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Linear and nonlinear stability of a reaction diffusion system of P.D.Es, via Liapunov direct method. Application to a chemical autocatalytic reaction.

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Contents

Introduction

This thesis deals with the linear and nonlinear stability analysis of the equilibrium state of a binary reaction-diffusion system of partial differential equations (PDEs) modeling a chemical autocatalytic reaction.
Indeed, the dynamics of chemical mixtures may be described by equations that often take the form of system of nonlinear parabolic PDEs with diffusion of the involved chemical substances. One of the main contributions to the mathematical theory for the chemical reactors is given by the monograph of Aris [2]. In [2] "it is the theory, than the practice, of catalysis and the mathematical, rather than physical or chemical, theory" that he discusses. According to [2], "this is appropriate, for the practical aspects of the matter have been ably treated by Satterfield in his *Mass transfer in heterogeneous catalysis*, but it does not mean that the theory is divorced from practice, or that the mathematical models are not based on the physics and chemistry of the situation."
Nevertheless, there are analogies between chemical system and the dynamic structure of living organism in certain biological system, such as on the level of populations, where individuals interact and move around. It follows that the space-time interaction-migration model have the same general appearance as those for diffusing and reacting chemical system, i.e.

$$\frac{\partial \mathbf{u}}{\partial t} = D\Delta\mathbf{u} + F$$

where $\mathbf{u}$ is an n-vector whose components represent the diffusing quantities, D is the diffusion matrix, Δ is the Laplace operator in the spatial coordinates and F is the term describing all reactions and interactions between the

components of $\mathbf{u}$.
Besides, the rich spectrum of solutions which gives rise reaction-diffusion system is reflected also in a very wide variety of applications, let's think that such a mechanism was proposed as a model for the chemical basis of morphogenesis by Turing in one of the most important papers in theoretical biology [76], since stable, stationary, nonconstant solutions can exist for certain system;

a wide variety of spatio-temporal wave phenomena may be exhibited, from travelling wavefronts that join different steady states of F, like in the Belousov-Zhabotinskii reaction, to stable periodic limit cycle which bifurcate from a stable steady state as a parameter increases through a critical value; also, if we consider two or three space dimensions, there may be travelling waves train of concentric circles, called target patterns, that were originally found experimentally by Zaikin and Zhabotinskii [83], [77], such as spiral waves, spherical waves, chaotic oscillations and so on.
Good references for applications to biological and ecological context, are the classical books of Murray [53], [54], Okubo [55], Grindrod [31], Britton [5], such as a good survey of mathematical modeling of biological and chemical phenomena using RD systems is given in Maini et al. [47].
It results, therefore, of great interest the study of the long time behaviour of solutions for system of reaction-diffusion type. This study can be reduced to analyse the stability property of the uniform steady state.
The method we will apply is the direct method of Lyapunov which, unlike approximate methods that are often involved in the stability study of PDEs, works directly with the system and it is potentially applicable when nonlinearities are involved. However, for the stability analysis, the central problem in using the direct method remains the construction of a Lyapunov function and, often, to find conditions ensuring coincidence between linear and nonlinear analysis.
In this thesis we employ a peculiar Lyapunov functional, introduced by Rionero (see [65], [66]), which is a direct link between linear and nonlinear

stability. As we will see, it is construct in such a way that the functional, with its derivative along the perturbations, depends directly on the eigenvalues of the linear part of the involved operator.
We provide, in the first three chapters, some backgrounds useful to face the study of the stability properties for reaction diffusion system considered in the subsequent part of the thesis where, among some recent results, there are some original contributions of the author.
In particular:
in the first chapter, since we may look at reaction diffusion equations from a dynamical point of view, we recall some basic properties of dynamical system and the basic tools for Lyapunov's direct method;
reaction diffusion system are coupled nonlinear equations of parabolic type and so, in the second chapter, we consider general properties of the parabolic operator: starting with the second order linear operator we underline one of the most important qualitative technique, the maximum principle, which enable us to obtain comparison theorems for nonlinear parabolic operator and that we will use in order to obtain boundness of solution for the system object of our study;
chapter III is essentially devoted to the derivation of reaction diffusion equations, on fixed or moving domain, and to gather results about existence and asymptotic behaviour of solution for reaction diffusion system obtained using comparison theorems, etc... ;
with the fourth chapter we pose the problem of the stability study for a reaction diffusion system: the connection between linear and nonlinear stability analysis opens the chapter and then we put in evidence the advantage of determining and using Lyapunov functionals depending on the eigenvalues of the operator $-\Delta$. This new methodology is applied to a binary chemical reaction diffusion system of P.D.Es which models a chemical autocatalytic reaction and is capable of generating Turing type spatial pattern.
The stability analysis, on fixed and moving domain, is carried on through fourth and fifth chapter.

In particular, in chapter IV the linear stability-instability of the equilibrium state is studied and the onset of Turing instability is obtained, while chapter V is devoted to nonlinear stability theorems which are get in two different ways: from one hand we use the boundedness of regular solutions and from the other we introduce auxiliary cross-diffusion terms in order to control perturbations in the Sobolev space H_0^1.

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...ci son persone che ringrazi per quel che ti han dato, altre perché ti sono state date...(Anonimo)

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e volta nostra poppa nel mattino
de' remi facemmo ali al folle volo,
sempre acquistando dal lato mancino.
...
Tre volte il fe' girar con tutte l'acque;
a la quarta levar la poppa in suso
e la prora ire in giù, com'altrui piacque,
infin che 'l mar fu sovra noi richiuso.
Dante Alighieri, Inferno, canto XXVI

Chapter 1

Lyapunov stability. Direct method

1.1 Introduction

Many physical system are represented by partial differential equations (PDEs) that involve time. These PDEs are also called *evolution equations*, the idea being that the solution evolve in time from a given initial configuration. Therefore, the study of properties of solutions to these equations is very important. One of these properties is the stability or instability of certain solutions, that means knowing, in such a way, how errors, contained in a mathematical model of a real phenomenon, may influence the solution.
Over the years a number of methods have been developed for investigating the stability properties of solutions to PDEs. Most of these require linearization, truncation or other approximations of the original equations. As distinct from these approximate methods, Lyapunov's Direct Method deals directly with the system without resorting to approximation.
In this chapter, following nearly [24], [80], we describe some basic features of the Lyapunov's Direct Method, since, throughout this thesis, we'll use a peculiar Lyapunov functional in order to investigate the stability properties of the solutions of evolution equations.

1.2 Preliminaries to evolution equation

In order to describe a phenomenon, a mathematical model -called **evolution equation**- is constructed, whose solutions are required to reflect the behaviour of that phenomenon.

Let $\mathcal{F}$ be a phenomenon taking place on a domain Ω of the physical three dimensional space $\mathbf{R}^3$ and $\mathbf{u}(\mathbf{x}, t)$ a vectorial function of space ($\mathbf{x} \in \Omega$) and time ($t \in \mathbf{R}$ or $t \in [0, T],$ with $T \in [0, \infty]$) whose components, $u_i(\mathbf{x}, t), i = 1, \ldots n$ ($n < \infty$), are the relevant quantities describing the state of $\mathcal{F}$.

The vector $\mathbf{u}$ is called **state vector**.

If one finds (by experimental data, physical law, etc.) that there exists a function

$$\mathbf{F}\Big(\mathbf{x}, t, \mathbf{u}, \frac{\partial u_i}{\partial x_r}, \frac{\partial^2 u_j}{\partial x_r \partial x_s}\Big), \qquad i, j = 1.2 \ldots, n;\ r, s = 1, 2, 3;$$

which governs the behaviour of the time derivative of $\mathbf{u}$, such that, at any finite positive T:

$$\mathbf{u}_t = \mathbf{F}, \qquad \text{in } \Omega \times (0, T) \tag{1.1}$$

holds, then it is said that $\mathcal{F}$ is modelled by the PDE (1.1) to which we append, also motivated by physical reality, prescribed initial data

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \qquad \text{in } \Omega \tag{1.2}$$

and appropriate boundary conditions

$$\mathbf{B}(\mathbf{u}, \nabla \mathbf{u}) = \mathbf{u}^* \qquad \text{on } \partial\Omega \times [0, T], \tag{1.3}$$

where $\mathbf{B}$ is a given operator and $\mathbf{u}^*(\mathbf{x}, t)$ is prescribed.

The initial-boundary value problem (I.B.V.P.) obtained is the mathematical model called **evolution equation** of $\mathcal{F}$.

At this stage we would prefer that our (unique) solution changes only a little when the conditions specifying the problem change a little.

According to the definition due to Hadamard, we say that a given PDE, supplemented with boundary data and initial conditions, is well posed in the state space X, endowed with a suitable topology, if:

a there exist a solution;

b this solution is unique;

c the solution depends continuously on the data given in the problem.

A problem which is not well posed is said to be ill posed.

Remark 1. These requirements depend strongly on the choice of the underlying function spaces in which the data is given and in which we are seeking the solution. Depending on the problem one might use spaces of continuously differentiable functions $C^k((0,T)\times\Omega)$ or spaces of integrable functions $L^p((0,T)\times\Omega)$. Thus, the choice of functional topology in the state space is very important: it has to be linked to the physics of the phenomenon.

1.3 Dynamical system: basic properties

In this section we recall some basic concepts of the theory of dynamical system referring, among the wide literature on the subject, to [24], [80], [3], [75], and assuming that IBVP (1.1)-(1.3) is well-posed.

Definition 1.1. - A **dynamical system** on a metric space X is a mapping

$$v : (v_0, t) \in X \times \mathbf{R} \to v(v_0, t) \in X$$

such that

$$v(v_0, 0) = v_0. \tag{1.4}$$

Usually, the following additional property is required for a dynamical system (**semigroup property**):

$$v(v_0, t+\tau) = v(v(v_0,\tau), t), \qquad v_0 \in X,\ t, \tau \in \mathbf{R}^+. \tag{1.5}$$

Example: let $u(u_0, t)$, with $u(u_0, 0) = u_0$ be a global solution to the problem (1.1)-(1.3). Then u is a dynamical system.

The property (1.4) and (1.5) give to the one parameter family of operators $v(v_0, \cdot)$ the semigroup structure, according to the following definition:

Definition 1.2. A **semigroup of operators** on a metric X is a one parameter family $\{S(t)\}_{t\geq 0}$ of operators $S(t) : X \to X$ such that

$$S(t+s) = S(t)S(s)$$

$$S(0) = I, \qquad (I \text{ is the identity } \in X)$$

The equivalence between the semigroup of operators $\{S(t)\}_{t\geqslant 0}$ and the dynamical system is immediately seen by setting

$$v(v_0, t) = S(t)v_0 \qquad v_0 \in X,\ t \in \mathbf{R}^+.$$

Definition 1.3. - Given a dynamical system v, the function

$$v(v_0, \cdot) : t \in \mathbf{R} \to v(v_0, t) \in X$$

for a prescribed $v_0 \in X$, is called **motion** associated with the initial condition v_0 and is denoted by $v(v_0, t)$ or $v(t)$.

If $v(v_0, t) = v_0, \quad \forall t \in \mathbf{R}$, the motion is **stationary** (or **steady**) and v_0 is an **equilibrium point**.
Let v and w be two motions. If

$$v(0) = w(0) \Rightarrow v(t) = w(t) \quad \forall t > 0 (t < 0)$$

then the motion is **unique forward** (respectively **backward**) in time with respect to the initial data.
The forward uniqueness ensure the semigroup property.

The set $\{t, v(t)\}$, with $t \in \mathbf{R}^+$, is the **positive graph** of the motion v and its projection into X, that is the subset $\gamma^+ = \{v(t) : t \in \mathbf{R}^+\}$ is the **positive orbit** or **trajectory** starting at v_0.

Given a dynamical system v we will say that it is a C^0-semigroup according to the following definition:

Definition 1.4. - A dynamical system on a metric space X is a C^0**-semigroup** if (1.4)-(1.5) and the following properties hold:

$$v(t,\cdot): X \to X \text{ is continuous } \forall t \geq 0; \tag{1.6}$$

$$v(\cdot, v_0): \mathbf{R}^+ \to X \text{ is continuous } \forall v_0 \in X. \tag{1.7}$$

Remark 2. As we have just seen, a dynamical system may be generated by the evolution equation. In the study of dynamical system generated by PDEs, the existence of the operators $S(t)$ and their properties is linked to the problem of the existence of solution for PDEs, and so, like for the uniqueness, it must be proved case by case.

We end this section with the important notion of continuous dependence (third requirement for the wellposedness) with respect to initial data, that is, given a particular (basic) motion, $v(v_0,\cdot)$, will any other motion, $v(v_1,\cdot)$ -starting at the same initial time from a position v_1 sufficiently closed to v_0- remain as closed as desired to $v(v_0,\cdot)$ for every finite time $T > 0$?
This is a necessary requirement if the mathematical formulation is to describe observable natural phenomena. Data in nature cannot possibly be conceived as rigidly fixed; the mere process of measuring them involves small errors. For example, prescribed values for space or time coordinates are always given within certain margins of precision. Therefore, a mathematical problem cannot be considered as realistically corresponding to physical phenomena unless a variation of the given data in a sufficiently small range leads to an arbitrary small change in the solution. This requirement of "stability" is not only essential for meaningful problems in mathematical physics, but also for approximation methods.
Let v be a dynamical system on a metric space (X,d) and let $B(x,r)$, with $x \in X$ and $r > 0$, be the open ball centered at x and having radius r.

Definition 1.5. - A motion $v(v_0,\cdot)$ of a dynamical system **depends continuously on the initial data** if and only if:

$$\forall T, \varepsilon > 0, \exists \delta(\varepsilon, T) > 0 : v_1 \in B(v_0, \delta) \Rightarrow v(v_1, t) \in B(v(v_0,t), \varepsilon), \quad \forall t \in [0,T].$$

The following theorems hold

Theorem 1.3.1. *Let v be a dynamical system on a metric space X having the C^0-semigroup properties. Then any motion depends continuously on the initial data.*

Proof. See [24]

Theorem 1.3.2. *A motion which is not unique cannot depend continuously on the initial data.*

Proof. See [24]

1.4 Lyapunov stability

The concept of stability can be interpreted in many different ways. In the following stability will be referred to in the sense of Lyapunov, that is, roughly speaking, for a sufficiently small perturbation the system will remain close to the original solution for all future time.

Definition 1.6. A motion $v(v_0, t)$ is **Lyapunov stable** (with respect to perturbations in the initial data) if and only if

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 : v_1 \in B(v_0, \delta) \rightarrow v(v_1, t) \in B(v(v_0, t), \varepsilon), \quad \forall t \in [0, \infty).$$

A motion is **unstable** if it is not stable.
It results that the Lyapunov stability extends the requirements of continuous dependence to the infinite interval of time $(0, \infty)$.

Definition 1.7. A motion of a dynamical system $v(v_0, \cdot)$ is said to be an **attractor** or **attractive** on a set $Y \subset X$ if

$$v_1 \in Y \Rightarrow \lim_{t \to \infty} d[v(v_0, t), v(v_1, t)] = 0 \tag{1.8}$$

The largest set Y satisfying (1.8) is called the **basin** (or **domain**) **of attraction** of $v(v_0, \cdot)$.

Definition 1.8. A motion $v(v_0, \cdot)$ of a dynamical system is **asymptotically stable** if it is stable and if there exists $\delta_1 > 0$ such that $v(v_0, \cdot)$ is attractive on $B(v_0, \delta_1)$.
In particular $v(v_0, \cdot)$ is **exponentially stable** if there exist $\delta_1 > 0$, $\lambda(\delta_1) > 0$, $M(\delta_1) > 0$ such that:

$$v_1 \in B(v_0, \delta_1) \Rightarrow d[v(v_0, t), v(v_1, t)] \leq Me^{-\lambda t} d(v_1, v_0), \quad \forall t \geq 0$$

If $\delta_1 = \infty$, then $v(v_0, \cdot)$ is asymptotically (exponentially) **unconditionally** (or **globally**) stable.

Let X be a metric linear space. It is always possible to express the stability of a given basic motion $v(v_0, t)$ through the stability of the zero solution of the perturbed dynamical system

$$u : (u_0, t) \in X \times \mathbf{R}^+ \to v(v_0 + u_0, t) - v(v_0, t),$$

where

$$u(u_0, t) = v(v_1, t) - v(v_0, t) \qquad \big(v_1 = v_0 + u_0\big)$$

is the perturbation at time t to the basic motion $v(v_0, t)$. Indeed the definition of stability of $v(v_0, \cdot)$ is equivalent to

$$\forall \varepsilon > 0, \, \exists \delta(\varepsilon) > 0 : u_0 \in B(O, \delta) \to u(u_0, t) \in B(O; \varepsilon), \quad \forall t \geq 0$$

where O is the origin of X.

Remark 3. If the dynamical system $v(\cdot, t)$ is a linear operator of X on X, $\forall t \in \mathbf{R}^+$, then the stability of every motion is determined by the stability of the zero solution, but if v is nonlinear, the stability of the trivial solution doesn't determine the stability of every motion.

1.5 The Lyapunov's Direct Method

For a conservative system

- a rest point is stable if the potential energy is a local minimum, otherwise it is unstable;
- the total energy is a constant during any motion;

hold.

Basically, Lyapunov's Direct Method is a generalization of these two physical principles for conservative system. The technique is based on the definition of an "auxiliary" function of the system states which is decreasing along the system trajectories. It was introduced by the Russian mathematician A. M. Lyapunov in 1893 [45] for the stability analysis of solution of ordinary differential equations (ODEs) and is referred to as second method or Direct Method because no knowledge of solution of the evolution equation is required.

It is well established in the qualitative theory of ODEs [42],[82], [32],[11]. Perhaps a first step toward applying Lyapunov's direct method to PDEs was made by Massera [49], who extended this method to denumerably infinite system of ODE. A general stability theory based on the existence of a Lyapunov functional for the invariant sets of dynamical systems in general metric spaces is established by Zubov [84] who employs this theory to derive results for systems of partial differential equations. Others first attempts are due to Movchan [51].

