the class of all martingales M such that $\langle M\rangle_\infty$ is bounded. Since

$$\|M\|_{BMO} \leq 2\|M\|_\infty \quad \|M\|_{BMO_2} \leq \|\langle M\rangle\|_\infty^{\frac{1}{2}},$$

then L^∞ and H^∞ are contained in BMO, but in general there is not an inclusion relation between L^∞ and H^∞. In fact, for example, if $B = (B_t)$ is a one dimensional Brownian motion, the process B stopped at τ, where $\tau = \inf\{t : |B_t| = 1\}$, belongs to $L^\infty \backslash H^\infty$. On the other hand, one can see that $(B_{t\wedge 1}) \in H^\infty \backslash L^\infty$.

The following Theorem is the John-Nirenberg inequality in BMO-martingales space.

Theorem 5.2. *[Ka] (John-Nirenberg inequality) If $\|M\|_{BMO} < \frac{1}{4}$, then for any stopping time τ*

$$E[\exp(|M_\infty - M_\tau|)|\mathcal{F}_\tau] \leq \frac{1}{1 - 4\|M\|_{BMO}}. \tag{5.2}$$

Corollary 5.3. *[Ka] Let $1 < p < \infty$. There is a positive constant C_p such that for any uniformly integrable martingale M*

$$\|M\|_{BMO} \leq \|M\|_{BMO_p} \leq C_p\|M\|_{BMO}.$$

The following inequality, which is also called the John-Nirenberg inequality, was given by Garsia [Gar] for discrete parameter martingales and by Meyer [Me] for general martingales.

Theorem 5.4. *([Gar], [Me]) If $\|M\|_{BMO_2} < 1$, then for every stopping time τ*

$$E[\exp(\langle M\rangle_\infty - \langle M\rangle_\tau)|\mathcal{F}_\tau] \leq \frac{1}{1 - \|M\|^2_{BMO_2}}. \tag{5.3}$$

The following Remark shows the connection between BMO-functions and BMO-martingales.

Remark 5.1. *Let $D = \{z : |z| < 1\}$ be the unit disc in the complex plane, ∂D its boundary and $m(d\theta)$ the normalized Lebesgue measure on ∂D. An integrable* *real valued function f is in $BMO(\mathbb{R})$ if there exists a positive constant C such that for all intervals $I \subset \partial D$,*

$$\frac{1}{m(I)} \int_I |f - f_I| \, m(d\theta) \leq C,$$

where $f_I = \frac{1}{m(I)} \int_I f \, dm$ and the smallest constant with the previous property is denoted by $\|f\|_*$ is the BMO-norm of a function. Now, let

$$h(z) = \int_0^{2\pi} f(t) P(r, \theta - t) \, m(dt) \quad (z = re^{i\theta} \in D),$$

where $P(r, \eta) = \frac{1-r^2}{1-2r\cos(\eta)+r^2}$ is the Poisson kernel. Then h is the harmonic function in D with boundary function f. Let now $B = B(B_t, \mathcal{F}_t)$ be the complex Brownian motion starting at 0 and let $\tau = \inf\{t : |B_t| = 1\}$. The process $(h(B_{t\wedge\tau}), \mathcal{F}_{t\wedge\tau})$ is a uniformly integrable martingale. In particular, if f is in BMO, then the process $h(B^\tau)$ is a BMO-martingale and there are constants $C_1, C_2 > 0$, independent of f, such that

$$C_1 \|f\|_* \leq \|h(B^\tau)\|_{BMO} \leq C_2 \|f\|_*.$$

Conversely, if X is a uniformly integrable martingale adapted to the filtration $(\mathcal{F}_{t\wedge\tau})$, then there is a unique Borel measurable function f defined on ∂D such that $f(B_\tau) = E[X_\infty | \sigma(B_\tau)]$. Let us consider the mapping $J : X \longrightarrow f$. Then there is a constant C such that

$$\|J(X)\|_* \leq C \|X\|_{BMO}$$

for all BMO-martingales X adapted to the filtration $(\mathcal{F}_{t\wedge\tau})$. The family of all real-valued BMO-functions on ∂D is identified in this way with the family of all BMO-martingales X which have X_∞ measurable with respect to $\sigma(B_\tau)$.

