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Geometry in tensor products

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Daniele Puglisi
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To Alexander Grothendieck.

Introduction

... ο δε ανεξεταστος
βιος ου βιωος
ανθρωπω ...
(**Plato**, *Apology*- 38.a)

The notion of tensor product was first given by R. Schatten [93], [94], but A. Grothendieck was the first one which studied deeply this object. Indeed he introduced the notion of "reasonable" crossnorm, and defined the greatest and the least reasonable crossnorm: the *projective* and the *injective* tensor norm. He also introduced the notion of *integral operator* between Banach spaces, that is still very useful in the theory of Banach spaces. The strongest impact of the tensor products in the geometry of Banach space, it was due for the following facts:

$$\mathcal{L}(X, Y^*) = (X \overset{\wedge}{\otimes}_\pi Y)^* \quad , \quad \mathcal{I}(X, Y^*) = (X \overset{\wedge}{\otimes}_\epsilon Y)^*$$

where $\mathcal{L}(X, Y^*)$ (resp. $\mathcal{I}(X, Y^*)$) denoted the Banach space of bounded (resp. integral) linear operators from X to Y^*.
In the begining the Grothendieck's work was not very well understood; only after a period of sixteen years , Joram Lindenstrauss and Alexander Pelczynski reviewed the tensor product theory, extending many Grothendieck results (especially Grothendieck's Fundamental Inequality). Lindenstrauss and Pelczynski commented: *" The paper of Grothendieck is quite hard to read and its results are not generally known even to experts in Banach space theory "*.
After them, many other people attacked the tensor product theory, understanding the power of this object, establishing how the subject come in aid to many problems of more classical aspects of mathematics, like harmonic analysis, probability, complex analysis, geometry of convex bodies, real analysis and operator theory.

The first chapter of this dissertations is an outline of the theory of tensor products, introducing the notion of reasonable tensor norm, the greatest and the least reasonable tensor norms, how the projective tensor norm can be *injective* and how the injective tensor norm can be *projective* (presenting for that a deep result of Heinrich-Manckiewicz and Radrianantoanina). We close the chapter with some applications of the approximation property (another notion introduced by A. Grothendieck) in duality between the injective and projective tensor product. After the injective and projective tensor norms, other useful crossnorm were introduced, i.e. the Chevet-Saphar (see [11], [92]) tensor norm was crucial to study the theory of p-summing operator, but equally important was the Fremlin tensor norm. Indeed in the chapter 2, after giving an outline of the theory of Banach lattices and semi-embeddings notions, we introduce the Fremlin tensor product. In 1974 D. Fremlin introduced a new tensor norm between two Banach lattices, and his norm was useful specially for the fact that, if X and Y are two Banach lattices then

$$\mathcal{L}^r(X, Y^*) = (X \overset{\wedge}{\otimes}_F Y)^*$$

where $\mathcal{L}^r(X, Y^*)$ denote the the Banach lattice of the regular operators from X to Y^*. In the last section we solve an old question raised by Diestel and Uhl (see [26]): given a tensor norm , is there stability of the Radon-Nikodym property in the tensor product equipped with a such norm? indeed we close the question relative to the Fremlin tensor norm, with the following

Theorem[83]
If X and Y are two Banach lattices, one of them atomic, then the Fremlin tensor product $X \overset{\wedge}{\otimes}_F Y$ of X and Y has the Radon-Nikodym property if both posses this property.

Moreover we noted that the hypothesis of atomicity is crucial for the theorem; indeed it was proved that $L_2[0,1] \overset{\wedge}{\otimes}_F L_2[0,1]$ cannot have the Radon-Nikodym theorem, even though $L_2[0,1]$ a Hilbert space.
In the chapter 3 we come back to the projective and injective tensor norms, studying two special geometry properties: the *l.u.st.* property and the *Gordon Lewis* (or GL) property. First of all, we recall meant when a property is local, introducing a very special class of Banach spaces, for which locally are the same like the ℓ_p space: so called $\mathcal{L}_p$ spaces. In some sense (see 3.1.6 for a more precise definition) a Banach space is said to be an $\mathcal{L}_p$ space if any finite dimensional subspace can be contained in some ℓ_p^n space, for some n. Then how the reader can note the meaning of the local property come out from the behavior of its finite dimensional subspaces. In the local geometry of Banach

space l.u.st. and GL properties are really important; l.u.st. was first introduced in a paper by E. Dubinsky, A. Pełczyński and H.P. Rosenthal ([27]) in a pretty difficult way that is equivalent to the more natural condition: X *has l.u.st. iff there exists a constant* $\lambda > 1$ *such that, for each* $E \in \mathcal{F}_X$*, there is an* $F \in \mathcal{F}_X$ *with* $E \subseteq F$ *and* $ub(F) < \lambda$ (where $\mathcal{F}_X$ denoted the family of all finite dimensional subspaces of X and $ub(F)$ denoted the unconditional basis constant; see the section 3.2 for that notations). Our definition is due to Y. Gordon and D.R. Lewis ([40]).

The GL-property was for first isolated in 1979 by S. Reisner ([88]) in a different way from us: for him a Banach space has (GL) if $\Pi_1(X,Y) \subseteq \Gamma_1(X,Y)$ for every Banach space Y and it's not known if the two definitions are equivalent (where $\Pi_1(X,Y)$ and $\Gamma_1(X,Y)$ denoted the absolutely summing and 1-factorable respectively operators space from X to Y; see section 3.2 for the definition). Actually Gordon and Lewis [40] gave an elegant proof that every Banach space having the l.u.st. property has the GL property (hence the name of the property). To have an example of a GL-space failing l.u.st. we had to wait for the 1980's paper by W.B. Johnson, J. Lindenstrauss and G. Schechtman ([49]). The following year T. Ketonen ([52]) did even better: he found a subspace of an $L_1(\mu)$-space without l.u.st.. But examples of Banach spaces space failing (GL) is due to Gordon and Lewis (1974): the space $\mathcal{L}(\ell_2, \ell_2)$ of all linear bounded operators from ℓ_2 to ℓ_2.
There is a vast literature on the subject, mainly paper of Junge, Gordon, Lewis, Maurey, N.J. Nielsen, Pisier, Tomczak-Jaegermann, who have published intensively on the gl and $l.u.st$ constants and somming operators (see [40], [39], [38], [80], [10], [55], to get other references there). Especially, a very related to GL-property in injective tensor product result is the following:

Theorem([38])
Let X *and* Y *be Banach spaces. Then* $gl(X_k \overset{\vee}{\otimes} Y_k) \overset{k\to\infty}{\longrightarrow} \infty$ *for every increasing sequence* $\{X_k\}_{k=1}^{\infty}$ *and* $\{Y_k\}_{k=1}^{\infty}$ *of finite-dimensional subspaces of* X *and* Y *respectively, if, and only if,* X *and* Y *not contain subspaces uniformly isomorphic to* ℓ_∞^n *'s (i.e.* X *and* Y *have finite cotype).*

After that, Y.Gordon gave a more general definition of GL property: the $GL(p,q)$ property (see [39]), giving many interesting application in Banach space theory.
In the chapter 3 we prove the following
Theorem[86]
Let X *and* Y *be Banach spaces. We have*

1. *If* X *is a* $\mathcal{L}_\infty$*-space, then*

$X \overset{\wedge}{\otimes}_\epsilon Y$ *has l.u.st. (or GL) property if* Y *does;*

2. *If* X *is a* $\mathcal{L}_1$*-space, then*
   $X \overset{\wedge}{\otimes}_\pi Y$ *has l.u.st. (or GL) property if* Y *does.*

In some sense the result above complete the work of Gordon and Lewis; they proved that if E and F are $\mathcal{L}_p$-spaces ($1 < p < \infty$), then none of $E \overset{\wedge}{\otimes}_\epsilon F$, $E \overset{\wedge}{\otimes}_\pi F$, $(E \overset{\wedge}{\otimes}_\epsilon F)^*$, $(E \overset{\wedge}{\otimes}_\pi F)^*$, $(E \overset{\wedge}{\otimes}_\epsilon F)^{**}$, $(E \overset{\wedge}{\otimes}_\pi F)^{**}$, etc... has l.u.st.
In the chapter 4 we continue a job started by A. Grothendieck about the compactness in the projective tensor product. Indeed he was able to show that if X and Y are Banach spaces then substantially the compact set in $X \overset{\wedge}{\otimes}_\pi Y$ have the following lovely form: $\overline{co}(K_X \otimes K_Y)$, where K_X is a compact subset of X and K_Y is a compact subset of Y. That was very useful in the real life; i.e. using that in [1] the authors proved that

Theorem
Let X *and* Y *be Banach spaces. For every relatively compact subset* H *of* $\mathcal{K}(X^*, Y)$, *there exist an universal Banach space* Z, *an operator* $u \in \mathcal{K}(X^*, Z)$, *a relatively compact subset* $\{B_T : T \in H\}$ *of* $K(Z)$ *and an operator* $v \in \mathcal{K}(Z, Y)$ *such that* $T = v \circ B_T \circ u$ *for all* $T \in H$.

Then in projective tensor product the compact subsets have a nice form, but what about the weakly compact subsets? Actually in projective tensor products weakly compact subsets look like very different that the compact subsets; indeed if W_X and W_Y are two weakly compact subsets of X and Y respectively, then in general $W_X \otimes W_Y$ is not weakly compact in $X \overset{\wedge}{\otimes}_\pi Y$ (i.e. $\overline{B_{\ell_2}} \otimes \overline{B_{\ell_2}}$ is not weakly compact in $\ell_2 \overset{\wedge}{\otimes}_\pi \ell_2$). Then we cannot have any representation of weakly compact in projective tensor product as $\overline{co}(W_X \otimes W_Y)$, where W_X and W_Y are weakly compact subsets of X and Y. But not every thing is lost. The following theorem explain why.

Theorem[24]
Let X *and* Y *be two Banach spaces. Every weakly compact subset in* $X \overset{\wedge}{\otimes}_\pi Y$ *can be written as the intersection of a finite union of sets of the form* $\overline{co}(U \otimes V)$, *where* U *and* V *are weakly compacts subsets of* X *and* Y *respectively.*

In case either X or Y has the Dunford-Pettis property, then this condition is also sufficient for the weak compactness of a subset of the projective tensor product. In the same chapter, it is noted that if one of the space has the Dunford-Pettis property, then we get even better.

Proposition[24]
Let X , Y be Banach spaces, with X having the Dunford-Pettis property. If $W_X \subseteq X$ and $W_Y \subseteq Y$ are weakly compact subsets then $W_X \otimes W_Y$ is weakly compact subset of $X \overset{\wedge}{\otimes}_\pi Y$.

Corollary
Let X, Y be Banach spaces, such that X has the DP property. Then every weakly compact subset in $X \overset{\wedge}{\otimes}_\pi Y$ can be written as the intersection of a finite unions of sets of the form $co(U \otimes V)$, where U and V are weakly compacts subsets of X and Y respectively.

In [18] the authors studied the injective tensor product of two weakly compact operators. Indeed they proved that

Theorem
Let X_1, X_2, Y_1, Y_2 be Banach spaces. Let $T_1 : X_1 \longrightarrow Y_1$ and $T_2 : X_2 \longrightarrow Y_2$ be two bounded linear operators. Then

1. *If T_1 is compact and T_2 is weakly compact, then the injective tensor product $T_1 \overset{\wedge}{\otimes}_\epsilon T_2 : X_1 \overset{\wedge}{\otimes}_\epsilon X_2 \longrightarrow Y_1 \overset{\wedge}{\otimes}_\epsilon Y_2$ of T_1 and T_2 is weakly compact.*

2. *Suppose X_1 be a Banach space whose dual space possesses the approximation property and the Dunford-Pettis property. If T_1 and T_2 are both weakly compact operators, then then the injective tensor product $T_1 \overset{\wedge}{\otimes}_\epsilon T_2 : X_1 \overset{\wedge}{\otimes}_\epsilon X_2 \longrightarrow Y_1 \overset{\wedge}{\otimes}_\epsilon Y_2$, of T_1 and T_2, is weakly compact.*

As an easily consequence of the results above we have

Corollary
Let X_1, X_2, Y_1, Y_2 be Banach spaces. Let $T_1 : X_1 \longrightarrow Y_1$ and $T_2 : X_2 \longrightarrow Y_2$ be two weakly compact operators. Suppose either Y_1 or Y_2 has the Dunford-Pettis property, then the projective tensor product $T_1 \overset{\wedge}{\otimes}_\pi T_2 : X_1 \overset{\wedge}{\otimes}_\pi X_2 \longrightarrow Y_1 \overset{\wedge}{\otimes}_\pi Y_2$, of T_1 and T_2, is weakly compact.

Daniele Puglisi
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Contents

Notations and Basic Facts

We will use standard notations about Banach spaces (see [72], [16], [62], [63]), measure theory (see [46] , [13], [26]), tensor products theory (see [19], [89], [14]), operator theory (see [22]), descriptive set theory (see [52]), and topology (see [58], [30]); anyway some clarification is in order.
In the sequel $\mathbb{K}$ will denote the field of scalars $\mathbb{R}$ or $\mathbb{C}$ unless explicitly mentioned to the contrary. If X is a Banach space, then X^* denotes the dual or conjugate space of X. If Y is a closed linear subspace of X then $Y^\perp$, *the annihilator of* Y, is defined by $Y^\perp = \{x^* \in X^* \ : \ x^*(y) = 0 \quad \text{for all } y \in Y\}$. Clearly $Y^\perp$ is a closed linear subspace of X^*. For us $L(X,Y)$ will be the vector space of all the linear maps between two vector spaces X and Y, instead $\mathcal{L}(X,Y)$ will be the Banach space of all the bounded linear maps between two Banach spaces X and Y. A bounded linear operator $T : X \longrightarrow Y$ is called an *isomorphism* if it is one-one with closed range (thus an isomorphism from X to Y need not to be a surjection). X and Y are *isometrically isomorphic* if there is an isomorphism T from X onto Y with $\|T\| = \|T^{-1}\| = 1$. In the same way $Bil(X,Y;Z)$ will be the vector space of all the bilinear maps from two vector spaces X and Y to a vector space Z, and $\mathcal{B}(X,Y;Z)$ will be the Banach space of all the bounded bilinear maps from two Banach spaces X and Y to a Banach space Z. A closed linear subspace Z of X is said to be *complemented* in X if there is a projection (i.e. a bounded linear projection) from X onto Z. Note that an bounded linear operator $P : X \longrightarrow X$ is a projection onto Z if and only $P(X) \subseteq Z$ and $P(z) = z$ for all $z \in Z$.

We will indicate with $\mathcal{F}$ the family of all finite dimensional Banach spaces and with $\mathcal{F}_X$ the family of all finite dimensional Banach subspaces of a certain Banach space X. If $E \in \mathcal{F}_X$ we label the canonical inclusion as

$$i_E : E \longrightarrow X.$$

The symbol $\mathcal{C}_X$ means the collection of all closed, finite codimensional subspaces of X and, if $Z \in \mathcal{C}_X$, the natural quotient map will be:

$$q_Z : X \longrightarrow X / Z.$$

Let X be a Banach space, and $\{x_\alpha \ : \ \alpha \in \Gamma\}$ a subset of X. The closed linear span of $\{x_\alpha \ : \ \alpha \in \Gamma\}$, denoted by $span\{x_\alpha \ : \ \alpha \in \Gamma\}$ or $span\{x_\alpha\}$, is the smallest closed linear subspace of X containing $\{x_\alpha \ : \ \alpha \in \Gamma\}$.
As usual, if $1 \leq p < \infty$, ℓ_p is the Banach space of all sequence $(x_n)_n$ such that $\|(x_n)_n\|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}} < \infty$. Instead ℓ_∞ is the Banach space of all bounded sequences $(x_n)_n$ with $\|(x_n)_n\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$, and c_0 is the closed linear subspace of ℓ_∞ consisting of all sequences $(x_n)_n$ in ℓ_∞ converging to 0. If $1 \leq p < \infty$, then ℓ_p^n is the n-dimensional Banach space having norm $\|(x_1, ..., x_n)\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$ if $p < \infty$ and $\|(x_1, ..., x_n)\|_\infty = \max_{1\leq i\leq n} |x_i|$.
Let Ω be a set. Recall that an field (or algebra) on Ω is a collection of subsets of Ω containing $\emptyset$, Ω, and closed under complements and finite unions (so also under finite intersections). It is a σ-field (or σ-algebra) if it is also closed under countable unions (so also under countable intersections). A pair (Ω, Σ) is called a measurable space, where Ω is a set and Σ is a σ-field on Ω. The members of Σ are called measurable.
A measure in a measurable space (Ω, Σ) is a function $\mu : \Sigma \longrightarrow \mathbb{R} \cup \{\infty\}$ which is non negative with $\mu(\emptyset) = 0$ and $\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$, for every pairwise disjoints sequence $(E_n)_n$ of measurable sets. If (Ω, Σ) is a measurable space and μ as a measure on (Ω, Σ) then (Ω, Σ, μ) will be called *measure space*. If the measure μ is so that $\mu(\Omega) = 1$ then (Ω, Σ, μ) is called *probability space*.
If (Ω, Σ, μ) is a measure space and $1 \leq p < \infty$, then $L_p(\mu)$ is the Banach space of equivalence classes of μ-measurable function on Ω whose pth power is absolutely integrable, with $\|f\|_p = (\int_\Omega |f(t)|^p d\mu(t))^{\frac{1}{p}}$ for $f \in L_p(\mu)$. Instead $L_\infty(\mu)$ is the Banach space of equivalence classes of μ-measurable functions which are essentially bounded, with $\|f\|_\infty = ess\sup\{|f(t)| \ : \ t \in \Omega\} = \inf\{c \in \mathbb{R} \ : \ \mu(\{t \in \Omega : |f(t)| > c\}) = 0\}$. Usual we will be interested in measure spaces (Ω, Σ, μ) such that $L_\infty(\mu) = L_1(\mu)^*$ (actually Pelczyński proved that for any measure space (Ω, Σ, μ) $L_\infty(\mu)$ is isometrically isomorphic to $L_1(\nu)$ for some measure ν on Σ. However, for $1 < p < \infty$ we have always $L_p(\mu)^* = L_q(\mu)$, with $\frac{1}{p} + \frac{1}{q} = 1$, for any measure space (Ω, Σ, μ); that follows from a more geometrical reason: the *uniformly convexity of the $L_p(\mu)$'s spaces*; but that is another story!).
If K is a compact Hausdorff space, then $C(K)$ is the Banach space of continuous functions on K with $\|f\| = \sup\{|f(x)| \ : \ x \in K\}$ for $f \in C(K)$. Mention of "$C(K)$ space" always refers to such a space.
Let X be a Banach space, a function $f : \Omega \longrightarrow X$ is called *μ-measurable* if there exists a sequence of simple functions $(f_n)_n$ with $\lim_n \|f_n - f\| = 0$ μ-almost everywhere. A function $f : \Omega \longrightarrow X$ is called *weakly μ-measurable* if for each $x^* \in X^*$ the numerical function $x^* f$ is μ-measurable.

Let X be a Banach space, let (Ω, Σ, μ) be a measure space. A measurable function $f : \Omega \longrightarrow X$ is called Bochner integrable if there exists a sequence of simple functions $(f_n)_n$ such that

$$\lim_n \int_\Omega \|f_n - f\| d\mu = 0.$$

In this case, for any $E \in \Sigma$

$$\int_E f d\mu = \lim_n \int_E f_n d\mu,$$

where $\int_E f_n d\mu$ is defined in the obvious way. The space of the Bochner integrable functions will be denoted by $L_1(\mu, X)$. More general, for $1 \leq p < \infty$ $L^p(\mu, X)$ will denote the Banach space of equivalence classes of μ-measurable X valued on Ω whose pth power is Bochner integrable functions, i.e. such that

$$\|f\|_p = (\int_\Omega \|f(t)\|_X^p d\mu(t))^{\frac{1}{p}} < \infty$$

for $f \in L_p(\mu, X)$. Instead $L_\infty(\mu, X)$ stand for the space of all (equivalence classes of μ-measurable) X valued Bochner integrable functions defined on Ω which are essentially bounded, i.e. such that

$$\|f\|_\infty = ess\sup\{\|f(t)\| \ : \ t \in \Omega\} < \infty.$$

This also is a Banach space under the norm $\| \cdot \|_\infty$ and the countably valued functions in $L_\infty(\mu, X)$ are dense in $L_\infty(\mu, X)$. Finally if K is a compact Hausdorff space, then $C(K, X)$ is the Banach space of all continuous functions from K to X with the norm

$$\|f\|_\infty = \sup\{\|f(t)\|_X \ : \ t \in K\}$$

for $f \in C(K, X)$. Now we recall some other classical integration.

Lemma 0.0.1. *(Dunford) Let X be a Banach space and (Ω, Σ, μ) be a measure space. Suppose f is weakly μ-measurable function on Ω and $x^* f \in L_1(\mu)$ for each $x^* \in X^*$. Then for each $E \in \Sigma$ there exists $x_E^{**} \in X^{**}$ so that*

$$x_E^{**}(x^*) = \int_E x^*(f) d\mu$$

for all $x^* \in X^*$.

From this lemma Dunford called a weakly μ-measurable X valued function f on Ω such that $x^* f \in L_1(\mu)$, to be *Dunford integrable*, and the Dunford integral of f over $E \in \Sigma$ is defined by the element $x_E^{**} \in X^{**}$ such that

$$x_E^{**}(x^*) = \int_E x^* f d\mu$$

for all $x^* \in X^*$, the Dunford integral will be denoted by $x_E^{**} = (D) - \int_E f d\mu$. In case $(D) - \int_E f d\mu \in X$ for each $E \in \Sigma$, f is called *Pettis integrable*. Using the same ideas if $f : \Omega \longrightarrow X^*$ is a function so that $xf \in L_1(\mu)$ for all $x \in X$, then for each set $E \in \Sigma$ there is a vector $x_E^* \in X^*$ such that

$$x_E^*(x) = \int_E x f d\mu$$

for all $x \in X$. The element $x_E^* \in X^*$ is called the *Gel'fand integral* of f over E.