Definition 1.9. Let v be a dynamical system on a metric space X. A functional $V : X \to \mathbf{R}$ is a **Lyapunov function** on a subset $I \subset X$ if V is continuous on I and a nonincreasing function of time along the motions having the initial data in I.

In order to assure that $V[v(x,\cdot)]$ is a nonincreasing function of time, we assume that V is differentiable with respect to time and that the derivative is non-positive. It is standard, in literature, to require that the generalized time derivative

$$\dot{V} := \lim_{t\to 0^+} \inf \frac{1}{t}\{V[v(x,t)] - V(x)\}, \quad x \in I$$

is non-positive.

Assume that X is a normed linear space. As we have seen the stability of a given motion can be expressed through the stability of the zero solution of the perturbed dynamical system. Therefore, one can introduce the direct method for investigating the stability of an equilibrium position only.
Denoting by $\mathcal{F}_r, r > 0$, the set of function $\phi : [0, r) \to [0, \infty)$ which are continuous, strictly increasing and such that $\phi(0) = 0$, then the Lyapunov direct method can be summarized by the following theorems.

Theorem 1.5.1. *Let u be a dynamical system on a normed space X and let O be an equilibrium point. If V is a Lyapunov function on the open ball $B(O, r)$, for some $r > 0$, such that*

i) $V(O) = 0$

ii) $\exists f \in \mathcal{F}_r : V(u) \geq f(\|u\|), \quad u \in B(O, r)$

then O is stable.
If, in addition,

iii) $\exists g \in \mathcal{F}_r : \dot{V}(u) \leq -g(\|u\|), \quad \forall u \in B(O; r),$

then O is asymptotically stable.

Proof. See [24]

In particular, if we replace assumption **ii**) with $V(u) > 0 \quad u \neq O$, then we have the stability with respect to the measure V. When assumption **i**) and **ii**) hold we say that V is positive definite. If, moreover, there exists a positive constant c such that along the motions

$$\dot{V} \leq -cV$$

then one obtain the exponential stability in the measure of V, i.e.

$$V \leq V(u_0)e^{-ct}.$$

Set $\Sigma(X, \alpha) = \{x \in X : V(x) < \alpha\}$. The following theorem holds.

Theorem 1.5.2. *Let u be a dynamical system on $X \times \mathbf{R}^+$ and let O be an equilibrium point. If V is a Lyapunov function on the open set $A_r = B(O,r) \cap \Sigma(X,0)$, for some $r > 0$, and*

i) *V(O)=0*

ii) $\exists g \in \mathcal{F}_r : V(u) \leq -g[-V(u)], u \in A_r$,

iii) $A_\varepsilon \neq \emptyset, \quad \forall \varepsilon > 0$,

then O is unstable.

Proof. See [24]

All the above theorems are set in a normed linear space X where one can introduce many other norms. Recall that two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X are equivalent, if there exist constants $c_1 \geq c_2 > 0$ such that $c_2\|x\|_2 \geq \|x\|_1 \geq c_1\|x\|_2, \forall x \in X$. Therefore stability (instability) properties are invariant under equivalent norms.
If $X = \mathbf{R}^n$, i.e. X is finite dimensional space, all possible norms are equivalent and so the stability doesn't depend on the chosen norm. This is the case of phenomena modeled by ODEs.
If we consider phenomena with an infinite degrees of freedom, and so modeled by PDEs, then it can turn out that a solution is stable with one choice of norm, and unstable with another choice. In this case stability depend on topology in the state space. This is a relevant difference between ODE and PDE. For a discussion about the importance of the choice of functional topology see [22], while for example of topology dependent stability see [24].
Finally we mention the fact that in order to discuss stability in a meaningful sense it is often necessary to put restrictions on the initial states. The idea of introducing a second metric for this purpose seems to have been originated by Movchan [52]. Stability is then defined in terms of the two metrics, rather than one: one, d for measuring initial data and another, d^* for the perturbation. The relationship generally required between d and d^* is that $d \to 0$ implies $d^* \to 0$.

An important application of Lyapunov functions is that they can be used for the determination of some positive invariant set

Definition 1.10. A set $A \subset X$ is **positively** (**negatively**) **invariant** for the dynamical system v if $v(v_0, t) \in A$ for any $v_0 \in A$ and $t \geq 0 (t \leq 0)$.

This role is important because if a bounded set $A \subset X$ can be shown to be positive invariant, then $x \in A \Rightarrow \gamma(x) \in A$ and hence the positive orbit $\gamma(x)$ is bounded.

Definition 1.11. A set A is **attractive** on an open set $B \supset A$ if it is positive invariant and

$$v_0 \in B \Rightarrow \lim_{t \to \infty} d[v(v_0, t), A] = 0$$

An important role in the study of the asymptotic behaviour of solutions is played by the positive limit set.

Definition 1.12. Let v be a dynamical system on a metric space X and let $x \in X$. A set $\Omega(x) \subset X$ is the **positive limit set** of the motion $v(x, t)$ if, $\forall y \in \Omega(x)$, there exists a sequence $\{t_n(y)\}, t_n \in \mathbf{R}^+$, such that:

$$\begin{cases} \lim_{n \to \infty} t_n = \infty \\ \\ \lim_{n \to \infty} d[v(x, t_n), y] = 0 \end{cases} \tag{1.9}$$

In particular, $\Omega(x) = x$ if x is an equilibrium point; if $v(x, t)$ is periodic in time (i.e. $\exists \tau : v(v_0, t + \tau) = v(v_0, t)$), then $\Omega(x) = \gamma(x)$, where $\gamma(x)$ is the orbit of $v(x, t)$. In general, $\Omega(x)$ belongs to the closure of $\gamma(x)$.

Information about the asymptotic behaviour of motions by means of Lyapunov functions are furnished by the following LaSalle Invariance Principle.

Theorem 1.5.3. *Let v be a dynamical system on a metric space X, with the C_0-semigroup properties and let V be a Lyapunov function on a set $A \subset X$. If*

i) $V(x) > -\infty \qquad \forall x \in \bar{A}$,

ii) $\gamma(x) \subset A$,

then $\Omega(x)$ *belongs to the largest positive invariant subset* $\mathcal{M}^+$ *of* $\Omega^* = \{x \in \bar{A} : \dot{V}(x) = 0\}$*. Further, if* X *is complete and* $\gamma(x)$ *is precompact, then*

$$\lim_{t\to\infty} d[v(x,t), \mathcal{M}^+] = 0.$$

Proof. See [42], or [80] where is allowed V to be lower semicontinuous and not only continuous.

As remarked in [24] the LaSalle Invariance Principle works very well when $X = \mathbf{R}^n$, but when X is infinite dimensional then, in using theorem (1.5.3) one needs also conditions ensuring precompactness of positive orbits. This is another fundamental difference between ODE and PDE: generally Lyapunov functions allow one to obtain boundedness of positive orbits and only if X is locally compact does boundedness imply precompactness. Now, given X a Banach space, it is locally compact if and only if it is finite dimensional.

Chapter 2

Partial Differential Equations of Parabolic Type

2.1 Introduction

In physical applications, PDEs are more ubiquitous than ODEs. This situation can be understood because physical quantities more often depend on space and time than on, say, time alone. A partial differential equation relates the variations of this physical quantity in time and in space. Of course, in mathematical abstraction, one does not need to assign the physical meaning of time to the symbol t, or space to the symbol x; one is simply concerned with the variations of the unknown with respect to more than one independent variable as governed by a PDE.

In this chapter we introduce some well known definitions and properties of second-order PDEs of parabolic type which, generally, can be seen as an equation describing physical phenomena known to be diffusive. Similar equations arise in biological, chemical system, in probability theory and in financial mathematics, modeling the price of an option, e.g. the Black-Scholes equation.

2.2 Classification of second order equation

Let $\Omega \subset \mathbf{R}^n$ a bounded domain and $u : \Omega \to \mathbf{R}$. We consider the general quasilinear second order PDE

$$\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = f(x, u, \nabla u), \tag{2.1}$$

where $(a_{ij}) = (a_{ij})(x, u, \nabla u) \in \mathbf{R}^{n \times n}$, a real symmetric matrix, and f are given functions.
Based on the spectrum of (a_{ij}), we can classify second order PDEs.

Definition 2.1. (Classification of second order PDEs). Let $x \in \Omega$ and let $\lambda_i = \lambda_i(x, u, \nabla u) \in \mathbf{R}^n$ be the eigenvalues of (a_{ij}). We call the PDE (2.1)

elliptic at x, if $\lambda_i(x, u(x), \nabla u(x)) > 0$ for all $i = 1, \dots n$ (or $\lambda_i < 0$ for all i);

hyperbolic at x, if one $\lambda_j(x, u(x), \nabla u(x)) > 0$ and $\lambda_i(x, u(x), \nabla u(x)) < 0$ for all $i \neq j$ (or the other way round)

parabolic at x, if at least one $\lambda_i(x, u(x), \nabla u(x)) = 0$.

If the PDE (2.1) is elliptic-hyperbolic-parabolic for all $x \in \Omega$, we call the PDE elliptic-hyperbolic-parabolic.

Remark 4. If $n \geq 4$, it can happen that two or more λ_j have one sign and two or more λ_i have the other sign. These cases are called ultra-hyperbolic.

The following are the archetypes for the more complicate equations of second order for a function $u(x, y, z)$:

- Poisson's equation (elliptic type)

$$-\Delta u = f;$$

- the wave equation (hyperbolic type)

$$u_{xx} + u_{yy} - u_{zz} = 0$$

- the heat equation (parabolic type)

$$u_z = u_{xx} + u_{yy}$$

There are phenomena which lead to equations of mixed type, for example the study of transonic flows. The prototype, in this case, is the **Tricomi equation** $u_{xx} + xu_{yy} = 0$: elliptic for $x > 0$, parabolic for $x = 0$, and hyperbolic for $x < 0$.

From a physical point of view PDE can be classified as equilibrium problems and marching problems. The first class, equilibrium or steady state problems include elliptic ones, the marching problems include both the parabolic and hyperbolic ones, i.e. those whose solution depends on time.

2.3 Linear second order parabolic operator

Let Ω be a bounded domain of $\mathbf{R}^n$, $T > 0$ and set

$$Q_T = \Omega \times (0, T]$$

the spatio-temporal cylinder, and

$$S_T = \bar{Q}_T - Q_T$$

the parabolic boundary of Q_T.
The general linear parabolic second order equation in n-space variable can be written in the form

$$Au + au = f, \tag{2.2}$$

where $u : \bar{Q}_T \to \mathbf{R}$, $u = u(x,t)$ is the unknown, $f : Q_T \to \mathbf{R}$ is given;
The operator A is defined by

$$Au = \sum_{i,j=1}^{n} a_{ij}(x,t) D_i D_j u + \sum_{i=1}^{n} a_i(x,t) D_i u - u_t \tag{2.3}$$

where $D_i = \dfrac{\partial}{\partial x_i}$; all the coefficients a_{ij}, a_i and the function $a = a(x,t)$, are given coefficient bounded in Q_T.

A is called **uniformly parabolic** if there exists a positive constant μ such that

$$\sum_{i,j=1}^{n} a_{i,j}(x,t)\xi_i\xi_j \geq \mu|\xi|^2 \quad \forall\xi \in \mathbf{R}^n, \forall(x,t) \in Q_T \tag{2.4}$$

We shall assume that the above conditions on A and a are valid and (without loss of generality) that $a_{ij} = a_{ji}$

General second order parabolic PDE describe the time evolution of the density of some quantities, say, a chemical concentration u, diffusing within a region. The second order term, $\sum_{i,j=1}^{n} a_{ij}(x,t)D_iD_ju$, describe diffusion of u, the first order term, $\sum_{i=1}^{n} a_i(x,t)D_iu$, describe transport while the zeroth-order term au is the reaction term. The matrix (a_{ij}), which generally is not a multiple of identity matrix, describe the anisotropic, heterogeneous nature of the medium in which diffusion holds.
The most simple equation of (2.2) is obtain by choosing

$$\begin{cases} a_{ij} = k\delta_{ij} \\ \\ \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \\ \\ a = 0, f = 0, k = \text{cost} > 0 \end{cases} \tag{2.5}$$

and it results

$$u_t = k\Delta u \tag{2.6}$$

which is called **heat equation** or - more generally - **diffusion equation** of a substance u, **without convection**.
It is quite clear that this equation is preserved under the transformation $(x,t) \to (\lambda x, \lambda^2 t)$, while it changes under the transformation $t \to -t$.
The invariance under the reversal of time means that there may be dissipation effects which lead to an increase in entropy since the "knowledge" about the past is lost as time increase.

Actually, parabolic equations arise in "irreversible" time-dependent processes.

Given f in (2.2), which u solves (2.2) in Q_T? For definiteness we must append to (2.2)

-initial condition

$$u(x,0) = u_0(x) \quad \text{in } \Omega;$$

and

-boundary condition

$$\mathbf{B}(u, \nabla u) = b(x,t) \qquad \text{on } \partial\Omega \times [0,T],$$

where $\mathbf{B}$ is an operator, $b(x,t)$ is prescribed. In particular we have

- Dirichlet condition: $\mathbf{B}(u, \nabla u) = u$;
- Neumann condition: $\mathbf{B}(u, \nabla u) = \nabla u \cdot \mathbf{n}$;
- Robin condition: $\mathbf{B}(u, \nabla u) = \lambda(x,t)u + \mu(x,t)\nabla u \cdot \mathbf{n}$, with $\lambda \geq 0\,, \mu \geq 0$ and $\lambda^2 + \mu^2 > 0$.

We will call **regular** or **classical** solution of (2.2), supplemented by initial and boundary conditions, in some region D, a function u such that all the derivative of u which occur in (2.2) are continuous functions in D and $Au(x,t) + au(x,t) = f$ at each point (x,t) of D.
There are problem in which, in order to recover the underlying physics (the formation and propagation of shock waves), we must allow for solutions which are not continuously differentiable or even continuous. Besides, for a given problem, one may prove well-posedness in a wider class of function and then try to "regularize" the so-called **weak** solution.
For a definition of a weak solution we will consider only the ingredient for a weak formulation of the initial-boundary value problem (see [21] for details).
First of all we will consider a second order parabolic operator with the principal part given in divergence form

$$A = \sum_{i,j=1}^{n} D_j\big(a_{i,j}(x,t)D_i u\big) + \sum_{i=1}^{n} a_i(x,t)D_i u - u_t$$

and then:

- boundary conditions.
  Let V be the suitable Hilbert space for the boundary condition (assume homogeneous ones), and so: $V = H_0^1(\Omega)$ for Dirichlet's conditions; $V = H^1(\Omega)$ for Neumann and Robin problem;

- bilinear form associated with the divergence form parabolic operator. Set

$$B(u, v; t) = \int_\Omega \Big\{ \sum_{i,j=1}^n a_{ij}(\cdot, t) D_i u D_j v + \sum_{i=1}^n a_i(\cdot, t) D_i u \, v + a(\cdot, t) uv \Big\} dx$$

  for a.e. $t \in (0, T)$ and $u, v \in V$;
  for Robin problem,

$$\bar{B}(u, v; t) = B(u, v; t) + \int_{\partial\Omega} \beta uv$$

  with $\beta \in L^\infty(\partial\Omega)$.
  It results that the above bilinear form satisfies the following conditions

  1. $\forall u, v \in V$, $\cdot : t \to B(u, v; t)$ is measurable;
  2. there exists a positive constant M such that
  $$|B(u, v; t)| \leq M \|u\|_V \|v\|_V \quad \forall u, v \in V, \; a.e. \, t \in (0, T);$$
  3. there exist $\alpha > 0$, $\lambda \geq 0$ such that
  $$B(u, u; t) \geq \alpha \|u\|_V^2 - \lambda \|u\|_{L^2}^2 \quad \forall u \in V, \; a.e. \, t \in (0, T).$$

- data u_0 and f.
  Assume $u_0 \in L^2(\Omega)$ and $f \in L^2(0, T; L^2(\Omega))$.

Considering u not as a function of space and time but as a mapping

$$\mathbf{u} : [0, T] \to V$$

defined by $[\mathbf{u}(t)](x) := u(x,t) \quad (x \in \Omega, 0 \leq t \leq T)$,
and similarly

$$\mathbf{f} : [0,T] \to V$$

setting $[f(t)](x) := f(x,t) \quad (x \in \Omega, 0 \leq t \leq T)$,
if we fix a function $v \in V$, we can multiply $Au + au = f$ by v and integrate by parts, to find

$$(u', v) + B(u, v; t) = (f, v) \quad \forall\, 0 \leq t \leq T \tag{2.7}$$

where the pairing $(\cdot, \cdot)$ denote the inner product in $L^2(\Omega)$ and $' = \dfrac{d}{dt}$.
It is reasonable to look for a weak solution with $u' \in V'$ (V' the dual space to V) since $B(u, \cdot, t)$ and f are linear and continuous functionals on V, so they are in V' too. As consequence the pairing (u', v) can be reexpressed as $< u', v >$, $< \cdot, \cdot >$ being the pairing of V and V'.
Finally we have

Definition 2.2. A function

$$u \in L^2(0,T;V), \text{ with } u' \in L^2(0,T;V')$$

is a weak solution of the parabolic initial/boundary value problem provided

1. $$< u', v > + B(u, v; t) = (f, v)$$

   for each $v \in V$ and a.e time $0 \leq t \leq T$

2. $u(0) = u_0$

Remark 5. It results([13], [21]) that if $u \in L^2(0,T;V)$ with $u' \in L^2(0,T;V')$, then $u \in C([0,T]; L^2(\Omega))$ and so the equality 2. in the above definition makes sense.