The following Theorem gives a sufficient condition to have a uniformly integrable martingale.

Theorem 5.5. *[Ka] Let M be a martingale in BMO, then $\varepsilon(M)$ is an uniformly integrable martingale.*

Moreover, in general, $\varepsilon(M)$ is not a true martingale, so the previous Theorem gives also a sufficient condition to have an exponential martingale. This fact is very useful when we have to solve mathematical finance problems where we need the Girsanov Theorem about the change of probability measure (see section 5.5).

5.3 Relation between A_p-condition and BMO-martingales in probabilistic setting

The following Definition is the analogous of A_p-condition (see Definition 1.1) in probabilistic setting.

Definition 5.16. Let $1 < p < \infty$. We say that $\varepsilon(M)$ satisfies A_p**-condition** if

$$\sup_{\tau} \|E[\{\varepsilon(M)_\tau/\varepsilon(M)_\infty\}^{1/(p-1)}|\mathcal{F}_\tau]\|_\infty < \infty,$$

where the supremum is taken over all stopping times τ. In particular, if $p = 1$

$$\sup_{\tau} \|\varepsilon(M)_\tau/\varepsilon(M)_\infty\|_\infty < \infty,$$

then we say that it satisfies A_1**-condition**.

The following Theorem shows connections between A_p and BMO-martingales.

Theorem 5.6. *[Ka] The following conditions are equivalent.*

(a) $M \in BMO$.

(b) *$\varepsilon(M)$ satisfies A_p for some $p \geq 1$.*

(c) $\sup_{\tau} \left\| E[\log^+ \frac{\varepsilon(M)_\tau}{\varepsilon(M)_\infty} |\mathcal{F}_\tau] \right\|_\infty < \infty.$

Remark 5.2. *In the case where M is right continuous local martingale satisfying $-1 < \Delta M \leq C$ for some constant $C > 0$, if $\varepsilon(M)$ satisfies A_p for some $p > 1$, then M is a BMO-martingale.*

In the following remark we report the connection between A_p-condition for functions (see Definition 1.1) and probabilistic A_p-condition (see Definition 5.16).

Remark 5.3. *Let $D = \{z : |z| < 1\}$ be the unit disc in the complex plane and let $0 < w \in L^1(\partial D, dm)$, where $m(d\theta)$ denotes the normalized Lebesgue measure on ∂D. Let $B = (B_t, \mathcal{F}_t)$ be the complex Brownian motion starting at 0 and let $\tau = \inf\{t : |B_t| = 1\}$. Then the positive martingale W given by*

$W_t = E[w(B_\tau)|\mathcal{F}_t]$, $(0 \leq t < \infty)$ satisfies $E[W_\infty] = 1$, and further from an important Theorem due to Kakutani it follows that

$$W_t = Pw \cdot (B_t) \text{ on } \{t < \tau\}$$

where Pw is the Poisson integral of w. In this setting Kazamaki [Ka] proved the following Theorem

Theorem 5.7. *If the martingale W satisfies A_p-condition, then the function w is in A_p.*

The converse of the previous Theorem in not true. For example, let $w(t) = |t|^\lambda$ where $1 < \lambda < \infty$. Then $w \in A_p$, for $p > 1 + \lambda$, but W doesn't satisfy any A_p-condition.

Note that only in the case $p = 2$ we have $w \in A_2 \Longleftrightarrow W \in A_2$.

5.4 About the distance in BMO to L^∞

In this section we will give comparable upper and lower bounds for the distance in BMO to L^∞ in probabilistic setting.

