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STOCHASTIC SELF-SIMILAR PROCESSES AND LARGE SCALE STRUCTURES

TESI DI DOTTORATO DI RICERCA

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IL COORDINATORE
Prof. Aldo De Luca

Per tre cose vale la pena di vivere:
la matematica, la musica e l'amore

Renato Caccioppoli

To my family

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Marta Chinnici

Naples, November 2007

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INTRODUCTION

La Natura é un libro scritto
in caratteri matematici

Galileo Galilei

The geometry of objects in Nature ranging in size from the atomic scale to the size of the universe is central to models we develop in order to "understand Nature". The geometry of particle trajectories of hydrodynamic (flow lines, waves, ships and shores), landscapes (mountains, islands, rivers, glaciers and sediments), grains in rock (metals and composite materials), plants, insects and cells, as well as the geometrical structure of crystals, chemicals and proteins, in short, the geometry of nature is so central to the various fields of natural science that we tend to take the geometrical aspects for granted. Each field tends to develop adapted concepts (e.g. morphology, four-dimensional spaces, texture, etc...) used intuitively by the scientists in that field. Then, in order to understand the geometry of natural objects, mathematicians have developed recently geometrical concepts that transcend traditional geometry. Hence, the traditional Euclidean lines, circles, spheres and tetrahedra are inappropriate to describe shapes in Nature.

The most relevant author in this field has been Benoit B. Mandelbrot, who conceived and developed a new geometry of Nature. Through his creative and monumental work, he has generated a widespread interest in *Fractal Geometry* - concept introduced by Mandelbrot himself. His book *The Fractal Geometry of Nature* [3] is the standard reference on such matter and contains both the elementary concepts and broad range of new and rather advanced ideas, such as fractals and multifractals, currently under active study. In particular he presented the concept of fractal, “a shape made of parts similar to the whole in some way", in an unusually inspring way.

Fractal geometry is an extension of classical geometry and provides a general framework for the study of irregular sets. It can be used to develop models of physical structures from ferns to galaxies.

In the last years, we find out an increasing interest in the use of *fractal* and *multifractal* concepts. Indeed, they are usually introduced with the help of rain and turbulent phenomenology, as well as with the help of very simple toy models. However, the fractals thanks to their original fascinating and beautiful form, are able to link "Art and Mathematics". Surely, in this term we want to talk about the beauty of mathematics. Indeed, we talk about the visual art, the architecture (like Hindi Prambanan Temple) and modern architecture (like Amsterdam housing by MVRDV in 1995), the painting (see Pollock's fractal painting), the wanderful drawings of M.C. Escher (1898-1972), the music as in a part of J. S. Bach (1685-1750) known as the "Trias Harmonica for 8 canon instruments", etc.

Fractals can often be regarded as special cases of continuous or discrete multifractals. The concept of multifractals, which are spatially interwined fractals, has replaced the concept of fractals which are now often referred to as unifractals (or mixing the Latin with the Greek:"monofractals"). Now that fractal and multifractals have become well established as practical tools, it has also become apparent that, in nature, there are often situations that a single, relatively simple fractal or multifractal model does not apply and mixtures of models or other types of generalizations are required.

In particular, we focus our attention on an important property of fractals, called "self-similarity". Indeed, a fractal is a geometric object that possesses the property of self-similarity, often combined with non-integer dimensions.

The term *self-similar* was formally defined by Mandelbrot (see [3]) to describe the phenomenon where a certain property of object is preserved with respect to scaling behavior in space and time. The scaling behavior can be defined as a property of scale invariance, that is, when there is no controlling characteristic or when all scales have equal importance. The basic feature of self-similar process is the scale invariance, in the sense that it is identical in terms of distribution to any of its rescaled version, up to some suitable renormalization factor, which depends on a self-similarity parameter. In other words, we can say that fractals may describe shapes by iteration of a very simple rule of self-similarity. For example, the classical fractals, as Von Koch's snow-flake and the third Cantor, are literally self-similar because their parts resemble the shape of the whole, that is, they are the smaller copies of themselves. For this reason, the concept of self-similarity is intimately linked to fractals. Many natural phenomena exhibit some sort of self-similarity and scientists have applied self-similarity models to many areas including image processing (fractal image compression and segmentation), dynamical systems (turbulence), biology and medicine (physical time series), etc..

The notion of self-similarity, however, is also widely used in literature; especially, in this dissertation, we consider the notion of self-similarity is linked to stochastic processes. Indeed, the self-similar stochastic processes are the main topic of this Ph.D. thesis.

Although *Self-Similar Stochastic Processes* were first introduced in a theoretical context by Kolmogorov in 1941, statisticians were made aware of the practical applicability of such processes through the work of B. B. Mandelbrot (Mandelbrot and Van Ness (1968)).

In detail, a "*stochastic process* $Y(t)$ is a self-similar process with self-similarity parameter H if for any positive stretching factor c, the distribution of the rescaled and reindexed process $c^{-H}Y(ct)$ is the same as that of the original process $Y(t)$". The value of the self-similarity parameter or scaling exponent H dictates the dynamic behavior of a self-similar process $Y(t)$. It is well known that *Brownian motion* is self-similar and also *Fractional Brownian motion*, which is a Gaussian self-similar process with stationary increments, was first discussed by Kolmogorov.

We had been personally motivated by recent applications of stochastic self-similar processes in cosmology. What is the geometry of the Universe? Has the Universe a memory of its quantum and relativistic origin? In the present work, we consider that the formation of structures of the Universe appears as if it were classically self-similar stochastic process at all astrophysical scales [50]. The observations shows that Universe has structures with scaling rules, where the clustering properties of cosmological objects reveal a form of hierarchy. In a number of valuable papers on the subject, the segregated Universe has been presented as the result of a fundamental self-similar law. This results and many other are seen in the context of El Naschie E-Infinity ($\epsilon^{(\infty)}$) Cantorian space-time.

In particular, reading El Naschie's papers, E-Infinity appears to be clearly a new framework for understanding and describing Nature. Indeed, Nature clearly appears not continuous, not periodic, but self-similar and Mohamed El Naschie with $\epsilon^{(\infty)}$ has introduced a mathematical formulation to describe phenomena that are resolution dependent. As reported by the author, $\epsilon^{(\infty)}$ space-time is an infinite dimensional fractal, which has $D = 4$ as the expectation value for the topological dimension [47], [48], [49]. The topological value $3 + 1$ means that in our low energy resolution, the world appears to us as if it were four-dimensional. This is a sweeping generalization of what Einstein did in his general theory of relativity. El Naschie introduced a new geometry for space-time which differs considerably from the space-time of our sensual experience. Consequently, entirely depends on the energy scale through which we are making our observation. Observations of large scale structures show that the dimension changes if we consider different energies, corresponding to different lengths-scale in Universe, as reported in [50], [51], [52].

The purpose of this Ph.D. thesis is to investigate the properties of the involved stochastic self-similar processes in the context of M. El Naschie's $\epsilon^{(\infty)}$ Cantorian space-time and to present new results obtained by using the fractal and multifractal properties within this scenario. In particular, this work focuses on the fractal aspect of Brownian motion which is the topic *pivot* of the discussion. In this dissertation, we analyze the stochastic processes in a fixed framework applying them to large-scale phenomena, modeling the natural scaling phenomenon in the context above mentioned.

The structure of Ph.D. thesis is as follows. Chapter 1 includes a short review of general self-similar stochastic processes theory; moreover, the self-similar theme is analyzed in order to connect brownian motion and fractals.

Chapter 2 presents the fractal dimensions along with some classical tools, such as Hausdorff dimension and multifractal formalism; it also includes an introduction to $\epsilon^{(\infty)}$ Cantorian space-time, together with some classical material needed to understand the other chapters. A number of excellent books and papers had been written on these subjects, some of them have been discussed briefly in the notes and comments where, in particular, references and credits are given.

In Chapter 3, some results are presented by using the context of $\varepsilon^{(\infty)}$ Cantorian space-time in connection with stochastic self-similar processes in order to give a possible explanation of the segregation of the Universe at fixed scale in terms of brownian motion.

Finally, Chapter 4 presents the analysis of Multifractals in the context of El Naschie's $\epsilon^{(\infty)}$ Cantorian space-time applied to cosmology. In detail, it summarizes some recent results concerning fractal structures and brownian paths in order to calculate the fractal dimensions and the characteristic parameters for large scale structures and for the atomic elements that live in El Naschie's $\epsilon^{(\infty)}$ Cantorian space-time. In this framework, brownian paths play a crucial role if considered as multifractals.

1. SELF-SIMILAR STOCHASTIC PROCESSES

The self-similarity of many natural phenomena has recently generated much interest in the representation and in the properties's analysis of random sets and functions scale invariant. Indeed, these natural objects are usually referred to as random fractal sets, self-similar processes and multifractals. Self-similar processes such as fractional Brownian motion are stochastic processes that are invariant in distribution under suitable scaling of time and space. These processes also enter in the analysis of random phenomena exhibiting certain forms of long-range dependence naturally. Obviously, these processes are closely related to the notion of renormalization in statistical and high energy physics. They are also increasingly important in many other fields of applications, such as economics and finance.

1.1 The mathematical tools of Self-similar stochastic processes

The concept of self-similarity is intimately linked to fractals; indeed, the term "self-similar" was formally defined by Mandelbrot [1] to describe the phenomenon where a certain property of object is preserved with respect to scaling behavior in space and time. The scaling behavior can be defined as a property of scale invariance, that is, when there is no controlling characteristic or when all scales have equal importance.

Further applications and references on the theory of self-similar processes, can be found Mandelbrot[3] and in the extensive bibliography within Taqqu in [2]'s work.

Indeed, the basic idea of self-similarity is simple: a set C is called a self-similar set if it is a union of smaller and smaller copies of itself.

Example 1.1.1: The classical fractals,

- The von Koch's snow flake consists of four copies of itself each one contracted by a factor of 1/3;

- The Sierpinski gasket consists of three copies of itself, each one contracted by a factor of 1/2;

are literally self-similar in the sense that their parts resemble the shape of the whole, that is, they are smaller copies of themselves.

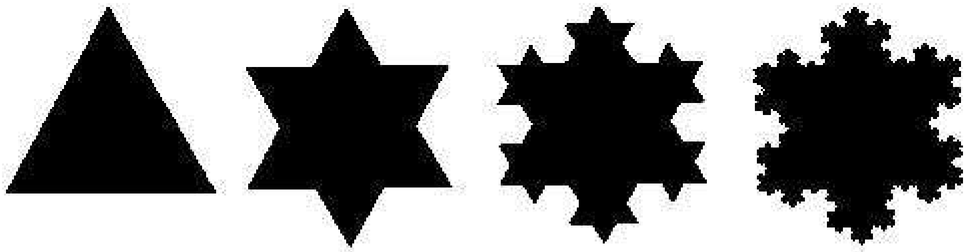

Fig. 1.1: Construction of a Fractal Snowflake: the first four stages in the construction of the Koch Snowflake.

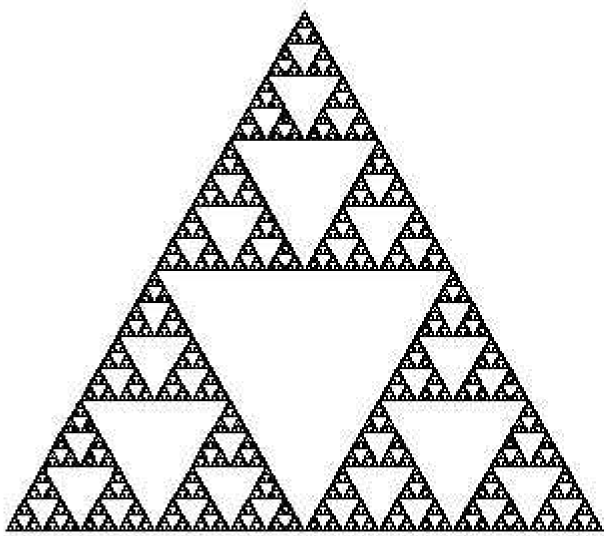

Fig. 1.2: The Sierpinski gasket: a simple fractal produced by breaking up a triangle into successively smaller ones.

In Nature, there are many examples of non-deterministic fractals that are self-similar in a statistical sense over a wide range of scales. The scaling processes, or self-similar processes are largely used to describe these natural phenomena, such as landscape structure and texture, network traffic [4], [5] and, lately, the hierarchy of Universe [50]. The basic feature of self-similar processes is the scale invariance, in the sense that, it is identical in terms of distribution, to any of its rescaled version, up to some suitable renormalization factor, that depends on a self-similarity parameter.

The mathematical and statistical properties of self-similar processes that we are going to outline are dealt with in [6].

The notion of self-similarity is not merely an intuitive description, but a precise concept captured by the following rigorous mathematical definition.

In all this chapter, $(\Omega, \mathcal{F}, P)$ is a fixed probability space. Let $\{X(t),\, t \geq 0\}$ and $\{Y(t),\, t \geq 0\}$ be $\Re^d$-valued stochastic processes on this space. In the following, by $\{X(t)\} \stackrel{d}{=} \{Y(t)\}$, we mean the equality of all finite-dimensional distributions (i.e. scaling of time is equivalent to an appropriate scaling of space).

Definition 1.1.1: An $\Re^d$-valued stochastic process $\{X(t), t \geq 0\}$, is said to be self-similar if for any $a > 0$, there exists $b > 0$ such that

$$\{X(at)\} \stackrel{d}{=} \{bX(t)\} \tag{1.1.1}$$

We recall that $\{X(t),\, t \geq 0\}$ is *stochastically continuous* at t, if for any $\epsilon > 0$,

$$\lim_{h \longrightarrow 0} P\{|X(t+h) - X(t)| \, > \epsilon\} = 0$$

and we also say that it is *trivial* if $X(t)$ is a constant almost surely for every t.

In more recent literature, self-similar processes are usually defined as stochastic processes $\{X(t),\, t \geq 0\}$ for which a parameter $H > 0$ exists such that for any $a > 0$, $\{X(at)\} \stackrel{d}{=} \{a^H X(t)\}$; in this case it is obvious that $X(0) = 0$ a.s., but the uniqueness of the exponent H is not obvious. Hence, one might want to consider a seemingly more general notion of self-similar processes, viz the definition 1.1.1, and to show that for an $\Re^d$-valued stochastic process, stochastically continuous at $t = 0$ and nontrivial, an unique scaling index H exists, in order to connect the coefficient a with b.

The parameter $H > 0$ is called the *Hurst index*; it is the exponent of self-similarity of the process. For this reason, we refer to such a process as H-selfsimilar (or H-ss, for short).

Before giving the Theorem, we start with an easy Lemma.

Lemma 1.1.1: If X is a nonzero random variable in $\Re^d$, and if $b_1 X \overset{d}{\sim} b_2 X$[1] with $b_1, b_2 > 0$, then $b_1 = b_2$.

Proof. Let us suppose that $0 < b_2 < b_1$, we consider the parameter $b = \frac{b_1}{b_2}$, then $X \overset{d}{\sim} bX$ with $b \in (0,1)$. Hence, for the mathematical induction, $X \overset{d}{\sim} b^n X$ [2], for any $n \in N$. For $n \longrightarrow \infty$ we have $X \overset{d}{\sim} 0 \Longrightarrow X = 0$ almost surely, which is a contradiction. ∎

Consequently, the following Theorem holds.

Theorem 1.1.1: If $\{X(t),\, t \geq 0\}$ is nontrivial, stochastically continuous at $t = 0$ and self-similar, then there exists a unique $H \geq 0$ such that b in (1.1.1) can be expressed as $b = a^H$.

Proof. Let us suppose $X(at) \overset{d}{\sim} b_1 X(t) \overset{d}{\sim} b_2 X(t)$. By non-triviality of $\{X(t)\}$ exists a t such that $X(t)$ is nonzero, and by previous Lemma we have $b_1 = b_2$. Consequently, the coefficient b in (1.1.1) is uniquely determined by a, then $b = b(a)$. Then we show the monotonicity of $b(a)$ and the following equality

$$b(aa') = b(a)b(a'). \tag{1.1.2}$$

Then

$$\begin{aligned} &X(aa't) \overset{d}{\sim} b(a)X(a't) \overset{d}{\sim} b(a)b(a')X(t), \\ &X(aa't) \overset{d}{\sim} b(aa')X(t) \end{aligned}$$

so the (1.1.2).

Consequently, we are going to prove only the monotonicity of $b(a)$. Let us suppose that $a < 1$, by (1.1.2) we obtain

$$X(a^n) \overset{d}{\sim} b(a)^n X(1),$$

[1] where $\overset{d}{\sim}$ means the equality of marginal distributions.

[2] **Proof.** For $n = 1$ is trivial, indeed we have $X \overset{d}{\sim} bX$. We suppose it true for some m, and we want it prove for $m+1$: $X \overset{d}{\sim} b^m X \Longrightarrow bX \overset{d}{\sim} b \cdot b^m X \Longrightarrow X \overset{d}{\sim} bX \overset{d}{\sim} b \cdot b^m X$, so $X \overset{d}{\sim} b^{m+1} X$. In detail, we have $X \overset{d}{\sim} b^n X$, $\forall x \in N$. ∎

but since for $n \longrightarrow \infty$, $X(a^n) \longrightarrow X(0)$ in probability. By the stochastic continuity of $\{X(t)\}$ at $t = 0$ result $b(a) \leq 1$.

If we consider $a_1 < a_2 \Longrightarrow \frac{a_1}{a_2} < 1$, according to previous considerations $b(\frac{a_1}{a_2}) \leq 1$; furthermore by (1.1.2), we have $b(\frac{a_1}{a_2}) = \frac{a_1}{a_2} \leq 1$, then $b(a_1) \leq b(a_2)$. We can now conclude that $b(a)$ is non-decreasing.

Thus $b(a) = a^H$ for some unique constant $H \geq 0$. Indeed, this exponent H is unique because if we consider another exponent $H' \geq 0$, we have $b(a) = a^H = a^{H'} \Longrightarrow H = H'$. ■

By Theorem 1.1.1 the following results stem from.

Proposition 1.1.1: If $\{X(t),\, t \geq 0\}$ is H-self-similar process and $H > 0$, then $X(0) = 0$ almost surely.

Proof. Taking into consideration (1.1.1), it follows that $X(0) \overset{d}{\sim} a^H X(0)$ and for $a \longrightarrow 0$, we obtain $X(0) = 0$. ■

Since the previous Proposition does not hold when $H = 0$, in this case we have the following Theorem.

Theorem 1.1.2: Under the same assumptions of Theorem 1.1.1, $H = 0$ if and only if $X(t) = X(0)$ almost surely for every $t > 0$.

Proof. Let us suppose $X(t) = X(0)$ a.s. for every $t > 0$, by property of H-ss, we have

$$\{X(at)\} \overset{d}{=} \{a^H X(t)\} = \{a^H X(0)\} \overset{d}{=} \{X(0)\} = \{X(t)\}, \qquad \forall a > 0$$

consequently, $\{X(at)\} \overset{d}{=} \{X(t)\}, \forall a > 0$, then $H = 0$.

Conversely, if $H = 0$, by the property 0-ss, $\{X(at)\} \overset{d}{=} \{X(t)\}, \forall a > 0$; then the joint distributions at $t = 0$ and $t = s/a$ are the same:

$$(X(0), X(s)) \overset{d}{\sim} (X(0), X(s/a)),$$

Then, for any $\varepsilon > 0$ and $a > 0$ we have

$$P\{|X(s) - X(0)| > \varepsilon\} = P\{|X(s/a) - X(0)| > \varepsilon\};$$

then from the stochastic continuity of X at 0, it follows that X is trivial and we have

$$\lim_{a\longrightarrow\infty} P\{|X(s/a) - X(0)|\ > \varepsilon\} = 0.$$

Hence, for each $s > 0$

$$P\{|X(s) - X(0)|\ > \varepsilon\} = 0, \qquad \forall\varepsilon > 0,$$

so that $X(s) = X(0)$ almost surely. ∎

From the above considerations, it seems natural to consider only self-similar processes that fulfill the assumptions of Theorem 1.1.1.

Self-similar processes cannot be stationary, but they are strongly related to stationary processes through a non linear time change, as the following *Lamperti trasformation* shows.

Theorem 1.1.3: If $\{X(t),\ t \geq 0\}$ is H-ss, then the process $Y(t) = e^{-tH}X(e^t)$, $t \in \Re$, is strictly stationary. Conversely, if $\{Y(t), t \in \Re\}$ is strictly stationary, then $X(t) = t^H Y(\log t)$, $t > 0$.

Proof. Let $c_1, ..., c_n \in \Re$ be a real number. If the process $\{X(t),\ t \geq 0\}$ is H-ss, then for any $t_1, ..., t_n > 0$ and $h \in \Re$, we have

$$\begin{aligned}
\sum_{j=1}^{n} c_j Y(t_j + h) &= \sum_{j=1}^{n} c_j e^{-t_j H} e^{-hH} X(e^{-t_j+h}) \\
&\stackrel{d}{=} \sum_{j=1}^{n} c_j e^{-t_j H} X(e^{t_j}) \\
&= \sum_{j=1}^{n} c_j Y(t_j)
\end{aligned}$$

thus the joint distributions of process $\{Y(t), t \in \Re\}$ are invariant under time shifts, hence $\{Y(t), t \in \Re\}$ is strictly stationary.

Conversely, for any $n \in N$ we consider $c_1, ..., c_n \in \Re$, $t_1, ..., t_n > 0$ and $a \in \Re$, then

$$\begin{aligned}\sum_{j=1}^{n} c_j X(at_j) &= \sum_{j=1}^{n} c_j a^H t_j^H Y(\log a + \log t_j)\\ &\stackrel{d}{=} \sum_{j=1}^{n} c_j a^H t_j^H Y(\log t_j)\\ &= \sum_{j=1}^{n} c_j a^H t_j^H Y(\log t_j)\\ &= \sum_{j=1}^{n} c_j a^H X(t_j).\end{aligned}$$

Thus $\{X(t),\, t \geq 0\}$ is H-ss. ■

1.1.1 Simple invariance properties of Brownian Motion and Fractional Brownian Motion

One of the topics of this chapter is about many natural sets which can be derived from the sample paths of Brownian motion; they are in some sense random fractals. An intuitive approach to fractals is that they are sets which have a nontrivial geometric structure at all scales. A key role in this behaviour is played by the very simple *scaling invariance* property of Brownian motion, which we are going to formulate. It identifies a transformation on the space of functions, which changes the individual Brownian random functions but leaves their distribution unchanged.

The fractional Brownian motion was originally introduced by Kolmogorov [7], in 1940. He was interested in modelling turbulence (see Kolmogorov [8], or Shiryaev [9] for more details of Kolmogorov's studies connected to turbulence). Kolmogorov did not use the name fractional Brownian motion. He called the process *Wiener spiral*. Kolmogorov studied the fractional Brownian motion within a Hilbert space framework and deduced its covariance function from a scaling property that we now call *self-similarity*. Among early works connected to fractional Brownian motion we would like to mention Hunt [10]. He was interested in almost sure convergence of random Fourier series and in the modulus of continuity of such series. He also considered random Fourier transformations and their continuity properties. In his work the fractional Brownian motion was implicitly introduced as a Fourier–Wiener transformation of a power function (nowadays we would call this a spectral representation of the fractional Brownian motion). Hunt proved some results concerning a Hölder-type modulus of continuity of the fractional Brownian motion. Let us also note that Lévy [11] considered a process that is similar to the fractional Brownian motion. He introduced a process that is obtained from the standard Brownian motion as a fractional integral in the Riemann–Liouville sense. Although this process shares many of the (path) properties of the fractional Brownian motion it does not have stationary increments. This process is sometimes called the "Lévy fractional Brownian motion" or the "Riemann-Liouville process". Yaglom [12] was interested in generalizing the spectral theory of stationary processes to processes from a more general class. Indeed, He was interested in linear extrapolation and linear filtering. Yaglom studied processes with "random stationary n th increments". In his work the fractional Brownian motion was considered as an example of a process with stationary first increments. It was defined through its spectral density. Lamperti studied semi-stable processes (which we nowadays call selfsimilar processes) [13]. The fractional Brownian motion appears implicitly in his work as an example of a Gaussian semi-stable process. Lamperti noted that the fractional Brownian motion cannot be Markovian, except in the standard Brownian case. He showed that each self-similar process can be obtained from a stationary process and *vice versa* by a time-change transformation. Also, Lamperti proved a "fundamental limit theorem" stating that each non-degenerate self-similar process can be understood as a time-scale limit of a stochastic process. Molchan and Golosov studied the derivative of fractional Brownian motion using generalized stochastic processes (in the sense of Gelfand-Ito) [14]. They called this derivative a "Gaussian stationary process with asymptotic power spectrum" (nowadays it is called fractional Gaussian noise or fractional white noise). Molchan and Golosov found a finite interval representation for the fractional Brownian motion with respect to the standard one (the more well-known Mandelbrot-Van Ness representation requires integration from minus infinity). In [14] there is also a reverse representation, i.e. a finite interval integral representation of the standard Brownian motion with respect to the fractional one. Molchan and Golosov noted the connection of these integral representations with deterministic fractional calculus. They also pointed out how it's possible to obtain the Girsanov theorem and prediction formulas for the fractional Brownian motion by using the integral representation. The name "fractional Brownian motion" comes from the influential paper by Mandelbrot and Van Ness [15]. They defined the fractional

Brownian motion as a fractional integral with respect to the standard one (whence the name). The notation for the index H and the current parameterization with range (0; 1) are due to Mandelbrot and Van Ness too. The parameter H is called the Hurst index after an English hydrologist studied the memory of Nile River maxima in connection with the designing of water reservoirs [16]. Mandelbrot and Van Ness considered an approximation of the fractional Gaussian noise by smoothing the fractional Brownian motion. They also studied simple interpolation and extrapolation of the smoothed fractional Gaussian noise and fractional Brownian motion.

Recently the fractional Brownian motion has paved the way to many applications. It (and its further generalizations) has been studied in connection to financial time series, fluctuations in solids, hydrology, telecommunications and generation of artificial landscapes, just to mention few. Moreover, these potential applications the study of the fractional Brownian motion is endorsed because it is one of the simplest processes that is neither a semimartingale nor a Markov process.

The Fractional Brownian motion is a prime example of a stochastic process that is statistically self-similar with stationary increments. It is a generalization of the well-known process of Brownian motion. It is the unique centred Gaussian self-similar process with stationary increments [7], [17], in the sense that the class of all fractional Brownian motion coincides with that of all Gaussian self-similar processes with stationary increments. It can be obtained via a stochastic fractional integration of the standard Brownian motion. However, the increments of the fractional Brownian motion are not independent, except in the standard Brownian case; it can be used as a model to describe a large number of natural phenomena and shapes such as the range of rivers, time series of economic action, terrain surfaces, etc...

The main disadvantages of Fractional Brownian motion is that it fails to model impulsiveness due to its Gaussianity.

Within applications, a self-similar process is often a (continuous time) model for a cumulative input of a system in steady state. Hence self-similar processes reflecting this feature, namely, *processes with stationary increments,* become particularly interesting.

In the following, we will focus on self-similar processes having stationary increments. In common abbreviation a self-similar process with stationary increments is SSSI (or $H-$sssi if one wants to emphasize the exponent of self-similarity).

We say that an $\Re^d$-valued stochastic process $\{X(t),\ t \geq 0\}$ is said to have *stationary increments*, if the distributions of $\{X(t+h) - X(h),\ t \geq 0\}$ are independent of $h \geq 0$, that is

$$\{X(t+h) - X(h), t \geq 0\} \stackrel{d}{=} \{X(t) - X(0), t \geq 0\}.$$

We also say that an $\Re^d$-valued stochastic process $\{X(t),\ t \geq 0\}$ has *independent increments*, if $\forall m \geq 1$, and for any partition $0 \leq t_0 < t_1 < ... < t_m$, $X(t_1) - X(t_0), ..., X(t_m) - X(t_{m-1})$ are independent.

We define the fractional Brownian motion (FBM, for short) by its scaling property and discuss some basic properties of the process. An indepth introduction to fractional Brownian motion can be found in the book by Samorodnitsky and Taqqu, [18], Chapter 7.2 (which is surprising given the name of the book), or in a recent book by Embrechts and Maejima [6].

As we show in the following (see Theorem 1.1.4) the Brownian motion is a special H-ss process. We consider on $(\Omega, \mathcal{F}, P)$ a stochastic continuous time stochastic process $\{B(t), t \geq 0\}$ which is to say that for each $t \geq 0$ one considers a real random variable $B(t)$.

Definition 1.1.2: The stochastic process $\{B(t), t \geq 0\}$ is said to be a standard brownian motion if

(a) $B(0) = 0$ almost surely;

(b) it is a process with independent and stationary increments;

(c) for each $t > 0$, $B(t)$ has a Gaussian distribution with mean zero and covariance matrix $E[B(t)B(t)'] = tI$, where I is the identity matrix, where $B(t)'$ is the mean value of $B(t)$;

(d) its sample paths $t \longrightarrow B(\omega)$ are continuous almost surely.

We will not talk about the construction of such a process which can be studied in most standard reference books.

We can observe that with the above definition, one gets that for $s < t$, $B(t) - B(s)$ is Gaussian, centered with variance $t - s$, which say that Brownian motion is stationary. Indeed, one has

$$B(t) = B(s) + [B(t) - B(s)].$$

Therefore, using characteristic functions[3] and the independence of $B(s)$ and $B(t) - B(s)$, one has

$$E(e^{ir[B(t)-B(s)]}) = e^{-\frac{r^2}{2}(t-s)}$$

As anticipated before, it is fundamental the following result.

Theorem 1.1.4: Brownian motion $\{B(t), t \geq 0\}$ is $\frac{1}{2}$-ss.

Proof. We will show that for each $a > 0$, $\{a^{-\frac{1}{2}}B(at)\}$ is a Brownian motion; indeed, if we prove it, then $\{a^{-\frac{1}{2}}B(at)\} \stackrel{d}{=} \{B(t)\}$; consequently, $\{B(t),\ t \geq 0\}$ is H-ss, with $H = \frac{1}{2}$. Conditions (a), (b), and (d), follow from the same conditions for $\{B(t)\}$. For the condition (c), it is trivial that if $\{B(t)\}$ has a Gaussian distribution with mean zero, then also $\{a^{-\frac{1}{2}}B(at)\}$ has the same distribution with mean zero. In particular, the covariance matrix is

$$\begin{aligned} E[(a^{-\frac{1}{2}}B(at))(a^{-\frac{1}{2}}B(at))'] &= \frac{1}{a}E[B(at)B(at)'] \\ &= \frac{1}{a}atI = tI \end{aligned}$$

thus, $\{a^{-\frac{1}{2}}B(at)\}$ is a Brownian motion. ■

Moreover, the covariance matrix of Brownian motion fulfills the following theorem.

