DOTTORATO DI RICERCA
in
SCIENZE COMPUTAZIONALI E INFORMATICHE
Ciclo XIX

Consorzio tra Università di Catania, Università di Napoli Federico II, Seconda Università di Napoli, Università di Palermo, Università di Salerno

SEDE AMMINISTRATIVA: UNIVERSITÀ DI NAPOLI FEDERICO II

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REGULARITY RESULTS FOR EVOLUTIONARY VARIATIONAL AND QUASI-VARIATIONAL INEQUALITIES AND APPLICATIONS TO DYNAMIC EQUILIBRIUM PROBLEMS

TESI DI DOTTORATO DI RICERCA

IL COORDINATORE
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In a scientific work, who refuses to go beyond the facts
very rarely succeeds to only arrive also to the facts.
Thomas Henry Huxley

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Preface

In recent years, the theory of variational inequalities has had more developments. This theory has originally been introduced in the seventies of the 20^{th} century as an innovative and effective method to solve a group of nonlinear boundary value problems for partial differential equations of elliptic or parabolic type, as, for example, the Signorini problem, the obstacle problem and the elastic-plastic torsion problem. The pioneer works in this field are due to G. Fichera (see [44]) and G. Stampacchia (see [94], [56], [57] and [95]).

The critical point for which the other theories, available in the literature, have revealed themselves unable to solve the above-mentioned problems is that these problems request a condition of complementarity type on the boundary or on a part of the set where the problems are defined and, in general, it is not possible to express them as an optimization problem.

After a period in which many fundamental results were been obtained about the theory of variational inequalities, maybe in consequence of the untimely death of G. Stampacchia on 1978, the interest for variational inequalities declined and it seemed that the theory had no more to say.

Some years later, after results proved by M.J. Smith (see [90]) and S. Dafermos (see [26]) on the formulation of the traffic network equilibrium problem in terms of a finite-dimensional variational inequality and, as a consequence, on the possibility to study in this way existence, uniqueness, stability of traffic equilibria and to compute the solutions, the theory of variational inequalities had more impulse.

In fact, the last decades have witnessed an exceptional interest for variational inequalities and an enormous amount of papers and books have been devoted to this topic. More and more problems arising from the economic world, as the spatial price equilibrium problem, the oligopolistic market equilibrium problem, the migration problem and many others (see [72]), are formulated in terms of a finite dimensional Variational Inequality and, by means of this theory, solved.

Moreover, in the end of the nineties of the 20^{th} century the last it is studied the traffic network equilibrium problem with feasible path flows which have to satisfy time-dependent capacity constraints and demands which has been formulated in [36] and [37] (see also [45]) as an evolutionary variational inequality. Then, existence theorems and computational procedures are given. After first dynamic problem, many other problems with time-dependent data have been formulated in the same terms. In [34] and [29] the authors consider the spatial price equilibrium problem when the prices and the commodity shipment bounds vary over the time. The dynamic spatial price equilibrium problem in which the variables are commodity shipments is studied in [31]. In [30] and [33] the authors consider a evolutionary financial network model consisting of multiple sectors, each of which seeks to determine its optimal portfolio given time-dependent supplies of the financial holdings. Moreover, we recall that analogous relationships have been obtained in physical oceanography (see [79]), in biology (see [47]), in electric power supply chain networks with known demands (see [73] and [74]) and in human migration (see [71] and [75]). All these problems have a common element: their equilibrium conditions can be handled as generalized complementarity problems and moreover the evolutionary variational inequality formulation can be expressed in a unified way (see [25]).

In this treatise we investigate the problem of regularity of solutions to evolutionary variational and quasi-variational inequalities. Such problem is very important for applications. In fact, as we have said, the dynamic traffic equilibrium problem, the spatial equilibrium problems with either quantity or price formulations, a variety of financial equilibrium problems (see [25]), are modeled by a common formulation by means of a evolutionary variational inequality. Then, we can apply regularity results to these problems to obtain that equilibrium solutions are continuous with respect to the time. This fact plays an important rule to solve numerically the dynamic equilibrium problems. In particular it is possible to introduce a method to compute dynamic equilibria by means of a discretization procedure.

Few authors studied properties of regularity for variational inequalities. Important results about this arguments was obtained by U. Mosco in 1969 (see [70]). In particular, it was proved the convergence of solutions to variational inequalities in Hilbert spaces. Moreover, in literature there are few works on the regularity of quasi-variational inequalities. This induce us to study regularity results for variational and quasi-variational inequalities which depend explicitly on the time. In order to archive our analytic results, the convergence for convex closed sets in Mosco's sense plays an important role. It generalize the classical Hausdorff definition of a metric for the space of closed subsets of a (compact) metric space. And in which both the strong and weak topologies of X are involved. Moreover, in literature very few methods for the calculation of the solution to dynamic equilibrium problems are available (see for instance the sub-gradient method presented in [37]). Then, our result seems to have a particular relevance. Applying our regularity results to dynamic equilibrium problems and to associated variational inequalities, we can discretize the time interval $[0, T]$ and hence to reduce the computational procedure to finite-dimensional problems. This allows us to use a method to solve static equilibrium problems and finally, by means of a interpolation procedure, we find the equilibrium solutions. After a brief introduction in which we introduce the preliminary concepts and the most important related results, in Chapter 7 and Chapter 6 we present a general model of evolutionary variational and quasi-variational inequalities and we study the regularity.

We conclude this preface with a short compendium of the structure of this thesis.

It consists of three parts.

Part one mainly concerns itself with the theory of variational and quasi-variational inequalities. In particular in Chapter 1 we review existence results for variational and quasi-variational inequalities both in the finite case and the infinite case. Then, Chapter 2 focuses on the traffic network models, starting from the static case and analyzing also the elastic model. For each one of them a variational formulation is studied, and existence theorems are presented in Chapter 3. And we remark that dynamic equilibrium problems can be grouped into a unified definition of constraint set.

In the second part, we show theoretical results on the regularity of solutions to evolutionary variational and quasi-variational inequalities. We make use on the property of the set convergence in Mosco's sense (see [70]), which is explained extensively in Chapter 4. We show that the set of feasible flows associated to the traffic equilibrium problem both in the time-dependent case and time-dependent elastic case verifies the property of the set convergence in Mosco's sense. Moreover, we prove that the analogous result holds for the constraint set of the dynamic equilibrium problems written in the unified formulation. In Chapter 5, after surveying the state-of-the-art, we present some regularity results for a general class of evolutionary variational inequalities. In particular, we prove the continuity with respect to the time of solutions to linear and nonlinear evolutionary variational inequalities under assumptions of strong monotonicity, degenerate condition and strict monotonicity. Minty's Lemma for variational inequalities and the notion of the sets convergence in Mosco's sense play an important role in the attainment of the continuity results for strongly monotone evolutionary variational inequalities. By means a regularization procedure, we obtain that the solutions to evolutionary variational inequalities associated to degenerate and strictly monotone operators are continuous mappings from the time interval $[0, T]$ to the Euclidian space $\mathbb{R}^m$. With analogous technique, in Chapter 6, we extend the results to a general class of evolutionary quasi-variational inequalities under similar monotonicity assumptions on the associated operator. Finally, Chapter 7 presents in detail the regularity for dynamic equilibrium problems making use of theoretical results shown above.

The third part of the present dissertation concerns the dynamical equilibrium problem from a computational point of view. In particular, in Chapter 8 we propose a method to solve the evolutionary variational inequalities which express dynamic equilibrium problems. More specifically, we propose a discretization procedure which reduces the infinite problem to the calculus of solutions to finite-dimensional variational inequality. In particular, taking into account of the continuity, we can discretize the time interval $[0, T]$ and then we can compute, by means of both the projection methods and descent methods, the solution of the finite-dimensional variational inequalities obtained using the discretization. At last, we construct an approximation solution with linear interpolation. To obtain numerical results, we use a MatLab computation.

This work is the synthesis of the support and the encouragement of my advisor Professor Antonino Maugeri, who I wish to thank warmly.

All the results have been present during conferences and workshops and some questions are still topic of research and further improvements.

1

Theoretical foundations on variational inequalities

1.1 Historical development

Variational inequalities proved to be a very useful and powerful tool for investigation and solution of many equilibrium type problems in Economics, Engineering, Operations Research and Mathematical Physics. In fact, variational inequalities for example provide a unifying framework for the study of such diverse problems as boundary value problems, price equilibrium problems and traffic network equilibrium problems. Besides, they are closely related with many general problems of Nonlinear Analysis, such as fixed point, optimization and complementarity problems. As a result, the theory and solution methods for variational inequalities have been studied extensively, and considerable advances have been made in these areas.

The theory of variational inequalities, born in Italy in the sixties, was introduced to study elliptic problems with unilateral conditions at the boundary (the celebrated Signorini problem [88]), the obstacle problem, the elastic plastic problem, and other similar problems of mathematical physics. The pioneer works in this field are due to G. Fichera (see [44]) and G. Stampacchia (see [94]) were motivated by concrete problems, the first in mechanics (a problem in elasticity with a unilateral boundary condition) and the second in potential theory (in connection with capacity, a basic concept from electrostatics). A further study of a special case of variational inequalities was done by J.L. Lions and G. Stampacchia in the joint papers, [56] and [57], with applications to elliptic and parabolic unilateral boundary value problems. In the same period, H. Brezis (see [17]) introduced evolutionary variational inequalities.

The existence theorem in the general form stated above (and its extension to semi-monotone operators) was obtained by F.E. Browder (see [22]) and P.H. Hartman and G. Stampacchia (see [49]) by using the “monotonicity” approach to nonlinear problems previously developed for operator equations in Hilbert space by E.H. Zarantonello (see [103]), G. Minty (see [67]) and F.E. Browder (see [18] and [19]) and for equations involving operators from a Banach space X to its dual X^* by F.E.

Browder (see [20] and [21]), G. Minty (see [68]) and J. Leray and J.L. Lions (see [55]).

In the following, many other authors worked on the theory of variational inequalities, as D. Kinderleher and G. Satmpacchia (see [51]).

In the same years, A. Bensoussan and J.L. Lions in a series of papers (see, e.g., [14]) introduced a more general mathematical tool, quasi-variational inequalities, in connection with impulse optimal control problems. Then they have been extensively studied in numerous publications, mainly from the viewpoints of existence of solutions and numerical methods; see [2], [23], [97] among others.

In the next sections we present various basic concepts in optimization and variational analysis and recall their properties.

1.2 Preliminary concepts

Let X be a real topological vector space and let S be a subset of X. Moreover let X' be the topological dual space of X.

Definition 1.2.1. A functional $f : S \to \mathbb{R} \cup \{\pm\infty\}$ is said to be *upper semi-continuous* (briefly *u.s.c.*) if for each x', we have

$$\limsup_{x \to x'} f(x) \leq f(x').$$

Definition 1.2.2. A functional $f : S \to \mathbb{R} \cup \{\pm\infty\}$ is said to be *lower semi-continuous* (briefly *l.s.c.*) if $-f(x)$ is upper semi-continuous.

Definition 1.2.3. An operator $f : S \to X'$ is *monotone* on S if

$$\langle f(x_1) - f(x_2), x_1 - x_2 \rangle \geq 0, \quad \forall x_1, x_2 \in S.$$

Definition 1.2.4. An operator $f : S \to X'$ is *strictly monotone* on S if

$$\langle f(x_1) - f(x_2), x_1 - x_2 \rangle > 0, \quad \forall x_1 \neq x_2.$$

Definition 1.2.5. An operator $f : S \to X'$ is *strongly monotone* on S if for some $\nu > 0$

$$\langle f(x_1) - f(x_2), x_1 - x_2 \rangle \geq \nu \|x_1 - x_2\|^2, \quad \forall x_1, x_2 \in S.$$

Definition 1.2.6. An operator $f : S \to X'$ is *pseudomonotone* on S if for all $x_1, x_2 \in S$

$$\langle f(x_1), x_1 - x_2 \rangle \geq 0 \Longrightarrow \langle f(x_2), x_1 - x_2 \rangle \leq 0.$$

Let X be a real topological vector space and let $\mathbf{K}$ be a convex subset of X.

Definition 1.2.7. An operator $f : \mathbf{K} \to X'$ is *hemicontinuous* if for any $x \in \mathbf{K}$, the function

$$\mathbf{K} \ni \xi \to \langle f(\xi), x - \xi \rangle$$

is upper semi-continuous on $\mathbf{K}$.

Definition 1.2.8. An operator $f : \mathbf{K} \to X'$ is *hemicontinuous along line segments* if and only if for any $x, y \in \mathbf{K}$, the function

$$\mathbf{K} \ni \xi \to \langle f(\xi), y - x \rangle$$

is upper semi-continuous on the line segment $[x, y]$.

Let X, Y be two Hausdorff topological vector spaces and let S be a subset of X. Moreover, let X' denote the dual space of X.

Definition 1.2.9. A set-valued map $F : S \to 2^Y$ is *upper semi-continuous* (briefly *u.s.c.*) in $x' \in S$ if for any open subset Ω of Y such that $F(x') \subseteq \Omega$, there exists a neighborhood V of x' such that for all $x \in V$

$$F(x) \subseteq \Omega.$$

Definition 1.2.10. A set-valued map $F : S \to 2^Y$ is *lower semi-continuous* (briefly *l.s.c.*) in $x' \in S$ if for any open subset Ω of Y such that $F(x') \cap \Omega \neq \emptyset$, there exists a neighborhood V of x' such that for all $x \in V$

$$F(x) \cap \Omega \neq \emptyset.$$

Definition 1.2.11. A set-valued map $F : S \to 2^Y$ is *continuous* if it is both u.s.c. and l.s.c.

Definition 1.2.12. A set-valued map $F : S \to 2^Y$ is called *closed* if its graph

$$G = \{(x, y) : \ x \in S, \ y \in F(x)\}$$

is a closed subset of $X \times Y$.

Remark 1.2.1. It is easy to show that if X and Y are real topological linear locally convex Hausdorff spaces the following statements hold:

1. F is closed if and only if for any sequence $\{x_n\}_{n\in\mathbb{N}}$, $x_n \to x$, and any $\{y_n\}_{n\in\mathbb{N}}$, $y_n \in F(x_n)$, $y_n \to y$, then it results that $y \in F(x)$;

2. F is l.c.s. in $x \in \mathbf{K}$ if and only if for any $y \in F(x)$ and any $\{x_n\}_{n\in\mathbb{N}}$, $x_n \to x$, there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ such that $y_n \in F(x_n)$ and $y_n \to y$.

1.3 Finite dimensional variational inequalities

Now, we introduce finite dimensional variational inequalities and we recall some existence results.

Definition 1.3.1. Let $\mathbf{K}$ be a nonempty, convex and closed set of the m-dimensional Euclidean space $\mathbb{R}^m$ and let $C : \mathbf{K} \to \mathbb{R}^m$ be a vector-function. The *finite dimensional variational inequality* is the problem to find a vector $x \in \mathbf{K}$, such that

$$\langle C(x), y - x \rangle \geq 0, \quad \forall y \in \mathbf{K}. \tag{1.3.1}$$

Geometrical meaning (1.3.1) states that $C(x)^T$is orthogonal to the set $\mathbf{K}$ at the point x.

Now, we recall some classic conditions showed by Stampacchia for existence of solutions to variational inequality (1.3.1).

Theorem 1.3.1. ([49]) *If $\mathbf{K}$ is a nonempty, convex and compact subset of $\mathbb{R}^m$ and $C : \mathbf{K} \to \mathbb{R}^m$ is a continuous operator, then variational inequality* (1.3.1) *admits at least one solution.*

Theorem 1.3.2. ([57]) *If $\mathbf{K}$ is a nonempty, convex and compact subset of $\mathbb{R}^m$ and C is continuous on $\mathbf{K}$, then the set of solutions to the variational inequality* (1.3.1) *is convex and compact.*

Theorem 1.3.3. ([58]) *If C is strictly monotone on $\mathbf{K}$, then the solution to variational inequality* (1.3.1)*, if it exists, is unique.*

Whenever the set $\mathbf{K}$ is unbounded, the existence of solutions may also be established under the coercivity condition, as shows the following result.

Theorem 1.3.4. ([51]) *If C satisfies the coercivity condition*

$$\lim_{\|x\|_m \to +\infty} \frac{\langle C(x) - C(x'), x - x' \rangle}{\|x - x'\|_m} = +\infty \tag{1.3.2}$$

for $x \in \mathbf{K}$ and some $x' \in \mathbf{K}$[1]*. Then variational inequality* (1.3.1) *admits a solution.*

1.4 Infinite dimensional variational inequalities

In this section we give some results for the existence of solutions to variational inequalities in infinite dimensional spaces.

Let X be a reflexive Banach space and let $\mathbf{K} \subseteq X$ be a convex and closed set. Let us denote by $\| \cdot \|$ the norm in X. Let B_R be the closed ball with center in O and radius R and let us consider the closed and convex set $\mathbf{K}_R = \mathbf{K} \cap B_R$. If R is large enough, then $\mathbf{K}_R$ is nonempty. We have the following result.

Theorem 1.4.1. ([95]) *Let $C : \mathbf{K} \to X'$ be a monotone and hemicontinuous along line segments function, the the variational inequality*

$$x \in \mathbf{K} : \ \langle C(x), y - x \rangle \geq 0, \quad \forall y \in \mathbf{K}, \tag{1.4.1}$$

[1]From here onward we always denote by $\| \cdot \|_m$ the norm in $\mathbb{R}^m$, for all $m \geq 1$.

admits a solution if and only if there exists a constant R such that at least one solution of the variational inequality

$$x_R \in \mathbf{K}_R : \; \langle C(x_R), y - x_R \rangle \geq 0, \quad \forall y \in \mathbf{K}_R, \tag{1.4.2}$$

satisfies the condition

$$\|x_R\| < R. \tag{1.4.3}$$

Remark 1.4.1. If the set $\mathbf{K}$ is unbounded, then the following conditions for the existence of solutions are provided:

1. let us suppose that $\exists x_0 \in \mathbf{K}$ and $R > \|x_0\|$ such that

$$\langle C(y), x_0 - y \rangle < 0,$$

   $\forall y \in \mathbf{K}$, $\|y\| = R$, then (1.4.3) is verified.

2. let us suppose that $\exists x_0$ such that C satisfies the coercivity condition (1.3.2), then (1.4.2) holds.

3. let us suppose that C satisfies the weak coercivity requirement:

$$\lim_{\|y\| \to +\infty} \frac{\langle C(y), y \rangle}{\|y\|} = +\infty$$

   $\forall y \in \mathbf{K}$, then (1.4.3) is fulfilled.

We recall Theorems 2 and 3 in [80].

Theorem 1.4.2. *Let X be a real topological vector space and let $\mathbf{K} \subseteq X$ be a nonempty and convex set. Let $C : \mathbf{K} \to X'$ be a given function such that:*

(i) there exist $A \subseteq \mathbf{K}$ nonempty, compact and $B \subseteq \mathbf{K}$ compact, convex such that, for every $y \in \mathbf{K} \setminus A$, there exists $\widehat{x} \in B$ with $\langle C(y), \widehat{x} - y \rangle < 0 \rangle$;

(ii) C is pseudomonotone and hemicontinuous along line segments.

Then, there exists $x \in A$ such that $\langle C(x), y - x \geq 0 \rangle$, for all $y \in \mathbf{K}$.

Theorem 1.4.3. *Let X be a real topological vector space and let $\mathbf{K} \subseteq X$ be a nonempty and convex set. Let $C : \mathbf{K} \to X'$ be a given function such that:*

(i) there exist $A \subseteq \mathbf{K}$ nonempty, compact and $B \subseteq K$ compact, convex such that, for every $y \in \mathbf{K} \setminus A$, there exists $\widehat{x} \in B$ with $\langle C(y), \widehat{x} - y \rangle < 0 \rangle$;

(ii) C is hemicontinuous.

Then, there exists $x \in A$ such that $\langle C(x), y - x \geq 0 \rangle$, for all $y \in \mathbf{K}$.

With a weakened coercivity assumption, we get the following theorem.

Theorem 1.4.4. ([85]) *Let X be a Hausdorff real topological vector space and $\mathbf{K} \subseteq X$ be a closed and convex subset with nonempty relative interior (that is the interior of $\mathbf{K}$ in its affine hull) and $C : \mathbf{K} \to X'$ a weakly* continuous function. Moreover, let $\mathbf{K}_1$ and $\mathbf{K}_2$ be two nonempty and compact subset of X with $\mathbf{K}_2 \subseteq \mathbf{K}_1$ and $\mathbf{K}_2$ having finite dimension, such that $\forall x \in X \setminus \mathbf{K}_1$, we have*

$$\sup_{y \in \mathbf{K}_2} \langle C(x), x - y \rangle > 0.$$

Then the variational inequality

$$\langle C(x), y - x \rangle \geq 0, \quad \forall y \in \mathbf{K}$$

admits solutions in $\mathbf{K}$.

In particular, if X is a real Hilbert space and the operator C is affine, the next result, due to Lions and Stampacchia (see [57]), holds.

Theorem 1.4.5. *Let X be a real Hilbert space, let $\mathbf{K}$ be a nonempty, convex and closed, subset of X and let $A : \mathbf{K} \to X'$ a Lipschitz and coercive operator (not necessarily linear), that is,*

$$\begin{aligned} \|Ax - Ay\|_* &\leq M\|x - y\|, \quad \forall x, y \in \mathbf{K}, \\ \langle Ax - Ay, x - y \rangle &\geq \nu \|x - y\|^2, \quad \forall x, y \in \mathbf{K}, \end{aligned}$$

for some constant $M, \nu > 0$. Then for each $B \in X'$, there exists a unique solution to the variational inequality

$$x \in \mathbf{K} : \ \langle Au + B, y - x \rangle \geq 0, \quad \forall y \in \mathbf{K}.$$

Moreover, the (nonlinear) solution mapping is Lipsichitz continuous, that is, if $x_1, x_2 \in \mathbf{K}$ are the solutions to the variational inequalities related to two different free terms $B_1, B_2 \in X'$, it results

$$\|x_1 - x_2\| \leq \frac{1}{\nu} \|B_1 - B_2\|_*. \tag{1.4.4}$$

1.5 Finite dimensional quasi-variational inequalities

Let us introduce finite dimensional quasi-variational inequalities.

Definition 1.5.1. Let D be a nonempty subset of $\mathbb{R}^m$, let $C : D \to \mathbb{R}^m$ and $\mathbf{K} : D \to 2^D$ be a function and a multifunction, respectively. The *quasi-variational inequality* is the problem to find a vector $x \in \mathbf{K}(x)$ such that

$$\langle C(x), y - x \rangle \geq 0, \quad \forall y \in \mathbf{K}(x). \tag{1.5.1}$$

Let us give some theorems concerning the existence of solutions to finite dimensional quasi-variational inequalities.

Theorem 1.5.1. ([48]) *Let D be a compact and convex set. Let C and $\mathbf{K}$ be a function and a multifunction, respectively, and, for all $x \in D$, let $\mathbf{K}(x)$ be a nonempty, closed and convex subset of $\mathbb{R}^m_+$. Then quasi-variational inequality* (1.5.1) *admits a solution.*

Theorem 1.5.2. ([42]) *Let D be a compact and convex set. Let $\mathbf{K}$ be a continuous multifunction such that, for all $x \in D$, $\mathbf{K}(x)$ is a nonempty, closed and convex subset of $\mathbb{R}^m_+$ and let C satisfy the condition*

$$\{x \in X : \ C(x)y \leq 0\} \text{ is closed } \forall y \in D - D.$$

Then quasi-variational inequality (1.5.1) *admits a solution.*

Theorem 1.5.3. ([38]) *Let D be a compact and convex set. Let $\mathbf{K}$ be a continuous multifunction such that, for all $x \in D$, $\mathbf{K}(x)$ is a nonempty, closed and convex subset of $\mathbb{R}^m_+$. Let $C : D \to 2^{\mathbb{R}^m_+}$ be a set-valued map (possibly discontinuous) such that:*

$$\forall y \in D - D \text{ the set } G_y = \left\{x \in D : \inf_{z \in C(x)} zy \leq 0\right\} \text{ is closed.}$$

Then, there exist $x \in \mathbf{K}(x) \cap D$ and $z \in C(y)$ such that $z(y - x) \geq 0$, for all $y \in \mathbf{K}(x) \cap D$.

1.6 Infinite dimensional quasi-variational inequalities

We may present problem (1.5.1) in an infinite dimensional setting by replacing $\mathbb{R}^m$ with a real topological vector space X and assuming that C is a operator from D to X', where X' is the topological dual of X.

In the following, we recall some results for the existence of solutions to the quasi-variational inequality in infinite dimensional spaces.

Theorem 1.6.1. ([97]) *Let X be a topological linear locally convex Hausdorff space and let $D \subset X$ be a convex, compact and nonempty subset. Let $C : D \to 2^{X'}$ be an u.s.c. multifunction with $C(y)$, $y \in C$, convex, compact and nonempty and let $\mathbf{K} : D \to 2^D$ be a closed l.s.c. set-valued mapping with $\mathbf{K}(y)$, $y \in D$, convex, compact and nonempty and let $\varphi : D \to \mathbb{R}$ a convex l.s.c. function. Then, there exists $x \in C(x)$ such that:*

1. *$x \in \mathbf{K}(x)$,*

2. *there exists* $y^* \in C(x)$ *for which*

$$\langle y - x, y^* \rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in \mathbf{K}(x).$$

The following theorem relaxes the hypothesis of compactness of the set D requiring the coercivity of the operator.

Theorem 1.6.2. ([98]) *Let D be a convex subset in a locally convex Hausdorff topological vector space X. Let us suppose that*

(i) $\mathbf{K} : D \to 2^D$ *is a closed l.s.c. correspondence with closed, convex and nonempty values,*

(ii) $C : D \to 2^{X'}$ *is a monotone, finite continuous and bounded single-valued map,*

(iii) there exist a compact, convex and nonempty set $Z \subset D$ and a nonempty subset $B \subset Z$ such that

(iii.a) $\mathbf{K}(B) \subset Z$*;*

(iii.b) $\mathbf{K}(z) \cap Z \neq \emptyset$*, for all $z \in Z$;*

(iii.c) for every $z \in Z \setminus B$ there exists $\widehat{z} \in \mathbf{K}(z) \cap Z$ with $\langle C(z), \widehat{z} - z \rangle < 0$.

Then there exists x such that

$$x \in \mathbf{K}(x) : \ \langle C(x), y - x \rangle \geq 0, \quad \forall y \in \mathbf{K}(x).$$

2

The equilibrium problems

2.1 Historical development

The study of the efficient operation on transportation networks dates to ancient Rome with a classical example being the publicly provided Roman road network and the time of day chariot policy, whereby chariots were banned from the ancient city of Rome at particular times of day.

But, the first authors who studied mathematically the traffic problem was Pigou in 1920 (see [82]). He had the idea of imposing some tolls on a network with only two paths in order to regulate the congestion. But it was only during most recent decades that traffic equilibrium problem have attracted the attention of several researches. In 1952, Wardrop (see [102]) laid the formulations for the study of traffic theory, starting two principles until now named after him:

First principle: *the journey times of paths are equal and less than those which would be experienced by a single vehicle on any unused path.*

Second principle: *the average journey time is minimal.*

In terms of travel costs, the first principle establishes that users seek to determine their minimal travel costs and, as consequence, costs on used paths for each origin-destination pair are equalized and minimal. Whereas the second principle states that the travel cost in the whole network is minimal. Only some years later, in 1956, the rigorous mathematical programming formulation of Wardrop's principles was elaborated. It was work of Beckmann, McGuire and Winsten (see [13]), who showed the equivalence between the traffic equilibrium conditions, as stated by Wardrop and the Kuhn-Tucker conditions of an appropriate optimization problem under some symmetry assumptions.

In 1969 Dafermos and Sparrow (see [27]), recognizing two different approaches suggested by Wardrop, coined the terms *user-optimized* and *system-optimized* transportation networks in order to distinguish between two distinct situations: the first one in which users act unilaterally, in their own self-interest, in selecting their routes; the second one in which users select routes according to what is optimal from a societal point of view, in that the total cost in the system are minimized.

Only later in 1979, it was proved by M.J. Smith (see [90]) that the traffic network equilibrium problem can be formulated in terms of a finite-dimensional variational inequality. This fact have render possible to study the problem with the powerful tool of variational inequalities, and to obtain existence, uniqueness, stability of traffic equilibria and to compute solutions.

To the end of the nineties of the 20^{th} century the last, in order of time, event: the traffic network equilibrium problem with feasible path flows which have to satisfy time-dependent capacity constraints and demands has been formulated by P. Daniele, A. Maugeri and W. Oettli (see [36] and [37]) and also by T.L. Friesz, D. Bernstein, T.E. Smith, R.L. Tobin and B.W. Wie (see [45]), as an evolutionary variational inequality, for which existence theorems and computational procedures are studied. In particular, they considered traffic networks in which the demand varied over the time horizon as well as the capacities on the flows on the paths connecting the origins to the destinations. The results therein demonstrated how traffic network equilibria evolve in the presence of such variations.

Subsequently, F. Raciti (see [83]) applied the results of [37] to construct a concrete numerical traffic network example in which the demand was a function of time and the equilibrium at each time instant could be computed exactly and in closed form. L. Scrimali (see [87]) developed an elastic-demand time-dependent traffic network model with delays and formulated the equilibrium conditions as a quasi-variational inequality problem. Then, she established existence results and also provided a numerical example.

P. Daniele and A. Maugeri (see [34]) developed a time-dependent spatial equilibrium model (price formulation) in which there were imposed bounds over time on the supply market prices, the demand market prices, and the commodity shipments between the supply and demand market pairs. Moreover, they presented existence results.

Static spatial price equilibrium problems of this form had been studied by numerous researchers (see [72] and the references therein) as well as through (as noted above) using projected dynamical systems; see also A. Nagurney and D. Zhang (see [77]). The contribution of P. Daniele and A. Maugeri (see [34]) allowed for the price and commodity shipment bounds to vary over time. Furthermore, the solution of the formulated evolutionary variational inequality traces the curve(s) of the resulting equilibrium price and commodity shipment patterns.

Then, P. Daniele (see [31]) addressed the time-dependent spatial price equilibrium problem in which the variables were commodity shipments. Not only did she provide existence results, but also she performed stability analysis of the model based on S. Dafermos and A. Nagurney (see [28]); see also A. Nagurney (see [72]).

In terms of evolutionary variational inequalities and financial equilibria, P. Daniele (see [30]) introduced a time-dependent financial network model consisting of multiple sectors, each of which seeks to determine its optimal portfolio given time-dependent supplies of the financial holdings. The work was motivated, in part, by the contributions of A. Nagurney and S. Siokos (see [76]) in the modeling of static and dynamic general financial equilibrium problems using, respectively, finite-dimensional variational inequality theory and projected dynamical system theory.

2.2 The traffic equilibrium problem

This section describes a variety of traffic network equilibrium models, and provides the variational inequality formulations of the governing equilibrium conditions.

In particular, it will be considered the static case and the dynamic case with travel demands fixed, and then it will be introduced the elastic case which arose whenever travel demands are not fixed but dependent on the equilibrium distribution.

2.2.1 Static model

For what we need in the sequel, we premise with a presentation of a general version of the traffic equilibrium problem in the static case, considering a model with capacity constraints on the flows. To this aim, let us consider a general traffic network is represented by a graph $G = [N, L]$, where N is the set of nodes (i.e. crossroads, airports, railway stations) and L is the set of directed links between the nodes (stretches of streets). Let a denote a link of the network connecting a pair of nodes and let r be a path consisting of a sequence of links which connect an Origin-Destination (O/D) pair of nodes. In the network there are n links and m paths. Let W denote the set of O/D pairs with typical O/D pair w_j, $|W| = l$ and $m > l$. The set of paths connecting the O/D pair w_j is represented by $\mathcal{R}_j$ and the entire set of paths in the network by $\mathcal{R}$. Let f_a be the flow on link a and let F_r be the non-negative flow on path r. Let $f = (f_1, \dots, f_a, \dots, f_n)^T$ denote the link flow vector and $F = (F_1, \dots, F_r, \dots, F_m)^T$ the path flow vector. The relationship between link and path flows is given by:

$$f_a = \sum_{r=1}^{m} \delta_{ar} F_r \quad \text{or} \quad f = \Delta F,$$

where Δ is the link-path incidence matrix, whose typical entry δ_{ar} is 1 if the link a is contained in the path r and 0 otherwise.

Let $\lambda, \mu \in \mathbb{R}^m_+$ denote the capacity constraints, and let us assume that the feasible flows have to satisfy some capacity restrictions

$$\lambda_r \leq F_r \leq \mu_r, \quad \text{for } r = 1, 2, \dots, m.$$

Let ρ_j represent the travel demand associated with the users travelling between O/D pair w_j and let $\rho = (\rho_1, \dots, \rho_j, \dots, \rho_l)^T$ be the total demand vector. The travel demand is the total number of movements from the origin to the destination. The relationship between path flows and travel demands is given by:

$$\sum_{r=1}^{m} \varphi_{jr} F_r = \rho_j \quad \text{or} \quad \Phi F = \rho,$$

where Φ is the O/D pairs-paths incidence matrix, whose typical entry φ_{jr} is 1 if path r connects the pair w_j and 0 otherwise. It is worth noting that each column of the matrix Φ has only one entry equal to 1, because otherwise the same path would connect different O/D pairs. The meaning of the conservation of the flows condition is that flows and hence travelers are not lost or generated in the network.

Let $c_a(f)$ denote the user travel cost associated with link a and group the link costs into the vector $c(f) = (c_1(f), \dots, c_a(f), \dots, c_n(f))^T$. In this paper, we are concerned with the general case, namely with the case of asymmetric costs, i.e. the cost on a link does not depend only on the flow on that link, but it is affected by the flows on all the links in the network.

Let $C_r(F)$ denote the user travel cost path r. It results:

$$C_r(F) = \sum_{a=1}^{n} \delta_{ar} c_a(f) \quad \text{or} \quad C(F) = \Delta^T c(f) = \Delta^T c(\Delta F),$$

where $C(F) = (C_1(F), \dots, C_r(F), \dots, C_m(F))^T$ is the path cost vector. The above relationship shows that the cost on a path is given by the sum of the costs on links which form the path.

Definition 2.2.1. The *set of feasible flows* is the set $\mathbf{K}$ of all the path flows in the network which satisfy the capacity constraints and the conservation law:

$$\mathbf{K} = \{F \in \mathbb{R}^m : \ \lambda \le F \le \mu, \quad \Phi F = \rho\}.$$

Analytically, the user-optimized equilibrium is expressed by the following definition due to A. Maugeri, W. Oettli and D. Schläger (see [64]) which is a generalization of Wardrop's principle (see [102]).

Definition 2.2.2. A flow $H \in \mathbf{K}$ is a *user traffic equilibrium flow* if and only if $\forall w_j \in W$ and $\forall q, s \in \mathcal{R}_j$ it results:

$$C_q(H) > C_s(H) \implies H_q = \lambda_q \text{ or } H_s = \mu_s.$$

Clearly the meaning of Definition 2.2.2 is that the users choose the less expensive routes and it perfectly agrees with the notion of user-optimization equilibrium. Moreover, Definition 2.2.2 is characterized by a variational inequality, by means of the following theorem (see [64], Theorem 1).

Theorem 2.2.1. *A flow $H \in \mathbf{K}$ is an equilibrium pattern if and only if it satisfies the following variational inequality:*

$$\langle C(H), F - H \rangle \geq 0, \quad \forall F \in \mathbf{K}.$$

Remark 2.2.1. If the symmetry condition $\dfrac{\partial c_a(f)}{\partial f_b} = \dfrac{\partial c_b(f)}{\partial f_a}$ holds, then the solution to the equilibrium problem (1.3.1) any be formulated as the solution of the following optimization problem:

$$\min \sum_{a \in L} \int_0^{f_a} c_a(x)dx,$$

subjected to

$$\begin{array}{rcl} \Phi F & = & \rho, \\ f & = & \Delta F, \\ F_p & \geq & 0, \ \forall p \in \mathcal{P}. \end{array}$$

In our approach, we assume that no symmetry holds. In fact, supposing that the above condition is verified would mean that the flow on link a affects the cost on the link b in the same way as the flow on link b affects the cost on link a. This is obviously an unrealistic requirement.

2.2.2 Dynamic model

We consider now the dynamic case. The traffic network, whose geometry remains fixed, is considered at all times $t \in [0, T]$.

For each time $t \in [0, T]$ let us assume that a route-flow vector $F(t) \in \mathbb{R}^m$. Let us suppose that the feasible flows have to satisfy time-dependent capacity constraints and demand requirements. Each component $F_r(t)$ of $F(t)$ gives the flow trajectory $F : [0, T] \to \mathbb{R}^m_+$ which has to satisfy almost everywhere on $[0, T]$ the capacity constraints

$$\lambda(t) \leq F(t) \leq \mu(t)$$

and the traffic conservation law

$$\Phi F(t) = \rho(t),$$

where the bounds $\lambda, \mu : [0, T] \to \mathbb{R}^m_+$ satisfying the next condition $\lambda < \mu$, and the demand $\rho : [0, T] \to \mathbb{R}^m_+$ are given and Φ is again the pair-route incident matrix.

We define the set of feasible flows

$$\mathbf{K} = \Big\{ F \in L^2([0, T], \mathbb{R}^m) : \quad \lambda(t) \leq F(t) \leq \mu(t), \quad \text{a.e. in } [0, T], \\ \Phi F(t) = \rho(t), \quad \text{a.e. in } [0, T] \Big\}. \tag{2.2.1}$$

We assume that λ and μ belong to $L^2([0,T],\mathbb{R}^m_+)$ and that ρ lies in $L^2([0,T],\mathbb{R}^m_+)$. Assuming in addition that

$$\Phi\lambda(t) \leq \rho(t) \leq \Phi\mu(t) \quad \text{a.e. in } [0,T],$$

we obtain that the set of feasible flows (2.2.1) is nonempty, as in [46]. It is easily seen that $\mathbf{K}$ is a convex, closed, bounded subset of $L^2([0,T],\mathbb{R}^m_+)$.

Furthermore, we give the cost trajectory C, which becomes function of the time $C : [0,T] \times \mathbb{R}^m_+ \to \mathbb{R}^m_+$.

The equilibrium condition is given by a generalized dynamic version of Wardrop's condition (see [36] and [37]), namely:

Definition 2.2.3. A flow $H \in \mathbf{K}$ is a *user traffic equilibrium flow* if $\forall w_j \in \mathcal{W}$, $\forall q, s \in \mathcal{R}_j$ and a.e. in $[0,T]$ it results:

$$C_q(t,H(t)) > C_s(t,H(t)) \implies H_q(t) = \lambda_q(t) \quad \textit{or} \quad H_s(t) = \mu_s(t). \tag{2.2.2}$$

The overall flow pattern obtained according with condition (2.2.2) fits very well in the framework of the theory of variational inequalities. In fact in [36] and [37] the following result is shown:

Theorem 2.2.2. *A flow $H \in \mathbf{K}$ is an equilibrium pattern if and only if it satisfies the following evolutionary variational inequality:*

$$\int_0^T \langle C(t,H(t)), F(t) - H(t)\rangle dt \geq 0, \quad \forall F \in \mathbf{K}. \tag{2.2.3}$$

It is worth observing that problem (2.2.3) (see [65]) is also equivalent to the following one:

Find $H \in \mathbf{K}$ such that

$$\langle C(t,H(t)), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T], \tag{2.2.4}$$

where

$$\mathbf{K}(t) = \Big\{F(t) \in \mathbb{R}^m : \ \lambda(t) \leq F(t) \leq \mu(t), \quad \Phi F(t) = \rho(t)\Big\}.$$

This remark is interesting because we can apply to (2.2.4), among the others, the direct method (see [29], [35] and [61]) in order to find solutions to the variational inequality (2.2.3).

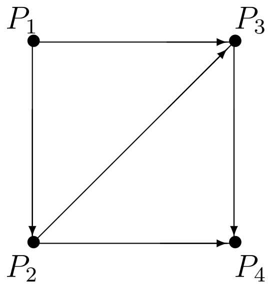

Figure 2.1: Network model for Example 2.2.1.

Example 2.2.1. Let us consider a network as Figure 2.1. The network consists of four nodes, P_1, P_2, P_3, P_4, and five links, $(P_1, P_2), (P_1, P_3), (P_2, P_3), (P_2, P_4), (P_3, P_4)$.

The origin-destination pair is $w = (P_1, P_4)$, which is connected by the paths

$$\begin{aligned} R_1 &= (P_1, P_3) \cup (P_3, P_4), \\ R_2 &= (P_1, P_2) \cup (P_2, P_4), \\ R_3 &= (P_1, P_2) \cup (P_2, P_3) \cup (P_3, P_4). \end{aligned}$$

We consider the cost operator on the path C defined by

$$C : L^2([0,2], \mathbb{R}^3_+) \to L^2([0,2], \mathbb{R}^3_+);$$

$$\begin{aligned} C_1(H(t)) &= \alpha H_1(t) + \beta H_2(t) + \gamma, \\ C_2(H(t)) &= \alpha H_1(t) + H_2(t), \\ C_3(H(t)) &= \alpha H_2(t) + H_3(t) + \sigma, \end{aligned}$$

where $\alpha > 2$, $0 \leq \beta < 1$, $\gamma, \delta, \sigma \geq 0$.

The set of feasible flows is given by

$$\mathbf{K} = \Big\{ F \in L^2([0,T], \mathbb{R}^3) : \quad F_1(t), F_2(t), F_3(t) \geq 0, \quad \text{a.e. in } [0,T] \\ F_1(t) + F_2(t) + F_3(t) = \varepsilon t + \zeta, \quad \text{a.e. in } [0,T] \Big\},$$

where $\varepsilon, \zeta \geq 0$.

The evolutionary variational inequality expressing the equilibrium problem has the following formulation

$$H \in \mathbf{K} : \sum_{p=1}^{3} C_p(H(t))(F_p(t) - H_p(t)) \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T]. \quad (2.2.5)$$

To compute the solution, we apply the direct method (see [29], [35] and [61]). Deducing $F_3(t)$ from $F_1(t) + F_2(t) + F_3(t) = \varepsilon t + \zeta$, the set of feasible flows becomes

$$\widetilde{\mathbf{K}} = \Big\{ \widetilde{F} \in L^2([0,T], \mathbb{R}^2 : \ F_1(t), F_2(t) \geq 0, \ F_1(t) + F_2(t) \leq \varepsilon t + \zeta, \quad \text{a.e. in } [0,T] \Big\}.$$

Let us consider:

$$\begin{aligned}\Gamma_1(\widetilde{F}(t)) &= C_1(\widetilde{F}(t)) - C_3(\widetilde{F}(t)) \\ &= (\alpha+1)F_1(t) + (1-\alpha+\beta)F_2(t) - \varepsilon t + \gamma - \sigma - \zeta, \\ \Gamma_2(\widetilde{F}(t)) &= C_2(\widetilde{F}(t)) - C_3(\widetilde{F}(t)) \\ &= (\alpha+1)F_1(t) + (2-\alpha)F_2(t) - \varepsilon t - \sigma - \zeta,\end{aligned}$$

then, evolutionary variational inequality (2.2.5) becomes

$$\widetilde{H} \in \widetilde{\mathbf{K}}: \sum_{p=1}^{2} \Gamma_p(\widetilde{H}(t))(\widetilde{F}_p(t) - \widetilde{H}_p(t)) \geq 0, \quad \forall \widetilde{F}(t) \in \widetilde{\mathbf{K}}(t), \quad \text{a.e. in } [0,T].$$

We recall that if $\widetilde{H}$ satisfies the following system:

$$\begin{cases} \Gamma_1(\widetilde{H}(t), \widetilde{H}(t)) = 0 \\ \Gamma_2(\widetilde{H}(t), \widetilde{H}(t)) = 0 \\ \widetilde{H} \in \widetilde{\mathbf{K}}(\widetilde{H}) \end{cases}$$

then it solves variational inequality 2.2.5. We find that:

$$\begin{aligned} H_1(t) &= \frac{\varepsilon t + \zeta + \sigma}{\alpha+1} + \frac{\gamma(\alpha-2)}{(\alpha+1)(1-\beta)}, \\ H_2(t) &= \frac{\gamma}{1-\beta}, \\ H_3(t) &= \frac{\alpha(\varepsilon t + \zeta) - \sigma}{\alpha+1} + \frac{\gamma(1-2\alpha)}{(\alpha+1)(1-\beta)}. \end{aligned}$$

2.2.3 Elastic model

In the previous section, we have dealt with traffic equilibrium models with the fixed travel demands, but this kind of approach corresponds to study only a first approximation of the problem. In fact, it is clear that the travel demands is influenced by the evaluation of the amount of traffic flows on the paths, namely by the forecasted equilibrium solutions. For this reason some authors (see, for instance [39], [42] and [78]), by means of different approaches, have interested in the so-called elastic model, in which travel demands associated with the users traveling between the O/D pairs depend on the equilibrium distribution.

Under this prospective, the set of feasible flows becomes as follows. Let D be a nonempty, compact and convex subset of $\mathbb{R}^m_+$. Let us consider the multifunction $\mathbf{K} : D \to 2^{\mathbb{R}^m_+}$ defined by

$$\mathbf{K}(H) = \{F \in D : \ \lambda \leq F \leq \mu, \ \ \Phi F = \rho(H)\},$$

with $\lambda, \mu \in \mathbb{R}^m_+$ and $\rho : D \to \mathbb{R}^l_+$, then the set of feasible flows is the set-value $\mathbf{K}(H)$ of the multifunction $\mathbf{K}$.

In this case the elastic equilibrium condition is given by the next way (see [42]).

Definition 2.2.4. A flow $H \in \mathbf{K}(H)$ is a *user traffic equilibrium flow* if $\forall w_j \in W$ and $\forall q, s \in \mathcal{R}_j$ it results:

$$C_q(H) > C_s(H) \implies H_q = \lambda_q \quad or \quad H_s = \mu_s.$$

The following theorem (see [42]) establishes a complete characterization of the elastic equilibrium flow by means of the variational formulation.

Theorem 2.2.3. *A flow $H \in \mathbf{K}(H)$ is an equilibrium pattern in the sense of Definition* 2.2.5 *if and only if it satisfies the following quasi-variational inequality:*

$$\langle C(H), F - H \rangle \geq 0, \quad \forall F \in \mathbf{K}(H). \tag{2.2.6}$$

2.2.4 Dynamic elastic model

Now, let us introduce the dynamic elastic traffic equilibrium problem and let us assume that the travel demand ρ depends on the equilibrium solutions $H(t)$ in the average sense with respect to the time, namely

$$\rho(H) = \frac{1}{T} \int_0^T \rho(t, H(\tau)) d\tau.$$

(see [34] and [87]). In fact, travel demands are supposed to depend on the user's evaluation of the flows. So one can expect that travelers evaluate the network practicability not instant, but by an average with respect to the whole time interval.

Also in this case, let $\lambda, \mu \in L^2([0,T], \mathbb{R}^m_+)$ be the capacity constraints such that $\lambda(\cdot) < \mu(\cdot)$ and let $\rho \in L^2([0,T] \times \mathbb{R}^m_+, \mathbb{R}^l_+)$ be the elastic travel demand function. Let $D \subseteq L^2([0,T], \mathbb{R}^m_+)$ be a nonempty, compact and convex subset and let $\mathbf{K} : D \to 2^{L^2([0,T],\mathbb{R}^m_+)}$ be a multifunction defined by

$$\mathbf{K}(H) = \Big\{ F \in L^2([0,T], \mathbb{R}^m) : \quad \lambda(t) \leq F(t) \leq \mu(t) \quad \text{a.e. in } [0,T],$$
$$\Phi F(t) = \frac{1}{T} \int_0^T \rho(t, H(\tau)) d\tau \quad \text{a.e. in } [0,T] \Big\},$$

the set of feasible flows is the set-value $\mathbf{K}(H)$ of the multifunction $\mathbf{K}$.

In this case the elastic equilibrium condition is given by the next way (see [34]).

Definition 2.2.5. A flow $H \in \mathbf{K}(H)$ is a *user traffic equilibrium flow* if $\forall w_j \in W$ and $\forall q, s \in \mathcal{R}_j$ it results:

$$C_q(t, H(t)) > C_s(t, H(t)) \implies H_q(t) = \lambda_q(t) \quad or \quad H_s(t) = \mu_s(t).$$

The following theorem (see [34]) establishes a complete characterization of the elastic equilibrium flow by means of the variational formulation.

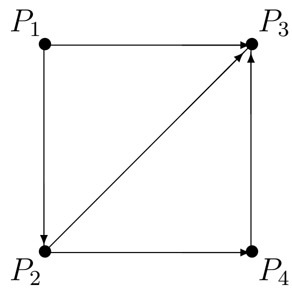

Figure 2.2: Network model for Example 2.2.2.

Theorem 2.2.4. *A flow $H \in \mathbf{K}(H)$ is an equilibrium pattern in the sense of Definition* 2.2.5 *if and only if it satisfies the following quasi-variational inequality:*

$$\int_0^T \langle C(t, H(t)), F(t) - H(t) \rangle dx \geq 0, \quad \forall F \in \mathbf{K}(H). \tag{2.2.7}$$

It is worth observing that problem (2.2.7) (see [65]) is also equivalent to the following one:

Find $H \in \mathbf{K}$ such that

$$\langle C(t, H(t)), F(t) - H(t) \rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t, H), \text{ a.e. in } [0, T]. \tag{2.2.8}$$

where

$$\mathbf{K}(t, H) = \left\{ F(t) \in \mathbb{R}^m : \ \lambda(t) \leq F(t) \leq \mu(t), \ \ \Phi F(t) = \frac{1}{T} \int_0^T \rho(t, H(\tau)) d\tau \right\}.$$

This remark is very important because we can apply to (2.2.4), among the others, the direct method (see [61], [38] [40] and [41]) in order to find solutions to the variational inequality (2.2.3).

Example 2.2.2. Let us consider the network as in Figure 2.2, where $N = \{P_1, P_2, P_3, P_4\}$ is the set of nodes and $L = \{(P_1, P_2), (P_1, P_3), (P_2, P_3), (P_2, P_4), (P_4, P_3)\}$ is the set of links.

The origin-destination pair is represented by $w = (P_1, P_3)$, so that the paths are the following:

$$\begin{aligned} R_1 &= (P_1, P_3), \\ R_2 &= (P_1, P_2) \cup (P_2, P_3), \\ R_3 &= (P_1, P_2) \cup (P_2, P_4) \cup (P_4, P_3). \end{aligned}$$

Let us assume that the path costs are the following:

$$\begin{aligned} C_1(H(t)) &= \alpha H_1(t) + \beta, \\ C_2(H(t)) &= \alpha H_2(t) + \gamma, \\ C_3(H(t)) &= \alpha H_2(t) + \alpha H_3(t) + \delta, \end{aligned}$$

where $\alpha, \beta, \gamma, \delta \geq 0$.