For every $M \in BMO$-martingales, let $a(M)$ be the supremum of the set of a for which

$$a(M) = \sup_a \{\sup_\tau \|E[\exp(a|M_\infty - M_\tau|)|\mathcal{F}_\tau]\|_\infty < \infty\},$$

and let d_p be the distance on the space BMO deduced from the norm $\|\cdot\|_{BMO_p}$, by usual procedure. Then there is a very beautiful relation between $a(M)$ and $d_1(M, L^\infty)$ as the following Theorem shows

Theorem 5.8. *([V], [E]) Let $M \in BMO$ be a martingale, then we have*

$$\frac{1}{4d_1(M, L^\infty)} \leq a(M) \leq \frac{4}{d_1(M, L^\infty)}. \tag{5.4}$$

Note that Theorem 5.8 is the probabilistic version of Theorem 4.7 where $a(M)$ "plays the role" of $\dfrac{1}{\epsilon(f)}$.

Next example show us that L^∞ is not dense in BMO.

Example 5.2. We know that there exists a BMO-martingale M such that $\exp M_\infty$ is not integrable. In such a case we have $a(M) \leq 1$, and so using Theorem 5.8, $d_1(M, L^\infty) \geq \frac{1}{4}$. This is an instance where L^∞ is not dense in BMO.

Dellacherie, Meyer and Yor proved in [DMY] that L^∞ is neither closed nor dense in BMO whenever $BMO \neq L^\infty$. In the classical setting Garnett and Jones [Ga] proved for locally integrable function f on $\mathbb{R}^n$ that

$$f \in BMO - \text{closure of } L^\infty \Longleftrightarrow e^f, e^{-f} \in A_p, \ \ \forall p > 1.$$

Now we report a probabilistic analogue of this result. For a uniformly integrable martingale M, let

$$p(M) = \inf\{p > 1 : E[\exp(M_\infty)|\mathcal{F}], E[\exp(-M_\infty)|\mathcal{F}] \in A_p\}. \tag{5.5}$$

From Hölder inequality it follows that $E[\exp(M_\infty)|\mathcal{F}] \in A_p$, $\forall p > p(M)$.

Lemma 5.9. *[Ka] If $p(M) < \infty$, then $p(M) \leq 2$, $M \in BMO$ and*

$$p(M) - 1 = \frac{1}{a(M)}.$$

Lemma 5.9 is the analogous of Lemma 4.8.

Theorem 5.10. *[Ka] Let $M \in BMO$, then*

$$M \in BMO\text{−}\textit{closure of } L^\infty \ (\overline{L^\infty}) \Longleftrightarrow E[\exp(M_\infty)|\mathcal{F}], E[\exp(-M_\infty)|\mathcal{F}] \in A_p, \forall p$$

.

Corollary 5.11. *[Ka] Let $M \in BMO$, then*

$$\varepsilon(M), \varepsilon(-M) \in A_p, \forall p \Longleftrightarrow E[\log \varepsilon(M)_\infty|\mathcal{F}] \in \overline{L^\infty}.$$

Following Theorems show that in general L^∞ is neither closed nor dense in BMO.

Theorem 5.12. *[Ka] The following conditions are equivalent:*

1. *$BMO = L^\infty$*
2. *The filtration $(\mathcal{F}_t)$ is constant, that is, $\mathcal{F}_t = \mathcal{F}_0$, $\forall t > 0$.*

Theorem 5.13. *[Ka] If $\mathcal{F}_t$ is not constant, then L^∞ is not closed in BMO.*

Theorem 5.14. *[Ka] If $\mathcal{F}_t$ is not constant, then L^∞ is not dense in BMO.*

5.5 Application: Mathematical Finance

The aim of this section is to give some ideas about the possible applications of previous results in mathematical finance.

Mathematical Finance is the mathematical theory of financial markets. It tries to develop theoretical models, that can be used by practitioners to evaluate certain data from real financial markets. Roughly speaking a **financial market** is a place where people can buy or sell financial derivatives. A **financial derivative** is a financial contract, whose value at expire is determined by the prices of the underlying financial assets (here we mean Stocks and Bonds).

The most important application of the Itô calculus in financial mathematics is that of option pricing. In this area, the most famous result is the Black-Scholes formula for pricing European put and call options. Options are so-called derivative securities, i.e. securities which are derived from underlying assets.

To understand the following we report some basic notions.