Theorem 1.1.5: $E[B(t)B(s)'] = \min\{t, s\}I$

[3] Let X a random variable, then it is convenient to define the *characteristic function* of X by

$$\widehat{\mu}(\theta) = E[e^{i\theta X}], \qquad -\infty < \theta < \infty$$

where $i = \sqrt{-1}$. It can be shown that $\widehat{\mu}(\theta)$ always exists and, like the moment generating function, uniquely determines the distribution of X (For more details, see [24]).

Proof. Suppose that $s < t$. Then by considering the stationarity of the increments, we have

$$
\begin{aligned}
E[B(t)B(s)'] &= \frac{1}{2}\{E[B(t)B(t)'] + E[B(s)B(s)'] - E[(B(t) - B(s))(B(t) - B(s))']\} \\
&= \frac{1}{2}\{E[B(t)B(t)'] + E[B(s)B(s)'] - E[B(|t-s|)B(|t-s|)']\} \\
&= \frac{1}{2}\{t + s - |t-s|\}I \\
&= \left(\left\{\begin{array}{cc} s & \text{for } t \geq s \\ t & \text{for } t \leq s \end{array}\right\}\right) I \\
&= \min\{t, s\}I.
\end{aligned}
$$

■

To define a Fractional Brownian motion we will show a basic result for general self-similar processes with stationary increments.

Theorem 1.1.6: Let $\{X(t)\}$ be real-valued $H-$ss with stationary increments and suppose that $E[X(1)^2] < \infty$. Then

$$
E[X(t)X(s)] = \frac{1}{2}\{t^{2H} + s^{2H} - |t-s|^{2H}\}E[X(1)^2].
$$

Proof. By self-similarity and stationarity of increments, we have

$$
\begin{aligned}
E[X(t)X(s)] &= \frac{1}{2}\{E[X(t)^2] + E[X(s)^2] - E[(X(t) - X(s))^2]\} \\
&= \frac{1}{2}\{E[X(t)^2] + E[X(s)^2] - E[X(|t-s|)^2]\} \\
&= \frac{1}{2}\{t^{2H}E[X(1)^2] + s^{2H}E[X(1)^2] - |t-s|^{2H}E[X(1)^2]\} \\
&= \frac{1}{2}\{t^{2H} + s^{2H} - |t-s|^{2H}\}E[X(1)^2].
\end{aligned}
$$

■

Consequently, we can give the following definition.

Definition 1.1.3: Let $0 < H \leq 1$. A real-valued Gaussian process $\{B_H(t), t \geq 0\}$ is called "fractional Brownian motion" (or FBM, for short) if $E[B_H(t)] = 0$ and

$$E[B_H(t)B_H(s)] = \frac{1}{2}\{t^{2H} + s^{2H} - |t-s|^{2H}\}E[B_H(1)^2]. \tag{1.1.3}$$

Hence, for $0 < H \leq 1$ there is a unique zero Gaussian process whose covariance function is consistent with self-similarity with exponent H and stationary increments and, hence, is given by (1.1.3) and the condition $E[B_H(t)] = 0$. Conversely, a Gaussian process with a covariance function given by (1.1.3) is, clearly, both self-similar with exponent H and has stationary increments. That is, for every $0 < H \leq 1$ there is a unique (up to a global multiplicative constant) $H-$sssi zero mean Gaussian process (this process is a FBM).

Theorem 1.1.7: $\{B_{1/2}(t)\}$ is a Brownian motion up to a multiplicative constant.

Proof. If we put in the equation (1.1.3) and in $E[B_H(t)] = 0$ the value $H = 1/2$, then

$$E[B_{\frac{1}{2}}(t)B_{\frac{1}{2}}(s)] = \frac{1}{2}\{t + s - |t-s|\}E[B_{\frac{1}{2}}(s)^2] = \min\,(t,s)I$$

Hence, the Fractional Brownian motion with $H = 1/2$ has the same mean and covariance structure of Brownian motion, as mentioned in Remark 1.1.1. ■

Remark 1.1.1: It is known that the distribution of a Gaussian process is determined by its mean and covariance structure. Indeed, the distribution of a process is determined by all joint distributions and the density of a multidimensional Gaussian distribution is explicitly given through its mean and covariance matrix. Thus, the previous conditions (see Definition 1.1.3) determine a unique Gaussian process.

The following Theorem is fundamental; indeed, it is very handy because it provides an useful criterion for checking whether a given process is FBM.

Theorem 1.1.8: A Fractional Brownian motion $\{B_H(t),\ t \geq 0\}$ is H-sssi. Moreover, the Fractional Brownian motion is unique in the sense that the class of all Fractional Brownian motion coincides with that of all Gaussian self-similar processes with stationary increments. Finally, $\{B_H(t)\}$ has independent increments iff $H = 1/2$.

Proof. Let $\{B_H(t),\, t \geq 0\}$ a Fractional Brownian motion.

1) Initially, we prove that it is a self-similar process.

Indeed, we have that

$$\begin{aligned} E[B_H(at)B_H(as)] &= \frac{1}{2}\{(at)^{2H} + (as)^{2H} - a(|t-s|)^{2H}\}E[B_H(1)^2] \\ &= a^{2H}E[B_H(t)B_H(s)] \\ &= E[(a^H B_H(t))(a^H B_H(s))]. \end{aligned}$$

Since, Fractional Brownian motion is mean zero Gaussian, this equality in covariance implies that $\{B_H(at)\} \stackrel{d}{=} \{a^H B_H(t)\}$.

2) We now show that it has stationary increments.

We must prove that $\{B_H(t+h) - B_H(h),\, t \geq 0\} \stackrel{d}{=} \{B_H(t) - B_H(0),\, t \geq 0\}$, but thank to the result of proposition 1.1.1, we can show that $\{B_H(t+h) - B_H(h),\, t \geq 0\} \stackrel{d}{=} \{B_H(t), t \geq 0\}$.

It is enough to consider only covariances:

$$\begin{aligned} E[(B_H(t+h) - B_H(h))(B_H(s+h) - B_H(h))] = \\ &= E[B_H(t+h)B_H(s+h)] - E[B_H(t+h)B_H(h)]+ \\ &- E[B_H(s+h)B_H(h)] + E[B_H(h)^2] \\ &= \frac{1}{2}\{((t+h)^{2H} + (s+h)^{2H} - |t-s|^{2H})+ \\ &- ((t+h)^{2H} + h^{2H} - t^{2H})+ \\ &- ((s+h)^{2H} + h^{2H} - s^{2H}) + 2h^{2H}\}E[B_H(1)^2] \\ &= \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})E[B_H(1)^2] \\ &= E[B_H(t)B_H(s)]. \end{aligned}$$

Thus, $\{B_H(t),\, t \geq 0\}$ is H-sssi.

3)To show the uniqueness of the class of all Fractional Brownian motions, it is enough to prove that any H-sssi process with Gaussian distribution and mean zero is a Fractional Brownian motion.

Let $\{X(t),\, t \geq 0\}$ be a mean zero Gaussian process, first note that once it is H-sssi and has stationary increments, by Theorem 1.1.6, it results that $\{X(t),\, t \geq 0\}$ has the following covariance structure

$$E[X(t)X(s)] = \frac{1}{2}\{t^{2H} + s^{2H} - |t-s|^{2H}\}E[X(1)^2],$$

then $X(t)$ is the same of $\{B_H(t)\}$ in law.

4)In conclusion, we prove that $\{B_H(t)\}$ has independent increments iff $H = 1/2$.

Let $\{B_H(t)\}$ be a Fractional Brownian motion with $H = 1/2$, by Theorem 1.1.7 it is a Brownian motion up to a multiplicative constant, then trivially it has independent increments. Conversely, if $\{B_H(t)\}$ has independent increments also, then for $0 < s < t$,

$$\begin{aligned}
E[B_H(s)(B_H(t) - B_H(s))] &= E[B_H(s)B_H(t)] - E[B_H(s)^2] \\
&= \frac{1}{2}\{t^{2H} + s^{2H} - |t-s|^{2H} - 2s^{2H}\}E[B_H(1)^2] \\
&= \frac{1}{2}\{t^{2H} - s^{2H} - |t-s|^{2H}\}E[B_H(1)^2] \\
&= 0
\end{aligned}$$

The latter however only holds for $H = 1/2$. ■

Remark 1.1.2: We can prove that if $H = 1, B_1(t) = tB_1(t)$. In this case, the covariance structure become $E[(B_1(t) - tB_1(1))^2] = tsE[(B_1(t))^2]$; consequently,

$$\begin{aligned}
E[(B_1(t) - tB_1(1))^2] &= E[B_1(t)^2] - 2tE[B_1(t)B_1(1)] + t^2E[B_1(t)^2] \\
&= (t^2 - 2t^2 + t^2)E[B_1(t)^2] \\
&= 0
\end{aligned}$$

so that $B_1(t) = tB_1(t)$ a.s.

In order to show the next result, we use the notion of a *Wiener integral*. Indeed, we give in the following Theorem the integral representation of Fractional Brownian motion through a Wiener integral (for details, see [6]).

Theorem 1.1.9: When $0 < H < 1$, Fractional Brownian motion $\{B_H(t),\ t \geq 0\}$ has a stochastic integral representation

$$C_H = \left\{ \int_{-\infty}^{0} ((t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}}) dB(u) + \int_{0}^{t} (t-u)^{H-\frac{1}{2}} dB(u) \right\}, \quad (1.1.4)$$

where

$$C_H = E[B_H(t)]^{\frac{1}{2}} \left\{ \int_{-\infty}^{0} ((t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}}) du + \frac{1}{2H} \right\}^{-\frac{1}{2}}$$

Sample path properties of Brownian motion have been well studied.

As Brownian motion, Fractional Brownian motion is also:

- sample continuous;
- nowhere differentiable;
- unbounded variation almost surely.

Several properties of trajectories of multidimensional Fractional Brownian motion with multiparameter have also been studied. Let $\{B_H(t),\ t \in \Re^N\}$ be a mean-zero Gaussian process with covariance

$$E[B_H(t)B_H(s)] = |t|^{2H} + |s|^{2H} - |t-s|^{2H},$$

where $|t|$ is the Euclidean norm of $t \in \Re^N$.

1.1.2 Stable Lévy Processes

Stable Lévy Processes (including Brownian motion) are the only selfsimilar processes with independent and stationary increments.

Definition 1.1.4: An $\Re^d-$values stochastic process $\{X(t),\ t \geq 0\}$ is called a Lévy process if

(a) $X(0) = 0$ almost surely,

(b) it is stochastically continuous at any $t \geq 0$,

(c) it has independent and stationary increments,

(d) its sample paths are right-continuous and have left limits almost surely.

Definition 1.1.5: A probability measure μ on $\Re^d$ is called "strictly stable", if it is not a delta measure, the characteristic function $\widehat{\mu}(\theta)$ does not vanish and for any $a > 0$ there exists $b > 0$ such that

$$\widehat{\mu}(\theta)^a = \widehat{\mu}(b\theta), \qquad \forall\theta \in \Re^d.$$

Each stable distribution has a unique index as follows.

Theorem 1.1.10: If μ on $\Re^d$ is stable, there exists a unique $0 < \alpha \leq 2$ such that $b = a^{1/\alpha}$. Such a μ is referred to as $\alpha-$stable. When $\alpha = 2$, μ is a mean zero Gaussian probability measure.

Non Gaussian stable distributions are, sometimes by physicists, called Lévy distributions [19]. In particular, the case with $\alpha = 1$, is called Cauchy distribution (or Lorentz distribution by physicists). A significant difference between Gaussian distributions and non-Gaussian stable ones like the Cauchy is that the latter have heavy tails, namely their variances are infinite. Such models were for a long time not accepted by physicists. More recently, the importance of modeling stochastic phenomena with heavy-tailed processes is dramatically increasing in many fields.

In the following Theorem, we show as self-similar processes with independent and stationary increments are the only stable Lévy processes.

Theorem 1.1.11: Suppose $\{X(t), t \geq 0\}$ is a Lévy process and let $0 < \alpha \leq 2$. Then $Ł(X(1))$ [4] is $\alpha-$stable if and only if $\{X(t)\}$ is selfsimilar. The index α of stability and the exponent H of selsimilarity satisfy $\alpha = 1/H$.

Proof. Let $\mu_t = (X(t))$ and $\mu = \mu_1$. Since $\{X(t)\}$ is a Lévy process, for each $t \geq 0$, the characteristic function $\widehat{\mu_t}$ satisfies $\widehat{\mu_t}(\theta) = \widehat{\mu}(\theta)^t$. Indeed, for any n and m,

[4] Where Ł is the Lebesgue measure.

$$X(\frac{m}{n}) = \left\{X(\frac{m}{n}) - X(\frac{m-1}{n})\right\} + ... + \left\{X(\frac{1}{n}) - X(0)\right\}, \tag{1.1.5}$$

where $X(\frac{k}{n}) = \left\{X(\frac{k}{n}) - X(\frac{k-1}{n})\right\}$, $k = 1, ..., m$, are independent and identically distributed (for short i.i.d.).

From (1.1.5) we have that $\widehat{\mu}_{m/n}(\theta) = \widehat{\mu}_{1/n}(\theta)^m$ and in particular that $\widehat{\mu}_{1/n}(\theta) = \widehat{\mu}(\theta)^{1/n}$. Thus

$$\widehat{\mu}_{m/n}(\theta) = \widehat{\mu}_{1/n}(\theta)^m = \widehat{\mu}(\theta)^{m/n}.$$

This, with the stochastic continuity of $\{X(t)\}$, implies that $\widehat{\mu}_t(\theta) = \widehat{\mu}(\theta)^t$ for any $t \geq 0$.

1) Initially, we prove that the measure the probability Ł$(X(1))$ is $\alpha-$stable.

By selfsimilarity, for some $H > 0$, $X(a) \overset{d}{\backsim} a^H X(1)$, $\forall a > 0$, hence $\widehat{\mu}(\theta)^a = \widehat{\mu}(a^H\theta), \forall \theta \in \Re^d, \forall a > 0$, implying that μ is stable with $\alpha = 1/H$, necessarily $H \geq \frac{1}{2}$.

2)We now show that Ł$(X(1))$ is a self-similar process.

Suppose that μ is $\alpha-$stable and $0 < \alpha \leq 2$. Since $\{X(t)\}$ has independent and stationary increments, it is enough to show that for any $a > 0$,

$$X(at) \overset{d}{\backsim} a^{1/\alpha} X(t).$$

Hence,

$$\begin{aligned} E[\exp\{i\langle\theta,\ X(at)\rangle\}] &= \widehat{\mu_{at}}(\theta) = \widehat{\mu}(\theta)^{at} = \widehat{\mu}(a^{1/\alpha}\theta)^t = \widehat{\mu_t}(a^{1/\alpha}\theta) \\ &= E[\exp\{i\langle\theta,\ a^{1/\alpha}X(t)\rangle\}]. \end{aligned}$$

This completes the proof. ■

If $\{X(t),\, t \geq 0\}$ is a Lévy process and Ł$(X(1))$ is $\alpha-$stable, then it is called an $\alpha-$stable Lévy process and denoted by $\{Z_\alpha(t),\, t \geq 0\}$. $\{Z_2(t)\}$ is Brownian motion.

The process $\{X_1(t),\, t \geq 0\}$ is H-sssi, $\{X_1(t)\}$ with $\alpha = 2$ is a fractional Brownian motion and $\{X_1(t)\}$ with $0 < \alpha < 2$ is an extension of $\{B_H(t)\}$ to infinite variance processes. It is called the linear fractional stable motion. Finally, $\{X_2(t)\}$ is $\frac{1}{\alpha}-$ss, si. This is called the log-fractional stable motion. Note that $\{X_2(t)\}$ with $\alpha = 2$ is no more Brownian motion.

Moreover, the self-similarity there is another property that makes the fractional Brownian motion a suitable model for many applications that we will show in following section.

1.2 Long and Short Range dependence

The notion of "long memory" or "Long Range Dependence" (LRD) has intrigued many at least since B. Mandelbrot brought it to the attention of the scientific community in the 1960s in a series of papers (Mandelbrot-1965 [21], Mandelbrot and Van Ness-1968 [22], Mandelbrot and Wallis-1968/1969 [23]) that, among other things, explained the so-called "Hurst phenomenon", having to do with unusual behaviour of the water levels in the Nile river.

Today this notion has become especially important as potentially crucial applications arise in new areas such as communication networks and finance [17], [20].

There is an agreement in probability that the notion of long range dependence should be considered in application to stationary processes only, i.e. only in the context of phenomena "in steady state". The point is, however, delicate.

First, in various applications of stochastic modeling this term is applied to non-stationary processes. Thus, for example, the usual Brownian motion is sometimes viewed as having LRD because it never really forgets where started from (this is very unreasonable to a probabilist who immediately thinks about independent increments of the Brownian motion).

Second, stationary processes with LRD (in whatever sense) sometimes resemble their non-stationary counterparts. It is, therefore, possible to think of LRD processes as being that layer among the stationary processes that is "near the boundary" with non-stationarity, or as the layer separating the non-stationary processes from the "well behaved, usual" stationary processes.

Let us consider a stationary process, the decrease's speed of its autocorrelation function describes the behavior of process to long or short range; then, we say that the process has a Long or Short range dependence. In general, in a stationary time series the Long range dependence occurs when the covariances tend to zero like a power function and so slowly that their sums diverge, vice versa (otherwise) the short range dependence happens when the covariance's sums converge. Through the increments process of a H-sssi process, there is a close relationship between long range or short range dependence and selfsimilar process with stationary increment. Consequently, to define this correlation we have to introduce the increments process of H-sssi process $\{X(t),\, t \geq 0\}$, and its autocorrelation function.

Let $\{X(t),\, t > 0\}$ be a H-sssi process, $0 < H < 1$, non degenerate for each $t > 0$ with $E[X(1)^2] < \infty$, then the *increments process* is

$$\xi(n) = X(n+1) - X(n), \qquad n \in N_0$$

Moreover, the self-similarity there is another property that makes the fractional Brownian motion a suitable model for many applications.

Definition 1.2.1: If $\{X(t), t > 0\}$ is a Fractional Brownian motion, then $\{\xi(n)\}_{n \in N_0}$ is called Fractional Gaussian Noise (FGN).

The autocorrelation function $r = r_H$ of the fractional Gaussian noise with $H \neq 1/2$ satisfies

$$r(n) \sim H(2H-1)n^{2H-2}$$

as n tends to infinity.

Therefore, $H > 1/2$ then the increments of the corresponding fractional Brownian motion are positively correlated and exhibit the long-range dependence property. The case $0 < H < 1/2$ corresponds to negatively correlated increments and short-range dependence. When $H = 1/2$ the FGN is simply the standard Brownian motion, so it has independent increments.

Let us further illustrate the dependence structure of the fractional Brownian motion.

Proposition 1.2.1: The fractional Brownian motion with Hurst index H is a Markov process if and only if $H = 1/2$.

Definition 1.2.2: If $\{X(t), t > 0\}$ is Brownian motion, then $\{\xi(n)\}_{n \in N_0}$ is called White Gaussian Noise.

Consequently, the *autocorrelation function* of $\{\xi(n)\}_{n \in N_0}$ is

$$r(n) = E[\xi(0)\xi(n)], \qquad n \in N_0$$

Remark 1.2.1: It is trivial to observe that $\{\xi(n)\}_{n\in N_0}$ is stationary process.

Definition 1.2.3: Let $\{X(t), t > 0\}$ be a H-sssi process, $0 < H < 1$, non degenerate for each $t > 0$ with $E[X(1)^2] < \infty$, then when autocorrelation function $r(n)$ decays hyperbolically such that the following condition holds

$$\sum_{n=0}^{+\infty} |r(n)| = \infty,$$

we call the increments process $\{\xi(n)\}_{n\in N_0}$ long-range dependent, while $\{\xi(n)\}_{n\in N_0}$ is short-range dependent if the autocorrelation function is summable.

Indeed, if r decays exponentially, i.e. $r(n) \sim \rho^n$ as n tends to infinity, then the stationary sequence $(\xi_n)_{n\in N}$ exhibits *short-range dependence.*

In conclusion, we can give the fundamental result of this subsection, which represents a characterization of long and short range dependence process in term of H, scaling exponent (Hurst parameter).

Proposition 1.2.2: Let $\{X(t), t > 0\}$ be a H-sssi process, $0 < H < 1$, non degenerate for each $t > 0$ with $E[X(1)^2] < \infty$, then when autocorrelation function $r(n)$ has following asymptotical course

- $r(n) \backsim H(2H-1)n^{2H-2}E[X(1)^2], \qquad \text{as } n \longrightarrow \infty, \text{ if } H \neq \frac{1}{2},$
- $r(n) = 0, \qquad \text{as } n \longrightarrow \infty, \text{ if } H = \frac{1}{2}.$

Consequently, it results

(a) if $0 < H < 1$, $\sum_{n=0}^{+\infty} |r(n)| < \infty$, so there is the short range dependence;

(b) if $H = \frac{1}{2}$, $\{\xi(n)\}_{n\in N_0}$ is uncorrelated;

(c) if $1/2 < H < 1$, $\sum_{n=0}^{+\infty} |r(n)| = \infty$, so there is the long range dependence.

Proof. Noticing that $X(0) = 0$ a.s. and by using Theorem 1.1.6, we have, for $n \geq 1$,

$$
\begin{aligned}
r(n) &= E[\xi(0)\xi(n)] \\
&= E[X(1)X(n+1) - X(n)] \\
&= E[X(1)X(n+1)] - E[X(1)X(n)] \\
&= \frac{1}{2}\{(n+1)^{2H} - 2n^{2H} + (n+1)^{2H}\}E[X(1)^2] \\
&= \frac{1}{2}n^{2H}\{(1+\frac{1}{n})^{2H} - 2 + (1-\frac{1}{n})^{2H}\}E[X(1)^2] \\
&= \frac{1}{2}n^{2H}\{1 + 2H\frac{1}{n} + 2H(2H-1)\frac{1}{n^2} - 2 + 1 - 2H\frac{1}{n} + 2H(2H-1)\frac{1}{n^2}\}E[X(1)^2] \\
&= 2H(2H-1)n^{2H-2}
\end{aligned}
$$

which implies the conclusion. ■

The previous result indicates that there are self-similar processes which are not long-range dependent and vice versa; for example Brownian motion is $\frac{1}{2}-$sssi with White Gaussian noise as its increments, but the latter is not long-range dependent.

2. FRACTALS DIMENSIONS, MULTIFRACTALS AND CANTORIAN SPACE

Dimensions are a tool to measure the size of mathematical objects on a crude scale. For example, in classical geometry dimension is able to tell us that in three-dimensional space a line segment (a one-dimensional object) is smaller than the surface of a ball (a two-dimensional object), but there is no difference between line-segments of different lengths. It may therefore come as a surprise that dimension is able to distinguish the size of so many objects in probability theory, for example:

- the ranges of stable processes with different indices,
- the boundaries of supercritical Galton-Watson trees with different mean offspring numbers,
- the n-fold self-intersection of a planar Brownian motion for different n.

Non integer dimensionalities have recently entered physics from at least two separate directions: continuous ϵ expansions near an integer d in the theory of critical phenomena, and *fractals*. Fractals might appear at first to be unrelated to our current studies. However, they are connected to dynamical systems in an interesting way: a number of dynamical systems have orbits that approach a set which is itself a fractal. This portion of the lecture will cover the definition of a fractal and a few examples of such.

In this chapter we introduce the notion of dimension. One can use a notion of dimension taking variations of the size in the different sets in a covering into account. This captures finer details of the set and leads to the notion of *Hausdorff dimension.*

We show how Hausdorff dimension is used to determine the size of a set and describe techniques to calculate the Hausdorff dimension.

In what follows, we will also introduce the concept of *Multifractals.*

Fractals can be often be regarded as special cases of continuous or discrete multifractals. The concept of multifractals which are spatially interwined fractals has succeeded the concept of fractals which are now often referred to as unifractals (or mixing the Latin with the Greek:"monofractals"). *The multiplicative cascade* model originally developed by physicists to explain characteristic behaviour of turbulence yields a spatial frequency distribution on the one hand, and a corresponding multifractal dimension spectrum on the other. The statistical estimation of fractal dimensions remains an important topic of investigation as well.

Now that fractal and multifractals have become well established as practical tools, it also has become apparent that, in Nature, there are often situations that a single, relatively simple fractal or multifractal model does not apply and mixtures of models or other types of generalizations are required.

In particular, reading El Naschie's papers, E-Infinity ($\epsilon^{(\infty)}$) appears to be clearly a new framework for understanding and describing Nature. Indeed, Nature appears clearly not continuous, not periodic, but self-similar and Mohamed El Naschie with $\epsilon^{(\infty)}$ has introduced a mathematical formulation to describe phenomena that are resolution dependent. As reported by the author, $\epsilon^{(\infty)}$ space-time is an infinite dimensional fractal, that happens to have $D = 4$ as the expectation value for the topological dimension [47], [48], [49]. The topological value $3 + 1$ means that in our low energy resolution, the world appears us as if it were four-dimensional. This is a sweeping generalization of what Einstein did in his general theory of relativity. El Naschie introduces a new geometry for space-time which differs considerably from the space-time of our sensual experience. Consequently, it all depends on the energy scale through which we are making our observation. Observations of large scale structures show that the dimension changes if we consider different energies, corresponding to different lengths-scale in Universe, as reported in [50], [51], [52].

2.1 Introduction to Fractals

The geometry of natural objects ranging in size from the atomic scale to the size of the universe is central to models we develop in order to "understand nature" [25]. The geometry of particle trajectories; of hydrodynamic flow lines, waves, ships and shores; of landscapes, mountains, islands, rivers, glaciers and sediments; of grains in rock, metals and composite materials; of plants, insects and cells, as well as the geometrical structure of crystals, chemicals and proteins - in short the geometry of nature is so central to the various fields of natural science that we tend to take the geometrical aspects for granted. Each field tends to develop adapted concepts (e.g. morphology, four-dimensional spaces, texture, etc...) used intuitively by the scientists in that field. Traditionally the Euclidean lines, circles, spheres and tetrahedra have served as the basis of the intuitive understanding of the geometry of nature.

Mathematicians have developed geometrical concepts that transcend traditional geometry, but unfortunately these concepts have failed in the past to gain acceptance in the natural sciences because of the rather abstract and "pedantic" presentations, and because of warnings that such geometries were "dangerous to use".

Benoit B. Mandelbrot, with his creative and monumental work, has generated a widespread interest in *Fractal Geometry*; a concept introduced by Mandelbrot himself. In particular he has presented what he has called fractals in an unsually inspring way. His book *The Fractal Geometry of Nature* [3] is the standard reference and contains both the elementary concepts and an unusually broad range of new and rather advanced ideas, such as *multifractals*, currently under active study (as we will see in the next section) .

There are many examples of sets that are commonly referred to as *fractals.* The word "fractal" was coined by Mandelbrot in his fundamental essay from the latin *fractus,* meaning broken, to describe objects that were too irregular to fit into a traditional geometrical setting.

Fractal geometry will make you see everything differently.

Fractal geometry is an extention of classical geometry and provides a general framework for the study of such irregular sets. It can be used to make precise models of physical structures from ferns to galaxies.

Fractal geometry is a new language. Once you can speak it, you can describe the shape of a cloud as precisely as an architect can describe a house [26].

A fractal, as defined by Mandelbrot, "is a shape made of parts similar to the whole in some way"; it is a geometric object that possesses the property of self-similarity and it can possess non-integer dimensions. Fractals can be classified in numerous manners, of which one stands out rather distinctly: *exact* (regular) fractals versus *statistical* (random) fractals. An exact fractal is an "object which appears self-similar under varying degrees of magnification in effect, possessing symmetry across scale, with each small part replicating the structure of the whole". Taken literally, when the same object replicates itself on successively smaller scales, even though the number of scales in the physical world is never infinite, we call this object an "exact fractal." When, on the other hand, the object replicates itself in its statistical properties only, it is defined as a "statistical fractal." Statistical fractals have been observed in many physical systems, ranging from material structures (polymers, aggregation, interfaces, etc.), to biology, medicine, electric circuits, computer interconnects, galactic clusters, and many other surprising areas, including stock market price fluctuations. In optics, fractals were identified in conjunction with the Talbot effect and diffraction from a binary grating and with unstable cavity modes. Exact fractals, on the other hand, such as the *Cantor set*, occur rarely in nature except as mathematical constructs. The distinction between "natural fractals" and the mathematical "fractal sets" that might be used to describe them was emphasized in Mandelbrot's original essay, but this distinction seems to have become somewhat blurred. There are no true fractals in nature.

2.2 Fractal Dimension

How big is a fractal? When are two fractals to one another in some sense? What experimental measurements might we make to tell if two different fractals may be metrically equivalent?

There are various numbers associated with fractals which can be used to compare them. They are generally referred to as *fractal dimensions.* They are attempts to quantify a subjective feeling which we have about how densely the fractal occupies the metric space in which it lies. Fractal dimensions provide an objective means for comparing fractals.

Fractal dimensions are important because they can be defined in connection with real world data, and they can be measured approximately by means of experiments. Fractal dimensions can be attached to clouds, trees, coastlines, feathers, networks of neurons in the body, dust in the air at an instant in time, the clothers you are wearning, the distribution of frequencies of light reflected by a flower, the colors emitted by the sun, and the wrinkled surface of the sea during a storm. These numbers allow us to compare sets in the real world with the laboratory fractals.

When we refer to a set F as a fractal, therefore, we will typically have the following in mind.

1) F has a fine structure, i.e. detail on arbitrarily small scales.

2) F is too irregular to be described in traditional geometrical language, both locally and globally.

3) Often F has some form of self-similarity, perhaps approximate or statistical.

4) Usually, the "fractal dimension" of F (defined in some way) is greater than its topological dimension.

5) In most cases of interest F is defined in a very simple way, perhaps *recursively.*

It is known that there are several ways of measuring a *fractal dimension.* This means that several alternative definitions exist, but in the extensively studied case of strictly self-similar fractals all these definitions yield the same value.

How can we capture the dimension of a geometric object? One requirement for a useful definition of dimension is that it should be *intrinsic.* This means that it should be independent of an embedding of the object in an ambient space like $\Re^d$. Intrinsic notions of dimension can be defined in arbitrary metric space.

2.2.1 The Minkowski dimension

Suppose S is a bounded metric space with metric d. Here bounded means that the diameter $|S| = \sup\{d(x,y)\ |x,y \in S\}$ of S is finite [27].

The example we have in mind is a bounded subset of $\Re^d$.