The set of feasible flows is given by:

$$\mathbf{K}(H) = \Bigg\{ F \in L^2([0,T], \mathbb{R}^3) : \; F_1(t), F_2(t), F_3(t) \geq 0 \quad \text{a.e. in } [0,T],$$

$$F_1(t) + F_2(t) + F_3(t) = \frac{1}{T} \int_0^T (\varepsilon t + \zeta H_1(\tau)) d\tau \quad \text{a.e. in } [0,T] \Bigg\},$$

where $\varepsilon \geq 0$, $\zeta \in [0, 2[$.

The equilibrium flow is the solution of the evolutionary quasi-variational inequality:

$$H \in \mathbf{K}(H) : \sum_{p=1}^{3} C_p(H(t))(F_p(t) - H_p(t)) \geq 0, \quad \forall F \in \mathbf{K}(H), \text{ a.e. in } [0,T]. \tag{2.2.9}$$

Following the procedure shown in [61], [38] [40] and [41], we have:

$$F_3(t) = \frac{1}{T} \int_0^T (\varepsilon t + \zeta H_1(\tau)) d\tau - F_1(t) - F_2(t);$$

$$\widetilde{\mathbf{K}}(H) = \Bigg\{ \widetilde{F} \in L^2([0,T], \mathbb{R}^2) : \; F_1(t), F_2(t) \geq 0, \quad \text{a.e. in } [0,T],$$

$$F_1(t) + F_2(t) \leq \frac{1}{T} \int_0^T (\varepsilon t + \zeta H_1(\tau)) d\tau \quad \text{a.e. in } [0,T] \Bigg\}.$$

Let us consider:

$$\begin{aligned} \Gamma_1(\widetilde{F}(t), \widetilde{H}(t)) &= C_1(\widetilde{F}(t), \widetilde{H}(t)) - C_3(\widetilde{F}(t), \widetilde{H}(t)) \\ &= 2\alpha F_1(t) - \frac{\alpha}{T} \int_0^T (\varepsilon t + \zeta H_1(\tau)) d\tau + \beta - \delta, \\ \Gamma_2(\widetilde{F}(t), \widetilde{H}(t)) &= C_2(\widetilde{F}(t), \widetilde{H}(t)) - C_3(\widetilde{F}(t), \widetilde{H}(t)) \\ &= \alpha F_1(t) + \alpha F_2(t) - \frac{\alpha}{T} \int_0^T (\varepsilon t + \zeta H_1(\tau)) d\tau + \gamma - \delta. \end{aligned}$$

Thus, the quasi-variational inequality problem 2.2.9 may be written as:

$$\widetilde{H} \in \widetilde{\mathbf{K}}(\widetilde{H}) : \sum_{p=1}^{2} \Gamma_p(\widetilde{H}(t))(\widetilde{F}_p(t) - \widetilde{H}_p(t)) \geq 0, \quad \forall \widetilde{F} \in \widetilde{\mathbf{K}}(\widetilde{H}), \text{ a.e. in } [0,T]. \tag{2.2.10}$$

It is immediate to show that if $\widetilde{H}$ satisfies the following system:

$$\begin{cases} \Gamma_1(\widetilde{H}(t), \widetilde{H}(t)) = 0 \\ \Gamma_2(\widetilde{H}(t), \widetilde{H}(t)) = 0 \\ \widetilde{H} \in \widetilde{\mathbf{K}}(\widetilde{H}) \end{cases}$$

then it solves the quasi-variational inequality 2.2.9. We find that:

$$\int_0^T H_1(\tau)d\tau = \frac{T}{2\alpha} \frac{\alpha\varepsilon T - 2\beta + 2\gamma}{2 - \zeta},$$

and

$$\begin{aligned} H_1(t) &= \frac{\varepsilon}{2}t + \frac{\zeta}{4\alpha}\frac{\alpha\varepsilon T - 2\beta + 2\gamma}{2 - \zeta} - \frac{\beta - \gamma}{2\alpha}, \\ H_2(t) &= \frac{\varepsilon}{2}t + \frac{\zeta}{4\alpha}\frac{\alpha\varepsilon T - 2\beta + 2\gamma}{2 - \zeta} + \frac{\beta + \delta - \gamma}{2\alpha}, \end{aligned}$$

under condition that:

$$H_1(t) + H_2(t) \leq \frac{1}{T}\int_0^T (\varepsilon t + \zeta H_1(\tau))d\tau.$$

From which, we obtain

$$H_3(t) = \frac{\gamma - \delta}{\alpha},$$

2.3 Common formulation of dynamic equilibrium problems

We provide here a novel unified definition of the constraint set $\mathbf{K}$, proposed in [36] and [30], for the evolutionary variational inequality arising in time-dependent traffic network problems, spatial equilibrium problems with either quantity or price formulations, and a variety of financial equilibrium problems.

We consider a nonempty, convex, closed, bounded subset of the reflexive Banach space $L^2([0,T],\mathbb{R}^q)$ given by

$$\mathbf{K} = \Bigg\{ u \in L^2([0,T],\mathbb{R}^q) : \quad \lambda(t) \le u(t) \le \mu(t), \quad \text{a.e. in } [0,T]; \\ \sum_{i=1}^{q} \xi_{ji} u_i(t) = \rho_j(t), \quad \text{a.e. in } [0,T], \tag{2.3.1} \\ \xi_{ji} \in \{-1,0,1\},\ i \in \{1,\dots,q\},\ j \in \{1,\dots,l\} \Bigg\}.$$

Let $\lambda, \mu \in L^2([0,T],\mathbb{R}^q)$ and $\rho \in L^2([0,T],\mathbb{R}^l)$ be convex functions in the above definition. For chosen values of the scalars ξ_{ji} , of the dimension q and l, and of the constraints λ and μ, we obtain each of the formulations of the constraint sets in the models cited above (see [25] and also [32]) as follows:

- for the traffic network problem (see [36], [37] and [32]), let $q = m$ $\xi_{ji} \in \{0,1\}$, $i \in \{1,2,\dots,q\}$, $j \in \{1,2,\dots,l\}$, and $\lambda(t) \ge 0$ for a.e. $t \in [0,T]$;
- for the quantity formulation of spatial price equilibrium (see [31] and [32]), let $q = n+m+nm$, $l = n+m$, $\xi_{ji} \in \{-1,0,1\}$, $i \in \{1,2,\dots,q\}$, $j \in \{1,2,\dots,l\}$, $\mu(t)$ large and $\lambda(t) = 0$ for a.e. $t \in [0,T]$;
- for the price formulation of spatial price equilibrium (see [34], [29] and [32]), let $q = n+m+nm$, $l = 1$, $\xi_{ji} = 0$, $i \in \{1,2,\dots,q\}$, $j \in \{1,2,\dots,l\}$, and $\lambda(t) \ge 0$ for a.e. $t \in [0,T]$;
- for the financial equilibrium problem (see [30] and [32]), let $q = 2nm + np$, $l = 2m$, $\xi_{ji} \in \{0,1\}$ for $i \in \{1,2,\dots,q\}$, $j \in \{1,2,\dots,l\}$, $\mu(t)$ large and $\lambda(t) = 0$ for a.e. $t \in [0,T]$.

Then, setting

$$\ll \phi, u \gg := \int_0^T \langle \phi(t), u(t) \rangle dt,$$

where $\phi \in (L^2([0,T],\mathbb{R}^q))^* = L^2([0,T],\mathbb{R}^q)$ and $u \in L^2([0,T],\mathbb{R}^q)$. If F is given such that $F : \mathbf{K} \to L^2([0,T],\mathbb{R}^q)$, we have the following standard form of the evolutionary variational inequality:

Find $u \in \mathbf{K}$ such that

$$\ll F(u), v-u\rangle \gg 0, \quad \forall v \in \mathbf{K}. \tag{2.3.2}$$

It was shown in [37] (see Theorem 5.1 and Corollary 5.1) that, if F satisfies either of the following conditions:

- F is hemicontinuous with respect to the strong topology on $\mathbf{K}$ and there exist $A \subseteq \mathbf{K}$ nonempty, compact, and $B \subseteq \mathbf{K}$ compact such that, for every $u \in \mathbf{K} \setminus A$, there exists $v \in B$ with $\langle F(u), v - u \rangle < 0$,
- F is hemicontinuous with respect to the weak topology on $\mathbf{K}$,
- F is pseudomonotone and hemicontinuous along line segments,

then the evolutionary variational inequality problem (2.3.2) admits a solution over the constraint set $\mathbf{K}$. In Stampacchia (see [95]), it is shown that, if in addition F is strictly monotone, then the solution to the evolutionary variational inequality is unique.

3

Theory of the existence

3.1 Introduction

An important aspect of theory of variational inequalities is the research of opportune conditions that imply the existence of solutions. More authors have obtained existence results for variational and quasi-variational inequalities (see for instance [56], [57], [2], [62], [39] and [37]).

The aim of this chapter is to present some existence results for evolutionary variational and quasi-variational inequalities which express dynamic traffic equilibrium problems. Then, we present analogous results that hold for general evolutionary variational and quasi-variational inequalities.

Moreover, we introduce a new class of evolutionary variational inequalities, that we call degenerate. For this class of evolutionary variational inequalities, we show existence and uniqueness results. An analogous definition is given for evolutionary quasi-variational inequalities. Also in this case, we prove existence results.

3.2 Existence results for evolutionary variational inequalities

Let us recall an existence result for the dynamic traffic equilibrium problem that can be obtained applying Theorems 1.4.2 and 1.4.3. Let $C : [0,T] \times \mathbb{R}^m_+ \to \mathbb{R}^m_+$ be the cost vector-function, let $\lambda, \mu : [0,T] \to \mathbb{R}^m_+$ be the capacity constraints and let $\rho : [0,T] \to \mathbb{R}^m_+$ be the travel demand. The evolutionary variational inequality that models the dynamic traffic equilibrium problem is

$$\langle C(t, H(t)), F(t) - H(t) \rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T], \tag{3.2.1}$$

where

$$\mathbf{K}(t) = \Big\{ F(t) \in \mathbb{R}^m : \ \lambda(t) \leq F(t) \leq \mu(t), \quad \Phi F(t) = \rho(t) \Big\}.$$

The following result holds (see [37]).

Theorem 3.2.1. *For* $\mathbf{K} \subseteq L^2([0,T],\mathbb{R}^m)$ *given by*

$$\mathbf{K} = \Big\{ F \in L^2([0,T],\mathbb{R}^m) : \ \lambda(t) \le F(t) \le \mu(t), \quad \Phi F(t) = \rho(t), \quad a.e. \ in \ [0,T] \Big\}.$$

and $C : [0,T] \times \mathbf{K} \to L^2([0,T],\mathbb{R}^m)$ *each of the following conditions is sufficient for the existence of a solution to evolutionary variational inequality* (3.2.1)*:*

(i) C *is hemicontinuous with respect to the strong topology on* $\mathbf{K}$*, and there exist* $A \subseteq \mathbf{K}$ *nonempty, compact, and* $B \subseteq \mathbf{K}$ *compact, convex such that, for every* $H \in \mathbf{K} \setminus A$*, there exists* $F \in B$ *with*

$$\langle C(t,H(t)), F(t) - H(t) \rangle < 0, \quad a.e. \ in \ [0,T];$$

(ii) C *is hemicontinuous with respect to the weak topology on* $\mathbf{K}$*;*

(iii) C *is pseudomonotone and hemicontinuous along line segments.*

From Theorem 3.2.1 it is possible to derive the following existence theorem, which gives a sufficient condition in terms of the operator $C(t,F)$ (see [63]).

Theorem 3.2.2. *Let* $C(t,F) : [0,T] \times \mathbb{R}^m_+ \to \mathbb{R}^m_+$ *be a vector-function measurable in* t*, continuous in* F *and such that*

$$\|C(t,F)\|_m \le A(t)\|F\|_m + B(t), \quad \forall F \in \mathbb{R}^m_+, \ a.e. \ in \ [0,T],$$

with $B \in L^2([0,T])$ *and* $A \in L^\infty([0,T])$*, and it results*

$$\langle C(t,H) - C(t,F), H - F \rangle \ge 0, \quad \forall H, F \in \mathbb{R}^m_+, \ a.e. \ in \ [0,T].$$

Let $\lambda, \mu \in L^2([0,T],\mathbb{R}^m_+)$ *and let* $\rho \in L^2([0,T],\mathbb{R}^l_+)$ *be vector-functions. Then, evolutionary variational inequality* (3.2.1) *admits some solutions.*

It is well known that if C is in addition strictly monotone, then the solution to the evolutionary variational inequality is unique.

More general existence result is given by the following theorem:

Theorem 3.2.3. *Let* $C(t,F) : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ *be a vector-function measurable in* t*, continuous in* F *and such that*

$$\|C(t,F)\|_m \le A(t)\|F\|_m + B(t), \quad \forall F \in \mathbb{R}^m, \ a.e. \ in \ [0,T],$$

with $B \in L^2([0,T])$ *and* $A \in L^\infty([0,T])$*, and it results*

$$\langle C(t,H) - C(t,F), H - F \rangle \ge 0, \quad \forall H, F \in \mathbb{R}^m, \ a.e. \ in \ [0,T].$$

Let $\mathbf{K} \subseteq L^2([0,T],\mathbb{R}^m)$ *be a nonempty, convex and closed set. Then, evolutionary variational inequality*

$$\langle C(t,H(t)), F(t) - H(t) \rangle \ge 0, \quad \forall F(t) \in \mathbf{K}(t), \ a.e. \ in \ [0,T], \tag{3.2.2}$$

admits some solutions.

For reader's convenience we report the proof of a type Minty's Lemma.

Lemma 3.2.1. *Let $C(t,F) : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ be a vector-function measurable in t, continuous in F and such that*

$$\|C(t,F)\|_m \leq A(t)\|F\|_m + B(t), \quad \forall F \in \mathbb{R}^m, \text{ a.e. in } [0,T],$$

with $B \in L^2([0,T],\mathbb{R}_+)$ and $A \in L^\infty([0,T],\mathbb{R}_+)$, and it results

$$\langle C(t,H) - C(t,F), H - F\rangle \geq 0, \quad \forall H,F \in \mathbb{R}^m, \text{ a.e. in } [0,T].$$

Let $\mathbf{K} \subseteq L^2([0,T],\mathbb{R}^m)$ be a nonempty, convex and closed set. Then, the time-dependent variational inequality (3.2.2) is equivalent to

$$\langle C(t,F(t)), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \quad \text{a.e. in } [0,T]. \tag{3.2.3}$$

Proof. The existence of solutions to (3.2.2) is ensured by Theorem 3.2.3.

Moreover, the monotonicity of C and (3.2.2) imply for any $F(t) \in \mathbf{K}(t)$

$$\begin{aligned}\langle C(t,F(t)), F(t) - H(t)\rangle &= \langle C(t,F(t)) - C(t,H(t)), F(t) - H(t)\rangle \\ &\quad + \langle C(t,H(t)), F(t) - H(t)\rangle \geq 0, \quad \text{a.e. in } [0,T].\end{aligned}$$

Conversely, taking in (3.2.3)

$$F(t) = H(t) + \theta(G(t) - H(t)) \in \mathbf{K}(t),$$

for arbitrary $\theta \in]0,1]$ and $G(t) \in \mathbf{K}(t)$ a.e. in $[0,T]$, it results

$$\langle C(t, H(t) + \theta(G(t) - H(t))), G(t) - H(t)\rangle \geq 0, \quad \text{a.e. in } [0,T],$$

and letting $\theta \to 0$ we obtain (3.2.3) by the continuity of C with respect to the second variable. □

Now, we assume that the cost $C(t,F)$ is a linear operator with respect to flows, namely it results

$$C(t,F(t)) = A(t)F(t) + B(t),$$

for each $t \in [0,T]$, where $A : [0,T] \to \mathbb{R}^{m\times m}$ and $B : [0,T] \to \mathbb{R}^m$ are two functions. In this case, the following result holds.

Theorem 3.2.4. *Let $A : [0,T] \to \mathbb{R}^{m\times m}$ be a bounded nonnegative definite matrix-function, that is,*

$$\exists M > 0 : \ \|A(t)\|_{m\times m} \leq M, \quad \text{a.e. in } [0,T], \tag{3.2.4}$$

$$\langle A(t)F, F\rangle \geq 0, \quad \forall F \in \mathbb{R}^m, \text{ a.e. in } [0,T], \tag{3.2.5}$$

and let $B \in L^2([0,T],\mathbb{R}^m)$. Let $\mathbf{K} \subseteq L^2([0,T],\mathbb{R}^m)$ be a nonempty, convex and closed set. Then, there exists some solutions to the evolutionary variational inequality

$$\langle A(t)H(t) + B(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T]. \tag{3.2.6}$$

Moreover, we remark that the set $\mathbf{X}$ of solutions is a closed convex subset of $\mathbf{K}$, for Theorem 3.1 in [57].

It is easy to see that if A is a bounded positive definite matrix-function, that is,

$$\exists M > 0: \ \|A(t)\|_{m\times m} = \Big(\sum_{r,s=1}^{m} A_{rs}^2(t)\Big)^{\frac{1}{2}} \leq M, \quad \text{a.e. in } [0,T], \tag{3.2.7}$$

$$\exists \nu > 0: \ \langle A(t)F, F\rangle \geq \nu \|F\|_m^2, \quad \forall F \in \mathbb{R}^m, \quad \text{a.e. in } [0,T], \tag{3.2.8}$$

and $B \in L^2([0,T],\mathbb{R}^m)$, then there exists a unique solution to evolutionary variational inequality (3.2.6).

Moreover, if $H_1, H_2 \in L^2([0,T],\mathbb{R}^m)$ are the solutions to the evolutionary variational inequalities related to two different free terms $B_1, B_2 \in L^2([0,T],\mathbb{R}^m)$, it results

$$\|H_1 - H_2\|_{L^2([0,T],\mathbb{R}^m)} \leq \frac{1}{\nu}\|B_1 - B_2\|_{L^2([0,T],\mathbb{R}^m)}. \tag{3.2.9}$$

The proof of these facts can be found in [86].

3.3 Existence results for degenerate evolutionary variational inequalities

In this section we introduce a new type of evolutionary variational inequalities. More precisely, we suppose that the operator C satisfies a more general condition than the strongly monotonicity condition, more precisely, we assume that ν is a function belonging to Lebesgue's space $L^\infty([0,T],\mathbb{R}_+^m)$.

3.3.1 Affine case

At first, we suppose that the operator C is affine

$$C(t, F(t)) = A(t)F(t) + B(t),$$

for each $t \in [0,T]$, where $A : [0,T] \to \mathbb{R}^{m\times m}$ such that

$$\langle A(t)F, F\rangle \geq \nu(t)\|F\|_m^2, \quad \forall F \in \mathbb{R}^m, \ \text{a.e. in } [0,T], \tag{3.3.1}$$

where $\nu \in L^\infty([0,T],\mathbb{R}_0^+)$ is such that

$$\nexists I \subseteq [0,T], \ \mu(I) > 0: \ \nu(t) = 0, \quad \text{a.e. in } I,$$

being μ Lebesgue's measure, namely A is a degenerate operator, and $B : [0,T] \to \mathbb{R}^m$. Now, we are able to prove the existence and uniqueness theorem for degenerate evolutionary variational inequalities associated to an affine operator (see [4]).

Theorem 3.3.1. *Let $A \in L^\infty([0,T], \mathbb{R}^{m\times m})$ be a bounded matrix-function satisfying condition* (3.3.1) *and let $B \in L^2([0,T], \mathbb{R}^m)$ be a vector-function. Let $\mathbf{K} \subseteq L^2([0,T], \mathbb{R}^m)$ be a nonempty, convex and closed set. Then, the degenerate evolutionary variational inequality*

$$\langle A(t)H(t) + B(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \quad \textit{a.e. in } [0,T], \tag{3.3.2}$$

admits a unique solution.

Proof. The existence of the solution to evolutionary variational inequality (3.3.2) is obvious. Now, let us show that the assumption (3.3.1) guarantees the uniqueness of the solution to the variational inequality. In fact, ab absurdum, let us suppose that there exist two solutions $H_1, H_2 \in \mathbf{K}$ such that

$$\langle A(t)H_1(t) + B(t), F(t) - H_1(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T], \tag{3.3.3}$$

$$\langle A(t)H_2(t) + B(t), F(t) - H_2(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T]. \tag{3.3.4}$$

From (3.3.3) and (3.3.4), we obtain

$$\langle A(t)H_1(t) + B(t), H_2(t) - H_1(t)\rangle \geq 0, \quad \text{a.e. in } [0,T],$$

$$\langle A(t)H_2(t) + B(t), H_1(t) - H_2(t)\rangle \geq 0, \quad \text{a.e. in } [0,T],$$

having chosen $F = H_2$ in (3.3.3) and $F = H_1$ in (3.3.4). Summing the last inequalities, it results

$$\langle A(t)[H_1(t) - H_2(t)], H_2(t) - H_1(t)\rangle \geq 0, \quad \text{a.e. in } [0,T],$$

then

$$\langle A(t)[H_1(t) - H_2(t)], H_1(t) - H_2(t)\rangle \leq 0, \quad \text{a.e. in } [0,T].$$

Since A satisfies condition (3.3.1), we get

$$\nu(t)\|H_1(t) - H_2(t)\|_m^2 dt \leq 0, \quad \text{a.e. in } [0,T].$$

By virtue of the assumptions on ν, it follows

$$H_1(t) = H_2(t), \quad \text{a.e. in } [0,T].$$

□

3.3.2 Nonlinear case

Let $C : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ be a nonlinear operator and let us consider the following evolutionary variational inequality

Find $H \in \mathbf{K}$ such that

$$\langle C(t,H(t)), F(t)-H(t)\rangle \geq 0, \quad \forall F(t)\in \mathbf{K}(t), \text{ a.e. in } [0,T], \tag{3.3.5}$$

where $\mathbf{K} \subseteq L^2([0,T],\mathbb{R}^m)$ is a nonempty, convex and closed set.

Let us suppose that the operator C is degenerate, namely

$$\langle C(t,H)-C(t,F), H-F\rangle \geq \nu(t)\|H-F\|_m^2, \tag{3.3.6}$$

$\forall H,F\in\mathbb{R}^m$, a.e. in $[0,T]$, where $\nu\in L^\infty([0,T],\mathbb{R}_0^+)$ is such that

$$\nexists I\subseteq[0,T],\ \mu(I)>0:\ \nu(t)=0, \quad \forall t\in I,$$

being μ Lebesgue's measure.

Now, we can prove the existence and uniqueness result (see [9], Theorem 3).

Theorem 3.3.2. *Let $C(t,F):[0,T]\times\mathbb{R}^m\to\mathbb{R}^m$ be a vector-function measurable in t, continuous in F and such that*

$$\|C(t,F)\|_m \leq A(t)\|F\|_m + B(t), \quad \forall F\in\mathbb{R}^m, \text{ a.e. in } [0,T], \tag{3.3.7}$$

with $B\in L^2([0,T])$ and $A\in L^\infty([0,T])$, and satisfying condition (3.3.6)*. Let $\mathbf{K}\subseteq L^2([0,T],\mathbb{R}^m)$ be a nonempty, convex and closed set. Then, evolutionary variational inequality* (3.3.5) *admits a unique solution.*

Proof. The existence of the solution to the evolutionary variational inequality (3.3.5) is guaranteed by Theorem 3.2.3. Now, we may prove that the assumption (3.3.6) implies the uniqueness of the solution to the variational inequality. We proceed ab absurdum. We suppose that there exist two solutions $H_1,H_2\in\mathbf{K}$ such that

$$\langle C(t,H_1(t)), F(t)-H_1(t)\rangle \geq 0, \quad \forall F(t)\in\mathbf{K}(t), \text{ a.e. in } [0,T], \tag{3.3.8}$$

$$\langle C(t,H_2(t)), F(t)-H_2(t)\rangle \geq 0, \quad \forall F(t)\in\mathbf{K}(t), \text{ a.e. in } [0,T], \tag{3.3.9}$$

we obtain

$$\langle C(t,H_1(t)), H_2(t)-H_1(t)\rangle \geq 0, \quad \text{a.e. in } [0,T],$$
$$\langle C(t,H_2(t)), H_1(t)-H_2(t)\rangle \geq 0, \quad \text{a.e. in } [0,T],$$

having set $F=H_2$ in (3.3.8) and $F=H_1$ in (3.3.9). Adding the last inequalities, we obtain

$$\langle C(t,H_1(t))-C(t,H_2(t)), H_2(t)-H_1(t)\rangle \geq 0, \quad \text{a.e. in } [0,T],$$

then

$$\langle C(t,H_1(t))-C(t,H_2(t)), H_1(t)-H_2(t)\rangle \leq 0, \quad \text{a.e. in } [0,T].$$

Since C satisfies condition (3.3.6), we get

$$\nu(t)\|H_1(t)-H_2(t)\|_m^2 dt \leq 0, \quad \text{a.e. in } [0,T].$$

Taking into account of the assumptions on ν, it results

$$H_1(t)=H_2(t), \quad \text{a.e. in } [0,T].$$

□

3.4 Existence results for evolutionary quasi-variational inequalities

In literature there are numerous results for the existence of solutions to quasi-variational inequalities, in both finite and infinite dimension, which mostly require the compactness of the set $\mathbf{K}(H)$ or, alternatively, some coercivity condition on the operator C.

In this section we give some sufficient conditions for the existence of solutions which follow by Theorem 1.6.1 (see [84]).

Theorem 3.4.1. *Let $C : [0,T] \times \mathbb{R}^m_+ \to \mathbb{R}^m_+$ be an operator measurable in t, $\forall F \in \mathbb{R}^m_+$, and continuous in F, a.e. in $[0,T]$, such that*

$$\exists \gamma \in L^2([0,T]) : \ \|C(t,F)\|_m \leq \gamma(t) + \|F\|_m, \quad \forall F \in \mathbb{R}^m_+, \ a.e. \ in \ [0,T],$$

Let $\lambda, \mu \in L^2([0,T], \mathbb{R}^m_+)$ be vector-functions and let $\rho \in L^2([0,T] \times \mathbb{R}^m_+, \mathbb{R}^l_+)$ be an operator verifying the following conditions

$$\exists \psi \in L^1([0,T]) : \ \|\rho(t,F)\|_l \leq \psi(t) + \|F\|^2_m, \quad \forall F \in \mathbb{R}^m_+, \ a.e. \ in \ [0,T],$$

$$\exists \nu \in L^2([0,T]) : \ \|\rho(t,H) - \rho(t,F)\|_l \leq \omega(t)\|H - F\|_m, , \quad \forall H, F \in \mathbb{R}^m_+, \ a.e. \ in \ [0,T],$$

and let us suppose that

$$\mathbf{K}(H) \subset D, \quad \forall H \in D.$$

Then, the quasi-evolutionary variational inequality

$$H \in \mathbf{K}(H) : \ \langle C(t, H(t)), F(t) - H(t) \rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t,H), \ a.e. \ in \ [0,T],$$

admits a solution.

Now, we present an existence theorem, which does not require the compactness of the set D, but it is based on the coercivity of the cost operator (see [84]).

Theorem 3.4.2. *Let $C : [0,T] \times \mathbb{R}^m_+ \to \mathbb{R}^m_+$ be an operator measurable in t, $\forall F \in \mathbb{R}^m_+$, continuous in F, a.e. in $[0,T]$, such that*

$$\exists \gamma \in L^2([0,T]) : \ \|C(t,F)\|_m \leq \gamma(t) + \|F\|_m, \quad \forall F \in \mathbb{R}^m_+, \ a.e. \ in \ [0,T],$$

$$\exists c \geq 0 : \ \|C(t,F)\|_m \leq c\|F\|_m, \quad \forall F \in \mathbb{R}^m_+, \ a.e. \ in \ [0,T],$$

$$\langle C(t,H) - C(t,F), H - F \rangle \geq 0, \quad \forall H, F \in \mathbb{R}^m_+, \ a.e. \ in \ [0,T].$$

Let $\lambda, \mu \in L^2([0,T], \mathbb{R}^m_+)$ be vector-functions and let $\rho \in L^2([0,T] \times \mathbb{R}^m_+, \mathbb{R}^l_+)$ be an operator verifying the following conditions

$$\exists \psi \in L^1([0,T]) : \ \|\rho(t,F)\|_l \leq \psi(t) + \|F\|^2_m, , \quad \forall F \in \mathbb{R}^m_+, \ a.e. \ in \ [0,T],$$

$$\exists \omega \in L^2([0,T]): \ \|\rho(t,H)-\rho(t,F)\|_l \le \omega(t)\|H-F\|_m,, \quad \forall H,F \in \mathbb{R}^m_+, \ a.e. \ in \ [0,T].$$

Then, the evolutionary quasi-variational inequality

$$H \in \mathbf{K}(H): \ \langle C(t,H(t)), F(t)-H(t)\rangle \ge 0, \quad \forall F(t) \in \mathbf{K}(t,H), \ a.e. \ in \ [0,T],$$

admits a solution.

This result has been extended by L. Scrimali (see [87]) to retarded quasi-variational inequalities.

More general existence result is given by the following.

Theorem 3.4.3. *Let $C : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ be an operator measurable in t, $\forall F \in \mathbb{R}^m$, continuous in F, a.e. in $[0,T]$, such that*

$$\exists \gamma \in L^2([0,T]): \ \|C(t,F)\|_m \le \gamma(t) + \|F\|_m, \quad \forall F \in \mathbb{R}^m, \ a.e. \ in \ [0,T],$$

$$\exists c \ge 0: \ \|C(t,F)\|_m \le c\|F\|_m, \quad \forall F \in \mathbb{R}^m, \ a.e. \ in \ [0,T],$$

$$\langle C(t,H) - C(t,F), H-F\rangle \ge 0, \quad \forall F \in \mathbb{R}^m, \ a.e. \ in \ [0,T].$$

Let D be a nonempty, compact, convex subset of $L^2([0,T],\mathbb{R}^m)$. Let $\mathbf{K} : D \to 2^{L^2([0,T],\mathbb{R}^m)}$ be a closed l.s.c. multifunction, with $\mathbf{K}(H)$, for each $H \in L^2([0,T],\mathbb{R}^m)$, nonempty, convex, closed of $L^2([0,T],\mathbb{R}^m)$. Then, the evolutionary quasi-variational inequality

$$H \in \mathbf{K}(H): \ \langle C(t,H(t)), F(t)-H(t)\rangle \ge 0, \quad \forall F(t) \in \mathbf{K}(t,H), \ a.e. \ in \ [0,T],$$

admits a solution.

It is well known that if C is in addition strictly monotone, then the solution H to the evolutionary quasi-variational inequality is unique in the set $\mathbf{K}(H)$.

Obviously, Theorem 3.4.3 holds if the operator is affine (namely $C(t,H(t)) = A(t)H(t) + B(t)$) supposing that A is a bounded nonnegative matrix-function.

For reader's convenience we report the proof of a type Minty's Lemma on evolutionary quasi-variational inequalities.

Lemma 3.4.1. *Let $C(t,F) : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ be an operator measurable in t, $\forall F \in \mathbb{R}^m$, and continuous in F, a.e. in $[0,T]$, such that*

$$\exists \gamma \in L^2([0,T]): \ \|C(t,F)\|_m \le \gamma(t) + \|F\|_m, \quad \forall F \in \mathbb{R}^m, \ a.e. \ in \ [0,T],$$

and

$$\exists \nu > 0: \ \langle C(t,H) - C(t,F), H-F\rangle \ge \nu\|H-F\|^2_m, \quad \forall H,F \in \mathbb{R}^m, \ a.e. \ in \ [0,T].$$

Let D be a nonempty, compact, convex subset of $L^2([0,T],\mathbb{R}^m)$. Let $\mathbf{K} : D \to 2^{L^2([0,T],\mathbb{R}^m)}$ be a closed l.s.c. multifunction, with $\mathbf{K}(H)$, for each $H \in L^2([0,T],\mathbb{R}^m)$, *nonempty, convex, closed of* $L^2([0,T],\mathbb{R}^m)$. *Then, the evolutionary quasi-variational inequality*

$$H \in \mathbf{K}(H): \langle C(t,H(t)), F(t)-H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t,H), \text{ a.e. in } [0,T], \tag{3.4.1}$$

is equivalent to

$$H \in \mathbf{K}(H): \langle C(t,F(t)), F(t)-H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t,H), \text{ a.e. in } [0,T]. \tag{3.4.2}$$

Proof. The existence of solutions to (3.4.1) is ensured by assumptions. Let $H \in \mathbf{K}(H)$ be a solution.

Then, the monotonicity of C and (3.4.1) imply for any $F \in \mathbf{K}(H)$

$$\begin{aligned}\langle C(t,F(t)), F(t)-H(t)\rangle &= \langle C(t,F(t)) - C(t,H(t)), F(t)-H(t)\rangle \\ &\quad + \langle C(t,H(t)), F(t)-H(t)\rangle \geq 0, \quad \text{a.e. in } [0,T].\end{aligned}$$

Conversely, setting in (3.4.2)

$$F(t) = H(t) + \theta(G(t)-H(t)) \in \mathbf{K}(t,H),$$

for arbitrary $\theta \in]0,1]$ and $G(t) \in \mathbf{K}(t,H)$ a.e. in $[0,T]$, it follows

$$\langle C(t, H(t)+\theta(G(t)-H(t))), G(t)-H(t)\rangle \geq 0, \quad \text{a.e. in } [0,T],$$

and letting $\theta \to 0$ it results (3.4.2) by the continuity of C with respect to the second variable. □

3.5 Existence results for degenerate evolutionary quasi-variational inequalities

In this section, we extend existence and uniqueness results to degenerate evolutionary quasi-variational inequalities.

3.5.1 Affine case

Let us consider a degenerate matrix-function, namely a function $A : [0,T] \to \mathbb{R}^{m\times m}$ such that

$$\langle A(t)F, F\rangle \geq \nu(t)\|F\|_m^2, \quad \forall F \in \mathbb{R}^m, \text{ a.e. in } [0,T], \tag{3.5.1}$$

where $\nu \in L^\infty([0,T],\mathbb{R}_0^+)$ is such that

$$\nexists I \subseteq [0,T],\ \mu(I) > 0 : \ \nu(t) = 0, \quad \text{a.e. in } I,$$

being μ Lebesgue's measure, and a vector-function $B : [0,T] \to \mathbb{R}^m$. Let $\mathbf{K} : [0,T] \to 2^{L^2([0,T],\mathbb{R}^m)}$ be a multifunction.

Now, we study under which assumptions the evolutionary quasi-variational inequality

Find $H \in \mathbf{K}(H)$ such that

$$\langle A(t)H(t) + B(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t, H), \text{ a.e. in } [0, T], \tag{3.5.2}$$

admits some solutions.

Theorem 3.5.1. *Let $A \in L^{\infty}([0, T], \mathbb{R}^{m\times m})$ be a bounded matrix-function satisfying condition (3.5.1) and let $B \in L^2([0, T], \mathbb{R}^m)$ be a vector-function. Let D be a nonempty, compact, convex subset of $L^2([0, T], \mathbb{R}^m)$. Let $\mathbf{K} : D \to 2^{L^2([0,T],\mathbb{R}^m)}$ be a closed l.s.c. multifunction, with $\mathbf{K}(H)$, for each $H \in L^2([0, T], \mathbb{R}^m)$, nonempty, convex, closed of $L^2([0, T], \mathbb{R}^m)$. Then, evolutionary quasi-variational inequality (3.5.2) admits a solution, and it is unique in the set $\mathbf{K}(H)$.*

Proof. The existence of a solution H to evolutionary quasi-variational inequality (3.5.2) in the set $\mathbf{K}(H)$ is guaranteed by Theorem 3.4.3. Now, let us prove that the assumption (3.5.1) implies the uniqueness of the solution in the set $\mathbf{K}(H)$. In fact, let us suppose, ab absurdum, that there exist two solutions $H_1, H_2 \in \mathbf{K}(H)$ such that

$$\langle A(t)H_1(t) + B(t), F(t) - H_1(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t, H), \text{ a.e. in } [0, T], \tag{3.5.3}$$

$$\langle A(t)H_2(t) + B(t), F(t) - H_2(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t, H), \text{ a.e. in } [0, T]. \tag{3.5.4}$$

Setting $F = H_2$ in (3.5.3) and $F = H_1$ in (3.5.4), we obtain

$$\langle A(t)H_1(t) + B(t), H_2(t) - H_1(t)\rangle \geq 0, \quad \text{a.e. in } [0, T],$$

$$\langle A(t)H_2(t) + B(t), H_1(t) - H_2(t)\rangle \geq 0, \quad \text{a.e. in } [0, T],$$

and, summing the last inequalities, we get

$$\langle A(t)[H_1(t) - H_2(t)], H_2(t) - H_1(t)\rangle \geq 0, \quad \text{a.e. in } [0, T],$$

then

$$\langle A(t)[H_1(t) - H_2(t)], H_1(t) - H_2(t)\rangle \leq 0, \quad \text{a.e. in } [0, T].$$

Under the assumption (3.5.1), it follows

$$\nu(t)\|H_1(t) - H_2(t)\|_m^2 dt \leq 0, \quad \text{a.e. in } [0, T].$$

By virtue of the assumptions on the function ν, it results

$$H_1(t) = H_2(t), \quad \text{a.e. in } [0, T].$$

□

3.5.2 Nonlinear case

Finally, we present the existence result for nonlinear degenerate evolutionary quasi-variational inequalities. To this aim, we consider a nonlinear operator $C : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ such that

$$\langle C(t,H) - C(t,F), H - F \rangle \geq \nu(t)\|H - F\|_m^2, \quad \forall H, F \in \mathbb{R}^m, \text{ a.e. in } \mathbb{R}^m, \quad (3.5.5)$$

where $\nu \in L^\infty([0,T], \mathbb{R}_0^+)$ is such that

$$\nexists I \subseteq [0,T],\ \mu(I) > 0 :\ \nu(t) = 0, \quad \forall t \in I,$$

being μ Lebesgue's measure. Under this assumptions we call that the operator is degenerate.

Let D be a nonempty, compact, convex subset of $L^2([0,T], \mathbb{R}^m)$. Let $\mathbf{K} : D \to 2^{L^2([0,T],\mathbb{R}^m)}$ be a closed l.s.c. multifunction, with $\mathbf{K}(H)$, for each $H \in L^2([0,T], \mathbb{R}^m)$, nonempty, convex, closed of $L^2([0,T], \mathbb{R}^m)$. Now, we study the following evolutionary quasi-variational inequality

Find $H \in \mathbf{K}(H)$ such that

$$\langle C(t, H(t)), F(t) - H(t) \rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t, H), \text{ a.e. in } [0,T], \quad (3.5.6)$$

showing the next existence result.

Theorem 3.5.2. *Let $C(t,F) : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ be a vector-function measurable in t, $\forall F \in \mathbb{R}^m$, continuous in F, a.e. in $[0,T]$, such that*

$$\exists \gamma \in L^2([0,T]) :\ \|C(t,F)\|_m \leq \gamma(t) + \|F\|_m, \quad \forall F \in \mathbb{R}^m, \text{ a.e. in } [0,T],$$

and satisfying condition (3.5.5)*. Let D be a nonempty, compact, convex subset of $L^2([0,T], \mathbb{R}^m)$. Let $\mathbf{K} : D \to 2^{L^2([0,T],\mathbb{R}^m)}$ be a closed l.s.c. multifunction, with $\mathbf{K}(H)$, for each $H \in L^2([0,T], \mathbb{R}^m)$, nonempty, convex, closed of $L^2([0,T], \mathbb{R}^m)$. Then, evolutionary quasi-variational inequality* (3.5.6) *admits a solution, which is unique in the set $\mathbf{K}(H)$.*

Proof. The existence of a solution H to evolutionary quasi-variational inequality (3.5.6) in the set $\mathbf{K}(H)$ is guaranteed by Theorem 3.4.3. Now, we show that the assumption (3.5.5) implies the uniqueness of the solution to the evolutionary quasi-variational inequality in the set $\mathbf{K}(H)$. We suppose that there exist two solutions $H_1, H_2 \in \mathbf{K}(H)$ such that

$$\langle C(t, H_1(t)), F(t) - H_1(t) \rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t, H), \text{ a.e. in } [0,T], \quad (3.5.7)$$

$$\langle C(t, H_2(t)), F(t) - H_2(t) \rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t, H), \text{ a.e. in } [0,T], \quad (3.5.8)$$

and, choosing $F = H_2$ in (3.5.7) and $F = H_1$ in (3.5.8), we get

$$\langle C(t, H_1(t)), H_2(t) - H_1(t)\rangle \geq 0, \quad \text{a.e. in } [0, T],$$

$$\langle C(t, H_2(t)), H_1(t) - H_2(t)\rangle \geq 0, \quad \text{a.e. in } [0, T],$$

We add the last inequalities, obtaining

$$\langle C(t, H_1(t)) - C(t, H_2(t)), H_2(t) - H_1(t)\rangle \geq 0, \quad \text{a.e. in } [0, T],$$

then

$$\langle C(t, H_1(t)) - C(t, H_2(t)), H_1(t) - H_2(t)\rangle \leq 0, \quad \text{a.e. in } [0, T].$$

The assumption (3.5.5) implies

$$\nu(t)\|H_1(t) - H_2(t)\|_m^2 dt \leq 0, \quad \text{a.e. in } [0, T].$$

and, for the assumptions on the function ν, it follows

$$H_1(t) = H_2(t), \quad \text{a.e. in } [0, T].$$

□

4

The convergence of convex sets

4.1 Historical development

The classical Hausdorff definition of a metric for the space of closed subsets of a (compact) metric space has been generalized by many authors, who have introduced a topology, or a pseudo-topology, or simply a convergence, in the space of closed subsets of a topological space, see for instance L. Vietoris [101], C. Kuratowski [54], C. Choquet [24] and E. Michael [66].

In the pioneering work [70], U. Mosco introduced a special convergence for convex closed subsets of a normed space X, in which both the strong and weak topologies of X are involved. Let us notice, incidentally, that this convergence can be defined in any locally convex topological vector space. Moreover, U. Mosco, in his work, dealt with the convergence of solutions of perturbed variational inequalities where both monotone operators and convex sets are perturbed using different topologies for upper and lower limits.

The set convergence in Mosco's sense can be refined to give an excellent tool for the analysis of the convergence of solutions to variational inequalities and of discretization methods, in particular finite element methods applied to variational inequalities.

4.2 Set convergence in Mosco's sense

In this section, we present the concept of Mosco [70] (see also [93] and [43]) of set convergence (or the set convergence in Mosco's sense) and we recall some stability results.

Definition 4.2.1. Let $(X, \|\cdot\|)$ be an Hilbert space and $\mathbf{K} \subset X$ a closed, nonempty, convex set. A sequence of nonempty, closed, convex sets $\mathbf{K}_n$ *converges in Mosco's sense* to $\mathbf{K}$ as $n \to +\infty$, i.e. $\mathbf{K}_n \to \mathbf{K}$, if and only if

(M1) for any $H \in \mathbf{K}$, there exists a sequence $\{H_n\}_{n\in\mathbb{N}}$ strongly converging to H in X such that H_n lies in $\mathbf{K}_n$ for all $n \in \mathbb{N}$, (M2) for any $\{H_{k_n}\}_{n\in\mathbb{N}}$ weakly converging to H in X, such that H_{k_n} lies in $\mathbf{K}_{k_n}$ for all $n \in \mathbb{N}$, then the weak limit H belongs to $\mathbf{K}$.

Conditions (M1)–(M2) can be shortly phased as

$$\mathrm{w} - \limsup_{n\to+\infty} \mathbf{K}_n \subseteq \mathbf{K}, \quad \mathrm{s} - \liminf_{n\to+\infty} \mathbf{K}_n \supseteq \mathbf{K},$$

where in the sense of Kuratowski (see [1] and [52]) $\mathrm{w}-\limsup$, respectively $\mathrm{s}-\liminf$ denotes the limit superior with respect to the weak convergence, respectively the limit inferior with respect to the strong convergence.

Let us remark that Stummel [96] obviously independently developed this concept of set convergence in his study of perturbations of linear elliptic boundary value problems in Sobolev spaces. He notes in [96], p.11, that these two hypotheses are equivalent to

$$\mathrm{w} - \limsup_{n\to+\infty} \mathbf{K}_n \subseteq \mathbf{K} \subseteq \mathrm{w} - \liminf_{n\to+\infty} \mathbf{K}_n, \quad \mathrm{s} - \limsup_{n\to+\infty} \mathbf{K}_n \subseteq \mathbf{K} \subseteq \mathrm{s} - \liminf_{n\to+\infty} \mathbf{K}_n,$$

since strongly convergence implies weak convergence.

Definition 4.2.2. ([70]) A sequence of operators $A_n : \mathbf{K}_n \to X'$ *converges* to an operator $A : \mathbf{K} \to X'$ if

$$\|A_n H_n - A_n F_n\|_* \leq M \|H_n - F_n\|, \quad \forall H_n, F_n \in \mathbf{K}_n, \tag{4.2.1}$$

$$\langle A_n H_n - A_n F_n, H_n - F_n\rangle \geq \nu \|H_n - F_n\|^2, \quad \forall H_n, F_n \in \mathbf{K}_n, \tag{4.2.2}$$

hold with fixed constants $M, \nu > 0$ and

(M3) the sequence $\{A_n H_n\}_{n\in\mathbb{N}}$ strongly converges to AH in X', for any sequence $\{H_n\}_{n\in\mathbb{N}}$, such that H_n lies in $\mathbf{K}_n$ for all $n \in \mathbb{N}$, strongly converging to $H \in \mathbf{K}$.

In (4.2.1) $\|\cdot\|_*$ is the norm in the dual space of X.

Now, we recall an abstract stability result due to Mosco (see [86], Theorem 4.1):

Theorem 4.2.1. *Let* $\mathbf{K}_n \to \mathbf{K}$ *in Mosco's sense* (M1)–(M2) , $A_n \to A$ *in the sense of* (M3) *and* $B_n \to B$ *in* V'. *Then the unique solutions* H_n *of*

$$H_n \in \mathbf{K}_n : \ \langle A_n H_n - B_n, F_n - H_n\rangle \geq 0, \quad \forall F_n \in \mathbf{K}_n \tag{4.2.3}$$

converge strongly to the solution H *of the limit problem* (2.2.3), *namely,*

$$H_n \to H \quad \textit{in } X.$$

The stability of equilibrium solutions is very important for the numerical approximation of evolutionary variational inequalities, where $n \to +\infty$ denotes a discretization parameter as, for example, the mesh size in the finite elements method.

Now, we remark that the set convergence may be expressed in terms of the projection operator, as the next theorem shows (see [86], Theorem 4.3).

Theorem 4.2.2. *The convergence of the convex sets* $\mathbf{K}_n \to \mathbf{K}$ *in the sense of* (M1)–(M2) *is equivalent to*

$$P_{\mathbf{K}_n}F \to P_{\mathbf{K}}F \quad \text{in } V,\ \forall F \in V.$$

In order to get a stability result, let us give the following definition about local "distance" between two closed convex nonempty subsets $\mathbf{K}$ and $\mathbf{K}_n$ of an Hilbert space $(X, \|\cdot\|)$.

Definition 4.2.3. For each $r > 0$ such that $\mathbf{K} \cap \{\|F\| \leq r\}$ and $\mathbf{K}_n \cap \{\|F\| \leq r\}$ are nonempty, let

$$\pi_r(\mathbf{K}, \mathbf{K}_n) = \sup_{\substack{G \in V \\ \|G\| \leq r}} \|P_{\mathbf{K}}G - P_{\mathbf{K}_n}G\|, \tag{4.2.4}$$

where $P_{\mathbf{K}}$ denotes the hilbertian projection on $\mathbf{K}$.

Analogously, for the operators $A, A_n : X \to X'$ one can introduce a local "distance" by defining for each $r > 0$.

Definition 4.2.4. For each $r > 0$, let

$$\delta_r(A, A_n) = \sup_{\substack{H \in V \\ \|H\| \leq r}} \|AH - A_nH\|_*. \tag{4.2.5}$$

It is clear, from (4.2.4) and (4.2.5), that $\pi_r(\mathbf{K}, \mathbf{K}_n) \to 0$ or $\delta_r(A, A_n) \to 0$ for each $r > 0$, as $n \to +\infty$, imply the convergence (M1) and (M2) for the convex sets or (M3) for the operators, respectively. Those conditions are stronger assumptions and they allow to refine Mosco's Theorem by estimating the rate of convergence in terms of those quantities, as the following result shows (see [86], Theorem 4.4):

Theorem 4.2.3. *Let* H *and* H_n *be the solutions of* (2.2.3) *and* (4.2.3), *respectively, with operators* $A, A_n : X \to X'$ *verifying* (4.2.1) *and* (4.2.2) *with constants* $M, \nu > 0$ *such that* $\|A0\|_*, \|A_n0\|_* \leq a_0$, *with* $\|B\|_*, \|B_n\|_* \leq b$ *and with convex sets* $\mathbf{K}, \mathbf{K}_n$ *such that* $\|P_{\mathbf{K}}0\|_*, \|P_{\mathbf{K}_n}0\|_* \leq d_0$. *Then the following estimate holds*

$$\|H - H_n\| \leq C\Big\{\|B - B_n\|_* + \delta_{r_0}(A, A_n) + \pi_{r_1}(\mathbf{K}, \mathbf{K}_n)\Big\}$$

where, for any $\rho \in]0, 2\nu/M^2[$,

$$C = \max(1, \rho)/(1 - \sqrt{1 - 2\rho\nu + \rho^2M^2}) > 0,$$

$$r_0 = (b + a_0)/\nu + d_0(1 + M/\nu)$$

and

$$r_1 = r_0 + \rho(a_0 + Mr_0 + b).$$

4.3 Applications to traffic equilibrium problem

In this section, we are proving that sets of feasible flows of the traffic equilibrium problem in the dynamic case and in the dynamic elastic case satisfy the property of the set convergence in Mosco's sense.

4.3.1 Dynamic case

At first, we fix the attention on the set of feasible flows of the dynamic traffic equilibrium problem (see [3], proof of Theorem 3.2).

Proposition 4.3.1. *Let $\lambda, \mu \in C([0,T], \mathbb{R}^m_+)$, let $\rho \in C([0,T], \mathbb{R}^l_+)$ and let $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,T]$ be a sequence such that $t_n \to t \in [0,T]$, as $n \to +\infty$. Then, the sequence of sets*

$$\mathbf{K}(t_n) = \Big\{F(t_n) \in \mathbb{R}^m) : \lambda(t_n) \le F(t_n) \le \mu(t_n),\ \Phi F(t_n) = \rho(t_n)\Big\},$$

$\forall n \in \mathbb{N}$, converges to

$$\mathbf{K}(t) = \Big\{F(t) \in \mathbb{R}^m) :\ \lambda(t) \le F(t) \le \mu(t),\ \Phi F(t) = \rho(t)\Big\},$$

as $n \to +\infty$, in Mosco's sense.

