Consequences of Schanuel's Conjecture in Exponential Algebra

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Preface

Model Theory is a branch of mathematical logic in which one studies mathematical structures by considering the true first order sentences. The fields of real and complex numbers have been long served as motivating examples for model theorists. The logical study of the field of real numbers began with the work of Tarski (see [18]). He proved that the theory of the reals is decidable.

The study of exponential rings (E-rings) starts with a problem left open by Tarski in the 30's about the decidability of the reals with exponentiation [19]. Only in the mid 90's Macintyre and Wilkie in [8] gave a positive answer to this question modulo a conjecture due to Schanuel (1960) concerning Transcendental Number Theory:

Schanuel's Conjecture (SC) Let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$. Then $\mathbb{Q}(\lambda_1, \ldots, \lambda_n, e^{\lambda_1}, \ldots, e^{\lambda_n})$ has transcendence degree (t.d.) at least n over $\mathbb{Q}$.

In this thesis we will examine some consequences of (SC) both in the well known exponential fields as the reals and the complexes, and in a more abstract context of the pseudo exponential fields introduced by Zilber [28].

However, independent of the above mentioned problem, the class of exponential rings and fields is a very interesting and fascinating subject of investigation, and it has been enriched by ideas from analytic geometry (see [28]) and differential topology. Indeed, in the last twenty five years, several people as A. Macintyre ([4], [5], [6]), L. Van den Dries ([22]), A. Wilkie ([23]), H. Wolter ([24], [25]), have been concerned with exponential rings and fields and obtained interesting results involving the real field and the complex field.

The first Chapter contains some introductory material and preliminary results on the Theory of Exponential Algebra.

Macintyre (1991) used Schanuel's Conjecture to prove that the exponential subring of the reals generated by 1 is free on no generators. This result implies that there are no hidden iterated exponential identities for exponential constants (modulo SC), that is there are no unexpected exponential algebraic relations on the integers $\mathbb{Z}$.

Following this line of research, in Chapter 2 we consider the exponential field $(\mathbb{C}, e^x)$ where the following well known identities hold

$$e^{\pi i} = -1 \text{ and } i^2 = -1.$$

A natural question is:

Are these the only identities in $(\mathbb{C}, e^x)$ involving π and i?

We show that, modulo Schanuel's Conjecture these are the only relations. This is obtained by characterizing the kernel of the E-morphism from the free E-ring on two generators x, y mapping $x \mapsto \pi$ and $y \mapsto i$.

We also show that such a kernel is not a principal E-ideal, where an E-ideal is an ideal I such that if $\alpha \in I$ then $E(\alpha) - 1 \in I$. This was the starting point to begin the study of E-ideals in the free E-ring on n generators as kernels of certain E-morphisms (see Section 2.6).

Assuming Schanuel's Conjecture we obtained also some information on the algebraic relations among elements of $(\mathbb{R}, e^x)$. We prove that the E-subring of $\mathbb{R}$ generated by π is isomorphic to the free E-ring on π (modulo (SC)).

In the third Chapter we study consequences of Schanuel's Conjecture concerning decidability issues, proving that, modulo (SC), there are algorithms which decide if two exponential polynomials in π are equal in $\mathbb{R}$ and if two exponential polynomials in π and i coincide in $\mathbb{C}$.

A connection between exponential fields and algebraic geometry comes from the construction of new structures due to Zilber (see [28]). He constructed the new structures getting inspiration by the complex exponential field and the new approach introduced by Hrushovski (1993) (see [14]) in order to construct strongly minimal sets. A Zilber's field is an algebraically closed field of characteristic 0 with an exponentiation defined on it with periods of a certain form. Moreover, it satisfies Schanuel's Conjecture and two other axioms, the strong exponential closure and the countable closure, concerning solutions of certain systems of exponential polynomials. His axioms provided a fascinating, novel approach to the complex exponential field. Zilber proved that the class of exponentially-algebraically closed structures with the countable closure property has a unique model in every uncountable cardinality. He conjectured that the complex exponential field is the unique model of cardinality $2^{\aleph_0}$. Until now no attempt to disprove Zilber's conjecture has succeeded.

In this context there are some recent results due to Marker [9] concerning the solvability of a simple exponential polynomial, Marker's results imply an instance of the strong exponential closure for $(\mathbb{C}, e^x)$, supporting in this way Zilber's conjecture.

In Chapter 4 we analyze a more complicated exponential polynomial, and under suitable hypothesis, we obtain an analogous result to Marker's.

On the opposite side it is very interesting to characterize when an exponential polynomial has no solutions. For the complex exponential field such characterization exists and it is due to Henson and Rubel in [12]:

Let $F(z_1, \ldots, z_n) \in \mathbb{C}[z_1, \ldots, z_n]^E$

$$F(z_1, \ldots, z_n) \text{ has no roots in } \mathbb{C} \text{ iff } F(z_1, \ldots, z_n) = e^{G(z_1, \ldots, z_n)},$$

where $G(z_1, \ldots, z_n) \in \mathbb{C}[z_1, \ldots, z_n]^E$.

The proof uses Nevanlinna Theory and, moreover, the authors claim that it is not possible to use a direct algebraic approach substituting Nevanlinna Theory. As we will see, in the last Chapter, we prove Henson and Rubel result we extend Henson and Rubel's result to Zilber's fields using purely algebraic methods. So, if Zilber's Conjecture is true, we find an alternative proof of Henson and Rubel's result with no use of Nevanlinna Theory.

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Chapter 1

E-rings and E-fields

1.1 Introduction

The goal of this chapter is to illustrate our framework. We will work with exponential rings, exponential polynomial rings, free exponential rings and exponential fields. In this chapter we introduce definitions and properties relative of these objects. We also introduce the notion of exponential ideal (E-ideal) which will be crucial in our main results.

1.1.1 E-rings and E-fields

Definition 1.1.1. *An* exponential ring, or E-ring*, is a pair* (R, E) *with* R *a commutative ring with 1 and* $E : R \to \mathcal{U}(R)$ *a map of the additive group of* R *into the multiplicative group of units of* R *satisfying:*

1. $E(x + y) = E(x) \cdot E(y)$ *for all* $x, y \in R$,

2. $E(0) = 1$.

Definition 1.1.2. *An* E-field *is a pair* (K, E) *where* K *is a field.*

Examples 1.1.3. *1.* $(\mathbb{R}, a^x)$, *with* $a > 0$, *and* $(\mathbb{C}, e^x)$.

2. *Any ring* R *can be expanded to an exponential ring by the trivial exponentiation,* $E(x) = 1$ *for all* x *in* R.

3. *If the ring* R *has no nilpotent elements different from 0 and has prime characteristic* $p > 0$ *then the only exponentiation definable over* R *is the trivial one (see [22]).*

4. $(S[t], E)$, *where* S *is an* E*-field of characteristic 0 and* $S[t]$ *is the ring of formal power series in* t *over* S. *Let* $f \in S[t]$, *where* $f = r + f_1$, *and* $r \in S$,

$$E(f) = E(r) \cdot \sum_{n=0}^{\infty} (f_1)^n / n!$$

5. $\mathbb{Z}_p$ *the ring of* p*-adic integers with* $(1+p)^x$ *if* $p > 2$, *and* 5^x *if* $p = 2$.

6. *On the ring* $\mathbb{Z}$ *we can define only two* E*-morphisms, the trivial one and the morphism:*

$$E(x) = \begin{cases} 1 & \text{if } x \text{ is even} \\ -1 & \text{if } x \text{ is odd.} \end{cases}$$

1.1.2 Group algebra

Starting with an E-ring we will construct exponential polynomials (E-polynomials), and we will equip such a set with an E-ring structure.

The construction is based on the concept of group ring. We start by recalling the definition of group ring and group algebra.

Definition 1.1.4. *Let G be a multiplicative group finite or infinite, and R a ring. The group ring of G over R is the set of all linear combinations of finitely many elements of G with coefficients in R,*

$$\sum_{g\in G} r_g g,$$

where $r_g = 0$ for all but finitely many elements of G, and the ring operations are defined as follows:

$$(\sum_{i=1}^{n} r_{g_i} g_i) + (\sum_{i=1}^{n} s_{g_i} g_i) = \sum_{i=1}^{n} (r_{g_i} + s_{g_i}) g_i$$

$$(\sum_{i=1}^{n} r_i g_i) \cdot (\sum_{j=1}^{m} s_j k_j) = \sum_{i=1}^{n} \sum_{j=1}^{m} (r_i s_j)(g_i k_j).$$

Remark 1.1.5.

1. *$R[G]$ is commutative if and only if both R and G are commutative.*

2. *If R has an identity 1_R, and e is the identity element of G, then $1_R e$ is the identity element of $R[G]$.*

3. *R is always a subring of $R[G]$,*

$$R \hookrightarrow R[G],$$

whereas, $R[G]$ contains a copy of G,

$$G \hookrightarrow R[G]$$

if and only if R is a ring with unity.

4. *On the group ring $R[G]$ (we define) the following R-module structure is defined*

$$r(\sum s_i g_i) = \sum (r s_i) g_i \ (r, s_i \in R, g_i \in G).$$

It is easy to show that $R[G]$ is a R-algebra which is called the group algebra of G over R.

We recall the following classical result on group rings:

Proposition 1.1.6. *Let R be an integral domain of characteristic 0 and G a group. Then the group ring $R[G]$ is an integral domain of characteristic 0 if and only if G is torsion free.*

It is possible to characterize the units of a group ring, indeed we have the following result:

Corollary 1.1.7. *If $R[G]$ is an integral domain of characteristic 0, then the units of $R[G]$ are of the form:*

$$rg \in \mathcal{U}(R[G]), \ \textit{where } r \in \mathcal{U}(R) \textit{ and } g \in G.$$

1.1.3 Construction of E-polynomial ring

In this section we review the construction of the E-polynomial ring as in [22].

Let (R, E) be an E-ring. We denote the ring of exponential polynomials over R and in the indeterminates $X_1, \ldots, X_m$ by $R[X_1, \ldots, X_m]^E$. The construction of $R[X_1, \ldots, X_m]^E$ is by recursion. The exponential ring of E-polynomials can be viewed as a group ring over the ordinary polynomial ring $R[X_1, \ldots, X_m]$, with the E-morphism an extension of the exponential function over R. At each step we construct a group ring over the ring obtained at the previous step. Also the exponential function is defined by steps. All together we will construct three sequences $(R_k)_{k\in\mathbb{N}}$, $(A_k)_{k\in\mathbb{N}}$ and $(E_k)_{k\in\mathbb{N}}$, where R_k's are group rings, A_k's are abelian groups, and E_k's are partial E-morphisms from R_k into the units of R_{k+1}.

Initial step: for $k=-1$ and $k=0$ define

$$R_{-1}=R, R_0=R[X_1,\dots,X_m] \text{ and } A_0= \text{ the ideal generated by } X_1,\dots,X_m.$$

So, as additive group R_0 is:

$$R_0=R\oplus A_0.$$

The morphism E_{-1} from R_{-1} into R_0 is the composition of the initial E-morphism over R with the embedding of the ring R into $R[X_1,\dots,X_m]$, i.e.:

$$E_{-1}:R_{-1}=R\xrightarrow{E}R\xhookrightarrow{i}R[X_1,\dots,X_m]=R_0.$$

Inductive step: suppose $k\geq 0$ and R_{k-1}, R_k, A_k and E_{k-1} have been defined in accordance with the description above: so $R_k=R_{k-1}\oplus A_k$, and E_{k-1} is a morphism from the additive group R_{k-1}, previously constructed, into the multiplicative group of units of R_k. For the inductive step it is convenient to consider a multiplicative copy of A_k which we denote by t^{A_k}, where t is a formal isomorphism

$$t:A_k\rightarrow t^{A_k}.$$

The ring R_{k+1} is construed as a group ring over the ring R_k and the group t^{A_k}, i. e.

$$R_{k+1}=R_k[t^{A_k}].$$

As additive group R_{k+1} is

$$R_{k+1}=R_k\oplus A_{k+1},$$

where A_{k+1} is the R_k-submodule of R_{k+1} freely generated by t^a, with $a\in A_k$ and $a\neq 0$ (this last condition ensures that A_{k+1} does not coincide with R_{k+1}). The definition of R_{k+1} as a group ring is convenient in order to see explicitly the exponential iterations of exponentiation, while the interpretation of R_{k+1} as a direct sum is used in order to define the morphism E_k at $k+1$ step. Now we extend the morphism E_{k-1} to the morphism E_k.

Let $r \in R_k$, we can write $r = r' + a$, with $r' \in R_{k-1}$, $a \in A_k$, and this decomposition of r is unique.

We define

$$E_k : R_k \to \mathcal{U}(R_{k+1}) \text{ by } E_k(r) = E_{k-1}(r') \cdot t^a.$$

The ring $R[X_1, \ldots, X_m]^E$ is the limit of R_k's, i.e. $R[X_1, \ldots, X_m]^E = \bigcup_k R_k$, and its exponential map is defined in the natural way as $E(x) = E_k(x)$ if $x \in R_k$. This end the construction.

Notice that at each step R_{k+1} as additive group is the direct sum $R \oplus A_0 \oplus A_1 \oplus \ldots \oplus A_{k+1}$. Moreover, as group ring $R_{k+1} = R_k[t^{A_k}] = R_{k-1}[t^{A_{k-1}}][t^{A_k}] \cong R_{k-1}[t^{A_{k-1} \oplus A_k}]$. More generally, the following is true

$$R_{k+1} \cong R_0[t^{A_0 \oplus A_1 \oplus \ldots \oplus A_k}].$$

We can then view the additive group of $R[X_1, \ldots, X_m]^E$ as $R \oplus A_0 \oplus A_1 \oplus \ldots \oplus A_k \oplus \ldots$, and as group ring $R[X_1, \ldots, X_m]^E$ is $R[X_1, \ldots, X_m][t^{A_0 \oplus A_1 \oplus \ldots \oplus A_k \oplus \ldots}]$

Remark 1.1.8. *The isomorphism $t : A_k \to t^{A_k}$ is the restriction of the exponential map E to A_k, and we will use either notation t^{A_k} or $E(A_k)$, freely.*

In the classical case there are properties which are preserved from a ring to the polynomial ring. Also in the case of E-polynomial rings some properties are preserved, for example being an integral domain.