In order to give the definition of Minkowski dimension we define for $\epsilon > 0$,

$$M(S,\epsilon) = \min\{k \geq 1 \text{ there exist } x_1, ..., x_k \in S \text{ with } S \subset \bigcup_{i=1}^{k} B(x_i,\epsilon)\}$$

where $B(x,\epsilon) = \{y \in S : d(x,y) < \epsilon\}$ is the open ball around x of radius ϵ. Intuitively, when S has dimension d the number $M(S,\epsilon)$ should be approximately C/ϵ^d. This can be verified in simple cases like line segments, planar squares, etc. This argument motivates the definition of *Minkowski dimension.*

Definition 2.2.1: For a bounded metric space S we define the lower Minkowski dimension as

$$\underline{\dim}_M S = \liminf_{\epsilon \downarrow 0} \frac{\log M(S,\epsilon)}{\log(1/\epsilon)}$$

and the upper Minkowski dimension as

$$\overline{\dim}_M S = \limsup_{\epsilon \downarrow 0} \frac{\log M(S,\epsilon)}{\log(1/\epsilon)}$$

We always have $\underline{\dim}_M S \leq \overline{\dim}_M S$, but equality need not hold. If it holds we write $\dim_M S = \underline{\dim}_M S = \overline{\dim}_M S$.

These definitions of Minkowski dimension have limitations. In particular, we shall see below that Minkowski dimension does not have the *countable stability property*

$$\dim \bigcup_{k=1}^{\infty} S_k = \sup\{\dim S_k : k \geq 1\}$$

this is one of the properties we expect from a reasonable concept of dimension there are two ways out of this problem.

(i) One can use a notion of dimension based on covering with balls of varying size this captures finer details of the set and leads to the notion of *Hausdorff dimension.*

(ii) One can enforce the countable stability property by subdividing every set in countably many bounded pieces and taking the maximal dimension of them. The infimum over the numbers such obtained leads to the notion of *packing dimension.*

The dimension (ii) was introduced surprisingly late by Tricot (1982) and should perhaps be called *Tricot dimension* can be founded on regularization of the upper Minkowski dimension (for details, see [27]).

2.3 The Hausdorff and Self-Similarity Dimension

The Hausdorff dimension and Hausdorff measure were introduced by Felix Hausdorff in 1919. Of the wide variety of "fractal dimensions" in use, the definition of Hausdorff, based on a construction of Carathèodory, is the oldest and probably the most important. Hausdorff dimension has the advantage of being defined for any set, and is mathematically convenient, as it based on measures, which are relatively easy to manipulate. A major advantage is that in many cases it is hard to calculate or to estimate by computational methods. However, for an understanding of the mathematics of fractals, familiarity with Hausdorff measure and dimension is essential [1].

Hausdorff dimension can be based on the notion of a *covering* of the metric space S by sets of finite diameter [28].

A covering of S is a finite or countable collection of sets $S_1, S_2, ...$ with

$$S \subset \bigcup_{i=1}^{\infty} S_i \tag{2.3.1}$$

Therefore, like the Minkowski dimension, in order to give the dimension of Hausdorff dimension we can use a covering of the metric space S by balls; indeed, a covering of S by balls is an at most countable collection of balls

$$B(x_1, r_1), B(x_2, r_2), B(x_3, r_3), ...$$

In this case the relation (2.3.1) became

$$S \subset \bigcup_{i=1}^{\infty} B(x_i, r_i) \tag{2.3.2}$$

(for details see [28]).

In the (2.3.1), for every $\alpha \geq 0$ we say that the $\alpha-$value of the covering is

$$\sum_{i=1}^{\infty} |S_i|^{\alpha}$$

where $|S_i|$ denotes the diameter of the set S_i.

Informally speaking, the $\alpha-$value of the most efficient covering by small sets is the $\alpha-$Hausdorff measure of the set.

The terminology of the $\alpha-$values of a covering allows to formulate a concept of dimension, which is sensitive to the effect that the fine features of this set occur in different scales at different places.

Definition 2.3.1: For every $\alpha \geq 0$ the $\alpha-$Hausdorff content of a metric space S is defined as

$$H^{\alpha}_{\infty}(S) = \inf\left\{\sum_{i=1}^{\infty} |S_i|^{\alpha} : S_1, S_2, ... \text{ is a covering of } S\right\}, \tag{2.3.3}$$

informally speaking the $\alpha-$value of the most efficient covering. If $0 \leq \alpha < \beta$, and $H^{\alpha}_{\infty}(S) = 0$, then also $H^{\beta}_{\infty}(S) = 0$. Thus we can define

$$\dim S = \inf\{\alpha \geq 0 : H^{\alpha}_{\infty}(S) = 0\} = \sup\{\alpha \geq 0 : H^{\alpha}_{\infty}(S) > 0\},$$

the Hausdorff dimension of the set S.

The definition (2.3.3) can be write in other terms. Indeed, for every $\alpha \geq 0$ the (spherical) $\alpha-$Hausdorff content of a metric space S is defined as

$$H^{\alpha}(S) = \inf\left\{\sum_{i=1}^{\infty} {r_i}^{\alpha} : (B(x_i, r_i)) \text{ is a covering of } S\right\},$$

informally speaking the $\alpha-$value of the most efficient covering.

The concept of the $\alpha-$Hausdorff content plays an important part in the definition of the Hausdorff dimension. However, it does not help distinguish the size of the sets of the same dimension. For example, a line segment of unit length and a plus consisting of two orthogonal line segments of unit length have the same 1-Hausdorff content. Therefore, one considers a refined concept, the *Hausdorff measure.* Here the idea is to consider only coverings by small sets, which need not be balls.

In the following we give the definition of the Hausdorff measure.

Let X be a metric space and $S \subset X$. For every $\alpha \geq 0$ and $\delta > 0$ define

$$H^{\alpha}_{\delta}(S) = \inf\left\{\sum_{i=1}^{\infty} |S_i|^{\alpha} : S_1, S_2, ... \text{ is a covering of } S, \text{ and } |S_i|^{\alpha} \leq \delta\right\},$$

i.e. we are considering covering of S by sets of diameter no more than δ. Then

$$H^{\alpha}(S) = \sup_{\delta > 0} H^{\alpha}_{\delta}(S) = \lim_{\delta \downarrow 0} H^{\alpha}_{\delta}(S)$$

is the $\alpha-$*Hausdorff measure* of the set S.

The $\alpha-$*Hausdorff measure* has two obvious properties which, together with $H^\alpha(\emptyset) = 0$, make it an *outer measure.* These are *countable subadditivity*,

$$H^\alpha\left(\bigcup_{i=1}^{\infty} S_i\right) \leq \sum_{i=1}^{\infty} H^\alpha(S_i), \text{ for any sequence } S_1, S_2, ... \subset X$$

and *monotonicity*,

$$H^\alpha(S) \subset H^\alpha(D), \qquad \text{if } S \subset D \subset X.$$

In the following, we recall the crucial properties of the Hausdorff dimension.

- if $A \subset B$, then $\dim A \leq \dim B$ in particular subsets of $\Re^d$ have Hausdorff dimension no longer than d;
- $\dim \bigcup\limits_{k=1}^{\infty} S_k = \sup\{\dim S_k : k \geq 1\}$, this is the countable stability property;
- if $f : A \longrightarrow B$ is Lipschitz, then $\dim f(A) \leq \dim A$.

From the definition of the Hausdorff dimension it is plausible that in many cases it is relatively easy to give an upper bound on the dimension: just find an efficient covering of the set.

However it looks more difficult to give lower bounds as we must obtain a lower bound on $\alpha-$values of *all* covering of the set. There are three important techniques to obtain lower bounds for the Hausdorff dimension:

1) the **mass distribution principle:** if it is possible to distribute a positive amount of mass on a set S in such manner that its local concentration is bounded above, then the set must be large in a suitable sense.

2) the **potential theoretic method:** it is particularly interesting in applications to random fractals and it is based on a localization of the mass distribution principle.

3) **stochastic co-dimension** (see [28]).

One can express the Hausdorff dimension in terms of the Hausdorff measure. Indeed, for every metric space S we have

$$\begin{aligned}\dim S &= \inf\{\alpha : H^{\alpha}(S) = 0\} = \inf\{\alpha : H^{\alpha}(S) < \infty\} \\ &= \sup\{\alpha : H^{\alpha}(S) > 0\} = \sup\{\alpha : H^{\alpha}(S) = \infty\}.\end{aligned}$$

The Hausdorff dimension was used by Mandelbrot in order to offer the following *tentative* definition of a fractal:

Definition 2.3.2: A fractal is by definition a set for which the Hausdorff-Beicovitch fractal dimension (D) strictly exceed the topological dimension (D_T)

$$D > D_T \tag{2.3.4}$$

The dimension D_T is *always an integer,* but D *need not be an integer*; the two dimensions need not coincide.

Every set with a non-integer D is a fractal.

Example 2.3.1: In the following we give some examples of classical fractals:

- The original **Cantor set** is a fractal because (we will see in the next section):

$$D = \log 2/\log 3 \sim 0.6309 > 0, \text{ while } D_T = 0$$

- The original **Koch curve** (see Fig. 1.1) is a fractal because

$$D = \log 4/\log 3 \sim 1.2618 > 1, \text{ while } D_T = 1$$

- The **trail of Brownian motion** is a fractal because

$$D = 2, \text{ while } D_T = 1$$

The striking fact that D need not be an integer deserves a terminological aside. If one uses fraction broadly, as synonymous with a non-integer real number, several of the above listed values of D are fractional, and indeed the Hausdorff-Besicovitch dimension is often called *fractional dimension*. Hence, D may be an integer which is called a *fractal dimension*.

The fractal we discuss may be considered to be sets of points embedded in space. For example, the set of points that make up a line in ordinary Euclidean space has the topological dimension $D_T = 1$, and the Hausdorff-Besicovitch dimension $D = 1$. The Euclidean dimension of space is $S = 3$. Since $D = D_T$ for the line it is not fractal according to Mandelbrot's definition, which is reassuring. The concept of a distance between points in space is central to the definition of the Hausdorff-Besicovitch dimension and therefore of the fractal dimension D.

How do we measure the "size" of a set S of points in space? A simple way to measure the length of curves, the area of surfaces or the volume of an object is to divide space into small cubes of side δ.

Indeed, we can utilize the relations (2.3.2) and (2.3.3) referred to the balls.

Mandelbrot [3] has retracted this tentative definition and proposes instead the following:

Definition 2.3.3: A fractal is a shape made of parts similar to the whole in some way.

A neat and complete characterization of fractals is still lacking. The point is that the first definition, although correct and precise, is too restrictive. It excludes many fractals that are useful in physics. The second definition contains the essential feature that is emphasized in this thesis, and seen in experiments: a fractal looks the same whatever the scale. In this definition we have the concept of *self-similarity*, that we have discussed in detail in the first chapter.

In order to understand the fractal dimension (see definition 2.3.3), we introduce the intuitive concept of the *topological dimension*. Topological dimension is a concept meant to correspond to our intuitive notion of the number of independent ways one can move within an object. We intuitively think of a line as one dimensional because there's only one independent way one can move on a line, and similarly, a plane would be two dimensional.

We define topological dimension as an inductive concept. First we have the base case:

Definition 2.3.4: A set S has topological dimension if every point has arbitrary small neighborhoods, that is, neighborhoods U with $\sup(d(x, y) \mid x, y \in U)$ arbitrarily small whose boundaries do not intersect the set.

The following Theorem is connected with topological dimension.

Theorem 2.3.1: Every connected component of a nonempty set of topological dimension zero is a point.

Proof. Say a connected component C of a set of topological dimension zero contains two distinct points, call them p and q, and let $d = d(p, q)$. Given an open set U containing p such that $\sup(d(x, y) \mid x, y \in U) < d$. Then U does not contain q, and thus the boundary U must intersect C or else U and the interior of U^c separate C, which is impossible since C is connected. But then the set is not of topological dimension zero. Hence C has either one or zero points. ■

Definition 2.3.5: The topological dimension of a subset S of $\Re^n$ is the least non-negative integer k such that each point of S has arbitrarily small neighborhoods whose boundaries meet S in a set of dimension $k - 1$.

Example 2.3.2: A line in a higher dimensional space has topological dimension 1, since p contained in the line implies that $B(p, r)$'s boundary intersects the line in two disjoint points, which comprise a set of topological dimension 0. The line is connected, and so is not of topological dimension 0.

However, the topological dimension of objects is not sensitive enough of a measure to describe the intrinsic properties of fractals. In fact, using the notion of topological dimension, we essentially get a lower bound for the dimension of a set.

The best way to approach the matter of non-integer dimensions is to examine examples of objects with integer dimensions that we are familiar with. Indeed, many fractals have some degree of self-similarity. They are made up of parts that resemble the whole in some way. Sometimes, the resemblance may be weaker than strict geometrical similarity; for example, the similarity may be approximate or statistical.

We are familiar enough with the idea that a (smooth) curve is a 1-dimensional object and a surface is 2-dimensional. It is less clear that, for many purposes, the Cantor set should be regarded as having dimension $\log 2/\log 3$ and the other examples that we have given before.

The following argument gives one (rather crude) interpretation of the meaning of these "dimensions" indicating how they reflect scaling properties and self-similarity.

Consider a line segment, which has one dimension. If we break this line segment up into N equally sized portions, where r is the scaling ratio, then we observe that $Nr = 1$. So, for instance, say a line segment one unit in length is divided into 4 equal pieces, then $r = \frac{1}{4}$, and $4(\frac{1}{4}) = 1$. Similarly, we can expand this to a two-dimensional square. If we use the scaling ratio $r = \frac{1}{3}$ such that we divide each of its sides into 3 equal pieces, then this will result in dividing the original square into 9 sub-squares. Note that $9(\frac{1}{3})^2 = 1$, and in general $Nr^2 = 1$. If we extend this notion to a three-dimensional object, one can derive the relation $Nr^3 = 1$. Notice that in each case, when we partition the original object into N equal sub-units, the exponent of r is the dimension of the object under scrutiny. In general, continuing along these lines, our relation will be

$$Nr^D = 1$$

We can find a formula for D by doing some simple algebra, and if we do we get

$$D_s = \frac{\log(N)}{\log(\frac{1}{r})} \tag{2.3.5}$$

There are any number of values of N and r such that when put into the formula above for D will generate non-integral solutions for D. The number obtained in this way is usually referred to as the *similarity dimension* of set. The Hausdorff-Besicovitch dimension D equals D_s for self-similar fractals and we drop the index s for such fractals. One such example we wish to explore is the Cantor "middle thirds" set (or Cantor Dust). Unfortunately, similarity dimension is meaningful only for a small class of strictly self-similar sets. Nevertheless, there are other definitions of dimension that are much more widely applicable. For example the Hausdorff dimension and the *box-counting* dimensions may be defined for any sets, and, in the example as **Cantor set** (see Fig. 2.1), **von Kock curve** (fitting together three suitably rotated copies of the Koch curve produces a figure, which for obvious reason is called the *snowflake curve* or the *Koch island*), as in the last stage of Fig. 1.1, **Sierpinski Gasket** (see

Fig. 1.2) and **Sierpinski Carpet** (see Fig. 2.2) be shown to equal the similarity dimension [1].

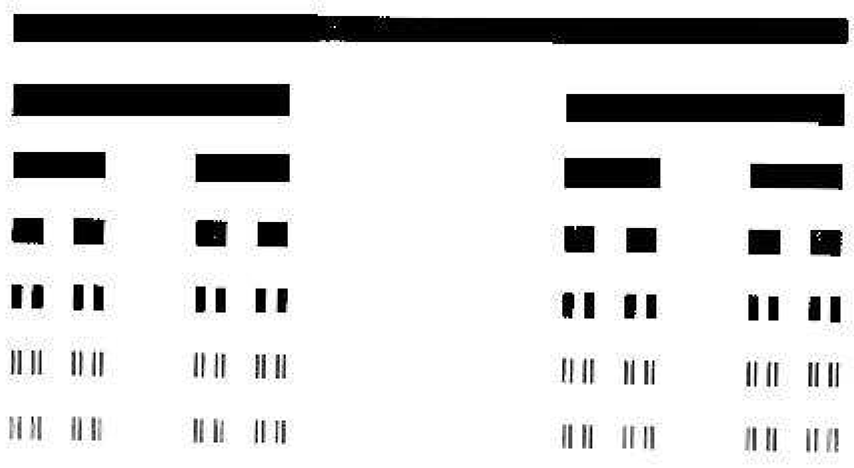

Fig. 2.1: The Cantor set emerges as the middle third is removed from a segment and iterated infinitely.

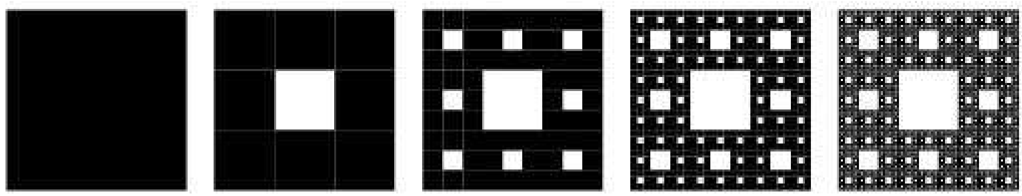

Fig. 2.2: The Sierpinski carpet is the intersection of all the sets in this sequence

In the next section we introduce only the classical fractal note as Cantor set.

2.3.1 The Cantor set

Georg Cantor, a mathematician from the late 19th to early 20th centuries, introduced us to a contrived fractal which has come to be known as *Cantor Set* (Fig. 2.1), or, for reasons which will become obvious, Cantor Dust. The set has some interesting properties which have led to further research and discovery in fractals and chaos theory. The so called Cantor set is a famous construction in mathematics, much older than the relatively recent interest in Chaos and Fractal Geometry. It is the simplest example of a fractal. A Cantor set is best characterized by describing its generation. In this section we will see in detail the first of example 2.3.1. The general approach for constructing the set is by starting with a line segment, and then removing the middle third of the segment. This leaves two sub-segments each with length one-third of the original one. The geometric construction is iterated by removing the middle thirds of these, and so on, *ad infinitum*. At every stage of the process, the result is *self-similar* to the previous stage, i.e., identical upon rescaling. This triplet set is not the only possible Cantor set: any arbitrary cascaded removal of portions of the line segment may form the repetitive structure.

Formally, the Cantor set has the following definition:

Definition 2.3.6: To define the Cantor set in $\Re$, we first define the following sequence of subset of $\Re$, $C_0, C_1, ...$,that satisfy the following conditions:

1. $C_0 = [0, 1]$;
2. Each C_n is the union of 2^n closed intervals and $C_0 \supset C_1 \supset ... \supset C_n \supset C_{n+1}...$
3. C_n is constructed by removing the open middle third of each interval in C_{n-1}, i.e., replacing each $[a, b]$ in C_{n-1} by two closed intervals $L[a, b] = [a, a + \frac{1}{3}(b - a)]$ and $R[a, b] = [a + \frac{2}{3}(b - a), b]$;
4. Then we let $C = \bigcap_{n}^{\infty} C_n$ be the Cantor set.

The object that remains when this geometric construction has been iterated into transfinite is the Cantor Set.

The middle thirds Cantor set or Cantor Dust. Since the Cantor set is totally disconnected, it has topological dimension equal to 0.

The construction of the Cantor dust is as follows: starting with the unit interval $[0, 1]$, use $r = \frac{1}{3}$ to scale the interval into 3 equal pieces. Now remove the open interval in the middle, that is $(\frac{1}{3}, \frac{2}{3})$. We continue repeating this process for each remaining segment, i. e., scale each remaining line segment by $r = \frac{1}{3}$, and remove the open middle third of the segment. Notice that since we are removing the middle third, we partition this segment into $N = 2$ equal sub-units.

We can calculate the fractal dimension of the Cantor Dust by plugging $N = 2$ and $r = \frac{1}{3}$ into formula (2.3.5). We obtain

$$D_s = \frac{\log(2)}{\log(3)} \approx 0.63092975... \tag{2.3.6}$$

Also, if we plug in $N = 2$, $r = \frac{1}{3}$, and $D = \frac{\log(2)}{\log(3)}$ into Nr^D then we will indeed obtain 1 for an answer. Thus, the Cantor dust has as its dimension $D_s \approx 0.63092975$. So, the Cantor set possesses non-integer dimension, which fulfills one condition for being a fractal. Since, in the Cantor set, the self-similar dimension is identical to Hausdorff dimension. In general, we can call (2.3.6) fractal dimension of the Cantor set.

We can examine its self-similarity. Notice if we "zoom in" or focus on half of the first iteration, it looks similar to the original set $[0, 1]$ that is, a straight line of one dimension. Now if we "zoom in" on the second iteration, focusing on half of one half of the first iteration, again we see a straight line, resembling the original set. if we only focus on one half of the entire second iteration, then we see an object similar to that of the first iteration. We can generalize this process like this: if we focus on half $(\frac{1}{2^1})$ of the n^{th} iteration, we will see an object similar to C_{n-1}. If we focus in on one fourth $(\frac{1}{2^2})$ of the n^{th} iteration, then we will see an object similar to C_{n-2}, and so on.

We list some of the features of the middle third Cantor set C; as we shall see, similar features are found in many fractals.

(1) C is self-similar. It is clear that the part of C in the interval $[0, 1/3]$ and the part of C in $[2/3, 1]$ are geometrically similar to C, scaled by a factor $1/3$. Again, the parts of C in each of the four intervals of C_2 are similar to C but scaled by a factor $1/9$, and so on. The Cantor set contains copies of itself at many different scales.

(2) The set C has a fine structure; that is, it contains detail at arbitrarily small scales. The more we enlarge the picture of the Cantor set, the more gaps become apparent to the eye.

(3) Although C has an intricate detailed structure, the actual definition of C is very straightforward.

(4) C is obtained by a recursive procedure. The construction consisted of repeatedly removing the middle thirds of intervals. Successive steps give increasingly good approximations C_n to the set C.

(5) The geometry of C is not easily described in classical terms: it is not the locus of the points that satisfy some simple geometric condition, nor is it the set of solutions of any simple equation.

(6) It is awkward to describe the local geometry of C. Near each of its points are a large number of other points, separated by gaps of varying lengths.

(7) Although C is in some ways quite a large set (it is uncountably infinite), its size is not quantified by the usual measures such as length by any reasonable definition C has length zero.

2.3.2 More on Scaling

A different point of view is often useful in discussing scale invariance. If we consider the Koch curve (see Fig. 1.1) to be the graph of a function $f(t)$.

The graph is the set of points (x_1, x_2) in the plane given by relation it is clear that the triadic Koch curve has the property

$$f(\lambda t) = \lambda^{\alpha} f(t)$$

with scaling exponent $\alpha = 1$. Note that for the Koch curve we have that $f(t)$ is not single-valued, but the scaling relation above still holds for any point in the set. The same type of construction may be used on functions defined over all real positive numbers. For example, the power law function $f(t) = bt^{\alpha}$, satisfies the *homogeneity* relation

$$f(\lambda t) = \lambda^{\alpha} f(t) \tag{2.3.7}$$

for all positive values of the scale factor λ. Functions that satisfy this relation are said to be *scaling.*

The power-law function and many other functions that exhibit scaling are not fractal curves. However, scaling fractals have nice scaling symmetry, and most of the fractals discussed by Mandelbrot are scaling in some sense.

2.4 Introduction to Multifractals

The concepts underlying the recent development of what are now called *multifractals* were originally introduced by Mandelbrot [30], [31] in the discussion of turbulence and expanded by Mandelbrot to many other contexts as in physical and mathematical [3]. The application to turbulence was further developed by Fisch and Parisi (1985) and Benzi et al. (1984) (for details of the authors see [25]). Much of recent interest started out with many works of general setting, for details see [32], [33]. Indeed, Multifractals are applied in many contexts such as DLA patterns investigation, earth quake distribution analysis, signal processing and internet data traffic modelling. The goal of this section is to sketch the theory of self similar measures which are usually called *multifractals.*

Let be μ a mass distribution over a region in such a way that the concentration of mass varies widely. If often happens that the sets where the mass concentration has a given density, say where $\mu(B_r(x)) \backsimeq r^\alpha$ for small r, display fractal-like features, with different sets corresponding to different α. A mass distribution or measure μ with this sort of property is called a *multifractal measure*. In detail, we can say that the **Multifractal measures** are related to the study of a distribution of physical or other quantities on a geometric *support.* The support may be an ordinary plane, the surface of a sphere or a volume, or it could itself be a fractal. The idea that a fractal measure may be represented in terms of intertwined fractals subsets having different scaling exponents opens a new realm for the applications of fractal geometry to physical systems.

As with fractals, an exact definition of multifractal measures tends to be avoided. Therefore, an important class of multifractal occurs in connection with attractors in dynamical systems [1].

In general, the **Multifractal analysis** is concerned with describing the local singular behavior of measures or functions in a geometrical and statistical fashion. Much of recent interest started out with many works, for details see [32], [33]. The Multifractals are introduced in order to give a new approach to dealing with data or with geometrical and/ or probabilistic objects and a new set of models for such is at hand.

At the beginning stands the discovery that on fractals local scaling behavior as measured by exponents is not uniform in general. In other words, the exponents are typically not constant in one variable but assume a whole range of values, thus imprinting a rich structure on the object of interest. This structure can be characterized either in geometrical terms making use of the concept of dimension, or in statistical terms based on sample moments. A tight connection between these two descriptions emerges from the *multifractal formalism.*

As we will see, as far as the validity of the multifractal formalism is concerned there is no restriction in choosing a singularity exponent which seems fit for describing scaling behavior of interest, as long as one is consistent in using the same exponents for both, the geometrical and statistical description.

2.4.1 Simple Examples of Multifractals

There are many examples of multifractal but in this section we give an easy construction of one of this.

Consider a geographical map of a continent or island [34]. An example of a measure μ on such a map is "the quantity of ground water". To each subset S of the map the measure attributes a quantity $\mu(S)$, which is the amount of ground water below S, down to some prescribed level. Now divide the map into two equally sized pieces S_1 and S_2. It will not come as a surprise if their respective ground water contents $\mu(S_1)$ and $\mu(S_2)$ are unequal. If S_1 is subdivided further into two equally sized pieces S_{11} and S_{12}, their ground water contents would again different. This subdivision could be extended until the pieces are the size of pores in rocks where some pores are found filled with water and others are found empty. This is a familiar story some countries have more ground water than others parts of a country contain more ground water than others you may drill a well and find flowing water, while your neighbor finds none and so on. Many other quantities exhibit the same behavior, that is, the quantity

$$\mu = \text{ the amount of ground water below } S$$

is an example of a *measure which is irregular at all scales.*

When the *irregularity is the same at all scales,* or at least statistically the same, one says that the measure is *self-similar* or that it is a *multifractal.* A Sierpinski gasket is a self similar set, in the sense that each small piece is identical to the whole after some rescaling and translation; something similar holds for *multifractal measures.*

The Binomial measure and singular behavior

The techniques of multifractal analysis are best illustrated with the *binomial measure* on the unit interval also called the Bernoulli or Besicovitch measures. The binomial measure is a probability measure μ which is defined conveniently via a recursive construction.

Let us consider a uniformly distributed unit of mass on the unit interval $I = [0, 1]$. Start by splitting $I = [0, 1]$ into two sub-intervals I_0 and I_1 of equal length and assign the masses m_0 uniformly on the left half $I_0 = [0, 1/2]$ of the unit interval and $m_1 = 1 - m_0$ uniformly on the right half $I_1 = [1/2, 1]$.

At this stage, the left half carries the measure $\mu(I_0) = m_0$ and the right half carriers the measure $\mu(I_1) = m_1$. In this process, because $\mu(I) = \mu(I_0) + \mu(I_1) = m_0 + m_1 = 1$, the original measure of the unit intervals is conserved; the μ's appear like probabilities, and one says that μ is a probability measure.

With the two sub-intervals one proceeds in the same manner and so forth (it is obvious that the condition $m_0+m_1 = 1$ continues to insure that the original unit of mass is conserved). Indeed, at the stage *two*, the sub-intervals I_{00}, I_{01}, I_{10} and I_{11} have masses m_0m_0, m_0m_1, m_1m_0 and m_1m_1 respectively. In other words, the intervals receive the same treatment as the original unit interval.

At stage n^{th} , the total mass 1 is distributed among the 2^n *dyadic intervals* of order n such that $I_{\epsilon 1}, ..., I_{\epsilon n}$ has mass $m_{\epsilon_1}, ..., m_{\epsilon_n}$. This defines a sequence of measures μ_n, all piecewise uniform. Since $\mu_k(I_{\epsilon_1}, ..., I_{\epsilon_n}) = \mu_n(I_{\epsilon_1}, ..., I_{\epsilon_n})$ for all $k \geq n$ we may define the limit measure μ by $\mu(I_{\epsilon_1}, ..., I_{\epsilon_n}) = m_{\epsilon 1}, ..., m_{\epsilon n}$, in other words, μ_n converges weakly towards μ. By construction, the restrictions of μ to the intervals I_0 and I_1 have the same structure as μ itself. Indeed, they are reduced copies of μ where the reductions in space and mass are by $1/2$ and m_i respectively.

Hence, μ is *self-similar* in a very strict way for all intervals $[a, b]$

$$\mu[a, b] = m_0\mu([2a, 2b]) + m_1\mu([2a - 1, 2b - 1]).$$

Another way of defining μ is the following. Let $x = .\sigma_1\sigma_2...\sigma_n$ be the *dyadic representation* (or x have the binary expansion) of a point in $[0, 1]$:

$$x = \sigma_1 2^{-1} + \sigma_2 2^{-2} + ... + \sigma_n 2^{-1} \qquad \text{with} \qquad \sigma_i \in [0, 1].$$

Here we don' t have to care about points with multiple expansion since our results concern *almost all points* x. Imagine that the digits n are picked randomly such that $P[\sigma_n = i] = m_i$ independently of n. With dyadic point it is possible to give an other definition of μ; in this way μ is the law or probability distribution of the corresponding point x on $[0,1]$.

In conclusion, this measure μ has no density, unless $m_0 = m_1 = 1/2$. More precisely, $M(x) = \mu([0,x])$ has zero derivative almost everywhere. Nevertheless any coarse graining of μ, e. g. through dyadic intervals $I_{\epsilon_1}, ..., I_{\epsilon_n}$ as above will naturally result in a distribution with density It is therefore essential to understand the limit behavior of such an approximation.

The absence of a density for μ is responsible for its erratic, or "fractal" appearance. It is the aim of multifractal analysis to characterize this erratic behavior. The multiplicative construction of μ plays a crucial role in the multifractal analysis.

The construction of the binomial measure is important in order to introduce the *cascade.*

The innovation of the multifractal analysis is found in the *multiplicative iterative schemes* that are essentially different from the additive ones. Indeed, if we consider a single process fragments a set into smaller and smaller components according to a fixed rule, and at the same time fragments the measure of the components by another rule. Such a process is called *multiplicative process* or *cascade.* Hence, multiplicative processes are a very important paradigm in the theory of multifractals.