Proof. In order to prove that the sequence $\{\mathbf{K}(t_n)\}_{n\in\mathbb{N}}$ converges to $\mathbf{K}(t)$ in Mosco's sense, for any sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,T]$ such that $t_n \to t \in [0,T]$, as $n \to +\infty$, it is enough to show that conditions (M1) and (M2) hold.

For the first condition, let $F(t) \in \mathbf{K}(t)$ be fixed and, for each j, $1 \le j \le l$, let us set

$$\begin{aligned} A_j &= \Big\{r \in \{1,2,\dots,m\} :\ \varphi_{jr} = 1,\ F_r(t) = \lambda_r(t)\Big\} \\ B_j &= \Big\{r \in \{1,2,\dots,m\} :\ \varphi_{jr} = 1,\ F_r(t) = \mu_r(t)\Big\} \\ C_j &= \Big\{r \in \{1,2,\dots,m\} :\ \varphi_{jr} = 1,\ \lambda_r(t) < F_r(t) < \mu_r(t)\Big\}. \end{aligned}$$

Let us assume that $C_j \ne \emptyset$ and let us observe that for each $r \in C_j$ it results

$$\lim_{n\to+\infty} \mu_r(t_n) - \Bigg[F_r(t) + \frac{\rho_j(t_n) - \rho_j(t)}{\sum_{r\in C_j}\varphi_{jr}} - \frac{\sum_{r\in A_j}[\lambda_r(t_n) - \lambda_r(t)]}{\sum_{r\in C_j}\varphi_{jr}} - \frac{\sum_{r\in B_j}[\mu_r(t_n) - \mu_r(t)]}{\sum_{r\in C_j}\varphi_{jr}}\Bigg] = \mu_r(t) - F_r(t) > 0,$$

$$\begin{aligned}\lim_{n\to+\infty} F_r(t) &+ \frac{\rho_j(t_n)-\rho_j(t)}{\sum_{r\in C_j}\varphi_{jr}} - \frac{\sum_{r\in A_j}[\lambda_r(t_n)-\lambda_r(t)]}{\sum_{r\in C_j}\varphi_{jr}} \\ &- \frac{\sum_{r\in B_j}[\mu_r(t_n)-\mu_r(t)]}{\sum_{r\in C_j}\varphi_{jr}} - \lambda_r(t_n) = F_r(t)-\lambda_r(t) > 0.\end{aligned}$$

Then there exists an index ν_j such that for $n > \nu_j$ and $r \in C_j$ we have

$$\lambda_r(t) \le F_r(t)+\frac{\rho_j(t_n)-\rho_j(t)}{\sum_{r\in C_j}\varphi_{jr}} - \frac{\sum_{r\in A_j}[\lambda_r(t_n)-\lambda_r(t)]}{\sum_{r\in C_j}\varphi_{jr}} - \frac{\sum_{r\in B_j}[\mu_r(t_n)-\mu_r(t)]}{\sum_{r\in C_j}\varphi_{jr}} \le \mu_r(t).$$

Hence we can consider a sequence $F(t_n)$ such that:

- for $n > \nu_j$ and $\varphi_{jr} = 1,\ j = 1, 2, \ldots, l$

$$F_r(t_n) = \begin{cases} \lambda_r(t_n) & \text{for } r \in A_j \\ \mu_r(t_n) & \text{for } r \in B_j \\ F_r(t) + \dfrac{\rho_j(t_n)-\rho_j(t)}{\sum_{r\in C_j}\varphi_{jr}} - \dfrac{\sum_{r\in A_j}[\lambda_r(t_n)-\lambda_r(t)]}{\sum_{r\in C_j}\varphi_{jr}} & \\ \quad - \dfrac{\sum_{r\in B_j}[\mu_r(t_n)-\mu_r(t)]}{\sum_{r\in C_j}\varphi_{jr}} & \text{for } r \in C_j \end{cases}$$

- and for $n \le \nu_j$, $\varphi_{jr} = 1,\ j = 1, 2, \ldots, l$

$$F_r(t_n) = P_{\mathbf{K}(t_n)F_r(t)},$$

  where $P_{\mathbf{K}(t_n)}$ denotes the hilbertian projection on $\mathbf{K}(t_n)$.

Obviously if $n \le \nu_j$ it results $F_r(t_n) \in \mathbf{K}(t_n)$, whereas for $n > \nu_j$ we have

$$\lambda_r(t_n) \le F_r(t_n) \le \mu_r(t_n)$$

and

$$\begin{aligned}\sum_{r=1}^{m}\varphi_{jr}F_r(t_n) &= \sum_{r=1}^{m}F_r(t) + \sum_{r\in A_j}[\lambda_r(t_n)-\lambda_r(t)] + \sum_{r\in B_j}[\mu_r(t_n)-\mu(t)] \\ &\quad + \sum_{r\in C_j}\frac{\rho_j(t_n)-\rho_j(t)}{\sum_{r\in C_j}\varphi_{jr}} - \sum_{r\in C_j}\frac{\sum_{r\in A_j}[\lambda_r(t_n)-\lambda_r(t)]}{\sum_{r\in C_j}\varphi_{jr}} \\ &\quad - \sum_{r\in C_j}\frac{\sum_{r\in B_j}[\mu_r(t_n)-\mu_r(t)]}{\sum_{r\in C_j}\varphi_{jr}} \\ &= \rho_j(t_n).\end{aligned}$$

So $F(t_n) \in \mathbf{K}(t_n)$, $\forall n \in \mathbb{N}$ and it results $\lim_{n\to+\infty} F(t_n) = F(t)$.

Now, let us assume that $C_j = \emptyset$ and let us observe that it results

$$\begin{aligned}\lim_{n\to+\infty}\Bigg[\lambda_r(t_n) &+ \frac{1}{\sum_{r\in A_j}\varphi_{jr}}\max\Big(0,\rho_j(t_n)-\sum_{r\in A_j}\lambda_r(t_n)-\sum_{r\in B_j}\mu_r(t_n)\Big)\Bigg]\\ &- \mu_r(t_n) = \lambda_r(t)-\mu_r(t)<0, \quad r\in A_j,\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to+\infty}\Bigg[\mu_r(t_n) &+ \frac{1}{\sum_{r\in B_j}\varphi_{jr}}\min\Big(0,\rho_j(t_n)-\sum_{r\in A_j}\lambda_r(t_n)-\sum_{r\in B_j}\mu_r(t_n)\Big)\Bigg]\\ &- \lambda_r(t_n) = \mu_r(t)-\lambda_r(t)>0, \quad r\in B_j.\end{aligned}$$

Then there exists an index ν_j such that for $n>\nu_j$ it results

$$\lambda_r(t_n)\le\lambda_r(t_n)+\frac{1}{\sum_{r\in A_j}\varphi_{jr}}\max\Big(0,\rho_j(t_n)-\sum_{r\in A_j}\lambda_r(t_n)-\sum_{r\in B_j}\mu_r(t_n)\Big)\le\mu_r(t_n)$$

$$\lambda_r(t_n)\le\mu_r(t_n)+\frac{1}{\sum_{r\in B_j}\varphi_{jr}}\min\Big(0,\rho_j(t_n)-\sum_{r\in A_j}\lambda_r(t_n)-\sum_{r\in B_j}\mu_r(t_n)\Big)\le\mu_r(t_n).$$

So, we can choose the sequence $F(t_n)$ in the following way:

- for $n>\nu_j$, $\varphi_{jr}=1$, $j=1,2,\ldots,l$

$$F_r(t_n)=\begin{cases}\lambda_r(t_n)+\dfrac{1}{\sum_{r\in A_j}\varphi_{jr}}\max\Big(0,\rho_j(t_n)-\sum_{r\in A_j}\lambda_r(t_n)-\sum_{r\in B_j}\mu_r(t_n)\Big)\\ \hfill\text{for } r\in A_j\\ \mu_r(t_n)+\dfrac{1}{\sum_{r\in B_j}\varphi_{jr}}\min\Big(0,\rho_j(t_n)-\sum_{r\in A_j}\lambda_r(t_n)-\sum_{r\in B_j}\mu_r(t_n)\Big)\\ \hfill\text{for } r\in B_j\end{cases}$$

- whereas for $n\le\nu_j$, $\varphi_{jr}=1$, $j=1,2,\ldots,l$

$$F_r(t_n)=P_{\mathbf{K}(t_n)F_r(t)}.$$

It results $\lim_{n\to+\infty}F(t_n)=F(t)$ and for $n>\nu_j$

$$\begin{aligned}\sum_{r=1}^m\varphi_{jr}F_r(t_n) &= \sum_{r\in A_j}\lambda_r(t_n)+\max\Big(0,\rho_j(t_n)-\sum_{r\in A_j}\lambda_r(t_n)-\sum_{r\in B_j}\mu_r(t_n)\Big)\\ &\quad+\sum_{r\in B_j}\mu_r(t_n)+\min\Big(0,\rho_j(t_n)-\sum_{r\in A_j}\lambda_r(t_n)-\sum_{r\in B_j}\mu_r(t_n)\Big)\\ &= \rho_j(t_n).\end{aligned}$$

The proof of the first condition (M1) has been obtained.

For the second one, let $\{F(t_n)\}_{n\in\mathbb{N}}$ be a fixed sequence, with $F(t_n) \in \mathbf{K}(t_n)$, $\forall n \in \mathbb{N}$, such that $F(t_n) \rightharpoonup F(t)$ (weakly) in $\mathbb{R}^m$. Since $\mathbb{R}^m$ is a finite-dimensional space, the weak convergence is equivalent to $F(t_n) \to F(t)$ (strongly) in $\mathbb{R}^m$. It remains to prove that $F(t) \in \mathbf{K}(t)$. From $F(t_n) \in \mathbf{K}(t_n)$, $\forall n \in \mathbb{N}$, we derive

$$\lambda(t_n) \leq F(t_n) \leq \mu(t_n), \quad \forall n \in \mathbb{N}, \tag{4.3.1}$$

$$\sum_{r=1}^{m} \varphi_{jr} F_r(t_n) = \rho_j(t_n), \quad \forall n \in \mathbb{N},\ j = 1, 2, \ldots, l. \tag{4.3.2}$$

Passing to the limit for $n \to +\infty$ in (4.3.1), and using the continuity of λ and μ on $[0, T]$, we obtain

$$\lambda(t) \leq F(t) \leq \mu(t),$$

and, from (4.3.2), we have for the continuity of ρ on $[0, T]$

$$\sum_{r=1}^{m} \varphi_{jr} F_r(t) = \rho_j(t), \quad j = 1, 2, \ldots, l.$$

Then

$$F(t) \in \mathbf{K}(t),$$

so the second condition (M2) has also been proved.

Hence, we conclude that

$$\mathbf{K}(t_n) \to \mathbf{K}(t) \text{ in Mosco's sense,}$$

when $t_n \to t \in [0, T]$. □

Now, we prove that the set $\mathbf{K}(t)$ is uniformly bounded in $[0, T]$.

Proposition 4.3.2. *Let $\lambda, \mu \in C([0, T], \mathbb{R}^m_+)$, let $\rho \in C([0, T], \mathbb{R}^l_+)$ be vector-functions. Then, the set*

$$\mathbf{K}(t) = \Big\{F(t) \in \mathbb{R}^m) : \ \lambda(t) \leq F(t) \leq \mu(t),\ \Phi F(t) = \rho(t)\Big\},$$

is uniformly bounded in $[0, T]$.

Proof. Let us fix an arbitrary vector-flow $H(t)$ in $\mathbf{K}(t)$, then

$$\lambda(t) \leq H(t) \leq \mu(t).$$

Since $\lambda, \mu \in C([0, T], \mathbb{R}^m_+)$, it follows

$$\|H(t)\|_m \leq \max_{t\in[0,T]} \mu(t) = C, \quad \text{in } [0, T],$$

where C is a constant independent on $t \in [0, T]$. □

4.3.2 Dynamic elastic case

Now, we extend the previous result to the dynamic elastic case (see [3], proof of Theorem 4.1).

Proposition 4.3.3. *Let $\lambda, \mu \in C([0,T], \mathbb{R}^m_+)$ be and let $\rho \in C([0,T] \times \mathbb{R}^m_+, \mathbb{R}^l_+)$ be such that*

$$\exists \psi \in C([0,T], \mathbb{R}_+): \ \|\rho(t,F)\|_l \leq \psi(t) + \|F\|^2_m, \tag{4.3.3}$$

and let $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,T]$ be a sequence such that $t_n \to t \in [0,T]$, as $n \to +\infty$. Then, the sequence of sets

$$\mathbf{K}(t_n, H) = \left\{ F(t_n) \in \mathbb{R}^m : \ \lambda(t_n) \leq F(t_n) \leq \mu(t_n), \ \Phi F(t_n) = \frac{1}{T}\int_0^T \rho(t_n, H(\tau))d\tau \right\},$$

$\forall n \in \mathbb{N}$, converges to

$$\mathbf{K}(t, H) = \left\{ F(t) \in \mathbb{R}^m : \ \lambda(t) \leq F(t) \leq \mu(t), \ \Phi F(t) = \frac{1}{T}\int_0^T \rho(t, H(\tau))d\tau \right\},$$

as $n \to +\infty$, in Mosco's sense.

Proof. Let $t \in [0,T]$ be fixed and let $\{t_n\}_{n\in\mathbb{N}} \subset [0,T]$ be a sequence, with $t_n \to t$. Owing to the continuity of λ, μ and ρ, $\lambda(t_n) \to \lambda(t)$, $\mu(t_n) \to \mu(t)$ and $\rho(t_n, F) \to \rho(t, F)$, $\forall F \in \mathbf{K}(t, H)$, respectively, follows. Let us prove that $\mathbf{K}(t_n, H) \to \mathbf{K}(t, H)$ in Mosco's sense, i.e. let us show that (M1) and (M2) hold.

Regard the first condition, let $F(t) \in \mathbf{K}(t, H)$ be fixed and, for each j, $1 \leq j \leq l$, let us set

$$\begin{aligned} A_j &= \left\{ r \in \{1, 2, \ldots, m\} : \ \varphi_{jr} = 1, \ F_r(t) = \lambda_r(t) \right\} \\ B_j &= \left\{ r \in \{1, 2, \ldots, m\} : \ \varphi_{jr} = 1, \ F_r(t) = \mu_r(t) \right\} \\ C_j &= \left\{ r \in \{1, 2, \ldots, m\} : \ \varphi_{jr} = 1, \ \lambda_r(t) < F_r(t) < \mu_r(t) \right\}. \end{aligned}$$

At first, we observe that, by the assumptions, we have

$$\exists \psi \in C([0,T], \mathbb{R}_+): \ \|\rho(t, H(\tau))\|_l \leq \psi(t) + \|H(\tau)\|^2_m,$$

for $t \in [0,T]$ and $\tau \in [0,T]$. Since $\psi \in C([0,T], \mathbb{R}^m_+)$ and $H \in L^2([0,T], \mathbb{R}^m_+)$, we obtain, for $t \in [0,T]$ and $\tau \in [0,T]$

$$\|\rho(t, H(\tau))\|_l \leq \psi(t) + \|H(\tau)\|^2_m \in L^1([0,T], \mathbb{R}_+)$$

and, by virtue of the continuity of ρ with respect to the first variable we also have

$$\lim_{n\to+\infty} \rho(t_n, H(\tau)) = \rho(t, H(\tau)),$$

for $\tau \in [0,T]$ and $H \in L^2([0,T],\mathbb{R}^m_+)$. From a well known generalization of Lebesgue's Theorem we get

$$\lim_{n\to+\infty} \int_0^T \rho(t_n, H(\tau))d\tau = \int_0^T \rho(t, H(\tau))d\tau, \tag{4.3.4}$$

for every $H \in L^2([0,T],\mathbb{R}^m_+)$.

Now, we assume that $C_j \neq \emptyset$ and we remark that for each $r \in C_j$ we obtain

$$\begin{aligned}
\lim_{n\to+\infty} \mu_r(t_n) \quad &- \quad \Bigg[F_r(t) + \frac{1}{T}\frac{\int_0^T \rho_j(t_n, H(\tau))d\tau - \int_0^T \rho_j(t, H(\tau))d\tau}{\sum_{r\in C_j}\varphi_{jr}} \\
&- \quad \frac{\sum_{r\in A_j}[\lambda_r(t_n)-\lambda_r(t)]}{\sum_{r\in C_j}\varphi_{jr}} - \frac{\sum_{r\in B_j}[\mu_r(t_n)-\mu_r(t)]}{\sum_{r\in C_j}\varphi_{jr}}\Bigg] = \mu_r(t) - F_r(t) > 0,
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to+\infty} F_r(t) \quad &+ \quad \frac{1}{T}\frac{\int_0^T \rho_j(t_n, H(\tau))d\tau - \int_0^T \rho_j(t, H(\tau))d\tau}{\sum_{r\in C_j}\varphi_{jr}} - \frac{\sum_{r\in A_j}[\lambda_r(t_n)-\lambda_r(t)]}{\sum_{r\in C_j}\varphi_{jr}} \\
&- \quad \frac{\sum_{r\in B_j}[\mu_r(t_n)-\mu_r(t)]}{\sum_{r\in C_j}\varphi_{jr}} - \lambda_r(t_n) = F_r(t) - \lambda_r(t) > 0.
\end{aligned}$$

Then, there exists an index ν_j such that for $n > \nu_j$ and $r \in C_j$ it results

$$\begin{aligned}
\lambda_r(t) \quad &\leq \quad F_r(t) + \frac{1}{T}\frac{\int_0^T \rho_j(t_n, H(\tau))d\tau - \int_0^T \rho_j(t, H(\tau))d\tau}{\sum_{r\in C_j}\varphi_{jr}} \\
&\quad -\frac{\sum_{r\in A_j}[\lambda_r(t_n)-\lambda_r(t)]}{\sum_{r\in C_j}\varphi_{jr}} - \frac{\sum_{r\in B_j}[\mu_r(t_n)-\mu_r(t)]}{\sum_{r\in C_j}\varphi_{jr}} \leq \mu_r(t).
\end{aligned}$$

So, we can consider a sequence $F(t_n)$ such that:

- for $n > \nu_j$ and $\varphi_{jr} = 1$, $j = 1, 2, \ldots, l$

$$F_r(t_n) = \begin{cases}
\lambda_r(t_n) & \text{for } r \in A_j \\
\mu_r(t_n) & \text{for } r \in B_j \\
F_r(t) + \frac{1}{T}\frac{\int_0^T \rho_j(t_n, H(\tau))d\tau - \int_0^T \rho_j(t, H(\tau))d\tau}{\sum_{r\in C_j}\varphi_{jr}} & \\
\quad -\frac{\sum_{r\in A_j}[\lambda_r(t_n)-\lambda_r(t)]}{\sum_{r\in C_j}\varphi_{jr}} - \frac{\sum_{r\in B_j}[\mu_r(t_n)-\mu_r(t)]}{\sum_{r\in C_j}\varphi_{jr}} & \text{for } r \in C_j
\end{cases}$$

- and for $n \le \nu_j$, $\varphi_{jr} = 1$, $j = 1, 2, \ldots, l$

$$F_r(t_n) = P_{\mathbf{K}(t_n)} F_r(t).$$

Obviously, if $n \le \nu_j$ we have $F_r(t_n) \in \mathbf{K}(t_n)$, whereas for $n > \nu_j$ it results

$$\lambda_r(t_n) \le F_r(t_n) \le \mu_r(t_n)$$

and

$$\begin{aligned}\sum_{r=1}^{m} F_r(t_n)\varphi_{jr} &= \sum_{r=1}^{m} F_r(t) + \sum_{r\in A_j}[\lambda_r(t_n) - \lambda_r(t)] + \sum_{r\in B_j}[\mu_r(t_n) - \mu(t)] \\ &\quad + \frac{1}{T}\sum_{r\in C_j} \frac{\int_0^T \rho_j(t_n, H(\tau))d\tau - \int_0^T \rho_j(t, H(\tau))d\tau}{\sum_{r\in C_j}\varphi_{jr}} \\ &\quad - \sum_{r\in C_j} \frac{\sum_{r\in A_j}[\lambda_r(t_n) - \lambda_r(t)]}{\sum_{r\in C_j}\varphi_{jr}} - \sum_{r\in C_j} \frac{\sum_{r\in B_j}[\mu_r(t_n) - \mu_r(t)]}{\sum_{r\in C_j}\varphi_{jr}} \\ &= \frac{1}{T}\int_0^T \rho_j(t_n, H(\tau))d\tau.\end{aligned}$$

Hence, $F(t_n) \in \mathbf{K}(t_n)$, $\forall n \in \mathbb{N}$ and we have $\lim_{n\to+\infty} F(t_n) = F(t)$.

We assume that $C_j = \emptyset$ and we remark that we obtain

$$\begin{aligned}\lim_{n\to+\infty}\Bigg[\lambda_r(t_n) &+ \frac{1}{\sum_{r\in A_j}\varphi_{jr}} \max\left(0, \frac{1}{T}\int_0^T \rho_j(t_n, H(\tau))d\tau - \sum_{r\in A_j}\lambda_r(t_n)\right. \\ &- \left.\left.\sum_{r\in B_j}\mu_r(t_n)\right)\right] - \mu_r(t_n) = \lambda_r(t) - \mu_r(t) < 0, \quad r \in A_j,\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to+\infty}\Bigg[\mu_r(t_n) &+ \frac{1}{\sum_{r\in B_j}\varphi_{jr}} \min\left(0, \frac{1}{T}\int_0^T \rho_j(t_n, H(\tau))d\tau - \sum_{r\in A_j}\lambda_r(t_n) - \sum_{r\in B_j}\mu_r(t_n)\right)\Bigg] \\ &- \lambda_r(t_n) = \mu_r(t) - \lambda_r(t) > 0, \quad r \in B_j.\end{aligned}$$

Then, there exists an index ν_j such that for $n > \nu_j$ it follows

$$\begin{aligned}\lambda_r(t_n) \le \lambda_r(t_n) + \frac{1}{\sum_{r\in A_j}\varphi_{jr}} \max\left(0, \frac{1}{T}\int_0^T \rho_j(t_n, H(\tau))d\tau - \sum_{r\in A_j}\lambda_r(t_n) - \sum_{r\in B_j}\mu_r(t_n)\right) \\ \le \mu_r(t_n) \\ \lambda_r(t_n) \le \mu_r(t_n) + \frac{1}{\sum_{r\in B_j}\varphi_{jr}} \min\left(0, \frac{1}{T}\int_0^T \rho_j(t_n, H(\tau))d\tau - \sum_{r\in A_j}\lambda_r(t_n) - \sum_{r\in B_j}\mu_r(t_n)\right) \\ \le \mu_r(t_n).\end{aligned}$$

So, we can choose the sequence $F(t_n)$ in the following way:

- for $n > \nu_j$, $\varphi_{jr} = 1$, $j = 1, 2, \dots, l$

$$F_r(t_n) = \begin{cases} \lambda_r(t_n) + \dfrac{1}{\sum_{r\in A_j}\varphi_{jr}} \\ \qquad \cdot \max\left(0, \dfrac{1}{T}\int_0^T \rho_j(t_n, H(\tau))d\tau - \sum_{r\in A_j}\lambda_r(t_n) - \sum_{r\in B_j}\mu_r(t_n)\right) \\ \qquad\qquad \text{for } r \in A_j \\ \mu_r(t_n) + \dfrac{1}{\sum_{r\in B_j}\varphi_{jr}} \\ \qquad \cdot \min\left(0, \dfrac{1}{T}\int_0^T \rho_j(t_n, H(\tau))d\tau - \sum_{r\in A_j}\lambda_r(t_n) - \sum_{r\in B_j}\mu_r(t_n)\right) \\ \qquad\qquad \text{for } r \in B_j \end{cases}$$

- and for $n \le \nu_j$, $\varphi_{jr} = 1$, $j = 1, 2, \dots, l$

$$F_r(t_n) = P_{\mathbf{K}(t_n)}F_r(t).$$

We have $\lim_{n\to+\infty} F(t_n) = F(t)$ and for $n > \nu_j$

$$\begin{aligned} \sum_{r=1}^m \varphi_{jr}F_r(t_n) &= \sum_{r\in A_j}\lambda_r(t_n) + \max\left(0, \frac{1}{T}\int_0^T \rho_j(t_n, H(\tau))d\tau - \sum_{r\in A_j}\lambda_r(t_n) - \sum_{r\in B_j}\mu_r(t_n)\right) \\ &\quad + \sum_{r\in B_j}\mu_r(t_n) + \min\left(0, \frac{1}{T}\int_0^T \rho_j(t_n, H(\tau))d\tau - \sum_{r\in A_j}\lambda_r(t_n) - \sum_{r\in B_j}\mu_r(t_n)\right) \\ &= \frac{1}{T}\int_0^T \rho_j(t_n, H(\tau))d\tau. \end{aligned}$$

Then the first condition has been proved.

For the second one, let $\{F(t_n)\}_{n\in\mathbb{N}}$ be a sequence, with $F(t_n) \in \mathbf{K}(t_n, H)$, $\forall n \in \mathbb{N}$ and $H \in L^2([0,T], \mathbb{R}^m_+)$, such that $F_n \rightharpoonup F$ (weakly) in $\mathbb{R}^m$. Since $\mathbb{R}^m$ is a finite-dimensional space, it is equivalent to $F(t_n) \to F(t)$ (strongly) in $\mathbb{R}^m$. Let us prove that $F(t) \in \mathbf{K}(t, H)$. Since $F(t_n) \in \mathbf{K}(t_n, H)$, $\forall n \in \mathbb{N}$, with $H \in L^2([0,T], \mathbb{R}^m_+)$, it results

$$\lambda(t_n) \le F(t_n) \le \mu(t_n), \quad \forall n \in \mathbb{N}, \tag{4.3.5}$$

$$\sum_{r=1}^m \varphi_{jr}F_r(t_n) = \frac{1}{T}\int_0^T \rho_j(t_n, H(\tau))d\tau, \quad j = 1, 2, \dots, l. \tag{4.3.6}$$

Passing to the limit for $n \to +\infty$ in (4.3.5), and using the continuity of λ and μ, we obtain

$$\lambda(t) \le F(t) \le \mu(t).$$

Now, passing to the limit for $n \to +\infty$ in the left-hand side of (4.3.6), we have

$$\lim_{n\to+\infty}\sum_{r=1}^m \varphi_{jr}F_r(t_n) = \sum_{r=1}^m \varphi_{jr}F_r(t), \quad j = 1, 2, \dots, l. \tag{4.3.7}$$

Then, from (4.3.7) and (4.3.4), we deduce

$$\sum_{r=1}^{m} \varphi_{jr} F_r(t) = \frac{1}{T}\int_0^T \rho_j(t, H(\tau))d\tau,$$

for every $j = 1, 2, \dots, l$ and $H \in L^2([0,T], \mathbb{R}^m_+)$, namely

$$F(t) \in \mathbf{K}(t, H),$$

and the second condition has been just proved.

We conclude that

$$\mathbf{K}(t_n, H) \to \mathbf{K}(t, H) \text{ in Mosco's sense,}$$

being $t_n \to t \in [0,T]$. □

Also in the elastic case, we show that the set $\mathbf{K}(t, H)$ is uniformly bounded in $[0, T]$.

Proposition 4.3.4. *Let $\lambda, \mu \in C([0,T], \mathbb{R}^m_+)$ be vector-functions and let $\rho \in C([0,T] \times \mathbb{R}^m_+, \mathbb{R}^l_+)$ be an operator satisfying condition (4.3.3). Then, the set*

$$\mathbf{K}(t, H) = \left\{ F(t) \in \mathbb{R}^m : \ \lambda(t) \le F(t) \le \mu(t) \ \ \Phi F(t) = \frac{1}{T}\int_0^T \rho(t, H(\tau))d\tau \right\},$$

is uniformly bounded in $[0, T]$.

Proof. Let us fix an arbitrary vector-flow $H(t)$ in $\mathbf{K}(t, H)$, then

$$\lambda(t) \le H(t) \le \mu(t).$$

Under assumptions $\lambda, \mu \in C([0,T], \mathbb{R}^m_+)$, it follows

$$\|H(t)\|_m \le \max_{t\in[0,T]} \mu(t) = C, \quad \text{in } [0,T],$$

where C is a constant independent on $t \in [0, T]$. □

4.4 Application to dynamic equilibrium problems in the common formulation

Now, we are showing that the set of feasible flows of dynamic equilibrium problems in the common formulation (see Section 2.3) satisfies the property of the set convergence in Mosco's sense.

Proposition 4.4.1. *Let* $\lambda, \mu \in C([0,T], \mathbb{R}^q_+)$*, let* $\rho \in C([0,T], \mathbb{R}^l_+)$ *such that*

$$\exists \psi \in C([0,T]): \ \|\rho(t, F(t))\|_l \leq \psi(t) + \|F(t)\|^2_m,$$

and let $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,T]$ *be a sequence such that* $t_n \to t \in [0,T]$*, as* $n \to +\infty$*. Then, the sequence of sets*

$$\mathbf{K}(t_n) = \Bigg\{ u(t_n) \in \mathbb{R}^q: \quad \lambda(t_n) \leq u(t_n) \leq \mu(t_n),\ \sum_{i=1}^{q} \xi_{ji} u_i(t_n) = \rho_j(t_n), \\ \xi_{ji} \in \{-1,0,1\},\ i \in \{1,\ldots,q\},\ j \in \{1,\ldots,l\} \Bigg\}.$$

$\forall n \in \mathbb{N}$*, converges to*

$$\mathbf{K}(t) = \Bigg\{ u(t) \in \mathbb{R}^q: \quad \lambda(t) \leq u(t) \leq \mu(t),\ \sum_{i=1}^{q} \xi_{ji} u_i(t) = \rho_j(t), \\ \xi_{ji} \in \{-1,0,1\},\ i \in \{1,\ldots,q\},\ j \in \{1,\ldots,l\} \Bigg\}.$$

as $n \to +\infty$*, in Mosco's sense.*

Proof. In order to prove that the sequence $\{\mathbf{K}(t_n)\}_{n\in\mathbb{N}}$ converges to $\mathbf{K}(t)$ in Mosco's sense, for any sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,T]$ such that $t_n \to t \in [0,T]$, as $n \to +\infty$, it is enough to show that conditions (M1) and (M2) hold.

For the first condition, let $u(t) \in \mathbf{K}(t)$ be fixed and, for each j, $1 \leq j \leq l$, let us set

$$\begin{aligned} A_j &= \Big\{ i \in \{1,2,\ldots,q\}: \ \xi_{ji} = 1,\ u_i(t) = \lambda_i(t) \Big\} \\ B_j &= \Big\{ i \in \{1,2,\ldots,q\}: \ \xi_{ji} = 1,\ u_i(t) = \mu_i(t) \Big\} \\ C_j &= \Big\{ i \in \{1,2,\ldots,q\}: \ \xi_{ji} = 1,\ \lambda_i(t) < u_i(t) < \mu_i(t) \Big\}. \end{aligned}$$

Let us assume that $C_j \neq \emptyset$ and let us observe that for each $i \in C_j$ it results

$$\lim_{n\to+\infty} \mu_i(t_n) - \Bigg[u_i(t) + \frac{\rho_j(t_n) - \rho_j(t)}{\sum_{i\in C_j} \xi_{ji}} - \frac{\sum_{i\in A_j}[\lambda_i(t_n) - \lambda_i(t)]}{\sum_{i\in C_j} \xi_{ji}} \\ - \frac{\sum_{i\in B_j}[\mu_i(t_n) - \mu_i(t)]}{\sum_{i\in C_j} \xi_{ji}} \Bigg] = \mu_i(t) - u_i(t) > 0,$$

$$\lim_{n\to+\infty} u_i(t) \quad + \quad \frac{\rho_j(t_n)-\rho_j(t)}{\sum_{i\in C_j}\xi_{ji}} - \frac{\sum_{r\in A_j}[\lambda_i(t_n)-\lambda_i(t)]}{\sum_{i\in C_j}\xi_{ji}}$$
$$- \quad \frac{\sum_{i\in B_j}[\mu_i(t_n)-\mu_i(t)]}{\sum_{i\in C_j}\xi_{ji}} - \lambda_i(t_n) = u_i(t)-\lambda_i(t) > 0.$$

Then there exists an index ν_j such that for $n > \nu_j$ and $i \in C_j$ we have

$$\lambda_i(t) \leq u_i(t) + \frac{\rho_j(t_n)-\rho_j(t)}{\sum_{i\in C_j}\xi_{ji}} - \frac{\sum_{i\in A_j}[\lambda_i(t_n)-\lambda_i(t)]}{\sum_{i\in C_j}\xi_{ji}} - \frac{\sum_{i\in B_j}[\mu_i(t_n)-\mu_i(t)]}{\sum_{i\in C_j}\xi_{ji}} \leq \mu_i(t).$$

Hence we can consider a sequence $F(t_n)$ such that:

- for $n > \nu_j$ and $\xi_{ji} = 1$, $j = 1, 2, \ldots, l$

$$u_i(t_n) = \begin{cases} \lambda_i(t_n) & \text{for } i \in A_j \\ \mu_i(t_n) & \text{for } i \in B_j \\ u_i(t) + \dfrac{\rho_j(t_n)-\rho_j(t)}{\sum_{i\in C_j}\xi_{ji}} - \dfrac{\sum_{i\in A_j}[\lambda_i(t_n)-\lambda_i(t)]}{\sum_{i\in C_j}\xi_{ji}} & \\ \qquad - \dfrac{\sum_{i\in B_j}[\mu_i(t_n)-\mu_i(t)]}{\sum_{i\in C_j}\varphi_{ji}} & \text{for } i \in C_j \end{cases}$$

- and for $n \leq \nu_j$, $\xi_{ji} = 1$, $j = 1, 2, \ldots, l$

$$u_i(t_n) = P_{\mathbf{K}(t_n)} u_i(t),$$

  where $P_{\mathbf{K}(t_n)}$ denotes the hilbertian projection on $\mathbf{K}(t_n)$.

Obviously if $n \leq \nu_j$ it results $u_i(t_n) \in \mathbf{K}(t_n)$, whereas for $n > \nu_j$ we have

$$\lambda_r(t_n) \leq u_i(t_n) \leq \mu_i(t_n)$$

and

$$\begin{aligned} \sum_{i=1}^{q} \xi_{ji} u_i(t_n) &= \sum_{i=1}^{q} u_i(t) + \sum_{i\in A_j}[\lambda_i(t_n)-\lambda_i(t)] + \sum_{i\in B_j}[\mu_i(t_n)-\mu(t)] \\ &\quad + \sum_{i\in C_j}\frac{\rho_j(t_n)-\rho_j(t)}{\sum_{i\in C_j}\xi_{ji}} - \sum_{i\in C_j}\frac{\sum_{i\in A_j}[\lambda_i(t_n)-\lambda_i(t)]}{\sum_{i\in C_j}\xi_{ji}} \\ &\quad - \sum_{i\in C_j}\frac{\sum_{i\in B_j}[\mu_i(t_n)-\mu_i(t)]}{\sum_{i\in C_j}\xi_{ji}} \\ &= \rho_j(t_n). \end{aligned}$$

So $u(t_n) \in \mathbf{K}(t_n)$, $\forall n \in \mathbb{N}$ and it results $\lim_{n\to+\infty} u(t_n) = u(t)$.

Now, let us assume that $C_j = \emptyset$ and let us observe that it results

$$\begin{aligned}\lim_{n\to+\infty}\left[\lambda_i(t_n)\right. &+ \left.\frac{1}{\sum_{i\in A_j}\xi_{ji}}\max\left(0,\rho_j(t_n)-\sum_{i\in A_j}\lambda_i(t_n)-\sum_{i\in B_j}\mu_i(t_n)\right)\right]-\mu_i(t_n)\\ &= \lambda_i(t)-\mu_i(t)<0,\quad i\in A_j,\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to+\infty}\left[\mu_i(t_n)\right. &+ \left.\frac{1}{\sum_{i\in B_j}\xi_{ji}}\min\left(0,\rho_j(t_n)-\sum_{i\in A_j}\lambda_i(t_n)-\sum_{i\in B_j}\mu_i(t_n)\right)\right]-\lambda_i(t_n)\\ &= \mu_i(t)-\lambda_i(t)>0,\quad i\in B_j.\end{aligned}$$

Then there exists an index ν_j such that for $n>\nu_j$ it results

$$\lambda_i(t_n)\le\lambda_i(t_n)+\frac{1}{\sum_{i\in A_j}\varphi_{ji}}\max\left(0,\rho_j(t_n)-\sum_{i\in A_j}\lambda_i(t_n)-\sum_{i\in B_j}\mu_i(t_n)\right)\le\mu_i(t_n)$$

$$\lambda_i(t_n)\le\mu_i(t_n)+\frac{1}{\sum_{i\in B_j}\varphi_{ji}}\min\left(0,\rho_j(t_n)-\sum_{i\in A_j}\lambda_i(t_n)-\sum_{i\in B_j}\mu_i(t_n)\right)\le\mu_i(t_n).$$

So, we can choose the sequence $u(t_n)$ in the following way:

- for $n>\nu_j$, $\varphi_{ji}=1$, $j=1,2,\dots,l$

$$u_i(t_n)=\begin{cases}\lambda_i(t_n)+\dfrac{1}{\sum_{i\in A_j}\xi_{ji}}\max\left(0,\rho_j(t_n)-\sum_{i\in A_j}\lambda_i(t_n)-\sum_{i\in B_j}\mu_i(t_n)\right)\\ \hfill\text{for } i\in A_j\\ \mu_i(t_n)+\dfrac{1}{\sum_{i\in B_j}\xi_{ji}}\min\left(0,\rho_j(t_n)-\sum_{i\in A_j}\lambda_i(t_n)-\sum_{i\in B_j}\mu_i(t_n)\right)\\ \hfill\text{for } i\in B_j\end{cases}$$

- whereas for $n\le\nu_j$, $\varphi_{ji}=1$, $j=1,2,\dots,l$

$$u_i(t_n)=P_{\mathbf{K}(t_n)}u_i(t).$$

It results $\lim_{n\to+\infty}u(t_n)=u(t)$ and for $n>\nu_j$

$$\begin{aligned}\sum_{i=1}^{q}\xi_{ji}u_i(t_n) &= \sum_{i\in A_j}\lambda_i(t_n)+\max\left(0,\rho_j(t_n)-\sum_{i\in A_j}\lambda_i(t_n)-\sum_{i\in B_j}\mu_i(t_n)\right)\\ &\quad+\sum_{i\in B_j}\mu_i(t_n)+\min\left(0,\rho_j(t_n)-\sum_{i\in A_j}\lambda_i(t_n)-\sum_{i\in B_j}\mu_i(t_n)\right)\\ &= \rho_j(t_n).\end{aligned}$$

The proof of the first condition (M1) has been obtained.

For the second one, let $\{u(t_n)\}_{n\in\mathbb{N}}$ be a fixed sequence, with $u(t_n) \in \mathbf{K}(t_n)$, $\forall n \in \mathbb{N}$, such that $u(t_n) \rightharpoonup u(t)$ (weakly) in $\mathbb{R}^m$. Since $\mathbb{R}^m$ is a finite-dimensional space, the weak convergence is equivalent to $u(t_n) \to u(t)$ (strongly) in $\mathbb{R}^m$. It remains to prove that $u(t) \in \mathbf{K}(t)$. From $u(t_n) \in \mathbf{K}(t_n)$, $\forall n \in \mathbb{N}$, we derive

$$\lambda(t_n) \leq u(t_n) \leq \mu(t_n), \quad \forall n \in \mathbb{N}, \tag{4.4.1}$$

$$\sum_{i=1}^{q} \xi_{ji} u_i(t_n) = \rho_j(t_n), \quad \forall n \in \mathbb{N},\ j = 1, 2, \ldots, l. \tag{4.4.2}$$

Passing to the limit for $n \to +\infty$ in (4.4.1), and using the continuity of λ and μ on $[0, T]$, we obtain

$$\lambda(t) \leq u(t) \leq \mu(t),$$

and, from (4.4.2), we have for the continuity of ρ on $[0, T]$

$$\sum_{i=1}^{q} \xi_{ji} u_i(t) = \rho_j(t), \quad j = 1, 2, \ldots, l.$$

Then

$$u(t) \in \mathbf{K}(t),$$

so the second condition (M2) has also been proved.

Hence, we conclude that

$$\mathbf{K}(t_n) \to \mathbf{K}(t) \text{ in Mosco's sense},$$

for all sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq [0, T]$, such that $t_n \to t$. □

At last, we prove the following boundedness result.

Proposition 4.4.2. *Let $\lambda, \mu \in C([0, T], \mathbb{R}^q)$ and let $\rho \in C([0, T], \mathbb{R}^l_+)$ be vector-functions. Then, the set*

$$\mathbf{K}(t) = \Big\{ u(t) \in \mathbb{R}^q : \quad \lambda(t) \leq u(t) \leq \mu(t),\ \sum_{i=1}^{q} \xi_{ji} u_i(t) = \rho_j(t), \quad \xi_{ji} \in \{-1, 0, 1\},\ i \in \{1, \ldots, q\},\ j \in \{1, \ldots, l\} \Big\}.$$

is uniformly bounded in $[0, T]$.

Proof. Let us fix an arbitrary vector-function $u(t)$ in $\mathbf{K}(t)$, then

$$\lambda(t) \leq u(t) \leq \mu(t).$$

Since $\lambda, \mu \in C([0, T], \mathbb{R}^m_+)$, it results

$$\|u(t)\|_q \leq \max_{t\in[0,T]} \mu(t) = C, \quad \text{in } [0, T],$$

where C is a constant independent on $t \in [0, T]$. □

5

Regularity results for evolutionary variational inequalities

5.1 Introduction

An aspect very important of the theory of evolutionary variational inequalities is the property of the regularity. Up till now, few authors take an interest in this. In 1969, U. Mosco (see [70]) studied the convergence of solutions to variational inequalities in Hilbert spaces. Now, we obtain similar results for variational inequalities which depend explicitly on the time and the set $\mathbf{K}$ satisfies the next assumption

(M) $\mathbf{K} \subseteq L^2([0,T], \mathbb{R}^m)$ is a nonempty convex, closed set, such that the set sequence $\{\mathbf{K}(t_n)\}_{n\in\mathbb{N}}$ converges to $\mathbf{K}(t)$ in Mosco's sense, for each $t \in [0,T]$, and the sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,T]$, such that $t_n \to t$, as $n \to +\infty$.

In particular, we prove that solutions are continuous with respect to the time. At first, we show results for strongly monotone evolutionary variational inequalities and, then, we generalize them for degenerate and strictly monotone evolutionary variational inequalities. The generalization will be possible making use of a regularization procedure. The results will be obtained both in the affine case and nonlinear case. For the prove of the second one, an important instrument will be a type Minty's Lemma.

5.2 Regularity results for strongly monotone evolutionary variational inequalities

In this section, a theorem of continuity for solutions to evolutionary strongly monotone variational inequalities will be proved. More precisely, at first we obtain the result for an affine operator. This is a consequence of the abstract theorem 4.2.1. Then, we generalize the result for nonlinear operators. The Minty's Lemma for variational inequalities and the notion of the sets convergence in Mosco's sense play an important role in the attainment of this result.

5.2.1 Affine case

Let us assume that the operator is affine with respect to the vector F, namely it results

$$C(t, F(t)) = A(t)F(t) + B(t),$$

for each $t \in [0, T]$, where $A : [0, T] \to \mathbb{R}^{m\times m}$ and $B : [0, T] \to \mathbb{R}^m$ are two functions. We study the continuity for solutions to the following evolutionary variational inequality

Find $H \in \mathbf{K}$ such that

$$\langle A(t)H(t) + B(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0, T], \tag{5.2.1}$$

where $\mathbf{K} \subseteq L^2([0, T], \mathbb{R}^m)$ satisfies condition (M).

The following result holds (see also [8], Theorem 3.2).

Theorem 5.2.1. *Let $A \in C([0, T], \mathbb{R}^{m\times m})$ be a positive definite matrix-function and let $B \in C([0, T], \mathbb{R}^m)$ be a vector function. Let $\mathbf{K} \subseteq L^2([0, T], \mathbb{R}^m)$ be a set satisfying condition* (M). *Then, evolutionary variational inequality* (5.2.1) *admits a unique solution $H \in \mathbf{K}$ such that $H \in C([0, T], \mathbb{R}^m)$.*

Proof. By virtue of Theorem 3.2.4 and being A a positive definite matrix-function, we have that (5.2.1) admits a unique solution $H(t) \in \mathbf{K}(t)$, for $t \in [0, T]$.

Now, we prove the continuity of the solution applying Theorem 4.2.1. Let $t \in [0, T]$ be fixed and let $\{t_n\}_{n\in\mathbb{N}} \subseteq [0, T]$ be a sequence, with $t_n \to t \in [0, T]$, as $n \to +\infty$. From the assumption of continuity of the function A, one has

$$A(t_n) \to A(t) \quad \text{in } \mathbb{R}^{m\times m},$$

moreover, if $\{F(t_n)\}_{n\in\mathbb{N}}$ is a sequence, with $F(t_n) \in \mathbf{K}(t_n)$, such that $F(t_n) \to F(t)$ in $\mathbb{R}^m$, it results

$$A(t_n)F(t_n) \to A(t)F(t) \quad \text{in } \mathbb{R}^m.$$

Finally, for the continuity of the function B we have

$$B(t_n) \to B(t) \quad \text{in } \mathbb{R}^m.$$

Taking into account that the set $\mathbf{K} \subseteq L^2([0, T], \mathbb{R}^m)$ satisfies condition (M) and using the stability Theorem 4.2.1, we can conclude that the unique solution $H(t_n)$ of

$$\langle A(t_n)H(t_n) + B(t_n), F(t_n) - H(t_n)\rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n),$$

converge strongly to the solution $H(t)$ of the limit problem (5.2.1), namely,

$$H(t_n) \to H(t) \quad \text{in } \mathbb{R}^m,$$

that implies $H \in C([0, T], \mathbb{R}^m)$. □

Now, we still assume that the cost $C(t, F(t))$ is a linear operator with respect to the flows, but the matrix-function A depends on time and on integral average of the flow vectors, namely,

$$C(t, F(t)) = A(t, F_{\mathcal{T}})F(t) + B(t),$$

for a.e. $t \in [0, T]$ and for every $F \in L^2([0, T], \mathbb{R}^m)$, where $A : [0, T] \times \mathbb{R}^m \to \mathbb{R}^{m\times m}$ and $B : [0, T] \to \mathbb{R}^m$ are two functions, $\mathcal{T} = [0, T]$ and $F_{\mathcal{T}}$ is the integral average, namely,

$$F_{\mathcal{T}} = \frac{\int_0^T F(\tau)d\tau}{T}.$$

We suppose that $A(t, u)$ is a bounded matrix, namely

$$\exists M > 0 : \ \|A(t, u)\|_{m\times m} \leq M, \quad \text{for a.e. } t \in [0, T], \ \forall u \in \mathbb{R}^m. \tag{5.2.2}$$

Then, we study the continuity of the solution to the following evolutionary variational inequality:

Find $H \in \mathbf{K}$ such that

$$\langle A(t, H_{\mathcal{T}})H(t) + B(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0, T], \tag{5.2.3}$$

where $\mathbf{K} \subseteq L^2([0, T], \mathbb{R}^m)$ satisfies condition (M).

Taking into account Theorem 5.2.1, we can derive for the variational inequality (5.2.3) the following result.

Theorem 5.2.2. *Let $A \in C([0, T] \times \mathbb{R}^m, \mathbb{R}^{m\times m})$ be a matrix-function verifying the condition*

$$\exists \nu > 0 : \ \langle A(t, F_{\mathcal{T}})F, F\rangle \geq \nu\|F\|^2_{\mathbb{R}^m}, \quad \forall F \in \mathbb{R}^m, in\ [0, T], \tag{5.2.4}$$

and let $B \in C([0, T], \mathbb{R}^m)$ be a vector function. Let $\mathbf{K} \subseteq L^2([0, T], \mathbb{R}^m)$ be a set satisfying condition (M). *Then, evolutionary variational inequality* (5.2.3) *admits a unique solution $H \in \mathbf{K}$ such that $H \in C([0, T], \mathbb{R}^m)$.*

Proof. The existence and the uniqueness of solution to evolutionary variational inequality (5.2.3) is guaranteed by Theorem 3.2.3 and by condition (5.2.4).

Following the proof of Theorem 5.2.1, we have the continuity of the solution to (5.2.3), since $A(t, H_\tau)$ is continuous in t and H_τ is a constant. □

5.2.2 Nonlinear case

Now, we extend Theorem 5.2.1 to the following nonlinear evolutionary variational inequality

Find $H \in \mathbf{K}$ such that

$$\langle C(t, H(t)), F(t) - H(t) \rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ in } [0, T]. \tag{5.2.5}$$

where $\mathbf{K} \subseteq L^2([0,T], \mathbb{R}^m)$ satisfies condition (M).

Let us assume that the operator $C : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ verifies the following assumptions:

$$\|C(t, F)\|_m \leq A(t)\|F\|_m + B(t), \quad \forall F \in \mathbb{R}^m, \text{ in } [0,T], \tag{5.2.6}$$

with $B \in L^2([0,T])$ and $A \in L^\infty([0,T])$, and it results

$$\exists \nu > 0 : \langle C(t, H) - C(t, F), H - F \rangle \geq \nu \|H - F\|_m^2, \quad \forall H, F \in \mathbb{R}^m, \textit{ in } [0,T]. \tag{5.2.7}$$

Then, the following result holds (see also [7], Theorem 6).

Theorem 5.2.3. *Let $C \in C([0,T] \times \mathbb{R}^m, \mathbb{R}^m)$ be an operator verifying conditions* (5.2.6) *and* (5.2.7)*. Let $\mathbf{K} \subseteq L^2([0,T], \mathbb{R}^m)$ be a set satisfying condition* (M)*. Then, evolutionary variational inequality* (5.2.5) *admits a unique solution $H \in \mathbf{K}$ such that $H \in C([0,T], \mathbb{R}^m)$.*

Proof. Taking into account Theorem 3.2.3 and condition (5.2.7), it results that (5.2.5) admits a unique solution $H(t) \in \mathbf{K}(t)$, for $t \in [0,T]$.