For example, for $V = H_0^1(\Omega)$, considering

$$u_t = g^0 + \sum_{j=1}^{n} D_j(g^j)$$

where we have set $g^0 = f + \sum_{i=1}^n a_i D_i u + au$ and $g^j = \sum_{i=1}^n a_{ij} D_i u \quad (j = 1, \ldots, n)$, by the characterization of $H^{-1}(\Omega)$ (the dual space to $H_0^1(\Omega)$, it results that the right hand side of (2.7) lies in $H^{-1}(\Omega)$ and the following estimate holds:

$$\|u_t\|_{H^{-1}} \leq \Big(\sum_{j=0}^{n} \|g^j\|^2_{L^2(\Omega)} \Big) \leq C\big(\|u\|_{H_0^1(\Omega)} + \|f\|_{L^2(\Omega)}\big)$$

with C a constant depending only on T, Ω. This estimate suggest it may be reasonable to look for a solution with $u' \in H^{-1}(\Omega)$ for a.e. $t \in [0, T]$.
The weak formulation is the natural basis for the implementation of the Faedo-Galerkin method in order to obtain existence and uniqueness of the weak solutions for initial-boundary value problem.

2.4 The maximum principle

Second-order linear parabolic equations retain many properties of the simplest equation of this type - the heat equation. One of the most important properties of (2.6) is the maximum principle [26] that enables us to obtain information about solutions without any explicit knowledge of the solutions themselves. Just as for the heat equation, for general second-order parabolic equations the maximum principle implies the uniqueness of solutions for the main boundary-value problems and the Cauchy problem.
In this section we will assume that the operator A has the nondivergence form (2.3) and the coefficient a_{ij}, a_i, a are continuous. We will always suppose uniform parabolicity condition and that $a_{ij} = a_{ji}, (i, j = 1, \ldots, n)$.

Theorem 2.4.1. *(Weak maximum principle). Assume $u \in C_1^2(Q_T) \cap C(\bar{Q}_T)$ and*

$$a \equiv 0 \quad \textit{in } Q_T$$

.

(i) *If*

$$Au \geq 0 \quad \text{in } Q_T, \tag{2.8}$$

then

$$\max_{\bar{Q}_T} u = \max_{S_T} u.$$

(ii) *Likewise, if*

$$Au \leq 0 \quad \text{in } Q_T, \tag{2.9}$$

then

$$\min_{\bar{Q}_T} u = \min_{S_T} u.$$

Proof. See [21]

Next, allowing zeroth-order term,

Theorem 2.4.2. *(Weak maximum principle for $a \geq 0$). Assume $u \in C_1^2(Q_T) \cap C(\bar{Q}_T)$ and*

$$a \leq 0 \quad \text{in } Q_T$$

.

(i) *If*

$$Au \geq 0 \quad \text{in } Q_T, \tag{2.10}$$

then

$$\max_{\bar{Q}_T} u \leq \max_{S_T} u^+.$$

(ii) *Likewise, if*

$$Au \leq 0 \quad \text{in } Q_T, \tag{2.11}$$

then

$$\min_{\bar{Q}_T} u \geq -\max_{S_T} u^-.$$

where $u^+ = \max\{u, 0\}$ while $u^- = -\max\{-u, 0\}$.

Proof. See [21]

Remark 6. In particular, if $Au + au = 0$ within Q_T, then

$$\max_{\bar{Q}_T} |u| = \max_{S_T} |u|$$

.

We will recall, now, the Harnack's inequality which state that the maximum of a nonnegative-regular solution of our parabolic equation, in some interior region, at a positive instant of time, can be estimate by the minimum of the solution in the same region, at a later time.

Theorem 2.4.3. *(Parabolic Harnack inequality). Assume $u \in C_1^2(Q_T)$ solve*

$$Au + au = 0 \quad \text{in } Q_T, \tag{2.12}$$

and

$$u \leq 0 \quad \text{in } Q_T.$$

Suppose $V \subset\subset \Omega$ is connected. Then for each $0 < t_1 < t_2 \leq T$, there exists a constant C such that

$$\sup_V u(\cdot, t_1) \leq C \inf_V u(\cdot, t_2).$$

The constant C depends only on V, t_1, t_2, and the coefficients of the equation (2.12).

This is true if the coefficients are continuous, or even merely bounded and measurable (see [21] [44])
This inequality may be employed in order to have a strong maximum principle, which substantially strengthen the foregoing assertion demonstrating that u cannot attains its maximum at an interior point of a bounded connected open set at all, unless u is constant.

Theorem 2.4.4. *(Strong maximum principle). Assume $u \in C_1^2(Q_T) \cap C(\bar{Q}_T)$ and*

$$a \equiv 0 \quad \text{in } Q_T$$

.

(i) *If*

$$Au \geq 0 \quad in\ Q_T, \tag{2.13}$$

and u attains its maximum over $\bar{Q}_T$ at a point $(x_0, t_0) \in Q_T$, then u is constant on Q_{t_0}.

(ii) *Likewise, if*

$$Au \leq 0 \quad in\ Q_T, \tag{2.14}$$

and u attains its maximum over $\bar{Q}_T$ at a point $(x_0, t_0) \in Q_T$, then u is constant on Q_{t_0}.

Proof.[21]

Theorem 2.4.5. *(Strong maximum principle for $a \leq 0$). Assume $u \in C_1^2(Q_T) \cap C(\bar{Q}_T)$ and*

$$a \leq 0 \quad in\ Q_T$$

.

(i) *If*

$$Au \geq 0 \quad in\ Q_T, \tag{2.15}$$

and u attains a nonnegative maximum over $\bar{Q}_T$ at a point $(x_0, t_0) \in Q_T$, then u is constant on Q_{t_0}.

(ii) *Likewise, if*

$$Au \leq 0 \quad in\ Q_T, \tag{2.16}$$

and u attains a nonpositive maximum over $\bar{Q}_T$ at a point $(x_0, t_0) \in Q_T$, then u is constant on Q_{t_0}.

Proof.[21]

2.5 Extension of the strong maximum principle

In this section we will consider the equation (2.2) in a domain $\mathcal{D} \subset \mathbf{R^n} \times \mathbf{R}^+$ assuming the coefficients $a_{ij}, a_i, a \in L^\infty(\mathcal{D})$ and the operator A uniformly

parabolic in $\mathcal{D}$.
We present, here, results, due to Nirenberg and Friedmann (for a complete proof see, [26], [73], [58]).

Given any two points in $\mathcal{D}$, (x_1, t_1) and (x_2, t_2), we will say that (x_1, t_1) is connected in $\mathcal{D}$ to (x_2, t_2) by a *horizontal* segment if $t_1 = t_2$ and the points can be joined by a line segment lying in $(t = t_1) \cap \mathcal{D}$. Similarly the points can be joined by an *upward vertical* segment if $x_1 = x_2$, $t_1 < t_2$ and the line segment joining them is contained in $\mathcal{D}$.

Theorem 2.5.1. *(Strong maximum principle). Suppose that A is uniformly parabolic in a domain $\mathcal{D}$, where $a \leq 0$ and $f \geq 0$ (resp.≤ 0) in $\mathcal{D}$. Let $\sup_{\bar{\mathcal{D}}} u = M \geq 0$ (resp. $\inf_{\bar{\mathcal{D}}} u = M \leq 0$), and suppose that $u(x_0, t_0) = M$ for some $(x_0, t_0) \in \mathcal{D}$. Then $u(x, t) = M$ at all points in $\mathcal{D}$ which can be connected to (x_0, t_0) by an arc in $\mathcal{D}$ consisting of a finite number of horizontal and upward vertical segments.*

The proof of the theorem is based on the following proposition.

Proposition 2.5.2. *Suppose $Au \geq 0$ (resp.≤ 0) in $\mathcal{D}$, $a \leq 0$ in $\mathcal{D}$ and $\sup_{\bar{\mathcal{D}}} u = M$ is attained at a point in $\mathcal{D}$. Then the conclusion of theorem(2.5.1) holds.*

This proposition follows from the next lemmas. We assume $A \geq 0$, the other case take a similar proof.

Lemma 2.5.3. *Let K be a ball with $\bar{K} \subset \mathcal{D}$, and suppose $u < M$ in K, and $u(x_1, t_1) = M$, with $(x_1, t_1) \in \partial K$. Then t_1 is either the largest or smallest t-value in K, that is (x_1, t_1) is either at the* top *or* bottom *of K.*

Now we have two lemmas that gives the result of proposition (2.5.2) pertaining, respectively, to horizontal and upward vertical segments.

Lemma 2.5.4. *Let $\mathcal{D}$ be the domain in the $x - t$ space and $A \geq 0$ in $\mathcal{D}$. Let $u \leq M$ in $\mathcal{D}$ and $u(x_0, t_0) < M$ for some $(x_0, t_0) \in \mathcal{D}$. Let Γ be the component of $\{t = t_0\} \cap \mathcal{D}$ which contains (x_0, t_0). Then $u < M$ on Γ.*

Lemma 2.5.5. *Let $A \geq 0$ in $\mathcal{D}$ and $u < M$ in $\mathcal{D} \cap \{t_0 < t < t_1\}$ for some $t_0 < t_1$. Then $u < M$ on $\mathcal{D} \cap \{t = t_1\}$.*

If we take, now, two point $p = (x_0, t_0)$ and q that can be connected by an arc in $\mathcal{D}$ made of a finite number of horizontal and/or upward vertical segments, by the foregoing lemmas the proposition (2.5.2) follows.

Remark 7. The strong maximum principle given in the previous section is a corollary of the theorem (2.5.1) with $\mathcal{D}$ a cylinder in $\mathbf{R}^n \times \mathbf{R}$

Another important point is the behaviour of the outward directional derivatives at those points of $\partial\mathcal{D}$ in which the maximum of u in $\mathcal{D}$ is achieved (analogously to the elliptic case): these derivatives are nonzero.

Theorem 2.5.6. *Suppose that u is a solution of (2.2) in $\mathcal{D}$ and that $a \leq 0$ in $\mathcal{D}$. Suppose $f \geq 0$ in $\mathcal{D}$ and $\max_{\bar{\mathcal{D}}} u = M$ is attained at $p \in \partial\mathcal{D}$. Assume that $\partial\mathcal{D}$ is so regular at p that a ball S can be constructed through p with $\in \mathcal{D}$ and $u < M$ on the interior of S. Suppose too that the radial direction from the center of S to p is not parallel to the t-axis. Then $du(p)/d\boldsymbol{n} > 0$ for every outward direction $\boldsymbol{n}$. (A similar statement holds in the case $f < 0$ in $\mathcal{D}$, where $M = \min_{\bar{\mathcal{D}}} u$ and we conclude $du(p)/d\boldsymbol{n} < 0$.)*

Proof.[73]

2.6 Parabolic nonlinear operator: comparison principles

The strong maximum principle for linear parabolic equations may be applied to nonlinear parabolic (as well as elliptic) ones to prove comparison theorems, i.e pointwise inequalities between different solutions (roughly speaking, if u and v are two solutions, with $u \leq v$ on $\partial\mathcal{D}$, then $u \leq v$ on $\mathcal{D}$).
Often comparison theorem are used to obtain information about the asymptotic behavior of solutions .

Following ([58], [23]) consider the vectors $x = (x_1, \dots, x_n)$ and $p = (p_1, \dots, p_n)$ and the matrix $R = (r_{ij})$, and let $F(x, t, u, p, R)$ be a continuously differentiable function of its $n^2 + 2n + 2$ variables. We shall use the notation $F(x, t, u, p_i, r_{ij})$ to denote the above function with p_i and r_{ij} denoting generic arguments of F.

We say that the nonlinear operator

$$L[u] = F(x, t, u, p_i, r_{ij}) - \frac{\partial u}{\partial t} \tag{2.17}$$

is **parabolic** with respect to a function $u(x, t)$ at a point (x_0, t_0) of a domain $\mathcal{D}$ in the (x, t)-space if, for any $u, p_1, p_2, \dots, p_n, r_{11}, \dots, r_{nn}$, the matrix

$$\Big(\frac{\partial F(x_0, t_0, u, p_i, r_{i,j})}{\partial r_{hk}}\Big) \tag{2.18}$$

is positive definite when the values $p_i = \dfrac{\partial u}{\partial x_i}$ and $r_{ij} = \dfrac{\partial^2 u}{\partial x_i \partial x_j}$ are substituted in the arguments of the partial derivatives of F appearing in (2.18).

The operator L is **parabolic in the domain** $\mathcal{D}$ if it is parabolic at each point of $\mathcal{D}$.

Remark 8. As in the linear case note, in particular, that for each fixed time $0 \le t \le T$, $L[u] = F(x, t, u, p_i, r_{ij})$ is **elliptic** with respect to a function $u(x, t)$ at a given point (x, t) , i.e., for all real vectors $\xi = (\xi_1, \xi_2, \dots, \xi_n)$, we have

$$\sum_{i,j=1}^{n} \frac{\partial F}{\partial r_{ij}} \xi_i \xi_j > 0 \qquad \text{for } \xi \neq \mathbf{0} \tag{2.19}$$

Now, let u be a (regular) solution of

$$L[u] = f(x, t) \quad \text{ in } \mathcal{D},$$

with L given by (2.17), and suppose that $w = w(x, t)$ satisfies

$$L[w] \le f(x, t) \quad \text{ in } \mathcal{D}.$$

Set

$$v(x, t) = u(x, t) - w(x, t)$$

and consider the inequality (using subscript for partial derivatives)

$$F(x,t,u,u_x,u_{x_i},u_{x_ix_j}) - F(x,t,w,w_x,w_{x_i},w_{x_ix_j}) - \frac{\partial u}{\partial t} \geq 0.$$

Applying the mean value theorem, evaluating the derivatives of F at the arguments $\theta u + (1-\theta)w$, $\theta u_{x_i} + (1-\theta)w_{x_i}$, $\theta u_{x_ix_j} + (1-\theta)w_{x_ix_j}$ for some mapping $\theta = \theta(x,t)$ such that $0 < \theta < 1$, we find

$$\sum_{i,j=1}^{n} \Big(\frac{\partial F}{\partial r_{ij}}\Big)\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \Big(\frac{\partial F}{\partial r_i}\Big)\frac{\partial v}{\partial x_i} + \Big(\frac{\partial F}{\partial u}\Big)v - \frac{\partial v}{\partial t} \geq 0. \tag{2.20}$$

We assume that F is elliptic in $\mathcal{D}$ for all the functions of the form $\theta u+(1-\theta)w$. Under this assumption, the left hand side of (2.20) is a linear parabolic operator for the function v. We may apply the maximum principle for such operators and conclude that if v is nonpositive initially and on the boundary, then v is nonpositive in $\mathcal{D}$.
The above discussion establishes the following result on approximation.

Theorem 2.6.1. *Let Ω be a bounded domain in n-dimensional space and let $\mathcal{D} = \Omega \times (0,T]$. Suppose that u satisfies the initial and boundary conditions*

$$u(x,0) = g_1(x)$$

$$u(x,t) = g_2(x,t) \qquad \text{on } \partial\Omega \times (0,T).$$

We assume that z and Z satisfy the inequalities

$$L[Z] \leq f(x,t) \leq L[z] \qquad \text{in } \mathcal{D}$$

and that L is parabolic with respect to the function $\theta u + (1-\theta)z$ and $\theta u + (1-\theta)Z$, $0 \leq \theta \leq 1$. If

$$\begin{cases} z(x,0) \leq g_1(x) \leq Z(x,0) & \text{in } \Omega, \\ z \leq g_2(x) \leq Z & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{2.21}$$

then

$$z(x,t) \leq u(x,t) \leq Z(x,t) \qquad \text{in } \mathcal{D}.$$

For example consider the nonlinear equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[k(u)\frac{\partial u}{\partial x}\right] \tag{2.22}$$

where k is a positive function with a bounded first derivative. The above equation governs the flow of heat through a homogeneous medium. Well, (2.22) is parabolic for all functions u and since it is satisfied by any constant, we may apply the last theorem to conclude that for any solution u, the maximum and minimum values must occur either at the initial time or on the boundary. The physical meaning immediately follows: if the temperature on the boundary and at the initial time is less then a certain value, say M, then, in the absence of sources, the inside of the body cannot be at a temperature bigger than M.

We want add, also, that the maximum principle can be used to prove existence theorems by using "upper" and "lower" solution, the solution being the limit of a monotone iteration schemes, where the convergence of the schemes is a consequence of the maximum principle. We omit details since in the next chapter we will consider existence theory for parabolic system. However, it is worth noting that when the elliptic operator is considered his solution are steady-state (i.e., equilibrium or time independent) solution of the associated parabolic equation.

Chapter 3

Reaction diffusion system

3.1 Introduction

In the class of nonlinear evolutions equations we find reaction diffusion systems, which are coupled partial differential equations of parabolic type.
Their most natural roots lie in the study of chemical systems.
Indeed, these mathematical models describe how concentration of one or more substance distributed in the space changes under the influence of two processes: **local** chemical **reactions**, in which the substances are converted into each other, and **diffusion**, which cause the substances spread out in space.
However, many others areas of life's sciences find in reaction diffusion equations a "natural" way to describe dynamical processes.
The starting point in this chapter is the derivations of these equations. Then we recall some qualitative technique that allows many problems to be attacked, such as existence of solutions or even delicate stability phenomena.