We recall the multiplicative construction of μ; in this way it is clear that the mass of a sequence of intervals $\mu(I_{\epsilon_1}, ..., I_{\epsilon_n})$ will decay roughly exponentially fast as the $I_{\epsilon_1}, ..., I_{\epsilon_n}$ shrinks down to a point x, say approximately as $2^{-n\alpha(x)}$. The $\alpha(x)$ exponent could be view as a generalization of the local degree of differentiability of $M(x) = \mu([0,x])$. Indeed, $M(x') - M(x) \backsimeq |x' - x|^\alpha$ is called *Holder continuity* of order α at x (where $M(x) \neq 0 = \int_0^x M(x')dt$). We can write the exponent as

$$\alpha_n(x) = \frac{\log \mu(I_{\epsilon_1}, ..., I_{\epsilon_n})}{\log |I_{\epsilon_1}, ..., I_{\epsilon_n}|} = -\frac{1}{n} \log_2 \mu(I_{\epsilon_1}, ..., I_{\epsilon_n}), \qquad \text{and } \alpha(x) = \lim_{n \longrightarrow \infty} \alpha_n(x)$$

whenever this limit exists.

In general, $\alpha(x)$ exist for many points x and that it takes quite different values depending on the dyatic expansion $x = \sum_k \epsilon_k 2^{-k}$. Let $l_n(x)$ be the number of ones among the first n binary digit of x. In this way we can find $\mu(I_{\epsilon_1}, ..., I_{\epsilon_n}) = m_0^{n-l_n(x)} m_1^{l_n(x)}$ and the exponent assuming this form:

$$\alpha(x) = - \lim_{n \longrightarrow \infty} \frac{n - l_n(x)}{n} \log_2 m_0 + \frac{l_n(x)}{n} \log_2 m_1$$

We conclude that the exponent $\alpha(x)$ can take all values between $\log_2 m_0$ and $\log_2 m_1$ and some of these values will be assumed more likely than others. In particular, the points x where $\alpha(x)$ assumes a given value α will typically form highly interwoven fractal sets, whence the term *multifractal*. Therefore, the term fractal is not so much referring to this fractured appearance as rather to the fact that aforementioned sets have a dimension which is not integer. For more details see [33].

2.4.2 Characterization of Multifractals: Multifractal spectra and formalism

Given a compact K in Euclidean space $\Re^d$, such as the attractor of the dynamical systems, the notion of Hausdorff dimension [1] has been used successfully to characterize K [35].

But one single number such as the dimension is usually too crude and can only describe a global aspect of the geometry of K. More subtle structures may be detected when considering an appropriate measure with support K. Moreover, fractals sets are often insufficient in order to model nature. In a dynamical system, e.g. many essential features such as the long time behaviour of orbits can not be represented by a set, but rather by a measure. To give a second example, fractal sets, may approximate porous media but not their content of some liquid. So, measures have become of increasing interest, in particular their local properties.

A priori we consider the metric space S of dimension d and a distribution of points in S that have a form of a Borel measure μ (with bounded support K). Hence, each point in a set S have probability $\mu(S)$. Therefore, if this distribution is singular one cannot describe it by means of a density and multifractal analysis proves useful in characterizing the complicated geometrical properties of μ.

Hence, the aim is to classify the singularities of μ by strength; this, strength is measured as a singularity exponent $\alpha(x)$, called *Holder exponent.* If the points have the equal strength than they lie on *interwoven fractal sets* K_α :

$$K_\alpha = \{x \in \Re^d : \alpha(x) = \lim_{B \longrightarrow \{x\}} \frac{\log \mu(B)}{\log |B|} = \alpha\}; \tag{2.4.1}$$

Hence, we can think K as the union of infinitely many interwoven subsets K_α, usually fractals, with homogeneous concentration of μ. Based on this motivation μ has been termed *multifractal* with the *multifractal decomposition* K_α.

In (2.4.1) B is a ball that containing x and that its diameter $|B| \longrightarrow 0$.

To be more precise let $B(x, \delta)$ denote the closed ball of radius δ centered in x. The quantities

$$\overline{d}_\mu(x) = \lim\sup_{\delta \longrightarrow 0} \frac{\log \mu(B(x,\delta))}{\log \delta}, \qquad \underline{d}_\mu = \lim\inf_{\delta \longrightarrow 0} \frac{\log \mu(B(x,\delta))}{\log \delta}$$

are called *upper (lower) pointwise dimension at x.* When they coincide, the common value is denoted by $d_\mu(x)$. In multifractal theory, one is interested in the Hausdorff dimension of sets like

$$K_\alpha = \{x : \overline{d}_\mu(x) = \underline{d}_\mu(x) = \alpha\}$$

The following expression give us the size of the sets K_α and in particular the characterization of the geometry of the singular distribution μ

$$f_H(\alpha) = \dim(K_\alpha) \tag{2.4.2}$$

in detail with (2.4.2) we obtain (as in [1]) the *Hausdorff dimension* of the singular distribution μ.

Thereby, the Legendre transform has turned out to be a useful tool linking (2.4.2) as a function of α, called the *multifractal spectrum* of μ, with the *singularity exponents* $\tau(q)$, which are given by

$$\tau(q) = \lim\sup_{\delta \longrightarrow 0} \frac{\log s_\delta(q)}{-\log \delta} \qquad \text{with} \qquad s_\delta(q) = \sum_{\mu(B) \neq 0} \mu(C)^q \tag{2.4.3}$$

In (2.4.3), the sum runs over a partition of $\Re^d$ into cubes C of size δ .

According to the particular interests different notions of singularity exponents and dimension distributions have been developed in various fields such as measure theory, dynamical systems and applied mathematics, i.e. with emphasis on box-counting methods.

The *self-similar measures* are probably the best known multifractals. The multifractal spectrum for a large class of such measures has been calculated in [37].

Thereby, the code space is an invaluable tool. As an interesting corollary (we can find it in subsection 3.3 of [36]) one has

$$\overline{d}_\mu(x) = \underline{d}_\mu(x) = \alpha_1 \qquad \text{for } \mu \text{ almost every } x \tag{2.4.4}$$

where $\alpha_1$does not depend on x. But note, that the range of $d_\mu(x)$ is a whole interval $[\alpha_\infty, \alpha_{-\infty}]$. The multifractal spectrum of a broader classes of invariant measures have been found by Falconer and al.

If we consider μ a measure supported by a bounded region of $\Re^d$, with total mass $\mu(\Re^d) = 1$. The support of μ itself may or may not be a fractal. For each $0 < \delta < 1$, let N_δ the number of cubes C of size δ with *coarse Holder exponent* $\alpha(C)$[1] roughly equal to α.

Then we consider the following object called *large deviation spectrum*

$$f_G(\alpha) = \lim_{\epsilon \longrightarrow 0} \limsup_{\delta \longrightarrow 0} \frac{\log N_\delta(\alpha, \varepsilon)}{\log 1/\delta}. \tag{2.4.5}$$

If we consider the footnote, we can write N_δ in this way:

[1] In order to give the form of *coarse Holder exponent*, we consider G_δ the set of all cubes of the form

$$C = [l_1\delta, (l_1 + 1)\delta) \times \ldots \times [l_d\delta, (l_d + 1)\delta)$$

with integer $l_1, \ldots, l_d$ and with $\mu(C) \neq 0$.Then we set

$$C^* = [(l_1 - 1)\delta), (l_2 + 2)\delta), \times \ldots \times [(l_d - 1)\delta, (l_d + 2)\delta).$$

Hence, we define with

$$\alpha(C) = \frac{\log \mu(C^*)}{\log \delta}$$

the *coarse Holder exponent*.

In order to understand the use of C^* instead of C you can see [36].

$$N_\delta(\alpha,\varepsilon) = \text{number}\{C \in G_\delta : \alpha(C) \in (\alpha-\varepsilon, \alpha+\varepsilon]\}.$$

In particular we remember that f_G is not the box dimension [2] of K_α but it is a function explained in statistical terms. Indeed, the number N_δ of cubes in G_δ behaves roughly as $N_\delta \backsimeq \delta^{-D_0}$ where D_0 denotes the box dimension of the support of μ. It follows that $f_G \leq D_0$ for all α. If we suppose that one picks a cube C out of G_δ randomly and its coarse Holder exponent is given by $\alpha(C) = \log\mu(C^*)/\log\delta$.Then the probability of finding $\alpha(C) \backsimeq \alpha$ behaves roughly like

$$N_\delta(\alpha,\varepsilon)/N_\delta = P_\alpha[\alpha(C) \backsimeq \alpha] \backsimeq \delta^{D_0 - f_G(\alpha)} \tag{2.4.6}$$

This is the statistical interpretation of f_G. In the limit $\delta \longrightarrow 0$ the only Holder exponent which is observed with non-vanishing probability is α_0, where $f_G(\alpha_0) = D_0$. For more details see [33].

Of great interest are the ergodic invariant measures in the theory of dynamical systems. Relations between dimension like quantities (such as generalized dimensions and Hausdorff dimension) and characteristics of dynamical systems (such as Lyapunov exponents, entropy and pressure) are given in many papers (as is pointed out in: [36], [40], [41], [39]). Thereby, some authors develop own notions of singularity exponents which serve as a powerful tool, but which only apply to the special situations under consideration. Pesin gives a survey of different notions of generalized spectra for dimensions which apply to arbitrary Borel measures μ.

The approach by Cutler [38] is tailored to measures theory and applies to finite Borel measures μ, providing a dimension distribution $\widehat{\mu}$ of a random variable $\widehat{\alpha}(x)$, which is related

[2] The box-counting dimension is perhaps the simplest notion amongst the variety of fractal dimensions in use; see Falconer [1]. For every non-empty bounded subset $S \subseteq [0,\infty)$, let $N_\varepsilon(S)$ be the smallest number of intervals of length (at most) $\varepsilon > 0$ which can cover S. The *lower* and *upper box-counting dimensions* of S are defined as

$$\underline{\dim}_B(S) = \lim_{\varepsilon\to 0^+}\inf \frac{\log N_\varepsilon(S)}{\log 1/\varepsilon} \qquad \overline{\dim}_B(S) = \lim_{\varepsilon\to 0^+}\sup \frac{\log N_\varepsilon(S)}{\log 1/\varepsilon}$$

respectively. When this two quantities are equal, their common value is referred to as the *box-dimension* (or also the Minkowski dimension) of S.

to $\underline{d}_\mu(x)$ through $\widehat{\mu}([0,\alpha]) = \mu(C_\alpha)$. In (2.4.4), $\widehat{\mu}$ reduces to the Dirac measure concentrated in α_1.

Finally, Falconer developed a *multifractal formalism* [1] which is based on box-counting methods [42]. The advantage of such an approach is its relevance in numerical simulations. Unfortunately, (2.4.3) turns out to be unsatisfactory for reasons of convergence as well as for an undesired dependence on coordinates. The difficulties (as with the notions of Pesin) are imperceptibly hidden in the negative q domain.

The multifractal formalism is not aim of this section but we will show some important result about it in the following.

In order to give some information about multifractal formalism, we will show an important tool in multifractal theory that is the Legendre transform.

First, we give the follows Ellis' theorem [46].

Theorem 2.4.1: Assume that the **moment generating function**

$$c(q) = \lim_{n\longrightarrow\infty} \frac{-1}{n} \log_2 E[\exp(q \log \mu(C_n^*(x)))]$$

exists and is convex and differentiable for all $q \in \Re$. Then,

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{n\log 2} P_n[| - \frac{1}{n}\log_2 \mu(C_n^*(x)) - \alpha| \ \leq \varepsilon] = c^*(\alpha)$$

where $c^*(\alpha) = \inf_q(q\alpha - c(q))$ is the Legendre transform of c.

So, it is natural to introduce the *partition function* $\tau(q)$ in (2.4.3); indeed,

$$\tau(q) = \lim_{\delta\longrightarrow 0} \frac{\log S_\delta(q)}{\log \delta} \qquad \text{with} \qquad S_\delta(q) = \sum_{C\in G_\delta} \mu(C^*)^q.$$

We can find the definition of C^* in footnote (1) of this Chapter.

As a matter of fact, $\tau(q)$ stands at the beginning of multifractal analysis and has since played a central role.

We can note that in (2.4.3) $s_\delta(0)$ simply counts the number of cubes with non-vanishing measure. Thus, $-\tau(0)$ is actually the box-dimension of the support of μ, i.e..

$$D_0 = -\tau(0)^3$$

It follows then the definitions that $c(q) = \tau(q) - \tau(0) = \tau(q) + D_0$. If we assume the Ellis' theorem (2.4.1), i.e assuming that $\tau(q)$ exists and is differentiable, it follows that (2.4.6) holds with $c^* = f_G(\alpha) - D_0$, i.e.

$$f_G(\alpha) = \tau^*(\alpha). \tag{2.4.7}$$

This has been termed the multifractal formalism. The similarity to the well-known thermo-dynamical formalism ([33] and [45]) is immediate.

We note that $\tau(q)$ is obtained by averaging, it depends more regularly on the data than $f_G(\alpha)$ is easier to compute; in particular, it contains in general less information than $f_G(\alpha)$. Indeed, the partition function $\tau(q)$ is always convex since $s_\delta(q)$ is convex for all δ.In general, it is not necessarily differentiable in every q and the multifractal formalism may not hold for all α.At this point, i is natural to introduce the *Legendre spectrum*

$$f_L(\alpha) = \tau^*(\alpha). \tag{2.4.8}$$

This spectrum is sometimes referred to as obtained by the *method of moments.*

While (2.4.8) may be wrong for certain α, the opposite relation holds for all q as in shown in [33] and [44]:

Theorem 2.4.2:

$$\tau(q) = f_G^*(q) = \inf_{\alpha\in\Re}(q\alpha - f_G(\alpha)) \tag{2.4.9}$$

By means of (2.4.9) it is easy to calculate the singularity exponents once the spectrum is known. In typical applications however one will meet the converse situation: one would like to be able to deduce the spectrum from the singularity exponents. This would be

[3] Where D_0 denotes the box dimension of the support of μ and represent the following :

$$D_0 = f_G(\alpha_0)$$

with $f_G(\alpha_0)$ *large deviation spectrum.*

straightforward if differentiability and concavity of the spectrum would be known *in advance.* Such properties can be established a priori only for a multifractal formalism.

As consequences of (2.4.9), we have :

1. D_0 is indeed the maximal value of f_G in general;
2. $f_L = \tau^* = f_G^{**}$ is the concave hull of $f_G \Longrightarrow f_G(\alpha) \leq f_L(\alpha)$.
3. it follows than even a not everywhere differentiable $\tau(q)$ determines $f_G(\alpha)$ at least in its concave points.

An alternative way of displaying the scaling of moments is through the so-called *generalized dimensions* $D_q = \tau(q)/(q-1)$.They are interesting of their own: in the case of dynamical system they are directly observable from the longtime behaviour of orbits. Moreover, they depend more regularly on the data $\mu(C)$ and are easier to handle analytically and numerically.

We conclude the section by noting that in all generality we have

$$f_H(\alpha) \leq f_G(\alpha) \leq f_L(\alpha) \tag{2.4.10}$$

if the equality (2.4.10) holds for a particular measure μ then the *multifractal formalism* is said to hold for μ.

For more details see [33] and [43].

2.5 Cantorian Space-time E infinity ($\varepsilon^{(\infty)}$): some fundamental concepts

In what follows we would like to give a short account of the so-called *E Infinity theory* starting from El Naschie in 1995 [53], [54], [55]; this $\varepsilon^{(\infty)}$ is a physical space-time, i.e. an infinite dimensional fractal space, where time is specialized and the transfinite nature manifests itself. El Naschie's Cantorian space-time is an arena where the physics laws appear at each scale in a self-similar way linked to the resolution of the act of observation.

The idea of using fractals to model the space goes back to some pioneering works by Feynmann, Zeldovich, Ord, Nottale, Svozil as well as Mohamed El Naschie [56], [57], [58], [59], [60], [61].

In the last 10 years, following El Naschie's vision of *E Infinity,* an International Community has studied various physical phenomena in the context of Cantorian space-time. Among these, Sidharth considered many aspects of *E Infinity* [62].

The main conceptual idea of Mahamed El Naschie [63] is a sweeping generalization of what Einstein did in his general theory of relativity, namely introducing a new geometry for space-time which differs considerably from the space-time of our sensual experience. This space-time is taken for granted to be Euclidean. By contrast, general relativity persuaded us that the Euclidean $3+1$ dimensional space-time is only an approximation and that the true geometry of the Universe in the large, is in reality a four dimensional curved manifold. In $\varepsilon^{(\infty)}$ El Naschie takes a similar step and allege that space-time at quantum scales is far from being the smooth, flat and passive space which we use in classical physics [64],[65], [66]. On extremely small scales, at very high observational resolution equivalent to a very high energy, space-time resembles a stormy ocean [64]. The picture of a stormy ocean is very suggestive and may come truly close to what we think the high energy regime of the quantum world probably looks like (see Figg. 2.3-2.7). However such a picture is not accessible to mathematical formulation, let alone an exacting solution. The crucial step in $\varepsilon^{(\infty)}$ formulation was to identify the stormy ocean with vacuum fluctuation and in turn to model this fluctuation using the mathematical tools of non-linear dynamics, complexity theory and chaos [64], [67], [68]. In particular the geometry of chaotic dynamics, namely fractal geometry is reduced to its quintessence, i.e. *Cantor sets* and employed directly in the geometrical description of the fluctuation of the vacuum. How this is done and how to proceed from there to calculating for instance the mass spectrum of high energy elementary particle is what the author explains in his papers [63].

As is well known, special relativity fused time and space together, then came general relativity and introduced a curvature to space-time. Subsequently Kaluza and later on Klein added one more dimension to the classical four in order to unify general relativity and electromagnetism. From this time on, the dimensionality of space-time played a paramount role in the theoretical physics of unification leading to the introduction of the 26 dimensions of string theory, the 10 dimensions of super string theory and finally the Heterotic string

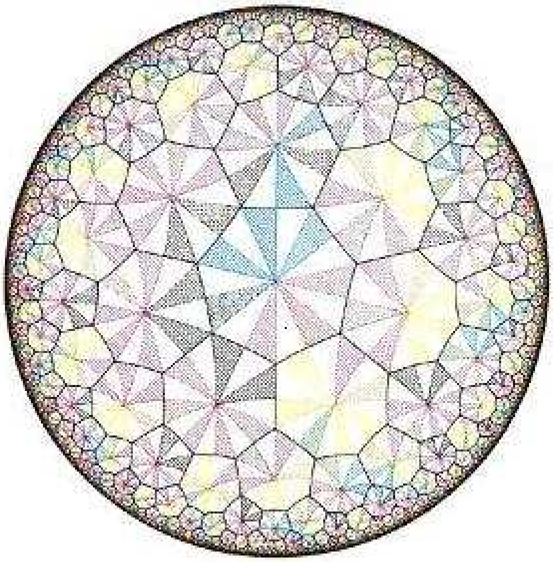

Fig. 2.3: Tiling the plane using Klein's modular curve in the Beltrami-Poincaré representation. *E* infinity theory alleges that the quantum gravity of space-time is a hyperbolic fractal on a Klein modular group akin to what in shown in the figure. The relevance to high energy physics is more direct than one may suspect.

theory dimensional hierarchy 4, 6, 10, 16 and 26 [69]. This is all apart from the so-called abstract or internal dimensions of various symmetry groups used.

By contrast, in $\varepsilon^{(\infty)}$ theory the author admits formally infinite dimensional "real" space-time. However, this infinity is hierarchical in a strict mathematical way and he shows that although $\varepsilon^{(\infty)}$ has formally infinitely many dimensions, seen from a distance, i.e. at low resolution or equivalently at low energy, it mimics the appearance of a four dimensional space-time manifold which has only four dimensions. Thus the four dimensionality is a probabilistic statement, a so-called expectation value.

It is remarkable that the Hausdorff dimension of this topologically four dimensional like "pre" manifold is also a finite value equal to $4+\phi^3$, where $\phi = (\sqrt{5}-1)/2$ with the remarkable self-similar continued fracture representation (which is in a sense self-similar) :

$$4 + \phi^3 = 4 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \dots}}} \tag{2.5.1}$$

There are various ways for deriving the result (2.5.1) which was given in detail in numerous El Naschie's publications. However, maybe the simplest and most direct way is to

Fig. 2.4: A depiction of T. Right's cosmos as a form of sphere packing on all scales.

proceed from the mathematical definition of $\varepsilon^{(\infty)}$.

2.5.1 Definition of the E-Infinity space and some fundamental concepts

If we focus our attention on hierarchy and self-similarity rather than on mathematical transfiniteness, then one may be surprised to see an unsuspected long history of ideas which bear a striking resemblance to the geometrical concept of E infinity.

Definition 2.5.1: *E Infinity refers to the limit set of a pre-geometry model of the transfinite extension of a projective Borel hierarchy.* [70]

From the definition of the above and in particular the definition of Borel sets and projective hierarchy [70], it follows that if the sets involved in the Borel set are taken themselves to be transfinite Cantor sets (Fig. 2.7), then the Hausdorff dimension of *E Infinity* could be written as

$$\langle DimE - \infty \rangle_H = \sum_0^\infty n(d_c^{(0)})^n \tag{2.5.2}$$

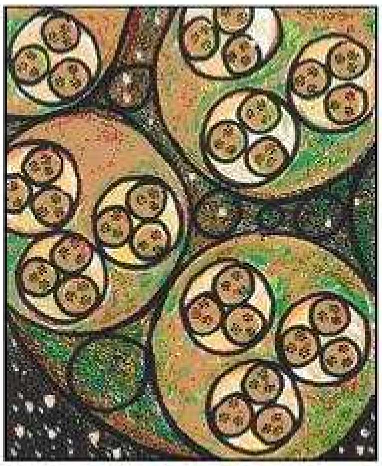

Fig. 2.5: A fractal-like universe, with clusters of clusters ad infinitum as envisaged by the Swedish astronomer C. Charlier who lived between 1862 and 1934.

where $d_c^{(0)}$ is the Hausdorff dimension of the involved transfinite sets where the superscript refers to the Menger- Urysohn dimension of the one dimensional Cantor set, namely $1-1=0$ (for more details see [64], [65], [67], [68]). Suspending the classical triadic Cantor set in n-dimensional space through amplification with a certain geometrical probability quotient, it was shown that this leads to the following Hausdorff capacity dimensions, $d_c^{(n)}$:

$$\begin{aligned} d_c^{(0)} &= \log 2/\log 3; & d_c^{(5)} &= 6.31067; \\ d_c^{(1)} &= 1; & d_c^{(6)} &= 10.00218; \\ d_c^{(2)} &= \log 3/\log 2 = 1.58496; & d_c^{(7)} &= 15.85309; \\ d_c^{(3)} &= 2.51210; & d_c^{(8)} &= 25.12655; \\ d_c^{(4)} &= 3.98159; \end{aligned}$$

where $n = d_M^{(n)}$ is the Menger-Urysohn dimension for $n \geq 0$ (for more details see [59]). These results were recently reinforced using statistical mechanics in conjunction with a cellular space setting for which the following Gibbs-Shanon entropies $S_S^{(n)}$ were found:

$$\begin{aligned} S_S^{(2)} &= 0.69314; & S_S^{(6)} &= 10.3837; \\ S_S^{(3)} &= 1.7351; & S_S^{(7)} &= 16.0277; \\ S_S^{(4)} &= 3.68148; & S_S^{(8)} &= 25.101 \cong d_c^{(8)} \\ S_S^{(5)} &= 6.1202; \end{aligned}$$

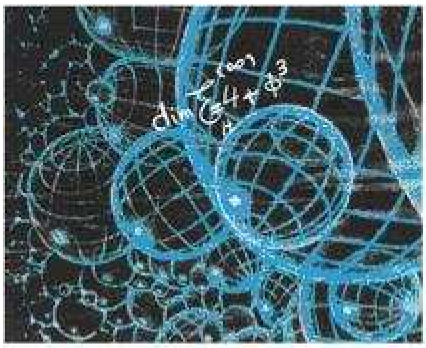

Fig. 2.6: An artistic impression of E- Infinity space-time published by El Naschie. The figure represents a form of space made up of turbulent disorderly packed 3D spheres. E infinity space is similar only it has infinitely more dimensions.

There are several interesting points here which may be relevant to the remarks of Hawking and Feyman. First for $n < 4$ we have $d_c^{(n)} < d_M^{(n)} = n$ while for $n > 4$ we have $d_c^{(n)} >> d_M^{(n)}$. The jump takes place exactly at $n = 4$ also for $S_S^{(n)}$. It is only when $n = 4$ that we have quasi ergodic behaviour for which $d_c^{(n)} \cong d_M^{(n)} = n = 4$. Putting it in *stability* terms we may say that for $n < 4$ our "world" set is stable but could not account for physical reality as we know it while for $n > 4$ the set is totally unstable, in fact, *chaotic* [59].

Now there is a well known theorem due to Mauldin and Williams which states that with a probability equal to one, a one dimensional randomly constructed Cantor set will have the Hausdorff dimension equal to $(\sqrt{5} - 1)/2 = 0.618033$, i.e. the **golden mean** ϕ [64], [65]. Setting $d_c^{(0)} = \phi$ one finds

$$\begin{aligned}\langle DimE - \infty\rangle_H &= (0)(\phi)^0 + (1)(\phi)^1 + (2)(\phi)^2 + (3)(\phi)^3 + ...\\ &= 4 + \phi^3\\ &= (1/\phi)^3\\ &= 4.236067977...\end{aligned}$$

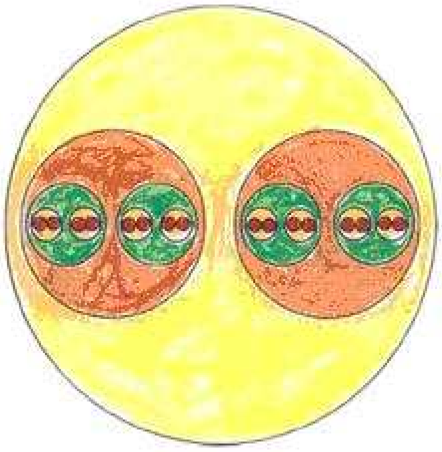

Fig. 2.7: An alternative two dimensional construction of a topological equivalent to the Cantor set using pairs of circles. The E infinity limit set is very similar but has infinitely many dimensions and not only two like her.

as anticipated. It is now instructive to contemplate the following. The intersection rule of sets shows that we can lift $d_c^{(0)}$ to any dimension n as follows

$$d_c^{(n)} = (1/d_c^{(0)})^{n-1}$$

If we consider $d_c^{(0)} = \phi$ and $n = 4$, one finds

$$d_c^{(4)} = (1/d_c^{(0)})^3 = 4 + \phi^3 = (1/\phi)^3 = 4.236067977...$$

In other words, we can write (2.5.2) as

$$\langle DimE - \infty \rangle_H = d_c^{(4)} = 4 + \phi^3 \tag{2.5.3}$$

The (2.5.3) shows that the expectation value of the Hausdorff dimension of E Infinity is $4+\phi^3$ but its intrinsic embedding "expectation" dimension is exactly 4 and that although the formal dimension is infinity. Indeed, the expression $\sum_0^\infty n(d_c^{(0)})^n$ in (2.5.2) may be regarded as the sum of the weighed $n = 1$, $n = 2$, $n = 3, ...$ dimensions where the weights are the golden mean and its power. That is why *E Infinity is hierarchical.* Note that intrinsic embedding is just another name for the Menger- Urysohn dimension and that our intuitive embedding dimension for $d_c^{(0)}$ is not zero, but one. The same is for $d_c^{(4)}$ it is 5 and not 4.

Now if we look closely at $d_c^{(2)} = 1.58496$ we clearly recognize it as the Hausdorff dimension of a *Serpinski triangular space* with its well-known infinite hierarchy of semi-loops. It is thus not particularly difficult to imagine how these loops evolve with increasing n to become for $n = 4$ almost identical to a space filling Peano like curve.

Remark 2.5.1: It then turns out that the limit set of any Kleinian like group is a set which is best described in terms of chaotic Cantor sets and E Infinity. This fact is clear from the work of Mumford et al. [68]. In particular, it was clear in many works the following result:

$$\langle DimE - \infty \rangle_H = 4 + \phi^3 = \langle d_c \rangle =\sim \langle n \rangle$$

is just twice the isomorphic length of the so called Penrose- hyperbolic fractal tiling

$$l \leq \frac{1}{2}(4 + \phi^3)(\rho)$$

where ρ is the radius of the circular region considered.

Indeed, if one projects the space-time of vacuum fluctuation on a Poincare circle we will see a hyperbolic tesselation of this circle with predominantly Klein curve like geometry which ramifies at the circular boundary exactly as in many of the famous pictures of the artist M. Escher. Fore more details see [59].

It is important to look closely at the continuous fraction representation of $\sim \langle n \rangle$ and $\langle d_c \rangle$:

$$\sim \langle n \rangle = \langle d_c \rangle = 4 + \cfrac{1}{4 + \frac{1}{4 + \frac{1}{4 + \ldots}}}$$

2.5.2 Cantorian space and E-Infinity Cantorian space

In descriptive set theory and the theory of polish spaces it is shown that [71],

Definition 2.5.2: When a space A^N is viewed as the product of infinitely many copies of A with discrete topology and is completely metrizable and if A is countable, then the space is said to be polish.

In particular, when $A = \{0,1\}$, $|A| = 2$, then we call $\mathcal{C} = 2^N$ Cantor space. For A^{-1} defined in an interval $A^{-1} \subset]0,1[$ then $C_F = A^N$ is called a *fuzzy Cantor space.* If $|A^{-1}| = (\sqrt{5}-1)/2$ and $N = n-1$, where $-\infty \leq n \leq \infty$, then $C_F = \varepsilon^{(n)}$ is the E-Infinity Cantorian space. Mohamed El Naschie showed the relationship between the Cantor space $\mathcal{C}$ and $\varepsilon^{(\infty)}$ in [72] (for details see [52]). As he reports: "the relationship comes from the cardinality problem of a Borel set in polish spaces; thus we call a subset of a topological space a Cantor set if it is homeomorphic to the Cantor space".

Preliminaries

Let Ω be a nonempty open set in $\mathbb{R}^m$ ([104]). We denote by $\mathcal{D}(\Omega)$ the set of $C^\infty(\Omega)$ functions with compact support in Ω, $\mathcal{D}(\Omega) := C_c^\infty(\Omega)$.

Definition 2.5.3: A distribution is a linear mapping $T \mapsto < T, \varphi >$ from $\mathcal{D}(\Omega)$ to $\mathbb{R}$, which is (sequentially) continuous, i.e. if $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$, then $< T, \varphi_n > \to < T, \varphi >$. The set of all distributions is called $\mathcal{D}'(\Omega)$.