Let $t \in [0,T]$ be fixed and let $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,T]$ be a sequence, with $t_n \to t$. Our statement is equivalent to say that the unique solution $H(t_n)$, for $n \in \mathbb{N}$, to the following variational inequality

$$H(t_n) \in \mathbf{K}(t_n) : \langle C(t_n, H(t_n)), F(t_n) - H(t_n) \rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n), \tag{5.2.8}$$

converges strongly, as $n \to +\infty$, to the solution $H(t)$ to the limit problem

$$H(t) \in \mathbf{K}(t) : \langle C(t, H(t)), F(t) - H(t) \rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \tag{5.2.9}$$

namely

$$\lim_{n \to +\infty} H(t_n) = H(t) \quad \text{in } \mathbb{R}^m.$$

For the solution $H(t) \in \mathbf{K}(t)$ to (5.2.9), we use the properties of the set convergence in Mosco's sense of $\{\mathbf{K}(t_n)\}_{n\in\mathbb{N}}$ to $\mathbf{K}(t)$, as $n \to +\infty$. Then, it is possible to choose a sequence $\{G(t_n)\}_{n\in\mathbb{N}}$, with $G(t_n) \in \mathbf{K}(t_n)$, $\forall n \in \mathbb{N}$, such that,

$$\lim_{n \to +\infty} G(t_n) = H(t) \quad \text{in } \mathbb{R}^m$$

and, by virtue of the assumption on the operator C, we obtain

$$\lim_{n\to+\infty} C(t_n, G(t_n)) = C(t, H(t)) \quad \text{in } \mathbb{R}^m.$$

Setting $F(t_n) = G(t_n)$ in (5.2.8), we have

$$\langle C(t_n, H(t_n)), G(t_n) - H(t_n)\rangle \geq 0, \tag{5.2.10}$$

and using the strong monotonicity of the operator C, it results

$$\langle C(t_n, H(t_n)) - C(t_n, G(t_n)), H(t_n) - G(t_n)\rangle \geq \nu \|H(t_n) - G(t_n)\|_m^2.$$

From (5.2.10) we derive that

$$\langle C(t_n, H(t_n)) - C(t_n, G(t_n)), H(t_n) - G(t_n)\rangle = \langle C(t_n, H(t_n)), H(t_n) - G(t_n)\rangle$$
$$-\langle C(t_n, G(t_n)), H(t_n) - G(t_n)\rangle \leq -\langle C(t_n, G(t_n)), H(t_n) - G(t_n)\rangle,$$

then

$$\begin{aligned} \nu\|H(t_n) - G(t_n)\|_m^2 &\leq -\langle C(t_n, G(t_n)), H(t_n) - G(t_n)\rangle \\ &\leq \|C(t_n, G(t_n))\|_m \|H(t_n) - G(t_n)\|_m, \end{aligned}$$

that is

$$\nu\|H(t_n) - G(t_n)\|_m \leq \|C(t_n, G(t_n))\|_m.$$

Hence, one deduces

$$\begin{aligned} \|H(t_n)\|_m &\leq \|H(t_n) - G(t_n)\|_m + \|G(t_n)\|_m \\ &\leq \frac{\|C(t_n, G(t_n))\|_m}{\nu} + \|G(t_n)\|_m. \end{aligned}$$

Since $\{C(t_n, G(t_n))\}_{n\in\mathbb{N}}$ is a sequence convergent then it is bounded, namely,

$$\exists h \in \mathbb{R}_+ : \ \|C(t_n, G(t_n))\|_m \leq h, \quad \forall n \in \mathbb{N},$$

for the same reason, the sequence $\{G(t_n)\}_{n\in\mathbb{N}}$ is bounded, namely,

$$\exists k \in \mathbb{R}_+ : \ \|G(t_n)\|_m \leq k, \quad \forall n \in \mathbb{N}.$$

From those conditions, it follows

$$\|H(t_n)\|_m \leq c, \quad \forall n \in \mathbb{N},$$

where the constant c is independent on n. Hence there exists a subsequence $\{H(t_{k_n})\}_{n\in\mathbb{N}}$ converging in $\mathbb{R}^m$ to an element $\widetilde{H}(t) \in \mathbb{R}^m$, and thus

$$\lim_{n\to+\infty} H(t_{k_n}) = \widetilde{H}(t).$$

Moreover, taking into account the second condition of the set convergence in Mosco's sense, we have

$$\widetilde{H}(t) \in \mathbf{K}(t).$$

By virtue of the set convergence in Mosco's sense, it follows

$$\forall F(t) \in \mathbf{K}(t),\ \exists F(t_n) \in \mathbf{K}(t_n)\ \forall n \in \mathbb{N} :\ \lim_{n\to+\infty} F(t_n) = F(t),\ \text{in } \mathbb{R}^m,$$

and from the assumption on the operator C, we get

$$\lim_{n\to+\infty} C(t_n, F(t_n)) = C(t, F(t)),\ \text{in } \mathbb{R}^m.$$

Now, we consider the following variational inequality

$$\langle C(t_{k_n}, F(t_{k_n})), F(t_{k_n}) - H(t_{k_n})\rangle \geq 0,$$

and passing to the limit as $n \to +\infty$, we obtain

$$\widetilde{H}(t) \in \mathbf{K}(t) :\ \langle C(t, F(t)), F(t) - \widetilde{H}(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t),$$

which, again by Lemma 3.2.1 and the uniqueness of the solution to (5.2.9), implies

$$\widetilde{H}(t) = H(t).$$

Then it follows that every subsequence of $\{H(t_n)\}_{n\in\mathbb{N}}$ converges to the same limit $\widetilde{H}(t)$ and hence

$$\lim_{n\to+\infty} H(t_n) = H(t).$$

□

5.3 Regularity results for degenerate evolutionary variational inequalities

In this section, we generalize continuity theorems for solutions to evolutionary variational inequalities associated to strongly monotone operators to these associated to degenerate operators. At first we obtain the result claimed for affine operators and, then, for nonlinear operators.

5.3.1 Affine case

We begin studying under which assumptions the continuity of the solution to affine degenerate evolutionary variational inequality is ensured. We consider the following evolutionary variational inequality

Find $H \in \mathbf{K}$ such that

$$\langle A(t)H(t) + B(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T], \tag{5.3.1}$$

where $\mathbf{K} \subseteq L^2([0,T], \mathbb{R}^m)$ satisfies condition (M).

Let us suppose that the following assumptions are satisfied

$$\exists M > 0 : \ \|A(t)\|_{m\times m} = \Big(\sum_{r,s=1}^{m} A_{rs}^2(t)\Big)^{\frac{1}{2}} \leq M, \quad \text{a.e. in } [0,T], \tag{5.3.2}$$

$$\langle A(t)F, F\rangle \geq \nu(t)\|F\|_m^2, \quad \forall F \in \mathbb{R}^m, \text{ a.e. in } [0,T], \tag{5.3.3}$$

where $\nu \in L^\infty([0,T], \mathbb{R}_0^+)$ is such that

$$\nexists I \subseteq [0,T], \ \mu(I) > 0 : \ \nu(t) = 0, \quad \forall t \in I,$$

being μ Lebesque's measure, in this case we call that A is a degenerate bounded matrix-function.

At first, we prove a preliminary lemma related to nonnegative matrix-function A, namely,

$$\langle A(t)F, F\rangle \geq 0, \quad \forall F \in \mathbb{R}^m, \text{ a.e. in } [0,T], \tag{5.3.4}$$

satisfying condition (5.3.2). Hence, let us observe that, under this assumption, the set $\mathbf{X}$ of solutions to time-dependent variational inequality (5.3.1) is closed, convex and nonempty. Let $I : L^2([0,T], \mathbb{R}^m) \to L^2([0,T], \mathbb{R}^m)$ be the identity operator and let us consider the following evolutionary variational inequality

$$\langle I(t)H(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{X}(t), \text{ a.e. in } [0,T], \tag{5.3.5}$$

which admits a unique solution $H(t) \in \mathbf{X}(t)$. Further, for every $\varepsilon > 0$, let us consider the following perturbed evolutionary variational inequality

$$\langle [A(t) + \varepsilon I(t)]H(t) + B(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T], \tag{5.3.6}$$

which also admits a unique continuous solution H_ε by virtue of Theorem 5.2.1. Then, we can prove the following preliminary result for nonnegative matrix-functionS:

Lemma 5.3.1. *Let $A \in C([0,T], \mathbb{R}^{m\times m})$ be a nonnegative matrix-function, let $B \in C([0,T], \mathbb{R}^m)$ be a vector-function. Let $\mathbf{K} \subseteq L^2([0,T], \mathbb{R}^m)$ be a set satisfying condition* (M) *and such that $\mathbf{K}(t)$ are uniformly bounded for $t \in [0,T]$. If $H_\varepsilon(t)$, $\forall \varepsilon > 0$, is the unique solution to* (5.3.6)*, it results*

$$\lim_{\varepsilon\to 0} H_\varepsilon(t) = H(t), \quad in \ [0,T],$$

and

$$\lim_{\varepsilon\to 0} \|H_\varepsilon(t) - H(t)\|^2_{L^2([0,T],\mathbb{R}^m)} = 0,$$

where H is a solution to evolutionary variational inequality (5.3.1)*.*

Proof. Let H be the unique solution to (5.3.5), therefore $H \in \mathbf{X}$ and

$$\langle I(t)H(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{X}(t), \text{ in } [0,T]. \tag{5.3.7}$$

Let H_ε be the unique solution to (5.3.6), namely $H_\varepsilon \in \mathbf{K}$ and

$$\langle [A(t) + \varepsilon I(t)]H_\varepsilon(t) + B(t), F(t) - H_\varepsilon(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ in } [0,T]. \tag{5.3.8}$$

Setting $F(t) = H_\varepsilon(t)$, for $t \in [0,T]$, in (5.3.1) and $F(t) = H(t)$, for $t \in [0,T]$, in (5.3.8) and adding we get

$$\langle A(t)[H(t) - H_\varepsilon(t)], H_\varepsilon(t) - H(t)\rangle + \varepsilon\langle H_\varepsilon(t), H(t) - H_\varepsilon(t)\rangle \geq 0, \tag{5.3.9}$$

in $[0,T]$. By assumption (5.3.4), it follows

$$\langle A(t)[H(t) - H_\varepsilon(t)], H_\varepsilon(t) - H(t)\rangle \leq 0, \quad \text{in } [0,T],$$

then, by (5.3.9), we obtain

$$\varepsilon\langle H_\varepsilon(t), H(t) - H_\varepsilon(t)\rangle \geq 0, \quad \text{in } [0,T],$$

and dividing by $\varepsilon > 0$, it results

$$\langle H_\varepsilon(t), H(t) - H_\varepsilon(t)\rangle \geq 0, \quad \text{in } [0,T]. \tag{5.3.10}$$

Taking into account (5.3.10), we have

$$\|H_\varepsilon(t)\|_m^2 \leq \langle H_\varepsilon(t), H(t)\rangle \leq \|H(t)\|_m \|H_\varepsilon(t)\|_m, \quad \text{in } [0,T],$$

then

$$\|H_\varepsilon(t)\|_m \leq \|H(t)\|_m, \quad \text{in } [0,T].$$

Since $H(t) \in \mathbf{X}(t) \subseteq \mathbf{K}(t)$, in $[0,T]$, and $\mathbf{K}(t)$, $t \in [0,T]$, is a family of uniformly bounded sets of $\mathbb{R}^m$, it results

$$\|H(t)\|_m \leq C, \quad \text{in } [0,T],$$

with C a constant independent on $t \in [0,T]$, then

$$\|H_\varepsilon(t)\|_m \leq C, \quad \forall \varepsilon > 0, \text{ in } [0,T].$$

Hence there exists a subsequence $\{H_\eta(t)\}_\eta$ converging in $\mathbb{R}^m$ to an element $\overline{H}(t)$ of $\mathbb{R}^m$, in [0,T], and thus

$$\lim_{\eta \to 0} H_\eta(t) = \overline{H}(t), \quad \text{in } [0,T].$$

Taking into account that $\mathbf{K}(t)$ is a closed set of $\mathbb{R}^m$ and $\{H_\eta(t)\}_\eta \subseteq \mathbf{K}(t)$, then

$$\overline{H}(t) \in \mathbf{K}(t), \text{ in } [0,T].$$

It remains to prove that
$$\overline{H}(t) = H(t), \quad \text{in } [0,T].$$
Hence, considering (5.3.8) with $\varepsilon = \eta$, we get
$$\langle A(t)H_\eta(t) + B(t), F(t) - H_\eta(t)\rangle + \eta\langle H_\eta(t), F(t) - H_\eta(t)\rangle \geq 0, \tag{5.3.11}$$
for all $F(t) \in \mathbf{K}(t)$, in $[0,T]$, and taking into account that
$$\lim_{\eta\to 0}\langle H_\eta(t), H_\eta(t)\rangle = \langle \overline{H}(t), \overline{H}(t)\rangle, \quad \text{in } [0,T],$$
and that
$$\lim_{\eta\to 0}\langle A(t)H_\eta(t) + B(t), H_\eta(t)\rangle = \langle A(t)\overline{H}(t) + B(t), \overline{H}(t)\rangle, \quad \text{in } [0,T],$$
from (5.3.11), we obtain
$$\langle A(t)\overline{H}(t) + B(t), F(t) - \overline{H}(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ in } [0,T]. \tag{5.3.12}$$
Then (5.3.12) implies that $\overline{H}$ is a solution to (5.3.1), in $[0,T]$, namely
$$\overline{H} \in \mathbf{X}.$$
If the solution to (5.3.1) is unique, then the proof is concluded. Now, we suppose that the solution to (5.3.1) is not unique. Setting $\varepsilon = \eta$ in (5.3.10) we get
$$\langle H_\eta(t), H(t) - H_\eta(t)\rangle \geq 0, \quad \text{in } [0,T],$$
and passing to the limit as $\eta \to 0$, we obtain
$$\langle \overline{H}(t), H(t) - \overline{H}(t)\rangle \geq 0, \quad \text{in } [0,T]. \tag{5.3.13}$$
Rewriting (5.3.7) with $F = \overline{H} \in \mathbf{X}$, it results
$$\langle H(t), \overline{H}(t) - H(t)\rangle \geq 0, \quad \text{in } [0,T], \tag{5.3.14}$$
and adding (5.3.13) and (5.3.14), we have
$$\langle \overline{H}(t) - H(t), H(t) - \overline{H}(t)\rangle \geq 0, \quad \text{in } [0,T].$$
Then
$$\langle \overline{H}(t) - H(t), H(t) - \overline{H}(t)\rangle = 0, \quad \text{in } [0,T],$$
that implies
$$\overline{H}(t) = H(t), \quad \text{in } [0,T].$$
In this way, we have shown that every convergent subsequence converges to the same limit $H(t)$ and hence
$$\lim_{\varepsilon\to 0} H_\varepsilon(t) = H(t), \quad \text{in } [0,T],$$

from this it follows

$$\lim_{\varepsilon\to 0} \|H_\varepsilon(t) - H(t)\|_m^2 = 0, \quad \text{in } [0,T].$$

Moreover, we remark that

$$\|H_\varepsilon(t) - H(t)\|_m^2 \le 2\big(\|H_\varepsilon(t)\|_m^2 + \|H(t)\|_m^2\big) \le 4C^2, \quad \text{in } [0,T],$$

then, by virtue of Lebesgue's Theorem we have

$$\lim_{\varepsilon\to 0} \|H_\varepsilon(t) - H(t)\|_{L^2([0,T],\mathbb{R}^m)}^2 = 0.$$

□

Now, we present the main result for degenerate evolutionary variational inequality (5.3.1), namely, when the matrix-function A verifies the conditions (5.3.2) and (5.3.3) (see also [4], Theorem 3.2).

Theorem 5.3.1. *Let $A \in C([0,T],\mathbb{R}^{m\times m})$ be a matrix-function satisfying condition* (5.3.3) *and let $B \in C([0,T],\mathbb{R}^m)$ be a vector-function. Let $\mathbf{K} \subseteq L^2([0,T],\mathbb{R}^m)$ be a set satisfying condition* (M) *and such that $\mathbf{K}(t)$ are uniformly bounded for $t \in [0,T]$. Then, the evolutionary variational inequality*

$$\langle A(t)H(t) + B(t), F(t) - H(t)\rangle \ge 0, \quad \forall F(t) \in \mathbf{K}(t), \quad \textit{in } [0,T], \tag{5.3.15}$$

admits a unique solution $H \in \mathbf{K}$ such that $H \in C([0,T],\mathbb{R}^m)$.

Proof. The existence and the uniqueness of solution to evolutionary variational inequality (5.3.15) is guaranteed by Theorem 3.3.1. Now, we prove that the solution is continuous in $[0,T]$.

Let $t \in [0,T]$ be fixed and let $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,T]$ be a sequence, with $t_n \to t$, as $n \to +\infty$.

Let us consider the solution $H(t)$ to variational inequality (5.3.15) and the solution $H(t_n)$, $\forall n \in \mathbb{N}$, to the following variational inequalities

$$\langle A(t_n)H(t_n) + B(t_n), F(t_n) - H(t_n)\rangle \ge 0, \quad \forall F(t_n) \in \mathbf{K}(t_n),\ \forall n \in \mathbb{N}. \tag{5.3.16}$$

Let $H_\varepsilon(t)$ be the unique solution to the strongly monotone perturbed variational inequality (5.3.6), namely $H_\varepsilon(t) \in \mathbf{K}(t)$ and

$$\langle [A(t) + \varepsilon I(t)]H_\varepsilon(t) + B(t), F(t) - H_\varepsilon(t)\rangle \ge 0, \quad \forall F(t) \in \mathbf{K}(t),\ \text{in } [0,T]. \tag{5.3.17}$$

Since $H_\varepsilon(t)$ is continuous in $[0,T]$, we have that solutions $H_\varepsilon(t_n)$, $\forall n \in \mathbb{N}$, to the following evolutionary variational inequalities

$$\langle [A(t_n) + \varepsilon I(t_n)]H_\varepsilon(t_n) + B(t_n), F(t_n) - H_\varepsilon(t_n)\rangle \ge 0, \quad \forall F(t_n) \in \mathbf{K}(t_n), \tag{5.3.18}$$

$\forall n \in \mathbb{N}$, converge to $H_\varepsilon(t)$, as $n \to +\infty$. Setting $F(t_n) = H(t_n)$, $\forall n \in \mathbb{N}$, in (5.3.18) and $F(t_n) = H_\varepsilon(t_n)$, $\forall n \in \mathbb{N}$, in (5.3.16) and adding we get, $\forall n \in \mathbb{N}$

$$\langle A(t_n)[H_\varepsilon(t_n) - H(t_n)], H(t_n) - H_\varepsilon(t_n)\rangle + \varepsilon\langle H_\varepsilon(t_n), H(t_n) - H_\varepsilon(t_n)\rangle \geq 0. \quad (5.3.19)$$

We remark that for condition (5.3.3) on the matrix-function A we have

$$\langle A(t_n)[H_\varepsilon(t_n) - H(t_n)], H(t_n) - H_\varepsilon(t_n)\rangle \leq 0, \quad \forall n \in \mathbb{N}.$$

Then, from (5.3.19) it follows

$$\varepsilon\langle H_\varepsilon(t_n), H(t_n) - H_\varepsilon(t_n)\rangle \geq 0, \quad \forall n \in \mathbb{N},$$

and proceeding as in the proof of Lemma 5.3.1, we get

$$\|H_\varepsilon(t_n)\|_m \leq C, \quad \forall \varepsilon > 0, \ \forall n \in \mathbb{N}, \quad (5.3.20)$$

where C is a constant independent on ε and on $n \in \mathbb{N}$.

For Lemma 5.3.1, it follows

$$\lim_{\varepsilon \to 0} H_\varepsilon(t_n) = \widetilde{H}(t_n), \quad \forall n \in \mathbb{N},$$

with $\widetilde{H}(t_n) \in \mathbf{K}(t_n)$, $\forall n \in \mathbb{N}$, and such that

$$\langle A(t_n)\widetilde{H}(t_n) + B(t_n), F(t_n) - \widetilde{H}(t_n)\rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n), \ \forall n \in \mathbb{N}.$$

Since the solution to (5.3.16) is unique, one has

$$\widetilde{H}(t_n) = H(t_n), \quad \forall n \in \mathbb{N},$$

and, passing to the limit as $\varepsilon \to 0$ in (5.3.20), it results

$$\|H(t_n)\|_m \leq C, \quad \forall n \in \mathbb{N}.$$

Hence the sequence $\{H(t_n)\}_{n\in\mathbb{N}}$ is bounded, then there exists a subsequence $\{H(t_{k_n})\}_{n\in\mathbb{N}}$, with $H(t_{k_n}) \in \mathbf{K}(t_{k_n})$, $\forall n \in \mathbb{N}$, converging in $\mathbb{R}^m$ to an element $\overline{H}(t)$ of $\mathbb{R}^m$, namely

$$\lim_{n \to +\infty} H(t_{k_n}) = \overline{H}(t).$$

Moreover, by (5.3.16) it obtains

$$\langle A(t)\overline{H}(t) + B(t), F(t) - \overline{H}(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t),$$

and, for the uniqueness of the solution to (5.3.15), it follows

$$\overline{H}(t) = H(t).$$

The same result holds for each subsequence and therefore

$$\lim_{n \to +\infty} H(t_n) = H(t),$$

namely our assert. The proof is now complete. □

5.3.2 Nonlinear case

The previous result can be extended to nonlinear degenerate evolutionary variational inequalities, as we can see in [9]. More precisely, we prove the continuity result supposing that the operator $C : [0,T]\times\mathbb{R}^m \to \mathbb{R}^m$ verifies the following assumptions:

$$\|C(t,F)\|_m \leq A(t)\|F\|_m + B(t), \quad \forall F \in \mathbb{R}^m, \text{ in } [0,T], \tag{5.3.21}$$

with $B \in L^2([0,T])$ and $A \in L^\infty([0,T])$, and

$$\langle C(t,H) - C(t,F), H - F\rangle \geq \nu(t)\|H - F\|_m^2, \quad \forall H, F \in \mathbb{R}^m, \ a.e.\ in\ [0,T], \tag{5.3.22}$$

where $\nu \in L^\infty([0,T],\mathbb{R}_0^+)$ is such that

$$\nexists I \subseteq [0,T],\ \mu(I) > 0 :\ \nu(t) = 0, \quad \forall t \in I,$$

being μ Lebesgue's measure.

Analogously to the linear case, to prove that the unique solution to a nonlinear degenerate evolutionary variational inequality is continuous, we need a preliminary Lemma related to monotone operator, namely C satisfies the following condition

$$\langle C(t,H) - C(t,F), H - F\rangle \geq 0, \quad \forall H, F \in \mathbb{R}^m, \ a.e.\ in\ [0,T]. \tag{5.3.23}$$

We observe that, under this assumption, the set $\mathbf{X}$ of solutions to the evolutionary variational inequality

Find $H \in \mathbf{K}$ such that

$$\langle C(t,H(t)), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T], \tag{5.3.24}$$

where $\mathbf{K} \in L^2([0,T],\mathbb{R}^m)$ satisfies condition (M),

is closed, convex and nonempty. Let $I : L^2([0,T],\mathbb{R}^m) \to L^2([0,T],\mathbb{R}^m)$ be the identity operator and let us consider the following evolutionary variational inequality

$$\langle IH(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{X}(t), \text{ in } [0,T]. \tag{5.3.25}$$

Then, variational inequality (5.3.25) admits a unique solution. Further, for every $\varepsilon > 0$, let us consider the following perturbed evolutionary variational inequality

$$\langle C(t,H(t)) + \varepsilon IH(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ in } [0,T], \tag{5.3.26}$$

which also admits a unique continuous solution H_ε by virtue of Theorem 5.2.3. Then, we can prove the following preliminary result for a monotone operator:

Lemma 5.3.2. *Let $C \in C([0,T] \times \mathbb{R}^m, \mathbb{R}^m)$ be an operator satisfying conditions* (5.3.21) *and* (5.3.23). *Let $\mathbf{K} \subseteq L^2([0,T], \mathbb{R}^m)$ be a set satisfying condition* (M) *and such that $\mathbf{K}(t)$ are uniformly bounded for $t \in [0,T]$. If $H_\varepsilon(t)$, $\forall \varepsilon > 0$, is the unique solution to* (5.3.26), *it results*

$$\lim_{\varepsilon \to 0} H_\varepsilon(t) = H(t), \quad \textit{in } [0,T],$$

and

$$\lim_{\varepsilon \to 0} \|H_\varepsilon(t) - H(t)\|^2_{L^2([0,T],\mathbb{R}^m)} = 0,$$

where H is a solution to evolutionary variational inequality (5.3.24) *(and to* (5.3.25)*).*

Proof. Let H be the unique solution to (5.3.25), therefore $H \in \mathbf{X}$ and

$$\langle IH(t), F(t) - H(t) \rangle \geq 0, \quad \forall F(t) \in \mathbf{X}(t), \text{ in } [0,T]. \tag{5.3.27}$$

Let H_ε be the unique solution to (5.3.26), namely $H_\varepsilon \in \mathbf{K}$ and

$$\langle C(t, H(t)) + \varepsilon I H_\varepsilon(t), F(t) - H_\varepsilon(t) \rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ in } [0,T]. \tag{5.3.28}$$

Setting $F(t) = H_\varepsilon(t)$, for $t \in [0,T]$, in (5.3.26) and $F(t) = H(t)$, for $t \in [0,T]$, in (5.3.25) and adding we obtain

$$\langle C(t, H(t)) - C(t, H_\varepsilon(t)), H_\varepsilon(t) - H(t) \rangle + \varepsilon \langle H_\varepsilon(t), H(t) - H_\varepsilon(t) \rangle \geq 0, \tag{5.3.29}$$

in $[0,T]$. By assumption (5.3.23), it follows

$$\langle C(t, H(t)) - C(t, H_\varepsilon(t)), H_\varepsilon(t) - H(t) \rangle \leq 0, \quad \text{in } [0,T],$$

then, by (5.3.29), we have

$$\varepsilon \langle H_\varepsilon(t), H(t) - H_\varepsilon(t) \rangle \geq 0, \quad \text{in } [0,T],$$

and dividing by $\varepsilon > 0$, it results

$$\langle H_\varepsilon(t), H(t) - H_\varepsilon(t) \rangle \geq 0, \quad \text{in } [0,T]. \tag{5.3.30}$$

Taking into account (5.3.30), we get

$$\|H_\varepsilon(t)\|_m^2 \leq \langle H_\varepsilon(t), H(t) \rangle \leq \|H(t)\|_m \|H_\varepsilon(t)\|_m, \quad \text{in } [0,T],$$

then

$$\|H_\varepsilon(t)\|_m \leq \|H(t)\|_m, \quad \text{in } [0,T].$$

Since $H(t) \in \mathbf{X}(t) \subseteq \mathbf{K}(t)$, in $[0,T]$, and $\mathbf{K}(t)$, $t \in [0,T]$, is a family of uniformly bounded sets of $\mathbb{R}^m$, it results

$$\|H(t)\|_m \leq C, \quad \text{in } [0,T],$$

with C a constant independent on ε and of $t \in [0, T]$, namely

$$\|H_\varepsilon(t)\|_m \leq C, \quad \forall \varepsilon > 0, \text{ in } [0, T].$$

Then, there exists a subsequence $\{H_\eta(t)\}_\eta$ converging in $\mathbb{R}^m$ to an element $\overline{H}(t)$ of $\mathbb{R}^m$, in [0,T], so

$$\lim_{\eta \to 0} H_\eta(t) = \overline{H}(t), \quad \text{in } [0, T].$$

Since $\mathbf{K}(t)$ is a closed set of $\mathbb{R}^m$ and $\{H_\eta(t)\}_\eta \subseteq \mathbf{K}(t)$, it results

$$\overline{H}(t) \in \mathbf{K}(t), \text{ in } [0, T].$$

It remains to prove that

$$\overline{H}(t) = H(t), \quad \text{in } [0, T].$$

Then, considering (5.3.28) with $\varepsilon = \eta$, we obtain

$$\langle C(t, H_\eta(t)), F(t) - H_\eta(t)\rangle + \eta\langle H_\eta(t), F(t) - H_\eta(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \tag{5.3.31}$$

in $[0, T]$, and making use that

$$\lim_{\eta \to 0}\langle H_\eta(t), H_\eta(t)\rangle = \langle \overline{H}(t), \overline{H}(t)\rangle, \quad \text{in } [0, T],$$

and that

$$\lim_{\eta \to 0}\langle C(t, H_\eta(t)), H_\eta(t)\rangle = \langle C(t, \overline{H}(t)), \overline{H}(t)\rangle, \quad \text{in } [0, T],$$

from (5.3.31), we have

$$\langle C(t, \overline{H}(t)), F(t) - \overline{H}(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ in } [0, T]. \tag{5.3.32}$$

Hence from (5.3.32) it follows that $\overline{H}$ is a solution to (5.3.24), in $[0, T]$, namely

$$\overline{H} \in \mathbf{X}.$$

If the solution to (5.3.24) is unique, then the proof is concluded, because each subsequence $\{H_\eta(t)\}_\eta$ converges to the same $\overline{H}(t)$, as $\eta \to 0$. And, hence, the whole sequence $\{H_\varepsilon(t)\}_\varepsilon$ converges to $\overline{H}(t)$, as $\varepsilon \to 0$.

Now, we suppose that the solution to (5.3.24) is not unique. Setting $\varepsilon = \eta$ in (5.3.30) we obtain

$$\langle H_\eta(t), H(t) - H_\eta(t)\rangle \geq 0, \quad \text{in } [0, T],$$

and passing to the limit for $\eta \to 0$, we get

$$\langle \overline{H}(t), H(t) - \overline{H}(t)\rangle \geq 0, \quad \text{in } [0, T]. \tag{5.3.33}$$

Setting $F = \overline{H} \in \mathbf{X}$ in (5.3.27), we have

$$\langle H(t), \overline{H}(t) - H(t)\rangle \geq 0, \quad \text{in } [0, T], \tag{5.3.34}$$

and adding (5.3.33) and (5.3.34), we obtain

$$\langle \overline{H}(t) - H(t), H(t) - \overline{H}(t)\rangle \geq 0, \quad \text{in } [0,T].$$

Then

$$\langle \overline{H}(t) - H(t), H(t) - \overline{H}(t)\rangle = 0, \quad \text{in } [0,T],$$

that implies

$$\overline{H}(t) = H(t), \quad \text{in } [0,T].$$

We have proved that every subsequence converges to the same limit $H(t)$ and then

$$\lim_{\varepsilon\to 0} H_\varepsilon(t) = H(t), \quad \text{in } [0,T].$$

Moreover, it results

$$\|H_\varepsilon(t) - H(t)\|_m^2 \leq 2\big(\|H_\varepsilon(t)\|_m^2 + \|H(t)\|_m^2\big) \leq 4C^2, \quad \text{in } [0,T],$$

hence, by virtue of Lebesgue's Theorem we get

$$\lim_{\varepsilon\to 0} \|H_\varepsilon(t) - H(t)\|^2_{L^2([0,T],\mathbb{R}^m)} = 0.$$

□

Now, we can prove the continuity result for nonlinear degenerate evolutionary variational inequalities.

Theorem 5.3.2. *Let $C \in C([0,T]\times\mathbb{R}^m,\mathbb{R}^m)$ be an operator satisfying conditions* (5.3.21) *and* (5.3.22). *Let $\mathbf{K} \subseteq L^2([0,T],\mathbb{R}^m)$ be a set satisfying condition* (M) *and such that $\mathbf{K}(t)$ are uniformly bounded for $t \in [0,T]$. Then, the nonlinear degenerate evolutionary variational inequality*

$$H \in \mathbf{K}: \ \langle C(t,H(t)), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \ \textit{in } [0,T], \tag{5.3.35}$$

admits a unique solution $H \in \mathbf{K}$ such that $H \in C([0,T],\mathbb{R}^m)$.

Proof. The existence and the uniqueness of solution to evolutionary variational inequality (5.3.35) is guaranteed by Theorem 3.3.2. Now, we prove that the solution is continuous in $[0,T]$.

Let $t \in [0,T]$ be fixed and let $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,T]$ be a sequence, with $t_n \to t$, as $n \to +\infty$.

Let us consider the solution $H(t)$ to variational inequality (5.3.35) and the solution $H(t_n)$, $\forall n \in \mathbb{N}$, to the following variational inequality

$$\langle C(t_n,H(t_n)), F(t_n) - H(t_n)\rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n), \ \forall n \in \mathbb{N}. \tag{5.3.36}$$

Let $H_\varepsilon(t)$ be the unique solution to perturbed strongly monotone variational inequality (5.3.26), namely $H_\varepsilon(t) \in \mathbf{K}(t)$ and

$$\langle C(t, H_\varepsilon(t)) + \varepsilon I H_\varepsilon(t), F(t) - H_\varepsilon(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ in } [0,T]. \tag{5.3.37}$$

Taking into account Theorem 5.2.3, it results that $H_\varepsilon(t)$ is a continuous function in $[0,T]$, then the solutions $H_\varepsilon(t_n)$, $\forall n \in \mathbb{N}$, to the following evolutionary variational inequalities

$$\langle C(t_n, H_\varepsilon(t_n)) + \varepsilon I(t_n) H_\varepsilon(t_n), F(t_n) - H_\varepsilon(t_n)\rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n), \tag{5.3.38}$$

$\forall n \in \mathbb{N}$, converge to $H_\varepsilon(t)$, as $n \to +\infty$. Setting $F(t_n) = H(t_n)$, $\forall n \in \mathbb{N}$, in (5.3.38) and $F(t_n) = H_\varepsilon(t_n)$, $\forall n \in \mathbb{N}$, in (5.3.36) and adding we obtain, $\forall n \in \mathbb{N}$

$$\langle C(t_n, H_\varepsilon(t_n)) - C(t_n, H(t_n)), H(t_n) - H_\varepsilon(t_n)\rangle + \varepsilon\langle H_\varepsilon(t_n), H(t_n) - H_\varepsilon(t_n)\rangle \geq 0. \tag{5.3.39}$$

Moreover, from the strongly monotonicity of the operator C it results

$$\langle C(t_n, H_\varepsilon(t_n)) - C(t_n, H(t_n)), H(t_n) - H_\varepsilon(t_n)\rangle \leq 0, \quad \forall n \in \mathbb{N}.$$

Then, (5.3.39) implies

$$\varepsilon\langle H_\varepsilon(t_n), H(t_n) - H_\varepsilon(t_n)\rangle \geq 0, \quad \forall n \in \mathbb{N},$$

and proceeding as in the proof of Lemma 5.3.2, we have

$$\|H_\varepsilon(t_n)\|_m \leq C, \quad \forall \varepsilon > 0, \ \forall n \in \mathbb{N}, \tag{5.3.40}$$

where C is a constant independent on ε and of $n \in \mathbb{N}$.

Lemma 5.3.2 implies

$$\lim_{\varepsilon \to 0} H_\varepsilon(t_n) = \widetilde{H}(t_n), \quad \forall n \in \mathbb{N},$$

with $\widetilde{H}(t_n) \in \mathbf{K}(t_n)$, $\forall n \in \mathbb{N}$, and such that

$$\langle C(t_n, \widetilde{H}(t_n)), F(t_n) - \widetilde{H}(t_n)\rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n), \ \forall n \in \mathbb{N}.$$

Since the solution to (5.3.36) is unique, it results

$$\widetilde{H}(t_n) = H(t_n), \quad \forall n \in \mathbb{N},$$

and, passing to the limit as $\varepsilon \to 0$ in (5.3.40), we have

$$\|H(t_n)\|_m \leq C, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{H(t_n)\}_{n\in\mathbb{N}}$ is bounded, that implies the existence of a subsequence $\{H(t_{k_n})\}_{n\in\mathbb{N}}$, with $H(t_{k_n}) \in \mathbf{K}(t_{k_n})$, $\forall n \in \mathbb{N}$, converging in $\mathbb{R}^m$ to an element $\overline{H}(t)$ of $\mathbb{R}^m$, namely

$$\lim_{n \to +\infty} H(t_{k_n}) = \overline{H}(t).$$

Taking into account (5.3.36) it follows

$$\langle C(t, \overline{H}(t)), F(t) - \overline{H}(t) \rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t),$$

and, for the uniqueness of the solution to (5.3.35), it results

$$\overline{H}(t) = H(t).$$

The same result holds for each subsequence and therefore

$$\lim_{n \to +\infty} H(t_n) = H(t),$$

namely our assert. The proof is now complete. □

5.4 Regularity results for strictly monotone evolutionary variational inequalities

In this section, we present results about the continuity of solutions to strictly monotone evolutionary variational inequalities.

5.4.1 Affine case

At first, we prove the continuity result for affine strictly monotone evolutionary variational inequalities, namely

Find $H \in \mathbf{K}(t)$ such that

$$\langle A(t)H(t) + B(t), F(t) - H(t) \rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ in } [0, T], \tag{5.4.1}$$

where $\mathbf{K} \in L^2([0,T], \mathbb{R}^m)$ satisfies condition (M),

under the assumption

$$\langle A(t)[H - F], H - F \rangle > 0, \quad \forall H, F \in \mathbb{R}^m, \ H \neq F, \text{ a.e. in } [0, T]. \tag{5.4.2}$$

During the proof of the following result, we make use of Lemma 5.3.1.

Theorem 5.4.1. *Let $A \in C([0,T], \mathbb{R}^{m \times m})$ be a bounded matrix-function verifying condition* (5.4.2) *and let $B \in C([0,T], \mathbb{R}^m)$ be a vector-function. Let $\mathbf{K} \subseteq L^2([0,T], \mathbb{R}^m)$ be a set satisfying condition* (M) *and such that $\mathbf{K}(t)$ are uniformly bounded for $t \in [0,T]$. Then, the evolutionary variational inequality* (5.4.1) *admits a unique solution $H \in \mathbf{K}$ such that $H \in C([0,T], \mathbb{R}^m)$.*

Proof. The existence and the uniqueness of solution to evolutionary variational inequality (5.4.1) is guaranteed by Theorem 3.2.4 and by condition (5.4.2). Now, we prove that the solution is continuous in $[0,T]$.

Let $t \in [0,T]$ be fixed and let $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,T]$ be a sequence, with $t_n \to t$, as $n \to +\infty$.

Let us consider the solution $H(t)$ to variational inequality (5.4.1) and the solution $H(t_n)$, $\forall n \in \mathbb{N}$, to the following variational inequalities

$$\langle A(t_n)H(t_n) + B(t_n), F(t_n) - H(t_n)\rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n),\ \forall n \in \mathbb{N}. \tag{5.4.3}$$

Let $H_\varepsilon(t) \in \mathbf{K}(t)$ be the unique solution to the strongly monotone perturbed variational inequality

$$\langle [A(t) + \varepsilon I(t)]H_\varepsilon(t) + B(t), F(t) - H_\varepsilon(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t),\ \text{in } [0,T]. \tag{5.4.4}$$

Being $H_\varepsilon(t)$ continuous in $[0,T]$, we have that solutions $H(t_n)$, $\forall n \in \mathbb{N}$, to the following evolutionary variational inequalities

$$\langle [A(t_n) + \varepsilon I(t_n)]H_\varepsilon(t_n) + B(t_n), F(t_n) - H_\varepsilon(t_n)\rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n), \tag{5.4.5}$$

$\forall n \in \mathbb{N}$, converge to $H_\varepsilon(t)$, as $n \to +\infty$. Setting $F(t_n) = H(t_n)$, $\forall n \in \mathbb{N}$, in (5.4.5) and $F(t_n) = H_\varepsilon(t_n)$, $\forall n \in \mathbb{N}$, in (5.4.3) and adding we get, $\forall n \in \mathbb{N}$

$$\langle A(t_n)[H_\varepsilon(t_n) - H(t_n)], H(t_n) - H_\varepsilon(t_n)\rangle + \varepsilon\langle H_\varepsilon(t_n), H(t_n) - H_\varepsilon(t_n)\rangle \geq 0. \tag{5.4.6}$$

From assumption (5.4.2) on the matrix-function A, it follows

$$\langle A(t_n)[H_\varepsilon(t_n) - H(t_n)], H(t_n) - H_\varepsilon(t_n)\rangle \leq 0, \quad \forall n \in \mathbb{N}.$$

Hence, for (5.4.6) we obtain

$$\varepsilon\langle H_\varepsilon(t_n), H(t_n) - H_\varepsilon(t_n)\rangle \geq 0, \quad \forall n \in \mathbb{N},$$

and proceeding as in the proof of Lemma 5.3.1, we get

$$\|H_\varepsilon(t_n)\|_m \leq C, \quad \forall \varepsilon > 0,\ \forall n \in \mathbb{N}, \tag{5.4.7}$$

where C is a constant independent on ε and on $n \in \mathbb{N}$.

For Lemma 5.3.1, it follows

$$\lim_{\varepsilon \to 0} H_\varepsilon(t_n) = \widetilde{H}(t_n), \quad \forall n \in \mathbb{N},$$

with $\widetilde{H}(t_n) \in \mathbf{K}(t_n)$, $\forall n \in \mathbb{N}$, and such that

$$\langle A(t_n)\widetilde{H}(t_n) + B(t_n), F(t_n) - \widetilde{H}(t_n)\rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n),\ \forall n \in \mathbb{N}.$$

Since the solution to (5.4.3) is unique, one has

$$\widetilde{H}(t_n) = H(t_n), \quad \forall n \in \mathbb{N},$$

and, passing to the limit as $\varepsilon \to 0$ in (5.4.7), it results

$$\|H(t_n)\|_m \leq C, \quad \forall n \in \mathbb{N}.$$

Hence the sequence $\{H(t_n)\}_{n\in\mathbb{N}}$ is bounded, then there exists a subsequence $\{H(t_{k_n})\}_{n\in\mathbb{N}}$, with $H(t_{k_n}) \in \mathbf{K}(t_{k_n})$, $\forall n \in \mathbb{N}$, converging in $\mathbb{R}^m$ to an element $\overline{H}(t)$ of $\mathbb{R}^m$, namely

$$\lim_{n\to+\infty} H(t_{k_n}) = \overline{H}(t).$$

Moreover, by (5.4.3) it obtains

$$\langle A(t)\overline{H}(t) + B(t), F(t) - \overline{H}(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t),$$

and, for the uniqueness of the solution to (5.4.1), it follows

$$\overline{H}(t) = H(t).$$

The same result holds for each subsequence and therefore

$$\lim_{n\to+\infty} H(t_n) = H(t).$$

Then, the assertion is achieved. □

5.4.2 Nonlinear case

Finally, we generalize Theorem 5.4.1 to the nonlinear case. We suppose that the operator $C : [0, T] \times \mathbb{R}^m \to \mathbb{R}^m$ verifies the following assumptions:

$$\|C(t, F)\|_m \leq A(t)\|F\|_m + B(t), \quad \forall F \in \mathbb{R}^m, \text{ a.e. in } [0, T], \tag{5.4.8}$$

where $B \in L^2([0, T])$ and $A \in L^\infty([0, T])$, and

$$\langle C(t, H) - C(t, F), H - F\rangle > 0, \quad \forall H, F \in \mathbb{R}^m, \text{ a.e. in } [0, T], \tag{5.4.9}$$

We consider the following evolutionary variational inequality

Find $H \in \mathbf{K}$ such that

$$\langle C(t, H(t)), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0, T]. \tag{5.4.10}$$

where $\mathbf{K} \in L^2([0, T], \mathbb{R}^m)$ satisfies condition (M),

Then, we can prove the next result (see also [10], Theorem 3.2).

Theorem 5.4.2. *Let $C \in C([0,T] \times \mathbb{R}^m, \mathbb{R}^m)$ be an operator satisfying conditions* (5.4.8) *and* (5.4.9). *Let $\mathbf{K} \subseteq L^2([0,T], \mathbb{R}^m)$ be a set satisfying condition* (M) *and such that $\mathbf{K}(t)$ are uniformly bounded for $t \in [0,T]$. Then, evolutionary variational inequality* (5.4.10) *admits a unique solution $H \in \mathbf{K}$ such that $H \in C([0,T], \mathbb{R}^m)$.*

Proof. The existence and the uniqueness of solution to evolutionary variational inequality (5.4.10) is guaranteed by Theorem 3.2.3 and by condition (5.4.9). Now, we prove that the solution is continuous in $[0,T]$.

Let $t \in [0,T]$ be fixed and let $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,T]$ be a sequence, with $t_n \to t$, as $n \to +\infty$.

Let us consider the solution $H(t)$ to variational inequality (5.4.10) and solutions $H(t_n)$, $\forall n \in \mathbb{N}$, to the following variational inequalities

$$\langle C(t_n, H(t_n)), F(t_n) - H(t_n)\rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n),\ \forall n \in \mathbb{N}. \tag{5.4.11}$$

We denote by $H_\varepsilon(t) \in \mathbf{K}(t)$ the unique solution to perturbed strongly monotone variational inequality

$$\langle C(t, H_\varepsilon(t)) + \varepsilon I H_\varepsilon(t), F(t) - H_\varepsilon(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t),\ \text{in } [0,T].$$

We have just remarked that $H_\varepsilon(t)$ is continuous in $[0,T]$, then we have that solutions $H_\varepsilon(t_n)$, $\forall n \in \mathbb{N}$, to the following evolutionary variational inequalities

$$\langle C(t_n, H_\varepsilon(t_n)) + \varepsilon I(t_n) H_\varepsilon(t_n), F(t_n) - H_\varepsilon(t_n)\rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n), \tag{5.4.12}$$

$\forall n \in \mathbb{N}$, converge to $H_\varepsilon(t)$, as $n \to +\infty$. Setting $F(t_n) = H(t_n)$, $\forall n \in \mathbb{N}$, in (5.4.12) and $F(t_n) = H_\varepsilon(t_n)$, $\forall n \in \mathbb{N}$, in (5.4.11) and adding it results, $\forall n \in \mathbb{N}$

$$\langle C(t_n, H_\varepsilon(t_n)) - C(t_n, H(t_n)), H(t_n) - H_\varepsilon(t_n)\rangle + \varepsilon\langle H_\varepsilon(t_n), H(t_n) - H_\varepsilon(t_n)\rangle \geq 0. \tag{5.4.13}$$

Moreover, from the strictly monotonicity of the function C it follows

$$\langle C(t_n, H_\varepsilon(t_n)) - C(t_n, H(t_n)), H(t_n) - H_\varepsilon(t_n)\rangle \leq 0, \quad \forall n \in \mathbb{N}. \tag{5.4.14}$$

Then, using (5.4.13) and (5.4.14) we obtain

$$\varepsilon\langle H_\varepsilon(t_n), H(t_n) - H_\varepsilon(t_n)\rangle \geq 0, \quad \forall n \in \mathbb{N},$$

and dividing by $\varepsilon > 0$, we have

$$\langle H_\varepsilon(t_n), H(t_n) - H_\varepsilon(t_n)\rangle \geq 0, \quad \forall n \in \mathbb{N}. \tag{5.4.15}$$

Taking into account (5.4.15), it results

$$\|H_\varepsilon(t_n)\|_m^2 \leq \langle H_\varepsilon(t_n), H(t_n)\rangle \leq \|H(t_n)\|_m \|H_\varepsilon(t_n)\|_m, \quad \forall n \in \mathbb{N},$$

then
$$\|H_\varepsilon(t_n)\|_m \leq \|H(t_n)\|_m, \quad \forall n \in \mathbb{N}.$$
Since $H(t_n) \in \mathbf{X}(t_n) \subseteq \mathbf{K}(t_n)$, in $[0,T]$, and $\mathbf{K}(t_n)$, $t \in [0,T]$, is a family of uniformly bounded sets of $\mathbb{R}^m$, it results

$$\|H(t_n)\|_m \leq C, \quad \forall n \in \mathbb{N},$$

where C is a constant independent on ε and of $n \in \mathbb{N}$, then

$$\|H_\varepsilon(t_n)\|_m \leq C, \quad \forall \varepsilon > 0, \ \forall n \in \mathbb{N}. \tag{5.4.16}$$

Taking into account of Lemma 5.3.2, we obtain

$$\lim_{\varepsilon \to 0} H_\varepsilon(t_n) = \widetilde{H}(t_n), \quad \forall n \in \mathbb{N},$$

where $\widetilde{H}(t_n) \in \mathbf{K}(t_n)$, $\forall n \in \mathbb{N}$, and such that

$$\langle C(t_n, \widetilde{H}(t_n)), F(t_n) - \widetilde{H}(t_n) \rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n), \ \forall n \in \mathbb{N}.$$

For the uniqueness of the solution to (5.4.11), it results

$$\widetilde{H}(t_n) = H(t_n), \quad \forall n \in \mathbb{N},$$

and, passing to the limit as $\varepsilon \to 0$ in (5.4.16), it follows

$$\|H(t_n)\|_m \leq C, \quad \forall n \in \mathbb{N},$$

namely the sequence $\{H(t_n)\}_{n\in\mathbb{N}}$ is bounded. Then, there exists a subsequence $\{H(t_{k_n})\}_{n\in\mathbb{N}}$, with $H(t_{k_n}) \in \mathbf{K}(t_{k_n})$, $\forall n \in \mathbb{N}$, converging in $\mathbb{R}^m$ to an element $\overline{H}(t)$ of $\mathbb{R}^m$, namely
$$\lim_{n \to +\infty} H(t_{k_n}) = \overline{H}(t).$$
Moreover, by (5.4.11) it obtains

$$\langle C(t, \overline{H}(t)), F(t) - \overline{H}(t) \rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t),$$

and, for the uniqueness of the solution to (5.4.10), it follows

$$\overline{H}(t) = H(t).$$

The same result holds for each subsequence and so

$$\lim_{n \to +\infty} H(t_n) = H(t),$$

namely our assert. □

6

Regularity results for evolutionary quasi-variational inequalities

6.1 Introduction

In this chapter, we investigate on the regularity of solutions to evolutionary quasi-variational inequalities. In the literature there are not results about this problem, then our work seems to represent an important contribution.

Let D be a nonempty, compact, convex subset of $L^2([0,T],\mathbb{R}^m)$. We consider a multifunction $\mathbf{K} : D \to 2^{L^2([0,T],\mathbb{R}^m)}$ satisfying the following assumption

(MM) $\mathbf{K}$ is closed l.s.c. with $\mathbf{K}(H)$, for each $H \in L^2([0,T],\mathbb{R}^m)$, nonempty, convex, closed of $L^2([0,T],\mathbf{R}^m)$ such that the sequence $\{\mathbf{K}(t_n,H)\}_{n\in\mathbb{N}}$ converges to $\mathbf{K}(t,H)$ in Mosco's sense, for each sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,T]$, with $t_n \to t$, as $n \to +\infty$.

Now, we prove that solutions to associated evolutionary quasi-variational inequalities are continuous with respect to the time. More precisely, we show results supposing that the operator is strongly monotone and, then, we extend them for degenerate and strictly monotone operators.

6.2 Regularity results for strongly monotone evolutionary quasi-variational inequalities

We consider strongly monotone evolutionary quasi-variational inequalities and we study under which assumptions the continuity of solutions is ensured. In particular, we obtain the regularity result for affine evolutionary quasi-variational inequalities and, making use of a type of Minty's Lemma, we obtain the analogous result in the nonlinear case.

