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Existence results for some classes of nonlinear elliptic problems

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Contents

Introduction

The main objective of this thesis is to obtain existence and comparison results for some classes of nonlinear elliptic Dirichlet problems. A leading role in such results is played by a priori estimates of solutions, which will be proved mainly by means of Schwarz symmetrization.

The symmetrization methods, a compound of techniques essentially referable to the use of isoperimetric inequalities and of the properties of rearrangements, allow to obtain explicit and sharp estimates of solutions of boundary value problems.

So we are devoted to the study of some classes of nonlinear elliptic Dirichlet problems in divergence form, which can be formally written as

$$\begin{cases} -\operatorname{div}(a(x,u,Du)) = H(x,u,Du) + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded open set of $\mathbb{R}^n$, $n \geq 2$, $a(x,s,\xi)$ is a Leray–Lions operator, and $H(x,s,\xi)$ and f verify suitable conditions. We look for a priori estimates for solutions u of problems like (1), in the sense that

$$\|u\| \leq K \|f\| , \tag{2}$$

where $\|u\|$ and $\|f\|$ are suitable norms of u and f. Usually, Schwarz symmetrization permits to obtain the best constant in (2), in the sense that one can find the supremum of the ratio

$$\frac{\text{a norm of } u}{\text{another norm of } f},$$

where the data of (1) are supposed to verify appropriate constraints, and Ω varies among the domains of fixed Lebesgue measure.

The thesis is organized as follows.

Chapter 1 is devoted to recall some basic facts about rearrangements of functions. The theory of rearrangements, whose beginning goes back to the works of Schwarz and Steiner at the end of XIX century, was popularized and systematically developed by the celebrated book of Hardy, Littlewood and Pólya of 1934 (see [**64**]). Referring to [**64**], [**20**], [**37**], [**67**], [**82**], [**84**], [**91**], [**93**], we recall some fundamental definitions and properties.

In Chapter 2 we deal with the classical theory of pseudo–monotone operators in Banach spaces. The theory of monotone operators applied to boundary value problems has its origins in the works of Minty and Browder, then it was generalized by Leray and Lions

([**73**]) to a wider class of operators, namely the pseudo–monotone ones. Such generalization allows to treat several classes of boundary value problems, some of them deriving from problems of calculus of variations. Referring to [**73**] and [**76**] (see also [**72**], [**87**]), we recall some results on this theory and we show how it can be applied to Dirichlet boundary value problems.

In Chapter 3 we deal with comparison results for nonlinear elliptic problems with lower–order terms.

Our aim is to establish a comparison, in some sense, between a solution of a given problem with the solution of a "symmetrized" one, whose data are spherically symmetric.

A first result in this direction is due to Weinberger (see [**98**]). Here we recall a more general result given by Talenti in [**89**] (see also Maz'ja [**80**]). In [**89**] Talenti proved that if $u \in H_0^1(\Omega)$ is a solution of the problem

$$-\left(a_{ij}(x)u_{x_i}\right)_{x_j} + c(x)u = f, \quad u \in H_0^1(\Omega), \tag{3}$$

where Ω is a bounded open set of $\mathbb{R}^n$ (supposing $n \geq 3$ for sake of simplicity),

$$a_{ij}(x)\xi_i\xi_j \geq |\xi|^2, \text{ for a.e. } x \in \Omega, \forall \xi \in \mathbb{R}^n,$$

$c(x) \geq 0$ and f belongs to $L^r(\Omega)$, with $r = 2n/(n+2)$, and if v is the solution of the problem

$$-\Delta v = f^{\#}, \quad v \in H_0^1(\Omega^{\#}), \tag{4}$$

then

$$u^{\#}(x) \leq v(x), \quad \forall x \in \Omega^{\#}, \tag{5}$$

and

$$\int_{\Omega} (a_{ij}(x)u_{x_i}u_{x_j})^{q/2} dx \leq \int_{\Omega^{\#}} |Dv|^q \, dx,$$

for $0 < q \leq 2$, where $\Omega^{\#}$ is the ball centered at the origin having the same measure of Ω and $f^{\#}$ denotes the spherically symmetric decreasing rearrangement of f (see Chapter 1 for definitions and properties of rearrangements).

Talenti's result provides the "largest" solution in the class of equations (3) where the measure of Ω is fixed and f has prescribed rearrangement. Moreover, the estimate (5) allows to obtain, for example, that for any $s \in [1, +\infty]$

$$\|u\|_{L^s(\Omega)} \leq \|v\|_{L^s(\Omega^{\#})} .$$

In this order of ideas, here we present a comparison result for equations whose prototype can be written in the form

$$\begin{cases} -\Delta_p u + h(x)|Du|^{p-1} = c(x)|u|^{p-2}u + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where Ω is a bounded open set of $\mathbb{R}^n$, Δ_p is the p–laplacian operator (namely, $\Delta_p u = \operatorname{div}(|Du|^{p-2}Du)$), $1 < p < +\infty$, $\|h\|_{L^\infty(\Omega)} \leq \beta$ (here the case $p = 2$ models the linear one), and c and f verify suitable summability hypotheses.

Our aim is to compare a solution of (6) with the radial solution $v = v^\#$ of the problem

$$\begin{cases} -\Delta_p v + \beta |Du|^{p-2} Du \cdot \frac{x}{|x|} = \left((c^+)^\# - (c^-)_\#\right) |v|^{p-2} v + f^\# & \text{in } \Omega^\# \\ v = 0 & \text{su } \partial\Omega^\#. \end{cases} \tag{7}$$

Observe that, differently from Talenti's result, in the "symmetrized" problem (7) we take into account also of the zero–order term. In general, this leads to loose a pointwise comparison like (5) in all $\Omega^\#$ (see [**36**] for a counterexample). More precisely, if u is a solution of (6), and $v = v^\#$ is the solution of (7), then given $s_0 = \inf\{s \in [0, |\Omega|] \colon (c^-)_* > 0\}$ (if $c^- \equiv 0$, we put $s_0 = |\Omega|$), we have

$$u^*(s) \leq v^*(s), \quad \forall s \in [0, s_0],$$

and

$$\int_0^s (u^*(t))^{p-1} \exp\left(-\frac{\beta t^{1/n}}{\omega_n^{1/n}}\right) dt \leq \int_0^s (v^*(t))^{p-1} \exp\left(-\frac{\beta t^{1/n}}{\omega_n^{1/n}}\right) dt, \ \forall s \in [s_0, |\Omega|].$$

Several results of this type can be found in literature. For linear case see, for example, [**7**], [**8**], [**9**], [**10**],[**15**], [**36**], [**77**], [**91**], [**96**], and for nonlinear case see [**21**], [**45**], [**50**], [**51**], [**81**], [**90**].

Here we present a result obtained in [**42**], where both the coefficients of zero and first order are considered, $c \in L^r(\Omega)$, with $r > n/p$, and $f \in L^q(\Omega)$, with $q > n/p$. We remark that the comparison result holds if there exists an unique (radial) solution of (7). This is not guaranteed for any choice of the coefficients $\beta \geq 0$ and $c \in L^r(\Omega)$. So we find a structure condition on the the equation in order to obtain an existence and uniqueness result for the "symmetrized" problem (7).

In Chapter 4 we deal with Dirichlet problems whose prototype is

$$\begin{cases} -\operatorname{div}(b(|u|)\, |Du|^{p-2}\, Du) = k(|u|)\, |Du|^q + f, & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$

where Ω is a bounded open set in $\mathbb{R}^n$, $0 < p - 1 < q \leq p < +\infty$, k and b are continuous functions such that $k \geq 0$ and $b > 0$, and $f \in L^r(\Omega)$, $r > \max\{n/p, 1\}$.

The study of (8) is also motivated by the fact that, in some particular cases, (8) is equivalent to a problem like (6), with $\beta = 0$. This can be easily seen by means of a well–known example, due to Kazdan and Kramer (see [**66**]). Furthermore, the same example well explains the typical behaviour of problems like (8). Now we briefly sketch it (see Chapter 4 for details).

Let us consider the semilinear problem

$$(9) \qquad \begin{cases} -\Delta u = |Du|^2 + \lambda, & u \in H_0^1(\Omega) \cap L^\infty(\Omega), \\ u > 0 & \text{in } \Omega, \end{cases}$$

where λ is a positive constant and Ω is a bounded open set in $\mathbb{R}^n$. Performing the change of variable $v = e^u - 1$, we obtain that (9) is equivalent to the problem

$$(10) \qquad \begin{cases} -\Delta v = \lambda(v+1), & v \in H_0^1(\Omega) \cap L^\infty(\Omega), \\ v > 0 & \text{in } \Omega. \end{cases}$$

By Fredholm alternative, Problem (10) (and then (9)) admits a solution if λ is smaller than $\lambda_1(\Omega)$, the first eigenvalue of the laplacian in Ω. Moreover, if $\lambda \geq \lambda_1(\Omega)$, is not difficult to show that (10) (and then (9)) does not admit a solution. In particular this means that in general one can expect that, in order to prove the existence of a solution to problem (8), one has to assume some smallness condition on the source term f.

Problems like (8) have been widely studied in literature under various hypotheses. In a series of papers by Boccardo, Murat and Puel (see [**29**], [**30**], [**31**], [**32**]) the existence of solutions of problems like (8), with $p = q$ and b constant, is proved under sign condition on the lower order terms or assuming the existence of sub and super solutions. Moreover, similar problems have been treated, for example, for $k \equiv 0$ in [**3**], [**2**], [**26**], and for $k \not\equiv 0$ in [**7**], [**62**], [**79**], [**2**], [**34**], [**94**], [**34**], [**53**], [**54**], [**84**], [**63**] in the case $p = q$, and in [**50**], [**33**], [**63**] in the case $p - 1 < q \leq p$.

The quoted results are of two kinds: the first one establishes existence of solutions without imposing any additional condition on f; the second one requires conditions on the smallness of some norm of f. More precisely, when it is possible to remove the smallness hypotheses on f, appropriate hypotheses on the structure of the equation are needed, like sign conditions or particular hypotheses on the functions $k(s)$ and $b(s)$.

Here we present an existence result for problems like (8) obtained in [**43**]. Our approach permits to treat in a unified way both the cases in which it is required a particular hypothesis on f and the cases in which such hypothesis it is not necessary.

For example, when $b \equiv 1$ in (8), our result reads as follows: if

$$(11) \qquad c\,\|f\|_{L^r(\Omega)} < \sup_{s>0} W(s),$$

where

$$W(s) = \int_0^s e^{-C\int_r^s k(y)^{\frac{1}{1-p+q}}\,dy}\,dr$$

and the constants c and C depend only on p, q, n and $|\Omega|$, there exists a solution of (8).

Depending on $k(s)$, the function $W(s)$ can be bounded or not. In the first case the condition (11) is an hypothesis on the norm of f. In the second case, that is $\sup_{s>0} W(s) = +\infty$, our result does not require any smallness assumption on f, because (11) is always satisfied. On the other hand, our approach allows us to consider the more general condition

$\sup_{s>0} W(s) = +\infty$ instead of $\lim_{s\to+\infty} W(s) = +\infty$ considered in previous papers (see [**34**], [**85**]).

The central point is to obtain an a priori estimate for solutions of (8), that we prove, by means of symmetrization methods, in terms of the function W, namely

$$W(\|u\|_{L^\infty(\Omega)}) \le c\,\|f\|_{L^r(\Omega)}\,, \tag{12}$$

where c is the same constant which appears in (11). If the condition (11) is not satisfied, it is clear that (12) cannot give any information on $\|u\|_{L^\infty(\Omega)}$, since it is trivially verified. On the other hand, if, for example, $W(s)$ is monotone, (11) and (12) immediately give us a L^∞−estimate for u. In general, under our assumptions, $W(s)$ is not monotone, and then (11) and (12) do not imply directly an estimate for u. The main point in the proof of the existence result consists in showing that the only hypothesis (11) allows us to obtain an uniform estimate for the solutions to suitable problems which approximate (8). After passing to the limit such approximate problems we get the result.

We observe that, if $k \equiv 0$, the problem (8) takes the form

$$\begin{cases} -\operatorname{div}(b(|u|)\,|Du|^{p-2}\,Du) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{13}$$

A typical example is given by a function $b(s)$ which goes to zero when s goes to $+\infty$. In this case the operator $u \mapsto -\operatorname{div}(b(|u|)|Du|^{p-2}Du))$ is, in general, not coercive. Problems like (13) have been investigated by several authors (see, for example, [**4**], [**2**], [**26**], [**22**]).

Another kind of degeneracy can be given, for instance, by means of functions $b(s)$ which blow up at finite values of s. A typical model of such problem is the following:

$$\begin{cases} -\operatorname{div}\left(\dfrac{|Du|^{p-2}\,Du}{(1-|u|)^\alpha}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{14}$$

Here Ω is a bounded open set in $\mathbb{R}^n$, $1 < p < +\infty$, $\alpha > 0$ and f verifies suitable summability hypotheses. In the last chapter we study the existence of solutions of problems like (14). So the main feature of this equation is the fact that the term $b(u) = 1/(1-|u|)^\alpha$ blows up when u approaches the values ± 1.

Existence results for such kind of problems have been obtained by several authors in the case $p = 2$ under different assumptions on the equation (see, for example, [**23**], [**24**], [**25**], [**58**], [**59**], [**83**]). More precisely, under suitable hypotheses, in [**83**] it is proved the existence of weak solutions $u \in H_0^1(\Omega)$ of problem (14) such that $\|u\|_{L^\infty(\Omega)} < m$ or, in a more general setting, the existence of distributional solutions which can approach the values ± 1 on a set of Lebesgue measure zero. In [**24**], [**25**], [**58**], [**59**], [**23**], problems similar to (14) are considered. In this papers the idea of weak solutions is not well–suited, because in such cases the solutions can achieve the critical values on a set of positive

Lebesgue measure. For this reason, the authors adapt the definition of renormalized solution to the structure of their problems, in order to obtain existence results.

Here we present some results obtained in [**44**], where we focus our attention on the more general case $1 < p < +\infty$, obtaining existence results with respect to the summability of b and f.

First of all, let us consider the case $b^{1/(p-1)} \notin L^1(0,1)$. If $f \in L^r(\Omega)$, $r > n/p$, it possible to prove that there exists a weak solution u of (14), belonging to $\in W_0^{1,p}(\Omega)$, such that $\|u\|_{L^\infty(\Omega)} < m$ (see also [**83**] for $p = 2$). We observe that this is the same behavior of well–known standard elliptic problems. If we decrease the summability assumption on f, it is possible to obtain solutions still in $W_0^{1,p}(\Omega)$, but which can reach the critical values ± 1 on a set of Lebesgue measure zero. More precisely, if $f \in L^r(\Omega)$, $np/(np - n + p) \leq r < n/p$, then we get weak solutions, otherwise if $f \in L^1(\Omega)$, we adapt the definition of entropy solution (see [**19**]) in order to have existence results. In the former case, our result improves the one obtained in [**83**] when $p = 2$, where distributional solutions are considered.

The case $b \in L^1(0,1)$ is completely different from the previous one. Indeed, in order to obtain existence of weak solutions $u \in W_0^{1,p}(\Omega)$ such that $\|u\|_{L^\infty(\Omega)} < m$ we need to require a smallness assumption on the L^r–norm of f, with $r > n/p$. If such smallness hypothesis is not satisfied, it is not possible to obtain weak solutions because, in general, the measure of the set $\{|u| = 1\}$ is positive. To avoid this problem, we use the adapted definition of entropy solution. This adaptation allows to treat also the case of the datum f in $L^1(\Omega)$.

We emphasize that all the obtained solutions are in the energy space $W_0^{1,p}(\Omega)$, differently from the typical results on elliptic equations with datum in L^1.

We observe that the behavior of the function b does not allow to apply the classical methods for Leray–Lions operators. To overcome this problem, we approximate the equation (14) by cutting the function b near the critical value 1 and taking smooth data, in order to obtain operators which belong to the class of Leray–Lions ones. For the solutions of this class of approximated problems, using symmetrization methods, we will obtain a priori estimates in Lebesgue and Sobolev spaces. Such estimates, together with a result on a.e. convergence of the gradients, allow us to pass to the limit obtaining the existence results. We observe that, due to the presence of the function b, the a.e. convergence of the gradients of approximated solutions holds only on the set $\{|u| < 1\}$.

Finally, I want to thank Prof. Vincenzo Ferone, who introduced me in this subject, and who continuously supported and encouraged me during these years.

CHAPTER 1

Rearrangements

1. Definitions

Let Ω be a Lebesgue measurable set of $\mathbb{R}^n$, with $n \geq 1$. We denote with $|\Omega|$ its n–dimensional Lebesgue measure, which we suppose to be positive and, for the sake of simplicity, finite. Let $u : \Omega \to \mathbb{R}$ be a measurable function.

Definition 1.1. The *distribution function* of u is the map $\mu_u \colon [0,+\infty[\to [0,+\infty[$ defined by

$$\mu_u(t) \;=\; |\{x \in \Omega :\, |u(x)| > t\}|.$$

Such function represents the measure of the level sets of u.

Proposition 1.1. *The function μ_u enjoys the following properties:*

1. $\mu_u(t)$ *is monotone decreasing;*
2. $\mu_u(0) = |\operatorname{spt} u|$*;*
3. $\operatorname{spt} \mu_u = [0, \operatorname{ess\,sup} |u|]$*;*
4. $\mu_u(t)$ *is right–continuous;*
5. $\mu_u(t^-) - \mu_u(t) = |\{x \in \Omega :\, |u(x)| = t\}|$.

Proof. $(1), (2)$ and (3) are immediate from the definition. As regards (4) and (5), being

$$\{x \in \Omega\colon |u(x)| > t\} = \bigcup_{k=1}^{\infty} \left\{x \in \Omega\colon |u(x)| > t + \frac{1}{k}\right\},$$

and

$$\{x \in \Omega\colon |u(x)| \geq t\} = \bigcap_{k=1}^{\infty} \left\{x \in \Omega\colon |u(x)| > t - \frac{1}{k}\right\},$$

we get

$$\mu_u(t^+) = \lim_{k\to+\infty} \left|\left\{x \in \Omega\colon |u(x)| > t + \frac{1}{k}\right\}\right| = \mu_u(t),$$
$$\mu_u(t^-) = \lim_{k\to+\infty} \left|\left\{x \in \Omega\colon |u(x)| > t - \frac{1}{k}\right\}\right| = \mu_u(t) + |\{x \in \Omega\colon |u(x)| = t\}|.$$

□

We observe that, by (5), $\mu_u(t)$ is discontinuous only for t such that

$$|\{x \in \Omega\colon |u(x)| = t\}| \neq 0.$$

DEFINITION 1.2. The *decreasing rearrangement* of u is the map $u^*\colon [0,\infty[\to [0,\infty[$ defined by

$$u^*(s) := \sup\{t \geq 0\colon \mu_u(t) > s\}.$$

Essentially, u^* is the "generalized" inverse of μ_u, in the sense that if t is a point of discontinuity of μ_u, then the value of u^* is fixed as t in the interval $[\mu_u(t), \mu_u(t^-)]$; moreover, the flat zones of μ_u become jump discontinuities in u^*.

Observe that u^* is the distribution function of μ_u; therefore u^* enjoys the following properties:

(1) u^* is monotone decreasing;
(2) u^* is right–continuous;
(3) $u^*(0) = \operatorname{ess\,sup} u$;
(4) $\operatorname{spt} u^* = [0, |\operatorname{spt} u|]$;
(5) $u^*(\mu_u(t)) \leq t$ and $\mu_u(u^*(s)) \leq s$.

PROPOSITION 1.2. *The mapping $u \mapsto u^*$ has the following property: given u, v measurable functions on Ω, then*

$$|u| \leq |v| \Rightarrow u^* \leq v^*.$$

PROOF. Since $\{x \in \Omega\colon |u(x)| > t\} \subset \{x \in \Omega\colon |v(x)| > t\}$, we have that

$$\{t \geq 0\colon \mu_u(t) > s\} \subset \{t \geq 0\colon \mu_v(t) > s\};$$

then the result follows from the definition. $\square$

DEFINITION 1.3. Two real–valued functions are *equimeasurable* if they have the same distribution function. Equimeasurable functions are said to be rearrangement of each other.

PROPOSITION 1.3. *The functions $u\colon \Omega \to \mathbb{R}$ and $u^*\colon [0, |\Omega|] \to [0, +\infty[$ are equimeasurable, that is for all $t \geq 0$,*

$$|\{x \in \Omega\colon |u(x)| > t\}| = |\{s \in [0, \Omega]\colon u^*(s) > t\}|. \tag{1.1}$$

PROOF. By the definition of u^*, it follows that

$$\text{if } u^*(s) > t, \text{ then } s < \mu_u(t);$$
$$\text{if } u^*(s) \leq t, \text{ then } s \geq \mu_u(t).$$

Hence

$$\{s \geq 0\colon u^*(s) > t\} = [0, \mu_u(t)[,$$

and this gives (1.1). $\square$

COROLLARY 1.1. *With the preceding notations, we have*

$$|\{x \in \Omega \colon |u(x)| \geq t\}| = |\{s \in [0, \Omega] \colon u^*(s) \geq t\}| \tag{1.2}$$

$$|\{x \in \Omega \colon |u(x)| < t\}| = |\{s \in [0, \Omega] \colon u^*(s) < t\}| \tag{1.3}$$

$$|\{x \in \Omega \colon |u(x)| \leq t\}| = |\{s \in [0, \Omega] \colon u^*(s) \leq t\}| \tag{1.4}$$

PROOF. Since (1.4) is equivalent to (1.1) and (1.2) is equivalent to (1.3), it is sufficient to prove that (1.1) and (1.2) are equivalent. Indeed, being

$$\lim_{h \to 0^+} |\{x \in \Omega \colon |u(x)| \geq t + h\}| = |\{x \in \Omega \colon |u(x)| > t\}|$$

and

$$\lim_{h \to 0^+} |\{x \in \Omega \colon |u(x)| > t - h\}| = |\{x \in \Omega \colon |u(x)| \geq t\}|,$$

we get the thesis. □

By Proposition 1.3 we get an important property of rearrangements: if $u \in L^p(\Omega)$, with $1 \leq p \leq +\infty$, then $u^* \in L^p(0, |\Omega|)$, and

$$\|u\|_{L^p(\Omega)} = \|u^*\|_{L^p(0,|\Omega|)}\,; \tag{1.5}$$

indeed, if $p < +\infty$, a simple application of Fubini's theorem gives

$$\|u\|_{L^p(\Omega)} = p \int_\Omega t^{p-1} \mu_u(t) dt;$$

by equimeasurability of u and u^*, (1.5) follows. If $p = \infty$, the result follows from the definition of rearrangement.

In general, (1.1) allows us to observe that if an operator Φ, acting on a measurable function u, depends only on the measure of the level sets of u, then the operator is invariant with respect to the action of the rearrangement (see [**82**], [**92**]). For example, the operator Φ, defined as

$$\Phi(u) := \int_\Omega F(|u(x)|)\, dx,$$

where $F \colon \mathbb{R} \to [0, +\infty[$ is a Borel measurable function, is rearrangement invariant, as stated in the following proposition.

PROPOSITION 1.4. *Let $u \colon \Omega \to \mathbb{R}$ measurable. Let $F \colon \mathbb{R} \to \mathbb{R}$ be a non–negative Borel measurable function. Then*

$$\int_\Omega F(|u(x)|)\, dx = \int_0^{|\Omega|} F(u^*(s))\, ds.$$

Proof. First, suppose that $F = \chi_{[t,+\infty]}$[1]. Then

$$\int_\Omega \chi_{[t,+\infty]}(|u(x)|)\,dx = |\,\{x \in \Omega \colon |u(x)| > t\}\,| =$$

$$= |\,\{s \geq 0 \colon u^*(s) > t\}\,| = \int_0^\Omega \chi_{[t,+\infty]}(u^*(s))\,ds.$$

Similarly, the result holds for $F = \chi_E$, where E is any interval and then if E is any open set and, again, if E is any Borel set, by standard arguments. Hence the result is true for any non–negative simple function F. If F is any non–negative Borel function, it can be expressed as the limit of an increasing sequence $\{F_k\}$ of non negative simple functions. Thus, for each n we have

$$\int_\Omega F_k(|u(x)|)\,dx = \int_0^{|\Omega|} F_k(u^*(s))\,ds.$$

Using the monotone convergence theorem, we can pass to the limit as $k \to +\infty$ to get the thesis. □

We now prove another important property of rearrangements.

Proposition 1.5. *Let $u \colon [0,l] \to [0,+\infty[$ be non–increasing. Then u and u^* coincides a.e.*

Proof. If $t < u(s)$, then $\mu_u(t) \geq s$, being u non–increasing. Hence $u^*(s) \geq t$, by definition. This implies that

$$u^*(s) \geq u(s), \quad \text{for all } s \in [0,l]. \tag{1.6}$$

On the other hand, being u is non–increasing, we have that

$$\mu_u(u(s-h)) \leq s - h < s$$

for $h > 0$. Thus, by definition, $u^*(s) \leq u(s-h)$. Hence, if s is a point of continuity for u, as h tends to 0 we obtain

$$u^*(s) \leq u(s); \tag{1.7}$$

then by (1.6) and (1.7) we obtain that $u^*(s) = u(s)$ on all points of continuity of u; observing that the set of points of discontinuity of a monotone function is countable, we get the thesis. □

Here we have described only one kind of rearrangement, the decreasing rearrangement ï¿½ la Hardy & Littlewood. Nevertheless, in literature several ways of rearranging a

[1] We will denote with χ_E the characteristic function of a set $E \subset \mathbb{R}^n$, namely

$$\chi_E(\sigma) := \begin{cases} 1 & \sigma \in E \\ 0 & \sigma \notin E \end{cases}$$

function have been defined. A good reference on this topic is, for example, [**67**]. Now we just recall some definitions which will be useful in the following.

DEFINITION 1.4. The *increasing rearrangement* of u is the map $u_*\colon [0,\infty[\to [0,\infty[$ defined by

$$u_*(s) = u^*(|\Omega| - s), \quad s \in [0, |\Omega|];$$

The increasing rearrangement enjoys similar properties than the decreasing rearrangement; however, observe that u_* is monotone increasing and left–continuous.

DEFINITION 1.5. Let $\Omega^\#$ the ball centered at the origin having the same measure as Ω. The *spherically symmetric decreasing rearrangement* of u is the function $u^\#\colon \Omega^\# \to \mathbb{R}$ defined by

$$u^\#(x) = u^*(\omega_n |x|^n), \quad x \in \Omega^\#,$$

where ω_n is the measure of the unit ball in $\mathbb{R}^n$, namely

$$\omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}.$$

The function $u^\#$ is also known as the *Schwarz symmetrization* of u. Likewise, the *spherically symmetric increasing rearrangement* of u is the function $u_\# : \Omega^\# \to \mathbb{R}$ defined by

$$u_\#(x) = u_*(\omega_n |x|^n).$$

All the functions defined above are equimeasurable with u.