Each $L^1(\Omega)$ function, say $f \in L^1(\Omega)$ can be regarded as a distribution setting

$$< f, \varphi > = \int_\Omega \varphi(x) f(x) dx.$$

But $\mathcal{D}'(\Omega)$ is much larger, for instance one may consider the Dirac mass centered at 0, with $0 \in \Omega$, δ_0 defining

$$< \delta_0, \varphi > := \int_\Omega \varphi(x)\delta_0 = \varphi(0).$$

Definition 2.5.4: A sequence $\{T_n\}$ in $\mathcal{D}'(\Omega)$ converges to $T \in \mathcal{D}'(\Omega)$ if

$$< T_n, \varphi > \to < T, \varphi >, \quad \text{for every } \varphi \in \mathcal{D}(\Omega).$$

Definition 2.5.5: Let Ω be an open set in $\mathbb{R}^m$. Let $T \in \mathcal{D}'(\Omega)$. Then the derivative of T with respect to x_j is defined as

$$\left\langle \frac{\partial T}{\partial x_j}, \varphi \right\rangle = -\left\langle T, \frac{\partial \varphi}{\partial x_j} \right\rangle$$

for every $\varphi \in \mathcal{D}(\Omega)$.

If $T \in \mathcal{D}'(\Omega)$, the support of T is the smallest closed set K outside which T vanishes, in the sense that $\varphi = 0$ outside K, i.e. $< T, \varphi >= 0$.

We also recall that the derivative operator, defined above is (sequentially) continuous, in the sense that if a sequence of distributions $\{T_n\}$ converges to T in $\mathcal{D}'(\Omega)$, then the sequence $\{DT_n\}$ still converges to DT.

Assume that $S, T \in \mathcal{D}'(\mathbb{R}^m)$, either S or T has compact support, then the convolution of S and T is defined by

$$\langle S * T, \varphi \rangle = \langle S(x)T(y), \varphi(x+y) \rangle \tag{2.5.4}$$

and convolution is easily seen to be a commutative operation.

Theorem 2.5.1: Let S be in $\mathcal{D}'(\mathbb{R}^m)$. Assume that $T_n \to T$ in $\mathcal{D}'(\mathbb{R}^m)$ and one of the following holds:

i) The supports of all the T_n are contained in a common compact set;

ii) S has compact support;

iii) $m = 1$ and the supports of the T_n and of S are bounded on the same side, independently of n.

Then $T_n * S \to T * S$ in $\mathcal{D}'(\mathbb{R}^m)$.

For further details about the Theory of Distributions we refer to [73].

In the remainder of this section we recall well known facts of measure theory for reader's convenience. This section is very much inspired by [74].

Let X be a non empty set and $\mathcal{M}$ a σ-algebra in X (closed to $\emptyset, X$ complementation and countable union).

Definition 2.5.6: Let $(X, \mathcal{M})$ be a measure space and $\mu : \mathcal{M} \to [0, \infty]$. We say that μ is a positive measure if $\mu(\emptyset) = 0$ and μ is σ-additive, i.e., for any sequence $\{E_h\}$ of pairwise disjoint elements of $\mathcal{M}$,

$$\mu\left(\bigcup_{h=0}^{\infty} E_h\right) = \sum_{h=0}^{\infty} \mu(E_h).$$

A positive measure μ such that $\mu(X) = 1$ is called a *probability measure*.

Definition 2.5.7: Let X be a locally compact and separable metric space, $\mathcal{B}(X)$ its Borel σ-algebra (σ-algebra generated by open sets), and consider the measure space $(X, \mathcal{B}(X))$.

Definition 2.5.8: A positive measure on $(X, \mathcal{B}(X))$ is called a Borel measure. If a Borel measure is finite on compact sets, it is called a positive Radon measure.

By $[\mathcal{M}_{loc}(X)]^m$ it is usually denoted the space of the $\mathbb{R}^m$-valued Radon measures on X.

Definition 2.5.9: Let $\mu \in [\mathcal{M}_{loc}(X)]^m$ and let $\{\mu_h\}_h \subset [\mathcal{M}_{loc}(X)]^m$; the sequence $\{\mu_h\}_h$ locally weakly $*$ converges to μ if

$$\lim_{h \to +\infty} \int_X u d\mu_h = \int_X u d\mu$$

for every $u \in C_c(X)$; if μ and μ_h are finite, we say that $\{\mu_h\}_h$ weakly $*$ converges to μ if

$$\lim_{h \to +\infty} \int_X u d\mu_h = \int_X u d\mu$$

for every $u \in C_0(X)$, where we recall that $C_0(X)$ is the space of continuous functions with compact support in X and $C_0(X)$ its completion with respect to the sup norm.

It can be useful to recall that weak $*$ convergence of a sequence $\{\mu_h\}_h$ of finite Radon measures is equivalent to the local weak $*$ convergence together with the condition $\sup_h |\mu_h|(X) < +\infty$.

Next we recall a very useful compactness criterion.

Theorem 2.5.2: If $\{\mu_h\}_h$ is a sequence of finite Radon measures on the locally compact and separable metric space X with $\sup\{|\mu_h|(X) : h \in \mathbb{N}\} < +\infty$, then it has a weakly $*$ converging subsequence. Moreover the map $\mu \mapsto |\mu|(X)$ is lower semicontinuous with respect to the weak $*$ convergence.

Remark 2.5.2: It is useful for our aims to recall that if X coincides with a non empty open set Ω in $\mathbb{R}^m$, then any Radon measure in $\mathcal{M}(\Omega)$ is a distribution, ($< \mu, \varphi >= \int_X \varphi d\mu$) for every $\varphi \in \mathcal{D}(\Omega)$.

Definition 2.5.10: Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be measure spaces, and let $\phi : X \to Y$ be such that $\phi^{-1}(F) \in \mathcal{E}$ whenever $F \in \mathcal{F}$. For any positive measure μ on $(X, \mathcal{E})$ we define a measure $\phi_\sharp \mu(F)$ in $(Y, \mathcal{F})$ by

$$\phi_\sharp \mu(F) := \mu(\phi^{-1}(F)) \quad \text{for every } F \in \mathcal{F}.$$

Given any Radon measure ν on the measure space $(X, \mathcal{E})$, and any subset G in $\mathcal{E}$, with the symbol $\nu \lfloor F$, we mean the measure ν acting on $G \cap E$, for any $E \in \mathcal{E}$.

Hausdorff measure and dimension

The notions of Hausdorff measure and dimension will be needed in the sequel.

Consider the metric space $(\mathbb{R}^m, d)$, where d is the metric induced from the Euclidean norm. Let $A \subset \mathbb{R}^m$ be bounded. By $\mathcal{A}$ we denote the set of sequences of subsets $\{A_i \subset A\}$, such that $A = \cup_i^\infty A_i$.

Let $0 < \varepsilon < +\infty$, and $0 \le s < +\infty$. We define

$$\mathcal{H}^s_\varepsilon(A) = \inf \left\{ \sum_{i=1}^\infty (\mathrm{diam} A_i)^s : \{A_i\} \in \mathcal{A}, \mathrm{diam} A_i < \varepsilon \text{ for every } i \in \mathbb{N} \right\}$$

Clearly $\mathcal{H}^s_\varepsilon(A)$ increases as $\varepsilon \to 0$, hence

$$\mathcal{H}^s(A) = \lim_{\varepsilon \to 0} \mathcal{H}^s_\varepsilon(A) \tag{2.5.5}$$

is well posed.

The next Theorem is proven in [75].

Theorem 2.5.3: Let m be a positive integer. Let A be a bounded subset of $(\mathbb{R}^m, d)$. Then there exists a unique real number $\dim_H \in [0, m]$ such that

$$\mathcal{H}^s(A) = \begin{cases} \infty & \text{if } s < \dim_H \text{ and } s \in [0, +\infty[\\ 0 & \text{if } s > \dim_H \text{ and } s \in [0, +\infty[. \end{cases}$$

Convergence

Let C be the Cantor middle third set. The well known construction we adopt here, is based on an IFS scheme (see [76] for details and [88] where an analogous construction has been performed).

Consider first two points $\{0, 1\} =: K_1$, and two contractive similitudes s_1, s_2 defined as

$$\begin{aligned} s_1 &: x \to \tfrac{1}{3}x \\ s_2 &: x \to \tfrac{1}{3}x + \tfrac{2}{3} \end{aligned} \tag{2.5.6}$$

then applying these functions to $\{0, 1\}$ one obtains

$$\left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\} =: K_2,$$

next one evaluate s_1 and s_2 to the last four points and gets

$$\left\{0, \frac{1}{9}, \frac{2}{9}, \frac{1}{3}, \frac{2}{3}, \frac{7}{9}, \frac{8}{9}, 1\right\} =: K_3,$$

and so on

$$\left\{0, \frac{1}{3^n}, \ldots, c_i^n, \ldots, \frac{3^n - 1}{3^n}, 1\right\} := K_n. \tag{2.5.7}$$

An inductive process leads from the sequence $\{K_n\}$ to C, (see the Fig. 2.8). We also

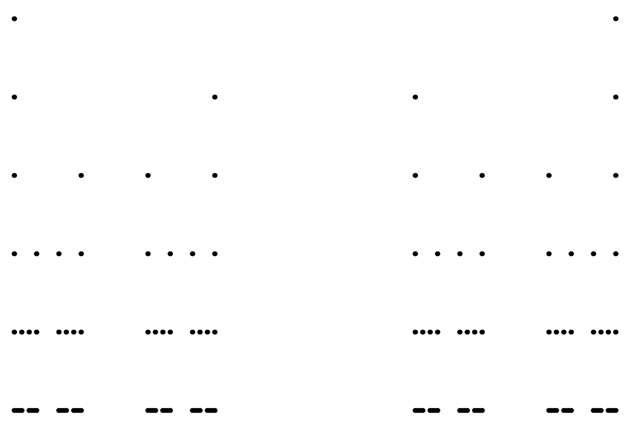

Fig. 2.8: Pre Cantor set

recall that $\dim_H(C) = \frac{\log 2}{\log 3}$ and $\mathcal{H}^{\dim_H}(C) = 1$, (see [1]), where C is the closure of the union of the K_n. Starting from the construction of C presented above we can define a sequence of probability measures on the (locally compact and separable) metric space $(\mathbb{R}, d)$, where d is the Euclidean metric, namely

$$
\begin{aligned}
&\mu_1 = \tfrac{1}{2}(\delta_0 + \delta_1)\\
&\mu_2 = \tfrac{1}{2^2}(\delta_0 + \delta_{\frac{1}{3}} + \delta_{\frac{2}{3}} + \delta_1)\\
&\mu_3 = \tfrac{1}{2^3}(\delta_0 + \delta_{\frac{1}{9}} + \delta_{\frac{2}{9}} + \delta_{\frac{1}{3}} + \delta_{\frac{2}{3}} + \delta_{\frac{7}{9}} + \delta_{\frac{8}{9}} + \delta_1)\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\mu_n = \tfrac{1}{2^n}\textstyle\sum_{i=1}^{2^n}\delta_{c_i^n}\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
\tag{2.5.8}
$$

where c_i^n is the i-th point in 2.5.7 which leads to the construction of C at level n.

By virtue of Theorem 2.5.2 the sequence $\{\mu_n\}$ admits a weakly $*$ converging subsequence $\{\mu_{k_n}\}$. From Definitions 2.5.3 and 2.5.7 it follows that the sequence $\{\mu_{n_k}\}$ converges also in the sense of distributions.

In order to identify the limit probability measure μ, we consider the primitives $\int_\cdot d\mu_n = f_n$, (we recall that if a sequence $\{T_n\} \subset \mathcal{D}'(\mathbb{R})$ converges to T in $\mathcal{D}'(\mathbb{R})$ then the sequence $\int_\cdot T_n$ still converges to $\int_\cdot T$, cf. [73]) where $f_n : \mathbb{R} \to [0,1]$ is the step function below:

$$
f_n(x) = \begin{cases}
0 & \text{if } x < 0\\
\frac{1}{2^n} & \text{if } 0 \le x < \frac{1}{3^n}\\
\frac{1}{2^{n-1}} & \text{if } \frac{1}{3^n} \le x < \frac{2}{3^n}\\
\dots\dots\dots\dots & \\
\dots\dots\dots\dots & \\
1-\frac{1}{2^n} & \text{if } 1-\frac{1}{3^n} \le x < 1\\
1 & \text{if } x \ge 1
\end{cases}
\tag{2.5.9}
$$

The sequence $\{f_n\}$ converges uniformly to the Cantor-Vitali function f.

It can be easily shown that f is increasing and continuous with 'classical' derivative coinciding with 0 a.e. On the other hand one can prove that the distributional derivative of f, namely Df is a probability measure μ supported on C, and it results

$$
\mu = \mathcal{H}^{\frac{\log 2}{\log 3}} \lfloor C. \tag{2.5.10}
$$

Hence

$$
f(t) = \mathcal{H}^{\frac{\log 2}{\log 3}}([0,t] \cap C) \text{ for any } t \ge 0 \tag{2.5.11}
$$

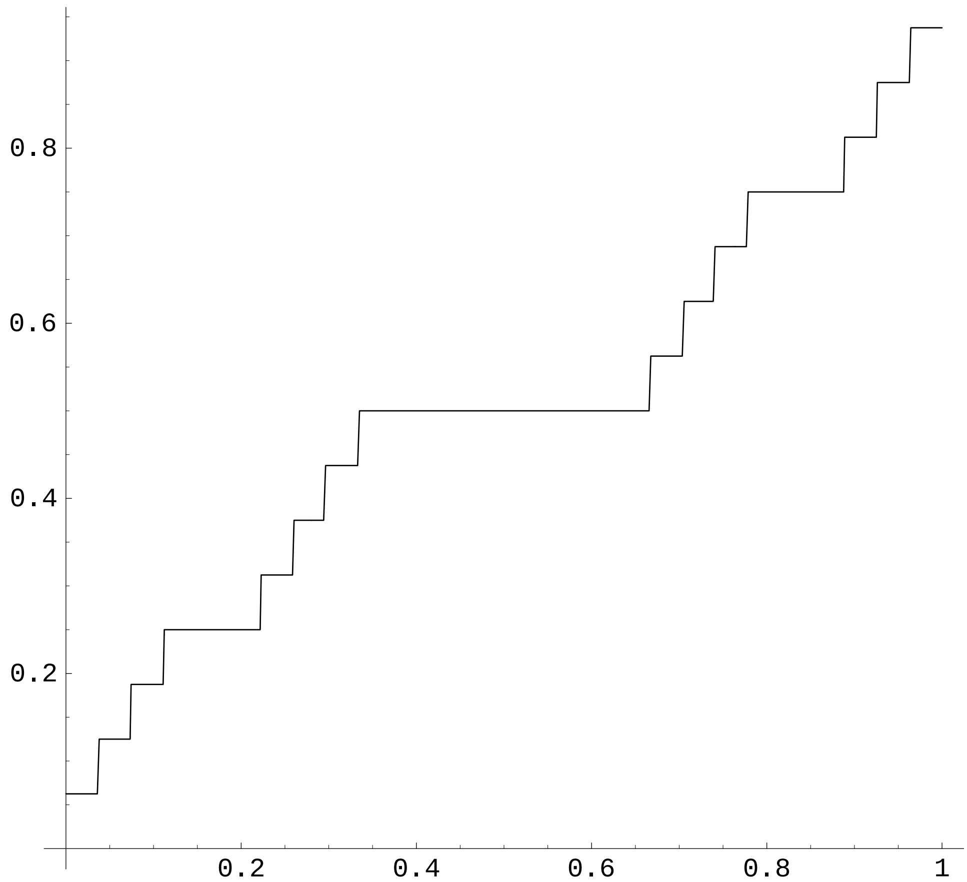

Fig. 2.9: Cantor Vitali function approximant

(see [74] for details).

In Fig. 2.9 there is f_4.

Consequently we can say that the sequence of distributional derivatives $\{Df_n\}$, namely $\{\mu_n\}$ converges in the sense of distributions to the derivative of f, the probability measure Df in 2.5.10, i.e.

$$Df_n = \mu_n \to Df = \mathcal{H}^{\frac{\log 2}{\log 3}} \lfloor C \text{ in } \mathcal{D}'(\mathbb{R}) \tag{2.5.12}$$

We also emphasize that this measure is the only probability measure on C which satisfies a scaling property as C itself does, namely

$$\mu = \frac{1}{2}[s_{1\sharp}(\mu) + s_{2\sharp}(\mu)] \tag{2.5.13}$$

As a consequence we can also say that $\mathcal{H}^{\frac{\log 2}{\log 3}} \lfloor C$ is the limit in the sense of Definition 2.5.10 of the whole sequence $\{\mu_n\}$.

Next taking any distribution (potential) $\Pi \in \mathcal{D}'(\mathbb{R})$ satisfying the assumptions of Theorem 2.5.1, we may define, again keeping in mind the IFS scheme above, a sequence of 'potentials' $\{\Pi_n\}$, defined as

$$\Pi_n(x) := \frac{1}{2^n} \sum_{i=1}^{2^n} \Pi(x - c_i) \tag{2.5.14}$$

where c_i is a generic point as in 2.5.7.

Clearly we may rewrite 2.5.14 as

$$\Pi_n = \Pi * \frac{1}{2^n} \sum_{i=1}^{2^n} \delta_{c_i} = \pi * \mu_n \tag{2.5.15}$$

Again Theorem 2.5.1 and convergence 2.5.12 give us that

$$\Pi_n \to \Pi * \mathcal{H}^{\frac{\log 2}{\log 3}} \lfloor C \text{ in } \mathcal{D}'(\mathbb{R}). \tag{2.5.16}$$

This argument proves the following theorem

Theorem 2.5.4: Let $\{\mu_n\}$ be the sequence of probability measures in 2.5.8 and let Π be any distribution in $\mathcal{D}'(\mathbb{R})$ satisfying the assumption of 2.5.1. Then 2.5.16 holds, with Π_n defined in 2.5.14.

Remark 2.5.3: We stress that if the potential Π is more regular than required by Theorem 2.5.1, the convergence in 2.5.16 can be shown to be much stronger.

Remark 2.5.4: It is worthwhile to mention that the argument above can be easily adapted to other kinds of Fractals, more general than C. The potential Π can be, as already mentioned, very general, thus leaving the opportunity to describe many physical problems. For instance a Gaussian potential will work for describing a barrier or an obstacle on the support where the motion happen. Furthermore the sequence $\{\mu_n\}$ presented in 2.5.8 can be replaced by any other probability measures' sequence converging to $\mathcal{H}^{\frac{\log 2}{\log 3}} \lfloor C$. Our choice was aimed just by the reason of working out a basic case. Clearly other choices are possible, even not probability measures' sequences, but just uniformly bounded ones, thus leading to describe other limit measures μ still supported on the same fractals but with different weights.

2.5.3 E-Infinity Cantorian space-time and stochastic self-similar random processes

If we consider the considerations in the previous section, we have the following scenario with respect to stochastic self-similar processes.

Let $\Re$ be real space and $\gamma_r \in \Re^+$, then we define a self-similar (ss) random process for every $r > 0$,

$$X(s) \stackrel{d}{=} \gamma_r X(rs), \qquad \text{with } s \in \Re \tag{2.5.17}$$

where $\stackrel{d}{=}$ denotes equality as distributions.

The relation (2.5.17) is invariant under the group of positive affine transformations,

$$X \longrightarrow \gamma X, \qquad s \longrightarrow rs, \qquad \gamma_r > 0. \tag{2.5.18}$$

We recall that γ_r satisfies the properties

$$\gamma_{r_1 r_2} = \gamma_{r_1}\gamma_{r_2}, \quad \forall\, r_1, r_2 > 0 \quad \text{with } \gamma_1 = 1 \tag{2.5.19}$$

then it must have the form

$$\gamma_r = r^{-\delta}, \quad \text{with} \quad \delta \in \Re \tag{2.5.20}$$

Thanks to (2.5.20), the relation (2.5.21) becomes

$$X(s) \stackrel{d}{=} r^{-\delta} X(rs), \quad \text{with} \quad s \in \Re \tag{2.5.21}$$

When a process satisfies (2.5.17) or (2.5.21), it is said to be self-similar or $\delta-$self-similar.

A generalization of self-similar random process is obtained by replacing the deterministic scaling factor $\gamma_r = r^{-\delta}$ in (2.5.17) or (2.5.21) with a random variable $\widetilde{\gamma}_r \in \Re_0^+$. This variable is independent of the process to which such a variable is multiplied. Then equation (2.5.21) becomes

$$X(s) \stackrel{d}{=} \widetilde{\gamma}_r X(rs), \qquad \text{with} \quad s \in \Re.$$

Veneziano demonstrated in [77] that $\widetilde{\gamma}_r$ can also be written as $\gamma_r = r^{-\widetilde{\delta}}$ with $\widetilde{\delta}$ real random variable.

We call this kinds of processes *stochastic self-similar random processes* (sss) and the previous ones (ss). This type of processes can be treated in the same theory.

In detail Gupta and Waymire showed that for $0 < r \leq 1$ the sss processes are dilations, while for $r > 1$ the sss processes are contractions [78] and [79].

In [77] the author proved the following relevant theorem:

$$\text{if} \quad \widetilde{\delta}_{r_1} \stackrel{d}{=} \widetilde{\delta}_{r_2} \qquad \text{for some } r_1 \neq r_2 \quad \Longrightarrow \widetilde{\delta} \text{ must be a deterministic constant } \delta.$$

For this fact, one can treat ss and sss random processes in a unique scheme. In [77] the author gives many relevant properties and generalizations to a d-dimensional space.

3. STOCHASTIC SELF-SIMILAR PROCESSES AND RANDOM WALK IN NATURE

During the last few years, the idea of hierarchy and Self-similarity in science first started in cosmology before moving to the realm of quantum and particle physics. Actually, many consequences of a stochastic self-similar and fractal Universe are studied; indeed, it was demonstrated that the observed segregated Universe is the result of a fundamental Self-similar law that we will show in this chapter.

In particular, it is well known the link between the universal scaling law and the Random Walk.

Because of the extraordinary importance of Random Walk and Brownian motion, the most important of all stochastic processes, which can be studied as Fractals, we will start the chapter with a discussion of this processes and their fractal properties; we discuss some aspects of the relation between random walk and Brownian motion.

Moreover, we will show some results obtained by using the context of $\varepsilon^{(\infty)}$ Cantorian space-time in connection with stochastic Self-similar processes in order to give a possible explanation of the segregation of the Universe at fixed scale in terms of Brownian motion.

3.1 Preliminaries: Random Walks and Fractals

Randomness is inherent in all natural phenomena. Even the most perfect crystal has many impurities and other placed at random. Therefore the actual state of even the most perfect system has elements of randomness. There is good evidence that many natural phenomena are best described as fractals. However, if fractals are to be useful in the description of nature we must develop the concepts of *random fractals* (for more details see [25]).

First we consider how to define the Brownian motion the simplest and more interesting stochastic process is called *Random Walk* [[95], [96], [97], [98], [99]]. The simplest version is the one-dimensions random walk, which then may be extended to higher dimensions.

The term "random walk" was originally proposed by Karl Pearson in 1905 [1] in a letter to *Nature* he gave a simple model to describe a mosquito infestation in a forest. Pearson wanted to know the distribution of the mosquitos after many steps had been taken. The letter was answered by Lord Rayleigh, who had already solved a more general form of this problem in 1880, in the context of sound waves in heterogeneous materials.

As its historical origins demonstrate, the concept of the random walk has incredibly broad applicability, and today it is nearly ubiquitous in science and engineering.

We wish to find the probability density function of the sound waves after many steps have been taken. We let $P_N(R)dR$ be the probability of traveling a distance between R and $R + dR$ in N steps. For steps of unit length, Rayleigh showed that as $N \to \infty$,

$$P_N(R) \sim \frac{2R}{N} e^{-R^2/N} \qquad \text{(essentially the Central Limit Theorem)} \qquad \text{(Gaussian behavior)} \tag{3.1.1}$$

Rayleigh had also found the solution for a random walk in $1 - D$, with steps of unit length

$$P_N(R)dR = \frac{1}{\sqrt{2\pi N}} e^{-R^2/2N} dR \qquad \text{(Gaussian process with } \sigma^2 \equiv N)$$

so, spreading about the origin $\sim \sigma$ (standard deviation) $\sim \sqrt{N}$.

We see that the expected distance traveled scales according to the square root of the number of steps, $\langle R^2 \rangle \sim N$, which is typical of "diffusion" phenomena.

Note the "square-root scaling" of the width of the probability density function (pdf), which grows like

$$R \propto \sqrt{\langle r^2 \rangle \, N} \tag{3.1.2}$$

which is characteristic of spreading by "normal diffusion".

[1] See B. Hughes, *Random Walk and Random Environments*, Vol. I, Sec. 2.1 (Oxford, 1995), for excepts and an entertaining historical discussion.

As well known, the random walk represents the movement of a particle in the space, identifying its position to the time n. Such a position depends on the preceding position and on an independent random variable; formally, it is defined as the sum of a sequence $\{Y_i\}$ of independent and identically distributed random variables, for which the total path $X_n = \sum_{i=1}^{n} Y_i$. Alternatively, the random walk process consists of a sequence of discrete steps of fixed length.
The state space of the process X_n will be discrete or continuous corresponding to the variables Y_i (discrete or continuous).

Let us consider specifically the *random walk* in one-dimension.
Suppose that the particle moves on the x-axis in the following hypotheses:

1. the particle occupies the position $X_0 = 0$ to the $n = 0$ instant, with $n \in N_0$
2. the particle occupies the following X_n position for all of the aforesaid moments:

$$X_n = X_0 + Y_1 + ... + Y_n, \tag{3.1.3}$$

where $\{Y_i\}$ is a sequence of independent and identically distributed random variables.
Alternatively, we may write (3.1.3) as:

$$X_n = X_{n-1} + Y_n \qquad (n = 1, 2, ...) \tag{3.1.4}$$

Therefore, the relation (3.1.4) gives the equation of the particle motion; consequently, we obtain the following system of difference equations of the first order:

$$X_0 = 0$$

$$X_1 = Y_1$$

$$X_2 = Y_1 + Y_2$$

$$\vdots$$

By iterating the procedure we have from (3.1.4):

$$X_n = \sum_{i=1}^{n} Y_i \tag{3.1.5}$$

In the specific case in which the random variables Y_i can only take the value 1, 0, -1 with distribution:

$$P(Y_i = 1) = p, \qquad P(Y_i = -1) = q, \qquad P(Y_i = 0) = 1 - p - q,$$

we name the process a *simple random walk*. Sometimes in literature, it is usual to find a simple random walk as one for which each step is either + 1 or - 1 with $p + q = 1$. However, we will assume that $p + q \leq 1$ with $1 - p - q$ as the probability of a zero step.
Besides, we denote with μ and σ^2 the mean value and variance of a step respectively:

1. $\mathrm{E}[Y_i] = \mu = p - q$;
2. $\mathrm{Var}[Y_i] = \sigma^2 = p + q - (p - q)^2 = 4pq$.

where, to calculate the variance we have used the relation $p + q = 1$.
Consequently, we obtain for the entire process

1. $\mathrm{E}[X_n] = n\mu = n(p - q)$;
2. $\mathrm{Var}[X_n] = n\sigma^2 = 4npq$.

If the distribution of the steps -1 and 1 assumes value 1/2:

$$P(Y_i = 1) = P(Y_i = -1) = 1/2, \qquad (n = 1, 2, ...), \tag{3.1.6}$$

it is well known that

$$E[X_n] = 0, \quad \mathrm{Var}[X_n] = 1. \tag{3.1.7}$$

The X_n process with probabilities (3.1.6) is called *symmetric random walk* (see for example [96] pag.321 for more details).

3.1.1 Unrestricted

In the present section we briefly introduce the *unrestricted* random walk ([95], pag. 25). We consider the equation (3.1.5) and suppose that the random walk starts at the origin and that the particle is free to move indefinitely in either directions.
The possible positions of the particle at some times n are $k = 0, \pm 1, ..., \pm n$. To obtain the position of the particle we fix N_1^+, N_2^- and N_3^0 non-negative integers that represent

respectively positive steps, negative steps and zero steps. The integers have to fulfill the simultaneous equalities:

$$N_1^+ + N_2^- + N_3^0 = n, \qquad N_1^+ - N_2^- = k \tag{3.1.8}$$

Let p be the probability of taking a step to the right, q the probability of taking a step to the left, $(1-p-q)$ the probability of zero steps.
Hence, the probability that $X_n = k$ is conditioned from initial condition $X_0 = 0$, where $\mathrm{P}(X_0 = 0) = 1$, is given by:

$$P(X_n = k | X_0 = 0) = \sum \frac{n}{N_1^+! N_2^-! N_3^0!} p^{N_1^+} q^{N_2^-} (1-p-q)^{N_3^0}, \tag{3.1.9}$$

where the summation is over the values of N_1^+, N_2^- and N_3^0, satisfying (3.1.8).
We notice that for all integers j it follows:

$$P(X_n = k | X_0 = 0) = P(X_n = k + j | X_0 = j).$$

In general, the relation (3.1.9) gives the probability for any initial state (j) too.
However, the summation (3.1.9) of multinomial probabilities introduces a lot of difficulties above all when n takes great values. In order to find an approximation, for this inconvenient summation of a large number of multinomial probabilities, we introduce the *central limit theorem*. Using such theorem X_n will be approximate by a normal distribution with mean equal to $n\mu$ and variance equal to $n\sigma^2$ (when n is sufficiently large).
Hence, we have:

$$\sum_i^n Y_i \approx N(n\mu; n\sigma^2).$$

In general, for the central limit theorem the succession Z_n of random variables which are defined as follows:

$$Z_n = \frac{X_n - n\mu}{\sqrt{n}\sigma}$$

convergence in law to $N(0,1)$ random variable; $\forall \epsilon > 0 \quad \exists \quad n_0 \in Z : \quad \forall n \geq n_0$ we have:

$$|P(Z_n \leq x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy| < \epsilon$$

Thus, with n sufficiently large:

$$P(Z_n \leq x) \simeq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy.$$

Hence

$$P(\frac{X_n - n\mu}{\sqrt{n}\sigma} \leq x) = P(X_n \leq n\mu + x\sqrt{n\sigma}) \simeq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy,$$

where, for $k = n\mu + x\sqrt{n}\sigma$ it follows:

$$P(X_n \leq k) \simeq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{k-n\mu}{\sqrt{n}\sigma}} e^{-y^2/2} dy. \tag{3.1.10}$$

The equation (3.1.10) allows us to give immediately an answer to some interesting questions:

1. the determination of the probability $P(X_n > a)$ that (for n sufficiently large) the particle occupies a position with an abscissa greater than a fixed real number a, with $a \in \Re$, arbitrarily;

2. the determination of the probability $P(-b < X_n < a)$ that (with n sufficiently large) the particle is found in an interval $(-b; a)$.