Definition 0.0.2. A topological space is *completely metrizable* if it admits a complatible metric d such that (X, d) is complete. A separable completely metrizable space is called *Polish*.

The following facts are easy to verify

Proposition 0.0.3. i) *The completion of a separable metric space is Polish;*

ii) *A closed subspace of a Polish space is Polish;*

iii) *The product of a sequence of completely metrizable (resp. Polish) spaces is completely metrizable (resp. Polish);*

iv) *A subspace of a Polish space is Polish if and only if it is a G_δ (intersection of countable many open sets).*

Indeed, let X be a Polish space, and let $Y = \bigcap_n U_n$ with U_n open in X. Let d be a complete compatible metric for X and consider $F_n = X \setminus U_n$ for all $n \in \mathbb{N}$. Define a new metric on Y

$$d'(x, y) = d(x, y) + \sum_{n=0}^{\infty} \min\{\frac{1}{2^{n+1}}, |\frac{1}{d(x, F_n)} - \frac{1}{d(y, F_n)}|\}$$

It is easy to see that this metric is complete compatible on Y and so Y is complete. In particular from *iv)* every open set in a Polish space is Polish.

If we consider $\mathbb{N}$ with the discrete topology, from *iii)* we can say that $\mathbb{N}^{\mathbb{N}}$ is Polish too. This last space (called *Baire space*) plays an important rule in the Polish spaces theory because of

Proposition 0.0.4. *If X is a non empty Polish space then there exists a continuous and open surjection $f : \mathbb{N}^{\mathbb{N}} \longrightarrow X$ (open means that the image of each open is open).*

Definition 0.0.5. Let (X, Θ) be a topological space. The class of *Borel set* of X is the σ-algebra generated by the open sets.
Let X, Y be topological spaces. A map $f : X \longrightarrow Y$ is *Borel* if the inverse image of a Borel (equivalently open or closed) set is Borel.

Very useful in the theory of geometry of Banach spaces is the following

Definition 0.0.6. A Schauder basis $(u_n)_{n\in\mathbb{N}}$ for a Banach space U is said to be an *unconditional basis* for the space if, calling $(u_n^*)_{n\in\mathbb{N}}$ the biorthogonal sequence in U^* of the basis, there exists a constant $\lambda \geq 1$ so that $\sum_{n=1}^{\infty} t_n \langle u_n^*, x\rangle u_n$ converges for every $(t_n)_{n\in\mathbb{N}} \in \ell_\infty$, $x \in U$ and

$$\left\| \sum_{n=1}^{\infty} t_n \langle u_n^*, x\rangle u_n \right\| \leq \lambda \left\| \sum_{n=1}^{\infty} \langle u_n^*, x\rangle u_n \right\| \qquad \forall (t_n)_{n\in\mathbb{N}} \in B_{\ell_\infty}.$$

Also we need to recall the following

Definition 0.0.7. A Banach space X has the *approximation property*, if for every Banach space Y, the set of finite-rank members of $\mathcal{L}(Y, X)$ is dense in space of compact operators $\mathcal{K}(Y, X)$.

The above definition was given by Grothendieck ([42]); indeed he deduced some equivalence of such a property

Theorem 0.0.8. *Let X be a Banach space. Then the following are equivalent*

(i) *The space X has the approximation property.*

(ii) *For every compact subset K of X and for $\epsilon > 0$ there is a bounded finite-rank linear operator $T_{K,\epsilon} : X \longrightarrow X$ such that $\|T_{K,\epsilon}(x) - x\| < \epsilon$ whenever $x \in K$.*

Grothendieck found nice interpretations of the approximation property in the tensor product theory, as the reader can see in the section 1.1. Every of the classical Banach space has the approximation property, for that Grothendieck thought that every Banach space has such property. For a period of twenty years people tried to prove that, but only in the 1973 Per Enflo gave a beautiful counterexample (see [29]).

Definition 0.0.9. Let X be a Banach space. Suppose that there is a positive constant t having the property that, for every compact subset K of X and $\epsilon > 0$, there is a bounded finite-rank linear operator $T_{K,\epsilon} : X \longrightarrow X$ with $\|T_{K,\epsilon}\| \leq t$ and such that $\|T_{K,\epsilon}(x) - x\| < \epsilon$ whenever $x \in K$. Then X has the *bounded approximation property.* If X satisfies the condition above with $t = 1$ then X is said to have the *metric approximation property.* It is clear that the bounded approximation property implies the approximation property, but they are not equivalent (see [31]).

Proposition 0.0.10. *Suppose that there exists a uniformly bounded net $\{T_\alpha\}$ of finite-rank operators on X such that $T_\alpha(x) \to x$ for every $x \in X$. Then X has the bounded approximation property.*

Proof. Let $\{T_\alpha\}$ be a net of finite-rank operator such that $C = \sup_\alpha \|T_\alpha\| < \infty$ and $T_\alpha(x) \to x$ for every $x \in X$. Let K be a compact subset of X and let $\epsilon > 0$. Consider $\{x_1, ..., x_n\}$ in X so that for every $x \in K$ there exists $i \in \{1, ..., n\}$ so that $\|x - x_i\| < \delta = \min \frac{\epsilon}{3}, \frac{\epsilon}{3C}$. For the hypothesis of convergence there exists α_0 so that if $\alpha \geq \alpha_0$ then $\|T(x_i) - x_i\| \leq \frac{\epsilon}{3}$ for each $i \in \{1, ..., n\}$. Let $x \in K$ and choose i such that $\|x - x_i\| < \delta$. Then

$$\|x - T_{\alpha_0}(x)\| \leq \|x - x_i\| + \|x_i - T_{\alpha_0}(x_i)\| + \|T_{\alpha_0}(x_i) - T_{\alpha_0}(x)\| < \epsilon$$

□

Of course, if every operator in the net of approximating finite-rank operators has norm at most one, then the space has the metric approximation property. For an application of the proposition above see 1.1.15.
To finish this section we recall the concept of *compactifications.* Let X be a topological space. A pair (Y, c), where Y is a compact space and $c : X \longrightarrow Y$ is a homeomorphic embedding of the space X into Y such that $\overline{c(X)} = Y$ is called a *compactification of the space* X. Of course, if a topological space is embeddable in a compact space Y, then the pair $(\overline{f(X)}, f)$ is a compactification of X (where $f : X \longrightarrow Y$ is the embedding). Usually not every topological space admits compactification; then, for what we said above, every space X embeddable in a compact space has a compactification. A more general theorem holds

Theorem 0.0.11. *A topological space has a compactification if and only if it is a Tychonoff space.*

Two *compactifcations* c_1X and c_2X are said to be *equivalent* if there exists a homeomorphism $f : c_1X \longrightarrow c_2X$ so that the following diagram is commutative

$$\begin{array}{ccc} c_1X & \xrightarrow{f} & c_2X \\ {\scriptstyle c_1}\uparrow & & \uparrow{\scriptstyle c_2} \\ X & \xrightarrow{id_X} & X \end{array}$$

i.e. $fc_1(x) = c_2(x)$ (where c_1, c_2 are the embeddings from X to the compactifications c_1X, c_2X respectively). Of course, every compactification of a compact space are equivalent, and then a compact space has a unique compactification (”modulo equivalences”).
Let $\mathfrak{C}(X)$ be denoted the family of all compactifications (up to equivalence) of a given Tychonoff space X. Now we define a partial ordering in the family $\mathfrak{C}(X)$. We say that $c_2X \leq c_1X$ if and only if there exists a mapping $f : c_1X \longrightarrow c_2X$ such that $f \circ c_1 = c_2$. It seems that two compactifications c_1X, c_2X of a Tychonoff space X are equivalent if and only if $c_1X \leq c_2X$ and $c_2X \leq c_1X$. What is important about that partial ordering is that $(\mathfrak{C}(X), \leq)$ is a complete semi-lattice. Indeed we have

Theorem 0.0.12. *For any subfamily $\mathfrak{C}_0 \subseteq \mathfrak{C}(X)$ there exists in $\mathfrak{C}(X)$ a least upper bound with respect to the partial ordering $\leq$.*

From the last theorem follows that for every Tychonoff space X, there exists a greatest element of $\mathfrak{C}(X)$, called the *Chech-Stone compactification of* X, usually denoted by $\beta(X)$. This compactification is very useful and it had found many application in functional analysis (we will find an application in the chapter 2). But what distinguish that compactification form the other is the following

Theorem 0.0.13. *Let X be a Tychonoff space and Z be a compact space. Then*

1. *Every continuous map $f : X \longrightarrow Z$ is extendable to a continuous map $F : \beta X \longrightarrow Z$.*

2. *If every continuous map defined on X with values in a compact space can be extended (as a continuous map) over a compactification αX of the space X, then αX is equivalent to the Cech-Stone compactification of X.*

Very useful in functional analysis is the fact that if $\mathbb{N}$ is equipped of the discrete topology then $\beta\mathbb{N}$ then $\ell_\infty = C(\beta\mathbb{N})$ (the space of continuous functions form $\beta\mathbb{N}$ to the real field). One of the many questions posed form Banach in his book (see [2]) was to see whether a smaller compact space S

corresponds to a smaller space $C(S)$ (the term "smaller space" will mean a space which is isomorphically contained in the comparized one). A counterexample was found by Pelczynski (see [76]) which proved that $C(\beta\mathbb{N} \setminus \mathbb{N})$ is not isomorphic to any subspace of $C(\beta\mathbb{N})$ (actually he proved that $\beta\mathbb{N} \setminus \mathbb{N}$ does not admit a *simultaneous extension* on $\beta\mathbb{N}$). Also he proved that if N_1 is an isolated set of power $\aleph_1$, then it is topologically contained in a Tychonoff cube $I^{\aleph_c}$, although $C(\beta N_1)$ is not isomorphic to any subspace of $C(I^{\aleph_c})$, for $C(I^{\aleph_c})$ is isomorphic to a *strictly convex space* and $C(\beta N_1)$ is not.
But these are other stories!

Chapter 1

Tensor Product of Banach Spaces

1.1 Tensor products of Banach Spaces and Tensor Norms

For the sake of completeness, We will start with a summary of the principal notions and results in the theory of Tensor Products.

Definition 1.1.1. Let E and F be $\mathbb{K}$-vector spaces. A pair (H, Ψ) of a $\mathbb{K}$-vector space H and a bilinear map $\Psi : E \times F \longrightarrow H$ is called a *Tensor Product* of the pair (E, F) if for each $\mathbb{K}$-vector space G and for each $\Phi \in Bil(E, F; G)$ there is a unique $T \in L(H, G)$ such that $\Phi = T \circ \Psi$, i.e. the following diagram is commutative:

$$\begin{array}{ccc} E \times F & \xrightarrow{\Phi} & G \\ {\scriptstyle\Psi}\downarrow & \nearrow_{T} & \\ H & & \end{array}$$

Follows by the definition the easy properties:

(1) If (H, Ψ) is a tensor product of (E, F), then *span* $\{\Psi(E \times F))\} = H$

(2) If (H_1, Ψ_1) and (H_2, Ψ_2) are two tensor products of (E, F), then there is a unique linear isomorphism $S : H_1 \longrightarrow H_2$ such that $S \circ \Psi_1 = \Psi_2$

(3) If $T_1 : E_1 \longrightarrow F_1$ and $T_2 : E_2 \longrightarrow F_2$ are linear isomorphisms and (H, Ψ) is a tensor product of (F_1, F_2), then $(H, \Psi \circ (T_1 \times T_2))$ is a tensor product of (E_1, E_2)

By (2) it seems that every pair (E, F) of vector spaces has (up to a linear isomorphism) a unique tensor product (H, Ψ). For that reason the *algebraic tensor product* of $\mathbb{K}$-vector spaces E, F is denoted by $H = E \otimes F$, with bilinear form associate $\Psi = \otimes$.
By (1) every element $z \in E \otimes F$ has the form

$$z = \sum_{k=1}^{n} x_k \otimes y_k;$$

The elements of the form $x \otimes y$ are called *elementary tensors.* If $\Phi \in Bil(E, F; G)$, then the linear map $T \in L(E \otimes F, G)$ such that

$$\Phi(x, y) = T(x \otimes y)$$

is called the *linearization of* Φ.
Now, let X and Y be Banach spaces, a norm α on the algebraic tensor product $X \otimes Y$ is called a *reasonable crossnorm* if α satisfies:

(a) $\alpha(x \otimes y) \leq ||x||\,||y|| \qquad$ for $x \in X$ and $y \in Y$

(b) $x^* \otimes y^* \in (X \otimes Y, \alpha)^*$ and

$$||x^* \otimes y^*||_{(X \otimes Y, \alpha)^*} \leq ||x^*||\,||y^*||$$

for any $x^* \in X^*$ and $y^* \in Y^*$.

If X and Y are Banach space and α is a reasonable crossnorm on $X \otimes Y$, then we will denote by $X \overset{\wedge}{\otimes}_\alpha Y$ the completion of $X \otimes Y$ equipped with the norm α. Note that, by the definition of reasonable crossnorm, every element $u^* \in X^* \otimes Y^*$ is a member of $(X \otimes Y, \alpha)^*$. This remark suggest us some natural example o reasonable crossnorm:

Example 1.1.2. *(1)* Let X and Y be Banach spaces. Consider $X \otimes Y$ as a subspace of $\mathcal{B}(X^*, Y^*)$ (i.e. it easy to see that the map $\Phi : X \times Y \longrightarrow \mathcal{B}(X^*, Y^*)$ given by $\Phi(x, y)(x^*, y^*) = x^*(x)y^*(y)$ is a bilinear map, then using the linearization of Φ we can see, in a natural way, $X \otimes Y$ as a subspace of $\mathcal{B}(X^*, Y^*)$), then a first natural norm on $X \otimes Y$ could be that induced by $\mathcal{B}(X^*, Y^*)$; that is, for $u \in X \otimes Y$

$$\|u\|_\epsilon = \sup\{|u(x^*, y^*)| \ : \ x^* \in X^* \ , \ y^* \in Y^*\}$$

It seems easy that ϵ is a reasonable crossnorm on $X \otimes Y$, and it will be denoted by *Injective Tensor Norm.* The completion of $X \otimes Y$ equipped with ϵ-norm will be called the *Injective tensor Product of the Banach spaces* X and Y, denoted by $X \overset{\wedge}{\otimes}_\epsilon Y$.

(2) The *projective tensor norm* is the norm induced on $X \otimes Y$ by duality with $\mathcal{B}(X, Y)$; that is, for $u \in X \otimes Y$

$$\|u\|_\pi = \sup\{|v(u)| \ : \ v \in \mathcal{B}(X, Y) \ , \ \|v\| \leq 1\}$$

It seems easy that π is a reasonable crossnorm on $X \otimes Y$. The completion of $X \otimes Y$ equipped with π-norm will be called the *Projective tensor Product of the Banach spaces* X and Y, denoted by $X \overset{\wedge}{\otimes}_\pi Y$.

Grothendieck (see [41]and [42]) defined for first the injective and the projective tensor norms, showing that the Injective and the Projective tensor norms are the least and the greatest reasonable crossnorm (i.e. for every reasonable crossnorm α it happens: $\| \cdot \|_\epsilon \leq \alpha(\cdot) \leq \| \cdot \|_\pi$), giving a more simple expression to the projective tensor norm. Indeed it can be seen as

$$\|u\|_\pi = \inf\{\sum_{i=1}^{n} \|x_i\|\|y_i\| \ : \ u = \sum_{i=1}^{n} x_i \otimes y_i\}, \quad u \in X \otimes Y$$

where the infimum is taken over all possible representations of u. He described the projective tensor product of X and Y in the following way: an element $u \in X \overset{\wedge}{\otimes}_\pi Y$ has the representation

$$u = \sum_{n=1}^{\infty} x_n \otimes y_n, \quad \text{with} \quad \sum_{n=1}^{\infty} \|x_n\|\|y_n\| < \infty$$

and the projective tensor norm of u as

$$\|u\|_\pi = \inf\{\sum_{n=1}^{\infty} \|x_n\|\|y_n\| \ : \ u = \sum_{n=1}^{\infty} x_n \otimes y_n\}$$

where the infimum is taken over all possible representations of u as above. For a good reference about the projective and injective tensor products see [19].

Now I recall some essential property concerning the Projective and Injective tensor norms. To start we recall some natural, but very important, properties for injective and projective tensor products of Banach spaces.

Proposition 1.1.3. *If X and Y are Banach spaces, then $X^* \overset{\wedge}{\otimes}_\epsilon Y$ is a closed linear subspace of the space $\mathcal{L}(X, Y)$ of all bounded linear operators from X to Y.*

Theorem 1.1.4. *(The universal mapping Theorem)[Grothendieck [41], Theorem 2]*
For any Banach spaces X, Y and Z, the space $\mathcal{L}(X \overset{\wedge}{\otimes}_\pi Y; Z)$ of all bounded linear operators from $X \overset{\wedge}{\otimes}_\pi Y$ to Z is isometrically isomorphic to the space $\mathcal{B}(X, Y; Z)$ of all bounded bilinear transformations taking $X \times Y$ to Z. The natural correspondence establishing this isometric isomorphism is given by

$$v \in \mathcal{L}(X \overset{\wedge}{\otimes}_\pi Y; Z) \;\Leftrightarrow\; \phi \in \mathcal{B}(X, Y; Z)$$

via

$$v(x \otimes y) = \phi(x, y)$$

Theorem 1.1.5. *If X_1, X_2, Y_1 and Y_2 are $\mathbb{K}$-Banach spaces and $T : X_1 \longrightarrow X_2$ and $S : Y_1 \longrightarrow Y_2$ are bounded linear operators, then $T \otimes S : X_1 \overset{\wedge}{\otimes}_\pi Y_1 \longrightarrow X_2 \overset{\wedge}{\otimes}_\pi Y_2$ is a bounded linear operator with $\|T \otimes S\| \leq \|T\| \cdot \|S\|$*

The names Injective and Projective come from the following two peculiarities of that norms:

Proposition 1.1.6. *(The injectivity of $\|\cdot\|_\epsilon$) Let X and Y be Banach spaces. If Z is a closed linear subspace of X, then $Z \overset{\wedge}{\otimes}_\epsilon Y$ is a closed linear subspace of $X \overset{\wedge}{\otimes}_\epsilon Y$*

Proposition 1.1.7. *(The projectivity of $\|\cdot\|_\pi$) Let X and Y be Banach spaces. If Z is a closed linear subspace of X, then $X/Z \overset{\wedge}{\otimes}_\pi Y$ is a quotient of of $X \overset{\wedge}{\otimes}_\pi Y$*

It seems very interesting to see when the behaviour of the projective tensor norm is "injective" (i.e. it is closed for linear subspaces), and when the behaviour of the injective tensor norm is "projective" (i.e. it is closed for quotients). A first simple example is the following

Proposition 1.1.8. *If X and Y are Banach spaces, and Z is a* complemented subspace *of X then $Z \overset{\wedge}{\otimes}_\pi Y$ is a* complemented subspace *of $X \overset{\wedge}{\otimes}_\pi Y$ (in particular is a closed linear subspace).*

The proof is very simple; indeed it is enough to consider $P \otimes id_Y : X \overset{\wedge}{\otimes}_\pi Y \longrightarrow X \overset{\wedge}{\otimes}_\pi Y$, where $P : X \longrightarrow X$ is a bounded linear projection of X onto Z and $id_Y : Y \longrightarrow Y$ is the identity map, and to note that $P \otimes id_Y$ is a bounded linear projection of $X \overset{\wedge}{\otimes}_\pi Y$ onto $Z \overset{\wedge}{\otimes}_\pi Y$.
Another example come form Grothendieck [41]

Theorem 1.1.9. *Let X and Y be Banach spaces. Then $X \overset{\wedge}{\otimes}_\pi Y$ is a closed linear subspace os $X^{**} \overset{\wedge}{\otimes}_\pi Y$*

The next result, concerning the injectivity of the projective tensor product of Banach spaces, cames from two beautiful theorems:

Theorem 1.1.10. *(Heinrich, Mankiewicz [47])*
If Z is a separable closed linear subspace of a Banach space X, then there is a separable closed linear subspace $\widetilde{Z}$ of X that contains Z and a linear operator $T : \widetilde{Z}^* \longrightarrow X^*$ of norm one so that $Tz^*|_{\widetilde{Z}} = z^*$ for each $z^* \in \widetilde{Z}^*$

Theorem 1.1.11. *(Randrianantoanina [87])*
If Z is a closed linear subspace of a Banach space X and if there is a linear operator $T : Z^* \longrightarrow X^*$ of norm one such that $Tz^*|_Z = z^*$ for each $z^* \in Z^*$, then $Z \overset{\wedge}{\otimes}_\pi Y$ is a closed subspace of $Z \overset{\wedge}{\otimes}_\pi X$, regardless of the Banach space Y

Actually the Radrianantoanina 's theorem say us that if Z is a *local dual* of X^* then $Z \overset{\wedge}{\otimes}_\pi Y$ is a closed linear subspace of $X \overset{\wedge}{\otimes}_\pi Y$ (see [37] for the local dual notion and [34] for a more general notion of local dual).

The following unexpected result will be very useful in the sequel:

Corollary 1.1.12. *Let X and Y be Banach spaces. If Z is a separable closed linear subspace of X, then there is a separable closed subspace $\widetilde{Z}$ of X contains Z so that $\widetilde{Z} \overset{\wedge}{\otimes}_\pi Y$ is a closed subspace of $X \overset{\wedge}{\otimes}_\pi Y$*

Another beautiful result, about this theme, is a consequence of the spectacular Maurey's extension theorem (see [67])

Theorem 1.1.13. *Let X and Y be Banach spaces. If X_0 and Y_0 are closed linear subspaces of X and Y respectively having type 2 then $X_0 \overset{\wedge}{\otimes}_\pi Y_0$ is a closed linear subspace of $X \overset{\wedge}{\otimes}_\pi Y$.*

After to have recalled the above result, we want to note that there are two spacial class of Banach spaces in which the projective and injective tensor norms behave in a very nice way.