2. Some fundamental rearrangement inequalities

THEOREM 2.1 (Hardy & Littlewood, [**64**]). *Let u and v be two measurable functions defined on a bounded open set $\Omega \subset \mathbb{R}^n$. Then:*

$$\int_0^{|\Omega|} u^*(s)v_*(s)ds \le \int_\Omega |u(x)v(x)|dx \le \int_0^{|\Omega|} u^*(s)v^*(s)ds \tag{2.1}$$

PROOF. Let us start to prove (2.1) in the case u and v are the characteristic functions of two measurable subsets A and B of Ω, that is $u(x) = \chi_A(x)$, and $v(x) = \chi_B(x)$. Being

$$\chi_E^*(s) = \chi_{[0,|E|[}(s), \qquad \forall E \subset \Omega,$$

we get

$$\begin{aligned} \int_\Omega |u(x)v(x)|dx &= \int_\Omega \chi_{A\cap B}(x)dx = |A \cap B| \le \\ &\le \int_0^{\min\{|A|,|B|\}} ds = \int_0^{|\Omega|} \chi_{[0,|A|[}(s)\chi_{[0,|B|[}(s)ds = \\ &= \int_0^{|\Omega|} u^*(s)v^*(s)ds, \end{aligned} \tag{2.2}$$

and this proves the right–hand side inequality in (2.1). As regards the other inequality, being $A \cap B = A \setminus (\Omega \setminus B)$, we have

$$|A \cap B| \geq \max\{0, |A| + |B| - |\Omega|\}.$$

Hence, by using similar arguments as above, we get

$$\begin{aligned}
\int_\Omega |u(x)v(x)|dx \geq \int_0^{\max\{0,|A|+|B|-|\Omega|\}} ds = \\
&= \int_0^{|\Omega|} \chi_{[0,|A|[}(s)\chi_{]||\Omega|-|B||,|\Omega|]}(s)\, ds = \\
&= \int_0^{|\Omega|} \chi_{[0,|A|[}(s)\chi_{[0,|B|]}(|\Omega| - s)\, ds = \\
&= \int_0^{|\Omega|} f^*(s)g_*(s)\, ds.
\end{aligned} \tag{2.3}$$

If u and v are any measurable functions, applying Fubini's Theorem and (2.2), we obtain:

$$\begin{aligned}
&\int_0^{|\Omega|} u^*(s)v^*(s)ds = \\
&= \int_0^{|\Omega|} \left(\int_0^{+\infty} \chi_{\{z \geq 0: u^*(z) > r\}}(s)dr \right) \left(\int_0^{+\infty} \chi_{\{z \geq 0: v^*(z) > t\}}(s)dt \right) ds = \\
&= \int_0^{+\infty} dr \int_0^{+\infty} dt \int_0^{|\Omega|} \chi_{\{z \geq 0: u^*(z) > r\}}(s)\chi_{\{z \geq 0: v^*(z) > t\}}(s)ds = \\
&= \int_0^{+\infty} dr \int_0^{+\infty} dt \int_0^{|\Omega|} \chi^*_{\{y \in \Omega: |u(y)| > r\}}(s)\chi^*_{\{y \in \Omega: |v(y)| > t\}}(s)ds \geq \\
&\geq \int_0^{+\infty} dr \int_0^{+\infty} dt \int_0^{|\Omega|} \chi_{\{y \in \Omega: |u(y)| > r\}}(s)\chi_{\{y \in \Omega: |v(y)| > t\}}(s)ds = \\
&= \int_\Omega |u(x)v(x)|dx.
\end{aligned}$$

This completes the proof of the right–hand side inequality of (2.1). Using (2.3) and applying similar argument as above, we get the other inequality and this completes the proof. □

REMARK 2.1. If u is a measurable function defined on a open bounded set Ω, and E is a measurable subset of Ω, applying Theorem 2.1 with $v = \chi_E$ we get

$$\int_E |u(x)|\, dx \leq \int_0^{|E|} u^*(s)ds;$$

nevertheless, it is possible to prove that for any $a \in [0, |\Omega|]$ there exists a set $E_a \subset \Omega$, such that $|E_a| = a$ and

$$\int_{E_a} |u(x)|\, dx = \int_0^a u^*(s)ds.$$

REMARK 2.2. Theorem 2.1 allows to prove easily that the decreasing rearrangement is a contraction mapping from $L^p(\Omega)$ to $L^p(0,|\Omega|)$, that is

$$\|u^* - v^*\|_{L^p(0,|\Omega|)} \leq \|u - v\|_{L^p(\Omega)}\,; \tag{2.4}$$

for example, if $p = 2$, we get:

$$\begin{aligned} \|u^* - v^*\|^2_{L^2(0,|\Omega|)} &= \int_0^{|\Omega|} u^*(s)^2 ds + \int_0^{|\Omega|} v^*(s)^2 ds - 2\int_0^{|\Omega|} u^*(s)v^*(s)ds \leq \\ &\leq \int_\Omega |u(x)|^2 dx + \int_\Omega |v(x)|^2 dx - 2\int_\Omega |u(x)v(x)|dx = \\ &= \|u - v\|^2_{L^2(\Omega)}\,. \end{aligned}$$

Observe that (2.4) implies that the rearrangement is a continuous mapping from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$; indeed, if $\|f_k - f\|_{L^p} \to 0$ as $k \to 0$, also $\|f_k^\# - f^\#\|_{L^p} \to 0$.

The following two technical lemmas will be useful in the next chapters.

LEMMA 2.1. *Let $f_1(s)$, $f_2(s)$ be measurable, nonnegative functions such that*

$$\int_0^r f_1(s)ds \leq \int_0^r f_2(s)ds, \quad \forall r \in [0,\delta].$$

If $\varphi \geq 0$ is a decreasing function then:

$$\int_0^r f_1(s)\varphi(s)ds \leq \int_0^r f_2(s)\varphi(s)ds, \quad \forall r \in [0,\delta].$$

PROOF. The result follows immediately from the following identity:

$$\int_0^r f(s)\varphi(s)ds = -\int_0^r \left(\int_0^t f(s)ds\right) d\varphi(t) + \varphi(r)\int_0^r f(t)dt, \quad r \in [0,\delta].$$

□

LEMMA 2.2. *If f and g belong to $L^1_+(\Omega)$, and*

$$\int_0^r f^*(s)ds \leq \int_0^r g^*(s)ds \quad \forall r \in [0,|\Omega|], \tag{2.5}$$

then

$$\int_\Omega F(f(x))dx \leq \int_\Omega F(g(x))dx,$$

for all convex, non–negative functions F such that $F(0) = 0$, F Lipschitz.

PROOF. It is sufficient to show the result for F convex and C^1. By Lemma 2.1 we get

$$\int_0^{|\Omega|} f^*(s)F'(f^*(s))ds \leq \int_0^{|\Omega|} g^*(s)F'(f^*(s))ds,$$

being $F'(f)^* = F'(f^*)$; hence

$$\int_0^{|\Omega|} F(g^*(s)) - F(f^*(s))ds \geq \int_0^{|\Omega|} F'(f^*(s))(g^*(s) - f^*(s))ds \geq 0,$$

and by Proposition 1.4 we get the result. □

REMARK 2.3. Applying the above Lemma, if $F = |s|^p$, and $f, g \in L^p(\Omega)$, $1 \leq p < +\infty$, verify (2.5), then we get

$$\|f\|_{L^p(\Omega)} \leq \|g\|_{L^p(\Omega)} .$$

The following result states another important rearrangement inequality.

THEOREM 2.2 (Riesz inequality). *If f,g and h are non–negative measurable functions the following inequality holds:*

$$\int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} f(x)g(y)h(x-y)dy \leq \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} f^{\#}(x)g^{\#}(y)h^{\#}(x-y)dy$$

As regard the effect of symmetrization on Sobolev function, we have the following important theorem, which affirms that a function in $W_0^{1,p}(\Omega)$ is in $W_0^{1,p}(\Omega^{\#})$ and the L^p-norm of the gradient decreases under the effect of rearrangement:

THEOREM 2.3 (Pólya–Szegö principle). *If $u \in W^{1,p}(\mathbb{R}^n)$, $1 \leq p < +\infty$, is a non–negative function with compact support, then $u^{\#} \in W^{1,p}(\mathbb{R}^n)$ and*

$$\int_{\mathbb{R}^n} |Du^{\#}|^p dx \leq \int_{\mathbb{R}^n} |Du|^p dx.$$

A first proof of this result, not in its full generality, was given by Pólya and Szegö in [**84**]. Another proof can be found, for example, in [**88**].

As regards the case of equality in the Pólya–Szegö principle, the following theorem holds:

THEOREM 2.4. *Suppose that $u \in W^{1,p}(\mathbb{R}^n)$, $1 < p < +\infty$, is a non–negative function with compact support, and*

$$\left|\{|Du^{\#}| = 0\} \cap (u^{\#})^{-1}(0, \operatorname{ess\,sup} u)\right| = 0.$$

If

$$\int_{\mathbb{R}^n} |Du^{\#}|^p dx = \int_{\mathbb{R}^n} |Du|^p dx$$

then Ω is equivalent to a ball, and $u = u^{\#}$ a.e., up to translations.

The first proof of this theorem for Sobolev functions was given by Brothers and Ziemer in [**35**]. Another proof can be found in [**49**] (see also [**38**]).

We observe that the Brothers and Ziemer result is false when $p = 1$. Indeed, the equality $\int_{\mathbb{R}^n} |Du^{\#}|dx = \int_{\mathbb{R}^n} |Du|dx$ holds for any function whose level sets are balls, but $u \neq u^{\#}$ if the level sets of u are non concentric balls.

3. Lorentz spaces

In this section we introduce some functional spaces, called Lorentz spaces, which are "intermediate" between Lebesgue spaces.

Given a measurable function $u : \Omega \to \mathbb{R}$, we denote with u^{**} the function

$$u^{**}(s) = \frac{1}{s} \int_0^s u^*(\sigma)\, d\sigma.$$

Definition 3.1. A measurable function $u : \Omega \to \mathbb{R}$ belongs to the Lorentz space $L^{(p,q)}(\Omega)$, $1 \leq p, q \leq +\infty$, if the quantity

$$\|u\|_{p,q} = \begin{cases} \left\{ \displaystyle\int_0^{+\infty} \left[t^{1/p} u^{**}(t) \right]^q \frac{dt}{t} \right\}^{1/q}, & 1 \leq q < +\infty, \\ \displaystyle\sup_{0<t<+\infty} t^{1/p} u^{**}(t), & q = +\infty, \end{cases} \tag{3.6}$$

is finite.

The map $\| \cdot \|_{p,q}$ defines a norm, and the space $(L^{(p,q)}(\Omega), \|\cdot\|_{p,q})$ is a rearrangement–invariant Banach space. We observe that $L^{(p,p)} = L^p$, for any $1 \leq p \leq +\infty$, and $\|u\|_{L^p} = \|u\|_{p,p}$. The space $L^{(p,\infty)}(\Omega)$, with $1 \leq p < +\infty$, is also known as the Marcinkiewicz space (or weak L^p), and denoted with $M^p(\Omega)$. Such space contains all the functions u such that

$$\mu_u(t) \leq \frac{1}{t^p}, \quad \forall t > 0.$$

The following estimate establishes a relation between Lorentz spaces.

Proposition 3.1. *If* $1 \leq p \leq +\infty$ *and* $1 \leq q \leq r \leq +\infty$, *then*

$$\|u\|_{p,r} \leq \left(\frac{q}{p} \right)^{1/q - 1/r} \|u\|_{p,q}.$$

In particular, we get the inclusions

$$L^r \subset L^{(p,1)} \subset L^{(p,q)} \subset L^{(p,p)} = L^p \subset L^{(p,r)} \subset L^{(p,\infty)} \subset L^q,$$

for $1 < q < p < r < +\infty$.

More details on Lorentz spaces can be found, for example, in [**65**], or in [**20**].

An useful tool when dealing with Lorentz spaces is the Hardy inequality, that we recall in the following form:

Proposition 3.2 (Hardy). *Suppose* $\lambda > 0$, $1 \leq \gamma < +\infty$. *Let* f *a nonnegative measurable function on* $(0, +\infty)$. *The following inequalities hold:*

$$\int_0^{+\infty} \left(t^{-\lambda} \int_0^t f(s) ds \right)^\gamma \frac{dt}{t} \leq c \int_0^{+\infty} (t^{1-\lambda} f(t))^\gamma \frac{dt}{t} \tag{3.7}$$

and

$$\int_0^{+\infty} \left(t^{\lambda} \int_t^{+\infty} f(s) ds \right)^\gamma \frac{dt}{t} \leq c \int_0^{+\infty} (t^{1+\lambda} f(t))^\gamma \frac{dt}{t}. \tag{3.8}$$

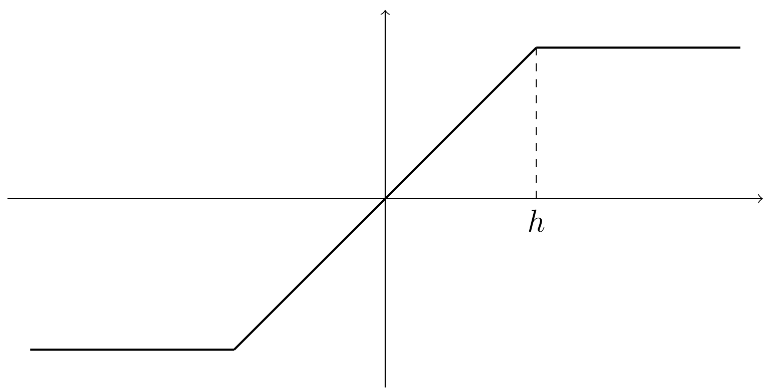
Remark 3.1. By Hardy inequality and the fact that $u^* \leq u^{**}$, it follows easily that the definition 3.1 can be equivalently stated replacing u^{**} with u^* in (3.6). We observe that in this case, if $p < q < +\infty$, (3.6) is not a norm.

Now we want to recall a property of Marcinkiewicz functions which will be useful in the following.

First of all, we give a sense to the gradient Du of a function $u \in L^1(\Omega)$ such that the truncates of u are Sobolev functions (see [**19**]).

Given $h > 0$, we denote with $T_h(s)$ the truncation function at level $\pm h$, defined as

$$T_h(s) = \begin{cases} s & \text{if } |s| \le h, \\ h \,\text{sign}\, s & \text{if } |s| > h. \end{cases}$$



The following result holds (see [**19**]):

Lemma 3.1. *Given a measurable function $u : \Omega \to \mathbb{R}$ such that for every $k > 0$ the truncated function $T_k(u)$ belongs to $W^{1,1}_{loc}(\Omega)$, there exists a unique measurable function $v : \Omega \to \mathbb{R}^n$ such that*

$$DT_k(u) = v\chi_{\{|u|<k\}} \qquad a.e. \tag{3.9}$$

Furthermore, $u \in W^{1,1}_{loc}(\Omega)$ if and only if $v \in L^1_{loc}(\Omega)$, and then $v = Du$ in the usual weak sense.

Therefore, if $u : \Omega \to \mathbb{R}$ is such that for every $k > 0$ the truncated function $T_k(u)$ belongs to $W^{1,1}_{loc}(\Omega)$ we *define* the weak gradient Du of u as the unique function v which verifies (3.9).

The following technical lemma gives a sufficient condition in order that the gradient of a function belongs to a Marcinkiewicz space.

Lemma 3.2 ([**19**]). *Let v be a measurable function belonging to $M^\gamma(\Omega)$ for some $\gamma \ge 1$, such that, for every $k \ge 0$, $T_k(v)$ belongs to $W^{1,p}_0(\Omega)$, $p > 1$. Suppose that*

$$\int_{\{|v|\le k\}} |Dv|^p dx \le c\, k^\lambda, \qquad \forall\, k > k_0, \tag{3.10}$$

for some non–negative λ, c and k_0. Then the weak gradient of v (in the sense of the above definition) is such that $|Dv|$ belongs to $M^q(\Omega)$, with $q = \gamma p/(\gamma + \lambda)$.

PROOF. Let $k > 0$ fixed. For every $t > 0$, we can write

$$|\{|Dv| > k\}| \leq |\{|Dv| > k, |v| \leq t\}| + |\{|v| > t\}|$$

Using (3.10) and the fact that $v \in M^\gamma(\Omega)$, we have, for $t > k_0$,

$$|\{|Dv| > k\}| \leq \frac{1}{k^p} \int_\Omega |DT_t(v)|^p \, dx + |\{|v| > t\}| \leq c \left(\frac{t^\lambda}{k^p} + \frac{1}{t^\gamma} \right). \tag{3.11}$$

For k sufficiently large, a minimization of the right–hand side of (3.11) gives

$$|\{|Dv| > k\}| \leq \frac{c}{t^{\gamma p/(\gamma+\lambda)}}.$$

Observing that, for any value of k, $|\{Dv\} > k| \leq |\Omega|$, we obtain the assertion. $\square$

CHAPTER 2

Monotone operators

1. Definitions and first properties

In this section we give a review of some classical results about boundary value problem for nonlinear elliptic equations. We will refer to equations involving monotone operators, or, more generally, pseudo–monotone operators.

First, we recall some basic definition. In what follows we will denote with V a real reflexive and separable Banach space and with V' its dual space. Let $A\colon V \to V'$ be a mapping (nonlinear, in general).

Definition 1.1. We say that A is *monotone* if

$$(A(u) - A(v), u - v) \geq 0, \quad \text{for any } u, v \in V; \tag{1.1}$$

A will be *strictly monotone* if the inequality in (1.1) is strict whenever $u = v$.

Definition 1.2. We say that A is *hemicontinuous* if for any $u, v \in V$, the mapping

$$t \to (A(u + tv), v)$$

is continuous from $\mathbb{R}$ to $\mathbb{R}$.

Example 1.1. The differential $DJ\colon V \to V'$ of a convex Gâteaux differentiable mapping[1] $J\colon V \to \mathbb{R}$ is monotone. Indeed, let $w = u - v$. The function $j(t) = J(v + tw)$, $t \in \mathbb{R}$, is convex and differentiable (therefore C^1), with $j'(t) = (DJ(v + tw), v)$. Hence j' is non–decreasing, that implies

$$(DJ(v), v) = j'(0) \leq j'(1) = (DJ(u), v).$$

By reversing the roles of u and v, we get the monotonicity of DJ. Moreover, the continuity of j' implies that DJ is also hemicontinuous.

We observe that if A is continuous from V strongly to V' weakly, then A is hemicontinuous; conversely, we have the following surprising result, being the hemicontinuity a very weak condition:

[1]We recall that a $f\colon V \to \mathbb{R}$ is Gâteaux differentiable if there exists a mapping $Df\colon V \to V'$ such that for any $x \in V$

$$\lim_{\varepsilon \to 0} \frac{f(x + \varepsilon y) - f(x)}{\varepsilon} = (Df(x), y), \quad \text{for any } y \in V.$$

Lemma 1.1. *If A be a bounded, hemicontinuous and monotone, then A is continuous from V strongly into V' weakly.*

Proof. Let u_j a sequence such that $u_j \to u$ strongly in V. Being A bounded, $A(u_j)$ is bounded in V'. Hence by reflexivity of V' we can extract a subsequence u_{j_k} such that $A(u_{j_k}) \rightharpoonup \xi$ weakly in V' . By monotonicity, we get that for any $v \in V$,

$$0 \leq (A(u_{j_k}), u_{j_k} - v) - (A(v), u_{j_k} - v).$$

Clearly, $(A(v), u_{j_k} - v)$ converges to $(A(v), u - v)$. Moreover, by weak convergence of $A(u_{j_k})$ in V' and strong convergence of u_{j_k} in V, it follows that

$$(A(u_{j_k}), u_{j_k} - v) \to (\xi, u - v);$$

consequently, we obtain that for any $v \in V$,

$$0 \leq (\xi - A(v), u - v). \tag{1.2}$$

Let $w \in V$, and $t > 0$. Applying (1.2) with $v = u + tw$, we get

$$(\xi - A(u + tw), w) \leq 0.$$

By hemicontinuity of A, it follows that for any $w \in V$, $(\xi - A(u), w) \leq 0$, that means

$$(\xi - A(u), w) = 0, \quad \forall w \in V.$$

Hence $\xi = A(u)$, and this completes the proof. □

2. Existence results for monotone operators: the Browder and Minty method

Here we prove a general existence result, which method of proof is due to Browder and Minty.

First, let us recall that an operator A is said to be *coercive* if

$$\lim_{\|v\| \to +\infty} \frac{(A(v), v)}{\|v\|} = +\infty.$$

Theorem 2.1 (Minty–Browder). *Let V a reflexive and separable Banach space. Let $A : V \to V'$ a bounded, hemicontinuous, monotone and coercive mapping. Then A is surjective, i.e. for any $f \in V'$ there exists a solution $u \in V$ of the equation*

$$A(u) = f. \tag{2.1}$$

To prove Theorem 2.1, it is fundamental the following technical lemma, which is a variant of the Brouwer fixed point Theorem:

Lemma 2.1 (Zeros of a vector field). *Assume the continuous function $P \colon \mathbb{R}^m \to \mathbb{R}^m$ satisfies*

$$P(x) \cdot x \geq 0, \quad \textit{if } |x| = r,$$

for some $r > 0$.[2] *Then there exists a point* $x \in B_r(0)$ *such that*

$$P(x) = 0.$$

PROOF OF THEOREM 2.1. We intend to build a solution of the equation (2.1) by first constructing solutions of certain finite–dimensional approximations to (2.1) and then passing to the limit[3]. Let $w_1, \dots, w_m, \dots$ be a basis of V; for each $m \in \mathbb{N}$, there exists $u_m \in \operatorname{span}\{w_1, \dots, w_m\}$ such that

$$(A(u_m), w_m) = (f, w_j), \quad 1 \le j \le m. \tag{2.2}$$

Indeed, first observe that

$$(A(u_m), u_m) - (f, u_m) \ge (A(u_m), u_m) - c\|u_m\|;$$

the coercivity implies that for $\|u_m\|$ sufficiently large, $(A(u_m), u_m) - c\|u_m\| \ge 0$. On the other hand, the function $v \to (A(v), v)$ is continuous on $\operatorname{span}\{w_1, \dots, w_m\}$, by Lemma 1.1. Applying Lemma 2.1 with $P(\eta) = (P_1(\eta), \dots, P_m(\eta))$ such that $P_j(\eta) = \left(A\left(\sum_{i=1}^m \eta_i w_i\right), w_j\right) - (f, w_j)$, $1 \le j \le m$, there exists $u_m \in \operatorname{span}\{w_1, \dots, w_m\}$ that solves (2.2). Hence, by (2.2) we get

$$(A(u_m), u_m) = (f, u_m) \le \|f\|_{V'} \|u_m\|;$$

by coercivity, and being A bounded, it follows that

$$\|u_m\| \le C, \quad \|A(u_m)\|_{V'} \le C.$$

Hence, up to subsequences,

$$\begin{aligned} u_m &\rightharpoonup u \text{ weakly in } V, \\ A(u_m) &\rightharpoonup \xi \text{ weakly in } V'. \end{aligned} \tag{2.3}$$

Passing to the limit into (2.2), we get that for any $1 \le j \le m$

$$(\xi, w_j) = (f, w_j),$$

that implies

$$\xi = f. \tag{2.4}$$

Moreover, by (2.2) we obtain $(A(u_m), u_m) = (f, u_m) \to (f, u)$, and by (2.4) we get

$$(A(u_m), u_m) \to (\xi, u). \tag{2.5}$$

Hence the Theorem is proved if we show that

$$\xi = A(u). \tag{2.6}$$

[2] Here $x \cdot y = \sum_1^n x_i y_i$ denotes the scalar product of two vectors $x, y \in \mathbb{R}^n$.
[3] This is the so–called *Galerkin's method*.

Starting from

$$0 \leq (A(u_m) - A(v), u_m - v), \quad \forall v \in V, \tag{2.7}$$

by (2.3) and (2.5) we can pass to the limit in (2.7) in order to obtain

$$0 \leq (\xi - A(v), u - v), \quad \forall v \in V. \tag{2.8}$$

Then we can reason exactly as in the proof of Lemma 1.1 in order to get (2.6). □

REMARK 2.1. We observe that under the hypotheses of Theorem 2.1, if the mapping A is strictly monotone, the solution of (2.1) is unique.

3. An example of monotone operator

DEFINITION 3.1. Let $\Omega \subset \mathbb{R}^n$ a bounded open set, and $d \in \mathbb{N}$. We say that a function

$$(x, \xi) \in \Omega \times \mathbb{R}^d \mapsto f(x, \xi) \in \mathbb{R}$$

is a *Carathéodory* function if

$$f(x, \cdot) \text{ is continuous for a.e. } x \in \Omega,$$

and

$$f(\cdot, \xi) \text{ is measurable for every } \xi \in \mathbb{R}^d.$$

Given f a Carathï¿½odory function, and $u \colon \Omega \to \mathbb{R}^d$, the mapping

$$A(u)(x) := f(x, u(x)) \tag{3.9}$$

is called the *Nemytskii* operator of f.

For such operators the following important result holds (see [**69**]):

THEOREM 3.1. *Let $f \colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a Carathéodory function such that*

$$|f(x, \xi)| \leq g(x) + C\,|\xi|^{p-1},$$

with $1 < p < +\infty$ and $g \in L^{p'}(\Omega)$, where p' is such that $1/p + 1/p' = 1$. Then the Nemytskii operator defined by (3.9) *is a bounded and continuous mapping from $L^p(\Omega)$ to $L^{p'}(\Omega)$.*

EXAMPLE 3.1. Let $\Omega \subset \mathbb{R}^n$ an open bounded set, $1 < p < +\infty$ and $V = W_0^{1,p}(\Omega)$. Suppose that $F \colon \mathbb{R}^n \to \mathbb{R}^n$ is a continuous monotone mapping, in the sense that for any $\xi, \eta \in \mathbb{R}^n$,

$$(F(\xi) - F(\eta)) \cdot (\xi - \eta) \geq 0, \tag{3.10}$$

and F satisfies the growth condition

$$|F(\xi)| \leq C(1 + |\xi|^{p-1}), \tag{3.11}$$

for any ξ in $\mathbb{R}^n$ and for some constant C. Then, the mapping

$$u \in W_0^{1,p}(\Omega) \mapsto A(u) = -\operatorname{div}(F(Du)) \in W^{-1,p'}(\Omega)$$

is bounded, hemicontinuous and monotone. Indeed, if $u \in W_0^{1,p}(\Omega)$, then $F(Du) \in L^{p'}(\Omega)$,

$$|F(Du)|^{p'} \le C(1 + |Du|^{p-1})^{p/(p-1)} \le C'(1 + |Du|^p) \in L^1(\Omega);$$

hence $-\operatorname{div}(F(Du)) \in W^{-1,p'}(\Omega)$ with the duality

$$(-\operatorname{div}(F(Du)), v) = \int_\Omega F(Du) \cdot Dv\, dx,$$

for any $v \in W_0^{1,p}(\Omega)$. The same computation as above gives that $F(Du)$ is bounded in $L^{p'}(\Omega;\mathbb{R}^n)$, therefore $A(u)$ is bounded in $W^{-1,p'}(\Omega)$. By theorem 3.1, the mapping $z \mapsto F(z)$ is continuous from $L^p(\Omega;\mathbb{R}^n)$ strongly to $L^{p'}(\Omega;\mathbb{R}^n)$ strongly, hence we deduce that A is continuous from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$, and *a fortiori* hemicontinuous. Finally, for any $u, v \in W_0^{1,p}(\Omega)$, we have

$$(A(u) - A(v), u - v) = \int_\Omega (F(Du) - F(Dv)) \cdot (u - v)\, dx \ge 0.$$

Moreover, if we add the hypothesis that

$$F(\xi) \cdot \xi \ge \alpha\, |\xi|^p\,, \quad \forall \xi \in \mathbb{R}^n, \tag{3.12}$$

then A is also coercive, being

$$\frac{(A(u), u)}{\|u\|} = \frac{\int_\Omega F(Du) \cdot Du\, dx}{\||Du|\|_{L^p(\Omega)}} \ge \||Du|\|_{L^p(\Omega)}^{p-1} \to \infty,$$

as $\||Du|\|_{L^p(\Omega)} \to +\infty$. Applying Theorem 2.1, we get that if $f \in W^{-1,p'}(\Omega)$, there exists a solution $u \in W_0^{1,p}(\Omega)$ of the problem

$$\begin{cases} -\operatorname{div}(F(Du)) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in the sense that

$$\int_\Omega F(Du) \cdot D\varphi\, dx = (f, \varphi) \quad \text{for any } \varphi \in W_0^{1,p}(\Omega).$$

An example of function verifying (3.10), (3.11) and (3.12) is given by $F(\xi) = |\xi|^{p-2}\xi$, $1 < p < +\infty$; then $A(u) = -\Delta_p u$, where $\Delta_p u = \operatorname{div}(|Du|^{p-2}\, Du)$ is the so–called p–laplace operator.