- **call option** is a contract that gives the holder the right (but not the obligation) to buy a fixed amount of an asset at a specified time in future for an already agreed price, the **strike price**, from the seller, also called **writer** of the option.

- **put option** is a contract that gives the holder the right to sell a fixed amount of an asset to the writer of the option for the strike price. Here the writer of the put option is obliged to buy the asset while the holder can decide on selling or not.

- **expiration date** is the date when the contact ceases to exist.

- **American option** is when the holder of the option is free to sell or to buy the asset during the whole timspan of the contract.

- **European option** is when the holder of the option can only exercise his option at maturity of the contract (expiration date).

The extension of BMO functional spaces results to BMO probability spaces has lot of applications in mathematical finance. A first possible application is about the application of Girsanov Theorem about the change of probability measure.

In probability theory, the Girsanov Theorem tells how stochastic processes change under changes in measure. The Theorem is especially important in the theory of financial mathematics as it tells how to convert from the physical measure which describes the probability that an underlying instrument (such as a share price or interest rate) will take a particular value or values to the risk-neutral measure which is a very useful tool for evaluating the value of derivatives on the underlying.

Theorem 5.15. *[O] (**Girsanov Theorem**) Let $B(t)$, $0 \leq t \leq T$, be a Brownian motion on $(\Omega, \mathcal{F}, P)$ and let $\Theta(t)$, $0 \leq t \leq T$, be a stochastic process adapted to the filtration $\mathcal{F}_t$. Let us set*

$$\widetilde{B}(t) = \int_0^t \Theta(s)ds + B(t), \quad 0 \leq t \leq T$$

and

$$Z(t) = e^{\left\{-\int_0^t \Theta(s)ds - \frac{1}{2}\int_0^t \Theta^2(s)ds\right\}}$$

Assume that $\Theta(s)$ satisfies "Novikov's condition"

$$E\left[e^{\left(\frac{1}{2}\int_0^T \Theta^2(s)ds\right)}\right] < \infty.$$

Define the new probability measure $\widetilde{P}$:

$$\forall A \in \mathcal{F},\ \widetilde{P}(A) = \int_A Z(t)dP.$$

Then $\widetilde{B}(t)$ is a Brownian motion with respect to the new probability law $\widetilde{P}$, $\forall t \leq T$.

Remark 5.4. *Novikov's condition is sufficient to guarantee that Z_t is a martingale. Actually, the result holds if we only assume that Z_t is a martingale (see [KS]).*

By previous Remark we can write Girsanov Theorem in the following way:

Theorem 5.16. *Let $B(t)$, $0 \leq t \leq T$, be a Brownian motion on $(\Omega, \mathcal{F}, P)$ and let $\Theta(t)$, $0 \leq t \leq T$, be a stochastic process adapted to the filtration $\mathcal{F}_t$. Let us set*

$$\widetilde{B}(t) = \int_0^t \Theta(s)ds + B(t), \quad 0 \leq t \leq T$$

and

$$Z(t) = e^{\left\{-\int_0^t \Theta(s)ds - \frac{1}{2}\int_0^t \Theta^2(s)ds\right\}}.$$

If Z_t is a martingale, then $\widetilde{B}(t)$ is a Brownian motion with respect to the new probability law $\widetilde{P}$, $\forall t \leq T$, where $\widetilde{P}(A) = \int_A Z(t)dP$, $\forall A \in \mathcal{F}$.

Remark 5.5. *So the important hypothesis in Girsanov Theorem is that Z_t is a martingale. We know from Theorem 5.5 that if $\Theta(t) \in BMO$, then Z_t is an uniformly integrable martingale.*

Why Girsanov Theorem is so important in Mathematical finance? Because by Girsanov transformation we can pass to the risk-neutral measure. A risk-neutral measure is the probability measure that results when one assumes that the future expected value of all financial assets are equal to the future payoff of the asset discounted at the risk-free rate. In other words, in an actual economy, the price of assets is affected by the amount investors are willing to pay to assume or eliminate risk. However, it is sometimes possible to calculate the prices of asset assuming that there was no risk. When the asset prices are corrected so that there is no risk, the probability that result are those of the risk-neutral measure. The measure is so-called because, under that measure, all financial assets in the economy have the same expected rate of return, regardless of the riskiness - i.e. the variability in the price - of the asset. This is in contrast to the physical measure - i.e. the actual probability distribution of prices where (almost universally) more risky assets (those assets with a higher price volatility) have a greater expected rate of return than less risky assets.