3.2 Derivations of the equations

Under the continuum hypothesis, the spatio-temporal state of a chemical system is described by PDEs derived from mass balance low.
Let $\Omega \subseteq \mathbf{R}^n$ be the reaction space, that is a bounded region which we will call the "domain", with boundary $\partial\Omega$, and B an elemental volume at fixed location within the domain.
The change of the amount of a "substance" within the elemental volume is given by the flux of matter through the elemental volume boundary ∂B plus the net production rate of a chemical species (the reaction kinetics) in B, and so, in mathematical terms

$$\frac{d}{dt}\int_B u(x,t)dx = \int_{\partial B} -\mathbf{J}\cdot\mathbf{n}\,dS + \int_B f\,dx \tag{3.1}$$

where

- $u(x,t) : \Omega \times \mathbf{R}^+ \rightarrow \mathbf{R}$ is the concentration of a chemical species U or, more general, the "particle" density function;

- $\mathbf{J}$ is the flux density, i.e. the scalar product $\mathbf{J}\cdot\mathbf{n}$ is the net rate at which particle cross a unit area in a plane perpendicular to $\mathbf{n}$ (positive in the $\mathbf{n}$ direction, $\mathbf{n}$ being the outward-oriented normal to B on ∂B);

- f, the reaction kinetics, is the rate of production and degradation of the reactant U. Generally they are described by polynomial or rational function in u and parameters that represent interaction with other chemicals and external factors.

Using the divergence theorem (assuming the underlying fields are smooth), (3.1) becomes

$$\frac{d}{dt}\int_B u(x,t)dx = \int_B \big[-\nabla \cdot \mathbf{J} + f \big]\, dx. \tag{3.2}$$

The domain is fixed in time, so we may differentiate through the integral and, by the arbitrary choice of the elemental volume B in Ω, the following local conservation equation

$$\frac{\partial u}{\partial t} = -\nabla \cdot \mathbf{J} + f \tag{3.3}$$

holds for any flux transport $\mathbf{J}$ and any "supply" f.
Of course, these last terms may depend on u, its derivatives, such as on position x and time t.
If we suppose that the instantaneous flux $\mathbf{J}$ is due to isotropic Fickian diffusion, then $\mathbf{J} = -D\nabla u$, where the diffusivity D is a constant, and we have the reaction-diffusion equation for species U on a fixed domain Ω,

$$\frac{\partial u}{\partial t} = D\Delta u + f. \tag{3.4}$$

Generally one is interested in the interaction of several particles species, for example several chemicals $\{U_1, \dots, U_n\}$. Then, the equation (3.4) is replaced by a system which describes the evolution of a vector of concentrations $\mathbf{u} = (u_1, \dots, u_n)$ and now the kinetic term, $\mathbf{f}(\mathbf{u}, x, t) = (f_1, \dots, f_n)$ is a vector describing the interaction of the species. In general the function f_i, for chemical systems, comes from the applications of the low of mass action to reaction taking place, that is the rate of a reaction is proportional to the product of the concentrations of the reactants.

Until now we have considered Ω like a fixed domain. Now we will obtain the reaction diffusion equations within a changing (with respect to time) domain.
Let $\Omega(t)$ be the reaction domain with boundary $\partial\Omega(t)$ and $B(t)$ an elemental

volume which moves with the flow due to domain change.
Applying the conservation of matter and the divergence theorem (being instantaneously valid at all time) to any measurable $B(t)$, we obtain

$$\frac{d}{dt}\int_{B(t)} u(x,t)dx = \int_{B(t)} \big[-\nabla\cdot\mathbf{J} + f \big]\, dx \tag{3.5}$$

Now, the Reynolds Transport theorem gives

$$\frac{d}{dt}\int_{B(t)} u(x,t)dx = \int_{B(t)} \Big[\frac{\partial u}{\partial t} + \nabla\cdot(\mathbf{v}u) \Big]\, dx \tag{3.6}$$

where $\mathbf{v}$ is the velocity field of the flow.
Also in this case, the arbitrary choice of $B(t)$ implies

$$\frac{\partial u}{\partial t} = -u\nabla\cdot\mathbf{v} - \nabla u\cdot\mathbf{v} - \nabla\cdot\mathbf{J} + f, \tag{3.7}$$

that, for isotropic fickian flux, becomes

$$\frac{\partial u}{\partial t} = -u\nabla\cdot\mathbf{v} - \nabla u\cdot\mathbf{v} + D\Delta u + f, \tag{3.8}$$

which is the local form of the diffusion equation with convection. The term $\nabla\cdot\mathbf{v}$ gives the local rate of volume expansion or contraction. In particular, for incompressible flows, $\nabla\cdot\mathbf{v} = 0$.
Moreover there is a convection or advection term, $\nabla u\cdot\mathbf{v}$, which represents the transport of chemicals within the domain as it moves and no relative movement of the chemicals with respect to the domain is present.

In the last year several (and almost numerical) studies have incorporated growing domain in the study of pattern formation: from Kondo and Asai [36], who model the growth increasing the numerical grid mesh spacing during a computation, to Varea et al.[78], where it is argued that domain growth reduces the effective diffusion and so it is assumed $D(t) = \dfrac{D_0}{(\alpha t)^2}$; furthermore Arcuri and Murray [1] use an appropriate scalings suggesting that domain growth influence reaction and diffusion. Other attempts in modeling domain growth are due to Kulesa et al.[37], but a more general framework, which allow for the subsequent inclusion of the properties of any specific tissue in which reactions are taking place, is due to Crampin et al.[16], [17], [18], and the reference quoted therein.

The starting point is (3.7).

It is assumed that the **tissue** is **incompressible**, that is domain undergoes deformation and expansion with no accompanying change in density. Specifically, growth consist of local directional volume expansion (possibly nonuniform) resulting in convection of material, the term $\nabla u \cdot \mathbf{v}$, while $u \nabla \cdot \mathbf{v}$ gives a "dilution term", since it may be read in the following way: neglecting the production due to the kinetic terms, the **local** concentration is decreasing while the containing volume is increasing.

In general, the flow $\mathbf{v}$ may be specified by some system of constitutive equations describing the properties of the medium or the tissue in which reactions are taking place.

One assume that growth properties are determined locally and are specified on an initial position and subsequently follow the flow due to tissue growth. The deformation of the medium due to the growth is given by the rate of deformation tensor which can be decomposed into symmetric $\mathbf{D} = \dfrac{1}{2}[\nabla v + (\nabla v)^T]$, where ^{T} denotes the transpose, and antisymmetric part $\mathbf{S} = \dfrac{1}{2}[\nabla v - (\nabla v)^T]$. The tensor $\mathbf{D}$ is the so called strain tensor, and because it is symmetric, there is an orthonormal basis in which $\mathbf{D}$ is diagonal. The trace of $\mathbf{D} = \nabla \cdot \mathbf{v}$ gives the rate of volumetric change per unit volume (and so, for domain growth, one requires $\nabla \cdot \mathbf{v} > 0$).

The antisymmetric part, $\mathbf{S}$, gives the vorticity, $\nabla \times \mathbf{v} = \omega$, which is associated with the rigid body rotation.

This last term may be considered not relevant in the analysis of pattern formation, since solid body translation and rotations of the domain leave the pattern generated by reaction and diffusion within the tissue unaffected.

3.3 Nondimensionalisation and statements of the mathematical problems

In order to nondimensionalise the coupled equations of type (3.4) we will consider the scaling parameters

$$\bar{u}_i = \frac{u_i}{U_i^*}, \quad \bar{x} = \frac{x}{L}$$

where U_i^* is a reference concentration for the chemical species U_i and L is a length scale.

The reaction term is nondimensionalised by using a reaction rate ω characteristic of the kinetic scheme. It is present as $\bar{f}$ in nondimensional form and, in general, has the same functional form of f but different coefficients.

Another important scaling parameter is

$$\gamma = \frac{\omega L^2}{D_1}$$

with $D_1 = \max\{D_i\}$, D_i the diffusivity coefficient for i-th species, which represents the ratio of diffusive T_D to kinetic T_R relaxation times, where

$$T_D = \frac{L^2}{D_1} \text{ and } T_R = \frac{1}{\omega}$$

Both of these timescale may be used to nondimensionalise the time variable, writing $\bar{t} = \frac{D_1}{L^2}t$ or $\bar{t} = \omega t$ respectively.

Omitting the bars we obtain, in nondimensional form, the system

$$u_t = \Gamma D \Delta u + f(u) \tag{3.9}$$

where $u = (u_1, \ldots, u_m)$, $f = (f_1, \ldots, f_m)$, D is a dimensionless diagonal matrix and $\Gamma = \gamma$ or $\Gamma = \frac{1}{\gamma}$ according to wether we use timescale T_D and T_R

respectively (see [20], [53]-[54]).
We will set the problem in a spatio-temporal domain $Q_T = \Omega \times (0,T]$, for $T > 0$ and Ω a bounded or an unbounded open domain in $\mathbf{R}^n$ whose boundary, we assume, to have a unit normal which is a smooth function of the position on $\partial\Omega$, when Ω is not the whole space.
To (3.9) we associate initial conditions

$$\mathbf{u}(x,0) = u_0(x) \quad \text{on } \Omega \times \{0\} \tag{3.10}$$

and, if $\Omega \neq \mathbf{R}^n$, the boundary conditions on $\partial\Omega \times (0,T]$ that we may take in the general form

$$\mathbf{B}u = b(x,t) \quad \text{on } \partial\Omega \times (0,T) \tag{3.11}$$

where $\mathbf{B}$ is a diagonal boundary operator, i.e. the i-th component of the vector $\mathbf{B}u$ depends only on the i-th component of u.
The diffusivity matrix D is diagonal when there is no cross diffusion among the species.
The vector $u = (u_1, \ldots, u_m)$ of chemical concentrations must be an element of the nonnegative cone C_m^+ of an m-dimensional real euclidean vector space. In order to be well-posed from the physical standpoint, the solution should exist and be nonnegative and bounded for $t \in (0, \infty)$.
Nonnegativity is guaranteed by the hypothesis that

$$f_i(u_1, u_2, \ldots, u_{i-1}, 0, \ldots, u_m) \geq 0$$

for $u_j \geq 0$, $j \neq i$.
The solution through any initial point in C_m^+ will be unique if the functions f_i are locally Lipschitz continuous in u throughout C_m^+.
In the next section we recall some existence results.

3.4 Existence theorems

Theorem of (local in time) existence and uniqueness of generalized and smooth solutions for reaction-diffusion system are well known in the literature and, further, when solutions are a-priori bounded, global existence can be obtained (see, for instance, [41], [73], [33], [70]). Very different techniques may be used to obtain existence results, for example comparison theorem or more topological-functional approach.

In this section we will refer only to a comparison-existence theorem for smooth solution and to an application of topological fixed point theorem.

Comparison theorem, based on the maximum principles, is a qualitative technique which, in the case of a single nonlinear equation, gives existence and uniqueness theorems for initial-boundary value problem by supplying a-priori bounds on the solution of the equation. It is capable to extension to certain system of parabolic PDEs, but, in general, gives weaker results (for a deep discussion we refer to [23], [5], [73] and the reference quoted therein).

Among the various existence and comparison theorem that can be established by both functional and classical methods (see [12], [38], [81]), we recall the approach due to Pao [57], since the monotone argument he adopts is constructive and in the mean time it leads to an existence-comparison theorem for the corresponding steady-state problem.

We will consider the more general coupled system of parabolic PDEs

$$(u_i)_t - L_i u_i = f_i(t, x, u_1, \ldots, u_n) \quad \text{on} Q_T \quad i = 1 \ldots, m \tag{3.12}$$

with

$$L_i = \sum_{j,k=1}^{n} a_{j,k}^{(i)}(x,t) \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j=1}^{n} b_j^{(i)}(x,t) \frac{\partial}{\partial x_j}$$

uniformly elliptic operators with smooth coefficients, and provided with initial and boundary conditions (3.10)-(3.11) respectively. We assume, also, that the functions defining the initial and boundary conditions are smooth nonnegative functions, while f_i are assumed to be Hölder continuous in $\bar{Q}_T \times J^m$, where J^m is a subset of $\mathbf{R}$.

We'll take the splitted form of $\mathbf{u}$,

$$\mathbf{u} = (u_i, [\mathbf{u}]_{a_i}, [\mathbf{u}]_{b_i})$$

with a_i, b_i nonnegative integer such that $a_i + b_i = m - 1$, so that $[\mathbf{u}]_{a_i}$ denote the a_i-components of $\mathbf{u}$ and the same holds for b_i.
The following is the definition of the **quasi-monotone** property which play a key role in the determination of the comparison function employed for the monotone argument.

Definition 3.1. A vector function $\mathbf{f} = (f_1, \ldots, f_m)$ is said to possess a quasi-monotone property if for each i there exist nonnegative integers a_i, b_i, $(a_i + b_i = m - 1)$ such that $f_i(\cdot, u_i, [u]_{a_i}, [u]_{b_i})$ is monotone nondecreasing in $[u]_{a_i}$, and is monotone nonincreasing in $[u]_{b_i}$.

The monotone nondecreasing of f_i in $[u]_{a_i}$ means that f_i is nondecreasing with respect the a_i-components.
When $a_i = 0$ $(b_i = 0)$ the function $\mathbf{f}$ is said to be nonincreasing (nondecreasing).

In order to construct convergent monotone sequences, one introduces the definition of coupled upper and lower solutions.

Definition 3.2. A pair of functions $\bar{\mathbf{u}} = (\bar{u}_1, \ldots, \bar{u}_m)$, $\underline{\mathbf{u}} = (\underline{u}_1, \ldots, \underline{u}_m)$ in $C(\bar{Q}_T) \cap C^{1,2}(Q_T)$ is called couple of upper and lower solutions of (3.12)-(3.10)-(3.11), if $\bar{\mathbf{u}} \geq \underline{\mathbf{u}}$ in $\bar{Q}_t$ and if

$$(\bar{u}_i - L_i\bar{u}_i) \geq f_i(t, x, \bar{u}_i, [\bar{\mathbf{u}}]_{a_i}, [\underline{\mathbf{u}}]_{b_i}),$$

$$(\underline{u}_i - L_i\underline{u}_i) \leq f_i(t, x, \underline{u}_i, [\underline{\mathbf{u}}]_{a_i}, [\bar{\mathbf{u}}]_{b_i}),$$

$$B_i\bar{u}_i \geq b_i(x,t) \geq B_i\underline{u}_i,$$

$$\bar{u}_i(x, 0) \geq u_{i,0}(x) \geq \underline{u}_i(x, 0)$$

for each $i = 1, \ldots, m$.

Given a pair of coupled upper and lower solutions, define the set

$$< \underline{\mathbf{u}}, \bar{\mathbf{u}} >\equiv \left\{ \mathbf{u} \in C(Q_T) : \underline{\mathbf{u}} \leq \mathbf{u} \leq \bar{\mathbf{u}} \right\}$$

and assume that for each $i = 1, \ldots, m$ there exist a function $c_i \in C(Q_T)$ such that

$$f_i(t, x, u_i, [\mathbf{u}]_{a_i}, [\mathbf{u}]_{b_i}) - f_i(t, x, v_i, [\mathbf{u}]_{a_i}, [\mathbf{u}]_{b_i}) \geq -c_i(u_i - v_i) \tag{3.13}$$

for $\underline{u}_i \leq v_i \leq u_i \leq \bar{u}_i$.
To ensure the uniqueness of the solutions we also assume the (Lipschitz) condition

$$|f_i(t, x, \mathbf{u}) - f_i(t, x, \mathbf{v})| \leq \|\mathbf{u} - \mathbf{v}\| \qquad \text{for } \mathbf{u}, \mathbf{v} \in < \underline{\mathbf{u}}, \bar{\mathbf{u}} > . \tag{3.14}$$

The following theorem hods.

Theorem 3.4.1. *Let $\underline{\boldsymbol{u}}$, $\bar{\boldsymbol{u}}$ be a coupled upper and lower solution of (3.12)-(3.10)-(3.11), and let $\boldsymbol{f}$ be quasimonotone in $< \underline{\boldsymbol{u}}, \bar{\boldsymbol{u}} >$ and satisfy the conditions (3.13)and (3.14).*
Then there exist an unique solution $\boldsymbol{U}$ to the i.b.v.p. (3.12)-(3.10)-(3.11) and $\boldsymbol{U} \in < \underline{\boldsymbol{u}}, \bar{\boldsymbol{u}} >$.
Moreover, two sequences $\left\{ \underline{\boldsymbol{u}}^{(k)} \right\}$, $\left\{ \bar{\boldsymbol{u}}^{(k)} \right\}$ can be constructed, with $\underline{\boldsymbol{u}}^0 = \underline{\boldsymbol{u}}$ and $\bar{\boldsymbol{u}}^0 = \bar{\boldsymbol{u}}$, both converging monotonically to $\boldsymbol{U}$.

Proof. See [57].

Assuming quasimonotone property for $\mathbf{f}$ one may give sufficient conditions for the local stability of a nonzero constant steady solution. For the sake of simplicity, take $m = 2$ and $f_i = f_i(u, v)$. Let (μ_1, μ_2) be the steady state solution such that $f_1(\mu_1, \mu_2) = f_2(\mu_1, \mu_2) = 0$ under homogeneous Neumann boundary condition. Assume (f_1, f_2) is a quasimonotone C^1-function in a neighborhood $\mathfrak{N}_\delta$ of (μ_1, μ_2) where the initial data are restricted.

Theorem 3.4.2. *Consider the following inequalities*

$$\frac{\partial f_1}{\partial u}(\mu_1,\mu_2)+\gamma\Big|\frac{\partial f_1}{\partial v}(\mu_1,\mu_2)\Big|<0$$

$$\gamma^{-1}\Big|\frac{\partial f_2}{\partial u}(\mu_1,\mu_2)\Big|+\frac{\partial f_2}{\partial v}(\mu_1,\mu_2)<0$$

and let be (f_1, f_2) either quasimonotone nondecreasing or quasimonotone nonincreasing in $\mathfrak{N}_\delta$. Then (μ_1, μ_2) is asymptotically stable if the last inequalities hold for all $x \in \bar{\Omega}$ and for some $\gamma > 0$. It is unstable if the reversed inequalities hold.

Proof. See [57].