4. Pseudo–monotone operators

In this section we want to examine a more general class of nonlinear mappings, namely the *pseudo–monotone* operators. In applications, it often occurs that the hypotheses made on the operator A in Theorem 2.1 are unnecessarily strong. In particular, the monotonicity assumption on A involves both the first–order derivatives and the function itself. Actually, it is necessary only to have a monotonicity assumption on the first–order derivatives.

Definition 4.1. We say that the mapping $A\colon V \to V'$ is said to be of *M–type* if

$$\left.\begin{array}{c} u_j \rightharpoonup u \text{ weakly in } V, \\ A(u_j) \rightharpoonup \xi \text{ weakly in } V', \\ \limsup_{j\to+\infty}(A(u_j), u_j) \le (\xi, u), \end{array}\right\} \Rightarrow \xi = A(u).$$

Remark 4.1. Such definition is suggested by the fact that, looking at the proof of Theorem 2.1, if we substitute the monotonicity assumption on A with the assumption of being of M–type, the Theorem continues to hold. Indeed, it is not difficult to show that A is still continuous from V strongly into V' weakly, and the monotonicity used at the end of the proof is replaced by the fact that (2.6) follows, by definition, from (2.5).

So the M–type assumption allows to avoid the monotonicity condition, but it is not easy to test that such assumption holds; to avoid this problem we introduce the class of *pseudo–monotone* operators. This class of mappings is intermediate between the class of monotone operators and the one of M–type.

Definition 4.2. A mapping $A : V \to V'$ is called *pseudo–monotone* if whenever $u_j \rightharpoonup u$ weakly in V and $\limsup_{j\to+\infty}(A(u_j), u_j - u) \le 0$, it follows that

$$\liminf_{j\to+\infty}(A(u_j), u_j - v) \ge (A(u) - v) \text{ for all } v \in V.$$

Remark 4.2. We observe that a pseudo–monotone operator is continuous from V strongly to V' weakly; indeed, suppose that there exists $u_j \to u$ strongly such that $A(u_j)$ does not converges weakly to $A(u)$ in V'. Then, up to a subsequence, $A(u_j) \rightharpoonup f$ in V' weakly, with $f \ne A(u)$. Then

$$\limsup_{j\to+\infty}(A(u_j), u_j - u) = 0,$$

hence, for any $v \in V$,

$$\liminf_{j\to+\infty}(A(u_j), u_j - v) = (f, u - v) \ge (A(u), u - v),$$

and then $f = A(u)$; this is a contradiction.

The class of pseudo–monotone operators is an “intermediate” class:

Proposition 4.1. *A bounded, hemicontinuous and monotone operator is pseudo–monotone, and a pseudo–monotone operator is of M–type.*

Finally, combining Proposition 4.1 with Remark 4.1, we get the following existence result.

Theorem 4.1. *Let $A\colon V \to V'$ a coercive pseudo–monotone mapping. Then, for any $f \in V'$, the equation $A(u) = f$ admits at least a solution.*

DEFINITION 4.3. A mapping $A\colon V \to V'$ is said to be of the *calculus of variations type* if it is bounded and it admits the representation

$$A(u) = \mathcal{A}(u, u),$$

where the mapping

$$(u, v) \in V \times V \mapsto \mathcal{A}(u, v) \in V'$$

has the following properties:

$$\begin{cases} \forall u \in V, \text{ the mapping } v \mapsto \mathcal{A}(u, v) \text{ is bounded and hemicontinuous} \\ \text{from } V \text{ to } V', \text{ and } (\mathcal{A}(u, u) - \mathcal{A}(u, v), u - v) \geq 0; \end{cases} \tag{4.13}$$

$$\begin{cases} \forall v \in V, \text{ the mapping } u \mapsto \mathcal{A}(u, v) \text{ is bounded} \\ \text{and hemicontinuous from } V \text{ to } V'; \end{cases} \tag{4.14}$$

$$\left.\begin{matrix} u_j \rightharpoonup u \text{ weakly in } V, \\ (\mathcal{A}(u_j, u_j) - \mathcal{A}(u_j, u), u_j - u) \to 0, \end{matrix}\right\} \Rightarrow \begin{cases} \forall v \in V, \\ \mathcal{A}(u_j, v) \rightharpoonup \mathcal{A}(u, v) \text{ weakly in } V'; \end{cases} \tag{4.15}$$

$$\left.\begin{matrix} u_j \rightharpoonup u \text{ weakly in } V, \\ \mathcal{A}(u_j, v) \rightharpoonup \Psi \text{ weakly in } V', \end{matrix}\right\} \Rightarrow (\mathcal{A}(u_j, v), u_j) \to (\Psi, u). \tag{4.16}$$

PROPOSITION 4.2. *An operator of calculus of variations type is pseudo–monotone.*

As before, by the above Proposition, the Proposition 4.1 and Remark 4.1 we get the following result.

THEOREM 4.2. *Let $A : V \to V'$ be an operator of calculus of variations type. Then, for any $f \in V'$, the equation $A(u) = f$ admits at least a solution.*

5. Leray–Lions operators

Let Ω be a bounded open set in $\mathbb{R}^n$. Given $f \in W^{-1,p'}(\Omega)$, we consider Dirichlet problems of the form

$$\begin{cases} -\operatorname{div}(a(x, u, Du)) + H(x, u, Du) = f, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.17}$$

where the mappings $a(x, s, \xi)\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $H(x, s, \xi)\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ have the following properties:

(i) a and H are Carathéodory functions, i.e. they are measurable with respect to $x \in \Omega$, and continuous with respect to $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$;

(ii) there exist $g \in L^{p'}(\Omega)$ and a constant C such that

$$|a(x, s, \xi)| \leq g(x) + C(|s|^{p-1} + |\xi|^{p-1}), \tag{ii_1}$$

$$|H(x, s, \xi)| \leq g(x) + C(|s|^{p-1} + |\xi|^{p-1}), \tag{ii_2}$$

with $1 < p < +\infty$, for a.e. $x \in \Omega$, and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$;

(iii) if $(u, w) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega) \mapsto B(u, w) \in \mathbb{R}$ is the form defined as

$$B(u, w) = \int_\Omega a(x, u, Du) \cdot Dw \, dx + \int_\Omega H(x, u, Du) w \, dx,$$

then

$$\frac{B(v, v)}{\|v\|_{W_0^{1,p}}} \longrightarrow +\infty \quad \text{as} \quad \|v\|_{W_0^{1,p}} \longrightarrow +\infty;$$

(iv) for a.e. $x \in \Omega$ and uniformly for s on bounded sets, we have

$$\frac{a(x, s, \xi) \cdot \xi}{|\xi| + |\xi|^{p-1}} \longrightarrow +\infty \quad \text{as} \quad |\xi| \longrightarrow +\infty;$$

(v) the function a is strictly monotone with respect to ξ, that is

$$(a(x, s, \xi) - a(x, s, \xi')) \cdot (\xi - \xi') > 0,$$

for a.e. $x \in \Omega$, $s \in \mathbb{R}$ and for all $\xi, \xi' \in \mathbb{R}^n$ such that $\xi \neq \xi'$.

Observe that the form $w \mapsto B(u, w)$ is linear and continuous on $W_0^{1,p}(\Omega)$; hence we can write

$$B(u, w) = (A(u), w), \quad A(u) \in W_0^{-1,p'}(\Omega),$$

where $A(u)$, $u \in \mathcal{D}(\Omega)$, is given by

$$A(u) = -\operatorname{div}(a(x, u, Du)) + H(x, u, Du). \tag{5.18}$$

THEOREM 5.1 (Leray & Lions, [**73**]). *Let $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ be the operator defined as in* (5.18)*, where $a(x, s, \xi)$ and $H(x, s, \xi)$ verify $(i) - (v)$. If $f \in W^{-1,p'}(\Omega)$, then there exists $u \in W_0^{1,p}(\Omega)$ such that*

$$A(u) = f.$$

To prove Leray–Lions theorem, we need some preliminary lemmas.

First, we recall the following result:

LEMMA 5.1. *Let h_j a bounded sequence in $L^q(\Omega)$, $1 < q < +\infty$ such that $h_j \to h$ a.e; then $h_j \rightharpoonup h$ weakly in $L^q(\Omega)$.*

Next result is crucial in what follows.

LEMMA 5.2. *Suppose that (i), (ii), (iv) and (v) hold. Let*

$$u_j, u \in W_0^{1,p}(\Omega) \text{ such that } u_j \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega). \tag{5.19}$$

Put

$$F_j = (a(x, u_j, Du_j) - a(x, u, Du)) \cdot (Du_j - Du),$$

and suppose that

$$\int_\Omega F_j(x) dx \to 0.$$

Then, up to a subsequence,

$$Du_j \to Du \qquad a.e.\ in\ \Omega,$$

and

$$H(x, u_j, Du_j) \rightharpoonup H(x, u, Du) \quad weakly\ in\ L^{p'}(\Omega).$$

PROOF. By the monotonicity assumption, $F_j \geq 0$; moreover, being $u_j \to u$ in $L^p(\Omega)$, by (5.19), up to a subsequence we have that

$$u_j(x) \to u(x),\ F_j(x) \to 0 \text{ in } \Omega \setminus \Omega_0, \tag{5.20}$$

with $|\Omega_0| = 0$. Fix $x \notin \Omega_0$ such that $g(x) < +\infty$, and let ξ^* be a limit of $Du_j(x)$. We prove that $|\xi^*|$ is finite. By (*ii*) we have:

$$F_j(x) \geq a(x, u_j(x), Du_j(x)) \cdot Du_j(x) - c(|Du_j(x)|^{p-1} + |Du_j(x)| + 1),$$

where the constant c involves the all the terms which does not depend on j; hence (*iv*) implies that if $|\xi^*| = +\infty$, then $F_j(x) \to +\infty$, and this contradict the fact that $F_j(x) \to 0$. Therefore, $|\xi^*| < +\infty$. Moreover, (5.20) together with the continuity of $a(x, s, \xi)$ with respect to (s, ξ) gives

$$(a(x, u(x), \xi^*) - a(x, u(x), Du(x))) \cdot (\xi^* - Du(x)) = 0,$$

that implies

$$Du(x) = \xi^*$$

by the strict monotonicity. Being this limit independent of the extracted sequence, $Du_j(x)$ converges to $Du(x)$. So we have

$$H(x, u_j, Du_j) \to H(x, u, Du) \qquad \text{a.e. in } \Omega;$$

being $H(x, u_j, Du_j)$ bounded in $L^{p'}(\Omega)$, by Lemma 5.1 we get that

$$H(x, u_j, Du_j) \rightharpoonup H(x, u, Du) \quad \text{weakly in } L^{p'}(\Omega). \tag{5.21}$$

□

PROOF OF THEOREM 5.1. We will prove that the operator

$$A(u) = -\operatorname{div}(a(x, u, Du)) + H(x, u, Du)$$

is of calculus of variations type. Then the result follows from Theorem 4.2.

First, we introduce the operator $A(u, v)$. Put

$$A_1(u, v, w) = \int_\Omega a(x, u, Dv) \cdot Dw\, dx,$$

$$A_2(u, w) = \int_\Omega H(x, u, Du) w\, dx.$$

The form $w \mapsto B_1(u,v,w) + B_2(u,w)$ is continuous on $W_0^{1,p}(\Omega)$, hence

$$A_1(u,v,w) + A_2(u,w) = \tilde{A}(u,v,w) = (A(u,v),w), \quad A(u,v) \in W^{-1,p'}(\Omega),$$

so we have

$$A(u,u) = A(u).$$

Proof of (4.13), (4.14). By *(v)*, we have

$$(A(u,u) - A(u,v), u-v) = (A_1(u,u,u-v) - A_1(u,v,u-v)) \geq 0;$$

moreover the mapping $v \mapsto A(u,v)$ is bounded and hemicontinuous from V to V'; indeed for $u, v_1, v_2 \in W_0^{1,p}(\Omega)$ we have, for $\lambda \to 0$,

$$\begin{aligned} &a(x,u,D(v_1+\lambda v_2)) \rightharpoonup a(x,u,Dv_1) \quad \text{weakly in } L^{p'}(\Omega), \\ &H(x,u,D(v_1+\lambda v_2)) \rightharpoonup H(x,u,Dv_1) \quad \text{weakly in } L^{p'}(\Omega), \end{aligned}$$

hence for any $w \in W_0^{1,p}(\Omega)$ we have

$$\tilde{A}(x, v_1+\lambda v_2, w) \to A(x,v_1,w) \quad \text{ if } \lambda \to 0,$$

and this proves (4.13). We can reason analogously to prove (4.14).

Proof of (4.15). Using the notation of Lemma 5.2, we get

$$(A(u_j,u_j) - A(u_j,u), u_j - u) = \int_\Omega F(x)dx;$$

then if $u_j \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ and $(A(u_j,u_j) - A(u_j,u), u_j-u) \to 0$, by Lemma 5.2 we get $H(x,u_j,Du_j) \rightharpoonup H(x,u,Du)$ weakly in $L^{p'}$; moreover, being

$$a(x,u_j,Dv) \rightharpoonup a(x,u,Dv) \quad \text{weakly in } L^{p'}(\Omega)$$

we have

$$\tilde{A}(u_j,v,w) \to \tilde{A}(u,v,w) \quad \text{for any } w \in W_0^{1,p}(\Omega);$$

hence $A(u_j,v) \to A(u,v)$ weakly in $W^{-1,p'}(\Omega)$.

Proof of (4.16). Let $u_j \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ and $A(u_j,v) \rightharpoonup \Psi$ weakly in $W^{-1,p'}(\Omega)$. So $u_j \to u$ strongly in L^p, hence by Carathéodory theorem

$$a(x,u_j,Dv) \to a(x,u,Dv) \text{ strongly in } L^{p'};$$

hence

$$A_1(u_j,v,u_j) \to A_1(u,v,u).$$

Moreover, being

$$|A_2(u_j,u_j-u)| \leq c\,\|u_j-u\|_{L^p}$$

it follows that

$$A_2(u_j,u_j-u) \to 0. \tag{5.22}$$

But

$$A_2(u_j, u) = (A(u_j, v), u) - A_1(u_j, v, u) \to (\Psi, u) - A_1(u, v, u),$$

hence by 5.22 we get

$$A_2(u_j, u_j) \to (\psi, u) - A_1(u, v, u)$$

and finally

$$(A(u_j, v), u_j) = A_1(u_j, v, u_j) + A_2(u_j, u_, j) \to (\Psi, u).$$

□

REMARK 5.1. A typical condition in order that (iv) holds is

$$(iv') \qquad a(x, s, \xi) \cdot \xi \geq \alpha\, |\xi|^p\,,$$

for a.e. $x \in \Omega$ and for any $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, with $\alpha > 0$. An operator

$$u \mapsto -\operatorname{div}(a(x, u, Du))$$

where $a(x, s, \xi)$ is a Carathéodory function verifying (ii_1), (iv'), (v), is called a *Leray–Lions* operator; by Theorem 5.1 the equation

$$-\operatorname{div}(a(x, u, Du)) = f, \quad u \in W_0^{1,p}(\Omega)$$

has a solution for any $f \in W^{-1,p'}(\Omega)$. Clearly, the operator in the Example 3.1 is of Leray–Lions type.

REMARK 5.2. We observe that the condition (iv') implies that $a(x, s, 0) = 0$. Indeed $a(x, s, t\xi) \cdot \xi > 0$ if $t > 0$, and $a(x, s, t\xi) \cdot \xi < 0$ if $t < 0$. Being $a(x, s, \xi)$ a Carathéodory function,

$$a(x, s, 0) = \lim_{t \to 0} a(x, s, t\xi) \cdot \xi = 0.$$

EXAMPLE 5.1. Let $u \mapsto -\operatorname{div}(a(x, u, Du))$ be a Leray–Lions operator, and suppose that $H(x, s, \xi)$ is a Carathéodory function such that (ii_2) holds. If we consider the Dirichlet problem

$$-\operatorname{div}(a(x, u, Du)) + H(x, u, Du) = f, \quad u \in W_0^{1,p}(\Omega)$$

with $f \in W^{-1,p'}(\Omega)$, we can apply Theorem 5.1 if, for example, $H(x, s, \xi)$ is bounded, or if it satisfies a sign condition.

CHAPTER 3

Nonlinear elliptic problems with lower–order terms

1. Comparison results

In this chapter we deal with Dirichlet problems for nonlinear elliptic equations, whose prototype can be written in the form

$$
\begin{cases} -\Delta_p u + h(x)|Du|^{p-1} = c(x)|u|^{p-2}u + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}
$$

where Ω is a bounded open set in $\mathbb{R}^n$, Δ_p is the p–laplacian operator, $1 < p < +\infty$, $\|h\|_{L^\infty(\Omega)} \leq \beta$ and c and f satisfy suitable conditions.

Our aim is to give some a priori estimates for solutions of equations like (1.1). More precisely, we want to establish a comparison, in some sense, between a solution of a given problem with the solution of a "symmetrized" one, whose data are spherically symmetric. Such comparison allows us to obtain sharp estimates for the solutions of (1.1).

The first result in this order of ideas is due to Weinberger (see [**98**]), who proved that if u is a solution of the problem

$$
-\left(a_{ij}(x)u_{x_i}\right)_{x_j} = f, \quad u \in H_0^1(\Omega),
$$

where $\left(a_{ij}(x)u_{x_i}\right)_{x_j}$ are bounded measurable coefficients such that

$$
a_{ij}(x)\xi_i\xi_j \geq \nu\left|\xi\right|^2, \qquad \text{for a.e. } x \in \Omega, \forall \xi \in \mathbb{R}^n, \nu > 0,
$$

and $f \in L^p(\Omega)$, $p > n/2$, then the following estimate holds:

$$
\|u\|_{L^\infty(\Omega)} \leq K\,\|f\|_{L^p(\Omega)}\,, \tag{1.2}
$$

where K is the best possible constant. The first general result is due to Talenti, in his pioneering work [**89**]. He proved the following theorem:

Theorem 1.1 (Talenti). *Let $u(x)$ be a solution of the problem*

$$
-\left(a_{ij}(x)u_{x_i}\right)_{x_j} + c(x)u = f, \quad u \in H_0^1(\Omega) \tag{1.3}
$$

where $a_{ij}(x)$ are bounded measurable coefficients such that

$$
a_{ij}(x)\xi_i\xi_j \geq \nu\left|\xi\right|^2, \quad \textit{for a.e. } x \in \Omega, \forall \xi \in \mathbb{R}^n, \nu > 0,
$$

$c(x) \geq 0$, $f \in L^r(\Omega)$, $r = 2n/(n+2)$ *if* $n > 2$, $r > 1$ *if* $n = 2$. *If* v *is the solution of the problem*

$$-\Delta v = f^{\#}, \quad v \in H_0^1(\Omega^{\#}), \tag{1.4}$$

then

$$u^{\#}(x) \leq v(x) \quad \forall x \in \Omega^{\#}, \tag{1.5}$$

and

$$\int_{\Omega} (a_{ij}(x) u_{x_i} u_{x_j})^{q/2} dx \leq \int_{\Omega^{\#}} |Dv|^q \, dx,$$

for $0 < q \leq 2$.

Remark 1.1. We observe that Talenti's result immediately gives estimates of the type

$$\|u\| \leq C \, \|f\| \, ;$$

to this aim, let us observe that the solution of (1.4) can be explicitly written as

$$v(x) = \frac{1}{n^2 \omega_n^{2/n}} \int_{\omega_n |x|^n}^{|\Omega|} r^{-2+2/n} dr \int_0^r f^*(s) ds; \tag{1.6}$$

hence,

$$\operatorname{ess\,sup} |u| = u^{\#}(0) \leq v^{\#}(0) = \frac{1}{n^2 \omega_n^{2/n}} \int_0^{|\Omega|} s^{-2+2/n} ds \int_0^s f^*(\sigma) d\sigma,$$

that gives an optimal estimate in the Lorentz space $L^{(n/2,1)}(\Omega)$, or, if $n > 3$,

$$\begin{aligned}\operatorname{ess\,sup} |u| = u^{\#}(0) \leq v^{\#}(0) &= \frac{1}{n^2 \omega_n^{2/n}} \int_0^{|\Omega|} \frac{s^{-1+2/n} - |\Omega|^{-1+2/n}}{n(n-2)\omega_n^{2/n}} f^*(s) ds \leq \\ &\leq \frac{|\Omega|^{2/n - 1/r}}{n(n-2)\omega_n^{2/n}} \left(\frac{n(r-1)}{2r - n} \right)^{1/p'} \|f\|_{L^p(\Omega)} \, .\end{aligned}$$

In Talenti's theorem, the influence of zero–order term $c(x)u$ is disregarded. This is due to the sign condition on c, which allows to get rid of it.

Nevertheless, it is possible to obtain comparison results with problems which "remind" the zero–order term.

For example, under the hypotheses of Theorem 1.1, if u is a solution of (1.3), and v is the solution of the problem

$$-\Delta v + c_{\#} v = f^{\#}, \quad v \in H_0^1(\Omega^{\#}),$$

we have

$$\int_0^{|\Omega|} u^*(s) ds \leq \int_0^{|\Omega|} v^*(s) ds \tag{1.7}$$

(see [**7**], [**36**], [**77**]). Actually, we loose pointwise comparison; indeed it is possible to show that, in general, (1.5) does not hold (see [**36**] for a counterexample). Nevertheless, we have an integral comparison. The quantities involved in (1.7) are sometimes called the concentrations of u and v.

In the linear case, a very general result, with no sign hypothesis on c and with additional lower–order terms, was given by Alvino, Lions and Trombetti in [**7**]:

THEOREM 1.2 (Alvino, Lions & Trombetti). *Let*

$$-\left(a_{ij}(x)u_{x_i}\right)_{x_j}+(b_i(x)u)_{x_i}+d(x)\cdot Du+c(x)u=f(x), \tag{1.8}$$

where $a_{ij}(x)$ are measurable functions in Ω, and

$$a_{ij}(x)\xi_i\xi_j\geq\nu\left|\xi\right|^2,\qquad \textit{for a.e. } x\in\Omega,\forall\xi\in\mathbb{R}^n,\nu>0,$$

$b(x)=(b_1(x),\ldots,b_n(x))$ and $d(x)=(d_1(x),\ldots,d_n(x))$ are such that

$$\sum_{i=1}^n\left|b_i(x)+d_i(x)\right|^2\leq R^2,$$
$$\sum_{i=1}^n(b_i(x))_{x_i}+c(x)\geq c_0(x)\quad \textit{in } \mathcal{D}'(\Omega),$$

where R is a constant and $c_0\in L^\infty(\Omega)$. If $v(x)=v(|x|)$ is a weak solution of the problem

$$-\nu\Delta u+R\frac{x}{|x|}\cdot Dv+(c_0^+)_\#(x)v-(c_0^-)^\#(x)v=f^\#(x)\quad \textit{in } \Omega^\#,\ v\in H_0^1(\Omega^\#), \tag{1.9}$$

where c_0^+ and c_0^- are respectively the positive and negative part of c_0, then the problem (1.8) has a solution $u\in H_0^1(\Omega)$ and we have that:

(1) *if $c_0\geq 0$ and $c_0\not\equiv 0$, then*

$$u^*(s)\leq v^*(s) \tag{1.10}$$

for any $s\in[0,s_1]$, where $s_1=\sup\{s\colon (c_0)_*(s)=0\}$, and

$$\int_\Omega \exp\left(-\frac{R}{\nu\omega_n^{1/n}}\sigma^{1/n}\right)u^*(\sigma)\,d\sigma\leq\int_\Omega \exp\left(-\frac{R}{\nu\omega_n^{1/n}}\sigma^{1/n}\right)v^*(\sigma)\,d\sigma \tag{1.11}$$

holds for any $s\in[s_1,|\Omega|]$;

(2) *if $c_0\leq 0$, then (1.10) holds for any $s\in[0,|\Omega|]$;*

(3) *if $c_0^+,c_0^-\neq 0$, then (1.10) holds in $[0,s_2]$, and (1.11) holds in $]s_2,|\Omega|]$, where $s_2=\inf\{s\colon (c_0^+)_*(s)>0\}$.*

The nonlinear case, first studied by Talenti in [**90**], has been treated by several authors with different conditions on the lower order terms, and recently studied in a series of papers by Ferone and Messano (see [**50**], [**51**], [**81**]).

The main aim is to prove a comparison result which allows to estimate a solution of (1.1) with the solution of the following problem:

$$\begin{cases} -\Delta_p v + \beta |Dv|^{p-2} Dv \cdot \frac{x}{|x|} = \hat{c}(x)|v|^{p-2}v + f^{\#} & \text{in } \Omega^{\#} \\ v = 0 & \text{on } \partial\Omega^{\#} \end{cases} \tag{1.12}$$

where $\hat{c}$ denotes the function $\hat{c}(x) = (c^+)^{\#}(x) - (c^-)_{\#}(x)$.

If c and f are bounded, it is possible to prove a comparison result if the positive part of the function $c(x)$ is small enough (see [**51**], [**81**]).

More precisely, if one defines the eigenvalue problem:

$$\begin{cases} -\Delta_p^{\beta} \psi = \lambda e^{-\beta|x|} |\psi|^{p-2}\psi & \text{in } \Omega^{\#} \\ \psi = 0 & \text{on } \partial\Omega^{\#}, \end{cases} \tag{1.13}$$

where Δ_p^{β} is the operator $\Delta_p^{\beta} u = \operatorname{div}(e^{-\beta|x|}|Du|^{p-2}Du)$, it has been proved that if c^+ is smaller than the first eigenvalue of (1.13), then

$$u^*(s) \le v^*(s), \quad \forall s \in [0, s_0], \tag{1.14}$$

and

$$\int_0^s u^*(t) \exp\left(-\frac{\beta t^{1/n}}{\omega_n^{1/n}}\right) dt \le \int_0^s v^*(t) \exp\left(-\frac{\beta t^{1/n}}{\omega_n^{1/n}}\right) dt, \quad \forall s \in [s_0, |\Omega|], \tag{1.15}$$

where $s_0 = \inf\{s \in [0, |\Omega|] : (c^-)_*(s) > 0\}$.

We are interested in studying what happens in the more general case when $c \in L^r(\Omega)$, with $r > \frac{n}{p}$, and $f \in L^q(\Omega)$, with $q > \frac{n}{p}$ (see [**42**]).

2. Some basic tools

Now we recall some basic definitions and properties which will be fundamental tools in what follows, namely the isoperimetric property of the sphere and the coarea formula for Sobolev mappings.

Definition 2.1. Let Ω an open subset of $\mathbb{R}^n$ and $E \subset \Omega$ measurable. We define the *perimeter* of E in Ω as

$$P_\Omega(E) = \sup\left\{ \int_E \operatorname{div} \varphi \, dx : \varphi \in C_0^\infty(\Omega, \mathbb{R}^n), \ \||\varphi|\|_{L^\infty(\Omega)} \le 1 \right\}. \tag{2.16}$$

We will denote $P_{\mathbb{R}^n}(E)$ with $P(E)$.

The definition for $\Omega = \mathbb{R}^n$ coincides with the one given by De Giorgi in [**39**].

Theorem 2.1 (Isoperimetric inequality). *If E is a measurable set of $\mathbb{R}^n$, one of the following inequalities holds:*

$$P(E) \ge n\omega_n^{1/n} |E|^{1-1/n}$$

or

$$P(E) \ge n\omega_n^{1/n} |\mathbb{R}^n \setminus E|^{1-1/n}.$$

A proof without particular restrictions of the isoperimetric inequality was given by De Giorgi in [**39**].