Another name for the risk-neutral measure is the **equivalent martingale measure**. A particular financial market may have one or more risk-neutral measures. If there is just one then there is a unique arbitrage-free price for each asset in the market. This is the fundamental theorem of arbitrage-free pricing. If there is more than one such measure then there is an interval of prices in which no arbitrage is possible. In this case the equivalent martingale measure terminology is more commonly used.

Another application of BMO-martingales results to mathematical finance is, for example, the approximation of stochastic integrals by integrals over piece-wise constant integrands. This necessity is very useful in financial problems because of, very often, one pass from continuous problems to discrete problems.

In a recent paper of Geiss [Ge] we can see how one can use BMO estimates in mathematical finance. The problem analyzed in [Ge] is a classic problem of stochastic finance in continuous time in a Black-Scholes context: an investor who has to continuously re-balance the payoff of his portfolio. The wealth process associated to the portfolio is analytically represented by a stochastic integral with respect to the discounted stock price process under the risk-neutral measure. In practise is impossible re-balanced continuously a portfolio so one can replaced it by the pay-off of a portfolio re-balanced at finitely many trading dates only. The approximation error between the stochastic integral and its approximation by discretization, can be interpreted as risk.

Usually the approximation error is measured in a distributional way with respect L^2 estimate. This approach gives some problems like the fact that the resulting tail-estimates are rather weak. So Geiss used spaces of weighted bounded mean oscillation (weighted BMO) that provide much more information for the purpose. He used the following norm

Definition 5.17. [Ge] Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{X_t\}_{t\in T}$ be an adapted stochastic process, with $X_0 = 0$ and $\{\Phi_t\}_{t\in T}$ be a geometrical Brownian motion with $\Phi_t > 0$, $\forall t \in T$, then

$$\|X\|_{BMO}^{\Phi} = \sup_{\tau} \left\| E\left[\frac{|X_T - X_\tau|^p}{\Phi_\tau^p} |\mathcal{F}_\tau \right] \right\|_\infty^{\frac{1}{p}},$$

where the supremum is taken over all stopping times τ.

In the paper was shown that BMO-spaces are of advantage because of two principal reasons. The first is that in general estimates with respect to BMO-spaces imply L^p estimates and secondly by a weighted John-Nirenberg type Theorem one can obtain significant better tail-estimates than we would get from L^2 estimates.

Bibliography

[ACS] Alberico, T., Corporente, R. and Sbordone, C., *Explicit bounds for composition operators preserving* $BMO(\mathbb{R})$, Georgian Mathematical Journal, 14 (2007), no. 1, 21-32.

[AS] Alberico, T. and Sbordone, C., *A precise interplay between BMO space and* A_2 *class in dimension one*, Mediterranean Journal of Math., 4 (2007), no. 1, 45-51.

[BDS] Bennett, C., De Vore, R.A. and Sharpley, R., *Weak-*L^∞ *and BMO*, Ann. Math., 113 (1981), no. 2, 601-611.

[Be] Besicovitch, A., *A general form of the covering principle and relative differentiation of additive functions*, Fund. Math. 41 (1945), 103-110.

[BA] Beurling, A. and Ahlfors, L., *The boundary correspondence under quasiconformal mappings*, Acta Math. 96 (1956), 125-142.

[Bo] Boiarsky, B., *Remarks on the stability of Reverse Hölder inequalities and quasiconformal mappings*, Ann. Acad. Sci. Fenn., A. Math., 10 (1985), 921-926.

[BSW] Boiarski, B., Sbordone, C. and Wik, I., *The Muckenhoupt class* $A_1(\mathbb{R})$, Studia Math., 101 (2) (1992), 155-163.