6.2.1 Affine case

We suppose that the cost vector-function is affine with respect to flows, and we study the continuity of solutions to the following evolutionary quasi-variational inequality

Find $H \in \mathbf{K}(H)$ such that

$$\langle A(t)H(t) + B(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t, H), \text{ a.e. in } [0, T], \tag{6.2.1}$$

where the multifunction $\mathbf{K} : D \to 2^{L^2([0,T],\mathbb{R}^m)}$ satisfies assumption (MM).

Now, we are able to show this continuity result (see also [8], Theorem 3.2).

Theorem 6.2.1. *Let $A \in C([0, T], \mathbb{R}^{m\times m})$ be a positive definite matrix-function and let $B \in C([0, T], \mathbb{R}^m)$ be a vector-function. Let D be a nonempty, compact, convex subset of $L^2([0, T], \mathbb{R}^m)$. Let $\mathbf{K} : D \to 2^{L^2([0,T],\mathbb{R}^m)}$ be a set-valued mapping satisfying condition* (MM)*. Then, evolutionary quasi-variational inequality* (6.2.1) *admits a solution $H \in \mathbf{K}(H)$ such that $H \in C([0, T], \mathbb{R}^m)$.*

Proof. Taking into account of Theorem 3.4.3, we have that (6.2.1) admits a solution $H \in \mathbf{K}(H)$ and the solution is unique in $\mathbf{K}(t, H)$ for each $t \in [0, T]$.

Now, we prove the continuity of the solution

$$H : [0, T] \ni t \to H(t) \in \mathbb{R}^m$$

making use of Theorem 4.2.1 on convergence in Mosco's sense.

Let $t \in [0, T]$ be fixed and let $\{t_n\}_{n\in\mathbb{N}} \subseteq [0, T]$ be a sequence, with $t_n \to t \in [0, T]$. From the assumption of continuity of the matrix-function A, we have

$$A(t_n) \to A(t) \quad \text{in } \mathbb{R}^{m\times m},$$

moreover, if $\{F(t_n)\}_{n\in\mathbb{N}}$ is a sequence, with $F(t_n) \in \mathbf{K}(t_n, H)$, for each $H \in L^2([0, T], \mathbb{R}^m)$, such that $F(t_n) \to F(t)$ in $\mathbb{R}^m$, we obtain

$$A(t_n)F(t_n) \to A(t)F(t) \quad \text{in } \mathbb{R}^m,$$

Finally, for the continuity of the function B we get

$$B(t_n) \to B(t) \quad \text{in } \mathbb{R}^m.$$

Taking into account that the multifunction $\mathbf{K}$ satisfies condition (MM) and using the stability Theorem 4.2.1, we can conclude that the solutions $H(t_n)$ of the quasi-variational inequalities

$$\langle A(t_n)H(t_n) + B(t_n), F(t_n) - H(t_n)\rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n, H),$$

converge strongly to the solution $H(t)$ of the limit problem (6.2.1), namely,

$$H(t_n) \to H(t) \quad \text{in } \mathbb{R}^m.$$

Consequently, it results that $H \in C([0, T], \mathbb{R}^m)$. □

Now, we still assume that the operator is affine with respect to the flows, but the matrix-function A depends on time and on integral average of the flow vectors, namely

$$C(t, F(t)) = A(t, F_{\mathcal{T}})F(t) + B(t),$$

for a.e. $t \in [0,T]$ and for every $F \in \mathbb{R}^m$, where $A : [0,T] \times \mathbb{R}^m \to \mathbb{R}^{m\times m}$ and $B : [0,T] \to \mathbb{R}^m$ are two functions, $\mathcal{T} = [0,T]$ and $F_{\mathcal{T}}$ is the integral average, that is

$$F_{\mathcal{T}} = \frac{\int_0^T F(\tau)d\tau}{T}.$$

We suppose that $A(t,u)$ is a bounded matrix, namely

$$\exists M > 0 : \ \|A(t,u)\|_{m\times m} \leq M, \quad \text{for a.e. } t \in [0,T], \ \forall u \in \mathbb{R}^m. \tag{6.2.2}$$

Then, we study the continuity of the solutions to the following evolutionary quasi-variational inequality:

Find $H \in \mathbf{K}(H)$ such that

$$\langle A(t, F_{\mathcal{T}})H(t) + B(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t,H), \text{ a.e. in } [0,T], \tag{6.2.3}$$

where $\mathbf{K} : D \to 2^{L^2([0,T],\mathbb{R}^m)}$ is a multifunction satisfying condition (MM).

We can obtain the regularity result for solutions to (6.2.3).

Theorem 6.2.2. *Let $A \in C([0,T] \times \mathbb{R}^m, \mathbb{R}^{m\times m})$ be a matrix-function verifying the condition* (6.2.2)*, and let $B \in C([0,T], \mathbb{R}^m)$ be a vector-function. Let D be a nonempty, compact, convex subset of $L^2([0,T],\mathbb{R}^m)$. Let $\mathbf{K}$ be a set-valued mapping satisfying condition* (MM)*. Then, evolutionary quasi-variational inequality* (6.2.3) *admits a solution $H \in \mathbf{K}(H)$ such that $H \in C([0,T],\mathbb{R}^m)$.*

Proof. The existence of solutions to the evolutionary quasi-variational inequality follows by Theorem 3.4.3. We remark that it needs to prove only that $A(t, F_{\mathcal{T}})$ is continuous in F, for $t \in [0,T]$. To this aim, let $F \in \mathbf{K}(H)$ be fixed and let $\{F_n\}_{n\in\mathbb{N}} \subseteq \mathbf{K}(H)$ be a sequence, such that $F_n \to F$, in $L^2([0,T],\mathbb{R}^m)$. It results

$$\lim_{n\to\infty} \int_0^T F_n(\tau)d\tau = \int_0^T F(\tau)d\tau,$$

and, taking into account that $A(t,v)$ is continuous in v and bounded for $t \in [0,T]$, we get the continuity of $A(t, F_{\mathcal{T}})$ in F, for $t \in [0,T]$. Then we have the existence of a solution and the continuity of the solution to the evolutionary quasi-variational inequality simply follows from Theorem 6.2.1. □

6.2.2 Nonlinear case

In this section, we present the continuity result for solutions to nonlinear evolutionary quasi-variational inequalities.

Let
$$C : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m,$$
be a nonlinear operator and let us consider the following evolutionary quasi-variational inequality

Find $H \in \mathbf{K}(H)$ such that
$$\langle C(t,H(t)), F(t) - H(t) \rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t,H), \text{ a.e. in } [0,T], \tag{6.2.4}$$
where $\mathbf{K} : D \to 2^{L^2([0,T],\mathbb{R}^m)}$ is a multifunction satisfying condition (MM).

The next result holds (see also [7], Theorem 8).

Theorem 6.2.3. *Let $C \in C([0,T] \times \mathbb{R}^m, \mathbb{R}^m)$ be a operator, satisfying conditions*
$$\exists \gamma \in L^2([0,T],\mathbb{R}) : \ \|C(t,F)\|_m \leq \gamma(t) + \|F\|_m, \quad \forall F \in \mathbb{R}^m, \ in \ [0,T],$$
$$\exists \nu > 0 : \ \langle C(t,H) - C(t,F), H - F \rangle \geq \nu \|F\|_m^2, \quad \forall H, F \in \mathbb{R}^m, \ in \ [0,T].$$
Let D be a nonempty, compact, convex subset of $L^2([0,T],\mathbb{R}^m)$. Let $\mathbf{K} : D \to L^2([0,T],\mathbb{R}^m)$ be a multifunction satisfying condition (MM). *Then, the evolutionary quasi-variational inequality* (6.2.4) *admits a unique solution $H \in \mathbf{K}(H)$ such that $H \in C([0,T],\mathbb{R}^m)$.*

Proof. From the assumption that C is a continuous operator, it follows that C verifies all conditions of Theorem 3.4.3. Then, the existence of $H \in \mathbf{K}(H)$ is guaranteed. Moreover, the assumption of strongly monotonicity on the operator ensures that the solution $H(t)$ is unique in the set $\mathbf{K}(t,H)$, for $t \in [0,T]$.

Now, let $t \in [0,T]$ be fixed and let $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,T]$ be a sequence, such that $t_n \to t$, as $n \to +\infty$. The thesis is equivalent to tell that the solutions $H(t_n)$, for $n \in \mathbb{N}$, to quasi-variational inequalities
$$H(t_n) \in \mathbf{K}(t_n,H) : \ \langle C(t_n,H(t_n)), F(t_n) - H(t_n) \rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n,H), \tag{6.2.5}$$
converges strongly to the solution $H(t)$ to the limit problem
$$H(t) \in \mathbf{K}(t,H) : \ \langle C(t,H(t)), F(t) - H(t) \rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t,H), \tag{6.2.6}$$
that is
$$\lim_{n\to+\infty} H(t_n) = H(t) \quad \text{in } \mathbb{R}^m.$$

Let $H(t) \in \mathbf{K}(t, H)$ be the solution to (6.2.6), for the properties of the convergence in Mosco's sense of $\{\mathbf{K}(t_n, H)\}_{n\in\mathbb{N}}$ to $\mathbf{K}(t, H)$, as $n \to +\infty$, it is possible to choose a sequence $\{G(t_n)\}_{n\in\mathbb{N}}$, with $G(t_n) \in \mathbf{K}(t_n, H)$, $\forall n \in \mathbb{N}$, such that,

$$\lim_{n\to+\infty} G(t_n) = H(t) \quad \text{in } \mathbb{R}^m$$

that implies

$$\lim_{n\to+\infty} C(t_n, G(t_n)) = C(t, H(t)) \quad \text{in } \mathbb{R}^m.$$

Setting $F(t_n) = G(t_n)$ in (6.2.5), we have

$$\langle C(t_n, H(t_n)), G(t_n) - H(t_n)\rangle \geq 0, \tag{6.2.7}$$

and for the strong monotonicity of the operator C, it results

$$\langle C(t_n, H(t_n)) - C(t_n, G(t_n)), H(t_n) - G(t_n)\rangle \geq \nu \|H(t_n) - G(t_n)\|_m^2.$$

From (6.2.7) it follows

$$\langle C(t_n, H(t_n)) - C(t_n, G(t_n)), H(t_n) - G(t_n)\rangle = \langle C(t_n, H(t_n)), H(t_n) - G(t_n)\rangle$$
$$-\langle C(t_n, H(t_n)), H(t_n) - G(t_n)\rangle \leq -\langle C(t_n, G(t_n)), H(t_n) - G(t_n)\rangle,$$

so

$$\begin{aligned} \nu\|H(t_n) - G(t_n)\|_m^2 &\leq -\langle C(t_n, G(t_n)), H(t_n) - G(t_n)\rangle \\ &\leq \|C(t_n, G(t_n))\|_m \|H(t_n) - G(t_n)\|_m, \end{aligned}$$

that is

$$\nu\|H(t_n) - G(t_n)\|_m \leq \|C(t_n, G(t_n))\|_m.$$

Then, we obtain

$$\begin{aligned} \|H(t_n)\|_m &\leq \|H(t_n) - G(t_n)\|_m + \|G(t_n)\|_m \\ &\leq \frac{\|C(t_n, G(t_n))\|_m}{\nu} + \|G(t_n)\|_m. \end{aligned}$$

Since $\{C(t_n, G(t_n))\}_{n\in\mathbb{N}}$ is a sequence convergent then it is bounded, i.e.,

$$\exists h \in \mathbb{R}_+ : \ \|C(t_n, G(t_n))\|_m \leq h, \quad \forall n \in \mathbb{N},$$

for the same reason, $\{G(t_n)\}_{n\in\mathbb{N}}$ is a sequence bounded, i.e.,

$$\exists k \in \mathbb{R}_+ : \ \|G(t_n)\|_m \leq k, \quad \forall n \in \mathbb{N}.$$

From those conditions, we have

$$\|H(t_n)\|_m \leq c, \quad \forall n \in \mathbb{N},$$

where the constant c is independent on n. Hence there exists a subsequence $\{H(t_{k_n})\}_{n\in\mathbb{N}}$ converging in $\mathbb{R}^m$ to an element $\widetilde{H}(t) \in \mathbb{R}^m$, and then

$$\lim_{n\to+\infty} H(t_{k_n}) = \widetilde{H}(t),$$

for the second condition of the convergence in Mosco's sense we have

$$\widetilde{H}(t) \in \mathbf{K}(t, H).$$

Moreover, for the convergence in Mosco's sense, it results

$$\forall F(t) \in \mathbf{K}(t, H)\ \exists F(t_n) \in \mathbf{K}(t_n, H)\ \forall n \in \mathbb{N} : \lim_{n\to+\infty} F(t_n) = F(t), \text{ in } \mathbb{R}^m,$$

and from the continuity of the operator C, it follows

$$\lim_{n\to+\infty} C(t_n, F(t_n)) = C(t, F(t)), \text{ in } \mathbb{R}^m.$$

We consider the following quasi-variational inequality

$$\langle C(t_{k_n}, F(t_{k_n})), F(t_{k_n}) - H(t_{k_n})\rangle \geq 0,$$

and passing to the limit as $n \to +\infty$, we deduce

$$\widetilde{H}(t) \in \mathbf{K}(t) : \langle C(t, F(t)), F(t) - \widetilde{H}(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t). \tag{6.2.8}$$

Taking into account of Lemma 3.4.1 and of the uniqueness of the solution to (6.2.6) in $\mathbf{K}(H, t)$, for $t \in [0, T]$, for quasi-variational inequality (6.2.8) it follows

$$\widetilde{H}(t) = H(t).$$

Every subsequence of $\{H(t_n)\}_{n\in\mathbb{N}}$ converges to the same limit $\widetilde{H}(t)$ and then

$$\lim_{n\to+\infty} H(t_n) = H(t).$$

□

6.3 Regularity results for degenerate evolutionary quasi-variational inequalities

In this section, we prove that regularity results hold also for degenerate evolutionary quasi-variational inequalities.

6.3.1 Affine case

Let $A : [0,T] \to \mathbb{R}^{m\times m}$ be a matrix-function and let $B : [0,T] \to \mathbb{R}^m$ be a vector-function. Let us consider the affine evolutionary quasi-variational inequality

Find $H \in \mathbf{K}(H)$ such that

$$\langle A(t)H(t) + B(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t,H), \text{ a.e. in } [0,T], \tag{6.3.1}$$

where $\mathbf{K} : D \to 2^{L^2([0,T],\mathbb{R}^m)}$ is a multifunction satisfying assumption (MM).

We prove the continuity of solutions to (6.3.1) under the following assumptions

$$\exists M > 0 : \; \|A(t)\|_{m\times m} = \Big(\sum_{r,s=1}^{m} A_{rs}^2(t)\Big)^{\frac{1}{2}} \leq M, \quad \text{a.e. in } [0,T], \tag{6.3.2}$$

and

$$\langle A(t)F, F\rangle \geq \nu(t)\|F\|_m^2, \quad \forall F \in \mathbb{R}^m, \text{ a.e. in } [0,T], \tag{6.3.3}$$

where $\nu \in L^\infty([0,T],\mathbb{R}_0^+)$ is such that

$$\nexists I \subseteq [0,T], \; \mu(I) > 0 : \; \nu(t) = 0, \quad \forall t \in I,$$

with μ Lebesgue's measure, namely A is a degenerate matrix-function.

We recall (see Theorem 3.5.1) that the evolutionary quasi-variational inequalities (6.3.1) admits a solution $\overline{H}$ and this solution is unique in the set $\mathbf{K}(\overline{H})$. Then, to prove the continuity of the solution to (6.3.1) means proving that the unique solution $\overline{H}$ in the set $\mathbf{K}(\overline{H})$ is continuous. Then, we fix the solution $\overline{H} \in \mathbf{K}(\overline{H})$ and we work in $\mathbf{K}(\overline{H})$.

Now, it is needed to prove a preliminary lemma related to evolutionary quasi-variational inequalities associated to a nonnegative matrix-function A. Hence, let us observe that, under this assumption, the set $\mathbf{X}(\overline{H})$ of solutions to the evolutionary quasi-variational inequality (6.3.1) is closed, convex and nonempty. We consider $I : L^2([0,T],\mathbb{R}^m) \to L^2([0,T],\mathbb{R}^m)$ the identity function and the evolutionary quasi-variational inequality

$$\langle I(t)H(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{X}(t,\overline{H}), \text{ a.e. in } [0,T], \tag{6.3.4}$$

which admits a solution $H(t)$ in the set $\mathbf{X}(t,\overline{H})$. Moreover, for every $\varepsilon > 0$, let us consider the following perturbed evolutionary quasi-variational inequality

$$\langle [A(t) + \varepsilon I(t)]H(t) + B(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t,\overline{H}), \text{ a.e. in } [0,T], \tag{6.3.5}$$

which admits a continuous solution H_ε, by virtue of Theorem 6.2.1, and is unique in the set $\mathbf{K}(t,\overline{H})$. Then, the following preliminary result holds:

Lemma 6.3.1. *Let $A \in C([0,T],\mathbb{R}^{m\times m})$ be a nonnegative matrix-function and let $B \in C([0,T],\mathbb{R}^m)$ be a vector-function. Let D be a nonempty, compact, convex subset of $L^2([0,T],\mathbb{R}^m)$. Let $\mathbf{K} : D \to L^2([0,T],\mathbb{R}^m)$ be a multifunction satisfying condition* (MM) *such that $\mathbf{K}(t,H)$ is uniformly bounded sets for $t \in [0,T]$. If $H_\varepsilon(t)$, $\forall \varepsilon > 0$, is a solution to* (6.3.5)*, it results*

$$\lim_{\varepsilon\to 0} H_\varepsilon(t) = H(t), \quad in\ [0,T],$$

and

$$\lim_{\varepsilon\to 0} \|H_\varepsilon(t) - H(t)\|^2_{L^2([0,T],\mathbb{R}^m)} = 0,$$

where H is a solution to evolutionary quasi-variational inequality (6.3.1)*.*

Proof. Let H be the solution to (6.3.4), then we have

$$\langle I(t)H(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{X}(t,\overline{H}), \text{ in } [0,T]. \tag{6.3.6}$$

Let H_ε be the solution to (6.3.5), namely $H_\varepsilon \in \mathbf{K}(\overline{H})$ and

$$\langle [A(t) + \varepsilon I(t)]H_\varepsilon(t) + B(t), F(t) - H_\varepsilon(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t,\overline{H}). \tag{6.3.7}$$

Setting $F(t) = H_\varepsilon(t)$, for $t \in [0,T]$, in (6.3.1) and $F(t) = H(t)$, for $t \in [0,T]$, in (6.3.7) and adding we get

$$\langle A(t)[H(t) - H_\varepsilon(t)], H_\varepsilon(t) - H(t)\rangle + \varepsilon\langle H_\varepsilon(t), H(t) - H_\varepsilon(t)\rangle \geq 0, \tag{6.3.8}$$

in $[0,T]$. Since A is a nonnegative matrix-function, it follows

$$\langle A(t)[H(t) - H_\varepsilon(t)], H_\varepsilon(t) - H(t)\rangle \leq 0, \quad \text{in } [0,T],$$

then, by (6.3.8), we obtain

$$\varepsilon\langle H_\varepsilon(t), H(t) - H_\varepsilon(t)\rangle \geq 0, \quad \text{in } [0,T],$$

and dividing by $\varepsilon > 0$, it results

$$\langle H_\varepsilon(t), H(t) - H_\varepsilon(t)\rangle \geq 0, \quad \text{in } [0,T]. \tag{6.3.9}$$

Taking into account (6.3.9), one has

$$\|H_\varepsilon(t)\|_m^2 \leq \langle H_\varepsilon(t), H(t)\rangle \leq \|H(t)\|_m \|H_\varepsilon(t)\|_m, \quad \text{in } [0,T],$$

then

$$\|H_\varepsilon(t)\|_m \leq \|H(t)\|_m, \quad \text{in } [0,T].$$

Since $H(t) \in \mathbf{X}(t,\overline{H}) \subseteq \mathbf{K}(t,\overline{H})$, in $[0,T]$, and $\mathbf{K}(t,\overline{H})$ is a family of uniformly bounded sets of $\mathbb{R}^m$ for $t \in [0,T]$, it results

$$\|H(t)\|_m \leq C, \quad \text{in } [0,T],$$

with C a constant independent on $t \in [0,T]$, then

$$\|H_\varepsilon(t)\|_m \leq C, \quad \forall \varepsilon > 0, \text{ in } [0,T].$$

Hence there exists a subsequence $\{H_\eta(t)\}_\eta$ converging in $\mathbb{R}^m$ to an element $\widehat{H}(t)$ of $\mathbb{R}^m$, in [0,T], and thus

$$\lim_{\eta \to 0} H_\eta(t) = \widehat{H}(t), \quad \text{in } [0,T].$$

Taking into account that $\mathbf{K}(t,\overline{H})$ is a closed set of $\mathbb{R}^m$ and $\{H_\eta(t)\}_\eta \subseteq \mathbf{K}(t,\overline{H})$, then

$$\widehat{H}(t) \in \mathbf{K}(t,\overline{H}), \text{ in } [0,T].$$

It remains to prove that

$$\widehat{H}(t) = H(t), \quad \text{in } [0,T].$$

Hence, considering (6.3.7) with $\varepsilon = \eta$, we get

$$\langle A(t)H_\eta(t) + B(t), F(t) - H_\eta(t)\rangle + \eta\langle H_\eta(t), F(t) - H_\eta(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t,\overline{H}), \tag{6.3.10}$$

for every $F(t) \in \mathbf{K}(t,\overline{H})$ and in $[0,T]$, and taking into account that

$$\lim_{\eta \to 0}\langle H_\eta(t), H_\eta(t)\rangle = \langle \widehat{H}(t), \widehat{H}(t)\rangle, \quad \text{in } [0,T],$$

and that

$$\lim_{\eta \to 0}\langle A(t)H_\eta(t) + B(t), H_\eta(t)\rangle = \langle A(t)\widehat{H}(t) + B(t), \widehat{H}(t)\rangle, \quad \text{in } [0,T],$$

from (6.3.10), we obtain

$$\langle A(t)\widehat{H}(t) + B(t), F(t) - \widehat{H}(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t,\overline{H}), \text{ in } [0,T]. \tag{6.3.11}$$

Then (6.3.11) implies that $\widehat{H}$ is a solution to (6.3.1), in $[0,T]$, namely

$$\widehat{H} \in \mathbf{X}(\overline{H}).$$

If the solution to (6.3.1) is unique in the set $\mathbf{K}(\overline{H})$, then the proof is concluded. Now, we suppose that the solution to (6.3.1) is not unique in the set $\mathbf{K}(\overline{H})$. Setting $\varepsilon = \eta$ in (6.3.9) we get

$$\langle H_\eta(t), H(t) - H_\eta(t)\rangle \geq 0, \quad \text{in } [0,T],$$

and passing to the limit as $\eta \to 0$, we obtain

$$\langle \widehat{H}(t), H(t) - \widehat{H}(t)\rangle \geq 0, \quad \text{in } [0,T]. \tag{6.3.12}$$

Rewriting (6.3.6) with $F = \widehat{H} \in \mathbf{X}(\overline{H})$, it results

$$\langle H(t), \widehat{H}(t) - H(t)\rangle \geq 0, \quad \text{in } [0,T], \tag{6.3.13}$$

and adding (6.3.12) and (6.3.13), we have

$$\langle \widehat{H}(t) - H(t), H(t) - \widehat{H}(t)\rangle \geq 0, \quad \text{in } [0,T].$$

Then

$$\langle \widehat{H}(t) - H(t), H(t) - \widehat{H}(t)\rangle = 0, \quad \text{in } [0,T],$$

that implies

$$\widehat{H}(t) = H(t), \quad \text{in } [0,T].$$

In this way, we have shown that every convergent subsequence converges to the same limit $H(t)$ and hence

$$\lim_{\varepsilon\to 0} H_\varepsilon(t) = H(t), \quad \text{in } [0,T],$$

from this it follows

$$\lim_{\varepsilon\to 0} \|H_\varepsilon(t) - H(t)\|_m^2 = 0, \quad \text{in } [0,T].$$

Moreover, we remark that

$$\|H_\varepsilon(t) - H(t)\|_m^2 \leq 2\big(|H_\varepsilon(t)\|_m^2 + \|H(t)\|_m^2\big) \leq 4C \in C([0,T],\mathbb{R}^m), \quad \text{in } [0,T],$$

then, by virtue of Lebesgue's Theorem we have

$$\lim_{\varepsilon\to 0} \|H_\varepsilon(t) - H(t)\|_{L^2([0,T],\mathbb{R}^m)}^2 = 0.$$

□

Now, we are able to prove the continuity result for degenerate evolutionary quasi-variational inequalities.

Theorem 6.3.1. *Let $A \in C([0,T],\mathbb{R}^{m\times m})$ be a matrix-function satisfying condition* (6.3.3) *and let $B \in C([0,T],\mathbb{R}^m)$ be a vector-function. Let D be a nonempty, compact, convex subset of $L^2([0,T],\mathbb{R}^m)$. Let $\mathbf{K} : D \to 2^{L^2([0,T],\mathbb{R}^m)}$ be a multifunction such that have uniformly bounded set-values and satisfies condition* (MM)*. Then, the evolutionary quasi-variational inequality*

$$\langle A(t)H(t) + B(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t,H), \quad \textit{in } [0,T], \tag{6.3.14}$$

admits a solution $H \in \mathbf{K}(H)$ such that $H \in C([0,T],\mathbb{R}^m)$.

Proof. Making use of Theorem 3.5.1, it results that (6.3.14) admits a solution $\overline{H} \in \mathbf{K}(\overline{H})$ and the solution is unique in $\mathbf{K}(t,\overline{H})$ for each $t \in [0,T]$.

Let $t \in [0,T]$ be fixed and let $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,T]$ be a sequence, such that $t_n \to t$, as $n \to +\infty$. Let $\overline{H}(t)$ be the solution to quasi-variational inequality (6.3.14) and let $\overline{H}(t_n)$, $\forall n \in \mathbb{N}$, be the solutions to the following quasi-variational inequalities

$$\langle A(t_n)H(t_n) + B(t_n), F(t_n) - H(t_n)\rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n,\overline{H}),\ \forall n \in \mathbb{N}. \tag{6.3.15}$$

Let $H_\varepsilon(t)$ be the solution to the perturbed quasi-variational inequality (6.3.5), namely $H_\varepsilon(t) \in \mathbf{K}(t, \overline{H})$ and

$$\langle [A(t) + \varepsilon I(t)]H_\varepsilon(t) + B(t), F(t) - H_\varepsilon(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t, \overline{H}). \tag{6.3.16}$$

Since $H_\varepsilon(t)$ is continuous in $[0, T]$ (see Theorem 6.2.1), it follows that the solutions $H(t_n)$, $\forall n \in \mathbb{N}$, to the following evolutionary quasi-variational inequalities

$$\langle [A(t_n) + \varepsilon I(t_n)]H_\varepsilon(t_n) + B(t_n), F(t_n) - H_\varepsilon(t_n)\rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n, \overline{H}), \tag{6.3.17}$$

$\forall n \in \mathbb{N}$, converge to $H_\varepsilon(t)$, as $n \to +\infty$. Setting $F(t_n) = H(t_n)$, $\forall n \in \mathbb{N}$, in (6.3.17) and $F(t_n) = H_\varepsilon(t_n)$, $\forall n \in \mathbb{N}$, in (6.3.15) and adding we obtain, $\forall n \in \mathbb{N}$,

$$\langle A(t_n)[H_\varepsilon(t_n) - H(t_n)], H(t_n) - H_\varepsilon(t_n)\rangle + \varepsilon\langle H_\varepsilon(t_n), H(t_n) - H_\varepsilon(t_n)\rangle \geq 0. \tag{6.3.18}$$

We remark that for condition (6.3.3) on the matrix-function A we have

$$\langle A(t_n)[H_\varepsilon(t_n) - H(t_n)], H(t_n) - H_\varepsilon(t_n)\rangle \leq 0, \quad \forall n \in \mathbb{N}.$$

Then, from (6.3.18) it follows

$$\varepsilon\langle H_\varepsilon(t_n), H(t_n) - H_\varepsilon(t_n)\rangle \geq 0, \quad \forall n \in \mathbb{N},$$

and proceeding as in the proof of Lemma 6.3.1, we get

$$\|H_\varepsilon(t_n)\|_m \leq C, \quad \forall \varepsilon > 0, \ \forall n \in \mathbb{N}, \tag{6.3.19}$$

where C is a constant independent on ε and on $n \in \mathbb{N}$.

Taking into account of Lemma 6.3.1, it results

$$\lim_{\varepsilon \to 0} H_\varepsilon(t_n) = \widetilde{H}(t_n), \quad \forall n \in \mathbb{N},$$

where $\widetilde{H}(t_n) \in \mathbf{K}(t_n, \overline{H})$, $\forall n \in \mathbb{N}$, and it is such that

$$\langle A(t_n)\widetilde{H}(t_n) + B(t_n), F(t_n) - \widetilde{H}(t_n)\rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n, \overline{H}), \ \forall n \in \mathbb{N}.$$

Since the solution to (6.3.15) is unique in $\mathbf{K}(t_n, \overline{H})$, we have

$$\widetilde{H}(t_n) = H(t_n), \quad \forall n \in \mathbb{N},$$

and, passing to the limit as $\varepsilon \to 0$ in (6.3.19), it results

$$\|H(t_n)\|_m \leq C, \quad \forall n \in \mathbb{N}.$$

Hence the sequence $\{H(t_n)\}_{n\in\mathbb{N}}$ is bounded, then there exists a subsequence $\{H(t_{k_n})\}_{n\in\mathbb{N}}$, with $H(t_{k_n}) \in \mathbf{K}(t_{k_n}, \overline{H})$, $\forall n \in \mathbb{N}$, converging in $\mathbb{R}^m$ to an element $\widehat{H}(t)$ of $\mathbb{R}^m$, namely

$$\lim_{n \to +\infty} H(t_{k_n}) = \widehat{H}(t).$$

Moreover, by (6.3.15) it obtains

$$\langle A(t)\widehat{H}(t) + B(t), F(t) - \widehat{H}(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t, \overline{H}),$$

and, for the uniqueness of the solution to (6.3.14) in $\mathbf{K}(t, \overline{H})$, it follows

$$\widehat{H}(t) = \overline{H}(t).$$

The same result holds for each subsequence and therefore

$$\lim_{n\to+\infty} H(t_n) = \overline{H}(t),$$

namely our assert. □

6.3.2 Nonlinear case

The aim of this section is to consider nonlinear degenerate evolutionary quasi-variational inequalities and to prove that they have some continuous solutions. More precisely, let $C : [0, T] \times \mathbb{R}^m \to \mathbb{R}^m$ be an operator satisfying the following assumptions:

$$\exists \gamma \in L^2([0, T], \mathbb{R}_+) : \ \|C(t, F)\|_m \leq \gamma(t) + \|F\|_m, \quad \forall F \in \mathbb{R}^m, \text{ a.e. in } [0, T]. \tag{6.3.20}$$

and

$$\langle C(t, F) - C(t, H), F - H\rangle \geq \nu(t)\|F - H\|_m^2, \quad \forall F, H \in \mathbb{R}^m, \text{ a.e. in } [0, T], \tag{6.3.21}$$

where $\nu \in L^\infty([0, T], \mathbb{R}_0^+)$ is such that

$$\nexists I \subseteq [0, T], \ \mu(I) > 0 : \ \nu(t) = 0, \quad \text{for a.e. } t \in I.$$

Let us consider the following evolutionary variational inequality

Find $H \in \mathbf{K}$ such that

$$\langle C(t, H(t)), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t, H), \text{ a.e. in } [0, T], \tag{6.3.22}$$

where the multifunction $\mathbf{K} : D \to 2^{L^2([0,T],\mathbb{R}^m)}$ satisfies condition (MM).

We observe that there exists a solution $\overline{H}$ to (6.3.22), and it is unique in $\mathbf{K}(\overline{H})$, for Theorem 3.5.2. Hence, to show the continuity of the solution to (6.3.22) means proving that the unique solution $\overline{H}$ in the set $\mathbf{K}(\overline{H})$ is continuous. Then, we fix the solution $\overline{H} \in \mathbf{K}(\overline{H})$ and we work in $\mathbf{K}(\overline{H})$.

The first step of the proof of the continuity result is to prove a regularization lemma. We recall that if the operator C is monotone it results that the set $\mathbf{X}(\overline{H})$ of solutions to evolutionary quasi-variational inequality (6.3.22) is closed, convex and nonempty.

Let $I : L^2([0,T],\mathbb{R}^m) \to L^2([0,T],\mathbb{R}^m)$ be the identity operator and let us consider the following evolutionary quasi-variational inequality

$$\langle IH(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{X}(t,\overline{H}), \text{ in } [0,T]. \tag{6.3.23}$$

Then, evolutionary quasi-variational inequality (6.3.23) admits a unique solution in the set $\mathbf{X}(\overline{H})$. Further, for every $\varepsilon > 0$, let us consider the perturbed evolutionary quasi-variational inequality

$$\langle C(t,H(t)) + \varepsilon IH(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t,\overline{H}), \text{ in } [0,T], \tag{6.3.24}$$

which admits a unique continuous solution H_ε by virtue of Theorem 6.2.3. Now, we can prove the following preliminary result.

Lemma 6.3.2. *Let $C \in C([0,T]\times\mathbb{R}^m,\mathbb{R}^m)$ be a monotone matrix-function satisfying condition* (6.3.20)*. Let D be a nonempty, compact, convex subset of $L^2([0,T],\mathbb{R}^m)$. Let $\mathbf{K} : D \to 2^{L^2([0,T],\mathbb{R}^m)}$ be a multifunction such that have uniformly bounded set-values and satisfies condition* (MM)*. If $H_\varepsilon(t)$, $\forall\varepsilon > 0$, is a solution to* (6.3.24)*, it results*

$$\lim_{\varepsilon\to 0} H_\varepsilon(t) = H(t), \quad \textit{in } [0,T],$$

and

$$\lim_{\varepsilon\to 0} \|H_\varepsilon(t) - H(t)\|^2_{L^2([0,T],\mathbb{R}^m)} = 0,$$

where H is a solution to the evolutionary quasi-variational inequality (6.3.22)*.*

Proof. Let H be the solution to (6.3.23), therefore $H \in \mathbf{X}(\overline{H})$ and

$$\langle IH(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{X}(t,\overline{H}), \text{ in } [0,T]. \tag{6.3.25}$$

Let H_ε be the solution to (6.3.24), namely $H_\varepsilon \in \mathbf{K}(\overline{H})$ and

$$\langle C(t,H(t)) + \varepsilon IH_\varepsilon(t), F(t) - H_\varepsilon(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t,\overline{H}). \tag{6.3.26}$$

Setting $F(t) = H_\varepsilon(t)$, for $t \in [0,T]$, in (6.3.24) and $F(t) = H(t)$, for $t \in [0,T]$, in (6.3.23) and adding we get

$$\langle C(t,H(t)) - C(t,H_\varepsilon(t)), H_\varepsilon(t) - H(t)\rangle + \varepsilon\langle H_\varepsilon(t), H(t) - H_\varepsilon(t)\rangle \geq 0, \tag{6.3.27}$$

in $[0,T]$. Being C monotone, we have

$$\langle C(t,H(t)) - C(t,H_\varepsilon(t)), H_\varepsilon(t) - H(t)\rangle \leq 0, \quad \text{in } [0,T],$$

then, by (6.3.27), we obtain

$$\varepsilon\langle H_\varepsilon(t), H(t) - H_\varepsilon(t)\rangle \geq 0, \quad \text{in } [0,T],$$

and dividing by $\varepsilon > 0$, it results

$$\langle H_\varepsilon(t), H(t) - H_\varepsilon(t)\rangle \geq 0, \quad \text{in } [0,T]. \tag{6.3.28}$$

Taking into account (6.3.28), one has

$$\|H_\varepsilon(t)\|_m^2 \leq \langle H_\varepsilon(t), H(t)\rangle \leq \|H(t)\|_m \|H_\varepsilon(t)\|_m, \quad \text{in } [0,T],$$

then

$$\|H_\varepsilon(t)\|_m \leq \|H(t)\|_m, \quad \text{in } [0,T].$$

We remark that $H(t) \in \mathbf{X}(t,\overline{H}) \subseteq \mathbf{K}(t,\overline{H})$, in $[0,T]$, and $\mathbf{K}(t,\overline{H})$, $t \in [0,T]$, is a family of uniformly bounded sets of $\mathbb{R}^m$, then

$$\|H(t)\|_m \leq C, \quad \text{in } [0,T],$$

with C a constant independent on ε and on $t \in [0,T]$, namely

$$\|H_\varepsilon(t)\|_m \leq C, \quad \forall \varepsilon > 0, \text{ for } t \in [0,T].$$

Then, there exists a subsequence $\{H_\eta(t)\}_\eta$ converging in $\mathbb{R}^m$ to an element $\widehat{H}(t)$ of $\mathbb{R}^m$, in [0,T], and thus

$$\lim_{\eta\to 0} H_\eta(t) = \widehat{H}(t), \quad \text{in } [0,T].$$

Under the assumption that $\mathbf{K}(t,\overline{H})$ is a closed set of $\mathbb{R}^m$, it results

$$\widehat{H}(t) \in \mathbf{K}(t,\overline{H}), \text{ in } [0,T].$$

It remains to prove that

$$\widehat{H}(t) = H(t), \quad \text{in } [0,T].$$

Then, setting $\varepsilon = \eta$ in (6.3.26), we obtain

$$\langle C(t,H_\eta(t)), F(t)-H_\eta(t)\rangle + \eta\langle H_\eta(t), F(t)-H_\eta(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t,\overline{H}), \tag{6.3.29}$$

in $[0,T]$, and taking account that

$$\lim_{\eta\to 0}\langle H_\eta(t), H_\eta(t)\rangle = \langle \widehat{H}(t), \widehat{H}(t)\rangle, \quad \text{in } [0,T],$$

and that

$$\lim_{\eta\to 0}\langle C(t,H_\eta(t)), H_\eta(t)\rangle = \langle C(t,\widehat{H}(t)), \widehat{H}(t)\rangle, \quad \text{in } [0,T],$$

from (6.3.29), we have

$$\langle C(t,\overline{H}(t)), F(t) - \widehat{H}(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t,\overline{H}), \text{ in } [0,T]. \tag{6.3.30}$$

Hence (6.3.30) implies that $\widehat{H}$ is a solution to (6.3.22), in $[0,T]$, namely

$$\overline{H} \in \mathbf{X}(\overline{H}).$$

If the solution to (6.3.22) is unique, then the proof is concluded. Now, we suppose that the solution to (6.3.22) is not unique. Setting $\varepsilon = \eta$ in (6.3.28) we obtain

$$\langle H_\eta(t), H(t) - H_\eta(t)\rangle \geq 0, \quad \text{in } [0,T],$$

and passing to the limit for $\eta \to 0$, we get

$$\langle \widehat{H}(t), H(t) - \widehat{H}(t)\rangle \geq 0, \quad \text{in } [0,T]. \tag{6.3.31}$$

Setting $F = \widehat{H} \in \mathbf{X}(\overline{H})$ in (6.3.25), it results

$$\langle H(t), \widehat{H}(t) - H(t)\rangle \geq 0, \quad \text{in } [0,T], \tag{6.3.32}$$

and adding (6.3.31) and (6.3.32), we have

$$\langle \widehat{H}(t) - H(t), H(t) - \widehat{H}(t)\rangle \geq 0, \quad \text{in } [0,T].$$

Then

$$\langle \widehat{H}(t) - H(t), H(t) - \widehat{H}(t)\rangle = 0, \quad \text{in } [0,T],$$

that implies

$$\widehat{H}(t) = H(t), \quad \text{in } [0,T].$$

We have proved that every convergent subsequence converges to the same limit $H(t)$ and then

$$\lim_{\varepsilon\to 0} H_\varepsilon(t) = H(t), \quad \text{in } [0,T].$$

Moreover, it results

$$\|H_\varepsilon(t) - H(t)\|_m^2 \leq 2\big(\|H_\varepsilon(t)\|_m^2 + \|H(t)\|_m^2\big) \leq 4C^2 \in C([0,T],\mathbb{R}^m), \quad \text{in } [0,T],$$

hence, by virtue of Lebesgue's Theorem we have

$$\lim_{\varepsilon\to 0} \|H_\varepsilon(t) - H(t)\|^2_{L^2([0,T],\mathbb{R}^m)} = 0.$$

□

Now, we can prove the continuity result for degenerate evolutionary quasi-variational inequalities.

Theorem 6.3.2. *Let $C \in C([0,T]\times\mathbb{R}^m,\mathbb{R}^m)$ be a vector-function satisfying conditions* (6.3.20) *and* (6.3.21). *Let D be a nonempty, compact, convex subset of $L^2([0,T],\mathbb{R}^m)$. Let $\mathbf{K} : D \to 2^{L^2([0,T],\mathbb{R}^m)}$ be a multifunction with uniformly bounded set-values and satisfying condition* (MM). *Then, the evolutionary quasi-variational inequality*

$$H \in \mathbf{K} : \ \langle C(t,H(t)), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t,H), \quad \textit{in } [0,T], \tag{6.3.33}$$

admits a solution $H \in \mathbf{K}(H)$ such that $H \in C([0,T],\mathbb{R}^m)$.

Proof. Theorem 3.5.2 ensures that (6.3.33) admits a solution $\overline{H} \in \mathbf{K}(\overline{H})$ and the solution is unique in $\mathbf{K}(t, \overline{H})$ for each $t \in [0, T]$.

Let $t \in [0, T]$ be fixed and let $\{t_n\}_{n\in\mathbb{N}} \subseteq [0, T]$ be a sequence, such that $t_n \to t$, as $n \to +\infty$.

Let us consider the solution $H(t)$ to evolutionary quasi-variational inequality (6.3.33) and the solution $H(t_n)$, $\forall n \in \mathbb{N}$, to the following quasi-variational inequalities

$$\langle C(t_n, H(t_n)), F(t_n) - H(t_n)\rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n, \overline{H}),\ \forall n \in \mathbb{N}. \tag{6.3.34}$$

Let $H_\varepsilon(t)$ be the solution to perturbed strongly monotone quasi-variational inequality (6.3.24), namely $H_\varepsilon(t) \in \mathbf{K}(t, H)$ and

$$\langle C(t, H_\varepsilon(t)) + \varepsilon I H_\varepsilon(t), F(t) - H_\varepsilon(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t, \overline{H}). \tag{6.3.35}$$

Taking into account Theorem 6.2.3, it results that $H_\varepsilon(t)$ is a continuous function in $[0, T]$, then the solutions $H_\varepsilon(t_n)$, $\forall n \in \mathbb{N}$, to the following evolutionary quasi-variational inequalities

$$\langle C(t_n, H_\varepsilon(t_n)) + \varepsilon I H_\varepsilon(t_n), F(t_n) - H_\varepsilon(t_n)\rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n, \overline{H}), \tag{6.3.36}$$

$\forall n \in \mathbb{N}$, converge to $H_\varepsilon(t)$, as $n \to +\infty$. Setting $F(t_n) = H(t_n)$, $\forall n \in \mathbb{N}$, in (6.3.36) and $F(t_n) = H_\varepsilon(t_n)$, $\forall n \in \mathbb{N}$, in (6.3.34) and adding we obtain, $\forall n \in \mathbb{N}$

$$\langle C(t_n, H_\varepsilon(t_n)) - C(t_n, H(t_n)), H(t_n) - H_\varepsilon(t_n)\rangle + \varepsilon\langle H_\varepsilon(t_n), H(t_n) - H_\varepsilon(t_n)\rangle \geq 0. \tag{6.3.37}$$

Moreover, for condition (6.3.21) it results

$$\langle C(t_n, H_\varepsilon(t_n)) - C(t_n, H(t_n)), H(t_n) - H_\varepsilon(t_n)\rangle \leq 0, \quad \forall n \in \mathbb{N}.$$

Hence, from (6.3.37) it follows

$$\varepsilon\langle H_\varepsilon(t_n), H(t_n) - H_\varepsilon(t_n)\rangle \geq 0, \quad \forall n \in \mathbb{N},$$

and proceeding as in the proof of Lemma 6.3.2, we have

$$\|H_\varepsilon(t_n)\|_m \leq C, \quad \forall \varepsilon > 0,\ \forall n \in \mathbb{N}, \tag{6.3.38}$$

where C is a constant independent on ε and of $n \in \mathbb{N}$.

From Lemma 6.3.2, it follows

$$\lim_{\varepsilon\to 0} H_\varepsilon(t_n) = \widetilde{H}(t_n), \quad \forall n \in \mathbb{N},$$

with $\widetilde{H}(t_n) \in \mathbf{K}(t_n, \overline{H})$, $\forall n \in \mathbb{N}$, and such that

$$\langle C(t_n, \widetilde{H}(t_n)), F(t_n) - \widetilde{H}(t_n)\rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n, \overline{H}),\ \forall n \in \mathbb{N}.$$

Since the solution to (6.3.34) is unique, it results

$$\widetilde{H}(t_n) = H(t_n), \quad \forall n \in \mathbb{N},$$

and, passing to the limit as $\varepsilon \to 0$ in (6.3.38), we have

$$\|H(t_n)\|_m \leq C, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{H(t_n)\}_{n\in\mathbb{N}}$ is bounded, that implies the existence of a subsequence $\{H(t_{k_n})\}_{n\in\mathbb{N}}$, with $H(t_{k_n}) \in \mathbf{K}(t_{k_n}, \overline{H})$, $\forall n \in \mathbb{N}$, converging in $\mathbb{R}^m$ to an element $\widehat{H}(t)$ of $\mathbb{R}^m$, namely

$$\lim_{n\to+\infty} H(t_{k_n}) = \widehat{H}(t).$$

Taking into account (6.3.34) it follows

$$\langle C(t, \widehat{H}(t)), F(t) - \widehat{H}(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t, \overline{H}),$$

and, for the uniqueness of the solution to (6.3.33), it results

$$\widehat{H}(t) = \overline{H}(t).$$

The same result holds for each subsequence and therefore

$$\lim_{n\to+\infty} H(t_n) = \overline{H}(t).$$

The proof is now complete. □

6.4 Regularity results for strictly monotone evolutionary quasi-variational inequalities

In this section, a theorem of continuity for solutions to evolutionary strictly monotone quasi-variational inequalities will be shown. More precisely, at first we obtain the result for an affine operator and, then, we generalize the result for nonlinear operators. The regularization Lemmas 6.3.1 and 6.3.2, proved in the previous section, play an important role in the attainment of results.

6.4.1 Affine case

Let us assume that the operator is affine with respect to the vector F, namely it results

$$C(t, F(t)) = A(t)F(t) + B(t),$$

for each $t \in [0,T]$, where $A : [0,T] \to \mathbb{R}^{m\times m}$ and $B : [0,T] \to \mathbb{R}^m$ are two functions. We study the continuity for solutions to the following evolutionary variational inequality

Find $H \in \mathbf{K}$ such that

$$\langle A(t)H(t) + B(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T], \tag{6.4.1}$$

where $\mathbf{K} : D \to 2^{L^2([0,T],\mathbb{R}^m)}$ is a multifunction satisfying condition (MM).

Let us suppose that the vector-function A satisfies the following assumption:

$$\langle A(t)[H - F], H - F\rangle > 0, \quad \forall H, F \in \mathbb{R}^m, \; H \neq F, \text{ a.e. in } [0,T]. \tag{6.4.2}$$

Theorem 6.4.1. *Let $A \in C([0,T],\mathbb{R}^{m\times m})$ be a matrix-function satisfying condition* (6.4.2) *and let $B \in C([0,T],\mathbb{R}^m)$ be a vector-function. Let D be a nonempty, compact, convex subset of $L^2([0,T],\mathbb{R}^m)$. Let $\mathbf{K} : D \to 2^{L^2([0,T],\mathbb{R}^m)}$ be a multifunction with uniformly bounded set-values and satisfying condition* (MM). *Then, evolutionary quasi-variational inequality* (6.4.1) *admits a solution $H \in \mathbf{K}(H)$ such that $H \in C([0,T],\mathbb{R}^m)$.*

Proof. Taking into account of Theorem 3.4.3, it results that (6.4.1) admits a solution $\overline{H} \in \mathbf{K}(\overline{H})$, furthermore the solution is unique in $\mathbf{K}(\overline{H})$. We fix the set $\mathbf{K}(\overline{H})$.

Now, let $t \in [0,T]$ be fixed and let $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,T]$ be a sequence, such that $t_n \to t$, as $n \to +\infty$.

Let us consider the solution $\overline{H}(t)$ to quasi-variational inequality (6.4.1) and the solution $H(t_n)$, $\forall n \in \mathbb{N}$, to the following quasi-variational inequalities

$$\langle A(t_n)H(t_n) + B(t_n), F(t_n) - H(t_n)\rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n,\overline{H}), \; \forall n \in \mathbb{N}. \tag{6.4.3}$$

Let us consider $H_\varepsilon(t) \in \mathbf{K}(t,\overline{H})$ such that

$$\langle [A(t) + \varepsilon I(t)]H_\varepsilon(t) + B(t), F(t) - H_\varepsilon(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t,\overline{H}), \text{ in } [0,T]. \tag{6.4.4}$$

We remark that $H_\varepsilon(t)$ is continuous in $[0,T]$, then the solutions $H(t_n)$, $\forall n \in \mathbb{N}$, to the following evolutionary quasi-variational inequalities

$$\langle [A(t_n) + \varepsilon I(t_n)]H_\varepsilon(t_n) + B(t_n), F(t_n) - H_\varepsilon(t_n)\rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n,\overline{H}), \tag{6.4.5}$$

$\forall n \in \mathbb{N}$, converge to $H_\varepsilon(t)$, as $n \to +\infty$. Setting $F(t_n) = H(t_n)$, $\forall n \in \mathbb{N}$, in (6.4.5) and $F(t_n) = H_\varepsilon(t_n)$, $\forall n \in \mathbb{N}$, in (6.4.3) and adding it follows, $\forall n \in \mathbb{N}$,

$$\langle A(t_n)[H_\varepsilon(t_n) - H(t_n)], H(t_n) - H_\varepsilon(t_n)\rangle + \varepsilon\langle H_\varepsilon(t_n), H(t_n) - H_\varepsilon(t_n)\rangle \geq 0. \tag{6.4.6}$$

We remark that for condition (6.4.2) on the matrix-function A we have

$$\langle A(t_n)[H_\varepsilon(t_n) - H(t_n)], H(t_n) - H_\varepsilon(t_n)\rangle \leq 0, \quad \forall n \in \mathbb{N}.$$

So, from (6.4.6) it results

$$\varepsilon\langle H_\varepsilon(t_n), H(t_n) - H_\varepsilon(t_n)\rangle \geq 0, \quad \forall n \in \mathbb{N},$$

and proceeding as in the proof of Lemma 6.3.1, we get

$$\|H_\varepsilon(t_n)\|_m \leq C, \quad \forall \varepsilon > 0,\ \forall n \in \mathbb{N}, \tag{6.4.7}$$

where C is a constant independent on ε and on $n \in \mathbb{N}$.