The following result is a very useful tool in analysis.

THEOREM 2.2 (Fleming & Rishel [**56**]). *Let $f \in W^{1,1}(\Omega)$, with Ω open set in $\mathbb{R}^n$. Then*

$$\int_\Omega |Df|\,dx = \int_{-\infty}^{+\infty} P_\Omega\left(\{x\in\Omega:\ f(x)>t\}\right)\,dt.$$

3. Nonlinear eigenvalue problems

In this section we want to recall some results about the following Dirichlet problem:

$$\begin{cases} -\Delta_p v + \beta|Dv|^{p-2}Dv\cdot\frac{x}{|x|} = m(x)|v|^{p-2}v + f & \text{in } \Omega,\\ v=0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $1<p<+\infty$, β is a nonnegative constant, $\Omega\subset\mathbb{R}^n$ is a bounded domain, $m\in L^r(\Omega)$, with $r>\max\{n/p,1\}$, $m\geq 0$, and $f\in L^q(\Omega)$, with $q>\max\{n/p,1\}$.

We observe that to solve the problem (3.1) is equivalent to solve the following one:

$$\begin{cases} -\Delta_p^\beta v = m(x)e^{-\beta|x|}|v|^{p-2}v + fe^{-\beta|x|} & \text{in } \Omega\\ v=0 & \text{on } \partial\Omega \end{cases} \tag{3.2}$$

where Δ_p^β is the operator defined as $\Delta_p^\beta u = \operatorname{div}(e^{-\beta|x|}|Du|^{p-2}Du)$. We want to study existence and uniqueness of (3.1).

In the linear case, with $p=2$ and $\beta=0$, if $m(x)\in L^\infty(\Omega)$, the problem (3.2) can be approached with the classical Fredholm alternative theory (see [**60**], for example).

In general, if the coefficient of the zero–order term is bounded, it is possible to relate the problem (3.1) with the eigenvalue problem

$$\begin{cases} -\Delta_p^\beta\psi = \lambda e^{-\beta|x|}|\psi|^{p-2}\psi & \text{in } \Omega,\\ \psi=0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

It is possible to prove existence and uniqueness results of related to a smallness assumption on the L^∞ norm of m (see [**14**], [**46**],[**51**]).

We want to focus our attention on the general case with m unbounded. In this case we have to consider a weighted eigenvalue problem which reminds the coefficients of the lower–order terms; namely, we consider the problem

$$\begin{cases} -\Delta_p^\beta\varphi = \lambda m(x)e^{-\beta|x|}|\varphi|^{p-2}\varphi & \text{in } \Omega\\ \varphi=0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

We recall that λ is an *eigenvalue* of (3.4) if there exists $v\in W_0^{1,p}(\Omega)$ such that $v\not\equiv 0$ and the couple (λ,v) satisfies

$$\int_B e^{-\beta|x|}\,|Dv|^{p-2}\,Dv\cdot D\varphi\,dx = \int_B m(x)e^{-\beta|x|}\,|v|^{p-2}\,v\varphi\,dx + \int_B e^{-\beta|x|}f\varphi\,dx,$$

and the function v is called *eigenfunction* relative to λ. We will denote with $\Lambda(\beta, m)$ the spectrum of (3.4).

We have the following result ([**11**], [**14**], [**57**], [**71**]):

THEOREM 3.1. *If* $|\{x \in \Omega : m(x) > 0\}| > 0$, *then there exists a sequence* $\lambda_k^\beta(\Omega)$ *of positive eigenvalues of* (3.2) *such that*

$$\lim_{k \to +\infty} \lambda_k^\beta(\Omega) = +\infty.$$

Such eigenvalues can be evaluated by using Lyusternik–Schnirelmann method (see [**78**]) as critical values of the functional

$$\mathcal{F}(w) = \frac{\displaystyle\int_\Omega e^{-\beta|x|}|Dw|^p dx}{\displaystyle\int_B e^{-\beta|x|}m(x)|w|^p dx}$$

on a family of subsets of $W_0^{1,p}(\Omega)$ with particular topological properties. One of these eigenvalues is characterized as the minimum of $\mathcal{F}$ on $W_0^{1,p}(\Omega)$:

$$\lambda_1^\beta(\Omega) = \min_{\substack{w \in W_0^{1,p}(\Omega) \\ w \neq 0}} \frac{\displaystyle\int_\Omega e^{-\beta|x|}|Dw|^p dx}{\displaystyle\int_\Omega e^{-\beta|x|}m(x)|w|^p dx}. \tag{3.5}$$

Clearly, $\lambda_1^\beta(\Omega)$ is such that no other eigenvalue belongs to the interval $[0, \lambda_1^\beta[$. We refer to $\lambda_1^\beta(\Omega)$ as the first eigenvalue of (3.2).

Such value enjoys some important properties ([**12**], [**74**]):

(1) $\lambda_1^\beta(\Omega)$ is simple; this means that if u and v are two eigenfunctions relative to $\lambda_1^\beta(\Omega)$, then $u = \alpha v$ for some $\alpha \in \mathbb{R}$;
(2) $\lambda_1^\beta(\Omega)$ is isolated, in the sense that no other eigenvalue is contained in the set $]\lambda_1^\beta(\Omega) - \varepsilon, \lambda_1^\beta(\Omega) + \varepsilon[$ for some $\varepsilon > 0$.

In the case $\beta = 0$, by symmetrization methods is possible to prove the following result (see, for example, [**40**], [**4**], [**1**]):

THEOREM 3.2 (Faber–Krahn inequality). *Let* Ω *be an open, bounded and connected subset of* $\mathbb{R}^n$, $1 < p < n$. *If* $\lambda_1(\Omega^\#)$ *is the first eigenvalue of the problem*

$$\begin{cases} -\Delta_p v = \lambda m^\#(x)|v|^{p-2}v & \text{in } \Omega^\# \\ v = 0 & \text{on } \partial\Omega^\#, \end{cases} \tag{3.6}$$

then

$$\lambda_1(\Omega^\#) \leq \lambda_1^0(\Omega),$$

and equality holds if and only if $\Omega = \Omega^\#$ *and* $m = m^\#$ *a.e. in* Ω, *modulo translations.*

As regards the solvability of problem (3.1), it is possible to prove a Fredholm alternative type result related to the spectrum $\Lambda(\beta, m)$ of the operator $u \mapsto -\Delta_p^\beta u - m(x)e^{-\beta|x|}\,|u|^{p-2}\,u$ (see, for example, [**14**]):

THEOREM 3.3. *If $\lambda \notin \Lambda(\beta, m)$, then the problem* (3.1) *admits a solution in $W_0^{1,p}(\Omega)$ for any $f \in W^{-1,p'}(\Omega)$.*

The proof relies on the Leray–Schauder degree theory (see, for example, [**99**]).

PROOF. Let $\lambda \notin \Lambda$. Applying for example the results of Chapter 2, we get that $-\Delta_p^\beta$ is an homeomorphism from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$; moreover, $W_0^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$, so the operator

$$(t,u) \in [0,1] \times W_0^{1,p}(\Omega) \mapsto A_t(u) = (-\Delta_p^\beta)^{-1}(\lambda m e^{-\beta|x|}|u|^{p-2}u + te^{-\beta|x|}f) \in W_0^{1,p}(\Omega)$$

is compact. If we prove that A_1 admits a fixed point, we get the theorem.

By degree theory, it is sufficient to show that the set

$$\{u \in W_0^{1,p}(\Omega) : \exists t \in [0,1] \text{ such that } (I - A_t)(u) = 0\} \tag{3.7}$$

is bounded. This means that the solutions in (3.7) are a priori bounded. By contradiction, there exists a sequence $u_j \in W_0^{1,p}(\Omega)$ and $t_n \in [0,1]$ such that

$$A_t(u_j) = u_j \text{ and } \|u_j\|_{W_0^{1,p}(\Omega)} \to +\infty \text{ as } j \to +\infty.$$

Let $v_j = u_j / \|u_j\|_{W_0^{1,p}(\Omega)}$. Up to a subsequence,

$$\begin{aligned} &v_j \rightharpoonup v \text{ weakly in } W_0^{1,p}(\Omega),\\ &v_j \to v \text{ strongly in } L^p(\Omega). \end{aligned}$$

Moreover, v_j satisfies

$$v_j = (-\Delta_p^\beta)^{-1}\left(\lambda m e^{-\beta|x|}|v_j|^{p-2}v_j + \frac{t_j e^{-\beta|x|}f}{\|u_j\|^{p-1}_{W_0^{1,p}(\Omega)}}\right),$$

hence we deduce that $v_j \to v$ in $W_0^{1,p}(\Omega)$, $\|v\|_{W_0^{1,p}(\Omega)} = 1$, and v solves the problem

$$\begin{cases} -\Delta_p^\beta v = \lambda m(x) e^{-\beta|x|}|v|^{p-2}v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

Consequently, $\lambda \in \Lambda$ and this is a contradiction. □

4. The radial case: existence and uniqueness

Now we want to study the Dirichlet problem (3.1) with radially symmetric data, that is

$$\begin{cases} -\Delta_p v + \beta|Dv|^{p-2}Dv \cdot \frac{x}{|x|} = b(x)|v|^{p-2}v + f & \text{in } B \\ v = 0 & \text{on } \partial B \end{cases} \tag{4.1}$$

where β is a nonnegative constant, B is a ball centered at the origin, and b and f are radially decreasing symmetric functions, $b(x) = b(|x|)$, $f(x) = f(|x|) \geq 0$ with $b \in L^r(B)$, $r > \max\{n/p, 1\}$, and $f \in L^q(B)$, with $q > \max\{n/p, 1\}$.

Equivalently, we will study the following problem:

$$\begin{cases} -\Delta_p^\beta v = b(x)e^{-\beta|x|}|v|^{p-2}v + fe^{-\beta|x|} & \text{in } B \\ v = 0 & \text{on } \partial B \end{cases} \tag{4.2}$$

where Δ_p^β is the operator defined in the previous section.

More precisely, we prove the following result:

Theorem 4.1. *If*

$$1 < \lambda_1^\beta(B), \tag{4.3}$$

then the problem (4.1) *admits a unique nonnegative solution* $v(x)$ *such that*

$$v(x) = v^{\#}(x). \tag{4.4}$$

Moreover, v *minimizes the functional*

$$E(w) = \frac{1}{p}\int_B |Dw|^p e^{-\beta|x|}dx - \frac{1}{p}\int_B b|w|^p e^{-\beta|x|}dx - \int_B fwe^{-\beta|x|}\,dx, \quad w \in W_0^{1,p}(B).$$

In order to prove Theorem 4.1, we recall a technical result (see [**74**]).

Lemma 4.1. *If* $p \geq 2$, *then*

$$|\xi_2|^p \geq |\xi_1|^p + p|\xi_1|^{p-2}\xi_1 \cdot (\xi_2 - \xi_1) + \frac{|\xi_2 - \xi_1|^p}{2^{p-1} - 1}$$

for every $\xi_1, \xi_2 \in \mathbb{R}^n$. *If* $1 < p < 2$, *then:*

$$|\xi_2|^p \geq |\xi_1|^p + p\,|\xi_1|^{p-2}\,\xi_1 \cdot (\xi_2 - \xi_1) + h(p)\frac{|\xi_2 - \xi_1|^2}{(|\xi_1| + |\xi_2|)^{2-p}}$$

for every $\xi_1, \xi_2 \in \mathbb{R}^n$, *where* $h(p)$ *is a positive constant depending on* p.

Proof of theorem 4.1. By the hypothesis (4.3), the operator

$$-\Delta_p^\beta u - b(x)e^{-\beta|x|}\,|u|^{p-2}\,u$$

is coercive with respect to the weighted norm $\|u\|_{\beta,1,p} = \int_\Omega e^{-\beta|x|}\,|Du|^p\,dx$; then, by Theorem 5.1 of Chapter 2, there exists a solution $v \in W_0^{1,p}(B)$ of (4.1). By well–known regularity results, the summability assumptions on b and f guarantee that the solutions are bounded (see [**61**], Theorem 7.5 and Remark 7.6).

Let $f \equiv 0$ in B. Suppose that there exists a function $v \in W_0^{1,p}(B)$ such that $v \not\equiv 0$ in B and

$$\int_B e^{-\beta|x|}|Dv|^{p-2}Dv \cdot D\varphi dx = \int_B e^{-\beta|x|}b(x)|v|^{p-2}v\varphi dx \quad \forall \varphi \in W_0^{1,p}(B).$$

We have

$$\int_B e^{-\beta|x|}|Dv|^p dx = \int_B e^{-\beta|x|} b(x)|v|^p dx \leq$$
$$\leq \int_B e^{-\beta|x|} b^+(x)|v|^p dx < \lambda_1^\beta(B) \int_B e^{-\beta|x|} b^+(x)|v|^p dx;$$

from the variational characterization (3.5) of $\lambda_1^\beta(B)$, we have a contradiction. So the theorem is completely proved when $f \equiv 0$.

Now let $f \not\equiv 0$ in B, and let v be a solution of (4.1). Suppose $v^- \not\equiv 0$ in B. We have $v^- \in W_0^{1,p}(B)$, and

$$\int_B e^{-\beta|x|}|Dv^-|^p dx = \int_B e^{-\beta|x|}(b^+(x) - b^-(x))|v^-|^p dx - \int_B e^{-\beta|x|} f v^- dx \leq$$
$$\leq \int_B e^{-\beta|x|} b^+(x)|v^-|^p dx < \lambda_1^\beta(B) \int_B e^{-\beta|x|} b^+(x)|v^-|^p dx;$$

again from (3.5), we have a contradiction. So, we have that any solution of (4.1) is nonnegative. Furthermore, we observe that v is positive and $\frac{\partial v}{\partial \nu}$ is negative, where ν is the exterior normal to ∂B, due to well known maximum principles (see, for example, [**61**], Chapter 7, and [**86**]).

As regards the uniqueness of solution of (4.1), we follow an argument which can be found in [**74**] (see also [**4**], [**12**], [**18**], [**46**], [**51**]), using suitable test functions in (4.1). If u and v are solutions of (4.1), we set

$$\varphi_1(x) = \frac{u^p - v^p}{u^{p-1}} e^{-\beta|x|}, \qquad \varphi_2(x) = \frac{v^p - u^p}{v^{p-1}} e^{-\beta|x|}.$$

Applying l'Hôpital rule, we have that u/v and v/u are bounded. So we can use φ_1 in the equation solved by u and φ_2 in the equation solved by v, obtaining

$$\int_\Omega e^{-\beta|x|}|Du|^{p-2}Du \cdot \left[Du - p\left(\frac{v}{u}\right)^{p-1} Dv + (p-1)\left(\frac{v}{u}\right)^p Du\right] dx =$$
$$= \int_\Omega e^{-\beta|x|}(u^p - v^p)\left[b + \frac{f}{u^{p-1}}\right] dx, \tag{4.5}$$

and

$$\int_\Omega e^{-\beta|x|}|Du|^{p-2}Du \cdot \left[Du - p\left(\frac{u}{v}\right)^{p-1} Du + (p-1)\left(\frac{u}{v}\right)^p Dv\right] dx =$$
$$= \int_\Omega e^{-\beta|x|}(v^p - u^p)\left[b + \frac{f}{v^{p-1}}\right] dx, \tag{4.6}$$

The sum of the right–hand side of (4.5) and (4.6) in less or equal than zero, hence by adding (4.5) and (4.6) we get

$$\begin{aligned}(4.7)\quad &\int_\Omega e^{-\beta|x|}u^p\left[|D(\log u)|^p - p|D(\log v)|^{p-2}D(\log u)\cdot D(\log v)+\right.\\ &\qquad\qquad\left. +(p-1)|D(\log v)|^p\right]dx+\\ &+\int_\Omega e^{-\beta|x|}v^p\left[|D(\log v)|^p - p|D(\log u)|^{p-2}D(\log u)\cdot D(\log v)+\right.\\ &\qquad\qquad\left.+(p-1)|D(\log u)|^p\right]dx\le 0.\end{aligned}$$

From (4.7), using lemma 4.1, we get that if $p\ge 2$,

$$\int_\Omega e^{-\beta|x|}(u^p+v^p)\frac{|D(\log u)-D(\log v)|^p}{2^{p-1}-1}\,dx\le 0, \tag{4.8}$$

or, if $1<p<2$, there exists a positive constant $h(p)$ such that:

$$\int_\Omega e^{-\beta|x|}(u^p+v^p)h(p)\frac{|D(\log u)-D(\log v)|^p}{(|D(\log u)|+|D(\log u)|)^{2-p}}\,dx\le 0. \tag{4.9}$$

Consequently, from (4.8) and (4.9) it follows, for $p>1$, that $|D(\log u)-D(\log v)|=0$ a.e. in B. Then, there exists a constant $a>0$ such that $u=av$ a.e. in B. Thus, being u and v solutions of (4.1) and $|\{x\in B: f(x)\ne 0\}|>0$, it follows that $u=v$ a.e. in B.

Finally, we have to prove that $v=v^\#$ in B (see [**50**], [**51**], [**81**]). An immediate consequence of uniqueness is that v is radially symmetric, $v(x)=v(|x|)$. So let us write $v(x)=\tilde v(\omega_n|x|^n)$, and set $s=\omega_n|x|^n$. Observing that $b(x)=(b^+)^\#(x)-(b^-)_\#(x)$, from (4.1) we get

$$-|\tilde v(s)|^{p-2}\tilde v'(s)=\exp\left(\frac{\beta s^{1/n}}{\omega_n^{1/n}}\right)\frac{s^{-(1-1/n)p}}{(n\omega_n^{1/n})^p}\psi(s), \tag{4.10}$$

where

$$\psi(s)=\int_0^s\exp\left(\frac{-\beta t^{1/n}}{\omega_n^{1/n}}\right)[f^*(t)+((b^+)^*(t)-(b^-)_*(t))(\tilde v(t))^{p-1}]dt.$$

To prove (4.4) it is enough to show that

$$\psi(s)\ge 0,\quad \text{for any } s\in(0,|B|). \tag{4.11}$$

To this aim, set $s_0=\inf\{s\in[0,|B|]:(b^-)_*(s)>0\}$ ($s_0=|B|$ if $b^-\equiv 0$); we have

$$\psi(s)=\begin{cases}\displaystyle\int_0^s\exp\left(\frac{-\beta t^{1/n}}{\omega_n^{1/n}}\right)[f^*(t)+(b^+)^*(t)(\tilde v(t))^{p-1}]dt, & \forall s\in[0,s_0],\\ \displaystyle\psi(s_0)+\int_{s_0}^s\exp\left(\frac{-\beta t^{1/n}}{\omega_n^{1/n}}\right)[f^*(t)-(b^-)_*(t)(\tilde v(t))^{p-1}]dt, & \forall s\in[s_0,|B|].\end{cases}$$

The condition (4.11) is obviously true when $s\le s_0$. So, let us suppose that there exists $s>s_0$ such that $\psi(s)<0$. Therefore, there exists $\bar s\in]s_0,|B|]$ such that $\psi(\bar s)=\min_{s\in[s_0,|B|]}\psi(s)<0$. Clearly $\bar s<|B|$, otherwise from (4.10) it follows that $\tilde v'(s)>0$

in some neighborhood of $|B|$, in contrast with the fact that $\tilde{v}(s) \geq 0$ in $[0, |B|]$ and $\tilde{v}(|B|) = 0$. Moreover, let us note that there is an $s \in]\bar{s}, |B|[$ such that $\psi(s) > 0$. Indeed, if $\psi(s) \leq 0$ in $[\bar{s}, |B|]$, from (4.10) it follows that $\tilde{v}(s)$ is increasing in $[\bar{s}, |B|]$ from which, being $\tilde{v}(|B|) = 0$, $\tilde{v}$ is equal to zero in $[\bar{s}, |B|]$. So $\psi(s) = 0$ in $[\bar{s}, |B|]$, but this is absurd.

Then, we can conclude that there exist $s_1, s_2 \in]s_0, |B|[$ such that $s_1 < \bar{s} < s_2$, $\psi(s_2) = 0$, $\psi(s) < 0$, $\forall s \in [s_1, s_2[$. In particular we have

$$\psi(\bar{s}) = \min_{s \in [s_1, s_2]} \psi(s).$$

As regards $\tilde{v}$, we can say that $\tilde{v}'(s) \geq 0$, $\forall s \in [s_1, s_2]$, then the function $h(s) = f^*(s) - (b^-)_*(s))^{p-1}$ is decreasing in $[s_1, s_2]$.

Moreover, let us show that $h(s) \geq 0$ in $[s_1, s_2]$. The assertion is obvious if $h(s_2) \geq 0$; otherwise, if $h(s_2) < 0$ there exists $t \in [s_1, s_2]$ such that

$$\psi'(s) < 0 \quad \text{and} \quad \psi(s) \leq 0, \qquad \forall s \in]t, s_2],$$

but this is absurd because $\psi(s_2) = 0$.

Consequently, $\psi'(s) = h(s) \exp(-\beta s^{1/n} / \omega_n^{1/n})$ is decreasing in $[s_1, s_2]$. So, $\psi(s)$ is concave in $[s_1, s_2]$, then $\psi(s)$ is constant in $[s_1, s_2]$, in contrast with the fact that $\psi(\bar{s}) < 0$ and $\psi(s_2) = 0$. This concludes the proof of (4.4). $\square$

Remark 4.1. Remind that v is a solution of (4.1) if and only if it is a solution of (4.2); so, let us consider $k \in]0, \sup v[$, and set $B_k = \{x \in B : v(x) > k\}$. If v is a solution of (4.1), then v is solution of the following problem:

$$\begin{cases} -\Delta_p^\beta v = e^{-\beta|x|} b |v|^{p-2} v + f e^{-\beta|x|} & \text{in } B_k \\ v = k & \text{on } \partial B_k \end{cases}$$

Moreover, v satisfies the condition below:

$$\frac{1}{p} \int_{B_k} e^{-\beta|x|} |Dv|^p dx - \frac{1}{p} \int_{B_k} e^{-\beta|x|} b |v|^p dx - \int_{B_k} e^{-\beta|x|} f v dx \leq$$
$$\leq \frac{1}{p} \int_{B_k} e^{-\beta|x|} |Dw|^p - \frac{1}{p} \int_{B_k} e^{-\beta|x|} b |w|^p dx - \int_{B_k} e^{-\beta|x|} f w dx,$$

for all $w \in W^{1,p}(B_k)$ such that $w - k \in W_0^{1,p}(B_k)$. So, if $w = w^\#$, and $s = \omega_n |x|^n$, we have:

$$\frac{(n\omega_n^{1/n})^p}{p} \int_0^{|B_k|} \exp\left(-\frac{\beta s^{1/n}}{\omega_n^{1/n}}\right) s^{(1-1/n)p} | -(v^*)'(s)|^p ds +$$
$$-\frac{1}{p} \int_0^{|B_k|} \exp\left(-\frac{\beta s^{1/n}}{\omega_n^{1/n}}\right) \tilde{b}(s) (v^*(s))^p ds - \int_0^{|B_k|} \exp\left(-\frac{\beta s^{1/n}}{\omega_n^{1/n}}\right) f^*(s) v^*(s) ds \leq$$

$$\le \frac{(n\omega_n^{1/n})^p}{p}\int_0^{|B_k|}\exp\left(-\frac{\beta s^{1/n}}{\omega_n^{1/n}}\right)s^{(1-1/n)p}|-(w^*)'(s)|^p ds+$$
$$-\frac{1}{p}\int_0^{|B_k|}\exp\left(-\frac{\beta s^{1/n}}{\omega_n^{1/n}}\right)\tilde{b}(s)(w^*(s))^p ds-\int_0^{|B_k|}\exp\left(-\frac{\beta s^{1/n}}{\omega_n^{1/n}}\right)f^*(s)w^*(s)ds,$$

where $b(x)=\tilde{b}(\omega_n|x|^n)$.

5. Main result

Consider the problem

$$\begin{cases} -\operatorname{div}(a(x,u,Du))=g(x,u)+H(x,u,Du) & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega \end{cases} \tag{5.1}$$

We assume that $a(x,s,\xi):\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$, $H(x,s,\xi):\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$, and $g(x,s):\Omega\times\mathbb{R}\to\mathbb{R}$ are Carathï¿½odory functions satisfying for some $p\in]1,+\infty[$ the following conditions:

$$|a(x,s,\xi)|\le\alpha(|\xi|^{p-1}+|s|^{p-1}+k(x)),\ \text{a.e. } x\in\Omega,\ \forall(s,\xi)\in\mathbb{R}\times\mathbb{R}^n, \tag{5.2}$$

$$a(x,s,\xi)\cdot\xi\ge|\xi|^p,\ \text{a.e. } x\in\Omega,\ \forall(s,\xi)\in\mathbb{R}\times\mathbb{R}^n, \tag{5.3}$$

$$g(x,s)s\le c(x)|s|^p,\ \text{a.e. } x\in\Omega,\ \forall s\in\mathbb{R}, \tag{5.4}$$

$$|g(x,s)|\le\theta(x)|s|^{p-1}\ \text{a.e. } x\in\Omega,\ \forall s\in\mathbb{R}, \tag{5.5}$$

$$|H(x,s,\xi)|\le\beta|\xi|^{p-1}+f(x)\ \text{a.e. } x\in\Omega,\ \forall(s,\xi)\in\mathbb{R}\times\mathbb{R}^n, \tag{5.6}$$

where $\alpha>0$, $k\in L_+^{p'}(\Omega)$, $c,\theta\in L^r(\Omega)$, with $r>\max\{n/p,1\}$, $f\in L^q(\Omega)$, with $q>\max\{n/p,1\}$, and β is a nonnegative constant.

THEOREM 5.1. *Let $u\in W_0^{1,p}(\Omega)$ be a solution of the problem* (5.1)*, under the assumptions* (5.2)–(5.6)*. If*

$$\lambda_1^\beta(\Omega^\#)>1 \tag{5.7}$$

and $v\in W_0^{1,p}(\Omega^\#)$ is the solution of (1.12)*, we have:*

$$u^*(s)\le v^*(s),\quad\forall s\in[0,s_0], \tag{5.8}$$

and

$$\int_0^s(u^*(t))^{p-1}\exp\left(-\frac{\beta t^{1/n}}{\omega_n^{1/n}}\right)dt\le\int_0^s(v^*(t))^{p-1}\exp\left(-\frac{\beta t^{1/n}}{\omega_n^{1/n}}\right)dt,\quad\forall s\in[s_0,|\Omega|], \tag{5.9}$$

where $s_0=\inf\{s\in[0,|\Omega|]:(c^-)_*>0\}$ (If $c^-\equiv0$, then $s_0=|\Omega|$).

Before giving the proof of the above theorem, we introduce some notation which will be used in the following. Let $\tilde{c}(s)=(c^+)^*(s)-(c^-)_*(s)$, $s\in[0,|\Omega|]$, and

$$U(s)=\int_0^s\tilde{c}(t)(u^*(t))^{p-1}\exp\left(-\frac{\beta t^{1/n}}{\omega_n^{1/n}}\right)dt,\quad\forall s\in[0,|\Omega|],$$

$$U_+(s) = \int_0^s (c^+)^*(t)(u^*(t))^{p-1} \exp\left(-\frac{\beta t^{1/n}}{\omega_n^{1/n}}\right) dt, \quad \forall s \in [0, s_0],$$
$$U_-(s) = \int_0^s (c^-)_*(t)(u^*(t))^{p-1} \exp\left(-\frac{\beta t^{1/n}}{\omega_n^{1/n}}\right) dt, \quad \forall s \in [s_0, |\Omega|],$$

where s_0 is defined in Theorem 5.1. Analogously, we define $V(s), V_+(s), V_-(s)$, related to v^*. Moreover, let us set:

$$F(s) = \int_0^s f^*(t) \exp\left(-\frac{\beta t^{1/n}}{\omega_n^{1/n}}\right) dt, \quad \forall s \in [0, |\Omega|],$$

and $\gamma(s) = \dfrac{s^{-(1-1/n)p'}}{(n\omega_n^{1/n})^{p'}}$, where $p' = \dfrac{p}{p-1}$.