Proposition 1.1.9. *If R is an integral domain of characteristic 0, then $R[X_1, \ldots, X_m]^E$ is an integral domain whose units are of the form $u \cdot E(p)$, where u is a unit of R and $p \in R[X_1, \ldots, X_m]^E$.*

Proof: By induction on $k \geq 0$ we show that R_k is an integral domain whose units are of the form $u \cdot E(p)$, where u is a unit of R and $p \in R_{k-1}$. For $k = 0$ it is clearly true. We assume the result true for $k > 0$, that is R_k is an integral domain of characteristic 0 whose units are as above. In these hypotheses A_k is a torsion free abelian group. Indeed, suppose by contradiction that there exists $a \in A_k$ such that:

$$n \cdot a = 0 \text{ for some } n > 0,$$

i.e.

$$(\underbrace{1 + 1 + \ldots + 1}_{n}) \cdot a = 0.$$

This gives immediately a contradiction since R_k is an integral domain of characteristic 0. This implies that t^{A_k} is also torsion free.

From $R_{k+1} = R_k[t^{A_k}]$ (i.e. R_{k+1} is a group ring over an integral domain), it follows that R_{k+1} is an integral domain of characteristic 0 (see Proposition 1.1.6).

Now we prove that $\mathcal{U}(R_{k+1}) = \{\alpha \cdot E(\beta) : \alpha \in \mathcal{U}(R_k), \beta \in A_k\}$. Clearly, if $\alpha \in \mathcal{U}(R_k)$ and $\beta \in A_k$ we have $\alpha \in \mathcal{U}(R_{k+1})$ and $E(\beta) \in \mathcal{U}(A_k) \subseteq \mathcal{U}(R_{k+1})$, and so $\alpha \cdot E(\beta) \in \mathcal{U}(R_{k+1})$. Hence $\{\alpha \cdot E(\beta) : \alpha \in \mathcal{U}(R_k), \beta \in A_k\} \subseteq \mathcal{U}(R_{k+1})$.

For the other inclusion, let $r \in \mathcal{U}(R_{k+1})$ and $r^{'} \in \mathcal{U}(R_{k+1})$ such that $r \cdot r^{'} = 1$. Then

$$r = \alpha_1 \cdot E(\beta_1) + \cdots + \alpha_n \cdot E(\beta_n), \text{ where } \alpha_i \in R_k, \beta_i \in A_k$$

and

$$r^{'} = \gamma_1 \cdot E(\delta_1) + \cdots + \gamma_h \cdot E(\gamma_h), \text{ where } \gamma_j \in R_k, \delta_j \in A_k.$$

From A_k being torsion free it follows that A_k can be made into an ordered group with order $<$. We can assume that

$$\beta_1 < \cdots < \beta_n$$

and

$$\delta_1 < \cdots < \delta_h.$$

Now we write

$$1 = [\alpha_1 \cdot E(\beta_1) + \cdots + \alpha_n \cdot E(\beta_n)] \cdot [\gamma_1 \cdot E(\delta_1) + \cdots + \gamma_h \cdot E(\delta_h)] = \sum_{i=1}^{n} \sum_{j=1}^{h} \alpha_i \cdot \gamma_j \cdot E(\beta_i + \delta_j).$$

We have, $\beta_1 + \delta_1 < \beta_n + \delta_h$, so necessarily $\beta_i + \delta_j = 0$ for $i = 1, \dots, n$, and $j = 1, \dots, h$. That is $\beta_1 = \beta_2 = \cdots = \beta_n$ and $\delta_1 = \delta_2 = \cdots = \delta_h$. So, without loss of generality we can assume $r = \alpha \cdot E(\beta)$ and $r' = \gamma \cdot E(\delta)$, where $\alpha, \gamma \in \mathcal{U}(R_k)$. By inductive hypothesis $\alpha = \alpha' \cdot E(\beta')$, where $\alpha' \in \mathcal{U}(R_{k-1})$. Repeating the procedure we write $r = u \cdot E(p)$, where $u \in \mathcal{U}(R)$ and $p \in R_k$, and the proof is completed.

□

In next chapter we will see that there are properties of the stating exponential ring which are not preserved in the exponential polynomial E-ring.

1.1.4 Degree

As in the classical case of polynomial rings, it is possible to associate a degree to any exponential polynomial in the following way:

Definition 1.1.10. *Let $p(X)$[1] $\in R[X]^E$. We define the height of $p(X)$ as:*

$$height(p) = \begin{cases} k & \text{if } p \in R_k \setminus R_{k-1}, k > 0 \\ 0 & \text{if } p \in R_0 = R[X] \end{cases} \tag{1.1}$$

Intuitively the height is the maximum number of iterated exponentiations in $p(X)$.

We focus our attention on $p(X) \in A_k$, $k > 0$. Then $p(X)$ is uniquely represented as

$$p(X) = \sum_{i=1}^{h} r_i \cdot E(a_i),$$

[1] where X stands for a tuple $X_1 \dots X_m$ of variables

where $a_i \in A_{k-1} \setminus \{0\}$ and $a_i \neq a_j$ for $i \neq j$, and $r_1, \ldots, r_h$ are non-zero elements of R_{k-1}. So, we obtain a normal form for the exponential polynomials. We define $t(p) = h$, i.e. t counts the number of summands with the same number of iterated exponentiations in p.

In case p is an ordinary polynomial, i.e. $p \in R_0$ we define:

$$t(p) = \begin{cases} 0 & \text{if } p = 0 \\ d+1 & \text{if } deg(p) = d \geq 0 \end{cases}$$

Now we are in a position to associate a degree to any exponential polynomial, and it will be an ordinal below ω^ω.

Let $p \in R[X]^E$, so $p \in R_k$ for some k. Recall that $R_k = R_0 \oplus A_1 \oplus \cdots \oplus A_k, k \geq 0$, hence any polynomial $p(X)$ in R_k has height $\leq k$, and can be written uniquely as:

$$p = p_0 + p_1 + \cdots + p_k, \text{ where } p_0 \in R_0, p_i \in A_i \text{ for } i > 0.$$

We define

$$ord(p) = \omega^k \cdot t(p_k) + \cdots + \omega \cdot t(p_1) + t(p_0),$$

where k is the maximum number of iterated exponentiations which appear in p (recall that $p \in R_k$). Note that $ord(p) = 0$ iff $p = 0$. We call p_0 the polynomial part of p. Since $ord(p)$ is an ordinal it allows proofs by induction on the degree.

Remark 1.1.11. *The degree can be considered as a map:*

$$ord : R[X]^E = \cup_k R_k \longrightarrow \omega^\omega.$$

This map is defined stepwise by recursion on k.

1.2 Solutions of exponential polynomials

The next natural step is to study solutions of exponential polynomials. In this respect there are differences between the behavior of classical polynomials and exponential polynomials. If $p(X)$ is an ordinary polynomial in one variable then $p(X)$ has only finitely many solutions in $\mathbb{C}$, while if we consider $p(X) = E(X) - 1$, we have infinitely many solutions given by $2ni\pi$, $n \in \mathbb{Z}$.

In order to study solutions of an exponential polynomial it is useful to associate to each exponential polynomial its corresponding function as follows. First all, let (R^{R^n}, E) be the E-ring of n-ary functions over R. It is very easy to show that (R^{R^n}, E) is an E-ring, where the operations are defined pointwise.

Consider the E-ring morphism

$$\hat{}: R[X_1, \ldots, X_n]^E \longrightarrow (R^{R^n}, E)$$

$$p \longmapsto \hat{p}.$$

The $\hat{p}$'s are called E-polynomial functions and we denote such an E-ring by $R[x_1, \ldots, x_m]^E$. In general the E-morphism $\hat{}$ may have a nontrivial kernel. Indeed, if we consider a ring R with the trivial exponentiation, that is $E(r) = 1$ for all $r \in R$, we have that the exponential polynomial $E(X_1) - 1$ is in the kernel of the morphism $\hat{}$.

Under suitable hypothesis this map is injective (see [22]). The following results holds:

Proposition 1.2.1. *Suppose the E-ring R is an integral domain of characteristic zero, and there are derivations $d_1, \ldots, d_m$ on a ring extension of $R[x_1, \ldots, x_m]^E$ which are trivial on R and satisfy $d_i(x_j) = \delta_{ij}$, $1 \le i, j \le m$, and $d_i(E(f)) = r \cdot d_i(f) \cdot E(f)$ for some $r \in R - \{0\}$ and for all f in $R[x_1, \ldots, x_m]^E$ and $i = 1, \ldots, m$. Then the map $\hat{}$ is an isomorphism.*

More generally the following result is true (see [22]):

Proposition 1.2.2. *Suppose R is an ordered E- field and its exponential map E satisfies $E(x) \geq 1 + rx$ for a fixed $r \in R$ and all $x \in R$. Then the map $p \longmapsto \hat{p}$ from $R[X_1, \ldots, X_n]^E$ into $R[x_1, \ldots, x_n]^E \subseteq (R^{R^n}, E)$ is an isomorphism.*

Using these results Henson, Macintyre, Van den Dries, and other people proved that for the exponential polynomial rings over $\mathbb{R}$ and over $\mathbb{C}$ the map $\hat{}$ is injective. In both cases the proof reduces to show that the kernel of $\hat{}$ is necessarily trivial since otherwise being closed under derivation it would coincide with $R[X_1, \ldots, X_n]^E$, where $R = \mathbb{C}, \mathbb{R}$.

Combining the degree associated to an exponential polynomial and derivations the following result follows from an observation of G.H. Hardy (see [11]).

Theorem 1.2.1. *Let R be an E-ring. If $p \in R[X]^E$, then there exists $q \in R[X]^E$ such that*

$$ord(\partial \frac{(E(q) \cdot p)}{\partial x}) < ord(p).$$

Remark 1.2.3. *Hardy used the above result in order to show that if p is an exponential polynomial with coefficients in $\mathbb{R}$ which is not identically null then p has only finitely many zeros. This result it is not obvious, since if we consider $p(X) \in \mathbb{C}[X]^E$ and $p(X) = E(X) - 1$, then $p(X)$ has infinitely many zeros given by $2ni\pi$, where $n \in \mathbb{Z}$.*

1.3 Free E-rings

The E-rings form an equational class relative to the language $\mathcal{L} = \{0, 1, +, \cdot, -, E\}$, hence free E-ring exist. For convenience we review the construction which is very similar to the construction of E-polynomial rings seen in Section 1.1.2.

Let $X_1, \ldots, X_m$ be distinct indeterminates, we also include the case $m = 0$, that is we allow also the free object on no generators.

Definition 1.3.1. *The free E-ring on $X_1, \ldots, X_m$, denoted by $[X_1, \ldots, X_m]^E$ is an E-ring containing $X_1, \ldots, X_m$ as elements, satisfying the universal property of "freeness", that is for each E-ring R and $r_1, \ldots, r_m \in R$ there is exactly one E-ring morphism*

$$f : [X_1, \ldots, X_m]^E \to R$$

such that $f(X_i) = r_i$ for $i = 1, \ldots, m$.

Notice that the universal property of freeness for the free E-ring on no generators $[\emptyset]^E$ is reduced to require for each E-ring R the existence of an E-morphism from $[\emptyset]^E$ to R (see [4]).

As in Section 1.1.2 also in this case we define three sequences $([X_1, \ldots, X_m]_k)_{k \in \mathbb{N}}$, $(B_k)_{k \in \mathbb{N}}$, $(E_k)_{k \in \mathbb{N}}$, where $[X_1, \ldots, X_m]_k$'s are group rings, B_k's are abelian groups and E_k's are partial E-morphisms.

The construction of $[X_1, \ldots, X_m]^E$ is by recursion. The difference with the construction of the ring of E-polynomials is only in the initial step of the recursion.

Initial step: $[X_1, \ldots, X_m]_{-1} = \{0\}$, $[X_1, \ldots, X_m]_0 = \mathbb{Z}[X_1, \ldots, X_m]$ as ring, $B_0 = \mathbb{Z}[X_1, \ldots, X_m]$ as additive group and $E_{-1}(0) = 1$.

Inductive step: Suppose $k \geq 0$ and $[X_1, \ldots, X_m]_{k-1}$, $[X_1, \ldots, X_m]_k$, B_k and E_{k-1} have been constructed in accordance with the description above, then the ring $[X_1, \ldots, X_m]_{k+1}$ is constructed as a group ring over $[X_1, \ldots, X_m]_k$, i.e

$$[X_1, \ldots, X_m]_{k+1} = [X_1, \ldots, X_m]_k[t^{B_k}],$$

where t is an isomorphism from the additive group B_k onto the multiplicative group t^{B_k}. As additive group $[X_1, \ldots, X_m]_{k+1}$ is

$$[X_1, \ldots, X_m]_{k+1} = [X_1, \ldots, X_m]_k \oplus B_{k+1}.$$

Now we extend the morphism E_{k-1} to the morphism E_k. Let $r \in [X_1, \ldots, X_m]_k$ so, we can write $r = r' + a$, with $r' \in [X_1, \ldots, X_m]_{k-1}$, $a \in B_k$, and this decomposition of r is unique and we define

$$E_k : [X_1, \ldots, X_m]_k \longrightarrow \mathcal{U}([X_1, \ldots, X_m]_{k+1})$$

by $E_k(r) = E_{k-1}(r') \cdot t^a$.

The free E-ring on $X_1, \ldots, X_m$ is defined as follows

$$[X_1, \ldots, X_m]^E = \lim_k [X_1, \ldots, X_m]_k = \bigcup_{k=0}^{\infty} [X_1, \ldots, X_m]_k \tag{1.2}$$

and its exponential morphism E is:

$$E(x) = E_k(x), \text{ if } x \in [X_1, \ldots, X_m]_k. \tag{1.3}$$

It is easy to prove that $[X_1, \ldots, X_m]^E$ satisfies the universal property of freeness:

if (R, E) is an E-ring and φ is a function from $\{X_1, \ldots, X_m\}$ to R, there is an unique E-morphism

$$f : [X_1, \ldots, X_m]^E \rightarrow (R, E)$$

defined as $f(1) = 1$ and $f(X_i) = \varphi(X_i), \text{ for } i = 1, \ldots, m.$

Notice that at each step of the construction $[X_1, \ldots, X_m]_{k+1}$ as additive group is the direct sum $B_0 \oplus B_1 \oplus \ldots \oplus B_{k+1}$. Moreover, as an additive group $[X_1, \ldots, X_m]^E$ can be considered as $B_0 \oplus B_1 \oplus \ldots \oplus B_{k+1} \oplus \ldots$.