Theorem 1.1.14. *(Grothendieck [41]) Let X and Y be Banach spaces. Then*

(1) *$X \overset{\wedge}{\otimes}_\pi Z$ is a closed linear subspace of $X \overset{\wedge}{\otimes}_\pi Y$, whenever Z is a closed linear subspace of Y, if and only if X is isometrically isomorphic to $L^1(\mu)$ for some measure μ.*

(2) *$X \overset{\wedge}{\otimes}_\epsilon Y/Z$ is a quotient of $X \overset{\wedge}{\otimes}_\epsilon Y$, whenever Z is a closed linear subspace of Y, if and only if X is isometrically isomorphic to $C(K)$ for some compact Hausdorff space K .*

Another interesting problem related to the projective and injective tensor product is to see when can be embed in the dual of the other. Indeed Grothendieck was the first to understand how the approximation property come out in help to that problem, giving the following

Theorem 1.1.15. *Let X be a Banach space. Then the following are equivalent*

(i) *X has the metric approximation property.*

(ii) *For every Banach space Y, the canonical mapping $X \overset{\wedge}{\otimes}_\pi Y \hookrightarrow (X^* \overset{\wedge}{\otimes}_\epsilon Y^*)^*$ is an isometric embedding.*

(iii) *The canonical map $X \overset{\wedge}{\otimes}_\pi X^* \hookrightarrow (X^* \overset{\wedge}{\otimes}_\epsilon X)^*$ is an isometric embedding.*

Proof. Suppose (i) holds. Since the canonical map $X \overset{\wedge}{\otimes}_\pi Y \hookrightarrow (X^* \overset{\wedge}{\otimes}_\epsilon Y^*)^*$ has norm one, it is enough to show that $\|u\|_{\epsilon^*} \geq \|u\|_\pi$ for every $u \in X \overset{\wedge}{\otimes}_\pi Y$ (where $\|\cdot\|_{\epsilon^*}$ is the dual norm of $\|\cdot\|_\epsilon$; actually such norm is called the integral norm (see [19])). Fix $u \in X \overset{\wedge}{\otimes}_\pi Y$ and $\epsilon > 0$. Choose a representation $\sum_{n=1}^\infty x_n \otimes y_n$ of u with $x_n \to 0$ and $\sum_{n=1}^\infty \|y_n\| < \infty$. Since the dual of the projective tensor product $X \overset{\wedge}{\otimes}_\pi Y$ of X and Y is just the operator space $\mathcal{L}(X, Y^*)$, there exists $T \in \mathcal{L}(X, Y^*)$, $\|T\| = 1$, such that $|\langle u, T\rangle| \geq \|u\|_\pi - \epsilon$. By the metric approximation property there exists a finite-rank operator $S : X \longrightarrow Y^*$, such that $\|S\| \leq 1$ and

$$\|T(x_n) - S(x_n)\| < \frac{\epsilon}{\sum_{n=1}^\infty \|y_n\|}$$

for every n. So $|\langle u, T - S\rangle| < \epsilon$ and therefore $|\langle u, S\rangle| \geq \|u\|_\pi - 2\epsilon$. Since S is a finite-rank operator, then S belongs to $X^* \overset{\wedge}{\otimes}_\epsilon Y^*$. Now, by the definition of the injective tensor norm, it is clear that the operator norm coincides with the injective tensor norm, and then $\|S\|_\epsilon \leq 1$. Therefore

$$\|u\|_{\epsilon^*} \geq |\langle u, S\rangle| \geq \|u\|_\pi - 2\epsilon$$

It is trivial that *(ii)* implies *(iii)*.
Suppose that *(iii)* holds. Since $X^* \otimes X$ is dense in $X^* \overset{\wedge}{\otimes}_\epsilon X$ then

$$\|u\|_\pi = \sup\{|\langle u, S\rangle| \ : \ S \in X^* \otimes X, \ \|S\|_\epsilon \leq 1\}$$

for every $u \in X^* \overset{\wedge}{\otimes}_\pi X$. On the other hand

$$\|u\|_\pi = \sup\{|\langle u, T\rangle| \ : \ T \in \mathcal{L}(X, X), \ \|T\| \leq 1\}$$

Therefore the set $F = \{S \in \mathcal{L}(X, X) \ : \ S$ has finite rank and $\|S\| \leq 1\}$ is weak*-dense in the closed unit ball of $\mathcal{L}(X, X)$ (where the set are considered in $(X \overset{\wedge}{\otimes}_\pi X^*)^*$). Thus the identity operator I on X can be weak* approximated by element of F: if $x \in X$, $x^* \in X^*$ and $\epsilon > 0$ then there exists $S \in F$ such that $|\langle S(x), x^*\rangle - \langle x, x^*\rangle| < \epsilon$. It follows that for every $x \in X$, x belongs to the weak closure of the set $F(x) = \{S(x) \ : \ S \in F\}$. But this set is convex, and so its weak and norm closure coincide. Thus, x lies in the norm closure of $F(x)$ for every x. Now consider the strong operator topology on $\mathcal{L}(X, X)$, generated by the seminorms $T \longmapsto \|T(x)\|$, as x ranges over X. Since the identity is in the closure of F for this topology, there exists a net $\{S_\alpha\}$ of finite-rank operators, each having norm at most one, such that $S_\alpha(x) \to x$ for every $x \in X$. Then the Theorem follows by 0.0.10 $\square$

Remark 1.1.16. As the reader can easily note, using the same proof as above, the following are equivalent:

(i) X has the bounded approximation property.

(ii) For every Banach space Y, the canonical mapping $X \overset{\wedge}{\otimes}_\pi Y \hookrightarrow (X^* \overset{\wedge}{\otimes}_\epsilon Y^*)^*$ is an isomorphic embedding.

(iii) The canonical map $X \overset{\wedge}{\otimes}_\pi X^* \hookrightarrow (X^* \overset{\wedge}{\otimes}_\epsilon X)^*$ is an isomorphic embedding.

The last result will be used specially in 3.3.13.

Chapter 2

Fremlin Tensor Product

2.1 A short introduction to Banach lattices

Throughout this section we recall some basic definition and property concerning the lattice structures (for the proofs or more about that we referee [73]).
Suppose X is a linear space and X is allowed with a partial order $\leq$. If $x \pm z \leq y \pm z$ and $px \leq py$ for all x, y, z in X and all positive numbers p whenever $x \leq y$; then we call X an ordered linear space. If the order of the ordered linear space X is a lattice order, that is, for any $x, y \in X$, $x \vee y =$ least upper bound of x and y as well as $x \wedge y =$ greatest lower bound of x and y coexist, then X is called a *vector lattice*. If x is an element of the vector lattice X, then we will use the standard notations

$$x^+ = x \vee 0, \quad x^- = (-x) \vee 0, \quad |x| = x^+ \vee x^-$$

The positive cone X^+ of X is the set

$$X^+ = \{x \in X \ : \ x \geq 0\}$$

A *Banach lattice* is a Banach space that is a vector lattice with $\|x\| \leq \|y\|$ whenever $0 \leq x \leq y$; alternatively, if $|x| \leq |y|$, then $\|x\| \leq \|y\|$. Naturally $\|x\| = \| \, |x| \, \|$.
If X and Y are Banach lattices and $T : X \longrightarrow Y$ is a linear mapping we say T is positive if $T(x) \geq 0$ whenever $x \geq 0$. Now, we recall some basic property about Banach lattices

Theorem 2.1.1. (1) *The lattice operations are continuous;*

(2) *X^+ is closed;*

(3) *If $(x_n)_n$ is an increasing convergent sequence, then $\sup_n x_n$ exists and is $= \lim_n x_n$;*

(4) *Positive linear mappings between Banach lattices are continuous;*

Definition 2.1.2. A vector lattice X is said to be *Dedekind complete* when, for every increasing net $(x_\alpha)_{\alpha \in I}$ (i.e. $x_\alpha \leq x_\beta$ if $\alpha, \beta \in I$, and $\alpha \leq \beta$) with $x_\alpha \leq x$ for each $\alpha \in I$, there exists $x_0 \in X$ such that $x_0 = \sup_{\alpha \in I} x_\alpha$.
Instead it is said to be *σ-Dedekind complete* when, for every increasing sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \leq x$ for all $n \in \mathbb{N}$, there exists $x_0 \in X$ such that $x_0 = \sup_{n \in \mathbb{N}} x_n$.
A norm of a Banach lattice X is called *order continuous* whenever $(x_\alpha)_\alpha$ is a decreasing net with $x_\alpha \geq 0$ $\forall \alpha \in I$ we have $\lim_\alpha ||x_\alpha|| = 0$.

Definition 2.1.3. Let E and F be Banach lattices. An linear bounded operator $T : E \longrightarrow F$ is called *positive* if $T(E^+) \subseteq F^+$ (where, as usual, we denote with E^+, F^+ the positive cone of E and F respectively). Let $\mathcal{L}^+(E, F)$ the collection of the positive operators from E to F. An operator $T \in \mathcal{L}(E, F)$ is called *regular* if there exist $T_1, T_2 \in \mathcal{L}^+(E, F)$ so that $T = T_1 - T_2$. Let $\mathcal{L}^r(E, F)$ the collection of the regular operators from E to F. It is known that if F is Dedekind complete then $\mathcal{L}^r(E, F)$ is a Banach lattice with the positive cone $\mathcal{L}^+(E, F)$ (see [73]), and norm

$$\|T\|_r = \inf\{\|S\| \ : \ S \in \mathcal{L}^+(E, F), \ |T(x)| \leq S(|x|), \ x \in E^+\}$$

and in such which case $\|T\|_r = \| \ |T| \ \|$.

2.2 Radon-Nikodým Property and Semi-embeddings

Let X be a Banach space, recall that a countable additive vector measure (or σ-additive vector measure) on a σ-field Σ of subsets of a set Ω to X is a function $F : \Sigma \longrightarrow X$ with the property: $F(\bigcup_{n=1}^\infty E_n) = \sum_{n=1}^\infty F(E_n)$ in the norm topology of X, for all sequences $(E_n)_n$ of pairwise disjoint members of Σ.

Definition 2.2.1. Let $F : \Sigma \longrightarrow X$ be a vector measure on a σ-field Σ. The *variation* of F is defined:

$$|F|(E) = \sup_\pi \sum_{A \in \pi} \|F(A)\|$$

where the supremum runs over all partition π of E into a pairwise disjoint members of Σ. If $|F|(\Omega) < \infty$, the F will be called a *measure of bounded variation*.

Let (Ω, Σ, μ) be a finite measure space, and let X be a Banach space. A countable additive vector measure $F : \Sigma \longrightarrow X$ is called *μ-continuous* if

$$\lim_{\mu(E)\to 0} F(E) = 0$$

If μ is a finite nonnegative real-valued measure on Σ the μ-continuity condition is equivalent to the fact that F vanished on set of μ-measure zero. (see [26], Theorem I.2.1)

Definition 2.2.2. A Banach space X has the *Radon-Nikodým property* whenever given a probability space (Ω, Σ, μ) and a countable additive, μ-continuous vector measure $F : \Sigma \longrightarrow X$ of bounded variation, there is a Bochner integrable $f : \Omega \longrightarrow X$ such that for each $E \in \Sigma$, we have

$$F(E) = \int_E f(w) d\mu(w)$$

By now the following theorem ought to be well known

Theorem 2.2.3. *Let X be a Banach spaces. Then the following are equivalents:*

(1) *X has the Radon-Nikodým property.*

(2) *Given a probability space (Ω, Σ, μ) and a vector measure $G : \Sigma \longrightarrow X$ such that $\|G(E)\| \leq \mu(E)$, for each $E \in \Sigma$, there is a (necessarily essentially bounded) Bochner integrable $g : \Omega \longrightarrow X$ such that for any $E \in \Sigma$,*

$$G(E) = \int_E g(\omega) d\mu(\omega).$$

(3) *Given a bounded linear operator $T : L_1[0,1] \longrightarrow X$, there is a (necessarily essentially bounded) Bochner integrable $h : [0,1] \longrightarrow X$ such that for any $f \in L_1[0,1]$,*

$$Tf = \int_{[0,1]} f(t)\ h(t)\ dt.$$

(4) *Any uniformly bounded martingale sequence* (X_n, Σ_n) *in* $L_1(\mu, X)$ *is almost surely convergent (given a monotone increasing net* $\{B_\tau\}_{\tau \in T}$ *of sub* σ*-fields of a field* Σ *(i.e.* $B_{\tau_1} \subseteq B_{\tau_2}$ *for* $\tau_1 \leq \tau_2$ *in* T*) a* martingale *is a net* $\{f_\tau\}_{\tau \in T}$ *in* $L_p(\mu, X)$ $(1 \leq p < \infty)$ *over the same directed set* T *such that*

$$\int_E f_\tau d\mu = \int_E f_{\tau_1} d\mu \qquad \text{for all } E \in B_{\tau_1}$$

for all $\tau \geq \tau_1$*).*

(5) *Every non void closed bounded convex subset* C *of* X *is dentable, i.e. given such a* C *and any* $\epsilon > 0$*, there is an* $x_\epsilon \in C$ *such that*

$$x_\epsilon \notin \overline{co}(C \setminus \{y \in C : \|y - x\| < \epsilon\}).$$

(6) *Every non void closed bounded convex subset* C *of* X *has a denting point, i.e. given such a* C*, there is a point* $x \in C$ *(called a denting point) such that regardless* $\epsilon > 0$

$$x \notin \overline{co}(C \setminus \{y \in C : \|y - x\| < \epsilon\}).$$

(7) *Every non void closed bounded convex subset* C *of* X *is the closed convex hull of its denting points.*

Remark 2.2.4. It is clear that the Radon-Nikodým property is an isomorphic invariant; also, a space with the Radon-Nikodým property shares that property with each of its closed linear subspaces; further, in light of (3) above, a Banach space, each of whose separable closed subspaces has the Radon-Nikodým property, enjoys the property, too.

It follows painlessly from the Radon-Nikodým property that all finite dimensional Banach spaces have the Radon-Nikodým property. Concerning infinite dimensional spaces, we feel it best to cite the following "observation" due to J. A. Clarkson.

Theorem 2.2.5. ℓ_1 *has the Radon-Nikodým property*

Proof. Let (Ω, Σ, μ) be a probability space and $F_0 : \Sigma \longrightarrow \ell_1$ be a vector measure satisfying $\|F_0(E)\| \leq \mu(E)$, for $E \in \Sigma$.
For each $n \geq 1$, define $P_n : \ell_1 \longrightarrow \ell_1$ by

$$P_n(\sum_{k=1}^{\infty} a_k e_k) = \sum_{k=1}^{n} a_k e_k$$

here, as usual, e_k denotes the k-th unit coordinate vector.
Naturally, each of the measures $F_n = P_n \circ F_0$ has the derivative f_n respect to μ; after all, the F_n' s have the finite dimensional ranges. Denoting by $|F_n|(E)$ the total variation of the measure F_n on the set E, we see that

$$\int_E \|f_n(\omega)\| d\mu(\omega) = |F_n|(E) \leq |F_0|(E) \leq \mu(E)$$

for all $n \in \mathbb{N}$ and all $E \in \Sigma$. It follows that for each $n \geq 1$

$$\|f_n(.)\| \leq \|f_{n+1}(.)\| \leq ... \leq 1$$

μ-almost all the time. Consequently, for the nature of ℓ_1' s norm we have

$$f_0(\omega) = \lim_{n\to\infty} f_n(\omega)$$

exists in norm for μ-almost all $\omega \in \Omega$. Plainly, f_0 is the sought after derivative of F_0 with the respect to μ $\square$

To take advantage of ℓ_1's enjoyment of the Radon-Nikodým property, we introduce the notion of a semi-embedding. With Heinrich Lotz, Tenny Peck and Horatio Porta [64], we say that a Banach space X is *semi-embedding* in a Banach space Y if there exists a bounded linear 1-1 operator $\sigma : X \longrightarrow Y$ for which $\sigma(B_X)$ is closed; naturally, the map σ is called a *semi-embedding.* To be sure, semi-embedding need not be embedding (= linear homomorphisms). After all if $1 \leq p < q \leq \infty$ then the natural inclusion map of $L_q[0,1]$ into $L_p[0,1]$ is a semi-embedding. In fact, to highlight the disparity between semi-embedding and embedding we make special note of the following assertion

Theorem 2.2.6. *The dual of any separable Banach space is semi-embeddable in ℓ_1*

Proof. Let X be a Banach space space and suppose that $\{x_n\}_{n\in\mathbb{N}}$ be a countable dense subset of X consisting, say, of non-zero vectors. Define $T : c_0 \longrightarrow X$ by

$$T\lambda = \sum_{n\in\mathbb{N}} \frac{1}{2^n}\lambda_n \frac{x_n}{\|x_n\|}.$$

Since for any $\lambda \in c_0$ we have

$$\|\frac{1}{2^n}\lambda_n \frac{x_n}{\|x_n\|}\| \leq \frac{\|\lambda\|}{2^n}$$

it is plain to see that $\|T\lambda\| \leq \|\lambda\|$ and T is a bounded linear operator with a dense range. T^* is therefore a weak*-continuous linear operator, $T^* : X^* \longrightarrow \ell_1$, which is 1-1. Of course $T^* B_{X^*}$ is a weak* compact in ℓ_1 and so is norm closed. T^* is a semi-embedding of X^* into ℓ_1 $\square$

To be a connection between Theorem 2.2.5 and Theorem 2.2.6 and the subject matter of this section we present the following elegant observation of Jean Bourgain and Haskell Rosenthal [6]

Theorem 2.2.7. *Suppose X is a separable Banach space semi-embeddable in a Banach space Y having the Radon-Nikodým property. Then X has the Radon-Nikodým property, too.*

To prove this theorem we need of the following

Remark 2.2.8. Let X be a separable Banach space and Y a Banach space. Let $\sigma : X \longrightarrow Y$ a semi-embedded, then

$$j = \sigma|_{B_X} : B_X \longrightarrow \sigma(B_X)$$

is a Borel isomorphism (i.e. j maps Borel sets in Borel sets).
To prove the claim above it is enough to prove that j map a closed in a Borel set of $\sigma(B_X)$.
Since X is separable then, if d is the metric generated of the norm, (X, d) is a Polish space. Let O be an open subset of B_X. By the proposition 0.0.3 there is a metric $\widetilde{d}$ on O so that $(O, \widetilde{d})$ is a Polish space, and the metrics d and $\widetilde{d}$ are equivalent on O.
Let A be an closed subset of B_X. Put $A_1 = A$ and $A_2 = B_X \setminus A$.
Claim: There exists continuous functions

$$f_1 : \mathbb{N}^{\mathbb{N}} \longrightarrow A_1$$
$$f_2 : \mathbb{N}^{\mathbb{N}} \longrightarrow A_2$$

so that

$$A_1 = f_1(\mathbb{N}^{\mathbb{N}}), \quad A_2 = f_2(\mathbb{N}^{\mathbb{N}})$$

see also proposition 0.0.4.
Proof of Claim: A_1, A_2 are Polish spaces, in particular $A_1 = \bigcup_{n_1 \in \mathbb{N}} C(n_1)$ (where with $C(n_1)$ we are denoting the balls of center in A_1 and radius 1). Since $C(n_1)$ is a Polish space (because open), for the same sake, $C(n_1) = \bigcup_{n_2 \in \mathbb{N}} C(n_1, n_2)$, where with $C(n_1, n_2)$ we are denoting the balls of center in $C(n_1)$ and radius $\frac{1}{2}$. By induction we can define $C(n_1, ..., n_k)$ so that

$$C(n_1, ..., n_{k-1}) = \bigcup_{n_k \in \mathbb{N}} C(n_1, ..., n_k), \quad \textit{and} \quad \textit{diam}(C(n_1, ..., n_k)) < \frac{1}{k}$$

Then we can define $f_1 : \mathbb{N}^{\mathbb{N}} \longrightarrow A_1$ by

$$f_1(\mathbf{n}) = \bigcap_{k \in \mathbb{N}} C(n_1, ..., n_k)$$

where we are denoting $\mathbf{n} = (n_1, ..., n_m, ...)$.
Let $\epsilon > 0$ arbitrary, by construction we can suppose that the open in A_1 is $C(n_1, ..., n_{k_1})$. Fix $k = \max\{k_1, \frac{1}{\epsilon}\}$. Let $\mathbf{n}$ so that $f_1(\mathbf{n}) \in C(n_1, ..., n_k)$ and we can consider the open of $\mathbf{n}$ in a such way

$$\mathcal{U} = \{\mathbf{m} : (m_1, ..., m_k) = (n_1, ..., n_k)\}$$

Then we have $f_1(\mathbf{m}) \in C(n_1, ..., n_k)$ for each $\mathbf{m} \in \mathcal{U}$, so

$$d(f_1(\mathbf{n}), f_1(\mathbf{m})) \leq diam C(n_1, ..., n_k) < \frac{1}{k} \leq \epsilon \qquad \forall \mathbf{m} \in \mathcal{U}$$

That imply that f_1 is continuous and by construction f_1 is onto.
Then we can put $g_i : \mathbb{N}^{\mathbb{N}} \longrightarrow j(A_i) = B_i$ by

$$g_i = j \circ f_i \qquad i = 1, 2$$

We show that B_1, B_2 are separated by Borel set in $j(B_X)$, i.e. there are E_1, E_2 Borel sets in $j(B_X)$ so that $B_i \subseteq E_i$ and $E_1 \cap E_2 = \emptyset$ (in particular that imply that $B_i = E_i$, and then we are done).
Suppose that we cannot separated with Borel set the B_i's. If we denoted

$$\mathcal{N}(n_1, ..., n_k) = \{\mathbf{m} \in \mathbb{N}^{\mathbb{N}} : (m_1, ..., m_k) = (n_1, ..., n_k)\}$$

we have

$$B_1 = g_1(\mathbb{N}^{\mathbb{N}}) = g_1(\bigcup_{n_1 \in \mathbb{N}} \mathcal{N}(n_1)) = \bigcup_{n_1 \in \mathbb{N}} g_1(\mathcal{N}(n_1))$$

and

$$B_2 = \bigcup_{m_1 \in \mathbb{N}} g_2(\mathcal{N}(m_1))$$

choose n_1, m_1 so that $g_1(\mathcal{N}(n_1))$ and $g_2(\mathcal{N}(m_1))$ cannot be separated by Borel sets (because if every $g_1(\mathcal{N}(n_1))$ and $g_2(\mathcal{N}(m_1))$can be separated by Borel sets then the union in n_1 and m_1 respectively can be separated by Borel sets too).
We write

$$g_1(\mathcal{N}(n_1)) = \bigcup_{n_2 \in \mathbb{N}} g_1(\mathcal{N}(n_1, n_2))$$

$$g_2(\mathcal{N}(m_1)) = \bigcup_{m_2 \in \mathbb{N}} g_1(\mathcal{N}(m_1, m_2))$$

and so by induction we can construction $g_1(\mathcal{N}(n_1, n_2, ..., n_k))$ and $g_2(\mathcal{N}(m_1, m_2, ..., m_k))$ so that they cannot be separated by Borel sets, for each $k \in \mathbb{N}$. Let $\mathbf{n} =$

$(n_1, ..., n_k, ...)$ and $\mathbf{m} = (m_1, ..., m_k, ...)$. If $g_1(\mathbf{n}) \neq g_2(\mathbf{m})$ than there are U, V open (in particular Borel set) disjoint so that

$$\mathbf{n} \in g_1^{-1}(U), \quad \mathbf{m} \in g_2^{-1}(V)$$

and than we can choose $k \in \mathbb{N}$ such that

$$\mathcal{N}(n_1, ..., n_k) \in g_1^{-1}(U) \quad \textit{and} \quad \mathcal{N}(m_1, ..., m_k) \in g_2^{-1}(V)$$

OOPS!
That's imply

$$g_1(\mathbf{n}) = g_2(\mathbf{m})$$

which contradict the injectivity of j.