To prove Theorem 5.1 we need some lemmas.

First, we recall the following generalization of well–known Gronwall lemma:

LEMMA 5.1. *If φ is a bounded function and $\varphi(t) \le \int_t^{+\infty} g(s)\varphi(s)ds + \psi(t)$, for a.e. $t > 0$, where $g \ge 0$ is an integrable function and ψ is a BV function so that $\psi(+\infty) = 0$, then, for a.e. $t > 0$, we have*

$$\varphi(t) \le \int_t^{+\infty} e^{\int_t^s g(r)dr}[-d\psi(s)].$$

LEMMA 5.2. *Under the hypotheses of Theorem 5.1, we have, a.e. in $(0, |\Omega|)$, the following relations:*

$$(-u^*(s))' \le \gamma(s) \left[\exp\left(\frac{\beta s^{1/n}}{\omega_n^{1/n}}\right)(F(s) + U(s))\right]^{1/(p-1)} \tag{5.10}$$

$$(-v^*(s))' = \gamma(s) \left[\exp\left(\frac{\beta s^{1/n}}{\omega_n^{1/n}}\right)(F(s) + V(s))\right]^{1/(p-1)} \tag{5.11}$$

PROOF. We will follow [**8**] and [**81**].

Let $u \in W_0^{1,p}(\Omega)$ be a weak solution of (5.1); hence,

$$\int_\Omega a(x, u, Du) \cdot D\varphi\, dx = \int_\Omega [g(x, u) + H(x, u, Du)]\, \varphi\, dx \quad \forall \varphi \in W_0^{1,p}(\Omega).$$

Using in (5.1) the truncation functions $T_t(u)$, $T_{t+h}(u)$, with $t, h > 0$, as test functions, and subtracting, we get:

$$\int_{\{t<|u|\le t+h\}} a(x, u, Du) \cdot Du\, dx =$$
$$= \int_{\{|u|>t\}} [g(x, u) + H(x, u, Du)] h \operatorname{sign} u\, dx +$$
$$+ \int_{\{t<|u|\le t+h\}} [g(x, u) + H(x, u, Du)](|u| - t) \operatorname{sign} u\, dx;$$

so, dividing both sides by h, using ellipticity condition (5.3), and the conditions (5.4), (5.6) we get

$$\frac{1}{h}\int_{\{t<|u|\le t+h\}}|Du|^p\,dx \le \\ \le \beta\int_{\{|u|>t\}}|Du|^{p-1}\,dx+\int_{\{|u|>t\}}[f(x)\operatorname{sign}u+c(x)\,|u|^{p-1}]dx. \tag{5.12}$$

On the other hand, by Hardy–Littlewood inequality it follows that

$$\int_{\{|u|>t\}}c(x)|u|^{p-1}\,dx\le\int_0^{\mu_u(t)}[(c^+)^*(s)-(c^-)_*(s)](u^*(s))^{p-1}\,ds;$$

hence, letting $h\to 0$ in (5.12) we get

$$-\frac{d}{dt}\int_{\{|u|>t\}}|Du|^p\,dx\le \\ \le\beta\int_{\{|u|>t\}}|Du|^{p-1}\,dx+\int_0^{\mu_u(t)}\{f^*(s)+[(c^+)^*(s)-(c^-)_*(s)](u^*(s))^{p-1}\}\,ds. \tag{5.13}$$

Denoting with $J(\sigma)$ the function

$$J(\sigma)=\int_0^{\sigma}\{f^*(s)+[(c^+)^*(s)-(c^-)_*(s)](u^*(s))^{p-1}\}\,ds,$$

we obtain

$$-\frac{d}{dt}\int_{\{|u|>t\}}|Du|^p\,dx\le \\ \le J(\mu_u(t))+\frac{\beta}{n\omega_n^{1/n}}\int_t^{+\infty}\left(-\frac{d}{d\sigma}\int_{|u|>\sigma}|Du|^p\,dx\right)(-\mu_u'(\sigma))\mu_u(\sigma)^{-1+1/n}\,d\sigma. \tag{5.14}$$

Indeed, using Hölder inequality we have that

$$\int_{\{|u|>t\}}|Du|^{p-1}\le\int_t^{+\infty}\left(-\frac{d}{d\sigma}\int_{\{|u|>\sigma\}}|Du|^pdx\right)^{1-1/p}(-\mu'(\sigma))^{1/p}\,d\sigma \tag{5.15}$$

and

$$\left(-\frac{d}{d\sigma}\int_{\{|u|>\sigma\}}|Du|^p\,dx\right)^{1/p}\ge\left(-\frac{d}{d\sigma}\int_{\{|u|>\sigma\}}|Du|\,dx\right)(-\mu'(\sigma))^{-1+\frac{1}{p}},$$

that is

$$\left(-\frac{d}{d\sigma}\int_{\{|u|>\sigma\}}|Du|^p\,dx\right)^{1-1/p}(-\mu'(\sigma))^{1/p}\le \\ \le\left(-\frac{d}{d\sigma}\int_{\{|u|>\sigma\}}|Du|\,dx\right)^{-1}\left(-\frac{d}{d\sigma}\int_{\{|u|>\sigma\}}|Du|^p\,dx\right)(-\mu'(\sigma)). \tag{5.16}$$

On the other hand, the coarea formula implies that

$$\int_{\{|u|>\sigma\}} |Du| dx = \int_{\sigma}^{+\infty} P(\{x \in \Omega : |u(x)| > \xi\});$$

hence, by isoperimetric inequality we obtain

$$-\frac{d}{d\sigma} \int_{\{|u|>\sigma\}} |Du| dx = P(\{x \in \Omega : |u(x)| > \sigma\}) \geq n\omega_n^{1/n} \mu_u(\sigma)^{1-1/n}.$$

Using the above inequality in (5.16) we get

$$\left(-\frac{d}{d\sigma} \int_{\{|u|>\sigma\}} |Du|^p dx\right)^{1-1/p} (-\mu'(\sigma))^{1/p} \leq \tag{5.17}$$

$$\leq \frac{1}{n\omega_n^{1/n}} \mu(\sigma)^{-1+1/n} \left(-\frac{d}{d\sigma} \int_{\{|u|>\sigma\}} |Du|^p dx\right) (-\mu'(\sigma)).$$

From (5.13) and (5.17) we deduce (5.14).

Using Gronwall lemma 5.1, and again Fleming–Rishel formula, the isoperimetric inequality and the properties of rearrangements, we get (see [**8**])

$$(-u^*(s))' \leq \frac{1}{(n\omega_n)^{p'} s^{p'-p'/n}} \times$$

$$\times \left[\exp\left(\frac{\beta s^{1/n}}{\omega_n^{1/n}}\right) \int_0^s \{[(c^+)^*(r) - (c^-)_*(r)](u^*(r))^{p-1} + f^*(r)\} \exp\left(-\frac{\beta r^{1/n}}{\omega_n^{1/n}}\right) dr\right]^{1/(p-1)}$$

a.e. in $(0, |\Omega|)$, that is (5.10).

As regards equality (5.11), we have proved that there exists a unique solution $v \in W_0^{1,p}(\Omega^\#)$ of problem (4.1), and this solution is positive, radially symmetric and coincides with his Schwarz symmetrization, namely $v(x) = v^\#(x)$. Hence we can proceed in the same way we did before, except that the inequalities are replaced by equalities, end we have the differential equality (5.11). This concludes the proof. □

Lemma 5.3. *Under the hypotheses of Theorem* 5.1*, the following inequality holds:*

$$U_+(s) \leq V_+(s), \quad s \in [0, s_0]. \tag{5.18}$$

Proof. Inequality (5.18) is trivial when $c^+ \equiv 0$, so, from now on, we suppose $c^+ \not\equiv 0$.

If $f \equiv 0$ in Ω, we are also in a trivial case. Indeed, it is easy to prove that

$$u^*(s) = v^*(s) = 0, \quad s \in [0, |\Omega|].$$

As regards v, we can argue as in Theorem 4.1. As regards u, we can use inequality (5.10).

If $u \not\equiv 0$, using (5.7), we have:

$$((-u^*(s))')^{p-1} < \lambda_1^\beta(\Omega^\#) \frac{s^{-(1-1/n)p}}{(n\omega_n^{1/n})^p} \exp\left(\frac{\beta s^{1/n}}{\omega_n^{1/n}}\right) \int_0^s \tilde{c}(t)(u^*(t))^{p-1} \exp\left(-\frac{\beta t^{1/n}}{\omega_n^{1/n}}\right) dt;$$

a straightforward calculation allows us to obtain that:

$$\begin{aligned}\int_\Omega |Du^\#|^p e^{-\beta|x|}dx &= (n\omega_n^{1/n})^p \int_0^{|\Omega|} |(-u^*(s))'|^p s^{(1-1/n)p} \exp\left(-\frac{\beta s^{1/n}}{\omega_n^{1/n}}\right) ds < \\ &< \lambda_1^\beta(\Omega^\#) \int_0^{|\Omega|} |(-u^*(s))'| \left[\int_0^s \tilde{c}(t)(u^*(t))^{p-1} \exp\left(-\frac{\beta t^{1/n}}{\omega_n^{1/n}}\right) dt\right] ds = \\ &= \lambda_1^\beta(\Omega^\#) \int_0^{|\Omega|} (u^*(s))^p\, \tilde{c}(s)\, \exp\left(-\frac{\beta s^{1/n}}{\omega_n^{1/n}}\right) ds = \\ &= \lambda_1^\beta(\Omega^\#) \int_\Omega |u^\#|^p\, \hat{c}\, e^{-\beta|x|}dx \le \lambda_1^\beta(\Omega^\#) \int_\Omega |u^\#|^p\, c^\# e^{-\beta|x|}dx,\end{aligned}$$

which gives a contradiction according to the variational characterization of $\lambda_1^\beta(\Omega^\#)$. It follows that $u \equiv 0$ in Ω.

Now suppose $f \not\equiv 0$. We argue as in [**51**]. Let us distinguish two different cases:

1) $U_+(s_0) \le V_+(s_0)$;
2) $U_+(s_0) > V_+(s_0)$.

Let us consider the case 1). If (5.18) is not satisfied, then there exists $\overline{s} \in]0, s_0[$ such that:

$$U_+(\overline{s}) - V_+(\overline{s}) = \max_{s\in[0,s_0]} (U_+(s) - V_+(s)) > 0.$$

Let us set:

$$\begin{aligned} s_1 &= \inf\{s \in [0, \overline{s}] : U_+(t) > V_+(t), \forall t \in [s, \overline{s}]\}, \\ s_2 &= \sup\{s \in [\overline{s}, s_0] : U_+(t) > V_+(t), \forall t \in [\overline{s}, s]\},\end{aligned}$$

and define the following functions:

$$\varphi_1(s) = \frac{(U_+(s))^{p'} - (V_+(s))^{p'}}{(U_+(s))^{p'-1}},$$

$$\varphi_2(s) = \frac{(U_+(s))^{p'} - (V_+(s))^{p'}}{(V_+(s))^{p'-1}}.$$

Observe that, in view of hypothesis made on f, u and v are bounded (see [**61**], Theorem 7.5 and Remark 7.6); consequently also the functions U_+/V_+ and V_+/U_+ are bounded in $[0, |\Omega|]$, and φ_1, φ_2 can be used as test functions in (5.10) and (5.11), respectively. Integrating between s_1 and s_2, we have:

$$\begin{aligned}(5.19)\quad \int_{s_1}^{s_2} (-u^*(s))'\varphi_1(s)ds \le \\ \le \int_{s_1}^{s_2} \gamma(s) \exp\left(\frac{\beta s^{1/n}}{\omega_n^{1/n}}\right) ((U_+(s))^{p'} - (V_+(s))^{p'}) \left(\frac{F(s)}{U_+(s)} + 1\right)^{p'-1} ds,\end{aligned}$$

and:

$$\int_{s_1}^{s_2}(-v^*(s))'\varphi_2(s)ds = \tag{5.20}$$

$$= \int_{s_1}^{s_2}\gamma(s)\exp\left(\frac{\beta s^{1/n}}{\omega_n^{1/n}}\right)((U_+(s))^{p'}-(V_+(s))^{p'})\left(\frac{F(s)}{V_+(s)}+1\right)^{p'-1}ds.$$

Being $U_+(s) > V_+(s)$ in $]s_1, s_2[$, we have that the difference of the second members of (5.19) and (5.20) is less than zero, so:

$$\int_{s_1}^{s_2}\left[(-u^*(s))'\varphi_1(s)ds + (v^*(s))'\varphi_2(s)\right]ds < 0. \tag{5.21}$$

Integrating by parts the first member of (5.21) and bearing in mind that $U_+(s_1)-V_+(s_1) = U_+(s_2) - V_+(s_2) = 0$, we have:

$$\int_{s_1}^{s_2}\left[u^*(s)(\varphi_1(s))' - v^*(s)(\varphi_2(s))'\right]ds < 0. \tag{5.22}$$

After some calculation (see [**51**] for details), setting $x = \frac{(u^*)^{p/p'}}{U_+}$ and $y = \frac{(v^*)^{p/p'}}{V_+}$, we have:

$$\int_{s_1}^{s_2}\left[u^*(s)(\varphi_1(s))' - v^*(s)(\varphi_2(s))'\right]ds =$$

$$= \int_{s_1}^{s_2}\exp\left(\frac{\beta s^{1/n}}{\omega_n^{1/n}}\right)\Big[(U_+)^{p'}\left(x^{p'}-p'xy^{p'-1}+(p'-1)y^{p'}\right)+$$

$$+(V_+)^{p'}\left(y^{p'}-p'yx^{p'-1}+(p'-1)x^{p'}\right)\Big]ds.$$

According to Lemma 4.1, if $p' \geq 2$:

$$\int_{s_1}^{s_2}\left[u^*(s)(\varphi_1(s))' - v^*(s)(\varphi_2(s))'\right]ds \geq \tag{5.23}$$

$$\geq \int_{s_1}^{s_2}\exp\left(\frac{\beta s^{1/n}}{\omega_n^{1/n}}\right)(c^+)^*((U_+)^{p'}+(V_+)^{p'})\frac{|y-x|^{p'}}{2^{p'-1}-1}ds$$

and, if $1 < p' < 2$, there exists a positive constant $h(p')$ such that:

$$\int_{s_1}^{s_2}\left[u^*(s)(\varphi_1(s))' - v^*(s)(\varphi_2(s))'\right]ds \geq \tag{5.24}$$

$$\geq \int_{s_1}^{s_2}\exp\left(\frac{\beta s^{1/n}}{\omega_n^{1/n}}\right)(c^+)^*((U_+)^{p'}+(V_+)^{p'})\frac{|y-x|^2}{(|x|+|y|)^{2-p'}}ds.$$

Consequently, being (5.23) and (5.24) greater than or equal to zero, from (5.22) we have a contradiction. So, condition (5.18) is verified if $U_+(s_0) \leq V_+(s_0)$.

Now, let us consider the case 2). If $u^*(s_0) \leq v^*(s_0)$, set:

$$s_1 = \inf\{s \in [0, s_0] : U_+(t) > V_+(t), \forall t \in [s, s_0]\}.$$

Let us observe that $U_+(s_1) = V_+(s_1)$; moreover, being $u^*(s_0) \le v^*(s_0)$ and $U_+(s_0) > V_+(s_0)$, we have:

$$\left(-\frac{u^*(s_0)}{(U_+(s_0))^{p'-1}} + \frac{v^*(s_0)}{(V_+(s_0))^{p'-1}}\right)\left((U_+(s_0))^{p'} - (U_+(s_0))^{p'}\right) \ge 0.$$

Then we can proceed as in case 1). Inserting φ_1 and φ_2 in the relations (5.10) and (5.11), respectively, and integrating between s_1 and s_0, we obtain a contradiction.

If $u^*(s_0) > v^*(s_0)$, we observe that necessarily $s_0 < |\Omega|$ and we set:

$$\overline{\overline{s}} = \inf\{s \in [s_0, |\Omega|] : u^*(t) > v^*(t), \forall t \in [s_0, s]\}.$$

Let us distinguish the following cases:

2a) $U(\overline{\overline{s}}) \ge V(\overline{\overline{s}})$;
2b) $U(\overline{\overline{s}}) < V(\overline{\overline{s}})$.

In the case 2a), being:

$$\frac{d}{ds}(U(s) - V(s)) = (c^-)_*(s)\exp\left(-\frac{\beta s^{1/n}}{\omega_n^{1/n}}\right)(-(u^*)^{p-1}(s) + (v^*)^{p-1}(s)) < 0,$$

a.e. in $[s_0, \overline{\overline{s}}]$,

we have that $U(s) - V(s)$ is decreasing in $[s_0, \overline{\overline{s}}]$. Thus, being $U(\overline{\overline{s}}) - V(\overline{\overline{s}}) \ge 0$, it follows:

$$U(s) \ge V(s), \qquad \forall s \in [s_0, \overline{\overline{s}}]. \tag{5.25}$$

Now let us consider the following function:

$$w(s) = \max\{u^*(s), v^*(s)\}, \qquad \forall s \in [0, \overline{\overline{s}}].$$

Set $W(s) = \displaystyle\int_0^s \tilde{c}(t)(w(t))^{p-1}\exp\left(-\frac{\beta t^{1/n}}{\omega_n^{1/n}}\right)dt$, $\forall s \in [0, \overline{\overline{s}}]$, it is easy to show that $w(s)$ satisfies the following relation:

$$-w'(s) \le \gamma(s)[F(s) + W(s)]^{\frac{1}{p-1}}, \qquad \text{a.e. in } [0, \overline{\overline{s}}]. \tag{5.26}$$

We also observe that a simple calculation gives:

$$W(s) \ge U(s), \qquad \forall s \in [s_0, \overline{\overline{s}}]. \tag{5.27}$$

Moreover, because $u^*(s_0) > v^*(s_0)$, we have that $v^* \not\equiv w$. Then, bearing in mind Remark 4.1, as v is the unique solution of the radial problem (1.12), we have:

$$\begin{aligned} &\frac{1}{p}\int_0^{\overline{\overline{s}}} (n\omega_n^{1/n})^p s^{(1-1/n)p}| - (v^*(s))'|^p \exp\left(-\frac{\beta s^{1/n}}{\omega_n^{1/n}}\right) ds + \\ &\quad -\frac{1}{p}\int_0^{\overline{\overline{s}}} \tilde{c}(s)(v^*(s))^p \exp\left(-\frac{\beta s^{1/n}}{\omega_n^{1/n}}\right) ds - \int_0^{\overline{\overline{s}}} f^*(s)v^*(s)\exp\left(-\frac{\beta s^{1/n}}{\omega_n^{1/n}}\right) ds < \end{aligned} \tag{5.28}$$

$$< \frac{1}{p}\int_0^{\overline{\overline{s}}} (n\omega_n^{1/n})^p s^{(1-1/n)p} |-w'(s)|^p \exp\left(-\frac{\beta s^{1/n}}{\omega_n^{1/n}}\right) ds +$$

$$-\frac{1}{p}\int_0^{\overline{\overline{s}}} \tilde{c}(s) w(s)^p \exp\left(-\frac{\beta s^{1/n}}{\omega_n^{1/n}}\right) ds - \int_0^{\overline{\overline{s}}} f^*(s) w(s) \exp\left(-\frac{\beta s^{1/n}}{\omega_n^{1/n}}\right) ds.$$

On the other hand, from (5.11) and (5.26), respectively, it follows:

$$|(-v^*(s))'|^p = -(v^*(s))'(n\omega_n^{1/n})^{-p} s^{(1-1/n)(-p)} \left[\exp\left(\frac{\beta s^{1/n}}{\omega_n^{1/n}}\right)(F(s)+V(s))\right] \tag{5.29}$$

$$|-w'(s)|^p \le -w'(s)(n\omega_n^{1/n})^{-p} s^{(1-1/n)(-p)} \left[\exp\left(\frac{\beta s^{1/n}}{\omega_n^{1/n}}\right)(F(s)+W(s))\right] \tag{5.30}$$

Substituting (5.29) and (5.30) in (5.28) and integrating by parts between 0 and $\overline{\overline{s}}$, from (5.25) and (5.27) we obtain:

$$\int_0^{\overline{\overline{s}}} v^*(s) f^*(s) \exp\left(-\frac{\beta s^{1/n}}{\omega_n^{1/n}}\right) ds > \int_0^{\overline{\overline{s}}} w(s) f^*(s) \exp\left(-\frac{\beta s^{1/n}}{\omega_n^{1/n}}\right) ds;$$

so we have a contradiction because $v^*(s) \le w(s)$ in $[0, \overline{\overline{s}}]$.

Finally, let us examine the case 2b). Being $u^*(s) > v^*(s)$ in $[s_0, \overline{\overline{s}}[$, we have again

$$\frac{d}{ds}(U(s) - V(s)) < 0 \qquad \text{a.e. in } [s_0, \overline{\overline{s}}[,$$

then $U(s) - V(s)$ is decreasing in $[s_0, \overline{\overline{s}}]$. So, there is $\tilde{s} \in]s_0, \overline{\overline{s}}[$ such that

$$U(\tilde{s}) = V(\tilde{s}) \text{ and } U(s) < V(s), \quad \forall s \in]\tilde{s}, \overline{\overline{s}}]. \tag{5.31}$$

As $\tilde{s} \in]s_0, \overline{\overline{s}}[$, thence:

$$u^*(\tilde{s}) > v^*(\tilde{s}). \tag{5.32}$$

On the other hand, integrating (5.10) and (5.11) between $\tilde{s}$ and $\overline{\overline{s}}$ and using (5.31) we have:

$$u^*(\tilde{s}) - u^*(\overline{\overline{s}}) < v^*(\tilde{s}) - v^*(\overline{\overline{s}}),$$

in contrast with (5.32) because $u^*(\overline{\overline{s}}) = v^*(\overline{\overline{s}})$. So we have a contradiction in the case 2b), too.

□

LEMMA 5.4. *Under the hypotheses of Theorem 5.1, the following inequality holds:*

$$U_-(s) \le V_-(s), \quad s \in [s_0, |\Omega|]. \tag{5.33}$$

PROOF. If $f \equiv 0$ we are in a trivial case; indeed, as already observed in Lemma 5.3, $u^* = v^* = 0$ in $[0, |\Omega|]$.

Let $f \not\equiv 0$. Suppose that $U_-(s) > V_-(s)$, for some $s \in [s_0, |\Omega|]$.
Let us set $Z(s) = U_-(s) - V_-(s)$, $s \in [0, |\Omega|]$. We observe that $Z(s_0) = 0$; moreover, there

exists $\overline{s} \in]s_0, |\Omega|]$ such that $Z(\overline{s}) = \max_{[s_0,|\Omega|]} Z(s) > 0$. Observe that by the hypothesis made on c, the derivative of $Z(s)$ can be not defined in $s = |\Omega|$. We have to distinguish two cases:

a) $\overline{s} = |\Omega|$,
b) $\overline{s} < |\Omega|$.

Let us consider case a). We put

$$s_1 = \inf\{s \in [s_0, |\Omega|] : Z(t) > 0, \forall t \in [s, |\Omega|]\}.$$

Observe that $s_1 \geq s_0$, $Z(s_1) = 0$ and $Z(s) > 0$, $\forall s \in]s_1, |\Omega|]$. If $s \in]s_1, |\Omega|]$, integrating (5.10), (5.11) between s and $|\Omega|$, and using Lemma 5.3, we have:

$$\begin{aligned} u^*(s) \leq \int_s^{|\Omega|} \gamma(t) \exp\left(\frac{\beta t^{1/n}}{\omega_n^{1/n}}\right) [F(t) + U(t)]^{\frac{1}{p-1}} dt = \\ = \int_s^{|\Omega|} \gamma(t) \exp\left(\frac{\beta t^{1/n}}{\omega_n^{1/n}}\right) [F(t) + U_+(s_0) - U_-(t)]^{\frac{1}{p-1}} dt \leq \\ \leq \int_s^{|\Omega|} \gamma(t) \exp\left(\frac{\beta t^{1/n}}{\omega_n^{1/n}}\right) [F(t) + V_+(s_0) - V_-(t)]^{\frac{1}{p-1}} dt = \int_s^{|\Omega|} (-v^*(t))' \, dt = v^*(s). \end{aligned}$$

So, because s is a generic element of $]s_1, |\Omega|]$, we have, recalling $Z(s_1) = U_-(s_1) - V_-(s_1) = 0$,

$$\begin{aligned} U_-(s) = U_-(s_1) + \int_{s_1}^s (c^-)_*(t)(u^*(t))^{p-1} \exp\left(-\frac{\beta t^{1/n}}{\omega_n^{1/n}}\right) dt \leq \\ \leq V_-(s_1) + \int_{s_1}^s (c^-)_*(t)(v^*(t))^{p-1} \exp\left(-\frac{\beta t^{1/n}}{\omega_n^{1/n}}\right) dt = V_-(s), \quad \forall s \in]s_1, |\Omega|], \end{aligned}$$

but this is absurd because $Z(s) > 0$, $\forall s \in]s_1, |\Omega|]$.

Now consider case b). Since $Z(\overline{s}) > 0$, and being Z absolutely continuous, we can choose $s_1, s_2 \in]s_0, |\Omega|[$ such that $s_1 < \overline{s} \leq s_2$, and

$$Z(s) > 0, \ \forall s \in]s_1, s_2], \text{ and } Z(s_1) = 0; \tag{5.34}$$

moreover,

$$Z \text{ is differentiable in } s_2 \text{ with } Z'(s_2) \leq 0. \tag{5.35}$$

By (5.34) we have that $U_-(s_1) = V_-(s_1)$ and, moreover,

$$U_-(s) > V_-(s), \ \forall s \in]s_1, s_2]. \tag{5.36}$$

Integrating (5.10) and (5.11) between s and s_2, and using Lemma 5.3, we have:

$$u^*(s) - u^*(s_2) \leq \int_s^{s_2} \gamma(t) \exp\left(\frac{\beta t^{1/n}}{\omega_n^{1/n}}\right) [F(t) + U(t)]^{\frac{1}{p-1}} dt =$$

$$= \int_s^{s_2} \gamma(t) \exp\left(\frac{\beta t^{1/n}}{\omega_n^{1/n}}\right) [F(t) + U_+(s_0) - U_-(t)]^{\frac{1}{p-1}} dt \leq$$

$$\leq \int_s^{s_2} \gamma(t) \exp\left(\frac{\beta t^{1/n}}{\omega_n^{1/n}}\right) [F(t) + V_+(s_0) - V_-(t)]^{\frac{1}{p-1}} dt = v^*(s) - v^*(s_2).$$

Recalling (5.35) and the fact that $Z(s_1) = U_-(s_1) - V_-(s_1) = 0$, it follows

$$U_-(s) \leq V_-(s), \quad \forall s \in]s_1, s_2];$$

so by (5.36), we have again a contradiction. □

Proof of Theorem 5.1. From Lemma 5.4 we have:

$$\int_{s_0}^{s} (c^-)_*(t)(u^*(t))^{p-1} \exp\left(-\frac{\beta t^{1/n}}{\omega_n^{1/n}}\right) dt \leq \int_{s_0}^{s} (c^-)_*(t)(v^*(t))^{p-1} \exp\left(-\frac{\beta t^{1/n}}{\omega_n^{1/n}}\right) dt, \quad \forall s \in [s_0, |\Omega|].$$

So, according to Lemma 2.1 we have:

$$\int_{s_0}^{s} (u^*(t))^{p-1} \exp\left(-\frac{\beta t^{1/n}}{\omega_n^{1/n}}\right) dt \leq \int_{s_0}^{s} (v^*(t))^{p-1} \exp\left(-\frac{\beta t^{1/n}}{\omega_n^{1/n}}\right) dt, \quad \forall s \in [s_0, |\Omega|]. \tag{5.37}$$

From Lemma 2.2, we have

$$u^*(s_0) \leq v^*(s_0). \tag{5.38}$$

On the other hand, from (5.10), (5.11), (5.18), we have:

$$u^*(s) - u^*(s_0) \leq v^*(s) - v^*(s_0), \quad \forall s \in [0, s_0].$$

Then, from (5.38) we obtain (5.8).