As application of Banach's fixed point theorem we obtain an existence result for the following I.B.V.P.

$$\begin{cases} u_t = \Delta u + f(u) & \in Q_T \\ u = 0 & \text{on } \partial\Omega \times [0,T] \\ u = g & \text{on } \Omega \times \{t=0\} \end{cases} \tag{3.15}$$

where, as usual, $u = (u_1, \ldots, u_m)$, $g = (g_1, \ldots, g_m)$ and Ω is open, bounded and with smooth boundary.
We assume that:

i) $g \in H_0^1(\Omega; \mathbf{R}^m)$

ii) $f : \mathbf{R}^m \to \mathbf{R}^m$ is Lipschitz continuous.

Adapting the terminology introduced in the second chapter, we say that a function

$$u \in L^2(0,T; H_0^1(\Omega;\mathbf{R}^m)), \quad \text{with } u' \in L^2(0,T; H^{-1}(\Omega;\mathbf{R}^m))$$

is a weak solution of (3.15) provided

$$< u', v > + B[u,v] = (f(u), v) \quad a.e.\, 0 \le t \le T, \quad \forall v \in H_0^1(\Omega;\mathbf{R}^m)$$

and

$$u(0) = g,$$

where $< \cdot, \cdot >$ denotes the pairing of $H^{-1}(\Omega; \mathbf{R}^m)$ and $H_0^1(\Omega; \mathbf{R}^m)$, $B[\cdot, \cdot]$ is the bilinear form associated with $-\Delta$ in $H_0^1(\Omega; \mathbf{R}^m)$, and $(\cdot, \cdot)$ the inner product in $L^2(\Omega, \mathbf{R}^m)$. The norm in $H_0^1(\Omega; \mathbf{R}^m)$ is taken to be

$$\|u\|_{H_0^1(\Omega;\mathbf{R}^m)} = \Big(\int_\Omega |\nabla u|^2 \, dx\Big)^{1/2}.$$

We recall that after possible redefinition of u on a set of measure zero, $u \in C([0,T]; L^2(\Omega, \mathbf{R}^m))$.

Theorem 3.4.3. *There exists a unique weak solution of (3.15).*

We give now only a sketch of the proof (see [21] for a complete one): Banach's theorem is applied in the space

$$X = C([0,T]; L^2(\Omega, \mathbf{R}^m)),$$

with the norm

$$\|v\| := \max_{0 \le t \le T} \|v(t)\|_{L^2(\Omega, \mathbf{R}^m)}.$$

Given a function $u \in X$ and setting

$$h(t) := f(u(t)) \quad (0 \le t \le T)$$

it results from hypothesis **ii)** that $h \in L^2(0,T; L^2(\Omega, \mathbf{R}^m))$, and so the linear parabolic P.D.E.

$$\begin{cases} w_t = \Delta w + h & \text{in } Q_T \\ w = 0 & \text{on } \partial\Omega \times [0,T] \\ w = g & \text{on } \Omega \times \{t = 0\} \end{cases} \tag{3.16}$$

has a unique weak solution. Now, defining

$$A : X \to X$$

by setting $A[u] = w$, one finds that if $T > 0$ is small enough, then A is a strict contraction. Banach's fixed point theorem lead to a weak solution on a small time interval that one may extends, after finitely many steps, on the full interval $[0, T]$.
The uniqueness, consequence of the Lipschitz condition, is obtained by using the Gronwall's inequality.

Finally, if the kinetic vectorfield admits an invariant rectangle, then one can also show that the solution exists for all time pointwise in space [14].

3.5 Asymptotic homogeneization

In this section we compare the solution of the I.B.V.P.

$$\begin{cases} u_t = \Delta u + f(u) \qquad \text{in } Q_T \\ \nabla u \cdot \mathbf{n} = 0 \qquad \text{on } \partial\Omega \times [0, T] \\ u(x, 0) = u_0(x) \qquad \text{on } \Omega \times \{t = 0\} \end{cases} \tag{3.17}$$

to the solution of the kinetic equation

$$\frac{du}{dt} = f(u). \tag{3.18}$$

The case of Eq. (3.18) is often referred to as the homogeneous dynamics since it describes the situation of a well stirred reactor. Mixing ensures a uniform distribution of reactants so that diffusive transport is absent, while, in the context of chemical reaction-diffusion systems, Eq. $(3.17)_1$ can be regarded as a model for an unstirred reactor, where concentrations may vary between different locations inducing diffusive fluxes.

We denote by $U(t) = 1/|\Omega| \int_\Omega u(x, t)\, dx$ the spatial average of u, with $|\Omega|$ the measure of Ω, by d the smallest eigenvalue of the diffusion matrix D (which we assume to be positive definite) and by λ the first eigenvalue of $-\Delta$ on Ω with homogeneous boundary conditions. We assume that (3.17) admits

a bounded invariant region Σ, then we define the parameter $\sigma = \lambda d - M$, where $M = \max_\Sigma |\nabla u|$. It results $M < \infty$ since Σ is compact.
The following theorem holds

Theorem 3.5.1. *Consider (3.17) in Ω and assume that $(3.17)_1$ admits a bounded invariant region Σ, and that $\{u_0(x) : x \in \Omega\} \subset \Sigma$. If σ is positive, then there exist constants $c_i > 0$ $(1 \leq i \leq 4)$ such that the following estimates hold for $t > 0$:*

1. $||\nabla_x u(\cdot, t)||_{L_2(\Omega)} \leq c_1 e^{-\sigma t}$
2. $||u(\cdot, t) - U(t)||_{L_2(\Omega)} \leq c_2 e^{-\sigma t}$
   Further, U satisfies
   $$\frac{dU}{dt} = f(U) + g(t), \qquad U(0) = 1/|\Omega| \int_\Omega u_0(x)\, dx$$
   with $|g(t)| \leq c_3 e^{-\sigma t}$. If D is a diagonal matrix, then (2) holds for the L_∞-norm, i.e.
   $$||u(\cdot, t) - U(t)||_{L_\infty(\Omega)} \leq c_4 e^{-\sigma t}$$

Proof. See [73]

The assumption $\sigma > 0$ may be read in two different way:

$$\lambda > \frac{M}{d} \quad \text{or} \quad d > \frac{M}{\lambda}$$

and so in the first case, since λ is inversely proportional to the squared diameter of Ω, the condition $\sigma > 0$ tell us that the spatial region is "small". In the second case, $\sigma > 0$ is saying that the diffusion is "strong" relative to the reaction terms. In both of these cases, small domain and big diffusion, it is reasonable to expect that spatial inhomogeneities are insignificant and become quickly damped out.

The theorem shows that solutions u of the I.B.V.P. (3.17) decay exponentially fast to their spatial average U. Furthermore, U satisfies an equation which becomes a better and better approximation of the kinetic equation as

t goes to ∞.
As consequence [48], the ω-limit sets of the partial differential equations coincide with the one of the O.D.E. (3.18).
Moreover there cannot exist any nonconstant solution of the elliptic system

$$D\Delta U + f(u) = 0$$

with $x \in \Omega$ under homogeneous Neumann conditions. This follows from the fact that solutions of the last equation depend only on x, while the theorem implies that they must tend to solutions independent of x.

Chapter 4

A new approach to the stability study for reaction-diffusion system and applications

Among the first applications of Lyapunov Direct Method to PDE we quote the ones in magnetohydrodynamics due to Rionero [60], [61], [62], [63], applications in fluid mechanics and elasticity by Galdi and Rionero [27], Joseph [34], Straughan [74]. In the context of biological systems we find Brown [6], Blat and Brown [4], Leung [43], Conway et al. [15], for ecological and predator-prey models; De Mottoni and Rothe [19], Rothe [70] for the Lotka-Volterra equations; Capasso [7] and the reference quoted therein for epidemic systems and Rauch and Smoller [59] for the FitzHugh-Nagumo equations which model nerve impulse conduction.
Where these works involve Lyapunov's method, the central problem remains the construction of Lyapunov functionals and the link between results about linear and nonlinear stability. In this chapter, we underline the importance of the application of the Direct Method with functional depending on the eigenvalues of the linear part of the operator taken into account and then we introduce the model whose stability with respect infinitesimal perturbations, on a given domain, is studied.

4.1 Connection between linear and nonlinear stability

Let H be a Hilbert space endowed with a scalar product $< \cdot, \cdot >$ and associated norm $\| \cdot \|$.
Denoting by L a linear operator (possibly unbounded) and N a nonlinear operator with $N(0) = 0$, where 0 is the zero in H (this condition ensuring that $(4.1)_1$ admits the null solution), consider in H the initial-value problem

$$\begin{cases} u_t + Lu + Nu = 0 \\ u(0) = u_0 \end{cases} \tag{4.1}$$

where $u(u_0, t)$ is the perturbed dynamical system to the basic motion $v(v_0, t)$. We assume that

i) L is densely defined, closed and sectorial such that $(L - \lambda I)^{-1}$ is compact for some complex number λ, I being the identity operator in H (i.e. L has compact resolvent);

ii) the bilinear form associated with L is defined (and bounded) on a space H^*, which is compactly embedded in H.

The following theorem hold.

Theorem 4.1.1. *The spectrum of L consists entirely of an at most denumerable number of eigenvalues $\{\lambda_n\}_{n\in\mathbf{N}}$ with finite (both algebraic and geometric) multiplicities and, moreover, such eigenvalues can cluster only at infinity*

Proof. See [35]

The eigenvalues λ_n of

$$L\Phi = \lambda\Phi \tag{4.2}$$

can be ordered in a sequence $\{\lambda_n\}_{n\in\mathbf{N}^+}$ such that

$$Re(\lambda_1) \leq Re(\lambda_2) \leq \ldots \leq Re(\lambda_n) \leq \ldots \tag{4.3}$$

The zero solution to (4.1) is said to be linearly stable if and only if

$$Re(\lambda_1) > 0. \tag{4.4}$$

The last condition implies that the zero solution of $(4.1)_1$ is asymptotically exponentially stable and the bacin of attractivity is H (global stability).
As concerns the nonlinear stability of the zero solution to (4.1) with respect to $\|u\|$, if L is symmetric and

$$< Nu, u > \geq 0 \qquad \forall u \in D(N), \tag{4.5}$$

$D(\cdot)$ denoting the domain of the associated operator, then the eigenvalues λ_n are real numbers and it can be shown that $\lambda_1 > 0$ implies the global nonlinear exponential stability with respect to $\|u\|$ and hence there is **coincidence between linear and nonlinear stability conditions**.
When L is not symmetric, if $L = L_1 + L_2$, with L_1 symmetric and L_2 skew-symmetric, under (4.5) the global asymptotic exponential stability with respect to $\|u\|$ can be obtained under the condition

$$\bar{\lambda}_1 > 0$$

with $\bar{\lambda}_1$ principal eigenvalue of L_1 and generally $\bar{\lambda}_1 \neq Re(\lambda_1)$.
In this case, in order to reach the coincidence -instead of the energy $\|u\|$- generalized energy (i.e. Lyapunov functionals $V(u) \neq \|u\|$) may be introduced.

4.2 Lyapunov direct method with functionals depending on the "L" eigenvalues

We present in this section, following [64] some general theorems in order to underline the advantage of using Lyapunov functionals linked, with their time

derivative along the perturbations, to the eigenvalues of the linear operator $L^*(u) = L_1(u) - \bar{\alpha}u$, where $\bar{\alpha}$ is the principal eigenvalue of $-\Delta$.
This operator, trough the lowest eigenvalue and hence the principal eigenfunction of $-\Delta$, is linked to the lowest L^2-energy dissipated by diffusion.
For the local stability the following results holds.

Theorem 4.2.1. *Let $V = V(\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a positive definite functional, equivalent to $\|u\|$, such that along (4.1) it follows that*

$$\frac{dV}{dt} = f(\lambda_1, \lambda_2, \ldots, \lambda_n)\|u\|^2 + \Psi(u) \tag{4.6}$$

with:

i) $\{\lambda_n\}$ *sequence of eigenvalues of L;*

ii) f *real function such that $Re(\lambda_1) > 0 \Rightarrow f < 0$;*

iii) $\Psi(u) = o(\|u\|^2)$

Then the zero solution of (4.1) is locally asymptotically exponentially stable with respect to $\|u\|$.

Proof. By virtue of assumptions exist positive constants ϵ, k_i $(i = 1, 2, 3, 4)$ such that

$$k_1\|u\|^2 \leq V \leq k_2\|u\|^2, \quad f \leq -k_3, \quad o(\|u\|^2) \leq k_4\|u\|^{2+\epsilon}. \tag{4.7}$$

In view of (4.6)-(4.7) it turns out that

$$\frac{dV}{dt} \leq -\Big(\frac{k_3}{k_2} - \frac{k_4}{k_1^{1+\epsilon}}V^{\epsilon}\Big)V.$$

Then, by recursive argument, it follows that $V_0^{\epsilon} = \dfrac{k_3 k_1^{1+\epsilon}}{k_2 k_4} m$ with $m < 1$ implies $\dfrac{dV}{dt} \leq -\delta V$ with $\delta = \dfrac{k_3}{k_1}(1-m)$ and hence

$$V \leq V_0 e^{-\delta t}, \quad \|u\|^2 \leq \frac{k_2}{k_1}\|u_0\|^2 e^{-\delta t}.$$

□

Now we put in evidence the conditions guaranteeing the global nonlinear stability.

Theorem 4.2.2. *Let (4.6) and the assumptions* $i) - ii)$ *of theorem (4.2.1) hold. If*

$$|\Psi| \leq k\|u\|^2, \quad k < k_3 \tag{4.8}$$

with k *positive constant, then the zero solution of (4.1) is globally asymptotic exponentially stable with respect to* $\|u\|$.

Proof. (4.6)-(4.8) imply

$$\frac{dV}{dt} \leq -(k_3 - k)\|u\|^2 \leq -\frac{(k_3 - k)}{k_1} V$$

hence

$$V \leq V_0 e^{-\delta_1 t}, \quad \|u\|^2 \leq \frac{k_2}{k_1}\|u_0\|^2 e^{-\delta_1 t}, \quad \delta_1 = \frac{1}{k_1}(k_3 - k).$$

□

Theorem 4.2.3. *Let (4.6) and assumptions* $i) - ii)$ *of theorem (4.2.1) hold. If* $f \geq m =$ *positive constant, then the zero solution of (4.1) is unstable with respect to* $\|u\|$.

Proof. In fact one obtains

$$\frac{dV}{dt} \geq m\|u\| - k_4\|u\|^{1+\epsilon}$$

and hence

$$\frac{dV}{dt} \geq a_1 V - a_2 V^{1+\epsilon}$$

with $a_i (i = 1, 2)$ positive constants. Integrating one obtains

$$V^\epsilon \geq \frac{a_1 V_0^\epsilon e^{\epsilon a_1 t}}{a_1 + a_2 V_0^\epsilon e^{\epsilon a_1 t}}, \quad \lim_{t \to \infty} V^\epsilon \geq \frac{a_1}{a_2}, \quad \forall V_0.$$

□

4.3 The binary chemical reaction diffusion system

We will study the long-time behaviour of solutions for the following system

$$\begin{cases} U_t = \gamma(a - U + U^2V) + \Delta U \\ \\ V_t = \gamma(b - U^2V) + d\Delta V \end{cases} \tag{4.9}$$

in a given domain $\Omega \subset \mathbb{R}^3$ under Dirichlet boundary conditions

$$\begin{cases} U = a + b \\ \\ V = \dfrac{b}{(a+b)^2} \end{cases} \qquad \forall (x,t) \in \partial\Omega \times \mathbb{R}^+ \tag{4.10}$$

with a, b, γ and d constants such that

$$\begin{cases} a + b > 0 \\ b > 0, d > 0 \end{cases} \tag{4.11}$$

The initial data are assumed to be nonnegative and the functions $f(U,V) = \gamma(a - U + U^2V)$ and $g(U,V) = \gamma(b - U^2V)$ are continuously differentiable on $\mathbf{R}^+ \times \mathbf{R}^+$ satisfying $f(0,V) \geq 0$ and $g(U,0) \geq 0$ for all $U, V \geq 0$ which imply, via the maximum principle (see [73]), the positivity of the solution on its interval of existence.

System (4.9) is contained as particular case in the Segel-Jackson system [72], and contains the one introduced by Schnackenberg [71] for trimolecular autocatalytic reactions that could exhibit a limite cycle behaviour.

Indeed, if we consider a reaction mechanism involving only two species, it was shown by Hanusse [39] that limit cycle solutions can only exist if there are trimolecular reactions (see [5]). It must be said that trimolecular reactions are biochemically unrealistic if they are seen as the only reactions involved, but such two reactant models can arise naturally from a higher-order system if typical enzyme reactions, for example, are part of the mechanism being

considered.
The Schnackenberg reaction mechanism is

$$X \rightleftharpoons A,\ B \rightarrow Y,\ 2X + Y \rightarrow 3X.$$

where X and Y are the chemicals which come, respectively, from A, B, two different source kept at constant concentration level.
Using the law of mass action and passing to nondimensional variables one obtains (4.9).
It has been shown that the Schnackenberg model possesses at most one limit cycle in R^2_+ (see [40]).
Moreover, as we will see in the following section, system (4.9) is capable of generating spatial patterns and form, i.e. it give rise to diffusion-driven instability, also called Turing instability.

4.4 Preliminaries to the stability study

A key factor in the methodology we make use is a link between the L^2-stability of the solutions to a binary reaction-diffusion system of P.D.Es and the stability of the solutions to a binary system of O.D.Es associated to the first one and obtained adopting suitable scalings.
This section is devoted to introduce these scalings and some functional space governing the perturbations. The peculiar Lyapunov functional we will use is then introduced.
Our aim is to study the longtime behaviour of the solution of (4.9)-(4.10).
Setting

$$\begin{cases} U = u^* + C_1 \\ V = v^* + C_2 \end{cases} \tag{4.12}$$

with

$$\begin{cases} u^* = a + b \\ v^* = \dfrac{b}{(a+b)^2} \end{cases} \tag{4.13}$$

critical point of (4.9)-(4.10), the longtime behaviour of the solution of (4.9)-(4.10) is then reduced to the stability of (4.13).