For Lemma 6.3.1, it follows

$$\lim_{\varepsilon \to 0} H_\varepsilon(t_n) = \widetilde{H}(t_n), \quad \forall n \in \mathbb{N},$$

with $\widetilde{H}(t_n) \in \mathbf{K}(t_n, \overline{H})$, $\forall n \in \mathbb{N}$, and such that

$$\langle A(t_n)\widetilde{H}(t_n) + B(t_n), F(t_n) - \widetilde{H}(t_n)\rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n, \overline{H}),\ \forall n \in \mathbb{N}.$$

Since the solution to (6.4.3) is unique in the set $\mathbf{K}(\overline{H})$, one has

$$\widetilde{H}(t_n) = H(t_n), \quad \forall n \in \mathbb{N},$$

and, passing to the limit as $\varepsilon \to 0$ in (6.4.7), it results

$$\|H(t_n)\|_m \leq C, \quad \forall n \in \mathbb{N}.$$

Hence the sequence $\{H(t_n)\}_{n\in\mathbb{N}}$ is bounded, then there exists a subsequence $\{H(t_{k_n})\}_{n\in\mathbb{N}}$, with $H(t_{k_n}) \in \mathbf{K}(t_{k_n}, \overline{H})$, $\forall n \in \mathbb{N}$, converging in $\mathbb{R}^m$ to an element $\widehat{H}(t)$ of $\mathbb{R}^m$, namely

$$\lim_{n \to +\infty} H(t_{k_n}) = \widehat{H}(t).$$

Moreover, by (6.4.3) we obtain

$$\langle A(t)\widehat{H}(t) + B(t), F(t) - \widehat{H}(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t, \overline{H}),$$

and, for the uniqueness of the solution to (6.4.1) in the set $\mathbf{K}(\overline{H})$, it follows

$$\widehat{H}(t) = \overline{H}(t).$$

The same result holds for each subsequence and therefore

$$\lim_{n \to +\infty} H(t_n) = \overline{H}(t).$$

The proof is now complete. □

6.4.2 Nonlinear case

Let $C : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ be a nonlinear operator satisfying the following conditions

$$\exists \gamma \in L^2([0,T],\mathbb{R}_+): \ \|C(t,F)\|_m \leq \gamma(t) + \|F\|_m, \quad \forall F \in \mathbb{R}^m, \ in \ [0,T], \quad (6.4.8)$$

and

$$\langle C((t,H) - C(t,F), H - F\rangle > 0, \quad \forall H, F \in \mathbb{R}^m, \ H \neq F, \text{ a.e. in } [0,T]. \quad (6.4.9)$$

Let us consider the evolutionary quasi-variational inequality

Find $H \in \mathbf{K}(H)$ such that

$$\langle C(t,H(t)), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t,H), \text{ a.e. in } [0,T], \quad (6.4.10)$$

where $\mathbf{K} : D \to 2^{L^2([0,T],\mathbb{R}^m)}$ is a multifunction satisfying condition (MM).

Now, we are able to prove the continuity of solutions to nonlinear strictly monotone evolutionary quasi-variational inequalities.

Theorem 6.4.2. *Let $C \in C([0,T] \times \mathbb{R}^m, \mathbb{R}^m)$ be a vector-function satisfying conditions* (6.4.8) *and* (6.4.9)*. Let D be a nonempty, compact, convex subset of $L^2([0,T],\mathbb{R}^m)$. Let $\mathbf{K} : D \to 2^{L^2([0,T],\mathbb{R}^m)}$ be a multifunction with uniformly bounded set-values and satisfying condition* (MM)*. Then, evolutionary quasi-variational inequality* (6.4.10) *admits a solution $H \in \mathbf{K}(H)$ such that $H \in C([0,T],\mathbb{R}^m)$.*

Proof. By Theorem 3.4.3, it follows that (6.4.10) admits a solution $\overline{H} \in \mathbf{K}(\overline{H})$, furthermore the solution is unique in the set $\mathbf{K}(\overline{H})$. Then, we fix the set $\mathbf{K}(\overline{H})$.

Let $t \in [0,T]$ be fixed and let $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,T]$ be a sequence, such that $t_n \to t$, as $n \to +\infty$.

Let $\overline{H}(t)$ be the solution to quasi-variational inequality (6.4.10) in $t \in [0,T]$ and let $H(t_n)$, $\forall n \in \mathbb{N}$, be the solutions to quasi-variational inequalities

$$\langle C(t_n, H(t_n)), F(t_n) - H(t_n)\rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n, \overline{H}), \ \forall n \in \mathbb{N}. \quad (6.4.11)$$

Let $H_\varepsilon(t) \in \mathbf{K}(t,\overline{H})$ be the solution to the following perturbed strongly monotone quasi-variational inequality

$$\langle C(t, H_\varepsilon(t)) + \varepsilon I H_\varepsilon(t), F(t) - H_\varepsilon(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t,\overline{H}).$$

From Theorem 6.2.3, it follows that H_ε is continuous in $[0,T]$, then we have that the solutions $H_\varepsilon(t_n)$, $\forall n \in \mathbb{N}$, to the evolutionary quasi-variational inequalities

$$\langle C(t_n, H_\varepsilon(t_n)) + \varepsilon I(t_n) H_\varepsilon(t_n), F(t_n) - H_\varepsilon(t_n)\rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n, \overline{H}), \quad (6.4.12)$$

$\forall n \in \mathbb{N}$, converge to $H_\varepsilon(t)$, as $n \to +\infty$. Setting $F(t_n) = H(t_n)$, $\forall n \in \mathbb{N}$, in (6.4.12) and $F(t_n) = H_\varepsilon(t_n)$, $\forall n \in \mathbb{N}$, in (6.4.11) and adding it results, $\forall n \in \mathbb{N}$

$$\langle C(t_n, H_\varepsilon(t_n)) - C(t_n, H(t_n)), H(t_n) - H_\varepsilon(t_n)\rangle + \varepsilon\langle H_\varepsilon(t_n), H(t_n) - H_\varepsilon(t_n)\rangle \geq 0. \tag{6.4.13}$$

Moreover, from the strict monotonicity of the function C it follows

$$\langle C(t_n, H_\varepsilon(t_n)) - C(t_n, H(t_n)), H(t_n) - H_\varepsilon(t_n)\rangle \leq 0, \quad \forall n \in \mathbb{N}. \tag{6.4.14}$$

Then, using (6.4.13) and (6.4.14) we obtain

$$\varepsilon\langle H_\varepsilon(t_n), H(t_n) - H_\varepsilon(t_n)\rangle \geq 0, \quad \forall n \in \mathbb{N},$$

and dividing by $\varepsilon > 0$, we get

$$\langle H_\varepsilon(t_n), H(t_n) - H_\varepsilon(t_n)\rangle \geq 0, \quad \forall n \in \mathbb{N}. \tag{6.4.15}$$

From (6.4.15), it follows

$$\|H_\varepsilon(t_n)\|_m^2 \leq \langle H_\varepsilon(t_n), H(t_n)\rangle \leq \|H(t_n)\|_m \|H_\varepsilon(t_n)\|_m, \quad \forall n \in \mathbb{N},$$

then

$$\|H_\varepsilon(t_n)\|_m \leq \|H(t_n)\|_m, \quad \forall n \in \mathbb{N}.$$

Since $H(t_n) \in \mathbf{X}(t_n, \overline{H}) \subseteq \mathbf{K}(t_n, \overline{H})$, for $n \in \mathbb{N}$, and $\mathbf{K}(t_n, \overline{H})$, for $n \in \mathbb{N}$, are uniformly bounded sets of $\mathbb{R}^m$, it results

$$\|H(t_n)\|_m \leq C, \quad \forall n \in \mathbb{N},$$

where C is a constant independent on ε and on $n \in \mathbb{N}$, then

$$\|H_\varepsilon(t_n)\|_m \leq C, \quad \forall \varepsilon > 0, \ \forall n \in \mathbb{N}. \tag{6.4.16}$$

By Lemma 6.3.2, we get

$$\lim_{\varepsilon \to 0} H_\varepsilon(t_n) = \widetilde{H}(t_n), \quad \forall n \in \mathbb{N},$$

where $\widetilde{H}(t_n) \in \mathbf{K}(t_n, \overline{H})$, $\forall n \in \mathbb{N}$, and such that

$$\langle C(t_n, \widetilde{H}(t_n)), F(t_n) - \widetilde{H}(t_n)\rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n, \overline{H}), \ \forall n \in \mathbb{N}.$$

For the uniqueness of the solution to (6.4.11) in the set $\mathbf{K}(t_n, \overline{H})$, it results

$$\widetilde{H}(t_n) = H(t_n), \quad \forall n \in \mathbb{N},$$

and, passing to the limit as $\varepsilon \to 0$ in (6.4.16), it follows

$$\|H(t_n)\|_m \leq C, \quad \forall n \in \mathbb{N},$$

namely the sequence $\{H(t_n)\}_{n\in\mathbb{N}}$ is bounded. Hence, there exists a subsequence $\{H(t_{k_n})\}_{n\in\mathbb{N}}$, with $H(t_{k_n}) \in \mathbf{K}(t_{k_n}, \overline{H})$, $\forall n \in \mathbb{N}$, converging in $\mathbb{R}^m$ to an element $\widehat{H}(t)$ of $\mathbb{R}^m$, namely

$$\lim_{n\to+\infty} H(t_{k_n}) = \widehat{H}(t).$$

Moreover, by (6.4.11) it obtains

$$\langle C(t, \widehat{H}(t)), F(t) - \widehat{H}(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t, \overline{H}),$$

and, for the uniqueness of the solution to (6.4.10) in $\mathbf{K}(t, \overline{H})$, it follows

$$\widehat{H}(t) = \overline{H}(t).$$

The same result holds for each subsequence and so

$$\lim_{n\to+\infty} H(t_n) = \overline{H}(t),$$

namely our assert. □

7

Application to dynamic equilibrium problems

7.1 Introduction

The aim of this chapter is to show regularity results for dynamic equilibrium problems. The regularity results, shown for a general class of evolutionary variational and quasi-variational inequalities (see Chapter 7 and Chapter 6), find applications to dynamic equilibrium problems. In fact, these regularity results are been proved for evolutionary variational inequalities associated to sets $\mathbf{K} \subseteq L^2([0,T],\mathbb{R}^m)$ satisfying the condition

(M) $\mathbf{K}$ is a nonempty convex, closed set, such that the set sequence $\{\mathbf{K}(t_n)\}_{n\in\mathbb{N}}$ converges to $\mathbf{K}(t)$ in Mosco's sense, for each $t \in [0,T]$, and the sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,T]$, such that $t_n \to t$, as $n \to +\infty$,

and for evolutionary quasi-variational inequalities associated to multifunctions $\mathbf{K} : [0,T] \to 2^{L^2([0,T],\mathbf{R}^m_+)}$ satisfying the assumption

(MM) $\mathbf{K}$ is closed l.s.c. with $\mathbf{K}(H)$, for each $H \in L^2([0,T],\mathbb{R}^m_+)$, nonempty, convex, closed of $L^2([0,T],\mathbf{R}^m_+)$ such that $\mathbf{K}(t_n,H)$ converges to $\mathbf{K}(t,H)$ in Mosco's sense, for each sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,T]$, with $t_n \to t$, as $n \to +\infty$.

Having proved that the sets of feasible flows associated to dynamic traffic equilibrium problems verify the previous conditions, we able to apply continuity results to dynamic traffic equilibrium problems. The same results hold for other equilibrium problems, as the dynamic market equilibrium problem and the dynamic financial equilibrium problem, because their sets of feasible vectors satisfy condition (M) for Propositions 4.3.1 and 4.4.2.

7.2 Regularity results for the dynamic traffic equilibrium problem

In this section, theorems on the continuity of solutions to dynamic traffic equilibrium problem with fixed demand will be provided.

7.2.1 Affine case

Let us assume that the cost vector-function $C(t, F(t))$ is an affine operator with respect to the flow-vector, that is

$$C(t, F(t)) = A(t)F(t) + B(t),$$

for each $t \in [0, T]$, where $A : [0, T] \to \mathbb{R}_+^{m\times m}$ and $B : [0, T] \to \mathbb{R}_+^m$ are two functions.

We study the continuity of solutions to dynamic traffic equilibrium problem express by the evolutionary variational inequality

Find $H \in \mathbf{K}$ such that

$$\langle A(t)H(t) + B(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0, T], \tag{7.2.1}$$

where

$$\mathbf{K}(t) = \Big\{F(t) \in \mathbb{R}^m : \ \lambda(t) \leq F(t) \leq \mu(t), \quad \Phi F(t) = \rho(t)\Big\}.$$

If the cost vector-function is strongly monotone, the following result holds (see [3], Theorem 3.2).

Theorem 7.2.1. *Let $A \in C([0, T], \mathbb{R}_+^{m\times m})$ be a positive definite matrix-function and let $B \in C([0, T], \mathbb{R}_+^m)$ be a vector function. Suppose that $\lambda, \mu \in C([0, T], \mathbb{R}_+^m)$ and $\rho \in C([0, T], \mathbb{R}_+^l)$. Then, the evolutionary variational inequality (7.2.1) admits a unique solution $H \in \mathbf{K}$ such that $H \in C([0, T], \mathbb{R}_+^m)$. Moreover, the estimate*

$$\|H_1 - H_2\|_{C([0,T],\mathbb{R}_+^m)} \leq \frac{1}{\nu}\|B_1 - B_2\|_{C([0,T],\mathbb{R}_+^m)} \tag{7.2.2}$$

holds, where ν is the constant of positive definition of the matrix $A(t)$, for each $t \in [0, T]$.

Proof. Taking into account of Proposition 4.3.1 and of Theorem 5.2.1, we can obtain that there exists a unique solution to (7.2.1) and that it is continuous in $[0, T]$.

Moreover, the estimate (7.2.2) is a well-known consequence of the assumption that $A(t)$ is positive definite, for each $t \in [0, T]$, and of Theorem 1.4.5. □

The stability of equilibrium solutions is very important for the numerical approximation of evolutionary variational inequalities, where $n \to +\infty$ denotes a discretization parameter as, for example, the mesh size in the finite elements method.

Let us consider the following variational inequality

Find $H(t_n) \in \mathbf{K}(t_n)$ such that

$$\langle A(t_n)H(t_n) + B(t_n), F(t_n) - H(t_n) \rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n), \tag{7.2.3}$$

where

$$\mathbf{K}(t_n) = \Big\{ F(t_n) \in \mathbb{R}^m : \ \lambda(t_n) \leq F(t_n) \leq \mu(t_n), \ \ \Phi F(t_n) = \rho(t_n) \Big\}.$$

Taking into account Theorem 4.2.3, we can derive in our case the following result:

Theorem 7.2.2. *Under the assumptions of Theorem* 7.2.1*, let* $H(t)$ *and* $H(t_n)$ *be the solutions of* (7.2.1) *and* (7.2.3)*, respectively, then the following estimate holds, in* $[0,T]$

$$\|H(t) - H(t_n)\|_m \leq C \Big\{ \|B(t) - B(t_n)\|_m + \delta_{r_0}(A(t), A(t_n)) + \pi_{r_1}(\mathbf{K}(t), \mathbf{K}(t_n)) \Big\},$$

where, for any $\rho \in]0, 2\nu/M^2[$, $C = \max(1,\rho)/(1 - \sqrt{1 - 2\rho\nu + \rho^2 M^2}) > 0$, $r_0 = b/\nu + d_0(1 + M/\nu)$*, with* $d_0 \leq \max_r \max_{[0,T]} \mu_r(t)$*, and* $r_1 = r_0 + \rho(Mr_0 + b)$*, with* $M = \|A\|_{C([0,T],\mathbb{R}^{m\times m})}$ *and* ν *is the constant of positive definition of* $A(t)$*, for each* $t \in [0,T]$.

Proof. Let $t \in [0,T]$ be fixed and let $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,T]$ be a sequence, such that $t_n \to t$, as $n \to +\infty$. Obviously, it results

$$\|A(t)0\|_m = \|A(t_n)0\|_m = 0.$$

Owing to the continuity of B, we have that $\{B(t_n)\}_{n\in\mathbb{N}}$ is a bounded sequence, namely

$$\exists b > 0 : \ \|B(t_n)\|_m \leq b, \quad \forall n \in \mathbb{N},$$

moreover, the limit $B(t)$ is bounded with the same constant, i.e.

$$\|B(t)\|_m \leq b.$$

Taking account that we proved that $\mathbf{K}(t_n) \to \mathbf{K}(t)$ in Mosco's sense (see Proposition 4.3.1), all the conditions of Theorem 4.2.2 are satisfied, and it follows

$$P_{\mathbf{K}(t_n)}F(t) \to P_{\mathbf{K}(t)}F(t), \quad \forall F(t) \in \mathbb{R}^m,$$

and in particular

$$P_{\mathbf{K}(t_n)}0 \to P_{\mathbf{K}(t)}0. \tag{7.2.4}$$

A direct calculation enables us to say that

$$\exists d_0 > 0 : \ \|P_{\mathbf{K}(t_n)}0\|_m, \|P_{\mathbf{K}(t)}0\|_m \leq d_0, \quad \forall n \in \mathbb{N},$$

where $d_0 \leq m \max_r \max_{[0,T]} \mu_r(t)$. Then, all conditions of the Theorem 4.2.3 are satisfied, and the conclusion follows directly from this theorem. □

Remark 7.2.1. In our case, if $\lambda_r(t) \geq \overline{\lambda}_r > 0$ $r = 1, 2, \dots, m$, $H(t)$ admits a low bound of the norm, for $t \in [0, T]$. In fact one has

$$\|H(t)\|_m \geq m\Lambda, \quad \text{in } [0, T],$$

where $\Lambda = \min_r \min_{[0,T]} \lambda_r(t)$. Then, under the assumptions of Theorem 7.2.2, we deduce, in $[0, T]$:

$$\frac{\|H(t) - H(t_n)\|_m}{\|H(t)\|_m} \leq \frac{C}{m\Lambda}\Big\{\|B(t) - B(t_n)\|_m + \delta_{r_0}(A(t), A(t_n)) + \pi_{r_1}(\mathbf{K}(t), \mathbf{K}(t_n))\Big\}.$$

Now, we still assume that the cost $C(t, F(t))$ is an affine operator with respect to the flows, but the matrix-function A depends only on the time but only on the integral average of the vector-flow, namely

$$C(t, F(t)) = A(t, F_{\mathcal{T}})F(t) + B(t),$$

for a.e. $t \in [0, T]$ and for every $F \in L^2([0, T], \mathbb{R}^m_+)$, where $A : [0, T] \times \mathbb{R}^m_+ \to \mathbb{R}^{m \times m}_+$ and $B : [0, T] \to \mathbb{R}^m_+$ are two functions, $\mathcal{T} = [0, T]$ and $F_{\mathcal{T}}$ is the integral average, that is

$$F_{\mathcal{T}} = \frac{\int_0^T F(\tau)d\tau}{T}.$$

We suppose that $A(t, u)$ is a bounded matrix, namely

$$\exists M > 0 : \ \|A(t, u)\|_{m \times m} \leq M, \quad \text{for a.e. } t \in [0, T], \ \forall u \in \mathbb{R}^m_+. \tag{7.2.5}$$

Then we study the continuity of the solutions to the following evolutionary variational inequality:

Find $H \in \mathbf{K}$ such that

$$\langle A(t, H_{\mathcal{T}})H(t) + B(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0, T], \tag{7.2.6}$$

where

$$\mathbf{K}(t) = \Big\{F \in \mathbb{R}^m_+ : \ \lambda(t) \leq F(t) \leq \mu(t), \quad \Phi F(t) = \rho(t)\Big\},$$

Now, we can show the continuity result for the dynamic traffic equilibrium problem express by evolutionary variational inequality (7.2.6) (see also [3], Theorem 5.1).

Theorem 7.2.3. *Let* $A \in C([0,T] \times \mathbb{R}^m_+, \mathbb{R}^{m\times m}_+)$ *be a matrix-function satisfying conditions* (7.2.5) *and*

$$\exists \nu > 0: \ \langle A(t, F_T)F, F\rangle \geq \nu \|F\|^2_m, \quad \forall F \in \mathbb{R}^m_+, \ in\ [0,T],$$

let $B \in C([0,T], \mathbb{R}^m_+)$ *be a vector function. Suppose that* $\lambda, \mu \in C([0,T], \mathbb{R}^m_+)$ *and* $\rho \in C([0,T], \mathbb{R}^l_+)$*. Then, evolutionary variational inequality* (7.2.6) *admits a unique solution* $H \in \mathbf{K}$ *such that* $H \in C([0,T], \mathbb{R}^m_+)$*.*

Proof. The thesis follows from Proposition 4.3.1 and Theorem 5.2.2. □

Let us consider the dynamic traffic equilibrium problem expressed by a degenerate evolutionary variational inequality (see [4]), that is when the matrix-function A satisfies the condition

$$\langle A(t)F, F\rangle \geq \nu(t)\|F\|^2_m, \quad \forall F \in \mathbb{R}^m_+, \ \text{a.e. in } [0,T], \tag{7.2.7}$$

where $\nu \in L^\infty([0,T], \mathbb{R}^+_0)$ is such that

$$\nexists I \subseteq [0,T], \ \mu(I) > 0: \ \nu(t) = 0, \quad \forall t \in I,$$

with μ Lebesgue's measure. We investigate on the continuity of equilibrium solutions to this problem.

Theorem 7.2.4. *Let* $A \in C([0,T], \mathbb{R}^{m\times m}_+)$ *be a matrix-function verifying condition and* (7.2.7) *and let* $B \in C([0,T], \mathbb{R}^m_+)$ *be a vector-function. Suppose that* $\lambda, \mu, \rho \in C([0,T], \mathbb{R}^m_+)$*. Then, the evolutionary variational inequality*

$$\langle A(t)H(t) + B(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \ in\ [0,T], \tag{7.2.8}$$

admits a unique solution $H \in \mathbf{K}$ *such that* $H \in C([0,T], \mathbb{R}^m_+)$*.*

Proof. Propositions 4.3.1 and 4.3.2 and Theorem 7.2.5 imply that there exists a unique continuous solution to (7.2.8). □

Finally, using Propositions 4.3.1 and 4.3.2 and Theorem 5.4.1, we show a more general result for the dynamic traffic equilibrium problem expressed by strictly monotone evolutionary variational inequalities.

Theorem 7.2.5. *Let* $A \in C([0,T], \mathbb{R}^{m\times m}_+)$ *be a matrix-function verifying condition*

$$\langle A(t)[H - F], H - F\rangle \geq 0, \quad \forall H, F \in \mathbb{R}^m_+, \ H \neq F, \ \text{in } [0,T], \tag{7.2.9}$$

and let $B \in C([0,T], \mathbb{R}^m_+)$ *be a vector-function. Suppose that* $\lambda, \mu, \rho \in C([0,T], \mathbb{R}^m_+)$*. Then, the evolutionary variational inequality*

$$\langle A(t)H(t) + B(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \ in\ [0,T], \tag{7.2.10}$$

admits a unique solution $H \in \mathbf{K}$ *such that* $H \in C([0,T], \mathbb{R}^m_+)$*.*

7.2.2 Nonlinear case

Let us consider nonlinear cost operator

$$C : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m,$$

and let us study the continuity of the dynamic traffic equilibrium problem expressed by the evolutionary variational inequality

Find $H \in \mathbf{K}$ such that

$$\langle C(t,H(t)), F(t) - H(t) \rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T], \tag{7.2.11}$$

where

$$\mathbf{K}(t) = \Big\{ F(t) \in \mathbb{R}^m : \ \lambda(t) \leq F(t) \leq \mu(t), \ \ \Phi F(t) = \rho(t) \Big\}.$$

The following result holds (see also [11], Theorem 3.1).

Theorem 7.2.6. *Let $C \in C([0,T] \times \mathbb{R}^m_+, \mathbb{R}^{m \times m}_+)$ be a vector-function satisfying conditions*

$$\|C(t,F)\|_m \leq A(t)\|F\|_m + B(t), \quad \forall F \in \mathbb{R}^m_+, \ \textit{in } [0,T],$$

with $B \in C([0,T], \mathbb{R}_+)$ and $A \in C([0,T], \mathbb{R}_+)$ and

$$\exists \nu > 0 : \ \langle C(t,H) - C(t,F), H - F \rangle \geq \nu \|H - F\|^2_m, \quad \forall H, F \in \mathbb{R}^m_+, \ \textit{in } [0,T],$$

let $\lambda, \mu \in C([0,T], \mathbb{R}^m_+)$ and let $\rho \in C([0,T], \mathbb{R}^l_+)$ be vector-functions. Then, the evolutionary variational inequality

$$H \in \mathbf{K} : \ \langle C(t,H(t)), F(t) - H(t) \rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ in } [0,T]. \tag{7.2.12}$$

admits a unique solution $H \in \mathbf{K}$ such that $H \in C([0,T], \mathbb{R}^m_+)$.

Proof. Taking into account of Proposition 4.3.1 and of Theorem 5.2.3, we can obtain the assertion. □

An analogous result holds when the cost operator is degenerate (see also [11], Theorem 3.2).

Theorem 7.2.7. *Let $C \in C([0,T] \times \mathbb{R}^m_+, \mathbb{R}^{m \times m}_+)$ be an operator satisfying conditions*

$$\|C(t,F)\|_m \leq A(t)\|F\|_m + B(t), \quad \forall F \in \mathbb{R}^m_+, \ \textit{in } [0,T],$$

with $B \in C([0,T],\mathbb{R}_+)$ *and* $A \in C([0,T],\mathbb{R}_+)$ *and*

$$\langle C(t,H) - C(t,F), H - F\rangle \geq \nu(t)\|H - F\|_m^2, \quad \forall H, F \in \mathbb{R}_+^m, \text{ in } [0,T],$$

where $\nu \in L^\infty([0,T],\mathbb{R}_0^+)$ *is such that*

$$\nexists I \subseteq [0,T], \ \mu(I) > 0 : \ \nu(t) = 0, \ a.e. \text{ in } I.$$

Let $\lambda, \mu \in C([0,T],\mathbb{R}_+^m)$ *and let* $\rho \in C([0,T],\mathbb{R}_+^l)$ *be vector-functions. Then, the evolutionary variational inequality*

$$H \in \mathbf{K} : \ \langle C(t,H(t)), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ in } [0,T], \tag{7.2.13}$$

admits a unique solution $H \in \mathbf{K}$ *such that* $H \in C([0,T],\mathbb{R}_+^m)$.

Proof. Making use of Propositions 4.3.1 and 4.3.2 and of Theorem 5.2.3, we derive that the unique solution to (7.2.13) is continuous. □

At last, we are able to prove this theorem in a more general case. In fact, as a consequence of Theorem 5.4.2, using Proposition 4.3.1 and the uniformly boundedness of $\mathbf{K}(t)$, in $[0,T]$, we obtain the following continuity result.

Theorem 7.2.8. *Let* $C \in C([0,T] \times \mathbb{R}_+^m, \mathbb{R}_+^{m\times m})$ *be a vector-function satisfying conditions*

$$\|C(t,F)\|_m \leq A(t)\|F\|_m + B(t), \quad \forall F \in \mathbb{R}_+^m, \text{ in } [0,T],$$

with $B \in C([0,T],\mathbb{R}_+)$ *and* $A \in C([0,T],\mathbb{R}_+)$ *and*

$$\langle C(t,H) - C(t,F), H - F\rangle > 0, \quad \forall F \in \mathbb{R}_+^m, \ H \neq F, \text{ in } [0,T].$$

Let $\lambda, \mu \in C([0,T],\mathbb{R}_+^m)$ *and let* $\rho \in C([0,T],\mathbb{R}_+^l)$ *be vector-functions. Then, the evolutionary variational inequality*

$$H \in \mathbf{K} : \ \langle C(t,H(t)), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ in } [0,T],$$

admits a unique solution $H \in \mathbf{K}$ *such that* $H \in C([0,T],\mathbb{R}_+^m)$.

7.3 Regularity results for the dynamic elastic traffic equilibrium problem

In this section, we investigate on the continuity of the solution to the dynamic elastic equilibrium problem.

7.3.1 Affine case

Let us suppose that the cost vector-function is affine with respect to the flows, and we study the continuity of solutions to the evolutionary quasi-variational inequality which expresses the dynamic elastic traffic problem, that is

Find $H \in \mathbf{K}(H)$ such that

$$\langle A(t)H(t) + B(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t, H), \text{ a.e. in } [0, T], \tag{7.3.1}$$

where

$$\mathbf{K}(t, H) = \left\{ F(t) \in \mathbb{R}^m : \ \lambda(t) \leq F(t) \leq \mu(t) \ \ \Phi F(t) = \frac{1}{T}\int_0^T \rho(t, H(\tau))d\tau \right\}.$$

Now, we show the continuity result (see also [3], Theorem 4.1).

Theorem 7.3.1. *Let $A \in C([0,T], \mathbb{R}_+^{m\times m})$ be a positive definite matrix-function and let $B \in C([0,T], \mathbb{R}_+^m)$ be a vector-function. Let $\lambda, \mu \in C([0,T], \mathbb{R}_+^m)$ be and let $\rho \in C([0,T] \times \mathbb{R}_+^m, \mathbb{R}_+^l)$ be such that*

$$\exists \psi \in C([0,T]) : \ \|\rho(t, F)\|_l \leq \psi(t) + \|F\|_m^2, \tag{7.3.2}$$

$$\exists \omega \in C([0,T]) : \ \|\rho(t, H) - \rho(t, F)\|_l \leq \omega(t)\|H - F\|_m^2, \tag{7.3.3}$$

$\forall H, F \in \mathbb{R}_+^m$, in $[0,T]$. Then, evolutionary quasi-variational inequality (7.3.1) *admits a solution $H \in \mathbf{K}(H)$ such that $H \in C([0,T], \mathbb{R}_+^m)$.*

Proof. From Proposition 4.3.1, it follows that the set of feasible flows satisfies the property of convergence in Mosco's sense. This fact allows us to apply Theorem 6.2.1, then there exists a continuous solution to (7.3.1). □

Now, we still assume that the cost is an affine operator with respect to flows, but the matrix-function A depends on time and on integral average of the flow-vector, namely

$$C(t, F(t)) = A(t, F_{\mathcal{T}})F(t) + B(t),$$

for a.e. $t \in [0,T]$ and for every $F \in L^2([0,T], \mathbb{R}_+^m)$, where $A : [0,T] \times \mathbb{R}_+^m \to \mathbb{R}_+^{m\times m}$ and $B : [0,T] \to \mathbb{R}_+^m$ are two functions, $\mathcal{T} = [0,T]$ and $F_{\mathcal{T}}$ is the integral average, namely

$$F_{\mathcal{T}} = \frac{\int_0^T F(\tau)d\tau}{T}.$$

Let us suppose that $A(t, u)$ is a bounded matrix, namely

$$\exists M > 0 : \ \|A(t, u)\|_{m\times m} \leq M, \quad \text{for a.e. } t \in [0,T], \ \forall u \in \mathbb{R}_+^m. \tag{7.3.4}$$

Then we study the continuity of solutions to the following evolutionary quasi-variational inequality:

Find $H \in \mathbf{K}(H)$ such that

$$\langle A(t, F_T)H(t) + B(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t, H), \text{ a.e. in } [0, T], \quad (7.3.5)$$

where

$$\mathbf{K}(t, H) = \left\{ F(t) \in \mathbb{R}^m : \ \lambda(t) \leq F(t) \leq \mu(t) \ \ \Phi F(t) = \frac{1}{T}\int_0^T \rho(t, H(\tau))d\tau \right\}.$$

Also in this case, we can obtain a regularity result for solutions to (7.3.5).

Theorem 7.3.2. *Let $A \in C([0, T] \times \mathbb{R}^m_+, \mathbb{R}^{m\times m}_+)$ be a matrix-function verifying conditions* (7.3.4) *and*

$$\exists \nu > 0 : \ \langle A(t, F_T)F, F\rangle \geq \nu\|F\|^2_m, \quad \forall F \in \mathbb{R}^m_+, \ in\ [0, T],$$

let $B \in C([0, T], \mathbb{R}^m_+)$ be a vector function. Let $\lambda, \mu \in C([0, T], \mathbb{R}^m_+)$ and let $\rho \in C([0, T] \times \mathbb{R}^m_+, \mathbb{R}^l_+)$ verify conditions (7.3.2) *and* (7.3.3)*. Then, evolutionary quasi-variational inequality* (7.3.5) *admits a solution $H \in \mathbf{K}(H)$ such that $H \in C([0, T], \mathbb{R}^m_+)$.*

Proof. By Theorem 6.2.2 and by Proposition 4.3.3, it follows the assertion. □

The previous results can be extended to the dynamic elastic equilibrium problem which is expressed by a degenerate evolutionary quasi-variational inequality, namely the assumption

$$\langle A(t)F, F\rangle \geq \nu(t)\|F\|^2_m, \quad \forall F \in \mathbb{R}^m_+, \text{ a.e. in } [0, T], \quad (7.3.6)$$

where $\nu \in L^\infty([0, T], \mathbb{R}^+_0)$ is such that

$$\nexists I \subseteq [0, T], \ \mu(I) > 0 : \ \nu(t) = 0, \quad \forall t \in I,$$

with μ Lebesgue's measure, holds.

Theorem 7.3.3. *Let $A \in C([0, T], \mathbb{R}^{m\times m}_+)$ be a matrix-function satisfying condition* (7.3.6) *and let $B \in C([0, T], \mathbb{R}^m_+)$ be a vector-function. Let $\lambda, \mu \in C([0, T], \mathbb{R}^m_+)$ and let $\rho \in C([0, T] \times \mathbb{R}^m_+, \mathbb{R}^l_+)$ verify conditions* (7.3.2) *and* (7.3.3)*. Then, the evolutionary quasi-variational inequality*

$$H \in \mathbf{K}(H) : \ \langle A(t)H(t) + B(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t, H), \ in\ [0, T], \quad (7.3.7)$$

admits a solution $H \in \mathbf{K}(H)$ such that $H \in C([0, T], \mathbb{R}^m_+)$.

Proof. Propositions 4.3.3 and 4.3.4 and Theorem 6.3.1 imply that there exists a continuous solution to (7.3.7). □

We can obtain a more general result for strictly monotone evolutionary quasi-variational inequalities.

Theorem 7.3.4. *Let $A \in C([0,T], \mathbb{R}_+^{m\times m})$ be a matrix-function such that*

$$\langle A(t)[H-F], H-F\rangle > 0, \quad \forall H, F \in \mathbb{R}_+^m,\ H \neq F, \text{ in } [0,T],$$

and let $B \in C([0,T], \mathbb{R}_+^m)$ be a vector-function. Let $\lambda, \mu \in C([0,T], \mathbb{R}_+^m)$ and let $\rho \in C([0,T] \times \mathbb{R}_+^m, \mathbb{R}_+^l)$ verify conditions (7.3.2) and (7.3.3). Then, the evolutionary quasi-variational inequality

$$\langle A(t)H(t) + B(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t,H),\ in\ [0,T], \tag{7.3.8}$$

admits a solution $H \in \mathbf{K}(H)$ such that $H \in C([0,T], \mathbb{R}_+^m)$.

Proof. The assertion follows by Propositions 4.3.3 and 4.3.4 and Theorem 6.4.1. □

7.3.2 Nonlinear case

In this section, we analyze under which assumptions the regularity of the dynamic elastic traffic equilibrium problem associated to a nonlinear cost vector-function is ensures.

We can able to obtain the following result.

Theorem 7.3.5. *Let $C \in C([0,T] \times \mathbb{R}_+^m, \mathbb{R}_+^m)$ be an operator, satisfying conditions*

$$\exists \gamma \in C([0,T], \mathbb{R}_+):\ \|C(t,F)\|_m \leq \gamma(t) + \|F\|_m, \quad \forall F \in \mathbb{R}_+^m, \tag{7.3.9}$$

$$\exists \nu > 0:\ \langle C(t,H) - C(t,F), H-F\rangle \geq \nu\|H-F\|_m^2, \quad \forall H, F \in \mathbb{R}_+^m, \tag{7.3.10}$$

in $[0,T]$. Let $\lambda, \mu \in C([0,T], \mathbb{R}_+^m)$ be vector-functions and let $\rho \in C([0,T] \times \mathbb{R}_+^m, \mathbb{R}_+^l)$ be an operator satisfying conditions

$$\exists \psi \in C([0,T], \mathbb{R}_+):\ \|\rho(t,F)\|_l \leq \psi(t) + \|F\|_m^2, \quad \forall F \in \mathbb{R}_+^m, \tag{7.3.11}$$

$$\exists \omega \in C([0,T]):\ \|\rho(t,H) - \rho(t,F)\|_l \leq \omega(t)\|H-F\|_m^2, \quad \forall H, F \in \mathbb{R}_+^m, \tag{7.3.12}$$

in $[0,T]$. Then, the evolutionary quasi-variational inequality

$$H \in \mathbf{K}(H) : \langle C(t, H(t)), F(t) - H(t)\rangle \geq 0,\ \forall F(t) \in \mathbf{K}(t,H),\ in\ [0,T]$$

admits a solution $H \in \mathbf{K}(H)$ such that $H \in C([0,T], \mathbb{R}_+^m)$.

Proof. The assertion is immediate consequence of Proposition 4.3.3 and Theorem 6.2.3. □

This result can be generalized supposing that ν is a function belonging to $L^\infty([0,T],\mathbb{R}^m)$, that is

$$\langle C(t,F)-C(t,H),F-H\rangle \geq \nu(t)\|F-H\|_m^2, \quad \forall F,H \in \mathbb{R}_+^m, \text{ a.e. in } [0,T], \tag{7.3.13}$$

where $\nu \in L^\infty([0,T],\mathbb{R}_0^+)$ is such that

$$\nexists I \subseteq [0,T], \; \mu(I)>0: \; \nu(t)=0, \quad \text{for a.e. } t \in I.$$

Theorem 7.3.6. *Let $C \in C([0,T]\times\mathbb{R}_+^m,\mathbb{R}_+^m)$ be an operator, satisfying conditions* (7.3.9) *and* (7.3.13). *Let $\lambda,\mu \in C([0,T],\mathbb{R}_+^m)$ be vector-functions and let $\rho \in C([0,T]\times\mathbb{R}_+^m,\mathbb{R}_+^l)$ be an operator satisfying conditions* (7.3.11) *and* (7.3.12). *Then, the evolutionary quasi-variational inequality*

$$H \in \mathbf{K}(H): \langle C(t,H(t)),F(t)-H(t)\rangle \geq 0, \; \forall F(t) \in \mathbf{K}(t,H), \; \textit{in } [0,T]$$

admits a solution $H \in \mathbf{K}(H)$ such that $H \in C([0,T],\mathbb{R}_+^m)$.

Proof. Making use of Propositions 4.3.3 and 4.3.4 and of Theorem 6.3.2, we can derive the thesis. □

But, a more general theorem holds for strictly monotone evolutionary quasi-variational inequalities, namely it holds

$$\langle C(t,F)-C(t,H),F-H\rangle > 0, \quad \forall F,H \in \mathbb{R}_+^m, \text{ a.e. in } [0,T]. \tag{7.3.14}$$

Theorem 7.3.7. *Let $C \in C([0,T]\times\mathbb{R}_+^m,\mathbb{R}_+^m)$ be an operator, satisfying conditions* (7.3.9) *and* (7.3.14). *Let $\lambda,\mu \in C([0,T],\mathbb{R}_+^m)$ be vector-functions and let $\rho \in C([0,T]\times\mathbb{R}_+^m,\mathbb{R}_+^l)$ be an operator satisfying conditions* (7.3.11) *and* (7.3.12). *Then, the evolutionary quasi-variational inequality*

$$H \in \mathbf{K}(H): \langle C(t,H(t)),F(t)-H(t)\rangle \geq 0, \; \forall F(t) \in \mathbf{K}(t,H), \; \textit{in } [0,T]$$

admits a solution $H \in \mathbf{K}(H)$ such that $H \in C([0,T],\mathbb{R}_+^m)$.

Proof. The assertion follows by Propositions 4.3.3 and 4.3.4 and by Theorem 6.4.2. □

7.4 Regularity results for dynamic equilibrium problems in the common formulation

In this section we show that theoretical results, proved for general variational inequalities, can be apply to dynamic equilibrium problems written in the common formulation. In particular, regularity results for the dynamic spatial price equilibrium problem can be found in [12].

We consider the following evolutionary variational inequality

Find $u \in \mathbf{K}$ such that

$$\ll F(u), v-u \gg \geq 0, \quad \forall v \in \mathbf{K}, \tag{7.4.1}$$

where

$$\mathbf{K} = \Bigg\{ u \in L^2([0,T],\mathbb{R}^q): \quad \lambda(t) \leq u(t) \leq \mu(t), \ \text{ a.e. in } [0,T],$$
$$\sum_{i=1}^{q} \xi_{ji} u_i(t) = \rho_j(t), \ \text{ a.e. in } [0,T],$$
$$\xi_{ji} \in \{-1,0,1\}, \ i \in \{1,\dots,q\}, \ j \in \{1,\dots,l\} \Bigg\},$$

which expresses dynamic equilibrium problems (see Section 2.3). It is worth remarking that problem (7.4.1) (see [65]) is also equivalent to the following one:

Find $u \in \mathbf{K}$ such that

$$\langle F(t,u(t)), v(t)-u(t) \rangle \geq 0, \quad \forall v(t) \in \mathbf{K}(t), \ \text{a.e. in } [0,T], \tag{7.4.2}$$

where

$$\mathbf{K}(t) = \Bigg\{ u(t) \in \mathbb{R}^m: \quad \lambda(t) \leq u(t) \leq \mu(t), \ \sum_{i=1}^{q} \xi_{ji} u_i(t) = \rho_j(t),$$
$$\xi_{ji} \in \{-1,0,1\}, \ i \in \{1,\dots,q\}, \ j \in \{1,\dots,l\} \Bigg\}.$$

Now, we are able to present the following result.

Theorem 7.4.1. *Let $F \in C([0,T] \times \mathbb{R}^q, \mathbb{R}^q)$ be an operator such that*

$$\|F(t,v)\|_q \leq A(t)\|v\|_q + B(t), \quad \forall v \in \mathbb{R}^q, \ in \ [0,T],$$

with $B \in C([0,T],\mathbb{R}_+)$ and $A \in C([0,T],\mathbb{R}_+)$ and

$$\exists \nu > 0: \ \langle F(t,u) - F(t,v), u-v \rangle \geq \nu \|u-v\|_q^2, \quad \forall u,v \in \mathbb{R}^q, \ in \ [0,T].$$

Let $\lambda, \mu \in C([0,T],\mathbb{R}^q)$ and let $\rho \in C([0,T],\mathbb{R}^l)$ be vector-functions. Then, evolutionary variational inequality (7.4.2) admits a unique solution $u \in \mathbf{K}$ such that $u \in C([0,T],\mathbb{R}^q)$.

Proof. The assertion follows by Proposition 4.4.1 and Theorem 5.2.3. □

If the operator F is degenerate, the next theorem holds.

Theorem 7.4.2. *Let $F \in C([0,T] \times \mathbb{R}^q, \mathbb{R}^q)$ be an operator such that*

$$\|F(t,v)\|_q \leq A(t)\|v\|_q + B(t), \quad \forall v \in \mathbb{R}^q, \ in \ [0,T],$$

with $B \in C([0,T], \mathbb{R}_+)$ and $A \in C([0,T], \mathbb{R}_+)$ and

$$\langle F(t,u) - F(t,v), u - v \rangle \geq \nu(t)\|u - v\|_q^2, \quad \forall u, v \in \mathbb{R}^q, \ in \ [0,T].$$

where $\nu \in L^\infty([0,T], \mathbb{R}_0^+)$ is such that

$$\nexists I \subseteq [0,T], \ \mu(I) > 0 : \ \nu(t) = 0, \ a.e. \ in \ I.$$

Let $\lambda, \mu \in C([0,T], \mathbb{R}^q)$ and let $\rho \in C([0,T], \mathbb{R}^l)$ be vector-functions. Then, evolutionary variational inequality (7.4.2) *admits a unique solution $u \in \mathbf{K}$ such that $u \in C([0,T], \mathbb{R}^q)$.*

Proof. Taking into account Propositions 4.3.1 and 4.4.2 and Theorem 5.3.2, we can derive the continuity of the solution to (7.4.2). □

Finally, we are able to present a more general continuity result for dynamic equilibrium problems in the common formulation.

Theorem 7.4.3. *Let $F \in C([0,T] \times \mathbb{R}^q, \mathbb{R}^q)$ be an operator such that*

$$\|F(t,v)\|_q \leq A(t)\|v\|_q + B(t), \quad \forall v \in \mathbb{R}^q, \ in \ [0,T],$$

with $B \in C([0,T], \mathbb{R}_+)$ and $A \in C([0,T], \mathbb{R}_+)$ and

$$\langle F(t,u) - F(t,v), u - v \rangle > 0, \quad \forall u, v \in \mathbb{R}^q, \ u \neq v, \ in \ [0,T].$$

Let $\lambda, \mu \in C([0,T], \mathbb{R}^q)$ and let $\rho \in C([0,T], \mathbb{R}^l)$ be vector-functions. Then, evolutionary variational inequality (7.4.2) *admits a unique solution $u \in \mathbf{K}$ such that $u \in C([0,T], \mathbb{R}^q)$.*

Proof. Making use to Propositions 4.3.1 and 4.4.2 and Theorem 5.4.2, we obtain the assertion. □

8

Algorithms to solve dynamic equilibrium problems

8.1 Introduction

The development of efficient algorithms for the numerical computation of dynamic equilibria is a topic as important as the qualitative analysis of dynamic equilibria. In fact, the complexity of dynamic equilibrium problems, united with their increasing scale, is precluding their resolution via closed form analytics.

In this chapter we propose some algorithms for the computation of equilibria, by means of discretization methods. The continuity of the solution to dynamic equilibrium problems, proved in previous chapter, allows us to consider a partition of the time interval and hence to reduce the infinite-dimensional problem to some finite-dimensional problems that can be solved by means of a known method. Since in literature very few results for the calculation of solutions to dynamic equilibrium problems are available (see for instance the sub-gradient method presented in [37]), our results seem to have a particular relevance.

In particular, we make use to the projection method, the extragradient method, Solodov-Svaiter's method, Solodov-Tseng's method and descent methods to solve finite-dimensional variational problems associated to points of the discretization of the time interval. Then, interpolating numerical solutions with linear splines, we determine the equilibrium curve of models. Moreover, we give some numerical examples that have been implemented under MatLab version 6.1a R12.1. The codes run on a Netebook PC (AMD 64 Athlon).

8.2 The generalized projection method

Let $A : [0,T] \to \mathbb{R}^{q\times q}$ be a matrix-function and let $B : [0,T] \to \mathbb{R}^q$ be a vector-function. Let us consider the following evolutionary variational inequality

Find $u \in \mathbf{K}$ such that

$$\langle F(u(t)), v(t) - u(t) \rangle \geq 0, \quad \forall v(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T], \tag{8.2.1}$$

where $F(u(t)) = A(t)u(t) + B(t)$ and

$$\mathbf{K}(t) = \left\{ u(t) \in \mathbb{R}^q : \quad \lambda(t) \leq u(t) \leq \mu(t), \ \sum_{i=1}^{q} \xi_{ji} u_i(t) = \rho_j(t), \right.$$
$$\left. \xi_{ji} \in \{-1, 0, 1\}, \ i \in \{1, 2, \ldots, q\}, \ j \in \{1, 2, \ldots, l\} \right\},$$

which expresses dynamic equilibrium problems in the common formulation.

Now, we present a computational method to compute the dynamic equilibrium solution to (8.6.1).

We suppose that the assumptions of Theorem 7.4.1 are satisfied and hence (8.6.1) admits a unique solution u belonging to $C([0,T], \mathbb{R}^q)$. As a consequence, (8.6.1) holds for each $t \in [0,T]$, namely

$$\langle F(u(t)), v(t) - u(t) \rangle \geq 0, \qquad \forall v(t) \in \mathbf{K}(t), \ \forall t \in [0,T].$$

In the following, applying a discretization procedure, we use the projection method in order to compute the solution to (8.6.1).

We consider now a partition of $[0,T]$, such that:

$$0 = t_0 < t_1 < \ldots < t_r < \ldots < t_N = T,$$

and, for each value t_r, for $r = 0, 1, \ldots, N$, we apply the projection method to solve the finite-dimensional variational inequality

$$\langle F(u(t_r)), v(t_r) - u(t_r) \rangle \geq 0, \quad \forall v(t_r) \in \mathbf{K}(t_r), \tag{8.2.2}$$

where $F(u(t_r)) = A(t_r)u(t_r) + B(t_r)$ and

$$\mathbf{K}(t_r) = \left\{ u(t_r) \in \mathbb{R}^q : \quad \lambda(t_r) \leq u(t_r) \leq \mu(t_r), \ \sum_{i=1}^{q} \xi_{ji} u_i(t_r) = \rho_j(t_r), \right.$$
$$\left. \xi_{ji} \in \{-1, 0, 1\}, \ i \in \{1, 2, \ldots, q\}, \ j \in \{1, 2, \ldots, l\} \right\}.$$

We can compute now the solution to the finite-dimensional variational inequality (8.2.2) using the following procedure. The algorithm, as it is well known, starting from any $u^0(t_r) \in \mathbf{K}(t_r)$ fixed, iteratively updates $u(t_r)$ according to the formula

$$u^{k+1}(t_r) = P_{\mathbf{K}(t_r)}(u^k(t_r) - \alpha F(u^k(t_r))),$$

for $k \in \mathbb{N}$, where $P_{\mathbf{K}(t_r)}(\cdot)$ denotes the orthogonal projection map onto $\mathbf{K}(t_r)$ and α is a judiciously chosen positive steplength. Here, $P_{\mathbf{K}(t_r)}(u^k(t_r) - \alpha F(u^k(t_r)))$, for $k \in \mathbb{N}$, is the solution of the following quadratic programming problem

$$\min_{u(t_r)\in\mathbf{K}(t_r)} \frac{1}{2}(u(t_r))^T u(t_r) - (u^k(t_r) - \alpha F(u^k(t_r)))^T u(t_r),$$

for $k \in \mathbb{N}$. The projection method is based on the observation that $u^*(t_r) \in \mathbf{K}(t_r)$ is a solution of (8.2.2) if and only if

$$u^*(t_r) = P_{\mathbf{K}(t_r)}(u^*(t_r) - \alpha F(u^*(t_r))).$$

This method requires restrictive assumptions on C for the convergence. The convergence analysis for the projection methods is based on the contractive properties of the operator $u(t_r) \to u(t_r) - \alpha F(u(t_r))$: if $F(t_r)$ is strongly monotone (with constant ν) and Lipschitz continuous on $\mathbf{K}(t_r)$ (with Lipschitz constant L), and if $\alpha \in (0, 2\nu/L^2)$, the projection method determines a sequence $\{u^k(t_r)\}_{k\in\mathbb{N}}$ convergent to a solution of (8.2.2), for every $r = 0, 1, \ldots, N$ (see [81] and [89]).