Finally, the condition (5.9) easily follows from (5.8) and (5.37). □

CHAPTER 4

Non–uniformly elliptic equations with general growth in the gradient

1. Statement of the problem

In this chapter we deal with Dirichlet problems of the form

$$-\operatorname{div}(a(x,u,Du)) = H(x,u,Du) + f, \quad u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \tag{1.1}$$

where $1 < p < +\infty$, Ω is a bounded open set of $\mathbb{R}^n$, $n \geq 2$, $a : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $H : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are Carathéodory functions verifying the following assumptions:

$$(a(x,s,\xi) - a(x,s,\xi')) \cdot (\xi - \xi') > 0, \tag{1.2}$$

for a.e. $x \in \Omega$, $\forall\, s \in \mathbb{R}$ and $\forall\, \xi, \xi' \in \mathbb{R}^n$, $\xi \neq \xi'$,

$$a(x,s,\xi) \geq b(|s|)|\xi|^p, \tag{1.3}$$

for a.e. $x \in \Omega, \forall\, s \in \mathbb{R}$, $\forall\, \xi \in \mathbb{R}^n$, and $b : [0,+\infty[\to]0,+\infty[$ is a continuous function,

$$|a(x,s,\xi)| \leq c_0(|\xi|^{p-1} + |s|^{p-1} + g(x)), \tag{1.4}$$

for a.e. $x \in \Omega, \forall\, s \in \mathbb{R}, \forall\, \xi \in \mathbb{R}^n$, with $g \in L^{p'}(\Omega)$,

$$|H(x,s,\xi)| \leq k(|s|)|\xi|^q, \tag{1.5}$$

for a.e. $x \in \Omega, \forall\, s \in \mathbb{R}, \forall\, \xi \in \mathbb{R}^n$, with $p-1 < q \leq p$, and the function $k : [0,+\infty[\to [0,+\infty[$ is continuous. Moreover, assume

$$f \in L^r(\Omega),\ r > \max\left\{\frac{n}{p}, 1\right\}. \tag{1.6}$$

The prototype equation therefore is

$$\begin{cases} -\operatorname{div}(b(|u|)\,|Du|^{p-2}\,Du) = k(|u|)\,|Du|^q + f, & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

We look for bounded solutions of (1.1); namely, we say that $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a *weak solution* of (1.1) if

$$\int_\Omega a(x,u,Du) \cdot D\varphi\,dx = \int_\Omega H(x,u,Du)\varphi\,dx + \int_\Omega f\varphi\,dx, \quad \forall\, \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

The following example, due to Kazdan and Kramer (see [**66**]), explains the typical behaviour of this kind of equations.

EXAMPLE 1.1. Let us consider the Dirichlet problem for the semilinear equation

$$-\Delta u = K\,|Du|^2 + \lambda, \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega) \tag{1.8}$$

with K and λ nonnegative constants. Any solution u of (1.8) is nonnegative: using u^- as test function, we get that

$$-\int_\Omega \left|Du^-\right|^2 dx = K \int_\Omega \left|Du^-\right|^2 u^- \, dx + \int_\Omega u^- \, dx \geq 0,$$

hence $u^- \equiv 0$. Performing the change of variable

$$v = e^u - 1,$$

we obtain a nonnegative function $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$, whenever $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Moreover, v satisfies the linear equation

$$-\Delta v = \lambda K(v+1). \tag{1.9}$$

Now let λ_1 be the first eigenvalue of the $-\Delta$ operator, and $v_1 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ the associated positive first eigenfunction; therefore

$$\int_\Omega Dv_1 \cdot D\varphi \, dx = \lambda_1 \int_\Omega v_1 \varphi \, dx, \quad \forall \varphi \in H_0^1(\Omega).$$

By Fredholm alternative, if

$$\lambda K < \lambda_1,$$

the equation (1.9) (and thus (1.8)) admits a unique solution.

On the other hand, suppose that there exists a solution v of (1.9). Using v_1 as test function in (1.9), we get

$$\int_\Omega Dv \cdot Dv_1 \, dx = \lambda K \int_\Omega v_1(v+1)\, dx = \lambda_1 \int_\Omega v_1 v \, dx,$$

that is

$$(\lambda_1 - \lambda K) \int_\Omega v\, v_1 \, dx = \lambda \int_\Omega v_1 \, dx > 0,$$

so we have necessarily that $\lambda K < \lambda_1$. This means that if

$$\lambda K \geq \lambda_1,$$

the equation (1.9) (and thus (1.8)) has no solution.

REMARK 1.1. Similar examples have been given for b and k constant and for $p-1 < q \leq p$ (see [**52**]).

The Example 1.1 shows that, in general, we need some additional hypotheses on the data of the equation in order to get existence results.

Such kind of problems have been widely studied in literature under various hypotheses.

The first results for this type of equations are contained in a series of papers of Boccardo, Murat and Puel (see [**29, 31, 30, 32**]), where the existence of solutions of problems like (1.1), with $p = q$ and b constant, is proved.

For example, in [**29**] and [**30**] the existence of a solution of (1.1) is proved assuming the existence of a subsolution and a supersolution of (1.1); more precisely, the following theorem holds:

THEOREM 1.1. *Let $a(x, s, \xi)$, $H(x, s, \xi)$ and f verify the hypotheses* (1.2)–(1.6)*, with b constant, $p = q$ and k monotone increasing. If there exist two functions $\underline{u}$, $\overline{u}$ which belong to $W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $\underline{u} \le \overline{u}$ a.e. in Ω, and*

$$\begin{cases} -\operatorname{div}(a(x, \overline{u}, D\overline{u})) \ge H(x, \overline{u}, D\overline{u}) + f & \text{in } \Omega, \\ \overline{u} \ge 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\operatorname{div}(a(x, \underline{u}, D\underline{u})) \le H(x, \underline{u}, D\underline{u}) + f & \text{in } \Omega, \\ \underline{u} \le 0 & \text{on } \partial\Omega, \end{cases}$$

then there exists a solution u of (1.1) with $\underline{u} \le u \le \overline{u}$ a.e. in Ω.

The existence of subsolutions and supersolutions can be avoided, for example, assuming a sign condition on the lower order terms (see [**31**] and [**32**]).

Similar problems with p–growth in the gradient have been considered, always with b constant, in [**79**] (for $p = 2$) and [**62**] (see also [**7**] for the case in which b and k are constant). The case where b is not necessarily constant is treated, for $k \equiv 0$, in [**3**], [**2**], [**26**], and for $k \not\equiv 0$ in [**94**], [**34**]. Existence of (possibly unbounded) solutions for equations similar to (1.7) in the case $f \in L^{n/p}(\Omega)$, is obtained in [**53**], with $p = 2$, and, with an additional term in the equation, in [**54**], always with further hypotheses on f. Moreover, existence results which do not depend on f when $q = p$ are given in [**34**] (for $p = 2$) and in [**84**].

As far as it concerns with the case of q–growth in the gradient, $p-1 < q \le p$, existence results with b and k constant are given in [**50**], and in [**33**], [**63**].

The quoted results are of two kinds: the first one establishes existence of solutions without imposing any additional condition on f (see [**34**], [**84**]); the second one requires conditions on the smallness of some norm of f (see [**79**], [**62**], [**94**], [**52**], [**53**], [**54**], [**63**]). More precisely, when it is possible to remove the smallness hypotheses on f, it is needed appropriate hypotheses on the structure of the equation, like sign conditions or particular hypotheses on the functions $k(s)$ and $b(s)$.

Our aim is to obtain an existence result for problems like (1.7). Our approach permits us to treat in a unified way both the cases in which it is required a particular hypothesis on f and the cases in which such hypothesis it is not necessary.

The standard method to obtain existence results consists in defining approximate problems, then to obtain a priori estimates for their solutions and then passing to the limit in such approximate problems. The main goal is to prove an L^∞ estimate for bounded solutions; so we turn the qualitative information $u \in L^\infty(\Omega)$ into a quantitative estimate

$$\|u\|_{L^\infty(\Omega)} \le C.$$

2. Main result and comments

Let us define the following functions:

$$Q : s \in [0, +\infty[\to C \int_0^s \left[\frac{k(y)}{b(y)}\right]^{\frac{1}{1-p+q}} dy; \tag{2.1}$$

$$F : s \in [0, +\infty[\to \int_0^s e^{Q(t)} b(t)^{\frac{1}{p-1}} dt; \tag{2.2}$$

$$W : s \in [0, +\infty[\to \frac{F(s)}{e^{Q(s)}}, \tag{2.3}$$

where

$$C = \frac{1-p+q}{p-1}\left(\frac{|\Omega|}{\omega_n}\right)^{\frac{p-q}{n}}.$$

We can state the following existence theorem:

THEOREM 2.1. *Under assumptions* (1.2)–(1.6), *if*

$$C_0 V(0) < \sup_{s>0} W(s), \tag{2.4}$$

where C_0 is a constant depending only on the data, namely $C_0 = e^{\frac{p'}{p}(p-q)(|\Omega|/\omega_n)^{\frac{p-q}{n}}}$, and

$$V(x) = \left(n\omega_n^{1/n}\right)^{-p'} \int_{\omega_n|x|^n}^{|\Omega|} t^{-\left(1-\frac{1}{n}\right)p'} \left(\int_0^t f^*(r)dr\right)^{\frac{p'}{p}} dt \tag{2.5}$$

is the solution of the equation

$$\begin{cases} -\Delta_p V = f^\# & \text{in } \Omega^\#, \\ V = 0 & \text{on } \partial\Omega^\#, \end{cases} \tag{2.6}$$

then there exists a weak solution of (1.1).

REMARK 2.1. If $p < n$ the condition (2.4) is a smallness assumption on the norm of f in the Lorentz space $L^{(n/p, p'/p)}(\Omega)$. Indeed, in view of (2.5), the hypothesis (2.4) can be rewritten in the form

$$C_0 n^{-p'} \omega_n^{-p'/n} \|f\|_{L^{(n/p, p'/p)}(\Omega)} < \sup_{s>0} W(s);$$

as recalled in chapter 1, $L^{(n/p, p'/p)}(\Omega)$ contains $L^r(\Omega)$, for any $r > n/p$.

REMARK 2.2. We observe that if $p = q$ and b and k are respectively monotone decreasing and monotone increasing, the hypothesis (2.4) coincides with the existence condition given in [**94**].

REMARK 2.3. We explicitly note that if

$$\sup_{s>0} W(s) = +\infty, \tag{2.7}$$

no smallness assumption on f is needed, therefore a solution exists for any $f \in L^r(\Omega)$, $r > \max\{n/p, 1\}$. In this case the hypothesis (2.7) also weakens, when $q = p$, the existence condition given in [**84**] (see also [**34**]), in which it is required that $\lim_{s\to+\infty} W(s) = +\infty$.

As a matter of fact, if $b \equiv 1$, a simple condition which verifies (2.7) is, for example, the following:

$$\lim_{s\to+\infty} k(s) = 0.$$

In this case, $\sup_{s>0} W(s) = \lim_{s\to+\infty} W(s) = +\infty$. The condition (2.7) is obviously more general, as we explicitly see in the following example.

Let $b \equiv 1$, and $p = q = 2$. Let us define k in the following way:

$$k(s) = \begin{cases} \frac{\pi}{2} \tan \frac{\pi}{4} s, & \text{if } 0 \le s \le 1; \\ \\ \frac{3\pi - 2\left[\sqrt{2}\sin\left(\log s + \frac{\pi}{4}\right) + 1 - \frac{2}{s^2}\right]}{2s\left[\sin(\log s) + 1 + \frac{2}{s^2}\right]}, & \text{if } s > 1. \end{cases}$$

We have that k is continuous in $[0, +\infty[$ and positive for any $s > 0$, and

$$\limsup_{s\to+\infty} k(s) = +\infty, \qquad \liminf_{s\to+\infty} k(s) = 0.$$

If we compute $W(s)$, we have that for $s > 1$,

$$W(s) = \frac{2}{3\pi} s \left[\sin(\log s) + 1 + \frac{2}{s^2}\right];$$

so we easily find that

$$\limsup_{s\to+\infty} W(s) = +\infty,$$

and

$$\liminf_{s\to+\infty} W(s) = 0.$$

REMARK 2.4. As regards the problem of uniqueness of solutions for equations like (1.1), we recall some known results. In the semilinear case, it is possible to prove that the problem

$$-\Delta u = k(|u|)|Du|^2 + f, \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega),$$

for suitable hypotheses on k and f, has at most an unique solution. More precisely, the following result holds (see [**17**]): suppose that u_1, u_2 belong to $H^1(\Omega) \cap L^\infty(\Omega)$ and are respectively a subsolution and a supersolution of the Dirichlet problem

$$-\Delta u + H(x, u, Du) = 0, \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega), \tag{2.8}$$

where Ω is a bounded domain (connected open set) in $\mathbb{R}^n$, H is a Carathéodory function on $\Omega \times \mathbb{R} \times \mathbb{R}^n$ such that $(u, \xi) \in \mathbb{R} \times \mathbb{R}^n \to H(x, u, \xi)$ is continuously differentiable, $\left|\frac{\partial H}{\partial \xi}\right| \leq k_1(|u|)(1+|\xi|)$ and $|H(x, u, 0)| \leq k_2(|u|)$, where k_1 and k_2 are continuous functions of $|u|$, and $\frac{\partial H}{\partial u}(x, u, \xi) \geq \alpha_0$, where α_0 is a nonnegative constant, and $f \in H^{-1}(\Omega)$. If $(u_1 - u_2)^+ \in H_0^1(\Omega)$, then

$$u_1 \leq u_2 \quad \text{in } \Omega.$$

In particular, the equation (2.8) admits at most one solution in $H_0^1(\Omega) \cap L^\infty(\Omega)$.

The uniqueness property does not hold for unbounded solutions, as shown by some counterexamples (see [**70**], [**54**], [**52**], [**63**]). For instance, if we consider the problem

$$\begin{cases} -\Delta_p u = |Du|^q & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{2.9}$$

where $0 < \frac{p}{n} + p - 1 < q < p < n$ and B is the ball of $\mathbb{R}^n$ centered at the origin with radius R, it obviously admits the solution $u = 0$. Moreover, the function

$$u(x) = c(|x|^{-\frac{p-q}{1-p+q}} - R^{-\frac{p-q}{1-p+q}}),$$

where c is a suitable positive constant which depends on q, p and n, belongs to $W_0^{1,p}(B)$ and solves (2.9). However, the solution $u = 0$ is the unique bounded solution of (2.9).

In the case $p = q$, with $p < n$, it is possible to show that the function

$$u(x) = (p-1)\log\left(\frac{|x|^{-(n-p)/(p-1)} - m}{R^{-(n-p)/(p-1)} - m}\right),$$

where m is any constant such that $mR^{(n-p)/(p-1)} < 1$, is a solution of the equation

$$\begin{cases} -\Delta_p u = |Du|^p & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{2.10}$$

(see [**54**]), but (2.10) admits the unique bounded solution $u = 0$.

3. A priori estimates

The aim of this section is to prove *a priori* estimates of the solutions of (1.1). We prove a L^∞ estimate of u in terms of the function W.

THEOREM 3.1. *Let V be the solution of* (2.6). *Under assumptions* (1.2) – (1.6), *if u is a weak solution of* (1.1), *then*

$$W(u^*(0)) \leq C_0 V(0), \tag{3.1}$$

where C_0 is the constant defined in Theorem 2.1.

PROOF. Let u be a solution of (1.1). Using in (1.1) the test functions $T_t(u)$, $T_{t+h}(u)$, with $t, h > 0$, and subtracting, we have:

$$\int_{\{t<|u|\le t+h\}} a(x,u,Du)\cdot Du\,dx =$$
$$= \int_{\{t<|u|\le t+h\}} H(x,u,Du)(|u|-t)\operatorname{sign} u\,dx + \int_{\{t<|u|\le t+h\}} f(|u|-t)\operatorname{sign} u\,dx +$$
$$+ h\int_{\{|u|>t+h\}} H(x,u,Du)\operatorname{sign} u\,dx + h\int_{\{|u|>t+h\}} f\operatorname{sign} u\,dx;$$

so, dividing both sides by h and using (1.3) and (1.5), we get:

$$\frac{1}{h}\int_{\{t<|u|\le t+h\}} b(|u|)|Du|^p dx \le \int_{\{|u|>t\}} k(|u|)|Du|^q dx + \int_{\{|u|>t\}} |f|dx. \tag{3.2}$$

We note that from (3.2) and from the fact that u is a solution of (1.1), it follows that the function

$$\Phi(t) = \int_{\{|u|>t\}} b(|u|)|Du|^p dx$$

is Lipschitz continuous. If $h \to 0$, being b continuous, and using Hardy - Littlewood inequality, from (3.2) we get

$$-b(t)\frac{d}{dt}\int_{\{|u|>t\}} |Du|^p dx \le \int_{\{|u|>t\}} k(|u|)|Du|^q dx + \int_0^{\mu(t)} f^*(s)ds \tag{3.3}$$

By the continuity of k we have

$$\int_{\{|u|>t\}} k(|u|)|Du|^q dx = \int_t^{+\infty} k(s)\left(-\frac{d}{ds}\int_{\{|u|>s\}} |Du|^q dx\right) ds;$$

moreover, by Hölder inequality we have

$$-\frac{d}{ds}\int_{\{|u|>s\}} |Du|^q dx \le \left(-\frac{d}{ds}\int_{\{|u|>s\}} |Du|^p dx\right)^{\frac{q}{p}} (-\mu'(s))^{1-\frac{q}{p}},$$

and therefore

$$\int_{\{|u|>t\}} k(u)|Du|^q \le \int_t^{+\infty} k(s)\left(-\frac{d}{ds}\int_{\{|u|>s\}} |Du|^p dx\right)^{\frac{q}{p}} (-\mu'(s))^{1-\frac{q}{p}} ds. \tag{3.4}$$

Now, using again Hölder inequality, we get

$$\left(-\frac{d}{dt}\int_{\{|u|>t\}} |Du|^p dx\right)^{\frac{p-q}{p}} \ge \left[\left(-\frac{d}{dt}\int_{\{|u|>t\}} |Du| dx\right)(-\mu'(t))^{-1+\frac{1}{p}}\right]^{p-q},$$

that is

$$\left(-\frac{d}{dt}\int_{\{|u|>t\}}|Du|^p dx\right)^{\frac{q}{p}}(-\mu'(t))^{1-\frac{q}{p}} \le$$
$$\le\left(-\frac{d}{dt}\int_{\{|u|>t\}}|Du|dx\right)^{q-p}\left(-\frac{d}{dt}\int_{\{|u|>t\}}|Du|^p dx\right)(-\mu'(t))^{(1-\frac{1}{p})(p-q)+(1-\frac{q}{p})}.$$

Using Fleming and Rishel formula and isoperimetric inequality, and observing that $p-q=(1-1/p)\,(p-q)+(1-q/p)$, we get

$$\left(-\frac{d}{dt}\int_{\{|u|>t\}}|Du|^p dx\right)^{\frac{q}{p}}(-\mu'(t))^{1-\frac{q}{p}} \le \tag{3.5}$$
$$\le\left(n\omega_n^{1/n}\mu(t)^{1-\frac{1}{n}}\right)^{q-p}\left(-\frac{d}{dt}\int_{\{|u|>t\}}|Du|^p dx\right)(-\mu'(t))^{p-q}.$$

So, from (3.4) and (3.5) we have

$$\int_{\{|u|>t\}}k(u)|Du|^q dx \le \tag{3.6}$$
$$\le\left(n\omega_n^{1/n}\right)^{q-p}\int_t^{+\infty}k(s)\left(-\frac{d}{ds}\int_{\{|u|>s\}}|Du|^p dx\right)\left(\frac{-\mu'(s)}{\mu(s)^{1-\frac{1}{n}}}\right)^{p-q}ds.$$

From (3.3) and (3.6) we obtain

$$-b(t)\frac{d}{dt}\int_{\{|u|>t\}}|Du|^p dx \le$$
$$\le\left(n\omega_n^{1/n}\right)^{q-p}\int_t^{+\infty}k(s)\left(-\frac{d}{ds}\int_{\{|u|>s\}}|Du|^p dx\right)\left(\frac{-\mu'(s)}{\mu(s)^{1-\frac{1}{n}}}\right)^{p-q}ds+\int_0^{\mu(t)}f^*(s)ds.$$

The previous inequality and Gronwall lemma imply:

$$-b(t)\frac{d}{dt}\int_{\{|u|>t\}}|Du|^p dx \le$$
$$\le\int_t^{+\infty}\exp\left\{\left(n\omega_n^{1/n}\right)^{q-p}\int_t^s\frac{k(y)}{b(y)}\left(\frac{-\mu'(y)}{\mu(y)^{1-\frac{1}{n}}}\right)^{p-q}dy\right\}f^*(\mu(s))[-d\mu(s)].$$

and then

$$-b(t)\frac{d}{dt}\int_{\{|u|>t\}}|Du|^p dx \le \tag{3.7}$$
$$\le\int_0^{\mu(t)}\exp\left\{\left(n\omega_n^{1/n}\right)^{q-p}\int_t^{u^*(r)}\frac{k(y)}{b(y)}\left(\frac{-\mu'(y)}{\mu(y)^{1-\frac{1}{n}}}\right)^{p-q}dy\right\}f^*(r)dr.$$

On the other hand, if $p-1<q<p$, using Hölder inequality we have

$$\int_t^{u^*(r)} \frac{k(y)}{b(y)} \left(\frac{-\mu'(y)}{\mu(y)^{1-\frac{1}{n}}}\right)^{p-q} dy \le \tag{3.8}$$
$$\le \left[\int_t^{u^*(r)} \left(\frac{k(y)}{b(y)}\right)^{\frac{1}{1-p+q}} dy\right]^{1-p+q} \left[\int_t^{u^*(r)} \frac{-\mu'(y)}{\mu(y)^{1-\frac{1}{n}}} dy\right]^{p-q}.$$

(Observe that last inequality is trivial if $q=p$). Furthermore, by the properties of the distribution function μ of u, we have

$$\int_t^{u^*(r)} \frac{-\mu'(y)}{\mu(y)^{1-\frac{1}{n}}} dy \le \int_0^{+\infty} \frac{-\mu'(y)}{\mu(y)^{1-\frac{1}{n}}} dy \le n|\Omega|^{\frac{1}{n}}. \tag{3.9}$$

Using (3.8) and (3.9) in (3.7), we get

$$-b(t)\frac{d}{dt}\int_{\{|u|>t\}} |Du|^p dx \le \tag{3.10}$$
$$\le \int_0^{\mu(t)} \exp\left\{\left(\frac{|\Omega|}{\omega_n}\right)^{\frac{p-q}{n}} \left[\int_t^{u^*(r)} \left(\frac{k(y)}{b(y)}\right)^{\frac{1}{1-p+q}} dy\right]^{1-p+q}\right\} f^*(r)dr.$$

Now we recall that if $x \ge 0$, and $0 \le \alpha \le 1$, we have

$$x^\alpha \le \alpha x + (1-\alpha); \tag{3.11}$$

so applying (3.11) to (3.10), we have

$$-b(t)\frac{d}{dt}\int_{\{|u|>t\}} |Du|^p dx \le$$
$$\le \int_0^{\mu(t)} \exp\left\{\left(\frac{|\Omega|}{\omega_n}\right)^{\frac{p-q}{n}} \left[(p-q) + (1-p+q)\int_t^{u^*(r)} \left(\frac{k(y)}{b(y)}\right)^{\frac{1}{1-p+q}} dy\right]\right\} f^*(r)dr,$$

that is

$$b(t)\left(-\frac{d}{dt}\int_{\{|u|>t\}} |Du|^p dx\right) \le e^{(p-q)(|\Omega|/\omega_n)^{\frac{p-q}{n}}} \int_0^{\mu(t)} e^{(p-1)[Q(u^*(r))-Q(t)]} f^*(r)dr, \tag{3.12}$$

with

$$Q(t) = \left(\frac{|\Omega|}{\omega_n}\right)^{\frac{p-q}{n}} \frac{1-p+q}{p-1} \int_0^t \left(\frac{k(y)}{b(y)}\right)^{\frac{1}{1-p+q}} dy.$$

On the other hand, we have by Hölder inequality that

$$-\frac{d}{dt}\int_{\{|u|>t\}} |Du|^p dx \ge \left(-\frac{d}{dt}\int_{\{|u|>t\}} |Du| dx\right)^p (-\mu'(t))^{-\frac{p}{p'}}$$

Applying again the coarea formula and the isoperimetric inequality, we have

$$-\frac{d}{dt}\int_{\{|u|>t\}} |Du| dx \ge n\omega_n^{1/n}\mu(t)^{1-\frac{1}{n}},$$

and then

$$b(t)\left(n\omega_n^{1/n}\right)^p \mu(t)^{p-\frac{p}{n}}(-\mu'(t))^{-\frac{p}{p'}} \le e^{(p-q)(|\Omega|/\omega_n)^{\frac{p-q}{n}}} \int_0^{\mu(t)} e^{(p-1)[Q(u^*(r))-Q(t)]} f^*(r)dr,$$

that is

$$b(t)e^{(p-1)Q(t)} \le e^{(p-q)(|\Omega|/\omega_n)^{\frac{p-q}{n}}} \left(n\omega_n^{1/n}\right)^{-p} \left(\frac{-\mu'(t)^{\frac{1}{p'}}}{\mu(t)^{1-\frac{1}{n}}}\right)^p \int_0^{\mu(t)} e^{(p-1)Q(u^*(r))} f^*(r)dr.$$

recalling that u^* is a decreasing function, we have

$$\frac{b(t)^{1/(p-1)}e^{Q(t)}}{e^{Q(u^*(0))}} \le e^{\frac{p'}{p}(p-q)(|\Omega|/\omega_n)^{\frac{p-q}{n}}} \left(n\omega_n^{1/n}\right)^{-p'} \frac{-\mu'(t)}{\mu(t)^{p'-\frac{p'}{n}}} \left(\int_0^{\mu(t)} f^*(r)dr\right)^{\frac{p'}{p}}.$$

Integrating between 0 and σ,

$$\frac{F(\sigma)}{e^{Q(u^*(0))}} \le e^{\frac{p'}{p}(p-q)(|\Omega|/\omega_n)^{\frac{p-q}{n}}} \left(n\omega_n^{1/n}\right)^{-p'} \int_0^{\sigma} \frac{-\mu'(t)}{\mu(t)^{p'-\frac{p'}{n}}} \left(\int_0^{\mu(t)} f^*(r)dr\right)^{\frac{p'}{p}} dt,$$

where $F(s)$ and $Q(s)$ is the function defined in (2.1), (2.2). If $\sigma = u^*(s)$, using the properties of the rearrangements, we have

$$\frac{F(u^*(s))}{e^{Q(u^*(0))}} \le e^{\frac{p'}{p}(p-q)(|\Omega|/\omega_n)^{\frac{p-q}{n}}} \left(n\omega_n^{1/n}\right)^{-p'} \int_s^{|\Omega|} t^{-\left(1-\frac{1}{n}\right)p'} \left(\int_0^{t} f^*(r)dr\right)^{\frac{p'}{p}} dt. \tag{3.13}$$

that is

$$W(u^*(0)) \le e^{\frac{p'}{p}(p-q)(|\Omega|/\omega_n)^{\frac{p-q}{n}}} V(0).$$

□

From the proof of Theorem 3.1 we get easily a $W^{1,p}$ estimate of u:

Proposition 3.1. *Let V be the solution of* (2.6). *Under assumptions* (1.2) – (1.6), *if u is a solution of* (1.1), *then*

$$\int_\Omega |Du|^p dx \le \frac{u^*(0)}{m} \exp\left\{ \left(\frac{|\Omega|}{\omega_n}\right)^{\frac{p-q}{n}} \left[\int_0^{u^*(0)} \left(\frac{k(y)}{b(y)}\right)^{\frac{1}{1-p+q}} dy\right]^{1-p+q} \right\} \int_0^{|\Omega|} f^*(r)dr, \tag{3.14}$$

where $m = \min_{t\in[0,u^*(0)]} b(t)$.