It is easily seen that the equations governing the perturbations (C_1, C_2) to the basic state (u^*, v^*) are

$$\begin{cases} \dfrac{\partial C_1}{\partial t} = a_1 C_1 + a_2 C_2 + \Delta C_1 + f(C_1, C_2) \\ \\ \dfrac{\partial C_2}{\partial t} = a_3 C_1 + a_4 C_2 + d\Delta C_2 + g(C_1, C_2) \end{cases} \tag{4.14}$$

under Dirichlet homogeneous boundary conditions

$$C_1 \equiv C_2 \equiv 0, \qquad (x,t) \in \partial\Omega \times \mathbb{R}^+; \tag{4.15}$$

with

$$\begin{cases} a_1 = -1 + 2u^*v^* = \gamma \dfrac{b-a}{a+b}, \qquad a_2 = \gamma u^{*2} = \gamma(a+b)^2 \\ \\ a_3 = -2\gamma u^* v^* = -2\gamma \dfrac{b}{a+b}, \qquad a_4 = -\gamma u^{*2} = -\gamma(a+b)^2 \end{cases} \tag{4.16}$$

and

$$\begin{cases} f(C_1, C_2) = \gamma(C_1^2 v^* + C_1^2 C_2 + 2u^* C_1 C_2) \\ \\ g(C_1 C_2) = -f(C_1, C_2) \end{cases} \tag{4.17}$$

Following the methodology introduced by Rionero {[65], [66], [67], [68]} we use the scalings

$$C_1 = \alpha u, \qquad C_2 = \beta v \tag{4.18}$$

with α and β constants and set

$$\mu = \frac{\alpha}{\beta} \tag{4.19}$$

Then it turns out that

$$\begin{cases} \dfrac{\partial u}{\partial t} = a_1 u + \dfrac{a_2}{\mu} v + \Delta u + f^* \\ \\ \dfrac{\partial v}{\partial t} = \mu a_3 u + a_4 v + d\Delta v + g^* \end{cases} \tag{4.20}$$

under the boundary conditions

$$u = v = 0, \qquad (x,t) \in \partial\Omega \times \mathbb{R}^+ \tag{4.21}$$

with

$$\begin{cases} f^* = \dfrac{1}{\alpha}[f]_{\substack{C_1 = \alpha u \\ C_2 = \beta v}} \\ g^* = \dfrac{1}{\beta}[g]_{\substack{C_1 = \alpha u \\ C_2 = \beta v}} \end{cases} \tag{4.22}$$

Denoting by
$< \cdot, \cdot >$ the $L^2(\Omega)$ scalar product,
$\| \cdot \|$ the $L^2(\Omega)$-norm,
we study the problem in the variable functional space $W_0^{1,2}[\Omega(t)]$. In this space the Poincaré inequality

$$\|\nabla\varphi\|^2 \geq \bar{\alpha}\|\varphi\|^2 \tag{4.23}$$

holds. It is well known that $\bar{\alpha} = \bar{\alpha}(\Omega)$ is the lowest eigenvalue of

$$\Delta\varphi + \lambda\varphi = 0, \quad \varphi \in W_0^{1,2}(\Omega) \tag{4.24}$$

and an approximate value is given by $\bar{\alpha} = \pi^2/l^2$ ([24]). Setting

$$\begin{cases} b_1 = a_1 - \bar{\alpha} = \gamma\dfrac{(b-a)}{a+b} - \bar{\alpha} \\ \\ b_4 = a_4 - d\bar{\alpha} = -[\gamma(a+b)^2 + d\bar{\alpha}] \end{cases} \tag{4.25}$$

(4.20) becomes

$$\begin{cases} \dfrac{\partial u}{\partial t} = b_1 u + \dfrac{a_2}{\mu} v + f^* + f_1^* \\ \\ \dfrac{\partial v}{\partial t} = \mu a_3 u + b_4 v + g^* + g_1^* \end{cases} \tag{4.26}$$

with

$$\begin{cases} f_1^* = \Delta u + \bar{\alpha} u \\ \\ g_1^* = d(\Delta v + \bar{\alpha} v) \end{cases} \tag{4.27}$$

System (4.26) is the system on which we will work. To (4.26) we associate the binary system of O.D.Es

$$\begin{cases} \dfrac{d\xi}{dt} = b_1\xi + \dfrac{a_2}{\mu}\eta \\ \\ \dfrac{d\eta}{dt} = \mu a_3\xi + b_4\eta \end{cases} \tag{4.28}$$

having eigenvalues

$$\lambda_{1/2} = \frac{I \pm \sqrt{I^2 - 4A}}{2} \tag{4.29}$$

with

$$\begin{cases} I = b_1 + b_4 \\ \\ A = b_1 b_4 - a_2 a_3 \end{cases} \tag{4.30}$$

We introduce, now, the Rionero-Lyapunov functional ([65], [66])

$$W = \frac{1}{2}[A(\|u\|^2 + \|v\|^2) + \|b_1 v - \mu a_3 u\|^2 + \|\frac{a_2}{\mu}v - b_4 u\|^2], \tag{4.31}$$

which essentially depends on the eigenvalues (4.29).
Indeed, in view of the boundary conditions, along the solutions of (4.26) it turns out that

$$\frac{dW}{dt} = AI(\|u\|^2 + \|v\|^2) + \psi^* + {\psi_1}^* \tag{4.32}$$

with

$$\begin{cases} \psi^* = < \alpha_1 u - \alpha_3 v, f^* > + < \alpha_2 v - \alpha_3 u, g^* > \\ \\ {\psi_1}^* = < \alpha_1 u - \alpha_3 v, {f_1}^* > + < \alpha_2 v - \alpha_3 u, {g_1}^* > \\ \\ \alpha_1 = A + {b_4}^2 + \mu^2 {a_3}^2, \qquad \alpha_2 = A + {b_1}^2 + \dfrac{a_2^2}{\mu^2} \\ \\ \alpha_3 = \mu a_3 b_1 + \dfrac{1}{\mu} a_2 b_4 \end{cases} . \tag{4.33}$$

and hence it is clear that the time derivative of W along the solutions of (4.26) is related to the $L^2(\Omega)$-norm of the perturbations and is influenced in a simple direct way by the eigenvalues of the linear problem trough the product AI.

Remark 9. Note that

i) W is a positive definite functional of (u, v) if $A > 0$.

ii) W is a norm equivalent to the $L^2(\Omega)$-norm, i.e. there exist two positive constants, k_1 and k_2 such that

$$\begin{cases} \frac{k_1}{2}(\|u\|^2 + \|v\|^2) \leq W \leq \frac{k_2}{2}(\|u\|^2 + \|v\|^2) \\ \\ k_1 = A, \quad k_2 = A + 2({b_1}^2 + \frac{{a_2}^2}{\mu^2} + \mu^2 {a_3}^2 + {b_4}^2) \end{cases} \tag{4.34}$$

4.5 Linear stability

In this section we will consider the system (4.9)-(4.10) in a bounded smooth domain $\Omega \subset \mathbf{R}^3$ (see [29]).

If one linearize (4.26), then (4.32) reduces to

$$\frac{dW}{dt} = AI(\|u\|^2 + \|v\|^2) + {\psi_1}^* \tag{4.35}$$

The following theorem hold.

Theorem 4.5.1. *Let*

$$\begin{cases} I < 0 \\ A > 0 \end{cases} \tag{4.36}$$

then (u^*, v^*) is linearly asymptotically stable with respect to the $L^2(\Omega)$-norm according to

$$W \leq W_0 e^{-\delta t} \tag{4.37}$$

with $\delta \leq \dfrac{2A|I|}{k_2}$ and k_2 given by (4.34).

Proof. In the case $b_1 < 0$, on choosing

$$\mu^2 = \left|\frac{a_2 b_4}{a_3 b_1}\right| \tag{4.38}$$

it follows that $\alpha_3 = 0$, hence

$$\psi_1{}^* = -\alpha_1 \left[\|\nabla u\|^2 + \bar{\alpha}\|u\|^2\right] - d\alpha_2 \left[\|\nabla v\| + \bar{\alpha}\|v\|^2\right] \leq 0$$

which implies

$$\frac{dW}{dt} \leq AI(\|u\|^2 + \|v\|^2). \tag{4.39}$$

By virtue of (4.34) it turns out that

$$\frac{dW}{dt} \leq -\frac{2A|I|}{k_2} W \tag{4.40}$$

and hence (4.37) immediately follows.

In the case $b_1 > 0$, following ([68]), let $\{\bar{\alpha_n}\}, \{\varphi_n\}$ be respectively the sequence of the eigenvalues of (4.24) and the sequence of the associate eigenfunction in $W_0^{1,2}(\Omega)$. Assuming that:

i) $\{\varphi_n\}$ is complete and orthogonal in $W^{1,2}(\Omega)$;

ii) u, v and their first and second (spatial) derivatives can be expanded in a (Fourier) series absolutely and uniformly converging in Ω according to

$$\begin{cases} u = \sum_{n=1}^{+\infty} X_n(t)\varphi_n \\ \\ v = \sum_{n=1}^{+\infty} Y_n(t)\varphi_n \end{cases} \tag{4.41}$$

and differentiable term by term,

from the linearized system (4.26) (for $\mu = 1$) we obtain

$$\begin{cases} \dfrac{dX_n}{dt} = b_{1n} X_n + a_2 Y_n \\ \\ \dfrac{dY_n}{dt} = a_3 X_n + b_{4n} Y_n \end{cases} \tag{4.42}$$

with

$$\begin{cases} b_{1n} = a_1 - \bar{\alpha}_n \\ \\ b_{4n} = a_4 - d\bar{\alpha}_n \end{cases} \tag{4.43}$$

Then

$$\begin{cases} I_n < 0 \\ A_n > 0 \end{cases} \tag{4.44}$$

imply the stability of the zero solution with respect the perturbation (u_n, v_n) and for stability we would like to have (4.44) satisfied for each n.
But

$$\begin{cases} I < 0 \Rightarrow I_n < 0 \\ \\ A > 0 \Rightarrow A_n > 0 \end{cases} \tag{4.45}$$

In fact

$$\begin{cases} I_n = a_1 + a_4 - (1 + d\bar{\alpha}_n) = b_1 + b_4 - d(\alpha_n - \alpha_1) < I \\ \\ A_n = b_{1n}b_{4n} - a_2a_3 = [b_1 - (\alpha_n - \alpha_1)][b_4 - (\alpha_n - \alpha_1)] - a_2a_3 = \\ \\ \quad = A + (\alpha_n - \alpha_1)^2 - (b_1 + b_4)(\alpha_n - \alpha_1) = \\ \\ \quad = A + (\alpha_n - \alpha_1)^2 + |b_1 + b_4|(\alpha_n - \alpha_1) \end{cases} \tag{4.46}$$

Then, setting

$$W_n = \frac{1}{2}[A_n(\|u_n\|^2 + \|v_n\|^2) + \|b_{1n}v_n - a_3u_n\|^2 + \|a_2v_n - b_{4n}u_n\|^2]$$

it follows that

$$W_n \leq W_n(0)e^{-\delta_n t} \tag{4.47}$$

where

$$\delta_n = \frac{2A_n|I_n|}{k_2^{(n)}} \tag{4.48}$$

with

$$k_2{}^{(n)} = A_n + 2[{b_{1n}}^2 + {b_{4n}}^2 + {a_2}^2 + {a_3}^2]. \tag{4.49}$$

Setting

$$W = \sum_{n=1}^{+\infty} W_n \tag{4.50}$$

it follows that ([68])

$$W \leq W_0 e^{-\delta t} \tag{4.51}$$

with δ positive constant independent of n.

In the case $b_1 = 0$ it turns out that

$$\alpha_1 = A + {b_4}^2 + \mu^2 {a_3}^2, \qquad \alpha_2 = A + \frac{a_2^2}{\mu^2}, \quad \alpha_3 = \frac{a_2 b_4}{\mu} \tag{4.52}$$

and for μ such that

$$(1+d)|\alpha_3| \leq 2\sqrt{d\alpha_1\alpha_2} \tag{4.53}$$

one obtain {See [67] Lemmas $1-3$}

$$\psi^* < 0.$$

□

4.6 Instability

The instability is guaranteed by the existence of at least one destabilizing admissible perturbation.

Theorem 4.6.1. *Let either*

$$I > 0 \tag{4.54}$$

or

$$A < 0 \tag{4.55}$$

then (u^*, v^*) *is unstable.*

Proof. We refer for the sake of concreteness to the case (4.54). Then for $n = 1$, (4.42) gives

$$\begin{cases} \dfrac{dX_1}{dt} = b_1 X_1 + a_2 Y_1 \\ \\ \dfrac{dY_1}{dt} = a_3 X_1 + b_4 Y_1 \end{cases} \tag{4.56}$$

with the eigenvalues

$$\lambda_{1,2} = \frac{I \pm \sqrt{I^2 - 4A}}{2} \tag{4.57}$$

and at least one either has positive real part or is zero. □

4.7 Diffusion driven instability

Natural system exhibit an amazing diversity of structures in both living and nonliving systems. In order to capture some essential characteristic of the natural mechanism of growth, at least qualitatively, some models are built whose primary interest is not in genes, but in processes that follows the activation of a gene.

Inspired by the complexity of self-organizing biological system, in particular by the problem of how a fertilized egg becomes a structured organism, the British mathematician A. M. Turing in his seminal work [76] assumed that genes act only as catalysts for spontaneous chemical reactions, which regulate the production of other catalysts or *morphogens*. Finally cells differentiate according to the morphogen concentration in their surroundings.

Neglecting the mechanical and electrical properties of biological tissue, he showed that a simple mathematical model describing spontaneously spreading and reacting chemicals could give rise to stationary spatial concentration patterns and proposed that reaction-diffusion models might have relevance in describing *morphogenesis*, the growth of biological form.

The key point in his theory of patterns formation is that a chemical state, which is stable in absence of diffusion, becomes unstable to perturbations when diffusion is present. Starting by arbitrary random deviations of the stationary state, *Turing instability* or *diffusion-driven instability* leads to stationary spatially periodic variations in the chemical concentration.

Diffusion induces instability: this is the innovative idea, since one often believe that diffusion is a smoothing process. Indeed this is the case for a single diffusion equation.

If one takes the heat equation

$$\begin{cases} u_t = \Delta u \quad \text{in } \Omega \times (0, \infty), \\ \\ u(x, 0) = u_0(x) \quad \text{in } \Omega, \\ \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0, \infty), \end{cases} \tag{4.58}$$

with Ω a bounded smooth domain in $\mathbf{R}^n$, ν the outer normal to $\partial\Omega$, the initial heat distribution given by u_0, a real-valued continuous function (not identically zero), and where the boundary condition implies that we have an isolated system, it results that the solution $u(x, t)$ eventually converges to the constant average, $\dfrac{1}{|\Omega|} \displaystyle\int_\Omega u_0(x)\, dx$, as t goes to ∞.

Analogously, if one replace the heat equation with

$$u_t = \Delta u + f(u)$$

with f a smooth linear or nonlinear source (or sink), it has been proved by Matano [50] and Casten-Holland [9] that stable steady state must be constant provided that the domain is convex.

The situation drastically changes when system of reaction diffusion equations are taken into account.

Any two-component reaction-diffusion system, to be said a Turing system, must have a reaction vector field that give rise to a Jacobian at the kinetic steady state with one of the following patterns of signs [20]:

$$K_p \equiv \begin{bmatrix} - & + \\ - & + \end{bmatrix} \qquad K_c \equiv \begin{bmatrix} - & - \\ + & + \end{bmatrix}.$$

The kinetic mechanism for which the Jacobian is of type K_p is said to be a *pure* activator-inhibitor mechanism, while a *cross* activator-inhibitor system if the Jacobian is of type K_c.

In the case of the Schnackenberg kinetic vector field we have a cross activator-inhibitor system whenever $a_1 > 0$, i.e. $-1 + 2uv > 0$, and in particular, this is true at the steady state whenever $\dfrac{b-a}{b+a} > 0$;

furthermore $u(x,t)$ is self-activating while $v(x,t)$ is self-inhibiting.

We will consider, now, the conditions ensuring the onset of the Turing instability for the Schnackenberg system.

Let us consider the system inequalities

$$\begin{cases} I_0 = a_1 + a_4 = \gamma \dfrac{b-a}{a+b} - \gamma(a+b)^2 < 0 \\ \\ A_0 = \gamma^2 (a+b)^2 > 0 \\ \\ A = A_0 - \bar{\alpha}(a_1 d + a_4) + d\bar{\alpha}^2 < 0 \end{cases} \tag{4.59}$$

Condition (4.59) are the conditions for the onset of driven instability (Turing effect) i.e. (u^*, v^*) *is stable in the absence of diffusion, but is destabilized by diffusion.* It is easily seen that (4.59) hold if and only if ([68])

$$\begin{cases} b > a \\ \\ b - a < (a+b)^3 \\ \\ d > \dfrac{A_0 + \bar{\alpha}|a_4|}{\bar{\alpha}(|a_1| - \bar{\alpha})} \end{cases} \tag{4.60}$$

In view of (4.60), in particular it follows that the onset of Turing instability is guaranteed by $\{l = 1, a < 1, b > 1, \gamma < \pi^2\}$.

We want underline that the condition about the difference in the characteristic of the random movement of the chemical molecules due to thermal fluctuations, i.e. diffusion, like the most important prerequisite for pattern formation process is only a condition that depend on the boundary conditions, since it has been proved that diffusion driven instability can occurs also when the diffusion coefficients are equal, see [68].
The first confirmation of Turing's ideas comes in 1989 from an experimental observation of a stationary spotty pattern in a chemical system involving the reactions of chlorite ions, iodide ions and malonic acid (CIMA reaction) [10]. Among the wide literature about Turing instability and the spatial form and shapes that it generates in living organism (see [53], [54], and the references quoted therein for modeling animal coat pattern and [46] for an application of Schnackenberg system in modeling the ligament of arcoid bivalve) we end this section by considering that recently, Turing systems, have been proposed to explain the formation of convolutions found on the cerebral cortex ([8]). Also, many inanimate or social systems show self-organizing behaviour, for example vegetation pattern and desertification [79] or the dynamics of language competition and spreading in societies [56].