Proof. (of the theorem 2.2.7)
Let $T : L_1[0,1] \longrightarrow X$ be a bounded linear operator with $\|T\| \leq 1$, say. Let $\sigma : X \longrightarrow Y$ be a semi-embedding again with $\|\sigma\| \leq 1$. We need to find a measurable $g : [0,1] \longrightarrow X$ that represent T. To do so we look at the operator $S : L_1[0,1] \longrightarrow Y$ given by $S = \sigma \circ T$. Since Y has the Radon-Nikodým property, there exists a Borel measurable, essentially bounded function $h : [0,1] \longrightarrow Y$ such that for any $f \in L_1[0,1]$

$$Sf = \int_{[0,1]} f(t)\ h(t)\ dt$$

For almost any $t \in [0,1]$, we have

$$h(t) = \lim_{s \to 0} \frac{1}{2s} \int_{t-s}^{t+s} h(u) du = \lim_{s \to 0} S(\frac{\chi_{[t-s,t+s]}}{\|\chi_{[t-s,t+s]}\|})$$

Therefore, for almost all $t \in [0,1]$,

$$h(t) \in \overline{S(B_{L_1[0,1]})} = \overline{\sigma \circ T(B_{L_1[0,1]}} \subseteq \overline{\sigma(B_X)} = \sigma(B_X)$$

by adjusting h, we can assume $h(t) \in \sigma(B_X)$ for all t. But now notice σ takes B_X in a 1-1 continuous fashion onto $\sigma(B_X)$ and B_X is a Polish space. By remark above follows that σ is a *Borel equivalence* and so $g = \sigma^{-1} \circ h$ is measurable with little choose but to represent T. □

Close on the heels of the Theorem 2.2.5, 2.2.6 and 2.2.7 we have to following classical result of Dunford and Pettis.

Theorem 2.2.9. *Separable dual space have the Radon-Nikodým property.*

We want to make a few remarks about the Radon-Nikodým property, especially as it appears in separable Banach space setting. Theorem 2.2.9 indicates the dividends points in studying a property that has mixed origin. It precisely the hybrid character of Radon-Nikodým property that lends the property to widespread application in analysis. No double other stability results, similar to theorem 2.2.7, will be uncovered in efforts to use the Radon-Nikodým property in other aspects of mathematical analysis. Indeed, almost one such result exists and it is an extraordinary result at that. We cite it

Theorem 2.2.10. *(Piotr Mankiewicz)*
Suppose X and Y are separable Banach space. Suppose there is a surjective map $\phi : X \longrightarrow Y$ satisfying

$$\frac{1}{k}\|x-\overline{x}\| \leq \|\phi(x)-\phi(\overline{x})\| \leq k\|x-\overline{x}\|$$

for same $k > 0$ and all $x, \overline{x} \in X$. If either X or Y has the Radon-Nikodým property, then both do and each is isomorphic to a subspace of the other

2.3 The Fremlin Tensor Product of Banach Lattices

For two Banach lattices X and Y, D. Fremlin in [32], and [33] introduced a new tensor product of Banach lattices, called *positive projective tensor product*. Let X, Y, Z be Banach lattices, an bilinear map $\phi : X \times Y \longrightarrow Z$ is called a *positive bilinear map* if $\phi(X^+, Y^+) \subseteq Z^+$; the projective cone on tensor product $X \otimes Y$ is defined as:

$$X^+ \otimes Y^+ = \{\sum_{k=1}^{n} x_k \otimes y_k : \quad n \in \mathbb{N},\ x_k \in X^+,\ y_k \in Y^+\}.$$

The positive projective tensor norm on $X \otimes Y$ is defined as:

$$\|u\|_{|\pi|} = \sup\{|\sum_{i=1}^{n} \phi(x_i, y_i)| \ : u = \sum_{i=1}^{n} x_i \otimes y_i,$$
$$\phi \text{ is a positive bilinear function on } X \times Y,\ \|\phi\| \leq 1\}$$

Let $X \overset{\wedge}{\otimes}_F Y$ the completion of $X \otimes Y$ equipped with the norm $\|\cdot\|_{|\pi|}$. Then $X \overset{\wedge}{\otimes}_F Y$ is a Banach lattices, having as positive cone the closure in $X \overset{\wedge}{\otimes}_F Y$

of the cone generated by $X^+ \otimes Y^+$. Fremlin ([33], Theorem 1E (vii)) gave a more simple equivalent form of the positive projective norm, as

$$\|u\|_{|\pi|} = \inf\{\sum_{k=1}^{\infty} \|x_k\| \cdot \|y_k\| : \quad x_k \in X^+,\ y_k \in Y^+,\ |u| \leq \sum_{k=1}^{\infty} x_k \otimes y_k\}.$$

In the same paper Fremlin gave a very meaning property which the Fremlin tensor product enjoys

Lemma 2.3.1. *(Fremlin‘s theorem [33])*
Let X and Y be Banach lattices. Then for each Banach lattice Z and for each $\phi : X \times Y \longrightarrow Z$ continuous bilinear map there exists a unique continuous linear map $T : X\widehat{\otimes}_F Y \longrightarrow Z$ so that

(i) $\|T\| = \|\phi\|$

(ii) $T(x \otimes y) = \phi(x, y)$

(iii) ϕ is a positive if and only if T is a positive.

Throughout this section U will denote a Banach lattice with a normalized unconditional basis $(u_n)_{n\in\mathbb{N}}$ with normalized biorthogonal functionals $(u_n^*)_{n\in\mathbb{N}}$ (see definition 0.0.6). We recall that a Banach space U with an unconditional basis $(u_n)_{n\in\mathbb{N}}$ becomes a Banach lattice with the new norm

$$|||u||| = \sup\{\| \sum_{n\in\mathbb{N}} t_n u_n^*(u) u_n\| \quad : \quad (t_n)_{n\in\mathbb{N}} \in \ell_\infty\}$$

where we are denoting with $\|\cdot\|$ the original norm in U. Moreover

$$\|\cdot\| \leq |||\cdot||| \leq K\|\cdot\|,$$

where K is a constant depending only on the unconditional basis $(u_n)_{n\in\mathbb{N}}$. Throughout the sequel, U is endowed with the norm $|||\cdot|||$.

Let X and U be Banach lattices, and, as we said above, let $(u_n)_n$ be a Schauder basis of U. We define

$$U(X) = \{(x_n)_{n\in\mathbb{N}} \subseteq X : \sum_{n\in\mathbb{N}} \|x_n\|\, u_n \text{ is convergent in } U\}$$

endow with the norm

$$\|(x_n)\|_{U(X)} = \|\sum_{n\in\mathbb{N}} \|x_n\|\,\|_U$$

and the order defined defined by

$$(x_n)_{n\in\mathbb{N}} \leq (y_n)_{n\in\mathbb{N}} \quad \Leftrightarrow \quad x_n \leq_X y_n \quad \forall n \in \mathbb{N}$$

Then $U(X)$ is a Banach lattice, as is easily verified.

2.4 The Radon-Nikodým property in Fremlin Tensor Product

We start to get some confidence with a useful tool : the semi-embeddings.

Theorem 2.4.1. *Let X be a separable Banach lattice and U be a Banach lattice with an unconditional basis. If U and X have the Radon-Nikodým property then the Fremlin tensor product $U\widehat{\otimes}_F X$ of U and X can be semi-embedded in $U(X)$*

Proof. To start, consider the bilinear operator

$$\begin{array}{rcl} \widetilde{\Psi} \ : \ U \times X & \longrightarrow & U(X) \\ (u,x) & \longmapsto & (u_n^*(u)x)_{n\in\mathbb{N}} \end{array}$$

Note that $\widetilde{\Psi}$ is bilinear bounded and positive. If $u \in U^+$ and $x \in X^+$, then $\widetilde{\Psi}(u,x)_i = u_i^*(u)x \geq 0$ for each $i \in \mathbb{N}$; therefore, $\Psi(u,x) \geq 0$ in $U(X)$; moreover,

$$\begin{aligned} \| \sum_{n\in\mathbb{N}} \|\widetilde{\Psi}(u,x)\|_X \ u_n \ \|_U &= \| \sum_{n\in\mathbb{N}} \|u_n^*(u)x\|_X u_n \ \|_U \\ &= \|x\|_X \| \sum_{n\in\mathbb{N}} |u_n^*(u)| u_n\|_U \\ &\leq \|x\|_X \|u\|_U \end{aligned}$$

So $\widetilde{\Psi}$ is bounded with $\|\widetilde{\Psi}\| \leq 1$.

By Fremlin's theorem there exists a unique continuous linear map Ψ : $U\widehat{\otimes}_F X \longrightarrow U(X)$ such that

(i) $\|\Psi\| \leq 1$;

(ii) $\widetilde{\Psi}(u,x) \ = \ \Psi(u \otimes x)$ for each $u \in U, x \in X$;

(iii) Ψ is positive

Step 1. Ψ is injective.
First, consider Ψ on $U \otimes_F X$. If $v = \sum_{k=1}^p v_k \otimes x_k \in (U \otimes_F X)^+$ (with $v_k, x_k \geq 0$ for each k) so that $\Psi(v) = 0$, then

$$0 = \Psi(v) = \sum_{k=1}^p \widetilde{\Psi}(v_k, x_k) = (\sum_{k=1}^p u_n^*(v_k) x_k)_{n\in\mathbb{N}};$$

thus

$$\sum_{k=1}^{p} u_n^*(v_k)x_k = 0, \quad \forall n \in \mathbb{N};$$

but every $u_n^*(v_k)x_k$ is in X^+, so

$$u_n^*(v_k)x_k = 0 \quad \forall n \in \mathbb{N},\ k = 1, ..., p$$

This means: either $x_k = 0$ or $u_n^*(v_k) = 0$ for each $n \in \mathbb{N}$; hence, either $x_k = 0$ or $u_k = 0$. Either way we have $v = \sum_{k=1}^{p} v_k \otimes x_k = 0$.
If there is a $z > 0$ in $U\widehat{\otimes}_F X$ so that $\Psi(z) = 0$, then we can choose a sequence $(z_n)_{n\in\mathbb{N}}$, of positive elements of $U \otimes_F X$ such that $z_n \leq z$ for every $n \in \mathbb{N}$, convergent to z (see [45]). Therefore,

$$\begin{gathered} 0 \leq \Psi(z_n) \leq \Psi(z) = 0 \quad \forall n \in \mathbb{N}; \\ \text{so} \qquad z_n = 0, \quad \forall n \in \mathbb{N}, \quad \text{and so} \\ z = \|\cdot\|_{|\pi|} - \lim_n z_n = 0 \end{gathered}$$

a contradiction. This means Ψ is injective on the positive cone of $U\widehat{\otimes}_F X$ and so injective on $U\widehat{\otimes}_F X$.

We want to show that Ψ is a semi-embedding, or in other words for a sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq B_{U\widehat{\otimes}_F X}$ and $(y_i)_{i\in\mathbb{N}} \in U(X)$ so that $\lim_n \Psi(z_n) = (y_i)_i$ in $U(X)$ there exists $z \in B_{U\widehat{\otimes}_F X}$ such that $\Psi(z) = (y_i)_i$.

Step 2. Now, fix $T \in \mathcal{L}^r(U, X^*)$ and consider the series $\sum_{i\in\mathbb{N}}\langle y_i, T(u_i)\rangle$. First, we show that the series is absolutely convergent: We suppose that $(z_n)_n \subseteq B_{U\otimes X}$, so we can write each z_n as

$$z_n = \sum_{k=1}^{p(n)} v_{k,n} \otimes x_{k,n}$$

Then

$$\begin{aligned} (y_i)_i &= \lim_n \Psi(z_n) \\ &= \lim_n \sum_{k=1}^{p(n)} \widetilde{\Psi}(v_{k,n}, x_{k,n}) \\ &= \lim_n (\sum_{k=1}^{p(n)} (u_i^*(v_{k,n})x_{k,n})_i \end{aligned}$$

$$= (\lim_n \sum_{k=1}^{p(n)} (u_i^*(v_{k,n})x_{k,n})_i.$$

Hence

$$\lim_n \sum_{k=1}^{p(n)} u_i^*(v_{k,n})x_{k,n} = y_i \quad i \in \mathbb{N}$$

Then for fixed $m \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ so that for $i = 1, 2, ..., m$ we have

$$\| \sum_{k=1}^{p(n_0)} u_i^*(v_{k,n_0})x_{k,n_0}) - y_i\|_X \leq \frac{\epsilon}{m}$$

Therefore,

$$\sum_{i=1}^{m} |\langle y_i, T(u_i)\rangle|$$

$$\leq \sum_{i=1}^{m} |\langle y_i - \sum_{k=1}^{p(n_0)} u_i^*(v_{k,n_0})x_{k,n_0}, T(u_i)\rangle| + \sum_{i=1}^{m} |\langle \sum_{k=1}^{p(n_0)} u_i^*(v_{k,n_0})x_{k,n_0}, T(u_i)\rangle|$$

$$\leq \epsilon\|T\| + \sum_{i=1}^{m} | \sum_{k=1}^{p(n_0)} \langle u_i^*(v_{k,n_0})x_{k,n_0}, T(u_i)\rangle|$$

$$\text{(which, if } \theta_i = sign \sum_{k=1}^{p(n_0)} \langle u_i^*(v_{k,n_0})x_{k,n_0}, T(u_i)\rangle)$$

$$= \epsilon\|T\| + |\sum_{i=1}^{m} \theta_i \sum_{k=1}^{p(n_0)} \langle u_i^*(v_{k,n_0})x_{k,n_0}, T(u_i)\rangle|$$

$$= \epsilon\|T\| + |\sum_{i_1}^{m} \theta_i T(u_i) \otimes u_i^*(\sum_{k=1}^{p(n_0)} v_{k,n_0} \otimes x_{k,n_0})|$$

$$= \epsilon\|T\| + |\sum_{i=1}^{m} \theta_i T(u_i) \otimes u_i^*(z_{n_0})|$$

$$\leq \epsilon\|T\| + \|\sum_{i=1}^{m} \theta_i T(u_i) \otimes u_i^*\|_{\mathcal{L}(U,X^*)}\|z_{n_0}\|$$

$$(*) \qquad \leq \epsilon\|T\| + \|T\|,$$

for each $m \in \mathbb{N}$.

To see why is so, recall: if $z \in (U \overset{\wedge}{\otimes}_F X)^+$ then for each $\epsilon > 0$ there exists $(x_j)_{j=1}^n \subseteq X^+$, $(v_j)_{j=1}^n \subseteq U^+$ so that $z \leq \sum_{j=1}^n v_j \otimes x_j$ and

$$\sum_{j=1}^n \|v_j\| \; \|x_j\| \leq \|z\|_{U \otimes_F X} + \epsilon.$$

So

$$\begin{aligned}
|\sum_{i=1}^m \theta_i T(u_i) \otimes u_i^*(z)| &\leq \sum_{i=1}^m T(u_i) \otimes u_i^*(z) \\
&\leq \sum_{i=1}^m T(u_i) \otimes u_i^*(\sum_{j=1}^n v_j \otimes x_j) \\
&= \sum_{i=1}^m \sum_{j=1}^n T(u_i)(x_j) u_i^*(v_j) \\
&= \sum_{j=1}^n T(\sum_{i=1}^m u_i^*(v_j) u_i)(x_j) \\
&\leq \sum_{j=1}^n \|T(\sum_{i=1}^m u_i^*(v_j) u_i)\|_{X^*} \; \|x_j\|_X \\
&\leq \|T\| \; \sum_{j=1}^n \| \sum_{i=1}^m u_i^*(v_j) u_i\|_U \; \|x_j\|_X \\
&\leq \|T\| \; \sum_{j=1}^n \|v_j\|_U \|x_j\|_X \\
&\leq \|T\| \; (\|z\|_{U \overset{\wedge}{\otimes}_F X} + \epsilon).
\end{aligned}$$

Now $(*)$ follows and with it we see that

$$\sum_{i \in \mathbb{N}} |\langle y_i, T(e_i) \rangle| \leq \|T\|$$

Now it makes sense to define $\phi : \mathcal{L}^r(U, X^*) \longrightarrow \mathbb{K}$ by

$$\phi(T) := \sum_{i \in \mathbb{N}} \langle y_i, T(u_i) \rangle \qquad \textit{for each } T \in \mathcal{L}^r(U, X^*)$$

From the note above, ϕ is well defined, with

(a) $\phi \in \mathcal{L}^r(U, X^*)^*$; and

(b) $\|\phi\| \leq 1$.

Step 3. We show that there exists $z \in B_{U\widehat{\otimes}_F X}$ so that $\Psi(z) = (y_i)_i$
Let $K = \beta((B_U\ ,\ \|\cdot\|) \times (B_{X^{**}}\ ,\ weak^*))$ where $\beta(S)$ is the Ćech-Stone compactification of S. K is a compact Hausdorff space and, because $(B_U, \|\cdot\|)$ is a Polish space, $(B_U\ ,\ \|\cdot\|) \times (B_{X^{**}}\ ,\ weak^*)$ is universally measurable with respect to all Radon measures on K (see [12]). Define

$$J : \mathcal{L}^r(U, X^*) \longrightarrow C(K)$$

by

$$J(T)(u, x^{**}) = x^{**}(T(u))$$

on $(B_U \times B_{X^{**}})$ and extend using the Ćech-Stone nature of K. J is a bounded linear operator with $\|JT\|_{C(K)} = \|T\|$ on the positive cone $\mathcal{L}^r(U, X^*)$. Now consider

$$F_\phi : J(\mathcal{L}^r(U, X^*)) \longrightarrow \mathbb{K}$$

defined by

$$F_\phi(JT) = \langle T, \phi\rangle \quad \forall T \in \mathcal{L}^r(U, X^*)$$

Note $\|F_\phi\| = \|\phi\|$. So by the Hahn-Banach theorem and the Riesz representation theorem, there exists a regular Borel measure ν so that

$$(A) \qquad F_\phi(JT) = \int_K JT(\omega)d\nu(\omega) \qquad \forall T \in \mathcal{L}^r(U, X^*)$$

and

$$|\nu|(K) = \|F_\phi\| = \|\phi\|.$$

Define $h_1 : (B_U\ ,\ \|\cdot\|) \times (B_{X^{**}}\ ,\ weak^*) \longrightarrow B_U$ by $h_1(u, x^{**}) = u$; h_1 is continuous into $(B_{U^{**}}\ ,\ weak^*)$ and so extends to a continuous function, still called h_1, from K to $(B_{U^{**}}\ ,\ weak^*)$. B_U's Polish character now allows us to look at

$$k_1 = h_1 \cdot \chi_{B_U \times B_{X^{**}}};$$

k_1 is scalarly ν-measurable and U-valued; hence, strongly ν-measurable. Also, k_1 is bounded in norm by 1 so

$$\int_K \|k_1(\omega)\| d|\nu| \leq |\nu|(K)$$

that is, k_1 is Bochner ν-integrable.

Now we know that (see [26], p.172) for every $\epsilon > 0$ there exists a sequence $(v_n)_{n\in\mathbb{N}} \subseteq U$ and a sequence of Borel sets $(B_n)_{n\in\mathbb{N}} \subseteq K$ such that

$$k_1(\omega) = \sum_{n=1}^{\infty} v_n \chi_{B_n}(\omega) \qquad |\nu|\text{-a.e.}$$

with

$$\sum_{n=1}^{\infty} \|u_n\|_U \ |\nu|(B_n) \leq \int_K \|k_1(\omega)\| \ d|\nu| + \epsilon \leq |\nu|(K) + \epsilon.$$

Let

$$h_2 : K \longrightarrow X^{**}$$

be given by

$$h_2(u, x^{**}) := x^{**}.$$

Then h_2 is weak*-continuous and hence weak*-measurable. Moreover, for each $x^* \in X^*$,

$$\int_K |\langle x^*, h_2(\omega)\rangle d|\nu|(\omega) \leq \|x^*\| \int_K |h_2(\omega)| d|\nu|(\omega) \leq \|x^*\||\nu|(K) < \infty.$$

So h_2 is Gelfand integrable (see the section: Notations and Basic facts).