Proof. By (3.10) it follows that

$$-b(t)\frac{d}{dt}\int_{\{|u|>t\}} |Du|^p dx \le$$

$$\le \exp\left\{ \left(\frac{|\Omega|}{\omega_n}\right)^{\frac{p-q}{n}} \left[\int_0^{u^*(0)} \left(\frac{k(y)}{b(y)}\right)^{\frac{1}{1-p+q}} dy\right]^{1-p+q} \right\} \int_0^{|\Omega|} f^*(r)dr;$$

so integrating between 0 and $u^*(0)$ we obtain the estimate (3.14). □

4. Proof of Theorem 2.1

In order to prove Theorem 2.1, we need to define some auxiliary functions and to study their behaviour. For any $\lambda > 0$, we put

$$k_\lambda(s) = k(T_\lambda(s)), \qquad b_\lambda(s) = b(T_\lambda(s)).$$

Furthermore, we set

$$Q_\lambda : s \in [0, +\infty[\to C \int_0^s \left[\frac{k_\lambda(y)}{b_\lambda(y)}\right]^{\frac{1}{1-p+q}} dy; \tag{4.1}$$

$$F_\lambda : s \in [0, +\infty[\to \int_0^s e^{Q_\lambda(t)} b_\lambda(t)^{\frac{1}{p-1}} dt; \tag{4.2}$$

$$W_\lambda : s \in [0, +\infty[\to \frac{F_\lambda(s)}{e^{Q_\lambda(s)}}; \tag{4.3}$$

where

$$C = \frac{1-p+q}{p-1} \left(\frac{|\Omega|}{\omega_n}\right)^{\frac{p-q}{n}}.$$

We observe that W is continuously differentiable, $W(0) = 0$ and

$$W'(s) = b(s)^{\frac{1}{p-1}} - CW(s) \left(\frac{k(s)}{b(s)}\right)^{\frac{1}{1-p+q}}.$$

It follows that $W'(0) > 0$. When $s > 0$, we have

$$W'(s) > 0 \text{ if } k(s) = 0, \tag{4.4}$$

while if $k(s) \neq 0$ then

$$W'(s) \geq 0 \iff W(s) \leq C^{-1} \frac{b(s)^{\frac{1}{p-1}\left(\frac{q}{p-1}\right)'}}{k(s)^{\frac{1}{1-p+q}}}. \tag{4.5}$$

Clearly we have:

$$W_\lambda(s) = W(s) \qquad \text{if } 0 \leq s \leq \lambda,$$

while for $s > \lambda$, it holds:

$$\begin{aligned}
W_\lambda(s) &= \int_0^s e^{C \int_s^r \left[\frac{k_\lambda(y)}{b_\lambda(y)}\right]^{\frac{1}{1-p+q}} dy} b_\lambda(r)^{\frac{1}{p-1}} dr \\
&= W(\lambda) e^{C \int_s^\lambda \left[\frac{k_\lambda(y)}{b_\lambda(y)}\right]^{\frac{1}{1-p+q}} dy} + \int_\lambda^s e^{C \int_s^r \left[\frac{k_\lambda(y)}{b_\lambda(y)}\right]^{\frac{1}{1-p+q}} dy} b_\lambda(r)^{\frac{1}{p-1}} dr = \\
&= W(\lambda) e^{C(\lambda - s)\left(\frac{k(\lambda)}{b(\lambda)}\right)^{\frac{1}{1-p+q}}} + b(\lambda)^{\frac{1}{p-1}} \int_\lambda^s e^{(r-s)\left(\frac{k(\lambda)}{b(\lambda)}\right)^{\frac{1}{1-p+q}}} dr.
\end{aligned}$$

It follows that, for $s > \lambda$,

$$\text{(4.6)}\quad W_\lambda(s) =$$
$$= \begin{cases} W(\lambda) + (s-\lambda) b(\lambda)^{\frac{1}{p-1}} & \text{if } k(\lambda) = 0; \\ \left[W(\lambda) - C^{-1} \frac{b(\lambda)^{\frac{1}{p-1}} \left(\frac{q}{p-1}\right)'}{k(\lambda)^{\frac{1}{1-p+q}}} \right] e^{C(\lambda - s)\left(\frac{k(\lambda)}{b(\lambda)}\right)^{\frac{1}{1-p+q}}} + C^{-1} \frac{b(\lambda)^{\frac{1}{p-1}} \left(\frac{q}{p-1}\right)'}{k(\lambda)^{\frac{1}{1-p+q}}} & \text{if } k(\lambda) > 0. \end{cases}$$

Thus from (4.4), (4.5) and (4.6) it follows that $W'(\lambda) \geq 0$ implies

$$W_\lambda(s) \geq W(\lambda), \quad \forall\, s > \lambda. \tag{4.7}$$

Inequality (4.7) is immediate for $k(\lambda) = 0$, while for $k(\lambda) > 0$ we have to observe that, in view of (4.5), $W_\lambda(s)$ is increasing with respect to s.

For any fixed $\lambda > 0$, we consider the truncated problem:

$$-\operatorname{div}(a_\lambda(x, u_\lambda, Du_\lambda)) = H_\lambda(x, u_\lambda, Du_\lambda) + f, \quad u_\lambda \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \tag{4.8}$$

where $a_\lambda(x, \eta, \xi)$ and $H_\lambda(x, \eta, \xi)$ are defined in the following way:

$$a_\lambda(x, s, \xi) = \begin{cases} a(x, s, \xi) & \text{if } |s| \leq \lambda, \\ a(x, \lambda, \xi) & \text{if } s > \lambda, \\ a(x, -\lambda, \xi) & \text{if } s < -\lambda. \end{cases}$$

and

$$H_\lambda(x, s, \xi) = \begin{cases} H(x, s, \xi) & \text{if } |s| \leq \lambda, \\ H(x, \lambda, \xi) & \text{if } s > \lambda, \\ H(x, -\lambda, \xi) & \text{if } s < -\lambda. \end{cases}$$

The functions $a_\lambda(x, s, \xi)$ and $H_\lambda(x, s, \xi)$ verify the assumptions (1.2)$-$(1.5) with b_λ and k_λ instead of b and k. So by Theorem 3.1 it follows that if u_λ is a solution of (4.8), then

$$W_\lambda(u_\lambda^*(0)) \leq C_0 V(0), \tag{4.9}$$

where W_λ is the function defined in (4.3), $V(0)$ is given by (2.5) and C_0 is the constant of the Theorem 3.1.

Remark 4.1. We observe that if there exists $\lambda > 0$ such that

$$u_\lambda^*(0) \leq \lambda,$$

then u_λ also solves the problem (1.1).

Now we obtain an uniform estimate for the solutions of the approximate problems (4.8):

Proposition 4.1. *Under assumptions* (1.2) $-$ (1.6), *if*

$$C_0 V(0) < \sup_{s>0} W(s), \tag{4.10}$$

where $C_0 = C_0(p, q, n, |\Omega|)$ *is given in Theorem* 3.1*, then there exists* $\lambda > 0$ *such that if* u_λ *is a solution of* (4.8)*, it results*

$$u^*_\lambda(0) \le \lambda. \tag{4.11}$$

Proof. We define

$$\tau = \sup\{s \in [0, +\infty[: W(\sigma) \le C_0 V(0),\ \forall \sigma \in [0, s]\}; \tag{4.12}$$

because of assumption (4.10), we have $0 \le \tau < +\infty$, and obviously $W(\tau) = C_0 V(0)$; moreover, there exists $\lambda > \tau$ such that $W(\tau) < W(\lambda)$ and $W'(\lambda) \ge 0$. By (4.7), we get

$$C_0 V(0) = W(\tau) < W(\lambda) \le W_\lambda(s), \quad \forall s > \lambda.$$

On the other hand, if u_λ is a solution of the approximate equation (4.8) at the truncation value λ, by (4.9) we have $W_\lambda(u^*_\lambda(0)) \le C_0 V(0)$, so it follows that

$$u^*_\lambda(0) \le \lambda.$$

□

Remark 4.2. As a matter of fact, we can choose λ in order to get a better estimate of the solutions. The following two facts can happen:

a) there exists $\bar\lambda > \tau$ such that $W(s) > W(\tau)$ for any $s \in]\tau, \bar\lambda[$; then if $u_{\bar\lambda}$ is a solution of (4.8) at the truncation value $\bar\lambda$, by Proposition 4.1 we get $u^*_{\bar\lambda}(0) \le \bar\lambda$. On the other hand, $W_{\bar\lambda}(s) = W(s)$ for any $s \le \bar\lambda$, so being $W_{\bar\lambda}(u^*_{\bar\lambda}(0)) \le C_0 V(0)$ we obtain
$$u^*_{\bar\lambda}(0) \le \tau;$$
b) such value $\bar\lambda$ does not exist; in this case, for any $\varepsilon > 0$ small, we can choose λ in the proof of Proposition 4.1 such that $\tau < \lambda < \tau + \varepsilon$ and $W(\lambda) > W(\tau)$, obtaining (4.11).

Remark 4.3. We observe that in the proof of Proposition 4.1 only the functions W and W_λ are involved. Thus we deduce that if the condition (4.10) is satisfied, there exists $\lambda > 0$ such that every solution v of problem like (1.1) satisfying, for a.e. $x \in \Omega$, for every $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n$, instead of (1.3) and (1.5) the assumptions:

$$a(x, s, \xi)\xi \ge b_\lambda(|s|)|\xi|^p$$
$$|H(x, s, \xi)| \le k_\lambda(|s|)|\xi|^q$$

verifies

$$v^*(0) \le \lambda.$$

It is clear that in order to prove Theorem 2.1, we need only to prove existence for the problem (4.8), and then use Remark 4.1.

Proof of the Theorem 2.1. Given $\lambda > 0$ and $\varepsilon > 0$, we consider the approximate problem

$$-\operatorname{div}(a_\lambda(x, u_{\lambda,\varepsilon}, Du_{\lambda,\varepsilon})) = H_{\lambda,\varepsilon}(x, u_{\lambda,\varepsilon}, Du_{\lambda,\varepsilon}) + f, \quad u_{\lambda,\varepsilon} \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \tag{4.13}$$

where

$$H_{\lambda,\varepsilon}(x, s, \xi) = \frac{H_\lambda(x, s, \xi)}{1 + \varepsilon |H_\lambda(x, s, \xi)|}.$$

We have

$$|H_{\lambda,\varepsilon}(x, s, \xi)| \leq |H_\lambda(x, s, \xi)|$$

and

$$|H_{\lambda,\varepsilon}(x, s, \xi)| \leq \frac{1}{\varepsilon}.$$

It is well known (see [**76**]) that problem (4.13) admit a solution $u_{\lambda,\varepsilon}$ for every $\varepsilon > 0$. Let us observe that

$$a_\lambda(x, s, \xi)\xi \geq b_\lambda(|s|)|\xi|^p,$$
$$|H_{\lambda,\varepsilon}(x, s, \xi)| \leq k_\lambda(|s|)|\xi|^q.$$

By Proposition 4.1 and Remark 4.3 it follows that there exists $\lambda > 0$ such that

$$u^*_{\lambda,\varepsilon}(0) \leq \lambda \tag{4.14}$$

for every $\varepsilon > 0$. Thus $\{u_{\lambda,\varepsilon}\}_{\varepsilon>0}$ is bounded in $L^\infty(\Omega)$. By (3.14) and (4.14) it follows that

$$\int_\Omega |Du_{\lambda,\varepsilon}|^p dx \leq \frac{\lambda}{m_\lambda} \exp\left\{ \left(\frac{|\Omega|}{\omega_n}\right)^{\frac{p-q}{n}} \left[\int_0^\lambda \left(\frac{k_\lambda(y)}{b_\lambda(y)}\right)^{\frac{1}{1-p+q}} dy \right]^{1-p+q} \right\} \int_0^{|\Omega|} f^*(r)dr,$$

where $m_\lambda = \min_{t\in[0,\lambda]} b_\lambda(t)$. So $\{u_{\lambda,\varepsilon}\}_{\varepsilon>0}$ is bounded in $W^{1,p}(\Omega)$, and weakly converges to a function $u_\lambda \in W^{1,p}(\Omega)$. Using standard techniques (see [**29, 31, 30, 32**]), it is possible to extract a subsequence which strongly converges in $W^{1,p}(\Omega)$ to a function $u_\lambda \in W^{1,p}(\Omega)$ which solves (4.8). Being $u^*_\lambda(0) \leq \lambda$, this is a solution of (1.1). $\square$

CHAPTER 5

Nonlinear elliptic equations with unbounded coefficients

1. Statement of the problem and definitions of solutions

Let $b : [0, m) \to (0, +\infty)$, with $m > 0$, be a continuous function such that

$$\lim_{s \to m^-} b(s) = +\infty. \tag{1.1}$$

We deal with Dirichlet problem of the form

$$\begin{cases} -\operatorname{div}(a(x, u, Du)) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded open set in $\mathbb{R}^n$, $n \geq 2$, $1 < p < n$, and $a : \Omega \times (-m, m) \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function verifying the following assumptions:

$$b(|s|)|\xi|^p \leq a(x, s, \xi) \cdot \xi, \tag{1.3}$$

for a.e. $x \in \Omega$, $\forall\, s \in (-m, m)$ and $\forall\, \xi \in \mathbb{R}^n$,

$$|a(x, s, \xi)| \leq C(h(x) + b(|s|)|\xi|^{p-1}) \tag{1.4}$$

for a.e. $x \in \Omega$, $\forall\, s \in (-m, m)$ and $\forall\, \xi \in \mathbb{R}^n$,

$$(a(x, s, \xi) - a(x, s, \xi')) \cdot (\xi - \xi') > 0, \tag{1.5}$$

for a.e. $x \in \Omega$, $\forall\, s \in (-m, m)$ and $\forall\, \xi, \xi' \in \mathbb{R}^n$, $\xi \neq \xi'$. Moreover, f is a measurable function on whose summability we will make different assumptions.

In this context we deal with some classes of solutions.

Definition 1.1. We say that $u \in W_0^{1,p}(\Omega)$ is a weak solution to problem (1.2) if

$$\int_\Omega a(x, u, Du) \cdot D\varphi\, dx = \int_\Omega f\varphi\, dx, \quad \forall\, \varphi \in W_0^{1,p}(\Omega). \tag{1.6}$$

We emphasize that, in general, our assumptions on problem (1.2) do not assure the existence of a weak solution (see the counterexample 4.1 in Section 4). For this reason, we need to introduce a special class of entropy solutions. In this way, we will obtain existence results for solutions which can achieve the critical values $\pm m$.

Definition 1.2. A measurable function $u \in W_0^{1,p}(\Omega)$ is an entropy solution to problem (1.2) if

$$|u| \leq m \text{ a.e. in } \Omega \tag{1.7}$$

and u satisfies, for all $0 < k < m$,

$$\int_\Omega a(x,u,Du)\cdot DT_k(u-\varphi)\,dx \le \int_\Omega fT_k(u-\varphi)dx \tag{1.8}$$

for any $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ s.t. $\|\varphi\|_{L^\infty(\Omega)} < m-k$.

2. A priori estimates

Let us define

$$B(s) = \int_0^s b(t)^{\frac{1}{p-1}}dt, \quad s \in [0,m).$$

We consider, for any $\varepsilon > 0$ sufficiently small, the following problem:

$$\begin{cases} -\operatorname{div}(a_\varepsilon(x,u_\varepsilon,Du_\varepsilon)) = f_\varepsilon & \text{in } \Omega,\\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $a_\varepsilon(x,s,\xi) = a(x,T_{m-\varepsilon}(s),\xi)$, with $x \in \Omega$, $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$ and $f_\varepsilon \in L^\infty(\Omega)$. We observe that classical results (see, for example, [**73**], [**76**]) assure that problem 2.1 has at least one solution $u_\varepsilon \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. Moreover, we define

$$b_\varepsilon(t) = b(T_{m-\varepsilon}(t)), \quad \forall t \in [0,+\infty), \text{ and } \quad B_\varepsilon(t) = \int_0^t b_\varepsilon(s)^{\frac{1}{p-1}}ds.$$

First of all, we prove an integral inequality for weak solutions of problem (2.1).

Proposition 2.1. *Let u_ε be a weak solution of* (2.1)*. Then the following inequality holds:*

$$B_\varepsilon(u_\varepsilon^*(s)) \le \left(n\omega_n^{1/n}\right)^{-p'}\int_s^{|\Omega|} t^{-\frac{p'}{n'}}\left(\int_0^t f_\varepsilon^*(\sigma)d\sigma\right)^{\frac{p'}{p}}dt, \quad s \in [0,|\Omega|]. \tag{2.2}$$

Proof. Let be $t,h > 0$. Using as test function $\varphi = T_h(u_\varepsilon - T_t(u_\varepsilon))$, by hypothesis (1.2), we get

$$\frac{1}{h}\int_{t<|u_\varepsilon|\le t+h} b_\varepsilon(|u_\varepsilon|)|Du_\varepsilon|^p \le \int_{|u_\varepsilon|>t}|f_\varepsilon|dx. \tag{2.3}$$

Applying Hardy–Littlewood inequality and letting $h \to 0$, we obtain

$$b_\varepsilon(t)\frac{d}{dt}\int_{|u_\varepsilon|\le t}|Du_\varepsilon|^p dx \le \int_0^{\mu_{u_\varepsilon}(t)} f_\varepsilon^*(s)ds. \tag{2.4}$$

By Hölder inequality we have

$$b_\varepsilon(t)\left(\frac{d}{dt}\int_{|u_\varepsilon|\le t}|Du_\varepsilon|dx\right)^p(-\mu'_{u_\varepsilon}(t))^{-\frac{p}{p'}} \le \int_0^{\mu_{u_\varepsilon}(t)} f_\varepsilon^*(s)ds.$$

From isoperimetric inequality and Fleming–Rishel formula, it follows that

$$b_\varepsilon(t) \le \left(n\omega_n^{1/n}\right)^{-p}(\mu_{u_\varepsilon}(t))^{-\frac{p}{n'}}(-\mu'_{u_\varepsilon}(t))^{\frac{p}{p'}}\int_0^{\mu_{u_\varepsilon}(t)} f_\varepsilon^*(\sigma)d\sigma.$$

Finally, raising to the power $1/(p-1)$ and using the properties of rearrangements, we easily get (2.2). $\square$

PROPOSITION 2.2. *Let u_ε be a solution of* (2.1). *The following inequalities hold:*

(a) *if $1 < r < n/p$, then*

$$\|B_\varepsilon(|u_\varepsilon|)^q\|_{L^1(\Omega)} \le c\|f_\varepsilon\|_{L^r(\Omega)}^{q\frac{p'}{p}} \tag{2.5}$$

where $q = nr(p-1)/(n-rp)$;

(b) *if $r = 1$, then*

$$\|B_\varepsilon(|u_\varepsilon|)\|_{M^{\frac{n(p-1)}{n-p}}(\Omega)} \le c\|f_\varepsilon\|_{L^1(\Omega)}^{\frac{p'}{p}}. \tag{2.6}$$

PROOF. Let us first observe that, being B_ε monotone, by (2.2) and properties of rearrangements, we get

$$\|B_\varepsilon(|u_\varepsilon|)^q\|_{L^1(\Omega)} \le \left(n\omega_n^{1/n}\right)^{-qp'} \int_0^{|\Omega|} \left(\int_s^{|\Omega|} t^{-\frac{p'}{n'}} \left(\int_0^t f_\varepsilon^*(\sigma) d\sigma \right)^{\frac{p'}{p}} dt \right)^q ds;$$

applying the inequalities (3.7) and (3.8), we obtain

$$\|B_\varepsilon(|u_\varepsilon|)^q\|_{L^1(\Omega)} \le c \int_0^{|\Omega|} s^{\frac{qp'}{n}} f_\varepsilon^*(s)^{\frac{qp'}{p}} ds,$$

so, recalling that, in particular, $f_\varepsilon \in M^r(\Omega)$, we get (a).

Now let be $r = 1$. The inequality (2.2) implies

$$B_\varepsilon(u_\varepsilon^*(s)) \le \frac{n'\|f_\varepsilon\|_{L^1(\Omega)}^{\frac{p'}{p}}}{(p'-n')(n\omega_n^{1/n})^{p'} s^{p'/n'-1}}.$$

This proves part (b). $\square$

Now we use estimates of Proposition 2.2 to obtain some gradient estimates of the solutions of (2.1) with respect to some L^r–norm of f_ε.

PROPOSITION 2.3. *Let u_ε be a weak solution of* (2.1). *The following estimates hold:*

(1) *if r is such that*

$$\frac{np}{np-n+p} \le r < \frac{n}{p},$$

then

$$\||DB_\varepsilon(|u_\varepsilon|)|\|_{L^p(\Omega)} \le c_1, \tag{2.7}$$

where the constant c_1 continuously depends on the norm of f_ε in $L^r(\Omega)$;

(2) *if r is such that*

$$\max\left\{1, \frac{n}{np-n+1}\right\} < r < \frac{np}{np-n+p},$$

then

$$\| |DB_\varepsilon(|u_\varepsilon|)| \|_{L^\beta(\Omega)} \le c_2, \tag{2.8}$$

where $\beta = nr(p-1)/(n-r)$ *and the constant* c_2 *continuously depends on the norm of* f_ε *in* $L^r(\Omega)$*;*

(3) *if* r *is such that*

$$1 \le r \le \max\left\{1, \frac{n}{np-n+1}\right\},$$

then

$$\| |DB_\varepsilon(|u_\varepsilon|)| \|_{M^\beta(\Omega)} \le c_3, \tag{2.9}$$

with $\beta = nr(p-1)/(n-r)$ *where the constant* c_3 *continuously depends on the norm of* f_ε *in* $L^r(\Omega)$.

Proof. If u_ε is a solution of the equation (2.1), by the definition of B_ε we can use as test function $v = T_h(B_\varepsilon(u_\varepsilon) - T_t(B_\varepsilon(u_\varepsilon)))$ and obtain

$$\frac{1}{h}\int_{t<B_\varepsilon(|u_\varepsilon|)\le t+h} |D(B_\varepsilon(|u_\varepsilon|)|^p dx \le \int_{B_\varepsilon(|u_\varepsilon|)>t} |f_\varepsilon| dx. \tag{2.10}$$

Let us prove part (1). Passing to the limit in (2.10), we get

$$\frac{d}{dt}\int_{B_\varepsilon(|u_\varepsilon|)\le t} |DB_\varepsilon(|u_\varepsilon|)|^p dx \le \int_0^{\mu_\varepsilon(t)} f_\varepsilon^*(s) ds, \tag{2.11}$$

where we have denoted with $\mu_\varepsilon(t)$ the distribution function of $B_\varepsilon(|u_\varepsilon|)$. Integrating (2.11) between 0 and $+\infty$ and using Hölder inequality, we have

$$\begin{aligned}\int_\Omega |DB_\varepsilon(|u_\varepsilon|)|^p dx &\le \\ &\le \int_0^{+\infty} dt \int_0^{\mu_\varepsilon(t)} f_\varepsilon^*(s) ds = \int_0^{|\Omega|} B_\varepsilon(u_\varepsilon^*(s)) f_\varepsilon^*(s) ds \le \\ &\le \|f\|_{L^r(\Omega)} \|B_\varepsilon(|u_\varepsilon|)\|_{L^{r'}(\Omega)}.\end{aligned} \tag{2.12}$$

We observe that if r is such that $np/(np-n+p) \le r < n/p$, by (2.5) the right hand side of last inequality is controlled by a constant depending on the norm of f_ε in $L^r(\Omega)$; so by (2.12), the inequality (2.7) follows.

As regards part (2), applying Hölder inequality in (2.10) and reasoning as before, we get

$$\int_\Omega |DB_\varepsilon(|u_\varepsilon|)|^\beta dx \le \int_0^{+\infty} \left(\int_0^{\mu_\varepsilon(t)} f_\varepsilon^*(s) ds\right)^{\beta/p} (-\mu_\varepsilon'(t))^{1-\beta/p} dt \le \tag{2.13}$$

$$\leq \left(\int_0^{+\infty} (1+t)^q (-\mu_\varepsilon'(t))dt \right)^{1-\beta/p} \left(\int_0^{+\infty} (1+t)^{q(1-p/\beta)} \left(\int_0^{\mu_\varepsilon(t)} f_\varepsilon^*(s)ds \right) dt \right)^{\beta/p},$$

where $q = nr(p - 1/(n - rp))$. Using the properties of rearrangements, we can write the first integral of the right hand side of (2.13) as

$$\int_0^{+\infty} (1+t)^q (-\mu_\varepsilon'(t))dt = \int_0^{|\Omega|} (1 + B_\varepsilon(u_\varepsilon^*(s)))^q ds,$$

and by (2.5) this quantity is bounded by a constant depending on the norm of f_ε in $L^r(\Omega)$. On the other hand, integrating by parts the second integral of the right hand side of (2.13) we have

$$\int_0^{+\infty} (1+t)^{q(1-p/\beta)} \left(\int_0^{\mu_\varepsilon(t)} f_\varepsilon^*(s)ds \right) dt \leq$$

$$\leq c \int_0^{|\Omega|} f_\varepsilon^*(s) \left[(1 + B_\varepsilon(u_\varepsilon^*(s)))^{(q(1-p/\beta)+1)} - 1 \right] ds \leq$$

$$\leq c \|f_\varepsilon\|_{L^r(\Omega)} \left\{ \int_0^{|\Omega|} \left[(1 + B_\varepsilon(u_\varepsilon^*(s)))^q \right] ds \right\}^{1-1/r}.$$

Applying again (2.5), by (2.13) it follows the estimate (2.8).

Finally, we prove part (3). Integrating between 0 and k the inequality (2.11), we obtain

$$\int_{B_\varepsilon(|u_\varepsilon|)\leq k} |DB_\varepsilon(|u_\varepsilon|)|^p dx \leq \int_0^k dt \int_0^{\mu_\varepsilon(t)} f_\varepsilon^*(\tau)d\tau. \tag{2.14}$$

If $r = 1$, from (2.14) we get

$$\int_{B_\varepsilon(|u_\varepsilon|)\leq k} |DB_\varepsilon(|u_\varepsilon|)|^p dx \leq k \|f_\varepsilon\|_{L^1(\Omega)}.$$

By (2.6) and lemma 3.2 we get the assertion.