4.8 Linear stability on variable domain

In the previous section the stability analysis has been carried out on a fixed domain but this doesn't happen in the real chemical reactions.
Here we will consider the system (4.9) under boundary conditions (4.10) on a smoothly variable domain [28],

$$\Omega(t) \subset \mathbf{R}^3 \text{ such that } (x, y, z) \in \Omega \Rightarrow z \in [0, l]$$

with $l = \text{const.} > 0$.
We assume that the perturbations have an (x, y) behaviour that is repetitive in the (x, y) direction, that is C_1 and C_2 have an (x, y)-dependence consistent with one that has a repetitive shape that tiles the plane.

In particular the (x, y) dependence is consistent with a wave number a for which

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = -a^2.$$

The periodic cell is defined by such a repetitive shape and its Cartesian product with $(0, l)$.

The results reached in the foregoing sections about linear stability-instability and diffusion driven instability may be get also in this case.

Indeed, one introduce the Rionero-Lyapunov functional (4.31), and evaluating its time derivative along the solutions of (4.26), one applies the Reynolds transport theorem.

This theorem concerns the rate of change of volume integrals over the finite but time varying fluid element $\Omega(t)$, that is

$$\frac{d}{dt}\int_{\Omega(t)} G(\mathbf{x}, t)\, d\mathbf{x} = \int_{\Omega(t)} \left[\frac{DG}{Dt} + G\nabla \cdot \mathbf{v}\right] d\mathbf{x}$$

where G is any scalar or vector function; $\Omega(t)$ is a region of space occupied by a finite deforming fluid element; $\dfrac{DG}{Dt}$ is the material derivative that give the rate of change of G following a fluid element,

$$\frac{DG}{Dt} = \frac{\partial G}{\partial t} + \mathbf{v} \cdot \nabla G$$

with $\mathbf{v}$ the flow velocity field.

Then, for $\int_{\Omega(t)} G(\mathbf{x}, t)\, d\mathbf{x} = W$ and in view of the boundary conditions, it turns out that, again,

$$\frac{dW}{dt} = AI(\|u\|^2 + \|v\|^2) + \psi^* + {\psi_1}^* \tag{4.61}$$

and one may proceed to the stability analysis as in the previous case.

Chapter 5

Nonlinear stability analysis

5.1 Introduction

Our aim, in this chapter, is to study the L^2-stability of the uniform steady state with respect perturbations of finite amplitude to the equilibrium concentrations of chemicals [30]. Before to face the problem we recall the system studied, that is

$$\begin{cases} \dfrac{\partial U}{\partial t} = \gamma(a - U + U^2 V) + \Delta U, \\ \\ \dfrac{\partial V}{\partial t} = \gamma(b - U^2 V) + d\Delta V, \end{cases} \tag{5.1}$$

considered in an open bounded domain $\Omega \subset \mathbf{R}^3$, with boundary at least C^2, under the initial conditions:

$$\begin{cases} U(\mathbf{x}, 0) = U_0(\mathbf{x}), \\ V(\mathbf{x}, 0) = V_0(\mathbf{x}), \end{cases} \qquad \forall \mathbf{x} \in \Omega \tag{5.2}$$

and the boundary conditions

$$\begin{cases} U(\mathbf{x}, t) = U^*(\mathbf{x}, t), \\ V(\mathbf{x}, t) = V^*(\mathbf{x}, t), \end{cases} \qquad \forall (\mathbf{x}, t) \in \partial\Omega \times \mathbf{R}^+ \tag{5.3}$$

where U^*, V^*, U_0, V_0 are regular functions.

The nonlinear stability results are obtained in two different way: the first one, referring to regular solutions, make use of the boundedness of the perturbation fields, while the other is based on the introduction of an auxiliary cross-diffusion term.

5.2 General properties of regular solutions

We obtain boundedness for the chemical concentrations by applying the maximum principle for parabolic operator.
From now on, for any $T > 0$, we set

$$\begin{aligned} Q_T &= \Omega \times (0, T], \\ \Gamma_T &= \bar{\Omega}_T - \Omega_T. \end{aligned} \tag{5.4}$$

for the spatio-temporal cylinder and parabolic boundary respectively.
The following theorem holds:

Theorem 5.2.1. *Let $\{U, V \in C_1^2(\Omega_T) \cap C(\bar{\Omega}_T)\}$ be a positive solution of (5.1),(5.2),(5.3) where U^*, V^*, U_0, V_0 are positive continuous functions. Then*

$$\begin{aligned} U(\mathbf{x}, t) &\geq m = \inf\left\{a, \min_{\bar{\Omega}} U_0, \min_{\partial\Omega\times[0,T]} U^*\right\}, \\ V(\mathbf{x}, t) &\leq M = \sup\left\{\frac{b}{m^2}, \max_{\bar{\Omega}} V_0, \max_{\partial\Omega\times[0,T]} V^*\right\}. \end{aligned} \tag{5.5}$$

Proof. Let $U(\mathbf{x}_0, t_0) = \min_{\bar{\Omega}_T} U$. Two cases are possible

i) If $(\mathbf{x}_0, t_0) \in \Omega \times (0, T)$,then

$$\left(\frac{\partial U}{\partial t}\right)_{(\mathbf{x}_0, t_0)} = 0, \qquad (\Delta U)_{(\mathbf{x}_0, t_0)} \geq 0, \tag{5.6}$$

from $(5.1)_1$ it follows that

$$a - U(\mathbf{x}_0, t_0) \leq 0, \tag{5.7}$$

and hence

$$U(\mathbf{x}_0, t_0) \geq a. \tag{5.8}$$

ii) On the other hand, if $(\mathbf{x}_0, t_0) \in \partial(\Omega \times [0, T])$ then

$$U(\mathbf{x}_0, t_0) = \inf\{\min_{\bar{\Omega}} U_0, \min_{\partial\Omega\times[0,T]} U^*\}. \tag{5.9}$$

From i) and ii) we get $(5.5)_1$.

Passing to V, let $V(\bar{\mathbf{x}}, \bar{t}) = \max_{\bar{\Omega}\times[0,T]} V$.

i. If $(\bar{\mathbf{x}}, \bar{t}) \in \Omega \times (0, T)$, we find

$$\left(\frac{\partial V}{\partial t}\right)_{(\bar{\mathbf{x}},\bar{t})} = 0, \qquad (\Delta V)_{(\bar{\mathbf{x}},\bar{t})} \leq 0, \tag{5.10}$$

from $(5.1)_2$ it turns out

$$b - U^2 V \geq 0 \tag{5.11}$$

and finally

$$V(\bar{\mathbf{x}}, \bar{t}) \leq \frac{b}{m^2}. \tag{5.12}$$

ii. On the other hand, if $(\bar{\mathbf{x}}, \bar{t}) \in \bar{\Omega} \cup \Gamma_t$,

$$V(\bar{\mathbf{x}}, \bar{t}) = \sup\{\max_{\bar{\Omega}} V_0, \max_{\partial\Omega\times[0,T]} V^*\}. \tag{5.13}$$

As in the previous case, we find $(5.5)_2$. □

5.3 Nonlinear stability analysis via the boundedness of V

When nonlinear term are involved we must try to control them: here we do this by using Sobolev embedding theorems and the results of the previous

section.
The following is a nonlinear asymptotic stability theorem for regular perturbations.

Theorem 5.3.1. *Let*

$$\frac{b-a}{a+b} < \frac{\bar{\alpha}}{\gamma}, \tag{5.14}$$

then (u^*, v^*) is nonlinearly asymptotically stable with respect the $L^2(\Omega)$-norm.

Proof. Let us observe that:

$$\frac{b-a}{a+b} < \frac{\bar{\alpha}}{\gamma} \Longleftrightarrow b_1 < 0, \tag{5.15}$$

and that $b_1 < 0$ implies that

$$\begin{cases} I < 0, \\ \\ A > 0, \\ \\ b_1 b_4 a_2 a_3 < 0. \end{cases} \tag{5.16}$$

Following [66], for any constant $\bar{\varepsilon}$ such that

$$0 < \bar{\varepsilon} < \inf\left\{\frac{|b_1|}{\bar{\alpha}}, \frac{|b_4|}{\bar{\alpha}}, \frac{|I|}{2\bar{\alpha}}, \frac{A}{\bar{\alpha}|I|}, 1, d\right\}, \tag{5.17}$$

setting

$$\begin{cases} \bar{b}_i = b_i + \bar{\alpha}\bar{\varepsilon}, \quad (i = 1, 4) \\ \bar{\gamma}_1 = 1 - \bar{\varepsilon}, \\ \bar{\gamma}_2 = d - \bar{\varepsilon}, \end{cases} \tag{5.18}$$

we can write (4.26) as follows

$$\begin{cases} \dfrac{\partial u}{\partial t} = \bar{b}_1 u + \dfrac{a_2}{\mu} v + f^* + \bar{f}_1^*, \\ \\ \dfrac{\partial v}{\partial t} = \mu a_3 u + \bar{b}_4 v + g^* + \bar{g}_1^*, \end{cases} \tag{5.19}$$

where

$$\begin{cases} \bar{f}_1^* = \bar{\gamma}_1(\Delta u + \bar{\alpha} u) + \bar{\varepsilon}\Delta u, \\ \\ \bar{g}_1^* = \bar{\gamma}_2(\Delta v + \bar{\alpha} v) + \bar{\varepsilon}\Delta v, \end{cases} \tag{5.20}$$

and we observe that from conditions (5.16) the following inequalities hold:

$$\begin{cases} \bar{I} = \bar{b}_1 + \bar{b}_4 < 0, \\ \\ \bar{A} = \bar{b}_1 \bar{b}_4 - a_2 a_3 > 0. \end{cases} \tag{5.21}$$

Along the solutions of (5.19), it turns out:

$$\frac{d\bar{W}}{dt} = \bar{A}\bar{I}(\| u \|^2 + \| v \|^2) + \bar{\Psi}^* + \bar{\Psi}_1^*, \tag{5.22}$$

where

$$\begin{cases} \bar{\Psi}^* =< \bar{\alpha}_1 u - \bar{\alpha}_3 v, f^* > + < \bar{\alpha}_2 v - \bar{\alpha}_3 u, g^* >, \\ \\ \bar{\Psi}_1^* =< \bar{\alpha}_1 u - \bar{\alpha}_3 v, \bar{f}_1^* > + < \bar{\alpha}_2 v - \bar{\alpha}_3 u, \bar{g}_1^* >, \\ \\ \bar{\alpha}_1 = \bar{A} + \bar{b}_4^2 + \mu^2 a_3^2, \qquad \bar{\alpha}_2 = \bar{A} + \bar{b}_1^2 + \dfrac{a_2^2}{\mu^2}, \\ \\ \bar{\alpha}_3 = \mu a_3 \bar{b}_1 + \mu^{-1} a_2 \bar{b}_4. \end{cases} \tag{5.23}$$

Now choosing

$$\mu^2 = \frac{|a_2\bar{b}_4|}{|\bar{b}_1 a_3|}, \tag{5.24}$$

it follows that $\bar{\alpha}_3 = 0$ and hence

$$\begin{cases} \bar{\Psi}^* = \bar{\alpha}_1 < u, f^* > + \bar{\alpha}_2 < v, g^* >, \\ \\ \bar{\Psi}_1^* = \bar{\alpha}_1 < u, \bar{f}_1^* > + \bar{\alpha}_2 < v, \bar{g}_1^* > . \end{cases} \tag{5.25}$$

But

$$\bar{\Psi}_1^* = \bar{\alpha}_1 \bar{\gamma}_1 < u, \Delta u + \bar{\alpha} u > + \bar{\alpha}_2 \bar{\gamma}_2 < v, \Delta v + \bar{\alpha} v >$$

$$- \bar{\alpha}_1 \bar{\varepsilon} \parallel \nabla u \parallel^2 - \bar{\alpha}_2 \bar{\varepsilon} \parallel \nabla v \parallel^2 \tag{5.26}$$

$$\leq -k^* (\parallel \nabla u \parallel^2 + \parallel \nabla v \parallel^2)$$

with $k^* = \bar{\varepsilon} \inf(\bar{\alpha}_1, \bar{\alpha}_2)$.
Furthermore

$$\bar{\Psi}^* = \frac{\bar{\alpha}_1}{\alpha^2} < C_1, f(C_1, C_2) > - \frac{\bar{\alpha}_2}{\beta^2} < C_2, f(C_1, C_2) >$$

$$= \frac{\bar{\alpha}_1}{\alpha^2} \gamma \left[v^* < C_1^3 > + < C_1^3 C_2 > + 2u^* < C_1^2 C_2 > \right] \tag{5.27}$$

$$- \frac{\bar{\alpha}_2}{\beta^2} \gamma \left[v^* < C_1^2 C_2 > + < C_1^2 C_2^2 > + 2u^* < C_1 C_2^2 > \right].$$

In order to prove the decay of $\bar{W}$, and then the stability of (4.13), we must control suitably the nonlinear terms in (5.27).
By using the usual embedding theorems, this can be done for all the terms except, as far as we know, the strong nonlinear term $< C_1^3 C_2 >$.
In the present section, we solve this problem by using the boundedness of the perturbation fields. While in the next we will do this introducing auxiliary cross diffusion terms.
Coming back to (5.27), choosing $\beta = 1$, recall that:

a) from theorem (5.2.1), there exists a positive constant Γ_2 such that

$$|C_2(\mathbf{x},t)| \leq \Gamma_2, \qquad \forall(\mathbf{x},t) \in \Omega \times [0,T]; \tag{5.28}$$

b) from Sobolev embedding theorem, there exists a positive constant $k(\Omega)$ such that

$$(< \phi^4 >)^{1/2} \leq k(\Omega) \parallel \nabla\phi \parallel^2 . \tag{5.29}$$

By means of the above inequalities and the Cauchy-Schwarz inequality it turns out that:

$$\begin{cases} < C_1^3 > \leq k \parallel C_1 \parallel \parallel \nabla C_1 \parallel^2, \\ \\ < C_1^3 C_2 > \leq \Gamma_2 k \parallel C_1 \parallel \parallel \nabla C_1 \parallel^2, \\ \\ < C_1^2 C_2 > \leq k \parallel C_2 \parallel \parallel \nabla C_1 \parallel^2, \\ \\ < C_1 C_2^2 > \leq k \parallel C_1 \parallel \parallel \nabla C_2 \parallel^2 . \end{cases} \tag{5.30}$$

From (5.27) and inequalities (5.30), we find:

$$\begin{aligned} \Psi^* \leq \frac{\bar{\alpha}_1}{\alpha^2}\gamma[kv^* \parallel C_1 \parallel \parallel \nabla C_1 \parallel^2 &+ \Gamma_2 k \parallel C_1 \parallel \parallel \nabla C_1 \parallel^2 \\ &+ 2u^* k \parallel C_2 \parallel \parallel \nabla C_1 \parallel^2] \\ + \bar{\alpha}_2 \gamma \left[kv^* \parallel C_2 \parallel \parallel \nabla C_1 \parallel^2 + 2u^* k \parallel C_1 \parallel \parallel \nabla C_2 \parallel^2\right] & \\ \leq \sqrt{2}\Gamma \left(\parallel C_1 \parallel^2 + \parallel C_2 \parallel^2\right)^{\frac{1}{2}} \left(\parallel \nabla C_1 \parallel^2 + \parallel \nabla C_2 \parallel^2\right), & \end{aligned} \tag{5.31}$$

where

$$\Gamma = \gamma k \left[\frac{\bar{\alpha}_1}{\alpha^2}(v^* + 2u^* + \Gamma_2) + \bar{\alpha}_2(v^* + 2u^*)\right]. \tag{5.32}$$

Finally, from (5.22), (5.26), (5.31) it turns out that:

$$\frac{d\bar{W}}{dt} \leq -\frac{\bar{A}|\bar{I}|}{k_2}\bar{W} + \left(\Gamma^* \sqrt{\frac{2}{k_1}}\bar{W}^{\frac{1}{2}} - k^*\right) (\parallel \nabla u \parallel^2 + \parallel \nabla v \parallel^2), \tag{5.33}$$

where $\Gamma^* = \Gamma\left(\sup\{\alpha^2, 1\}\right)^{\frac{3}{2}}$. So, provided that

$$\bar{W}_0^{\frac{1}{2}} < \frac{k^*}{\Gamma^*}\sqrt{\frac{k_1}{2}}, \tag{5.34}$$

by means of recursive arguments it turns out:

$$\bar{W} \leq \bar{W}_0 \exp(-\delta t), \tag{5.35}$$

where

$$\delta = \frac{1}{k_2}\left[\bar{A}|\bar{I}| - \bar{\alpha}\left(k^* - \Gamma^*\sqrt{\frac{2}{k_1}}\bar{W}_0^{\frac{1}{2}}\right)\right]. \tag{5.36}$$

$\square$

Remark 10. We observe that if

$$b < a \tag{5.37}$$

then by virtue of theorem (5.3.1), (u^*, v^*) is nonlinearly asymptotically stable with respect the $L^2(\Omega)$-norm.