Now, if we consider, for each $i \in \mathbb{N}$ and $x^* \in (X^*)^+$, $T_i = u_i^* \otimes x^* \in \mathcal{L}^+(U, X^*)$ by (A) we have

$$\begin{aligned}
\langle y_i, x^*\rangle &= \langle T_i, \phi\rangle \\
&= \int_K \langle T_i u, x^{**}\rangle d\nu(u, x^{**}) \\
&= \int_K \langle x^*, h_2(u, x^{**})\rangle \langle k_1(u, x^{**}), u_i^*\rangle d\nu(u, x^{**}) \\
&= \int_K \langle x^*, h_2(u, x^{**})\rangle \langle \sum_{n=1}^{\infty} v_n \chi_{B_n}(u, x^{**}), u_i^*\rangle d\nu(u, x^{**}) \\
&= \sum_{n=1}^{\infty} u_i^*(v_n) \int_{B_n} \langle x^*, h_2(u, x^{**})\rangle d\nu(u, x^{**}) \\
&= \sum_{n=1}^{\infty} u_i^*(v_n) \langle x^*, a_n^{**}\rangle
\end{aligned}$$

where

$$a_n^{**} = Gelfand - \int_{B_n} h_2(u, x^{**}) d\nu(u, x^{**})$$

therefore

$$(B) \qquad \langle y_i, x^* \rangle = \sum_{n=1}^{\infty} u_i^*(v_n) \langle x^*, a_n^{**} \rangle.$$

For every $x^* \in (X^*)^+$ and $n \in \mathbb{N}$

$$\begin{aligned} |\langle x^*, a_n^{**} \rangle| &= |\int_{B_n} \langle x^*, h_2(u, x^{**}) d\nu(u, x^{**})| \\ &\leq \int_{B_n} |\langle x^*, h_2(u, x^{**}) \rangle| d\nu(u, x^{**}) \\ &\leq \|x^*\| |\nu|(B_n). \end{aligned}$$

Hence,

$$\|a_n^{**}\| \leq |\nu|(B_n).$$

Moreover

$$\begin{aligned} (C) \qquad \sum_{n \in \mathbb{N}} \|u_i^*(v_n) a_n^{**}\| &= \sum_{n \in \mathbb{N}} |u_i^*(v_n)| \|a_n^{**}\| \\ &\leq \sum_{n \in \mathbb{N}} \|v_n\| |\nu|(B_n) \\ &\leq |\nu|(K) + \epsilon \end{aligned}$$

That means the series $\sum_{n \in \mathbb{N}} u_i^*(v_n) a_n^{**}$ is absolutely convergent in X^{**}. Note, now, that since X is a Banach lattice with the Radon-Nikodým property, X is norm-one complemented in X^{**}. Therefore there exists a norm one linear projection $P : X^{**} \longrightarrow X^{**}$, so that $P(X^{**}) = X$. Let $a_n = P(a_n^{**})$ and $z = \sum_{n \in \mathbb{N}} v_n \otimes a_n$. We have

$$\begin{aligned} \|z\|_{U \hat{\otimes}_F X} &\leq \sum_{n \in \mathbb{N}} \|u_n\| \|a_n\| \leq \\ &\leq \|P\| \sum_{n \in \mathbb{N}} \|u_n\| \|a_n^{**}\| \leq \\ &\leq \sum_{n \in \mathbb{N}} \|u_n\| |\nu|(B_n) \leq \\ &\leq (|\nu|(K) + \epsilon) = \\ &= (\|\phi\| + \epsilon) \end{aligned}$$

so that $\|z\|_{U \hat{\otimes}_F X} \leq \|\phi\| \leq 1$. In particular $z \in B_{U \hat{\otimes}_F X}$. Here is the catch: from (B), (C) and from the definition of Ψ we have, for each $i \in \mathbb{N}$

$$y_i = P(y_i)$$

$$\begin{aligned}
&= P(\sum_{n\in\mathbb{N}} u_i^*(v_n)a_n^{**}) = \\
&= \sum_{n\in\mathbb{N}} u_i^*(v_n)P(a_n^{**}) = \\
&= \sum_{n\in\mathbb{N}} u_i^*(v_n)a_n = \\
&= \sum_{n\in\mathbb{N}} (\Psi(u_n\otimes a_n))_i = \\
&= (\sum_{n\in\mathbb{N}} \Psi(u_n\otimes a_n)\ )_i = \\
&= \Psi(\sum_{n\in\mathbb{N}} u_n\otimes z_n\ )_i = \Psi(z)_i.
\end{aligned}$$

Hence

$$\Psi(z) = (y_i)_i.$$

We are done. □

Corollary 2.4.2. *Let X be a separable Banach lattice and U be a Banach space with an unconditional basis. If U and X have the Radon-Nikodým property then the Fremlin tensor product $U\widehat{\otimes}_F X$ of U and X has the Radon-Nikodým property*

Proof. It is enough to note that from our hypothesis follow that $U(X)$ has the Radon-Nikodým property (a more general form was shown by Bu, Diestel, Dowling, Oja [9]). Now the result is a direct consequence of a theorem of Bourgain and Rosenthal 2.2.7 (see [6]) □

Now,using Heinrich-Mankiewicz [47] and N. Randrianantoanina [89] (see Corollary 1.1.12) and for the Remark 2.2.4 we have the following

Corollary 2.4.3. *If U and X are two Banach lattices, one of them atomic , then the Fremlin tensor product of U and X, $U\widehat{\otimes}_F X$, has the Radon-Nikodým property if both U and X possess this property.*

Remark 2.4.4. In [21] the authors proved that:

Theorem. Let U be a Banach lattice, and X be a Banach space. Then $U \overset{\wedge}{\otimes}_\pi X$ has the Radon-Nikodým theorem iff U and X do too.

Then , in the same visual, one could asks if that happens in the Fremlin tensor product, i.e. if U, X are Banach lattices with the Radon-Nikodým property then $U\widehat{\otimes}_F X$ has the Radon-Nikodym property. Unfortunately in the Fremlin tensor product situation that doesn't happens. Indeed, as the same Fremlin proved (see [33]), $L_2[0,1]\widehat{\otimes}_F L_2[0,1]$ is not Dedekind complete and so it cannot have the Radon-Nikodým property (see [73] for the last assertion).

Chapter 3

Local Unconditional Structure and Gordon-Lewis Property in Tensor Products

3.1 The Class of Lindenstrauss-Pelczyński: the $\mathcal{L}_p$'s spaces

Definition 3.1.1. A Banach space X is called a $\mathcal{P}_1$ space (a $\mathcal{P}_\lambda$ space) if for all Banach space Y and Z with $Z \subseteq Y$ and all bounded linear operator $T : Z \longrightarrow X$, there exists an operator $\widetilde{T} : Y \longrightarrow X$ such that $\widetilde{T}|_Z = T$ and $\|\widetilde{T}\| = \|T\|$ ($\|\widetilde{T}\| \leq \lambda \|T\|$).

It is clear that a Banach space X is a $\mathcal{P}_1$ space (a $\mathcal{P}_\lambda$ space) if and only if whenever W is a Banach space containing X, then there exists a projection from W onto X of norm 1 (of norm less or equal to λ). A classical example of $\mathcal{P}_1$ space is the following

Lemma 3.1.2. *Let Γ be a set and $\ell_\infty(\Gamma)$ the Banach space of bounded scalar valued functions on Γ, with $\|f\|_\infty = \sum_{\gamma\in\Gamma} |f(\gamma)|$. Then $\ell_\infty(\Gamma)$ is a $\mathcal{P}_1$ space*

Actually Nachbin [75], Kelley [53]and Goodner [36] proved that every $\mathcal{P}_1$ space is isometrically isomorphic to a $C(K)$ space, where K is an extremely disconnected compact Hausdorff space (a topological space is called *extremely disconnected* or *Stonian* if the closure of every open set is open). Until now, no characterization of $\mathcal{P}_\lambda$ spaces is known for $\lambda > 1$. In particular, it is not known if every $\mathcal{P}_\lambda$ space is isomorphic to a $\mathcal{P}_1$ space. For some partial result in this area see [91]

Definition 3.1.3. A Banach space X is said to be *injective* if for all Banach spaces Z and Y with $Z \subseteq Y$ and all bounded linear operators $T : Z \longrightarrow X$ there exists a bounded linear operator $\widetilde{T} : Y \longrightarrow X$ such that $\widetilde{T}|_Z = T$.

By the definition it is clear that

Remark 3.1.4. *(a)* An injective space is a $\mathcal{P}_\lambda$ space, for some $\lambda \geq 1$.

(b) If X is isomorphic to a complemented subspace of $\ell_\infty(\Gamma)$ for some set Γ, then X is injective.

(c) A complemented subspace of an injective space is injective.

An $\mathcal{L}_p$ space is a Banach space which has essentially the same finite dimensional structure as an $L_p(\mu)$ and $C(K)$ spaces. $\mathcal{L}_p$ spaces were introduced by Lindenstrauss and Pełczyński in [59] and were examined further by Lindenstrauss and Rosenthal in [60] (see also [4] and [5])

Definition 3.1.5. Let X and Y be isomorphic Banach spaces. The number $d(X, Y) = \inf\{\|T\| \; \|T^{-1}\| \; : \; T : X \longrightarrow Y$ is a surjective isomorphism$\}$ is called the *distance coefficient* (usually called the *Banach-Mazur distance*) of X and Y.

Definition 3.1.6. Let $1 \leq p \leq \infty$ and $\lambda \geq 1$. A Banach space X is called an $\mathcal{L}_{p,\lambda}$ *space* if whenever F is a finite dimensional subspace of X, then there exists a finite dimensional subspace G of X such that $F \subseteq G$ and $d(G, \ell_p^n) \leq \lambda$, whose $n = dimG$.
A Banach space is called an $\mathcal{L}_p$ *space* if it is an $\mathcal{L}_{p,\lambda}$ space for some $\lambda \geq 1$.

There are two important classes of Banach space which are $\mathcal{L}_p$ spaces.

Theorem 3.1.7. *Let (Ω, Σ, μ) be a measure space. For $1 \leq p < \infty$, $L_p(\mu)$ is an $\mathcal{L}_{p,1+\epsilon}$ space for all $\epsilon > 0$. For a compact Hausdorff space K, $C(K)$ is an $\mathcal{L}_{\infty,1+\epsilon}$ space for all $\epsilon > 0$*

Remark 3.1.8. Actually Lindenstrauss and Pełczyński have proved a partial converse to the previous theorem. Namely, they showed that if $1 \leq p < \infty$ and X is an $\mathcal{L}_{p,1+\epsilon}$ space for all $\epsilon > 0$, then X is isometrically isomorphic to an $L_p(\mu)$ space.

The $L_p(\mu)$ spaces and $C(K)$ space are not the unique examples of $\mathcal{L}_p$ spaces. Indeed we have

Proposition 3.1.9. *Let X be an infinite dimensional Banach space.*

(i) If X^{**} is injective space, then X is an $\mathcal{L}_\infty$ space;

(ii) Assume that X^{**} is isomorphic to a complemented subspace of an $L_1(\mu)$ space. Then there exists a $K < \infty$ such that whenever F is a finite dimensional subspace of X, there exists a finite dimensional subspace G of X containing F such that $d(G, \ell_1^{dimG}) < K$, and a projection form X onto G of norm at most K.

Here we recall the deeper result about the relationship between a Banach space and its bidual, proved by Lindenstrauss and Rosenthal [60] in an early version and upgraded by Johnson, Rosenthal and Zippin [50], a tool very useful in the modern function analysis theory. For a simpler proof see [95] and [66]

Theorem 3.1.10. (The Local Reflexivity Principle)
Let X be a Banach space, and let E and F be finite dimensional linear subspaces of X^{**} and X^* respectively. Then, for each $\epsilon > 0$ there exists a one-one bounded linear operator $T : E \longrightarrow X$ such that

(i) $T(x) = x$ for all $x \in E \cap X$;

(ii) $\|T\| \ \|T^{-1}\| \leq 1 + \epsilon$;

(iii) $\langle T(x^{**}, x^*\rangle = \langle x^{**}, x^*\rangle$ for all $x^{**} \in E$ and $x^* \in F$.

It follows immediately from the Local Reflexivity Principle that

Proposition 3.1.11. *If X^{**} is an $\mathcal{L}_p$ space, then so is X.*

Now, recall some meaning result to know the $\mathcal{L}_p$ spaces more close.

Lemma 3.1.12. *Let X be an infinite dimensional $\mathcal{L}_1$ space. Let F be a finite dimensional quotient space of some Banach space Y and $\phi : Y \longrightarrow F$ the natural map. Then there exists a $K < \infty$ (depending only on X) such that given an operator $T : X \longrightarrow F$, there exists an operator $\widetilde{T} : X \longrightarrow Y$ such that $\phi \circ \widetilde{T} = T$. ($\widetilde{T}$ is said to be a* lifting *of T).*

Proof. We first show that there exists a finite dimensional subspace Y_0 of Y such that $B_F \subseteq \phi((2\ B_{Y_0})$ (B_{Y_0} is the closed unit ball of Y_0). To prove this, let $\delta > 0$ be such that $\frac{1+\delta}{1-\delta} < 2$. Let $b_1, ..., b_m$ be elements in F such that $\|b_i\| = 1$ for $1 \leq i \leq m$ and if $b \in F$, $\|b\| = 1$, then $\|b - b_i\| < \delta$ for some i. Then it follows that $B_F \subseteq \{\sum_{i=1}^m a_i b_i \ : \ \sum_{i=1}^m |a_i| \leq \frac{1}{1-\delta}\}$ (indeed if $\{\sum_{i=1}^m a_i b_i \ : \ \sum_{i=1}^m |a_i| \leq \frac{1}{1-\delta}\}$ does not contains the closed unit ball of F, then there exists an $f \in X^*$ and $u_0 \in F$, $\|u_0\| = 1$ such that $f(u_0) > \sup |f(y)|$ for all $y \in \{\sum_{i=1}^m a_i b_i \ : \ \sum_{i=1}^m |a_i| \leq \frac{1}{1-\delta}\}$. In particular, $\|f\| \geq f(u_0) > \sup_{1\leq i\leq m} \frac{1}{1-\delta}\ |f(b_i)|$ or, equivalently $\sup_{1\leq i\leq m} |f(b_i)| < (1 -$

$\delta)\|f\|$. But if $y \in F$, $\|y\| = 1$ and we choose j such that $\|y - b_j\| < \delta$, then $|f(y)| \leq |f(y - b_j)| + |f(b_j)| \leq \delta\|f\| + \sup_{1\leq i\leq m} |f(b_i)|$. Therefore, $\|f\| \leq \delta\|f\| + \sup_{1\leq i\leq m} |f(b_i)| < \|f\|$, a contradiction).
Now for every i there exists a $y_i \in Y$ such that $\|y_i\| < 1 + \delta$ and $\phi(y_i) = b_i$. Put $Y_0 = span\{y_i \ : \ 1 \leq i \leq m\}$. If $b \in F$, $\|b\| \leq 1$, then $b = \sum_{i=1}^m a_i b_i$ for some scalars $a_1, ..., a_m$ with $\sum_{i=1}^m |a_i| \leq \frac{1}{1-\delta}$. But then $\phi(\sum_{i=1}^m a_i y_i) = b$, $\| \sum_{i=1}^m a_i y_i\| \leq \sup_{1\leq i\leq m} \|y_i\| \ \sum_{i=1}^m |a_i| \leq \frac{1+\delta}{1-\delta} < 2$ and $\sum_{i=1}^m a_i y_i \in Y_0$. Hence $B_F \subseteq \phi(2\ B_{Y_0})$.
Let G be a finite dimensional subspace of X such that $T(X) = T(G)$. Since X is an $\mathcal{L}_{1,k_0}$ space for some $k_0 < \infty$, there exists a family $\mathcal{F}$ of finite dimensional subspaces W of X (directed by inclusion) such that $G \subseteq W$ and $d(W, \ell_1^n) \leq k_0$. For each $W \in \mathcal{F}$, we construct an operator $\widetilde{T}_W : W \longrightarrow Y$ such that $\widetilde{T}_W(B_W) \subseteq 2\ k_0\ \|T\|\ B_{Y_0}$ and $\phi\widetilde{T}_W = T|_W$.
Fix $W \in \mathcal{F}$ and let $w_1, ..., w_n$ be a basis for W such that $\|w_i\| = 1$ for all i and such that, given any scalars $a_1, ..., a_n$ $\| \sum_{i=1}^n a_i w_i\| \geq \frac{1}{k_0} \sum_{i=1}^n |a_i|$. By the first part of this proof, there exists for each i a $z_i \in 2\ \|T\| B_{Y_0}$ such that $\phi(z_i) = T(w_i)$. Defining $\widetilde{T}_W$ by

$$\widetilde{T}_W(\sum_{i=1}^n a_i w_i) = \sum_{i=1}^n a_i z_i,$$

it follows that $\|\widetilde{T}_W(\sum_{i=1}^n a_i w_i)\| \leq \sup_{1\leq i\leq n} \|z_i\| \ \sum_{i=1}^n |a_i| \leq 2\ \|T\|\ k_0\ \| \sum_{i=1}^n a_i w_i\|$, and hence that $\|\widetilde{T}_W\| \leq 2\ k_0\ \|T\|$ (note that by a composing a projection onto W with $\widetilde{T}_W$, we may conclude that every $T : X \longrightarrow F$ has a continuous lifting $\widetilde{T} : X \longrightarrow Y$. However, we have no control over the norm of a projection onto an arbitrary $W \in \mathcal{F}$ and must present further argument to gain this control).
Since the space $2\ k_0\ \|T\|\ B_{Y_0}$ is compact, the space $\Pi = \prod_{x\in X} 2\ k_0\ \|T\|\ \|x\|\ B_{Y_0}$ is a compact space. We extend $\widetilde{T}_W$ to a (non-linear) function from X into Y by putting $\widetilde{T}_W(x) = 0$ if $x \in X \setminus W$. Then in any case $\|\widetilde{T}_W(x)\| \leq 2\ k_0\ \|T\|\ \|x\|$, so $\widetilde{T}_W \in \Pi$, thus $\{\widetilde{T}_W \ : \ W \in \mathcal{F}\}$ is a net in Π. Let $\widetilde{T}$ be a cluster point of this net. Then $\widetilde{T}$ is linear, $\widetilde{T}$ lifts T, and $\|\widetilde{T}\| \leq 2\ k_0\ \|T\|$. We prove this in detail.
Let $\mathcal{F}'$ be a directed subfamily of $\mathcal{F}$ such that the net $\{\widetilde{T}_W \ : \ W \in \mathcal{F}'\}$ converges to T. First, let $x_1, x_2 \in X$ and let a_1, a_2 be scalars. Pick $W_0 \in \mathcal{F}'$ containing x_1 and x_2. Then if $W \in \mathcal{F}'$ contains W_0, $\widetilde{T}_W(a_1x_1 + a_2x_2) = a_1\widetilde{T}_W(x_1) + a_2\widetilde{T}_W(x_2)$, and so $\widetilde{T}(a_1x_1 + a_2x_2) = a_1\widetilde{T}(x_1) + a_2\widetilde{T}(x_2)$. If $x \in X$, and $x \in W_0$ for some $W_0 \in \mathcal{F}'$, then for $W \in \mathcal{F}'$, $W_0 \subseteq W$, $\phi\circ\widetilde{T}_W(x) = T(x)$, so $\phi\circ\widetilde{T}(x) = T(x)$. Finally if $x \in X$, $\|\widetilde{T}_W(x)\| \leq 2\ k_0\ \|T\|\ \|x\|$ for all $W \in \mathcal{F}'$, so $\|\widetilde{T}\| \leq 2\ k_0\ \|T\|$. $\square$

Remark 3.1.13. Using essentially the same proof as above, Lindenstrauss and Rosenthal [60] proved the following generalization:
Let X be an $\mathcal{L}_1$ space. Then, whenever F is a quotient space of Y, and $T : X \longrightarrow F$ is a compact operator, there exists a compact lifting $\widetilde{T} : X \longrightarrow Y$ of T.

Theorem 3.1.14. *(i) If X is an infinite dimensional $\mathcal{L}_1$ space, then X^* is a $\mathcal{P}_1$ space;*

(ii) If X is an infinite dimensional $\mathcal{L}_\infty$ space, then X^{**} is injective.

Proof. (i) Suppose X is an $\mathcal{L}_1$ space. Let Y^* be a conjugate $\mathcal{P}_1$ space containing X^*. By virtue of $Remark$3.1.4(c) it suffices to show that X^* is complemented in Y^*. Let F be a finite dimensional subspace of X^*, $T : X \longrightarrow F^*$ the natural quotient map from X onto F^*. By the previous lemma, there exists a lifting $\widetilde{T}_F : X \longrightarrow Y$ of T with $\|\widetilde{T}_F\| \leq K$ (where K depends only on X). But then, $\widetilde{T}_F^* : Y^* \longrightarrow X^*$ has the property that $\widetilde{T}_F^*(f) = f$ for every $f \in F = F^{**}$ ($\widetilde{T}_F^*$ will not in general be a projection). Let U be the unit ball of X^* in its weak* topology. Then $\{\widetilde{T}_F^* \ : \ F$ is a finite dimensional subspace of $X^*\}$ is a net (direct by inclusion) in the compact space $\prod_{y^* \in Y^*} \|y^*\| \ K \ U$ (via Tychonoff 's theorem). A cluster point P of this net is easily verified to be a bounded linear projection from Y^* onto X^*.
(ii) Suppose that X is an $\mathcal{L}_\infty$ space. Let Y be a $\mathcal{P}_1$ space containing X^{**}. We will show that X^{**} is complemented in Y. Since X is an $\mathcal{L}_\infty$ space, there exists a $K < \infty$ and a family $\mathcal{F}$ of finite dimensional subspaces of X,0 directed by inclusion, such that

(1) $X = \bigcup_{F \in \mathcal{F}} F$;

(2) $d(F, \ell_\infty^{dimF}) \leq K$ for all $F \in \mathcal{F}$;

(3) For each $F \in \mathcal{F}$ there exists a projection $P_F : Y \longrightarrow F$ onto F with $\|P_F\| \leq K$.