[B] Boyd, D.W. , *Indices for the Orlicz spaces*, Pacific J. Math. 38 (1971), 315-323.

[Bu] Buckley, S.M., *Estimates for operator norms on weighted spaces and reverse Jensen inequalities*, Trans. Amer. Math. Soc. 340 (1993), 253-272.

[CKL] Capogna, L., Kenig, C. and Lanzani, L., *Harmonic Measure*, University Lecture Series, Amer. Math. Soc., 35 (2005).

[CF] Capone, C. and Fiorenza, A., *Maximal inequalities in weighted Orlicz spaces*, Rend. Acc. Sc. Fis. Mat. Napoli, (4) 62 (1995), 213-224.

[CS] Carozza, M. and Sbordone, C. *The distance to L^∞ in some function spaces and applications*, Diff. Integr. Equations, 10 (4) (1997), 599-607.

[CoFe] Coifman, R. and Fefferman, C., *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. 51 (1974), 241-250.

[C] Corporente, R., *A precise relation among A_∞ and G_1 constants in one dimension*, Rend. Acc. Sci. Fis. e Mat. di Napoli, LXXII (2005), 65-70.

[DS] D'Apuzzo, L. and Sbordone, C., *Reverse Hölder inequalities: a sharp result*, Rend. Mat. Appl. (7) 10 (1990), no. 2, 357-366.

[DMSSS] Delbaen, F., Monat, P., Schachermayer, W., Schweizer, M., Stricker, C., *Weighted norm inequalities and closedness of a space of stochastic integrals*, C. R. Acad. Sci. Paris Sr. I Math. 319 (1994), no. 10, 1079-1081.

[DeS] Delbaen, F. and Schachermayer, W., *Applications to mathematical finance*, Handbook of the geometry of Banach spaces, Vol. I, North-Holland, Amsterdam, (2001), 367-391.

[DMY] Dellacherie, C., Meyer, P.A. and Yor, M. *Sur certaines propriétés des espaces de Banach H^1 et BMO.* (French) Séminaire de Probabilités, XII (Univ. Strasbourg, Strasbourg, 1976/1977), pp. 98-113, Lecture Notes in Math., 649, Springer, Berlin, 1978.

[DV] Dragičević, O. and Volberg, A. , *Sharp estimate of the Ahlfors-Beurling operator via averaging martingale transform*, Michigan Math. J. 51 (2003), no. 2, 415-435.

[DGPP] Dragičević, O., Grafakos, L., Pereyra, M.C. and Petermichl, S., *Extrapolation and sharp norm estimate for classical operators on weighted Lebesgue spaces*, Publ. Mat. 49 (2005), no. 1, 73-91.

[D] Duoandikoetxea, J., *Fourier analysis*, Graduate Studies in Mathematics, 29. American Mathematical Society, Providence, RI, (2001). Translated and revised from the 1995 Spanish original by David Cruz-Uribe.

[E] Émery, M., *Le Théorème de Garnett-Jones d'après Varopoulos*, Séminaire de Probabilités XV, Université de Strasbourg (Lecture Notes in Math. 721), Berlin Heidelberg New York, Springer (1985), 278-284.

[ESY] Émery, M., Stricker, C. and Yan, J. A. *Valeurs prises par les martingales locales continues un instant donn*, [Values taken by continuous local martingales at a given moment] Ann. Probab. 11 (1983), no. 3, 635-641.

[Fe] Fefferman, C., *Characterizations of bounded mean oscillation*, Bull. Amer. Math. Soc., 77 (1971), 587-588.

[F] Fiorenza, A., *An inequality for Jensen means*, Nonlinear Anal. 16 (1991), no. 2, 191-198.

[F2] Fiorenza, A. and Krbec, M. ,*A formula for the Boyd indices in Orlicz Spaces*, Funct. Approx. Comment. Math. 26 (1998), 173-179.

[FK] Fiorenza, A. and Krbec, M. , *Indices of Orlicz spaces and some applications*. Comment. Math. Univ. Carolin. 38 (1997), no. 3, 433-451.