Remark 1.3.2. *We can interpret the free object $[X_1, \ldots, X_m]^E$ as $\mathbb{Z}[X_1, \ldots, X_m]^E$ that is, an element in the free E-rings is an exponential polynomial with coefficients in $\mathbb{Z}$.*

Example 1.3.3. *An element of $[x, y]^E$, is for example a polynomial $P(x, y)$ where*

$$P(x, y) = -3x^2y - x^5y^7 + (2xy + 5y^2)e^{(-7x^3+11x^5y^4-3y^2)} + (6 - 2xy^5 + xy)e^{(5x+2x^7y^2)e^{5x-10y^2}}.$$

Notice that $P(x,y) \in R_2$. *We have used the more intuitive notation* $e^{q(x,y)}$ *instead of* $E(q(x,y))$.

Remark 1.3.4. *In the case* $m = 0$ *we have the free object on no generators, which we denote by* $[\emptyset]^E$. *It is obtained by considering* $R_0 = \mathbb{Z}$ *at the initial step (see [4]). An element in* $[\emptyset]^E$ *is an exponential constant, e.g. it is of the form*

$$e^{e^2+3} + 4 - 5e^{3+e^{-3}}.$$

1.3.1 Free object via terms of the language

In this section we consider a model theoretic approach to E-rings and we will introduce the free object via terms of the language of E-rings. Moreover, we will prove that such object is isomorphic to the free object previously constructed.

Let $\mathcal{L} = \{0, 1, +, -, \cdot, E\}$ be the language of E-rings and let ER be the $\mathcal{L}$-theory of rings with the further axioms $E(0) = 1$ and $E(x+y) = E(x) \cdot E(y)$, and $\mathcal{T}(X)$ be the set of terms in the variables $X = X_1, \dots, X_m$. Let $\tau, \mu \in \mathcal{T}(X)$, we define the following relation on $\mathcal{T}(X)$

$$\tau \equiv \mu \text{ if and only if } ER \vdash \forall X(\tau(X) = \mu(X)).$$

It is easy to show that $\equiv$ is an equivalence relation.

We can equip $\mathcal{T}(X)/_{\equiv}$ with an E-ring structure as follows

$$(\tau/_{\equiv}) \pm (\mu/_{\equiv}) = (\tau \pm \mu)/_{\equiv} \text{ and } E(\tau/_{\equiv}) = E(\tau)/_{\equiv}.$$

It is very easy to prove that $(\mathcal{T}(X)/_{\equiv}, E)$ is an E-ring.

Now, we show that $\mathcal{T}(X)/_{\equiv}$ coincides with the free object on X, that is it satisfies the universal property of freeness:

Let $f : X \hookrightarrow \mathcal{T}(X)/_{\equiv}$ be the natural immersion. For all E-rings R and for all $\theta : X \to R$, there exists a unique E-morphism $\psi : \mathcal{T}(X)/_{\equiv} \to R$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \overset{f}{\hookrightarrow} & \mathcal{T}(X)/_{\equiv} \\ & \theta \searrow \quad \swarrow \psi & \\ & R & \end{array} \tag{1.4}$$

Indeed, let $\theta : X \to R$, and consider the natural extension of θ to $\mathcal{T}(X)$, $\overline{\theta} : \mathcal{T}(X) \to R$ defined as follows by recursion on the complexity of the terms

$$\overline{\theta}(0) = 0_R, \overline{\theta}(1) = 1_R, \overline{\theta}(X) = \theta(X), \overline{\theta}(\tau \dot{\pm} \mu) = \overline{\theta}(\tau) \dot{\pm} \overline{\theta}(\mu)$$

and

$$\overline{\theta}(E(\tau)) = E(\overline{\theta}(\tau)).$$

Now we define ψ in terms of $\overline{\theta}$ in the following way:

$$\psi(\tau/_{\equiv}) = \overline{\theta}(\tau) \text{ for all } \tau \in \mathcal{T}(X).$$

By induction on the complexity of the terms it is easy to prove that ψ is well defined, i.e.

$$\text{if } \tau, \mu \in \mathcal{T}(X) \text{ and } \tau \equiv \mu \text{ then } \overline{\theta}(\tau) = \overline{\theta}(\mu).$$

So $\mathcal{T}(X)/_{\equiv}$ is a free object.

Now we verify that it is isomorphic to the free object previously constructed. We will denote $[X_1, \ldots, X_m]^E$ by $[X]^E$. We show that the function

$$\varphi : [X]^E \longrightarrow \mathcal{T}(X)/_{\equiv}$$

defined as follows

$$\varphi(X_i) = X_i/_{\equiv}, \varphi(1) = 1/_{\equiv}, \varphi(0) = 0/_{\equiv}$$

$$\varphi(X_i \dot{\pm} X_j) = (X_i \dot{\pm} X_j)/_{\equiv} = X_i/_{\equiv} \dot{\pm} X_j/_{\equiv}$$

and

$$\varphi(E(0)) = \varphi(1) = 1 = E(0) = E(\varphi(0))$$

is an E-ring isomorphism. First we prove that φ is injective. Let

$$p(X), q(X) \in [X]^E,$$

and suppose

$$\varphi(p(X)) = \varphi(q(X)), \text{ that is } p(X)/_{\equiv} = q(X)/_{\equiv},$$

but this is so if and only if

$$ER \vdash \forall X(p(X) = q(X)).$$

$[X]^E$ is an E-ring, so it is a model of the theory of E-rings, hence $p(X) = q(X)$ in the free E-ring $[X]^E$. So function φ is injective, and it is trivial to prove that it is also surjective.

□

1.4 E-ideals

As in the classical case we can introduce a notion of ideal for E-ring, but in order to have a bijective correspondence between ideals and kernels of E-morphism one extra property has to be required.

Definition 1.4.1. *Let R be an E-ring, and I be an ideal of R (as a ring). I is an E-ideal if:*

$$\alpha \in I \to E(\alpha) \text{ implies } 1 \in I. \tag{1.5}$$

It is trivial to see that the kernel of an E-morphism satisfies (1.5). Given an E-ideal I, the quotient ring R/I can be equipped with an exponential morphism as follows:

$$E(\alpha + I) = E(\alpha) + I.$$

Note that this definition of E over R/I is well defined. So, I is the kernel of the natural E-morphism

$$pr : (R, E) \to (R/I, E),$$

$$a \longrightarrow a + I.$$

Remark 1.4.2. *We observe that some properties which are preserved from a ring R to its polynomial ring $R[\overline{x}]$, in the case of polynomial E-rings are not preserved anymore, e. g. being Noetherian. We show that $\mathbb{C}[x]^E$ is not Noetherian. Let $I = \langle E(\frac{x}{2^n}) - 1\rangle^E_{n\in\mathbb{N}}$. We prove that I is not finitely generated. Suppose by contradiction that I is finitely generated, e.g. there exists $n \in \mathbb{N}$ such that $I_n = \langle E(x) - 1, E(\frac{x}{2}) - 1, \ldots, E(\frac{x}{2^n}) - 1\rangle^E$. We observe that $E(\frac{x}{2^{n+1}}) - 1 \notin I_n$ since for $x = 2^{n+1}i\pi$, we have*

$$E(2^{n+1}i\pi) - 1 = E(\frac{2^{n+1}i\pi}{2}) - 1 = \ldots = E(\frac{2^{n+1}i\pi}{2^n}) - 1 = 0$$

and

$$E(\frac{2^{n+1}i\pi}{2^{n+1}}) - 1 \neq 0,$$

so we have a contradiction since $E(\frac{x}{2^{n+1}}) - 1 \notin I$. Then $\mathbb{C}[x]^E$ is not Noetherian. This implies that also $\mathbb{R}[x]^E$ is not Noetherian, since also in this case the E-ideal $I = \langle E(\frac{x}{2^n}) - 1\rangle^E_{n\in\mathbb{N}}$ is not finitely generated. So, $\mathbb{C}[x]^E$ is not a principal domain, and in the next Chapter we will show that also the free E-ring on two generators is not principal.

The theory of E-ideals is not completely understood. We would like to extend classical notions like maximal ideal, prime ideal to E-ideals. But it does not seem an easy task.

There are many interesting open questions about E-ideals.

Open Problems:

1. If $p(x)$ is an irreducible exponential polynomial over R, is $I = \langle p(x) \rangle^E$ a prime E-ideal?

2. If $I = \langle p(x) \rangle^E$ is a prime E-ideal, is I a maximal E-ideal?

We introduce the following definition which will need in Chapter 4.

Definition 1.4.3. *The E-ideal I of E-polynomial ring $R[x]^E$ is prime if and only if the quotient $\frac{R[x]^E}{I}$ is a domain.*

Chapter 2

Some consequences of Schanuel's Conjecture in exponential rings

2.1 Introduction

In this chapter we will examine some exponential algebraic relations among elements of the complex exponential field and the real exponential field. For the main results we use a famous conjecture in Transcendental Number Theory, the conjecture was formulated by Stephen Schanuel in the early 1960s:

Schanuel's Conjecture (SC) Let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$. Then $\mathbb{Q}(\lambda_1, \ldots, \lambda_n, e^{\lambda_1}, \ldots, e^{\lambda_n})$ has transcendence degree (t.d.) at least n over $\mathbb{Q}$.

Schanuel's Conjecture includes as special case the Lindemann-Weierstrass theorem, which says: if $\lambda_1, \ldots, \lambda_n$ are algebraic numbers which are linearly independent over $\mathbb{Q}$, then $e^{\lambda_1}, \ldots, e^{\lambda_n}$ are algebraically independent over $\mathbb{Q}$, and the Gelfond-Schneider theorem (1934), which says: if α and β are algebraic numbers with α different from $0, 1$ and β is irrational then α^β is transcendental over $\mathbb{Q}$, for details see [16].

Macintyre (1991) used Schanuel's Conjecture to prove that the exponential subring of $\mathbb{R}$ generated by 1 is free on no generators (see Theorem 2.2.1). This result implies that there are no hidden iterated exponential identities for exponential constants (modulo SC), that is there are no unexpected exponential algebraic relations on the integers $\mathbb{Z}$.

In this Chapter we consider the exponential field $(\mathbb{C}, e^x)$ where the following well known identities hold

$$e^{\pi i} = -1 \text{ and } i^2 = -1.$$

A natural question is:

Are these identities in $(\mathbb{C}, e^x)$ involving π and i?

Following the line of research of Macintyre in [4] we show assuming Schanuel's Conjecture that these are the only relations. This is obtained by characterizing the kernel of the E-morphism from the free E-ring on two generators x, y mapping $x \mapsto \pi$ and $y \mapsto i$.

We also show that such kernel is not a principal E-ideal. This was the starting point to begin the study of E-ideals in the free E-ring on n generators as kernels of certain E-morphisms (see Section 2.6).

Assuming Schanuel's Conjecture we obtain also information on algebraic relations among some elements of $(\mathbb{R}, e^x)$. More precisely, we prove that the E-subring of $\mathbb{R}$ generated by π is isomorphic to the free E-ring on π (modulo SC).

2.2 Schanuel's Conjecture

Before analyzing the consequences of Schanuel's Conjecture concerning the complex and real fields we examine some consequences of this conjecture in Transcendental Number Theory.

Schanuel's Conjecture can be used as a machine in order to produce algebraic independence among elements. The first two examples show that some instances of (SC) are true. We show some examples in this context.

1. From (SC) we have that the t.d.$\mathbb{Q}(1, e) \geq 1$, but it is exactly 1, so e is transcendental over $\mathbb{Q}$, [Hermite 1873].

2. We consider $2i\pi$, so (SC) implies the transcendence of π over $\mathbb{Q}$, since

$$t.d.\mathbb{Q}(2i\pi, e^{2i\pi}) \geq 1,$$

   but $e^{2i\pi} = 1$, so it is exactly 1, [Lindemann 1882].

3. The elements $\pi, i\pi$ are linearly independent over $\mathbb{Q}$ hence, from (SC) we have

$$t.d.\mathbb{Q}(\pi, i\pi, e^{\pi}, e^{i\pi}) \geq 2,$$

   but it is 2, so e, e^{π} are algebraically independent, [Nesterenko 1996].

4. We show the algebraic independence of π and e. Indeed, 1 and $i\pi$ are linearly independent over $\mathbb{Q}$, so (SC) implies that the transcendence degree of $\mathbb{Q}(1, \pi, e, e^{i\pi})$ over $\mathbb{Q}$ is at least 2. But $e^{i\pi} = -1$, and this implies that t.d.$\mathbb{Q}(1, \pi, e, e^{i\pi}) = 2$, so π and e are algebraically independent over $\mathbb{Q}$.

5. We now prove that π, e, e^e are algebraically independent over $\mathbb{Q}$. From 4 it follows that 1, $i\pi$, e are linearly independent over $\mathbb{Q}$ (since they are algebraically

independent). (SC) implies that t.d.$\mathbb{Q}(1, i\pi, e, e, e^{i\pi}, e^e) \geq 3$ and using again the identity $e^{i\pi} = -1$, we obtain that it is exactly 3.

With an easy induction it can be proved that π, e, e^e, e^{e^e}, $e^{e^{e^e}}$,... are algebraically independent over $\mathbb{Q}$, for any number of iterations.

6. We now show that $e^{i\pi^2}$ is transcendental over $\mathbb{Q}$. Indeed, $i\pi$ and $i\pi^2$ are linearly independent over $\mathbb{Q}$, so (SC) implies t.d.$\mathbb{Q}(i\pi, i\pi^2, e^{i\pi}, e^{i\pi^2}) \geq 2$. But $i\pi^2$ is algebraic over $\mathbb{Q}(i\pi)$ and $e^{i\pi} = -1$, hence $e^{i\pi^2}$ is transcendental over $\mathbb{Q}$.

7. We next prove that π, e, $e^{i\pi^2}$, e^e, e^{e^e}, $e^{e^{i\pi^2}}$ are algebraically independent over $\mathbb{Q}$. The elements 1, $i\pi$, $i\pi^2$, e, e^e, $e^{i\pi^2}$ are linearly independent over $\mathbb{Q}$, so from (SC) it follows that $t.d.$ $\mathbb{Q}(1, i\pi, i\pi^2, e, e^e, e^{i\pi^2}, e, e^{i\pi}, e^{i\pi^2}, e^e, e^{e^e}, e^{e^{i\pi^2}}) \geq 6$. But not all the elements adjoint to $\mathbb{Q}$ contribute to the transcendence degree since some are repeated; moreover, $i\pi^2$ is algebraic over $\mathbb{Q}(i\pi)$. So we have that the transcendence degree is exactly 6, and this implies that π, e, $e^{i\pi^2}$, e^e, e^{e^e}, $e^{e^{i\pi^2}}$ are algebraically independent over $\mathbb{Q}$.