Let U be the unit ball of X^{**} in its weak* topology. Then $P_F(y) \in \|y\| \ K \ U$ for all $y \in Y$. In particular $\{P_F \ : \ F \in \mathcal{F}\}$ is a net in the compact space $\prod_{y \in Y} \|y\| \ K \ U$ (via the Tychonoff's theorem). Let P the cluster point of this net. Then P maps Y onto X^{**}, P is linear, $\|P\| \leq K$, and moreover if $x \in X$ then $P(x) = x$ (to see this last assertion, choose $F_0 \in \mathcal{F}$ such that $x \in F_0$. Then $P_{F_0}(x) = x$. But $P_F(x) \to P(x)$ weak*. Hence $P(x) = x$).
Now consider the operator $P^{**} : Y^{**} \longrightarrow (X^{**})^{**}$. If $x^{**} in X^{**}$, then $P^{**}(x^{**}) = x^{**}$ (this follows from the facts that $P^{**}|_Y = P$, that P^{**} is weak* continuous, and that X is weak* dense in X^{**}). Also, regarding $X^{**} \subseteq (X^{**})^{**}$, there

exists a projection $Q : (X^{**})^{**} \longrightarrow X^{**}$ of norm one (simply let Q be the restriction of a linear functional on X^{***} to X^*). Then the operator $QP^{**}|_Y$ is a projection from Y onto X^{**}, and consequently X^{**} is injective. □

Theorem 3.1.15. *Let X an infinite dimensional Banach space.*

(i) Let $1 < p < \infty$. Then X is an $\mathcal{L}_p$ space or an $\mathcal{L}_2$ space if and only if X is isomorphic to a complemented linear subspace of an $L_p(\mu)$ space.

(ii) Let $p = 1$ or $p = \infty$. Then X is an $\mathcal{L}_p$ space if and only if X^{**} is isomorphic to a complemented linear subspace of an $L_p(\mu)$ space.

Proof. We will give only the proof of (ii). Indeed, assume first that X is an $\mathcal{L}_1$ space. By the previous theorem X^* is injective and hence is isomorphic to a complemented of $C(K)$ space, for some compact Hausdorff space K. Therefore X^{**} is isomorphic to a complemented subspace of $M(K)$ (the Banach space of countable additive regular Borel measure on K, with $\|\mu\| = |\mu|(K)$). It is not so hard to see that there exists a measure space with measure μ such that $M(K)$ is isometrically isomorphic to $L_1(\mu)$. Then X^{**} is isometrically isomorphic to a $L_1(\mu)$ space.
If X^{**} is complemented in an $L_1(\mu)$ space, then Proposition 3.1.9 (ii) asserts that X is an $\mathcal{L}_1$ space.
Assume now that X is an $\mathcal{L}_\infty$ space. By the previous theorem, X^{**} is injective. But then X^{**} is isometric to a complemented subspace of $\ell_\infty(\Gamma)$, where Γ is the unit ball of X^{***}.
Now assume that X^{**} is isomorphic to a complemented subspace of an $L_\infty(\mu)$ space. Since any $L_\infty(\mu)$ space is a $\mathcal{P}_1$ space (by Nachbin, Kelley and Goodner 's theorem and for the fact that $L_\infty(\mu)$ is isometrically isomorphic to a $C(K)$ space, for some extremely disconnected compact Hausdorff space K (see [28])). Then X^{**} is injective, and so X is an $\mathcal{L}_\infty$ space by Proposition 3.1.9 (i). □

Now it is simple to see the following

Theorem 3.1.16. *Let X be an infinite dimensional Banach space. Then*

(i) X is an $\mathcal{L}_1$ space if and only if X^* is an $\mathcal{L}_\infty$ space;

(ii) X is an $\mathcal{L}_\infty$ space if and only if X^* is an $\mathcal{L}_1$ space.

We finish this section with the following

Theorem 3.1.17. *Let X be an infinite dimensional Banach space* *(i) Let $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then X is an $\mathcal{L}_p$ space if and only if X^* is an $\mathcal{L}_q$ space.*

(ii) A complemented subspace of an $\mathcal{L}_p$ space is an $\mathcal{L}_p$ space or an $\mathcal{L}_2$ space.

(iii) If X is an $\mathcal{L}_p$ space, then there exists a $K < \infty$ such that whenever F is a finite dimensional subspace of X, there exists a finite dimensional subspace G of X such that $F \subseteq G$ and $d(G, \ell_p^{dimG}) \leq K$, and a projection from X onto G of norm at most K.

3.2 p-Summing and p- Factorable Operators

The class of p-summing operators were introduced formally by A. Pietsch [77] and [78], but before for $p = 1, 2$ were introduced and studied by A.Grothendieck [41].

Definition 3.2.1. If $1 \leq p < \infty$ and $T : X \longrightarrow Y$, then T is said to be p-summing if there is a constant $c \geq 0$ such that, for any $m \in \mathbb{N}$ and for any choice of $x_1, x_2, ..., x_m$ in X, it happens

$$\left(\sum_{i=1}^{m} ||T(x_i)||^p \right)^{\frac{1}{p}} \leq c \sup \left\{ \left(\sum_{i=1}^{m} |\langle x^*, x_i \rangle|^p \right)^{\frac{1}{p}} : x^* \in B_{X^*} \right\}.$$

The smallest of such c's is denoted by $\pi_p(T)$ and the set of all p-summing operators from X to Y will be called $\Pi_p(X, Y)$.

It is simple to see that $\Pi_p(X, Y)$ is a linear subspace of $\mathcal{L}(X, Y)$, and π_p defines a norm on $\Pi_p(X, Y)$ with

$$\|T\| \leq \pi_p(T) \qquad \text{for all } T \in \Pi_p(X, Y)$$

Actually for any Banach spaces X, Y the space $\Pi_p(X, Y)$ with the norm π_p is a Banach space. Some properties are in order:

Proposition 3.2.2. ***(Ideal Property of p-Summing Operators)***
If $v \in \mathcal{L}(X, Y)$ is p-summing $(1 \leq p < \infty)$, X_0 and Y_0 are Banach spaces and $u \in \mathcal{L}(Y, Y_0)$, $w \in \mathcal{L}(X_0, X)$, then the composition operator $u \circ v \circ w : X_0 \longrightarrow Y_0$ is p-summing with $\pi_p(u \circ v \circ w) \leq \|u\| \, \pi_p(v) \, \|w\|$.

Proposition 3.2.3. *Let $1 \leq p < \infty$. If $u : X \longrightarrow Y$ is a bounded linear operator between two Banach spaces, then u is p-summing if and only if its second adjoint $u^{**} : X^{**} \longrightarrow Y^{**}$ is. In this case it is $\pi_p(u^{**}) = \pi_p(u)$.*

The charming side of the p-summing operators is the finite dimensional character which they look like (see [85] for a nice application of this)

Proposition 3.2.4. *Let X, Y be Banach spaces, and $u : X \longrightarrow Y$ a bounded linear operator. Then there exists a constant C so that $u \in \Pi_p(X, Y)$ and $\pi_p(u) \leq C$ if and only if for each finite subspace X_0 of X the restriction $u|_{X_0} \in \Pi_1(X_0, Y)$ and $\pi_p(u|_{X_0}) \leq C$.*

Proposition 3.2.5. ***(Injectivity of** Π_p**)*** *If $i : Y \hookrightarrow Y_0$ is an isometry, then $v \in \Pi_p(X, Y)$ if and only if $i \circ v \in \Pi_p(X, Y_0)$. In this case we even have $\pi_p(i \circ v) = \pi_p(v)$.*

Proposition 3.2.6. ***(Inclusion Theorem)*** *If $1 \leq p < q < \infty$, then $\Pi_p(X, Y) \subseteq \Pi_q(X, Y)$. Moreover, for $u \in \Pi_p(X, Y)$ we have $\pi_q(u) \leq \pi_p(u)$.*

Proposition 3.2.7. ***(Composition Theorem)*** *Let $u \in \Pi_p(Y, Z)$ and $v \in \Pi_q(X, Y)$ with $1 \leq p, q < \infty$. Define $1 \leq r < \infty$ by $\frac{1}{r} = min\{1, \frac{1}{p} + \frac{1}{q}\}$. Then $u \circ v$ is r-summing, and $\pi_r(u \circ v) \leq \pi_p(u) \cdot \pi_q(v)$.*

Who got seriously to study the p-summing operators was A. Pietsch, in particular he get the important following (see [22] 2.12 for a proof)

Theorem 3.2.8. ***(Pietsch Domination Theorem)***
Suppose that $1 \leq p < \infty$, that $T : X \longrightarrow Y$ is a bounded linear operator between the Banach spaces X, Y, and that K is a weak compact norming subset of B_{X^*}. Then T is p-summing if and only if there exist a constant C and a regular probability Borel measure μ on K such that for each $x \in X$*

$$\|T(x)\| \leq C \cdot \left(\int_K |\langle x^*, x\rangle|^p d\mu(x^*) \right)^{\frac{1}{p}}$$

In such a case $\pi(T)$ is the least of all the constants C for which such a measure exists.

Actually the Pietsch Domination theorem in terms of operators means: $T : X \longrightarrow Y$ is p-summing if and only if there is a regular Borel measure μ on K, a closed linear subspace X_p of $L_p(\mu)$ and an operator $\widetilde{T} : X_p \longrightarrow Y$ such that the following diagram commuted:

$$\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
{\scriptstyle i_X}\downarrow & & \uparrow{\scriptstyle \widetilde{T}} \\
i_X(X) & \xrightarrow{j_p^X} & X_p \\
\downarrow & & \downarrow \\
C(K) & & L_p(\mu)
\end{array}$$

where $i_X : X \longrightarrow C(K)$ is the natural isometry (actually $i_X : X \longrightarrow \ell_\infty(K)$ is the natural map $i_X(x)(x^*) = \langle x^*, x\rangle$; remember that K is norming subset of B_{X^*}), and j_p^X is the restriction of the natural embeds $j_p : C(K) \hookrightarrow L_p(\mu)$ (which is p-summing) on the closed linear subspace $i_X(X)$ of $C(K)$. Now we know that every Banach space Y embeds naturally in $\ell_\infty(B_{Y^*})$ which is an injective space. Then using the the peculiarity of the injective spaces we get

Theorem 3.2.9. (Pietsch Factorization Theorem)
Let $1 \le p < \infty$, let X and Y be Banach spaces , let K be a weak* compact norming subset of B_{X^*}. Then a bounded linear operator $T : X \longrightarrow Y$ is p-summing if and only if there exist a regular Borel probability measure μ on K and bounded linear operators $u : X \longrightarrow C(K)$, $v : L_p(\mu) \longrightarrow \ell_\infty(B_{Y^*})$ such that the following diagram commutes:

$$\begin{array}{ccccc} X & \xrightarrow{T} & Y & \overset{i_Y}{\hookrightarrow} & \ell_\infty(B_{Y^*}) \\ {\scriptstyle u}\downarrow & & & \nearrow {\scriptstyle v} & \\ C(K) & \overset{j_p}{\hookrightarrow} & L_p(\mu) & & \end{array}$$

In addition, we may choose μ u, v such that $\|v\| = 1$ and $\|u\| = \pi(T)$

Actually the Pietsch Factorization Theorem say us that a p-summing operator factorize "almost" for a $L_p(\mu)$ space. In particular when Y is an injective Banach space a p-summing operator $T : X \longrightarrow Y$ factorize through an $L_p(\mu)$ space. The argument above suggest to Kwapień [56] the following

Definition 3.2.10. If $1 \le p \le \infty$, an operator $T : X \longrightarrow Y$ between Banach spaces is called *p-factorable* if there exists a measure space (Ω, Σ, μ) and two operators $a\colon L_p(\mu) \to Y^{**}$ and $b\colon X \to L_p(\mu)$ such that the following diagram commutes

$$\begin{array}{ccccc} X & \xrightarrow{T} & Y & \overset{k_Y}{\hookrightarrow} & Y^{**} \\ & {\scriptstyle b}\searrow & & \nearrow {\scriptstyle a} & \\ & & L_p(\mu) & & \end{array}$$

where k_Y is the natural embeds from Y into Y^{**}.
We will denote with $\gamma_p(u)$ the smallest of all products $\|a\| \cdot \|b\|$ where a and b run over all the possible factorizations of $k_Y u$ we have indicated. Instead $\Gamma_p(X, Y)$ will be the set of all p-factorable operators from X to Y.

It is simple to see that $\Gamma_p(X, Y)$ is a linear subspace of $\mathcal{L}(X, Y)$, and γ_p defines a norm on $\Gamma_p(X, Y)$ with

$$\|T\| \le \gamma_p(T) \qquad \text{for all } T \in \Gamma_p(X, Y)$$

Actually for any Banach spaces X, Y the space $\Gamma_p(X, Y)$ with the norm γ_p is a Banach space. Some properties are in order:

Proposition 3.2.11. ***(Ideal Property of p-Factorable Operators)*** *If $v \in \Gamma_p(X,Y)$ $(1 \le p < \infty)$, X_0 and Y_0 are Banach spaces and $u \in \mathcal{L}(Y, Y_0)$, $w \in \mathcal{L}(X_0, X)$, then the composition operator $u \circ v \circ w \in \Gamma_p(X,Y)$ with $\gamma_p(u \circ v \circ w) \le \|u\|\, \gamma_p(v)\, \|w\|$.*

Proposition 3.2.12. *Let $1 \le p < \infty$ and $p^* := \frac{p}{p-1}$. If $u : X \longrightarrow Y$ is abounded linear operator between two Banach spaces, then the following are equivalent*

(i) $u : X \longrightarrow Y$ *is p-factorable*

(ii) $u^* : Y^* \longrightarrow X^*$ *is p^*-factorable*

(iii) $u^{**} : X^{**} \longrightarrow Y^{**}$ *is p-factorable*

In this case it is $\gamma_p(u^{**}) = \gamma_{p^*}(u^*) = \gamma_p(u)$.

Analogously to the p- summing operators, the p- factorable operators have a finite dimensional character as the following show

Proposition 3.2.13. *Let $1 \le p \le \infty$. An operator $u \in \mathcal{L}(X,Y)$ is p-factorable if and only if*

$$c := \sup\{\gamma_p(q_F \circ u \circ i_E) : E \in \mathcal{F}_X, F \in \mathcal{C}_Y\} < \infty,$$

where recall that $\mathcal{F}_X$ is denoted the class of finite dimensional subspaces of X with the natural embeds i_E $(E \in \mathcal{F}_X)$, and $\mathcal{C}_Y$ is denoted the class of finite codimensional subspaces of Y with q_F the natural quotient $(F \in \mathcal{C}_Y)$.
In this case, $\gamma_p(u) = c$.

It was one of Grothendieck's Résumé question [41] the following: does every absolutely summing (or 1-summing) operator factor through an $L_1(\mu)$ space ? form this question Reisner [88] introduced the following notion:

Definition 3.2.14. A Banach space X has *the Gordon Lewis property* (or that X is a *GL-space*) if every 1-summing operator from X to ℓ_2 is 1-factorable. Obviously this is equivalent to require the existence of a constant c such that

$$\gamma_1(u) \le c\, \pi_1(u)$$

for every $u \in \Pi_1(X, \ell_2)$. In this cases we will call $gl(X)$ the smallest of all these constants.

Actually Reisner required that $\Pi_1(X,Y) \subseteq \Gamma_1(X,Y)$ for every Banach space Y (the above definition came from Gordon and Lewis's paper [40]). It is unknown if the Reisner definition is more restrictive than the above definition.
Before to continue we need to recall some concept between the Banach lattices structure and the GL property.
A Schauder basis $(x_n)_n$ is *unconditional basis* if for each $x \in X$ the series $\sum_{n\in\mathbb{N}}\langle x_n^*, x\rangle x_n$ converges unconditionally to x (where $(x_n^*)_n$ denoted the corresponding biorthogonal sequence in X^*). It seems that the unconditionally condition is equivalent to requiring the convergence of $\sum_n t_n\langle x_n^*, x\rangle x_n$ for every $t = (t_n)_n \in \ell_\infty$, and there is a constant $\lambda \geq 1$ such that

$$\|\sum_n t_n\langle x_n^*, x\rangle x_n\| \leq \lambda\|\sum_n\langle x_n^*, x\rangle x_n\|$$

for every $t \in B_{\ell_\infty}$. We write $\lambda_{(x_n)_n}$ for the smallest such λ. The *unconditional basis constant* of X is

$$ub(X) = \inf\{\lambda_{(x_n)_n} \ : \ (x_n)_n \ \text{ is an unconditional basis}\}$$

Note that if X is a Banach space with an unconditional basis $(x_n)_n$ with $\|x_n\| = 1$ for each n, we can build a natural lattices structure on X by defining $x \leq y$ if and only if $\langle x_n^*, x\rangle \leq \langle x_n^*, y\rangle$ for all n. By setting

$$|||x||| = \sup_{(t_n)\in\ell_\infty}\|\sum_n t_n\langle x_n^*, x\rangle x_n\|,$$

we get an equivalent norm on X with respect to which it is a Banach lattice. The condition that a Banach space has an unconditional basis is pretty strong, for example the familiar space $L_1[0,1]$ doesn't have any unconditional basis. We can anyway consider a weaker property:

Definition 3.2.15. We say that a Banach space X has *local unconditional structure (l.u.st.)* if there exists a constant $\Lambda \geq 1$ such that, for every $E \in \mathcal{F}_X$, the canonical embedding $E \hookrightarrow X$ factors through a Banach space Y with unconditional basis via two operators $E \xrightarrow{v} Y \xrightarrow{u} X$ satisfying $||u||\,||v||\,ub(Y) \leq \Lambda$. The smallest of these constants is called the *l.u.st. constant* of X and is denoted by $\Lambda(X)$.

It is clear that Banach spaces X with unconditional basis have obviously l.u.st. and $\Lambda(X) \leq ub(X)$. Even if $L_1[0,1]$ doesn't have any unconditional basis. But we have more

Theorem 3.2.16. *Every Banach lattice has local unconditional structure, and the l.u.st. constant equal to one.*

For what we have seen above every Banach space with unconditional basis is isomorphic to a Banach lattice, and the last one has local unconditional structure. To close the chain of implications we have the following (from that derive the name of such a property; see [40])

Theorem 3.2.17. (Gordon-Lewis Inequality) *Let X and Y be Banach spaces and suppose that X has local unconditional structure. Then every 1-summing operator $u : X \longrightarrow Y$ is 1-factorable, with*

$$\gamma_1(u) \leq \Lambda(X)\ \pi_1(u)$$

That means every Banach space with local unconditional structure has the GL property and $gl(X) \leq \Lambda(X)$.

It is interesting to observe that, if a real Banach space X has an unconditional basis $(\tilde{x}_n)_n$, then we can normalize it obtaining a new unconditional basis $(x_n)_n$ with biorthogonal sequence $(x_n^*)_n$; at this point we create a lattice structure on X defining the order relation:

$$x \leq y :\Leftrightarrow \langle x_n^*, x\rangle \leq \langle x_n^*, y\rangle \quad \forall n \in \mathbb{N}.$$

The space X with this order relation is a Riesz space and we can renorm our space setting

$$|||x||| := \sup \left\{ \left\| \sum_{n=1}^{\infty} t_n \langle x_n^*, x\rangle x_n \right\| : (t_n)_n \in B_{\ell_\infty} \right\};$$

what we get is a norm on X equivalent to the original norm that makes the space a Banach lattice. Because it is possible to do this even for complex Banach spaces, it follows that every Banach space X with an unconditional basis is isomorphic to a Banach lattice.

L.u.st. is clearly a property invariant by isomorphism so all we have just said leads us to this chain of implications:

$$X \text{ has an unconditional basis } \Rightarrow X \text{ is isomorphic to a Banach lattice } \Rightarrow$$

$$\Rightarrow X \text{ has local unconditional structure } \Rightarrow X \text{ is a GL-space .}$$

It seems that the chain of implications is not closed, actually one can think that a Banach space with GL-property is so far to be isomorphic to a Banach lattice. The next spectacular theorem (see [10]) shows that is not true

Theorem 3.2.18. *Let X be a Banach space. The following two statements are equivalent*

(i) *X has GL-property;*

(ii) *there is a Banach lattice $L \supseteq X$ so that every $T \in \Pi_1(X, \ell_2)$ admits an extension $\widetilde{T} \in \Pi_1(L, \ell_2)$.*

3.3 L.U.ST. and GL property in Tensor Products

The idea of extending l.u.st. from two Banach spaces to their injective and projective tensor product is already in the fundamental paper of Y. Gordon and D.R. Lewis [40]; in particular one of the results they get is the following: if E and F are $\mathcal{L}_p$-spaces $(1 < p < \infty)$, then none of $E \overset{\wedge}{\otimes}_\epsilon F$, $E \overset{\wedge}{\otimes}_\pi F$, $(E \overset{\wedge}{\otimes}_\epsilon F)^*$, $(E \overset{\wedge}{\otimes}_\pi F)^*$, $(E \overset{\wedge}{\otimes}_\pi F)^{**}$, $(E \overset{\wedge}{\otimes}_\pi F)^{**}$, etc... has l.u.st. In this section we study the case $p = 1$ and $p = \infty$. In fact we show that if X is a $\mathcal{L}_1$-space (resp. $\mathcal{L}_\infty$-space) and Y a Banach space, then $X \overset{\wedge}{\otimes}_\pi Y$ (resp. $X \overset{\wedge}{\otimes}_\epsilon Y$) has lust property (or GL-property) if Y does. In the same paper of Gordon-Lewis ([40]) it is possible to find the proof that other tensor products fail to have l.u.st.: $\ell_\infty \overset{\wedge}{\otimes}_\pi \ell_p$ for $1 < p \leq \infty$ and $\ell_1 \overset{\wedge}{\otimes}_\epsilon \ell_p$ for $1 \leq p < \infty$ don't have l.u.st. so it is impossible to prove the theorem even for $\mathcal{L}_\infty \overset{\wedge}{\otimes}_\pi X$ and $\mathcal{L}_1 \overset{\wedge}{\otimes}_\epsilon X$.

Now we recall the following well known lemma, which we are including the proof for sake of completeness.