[FLS] Fusco, N., Lions, P.L. and Sbordone, C., *Some remarks on Sobolev imbeddings in borderline cases*, Proc. Amer. Math. Soc., 124 (2) (1996), 561-565.

[GR] Garcia Cuerva, J. and Rubio de Francia, J., *Weighted Norm Inequalities and Related Topics*, North Holland, (1985).

[Ga] Garnett, J., *Bounded Analytic Functions*, Academic Press (1981).

[GJ] Garnett, J. and Jones, P., *The distance in BMO to* L^{∞}, Annals of Mathematics, 108 (1978), 373-393.

[Gar] Garsia, A. M., *Martingale inequalities: Seminar notes on recent progress*, Mathematics Lecture Notes Series., W. A. Benjamin, (1973).

[GRo] Garsia, A.M. and Rodemich, E. , *Monotonicity of certain functionals under rearrangement*, Ann. Inst. Fourier, Grenoble, 24 (1974), no. 2, 67-116.

[G] Gehring, F. W., *The L^p-integrability of the partial derivatives of a quasiconformal mapping*, Acta Math., 130 (1973), 265-277.

[Ge] Geiss, S., *Weighted BMO and discrete time hedging within the Black-Scholes model*, Probab. Theory Relat. Fields, 132 (2005), 13-38.

[GS] Getoor, R. K. and Sharpe, M. J., *Conformal martingales*, Invent. Math. 16 (1972), 271-308.

[Go] Gotoh, Y. , *On composition operators which preserve BMO*, Pacific Journal of Mathematics, 201 (2001), no. 2, 289-307.

[HL] Hardy, G. H. and Littlewood, J. E., *A maximal theorem with function-theoretic applications*, Acta Math., 54 (1930), no. 1, 81-116.

[HSV] Harboure, E. , Salinas, O. and Viviani, B., *Reverse-Hölder classes in the Orlicz spaces setting*, Studia Math., 130 (1998), no.3, 245-261.

[H] Hruscev, S. V. , *A description of A_∞ condition of Muckenhoupt*, Proc. Amer. Soc., 90 (1984), 253-257.

[Hu] Hukovic, S. , PhD Thesis, Brown University (1998).

[HTV] Hukovic, S., Treil, S. and Volberg, A., *The Bellman function and sharp weighted inequalities for square functions*, In "Complex analysis, operators and related topics", Oper. Theory Adv. Appl., 113 (2000) , 97-113.

[HMW] Hunt, R. , Muckenhoupt, B. and Wheeden, R. *Weighted norm inequalities for the conjugate function and Hilbert transform*, Trans. Amer. Math. Soc., 176 (1973), 227-251.

[J] Jawerth, B.,*Weighted inequalities for maximal operators: linearization, localization and factorization*, Amer. J. Math., 108 (1986) , no. 2, 361-414.

[JN] John, F. and Nirenberg, L., *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. 14 (1961), 415-426.

[JN1] Johnson, R. and Neugebauer, C. J. , *Homeomorphisms preserving A_p*, Rev. Mat. Iberoamericana, 3 (1987), 249-273.

[JN2] Johnson, R. and Neugebauer, C. J., *Change of variable results for A_p- and reverse Hölder* RH_r*-classes*, Trans. Amer. Math. Soc., 328 (1991), no. 2, 639-666.

[Jo] Jones, P.W., *Homeomorphisms of the line which preserve BMO*, Ark. Mat. 21 (1983), no. 2, 229-231.

[Ka] Kazamaki, N., *Continuous exponential martingales and BMO*, Lecture Notes in Mathematics, 1579. Springer-Verlag, Berlin, (1994).

[KS] Karatzas, I. and Shreve, S.E., *Brownian Motion and Stochastic Calculus*, Springer-Verlag (1991) .

[K] Kenig, C.E., *Harmonic analysis techniques for second order elliptic boundary value problems*, CBMS Regional Conference Series in Mathematics, 83. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence (1994).

[Ko] Korenovskii, A.A., *The exact continuation of a Reverse Holder Inequality and Muckenhoupt's conditions*, Math. Notes, 52 (1992), (5-6), 1192-1201.