5.4 Nonlinear stability analysis via cross diffusion auxiliary terms

Now, we will apply the cross diffusion method introduced in [69]. To this end in view of (4.14) we obtain:

$$\begin{cases} \dfrac{\partial(C_1 + C_2)}{\partial t} = (a_1 + a_3)C_1 + \Delta C_1 + d\Delta C_2, \\ \\ \dfrac{\partial C_2}{\partial t} = a_3 C_1 + a_4 C_2 + d\Delta C_2 + g(C_1, C_2). \end{cases} \tag{5.38}$$

Setting

$$\alpha X = C_1 + C_2, \quad C_2 = \beta Y, \tag{5.39}$$

it turns out that

$$C_1 = \alpha X - \beta Y, \tag{5.40}$$

with α and β positive constants, and hence:

$$\begin{cases} \dfrac{\partial X}{\partial t} = (a_1 + a_3)X - \mu^{-1}(a_1 + a_3)Y + \Delta X + \mu^{-1}(d-1)\Delta Y, \\ \\ \dfrac{\partial Y}{\partial t} = \mu a_3 X + (a_4 - a_3)Y + d\Delta Y + \bar{g}, \end{cases} \tag{5.41}$$

where

$$\bar{g} = \beta^{-1} g|_{(\alpha X, \beta Y)}, \qquad \mu = \alpha/\beta. \tag{5.42}$$

By following the procedure of section (4.4), we obtain:

$$\begin{cases} \dfrac{\partial X}{\partial t} = b_1 X + \mu^{-1} b_2 Y + f^* + g_1^*, \\ \\ \dfrac{\partial Y}{\partial t} = \mu b_3 X + b_4 Y + g^* + \bar{g}, \end{cases} \tag{5.43}$$

on the boundary

$$\begin{cases} X(\mathbf{x}, t) = 0, \\ Y(\mathbf{x}, t) = 0. \end{cases} \qquad \forall (\mathbf{x}, t) \in \partial\Omega \times \mathbf{R}^+ \tag{5.44}$$

with

$$\begin{cases} b_1 = -(\gamma + \bar{\alpha}), \qquad b_2 = \gamma - (d-1)\bar{\alpha}, \\ \\ b_3 = -\dfrac{2\gamma b}{a+b}, \qquad b_4 = \dfrac{\gamma}{a+b}[2b - (a+b)^3] - d\bar{\alpha}, \end{cases} \tag{5.45}$$

and

$$\begin{cases} f^* = \Delta X + \bar{\alpha} X, \qquad g_1^* = \mu^{-1}(d-1)(\Delta Y + \bar{\alpha} Y), \\ \\ g^* = d(\Delta Y + \bar{\alpha} Y). \end{cases} \tag{5.46}$$

Observe that the modified system (5.43) contains the term g_1^* which give rise to cross diffusion while the nonlinear terms are given only by $\bar{g}$.

5.5 The main theorem

The following asymptotic stability theorem for perturbations in the Sobolev space H_0^1 holds.

Theorem 5.5.1. *Let $d < 1 + \gamma/\bar{\alpha}$ and either*[1]

$$\begin{cases} b > a + (a+b)^3, \\[2ex] \bar{\alpha} > \dfrac{\gamma[b - a - (a+b)^3]}{a+b}, \\[2ex] \dfrac{\gamma[2b - (a+b)^3]}{\bar{\alpha}(a+b)} < d, \end{cases} \tag{5.48}$$

or

$$2b < (a+b)^3. \tag{5.49}$$

hold. Then, there exists $\bar{d} > 0$, such that if $d \in [0, \bar{d}]$, $(X^* = 0, Y^* = 0)$, and hence (u^*, v^*), is nonlinearly asymptotically stable with respect to the $L^2(\Omega)$-norm.

In order to prove theorem (5.5.1) we rewrite system (5.43) as follows:

$$\begin{cases} \dfrac{\partial X}{\partial t} = \bar{b}_1 X + \mu^{-1} b_2 Y + \bar{f}^* + g_1^*, \\[2ex] \dfrac{\partial Y}{\partial t} = \mu b_3 X + \bar{b}_4 Y + \bar{g}^* + \bar{g}, \end{cases} \tag{5.50}$$

where

$$\bar{b}_i = b_i + \bar{\alpha}\bar{\varepsilon}, \quad i = 1, 4 \tag{5.51}$$

[1] In order to verify $(5.48)_1$ we observe that

i. if $b > 1 > a$, then $(5.48)_1$ easily follows;

ii. if $a < b < 1$, setting $a = kb$ with $k < 1$, we have that

$$b - a > (a+b)^3 \Longleftrightarrow b(1-k) > b^3(1+k)^3 \Longleftrightarrow b^2 < \frac{1-k}{(1+k)^3}. \tag{5.47}$$

$$
\begin{cases}
\bar{f}^* = \bar{\gamma}_1(\Delta X + \bar{\alpha} X) + \bar{\varepsilon}\Delta X, \\
\\
\bar{g}^* = \bar{\gamma}_2(\Delta Y + \bar{\alpha} Y) + \bar{\varepsilon}\Delta Y, \\
\\
\bar{\gamma}_1 = 1 - \bar{\varepsilon}, \quad \bar{\gamma}_2 = d - \bar{\varepsilon},
\end{cases}
\tag{5.52}
$$

and $\bar{\varepsilon}$ is any constant such that

$$
0 < \bar{\varepsilon} < \inf\left\{\frac{|b_1|}{\bar{\alpha}}, \frac{|b_4|}{\bar{\alpha}}, \frac{|I|}{2\bar{\alpha}}, \frac{A}{\bar{\alpha}|I|}, 1, d\right\}. \tag{5.53}
$$

By introducing the Rionero-Liapunov functional

$$
\bar{V} = \frac{1}{2}\left[\bar{A}(\|X\|^2 + \|Y\|^2) + \|\bar{b}_1 Y - \mu b_3 X\|^2 + \|\mu^{-1} b_2 Y - \bar{b}_4 X\|^2\right], \tag{5.54}
$$

along the solutions of (5.50) it turns out that

$$
\frac{d\bar{V}}{dt} = \int_\Omega \bar{A}\bar{I}(X^2 + Y^2)d\Omega + \bar{\Psi}^* + \bar{\Psi}, \tag{5.55}
$$

where

$$
\bar{A} = \bar{b}_1\bar{b}_4 - b_2 b_3, \qquad \bar{I} = \bar{b}_1 + \bar{b}_4, \tag{5.56}
$$

$$
\begin{cases}
\bar{\Psi}^* =< \bar{\alpha}_1 X - \bar{\alpha}_3 Y, \bar{f}^* + g_1^* > + < \bar{\alpha}_2 Y - \bar{\alpha}_3 X, \bar{g}^* >, \\
\\
\bar{\Psi} =< \alpha_2 Y - \bar{\alpha}_3 X, \bar{g} >, \\
\\
\bar{\alpha}_1 = \bar{A} + \mu^2 b_3^2 + \bar{b}_4^2, \ \bar{\alpha}_2 = \bar{A} + \bar{b}_1^2 + \mu^{-2} b_2^2, \ \bar{\alpha}_3 = \mu\bar{b}_1 b_3 + \mu^{-1} b_2 \bar{b}_4.
\end{cases}
$$

Lemma 5.5.2. *Let* $\bar{b}_1 b_2 b_3 \bar{b}_4 < 0$ *. If*

$$
\frac{\bar{\alpha}_1}{4\bar{\mu}^2\bar{\alpha}_2} < \frac{(d - \bar{\varepsilon})(1 - \bar{\varepsilon})}{(d-1)^2}, \tag{5.57}
$$

Then

$$
\bar{\Psi}^* \le -\bar{\varepsilon}(\bar{\alpha}_1\|\nabla X\|^2 + \bar{\alpha}_2\|\nabla Y\|^2). \tag{5.58}
$$

Proof. The lemma immediately follows from lemma 1 of [25].
In our case condition (5.57) can be written in the following way (see appendix 1 for details):

$$\varphi_1(d)\varphi_2(d) < 1, \tag{5.59}$$

with

$$\begin{cases} \varphi_1(d) = \dfrac{K_1\bar{\alpha}^2 d^2 + K_2\bar{\alpha}d + K_3}{4(-C_1\bar{\alpha}^2 + C_2\bar{\alpha}d + C_3)\left(E_1\bar{\alpha}d + \frac{E_2 - E_3\bar{\alpha}d}{E_4 + \bar{\alpha}d} + E_5\right)}, \\[2em] \varphi_2(d) = \dfrac{(d-1)^2}{(d-\bar{\varepsilon})(1-\bar{\varepsilon})}, \end{cases} \tag{5.60}$$

where C_i, E_i, K_i are constants, and it can be proved that functions φ_1 is bounded for $d \to 1$, while, by the choice (5.53), φ_2 vanishes in the same limit. So, we can conclude that (5.59), and then (5.57), is satisfied in a neighbourhood of 1. □

Lemma 5.5.3. *Let* $\bar{\alpha}_3 = 0$*. Then*

$$\bar{\Psi} \leq k^*(\|X\| + \|Y\|)(\|\nabla X\|^2 + \|\nabla Y\|^2), \tag{5.61}$$

where k^* *is a positive constant.*

Proof. Since $\bar{\alpha}_3 = 0$, and by means of Sobolev inequality (5.29) it turns out:

$$\begin{aligned} \bar{\Psi} &= < \alpha_2 Y - \bar{\alpha}_3 X, \bar{g} > = < \alpha_2 Y, \bar{g} > \\ &\leq -\frac{\gamma\bar{\alpha}_2}{\beta}\left[v^* \int_\Omega (\alpha X - \beta Y)^2 Y d\Omega + 2u^*\beta \int_\Omega (\alpha X - \beta Y) Y^2 d\Omega\right] \\ &= -\frac{\gamma\bar{\alpha}_2}{\beta}\left[v^*\alpha^2 \int_\Omega X^2 Y d\Omega - \beta^2(v^* - 2u^*) \int_\Omega Y^3 d\Omega + 2\alpha\beta(u^* - v^*) \int_\Omega XY^2 d\Omega\right] \\ &\leq k\gamma\bar{\alpha}_2 \left[\beta^{-1}v^*\alpha^2\|Y\|\|\nabla X\|^2 + \beta|v^* - 2u^*|\|Y\|\|\nabla Y\|^2 + 2\alpha|u^* - v^*|\|X\|\|\nabla Y\|^2\right] \\ &\leq k^*(\|X\| + \|Y\|)(\|\nabla X\|^2 + \|\nabla Y\|^2). \end{aligned} \tag{5.62}$$

with

$$k^* = k\gamma\bar{\alpha}_2 \sup\left\{\beta^{-1}v^*\alpha^2, \beta|v^* - 2u^*|, 2\alpha|u^* - v^*|\right\}. \tag{5.63}$$

□

Now we are able to prove theorem (5.5.1).

Proof. If (5.48) or (5.49) hold, then

$$\begin{cases} b_2 > 0, \\ b_4 < 0, \end{cases} \tag{5.64}$$

we can choose

$$\mu^2 = -\frac{b_2\bar{b}_4}{\bar{b}_1 b_3}, \tag{5.65}$$

so that $\bar{\alpha}_3 = 0$. This implies that:

$$\begin{cases} b_2 > 0, \\ \\ b_4 < 0, \end{cases} \Longleftrightarrow \begin{cases} I < 0, \\ \\ A > 0, \end{cases} \Longrightarrow \begin{cases} \bar{I} < 0, \\ \\ \bar{A} > 0, \end{cases} \tag{5.66}$$

Now, from (5.55), Lemma 1 and Lemma 2 we find:

$$\begin{aligned} \frac{d\bar{V}}{dt} &\leq \int_\Omega \bar{A}\bar{I}(X^2 + Y^2)d\Omega - \bar{\varepsilon}(\bar{\alpha}_1\|\nabla X\|^2 + \bar{\alpha}_2\|\nabla Y\|^2) \\ &\quad + k^*(\|X\| + \|Y\|)(\|\nabla X\|^2 + \|\nabla Y\|^2) \\ &\leq -\frac{\bar{A}|\bar{I}|}{\bar{k}_2}\bar{V} - \hat{k}(\|\nabla X\|^2 + \|\nabla Y\|^2) \\ &\quad + \sqrt{2}k^*(\|X\|^2 + \|Y\|^2)^{\frac{1}{2}}(\|\nabla X\|^2 + \|\nabla Y\|^2) \\ &\leq -\frac{\bar{A}|\bar{I}|}{\bar{k}_2}\bar{V} - \left(\hat{k} - \sqrt{\frac{2}{\bar{k}_1}}k^*\bar{V}^{\frac{1}{2}}\right)(\|\nabla X\|^2 + \|\nabla Y\|^2), \end{aligned} \tag{5.67}$$

where $\hat{k} = \bar{\varepsilon}\inf(\bar{\alpha}_1, \bar{\alpha}_2)$. From the previous inequality, by recursive arguments, one obtains

$$\bar{V}_0^{\frac{1}{2}} < \frac{\hat{k}}{k^*}\sqrt{\frac{\bar{k}_1}{2}} \Longrightarrow \frac{d\bar{V}}{dt} \leq 0, \tag{5.68}$$

and finally

$$\frac{d\bar{V}}{dt} \leq -\delta \bar{V}, \tag{5.69}$$

with

$$\delta = \frac{1}{\bar{k}_2} \left[\bar{A}|\bar{I}| + \bar{\alpha} \left(\hat{k} - \sqrt{\frac{2}{\bar{k}_1}} k^* \bar{V}_0^{\frac{1}{2}} \right) \right]. \tag{5.70}$$

$\square$

Appendix

In order to estimate all the terms involving diffusion the following inequality must holds,

$$\frac{\bar{\alpha}_1}{4\bar{\mu}^2\bar{\alpha}_2} < \frac{(d-\bar{\varepsilon})(1-\bar{\varepsilon})}{(d-1)^2} \tag{5.71}$$

with

$$\begin{aligned}
&\bar{\alpha}_1(\bar{\mu}) = \bar{A} + b_3^2\bar{\mu}^2 + \bar{b_4}^2 > 0\\
&\bar{\alpha}_2(\bar{\mu}) = \bar{A} + b_2^2\bar{\mu}^{-2} + \bar{b_1}^2 > 0\\
&\bar{\mu}^2 = |\frac{b_2\bar{b}_4}{\bar{b}_1 b_3}|
\end{aligned}$$

In our case the previous condition may be written as

$$\varphi_1(d)\varphi_2(d) < 1, \tag{5.72}$$

with

$$\begin{cases}
\varphi_1(d) = \dfrac{K_1\bar{\alpha}^2 d^2 + K_2\bar{\alpha}d + K_3}{4(-C_1\bar{\alpha}^2 + C_2\bar{\alpha}d + C_3)\left(E_1\bar{\alpha}d + \frac{E_2 - E_3\bar{\alpha}d}{E_4 + \bar{\alpha}d} + E_5\right)},\\[2ex]
\varphi_2(d) = \dfrac{(d-1)^2}{(d-\bar{\varepsilon})(1-\bar{\varepsilon})},
\end{cases} \tag{5.73}$$

where

$$K_1 = \Big[1 - \frac{2\gamma b}{(\gamma + \bar{\alpha} - \bar{\alpha}\bar{\varepsilon})(a+b)}\Big]$$

$$K_2 = \gamma + \bar{\alpha} + 2\gamma(a+b)^2 - \frac{6\gamma b}{a+b} - 3\bar{\alpha}\bar{\varepsilon} + \frac{2\gamma b}{(a+b)(\gamma + \bar{\alpha} - \bar{\alpha}\bar{\varepsilon})}$$
$$\big[\gamma + \bar{\alpha} + \bar{\alpha}\bar{\varepsilon} + \frac{2\gamma b}{a+b} - \gamma(a+b)^2\big]$$

$$K_3 = \gamma(\gamma + \bar{\alpha})(a+b)^2 + (\bar{\alpha}\bar{\varepsilon})^2 + \bar{\alpha}\bar{\varepsilon}[-(\gamma + \bar{\alpha}) + \frac{2\gamma b}{a+b} - \gamma(a+b)^2] + \frac{2\gamma b(\gamma + \bar{\alpha})}{(\gamma + \bar{\alpha} - \bar{\alpha}\bar{\varepsilon})(a+b)}$$
$$\big[-\frac{2\gamma b}{a+b} + \gamma(a+b)^2 - \bar{\alpha}\bar{\varepsilon}\big] + [\frac{2\gamma b}{a+b} - \gamma(a+b)^2 + \bar{\alpha}\bar{\varepsilon}]^2$$

$$C_1 = \frac{a+b}{2\gamma b(\gamma + \bar{\alpha} - \bar{\alpha}\bar{\varepsilon}})$$

$$C_2 = C_1[\gamma + \bar{\alpha} + \frac{2\gamma b}{(a+b)} - \gamma(a+b)^2 + \bar{\alpha}\bar{\varepsilon}]$$

$$C_3 = C_1(\gamma + \bar{\alpha})\big[-\frac{2\gamma b}{(a+b)} + \gamma(a+b)^2 - \bar{\alpha}\bar{\varepsilon}\big]$$

$$E_1 = \big(\gamma + \bar{\alpha} - \frac{2\gamma b}{a+b} - \bar{\alpha}\bar{\varepsilon}\big)$$

$$E_2 = \frac{2\gamma b}{a+b}(\gamma + \bar{\alpha})(\gamma + \bar{\alpha} - \bar{\alpha}\bar{\varepsilon})$$

$$E_3 = \frac{2\gamma b}{(a+b)}(\gamma + \bar{\alpha} - \bar{\alpha}\bar{\varepsilon})$$

$$E_4 = -\frac{2\gamma b}{(a+b)} + \gamma(a+b)^2 - (\bar{\alpha}\bar{\varepsilon})$$

$$E_5 = \gamma(\gamma + \bar{\alpha})(a+b)^2 + (\bar{\alpha}\bar{\varepsilon})^2 +$$

$$+(\bar{\alpha}\bar{\varepsilon})\big[-(\gamma + \bar{\alpha}) + \frac{2\gamma b}{a+b} - \gamma(a+b)^2\big] + (-\gamma - \bar{\alpha} + \bar{\alpha}\bar{\varepsilon})^2$$

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