Lemma 3.3.1. *If $E \in \mathcal{F}$ then $ub(E) = ub(E^*)$.*

Proof. Consider a basis $(x_i)_{i=1}^n$ of E and the corresponding biorthogonal basis $(x_i^*)_{i=1}^n$ in E^*; then we have

$$x = \sum_{i=1}^n \langle x_i^*, x \rangle \, x_i \qquad \forall x \in E$$

and

$$x^* = \sum_{i=1}^n \langle x^*, x_i \rangle \, x_i^* \qquad \forall x^* \in E^*$$

It follows:

$$\lambda_{(x_i)_{i=1}^n} = \sup \left\{ \left\| \sum_{i=1}^n t_i x_i^*(x) x_i \right\|_E : (t_i) \in B_{\ell_\infty^n}, x \in B_E \right\}$$

$$
\begin{aligned}
&= \sup\left\{\left|x^*\left(\sum_{i=1}^n t_i x_i^*(x)x_i\right)\right| : (t_i)\in B_{\ell_\infty^n}, x\in B_E, x^*\in B_{E^*}\right\}\\
&= \sup\left\{\left|\sum_{i=1}^n t_i x_i^*(x)x^*(x_i)\right| : (t_i)\in B_{\ell_\infty^n}, x\in B_E, x^*\in B_{E^*}\right\}\\
&= \sup\left\{\left|\left(\sum_{i=1}^n t_i x^*(x_i)x_i^*\right)(x)\right| : (t_i)\in B_{\ell_\infty^n}, x\in B_E, x^*\in B_{E^*}\right\}\\
&= \sup\left\{\left\|\sum_{i=1}^n t_i x^*(x_i)x_i^*\right\|_{E^*} : (t_i)\in B_{\ell_\infty^n}, x^*\in B_{E^*}\right\} = \lambda_{(x_i^*)_{i=1}^n}.
\end{aligned}
$$

Thanks to this we can say that the bijective correspondence between the basis of E and the basis of E^* that sends each basis in the corresponding biorthogonal basis preserves the unconditional constant of the basis, so the thesis follows. □

Lemma 3.3.2. *If $F \in \mathcal{F}$ then $ub(\ell_\infty^n \overset{\wedge}{\otimes}_\epsilon F) \le ub(F)$.*

Proof. Let $m = dimF$. If $(t_{i,j}) \in B_{\ell_\infty^{nm}}$, $\alpha_{i,j} \in \mathbb{R}$, $i = 1, 2, ..., n$, $j = 1, 2, ..., m$, and $(y_j)_{j=1}^m$ is a basis of F, then

$$
\begin{aligned}
&\left\|\sum_{i=1}^n\sum_{j=1}^m t_{i,j}\,\alpha_{i,j}\,e_i\otimes y_j\right\|_\epsilon =\\
&= \sup_{1\le i\le n}\left\|\sum_{j=1}^m t_{i,j}\,\alpha_{i,j}\,y_j\right\|_F\\
&\le \sup_{1\le i\le n} ub(F)\left\|\left(\sum_{j=1}^m \alpha_{i,j}\,y_j\right)\right\|_F\\
&= ub(F)\left\|\sum_{i=1}^n\sum_{j=1}^m \alpha_{i,j}\,e_i\otimes y_j\right\|_\epsilon .
\end{aligned}
$$

□

An easy consequence of the previous lemmas is the following:

Corollary 3.3.3. *If $F \in \mathcal{F}$ then $ub(\ell_1^n \overset{\wedge}{\otimes}_\pi F) \le ub(F)$.*

Lemma 3.3.4. *Let X be a normed space and $\widetilde{X}$ its completion . If X has l.u.st. then $\widetilde{X}$ has l.u.st.*

Proof. For the lemma‘s proof we need of the following result ([22] Lemma 17.3):
Let $\{x_1, ..., x_n\}$ be a basis for the finite dimensional normed space E. Given $0 < \epsilon < 1$ there is a $\delta > 0$ such that if X is a Banach space containing E and if $\widetilde{x}_1, ..., \widetilde{x}_n \in X$ satisfy $\|\widetilde{x}_k - x_k\| \le \delta$ ($1 \le k \le n$), then there exists an operator $u \in \mathcal{L}(X, X)$ such that
(a) $u(\widetilde{x}_k) = x_k$ ($1 \le k \le n$);
(b) $(1-\epsilon)\|x\| \le \|u(x)\| \le (1+\epsilon)\|x\|$.
Finally, from (b) u is invertible with $\|u\| \le 1+\epsilon$ and $\|u^{-1}\| \le (1-\epsilon)^{-1}$.
Let $\widetilde{G}$ a finite dimensional subspace of $\widetilde{X}$. Suppose that $\widetilde{G} = span\{\widetilde{u_1}, ..., \widetilde{u_n}\}$. Now, fix $0 < \epsilon < 1$ and we consider δ as above. Since X is dense in $\widetilde{X}$, we choose $u_k \in X$ so that $\|u_k - \widetilde{u}_k\| < \delta$.
Let $G = span\{u_1, ..., u_n\}$. Then G is a finite dimensional subspace of X. Therefore there exists an Banach space F with unconditional basis, $a \in \mathcal{L}(G, F)$ and $b \in \mathcal{L}(F, X)$ so that

$$i = b \circ a \quad , \quad \|a\| \, \|b\| \; ub(F) \le \Lambda(X)$$

where we are denoting with i the natural map from G to X.
Let $u \in \mathcal{L}(\widetilde{X}, \widetilde{X})$ so that
(a) $u(u_k) = \widetilde{u}_k$ ($1 \le k \le n$);
(b) $\|u\| \le 1+\epsilon$ and $\|u^{-1}\| \le (1-\epsilon)^{-1}$.
Then $u|_X \circ b \circ a \circ u^{-1}|_{\widetilde{G}}$ is just the natural map from $\widetilde{G}$ to $\widetilde{X}$ with $\|a \circ u^{-1}|_{\widetilde{G}}\| \, \|u|_X \circ b\| \; ub(F) \le \Lambda(X)$ □

The two previous computations in the case of finite dimension lead us to the two following theorems.

Theorem 3.3.5. *Let X a Banach space and K a compact Hausdorff space. Then $C(K, X)$ has l.u.st. if X does and $\Lambda(C(K, X)) \le \Lambda(X)$.*

Proof. By previous lemma it is enough to show that $C(K) \otimes X$ has l.u.st when equipped with the injective norm $\| \, \|_\epsilon$.
$C(K)$ is a $\mathcal{L}_{\infty,}$-space, so if $\varepsilon > 0$ is given and $E \in \mathcal{F}_{C(K)}$, there is an $\widetilde{E} \in \mathcal{F}_{C(K)}$, $E \subseteq \widetilde{E}$, and an isomorphism $u\colon \widetilde{E} \to \ell_\infty^{dim \widetilde{E}}$ with $||u|| \, ||u^{-1}|| < 1+\varepsilon$. X has l.u.st. so, given $F \in \mathcal{F}_X$, there is an $\widetilde{F} \in \mathcal{F}_X$, $F \subseteq \widetilde{F}$, and $v\colon F \to \widetilde{F}$, $w\colon \widetilde{F} \to X$ with $w \circ v = i_F$ such that $||v|| \, ||w|| ub(\widetilde{F}) \le \Lambda(X)$. If $G \in \mathcal{F}_{C(K) \otimes X}$ we can find an $E \in \mathcal{F}_{C(K)}$ and $F \in \mathcal{F}_X$ such that $G \subseteq E \otimes F \subseteq C(K) \otimes X$, where the two inclusions are isometries when each of the two tensor products is equipped with the injective norm, because the injective norm is injective. Let's name our isometries:

$$I\colon G \hookrightarrow E \otimes F, \qquad i\colon E \hookrightarrow \widetilde{E}, \qquad j\colon \widetilde{E} \hookrightarrow C(K);$$

now put them together:

$$G \xrightarrow{I} E \otimes F \xrightarrow{i \otimes id_F} \widetilde{E} \otimes F \xrightarrow{u \otimes v} \ell_\infty^{dim \widetilde{E}} \otimes \widetilde{F} \xrightarrow{u^{-1} \otimes w} \widetilde{E} \otimes X \xrightarrow{j \otimes id_X} C(K) \otimes X$$

and call $U = (u \otimes v) \circ (i \otimes i_F) \circ I$ and $V = (j \otimes id_X) \circ (u^{-1} \otimes w)$; notice that $V \circ U$ is the natural inclusion of G into $C(K) \otimes X$ and, taking in account previous lemma 4,

$$\begin{aligned}
&||U||\,||V||\,ub(\ell_\infty^{dim \widetilde{E}} \otimes \widetilde{F}) \\
&\leq ||u||\,||v||\,||i||\,||i_F||\,||I||\,||j||\,||id_X||\,||u^{-1}||\,||w||\,ub(\widetilde{F}) \\
&= ||u||\,||u^{-1}||\,||v||\,||w||\,ub(\widetilde{F}) \leq (1+\varepsilon)\Lambda(X).
\end{aligned}$$

Because $\varepsilon > 0$ is arbitrary, we get: $C(K, X)$ has l.u.st. and $\Lambda(C(K, X)) \leq \Lambda(X)$. □

Theorem 3.3.6. *If (Ω, Σ, μ) is a measure space and X is a Banach space with l.u.st., then $L^1(\mu, X)$ has l.u.st.*

Proof. Again in this proof we will take lemma 3.3.4 into account.
Take $G \in \mathcal{F}_{L_1(\mu)\otimes X}$: we can find $E \in \mathcal{F}_{L_1(\mu)}$, $E' \in \mathcal{F}_X$, such that $G \subseteq E \otimes E'$. We know ([61], Proposition II.5.9) there is a constant $\rho \geq 0$ so that, for every $E \in \mathcal{F}_{L_1(\mu)}$, there exists a finite dimensional subspace E'' of $L_1(\mu)$ containing E and a projection P from $L_1(\mu)$ onto E'' of norm $\leq \rho$. Since we are working with L-space, we have $L_1(\mu) \otimes E'$ is a subspace of $L_1(\mu) \otimes X$, where each is equipped with projective norm $\|\ \|_\pi$, then there is an isometry $I \colon G \to L_1(\mu) \otimes E'$.
Moreover, following the demonstration of the previous theorem, we can find an $n \in \mathbb{N}$, an $F \in \mathcal{F}_X$ and two operators $U \colon E'' \otimes E' \to \ell_1^n \otimes F$, $V \colon \ell_1^n \otimes F \to L_1(\mu) \otimes X$, such that $V \circ U$ is the natural inclusion of $E'' \otimes E'$ in $L_1(\mu) \otimes X$ and $||U||\,||V||\,ub(\ell_1^n \otimes F) \leq (1+\varepsilon)\Lambda(X)$. Now the composition

$$G \xrightarrow{I} L_1(\mu) \otimes E' \xrightarrow{P \otimes id_{E'}} E'' \otimes E' \xrightarrow{U} \ell_1^n \otimes F \xrightarrow{V} L_1(\mu) \otimes X$$

is the natural inclusion of G in $L_1(\mu) \overset{\wedge}{\otimes} X$ and

$$\begin{aligned}
||V||\,||U \circ (P \otimes id_{E'}) \circ I||\,ub(\ell_1^n \otimes F) &\leq ||V||\,||U||\,||P||\,||id_{E'}||\,||I||\,ub(\ell_1^n \otimes F) \\
&\leq \rho(1+\varepsilon)\Lambda(X)
\end{aligned}$$

and so (again by the lemma above) $L^1(\mu, X) = L_1(\mu) \overset{\wedge}{\otimes}_\pi X$ has l.u.st. and

$$\Lambda(L^1(\mu, X)) \leq \rho\, \Lambda(X).$$

□

Now it is time for GL property.

Lemma 3.3.7. *If X is a Banach space with the GL-property and $n \in \mathbb{N}$, then $\ell_\infty^n(X)$ is a GL-space and $gl(\ell_\infty^n(X)) \leq gl(X)$.*

Proof. Recall that $\ell_\infty^n(X)$ can be viewed as a $C(K, X)$ space in the obvious way (where K is a finite set of points $\{k_1, k_2, ..., k_n\}$ considered with the discrete topology). From this, using a result of Pietsch (**III.19.5** of [77]), we see that every operator $u\colon \ell_\infty^n(X) \to Y$ is representable in the form

$$u(f) = \int_K f(k)dm(k) \qquad\qquad \forall f \in \ell_\infty^n(X)$$

where m is a vector measure of bounded semivariation from the collection of all subsets of K to $\mathcal{L}(X, Y^{**})$; then we can write u in this way:

$$\begin{aligned} u(x_1, x_2, ..., x_n) = u\left(f_{(x_1,x_2,...,x_n)}\right) &= \int_K f_{(x_1,x_2,...,x_n)}(k)dm(k) \\ &= \sum_{i=1}^n \int_{\{k_i\}} f_{(x_1,x_2,...,x_n)}(k)dm(k) \\ &= \sum_{i=1}^n m(\{k_i\})\left(f_{(x_1,x_2,...,x_n)}(k_i)\right) \\ &= \sum_{i=1}^n m(\{k_i\})(x_i). \end{aligned}$$

Now we can recall two theorems of Swartz: if $u \in \Pi_1\left(\ell_\infty^n(X), Y\right)$, then m's values are all in the space $\Pi_1\left(X, Y\right)$ (Theorem 7 of [96]) and m has finite variation with respect to the π_1-norm with

$$(\pi_1 - \mathrm{var})(m) \leq \pi_1(u)$$

(Theorem 8 of [96]) and so

$$\sum_{i=1}^n \pi_1(m(\{k_i\})) \leq \pi_1(u).$$

By hypothesis every $m(\{k_i\})$ is 1-factorable, so, if we call P_i the norm 1 projection from $\ell_\infty^n(X)$ onto X defined by

$$P_i((x_1, x_2, ..., x_n)) = x_i\,,$$

we have

$$u = \sum_{i=1}^{n} m(\{k_i\}) \circ P_i$$

so u is 1-factorable with

$$\begin{aligned}
\gamma_1(u) &\leq \sum_{i=1}^{n} \gamma_1 \left(m(\{k_i\}) \circ P_i \right) \\
&\leq \sum_{i=1}^{n} \gamma_1 \left(m(\{k_i\}) \right) \\
&\leq gl(X) \sum_{i=1}^{n} \pi_1 \left(m(\{k_i\}) \right) \\
&\leq gl(X) \pi_1(u).
\end{aligned}$$

□

Corollary 3.3.8. *$\ell_1^n(X)$ has the GL-property if X does and $gl(\ell_1^n(X)) \leq gl(X)$.*

Proof. Since X has G-L property iff X^* has G-L property with $gl(X) = gl(X^*)$ (see, for instance, [22] 17.9 Proposition), and the last lemma: $(\ell_1^n(X))^* = \ell_\infty^n(X^*)$ has the GL-property, that implies $\ell_1^n(X)$ has the GL-property and

$$gl(\ell_1^n(X)) = gl((\ell_1^n(X))^*) = gl(\ell_\infty^n(X^*)) \leq gl(X^*) = gl(X).$$

□

Remark 3.3.9. As lemma 6 above we can have: Let X is a normed space and $\widetilde{X}$ its completion. If X is an G-L space then $\widetilde{X}$ is an G-L space.
This is clear since, if $u \in \Pi_1(\widetilde{X}, \ell_2)$ then $u|_X \in \Pi_1(X, \ell_2) = \Gamma_1(X, \ell_2)$. That means $u \in \Gamma_1(\widetilde{X}, \ell_2)$

Theorem 3.3.10. *$C(K, X)$ has the GL-property if X does and $gl(C(K, X)) \leq gl(X)$.*

Proof. Fix $\varepsilon > 0$. For any $E \in \mathcal{F}_{C(K) \otimes X}$ we can find $F \in \mathcal{F}_{C(K)}$, $G \in \mathcal{F}_X$ and an isomorphism $v \colon F \to \ell_\infty^{dimF}$ with $||v|| \, ||v^{-1}|| \leq 1 + \varepsilon$ such that $E \subseteq F \otimes G$. Let $\omega \in \Pi_1(C(K) \otimes X, Y)$ and consider the canonical inclusions $I \colon E \to F \otimes G$, $j \colon G \to X$ and $i \colon F \to C(K)$; look at the diagram

$$E \xrightarrow{I} F \otimes G \xrightarrow{v \otimes j} \ell_\infty^{dimF} \otimes X \xrightarrow{v^{-1} \otimes id_X} F \otimes X \xrightarrow{i \otimes id_X} C(K) \otimes X \xrightarrow{\omega} Y$$

then

$$\begin{aligned}\gamma_1(\omega|_E) &= \gamma_1(\omega \circ (i \otimes id_X) \circ (v^{-1} \otimes id_X) \circ (v \otimes j) \circ I) \\ &\leq \gamma_1(\omega \circ (i \otimes id_X) \circ (v^{-1} \otimes id_X)) \, ||(v \otimes j) \circ I|| \\ &\leq gl(X)\pi_1(\omega \circ (i \otimes id_X) \circ (v^{-1} \otimes id_X)) \, ||v|| \\ &\leq gl(X)\pi_1(\omega) \, ||v^{-1}|| \, ||v|| \\ &\leq gl(X)\pi_1(\omega)(1+\varepsilon).\end{aligned}$$

Because $\varepsilon > 0$ and $E \in \mathcal{F}_{C(K)\otimes X}$ are arbitrary and since 1-factorable has finite dimensional nature we get: ω is 1-factorable and $\gamma_1(\omega) \leq gl(X)\pi(\omega)$, so the thesis follows. □

Theorem 3.3.11. *$L^1(\mu, X)$ has the GL-property if X does.*

Proof. Again in this proof we will take in account the remark above.
Let $\varepsilon > 0$. Using Proposition II.5.9 of [61] we get a constant $\rho \geq 0$ such that, for any $E \in \mathcal{F}_{L_1(\mu)\otimes X}$, we can find $F \in \mathcal{F}_{L_1(\mu)}$, $G \in \mathcal{F}_X$, an isomorphism $v\colon F \to \ell_1^{dimF}$ with $||v|| \, ||v^{-1}|| \leq 1 + \varepsilon$ and a projection P from $L_1(\mu)$ onto F of norm less or equal to ρ, such that $E \subseteq F \otimes G$.
Let $\omega \in \Pi_1(L^1(\mu) \otimes X, Y)$. Because we are working with L-space, $L^1(\mu) \otimes G$ is a subspace of $L^1(\mu) \otimes X$, and then E is a subspace of $L_1(\mu)\widehat{\otimes}_\pi G$, so we can consider the canonical inclusions $I\colon E \to L_1(\mu) \otimes G$, $j\colon G \to X$ and $i\colon F \to L_1(\mu)$ and their composition:

$$E \xrightarrow{I} L_1(\mu)\otimes G \xrightarrow{P\otimes id_G} F\otimes G \xrightarrow{v\otimes j} \ell_1^{dimF}\otimes X \xrightarrow{v^{-1}\otimes id_X} F\otimes X \xrightarrow{i\otimes id_X} L_1(\mu)\otimes X \xrightarrow{\omega} Y.$$

Then we get

$$\begin{aligned}\gamma_1(\omega|_E) &= \gamma_1(\omega \circ (i \otimes id_X) \circ (v^{-1} \otimes id_X) \circ (v \otimes j) \circ (P \otimes id_G) \circ I) \\ &\leq \gamma_1(\omega \circ (i \otimes id_X) \circ (v^{-1} \otimes id_X)) \, ||(v \otimes j) \circ (P \otimes id_G) \circ I|| \\ &\leq gl(X)\pi_1(\omega \circ (i \otimes id_X) \circ (v^{-1} \otimes id_X)) \, ||P|| \, ||v|| \\ &\leq gl(X)\pi_1(\omega) \, ||v^{-1}|| \, ||v|| \, ||P|| \\ &\leq gl(X)\pi_1(\omega)\rho(1+\varepsilon).\end{aligned}$$

This implies that ω is 1-factorable and $\gamma_1(\omega) \leq gl(X)\rho\,\pi(\omega)$, so $L^1_X(\mu)$ has the GL-property and $gl(L^1_X(\mu)) \leq \rho\, gl(X)$. □

To extend the above results on $\mathcal{L}_1$ and $\mathcal{L}_\infty$-spaces we need the following lemmas :

Lemma 3.3.12. *$X^{**} \overset{\wedge}{\otimes}_\epsilon Y$ is a closed subspace of $(X \overset{\wedge}{\otimes}_\epsilon Y)^{**}$*

Proof. Let $x^{**}\otimes y \in X^{**}\overset{\wedge}{\otimes}_\epsilon Y$ and $\phi \in (X \overset{\wedge}{\otimes}_\epsilon Y)^* = \mathcal{B}^\pi(X,Y)$ subspace of $\mathcal{B}^\pi(X^{**},Y)$, then we have $|\phi(x^{**}\otimes y)| \le \|\phi\|_\pi \|x^{**}\otimes y\|_\epsilon$. So $x^{**}\otimes y \in (X \overset{\wedge}{\otimes}_\epsilon Y)^{**}$. For this, it' s easy see that $X^{**}\overset{\wedge}{\otimes}_\epsilon Y \subseteq (X \overset{\wedge}{\otimes}_\epsilon Y)^{**}$.
Now, we have to show that $\|\sum_{i=1}^p x_i^{**}\otimes y_i\|_\epsilon = \|\sum_{i=1}^p x_i^{**}\otimes y_i\|_{(X\overset{\wedge}{\otimes}_\epsilon Y)^{**}}$. The inequality $\|\sum_{i=1}^p x_i^{**}\otimes y_i\|_\epsilon \le \|\sum_{i=1}^p x_i^{**}\otimes y_i\|_{(X\overset{\wedge}{\otimes}_\epsilon Y)^{**}}$ follows easily from the definition of $\|\cdot\|_\epsilon$ norm and Goldstine 's theorem. For the converse

$$\|\sum_{i=1}^p x_i^{**}\otimes y_i\|_{(X\overset{\wedge}{\otimes}_\epsilon Y)^{**}} = \sup\{|(\sum_{i=1}^p x_i^{**}\otimes y_i)(\phi)| \;:\; \phi \in \mathcal{B}^\pi(X,Y),\ \|\phi\|_\pi \le 1\} \le$$

$$\le \sup\{|(\sum_{i=1}^p x_i^{**}\otimes y_i)(\phi)| \;:\; \phi \in \mathcal{B}^\pi(X^{**},Y),\ \|\phi\|_\wedge \le 1\} = \|\sum_{i=1}^p x_i^{**}\otimes y_i\|_\epsilon$$

Because the elements of the type $\sum_{i=1}^p x_i \otimes y_i$ are dense in $X^{**}\overset{\wedge}{\otimes}_\epsilon Y$, we are done. □

Lemma 3.3.13. *Let X and Y be Banach spaces such that X^{**} or Y has the Bounded Approximation Property. Then $X^{**}\overset{\wedge}{\otimes}_\pi Y$ is isomorphic to a closed subspace of $(X\overset{\wedge}{\otimes}_\pi Y)^{**}$*

Proof. The trace duality $\Phi : X^{**}\overset{\wedge}{\otimes}_\pi Y \to (X\overset{\wedge}{\otimes}_\pi Y)^{**} = (\mathcal{L}(X,Y^*))^*$ by

$$\Phi(z)(\phi) = <\phi, z>$$

has norm 1. The canonical map $I : X^{**}\overset{\wedge}{\otimes}_\pi Y \to (X^*\overset{\wedge}{\otimes}_\epsilon Y^*)^* = (\mathcal{K}(X,Y^*))^*$ is a isomorphic embedding (because X^{**} or Y^* has B.A.P., see 1.1.16). If $\alpha : (\mathcal{L}(X,Y^*))^* \to (\mathcal{K}(X,Y^*))^*$ is the restriction map, we have $\alpha\Phi = I$. Then Φ is a isomorphic embedding as well. □

Theorem 3.3.14. *Let X and Y be Banach spaces. We have*

1. *If X is a $\mathcal{L}_\infty$-space, then*
   $X\overset{\wedge}{\otimes}_\epsilon Y$ has l.u.st. property if Y does;
2. *If X is a $\mathcal{L}_1$-space, then*
   $X\overset{\wedge}{\otimes}_\pi Y$ has l.u.st. property if Y does.