[Kor] Korenovskii, A.A., *Mean oscillations and equimeasurable rearrangements of functions*, Lecture Notes of the Unione Matematica Italiana, vol. 4, (2007).

[KR] Krasnosel'skiĭ, M. A. and Rutickiĭ, Ja. B. , *Convex functions and Orlicz spaces*, Translated from the first Russian edition by Leo F. Boron. P. Noordhoff Ltd., Groningen (1961).

[KT] Kerman, R.A. and Torchinsky, A., *Integral inequalities with weights for the Hardy maximal function.* Studia Math. 71 (1981/82), no. 3, 277-284.

[Me] Meyer, P. A., *Un cours sur les intgrales stochastiques*, Sminaire de Probabilits, X ,Lecture Notes in Math., 511, Springer, Berlin, (1976).

[M] Migliaccio, L., *Some characterizations of Gehring G_q-class*, Houston J. of Mathematics, 19 (1993) no.1, 89-95.

[MS] Moscariello, G. and Sbordone, C., *A_∞ as a limit case of reverse-Hölder inequalities when the exponent tends to* 1, Ricerche Mat., 44 (1995), 131-144.

[M1] Muckenhoupt, B., *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 165 (1972), 207-226.

[M2] Muckenhoupt, B., *The equivalence of two conditions for weight functions*, Studia Math., 49 (1974), 101-106.

[O] Oksendal, B. , *Stochastic Differential Equations*, Springer-Verlag Heidelberg New York (1998).

[P] Perez, C., *Weighted Norm Inequalities for Potential and Maximal Operators*, PhD Thesis, (1989).

[Pe] Petermichl, S., *The sharp bound for the Hilbert transform on weighted Lebesgue spaces in terms of the classical A_p-characteristic*, preprint, (2006).

[Pe2] Petermichl, S., *The sharp weighted bound for the Riesz transforms*, preprint, (2006).

[PP] Petermichl, S. and Pott, S. , *An esimate for weighted Hilbert transform via square functions*, Trans. Amer. Math. Soc. 351 (2001), 1699-1703.

[PV] Petermichl, S. and Volberg, A., *Heating of the Beurling operator: weakly quasiregular maps on the plane are quasiregular*, Duke Math. J. 112 (2002), no.2, 281-305.

[Po] Popoli, A. , *Optimal integrability in B_p^q classes*, Le Matematiche, LII (1997), no.1, 159-170.

[S] Sbordone, C., *Some reverse integral inequalities*, Atti Acc. Pontaniana, Napoli,(1984).

[S1] Sbordone, C., *Sharp embeddings for classes of weights and applications*, Rendiconti Accademia Nazionale delle Scienze, detta dei XL, (2006).

[S2] Sbordone, C., *Rearrangement of functions and reverse Hölder inequalities*, Pitman Research Notes in Math. 125 (1983), 139-148.

[SW] Sbordone, C. and Wik, I., *Maximal functions and related weight classes*, Publ. Math. 38 (1994), 127-155.

[StWh] Stromberg, J. O. and Wheeden, R. L., *Fractional integrals on weighted H^p and L^p spaces*, Trans. Amer. Math. Soc. 287 (1985), 293-321.

[T] Torchinski, A., *Real Variable Methods in Harmonic Analysis*, Acad. Press (1986).

[Va] Varopoulos, N. Th., *A probabilistic proof of the Garnett-Jones theorem on BMO*, Pacific J. Math. 90 (1980), no. 1, 201-221.

[V] Vasyunin, V., *The sharp constant in the reverse Hölder inequality for Muckenhoupt weights*, St. Petersburg Math. J., 15 (2004), 49-79.

[W] Wik, I., *On Muckenhoupt's classes of weight functions*, Studia Math., XCIV (1989), 245-255.

[Wi] Wik, I. , *Reverse Hölder inequalities*, in "Methods of Real Analysis and Partial Differential equations" Quaderno n.14 Accademia Pontaniana, Napoli (1992), 131-137.

[W1] Wick, B.D., *Little BMO and Muckenhoupt weights in product domains*, preprint.