It is not known if the last four results are true without SC.

A more interesting and less trivial consequence of (SC) is the following result in [4]. Before stating Macintyre's result we recall a generalization of Schanuel's Conjecture, that is:

Schanuel's Condition (SC) An E-ring R satisfies *Schanuel's Condition* if R is a characteristic 0 domain and whenever $\alpha_1, \ldots, \alpha_n$ in R are linearly independent over $\mathbb{Q}$ the ring $\mathbb{Z}[\alpha_1, \ldots, \alpha_n, E(\alpha_1), \ldots, E(\alpha_n)]$ has transcendence degree $\geq n$ over $\mathbb{Z}$.

Theorem 2.2.1. *[4] Suppose S is an E-ring satisfying Schanuel's Condition, and S_0 is the E-subring of S generated by 1. Then the natural E-morphism $\varphi : [\emptyset]^E \to S_0$ is an E-isomorphism, i.e. S_0 is isomorphic to E-free ring on the empty set.*

2.3 Operators of control

Recall that an element of $[X]^E$ (where $X = X_1, \ldots, X_m$) on m generators is just an E-polynomial in m variables with coefficients (at each level) in $\mathbb{Z}$.

In this Section we will define two operators $\mathcal{D}$ and $\mathcal{E}$ which are going to control all the "components" of any given E-polynomial, and in order to do this the only identity we will use is $E(x+y) = E(x) \cdot E(y)$. For our purposes it is enough to define the operators on the free E-ring on two generators, but they can be defined in a similar way on the free E-ring on m generators, for any $m \in \mathbb{N}$.

Let $[x, y]^E$ denote the free E-ring on x and y. From Section 1.1.2 and 1.3 we can write

$$[x, y]^E = \bigcup_k R_k.$$

Let $P(x, y) \in [x, y]^E$, then $P(x, y) \in R_k$ for some k, and the polynomial P decomposes uniquely as:

$$P = p_0 + p_1 + \ldots + p_k$$

where $p_i \in R_i$, $i \leq k$. In particular, $p_i = \sum_{d \in B_{i-1}} c_d E(d)$, where $c_d \in R_{i-1}$, $i \geq 1$. We define the operator $\mathcal{D}$ as follows:

$$\mathcal{D}(P) = \mathcal{D}(p_0) \cup \mathcal{D}(p_1) \cup \ldots \cup \mathcal{D}(p_k),$$

where

$$\mathcal{D}(p_0) = \{x^t y^l : x^t y^l \text{ is a monomial in } p_0, \text{ with } t, l \in \mathbb{N}\}$$

and

$$\mathcal{D}(p_i) = \{c_d : d \neq 0\} \cup \{d : c_d \neq 0\}, \text{ with } i > 0.$$

For our purposes we will need to iterate the operator $\mathcal{D}$ i times on p_i with $i > 0$ in order to have all the "components" of the E-polynomial, and successively we will reconstruct it as an E-polynomial from its "components". By iterating the operator $\mathcal{D}$ on each p_i, we will be able to identify all monomials which appear in the exponential polynomial $P(x, y)$ at each level.

We define the operator $\mathcal{E}$ in following way:

$$\mathcal{E}(p_0) = \emptyset;$$

$$\mathcal{E}(p_1) = \emptyset;$$

$$\mathcal{E}(p_i) = \{d \mid c_d \neq 0\}, \text{ per } i \geq 2.$$

The operator $\mathcal{E}$ controls the exponents which cannot be decomposed using the only available identity $E(x+y) = E(x) \cdot E(y)$. In order to obtain this, we will need to iterate $\mathcal{E}$. So if $p_i(x, y) \in R_i$, then we need to iterate $\mathcal{D}$ and $\mathcal{E}$ i times, i.e. the height of the polynomial p_i. We have the following inclusions:

$$\mathcal{D}(p_i) \subseteq R_{i-1} \text{ and } \mathcal{E}(p_i) \subseteq B_{i-1}$$

$$\mathcal{D}^2(p_i) \subseteq R_{i-2} \text{ and } \mathcal{E}^2(p_i) \subseteq B_{i-2}$$

$$\ldots$$

$$\ldots$$

$$\mathcal{D}^i(p_i) \subseteq \mathbb{Z}[x, y] \text{ and } \mathcal{E}^i(p_i) \subseteq \mathbb{Z}[x, y].$$

Let

$$\Gamma_0(P) = \mathcal{D}(p_0) \cup \bigcup_{i=1}^{k} \mathcal{D}^i(p_i) \subseteq \mathbb{Z}[x, y]. \tag{2.1}$$

We introduce the set Γ_0 in order to control the highest exponents of the monomials which are present in the polynomial $P(x, y)$.

Example 2.3.1. *In order to clarify the above notions we consider the following simple example:*

Let $P(x, y) \in [x, y]^E$ *where*

$$P(x, y) = -3x^2y - x^5y^7 + (2xy + 5y^2)e^{-7x^3+11x^5y^4} + (3 - 2xy)e^{(5x+2xy^2)e^{5x-10y^2}}.$$

So

$$p_0 = -3x^2y - x^5y^7,$$

$$p_1 = (2xy + 5y^2)e^{-7x^3+11x^5y^4},$$

$$p_2 = (3 - 2xy)e^{(5x+2xy^2)e^{5x-10y^2}}.$$

In this case we have

$$\mathcal{D}(p_0) = \{-3x^2y, -x^5y^7\};$$

$$\mathcal{D}(p_1) = \{2xy, 5y^2, -7x^3, 11x^5y^4\};$$

$$\mathcal{D}(p_2) = \{3, -2xy\} \cup \{(5x + 2xy^2)e^{5x-10y^2}\},\ \mathcal{D}^2(p_2) = \{3, -2xy, 5x, 2xy^2, 5x, -10y^2\};$$

$$\mathcal{D}(P) = \{-3x^2y, -x^5y^7\} \cup \{2xy, 5y^2, -7x^3, 11x^5y^4\} \cup \{3, -2xy, (5x + 2xy^2)e^{5x-10y^2}\};$$

So we have

$$\Gamma_0(P) = \{-3x^2y, -x^5y^7\} \cup \{2xy, 5y^2, -7x^3, +11x^5y^4\} \cup \{3, -2xy, 5x, 2xy^2, 5x, -10y^2\},$$

and

$$\mathcal{E}(p_0) = \emptyset;$$

$$\mathcal{E}(p_1) = \emptyset;$$

$$\mathcal{E}(p_2) = \{5x - 10y^2\}.$$

2.4 E-subring of $\mathbb{C}$ generated by π, i

In this section we prove an interesting consequence of Schanuel's Conjecture which is along the lines of Theorem 2.2.1. Let $\langle \pi, i\rangle^E$ denote the E-subring of $\mathbb{C}$ generated by π, i. Our main result is the following.

Theorem 2.4.1. *(SC) Let $[x, y]^E$ be the free E-ring generated by $\{x, y\}$ and let ψ be the E-morphism:*

$$\psi : [x, y]^E \to (\mathbb{C}, e^x)$$

$$x \mapsto \pi$$

$$y \mapsto i.$$

Then there exists a unique isomorphism

$$f : [x, y]^E / ker\psi \to \langle \pi, i\rangle^E$$

and

$$ker\psi = \langle e^{xy} + 1, y^2 + 1\rangle^E.$$

Remark 2.4.1. *Before embarking in the proof we notice that Theorem 2.4.1 implies that in $(\mathbb{C}, e^x)$ the only algebraic relations between π, i and e are the formal consequences of the known ones:*

$$i^2 = -1 \text{ and } e^{i\pi} = -1.$$

Proof of Theorem 2.4.1: From $[x, y]^E$ being the free object on two generators in the equational class of E-rings it follows that ψ is an E-morphism. The existence of the isomorphism f is guaranteed by the First Homomorphism Theorem on E-rings. We will characterize the kernel of the E-morphism ψ. In order to do this it will be necessary to have a complete control over the "components" of each E-polynomial, and for this we will use the operators $\mathcal{D}$ and $\mathcal{E}$ introduced in Section 2.3. We use the construction of the free E-ring generated by $\{x, y\}$ as $\bigcup R_k$, where R_k are the partial E-rings that we constructed in Section 1.1.2 and 1.3 of previous Chapter. We will determine the kernel of ψ by steps considering the kernels of the restrictions of ψ to each R_k. We will make no distinctions between ψ and its restrictions.

k = 0: Recall that $R_0 = \mathbb{Z}[x, y]$. Clearly, $\psi(\mathbb{Z}[x, y]) = \mathbb{Z}[\pi, i]$. From the transcendence of π over $\mathbb{Q}$ it follows immediately that $\ker \psi|_{\mathbb{Z}[x,y]} = \langle y^2 + 1\rangle$.

k = 1: From the construction, $R_1 = \mathbb{Z}[x, y][t^{\mathbb{Z}[x,y]}]$. We observe that $\mathbb{Z}[x, y] \cong \mathbb{Z} \bigoplus (x, y)$, where (x, y) is the ideal generated by $\{x, y\}$. So $\mathbb{Z}[x, y][t^{\mathbb{Z}[x,y]}] \cong \mathbb{Z}[x, y][t^{(x,y)}]$, and we now consider the restriction of ψ to $\mathbb{Z}[x, y][t^{(x,y)}]$. We map t to e. Hence ψ maps $\mathbb{Z}[x, y][t^{(x,y)}]$ into the subring of $\mathbb{C}$ of formal polynomials in $e^{(\pi,i)}$ with coefficient in $\mathbb{Z}[\pi, i]$, i.e.:

$$\psi : \mathbb{Z}[x, y][t^{(x,y)}] \to \mathbb{Z}[\pi, i][e^{(\pi,i)}].$$

We have to characterize the kernel of this function. For this purpose we have to identify the polynomials

$$P(x, y) \in \mathbb{Z}[x, y][t^{(x,y)}]$$

such that

$$P(\pi, i) = 0.$$

It is useful to write explicitly all exponents which are present in the polynomial P, that is we write the polynomial $P(x, y)$ in the following way:

$$P(x, y, t^{g_1(x,y)}, \ldots, t^{g_k(x,y)})$$

where $g_j(x, y) \in \mathbb{Z}[x, y]$. So,

$$\psi(P(x, y, t^{g_1(x,y)}, \ldots, t^{g_k(x,y)})) = P(\pi, i, e^{g_1(\pi,i)}, \ldots, e^{g_k(\pi,i)}). \quad (2.2)$$

Let L be the highest power of π which appears in P and consider all possible monomials, both real and complex, which can be constructed from i, π^l, with $l \leq L$. Such monomials are in the complex case $i\pi^l$, and π^l in the real case for all $l \leq L$. The number of all monomials which can be constructed from i and π is then $2(L+1)$. Let $N = 2(L+1)$, and $\mu_1, \ldots, \mu_N$ be such monomials.

Without loss of generality, using the identity $E(x+y) = E(x) \cdot E(y)$, we can assume that the exponentials $e^{g_j(\pi,i)}$ in π and i, look like:

$$e^{c_j \mu_j}, \text{ for some, } \mu_j \text{ and } c_j \in \mathbb{Z}.$$

So $P(\pi, i, e^{g_1(\pi,i)}, \ldots, e^{g_k(\pi,i)})$ in (2.2) reduces to the polynomial expression:

$$Q(\pi, i, e^{\mu_{j_1}}, \ldots, e^{\mu_{j_k}}), \tag{2.3}$$

where $\mu_{j_1}, \ldots, \mu_{j_k}$ vary among the monomials previously described. Now we use (SC), and we are going to prove much more than we actually need. All the monomials $\mu_1, \ldots, \mu_N$ are clearly linearly independent over $\mathbb{Q}$. So (SC) implies

$$t.d.\mathbb{Q}(\mu_1, \ldots, \mu_N, e^{\mu_1}, \ldots, e^{\mu_N}) \geq N,$$

and obviously

$$t.d.\mathbb{Q}(\mu_1, \ldots, \mu_N, e^{\mu_1}, \ldots, e^{\mu_N}) \leq 2N,$$

since the transcendence degree can be at most the number of elements added to $\mathbb{Q}$. From the transcendence of π it follows that the contribution of the N monomials to the transcendence degree over $\mathbb{Q}$ is only 1. Moreover, $e^{i\pi} = -1$, so

$$t.d.\mathbb{Q}(\mu_1, \ldots, \mu_N, e^{\mu_1}, \ldots, e^{\mu_N}) = N.$$

Then the identity $Q(i, \pi, e, e^{\mu_{j_1}}, \ldots, e^{\mu_{j_k}}) = 0$ is true if and only if the polynomial Q is identically zero. This implies that the only relations among π, i, and e are $i^2 = -1$ and

$e^{\pi i} = -1$, hence $\ker \psi|_{R_1} = \langle y^2+1, e^{xy}+1\rangle$. We notice that in fact we have proved much more than we needed. In particular, we have shown the algebraic independence of the elements $\pi, e^{\mu_{j_1}}, \ldots, e^{\mu_{j_k}}$ (for $e^{\mu_{j_s}} \neq -1$) which appear in Q.

Inductive step: We suppose that the statement is true for $k-1$ and we prove the result for k, that is we suppose that for any polynomial

$$P(x,y) \in R_{k-1} = R_{k-2}[t^{B_{k-2}}],$$

$P(\pi, i) = 0$ if and only if P is the polynomial identically zero. Now we have to characterize the polynomials

$$P(x,y) \in R_k = R_{k-1}[t^{B_{k-1}}],$$

such that

$$P(\pi, i) = 0.$$

We will use the operators $\mathcal{D}$ and $\mathcal{E}$ introduced in Section 2.3. As already remarked we can write P as follows, $P = p_0 + p_1 + \ldots + p_k$ where $p_i \in R_i$, $i \leq k$. We define:

$$f_i = \mathcal{E}^i(p_i), \text{ for } i = 2, \ldots, k,$$

so $f_i \in \mathbb{Z}[x,y]$.

Recall that $\Gamma_0(P)$ is the set of all monomials which appear in P (see Section 2.3). We distinguish two cases.