By using the relation (3.1.10) to calculate the probability distribution of the particle in the first case:

$$P(X_n > a) \simeq \frac{1}{\sqrt{2\pi}} \int_{\frac{a-n\mu}{\sqrt{n}\sigma}}^{+\infty} e^{-y^2/2} \mathrm{dy},$$

we have:

$$\lim_{n\to\infty} P(X_n > a) = \begin{cases} 1 & \text{for} \quad \mu > 0 \\ 1/2 & \text{for} \quad \mu = 0 \\ 0 & \text{for} \quad \mu < 0 \end{cases}$$

Thus, when $n \to \infty$ we can say that, for:

- $\mu > 0$ (with $p > q$) the particle occupies the positions over a predetermined level (spatial threshold);

- $\mu < 0$ (with $p < q$) the particle occupies the positions under a predetermined level (spatial threshold);

- $\mu = 0$ (with $p = q$) the particle asymptotically occupies, with equal probability, the positions over and under a predetermined level (spatial threshold).

These considerations justify the term *drift* to which it is designated the parameter $\mu = p - q$.

By using the relation (3.1.10), the distribution of the probability that the particle is found in the inclusive strip $(-b; a)$ (second case) is :

$$P(-b < X_n < a) \simeq \frac{1}{\sqrt{2\pi}} \int_{\frac{-b-n\mu}{\sqrt{n}\sigma}}^{\frac{a-n\mu}{\sqrt{n}\sigma}} e^{-y^2/2} dy.$$

When $n \to \infty$ the distribution of probability $P(-b < X_n < a) \to 0$
Hence, the study of the asymptotic behaviour of the random walk provides an application of central limit theorem.

Remark 3.1.1: The random walk shows the utility of the central limit theorem when we wish a quantitative analysis of the behaviour for a system of the described type, abdicating to quantitative evaluations. Besides, there are cases, such as the *Brownian motion*, in which it is natural and necessary to consider a sufficiently large n. The Brownian motion is characterized by a time constant smaller than the measurements or the observations time, so that, for example, between two following measurements the test particle suffers an extremely high number of impacts, and so it is subjected to possible displacements.

3.1.2 Absorbing Barriers

In this section, we briefly consider the equation (3.1.5) when the random walk starts at the origin and the particle moves in the presence of *two absorbing barriers* to the points $-b$ and a (with $a, b > 0$) [95] [98]; for our purpose we do not consider the case of reflecting barriers. The probability that the particle moves indefinitely between the two barriers is zero.
The particle moves in the strip $(-b; a)$ and its motion is stopped when it touches one of the two barriers. We calculate the probability that the particle is absorbed by barrier a (likewise $-b$) at exactly the time n under the hypothesis that the particle starts at the point $X_0 = j$ $(-b \leq j \leq a)$:

$$f_{j,a}^{(n)} = P(-b < X_i < a, i = 1, 2, ..., n-1; X_n = a | X_0 = j) \qquad (n = 1, 2, ...).$$

While, for $n = 0$ it follows the initial condition:

$$f_{j,a}^{(0)} = \begin{cases} 1 & \text{for} \quad j = a \\ 0 & \text{for} \quad j \neq a. \end{cases}$$

The hypothesis makes sense when the process is time- and spatial-homogeneous because it is invariant for temporal and spatial translations.
We still notice that:

$$\sum_{n=0}^{\infty}(f_{j,a}^{(n)} + f_{j,-b}^{(n)}) = 1.$$

The probabilities for each step are the following:

- p for a forward step (that is $+1$) when the particle stars at the point $X_0 = j + 1$;
- q for a backward step (that is -1) when it stars at the point $X_0 = j - 1$;
- $1 - p - q$ for a zero step (that is 0) when it stars at the point $X_0 = j$.

Hence, we can write $f_{j,a}^{(n)}$ as:

$$f_{j,a}^{(n)} = pf_{j+1,a}^{(n-1)} + qf_{j-1,a}^{(n-1)} + (1-p-q)f_{j,a}^{(n-1)} \tag{3.1.11}$$

where, $j = -b + 1, ..., a - 1$ and $n = 0, 1, ...$.
Again, together with the initial condition we must write the boundary condition:

$$f_{a,a}^{(n)} = 0$$

and

$$f_{-b,a}^{(n)} = 0$$

with $n \in N$.
Since, $f_{j,a}^{n}$ is a function of the two discrete variables (n and j) and we have a difference equation of the first order in n and of the second order in j, we can use generating functions in order to calculate the solution of (3.1.11). By using the generating functions, we eliminate one of the variables and we obtain:

$$F_{j,a}(s) = \sum_{n=0}^{\infty} f_{j,a}^{(n)} s^n = F_j(s). \tag{3.1.12}$$

Let us observe that when the barrier $b \to \infty$ we obtain the case of *one absorbing barrier.*
Let us suppose that the particle starts from the state $X_0 = j$ (or $X_0 = 0$ equivalently) and that an absorbing barrier is placed at the point $a > j$ (or $a > 0$), so that the particle is free to move among the states $x < a$ if and until it reaches the state a which, once entered, holds the particle permanently.

Let $g_{j,a}^{(n)}$ the probability that the particle is absorbed by the barrier which is posed in a at exactly the time n under the hypothesis that the particle is free to move among the states $x < a$ and it starts at the state $X_0 = j$:

$$g_{j,a}^{(n)} = P\left\{X_i < a, \quad i = 1, 2, ..., n-1; X_n = a | X_0 = j\right\} \tag{3.1.13}$$

For $n = 0$ we have the initial conditions:

$$g_{j,a}^{(0)} = \begin{cases} 1 & \text{for} \quad j = a \\ 0 & \text{for} \quad j \neq a. \end{cases}$$

Moreover, we have the boundary conditions:

$$g_{a,a}^{(n)} = 0, \qquad \text{if} \;\; n \neq 0.$$

It is possible to define the generating function as:

$$G_{j,a}(z) = \sum_{n=0}^{\infty} g_{j,a}^{(n)} z^n. \tag{3.1.14}$$

This case, corresponding to a single barrier, will be used in the following with respect to a practical application in cosmology.

3.1.3 Brownian motion

It is curious that in the same year as Pearson's letter (see before section), Albert Einstein also published his seminal paper on *Brownian motion* - the complicated path of a large dust particle in air (Robert Brown (1828) was the first to realize that the erratic motion of microscopic pollen was physical, not biological in nature as was believed before his time) - which he modeled as a random walk, driven by collisions with gas molecules. Einstein did not seem to be aware of the related work of Rayleigh and Bachelier, and he focused on a different issue: the calculation of the diffusion coefficient in terms of the viscosity and temperature of the gas. Similar theoretical ideas were also published independently by Smoluchowski in 1906.

The Random-Walk theory of *Brownian motion* had an enormous impact, because it gave strong evidence for discrete particles ("atoms") at a time when most scientists still believed that matter was a continuum. In particular, in Brownian motion it is not the position of the particle at one time that is independent of the position of the particle at another; it is the displacement of that particle in one time interval that is independent of the displacement of the particle during another time interval. Increasing the resolution of the microscope and the time resolution only produces a similar random walk. As we have seen in the first chapter the Brownian motion is self-similar.

In this section, we are going to show the main probabilistic tools generally used to formulate the Brownian motion.

The Brownian motion is the most fundamental continuous time stochastic process. Since, it is not our aim to expose this continuous-time stochastic process thoroughly, we will emphasize the main results and formulae useful for our purpose.
We begin by considering the *Brownian motion process* $\{(B_t), t \geq 0\}$, sometimes called *Wiener's process*. Typically, the term "Brownian motion" is used to describe a wide class of processes; $(B_t)_{t\geq 0}$ is only a particular case (i.e. motion of a free particle with negligible acceleration). In detail, the Brownian motion will be developed as a limit of a random walk.
In all this section, $(\Omega, \Im\ P)$ is a fixed probability space with Ω nonempty set, $\Im$ σ-algebra of subsets of Ω and P probability measure on $\Im$. On this space we consider a stochastic process $(B_t)_{t\geq 0}$ with $t \geq 0$ and a real random variable B_t.

Definition 3.1.1: The stochastic process $(B_t)_{t\geq 0}$ is said to be a Brownian motion if:

1. $B(0) = 0$ almost surely;
2. $(B_t)_{t\geq 0}$ is a process with independent increments;
3. B_t is a centered Gaussian random variable with $\sigma^2 t$ variance[2], $\forall t$ and for some positive constant σ;
4. $\gamma : t \longrightarrow B(t)_t$, with $t \geq 0$ is continuous almost surely.

A full development of the Brownian motion can be found in [95], [96], [97], [98], [99].
Thanks to the previous definitions, one obtains that for $s < t$, $B_t - B_s$ is of Gaussian type, centered with variance $t - s$, which says that Brownian motion is stationary.
Indeed, one has

$$B_t = B_s + (B_t - B_s)$$

[2] When $\sigma = 1$, the process is often called *standard Brownian motion* or standard Wiener process.

Therefore, using characteristic functions and the independence of B_s and $B_t - B_s$ one has

$$E[e^{ir(B_t - B_s)}] = e^{\frac{-r^2(t-s)}{2}}$$

because $s < t$, $B_t - B_s \stackrel{L}{=} N(0, t-s)$, where "L" means that we have a convergence in law.
One can also compute the covariance function of the Brownian motion: for all $(s,t) \in (\Re^+)^2$,

$$E(B_s B_t) = s \wedge t$$

where the symbol $\wedge$ denotes the minimum between s and t.
Indeed, suppose that $s < t$ then

$$E(B_s B_t) = E\left\{B_s[B_s + (B_t - B_s)]\right\} = E(B_s^2) + E[B_s(B_t - B_s)] =$$
$$= E(B_s^2) = \mathrm{Var}(B_s) = \mathrm{s}$$

Since, a standard Brownian motion $\{B_t\}$ is normal with mean 0 and variance t, its density function is given by

$$f_t(x) = \frac{1}{\sqrt{2\pi t}} \exp[-x^2/2t].$$

The Brownian motion process can also be defined as the limit of the random walk. The random walk model is a first approximation of the theory of diffusion and Brownian motion. In the limit, the process will appear as a continuous motion (for example see [98], pag. 323; [97], pag. 356; [96] pag.323).
In order to prove the intuitive properties (see definition 3.1.1) of this limiting process we consider the relation (3.1.7) and the central limit theorem (see symmetric random walk, relation 3.1.6). In detail, if we begin with a simple random walk we obtain a Brownian motion process with *drift* coefficient μ of the form:

$$X_t = B_t + \mu t \quad t \geq 0 \tag{3.1.15}$$

where $\{B_t\}$ is the standard Brownian motion [97].

Definition 3.1.2: The stochastic process $(X_t)_{t\geq 0}$ is said to be a Brownian motion with drift coefficient if:

1. $X(0) = 0$ almost surely;
2. $(X_t)_{t\geq 0}$ is a process with independent increments;
3. X_t is normally distributed with mean μt and variance $\sigma^2 t$, with μ and σ positive real constants.

3.2 Introduction to stochastic processes in cosmology

The observations show that Universe has structure with scaling rules, where the clustering properties of cosmological objects reveal a form of hierarchy (to clustering properties from cosmological to nuclear objects). Mohamed El Naschie has realized a very wide scientific production in few hundred papers on $\epsilon^{(\infty)}$ Cantorian space-time. Most relevant results were in theoretical physics and high energy physics. Nevertheless his point of view is applicable in many other contexts such as cosmology. Reading El Nachie's papers and other previous contributions it clearly appears that the E-Infinity theory is more than a new framework for understanding and describing nature and not only a set of equations. Probably the main point of the theory is the fact that everything we see or measure is resolution dependent. As reported by El Naschie, in the Cantorian E-Infinity view, space-time is an infinite dimensional fractal that happens to have $D = 4$ as the expectation value for the topological dimension [85]. In particular, the topological dimension $3 + 1 = 4$ means that in our low energy resolution, the world appears to s as if it were four-dimensional. As stated in the work [50], the observations of the large scale structures show that the dimension changes if we consider different energies, corresponding to different lengths scale in the Universe. There, it was reported that the spatial dimension for objects such as globular clusters, galaxies, cluster of galaxies and superclusters are typically greater than 3. In particular, if we also add the time dimension, we can obtain a manifold dimension, that can be approximated by $4+\ \phi^3$, where ϕ is the Golden Mean. This means that the dimension becomes resolution dependent; consequently it all depends on the energy scale with which we are making our observation.

Hence, El Naschie used an infinite dimensional space, that is a $\epsilon^{(\infty)}$ Cantorian space-time, and his approach shows us a new point of view represented by classical dynamical processes on the infinite dimensional Cantorian space E-Infinity. In other words, $\epsilon^{(\infty)}$ [[80], [51]] behaves as an infinite set of mirrors; this behaviour finds its foundation in the waveguide channel mechanism, which was anticipated in [50] and analyzed in detail in a companion paper.

If this interpretation is correct, then the difference between micro and macro-physics only depends on the resolution by which the observers look at the world. The confirmation of this question is given in [50] where it is pointed out that nature shows us structures with scaling rules; in the same paper the author considered the compatibility of Stochastic Self-Similar, Fractal Universe with the observation and the consequences of this model.

In [52] [80] [81] the author presented the observed segregated Universe as the result of a fundamental self-similar law, which generalizes the Compton wavelength relation, $R(N) = (h/Mc)N^{1+\phi}$, where R is the radius of the structures, h is the Planck constant, M is the total mass of the self-gravitating system, c the speed of light, N the number of the nucleons within the structures, and $\phi \cong 1/2$ [82].

As noted by Mohamed El Naschie, this expression agrees with the Golden mean and with the gross law of Fibonacci and Lucas [83] and [84].

In the context of El Nashie's $\epsilon^{(\infty)}$ Cantorian space-time [85] starting from a universal scaling law, the author showed its agreement with the well-known Random Walk equation or Brownian motion relation used by Eddington for the first time [86], [87]. Consequently, he arrived at a self-similar Universe. In [82], [88], [51] and [90] the relevant consequences of a stochastic self-similar and fractal Universe were presented. It appears that Universe has a memory of its quantum origin as suggested by Sir Roger Penrose with respect to quasi-crystal [91]. Particularly, the model is related to Penrose tiling and thus to $\epsilon^{(\infty)}$ theory (Cantorian space-time theory) as proposed by El Naschie [92] and [93] as well a Connes Noncommutative Geometry [94].

In order to understand the intricacies of Iovane's result, $R(N) = (h/Mc)N^{1+\phi} = (h/m_n c)N^{\phi}$, where m_n is the mass of a nucleon (proton or neutron), first of all we have presented the well-know stochastic process named *Random Walk*. In detail, in the paper [106] we use a Brownian motion process and we develop this type of process as limit of a Random Walk.

Let us consider this point of view, more specifically the relation $R(N) = (h/m_n c)N^{\phi}$ will be seen as a Brownian motion process. Indeed, in the paper [106] we consider the result which was obtained by the first author of the work to describe scaling rules in nature, $R(N) = (h/m_n c)N^{\phi}$, in connection with the well-know Random Walk. By adopting this prospect, we will present the result as a Brownian motion process developed as a limit of a

Random Walk in the context of Mohamed El Naschie's $\epsilon^{(\infty)}$ Cantorian space-time applied in cosmology. Consequently, the results give us a mathematical formulation of the model presented in [52] in terms of a deep analysis with respect to the Brownian motion.

3.2.1 Astrophysical scenario

In [88], [89] the authors presented a study on the dynamical systems on Cantorian space-time to explain some relevant stochastic and quantum processes, where the space acts a harmonic oscillating support, such as it often happens in Nature. The observations show a structure of Universe with scaling rules, where clustering properties, from cosmological to nuclear objects, reveals a form of hierarchy. As an example, it is possible to distinguish among globular cluster, galaxies, clusters and superclusters of galaxies through their spatial lengths [100], [101]. Table 1 recalls the dimensions and masses of the previous systems [102].

System Type	**Length**	**Mass**($M_\odot$)
Globular Clusters	$R_{GC} \sim 10pc$	$M_{GC} \sim 10^{6\div7}$
Galaxies	$R_G \sim 1 \div 10kpc$	$M_G \sim 10^{10\div12}$
Cluster of galaxies	$R_{CG} \sim 1.5h^{-1}Mpc$	$M_{CG} \sim 10^{15}h^{-1}$
Supercluster of galaxies	$R_{SCG} \sim 10 \div 100h^{-1}Mpc$	$M_{SCG} \sim 10^{15\div17}h^{-1}$

Table 1: Classification of astrophysical systems by length and mass, where h is the dimensionless Hubble constant whose value is in the range [0.5,1].

Sys Type	**N.of Nucleons**	**Eval. Length**
Glob. Clusters	$N_G \sim 10^{63\div64}$	$R_{GC} \sim 1 \div 10pc$
Galaxies	$N_G \sim 10^{68}$	$R_G \sim 1 \div 10kpc$
Cluster of gal.	$N_{CG} \sim 10^{72}$	$R_{CG} \sim 1h^{-1}Mpc$
Superc. of gal	$N_{SCG} \sim 10^{73}$	$R_{SCG} \sim 10 \div 100h^{-1}Mpc$

Table 2 : Evaluated Length for different self-gravitating systems

In [52], [82], [51], [90] the first author of the present paper considered the compatibility of a Stochastic Self-Similar Fractal Universe by the observation and the consequences of the model. In detail, it was demonstrated that the observed segregated Universe is the result of a fundamental self-similar law, which generalizes the Compton wavelength relation, $R(N) = (h/m_n c)N^{\phi}$. A typical interaction length can be defined as a quantity, which is proportional to the size of the system which contains the constituents [50]. In other words, consider a maximum length corresponding to its size for each a system. In 1985 Sakharov argued that quantum primordial fluctuations had to be related to cosmological evolution and to the dynamics of astrophysical systems [103]. Eddington and later on Weinberg wrote the relevant relationship between quantum quantities and the cosmological ones:

$$h \cong G^{1/2} m^{3/2} R^{1/2},$$

where h is the Plank constant, G is the gravitational constant, m is the mass of nucleon, and R is the radius of Universe.

By following Eddington Weinberg is (E−W) approach, the first author and his team wrote a general relationship between the radius R of the self-equilibrated system and its number of nucleons. While the E−W relationship was only written for the radius of Universe, they presented a relationship which is scale invariant, so adoptable for all types of self-gravitating systems (and also for the entire universe):

$$R(N) = \frac{h}{Mc} N^{\alpha} = \frac{h}{m_n c} N^{\phi} \tag{3.2.1}$$

with $\alpha = 3/2$, for $M = M_G \approx 10^{10-12} M_{\odot}$,$m_n$ mass of the nucleons, $\phi = 1/2$ $N = 10^{68}$ (this is approximately the number of nucleons in a galaxy), again they reproduce exactly $R \approx 1-10$ *kpc* [3]. In general, the authors evaluate the number of nucleons in a self-gravitating system as

$$N = \tfrac{M}{m_n},$$

where N is the number of nucleons of mass m_n into self-gravitating system of total mass M. Then, they obtain the relevant results recalled in Table 2. In the second column the number of evaluated nucleons is shown, while they find the expected radius of self-gravitating system

[3] The value $\phi = 1/2$ is what is found by the observation. If we assume a Cantorian $\epsilon^{(\infty)}$ space time, as suggested by Mohamed El Nashie the expectation value is $\phi = \frac{\sqrt{5}-1}{2}$, that is the golden Mean.

in the last column.

Moreover, the relation (3.2.1) can be written in terms of Plakian quatities:

$$R_p(N) = \frac{l_p}{m_p}\sqrt{\frac{hc}{G}}N^{(1+\phi)} \tag{3.2.2}$$

and from eq. (3.2.2) we obtain:

$$R_p(N) \propto l_p N^{3/2}$$

where we have assumed $\phi = 1/2$.

If we consider the R radius as a fixed quantities, equalizing the equations (3.2.1) and (3.2.2), we get:

$$\tfrac{l_p}{m_p}\sqrt{\tfrac{hc}{G}}N^{3/2} = \tfrac{h}{Mc}N^{3/2}$$

and so

$$M = \frac{m_p}{l_p}\sqrt{\frac{Gh}{c^3}}. \tag{3.2.3}$$

The mass M of the structure is written, through the relation (3.2.3), in terms of Plank's length.

As reported in [104], and [52], the following theorems can be obtained.

Theorem 3.2.1: The structures of the Universe appear as if they were a classically self-similar random process at all astrophysical scales. The characteristic scale length has a self-similar expression

$$R(N) = \frac{h}{Mc}N^{1+\phi} = \frac{h}{m_n c}N^{\phi},$$

where the mass M is the mass of the structure, $m_n$is the mass of a nucleon, N is the number of nucleons into the structure and ϕ is the Golden Mean value. In terms of Plankian quantities the scale length can be recast in

$$R_p(N) = \frac{l_p}{m_p}\sqrt{\frac{hc}{G}}N^{(1+\phi)}.$$

The previous expression reflects the quantum (stochastic) memory of Universe at all scales, which appears as a hierarchy in the clustering properties.

Theorem 3.2.2: The mass and the extension of a body are connected with its quantum properties, through to the relation

$$E_{E,N}(N) = E_P N^{1+\phi},$$

that links Plank's and Einstein's energies.
The quantum (stochastic) memory is reflected at all scales and it manifests itself through a clusterization principle of mass and extension of the body.

3.3 Application to dynamical system and cosmology

3.3.1 Random Walk process and the segregated Universe

In [50] the authors noted that all astrophysical scales have a particular length. For this reason, they obtained the exact lengths of the self-gravitation system just by using an interesting power law. An invariant scale relation, from the quantum lengths to the astrophysical ones, plays a fundamental role. As a macroscopic system, Universe shows a sort of quantum and relativistic memory of its primordial phase. The choice to start with a $\alpha = 1/2$ is suggested by the Statistical Mechanics.

Indeed eq. (3.2.1) is strictly equivalent to

$$R(N) = lN^{\alpha} \tag{3.3.1}$$

where $l = h/m_n c$. The relation (3.3.1) is the well-known Random Walk or Brownian motion developed as limit of a Random Walk, when $\alpha = 1/2$
In this paragraph, we consider a segregated Universe as the result of an aggregation process, in which a test particle in its motion in the Fractal Cantorian $\epsilon^{(\infty)}$ space-time can be captured or not.

The model. Let us consider a test particle with mass m_n, moving in a physical space S (like the entire Universe). Moreover, let us also consider S composed by some substructures S_i (like a galaxy, a cluster of galaxy, and so on) on the N-axis (see Figg. 3.1, 3.2).

Due to the interaction between the systems and the test particle, the probability that the particle is captured by a component S_i can be expressed through a Random Walk X_n process. In detail, we can make the hypothesis that X_n is the aggregation process of a fixed structure S_i.

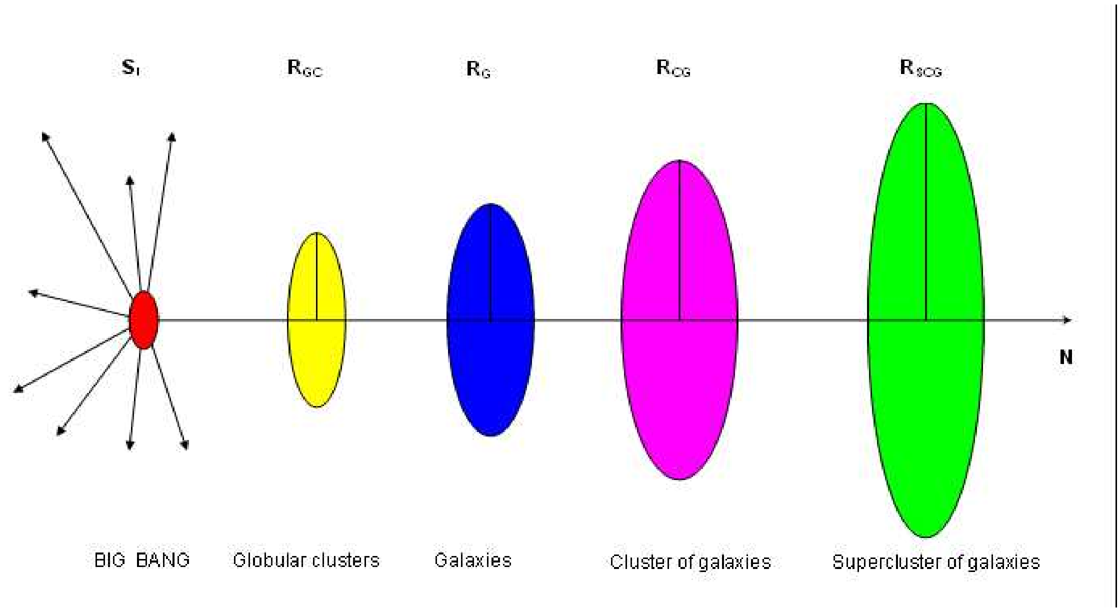

Fig. 3.1: A Sketch of a long bang in the Universe's Expansion

In this case, if we introduce a sequence of independent and identically distributed random variables $\{Y_i\}$, which are the single steps of aggregation, that is the single possibilities that a text particle is captured or not, then $X_N = \sum_{i=1}^{N} Y_i$ is the aggregation process, thanks to which a structure S_i after a fixed time (or a fixed number of steps) reaches the mass M_i and so the radius R_i.

We consider the simple random walk process (see previous section) since in this case, the mass-step Y_i can only take the value 1 or -1 with the distribution:

$$P(Y_i = 1) = p = \phi, \qquad P(Y_i = -1) = q = 1 - \phi,$$

where in the symmetric case $\phi = 1/2$; in the main context of our application $\phi \cong 1/2$. Specifically when $\phi = \frac{\sqrt{5}-1}{2}$, that is the Golden Mean, we have a Cantorian $\varepsilon^{(\infty)}$ Universe, according to Mohamed El Naschie is theory.

It is assumed that each mass-step is either + 1 (with probability ϕ) or - 1 (with probability $1 - \phi$).

In our context the steps assume the following meaning :

- corresponding to $Y_i = +1$ the test particle is captured by the system S_i, and m_{S_i} grows;
- corresponding to $Y_i = -1$ the test particle is not captured by S_i.

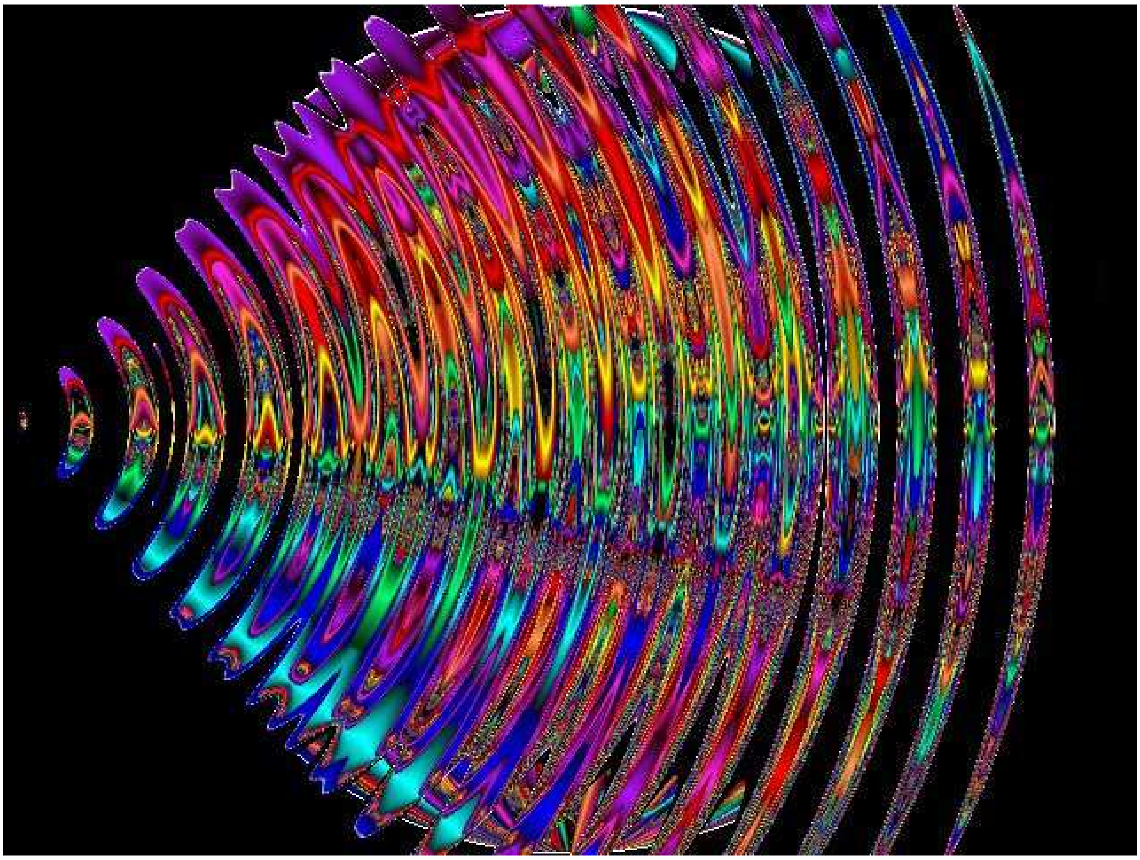

Fig. 3.2: A simulated result for the long bang in the Universe's Expansion

If the particle is not captured by S_i, it can be captured by an S_j, that is at the same scale (for example a galaxy scale) or at a different scale (for example globular cluster, cluster of galaxy and so on).

From the point of view of the structure S_i the particle can be captured (corresponding to $Y_i = +1$) or not captured (corresponding to $Y_i = -1$). Consequently, the case with $Y_i = 0$ becomes irrelevant.

Anyway, in a more realistic model, where we consider a space-time structure which is not homogeneous and isotropic and where the aggregation process follows a hierarchical approach, we will analyze if the case $Y_i = 0$ makes sense.

Remark 3.3.1: On the other hand, either there is a +1 mass-step linked to the energy of bounding or there is a -1 less-mass-step linked to the dissipative energy from the structure.

Remark 3.3.2: We notice that in our model, for all integer j, it remains the equality between the probability that the particle starts from zero and the probability that the particle starts from $k+j$:

$$P(X_N = k | X_0 = 0) = P(X_N = k + j | X_0 = j),$$

where $k = 0, \pm 1, \pm 2, ...$ and $N = 1, 2, ...$ While from a mathematical point of view this is not so interesting, from a physical point of view this is relevant due to the fact that we must start from a system with a mass m_n corresponding to a system composed by a single mass and not a zero mass system.

Let us denote μ and σ^2 the mean and variance of a mass-step.
Then $\mu = p - q$ and $\sigma^2 = p + q - (p-q)^2$ and hence in our case, the mean and variance are respectively:

$$E(Y_i) = \mu = p - q = (2\phi - 1) = \phi^3, \tag{3.3.2}$$

$$\mathrm{Var}(Y_i) = \sigma^2 = p + q - (p-q)^2 = 4pq = 4\phi(1-\phi) = 4/ < dim_H \epsilon^{(\infty)} > \tag{3.3.3}$$

with

$$\frac{1}{\phi(1-\phi)} = \frac{1}{\phi\phi^2} = \frac{1}{\phi^3} = 4 + \phi^3 =< dim_H \epsilon^{(\infty)} >$$

the Hausdorff's dimension.
We notice that $E(Y_i)$ is the mean of the number of mass-steps to obtain the structure of

radius R_i; consequently the mean and variance of our simple random walk for the structure S_i are respectively:

$$E(X_N) = \mu N = N(2\phi - 1) = N\phi^3$$

$$\mathrm{Var}(X_N) = \sigma^2 N = 4Npq = 4N\phi(1-\phi) = 4N/ < dim_H \epsilon^{(\infty)} > .$$

In the case of $\phi = \frac{\sqrt{5}-1}{2}$, it follows $p > q$; consequently, the probability that a new mass will be captured by the structure S_i is greater than the particle is not captured.