Marcotte and Wu in [60] have shown that the projection algorithm converges for cocoercive variational inequalities. We recall that the mapping F is cocoercive on $\mathbf{K}(t_r)$ if there exists a positive constant $\tilde{\nu}$ such that, $\forall u(t_r), v(t_r) \in \mathbf{K}(t_r)$ one has

$$\langle F(u(t_r)) - F(v(t_r)), u(t_r) - v(t_r)\rangle \geq \tilde{\nu}\|F(u(t_r)) - F(v(t_r))\|_q^2.$$

Any strongly monotone (with constant ν) and Lipschitz continuous mapping (with Lipschitz constant L) is cocoecive with the constant $\tilde{\nu} = \frac{\nu}{L^2}$. If $\mathbf{K}(t_r) \neq \emptyset$ and $\alpha \in (0, 2\tilde{\nu})$, the cocoercivity of the operator F is sufficient to assure the convergence of the projection algorithm.

A drawback is the choice of α when L is unknown. Indeed, if α is too small, the convergence is slow; when α is too large, there might be no convergence at all, this remark is confirmed by the numerical results shown in Table 8.2.

After iterative procedure, we can construct a function by performing a linear interpolation.

The complexity of this algorithm is $O(Nq^2)$. In fact, the algorithm repeats N cycles of complexity $O(q^2)$, being the operator F affine.

8.2.1 A numerical example

We now consider a transportational network pattern for the simple network shown in Figure 8.1, consisting of a single O/D pair of nodes A, B and five paths a_1, a_2, a_3, b_1, b_2, where a_1, a_2, a_3 are directed from A to B and b_1, b_2 are the return of a_1, a_2 respectively.

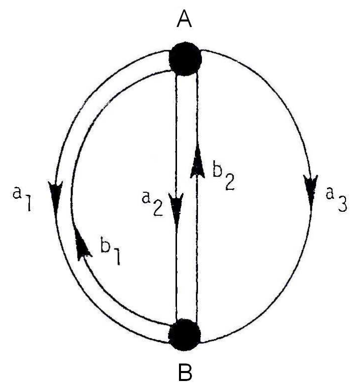

Figure 8.1: Network pattern of the numerical example

The set of feasible flows is given by:

$$\begin{aligned}\mathbf{K} = \Big\{ F \in L^2([0,2],\mathbb{R}^5): \quad & (5t,0,0,5t,0) \le (F_1(t),F_2(t),F_3(t),F_4(t),F_5(t)) \\ & \le (15t,20t,20t,20t+7,20t+7), \\ & F_1(t)+F_2(t)+F_3(t) = 52t, \\ & F_4(t)+F_5(t) = 32t+7 \text{ in } [0,2] \Big\}.\end{aligned}$$

We consider the cost vector-function on the path C defined by

$$C : L^2([0,2],\mathbb{R}^5_+) \to L^2([0,2],\mathbb{R}^5_+);$$

$$\begin{aligned} C_1(H(t)) &= 2H_1(t)+H_4(t)+20, \\ C_2(H(t)) &= 5H_2(t)+3H_5(t)+15, \\ C_3(H(t)) &= 4H_3(t)+30, \\ C_4(H(t)) &= 3H_1(t)+2H_4(t)+10, \\ C_5(H(t)) &= 2H_2(t)+5H_5(t)+5. \end{aligned}$$

The theory of evolutionary variational inequalities states that the problem has a unique equilibrium, since C is strongly monotone, for any arbitrarily fixed point $t \in [0,2]$. Indeed, one can easily see that

$$\begin{aligned}\langle C(H(t))-C(F(t)),(H(t)-F(t))\rangle &= 2(H_1(t)-F_1(t))^2+3(H_2(t)-F_2(t))^2 \\ &+4(H_3(t)-F_3(t))^2+2(H_4(t)-F_4(t))^2 \\ &+5(H_5(t)-F_5(t))^2+4(H_1(t)-F_1(t)) \\ &(H_4(t)-F_4(t))+5(H_2(t)-F_2(t)) \\ &(H_5(t)-F_5(t)) \ge 2\|H(t)-F(t)\|_m^2,\end{aligned}$$

for any $H(t), F(t) \in \mathbb{R}^5_+$, $H(t) \neq F(t)$ and for every $t \in [0,2]$. Moreover, it results that

$$\begin{aligned}
\|C(H(t)) - C(F(t))\|^2_m &= [2(H_1(t) - F_1(t)) + 3(H_4(t) - F_4(t))]^2 + [5(H_2(t) - F_2(t)) \\
&\quad +3(H_5(t) - F_5(t))]^2 + [4(H_3(t) - F_3(t))]^2 + [3(H_1(t) - F_1(t)) \\
&\quad +2(H_4(t) - F_4(t))]^2 + [2(H_2(t) - F_2(t)) + 5(H_5(t) - F_5(t))]^2 \\
&\leq 26(H_1(t) - F_1(t))^2 + 58(H_2(t) - F_2(t))^2 + 16(H_3(t) - F_3(t))^2 \\
&\quad +10(H_4(t) - F_4(t))^2 + 68(H_5(t) - F_5(t))^2 \\
&\leq 68\|H(t) - F(t)\|^2_m,
\end{aligned}$$

for any $H(t), F(t) \in \mathbb{R}^5_+$ and for every $t \in [0,2]$. As a consequence, the projection method is convergent for $\alpha \in (0, 0.058)$, for the property of C. We can compute an approximate curve of equilibria, by selecting $t_r \in \left\{\frac{k}{15} : \ k \in \{0,1,\dots,30\}\right\}$. Using a MatLab computation and choosing the initial point $H^0(t_r) = (15t_r, 20t_r, 17t_r, 16t_r + 3, 16t_r + 4)$ to start the iterative method, we obtain the equilibria consisting of the points, as shown in Table 8.1.

By implementing our algorithm, we find the approximate equilibrium solutions (see Table 8.2).

We report the number of iteration (iter), the number of function evaluations (nf), the number of projections (np) and the time of computation (CPUtime), expressed by seconds, for different choice of α when the projection method is applied on our problem. The stopping criterion is $\|R(H^k(t_r))\|_5 = \|H^k(t_r) - H^{k-1}(t_r)\|_5 \leq 10^{-6}$, for $r = 0, 1, \dots, 30$.

The interpolation of equilibria points yields the curves of equilibria shown in Figure 8.2.

8.3 The generalized extragradient method

Let us introduce a method to solve dynamic equilibrium problems formulated in terms of degenerate evolutionary variational inequalities.

We consider the following evolutionary variational inequality

Find $u \in \mathbf{K}$ such that

$$\langle F(u(t)), v(t) - u(t)\rangle \geq 0, \quad \forall v(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T], \tag{8.3.1}$$

where $F(u(t)) = A(t)u(t) + B(t)$, a.e. in $[0,T]$, with $A : [0,T] \to \mathbb{R}^{q\times q}$ satisfying the following condition

$$\langle A(t)v, v\rangle \geq \nu(t)\|v\|^2_q, \quad \forall v \in \mathbb{R}^q, \quad \text{a.e. in } [0,T], \tag{8.3.2}$$

Table 8.1: Numerical results

t_r	$H_1(t_r)$	$H_2(t_r)$	$H_3(t_r)$	$H_4(t_r)$	$H_5(t_r)$
0	0	0	0	4.2837012	2.7162988
1/15	1	1.3333333	1.1333333	5.7599347	3.3733986
2/15	2	2.4989472	2.4343861	7.1886468	4.0780199
1/5	3	3.4	4	8.5408460	4.8591540
4/15	4	4.5333333	5.3333333	9.9598888	5.5734445
1/3	5	5.6666667	6.6666667	11.378967	6.2877000
2/5	6	6.8	8	12.798003	7.0019972
7/15	7	7.9333333	9.3333333	14.217060	7.7162730
8/15	8	9.0666667	1.0666667	15.636135	8.4305316
3/5	9	10, 2	1.2	17.055156	9.1448435
2/3	10	11.333333	13.333333	18.474190	9.8591432
11/15	11	12.466667	14.666667	19.893234	10.573432
4/5	12	13.6	16	21.312288	11.287712
13/15	13	14.733333	17.333333	22.731349	12.001984
14/15	14	15.866667	18.666667	24.150418	12.716249
1	15	17	20	25.569425	13.430575
16/15	16	18.133333	21.333333	26.988505	14.144828
17/15	17	19.266667	22.666667	28.407523	14.859144
6/5	18	20.4	24	29.826613	15.573387
19/15	19	21.533333	25.333333	31.245640	16.287693
4/3	20	22.666667	26.666667	32.664670	17.001996
7/5	21	23.8	28	34.083704	17.716296
22/15	22	24.933333	29.333333	35.502741	18.430593
23/15	23	26.066667	30.666667	36.921849	19.144817
8/5	24	27.2	32	38.340892	19.859108
5/3	25	28.333333	33.333333	39.759938	20.573396
26/15	26	29.466667	34.666667	41.178985	21.287681
9/5	27	30.6	36	42.598036	22.001964
28/15	28	31.733333	37.333333	44.017088	22.716246
29/15	29	32.866667	38.666667	45.436074	23.430593
2	30	34	40	46.855130	24.144870

where $\nu \in L^\infty([0,T],\mathbb{R}_0^+)$ is such that $\nexists I \subseteq [0,T]$, $\mu(I) > 0$: $\nu(t) = 0$, $\forall t \in I$, $B : [0,T] \to \mathbb{R}^q$, and

$$\mathbf{K}(t) = \Bigg\{ u(t) \in \mathbb{R}^q : \quad \lambda(t) \le u(t) \le \mu(t), \quad \sum_{i=1}^{q} \xi_{ji} u_i(t) = \rho_j(t), \\ \xi_{ji} \in \{-1,0,1\},\ i \in \{1,2,\dots,q\},\ j \in \{1,2,\dots,l\} \Bigg\}.$$

We suppose that the assumptions of Theorem 7.4.2 are satisfied and hence the evolutionary variational inequality (8.3.1) admits a unique solution u belonging to $C([0,T],\mathbb{R}^q)$. As a consequence, (8.3.1) holds for each $t \in [0,T]$, namely

$$\langle F(u(t)), v(t) - u(t)\rangle \ge 0, \quad \forall v(t) \in \mathbf{K}(t),\ \forall t \in [0,T].$$

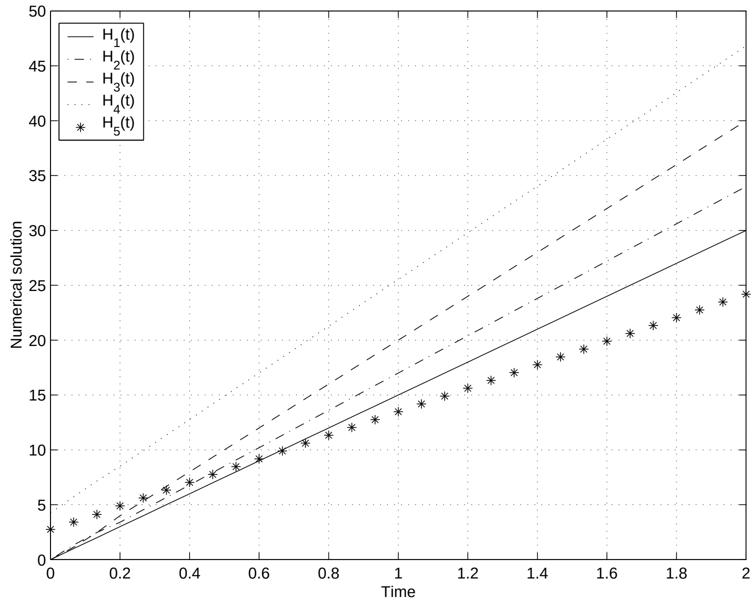

Figure 8.2: Curves of equilibria.

A more refined method for solving variational inequalities is the extragradient method, but it can be applied to evolutionary variational inequalities after that a discretization procedure has been made.

In the following, applying a discretization procedure, we will use the extragradient method and we will compute the solution of a variational inequality associated to a transportation network.

Let us consider a partition of $[0, T]$, such that:

$$0 = t_0 < t_1 < \ldots < t_r < \ldots < t_N = T,$$

hence, for each value t_r, for $r = 0, 1, \ldots, N$, we obtain the finite-dimensional variational inequality

$$\langle F(u(t_r)), v(t_r) - u(t_r)\rangle \geq 0, \quad \forall v(t_r) \in \mathbf{K}(t_r), \tag{8.3.3}$$

where $F(u(t_r)) = A(t_r)H(t_r) + B(t_r)$ and

$$\mathbf{K}(t_r) = \Bigg\{ u(t_r) \in \mathbb{R}^q : \quad \lambda(t_r) \leq u(t_r) \leq \mu(t_r), \ \sum_{i=1}^{q} \xi_{ji} u_i(t_r) = \rho_j(t_r), \\ \xi_{ji} \in \{-1, 0, 1\}, \ i \in \{1, 2, \ldots, q\}, \ j \in \{1, 2, \ldots, l\} \Bigg\}.$$

We compute now the solution to the finite-dimensional variational inequality (8.3.3) using the extragradient method. The algorithm, as it is well known, starting from any $u^0(t_r) \in \mathbf{K}(t_r)$ fixed, iteratively updates $u^{k+1}(t_r)$ from $u^k(t_r)$ according to the double projection formula

$$\overline{u}^k(t_r) = P_{\mathbf{K}(t_r)}(u^k(t_r) - \alpha F(u^k(t_r))), \quad u^{k+1}(t_r) = P_{\mathbf{K}(t_r)}(u^k(t_r) - \alpha F(\overline{u}^k(t_r))),$$

for $k \in \mathbb{N}$, where $P_{\mathbf{K}(t_r)}(\cdot)$ denotes the orthogonal projection map onto $\mathbf{K}(t_r)$.

In [15] and [100] the convergence of the extragradient method is proved under the following hypothesis: F is a monotone and Lipschitz continuous mapping and $\alpha \in (0, 1/L)$ where L is the Lipschitz constant.

A drawback is the choice of α when L is unknown. Indeed, if α is too small, the convergence is slow; when α is too large, there might be no convergence at all.

After an iterative procedure, we can construct a function by performing a linear interpolation.

We remark that a drawback is the choice of α when L is unknown. Indeed, if α is too small, the convergence is slow; when α is too large, there might be no convergence at all. Then, Khobotov in [50] introduced the idea to perform an adaptive choice of α, changing its value at each iteration. Now, we present a modification of Khoboton's algorithm obtained by Marcotte in [59].

The algorithm starting from any $u^0(t_r) \in \mathbf{K}(t_r)$ and a number $\alpha_0 > 0$ fixed, iteratively updates $u^{k+1}(t_r)$ from $u^k(t_r)$ according to the following projection formulas

$$u^{k+1}(t_r) = P_{\mathbf{K}(t_r)}(u^k(t_r) - \alpha_k F(\overline{u}^k(t_r))), \quad \overline{u}^k(t_r) = P_{\mathbf{K}(t_r)}(u^k(t_r) - \alpha_k F(u^k(t_r)))$$

for $k \in \mathbb{N}$, where α_k is chosen as following

$$\alpha_k = \min\left\{\frac{\alpha_{k-1}}{2}, \frac{\|u^k(t_r) - \overline{u}^k(t_r)\|_q}{\sqrt{2}\|F(u^k(t_r)) - F(\overline{u}^k(t_r))\|_q}\right\}.$$

If F is a monotone and Lipschitz continuous mapping, then, the convergence of the scheme is proved. This method was improved by Tinti in [99].

After the iterative procedure, we can construct the dynamic equilibrium solution by means of a linear interpolation of the obtained static equilibrium solutions.

The complexity of algorithms is $O(Nq^2)$. In fact, algorithms repeats N cycles of complexity $O(q^2)$, being the operator F affine.

8.3.1 A numerical example

Now, we apply the generalized extragradient method to solve a numerical example. Let us consider a transportation network pattern for the network shown in Figure 8.3. The network consists of six nodes and eight links. We assume that the O/D pairs are represented by $w_1 = (P_1, P_5)$ and $w_2 = (P_2, P_6)$, which are respectively connected by the following paths:

$$w_1 \ : \ \begin{cases} R_1 = (P_1, P_2) \cup (P_2, P_5) \\ R_2 = (P_1, P_4) \cup (P_4, P_5), \end{cases} \qquad w_2 \ : \ \begin{cases} R_3 = (P_2, P_3) \cup (P_3, P_6) \\ R_4 = (P_2, P_5) \cup (P_5, P_6) \\ R_5 = (P_2, P_5) \cup (P_5, P_3) \cup (P_3, P_6). \end{cases}$$

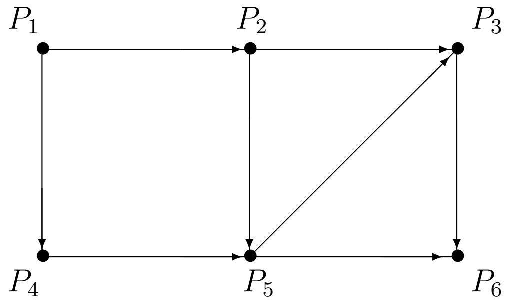

Figure 8.3: A network model.

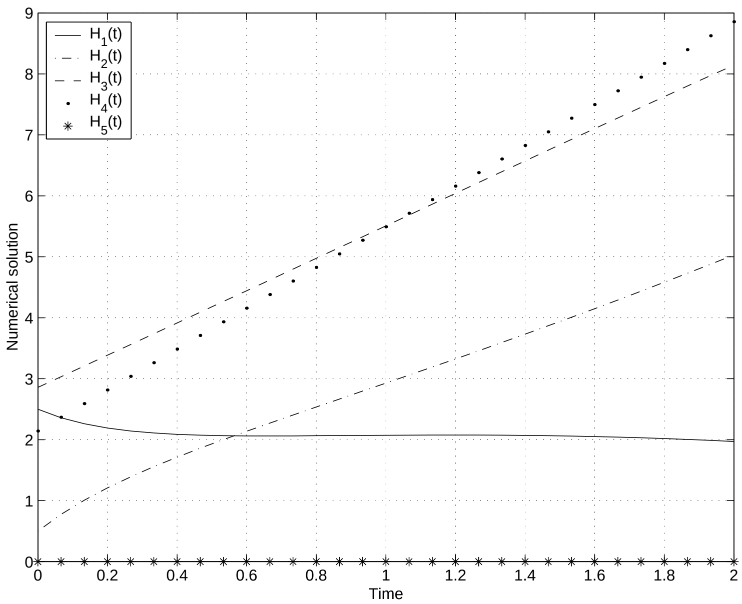

Figure 8.4: Curves of equilibria.

We consider the cost vector-function on path C defined by

$$C : L^2([0,2],\mathbb{R}^5_+) \to L^2([0,2],\mathbb{R}^5_+);$$

$$\begin{aligned}
C_1(H(t)) &= (2t+1)H_1(t) + t^2 H_4(t) + (t+3)H_5(t) + 3t + 1,\\
C_2(H(t)) &= (t+1)^2 H_2(t) + t^2 + 3,\\
C_3(H(t)) &= (2t+3)H_3(t) + (t^2+2)H_5(t) + 2t,\\
C_4(H(t)) &= t^2 H_1(t) + (t+4)H_4(t) + t^2 H_5(t),\\
C_5(H(t)) &= (t+3)H_1(t) + (t^2+2)H_3(t) + t^2 H_4(t) + (3t^2+2)H_5(t) + t + 2.
\end{aligned}$$

The set of feasible flows is given by:

$$\mathbf{K} = \Big\{ F \in L^2([0,2], \mathbb{R}^5_+) : \quad \begin{aligned}[t] &(0,0,0,0,0) \le (F_1(t), F_2(t), F_3(t), F_4(t), F_5(t)) \\ &\le (20t+15, 30t+10, 20t+15, 40t+19, 30t+21), \\ &F_1(t) + F_2(t) = 2t+3, \\ &F_3(t) + F_4(t) + F_5(t) = 6t+5, \quad \text{in } [0,2] \Big\}. \end{aligned}$$

It is easy to verify that the cost vector-function is degenerate. Moreover, it results that

$$\begin{aligned} \|C(H(t)) - C(F(t))\|_m^2 \;=\; & [(2t+1)(H_1(t) - F_1(t)) + t^2(H_4(t) - F_4(t)) + (t+3) \\ & (H_5(t) - F_5(t))]^2 + [(t+1)^2(H_2(t) - F_2(t))]^2 \\ & +[(2t+3)(H_3(t) - F_3(t)) + (t^2+2)(H_5(t) - F_5(t)]^2 \\ & +[t^2(H_1(t) - F_1(t)) + (t+4)(H_4(t) - F_4(t)) + t^2 \\ & (H_5(t) - F_5(t)]^2 + [(t+3)(H_1(t) - F_1(t)) + (t^2+2) \\ & (H_3(t) - F_3(t)) + t^2(H_4(t) - F_4(t)) + (3t^2+2) \\ & (H_5(t) - F_5(t))]^2 \\ \le\; & [3(2t+1)^2 + 3t^4 + 4(t+3)^2](H_1(t) - F_1(t))^2 + (t+1)^4 \\ & (H_2(t) - F_2(t))^2 + [2(2t+3)^2 + 4(t^2+2)](H_3(t) - F_3(t))^2 \\ & +[3t^4 + 3(t+4)^2 + 4t^4](H_4(t) - F_4(t))^2 + [3(t+3)^2 \\ & +2(t^2+2)^2 + 3t^4 + 4(3t^2+2)^2](H_5(t) - F_5(t))^2 \\ \le\; & 979\|H(t) - F(t)\|_m^2, \end{aligned}$$

for any $H(t), F(t) \in \mathbb{R}^5_+$ and for $t \in [0,2]$. As a consequence, the extragradient method is convergent for $\alpha \in (0, 0.031)$, for the property of C. We can compute an approximate curve of equilibria, by selecting $t_r \in \left\{\frac{k}{15} : k \in \{0,1,\dots,30\}\right\}$. Using a simple MatLab computation and choosing the initial point $H^0(t_r) = (t_r+1, t_r+2, 2t_r+2, 2t_r+2, 2t_r+1)$ to start the iterative method, we obtain the equilibria consisting of the points, as shown in Table 8.3 with the speed of the convergence of the method.

In detail, we report the number of iteration (iter), the number of function evaluations (nf), the number of projections (np) and the time of computation (CPUtime), expressed by seconds, for different methods. The stopping criterion is $\|R(H^k(t_r))\|_5 = \|H^k(t_r) - H^{k-1}(t_r)\|_5 \le 10^{-6}$, for $r = 0, 1, \dots, 30$.

The interpolation of equilibria points yields the curves of equilibria, how we can see in Figure 8.5.

8.3.2 A numerical example

In this subsection, we compute the equilibrium solution to a traffic network by means of the generalized version of Marcotte's method. Let us consider a network as Figure

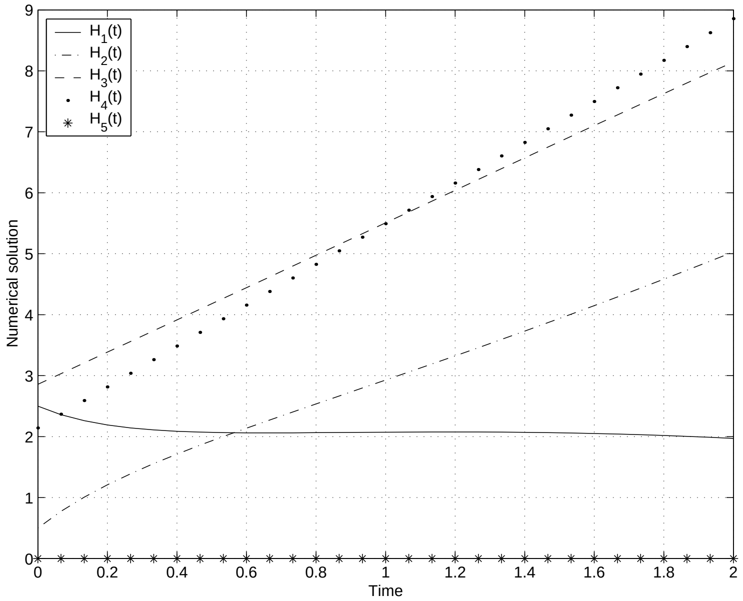

Figure 8.5: Curves of equilibria.

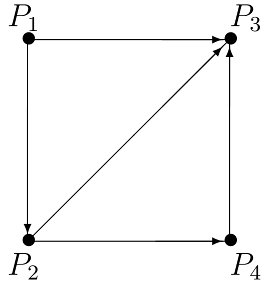

Figure 8.6: A network model.

8.6. The network consists of four nodes and five links. The origin-destination pair is $w = (P_1, P_3)$, which is connected by the paths $R_1 = (P_1, P_3)$, $R_2 = (P_1, P_2) \cup (P_2, P_3)$ and $R_3 = (P_1, P_2) \cup (P_2, P_4) \cup (P_4, P_3)$. We consider the cost operator on the path C defined by

$$C : L^2([0,2], \mathbb{R}^3_+) \to L^2([0,2], \mathbb{R}^3_+);$$

$$\begin{aligned} C_1(H(t)) &= (t+3)H_1(t) + 2t, \\ C_2(H(t)) &= (2t+4)H_2(t) + 1, \\ C_3(H(t)) &= 3tH_2(t) + (t+2)H_3(t) + t + 5. \end{aligned}$$

The set of feasible flows is given by

$$\begin{aligned} \mathbf{K} = \Big\{ F \in L^2([0,2], \mathbb{R}^3_+) : \quad & (2t, 2t, 0) \leq (F_1(t), F_2(t), F_3(t)) \leq (10t+5, 5t+3, 2t+1), \\ & F_1(t) + F_2(t) + F_3(t) = 5t + 3, \text{ in } [0,2] \Big\}. \end{aligned}$$

Now, we prove that the cost-vector function is strongly monotone:

$$\begin{aligned} \langle C(H(t)) - C(F(t)), H(t) - F(t) \rangle &= (t+3)[H_1(t) - F_1(t)]^2 + (2t+4) \\ & \quad [H_2(t) - F_2(t)]^2 + \big\{3t[H_2(t) - F_2(t)] \\ & \quad +(t+2)[H_3(t) - F_3(t)]\big\}[H_3(t) - F_3(t)] \\ &\geq \Big(2 - \frac{1}{2}t\Big)\|H(t) - F(t)\|_3^2 \\ &\geq \|H(t) - F(t)\|_3^2, \end{aligned}$$

for any $H(t), F(t) \in \mathbb{R}^3_+$.

Moreover, the cost vector-function is Lipschitz continuous, in fact it results that

$$\begin{aligned} \|C(H(t)) - C(F(t))\|_m^2 &= [(t+3)(H_1(t) - F_1(t))]^2 + [(2t+4)(H_2(t) - F_2(t))]^2 \\ & \quad +[3t(H_2(t) - F_2(t)) + (t+2)(H_3(t) - F_3(t))]^2 \\ &\leq 2(11t^2 + 8t + 8)\|H(t) - F(t)\|_m^2 \\ &\leq 136\|H(t) - F(t)\|_3^2, \end{aligned}$$

for any $H(t), F(t) \in \mathbb{R}^3_+$.

Now, we solve the numerical problem using the generalized Marcotte's version of the extragradient method. This methos is convergent for the property of C. Then, we can compute an approximate curve of equilibria, by selecting $t_r \in \big\{\frac{k}{15} : k \in \{0, 1, \ldots, 30\}\big\}$. Using a simple MatLab computation and choosing the initial point $H^0(t_r) = (2t_r + 1, 2t_r + 1, t_r + 1)$ to start the iterative method, we obtain the static equilibrium solutions, as shows Table 8.4. The stopping criterion is $\|R(H^k(t_r))\|_3 = \|H^k(t_r) - H^{k-1}(t_r)\|_3 \leq 10^{-6}$, for $r = 0, 1, \ldots, 30$.

The interpolation of equilibria points yields the curves of equilibria, as shows Figure 8.7.

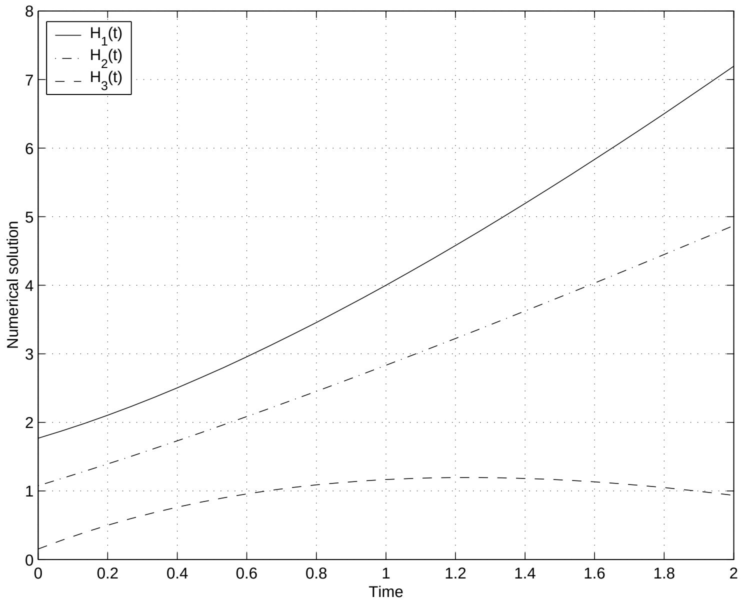

Figure 8.7: Curves of equilibria.

8.4 The generalized projection-contraction method

Let $F : [0,T] \times \mathbb{R}^q \to \mathbb{R}^q$ be a nonlinear operator, such that

$$\langle F(t,u) - F(t,v), u - v\rangle \geq \nu \|u - v\|_q^2, \quad \forall u, v \in \mathbb{R}^q, \text{ a.e. in } [0,T]. \tag{8.4.1}$$

Let us consider the following evolutionary variational inequality:

Find $u \in \mathbf{K}$ such that

$$\langle F(t,u(t)), v(t) - u(t)\rangle \geq 0, \quad \forall v(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T], \tag{8.4.2}$$

where

$$\mathbf{K}(t) = \Bigg\{ u(t) \in \mathbb{R}^q : \quad \lambda(t) \leq u(t) \leq \mu(t), \quad \sum_{i=1}^{q} \xi_{ji} u_i(t) = \rho_j(t),$$
$$\xi_{ji} \in \{-1,0,1\}, \ i \in \{1,2,\dots,q\}, \ j \in \{1,2,\dots,l\} \Bigg\}.$$

We suppose that the assumptions of Theorem 7.4.1 are satisfied, then the unique solution u to evolutionary variational inequality (8.4.2) belongs to $C([0,T],\mathbb{R}^q)$. As a consequence, (8.4.2) holds for each $t \in [0,T]$, namely

$$\langle F(t,u(t)), v(t) - u(t)\rangle \geq 0, \quad \forall v(t) \in \mathbf{K}(t), \ \forall t \in [0,T].$$

Now, we generalize a method for solving the static variational inequalities in the dynamic case.

In the beginning, we apply a discretization procedure in this way: we consider a partition of time interval $[0, T]$, such that:

$$0 = t_0 < t_1 < \ldots < t_r < \ldots < t_N = T.$$

Then, for each value t_r, for $r = 0, 1, \ldots, N$, we consider the static variational inequality

$$\langle F(t_r, u(t_r)), v(t_r) - u(t_r) \rangle \geq 0, \quad \forall v(t_r) \in \mathbf{K}(t_r), \tag{8.4.3}$$

where

$$\mathbf{K}(t_r) = \Bigg\{ u(t_r) \in \mathbb{R}^q : \quad \lambda_i(t_r) \leq F_i(t_r) \leq \mu_i(t_r), \ \sum_{i=1}^{q} \xi_{ji} u_i(t_r) = \rho_j(t_r),$$
$$\xi_{ji} \in \{-1, 0, 1\}, \ i \in \{1, 2, \ldots, q\}, \ j \in \{1, 2, \ldots, l\} \Bigg\}.$$

Now, we can compute the solution to the finite-dimensional variational inequality (8.4.3) using a class of *projection-contraction methods* proposed by Solodov and Tseng in [92] and improved by Tinti in [99].

The idea of these algorithms is to choose a symmetric positive definite matrix $M \in \mathbb{R}^{q \times q}$ and a starting point $u^0(t_r) \in \mathbf{K}(t_r)$, and to iteratively update $u^k(t_r)$, as follows:

$$\begin{aligned} \overline{u}^K(t_r) &= P_{\mathbf{K}(t_r)}(u^k(t_r) - \alpha F(t_r, u^k(t_r))), \\ u^{k+1}(t_r) &= u^k(t_r) - \gamma M^{-1}(T_\alpha(u^k(t_r)) - T_\alpha(\overline{u}^K(t_r)), \end{aligned}$$

where $\gamma \in \mathbb{R}_+$ and $T_\alpha = (I - \alpha F)$ in which I is the identity matrix, $\alpha \in (0, +\infty)$ is chosen dynamically (according to an Armijo type rule), so that T_α is strongly monotone. These methods converge under condition that a solution exists and the operator is monotone. Unlike the extragradient method, the procedure require only one projection per iteration, and they have an additional parameter, the scaling matrix M, that must be chosen as a symmetric positive matrix to accelerate the convergence.

After the iterative procedure, we can construct the equilibrium solution by performing a linear interpolation.

The complexity of this algorithm is $O(Nq^3)$. In fact, the algorithm repeats N cycles of complexity $O(q^3)$, being the operator F nonlinear.

8.4.1 A numerical example

Now, we consider a transportation network pattern for the network shown in Figure 8.8. The network consists of six nodes and eight links. We assume that the origin-destination pairs are represented by $w_1 = (P_1, P_2)$ and $w_2 = (P_3, P_4)$, which are

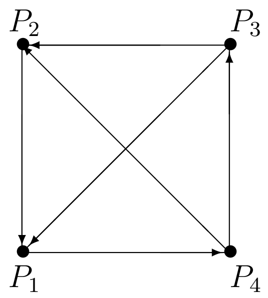

Figure 8.8: A network model.

respectively connected by the following paths:

$$w_1 \ : \ \begin{cases} R_1 = (P_1, P_4) \cup (P_4, P_2) \\ R_2 = (P_1, P_4) \cup (P_4, P_3), \end{cases} \qquad w_2 \ : \ \begin{cases} R_3 = (P_3, P_1) \cup (P_1, P_4) \\ R_4 = (P_3, P_2) \cup (P_2, P_1) \cup (P_1, P_4). \end{cases}$$

We consider the cost vector-function on the path C defined by

$$C : L^2([0,2], \mathbb{R}^4_+) \to L^2([0,2], \mathbb{R}^4_+);$$

$$\begin{aligned} C_1(H(t)) &= 4H_1^2(t) + 2H_2(t) + 5, \\ C_2(H(t)) &= 4\sqrt{H_1(t)} + 5H_2^3(t) + 1, \\ C_3(H(t)) &= 8\sqrt{H_2(t)} + 3H_3^2(t) + 2, \\ C_4(H(t)) &= 2\cos H_1(t) + 4\sqrt{H_3(t)} + 5H_4(t) + 4. \end{aligned}$$

The set of feasible flows is given by:

$$\begin{aligned} \mathbf{K} = \Big\{ F \in L^2([0,2], \mathbb{R}^4_+) : \quad & (t+1, t+1, t+1, 2t) \leq (F_1(t), F_2(t), F_3(t), F_4(t)) \\ & \leq (2t+1, 2t+4, 2t+3, 4t+5), \ F_1(t) + F_2(t) = 3t + 4, \\ & F_3(t) + F_4(t) = 2t + 3, \quad \text{a.e. in } [0,2] \Big\}. \end{aligned}$$

It is easy to verify that the cost vector-function is strongly monotone then the associated evolutionary variational inequality admits a unique solution. Indeed, it results

$$
\begin{aligned}
\langle C(H(t)) - C(F(t)), H(t) - F(t)\rangle &= 4(H_1(t) - F_1(t))^2(H_1(t) + F_1(t)) \\
&\quad +5(H_2(t) - F_2(t))^2(H_2^2(t) + H_2(t)F_2(t) + F_2^2(t)) \\
&\quad +3(H_3(t) - F_3(t))^2(H_3^2(t) + H_3(t)F_3(t) + F_3^2(t)) \\
&\quad +5(H_4(t) - F_4(t))^2 + \left(2 + \frac{4}{\sqrt{H_1(t)} + \sqrt{F_1(t)}}\right) \\
&\quad (H_1(t) - F_1(t))(H_2(t) - F_2(t)) + 2 \\
&\quad (\cos H_1(t) - \cos F_1(t))(H_4(t) - F_4(t)) + 8 \\
&\quad \frac{(H_2(t) - F_2(t))(H_3(t) - F_3(t))}{\sqrt{H_2(t)} + \sqrt{F_2(t)}} + 4 \\
&\quad \frac{(H_3(t) - F_3(t))(H_4(t) - F_4(t))}{\sqrt{H_3(t)} + \sqrt{F_3(t)}} \\
&\geq \left(7 - \frac{2}{\sqrt{H_1(t)} + \sqrt{F_1(t)}}\right)(H_1(t) - F_1(t))^2 \\
&\quad -(\cos H_1(t) - \cos F_1(t))^2 \\
&\quad + \left[5(H_2^2(t) + H_2(t)F_2(t) + F_2^2(t)) - 1\right. \\
&\quad \left. - \frac{2}{\sqrt{H_1(t)} + \sqrt{F_1(t)}} - \frac{4}{\sqrt{H_2(t)} + \sqrt{F_2(t)}}\right] \\
&\quad (H_2(t) - F_2(t))^2 + \left[3(H_3^2(t) + H_3(t)F_3(t) + F_3^2(t))\right. \\
&\quad \left. - \frac{4}{\sqrt{H_2(t)} + \sqrt{F_2(t)}} - \frac{2}{\sqrt{H_3(t)} + \sqrt{F_3(t)}}\right] \\
&\quad (H_3(t) - F_3(t))^2 \left(4 - \frac{2}{\sqrt{H_3(t)} + \sqrt{F_3(t)}}\right) \\
&\quad (H_1(t) - F_1(t))^2 \\
&\geq 5(H_1(t) - F_1(t))^2 + 11(H_2(t) - F_2(t))^2 \\
&\quad +6(H_3(t) - F_3(t))^2 + 3(H_4(t) - F_4(t))^2 \\
&\geq 3\|H(t) - F(t)\|_4^2,
\end{aligned}
$$

for any $H(t), F(t) \in \mathbb{R}_+^4$.

Now, we can compute an approximate curve of equilibria, by selecting $t_r \in \left\{\frac{k}{15} : k \in \{0, 1, \ldots, 30\}\right\}$. Using a simple MatLab computation and choosing the initial

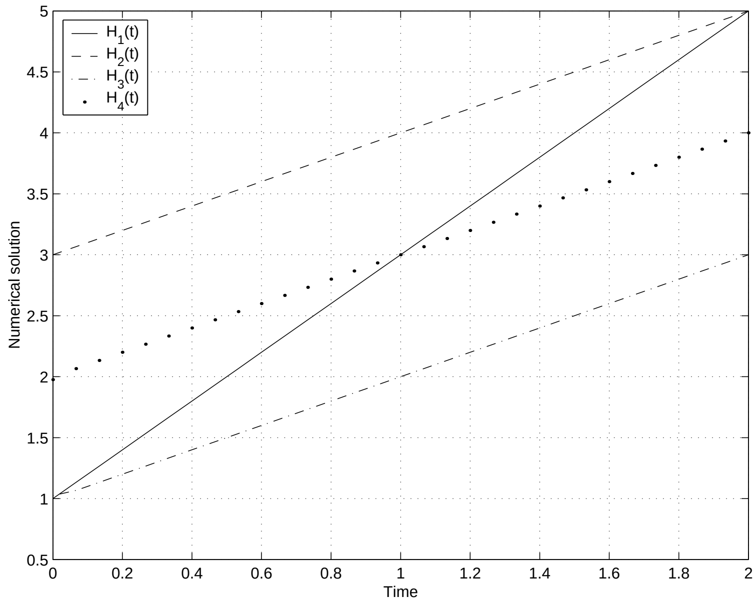

Figure 8.9: Curves of equilibria.

point $H^0(t_r) = (2t_r + 2, t_r + 2, t_r + 2, t_r + 1)$ to start the iterative method, we obtain the equilibria consisting of the points, as shown in Table 8.5. The stopping criterion is $\|R(H^k(t_r))\|_4 = \|H^k(t_r) - H^{k-1}(t_r)\|_4 \leq 10^{-6}$, for $r = 0, 1, \ldots, 30$.

The interpolation of equilibria points yields the curves of equilibria in Figure 8.9.

8.5 The generalized Solodov-Svaiter's method

In this section, we present a method to compute solutions to nonlinear degenerate evolutionary variational inequalities which model dynamic equilibrium problems.

Let $F : [0,T] \times \mathbb{R}^q \to \mathbb{R}^q$ be a nonlinear operator satisfying conditions

$$\|F(t,v)\|_q \leq A(t)\|v\|_q + B(t), \quad \forall v \in \mathbb{R}^q, \text{ a.e. in } [0,T], \qquad (8.5.1)$$

with $B \in L^2([0,T])$ and $A \in L^\infty([0,T])$, and

$$\langle F(t,u) - F(t,v), u - v\rangle \geq \nu(t)\|u - v\|_q^2, \quad \forall u, v \in \mathbb{R}^q, \text{ a.e. in } [0,T], \qquad (8.5.2)$$

where $\nu \in L^\infty([0,T], \mathbb{R}_0^+)$ is such that $\nexists I \subseteq [0,T]$, $\mu(I) > 0$: $\nu(t) = 0$, $\forall t \in I$.

Let us consider the nonlinear degenerate evolutionary variational inequality

Find $H \in \mathbf{K}$ such that

$$\langle F(t, u(t)), v(t) - u(t)\rangle \geq 0, \quad \forall v(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T], \qquad (8.5.3)$$

where

$$\mathbf{K}(t) = \Bigg\{ u(t) \in \mathbb{R}^q : \quad \lambda(t) \leq u(t) \leq \mu(t), \; \sum_{i=1}^{q} \xi_{ji} u_i(t) = \rho_j(t),$$
$$\xi_{ji} \in \{-1, 0, 1\}, \; i \in \{1, 2, \ldots, q\}, \; j \in \{1, 2, \ldots, l\} \Bigg\}.$$

If all assumptions of Theorem 7.4.2 are satisfied, the evolutionary variational inequality admits a unique solution u belonging to $C([0,T], \mathbb{R}^q)$. As a consequence, (8.7.3) holds for each $t \in [0,T]$, namely

$$\langle F(t, u(t)), v(t) - u(t) \rangle \geq 0, \quad \forall v(t) \in \mathbf{K}(t), \; \forall t \in [0,T].$$

Then, we can use a partition of $[0,T]$, as follows

$$0 = t_0 < t_1 < \ldots < t_r < \ldots < t_N = T,$$

and solve, for each value t_r, for $r = 0, 1, \ldots, N$, the following static variational inequality

$$\langle F(t_r, u(t_r)), v(t_r) - u(t_r) \rangle \geq 0, \quad \forall v(t_r) \in \mathbf{K}(t_r), \tag{8.5.4}$$

where

$$\mathbf{K}(t_r) = \Bigg\{ u(t_r) \in \mathbb{R}^q : \quad \lambda(t_r) \leq u(t_r) \leq \mu(t_r), \; \sum_{i=1}^{q} \xi_{ji} u_i(t_r) = \rho_j(t_r),$$
$$\xi_{ji} \in \{-1, 0, 1\}, \; i \in \{1, 2, \ldots, q\}, \; j \in \{1, 2, \ldots, l\} \Bigg\},$$

by means of a projection method and construct the approximate curve of equilibria with a linear interpolation.

Now, we present the projection algorithm that was proposed by Solodov and Svaiter, in [91].

Let $u^k(t_r)$ be the current approximation of the solution to (8.5.4); first, we compute the point $P_{\mathbf{K}(t_r)}(u^k(t_r) - \mu_k F(t_r, u^k(t_r)))$; next, we search the line segment between $u^k(t_r)$ and $P_{\mathbf{K}(t_r)}(u^k(t_r - \mu_k F(t_r, u^k(t_r))))$ for a point $G^j(t_r)$ such that the hyperplane

$$\partial u^k(t_r) = \Big\{ u(t_r) \in \mathbb{R}^q : \; \langle F(t_i, z^k(t_r)), u(t_r) - z^k(t_r) \rangle = 0 \Big\}$$

strictly separates $u^k(t_r)$ from the solution to (8.5.4). To compute $z^k(t_r)$, an Armijo-type procedure is used; after the hyperplane $\partial u^k(t_r)$ is constructed, the next iterate $u^{k+1}(t_r)$ is computing by projecting $u^k(t_r)$ onto the intersection between the feasible

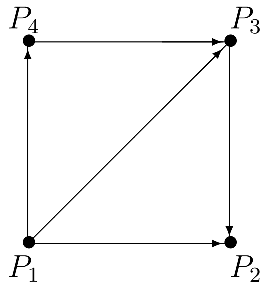

Figure 8.10: A network model.

set $\mathbf{K}(t_r)$ with the half-space $u^k(t_r) = \{u(t_r) \in \mathbb{R}^q : \langle F(t_r, z^k(t_r)), u(t_r) - z^k(t_r)\rangle \leq 0\}$ which contains the solution to (8.5.4).

In [91], it is shown that this method is convergent to a solution to the variational inequality problem under the only assumption that F is continuous and pseudomonotone.

After the iterative procedure, we can construct the dynamic equilibrium solution by means of a linear interpolation of the obtained static equilibrium solutions.

The complexity of this algorithm is $O(Nq^3)$. In fact, the algorithm repeats N cycles of complexity $O(q^3)$, being the operator F nonlinear.

8.5.1 A numerical example

Let us consider a network as Figure 8.10. The network consists of four nodes and five links. The origin-destination pair is $w = (P_1, P_2)$, which is connected by the paths $R_1 = (P_1, P_2)$, $R_2 = (P_1, P_3) \cup (P_3, P_2)$ and $R_3 = (P_1, P_4) \cup (P_4, P_3) \cup (P_3, P_2)$. We consider the cost operator on the path C defined by

$$C : L^2([0,2], \mathbb{R}^3_+) \to L^2([0,2], \mathbb{R}^3_+);$$

$$\begin{aligned} C_1(H(t)) &= \frac{1}{4} t H_1(t) + 3t + 1, \\ C_2(H(t)) &= 3t^2 H_2(t) + \sqrt{t^5} H_3^2(t) + t^2 + 2, \\ C_3(H(t)) &= t^2 \sqrt{H_1(t)} + \frac{7}{2} t^3 H_3(t) + \sqrt{t} + \frac{4}{3}. \end{aligned}$$

The set of feasible flows is given by

$$\begin{aligned} \mathbf{K} = \Big\{ F \in L^2([0,2], \mathbb{R}^3_+) : \quad & (t+1, 2t+1, t+2) \leq (F_1(t), F_2(t), F_3(t)) \\ & \leq (3(t+1), 4t+3, 3t+4), \\ & F_1(t) + F_2(t) + F_3(t) = 7t + 2, \text{ in } [0,2] \Big\}. \end{aligned}$$

In the following, we verify that the cost vector-function is degenerate

$$
\begin{aligned}
\langle C(H(t)) - C(F(t)), H(t) - F(t)\rangle &= \frac{1}{4}t(H_1(t) - F_1(t))^2 + 3t^2(H_2(t) - F_2(t))^2 \\
&\quad + \sqrt{t^5}(H_3(t) + F_3(t))(H_3(t) - F_3(t)) \\
&\quad (H_2(t) - F_2(t)) + \frac{t^2}{\sqrt{H_1(t)} + \sqrt{F_1(t)}}(H_1(t) - F_1(t)) \\
&\quad (H_3(t) - F_3(t)) + \frac{7}{2}t^3(H_3(t) - F_3(t))^2 \\
&\geq \left(\frac{1}{4}t - \frac{t}{2(\sqrt{H_1(t)} + \sqrt{F_1(t)})}\right)(H_1(t) - F_1(t))^2 \\
&\quad + \left[3t^2 - \frac{t^2}{2}(H_3(t) + F_3(t))\right](H_2(t) - F_2(t))^2 \\
&\quad + \left[\frac{7}{2}t^3 - \frac{t^3}{2}(H_3(t) + F_3(t)) - \frac{t^3}{12}\right](H_3(t) - F_3(t))^2 \\
&\geq \frac{1}{6}t(H_1(t) - F_1(t))^2 + t^2(H_2(t) - F_2(t))^2 \\
&\quad + \frac{17}{12}t^3(H_3(t) - F_3(t))^2 \\
&\geq \nu(t)\|H(t) - F(t)\|_3^2,
\end{aligned}
$$

for any $H(t), F(t) \in \mathbb{R}^3_+$ and for $t \in [0, 2]$, where

$$
\nu(t) = \begin{cases} \frac{17}{12}t^3, & \text{in } \left[0, \sqrt{\frac{2}{17}}\right], \\ \frac{1}{6}t, & \text{in } \left]\sqrt{\frac{2}{17}}, 2\right]. \end{cases}
$$

Then, by Theorem 5.3.2 it follows that the problem admits a unique continuous equilibrium solution.

Moreover, the cost vector-function is continuous, in fact it results

$$\begin{aligned}
\|C(H(t)) - C(F(t))\|_3^2 &= \left[\frac{1}{4}t(H_1(t) - F_1(t))\right]^2 + \Big[3t^2(H_2(t) - F_2(t)) + \sqrt{t^5} \\
&\quad (H_3^2(t) - F_3^2(t))\Big]^2 + \Big[t^2(\sqrt{H_1(t)} - \sqrt{F_1(t)}) \\
&\quad + \frac{7}{2}t^3(H_3(t) - F_3(t))\Big]^2 \\
&\le \left[\frac{1}{16}t^2 + \frac{2t^4}{(\sqrt{H_1(t)} + \sqrt{F_1(t)})^2}\right](H_1(t) - F_1(t))^2 \\
&\quad +18t^4(H_2(t) - F_2(t))^2 + 2\left[t^5(H_3(t) + F_3(t))^2 + \frac{49}{4}t^6\right] \\
&\quad (H_3(t) - F_3(t))^2 \\
&\le 27168\|H(t) + F(t)\|_3^2,
\end{aligned}$$

for any $H(t), F(t) \in \mathbb{R}^3_+$ and for $t \in [0, 2]$. Hence, Solodov-Svaiter's method is convergent, so we can use the generalized version of the method to solve the problem with the partition $t_i \in \left\{\frac{k}{15} : \ k \in \{0, 1, \dots, 30\}\right\}$. We use a simple MatLab computation and we choose the initial point $H^0(t_i) = (2t_i, 3t_i + 1, 2t_i + 1)$ to start the iterative method, then we obtain the static equilibrium solutions shown in Table 8.6. The stopping criterion is $\|R(H^k(t_r))\|_3 = \|H^k(t_r) - H^{k-1}(t_r)\|_3 \le 10^{-6}$, for $i = 0, 1, \dots, 30$.