Case k even: We define Δ_i, for $i = 0, \ldots, k$ in the following way:

$$\Delta_0 = \{\mu_1, \ldots, \mu_N\}$$

where μ_j's, $j = 1, \ldots, N$ are all the monomials which can be constructed from i, π^l, for $l \leq L$, where

$$L = max\{l : \pi^l \in \Gamma_0\}.$$

Let

$$\Delta_1 = \{e^{\mu_j} : j = 1, \ldots, N\}, \text{ i.e. } \Delta_1 = e^{\Delta_0}.$$

Now consider

$$\Delta_2 = \{\mu_j e^{f_s} : j = 1, \ldots, N \text{ and } s = 2, \ldots, k\};$$

$$\Delta_3 = \{e^{\mu_j e^{f_s}} : j = 1, \ldots, N \text{ and } s = 2, \ldots, k\}.$$

We observe explicitly that

$$\Delta_3 = e^{\Delta_2}.$$

More generally, we define

$$\Delta_{2j} = \Delta_0 \Delta_{2j-1} = \{\mu\delta : \mu \in \Delta_0, \delta \in \Delta_{2j-1}\} \text{ and } \Delta_{2j+1} = e^{\Delta_{2j}}, \text{ with } j = 0, \ldots, \frac{k}{2} - 1.$$

We now estimate the cardinalities of all Δ_i's for $i = 0, \ldots, k$.

$$\mid \Delta_0 \mid = N \text{ and } \mid \Delta_1 \mid = N,$$

$$\mid \Delta_{2j} \mid = N^j (k-2), \text{ with } j = 1, \ldots, \frac{k}{2},$$

and

$$\mid \Delta_{2j+1} \mid = \mid \Delta_{2j} \mid, \text{ with } j = 1, \ldots, \frac{k}{2} - 1.$$

So we have

$$\mid \Delta_0 + \Delta_1 + \cdots + \Delta_k \mid = 2N + 2 \sum_{t=1}^{\frac{k}{2}} N^t (k-2).$$

We denote such cardinality by S.

By inductive hypothesis, each of the sets $\Delta_0, \Delta_1, \ldots, \Delta_k$ is linearly independent over $\mathbb{Q}$, so (SC) implies:

$$t.d.\mathbb{Q}(\Delta_0, \Delta_1, \ldots, \Delta_k, e^{\Delta_0}, e^{\Delta_1}, \ldots, e^{\Delta_k}) \geq S.$$

Also,

$$t.d.\mathbb{Q}(\Delta_0, \Delta_1, \ldots, \Delta_k, e^{\Delta_0}, e^{\Delta_1}, \ldots, e^{\Delta_k}) \leq 2S.$$

The set Δ_0 gives a contribution of 2 to the transcendence degree and Δ_1 gives a contribution of $N-2$. Moreover, since $\Delta_{2j+1} = e^{\Delta_{2j}}$, for $j = 0, \ldots, \frac{k}{2} - 1$, there are some repetitions among the elements added to $\mathbb{Q}$ (as in the examples in section 6). The elements of Δ_{2j} are algebraically independent over $\mathbb{Q}$, so

$$t.d.\mathbb{Q}(\Delta_0, \Delta_1, \ldots, \Delta_k, e^{\Delta_0}, e^{\Delta_1}, \ldots, e^{\Delta_k}) = S,$$

that is, the identity $P(\pi, i) = 0$ is true if and only if the polynomial P is identically zero.

Case k odd: The proof for k odd follows the lines of the previous case for k even, but we have to pay attention to the indices. For completeness we prefer to go through the whole construction.

We define Δ_i for $i = 0, \ldots, k+1$ as before:

$$\Delta_0 = \{\mu_1, \ldots, \mu_N\}$$

where the μ_j's are all possible monomials which can be constructed from i, π^l, where $l \leq L$, and

$$L = max\{l : \pi^l \in \Gamma_0\}.$$

Let

$$\Delta_1 = \{e^{\mu_j} : j = 1, \ldots, N\}, \text{ i.e. } \Delta_1 = e^{\Delta_0}.$$

Now consider

$$\Delta_2 = \{\mu_j e^{f_s} : j = 1, \dots, N \text{ and } s = 2, \dots, k\}, \text{ i.e. } \Delta_2 = \Delta_2 = \Delta_0 \cdot \Delta_1;$$

$$\Delta_3 = \{e^{\mu_j e^{f_s}} : j = 1, \dots, N \text{ and } s = 2, \dots, k\}.$$

We observe explicitly that

$$\Delta_3 = e^{\Delta_2},$$

and, more generally, we define

$$\Delta_{2j} = \Delta_0 \Delta_{2j-1} = \{\mu\delta : \mu \in \Delta_0, \delta \in \Delta_{2j-1}\} \text{ and } \Delta_{2j+1} = e^{\Delta_{2j}}, \text{ with } j = 0, \dots, \frac{k-1}{2}.$$

We now estimate the cardinalities of all Δ_i's for $i = 0, \dots, k+1$:

$$|\Delta_0| = N \text{ and } |\Delta_1| = N,$$

$$|\Delta_{2j}| = N^j(k-2), \text{ with } j = 1, \dots, \frac{k+1}{2},$$

and

$$|\Delta_{2j+1}| = |\Delta_{2j}|, \text{ with } j = 1, \dots, \frac{k-1}{2}.$$

We have

$$|\Delta_0 + \Delta_1 + \cdots + \Delta_{k+1}| = 2N + 2\sum_{t=1}^{\frac{k+1}{2}} N^t(k-2) - 2N^{\frac{k+1}{2}}(k-2).$$

We denote this cardinality by T.

By the inductive hypothesis, all the sets $\Delta_0, \Delta_1, \dots, \Delta_k$ are linearly independent over $\mathbb{Q}$, and from (SC) it follows

$$t.d.\mathbb{Q}(\Delta_0, \Delta_1, \dots, \Delta_{k+1}, e^{\Delta_0}, e^{\Delta_1}, \dots, e^{\Delta_{k+1}}) \geq T.$$

Clearly,

$$t.d.\mathbb{Q}(\Delta_0, \Delta_1, \dots, \Delta_{k+1}, e^{\Delta_0}, e^{\Delta_1}, \dots, e^{\Delta_{k+1}}) \leq 2T.$$

The set Δ_0 gives a contribution of 2 to the transcendence degree over $\mathbb{Q}$ and Δ_1 gives a contribution of $N-2$. Moreover, since $\Delta_{2j+1} = e^{\Delta_{2j}}$, for $j = 0, \ldots, \frac{k-1}{2}$, as before, there are some repetitions among the elements added to $\mathbb{Q}$. The elements of Δ_{2j} are algebraically independent over $\mathbb{Q}$, so

$$t.d.\mathbb{Q}(\Delta_0, \Delta_1, \ldots, \Delta_k, e^{\Delta_0}, e^{\Delta_1}, \ldots, e^{\Delta_k}) = T.$$

This implies that the identity $P(\pi, i) = 0$ is true if and only if the polynomial P is identically zero. Now the proof is completed, since the kernel of ψ has stabilized at level $k = 2$, and so $\ker \psi = \langle y^2 + 1, e^{xy} + 1 \rangle$.

$\square$

2.4.1 Is the E-ideal $\langle e^{xy} + 1, y^2 + 1 \rangle^E$ principal?

In this section we will investigate the nature of the kernel of ψ, more precisely we want to understand if the number of generators can be reduced to one. In order to do this we recall the notion of augmentation which is a notion relative to any group algebra.

Definition 2.4.2. *For any group algebra $R[G]$ there exists a unique R-algebra morphism*

$$S : R[G] \to R$$

such that $S(\sum r_g g) = \sum r_g$. The morphism S is called the augmentation map and $kerS$ is called the augmentation ideal of $R[G]$.

It follows immediate by the definition that $S|_R = id_R$.

Now we define the augmentation ideal on the E-polynomial ring $R[X_1, \ldots, X_m]^E$. For this purpose we will define augmentation maps for all group rings in the construction of $R[X_1, \ldots, X_m]^E$.

Indeed, recall that for all n the group ring R_{n+1} can be rappresented in the following different ways

$$R_{n+1} \cong R_0[t^{B_0 \oplus B_1 \oplus \ldots \oplus B_n}];$$

$$R_{n+1} \cong R_1[t^{B_1 \oplus \ldots \oplus B_n}];$$

$$\ldots$$

$$\ldots$$

$$R_{n+1} \cong R_n[t^{B_n}].$$

Hence using the $n+1$ different expressions of R_{n+1} (as group ring) we can define $n+1$ different augmentation maps from R_{n+1} to R_i for each $0 \leq i \leq n$:

$$S_0 : R_{n+1} \to R_0;$$

$$S_1 : R_{n+1} \to R_1;$$

$$\ldots$$

$$\ldots$$

$$S_n : R_{n+1} \to R_n.$$

It turns out that $S_i|_{R_0} = S_0|_{R_0} = id_{R_0}$ for each $i \leq n$.

We now compare the augmentation maps defined on two successive R_i's. If we consider R_{n+2} and its $n+2$ different expressions as group ring (as for R_{n+1}) we define $n+2$ different augmentation maps $S'_0, S'_1, \ldots, S'_{n+1}$, and $S'_i|_{R_{n+1}} = S_i$ for all $i \leq n$.

For each n, let

$$J_{n+1} = \{f \in R_{n+1} : S_0(f) = 0\}.$$

It is very easy to verify that J_{n+1} is an ideal in R_{n+1} (it is the kernel of S_0). Moreover, it is an E-ideal since it contains $E(g) - 1$, for all $g \in R_n$. Let

$$J_\infty = \bigcup_n J_{n+1}.$$

J_∞ is an E-ideal of $R[X_1, \ldots, X_m]^E$ and we call it the augmentation ideal of $R[X_1, \ldots, X_m]^E$.

Now we have all the ingredients in order to prove the following theorem.

Theorem 2.4.2. *The E-ideal generated by $e^{xy} + 1$ and $y^2 + 1$ in $[x, y]^E$ is not principal.*

Proof: By contradiction we assume that $\langle e^{xy} + 1, y^2 + 1\rangle^E = \langle p(x, y)\rangle^E = I$, where $p(x, y) \in [x, y]^E$. We observe that $p(x, y) \neq e^{f(x,y)}$ for all $f(x, y) \in [x, y]^E$, since the elements $e^{f(x,y)}$ are invertible in $[x, y]^E$, (see Proposition 1.1.9). Moreover $I = \bigcup_n I_n$ where

$$I_0 = \langle p(x, y)\rangle \text{ (simply as ideal and not } E\text{-ideal)}$$

$$I_1 = \langle I_0, E(h) - 1\rangle_{h \in I_0} \subseteq I_0 + J_\infty$$

$$I_2 = \langle I_1, E(h) - 1\rangle_{h \in I_1} \subseteq I_1 + J_\infty$$

$$\ldots$$

$$I_n = \langle I_{n-1}, E(h) - 1\rangle_{h \in I_{n-1}} \subseteq I_{n-1} + J_\infty,$$

$$\ldots$$

In particular, we have

$$I_n \subseteq I_0 + J_\infty, \text{ for all } n \in \mathbb{N}.$$

We distinguish two cases, $p(x, y) \notin \mathbb{Z}[x, y]$, and $p(x, y) \in \mathbb{Z}[x, y]$.

In the first case, for $n \geq 1$, if $p(x,y) \in R_n - \mathbb{Z}[x,y]$ then $I \bigcap \mathbb{Z}[x,y] = \{0\}$. Indeed, if there is $\alpha \in I \bigcap \mathbb{Z}[x,y]$, and $\alpha \neq 0$, then $\alpha = \beta + \gamma$, where $\beta \in I_0$ and $S_n(\gamma) = 0$ for all $n \in \mathbb{N}$.

So, $\beta = a(x,y)p(x,y)$, where $a(x,y) \in [x,y]^E$. Since $\alpha \in \mathbb{Z}[x,y]$ we have

$$\alpha = S_0(\alpha) = S_n(\alpha) = S_n(a(x,y)p(x,y)) = S_n(a(x,y))S_n(p(x,y)) = S_n(a(x,y))p(x,y),$$

where the last equality follows from the hypothesis $p(x,y) \in R_n$. We clearly $\alpha = S_n(a(x,y))p(x,y)$ is impossible, and so get a contradiction. So, necessarily $I \bigcap \mathbb{Z}[x,y] = \{0\}$, and we have that $y^2 + 1 \notin I$.

Now we consider the case $p(x,y) \in \mathbb{Z}[x,y]$. By an easy induction on n we have $S_0(I_n) \subseteq \langle S_0(p(x,y)) \rangle$, as ideal of $\mathbb{Z}[x,y]$. This implies that for all $\alpha \in I$, $S_0(\alpha) \in \langle S_0(p(x,y)) \rangle$.

By the hypothesis $e^{xy} + 1, y^2 + 1 \in I$, and so $S_0(e^{xy} + 1), S_0(y^2 + 1) \in \langle S_0(p(x,y)) \rangle$. Hence, $2, y^2 + 1 \in \langle S_0(p(x,y)) \rangle$ and since

$$S_0(p(x,y)) = p(x,y)$$

necessarily $p(x,y) = \pm 1$. This gives a contradiction.

$\square$

2.5 E-subring of $\mathbb{R}$ generated by π

In this section we study a consequence of Theorem 2.4.1 in the exponential field $(\mathbb{R}, e^x)$. We will show that, modulo Schanuel's Conjecture, the E-subring of $\mathbb{R}$ generated by π is free on π, that is we have the following result:

Theorem 2.5.1. *(SC) Let $[x]^E$ be the free E-ring generated by $\{x\}$ and let R be the E-subring of $(\mathbb{R}, e^x)$ generated by π. Then the E-morphism φ*

$$\varphi : [x]^E \to (R, E)$$

$$x \mapsto \pi.$$

is an E-isomorphism.