Viceversa is for $p > q$.

In the case of $p = q$ we have the same probabilities that the particle is accepted or refused.

In a different vision for a fixed structure S_i we could have the above three cases at different time, corresponding to different eras in the life of S_i. By adopting this point of view $\phi = \frac{\sqrt{5}-1}{2}$ could be linked to the present, in which we carry out measurements.

This analysis is not conclusive; obviously this is just a toy model to start the investigation of cosmology in $\varepsilon^{(\infty)}$ Cantorian space-time by using a stochastic approach based on Random Walk.

Indeed, in future analysis we will consider some other parameters such as:

1. the homogeneity and the isotropic of the space-time;

2. the hierarchy between structures at different scale for accepting or not new mass in terms of new particles;

3. the presence of hidden variables for describing internal rules for a structure S_i to accept or do not accept a particle.

This hidden variables could represent the maximum capability to accept new mass with respect to the spatial dimension of the structure, or a changed number as it often it happens in particle physics, when we consider the colour, the flower and so on.

By using the result of the central limit theorem X_N will be approximated by a normal distribution with mean $N\mu$ and variance $N\sigma^2$ (when N is sufficiently large):

$$\sum_i^N Y_i \approx N(N\phi^3; 4N\phi(1-\phi)).$$

In general, for the central limit theorem the succession Z_N of random variable:

$$Z_N = \frac{X_N - N\mu}{\sqrt{n}\sigma}$$

converges in law to $N(0,1)$ random variable.
In detail, the particle will be within a distance of order N^{ϕ} from its starting point after $N = M/m_p$ mass-steps. From the previous considerations and by reading the earlier papers of first author, it clearly appears that the relation $R(N) = (h/m_n c)N^{\phi}$, is a Brownian motion process developed as a limit of the Random Walk.

Indeed, by using the central limit theorem, the Random Walk $R(N) = (h/m_n c)N^{\phi}$ will appear as a *Brownian motion with drift coefficient* ϕ^3 and from the relation (3.1.15) it follows:

$$(X_N)_N \xrightarrow{L} \widetilde{B}_N = B_N + \phi^3 N \tag{3.3.4}$$

where L is a convergence in Law and B_N is the standard Brownian motion process.
Hence, we notice that the process $R(N) = (h/m_n c)N^{\phi}$ is a *Brownian motion with drift coefficient* ϕ^3.
It is interesting to note that each scale (with radius R_i) of Universe has a Gaussian distribution with mean $\mu_i = N\phi^3$ and variance $\sigma_i^2 = 4N/ < dim_H \epsilon^{(\infty)} >$:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp[-\frac{(x-\mu_i)^2}{2\sigma_i^2}] \qquad x \in \Re.$$

In our model, we can see the different scales of Universe and so the segregation of Universe, as sequence of fundamental lengths. These lengths have gaussian distribution with mean μ_i and variance σ_i^2.

Moreover, we are assuming that there is no overlapping among different scales.
In other words, each scale has a Gaussian distribution, with the mean value that is R_i and with a dispersion that is given by the standard deviation.

For example, this implies that in the case of a system S_i on galaxy scale the best value will be of the order of $R_i = 10$kpc, corresponding to $\mu_i = 10^{68}$.

By taking into account the results in [105] we can easily understand that the present Universe can be obtained as a Brownian motion at some different scales. Equivalently, we can consider a fundamental length scale that has Gaussian distribution and that generates the segregated Universe thanks to translation and processes in scale.

3.3.2 Application to cosmology

From the previous analysis, the state of a system with scale length R_i appears to be more probable than the other with length $R_{i+\alpha}$, $\alpha \in \Re$, $\alpha \neq j$, where j is another scale with high

probability. The choice of fixed length (see table 1 and table 2) instead of others is more stable, since it is more probable.
In order to prove the stability of the structures, we introduce the specific case of an absorbing barrier. Let us suppose that the structure starts in the state $X_0 = j$ and that an absorbing barrier is placed at the point $a > j$ [95]. We suppose that the different scales of Universe are the absorbing barriers a_i and $\epsilon^{(\infty)}$ Cantorian space gives the minimal distance between two points among which it is possible to allocate an unitary mass (m_n), in connection with the Plank's length.

In other words, we have a process thanks to which a test system grows its mass step by step of quantities m_n . This goes on until the upper limit, fixed by the barriers value a_i, is reached. This means that if a particle arrives at the structure when this one has a mass smother than the value linked to a_i, then the particle with mass m_n is absorbed; otherwise the mass m_n is refused.

In order to have the barriers a_i, as dimensionaless number, we define:

$$a_i = \frac{R_i}{R_p(1)} = \frac{\frac{h}{m_n c} N_i^{\phi}}{\frac{l_p \sqrt{hc}}{m_p \sqrt{G}}}$$

$$= \frac{m_p}{m_n l_p} \sqrt{\frac{Gh}{c^3}} N_i^{\phi}.$$

Since, the barriers a_i are the absorbing barriers, the equation (3.1.14) gives the probability that the absorption occurs at a_i at the time N_i. Here we measure the time evolution in connection with the aggregation of the matter. In other words, N_i indicates both the number of components into a structure and a possible time scale, that is the number of steps to obtain the mass M_i corresponding to the length R_i.
Let us consider the random variable $T(a_i|j)$:

$$T(a_i|j) = inf\,\{N_i : X_{N_i} = a_i|X_0 = j\} \qquad (a_i > j)$$

that denotes the time to absorption at a_i or equivalently the first passage time to state a_i from j.
We can write:

$$T(a_i|j) = T_{j+1} + T_{j+2} + ... + T_{a_i}$$

where T_{j+i} $(i = 1, 2, ..., a_i - j)$ are independent and identically distributed random variables.

By taking into account that $T(a_i|0)$ is the sum of independent random variables with finite variance and by using the central limit theorem, in terms of the mean $\mu = p - q$ and variance $\sigma^2 = p + q - (p-q)^2$ of a single mass-addition-step, we have

$$E[T(a_i|0)] = \frac{a_i}{\mu} = \frac{a_i}{(2\phi - 1)},$$

$$\mathrm{Var}[T(a_i|0)] = a_i \frac{\sigma^2}{\mu^3} = a_i \frac{\frac{4}{<dim_H \epsilon(\infty)>}}{(2\phi-1)^3}.$$

By using the number of absorbed particles at a $T(a_i|j)$, we can write (see [95], pag.35):

$$G_{j,a_i}(z) = \sum_{N_i=1}^{\infty} z^{N_i} P[T(a_i|j) = N_i]$$

for $z = 1$ we have

$$G_{j,a_i}(1) = P[T(a_i|j) < \infty].$$

Thus

$$G_{j,a_i}(1) = P[T(a_i|j) < \infty] = \begin{cases} [\frac{p}{q}]^{a_i - j} & (p < q) \\ 1 & (p \geq q). \end{cases} \tag{3.3.5}$$

Example 3.3.1: The first passage time $T(1|0)$ denotes the number of absorbed particles (components) on the first length scale R_1 structure from state j (free-state) to state 1 (position of structure); it has a generating function equally to $G_1(z)$.

Corresponding to $T(a_i|j)$, we have the mass of the fixed structure equal to $M(a_i|j) = m_n T(a_i|j)$. Since, $T(a_i|j)$ denotes the *duration* of the Random Walk, (3.3.5) is the probability that the absorption occurs for each fixed barriers a_i.

Having posed

$$a_i = \frac{m_p}{m_n l_p}\sqrt{\frac{Gh}{c^3}} N_i^{\phi} = \alpha N_i^{\phi}$$

where $\alpha = \frac{m_p}{m_n l_p}\sqrt{\frac{Gh}{c^3}}$ and by taking into account the expression for the mean value $E_i(X_N)$, that we have introduced in the previous section, that is

$$E_i = (2\phi - 1) N_i$$

and passing to the logarithms, we obtain

$$E_i = \left\{ 2[\frac{\log a_i - \log \alpha}{\log N_i}] - 1 \right\} N_i$$

By simple calculations is gotten

$$a_i = \alpha N_i^{\frac{1}{2}(\frac{E_i}{N_i}+1)}$$

and since

$$R_i = a_i R_p(1)$$

it follows that

$$R_i = \frac{h}{m_n c} N_i^{\frac{1}{2}(\frac{E_i}{N_i}+1)}. \tag{3.3.6}$$

In the specific case $N_i = 1$ the relation (3.3.5) becomes

$$R_1 = \frac{h}{m_n c}.$$

The relation (3.3.6) represents the scaling law in terms of the means E_i in connection with the Random Walk process. Thanks to this approach, when the structure reaches the maximum permitted mass M_{i-max} that is compatible with the fractal structure with the fixed R_i, a free particle (test-particle) meets the barrier of that structure with probability equal to 1. This means that for:

$$M = M_i = m_n T(a_i|j)$$

the structure with length scale equal to R_i does not accept any other component (particle) but *reflects* it. Consequently, this free particle of mass m_n will tend to find a new accepting structure with the same scale length R_i or with R_j.
In this way, it is possible to create a segregated Universe at different length scales.

3.4 Conclusions

In the paper [106] we have considered the link between the scale invariant law, $R(N) = (h/m_n c)N^{\phi}$, introduced by the first author, and the *Random Walk*. The first law represents the observed segregated Universe as the result of a fundamental self-similar law, which generalizes the Compton wavelength relation, while the second is a well-known stochastic process. We have used a Brownian process in order to prove that this type of process, developed as a limit of a Random Walk, has the same form of the relation $R(N) = (h/m_n c)N^{\phi}$. Moreover, we have noticed that $R(N) = (h/m_n c)N^{\phi}$ is a Brownian motion with drift coefficient μ of the form: $\widetilde{B}_N = B_N + \phi^3 N$.

In the last section of this work, we have studied the state of the system with different length scales. In this way, we have seen that the system with a length R_i is more probable than other with $R_{i+\alpha}$ $(\alpha \in \Re)$ with a different radius. To be more precise, choosing a fixed length the masses take values, which are more stable. In order to prove this, we have considered the random walk with an absorbing barrier; here the different scales of Universe are considered in connection with some absorbing barriers and the $\epsilon^{(\infty)}$ Cantorian space-time is directly linked with the minimal distance between two massive points and Plank's length.

4. MULTIFRACTALS AND EL NASCHIE E-INFINITY CANTORIAN SPACE-TIME

In this chapter, we present the analysis of Multifractals in the context of El Naschie's $\epsilon^{(\infty)}$ Cantorian Space-Time applied to cosmology. In detail, we summarize some recent results concerning fractal structure and the Brownian paths in order to calculate fractal dimension and characteristic parameters for large scale structures and for the atomic elements that live in El Naschie's $\epsilon^{(\infty)}$ Cantorian Space-Time. As starting point we consider the results that describing scaling rules in nature. Then we use multifractal analysis to show that the result, already developed by the authors as Brownian motion in the Chapter 3, is a Multifractal process. Consequently, in the framework Brownian paths play a crucial role if considered to be a multifractal.

4.1 Introduction

Multifractal analysis has recently emerged as an important concept in various fields, including strange attractors of dynamical systems, stock market modelling, image processing, medical data, geophysics, probability theory and statistical mechanics, etc... [107].
Multifractal analysis is concerned with describing the local singular behaviour of measures or functions in a geometrical and statistical fashion. It was first introduced in the context of turbulence, and then studied as a mathematical tool in increasing general settings [32]. Multifractal theory has been discussed by numerous authors and it is developing rapidly [108]. Mandelbrot firstly mentioned multifractal theory in 1972. In 1986, T.C. Halsey drew attention to the concept of multifractal spectrum; that is an interesting geometric characteristic for discrete and continuous models of statistical physics. Olsen was motivated by the heuristic ideas of Halsey. In 1995 Olsen established a multifractal formalism. This formalism has been designed in order to account for the statistical scaling properties of singular

measures when it happens that a finite mass can be spread over a region of phase space in such a way that its distribution varies widely. The multifractal formalism is build on the definition of the singularity spectrum which is connected to the subset of the support of the measure where singularity has a given strength namely its *Hausdorff dimension.*

The purpose of this work is firstly to reconsider the case of the Bownian paths in the context of El Naschie's $\epsilon^{(\infty)}$ Cantorian space-time, from the point of view of the previous model of the authors [106] and by using multifractal analysis [109].

As shown in [52],[80] Nature shows us structures with scaling rules, where clustering properties from cosmological to nuclear objects reveals a form of hierarchy. In [81], [82], the consequences of a stochastic, self-similar, fractal model of Universe was compared with observations. Indeed, it was demonstrated that the observed segregated Universe is the result of a fundamental self-similar law, which generalizes the Compton wavelength relation, $R(N) = (h/Mc)N^{1+\phi}$, where R is the radius (characteristic length) of the structures, h is the Planck constant, M is the total mass of the self-gravitating system, c the speed of light, N the number of the nucleons within the structures and $\phi = \frac{\sqrt{5}-1}{2}$ is the Golden Mean. As noted by Mohamed El Naschie, this expression agrees with the Golden Mean and with the gross law of Fibonacci and Lucas [83], [84], [85].
Starting from an universal scaling law, the author showed its agreement with the well known Random Walk equation that was used by Eddington for the first time [86], [87]. In [82], [88], [51] and [90] the relevant implications of a stochastic self-similar and fractal Universe were presented.
The main aim of this paper is to investigate the link between our previous result $\widetilde{B}_N = B_N + \phi^3 N$ (indeed we will use the Brownian paths) and its multifractal nature in the context of El Naschie's $\epsilon^{(\infty)}$ Cantorian space-time [85].

Secondly, we are interested to evaluate, through the results in [90], the fractal dimension and characteristic parameters of the time for some objects.

4.2 Some fractal properties of Brownian paths

Models using *fractional Brownian motion* (fBm) have helped to advance the field through their ability to assess the impact of fractal features such as statistical self-similarity and

Long-Range Dependence (LRD) to performance [33]. Roughly speaking, a fractal entity is characterized by inherent, ubiquitous occurrence of irregularities which governs its shape and complexity.

First, we consider the fBm $B_H(t)$; its paths are almost surely continuous but not where $H \in (0, 1)$ is the self-similarity parameter:

$$B_H(at) = a^H B_H(t) \tag{4.2.1}$$

However, the scaling law (4.2.1) implies also that the oscillations of fBm at fine scales are uniform and this comes as a disadvantage in various situations. Indeed, fBm is a model with poor multifractal structure and does not contribute to a larger pool of stochastic processes with multifractal characteristics. Hence, in order to describe real world signals that often possess an erratically changing oscillatory behavior (*multifractals*) fBm is not an appropriate model.

The first "natural" multifractal stochastic process to be identified is *Lévy motion* [110], [111]. A Lévy process $\{(X_t), t \geq 0\}$ valued in $\Re^d$ is a stochastic process with stationary independent increments. Brownian motion and Poisson processes are examples of Lévy processes that can be qualified as *monofractal*; for instance the Holder exponent of the Brownian motion is everywhere 1/2 (the variations of its regularity are only of a logarithmic order of magnitude). Most Lévy processes are multifractal under the condition that their Lévy measure is neither too small nor too large near zero. Furthermore, their spectrum of singularities depends precisely on the growth of the Lévy measure near the origin.

Before applying the multifractal analysis to our cosmological scenario, we need recall some basic definitions and results about Brownian path [112].

Let $B_d = \{B_d(t)\}$ be a *standard d-dimensional Brownian motion*; this process (also named Wiener process) is (stochastically) self-similar with index 1/2 by which it means that, for any $c \geq 0$, the time-scaled process $\{B_d(ct)\}$ and the space-scaled process $\{\sqrt{c}B_d(t)\}$ are equivalent in the sense of finite-dimensional equivalence. This self-similar property is central in our study, from which various dimension formulae concerning Brownian paths can be figured out.

Brownian sample paths exhibit highly erratic patterns, despite the continuity. Thus, this should be rich source of fractal analysis/geometry and be one prevailing topic in nonlinearity. As well known fractal sets connected with Brownian motion can be written as:

$$[B_d] = \{x : x = B_d(t), \text{some t}\},$$
$$Z = \{t : B_1(t) = 0\},$$
$$I_d = [B_d] \cap [B'_d],$$

where B'_d denotes an independent copy of B_d. Thus, the three sets are simply the trail (range), the zero set, and the set of intersections of Brownian paths. Note that the zero set is meaningful only for the 1-dimensional case, while the trail and the intersection are meaningful only for the multidimensional case. These are due to the fact that the 1-dimensional Brownian motion is point-recurrent while it is not so for the multi-dim case. These sets are random, since they depend on a particular sample path realization $B_d(t, \omega)$, and so we must interpret any statement about these sets and their associated measures (random too) as being true "with probability one".

Let dim K denote the Hausdorff dimension of a Borel K. The following results are well-Known:

$$d = 1 \qquad \dim Z = 1/2,$$

$$d \geq 2 \qquad \dim[B_d] = 2,$$

$$d = 2, 3 \qquad \dim I_d = d - 2(d - 2).$$

There are natural measures associated with the above fractals Z, $[B_d]$ and I_d; there are respectively Brownian *local time measure, occupation measure* and *intersection measure.* These measure are regarded as *fractal measures*, since each of them is singularly continuous (non-atomic and supported by a set of Lebesgue measure zero) and exhibits a certain self-similarity which is inherited from the self-similarity of the process. The main difference (and difficulty) from pure analysis is that the self-similarity is now always in the distributional sense rather than the strict (analytic) sense.

4.2.1 Intersection of exponents

In the present section, we give a recent results on the intersection exponents and the multifractal spectrum for measures on Brownian paths (for more details see [113]). Intersections of Brownian motion or random walk paths were studied for quite a long time in probability theory and statistical mechanics. There is trivial behaviour in all dimensions exceeding a critical exponent which determines the universality class of the model and enter into most of its quantitative studies. Finding the intersection exponents of planar Brownian motion was one of the first problems solved by the rigorous techniques based on the stochastic Lowner evolution devised by Lawler, Schramm and Werner (for more details see [114], [115], [116]). Above, we have seen that an interesting geometric characteristic for discrete and continuous models of statistical physics is the *multifractal spectrum.* This evaluates the degree of variation in the intensity of a spatial distribution. A multifractal formalism is used for computing multifractal spectrum, based on a large-deviation heuristic (see [117]).
Let us give the definition of multifractal spectrum; suppose that μ is a (fractal) measure on $\Re^d$. The value $f(a)$ of the multifractal spectrum is the Hausdorff dimension of the set of points $x \in \Re^d$ with $d \geq 2$

$$lim_{r\downarrow 0}\frac{log\mu(B(x,r))}{logr} = a \tag{4.2.2}$$

where $B(x,r)$ denotes the open ball of radius r centered in x. In many cases of interest, the limit in (4.2.2) has to be replaced by liminf or limisup to obtain an interesting nontrivial spectrum. in detail, in fractal geometry the relation between Hausdorff dimension and the critical exponents of statistical physics is shown with *intersection exponents.* To define the intersection exponents for Brownian motion with $d = 2, 3$, suppose $n, m \geq 1$ are integers and let $W_1, ..., W_{m+n}$ be a family of independent Brownian motion in $\Re^d$ started uniformly on $\partial B(0,1)$ and running up to the first exit time $T^i(r)$ from a large ball $B(0,r)$ [1]. We divide the motions into two packets and look at the union of the paths in each family

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[1] We refer to (see [99] pag. 344)

$$T_i(r) = \inf\{t \geq 0 : \quad W_t^i \notin B(0,r)\} \qquad (W^i \text{ is a d-dimensional Brownian motion})$$

as the exit time from the open ball $B = (0,r) = \{x \in \Re^d : q(0,x) < r\}$, where q is the distance in $\Re^d$.

$$ß_1(r) = \bigcup_{i=1}^{m} W_i([0, T^i(r)]) \qquad ß_2(r) = \bigcup_{i=m+1}^{n+m} W_i([0, T^i(r)]). \tag{4.2.3}$$

The event that two packets of Brownian paths fail to intersect has a decreasing probability as $r \uparrow \infty$. Using subadditivity, it can be shown that there exists a constant $0 < \xi_d(m,n) < \infty$ such that

$$P\{ß_1(r) \cap ß_2(r) = \emptyset\} = r^{-\xi_d(m,n)+o(1)}, \tag{4.2.4}$$

as $r \uparrow \infty$.

The numbers $\xi_d(m,n)$ are called the *intersection exponents* (for more details see [113]).

Then, we consider a number of Brownian motion paths started at different points. As known the intersection among the paths depends on the dimension.

From a mathematical point of view our aim can be formulated in term of the following theorem ([118] pag. 172)

Theorem 4.2.1: (Dvoretsky, Erdös, Kakutani, Taylor). Suppose $d \geq 2$ and

$$\{B_1(t) : t \geq 0\}, ..., \{B_p(t) : t \geq 0\}$$

are p independent d-dimensional Brownian motions started in the origin. Let $S_1 = B_1(0,\infty), ..., S_p = B_p(0,\infty)$ be their ranges. Then, almost surely,

$$S_1 \cap ... \cap\ S_p = \{0\} \qquad \text{if and only if} \qquad p(d-2) \geq d$$

and otherwise

$$\dim(S_1 \cap ... \cap\ S_p) = d - p(d-2) > 0.$$

4.3 Astrophysical context and fundamental scale invariant law

In [88], [89] the authors presented a study on the dynamical systems on Cantorian space-time to explain some relevant stochastic and quantum processes, where the space acts as harmonic oscillating support, such as it often happens in Nature. The role of oscillating structures is played by cosmological objects as globular cluster, galaxies, clusters and superclusters of galaxies through their spatial lengths [100], [101]. Table 1 in the Chapter 3 recalls the dimensions and masses of the previous systems [102].

In [52], [82], [51], [90] the first author of the present paper considered the compatibility of a Stochastic Self-Similar Universe with the observation and the consequences of the model. Indeed, it was demonstrated that the observed segregated Universe is the result of a fundamental self-similar law, which generalizes the Compton wavelength relation, $R(N) = (h/m_n c)N^{\phi}$. A typical interaction length can be defined as a quantity, which is proportional to the size of the system which contains the constituents N (nucleons)[50]. In general, the authors evaluate the number of nucleons in a self-gravitating system as

$$N = \frac{M}{m_n},$$

where N is the number of nucleons of mass m_n into self-gravitating system of total mass M. Then, they obtain the relevant results recalled in Table 2 in Chapter 3. In the second column the number of evaluated nucleons is shown, while we find the expected radius of self-gravitating system in the last column.
Moreover, for more details we recall that the equation (3.2.1) in Chapter 3,

$$R(N) = \frac{h}{Mc} N^{\alpha} = \frac{h}{m_n c} N^{\phi}$$

can be written in terms of Plankian quatities (3.2.2).

As reported in [104], and [52], we can obtain the Theorems (3.2.1) and (3.2.2) that we can find in the Chapter 3.

4.4 Application to cosmology: Brownian Multifractals Structures

Brownian paths have rich fractal structure, as we have seen in previous section. However, the path is usually qualified as a monofractal, in view that the Holder exponent of the path is everywhere 1/2 (the variations of the regularity are only of a logarithmic order of magnitude). Thus, it is a first approximation to use Brownian path as a curve fitting to those data exhibiting the intermittence. In [50] the authors noted that all astrophysical scales have a particular length. For this reason, they obtained the lengths of the self-gravitation system just by using the previous power law. As a macroscopic system, Universe shows a sort of quantum and relativistic memory of its primordial phase. The choice to start with a $\alpha = 1/2$ is suggested by the Statistical Mechanics.

Indeed eq. (3.2.1) is strictly equivalent to eq. (3.3.1) that represent the well-known Random Walk or Brownian motion.

In a previous article [106], the authors considered a segregated Universe as the result of an aggregation process, in which a test particle in its motion in the Fractal Cantorian $\epsilon^{(\infty)}$ space-time can be captured or not.
They shown that $R(N) = (h/m_n c)N^{\phi}$, the law represents the observed segregated Universe as the result of a fundamental self-similar law, is a Brownian motion with drift as to show us the equation (3.3.4):

$$\widetilde{B}_N = B_N + \phi^3 N$$

where $\phi = (\sqrt{5} - 1)/2$.
Let $B(N)$ be a real-valued Brownian motion (or a fractional Brownian motion if one count the long range dependent), and let $M(N)$ be an increasing process (that is, a process which is pathwise increasing in N) [112]. Assume that B and M are totally independent (quite rough from the viewpoint of practical applications). The application $N \to B(M(N))$ is named *Brownian motion in multifractal time*. The path of the new process has some multifractal (= intermittent) structure and some dimension spectrum can be computed.
In order to construct a multifractal cosmological's scenario the ingredients are: a multifractal "time warp", i.e., an increasing function or process $M(N)$, for which the multifractal formalism is known, and a function or process B with strong mono-fractal scaling properties such as considered Brownian motion (equation (3.3.4)). Let us recall the method of *mid-point displacement* which can be used to define simple Brownian motion $B_{1/2}$ iteratively at dyadic points. Indeed, the increments of the Brownian motion in multifractal time become independent Gaussian once the path of $M(N)$ is realized.

In case that M is a subordinator, then the resulting process is a Lévy process. This case is also known in probability as *Brownian (time) substitution.* We recall that a Lévy process is a stochastic process (real-valued or vector-valued) with stationary and independent increments, and that a subordinator is a real-valued Lévy process with increasing paths.
Jaffard in [110] proved that the paths of "most" Lévy processes are multifractals and he also

determined their spectrum of Holder exponents.

Since, the Brownian motion is an example of Lévy processes that can be defined as a monofractal (see Paragraph 4.2), here, we continue that study by considering the intersection of the exponents of the Brownian motion paths to obtain the multifractal model in the context of Cantorian $\epsilon^{(\infty)}$ spacetime.

4.4.1 Multifractal Universe

We consider a Boundaryless Fractal Model of Universe in (3+1)-dimension (for more details see "The Origin of Universes" by Nagasawa [119]). Assume that a distribution of points in i-space, fractal set, (like a structures S_i of the physical space S) is given from the measure μ: the probability for a point to fall in a set S_i is $\mu(S_i)$. The spatial distribution of points is on a ball of radius $R_i(t)$ and hence, the different structures have a fixed radius $R_i(N)$ and thanks an aggregation process (like Brownian motion) after a fixed time it reaches the mass M_i. Each structure belongs to one different scale on the N-axis (see Fig. 4.1).

If this distribution is singular one cannot describe it by means of a density and multifractal analysis useful in characterizing the complicated geometrical properties of μ. In our case, we have more Brownian paths, indeed, everyone constructs a specific length scale (like galaxies, globular cluster, etc...). Let us consider a bunch of p independent Brownian motions $W_1, ..., W_p$ starting uniformly on $\partial S(0,0)$ in $\Re^d$ (with $d \geq 2$); we call their first exit times $N_1, ..., N_p$ the time to emerge from a large ball, that is a sort of segregated inflation. Indeed, in our model (as in Fig. 4.1) we start with four Brownian motions and in different time steps they arrive on the different balls. For this reason we have in the first structure the intersection of four paths, in the second structure the intersection of three paths and so on. In relation to Universe's structure at the end we obtained just one Brownian motion. By classical results of Dvoretzky, Erdos, Kakutani and Taylor (see Theorem 4.2.1) the intersection of the paths of these motions is

$$S_i = \bigcap_{k=1}^{p} \left\{ x \in \Re^d : x = W_k(N) \text{ for some } N \in [0, N_i) \right\}, \tag{4.4.1}$$

with $i = 1, ..., 4$ and contains different points from the starting point if and only if

$p < d/(d-2)$.

This means that in our fractal model, in (3+1)-dimensional space we obtain for different ball-structures:

- Globular clusters $S_1(0, R_{GC}(N))$: four intersections of Brownian paths
- Galaxies $S_2(0, R_G(N))$: three intersections of Brownian paths
- Cluster of galaxies $S_3(0, R_{CG}(N))$: two intersections of Brownian paths
- Supercluster of galaxies $S_4(0, R_{SCG}(N))$: just one Brownian path.

In order to classify the singularities of μ by strength, we use a Holder exponent and consider the value of the multifractal spectrum (4.2.2) which represents the Hausdorff dimension of structures:

$$f_i(a) = dim\left\{x \in \Re^d : a = lim_{R_i(N) \longrightarrow 0} \frac{log\mu(S_i(x, R_i(N)))}{logR_i(N)}\right\} \tag{4.4.2}$$

where $S_i(x, R_i(N))$ denotes the open ball of radius R_i of the structures centered in x and local dimension a. The structures have the same center $x = 0$ that correspond to the origin of the system (like Big Bang).

Example. For instance, we consider the structure S_i corresponding to a Galaxy with $i = G$ and $R_G \cong 1 \div 10kpc$. The relation (4.4.2) becomes

$$S_G^{[a]} = \left\{x \in \Re^d : a = lim_{R_G \longrightarrow 0} \frac{log\mu(S_G(0, R_G))}{logR_G}\right\}$$

The center x of the open ball $S(0, R_G)$ is the origin of Universe and it is the same for all structures.

Let us recall that our space-time domain (structure's domain) is

$$D = \{(x, N); N \geq 0,\ x \in [-R_i(N), R_i(N)]\}. \tag{4.4.3}$$

and thank to the result in [50], we can write the dimension (4.4.3) as

$$D = lim_{R_i \to \infty} \frac{log(N < R_i)}{logR_i} \tag{4.4.4}$$

where $N < R_i$ is the number of nucleons inside the radius of the structure R_i.

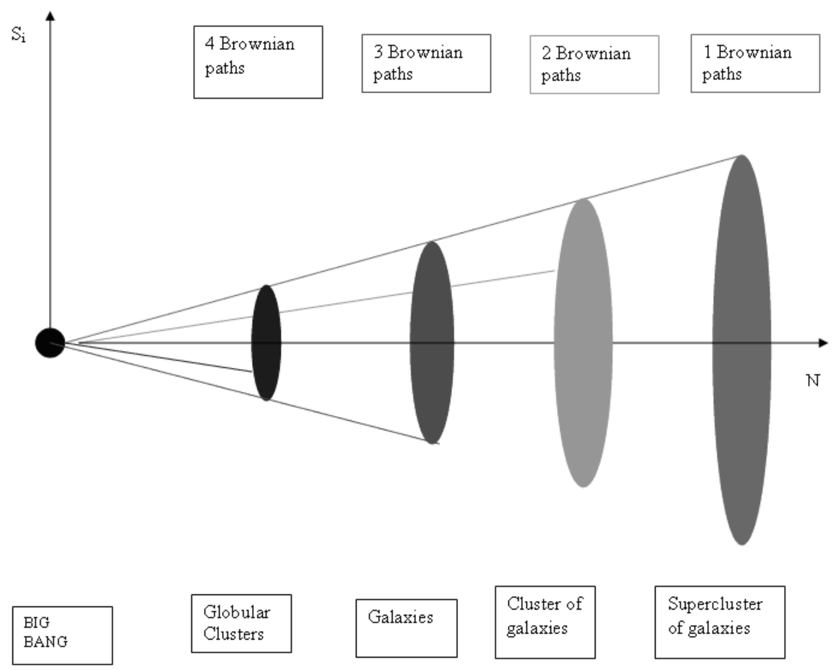

Fig. 4.1: Model of Universe with Brownian paths.