The interpolation of equilibria points yields the curves of equilibria, as shows Figure 8.11.

8.6 The generalized descent method: first version

Let $F : [0, T] \times \mathbb{R}^q \to \mathbb{R}^q$ be an operator, and let us consider dynamic equilibrium problems which are modeled by the following evolutionary variational inequality

Find $u \in \mathbf{K}$ such that

$$\langle F(t, u(t)), v(t) - u(t)\rangle \ge 0, \quad \forall v(t) \in \mathbf{K}(t), \ \text{a.e. in } [0, T], \tag{8.6.1}$$

where

$$\mathbf{K}(t) = \Bigg\{u(t) \in \mathbb{R}^q : \quad \lambda(t) \le u(t) \le \mu(t), \ \sum_{i=1}^q \xi_{ji} u_i(t) = \rho_j(t), \\ \xi_{ji} \in \{-1, 0, 1\}, \ i \in \{1, 2, \dots, q\}, \ j \in \{1, 2, \dots, l\}\Bigg\}.$$

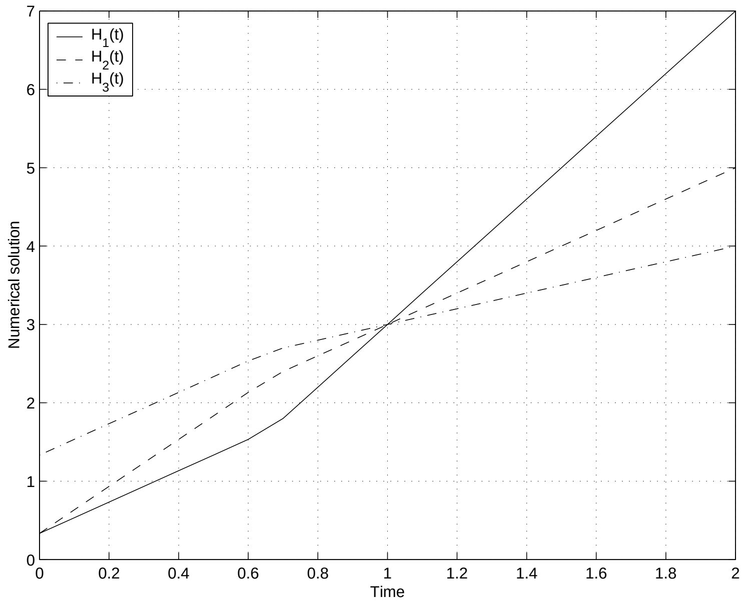

Figure 8.11: Curves of equilibria.

We suppose that the assumptions of Theorem 7.4.2 are satisfied and hence the solution u to (8.6.1) belongs to $C([0,T],\mathbb{R}^q)$. As a consequence, (8.6.1) holds for each $t \in [0,T]$, namely

$$\langle F(t,u(t)), v(t)-u(t)\rangle \geq 0, \quad \forall v(t) \in \mathbf{K}(t),\ \forall t \in [0,T].$$

In the following, applying a discretization procedure, we will use a combined relaxation method to compute the solution to the evolutionary variational inequality. This method (see [53]) runs as follows. After a partition of real interval $[0,T]$, such that:

$$0 = t_0 < t_1 < \ldots < t_r < \ldots < t_N = T.$$

the algorithm, to solve the finite-dimensional variational inequality

$$\langle F(t_r,u(t_r)), v(t_r)-u(t_r)\rangle \geq 0, \quad \forall v(t_r) \in \mathbf{K}(t_r),$$

where

$$\mathbf{K}(t_r) = \Bigg\{ u(t_r) \in \mathbb{R}^q : \quad \lambda_i(t_r) \leq F_i(t_r) \leq \mu_i(t_r),\ \sum_{i=1}^{q} \xi_{ji} u_i(t_r) = \rho_j(t_r),$$
$$\xi_{ji} \in \{-1,0,1\},\ i \in \{1,2,\ldots,q\},\ j \in \{1,2,\ldots,l\} \Bigg\},$$

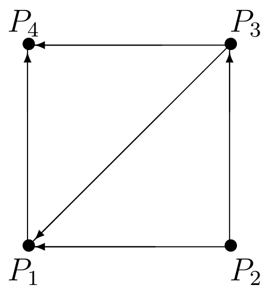

Figure 8.12: A network model.

starts from any $u^0(t_r) \in \mathbf{K}(t_r)$ fixed, from a sequence $\{\gamma_k\}$ satisfying the following conditions

$$\gamma_k \in [0,2],\ k = 0,1,\ldots; \quad \sum_{k=0}^{+\infty} \gamma_k(2-\gamma_k) = \infty,$$

and from numbers $\alpha \in (0,1)$, $\beta \in (0,1)$, $\widetilde{\theta} > 0$ chosen. Set $k := 0$. It finds m as the smallest number in $\mathbb{Z}_+$ such that

$$\begin{aligned} \langle F(t_r, u^k(t_r)) - F(t_r, z^{k,m}(t_r)), u^k(t_r) - z^{k,m}(t_r)\rangle \quad &\le \quad (1-\alpha)(\widetilde{\theta}\beta^m)^{-1} \\ &\quad\ \|z^{k,m}(t_r) - u^k(t_r)\|_q^2, \end{aligned} \tag{8.6.2}$$

where $z^{k,m}(t_r)$ is a solution to the auxiliary problem of finding $\overline{z}(t_r) \in \mathbf{K}(t_r)$ such that

$$\langle F(t_r, u^k(t_r)) + (\widetilde{\theta}\beta^q)^{-1}(\overline{z}(t_r) - u^k(t_r)), u(t_r) - \overline{z}(t_r)\rangle \ge 0, \quad \forall u(t_r) \in \mathbf{K}(t_r).$$

Set $\theta_k := \beta^m\widetilde{\theta}$, $v^k(t_r) := z^{k,m}(t_r)$. If $u^k(t_r) = v^k(t_r)$ or $F(t_r, v^k(t_r)) = 0$, the algorithm stops, else, setting

$$\begin{aligned} t^k(t_{ri}) &:= F(t_r, F^k(t_r)) - F(t_r, u^k(t_r)) - \theta_k^{-1}(v^k(t_r) - u^k(t_r)), \\ f^k(t_r) &:= F(t_r, v^k(t_r)), \qquad \sigma^k(t_r) := \alpha\theta_k^{-1}\|v^k(t_r) - u^k(t_r)\|_q^2 / \|t^k(t_r)\|_q^2, \\ u^{k+1}(t_r) &:= P_{\mathbf{K}(t_r)}(u^k(t_r) - \gamma_k\sigma^k(t_r)f^k(t_r)), \end{aligned}$$

the iteration repeats itself. After a linear interpolation, the approximate equilibrium solution is constructed.

In [53] it is shown that this method is convergent to a solution of the finite-dimensional variational inequality problem under the only assumption that F is locally Lipschitz continuous and monotone.

The complexity of this algorithm depends on the choose of the procedure to compute the small number in $\mathbb{Z}_+$ such that (8.6.2) holds.

8.6.1 A numerical example

Let us consider a network (see Figure 8.12) where $N = \{P_1, P_2, P_3, P_4\}$ is the set of the nodes and $L = \{(P_1,P_4),(P_2,P_1),(P_2,P_3),(P_3,P_1),(P_3,P_4)\}$ is the set of links.

We assume that the origin-destination pair is $w = (P_2, P_4)$, so that the routes are the following:

$$\begin{aligned} R_1 &= (P_2, P_4) \cup (P_1, P_4), \\ R_2 &= (P_2, P_3) \cup (P_3, P_4), \\ R_3 &= (P_2, P_3) \cup (P_3, P_1) \cup (P_1, P_4). \end{aligned}$$

Let us consider that the route costs are the following:

$$C : L^2([0,2], \mathbb{R}^3_+) \to L^2([0,2], \mathbb{R}^3_+);$$

$$\begin{aligned} C_1(t, H(t)) &= 3tH_1^2(t) + t^4 + 1, \\ C_2(t, H(t)) &= 2t^2 H_2^3(t) + 2, \\ C_3(t, H(t)) &= \sin t \cdot H_3(t) + 3t + 1. \end{aligned}$$

The set of feasible flows is given by

$$\begin{aligned} \mathbf{K} = \Big\{ F \in L^2([0,2], \mathbb{R}^3_+) : \quad & (\sqrt{t} + 1, 2t + 1, t^2 + 2) \leq (F_1(t), F_2(t), F_3(t)) \\ & \leq (5t + 4, 4t^3 + 10t, 3t^2 + 4), \\ & F_1(t) + F_2(t) + F_3(t) = 5t^2 + 2t + 1, \text{ in } [0,2] \Big\}. \end{aligned}$$

Now, we prove that the cost vector-function is degenerate

$$\begin{aligned} \langle C(t, H(t)) - C(t, F(t)), H(t) - F(t) \rangle \;=\; & 3t(H_1^2(t) - F_1^2(t))(H_1(t) - F_1(t)) + 2t^2 \\ & (H_2^3(t) - F_2^3(t))(H_2(t) - F_2(t)) + \sin t \\ & (H_3(t) - F_3(t))^2 \geq 3t(H_1(t) + F_1(t)) \\ & (H_1(t) - F_1(t))^2 + 2t^2 \\ & (H_2^2(t) + H_2(t)F_2(t) + F_2^2(t))(H_2(t) - F_2(t))^2 \\ & + \sin t (H_3(t) - F_3(t))^2 \geq 6t(H_1(t) - F_1(t))^2 \\ & + 6t^2(H_2(t) - F_2(t))^2 + \sin t (H_3(t) - F_3(t))^2 \\ \geq\; & \nu(t) \| H(t) - F(t) \|_3^2, \end{aligned}$$

for any $H(t), F(t) \in \mathbf{R}^3_+$ and for every $t \in [0,2]$, where

$$\nu(t) = \begin{cases} 6t, & \text{in } [0,1], \\ 6t^2, & \text{in }]1,2]. \end{cases}$$

So, all the assumptions of Theorem 7.2.7 are verified. Then, the problem admits a unique continuous equilibrium solution.

Moreover, the cost vector-function is obviously Lipschitz continuous. Hence, the descent method is convergent, so we can use the generalized version of the method to

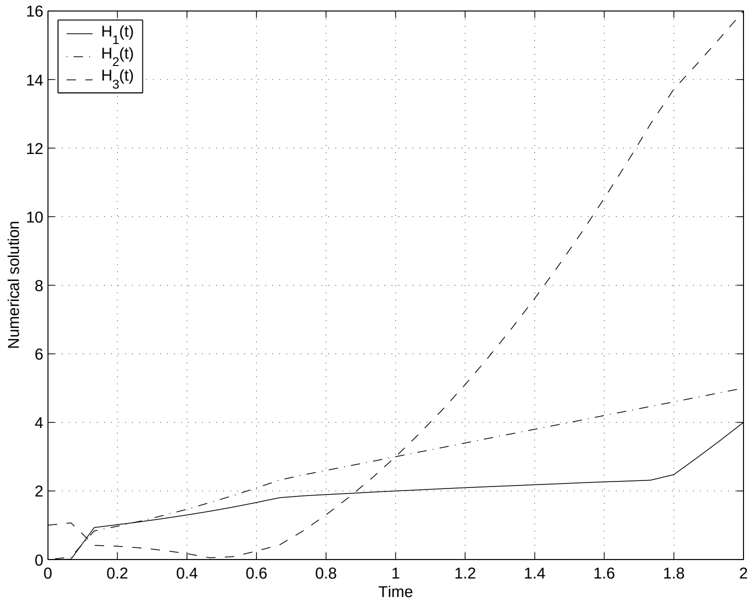

Figure 8.13: Curves of equilibria.

solve the problem with the partition $t_r \in \left\{\frac{k}{15} : k \in \{0, 1, \dots, 30\}\right\}$. We use a simple MatLab computation and we choose the initial point $H^0(t_r) = (2t_r^2, 2t_r^2+t_r, t_r^2+t_r+1)$ to start the iterative method. Then, we obtain the equilibrium solutions for every time instant which are shown in Table 8.7. The stopping criterion is $\|R(H^k(t_r))\|_3 = \|H^k(t_r) - H^{k-1}(t_r)\|_3 \leq 10^{-6}$, for $r = 0, 1, \dots, 30$.

The interpolation of equilibria points yields the curves of equilibria, as Figure 8.13 shows.

8.7 The generalized descent method: second version

Now, we present a method to compute solutions to dynamic equilibrium problems which are expressed to nonlinear strictly monotone evolutionary variational inequalities.

Let $F : [0, T] \times \mathbb{R}^q \to \mathbb{R}^q$ be an operator satisfying conditions

$$\|F(t, v)\|_q \leq A(t)\|v\|_q + B(t), \quad \forall v \in \mathbb{R}^q_+, \text{ a.e. in } [0, T], \tag{8.7.1}$$

with $B \in L^2([0, T])$ and $A \in L^\infty([0, T])$, and

$$\langle F(t, u) - F(t, v), u - v\rangle > 0, \quad \forall u, v \in \mathbf{R}^q, \; u \neq v, \text{ a.e. in } [0, T]. \tag{8.7.2}$$

Let us consider the evolutionary variational inequality

Find $H \in \mathbf{K}$ such that

$$\langle F(t,u(t)), v(t)-u(t)\rangle \geq 0, \quad \forall v(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T], \tag{8.7.3}$$

where

$$\mathbf{K}(t) = \Bigg\{ u(t) \in \mathbb{R}^q : \quad \lambda(t) \leq u(t) \leq \mu(t), \ \sum_{i=1}^{q} \xi_{ji} u_i(t) = \rho_j(t),$$
$$\xi_{ji} \in \{-1,0,1\}, \ i \in \{1,2,\ldots,q\}, \ j \in \{1,2,\ldots,l\} \Bigg\},$$

Let us suppose that assumptions of Theorem 7.4.3 are satisfied, then the unique solution u to (8.7.3) belongs to $C([0,T],\mathbb{R}^q)$. Then, (8.7.3) holds for each $t \in [0,T]$, namely

$$\langle F(t,u(t)), v(t)-u(t)\rangle \geq 0, \quad \forall v(t) \in \mathbf{K}(t), \ \forall t \in [0,T].$$

Now, we extend a combined relaxation method to the calculus of solution to the last evolutionary variational inequality. The method (see [53]) runs as follows. After a partition of real interval $[0,T]$, such that:

$$0 = t_0 < t_1 < \ldots < t_r < \ldots < t_N = T.$$

the algorithm, to solve the finite-dimensional variational inequality

$$\langle F(t_r,u(t_r)), v(t_r)-u(t_r)\rangle \geq 0, \quad \forall v(t_r) \in \mathbf{K}(t_r),$$

where

$$\mathbf{K}(t_r) = \Bigg\{ u(t_r) \in \mathbb{R}^q_+ : \quad \lambda_i(t_r) \leq F_i(t_r) \leq \mu_i(t_r), \ \sum_{i=1}^{q} \xi_{ji} u_i(t_r) = \rho_j(t_r),$$
$$\xi_{ji} \in \{-1,0,1\}, \ i \in \{1,2,\ldots,q\}, \ j \in \{1,2,\ldots,l\} \Bigg\},$$

starts from any $u^0(t_r) \in \mathbf{K}(t_r)$ fixed, from a sequence $\{\gamma_k\}_{k\in\mathbb{N}}$ satisfying the following conditions

$$\gamma_k \in [0,2], \ k = 0,1,\ldots; \quad \sum_{k=0}^{+\infty} \gamma_k(2-\gamma_k) = \infty,$$

and from numbers $\alpha \in (0,1)$, $\beta \in (0,1)$, $\widetilde{\theta} > 0$ chosen. Set $k := 0$. It finds m as the smallest number in $\mathbb{Z}_+$ such that

$$\langle F(t_r,u^k(t_r)) - F(t_r,z^{k,m}(t_r)), u^k(t_r) - z^{k,m}(t_r)\rangle \leq (1-\alpha)(\widetilde{\theta}\beta^m)^{-1} \tag{8.7.4}$$

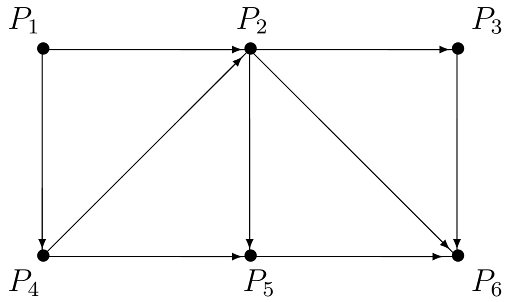

Figure 8.14: A network model.

$$\langle A_k[z^{k,m}(t_r) - u^k(t_r)], z^{k,m}(t_r) - u^k(t_r)\rangle,$$

where A_k is a $n \times n$ definite positive matrix and $z^{k,m}(t_r)$ is a solution to the auxiliary problem of finding $\overline{z}(t_r) \in \mathbf{K}(t_r)$ such that

$$\langle F(u^k(t_r)) + (\widetilde{\theta}\beta^m)^{-1}(\overline{z}(t_r) - u^k(t_r)), u(t_r) - \overline{z}(t_r)\rangle \geq 0, \quad \forall u(t_r) \in \mathbf{K}(t_r).$$

Set $\theta_k := \beta^m\widetilde{\theta}$, $v^k(t_r) := z^{k,m}(t_r)$. If $u^k(t_r) = v^k(t_r)$ or $F(t_r, v^k(t_r)) = 0$, the algorithm stops, else, setting

$$\begin{aligned}
t^k(t_r) &:= F(t_r, v^k(t_r)) - F(t_r, u^k(t_r)) - \theta_k^{-1} A_k(v^k(t_r) - u^k(t_r)),\\
f^k(t_r) &:= F(t_r, v^k(t_r)),\\
\sigma^k(t_r) &:= \alpha\theta_k^{-1}\ \langle A_k[z^{k,m}(t_r) - u^k(t_r)], z^{k,m}(t_r) - u^k(t_r)\rangle / \|t^k(t_r)\|_q^2,\\
u^{k+1}(t_q) &:= P_{\mathbf{K}(t_r)}(u^k(t_r) - \gamma_k\sigma^k(t_r) f^k(t_r)),
\end{aligned}$$

the iteration repeats itself. After a linear interpolation, the approximate equilibrium solution is constructed.

In [53] it is shown that this method is convergent to a solution of the finite-dimensional variational inequality problem under the only assumption that F is locally Lipschitz continuous and monotone.

As for the previous algorithm, the complexity of this depends on the choose of the procedure to compute the small number in $\mathbb{Z}_+$ such that (8.7.4) holds.

8.7.1 A numerical example

Let us consider a transportation network pattern for the network shown in Figure 8.14. The network consists of six nodes and nine links. We assume that the O/D pairs are represented by $w_1 = (P_1, P_2)$ and $w_2 = (P_4, P_6)$, which are respectively connected by the following paths:

$$w_1 \ : \ \begin{cases} R_1 = (P_1, P_2) \\ R_2 = (P_1, P_4) \cup (P_4, P_2), \end{cases} \qquad w_2 \ : \ \begin{cases} R_3 = (P_4, P_2) \cup (P_2, P_6) \\ R_4 = (P_4, P_5) \cup (P_5, P_6) \\ R_5 = (P_4, P_2) \cup (P_2, P_3) \cup (P_3, P_6) \\ R_6 = (P_4, P_2) \cup (P_2, P_5) \cup (P_5, P_6). \end{cases}$$

Let us suppose that the cost vector-function on the paths is the next one

$$C : L^2([0,2],\mathbb{R}^6_+) \to L^2([0,2],\mathbb{R}^6_+);$$

$$\begin{aligned}
C_1(t,H(t)) &= (3t+4)H_1(t)+t,\\
C_2(t,H(t)) &= H_1^3(t)+(t+3)H_2(t)+3t^3+1,\\
C_3(t,H(t)) &= (t+1)H_1(t)+3H_3^3(t)+5t+2,\\
C_4(t,H(t)) &= (t+4)H_4^2(t)+2H_5(t)+t+1,\\
C_5(t,H(t)) &= 2H_3(t)+(5t+2)H_5^2(t)+7t,\\
C_6(t,H(t)) &= (t+3)H_4^2(t)+(2t+5)H_6(t),
\end{aligned}$$

and the set of feasible flows is given by

$$\mathbf{K} = \Big\{ F \in L^2([0,2],\mathbb{R}^6_+) : \quad (t+1,0,2t+1,t+1,2t+1,0) \le (F_1(t),F_2(t),F_3(t),$$
$$F_4(t),F_5(t),F_6(t)) \le (5t,10t+5,5t+12,5t+9,10t+2),$$
$$F_1(t)+F_2(t) = 2t+5,$$
$$F_3(t)+F_4(t)+F_5(t)+F_6(t) = 5t+3 \text{ in } [0,2] \Big\}.$$

Now, we verify that the cost vector-function is strictly monotone:

$$\begin{aligned}
\langle C(t,H(t))-C(t,F(t)),H(t)-F(t)\rangle &= (3t+4)(H_1(t)-F_1(t))^2\\
&\quad +(H_1^2(t)+F_1(t)H_1(t)+F_1^2(t))(H_1(t)-F_1(t))\\
&\quad (H_2(t)-F_2(t))+(t+3)(H_2(t)-F_2(t))^2\\
&\quad +(t+1)(H_1(t)-F_1(t))(H_3(t)-F_3(t))\\
&\quad +3(H_3^2(t)+H_3(t)F_3(t)+F_3^2(t))(H_3(t)-F_3(t))^2\\
&\quad +(t+4)(H_4(t)+F_4(t))(H_4(t)-F_4(t))^2\\
&\quad +2(H_5(t)-F_5(t))(H_4(t)-F_4(t))\\
&\quad +2(H_3(t)-F_3(t))(H_5(t)-F_5(t))\\
&\quad +(5t+2)(H_5(t)+F_5(t))(H_5(t)-F_5(t))^2\\
&\quad +(t+3)(H_4(t)+F_4(t))(H_4(t)-F_4(t))\\
&\quad (H_6(t)-F_6(t))+(2t+5)(H_6(t)-F_6(t))^2\\
&\ge \Big(\frac{5}{2}t+2\Big)(H_1(t)-F_1(t))^2+\Big(t+\frac{3}{2}\Big)\\
&\quad (H_2(t)-F_2(t))^2+\Big(\frac{15}{2}-\frac{t}{2}\Big)(H_3(t)-F_3(t))^2\\
&\quad +(t+4)(H_4(t)-F_4(t))^2+(10t+2)\\
&\quad (H_5(t)-F_5(t))^2+(t+2)(H_6(t)-F_6(t))^2 > 0
\end{aligned}$$

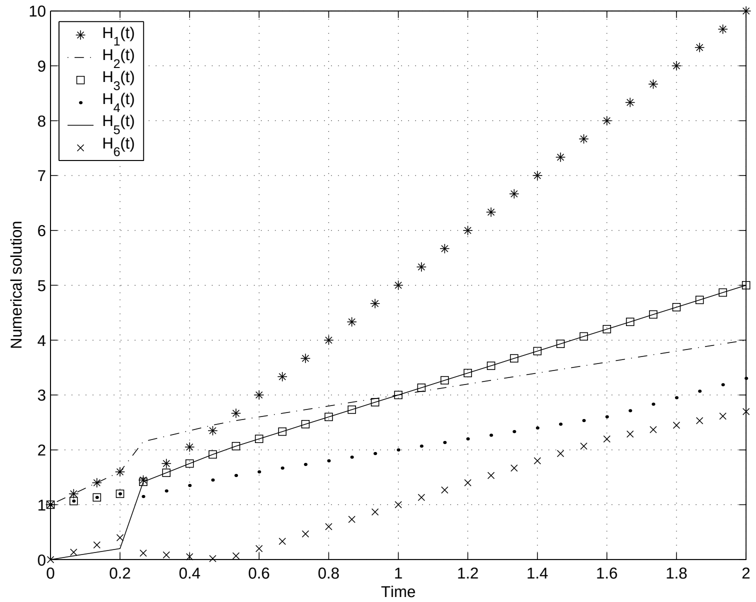

Figure 8.15: Curves of equilibria.

$\forall H(t), F(t) \in \mathbb{R}^6_+$, with $H(t) \neq F(t)$, in $[0, T]$. Moreover, the operator is continuous, then the descent method determines a sequence $\{H^k(t_r)\}_{k\in\mathbb{N}}$ convergent to the solution to the static variational inequality for each i.

Hence, we can compute the curve of equilibria, by selecting $t_r \in \left\{\frac{k}{15} : k \in \{0, 1, \ldots, 30\}\right\}$. Using a MatLab computation and choosing the initial point $H^0(t_r) = (t_r + 3, t_r + 2, 2t_r + 1, t_r + 1, t_r + 1, t_r)$ to start the iterative method, we find the equilibrium points in the Table 8.8. The stopping criterion is $\|R(H^k(t_r))\|_6 = \|H^k(t_r) - H^{k-1}(t_r)\|_6 \leq 10^{-6}$, for $i = 0, 1, \ldots, 30$.

The interpolation of equilibria points yields the curves of equilibria, as Figure 8.15 shows.

8.8 The convergence study

In this section, we investigate on the convergence of algorithms presented in the previous sections. We want to prove that under appropriate assumptions the sequence, generated by algorithms, converges in L^1-sense to the equilibrium solution to the evolutionary variational inequality

Find $u \in \mathbf{K}$ such that

$$\langle F(t, u(t)), v(t) - u(t) \rangle \geq 0, \quad \forall v(t) \in \mathbf{K}(t), \text{ a.e. in } [0, T], \tag{8.8.1}$$

where $F : [0,T] \times \mathbb{R}^q \to \mathbb{R}^q$ and

$$\mathbf{K}(t) = \Bigg\{ u(t) \in \mathbb{R}^q : \quad \lambda(t) \leq u(t) \leq \mu(t), \ \sum_{i=1}^{q} \xi_{ji} u_i(t) = \rho_j(t), \\ \xi_{ji} \in \{-1,0,1\}, \ i \in \{1,2,\ldots,q\}, \ j \in \{1,2,\ldots,l\} \Bigg\}.$$

At first, we remark that algorithms have the following common structure:

Step 1: Discretization of the time interval $[0,T]$:

$$0 = t_0 < t_1 < \ldots < t_r < \ldots < t_N = T.$$

Step 2: Solve static variational inequalities:

$$\langle F(t_r, u(t_r)), v(t_r) - u(t_r) \rangle \geq 0, \quad \forall v(t_r) \in \mathbf{K}(t_r), \tag{8.8.2}$$

where

$$\mathbf{K}(t_r) = \Bigg\{ u(t_r) \in \mathbb{R}^q_+ : \quad \lambda(t_r) \leq u(t_r) \leq \mu(t_r), \ \sum_{i=1}^{q} \xi_{ji} u_i(t_r) = \rho_j(t_r), \\ \xi_{ji} \in \{-1,0,1\}, \ i \in \{1,2,\ldots,q\}, \ j \in \{1,2,\ldots,l\} \Bigg\},$$

by means of a convergent method.

Step 3: Interpolate equilibrium solutions to (8.8.2).

Let us assume that all hypotheses to have the continuity of solution to (8.8.1) and the convergence of the method to compute solutions to finite-dimensional variational inequalities hold. Let us introduce a sequence $\{\pi_n\}_{n \in \mathbb{N}}$ of (not necessarily equidistant) partitions of the time interval $[0,T]$ such that $\pi_n = (t_n^0, t_n^1, \ldots, t_n^{N_n})$, where $0 = t_n^0 < t_n^1 < \ldots < t_n^{N_n} = T$. We consider a sequence of equidistant partitions, in the sense that

$$k_n := \max\{t_n^r - t_n^{r-1} | \ r = 1,2,\ldots,N_n\},$$

approaches zero for $n \to +\infty$.

By the interpolation theory, we know that if we construct the approximate solution to (8.8.1) by means of Hermite's polynomial, using known values of the solution $u(t)$, the sequence converges uniformly to the exact solution. We not use Hermite's polynomial, but we consider an approximation by means of a piecewise constant functions, we can prove that the convergence is in L^1-sense.

Let us consider the approximate solutions to (8.8.1) given by the following formula:

$$u_n(t) = \sum_{r=1}^{n^2} u(t_n^r)\chi_{[t_n^{r-1}, t_n^r[}(t), \tag{8.8.3}$$

where $u(t_n^r)$ is the solution to the finite-dimensional variational inequality which is obtained to (8.8.1) for $t = t_n^r$, which can be compute by means of a projection method or a descent method.

Let us estimate the following integral

$$\begin{aligned}\int_0^T \left\| u(t) - \sum_{r=1}^{N_n} u(t_n^r)\chi_{[t_n^{r-1}, t_n^r[}(t) \right\|_q dt \\ = \int_0^T \left\| \sum_{r=1}^{N_n} u(t)\chi_{[t_n^{r-1}, t_n^r[}(t) - \sum_{r=1}^{N_n} u(t_n^r)\chi_{[t_n^{r-1}, t_n^r[}(t) \right\|_q dt \\ \leq \sum_{r=1}^{N_n} \int_{t_{r-1}^n}^{t_r^n} \| u(t) - u(t_n^r) \|_q dt \qquad .\end{aligned}$$

Since u is uniformly continuous, we have that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $t \in [t_{r-1}^n, t_r^n]$ satisfies the condition $|t - t_n^r| < \delta$ it results

$$\| u(t) - u(t_n^r) \|_q^2 < \frac{\varepsilon}{T}, \quad \text{for } r = 1, 2, \ldots, n^2, \ \forall n \in \mathbb{N}.$$

Choosing n large enough in such way that $k_n < \delta$, we reach

$$\int_0^T \left\| u(t) - \sum_{r=1}^{N_n} u(t_n^r)\chi_{[t_n^{r-1}, t_n^r[}(t) \right\|_q dt < \sum_{r=1}^{N_n} \frac{\varepsilon}{T}(t_r^n - t_{r-1}^n) = \varepsilon. \tag{8.8.4}$$

The last estimate implies that sequence (8.8.3) converges to the solution to evolutionary variational inequality (8.8.1)

Table 8.2: Analysis of the convergence of the generalized projection method for different values of α.

	$\alpha = 0.05$			$\alpha = 0.005$			$\alpha = 0.0005$		
t_r	iter	np/nf	CPUtime	iter	np/nf	CPUtime	iter	np/nf	CPUtime
0	52	104/105	0.406	333	666/667	1.328	1979	3958/3959	6.829
1/15	54	108/109	0.172	349	698/699	1.157	2145	4290/4291	6.953
2/15	66	132/133	0.203	437	874/875	1.328	2610	5220/5221	7.937
1/5	56	112/113	0.187	366	732/733	1.204	2312	4624/4625	7.609
4/15	57	114/115	0.188	374	748/749	1.235	2392	4784/4785	7.859
1/3	58	116/117	0.188	381	762/763	1.265	2462	4924/4925	8.125
2/5	58	116/117	0.203	388	776/777	1.297	2524	5048/5049	8.375
7/15	59	118/119	0.203	393	786/787	1.297	2580	5160/5161	8.515
8/15	59	118/119	0.203	399	798/799	1.328	2631	5262/5263	8.656
3/5	60	120/121	0.203	403	806/807	1.328	2678	5356/5357	8.891
2/3	60	120/121	0.219	408	816/817	1.344	2721	5442/5443	9.015
11/15	61	122/123	0.219	412	824/825	1.359	2761	5522/5523	9.094
4/5	61	122/123	0.203	415	830/831	1.375	2799	5598/5599	9.235
13/15	62	124/125	0.203	419	838/839	1.375	2834	5668/5669	9.437
14/15	62	124/125	0.203	422	844/845	1.406	2867	5734/5735	9.5
1	62	124/125	0.219	426	852/853	1.406	2899	5798/5799	9.563
16/15	63	126/127	0.234	429	858/859	1.406	2929	5858/5859	9.688
17/15	63	126/127	0.203	431	862/863	1.422	2957	5914/5915	9.75
6/5	63	126/127	0.203	434	868/869	1.437	2984	5968/5969	9.86
19/15	64	128/129	0.219	437	874/875	1.453	3010	6020/6021	9.953
4/3	64	128/129	0.219	439	878/879	1.453	3034	6068/6069	10.11
7/5	64	128/129	0.203	442	884/885	1.468	3058	6116/6117	10.094
22/15	64	128/129	0.219	444	888/889	1.484	3080	6160/6161	10.172
23/15	65	130/131	0.219	446	892/893	1.469	3102	6204/6205	10.25
8/5	65	130/131	0.219	446	892/893	1.485	3123	6246/6247	10.328
5/3	65	130/131	0.219	448	896/897	1.485	3144	6288/6289	10.375
26/15	65	130/131	0.218	450	900/901	1.5	3163	6326/6327	10.547
9/5	66	132/133	0.219	452	904/905	1.516	3182	6364/6365	10.547
28/15	66	132/133	0.219	454	908/909	1.516	3200	6400/6401	10.562
29/15	66	132/133	0.218	456	912/913	1.515	3218	6436/6437	10.641
2	66	132/133	0.219	459	918/919	1.515	3235	6470/6471	10.688

Table 8.3: Numerical results and analysis of the convergence of the generalized extragradient method

t_r	$H_1(t_r)$	$H_2(t_r)$	$H_3(t_r)$	$H_4(t_r)$	$H_5(t_r)$	iter	np/nf	CPUtime
0	2.4997175	0.5002824	2.8571429	2.1428571	0	3551	7104/7105	10.922
1/15	2.3593725	0.7739608	3.0329378	2.3670622	0	3185	6370/6371	9.594
2/15	2.2605846	1.0060820	3.2090342	2.5909658	0	2913	5826/5827	8.750
1/5	2.1909429	1.2090571	3.3852142	2.8147858	0	5043	10087/10088	15.906
4/15	2.1425800	1.3907534	3.5614116	3.0385884	0	4819	9638/9639	15.172
1/3	2.1094881	1.5571786	3.7376283	3.2623717	0	4691	9382/9383	13.781
2/5	2.0875250	1.7124750	3.9138980	3.4861020	0	4616	9232/9233	13.5
7/15	2.0736694	1.8596639	4.0902629	3.7097371	0	4532	9064/9065	13.235
8/15	2.0656801	2.0009866	4.2667629	3.9332371	0	4378	8756/8757	12.687
3/5	2.0618720	2.1381280	4.4434288	4.1565712	0	4134	8268/8269	12.203
2/3	2.0609642	2.2723691	4.6202803	4.3797197	0	3826	7652/7653	11.219
11/15	2.0619749	2.4046918	4.7973251	4.6026749	0	3498	6996/6997	10.328
4/5	2.0641447	2.5358553	4.9745592	4.8254408	0	3186	6372/6373	9.266
13/15	2.0668817	2.6664516	5.1519674	5.0480326	0	2909	5818/5819	8.484
14/15	2.0697226	2.7969441	5.3295248	5.2704752	0	2673	5346/5347	7.875
1	2.0723025	2.9276975	5.5071972	5.4928028	0	2475	4950/4951	7.531
16/15	2.0743332	3.0590001	5.6849427	5.7150573	0	2309	4618/4619	6.844
17/15	2.0755860	3.1910807	5.8627124	5.9372876	0	2169	4338/4339	6.578
6/5	2.0758771	3.3241229	6.0404507	6.1595493	0	2048	4096/4097	6.578
19/15	2.0750606	3.4582727	6.2180975	6.3819025	0	1943	3886/3887	5.875
4/3	2.0730165	3.5936502	6.3955863	6.6044137	0	1848	3696/3697	5.516
7/5	2.0696484	3.7303516	6.5728470	6.8271530	0	1760	3520/3521	5.234
22/15	2.0648773	3.8684560	6.7498056	7.0501944	0	1677	3354/3355	5
23/15	2.0586367	4.0080300	6.9263838	7.2736162	0	1595	3190/3191	4.766
8/5	2.0508703	4.1491297	7.1025000	7.4975000	0	1509	3018/3019	4.625
5/3	2.0415295	4.2918039	7.2780691	7.7219309	0	1410	2820/2821	4.328
26/15	2.0305705	4.4360962	7.4530019	7.9469981	0	1271	2542/2543	3.766
9/5	2.0178891	4.5821109	7.6271046	8.1728954	0	1540	3081/3082	4.828
28/15	2.0038977	4.7294357	7.8008529	8.3991471	0	2132	4264/4265	6.375
29/15	1.9878496	4.8788171	7.9733178	8.6266822	0	2367	4734/4735	7.078
2	1.9700377	5.0299623	8.1447570	8.8552430	0	2486	4972/4973	7.438

Table 8.4: Numerical results and analysis of the convergence of the generalized Marcotte's method

t_r	$H_1(t_r)$	$H_2(t_r)$	$H_3(t_r)$	CPUtime
0	1.7692257	1.0769299	0.1538443	0.4210
1/15	1.8737438	1.1805298	0.2790597	0.2350
2/15	1.9858712	1.2865091	0.3942864	0.2190
1/5	2.1052578	1.3947422	0.5	0.2190
4/15	2.2315727	1.5051124	0.5966482	0.2030
1/3	2.3645034	1.6175103	0.6846530	0.1880
2/5	2.5037551	1.7318330	0.7644119	0.1880
7/15	2.6490475	1.8479856	0.8363002	0.1710
8/15	2.8001179	1.9658772	0.9006715	0.1720
3/5	2.9567160	2.0854239	0.95786	0.1720
2/3	3.1186055	2.2065464	1.0081814	0.1560
11/15	3.2855635	2.3291697	1.0519335	0.1560
4/5	3.4573773	2.4532243	1.0893985	0.1560
13/15	3.6338455	2.5786444	1.1208434	0.1560
14/15	3.8147791	2.7053674	1.1465202	0.1410
1	3.9999967	2.8333350	1.1666683	0.1570
16/15	4.1893269	2.9624921	1.1815144	0.1560
17/15	4.3826068	3.0927866	1.1912732	0.1560
6/5	4.5796812	3.2241697	1.1961491	0.1560
19/15	4.7804043	3.3565943	1.1963347	0.1570
4/3	4.9846357	3.4900168	1.1920143	0.1410
7/5	5.1922423	3.6243955	1.1833622	0.1410
22/15	5.4030976	3.7596912	1.1705446	0.1400
23/15	5.6170810	3.8958665	1.1537192	0.1560
8/5	5.8340776	4.0328860	1.1330364	0.1560
5/3	6.0539780	4.1707163	1.1086390	0.1570
26/15	6.2766778	4.3093255	1.0806633	0.1560
9/5	6.5020774	4.4486834	1.0492392	0.1400
28/15	6.7300817	4.5887612	1.0144904	0.1400
29/15	6.9606	4.7295316	0.9765350	0.1410
2	7.1935454	4.8709687	0.9354859	0.1410

Table 8.5: Numerical results and analysis of the convergence of the generalized Solodov-Tseng's method

t_r	$H_1(t_r)$	$H_2(t_r)$	$H_3(t_r)$	$H_4(t_r)$	CPUtime
0	1	3	1.0247146	1.9752854	23.1880
1/15	1.1333333	3.0666667	1.0666674	2.0666660	10.0000
2/15	1.2666667	3.1333333	1.1333341	2.1333327	6.1560
1/5	1.4	3.2	1.2	2.1999994	4.8130
4/15	1.5333333	3.2666667	1.2666674	2.2666660	4.2960
1/3	1.6666667	3.3333333	1.3333341	2.3333327	3.8910
2/5	1.8	3.4	1.4	2.3999994	3.3280
7/15	1.9333333	3.4666667	1.4666674	2.4666661	2.6560
8/15	2.0666667	3.5333333	1.5333341	2.5333327	2.3910
3/5	2.2	3.5999993	1.6	2.5999996	2.1090
2/3	2.3333333	3.6666658	1.6666667	2.6666667	1.8130
11/15	2.4666667	3.7333324	1.7333333	2.7333333	1.5000
4/5	2.6	3.7999993	1.7999999	2.8	1.1410
13/15	2.7333333	3.8666657	1.8666667	2.8666667	0.9530
14/15	2.8666667	3.9333324	1.9333333	2.9333333	0.6250
1	3	3.9999990	2	3	0.4680
16/15	3.1333333	4.0666667	2.0666674	3.0666660	3.8600
17/15	3.2666667	4.1333333	2.1333341	3.1333327	4.1880
6/5	3.4	4.2	2.2	3.1999994	3.7810
19/15	3.5333333	4.2666667	2.2666674	3.2666660	3.7190
4/3	3.6666667	4.3333333	2.3333341	3.3333327	3.5160
7/5	3.8	4.4	2.4	3.3999993	3.1250
22/15	3.9333333	4.4666667	2.4666674	3.4666660	2.7970
23/15	4.0666667	4.5333333	2.5333341	3.5333327	2.4840
8/5	4.2	4.6	2.6	3.5999994	2.1720
5/3	4.3333333	4.6666667	2.6666674	3.6666660	1.9060
26/15	4.4666667	4.7333333	2.7333341	3.7333327	1.6100
9/5	4.6	4.8	2.8	3.7999994	1.2810
28/15	4.7333333	4.8666665	2.8666674	3.8666661	0.9840
29/15	4.8666667	4.9333324	2.9333333	3.9333333	0.5630
2	5	4.9999990	3	4	0.3430

Table 8.6: Numerical results and analysis of the convergence of the generalized Solodov-Svaiter's method

t_r	$H_1(t_r)$	$H_2(t_r)$	$H_3(t_r)$	CPUtime
0	0.3333333	0.3333333	1.3333333	2.8184
1/15	0.4666666	0.5333333	1.4666667	2.8185
2/15	0.6	0.7333333	1.6	2.8185
1/5	0.7333333	0.9333333	1.7333333	2.8185
4/15	0.8666666	1.1333333	1.8666667	2.8185
1/3	1	1.3333333	2	2.8186
2/5	1.1333333	1.5333333	2.1333333	2.8186
7/15	1.2666667	1.7333333	2.2666667	2.8186
8/15	1.4	1.9333333	2.4	2.8186
3/5	1.5333333	2.1333333	2.5333333	2.8186
2/3	1.6666667	2.3333333	2.6666667	2.8186
11/15	1.9333333	2.4666667	2.7333333	2.8186
4/5	2.2	2.6	2.8	2.8186
13/15	2.4666667	2.7333333	2.8666667	2.8187
14/15	2.7333333	2.8666667	2.9333333	2.8187
1	3	3	3	2.8187
16/15	3.2666667	3.1333333	3.0666667	2.8187
17/15	3.5333333	3.2666667	3.1333333	2.8187
6/5	3.8	3.4	3.2	2.8187
19/15	4.0666667	3.5333333	3.2666667	2.8187
4/3	4.3333333	3.6666667	3.3333333	2.8187
7/5	4.6	3.8	3.4	2.8188
22/15	4.8666667	3.9333333	3.4666667	2.8188
23/15	5.1333333	4.0666667	3.5333333	2.8188
8/5	5.4	4.2	3.6	2.8188
5/3	5.6666667	4.3333333	3.6666667	2.8188
26/15	5.9333333	4.4666667	3.7333333	2.8188
9/5	6.2	4.6	3.8	2.8188
28/15	6.4666667	4.7333333	3.8666667	2.8188
29/15	6.7333333	4.8666667	3.9333333	2.8188
2	7	5	4	2.8188

Table 8.7: Numerical results and analysis of the convergence of the generalized descent method: first version

t_3	$H_1(t_3)$	$H_2(t_3)$	$H_3(t_3)$	CPUtime
0	0	0	1	0.3290
1/15	0.0088888	0.0755555	1.0711111	0.0010
2/15	0.9338026	0.8353209	0.4135679	0.6090
1/5	1.0181424	0.9709288	0.3890712	0.2340
4/15	1.1057467	1.1226822	0.3395400	0.1250
1/3	1.1997150	1.2890314	0.2665241	0.3440
2/5	1.3016370	1.4691815	0.1708185	0.2970
7/15	1.4124571	1.6626604	0.0528952	0.2500
8/15	1.5327904	1.8691603	0.0869381	0.1250
3/5	1.6630644	2.0884678	0.2484677	0.3280
2/3	1.8035903	2.3204271	0.4315381	0.2190
11/15	1.8563488	2.4666667	0.8325400	0.2350
4/5	1.8944272	2.6	1.3055728	0.3910
13/15	1.9309493	2.7333333	1.8246062	0.3440
14/15	1.9660918	2.8666667	2.3894638	0.2650
1	2	3	3	0.0010
16/15	2.0327956	3.1333333	3.6560933	0.4060
17/15	2.0645813	3.2666667	4.3576409	1.3280
6/5	2.0954451	3.4	5.1045549	1.5310
19/15	2.1254629	3.5333333	5.8967594	1.6090
4/3	2.1547005	3.6666667	6.7341884	1.5780
7/5	2.1832160	3.8	7.6167840	5.1100
22/15	2.2110601	3.9333333	8.5444954	5.5790
23/15	2.2382784	4.0666667	9.5172772	5.6250
8/5	2.2909944	4.2	10.535089	5.4530
5/3	5.6666667	4.3333333	11.597894	5.2180
26/15	2.3165612	4.4666667	12.705661	4.9840
9/5	2.48	4.6	13.72	14.6250
28/15	2.9688889	4.7333333	14.453333	9.8900
29/15	3.4755556	4.8666667	15.213333	7.2190
2	4	5	1.6	5.7500

Table 8.8: Numerical results and analysis of the convergence of the generalized descent method: second version

t_r	$H_1(t_r)$	$H_2(t_r)$	$H_3(t_r)$	$H_4(t_r)$	$H_5(t_r)$	$H_6(t_r)$	CPUtime
0	1	1	1	1	0	0	0.1100
1/15	1.2	1.2	1.0666674	1.0666667	0.0666666	0.1333333	0.0160
2/15	1.4	1.4	1.1333333	1.1333333	0.1333333	0.2666666	0.0150
1/5	1.6	1.6	1.2	1.2	2	4	0.0160
4/15	1.45	2.15	1.4166667	1.15	1.4166667	0.1166666	0.1400
1/3	1.75	2.25	1.5833333	1.25	1.5833333	0.0833333	0.0310
2/5	2.05	2.35	1.75	1.35	1.75	0.05	0.0160
7/15	2.35	2.45	1.9166667	1.45	1.9166667	0.0166666	0.6250
8/15	2.6666667	2.5333333	2.0666667	1.5333333	2.0666667	0.0666666	1.9530
3/5	3	2.6	2.2	1.6	2.2	0.2	3.4060
2/3	3.3333333	2.6666667	2.3333333	1.6666667	2.3333333	0.3333333	3.6560
11/15	3.6666667	2.7333333	2.4666667	1.7333333	2.4666667	4.6666667	3.7030
4/5	4	2.8	2.6	1.8	2.6	0.6	5.1720
13/15	4.3333333	2.8666667	2.7333333	1.8666667	2.7333333	0.7333333	7.6250
14/15	4.6666667	2.9333333	2.8666667	1.9333333	2.8666667	0.8666666	6.5940
1	5	3	3	2	3	1	6.2810
16/15	5.3333333	3.0666667	3.1333333	2.0666667	3.1333333	1.1333333	6.0470
17/15	5.6666667	3.1333333	3.2666667	2.1333333	3.2666667	1.2666667	10.9840
6/5	6	3.2	3.4	2.2	3.4	1.4	10.3290
19/15	6.3333333	3.2666667	3.5333333	2.2666667	3.5333333	1.5333333	10.2030
4/3	6.6666667	3.3333333	3.6666667	2.3333333	3.6666667	1.6666667	10.1230
7/5	7	3.4	3.8	2.4	3.8	1.8	9.1870
22/15	7.3333333	3.4666667	3.9333333	2.4666667	3.9333333	1.9333333	8.6720
23/15	7.6666667	3.5333333	4.0666667	2.5333333	4.0666667	2.0666667	8.4530
8/5	8	3.6	4.2	2.6020369	4.2	2.1979631	7.6560
5/3	8.3333333	3.6666667	4.3333333	2.7131884	4.3333333	2.2868116	161.5320
26/15	8.6666667	3.7333333	4.4666667	2.8320098	4.4666667	2.3679902	248.5150
9/5	9	3.8	4.6	2.9503770	4.6	2.4496230	292.1880
28/15	9.3333333	3.8666667	4.7333333	3.0683231	4.7333333	2.5316769	311.1400
29/15	9.6666667	3.9333333	4.8666667	3.1858802	4.8666667	2.6141198	329.7040
2	1	4	5	3.3030739	5	2.6969261	344.1400

Conclusions

The results presented give a theoretical justification for introducing methods to solve evolutionary variational inequalities which express dynamic equilibrium problems. We have fixed our attention to consider evolutionary variational and quasi-variational inequalities and to study under which assumptions the continuity of solutions can be ensured. In order to archive our analytic results, the set convergence in Mosco's sense plays a central role. Then, we apply continuity results to dynamic equilibrium problems, having proved that they satisfy all assumptions of general results. An important step is to show that the set of constraints of dynamic equilibrium problems satisfy the conditions of the set convergence in Mosco's sense. The continuity of solutions to dynamic equilibrium problems allows us to introduce methods for the calculation of equilibrium solutions. Until up now, very few methods are been given (see for instance the sub-gradient method presented in [37]), so our result seems to have a particular relevance.

After introducing the theory of evolutionary variational inequalities and dynamic equilibrium problems, we have focused on the connection between evolutionary variational inequalities and dynamic equilibrium problems. In the last years, this fact has given more impulse to the study of existence, uniqueness, stability for evolutionary variational inequalities which express dynamic equilibrium problems and to introduce methods for the computational of equilibrium solutions. All this behaves us to provide methods for the calculation of equilibrium solutions. To this aim, it is important to get notices about the continuity of solutions to dynamic equilibrium problems. In the detail, we have proved that the property of set convergence in Mosco's sense with assumptions of continuity of data provide the continuity of the unique solution to a general strongly monotone evolutionary variational inequality is continuous. This result is been generalized to degenerate and strictly monotone evolutionary inequalities. We have obtained analogous results for evolutionary quasi-variational inequalities. Having proved that sets of constraints of dynamic equilibrium problems, in the common formulation, satisfy the property of set convergence in Mosco's sense, we have applied the continuity results to equilibrium problems. In particular, the continuity allows us to reduce the computational procedure to finite-dimensional problems by means of a partition of the time interval and to use a method to solve static equilibrium problems then, by means of a interpolation procedure, we are able to find the dynamic equilibrium solutions. We have studied the complexity of algorithms and the convergence of the scheme in $L^1([0,T],\mathbb{R}^m)$. It still remains to investigate the possibility to apply the continuity of solutions to evolutionary quasi-variational inequalities in order to compute solutions to the dynamic elastic traffic equilibrium problem, which is expressed by a quasi-variational inequality.

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