Proof: The proof is very similar to the proof of Theorem 2.4.1, so we will sketch only the main steps, pointing out the differences with the previous proof. From the property of the free object it follows that the function φ is an E-morphism. As in the previous proof we will use induction on k in order to characterize the kernels of the restrictions φ to R_k's, and for each k we will show that the kernel is trivial. Since the E-morphism φ is trivially surjective, φ is an isomorphism.

k=0: Recall that $R_0 = \mathbb{Z}[x]$. Clearly, $\varphi(\mathbb{Z}[x]) = \mathbb{Z}[\pi]$. From the transcendence of π over $\mathbb{Q}$ it follows easily that $ker\varphi$ is trivial.

k=1: Since $R_1 = \mathbb{Z}[x][t^{(x)}]$ the function φ is the following:

$$\varphi : \mathbb{Z}[x][t^{(x)}] \to \mathbb{Z}[\pi][e^{(\pi)}].$$

We have to describe the kernel of this function, and for this purpose we have to characterize the polynomials

$$P(x) \in \mathbb{Z}[x][t^{(x)}] \text{ such that } P(\pi) = 0.$$

It is useful to write explicitly the exponents which appear in the polynomial $P(x)$, that is we write the polynomial $P(x)$ in the following way

$$P(x, t^{g_1(x)}, \ldots, t^{g_s(x)})$$

where $g_j(x) \in \mathbb{Z}[x]$. So the image of P via φ is:

$$\varphi(P(x, t^{g_1(x)}, \ldots, t^{g_s(x)})) = P(\pi, e^{g_1(\pi)}, \ldots, e^{g_s(\pi)}).$$

Let K be the highest power of π which appears in $P(\pi)$, so all the monomials in $P(\pi)$ are among the powers of π

$$\pi^k, \text{ where } k \leq K. \tag{2.4}$$

Without loss of generality, we can assume that the exponential polynomials $e^{g_j(\pi)}$ look like:

$$e^{c_j \pi^j}, \text{ where } c_j \in \mathbb{Z}, j \leq K$$

So $P(\pi)$ reduces to the exponential expression

$$Q(\pi, e^{\pi^{j_1}}, \ldots, e^{\pi^{j_t}}). \tag{2.5}$$

The elements $1, \pi, \pi^2, \ldots, \pi^K$ are linearly independent over $\mathbb{Q}$. Schanuel's Conjecture implies

$$t.d.\mathbb{Q}(1, \pi, \ldots, \pi^K, e, e^\pi, \ldots, e^{\pi^K}) \geq K + 1,$$

and obviously

$$t.d.\mathbb{Q}(1, \pi, \ldots, \pi^K, e, e^\pi, \ldots, e^{\pi^K}) \leq 2(K + 1).$$

From transcendence of π over $\mathbb{Q}$ it follows that the contribution of the $K+1$ monomials $i\pi$ to the transcendence degree is 1. Moreover, Theorem 2.4.1 implies that the elements $e, e^\pi, \ldots, e^{\pi^K}$ are algebraically independent over $\mathbb{Q}$. So

$$t.d.\mathbb{Q}(1, \pi, \ldots, \pi^K, e, e^\pi, \ldots, e^{\pi^K}) = K + 2,$$

then the identity $Q(\pi, e, e^{\pi^{j_1}}, \ldots, e^{\pi^{j_t}}) = 0$ holds if and only if $P(x) = 0$.

In the inductive step we proceed in a similar way as in the proof of Theorem 2.4.1 using the operators $\mathcal{D}$ and $\mathcal{E}$ introduced in Section 2.3.

□

Remark 2.5.1. *Theorem 2.5.1 can be interpreted by saying that π is exponentially transcendental over $\mathbb{Q}$.*

2.6 Other algebraic relations in $(\mathbb{C}, e^x)$

In this section we continue to investigate the nature of the kernels of some E-morphism, in order to understand the E-ideals in the free object. We have seen examples of non principal E-ideals in $\mathbb{C}[x]^E$ in Section 1.4.

We now consider the following question:

Is there $\alpha \in \mathbb{C}$ such that the E-morphism

$$\psi : [x]^E \to (\mathbb{C}, e^x)$$

$$x \mapsto \alpha$$

has kernel which is a principal E-ideal?

The answer is yes modulo Schanuel's Conjecture.

Theorem 2.6.1. *(SC) Let ψ be the E-morphism:*

$$\psi : [x]^E \to (\mathbb{C}, e^x)$$

$$x \mapsto \alpha$$

where $e^\alpha = \alpha$. Then $\ker \psi = \langle e^x - x \rangle^E$.

Proof: The equation $e^x = x$ has infinitely many zeros in $\mathbb{C}$ (see [9]). If α satisfies $e^\alpha = \alpha$ then α is transcendental over $\mathbb{Q}$, since otherwise α, e^α were both algebraic over $\mathbb{Q}$ contradicting the Lindemann-Weierstrass Theorem.

As in the previous proofs we have to characterize the kernel of ψ and we will proceed by recursion. The proof differs from the previous ones only in the initial steps. So we will focus our attention only on these.

k=0: Consider $\psi : \mathbb{Z}[x] \to \mathbb{Z}[\alpha]$. From transcendence of α over $\mathbb{Q}$ it follows easily that $ker\psi$ is trivial.

k=1: Consider $\psi : \mathbb{Z}[x][t^{(x)}] \to \mathbb{Z}[\alpha][e^{(\alpha)}]$. We have to characterize the polynomials

$$P(x, t^{g_1(x)}, \ldots, t^{g_s(x)})$$

such that

$$\psi(P(x, t^{g_1(x)}, \ldots, t^{g_s(x)})) = P(\alpha, e^{g_1(\alpha)}, \ldots, e^{g_s(\alpha)}) = 0.$$

Let K be the highest power of α which appears in P, so at most all the monomials α^k, where $k \leq K$ appear in P. Using the identity $E(x+y) = E(x) \cdot E(y)$ we can reduce the polynomial $P(\alpha, t^{g_1(\alpha)}, \ldots, t^{g_s(\alpha)})$ to the polynomial expression $Q(\alpha, e^{\alpha_{j_1}}, \ldots, e^{\alpha_{j_n}})$.

The monomials $1, \alpha, \ldots, \alpha^K$ are linearly independent over $\mathbb{Q}$, so from (SC) we have

$$t.d.\mathbb{Q}(1, \alpha, \ldots, \alpha^K, e, e^\alpha, \ldots, e^{\alpha^K}) \geq K + 1.$$

The contribution to the transcendence degree of the $K+1$ monomials is 1, since α is transcendental over $\mathbb{Q}$. Moreover, from $e^\alpha = \alpha$, it follows

$$t.d.\mathbb{Q}(1, \alpha, \ldots, \alpha^K, e, e^\alpha, \ldots, e^{\alpha^K}) = K + 1.$$

Hence the identities $Q(\alpha, e^{\alpha_{j_1}}, \ldots, e^{\alpha_{j_n}}) = 0$ holds if and only if the polynomial $P(x)$ is identically zero.

For the inductive step we use again the operators of control $\mathcal{D}$ and $\mathcal{E}$ on the free E-ring generated by x.

□

Theorem 2.6.1 says that if $\alpha \in \mathbb{C}$ satisfies the relation $e^\alpha = \alpha$ then α does not satisfy any further algebraic relation in terms of $+, \cdot$ and E, except those which are formal consequences of $e^\alpha = \alpha$.

We can generalize the previous result to the case of n elements. For the proof of the next theorem we need the following recent result due to Marker [9].

Theorem 2.6.2. *(SC) Suppose $p(x, y) \in \mathbb{Q}[x, y]$ is irreducible and depends on x and y. Then there are infinitely many algebraically independent zeros of $f(z) = p(z, e^z)$, in $\mathbb{C}$.*

Using the techniques introduced in the previous sections the following theorem can be proved.

Theorem 2.6.3. *(SC) Let φ be the E-morphism*

$$\varphi : [x_1, \ldots, x_n]^E \to (\mathbb{C}, e^x)$$

$$x_i \mapsto \alpha_i$$

where $e^{\alpha_i} = \alpha_i$, for all $i = 1, \ldots, n$ and $\alpha_i \neq \pm\alpha_j$, for $i < j$. Then

$$ker\varphi = \langle e^{x_1} - x_1, \ldots, e^{x_n} - x_n \rangle^E.$$

Proof: The proof proceeds by recursion as in the previous cases. We only notice that in the initial step we use the algebraic independence of α_i's (see [9]).

□

The last consequence of Schanuel's Conjecture which we examine in this section is the following result.

Theorem 2.6.4. *(SC) Let $u \in \mathbb{C}$ be algebraic over $\mathbb{Q}$, $u \neq 1$ and $t.d.\mathbb{Q}(\log u, u) = 1$. Let φ be the E-morphism*

$$\varphi : [x, y]^E \to (\mathbb{C}, e^x)$$

$$x \mapsto \log u$$

$$y \mapsto u.$$

Then $ker\varphi = \langle p(y), e^x - y\rangle^E$, where $p(y) \in \mathbb{Q}[y]$ is the minimal polynomial of u.

Proof: First of all notice that if u is algebraic over $\mathbb{Q}$, $u \neq 1$ and $t.d.\mathbb{Q}(\log u, u) = 1$ then $\log u$ is transcendental over $\mathbb{Q}$.

We now proceed by induction in order to characterize $ker\varphi$. We give the details only of the first steps.

k=0: Let $\varphi : \mathbb{Z}[x, y] \to \mathbb{Z}[\log u, u]$. From transcendence of $\log u$ it is trivial to prove that $ker\varphi = \langle p(y)\rangle$, where $p(y)$ is the minimal polynomial of u over $\mathbb{Q}$.

k=1: We have to characterize the polynomials $P(x, y) \in Z[x, y][t^{(x,y)}]$ such that $\varphi(P(x, y)) = P(\log u, u) = 0$. It is useful to write explicitly the exponents which appear in $P(x, y)$ in the following way

$$P(x, y, t^{g_1(x,y)}, \dots, t^{g_k(x,y)}),$$

where $g_j(x, y) \in \mathbb{Z}[x, y]$. So

$$\varphi(P(x, y, t^{g_1(x,y)}, \dots, t^{g_k(x,y)})) = P(\log u, u, e^{g_1(\log u,u)}, \dots, e^{g_k(\log u,u)}),$$

where

$$g_j(\log u, u) = \sum c_{m,n}(\log u)^m u^n, \text{ with } c_{m,n} \in \mathbb{Z} \text{ and } n < deg(p(y)).$$

As in the previous proofs without loss of generality we can assume $P(\log u, u)$ has the following polynomial expression

$$Q(\log u, u, e^{\delta_1}, \dots, e^{\delta_s})$$

where

$$\delta_j = (\log u)^m u^n \text{ with } n < deg(p(y)) = N, m \leq M, \tag{2.6}$$

and M is highest power of $\log u$ which appears in Q. Let $L = (M+1)(N+1)$, and let

$$\delta_1, \dots, \delta_L$$

be the L monomials constructed in (2.6). Clearly, $\delta_1, \dots, \delta_L$ are linearly independent over $\mathbb{Q}$, so by (SC)

$$t.d.\mathbb{Q}(\delta_1, \dots, \delta_L, e^{\delta_1}, \dots, e^{\delta_L}) \geq L$$

and

$$t.d.\mathbb{Q}(\delta_1, \dots, \delta_L, e^{\delta_1}, \dots, e^{\delta_L}) \leq 2L.$$

We observe that from transcendence of $\log u$, the transcendence degree of the L monomials over $\mathbb{Q}$ is 1. Moreover, from $u = \log u$ it follows

$$t.d.\mathbb{Q}(\delta_1, \dots, \delta_L, e^{\delta_1}, \dots, e^{\delta_L}) = L;$$

and so $ker\varphi = \langle p(y), e^x - y\rangle^E$.

For the inductive step we use the control operators previously introduced, and it can be proved that the kernel has stabilized at step $k = 1$.

□

Chapter 3

Decidability issues in exponential rings

3.1 Introduction

As already mentioned in the first Chapter Schanuel's Conjecture (SC) has played a fundamental role also in decidability issues.

Indeed in mid 90's Macintyre and Wilkie in [8] used Schanuel's Conjecture in order to prove the decidability of the theory of the reals with exponentiation, a problem left open by Tarski in the 30's. They reduce the proof to showing that the theory of the exponential real field is axiomatized by two subtheories, where the first is recursive unconditionally and the second, which is the existential theory, is decidable modulo (SC).

Even if the statement of (SC) does not make explicit mention of iterated exponentials, in fact it has consequences for exponential terms.

In this Chapter we prove, modulo Schanuel's Conjecture, that there are algorithms which decide if two exponential polynomials in π are equal in $\mathbb{R}$ and if two exponential polynomials in π and i coincide in $\mathbb{C}$.

We also consider the problems of the decidability of the universal and existential theories of ER, and of some completions of it. These last problems seem very difficult since they are related to hard open problems such as Hilbert Tenth Problem for $\mathbb{Q}$, and to the theory of E-ideals which is still to be understood.

3.2 Decidability on $(\mathbb{C}, e^x)$

The following result is well known :

Theorem 3.2.1. *The theory of the exponential complex field $Th(\mathbb{C}, e^x)$ is undecidable.*

Indeed, we can define the integers in $(\mathbb{C}, e^x)$ as:

$$\{x : \forall y(E(y) = 1 \rightarrow E(xy) = 1)\}.$$

So, $Th(\mathbb{C}, e^x)$ is subject to all of Gödel's phenomena.

Remark 3.2.1. *Using again the above formula defining the integers in $(\mathbb{C}, e^x)$ the undecidability of ER follows by interpreting Robinson Arithmetic Q.*

Notice that modulo (SC), ER is not essentially undecidable because of Macintyre and Wilkie's result.

Now we consider the existential theory of $(\mathbb{C}, e^x)$, which we denote by $Th_\exists(\mathbb{C}, e^x)$. It is not known if $Th_\exists(\mathbb{C}, e^x)$ is decidable or not, but we now explain why an undecidability result is more plausible than a decidable one.

We recall a fundamental undecidability result.

Hilbert Tenth Problem: Is there an algorithm which decides for any given polynomial $p(\overline{x}) \in \mathbb{Z}[\overline{x}]$ if there are integer solutions?

The negative answer to this question was given in 1971 by Matijasevic, Robinson, Davis, and Putnam (MRDP-Theorem).

Theorem 3.2.2. *$Th_\exists(\mathbb{C}, e^x)$, with π as a distinguished constant, is undecidable.*

Proof: Having a constant for π in the language allows us to existentially define the integers in $(\mathbb{C}, e^x)$ as follows

$$\varphi(\pi, x) = \exists y(E(2\pi xy) - 1 = 0).$$

Suppose now by contradiction that $Th_\exists(\mathbb{C}, e^x)$ is decidable, so we have that there exists an algorithm for deciding:

$$\exists x(\varphi(\pi, x) \wedge p(x) = 0),$$

where $p(x) \in \mathbb{Z}[x]$. But this contradicts MRDP-Theorem.

□

A natural question is:

Open Problem: Is $\mathbb{Z}$ existentially definable in $(\mathbb{C}, e^x)$?

The goal is to remove the constant for π, but this does not seem an easy task.