4.5 Fractal Dimension and Time for astrophysical structures

4.5.1 Fractal Dimension

Thanks to (4.4.4), we can estimate the fractal dimensions of all astrophysical structure and of Universe too. Hence, the relation (4.4.4) represents in this paper our multifractal spectrum.

By recasting (4.4.4) in

$$D = \frac{log(N < R_i)}{logR_i} \tag{4.5.1}$$

we obtain the fractal dimension of different structures. This recast makes sense due to we are summing that for $R > R_i$ the structure of length scale R_i do not take other N and so other matter. Indeed, in the following Tables we summarize these results.

Thank to relation (4.5.1) and values of Table 1 and Table 2 we can calculate the Fractal dimension of astrophysical objects (see Table 3).

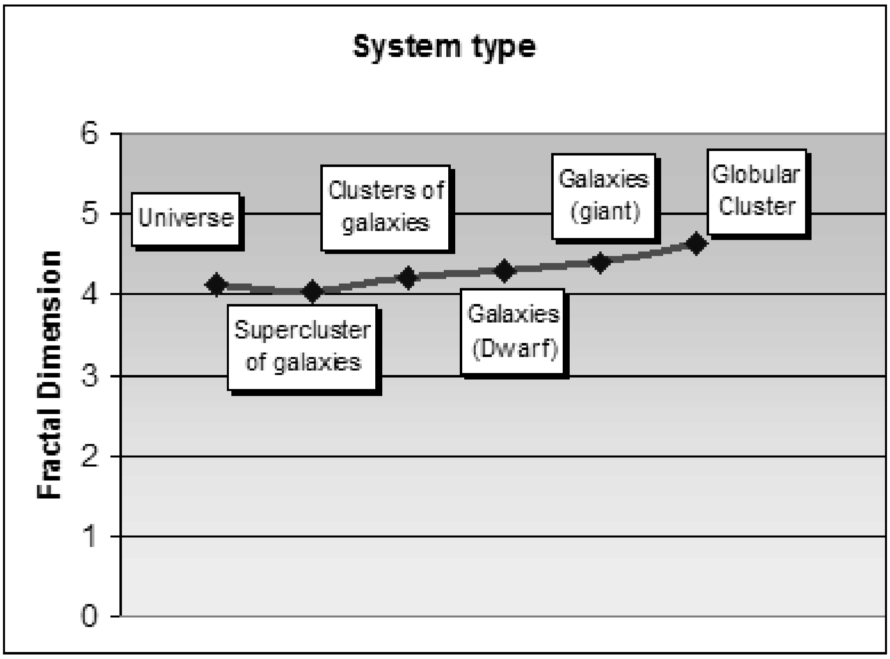

Fig. 4.2: Trend of Space-time Fractal Dimension of astrophysical objects.

System Type	D
Globular Clusters	$4.61 \div 4.66$
Galaxies (giant)	$4.27 \div 4.54$
Galaxies (dwarf)	$4.18 \div 4.39$
Clusters of galaxies	4.20
Superclusters of galaxies	$3.94 \div 4.15$
Universe	4.13

Table 3: Space-time Fractal Dimension of astrophysical objects

In the following graph (Fig. 4.2) we have ordered the objects for considering their Mass in terms of Solar Mass $M_{\odot}$ (see Table 1) where:

$$M_{\odot} = 1.98892 \times 10^{30} kg$$

In Table 4, we summarize the results with respect to solar system objects.

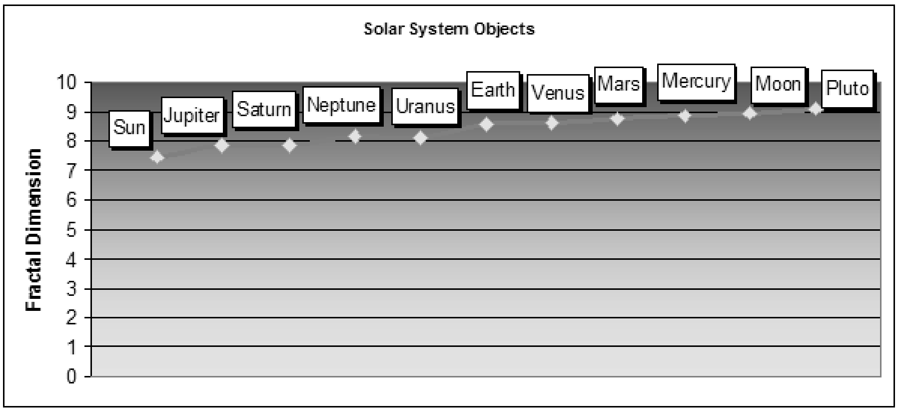

Fig. 4.3: Graph of Fractal Dimension of Solar System Objects.

Solar system objects	**Radius (10^6m)**	**Mass (kg)**	**N**	**D**
Sun	6.96×10^2	$M_\odot$	1.1892×10^{57}	7.4546
Mercury	2.439	3.2868×10^{23}	1.9650×10^{50}	8.8741
Venus	6.052	4.8704×10^{24}	2.9112×10^{51}	8.5884
Earth	6.378	5.976×10^{24}	3.5728×10^{51}	8.5761
Mars	3.3935	6.3943×10^{23}	3.8229×10^{50}	8.7454
Jupiter	71.4	1.8997×10^{27}	1.1358×10^{54}	7.8828
Saturn	59.65	5.6870×10^{26}	3.4000×10^{53}	7.8845
Uranus	25.6	8.6652×10^{25}	5.1806×10^{52}	8.1156
Neptune	24.75	1.0279×10^{26}	6.1453×10^{52}	8.1398
Pluto	1.1450	1.7928×10^{22}	1.0718×10^{49}	9.0924
Moon	1.738	7.3505×10^{22}	4.3946×10^{49}	8.9555

Table 4: Space-time Fractal Dimension of Solar system objects

In the last column in Table 4 we have calculated the Fractal dimension of the Solar system objects.

In the following graph (Fig. 4.3) we have ordered the objects to considering their Solar Mass.

The previous results suggest a Solar System that lives in a fractal space-time with extra-dimension or in a conventional (3+1) space-time but with a presence of dark energy to reduce the extra-dimensions. In Table 5 we calculate Space-time Fractal dimensions for atomic elements. In Fig. 4.4, we show the fractal dimension of 73 elements of the table

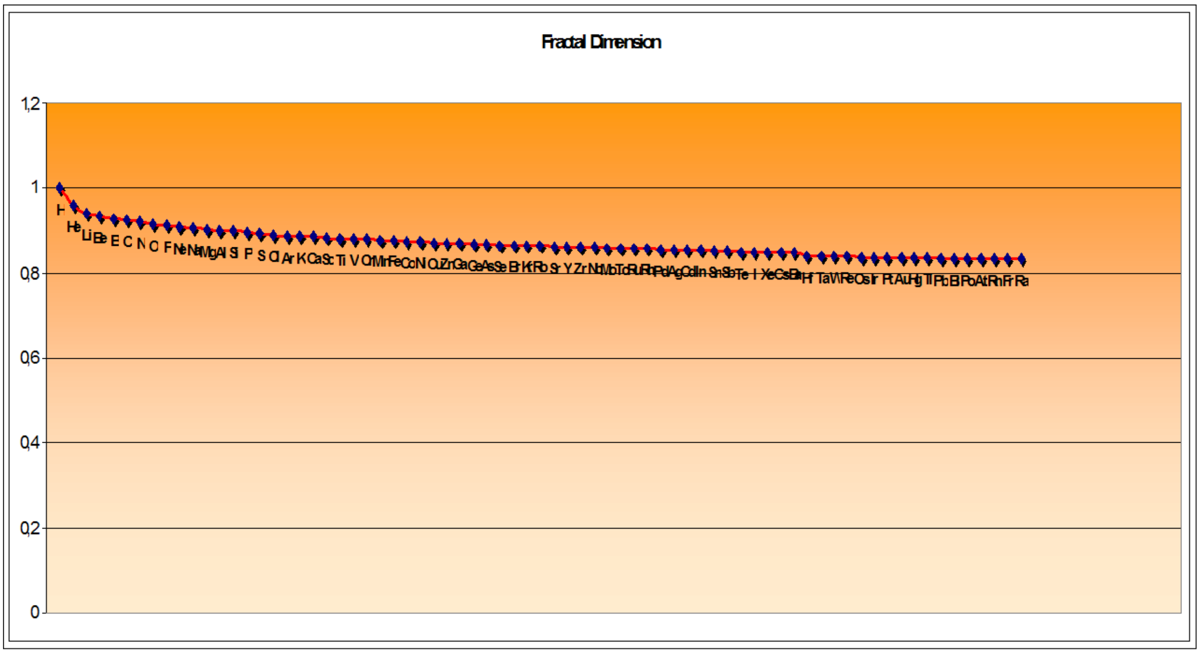

Fig. 4.4: Graph of Space-time Fractal Dimension of atomic elements.

because we have not considered the Lanthanoids, Actinoids and the elements from 89 and so on, while Fig. 4.5. gives us the Fractal dimensions at different scales.

The fact that we obtain a Space-time Fractal Dimension with a value that is smaller than 1 suggests that these elements are not stable but tend to create stable chemical links.

Elements	D	Elements	D	Elements	D
H	1	Mn	0, 87782	ln	0, 85424
He	0, 95883	Fe	0, 87725	Sn	0, 85314
Li	0, 94189	Co	0, 87559	Sb	0, 85233
Be	0, 93422	Ni	0, 87559	Te	0, 85078
B	0, 92808	Cu	0, 87301	I	0, 85103
C	0, 92539	Zn	0, 8725	Xe	0, 85003
N	0, 92064	Ga	0, 87015	Cs	0, 84954
O	0,91651	Ge	0,86881	Ba	0, 84858
F	0, 91118	As	0, 86794	Hf	0, 84009
Ne	0, 90958	Se	0, 86628	Ta	0, 83956
Na	0, 90524	Br	0, 86588	W	0, 83902
Mg	0, 90390	Kr	0, 86432	Re	0, 83867
Al	0, 90023	Rb	0, 86393	Os	0, 83797
Si	0, 89909	Sr	0, 86284	Ir	0, 83763
P	0, 8959	Y	0, 86247	Pt	0, 83713
S	0, 89491	Zr	0, 86175	Au	0, 8368
Cl	0, 89208	Nb	0, 86106	Hg	0, 83615
Ar	0, 88789	Mo	0, 86004	Tl	0, 83566
K	0, 88868	Tc	0, 85971	Pb	0, 83518
Ca	0, 88789	Ru	0, 85841	Bi	0, 83487
Sc	0, 88417	Rh	0, 85778	Po	0, 83487
Ti	0, 88213	Pd	0, 85685	At	0, 83472
V	0, 88021	Ag	0, 85626	Rn	0, 8329
Cr	0, 87959	Cd	0, 85508	Fr	0, 83276
				Ra	0, 83232

Table 5: Fractal dimension of atomic elements.

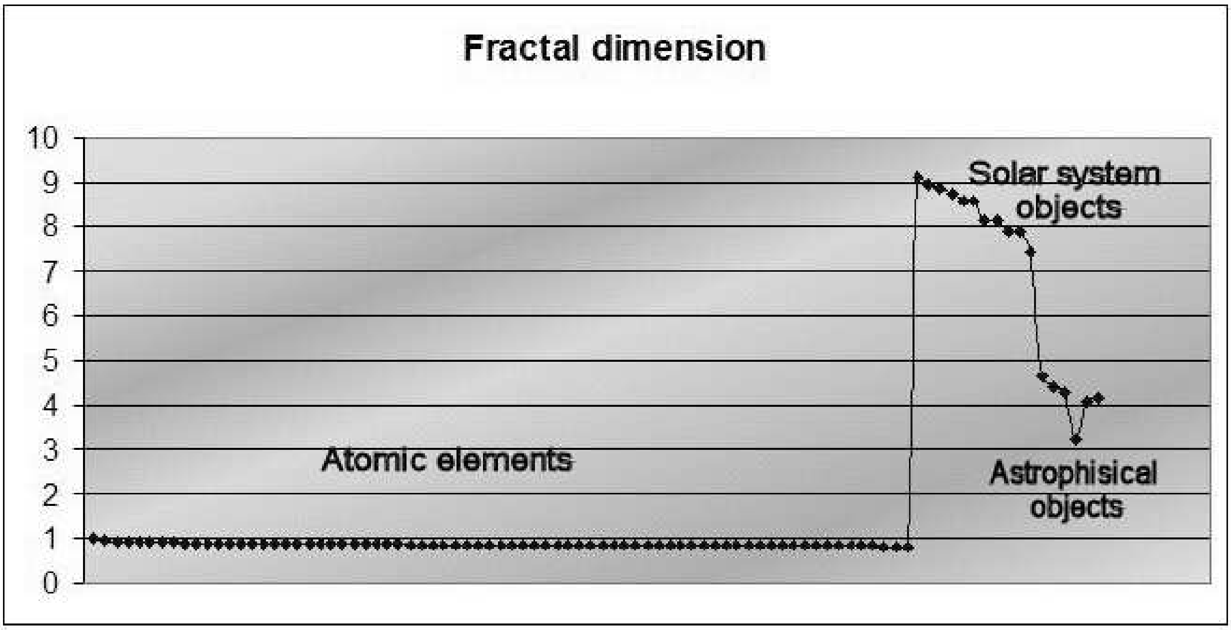

Fig. 4.5: The Space-time Fractal Dimension model: atomic elements, Solar System Objects, astrophysical objects.

4.5.2 *Characteristic parameters of time*

In addition, to the Theorems (3.2.1) and (3.2.2) in Chapter 3, we can evaluate the characteristic parameters of the time of Universe. As we see below it can be written as a function of its components N. Indeed, starting from the relation

$$R_U = cT_U \tag{4.5.2}$$

where R_U is the radius of Universe, T_U its time and $c = 2.99792458 \times 10^8 ms^{-1}$ by using the relation (3.2.3), we easily obtain

$$T = \frac{h}{E_E} N^{\phi} \tag{4.5.3}$$

where E_E is the Einstein energy for a nucleon, that is $E_E = m_c c^2$.

The relation (4.5.3) appears interesting since it connects the characteristic parameters of the time of Universe with the number of its components trough quantum and relativistic

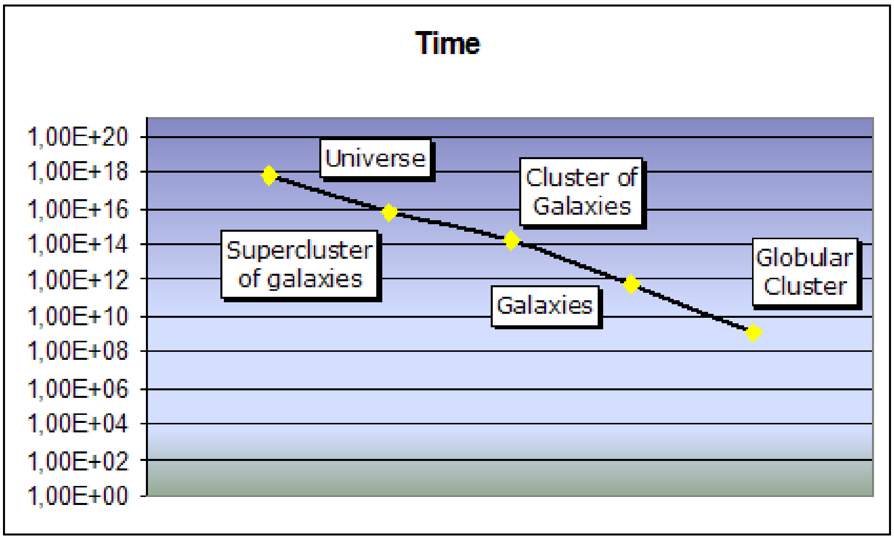

Fig. 4.6: The characteristic time parameter for Cosmological structures

contents. In the following Table 6 we calculate the the characteristic time parameter for Cosmological structures.

System type	**Time(s)**
Globular Cluster	1.0293×10^{9}
Galaxies	$1.0293 \times 10^{11} \div 1.0293 \times 10^{12}$
Cluster of galaxies	$1.5439 \times 10^{14} h^{-1}$
Supercluster of galaxies	$1.0293 \times 10^{15} h^{-1} \div 1.0293 \times 10^{16} h^{-1}$
Universe	6.1756×10^{17}

Table 6: Characteristic parameter time of cosmological structures

where $h = 6.6260755 \times 10^{-34} Js$ (Fig. 4.5).

Table 7 summarizes the results with respect to the Solar System objects (Fig. 4.6)

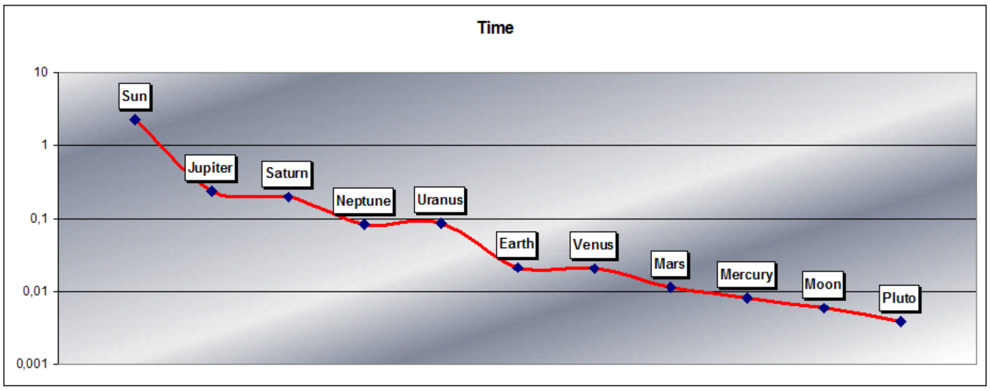

Fig. 4.7: Characteristic parameter time for Solar system objects

Solar system objects	**Time(s)**
Sun	2.3216
Mercury	8.1356×10^{-3}
Venus	2.0187×10^{-2}
Earth	2.1275×10^{-2}
Mars	1.1319×10^{-2}
Jupiter	0.23816
Saturn	0.19897
Uranus	8.5392×10^{-2}
Neptune	8.2557×10^{-2}
Pluto	3.8193×10^{-3}
Moon	5.7973×10^{-3}

Table 7: Characteristic parameter time for Solar system objects

Table 8 summarizes the results with respect to the periodic table of elements.

What is the meaning of this characteristic time? If we assume T_U as the age of Universe the other times can be seen as a thermalization time, that is the time which different structures take to start the dynamics as we know now. In other words, it is a sort of fluctuation before the structures born or emerge from their quantum or chaotic status (see Fig. 4.7-4.9)

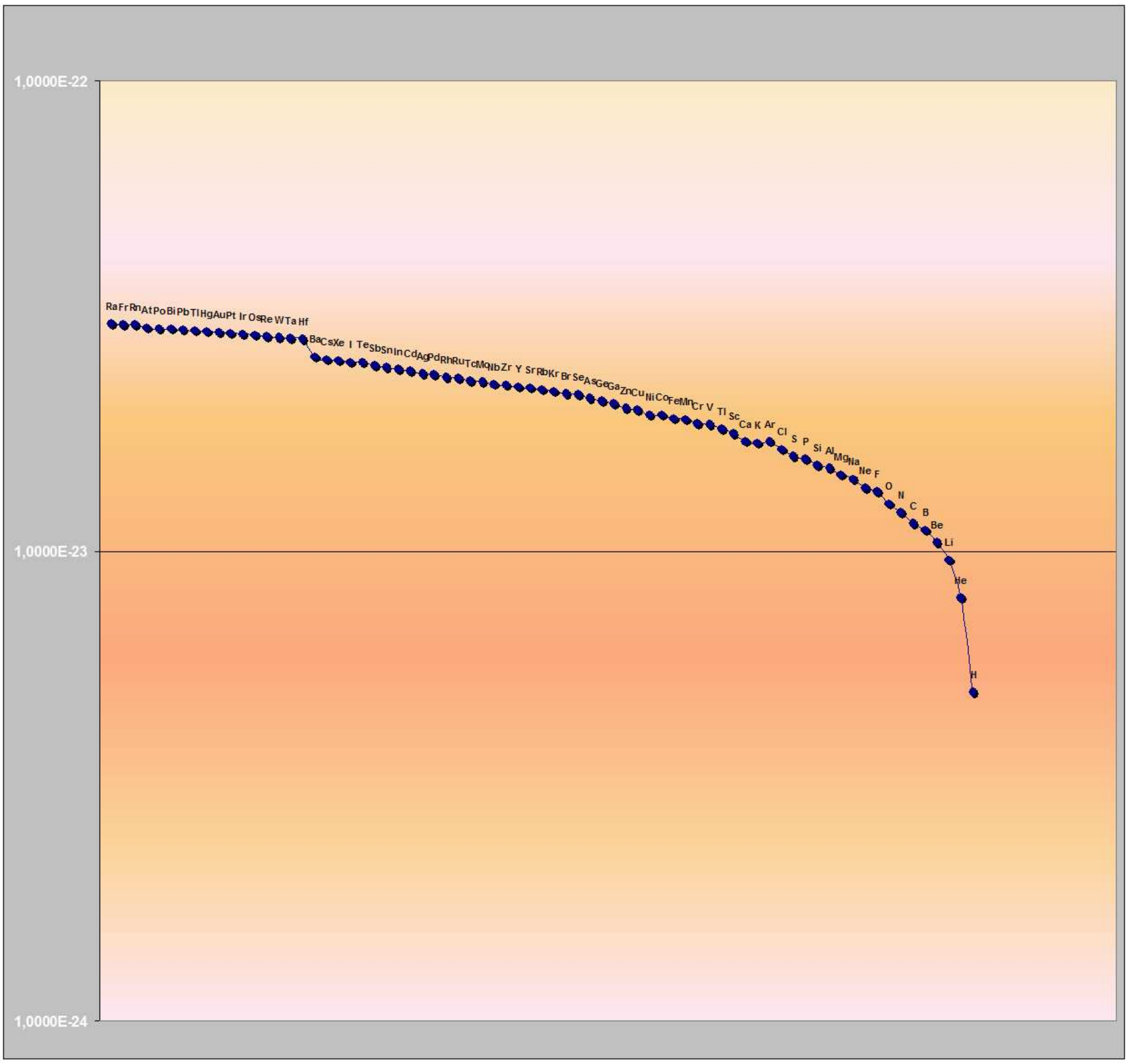

Fig. 4.8: Characteristic parameter time for atomic elements

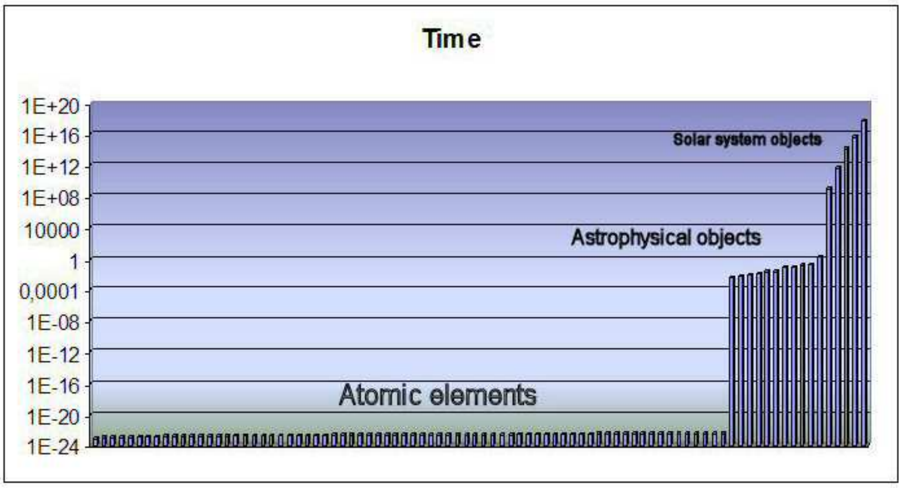

Fig. 4.9: Characteristic parameter time at all scales.

4.6 Conclusions

In the paper [109] we have considered the link between the scale invariant law, $R(N) = (h/m_n c)N^{\phi}$, introduced by authors as a Brownian motion, with drift of the form: $\widetilde{B}_N = B_N + \phi^3 N$, and *Multifractal processes*.

We have used the tools of multifractal in the context of Mohamed El Naschie's $\epsilon^{(\infty)}$ Cantorian space-time applied as to cosmology. In order to prove the multifractal nature of the cosmological's scenario we built a multifractal process. Brownian motion is an example of Lévy process. Thus it is qualified as *monofractal*, because its Hölder exponent is everywhere 1/2. However in general, most Lévy processes are multifractal. Furthermore, we have considered a Brownain motion in a *multifractal time* and calculated the intersection of the exponent for obtaining a multifractal process starting from a Brownian motion.

In conclusion, we showed some physical consequences with respect to the Fractal Dimension of Cantorian time and we discovered a characteristic parametric time linked to the quantum or chaotic fluctuation preceding the inciption of the structures.

Elements	Time (s)	Elements	Time (s)	Elements	Time (s)
H	5.0165×10^{-24}	Mn	1.9021×10^{-23}	ln	2.4319×10^{-23}
He	7.9442×10^{-24}	Fe	1.9125×10^{-23}	Sn	2.4589×10^{-23}
Li	9.5443×10^{-24}	Co	1.9471×10^{-23}	Sb	2.4798×10^{-23}
Be	1.0412×10^{-23}	Ni	1.9445×10^{-23}	Te	2.5189×10^{-23}
B	1.1063×10^{-23}	Cu	1.9966×10^{-23}	I	2.5144×10^{-23}
C	1.1459×10^{-23}	Zn	2.0157×10^{-23}	Xe	2.5431×10^{-23}
N	1.2061×10^{-23}	Ga	2.0593×10^{-23}	Cs	2.5534×10^{-23}
O	1.2608×10^{-23}	Ge	2.0872×10^{-23}	Ba	2.5815×10^{-23}
F	1.3351×10^{-23}	As	2.1093×10^{-23}	Hf	2.8171×10^{-23}
Ne	1.3622×10^{-23}	Se	2.1466×10^{-23}	Ta	2.83×10^{-23}
Na	1.4227×10^{-23}	Br	2.1551×10^{-23}	W	2.8451×10^{-23}
Mg	1.4493×10^{-23}	Kr	2.1895×10^{-23}	Re	2.8572×10^{-23}
Al	1.5007×10^{-23}	Rb	2.2040×10^{-23}	Os	2.8775×10^{-23}
Si	1.5209×10^{-23}	Sr	2.2223×10^{-23}	Ir	2.8876×10^{-23}
P	1.5713×10^{-23}	Y	2.2331×10^{-23}	Pt	2.9019×10^{-23}
S	1.5895×10^{-23}	Zr	2.2523×10^{-23}	Au	2.9112×10^{-23}
Cl	1.6437×10^{-23}	Nb	2.2661×10^{-23}	Hg	2.9289×10^{-23}
Ar	1.7104×10^{-23}	Mo	2.2906×10^{-23}	Tl	2.9472×10^{-23}
K	1.6982×10^{-23}	Tc	2.2990×10^{-23}	Pb	2.9607×10^{-23}
Ca	1.7123×10^{-23}	Ru	2.3306×10^{-23}	Bi	2.9692×10^{-23}
Sc	1.7791×10^{-23}	Rh	2.3447×10^{-23}	Po	2.9693×10^{-23}
Ti	1.8171×10^{-23}	Pd	2.3709×10^{-23}	At	2.974×10^{-23}
V	1.8548×10^{-23}	Ag	2.3818×10^{-23}	Rn	3.0296×10^{-23}
Cr	1.8675×10^{-23}	Cd	2.4147×10^{-23}	Fr	3.0342×10^{-23}
				Ra	3.0478×10^{-23}

Table 8: Characteristic parameter time for atomic elements.

5. CONCLUSIONS

In this thesis we presented a specific scenario of stochastic self-similar processes in the context of large scale structure and cosmology. Thanks to El Naschie's works, $\epsilon^{(\infty)}$ become the natural framework for describing the scenario of High Energy Physics. In other papers (e.g. [50], [51], [52]) fractals and multifractals become the natural language for describing Nature at all the scales in terms of its dynamical processes. In this work, we presented the stochastic self-similar processes in a fixed framework applying them to large-scale phenomena, modeling the natural scaling phenomenon in the context of M. El Naschie's $\epsilon^{(\infty)}$ Cantorian space-time. In particular, after an introduction on the mathematical context with respect to stochastic self-similar processes, fractals and multifractals concepts, our focus was on the following issues: we studied different analytical and numerical approaches to get stochastic self-similar processes applications for describing $\epsilon^{(\infty)}$ Cantorian space-time and presented new results obtained by using the fractal and multifractal properties within the above mentioned context.

In Chapter 3, we studied the implications of a stochastic self-similar and fractal Universe; indeed, it was demonstrated that the observed segregated Universe is the result of a fundamental Self-similar law (as anticipated in previous papers, e.g. [50], [51], [52]). In particular, we studied some aspects of the relation between the universal scaling law and the Random Walk; specifically, we used a Brownian motion developed as a limit of a Random Walk.

Moreover, we have showed some results obtained by using the context of $\varepsilon^{(\infty)}$ Cantorian space-time in connection with stochastic Self-similar processes in order to provide for a possible explanation of the segregation of the Universe at fixed scale in terms of Brownian motion.

In Chapter 4, we presented the analysis of Multifractals in the context of El Naschie's $\epsilon^{(\infty)}$ Cantorian Space-Time applied to cosmology. In detail, we summarized some recent results concerning fractal structure and the Brownian paths, in order to calculate fractal dimensions and characteristic parameters for large scale structures and for the atomic elements that live in El Naschie's $\epsilon^{(\infty)}$ Cantorian Space-Time. At the very beginning, we considered the results describing scaling rules in Nature as our starting point. Then, we used multifractal analysis to show that the result obtained by the author as Brownian motion in the Chapter 3, is a Multifractal process. In conclusion, we showed some physical consequences with respect to the Fractal Dimension of Cantorian time and we discovered a characteristic parametric time linked to the quantum or chaotic fluctuation preceding the inciption of the structures.

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