If $1 < r \leq \max\{1, n/(np - n + 1)\}$, then by (2.5) it follows that $B_\varepsilon(|u_\varepsilon|) \in M^q(\Omega)$, with $q = nr(p-1)/(n - rp)$ as in Proposition 2.2; so we obtain

$$\int_{B_\varepsilon(|u_\varepsilon|)\leq k} |DB_\varepsilon(|u_\varepsilon|)|^p dx \leq c \int_0^k \mu_\varepsilon(t)^{\frac{1}{r'}} dt \leq c \int_0^k \frac{1}{t^{\frac{q}{r'}}} dt,$$

where the constant c depends on the norm of f_ε in $L^r(\Omega)$. Applying Lemma 3.2, the inequality (2.9) follows. $\square$

Remark 2.1. We observe that the achieved estimates hold also replacing $DB_\varepsilon(|u_\varepsilon|)$ by Du_ε; furthermore it follows that

$$\int_\Omega |Du_\varepsilon|^\gamma dx \leq c,$$

with $\gamma < n(p-1)/(n-1)$, c is a constant depending on the L^1 norm of f_ε. Using (2.4), $T_k(u_\varepsilon)$ are uniformly bounded in $W_0^{1,p}(\Omega)$ for any $k > 0$. Hence, there exists a function $u \in W_0^{1,\gamma}(\Omega)$ such that

$$u_\varepsilon \to u \text{ a.e. in } \Omega, \tag{2.15}$$

and, for any $k > 0$,

$$T_k(u_\varepsilon) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,p}(\Omega).$$

Now let us suppose $f \in L^1(\Omega)$. Using $T_{2m}(|u_\varepsilon|) - T_m(|u_\varepsilon|)$ as test function in (2.1) and by Poincaré's inequality, we deduce that

$$b(m-\varepsilon)\int_\Omega (T_{2m}(|u_\varepsilon|) - T_m(|u_\varepsilon|))^p dx \le m \|f_\varepsilon\|_{L^1(\Omega)}.$$

Letting $\varepsilon \to 0$, from the condition (1.1) we conclude that, for almost all $x \in \Omega$,

$$|u(x)| \le m.$$

Moreover, choosing $k \ge m$, we get

$$u_\varepsilon \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega). \tag{2.16}$$

3. Almost everywhere convergence of the gradients

In this section we will prove the almost everywhere convergence of the gradients.

THEOREM 3.1. *Let u_ε be a weak solution to* (2.1)*. Suppose $f \in L^1(\Omega)$, and let $f_\varepsilon \in L^\infty(\Omega)$ be such that $f_\varepsilon \to f$ in $L^1(\Omega)$. Then*

$$Du_\varepsilon \to Du \quad \text{a.e. in } \{|u| < m\}.$$

In order to prove Theorem 3.1, we need the following result (see [**27**]):

LEMMA 3.1. *Let (X, T, m) a measurable space, such that $m(X) < +\infty$. Let γ be a measurable function $\gamma : X \to [0, +\infty)$ such that $m(\{x \in X : \gamma(x) = 0\}) = 0$. Then for any $\sigma > 0$ there exists $\delta > 0$ such that:*

$$\int_A \gamma\, dm \le \delta \Rightarrow m(A) \le \sigma.$$

PROOF OF THEOREM 3.1. We will follow the proof contained in [**27**] (see also [**41**]). By Remark 2.1, we get that

$$u_\varepsilon \to u \text{ in measure}.$$

We will prove that

$$Du_\varepsilon \to Du \text{ in measure on } \{|u| < m\}. \tag{3.1}$$

In order to prove (3.1), given $\lambda > 0$ and $\eta > 0$, we set for some $r < m$, and M, $k > 0$,

$$\begin{aligned}
&E_1 = \{|u| < m\} \cap (\{|Du_\varepsilon| > M\} \cup \{|Du| > M\} \cup \{|u_\varepsilon| > k\} \cup \{|u| > k\}) \,, \\
&E_2 = \{|u| < m\} \cap \{|u_\varepsilon - u| > \eta\}, \\
&E_3 = \{|u| < m\} \cap \{|u_\varepsilon - u| \le \eta,\ |Du_\varepsilon| \le M,\ |Du| \le M,\ |u_\varepsilon| \le k,\ |u| \le k,\ |D(u_\varepsilon - u)| \ge \lambda\}.
\end{aligned}$$

We remark that

$$\{|u| < m\} \cap \{|D(u_\varepsilon - u)| \ge \lambda\} \subset E_1 \cap E_2 \cap E_3.$$

Since u_ε and Du_ε are bounded in $L^1(\Omega)$, for any $\sigma > 0$ we can fix M and $k < m$ such that

$$|E_1| < \frac{\sigma}{3}$$

independently of ε.

As regards the measure of E_3, the monotonicity assumption (1.5) assures that there exists a real–valued function $\gamma(x)$ such that

$$|\{x \in \Omega :\ \gamma(x) = 0\}| = 0$$

and

$$(a(x, s, \xi) - a(x, s, \xi')) \cdot (\xi - \xi') \ge \gamma(x),$$

for any $s \in (-m, m)$, $\xi, \xi' \in \mathbb{R}^n$, $|s| \le k$, $|\xi|, |\xi'| \le M$, and $|\xi - \xi'| \ge \lambda$. Then, denoting with χ_η the characteristic function of $[0, \eta]$, we get

$$\begin{aligned}
&\int_{E_3} \gamma(x) dx \le \int_{E_3} [a_\varepsilon(x, u_\varepsilon, Du_\varepsilon) - a_\varepsilon(x, u_\varepsilon, Du)] \cdot D(u_\varepsilon - u) dx \le \\
&\le \int_{\{|u_\varepsilon| \le k,\, |u| \le k\}} [a_\varepsilon(x, u_\varepsilon, Du_\varepsilon) - a_\varepsilon(x, u_\varepsilon, DT_k(u))] \cdot D(u_\varepsilon - T_k(u)) \chi_\eta(|u_\varepsilon - T_k(u)|) dx \le \\
&\qquad \le \int_\Omega [a_\varepsilon(x, u_\varepsilon, Du_\varepsilon) - a_\varepsilon(x, u_\varepsilon, DT_k(u))] \cdot D(u_\varepsilon - T_k(u)) \chi_\eta(|u_\varepsilon - T_k(u)|) dx \le \\
&\le \int_\Omega a_\varepsilon(x, u_\varepsilon, Du_\varepsilon) \cdot DT_\eta(u_\varepsilon - T_k(u)) dx - \int_\Omega a_\varepsilon(x, u_\varepsilon, DT_k(u)) \cdot DT_\eta(u_\varepsilon - T_k(u)) dx = I_1 - I_2.
\end{aligned}$$

As regards I_1, using as test function $T_\eta(u_\varepsilon - T_k(u))$, we have that

$$|I_1| = \left| \int_\Omega f_\varepsilon T_\eta(u_\varepsilon - T_k(u)) dx \right| \le \eta \, \|f\|_{L^1(\Omega)} \,.$$

In order to evaluate the term I_2, we observe that

$$\{x :\ |u_\varepsilon - T_k(u)| \le \eta\} \subset \{x :\ |u_\varepsilon| \le k + \eta\};$$

therefore choosing $\eta > 0$ such that $k + \eta < m$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$

$$a_\varepsilon(x, u_\varepsilon, DT_k(u)) = a(x, u_\varepsilon, DT_k(u)) \text{ in } \{x :\ |u_\varepsilon - T_k(u)| \le \eta\};$$

hence

$$I_2 = \int_\Omega a(x, u_\varepsilon, DT_k(u)) \cdot DT_\eta(u_\varepsilon - T_k(u))dx =$$
$$= \int_\Omega a(x, T_{k+\eta}(u_\varepsilon), DT_k(u)) \cdot D(T_{k+\eta}(u_\varepsilon) - T_k(u))\chi_\eta(|u_\varepsilon - T_k(u)|)\, dx.$$

By Remark 2.1 it follows that

$$T_{k+\eta}(u_\varepsilon) \rightharpoonup T_{k+\eta}(u) \text{ weakly in } W_0^{1,p}(\Omega);$$

moreover, being

$$|a(x, T_{k+\eta}(u_\varepsilon), DT_k(u))| \leq b(|T_{k+\eta}(u_\varepsilon)|)|DT_k(u)|^{p-1},$$

by Vitali's theorem we have that

$$a(x, T_{k+\eta}(u_\varepsilon), DT_k(u)) \to a(x, T_{k+\eta}(u), DT_k(u)) \text{ strongly in } L^{p'}(\Omega).$$

Passing to the limit in I_2, we get

$$\lim_{\varepsilon\to 0} \int_\Omega a(x, u_\varepsilon, DT_k(u)) \cdot DT_\eta(u_\varepsilon - T_k(u))dx =$$
$$= \int_\Omega a(x, T_{k+\eta}(u), DT_k(u)) \cdot D(T_{k+\eta}(u) - T_k(u))\chi_\eta(|u - T_k(u)|)\, dx,$$

and

$$\lim_{\eta\to 0} \int_\Omega a(x, T_{k+\eta}(u), DT_k(u)) \cdot D(T_{k+\eta}(u) - T_k(u))\chi_\eta(|u - T_k(u)|)\, dx = 0.$$

Choosing η such that

$$\eta\, \|f\|_{L^1(\Omega)} < \frac{\delta}{2}$$

where δ is given from Lemma 3.1, we have

$$\int_{E_3} \gamma(x)dx \leq \delta,$$

and we can deduce that $|E_3| < \sigma$ independently of ε.

Finally, we fix η and thanks to the fact that $u_\varepsilon \to u$ in measure, we can choose ε_1 such that

$$|E_2| \leq \eta, \qquad \text{for } \varepsilon \leq \varepsilon_1.$$

This implies that $Du_\varepsilon \to Du$ in measure in $\{|u| < m\}$, consequently

$$Du_\varepsilon \to Du \text{ a.e. in } \{|u| < m\}.$$

$\square$

4. Existence results

Now we can apply the results of the previous sections in order to obtain existence results, which depend on the behaviour of b near m. First of all, we emphasize that a prevalent role is played by the set

$$\{x \in \Omega\colon |u(x)| = m\}$$

where u is the limit function of the solutions u_ε of approximated problems (2.1). In this direction, we observe that, being $u_\varepsilon \to u$ a.e. in Ω (see Remark 2.1),

$$\{x \in \Omega\colon |u(x)| = m\} = \left\{ x \in \Omega\colon \lim_{\varepsilon \to 0} \int_0^{|u_\varepsilon(x)|} b_\varepsilon(t)dt \geq \int_0^m b(t)dt \right\}. \tag{4.1}$$

4.1. The case $b^{1/(p-1)} \notin L^1(0,m)$.

THEOREM 4.1. *Let f be a function in $L^r(\Omega)$, with $r > \frac{n}{p}$. Assume that* (1.2) – (1.5) *hold, with $b^{1/(p-1)} \notin L^1(0,m)$. Then there exists a weak solution $u \in W_0^{1,p}(\Omega)$ of problem* (1.1) *such that*

$$\|u\|_{L^\infty(\Omega)} < m. \tag{4.2}$$

PROOF. Obviously, the estimate (2.2) holds for $f_\varepsilon \equiv f$ for any $\varepsilon > 0$. Since a_ε is bounded from above, by classical results (see [**73**], [**76**]) there exists a solution $u_\varepsilon \in W_0^{1,p}(\Omega)$ of the approximated problem (2.1). Estimate (2.2) implies

$$B_\varepsilon(\|u_\varepsilon\|_{L^\infty(\Omega)}) \leq C(f) = \left(n\omega_n^{1/n}\right)^{-p'} \int_0^{|\Omega|} t^{-\frac{p'}{n'}} \left(\int_0^t f^*(\sigma)d\sigma \right)^{\frac{p'}{p}} dt. \tag{4.3}$$

Being $B(s)$ unbounded in $[0,m)$, we can take $A = B^{-1}(C(f))$ and then we choose $\varepsilon_0 > 0$ such that $b(s) \leq b(m - \varepsilon_0)$ for any $s \in [0,A]$. By definition of b_ε and B_ε we have, for any $\varepsilon < \varepsilon_0$,

$$B_\varepsilon(s) = B(s), \quad s \in [0,A].$$

Moreover, being B_ε increasing, it follows that, for any $\varepsilon < \varepsilon_0$,

$$B_\varepsilon(s) \leq C(f) \iff s \in [0,A],$$

so by (4.3) we get

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \leq A < m. \tag{4.4}$$

Then there exists $\varepsilon_1 < \varepsilon_0$ such that for any $\varepsilon < \varepsilon_1$

$$a_\varepsilon(x, u_{\varepsilon_1}(x), Du_{\varepsilon_1}(x)) = a(x, u_{\varepsilon_1}(x), Du_{\varepsilon_1}(x))$$

for a.e. $x \in \Omega$; this implies that u_{ε_1} is a solution of (1.1), which obviously verifies (4.2). $\square$

REMARK 4.1. We want to emphasize that in the case $b^{1/(p-1)} \notin L^1(0,m)$ we can obtain directly the same result of Remark 2.1. Indeed, if we consider $f \in L^1(\Omega)$ and $f_\varepsilon \in L^\infty(\Omega)$

such that $f_\varepsilon \to f$ in $L^1(\Omega)$, by Theorem 4.1, there exists a weak solution u_ε of the problem

$$-\operatorname{div}(a(x,u_\varepsilon,Du_\varepsilon)) = f_\varepsilon, \quad u \in W_0^{1,p}(\Omega), \tag{4.5}$$

such that $|u_\varepsilon| \le c(\varepsilon) < m$; consequently we can integrate between 0 and m the inequality

$$b(0)\frac{d}{dt}\int_{|u_\varepsilon|\le t} |Du_\varepsilon|^p dx \le \int_0^{\mu_{u_\varepsilon}(t)} f_\varepsilon^*(s)ds$$

in order to obtain that

$$\int_\Omega |Du_\varepsilon|^p dx \le C. \tag{4.6}$$

Last estimate gives that

$$u_\varepsilon \to u \text{ a.e. in } \Omega, \quad |u| \le m \text{ a.e. in } \Omega, \quad u_\varepsilon \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega), \tag{4.7}$$

as in Remark 2.1.

Theorem 4.2. *Let $f \in L^r(\Omega)$, with $np/(np-n+p) \le r < n/p$. Under the hypotheses* (1.2) – (1.5)*, with $b^{1/(p-1)} \notin L^1(0,m)$, there exists a weak solution $u \in W_0^{1,p}(\Omega)$ of problem* (1.1)*, such that*

$$|\{|u| = m\}| = 0.$$

Proof. Let $u_\varepsilon \in W_0^{1,p}(\Omega)$ be a weak solution to the approximated problem (2.1).

Recalling Remark 2.1, $u_\varepsilon \to u$ a.e. in Ω; being $B(m) = +\infty$, (4.1) allows to obtain that

$$B_\varepsilon(|u_\varepsilon|) \to B(|u|) \quad \text{a.e. in } \Omega. \tag{4.8}$$

By estimate (2.7) and (4.8), it follows that

$$B_\varepsilon(|u_\varepsilon|) \rightharpoonup B(|u|) \text{ weakly in } W_0^{1,p}(\Omega), \tag{4.9}$$

hence $B(|u|)$ is bounded in $L^1(\Omega)$ and

$$|\{|u| = m\}| = 0. \tag{4.10}$$

Combining (4.10) with Theorem 3.1 we get

$$a_\varepsilon(x,u_\varepsilon,Du_\varepsilon) \to a(x,u,Du) \text{ a.e. in } \Omega.$$

Moreover, by (2.7) we get that

$$|a_\varepsilon(x,u_\varepsilon,Du_\varepsilon)| \text{ is bounded in } L^{p'}(\Omega);$$

these conditions allow to pass to the limit in the weak formulation of approximated problems,

$$\int_\Omega a_\varepsilon(x,u_\varepsilon,Du_\varepsilon)\cdot D\varphi = \int_\Omega f_\varepsilon \varphi\, dx, \quad \varphi \in W_0^{1,p}(\Omega),$$

obtaining that u is a weak solution of (1.1). $\square$

THEOREM 4.3. *Let $f \in L^r(\Omega)$, with $1 \le r < np/(np-n+p)$. Under the hypotheses* (1.2) – (1.5)*, with $b^{1/(p-1)} \notin L^1(0,m)$, there exists a solution $u \in W_0^{1,p}(\Omega)$, in the sense of* Definition 1.2*, of problem* (1.1)*, and*

$$|\{|u| = m\}| = 0.$$

PROOF. As before, we consider a weak solution u_ε to the approximated problem (2.1). By Remark 2.1, the limit function u satisfies (1.7). The argument used in the proof of Theorem 4.2 allows to claim that $u < m$ a.e. in Ω.

If we choose $T_k(u_\varepsilon - \varphi)$, $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, as test function, we get

$$\int_{|u_\varepsilon - \varphi| \le k} a(x, T_{m-\varepsilon}(u_\varepsilon), Du_\varepsilon) \cdot Du_\varepsilon \, dx - \int_{|u_\varepsilon - \varphi| \le k} a(x, T_{m-\varepsilon}(u_\varepsilon), Du_\varepsilon) \cdot D\varphi \, dx = \tag{4.11}$$
$$= \int_\Omega f_\varepsilon T_k(u_\varepsilon - \varphi) \, dx.$$

First of all, being f_ε strongly convergent in L^1 to f, we can pass to the limit in the right–hand side of the above equality.

We observe that $\{|u_\varepsilon - \varphi| \le k\} \subseteq \{|u_\varepsilon| \le k + \|\varphi\|_{L^\infty(\Omega)} = M\}$, hence taking $0 < k < m$ and $\|\varphi\|_{L^\infty(\Omega)} < m - k$, we get $M < m$ and then $|a(x, T_M(u_\varepsilon), DT_M(u_\varepsilon)|$ is bounded in $L^{p'}(\Omega)$. As before, we can pass to the limit in the second integral of (4.11).

As regard the first integral, being the integrand non–negative (by the ellipticity condition) and almost everywhere convergent, we can apply Fatou's lemma. Putting all the terms together, we obtain

$$\int_\Omega a(x, u, Du) \cdot DT_k(u - \varphi) \, dx \le \int_\Omega f T_k(u - \varphi) dx, \tag{4.12}$$

and the Theorem is proved. □

4.2. The case $b^{1/(p-1)} \in L^1(0,m)$. The case B bounded is completely different from the previous one. In order to obtain existence of weak solutions, it is necessary to require a smallness assumption on the datum.

THEOREM 4.4. *Let f be a function in $L^r(\Omega)$, with $r > n/p$. Assume that* (1.2) – (1.5) *hold, with $b^{1/(p-1)} \in L^1(0,m)$. If*

$$\left(n\omega_n^{1/n}\right)^{-p'} \int_0^{|\Omega|} t^{-\frac{p'}{n'}} \left(\int_0^t f^*(\sigma) d\sigma \right)^{\frac{p'}{p}} dt < B(m), \tag{4.13}$$

then there exists a weak solution $u \in W_0^{1,p}(\Omega)$ of problem (1.1) *such that*

$$\|u\|_{L^\infty(\Omega)} < m.$$

PROOF. We can proceed as in the proof of Theorem 4.1. By the estimate

$$B_\varepsilon(\|u_\varepsilon\|_{L^\infty(\Omega)}) \le \left(n\omega_n^{1/n}\right)^{-p'} \int_0^{|\Omega|} t^{-\frac{p'}{n'}} \left(\int_0^t f^*(\sigma) d\sigma \right)^{\frac{p'}{p}} dt$$

and using the smallness assumption (4.13), we get a solution $u \in W_0^{1,p}(\Omega)$ such that $|u| \le c < m$.

$\square$

REMARK 4.2. We emphasize that the condition (4.13) is a smallness assumption on the norm of f in the Lorentz space $L^{(n/p,\,p'/p)}(\Omega)$. Indeed (4.13) can be rewritten in the form

$$n^{-p'}\omega_n^{-p'/n}\,\|f\|_{L^{(n/p,\,p'/p)}(\Omega)} < B(m);$$

it is well–known that $L^{(n/p,\,p'/p)}(\Omega)$ contains $L^r(\Omega)$, for any $r > n/p$.

REMARK 4.3. We observe that the (4.13) means also that

$$V(0) < B(m),$$

where

$$V(x) = \left(n\omega_n^{1/n}\right)^{-p'}\int_{\omega_n|x|^n}^{|\Omega|} t^{-\left(1-\frac{1}{n}\right)p'}\left(\int_0^t f^*(r)dr\right)^{\frac{p'}{p}} dt \tag{4.14}$$

is the solution of the Dirichlet problem

$$\begin{cases} -\Delta_p V = f^{\#} & \text{in } \Omega^{\#} \\ V = 0 & \text{on } \partial\Omega^{\#}. \end{cases} \tag{4.15}$$

THEOREM 4.5. *Let $f \in L^r(\Omega)$, with $1 \le r < n/p$. Under the hypotheses* (1.2) – (1.5), *with $b^{1/(p-1)} \in L^1(0,m)$, there exists a solution $u \in W_0^{1,p}(\Omega)$, in the sense of* Definition 1.2, *of problem* (1.1).

PROOF. The proof follows using similar arguments contained in the proof of Theorem 4.3, with the only difference that the limit function u can be equal to $\pm m$ on a set of positive measure, and then $a_\varepsilon(x,u_\varepsilon,Du_\varepsilon)$ converges to $a(x,u,Du)$ in $\{|u| < m\}$. $\square$

In the following result we analyse the limit case in condition (4.13).

THEOREM 4.6. *Let f be a function in $L^r(\Omega)$, with $r > n/p$. Assume that* (1.2) – (1.5) *hold, with $b^{1/(p-1)} \in L^1(0,m)$. If*

$$\left(n\omega_n^{1/n}\right)^{-p'}\int_0^{|\Omega|} t^{-\frac{p'}{n'}}\left(\int_0^t f^*(\sigma)d\sigma\right)^{\frac{p'}{p}} dt = B(m), \tag{4.16}$$

then there exists a weak solution $u \in W_0^{1,p}(\Omega)$ of problem (1.1) *such that*

$$|\{|u| = m\}| = 0. \tag{4.17}$$

PROOF. Let $u_\varepsilon \in W_0^{1,p}(\Omega)$ be a weak solution to the approximated problem (2.1). Reasoning as in the proof of Theorem 4.5, we obtain that u satisfies the conditions of Definition 1.2.

If we show that (4.17) holds, the summability hypothesis on f allows to get that

$$B_\varepsilon(|u_\varepsilon|) \to B(|u|) \text{ a.e. in } \Omega$$

and

$$B_\varepsilon(|u_\varepsilon|) \rightharpoonup B(|u|) \text{ weakly in } W_0^{1,p}(\Omega);$$

so we can proceed analogously to the proof of Theorem 4.2 and get that u is a weak solution of (1.1).

In order to prove (4.17), let h be such that $0 < h < m$; we can use as test function $\varphi = T_t(u)$ in (1.1), with $0 < t < m - h$, obtaining

$$\int_\Omega a(x,u,Du)\cdot DT_h(u - T_t(u))dx \le \int_\Omega f T_h(u - T_t(u))dx. \tag{4.18}$$

From (4.18), it follows that

$$\frac{1}{h}\int_{t<|u|\le t+h} b(|u|)|Du|^p \le \int_{|u|>t} |f|dx,$$

for $t < m - h$, similarly to (2.3). Therefore, arguing as in the proof of Proposition 2.1, we get that

$$B(u^*(s)) \le V(s) = \left(n\omega_n^{1/n}\right)^{-p'} \int_s^{|\Omega|} t^{-\frac{p'}{n'}} \left(\int_0^t f^*(\sigma)d\sigma\right)^{\frac{p'}{p}} dt, \tag{4.19}$$

for any $s \ge 0$ such that $u^*(s) < m$. We observe that being $u^*(0) \le m$, we can suppose $u^*(0) = m$, otherwise (4.17) is trivially satisfied. Now take $s_0 = \inf\{s \ge 0\colon u^*(s) < m\}$. If we show that $s_0 = 0$, by equimeasurability of u and u^* we can conclude that (4.17) holds. To this aim, let s_0 be positive. By (4.19) and the monotonicity of $V(s)$ we have

$$B(m) = B(u^*(s_0)) \le V(s_0) < V(0);$$

but this contradicts the hypothesis (4.16). □

Remark 4.4. We emphasize that if $b^{1/(p-1)} \in L^1(0,m)$ and f does not verify the smallness hypotheses (4.13) or (4.16), we cannot have existence of weak solutions. This is due to the fact that the limit function u can be equal to $\pm m$ on a set of positive measure, and then $a_\varepsilon(x,u_\varepsilon,Du_\varepsilon)$ does not converge to $a(x,u,Du)$ in Ω, as shown by the following example (see [**83**] in the case $p = 2$).

Example 4.1. Let us consider the following problem:

$$\begin{cases} -\operatorname{div}\left(\dfrac{|Du|^{p-2}Du}{(1-|u|)^{(p-1)/2}}\right) = \lambda & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.20}$$

where $\Omega = \{x \in \mathbb{R}^n \colon |x| < 1\}$, λ is a positive constant and $1 < p < +\infty$. We can approximate problem (4.20) with

$$\begin{cases} -\operatorname{div}\left(b_{\varepsilon_k}(|u_{\varepsilon_k}|)|Du_{\varepsilon_k}|^{p-2}Du_{\varepsilon_k}\right) = \lambda & \text{in } \Omega, \\ u_{\varepsilon_k} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.21}$$

where $\varepsilon_k = k^{-2/(p-1)}$ and

$$b_{\varepsilon_k}(s) = \begin{cases} (1-|s|)^{-\frac{p-1}{2}} & \text{if } |s| \le 1-\varepsilon_k, \\ k & \text{if } |s| > 1-\varepsilon_k. \end{cases}$$

Hence,

$$B_{\varepsilon_k}(s) = \begin{cases} 2(1-(1-|s|)^{1/2})\operatorname{sign} s & \text{if } |s| \le 1-\varepsilon_k, \\ \left(2 + sk^{1/(p-1)} - k^{1/(p-1)} - k^{-1/(p-1)}\right)\operatorname{sign} s & \text{if } |s| > 1-\varepsilon_k. \end{cases}$$

Performing the change of variable $v = B_{\varepsilon_k}(u_{\varepsilon_k})$, we get that (4.21) is equivalent to the problem

$$\begin{cases} -\Delta_p v = \lambda & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

solved by the function

$$v(x) = \frac{\lambda^{\frac{p'}{p}}}{p' n^{p'-1}}(1-|x|^{p'}).$$

If we choose $\lambda > n(2p')^{p/p'}$, then

$$|\{x \in \Omega \colon v(x) > 2\}| > 0. \tag{4.22}$$

Being $u_{\varepsilon_k} = B_{\varepsilon_k}^{-1}(v)$, it follows that

$$u_{\varepsilon_k}(x) = \begin{cases} v(x) - \dfrac{v^2(x)}{4} & \text{if } 0 \le v(x) \le 2(1-\varepsilon_k^{1/2}), \\ \varepsilon_k^{1/2}(v(x) - 2 + \varepsilon_k^{1/2} + \varepsilon_k^{-1/2}) & \text{if } v(x) > 2(1-\varepsilon_k^{1/2}), \end{cases}$$

and for $k \to +\infty$, u_{ε_k} converges to the function

$$u(x) = \begin{cases} v(x) - \dfrac{v^2(x)}{4} & \text{if } 0 \le v(x) \le 2, \\ 1 & \text{if } v(x) > 2. \end{cases}$$

By (4.22), the function u is equal to the critical value 1 on a set of positive Lebesgue measure. This means that u cannot be a weak solution of problem (4.20). Nevertheless, u is an entropy solution in the sense of Definition 1.2.

Observe that in the case $\lambda < n(2p')^{p/p'}$ then $u = v - v^2/4$ is the weak solution of (4.20) and $\operatorname{ess\,sup}|u| < 1$.

Finally, in the limit case $\lambda = n(2p')^{p/p'}$, the function $u = v - v^2/4$ is such that $|\{u = 1\}| = 0$, and by Theorem 4.6, u is a weak solution of (4.20).

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