Remark 3.2.2. *For what concerns the decidability of $Th_\exists(\mathbb{C}, e^x)$, we observe that the rational field is existentially definable in $(\mathbb{C}, e^x)$ by the formula:*

$$\varphi(x) = \exists t \exists y \exists z(E(y) = E(z) = 1 \wedge t(y - z) = 1 \wedge xz = y).$$

So, the decidability of the existential theory of $(\mathbb{C}, e^x)$, would imply a positive answer to Hilbert's Tenth Problem for $\mathbb{Q}$, which is still one of the most important Open Problems.

3.3 Decidability problems for subtheories of ER

In the next sections we will examine the universal and existential theories of E-rings which we denote by $ER_\forall$ and $ER_\exists$, respectively.

3.3.1 The universal theory of E-rings

The general universal sentence in the language of E-rings has the form:

$$\forall x_1 \dots \forall x_n[(p_1(\overline{x}) = \dots = p_k(\overline{x}) = 0 \rightarrow q(\overline{x}) = 0)],$$

where $p_i(\overline{x}), q(\overline{x}) \in [\overline{x}]^E$, $i = 1, \dots, k$. We want to know if there is an algorithm which decides on the following:

$$ER \vdash \forall x_1 \dots \forall x_n[(p_1(\overline{x}) = \dots = p_k(\overline{x}) = 0 \rightarrow q(\overline{x}) = 0)].$$

Remark 3.3.1. *The decidability of $ER_\forall$ is reducible to the* ***word problem*** *for E-rings. We recall that the word problem for commutative rings is solvable and it reduces to ideal membership (see [17]).*

Open Problem: Is $ER_\forall$ decidable?

Macintyre conjectures a positive answer to this question. It does not seem an easy problem to solve, since it reduces to E-ideal membership, and as already mentioned in Chapter 2, the theory of E-ideals is not completely understood yet.

3.3.2 The existential theory of E-rings

We are interested in the existential theory of E-rings, i.e. $ER_\exists$. The general existential sentence has the form

$$\exists \overline{x}(p(\overline{x}) = 0 \wedge q(\overline{x}) \neq 0),$$

where $p(\overline{x}), q(\overline{x}) \in [\overline{x}]^E$, the free E-ring on $\overline{x}$. We want to know if there is an algorithm which decides on the following:

$$ER \vdash \exists \overline{x}(p(\overline{x}) = 0 \wedge q(\overline{x}) \neq 0). \tag{3.1}$$

There is not yet any definite answer to this question. We only have the following remarks. ER is an equational theory and for equational theories the following result due Herbrand holds.

Theorem 3.3.1. *(Herbrand) Let T be a universal theory. Then $T \vdash \exists \overline{x} \varphi(\overline{x})$ iff there exist tuples of closed terms $\overline{\tau_1}, \ldots, \overline{\tau_k}$ such that $T \vdash \varphi(\overline{\tau_1}) \vee \ldots \vee \varphi(\overline{\tau_k})$.*

Then we have

$$ER \vdash \exists \overline{x}(p(\overline{x}) = 0 \wedge q(\overline{x}) \neq 0) \text{ iff}$$

there are tuples of closed terms $\overline{\tau}_1, \ldots, \overline{\tau}_k$ such that

$$ER \vdash (p(\overline{\tau}_1) = 0 \wedge q(\overline{\tau}_1) \neq 0) \vee \ldots \vee (p(\overline{\tau}_k) = 0 \wedge q(\overline{\tau}_k) \neq 0).$$

Clearly $ER \vdash \exists \overline{x}(p(\overline{x}) = 0 \wedge q(\overline{x}) \neq 0)$ holds iff in all E-rings $\exists \overline{x}(p(\overline{x}) = 0 \wedge q(\overline{x}) \neq 0)$ is true, and in particular in the free E-ring on no generators. If the existential formula is positive (i.e. no inequalities are present) then a term, which instantiates the formula in the free E-ring on no generators, instantiates the formula in all E-rings. Hence we have that the positive $ER_\exists$ theory coincides with the positive $Th_\exists([\emptyset]^E)$.

3.4 Decidability of exponential terms

Now we focus on the decidability of exponential terms, or equivalently of exponential polynomials over $\mathbb{Z}$.

3.4.1 Normal form

Any term in the language of rings is canonically equivalent to a polynomial over $\mathbb{Z}$, modulo the axioms for commutative rings with 1. The identity $p(X_1, \ldots, X_n) = q(X_1, \ldots, X_n)$ holds in $\mathbb{Z}$ if and only if it holds in all commutative rings with 1, and this can be effectively tested. So, the free commutative ring $\mathbb{Z}[X_1, \ldots, X_n]$ has solvable word problem. This follows from the construction of the free object which gives a normal form for any element of $\mathbb{Z}[X_1, \ldots, X_n]$.

The word problem for a polynomial ring over a commutative ring R, can be stated as follows:

$$\text{If } p(X_1, \ldots, X_n), q(X_1, \ldots, X_n) \in R[X_1, \ldots, X_n] \text{ can we decide if}$$
$$p(X_1, \ldots, X_n) = q(X_1, \ldots, X_n)?$$

The answer is positive, modulo the diagram of R.

Analogous results have been obtained in the case of exponential polynomials. From the construction of the free exponential ring any element of $[X_1, \ldots, X_n]^E$ has a normal form see the construction in 1.3 of Chapter 1, and this gives immediately the following result in [7]:

Theorem 3.4.1. *The word problem for the free E-ring $[X_1, \ldots X_m]^E$ is primitive recursive.*

Let (R, E) be an E-ring, let $[\overline{X}, \overline{Y}]^E$ be the free E-ring on $\overline{X} = X_1, \ldots, X_n$ and $\overline{Y} = Y_1, \ldots, Y_m$ and let $r_1, \ldots, r_m \in R$. Consider the function:

$$[X_1, \ldots, X_n, Y_1, \ldots, Y_m]^E \longmapsto R[X_1, \ldots, X_n, r_1, \ldots, r_m]^E$$

$$p(X_1, \ldots, X_n, Y_1, \ldots, Y_m) \longmapsto p(X_1, \ldots, X_n, r_1, \ldots, r_m).$$

From the data $r_1, \dots, r_m$ we want to decide, if $p(X_1, \dots, X_n, r_1, \dots, r_m) = 0$. In [7] the following theorem is proved:

Theorem 3.4.2. *For any countable R, $R[X_1, \dots X_m]^E$ the word problem is primitive recursive in the diagram of R.*

3.4.2 Decidability over $\mathbb{R}$

Now we examine the role played by Schanuel's Conjecture in decidability issues relatively to exponential terms over $\mathbb{R}$.

If we consider the elements in the free E-ring on no generators, it is not obvious to decide if for example,

$$e^{e^2-2} - e^5 = e^{2+e^{-5}}$$

since there may be some hidden exponential algebraic relations. In [4] it is proved that there are no hidden exponential algebraic relations among exponential constants assuming a suitable generalization of Schanuel's Conjecture for an exponential ring of characteristic 0, (see Theorem 2.2.1).

As a consequence the following corollary holds.

Corollary 3.4.1. (SC) *The E-subring of $\mathbb{R}$ generated by 1 has solvable word problem.*

We can interpret this result by saying that we can decide modulo (SC) when two exponential constants are equal in $\mathbb{R}$.

We have an analogous result for exponential polynomials in π. This result follows from Theorem 2.5.1. Indeed, Theorem 2.5.1 can be stated also by saying that, modulo

Schanuel's Conjecture, there are no algebraic relations among the elements in the E-subring of $\mathbb{R}$ generated by π. So we can decide if two exponential polynomials in π are equal in $\mathbb{R}$. We have the following result:

Corollary 3.4.2. *(SC) There is an algorithm for deciding if two exponential polynomials in π are equal in $\mathbb{R}$.*

Proof: Recall that

$$\varphi : [x]^E \longrightarrow (\mathbb{R}, E),$$

such that $\varphi(x) = \pi$, is injective. Let $p(\pi), q(\pi) \in \langle\pi\rangle^E$, $p(\pi) = q(\pi)$ if and only if $p(x) - q(x) \in Ker\varphi$. But the kernel of φ is $\{0\}$, that is π does not satisfy any exponential algebraic relation over $\mathbb{Z}$. Hence, we can decide if $p(\pi) = q(\pi)$ since $[x]^E$ has decidable has word problem.

□

3.4.3 Decidability over $\mathbb{C}$

We now examine a decidability result in the case of exponential polynomials in π and i over $\mathbb{C}$, as consequence of Schanuel's Conjecture. Theorem 2.4.1, proved in the Chapter 2, implies that (modulo (SC)) the only exponential algebraic relations between i and π are the known ones, i. e. $e^{i\pi} = -1$ and $i^2 = -1$. As a consequence we have the following decidability result.

Corollary 3.4.3. (SC) *There exists an algorithm for deciding if two exponential polynomials in π and i are equal in $\mathbb{C}$.*

Proof: We need a normal form for elements of $\langle\pi, i\rangle^E$, the free E-subring of $\mathbb{C}$ generated by i and π. This is obtained by recursion from the construction of the E-ring generated by i and π, and it proceeds as in Chapter 1. We define $R_0 = \mathbb{Z}[i,y]/\langle i^2+1\rangle$ (thinking of y as π). It is easy to verify that R_0 is isomorphic to $\mathbb{Z}[i,\pi]$ where the isomorphism is

$$\varphi : R_0 \to \mathbb{Z}[i,\pi]$$

$$y \mapsto \pi.$$

The morphism is surjective and from transcendence of π it is also injective. In this ring we consider the additive subgroup $A = \langle iy\rangle$ and its complement

$$B = \langle iy^n, y^m\rangle_{n,m\in\mathbb{N}} - \{iy\}.$$

So $\mathbb{Z}[i,\pi]$ is isomorphic to $A \oplus B$. Now we define the exponential function on $A \oplus B$ as follows:

Let $\alpha \in A \oplus B$, so $\alpha = a + b$ where $a \in A$ and $b \in B$, if $a = kiy$ for some $k \in \mathbb{Z}$, then we define $E(\alpha) = (-1)^k \cdot t^b$.

Now we can construct the object in the next step, $R_1 = R_0[t^{A\oplus B}]$. From Theorem 3.4.3 the only relations between i and π are $i^2 = -1$ and $e^{\pi i} = -1$ modulo Schanuel's Conjecture, so we have

$$R_1 = R_0[t^{A\oplus B}] \cong R_0[t^B] \cong \mathbb{Z}[i,\pi][e^{\mathbb{Z}[i,\pi]}].$$

In the next step we construct the group ring $R_2 = R_1[t^{R_1}]$, so we have immediately

$$R_2 \simeq \mathbb{Z}[i,\pi][e^{Z[i,\pi]}][e^{Z[i,\pi][e^{Z[i,\pi]}]}].$$

At each step we repeat the same construction freely since we have already dealt with the only two relations which exist between i and π. We have that the E-ring generated by i and π is isomorphic to the limit of R_k with $k \in \mathbb{N}$. So we have a complete description of the elements of $\langle i,\pi\rangle^E$. Then we can decide when two exponential polynomials in π

and i are equal in $\mathbb{C}$.

$\square$

Chapter 4

Exponential polynomials in a Zilber's field

4.1 Introduction

In this chapter we will work with "particular" exponential fields, that is with the class of exponentially algebraically closed fields with pseudo exponentiation introduced by Zilber in [28]. He constructed and studied the new structures inspired by the complex exponential field and by the Hrushovski construction of strongly minimal structures (see [13]). Zilber's main result concerning these structures is that the class of exponentially algebraically closed fields with pseudo exponentiation has a unique model in every uncountable cardinality. He conjectures that the complex exponential field is the unique model of cardinality $2^{\aleph_0}$. This conjecture can be interpreted into two different conjectures: the first is Schanuel's Conjecture, and the second is a conjecture involving solutions of exponential polynomials over $\mathbb{C}$. So, this would entail Schanuel's Conjecture, hence it is very hard to prove.

In support of Zilber's conjecture Marker proved in [9] that in the simple case of an exponential polynomial with only one iteration of exponentiation, it is true (see Theorem 2.6.2). Following this line of research we examine the case of polynomials over $\mathbb{C}$ with two iterations of exponentiation.

Moreover, we characterize when an exponential polynomial over a Zilber's field has no solutions in the field, obtaining an analogous result of Henson and Rubel for $(\mathbb{C}, e^x)$ (see [12]).

4.2 Zilber's fields

In this section we examine the new structures introduced by Zilber. The main ideas come from a construction of Hrushovski (1993) (see [14]) in order to refute Zilber's trichotomy conjecture (see [26]). The key idea of Hrushovski is to construct an expansion of a structure with a "good" notion of dimension by adding a function (or relation) which still has a "good" notion of dimension.

We will study the case where the new function added to the language is exponentiation. For this purpose we review the set of axioms due to Zilber (see [28]).

We start working with structures that are not closed under multiplication or exponentiation. In the language $\mathcal{L}^- = \{0, 1, \omega, +, \frac{1}{m}\cdot, V\}$, where $0, 1$ and ω are constants, $+$ is a binary function symbol, $\frac{1}{m}\cdot$, $m \in \mathbb{N} - \{0\}$ are unary functions, and V denotes a collection of n-predicate symbols for irreducible algebraic varieties over $\mathbb{Q}$.

Let $\mathcal{L} = \mathcal{L}^- \cup E$ where E is a binary relation symbol. Now we recall the definitions of some classes of structures.

Definition 4.2.1. *Let $\mathcal{E}$ be the class of all algebraically closed fields K of characteristic 0, where the symbols $0, 1, +, \frac{1}{m}$ $m \in \mathbb{N} - \{0\}, V$ have their natural interpretations, ω*

is interpreted in a transcendental element over $\mathbb{Q}$ *and* E *is the graph of a surjective homomorphism* $ex : (K, +) \longrightarrow (K^{\times}, \cdot)$.

We notice that contrary to previous chapters we take exponentiation only as the graph of a partial function.

Definition 4.2.2. *An* $\mathcal{L}$*-structure* A *is in* $sub\mathcal{E}$ *if there is a structure* $K \in \mathcal{E}$ *such that* $A \subseteq K$ *and the domain* D_A *of the restriction of exponentiation to* A *is a divisible subgroup of* A.

We now introduce the notion of predimension on the new structures $A \in sub\mathcal{E}$ following Hrushovski's original definitions.

Definition 4.2.3. *Let* $A \in sub\mathcal{E}$ *and* X *be a finite subset of* A. *We define the predimension*

$$\delta_A(X) = t.d.(X \cup ex_A(X)) - l.d.(X),$$

where $t.d.(X)$ is the transcendence degree of X over $\mathbb{Q}$ and $l.d.(X)$ is the linear dimension of X over $\mathbb{Q}$.