Proof. Let S a subspace finite-dimensional of $X\overset{\wedge}{\otimes}_\epsilon Y$. Since X is a $\mathcal{L}_\infty$-space then X^{**} is complemented in a $C(K)$ space, for some compact Hausdorff space K. Therefore $X^{**} \overset{\wedge}{\otimes}_\epsilon Y$ is complemented in $C(K,Y)$ (see [4]) by a projection P. We have

$$S \xrightarrow{i} X \overset{\wedge}{\otimes}_\epsilon Y \overset{i_X \otimes id_Y}{\longrightarrow} X^{**} \overset{\wedge}{\otimes}_\epsilon Y \xrightarrow{\Psi} C(K,Y)$$

where i is the canonical embedding, and Ψ the natural inclusion. Then $\Psi \circ i_X \otimes id_Y \circ i$ is the canonical embedding from S into $C(K,Y)$. Since $C(K,Y)$ has l.u.st., there is a Banach space Z finite-dimensional with unconditional bases such that

$$S \xrightarrow{v} Z \xrightarrow{u} C(K,Y)$$

$$\Psi \circ i_X \otimes id_Y \circ i = u \circ v$$

Moreover $P \circ u(Z)$ is a subspace finite dimensional of $X^{**} \overset{\wedge}{\otimes}_\epsilon Y$ hence, for lemma 16, $P \circ u(Z)$ is a finite dimensional subspace of $(X \overset{\wedge}{\otimes}_\epsilon Y)^{**}$.
By the Principle of local reflexivity there exists a injective operator $s : P \circ u(Z) \longrightarrow X \overset{\vee}{\otimes} Y$ such that

$$s(e) = e \quad \forall e \in (P \circ u(Z)) \cap (X \overset{\vee}{\otimes} Y)$$

Then $\widetilde{u} = s \circ P \circ u$ is such that

$$\widetilde{u} : Z \longrightarrow X \overset{\vee}{\otimes} Y$$

and

$$v \circ \widetilde{u} = i$$

and

$$\|v\| \|\widetilde{u}\| ub(Z) \leq \|P\| \Lambda(C(K,Y))$$

.
(2) it's the same as (1), where we considered $L_1(\mu, Y)$-space instead of $C(K,Y)$, and lemma 3.3.13 instead of lemma 3.3.12. □

Theorem 3.3.15. *Let X and Y be Banach spaces. We have*

1. *If X is a $\mathcal{L}_\infty$-space, then*
   $X \overset{\wedge}{\otimes}_\epsilon Y$ has GL property if Y does;

2. *If X is a $\mathcal{L}_1$-space, then*
   $X \overset{\wedge}{\otimes}_\pi Y$ has GL property if Y does.

Proof. Let
$$u : X \overset{\wedge}{\otimes}_\epsilon Y \to \ell_2$$
an 1-summing operator, then $u^{**} \in \Pi_1((X \overset{\wedge}{\otimes}_\epsilon Y)^{**}, \ell_2)$. Now consider $\widetilde{u} = u^{**}|_{X^{**}\overset{\wedge}{\otimes}_\epsilon Y}$ (we note that for lemma 16 $X^{**} \overset{\wedge}{\otimes}_\epsilon Y$ is a closed subspace of $(X \overset{\wedge}{\otimes}_\epsilon Y)^{**}$ so $\widetilde{u} \in \Pi_1(X^{**} \overset{\wedge}{\otimes}_\epsilon Y, \ell_2)$). As in the proof above $X^{**} \overset{\wedge}{\otimes}_\epsilon Y$ is complemented in $C(K, Y)$. Since $C(K, Y)$ has GL property, we have
$$\widetilde{u} \circ P \in \Gamma_1(C(K, Y), \ell_2)$$
Then $u = \widetilde{u} \circ P|_{X\overset{\wedge}{\otimes}_\epsilon Y}$ is in $\Gamma_1(X \overset{\wedge}{\otimes}_\epsilon Y, \ell_2)$, with
$$\gamma_1(u) \leq \|P\| gl(C(K, Y))\pi_1(u)$$
(2) it is the same as (1), where using Lemma 3.3.13 instead of Lemma 3.3.12. We are done $\square$

We end this section with two natural questions:

Questiones 3.3.16.

(1) If X is a Banach space so that $X \overset{\wedge}{\otimes}_\pi Y$ has l.u.st. (GL-property) whenever Y does, is X a $\mathcal{L}_1$-space?

(2) If X is a Banach space so that $X \overset{\wedge}{\otimes}_\epsilon Y$ has l.u.st. (GL-property) whenever Y does, is X a $\mathcal{L}_\infty$-space?

Chapter 4

Weakly Compact Subsets in Projective Tensor Products

4.1 Some Preliminaries Facts

The problem to understand the compactness in any Banach space it is related to many classical problems in many areas of analysis. In projective tensor products the first signal of norm-compactness came from a classical result of Grothendieck (see [41]). He showed that

Proposition 4.1.1. *Let X and Y be Banach spaces. A subset K of the projective tensor product $X \overset{\wedge}{\otimes}_\pi Y$ of X and Y is compact if and only if there exist K_X and K_Y compacts subsets of X and Y respectively such that $K \subseteq \overline{co}(K_X \otimes K_Y)$, where with $K_X \otimes K_Y$ is denoted the set $\{x \otimes y \ : \ x \in K_X, \ y \in K_Y\}$.*

Using this fact Grothendieck (see [41], p.51) deduced

Proposition 4.1.2. *Let $K \subseteq X \overset{\wedge}{\otimes}_\pi Y$ be a compact subset of the projective tensor product of X and Y. Then there exist two norm null sequences $\{x_n\}_n$ and $\{y_n\}_n$ in X and Y respectively, and a compact subset $\overline{K}^{\ell_1}$ of ℓ_1 so that every element of $u \in K$ can be written as $u = \sum_{i=1}^{\infty} \lambda_i^u x_i \otimes y_i$ where $\lambda^u = \{\lambda_n^u\}_n \in \overline{K}^{\ell_1}$.*

This was the main topic to understand the norm-compact subsets of $\mathcal{K}(X,Y)$ and $\mathcal{W}(X,Y)$ (the space of compact and weakly compact operators from X to Y respectively, see [1], [65] and [74]). Essentially Grothendieck used only the following facts:

if K is a compact subset of $X \overset{\wedge}{\otimes}_\pi Y$, from the proposition above, there exist two compact subsets K_X, K_Y of X and Y respectively so that $K \subseteq \overline{co}(K_X \otimes K_Y)$. Now, for a beautiful Grothendieck 's result (i.e. see [72] for example) we know that there exist norm-null sequences $(x_n)_n \subseteq X$ and $(y_n)_n \subseteq Y$ so that

$$K_X \subseteq \overline{co}\{x_n \ ; \ n \in \mathbb{N}\} \quad \text{and} \quad K_Y \subseteq \overline{co}\{y_n \ : \ n \in \mathbb{N}\}$$

Now it is clear who are $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ and $\overline{K}^{\ell_1}$.
In the study of weakly compact subsets of the projective tensor product the singular result of Ülger [100] practically settled the problem in case one was an $L_1(\mu)$-space. Ülger's result was polished into final form in [25].
Previous to Ülger's work, Michel Talagrand [97] offered a profound analysis of conditionally weakly compact subsets of $X \overset{\wedge}{\otimes} Y$ when X is an $L_1(\mu)$-space. Talagrand's work influenced Ülger and so all that's came since. Here is the end result of Ülger, Diestel-Ruess-Schachermayer.

Theorem 4.1.3. ***(Ülger, Diestel, Ruess, Schachermayer)*** *Let (Ω, Σ, μ) be a finite measure space, and let X be a Banach space. Let A be a bounded subset of $L_1(\mu, X)$. Then the following are equivalent:*

(i) *A is relatively weakly compact;*

(ii) *A is uniformly integrable, and, given any sequence $(f_n)_n \subseteq A$ there exists a sequence $(g_n)_n$ with $g_n \in co\{f_k,\ k \geq n\}$ such that $(g_n(\omega))_n$ is norm convergent in X for a.e. $\omega \in \Omega$;*

(iii) *A is uniformly integrable, and, given any sequence $(f_n)_n \subseteq A$ there is a sequence $(g_n)_n$ with $g_n \in co\{f_k,\ k \geq n\}$ such that $(g_n(\omega))_n$ is weakly convergent in X for a.e. $\omega \in \Omega$;*

For the weak-compactness in projective tensor product almost nothing is known. First of all a big difference with the norm-compact in projective tensor norm is that, if K_X and K_Y are norm-compact subsets of the Banach spaces X and Y respectively, then $K_X \otimes K_Y$ is a norm-compact subset of $X \overset{\wedge}{\otimes}_\pi Y$. For the weak-compact subsets in projective tensor products the story changes completely. Indeed if X and Y are reflexive Banach spaces then B_X, B_Y are two weak-compacts subsets of X and Y respectively. But since $B_{X \overset{\wedge}{\otimes}_\pi Y} = \overline{co}(B_X \otimes B_Y)$, if $B_X \otimes B_Y$ was weak-compact in $X \overset{\wedge}{\otimes}_\pi Y$ then by the Krein-Smulian's theorem (i.e. see [72], 2.8.14) $B_{X \overset{\wedge}{\otimes}_\pi Y}$ should be weak-compact; in particular we should fin that $X \overset{\wedge}{\otimes}_\pi Y$ reflexive, a fact very rare in projective tensor product (that happens only when $\mathcal{L}(X, Y^*) = \mathcal{K}(X, Y^*)$, see [15], I.6; indeed $\ell_2 \overset{\wedge}{\otimes}_\pi \ell_2$ is not reflexive). What we can say is just a little bit, but first we need to recall some definition.

Definition 4.1.4. (A. Grothendieck, [44]**)** Let X and Y be Banach spaces. A bounded linear operator $T : X \longrightarrow Y$ is called *completely continuous* if maps weakly convergent sequences to norm convergent sequence.
A Banach space X has the *Dunford-Pettis property* (DPP) if, for every Banach space Y, every weakly compact operator from X to Y is completely continuous (see [17] for a good source).
It is know this condition to be equivalents to:

(i) Every weakly compact linear operator from X into c_0 is completely continuous;

(ii) For every sequence $(x_n)_n$ in X converging weakly to some x and every sequence $(x_n^*)_n$ in X^* converging weakly to some x^*, the sequence $\{x_n^*(x_n)\}_n$ converges to $x^*(x)$;

(iii) For every sequence $(x_n)_n$ in X converging weakly to 0 and every sequence $(x_n^*)_n$ in X^* converging weakly to 0, the sequence $\{x_n^*(x_n)\}_n$ converges to 0.

Proposition 4.1.5. *Let X , Y be Banach spaces, with X having the Dunford-Pettis property. If $W_X \subseteq X$ and $W_Y \subseteq Y$ are weakly compact subsets then $W_X \otimes W_Y$ is a weakly compact of $X \overset{\wedge}{\otimes} Y$*

Proof. By the Eberleiň-Smuliàn theorem it is suffices to show that $W_X \otimes W_Y$ is weakly sequentially compact. Let $(u_n = x_n \otimes y_n)_n$ be a sequence in $W_X \otimes W_Y$. Let $(n_k)_k$ be a strictly increasing sequence of positive integers such that for some $x \in W_X$ and $y \in W_Y$

$$x = weak - \lim_{k\to\infty} x_{n_k} \quad ,\text{and} \quad y = weak - \lim_{k\to\infty} y_{n_k}$$

We need to test $(u_{n_k})_k$ vis-a-vis members of $(X \overset{\wedge}{\otimes} Y)^*$. Since $(X \overset{\wedge}{\otimes} Y)^* = \mathcal{B}(X, Y)$, the space of bilinear continuous functionals on $X \times Y$, take a continuous bilinear functional F on $X \times Y$. If $x_k^* = F(\cdot, y_{n_k})$, then $x_k^* \in X^*$ and $x^* = F(\cdot, y) \in X^*$. Define $T_F : Y \longrightarrow X^*$ by

$$T_F(y)(x) = F(x, y)$$

T_F is a bounded linear operator and $T_F(y_{n_k}) = x_{n_k}^*$ as well as $T_F(y) = x^*$. Since T_F is also weak-to-weak continuous, the fact that $(y_{n_k})_k$ converges weakly to y soon reveals that $(x_{n_k}^*)_k$ converges weakly to x^*. Now we are in business: $x = weak - \lim_{k\to\infty} x_{n_k}$ and $x^* = weak - \lim_{k\to\infty} x^*_{n_k}$. Hence, thanks to X's enjoyment of the Dunford-Pettis property, $F(x,y) = T_F(y)(x) = x^*(x) = \lim_k x^*_{n_k}(x_{n_k}) = T_F(y_{n_k})(x_{n_k}) = F(x_{n_k}, y_{n_k})$, which is as it should be. □

By the previous proposition we easily have

Corollary 4.1.6. *Let X_1, X_2, Y_1, Y_2 be Banach spaces. Let $T_1 : X_1 \longrightarrow Y_1$ and $T_2 : X_2 \longrightarrow Y_2$ be two weakly compact operators. Suppose either Y_1 or Y_2 has the Dunford-Pettis property, then the projective tensor product $T_1 \overset{\wedge}{\otimes} T_2 : X_1 \overset{\wedge}{\otimes} X_2 \longrightarrow Y_1 \overset{\wedge}{\otimes} Y_2$, of T_1 and T_2, is weakly compact.*

In the next section we will give a representation theorem of weakly compact subsets in the projective tensor products ([24]).

4.2 Weakly Compact Subsets in Projective Tensor Products

In order to study this question let us introduce a topology in $X \overset{\wedge}{\otimes}_\pi Y$, which we will call in the sequel the τ-topology. A base of neighborhoods for the τ-topology has the form:

$$\mathcal{A} = X \overset{\wedge}{\otimes}_\pi Y \setminus \cup_{i=1}^n \overline{co}(U_i \otimes V_i)$$

where U_i and V_i are weakly compact subsets of X and Y respectively, for $i = 1, ..., n$. As the reader can note τ is the coarsest topology so that the sets $\overline{co}(U \otimes V)$ (with U and V weakly compact subsets of X and Y respectively) are τ-closed. Since such subsets are weakly closed (because every convex norm closed set in a Banach space is weakly closed) then the weakly topology is finer than the τ-topology on $X \overset{\wedge}{\otimes}_\pi Y$ (recall that if θ_1, θ_2 are two topologies in X then θ_2 is finer than θ_1 if $\theta_1 \subseteq \theta_2$). At first glance the τ-topology doesn't look very beautiful (because she is not Hausdorff in general), but the key idea is to study the restriction of τ to certain bounded subsets of $X \overset{\wedge}{\otimes} Y$ (especially the weak compact subsets) to get a "reasonable" topology (in particular we are interested to see when such a restriction τ is Hausdorff). We will not study the τ-topology on $X \overset{\wedge}{\otimes}_\pi Y$ in detail, but we will use it only to derive the result. Note that for the topology τ we have:

1. For fixed $v \in X \overset{\wedge}{\otimes} Y$ the map $u \longmapsto u + v$ is τ-continuous.

2. For fixed $\lambda > 0$ the map $u \longmapsto \lambda u$ is τ continuous.

3. The map $u \longmapsto -u$ is τ-continuous.

A topology which satisfies (1) and (2) is called a *prelinear topology*(see [35]). So τ is a prelinear topology.

Theorem 4.2.1. *Let X and Y be two Banach spaces. Every weakly compact subset in $X \overset{\wedge}{\otimes}_\pi Y$ can be written as the intersection of a finite union of sets of the form $\overline{co}(U \otimes V)$, where U and V are weakly compact subsets of X and Y respectively.*

Proof. Let W be a weakly compact subset of $X \overset{\wedge}{\otimes}_\pi Y$. Since the weak topology is finer than the topology τ, our theorem will be proved once it is shown that the restriction of τ to W is a Hausdorff topology; that means that W is closed for the topology τ, and so W will be as wished.

Let $u, v \in W$ so that $u \neq v$. Without lost of generality we can assume $u = 0$ (otherwise consider $\{u - w \ : \ w \in W\}$ which is still weakly compact in $X \overset{\wedge}{\otimes}_\pi Y$, and by (1) and (3) above, the translation is a τ-homeomorphism). Moreover using (2) we can assume $\|v\|_\wedge = 1$.

We need to distinguish two cases:

Case 1. $v = \sum_{k=1}^n \lambda_k x_k \otimes y_k$ with $\sum_{k=1}^n \lambda_k = 1$ and $\|x_k\|, \|y_k\| = 1$ for all $1 \leq k \leq n$; i.e. v is a simple vector of $X \otimes Y$. Now using the Hahn-Banach theorem there exist $x^* \in X^*$ and $y^* \in Y^*$ so that

$$x^* \otimes y^*(0) = 0 < \delta^2 < x^* \otimes y^*(v).$$

Since X and Y are norm one complemented in $X \overset{\wedge}{\otimes} Y$, let P_X, P_Y be the projections from $X \overset{\wedge}{\otimes} Y$ to X and Y respectively. Define

$$\begin{aligned} K_1^u &= [x^* \geq \delta] \cap P_X(W) \\ K_2^u &= [y^* \geq \delta] \cap P_Y(W) \\ K_1^v &= [x^* \leq \delta] \cap P_X(W) \\ K_2^v &= [y^* \leq \delta] \cap P_Y(W) \end{aligned}$$

where if $\alpha \in \mathbb{R}$ we are denoting by $[x^* \leq \alpha] = \{x \in X : x^*(x) \leq \alpha\}$ and $[y^* \geq \alpha] = \{y \in Y : y^*(y) \geq \alpha\}$. Then K_1^u, K_1^v are weakly compact subsets of

X, and K_2^u, K_2^v are weakly compact subsets of Y. By construction and by the definition of the topology τ, we get that $W \setminus \overline{co}(K_1^u \otimes K_2^u)$ is a τ-neighborhood of 0 and $W \setminus \overline{co}(K_1^v \otimes K_2^v)$ is a τ-neighborhood of v. Since

$$W \subseteq \overline{co}(K_1^u \otimes K_2^u) \cup \overline{co}(K_1^v \otimes K_2^v)$$

we get

$$[W \setminus \overline{co}(K_1^u \otimes K_2^u)] \cap [W \setminus \overline{co}(K_1^v \otimes K_2^v)] = \emptyset$$

hence when v is a simple tensor we can always separate 0 and v by two disjoint τ-neighborhoods in W.

Case 2. Suppose that $0 \neq v = \sum_{k=1}^{\infty} \lambda_k x_k \otimes y_k$; we can assume that for all $n \geq N$, $\sum_{k=1}^{n} \lambda_k x_k \otimes y_k \neq 0$, as well.

Suppose v and 0 cannot be separated by disjoint τ-open sets; this means that for any τ-open sets U, V with $0 \in U$ and $v \in V$ we have

$$(*) \quad U \cap V \neq \emptyset.$$

By *case 1* we know that for each $n \geq N$ there are τ-open sets U_n, V_n containing 0 so that

$$(U_n) \cap (\sum_{k=1}^{n} \lambda_k x_k \otimes y_k + V_n) = \emptyset.$$

But U_n and V_n, being τ-open, are norm open so there is a $n_0 > N$ so

$$v - \sum_{k=1}^{n_0} \lambda_k x_k \otimes y_k \in U_{n_0} \cap V_{n_0}$$

or

$$(**) \quad v \in \left(\sum_{k=1}^{n_0} \lambda_k x_k \otimes y_k + (U_{n_0} \cap V_{n_0}) \right)$$

In tandem $(*)$ and $(**)$ tell us that

$$\emptyset \neq \{U_{n_0} \cap V_{n_0}\} \bigcap \left(\sum_{k=1}^{n_0} \lambda_k x_k \otimes y_k + (U_{n_0} \cap V_{n_0}) \right)$$

(after all, $U_{n_0} \cap V_{n_0}$ is τ-open and contains 0 while $\sum_{k=1}^{n_0} \lambda_k x_k \otimes y_k + (U_{n_0} \cap V_{n_0})$ is τ-open and contains v, so $(*)$ is in effect)

$$\subseteq U_{n_0} \cup \left((\sum_{k=1}^{n_0} \lambda_k x_k \otimes y_k) + V_{n_0} \right) = \emptyset$$

OOPS!

$\square$

Using Proposition 4.1.4 we get

Corollary 4.2.2. *Let X, Y be Banach spaces, such that X has the DP property. Then every weakly compact subset in $X \overset{\wedge}{\otimes}_\pi Y$ can be written as the intersection of a finite unions of sets of the form $co(U \otimes V)$, where U and V are weakly compacts subsets of X and Y respectively.*

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