If X, Y are finite subsets of A we define by $\delta_A(X/Y) = \delta_A(X \cup Y) - \delta_A(Y)$.

We can extend in a natural way the Schanuel property also to fields in the class $\mathcal{E}$.

Definition 4.2.4. *We say that* $A \in sub\mathcal{E}^0$ *if* $A \in sub\mathcal{E}$ *and* $\delta_A(X) \geq 0$, *for all finite subset* X *of* D_A. *We say that* $K \in \mathcal{E}^0$ *if* $K \in \mathcal{E}$ *and* $K \in sub\mathcal{E}^0$, *i.e.*

$$\mathcal{E}^0 = \mathcal{E} \cap sub\mathcal{E}^0.$$

Remark 4.2.5. *Requiring* $\delta_A(X) \geq 0$ *is equivalent to the following version of Schanuel's Conjecture.*

(**Generalized Schanuel's Conjecture**) If $z_1, \ldots, z_n \in K$ are linearly independent over $\mathbb{Q}$, then the transcendence degree of $\mathbb{Q}(z_1, \ldots, z_n, E(z_1), \ldots, E(z_n))$ over $\mathbb{Q}$ is at least n.

We observe that Schanuel's original conjecture says that $(\mathbb{C}, e^x) \in \mathcal{E}^0$.

Definition 4.2.6. *Let $A \in sub\mathcal{E}^0$ and X be a subset of D_A. The dimension of X in A is*

$$dim_A(X) = min\{\delta_A(Y) : X \subseteq Y \subseteq_f A\}$$

Now we recall the concept of strong extension for partial E-domain.

Definition 4.2.7. *For $A, B \in sub\mathcal{E}$, we say that B is a strong extension of A if $A \subseteq B$ and the following two conditions hold:*

1. *$\delta_A(Y/Z) \leq \delta_B(Y/Z)$ for any $Y, Z \subseteq A$ and Y, Z finite;*

2. *$\delta_B(X/D_A) \geq 0$ where X is a finite subset of D_B.*

   If B is a strong extension of A we will write $A \leq B$.

Remark 4.2.8. *The first condition is necessary since A and B are only partial exponential domains: it could happen that $D_B \cap A \neq D_A$, that is the predimension δ_B of a finite subset of A is less than the corresponding predimension δ_A.*

Regarding condition 2), let the predimension of D_A be h. Let Y be the set obtained by extending D_A with finitely many linearly independent elements of D_B. Then the predimension of Y is greater or equal than h, that is the dimension does not change. This condition also assures that if A satisfies Schanuel's Conjecture so does B.

Zilber proved that if B is a strong extension of A then $dim_A(X) = dim_B(X)$ for all $X \subseteq D_A$ and X finite. If $D_A \subseteq D_B$, the converse is also true since in this case condition 1) of the definition is obvious.

If exponentiation is total we have the following equivalent definition of strong extension:

Definition 4.2.9. *Let K, F be exponential fields. We say that F is a strong extension of K if $K \subseteq F$ and $dim_K(X) = dim_F(X)$ for all $X \subseteq K$ where X is finite.*

For completeness we recall the following properties of a strong extension proved in [28]:

Lemma 4.2.10. *1. If $A \leq B$ and $B \leq C$, then $A \leq C$;*

2. If $(I, <)$ is a chain and $A_i \leq A_j$ for $i \leq j$, then $A_i \leq \cup_{i \in I} A_i$ for all $i \in I$.

In order to complete Zilber's axiomatization of his fields we need the following definitions.

Definition 4.2.11. *Let $A \in sub\mathcal{E}$. We say that A has a standard kernel if $\ker ex_A = \mathbb{Z} \cdot \omega$, where ω is transcendental over $\mathbb{Q}$. A has a full kernel if D_A contains all nth-roots of unity, for each $n \geq 1$.*

We will denote the structures in $\mathcal{E}^0$ with full standard kernels by $\mathcal{E}^0_{st}$, and those of $sub\mathcal{E}^0$ with full standard kernel by $sub\mathcal{E}^0_{st}$.

We observe that in the case of the exponential complex field, if $A = \mathbb{C}$ and $\omega = 2\pi i$, and if we consider ex_A the usual exponentiation in $\mathbb{C}$ but restricted to $D_A = \mathbb{Q} \cdot \omega$, we obtain that such structure is in $sub\mathcal{E}^0_{st}$.

Now we introduce an example of an extension which is not a strong extension, that is we prove the following result:

Proposition 4.2.12. *(Assuming Schanuel's Conjecture) $(\mathbb{C}, e^x)$ is not a strong extension of $(\mathbb{R}, e^x)$.*

Proof: We will show that

$$dim_{\mathbb{R}}(\pi) \neq dim_{\mathbb{C}}(\pi).$$

We first prove that $dim_{\mathbb{C}}(\pi) = min\{\delta_{\mathbb{C}}(X) : \pi \in X \subseteq \mathbb{C}, X \text{ finite }\} = 0$. Indeed, it is enough to consider $X = \{\pi, i\pi\}$, so we have $\delta_{\mathbb{C}}(\pi, i\pi) = t.d.(\pi, i\pi, e^{\pi}, e^{i\pi}) - l.d.(\pi, i\pi) = 0$ (here we use Schanuel's Conjecture). Suppose by contradiction that $(\mathbb{C}, e^x)$ is a strong extension of $(\mathbb{R}, e^x)$, so we have that $dim_{\mathbb{R}}(\pi) = 0$, that is $\delta_{\mathbb{R}}(Y) = 0$, for some $Y \subseteq \mathbb{R}$, and $\pi \in Y$. Let $Y = \{\pi, b_1, \ldots, b_n\}$ and suppose $l.d.(\pi, b_1, \ldots, b_n) = k+1$. We have that

$$\delta_{\mathbb{R}}(\pi, b_1, \ldots, b_n) = t.d.(\pi, b_1, \ldots, b_n, e^{\pi}, e^{b_1}, \ldots, e^{b_n}) - l.d.(\pi, b_1, \ldots, b_n) = 0,$$

that is $t.d.(\pi, b_1, \ldots, b_n, e^{\pi}, e^{b_1}, \ldots, e^{b_n}) = l.d.(\pi, b_1, \ldots, b_n) = k+1$. Then

$$t.d.(\pi, i\pi, b_1, \ldots, b_n, e^{\pi}, e^{i\pi}, e^{b_1}, \ldots, e^{b_n}) = k+1$$

since $i\pi$ is algebraic over $\mathbb{Q}(\pi)$ and $e^{i\pi} = -1$. But $l.d.(\pi, i\pi, b_1, \ldots, b_k) = k+2$ and we get a contradiction since we are assuming Schanuel's Conjecture. Hence $(\mathbb{C}, e^x)$ is not a strong extension of $(\mathbb{R}, e^x)$.

□

Remark 4.2.13. *In general, if we consider an E-field K the quotient over an E-ideal is not a strong extension of K.*

Indeed, if we consider $(\mathbb{R}, e^x)$ *and the E-ideal generated by* y^2+1*, the quotient*

$$R^* = \frac{\mathbb{R}[x]^E}{\langle y^2+1 \rangle^E}$$

is not a strong extension of $(\mathbb{R}, e^x)$*, since*

$$\dim_{\mathbb{R}}(\pi) \neq \dim_{R^*}(\pi).$$

Indeed, we saw that $\dim_{\mathbb{R}}(\pi) \neq 0$*, while* $\dim_{R^*}(\pi) = 0$*, since* $\delta_{R^*}(\pi, i\pi) = 0$.

In order to define the class of exponentially algebraically closed fields with pseudo exponentiation we now introduce some conditions on varieties $V \subseteq K^{2n}$ which will ensure that there is $(z_1, \ldots, z_n) \in K^n$ such that $(z_1, \ldots, z_n, ex(z_1), \ldots, ex(z_n)) \in V$.

We will denote the algebraic group by $G_n(K) = K^n \times (K^n)^*$. Given a $k \times n$ matrix of integers $T = (a_{ij})$, we denote

$$[T] : G_n(K) \longrightarrow G_k(K)$$

the homomorphism map given by

$$\langle z_1, \ldots, z_n, w_1, \ldots, w_n \rangle \longmapsto \langle z_1', \ldots, z_k', w_1', \ldots, w_k' \rangle$$

where

$$z_i' = a_{i1} z_1 + \ldots + a_{in} z_n$$

and

$$w_i' = w_1^{a_{i1}} \cdot \ldots \cdot w_n^{a_{in}}$$

for $i = 1, \ldots, k$.

Definition 4.2.14. *The variety $V \subseteq G_n(K)$ is normal if $dim V' \geq k$, where $V' = [T](V)$ for any $k \times n$ matrix T of rank k where $1 \leq k \leq n$, or equivalently $t.d.\mathbb{Q}(z_1', \ldots, z_k', w_1', \ldots, w_k') \geq k$.*

Definition 4.2.15. *The variety $V \subseteq G_n(K)$ is free if we cannot find $a_1, \ldots, a_n \in \mathbb{Z}$ and $b, d \in K$ with $d \neq 0$ such that V is contained in either variety*

$$\{(\overline{z}, \overline{w}) : a_1 z_1 + \ldots + a_n z_n = b\}$$

or

$$\{(\overline{z}, \overline{w}) : w_1^{a_1} \cdot \cdots \cdot \ldots \cdot \cdots \cdot w_n^{a_n} = d\}.$$

After introducing these conditions on varieties we can define a new class of structure.

Definition 4.2.16. *Let $K \in \mathcal{E}^0_{st}$. K is exponentially algebraically closed if whenever $W \subset V \subseteq G_n(K)$ are irreducible varieties defined over K and $F \in \mathcal{E}^0_{st}$ with F strong extension of K and $\overline{z} \in F$ such that $(\overline{z}, ex(\overline{z})) \in V \backslash W$ then there is $\overline{a} \in K$ with $(\overline{a}, ex(\overline{a})) \in V \backslash W$.*

Zilber proves the following characterization of an exponentially algebraically closed field.

Proposition 4.2.17. *Let $K \in \mathcal{E}^0_{st}$. K is exponentially algebraically closed if and only if for every variety $V \subseteq G_n(K)$ defined over K that is irreducible, normal and free there is $\overline{z} \in K^n$ such that $(\overline{z}, ex(\overline{z})) \in V$.*

This means that K is exponentially algebraically closed if and only if the variety intersects the graph of exponentiation.

The last definition we consider is the class of strongly exponentially algebraically closed structures.

Definition 4.2.18. *If $K \in \mathcal{E}^0_{st}$ we say that K is strongly exponentially algebraically closed if for any irreducible, free, normal variety $V \subseteq G_n(K)$ defined over a finite subset A of K with $\dim V = n$ there is $\overline{a} \in K^n$ such that $(\overline{a}, ex(\overline{a})) \in V(K)$ and $(\overline{a}, ex(\overline{a}))$ is generic over K.*

This means that K is strongly exponentially algebraically closed if and only if the variety intersects the graph of exponentiation in a generic point.

Definition 4.2.19. *We say that a structure $K \in \mathcal{E}$ satisfies the countable closure property if for all $A \subseteq K$ if $V \subseteq G_n(K)$ irreducible, normal ad free with $\dim V = n$ and* *defined over the definable closure of* A, *then* $\{(\overline{a}, ex(\overline{a})) \in V : \text{ generic over } A\}$ *is countable.*

Let T denote the theory of the class of strongly exponentially algebraically closed fields which satisfy the countable closure property.

Remark 4.2.20. *T is not first order, since for example in the case of the countable closure axiom we need a quantifier Q for "there exist countably many". Thus T is an $L_{\omega_1,\omega}(Q)$-theory.*

In this context Zilber proved an important categoricity result for the class of models of T, that is he showed:

Theorem 4.2.1 (Zilber). *For all uncountable cardinals κ there is a unique model of T of cardinality κ.*

So, a very natural and fundamental question is:

Open Problem: Is $(\mathbb{C}, e^x)$ the unique model of T of cardinality $2^{\aleph_0}$?

Zilber's Conjecture: YES.

In support to his conjecture using Ax's work (see [1]) and Schanuel's Conjecture for differential fields, he proved in [28] the following result:

Theorem 4.2.2. $(\mathbb{C}, e^x)$ *satisfies the countable closure property.*

Remark 4.2.21. *Assuming Schanuel's Conjecture the axiom of strong exponential closure for the $(\mathbb{C}, e^x)$ is the only impediment to prove Zilber's Conjecture.*

In this context in the next section we will investigate some cases of the strong exponential closure for $(\mathbb{C}, e^x)$. The simplest case has been studied by Marker in [9]. He proved that the set of solutions of a polynomial $f(x, y) \in \mathbb{C}[x, y]$ intersects the graph of exponentiation in a countable infinite set of generic points, which are moreover algebraically independent over $\mathbb{Q}$.

4.3 Solutions of exponential polynomials over $\mathbb{C}$

In this section we make a further step along the line of Marker's result in [9].

We consider the exponential field $(\mathbb{C}, e^x)$. Let $\sum$ be a finite system in $z_1, \ldots, z_n$ variables and consisting of equations over $\mathbb{C}$ involving $(+, \cdot, -, 0, 1, e^x)$.

The questions are:

1. When does $\sum$ have solutions?
2. What is the structure of the solution set?

More in particular we want to answer the following question:

Let $p(x, y) \in \mathbb{Q}[x, y]$, does there exist a complex number z such that $p(z, e^{e^z}) = 0$?

For this purpose we consider the corresponding system in four variables:

$$\overline{\sum} = \begin{cases} p(z_1, w_2) = 0 \\ w_1 = z_2 \end{cases} \tag{4.1}$$

Theorem 4.3.1. *1. Let $f(z) = p(z, e^{e^z})$. The function f has always a solution in $\mathbb{C}$ unless $p(x, y) = \alpha y^k$, where $\alpha \in \mathbb{Q}$.*

2. *Assuming Schanuel's Conjecture, any point (w, e^{e^w}) such that $p(w, e^{e^w}) = 0$ with w, e^w linearly independent, is a generic point.*

Proof:

1. By contradiction we assume that $f(z) = p(z, e^{e^z}) \neq 0$, for all $z \in \mathbb{C}$. Henson and Rubel in [12] proved that if $t(z)$ is a term-function over $\mathbb{C}$ which is never 0 then $t(z) = e^{s(z)}$ for some term $s(z)$ over $\mathbb{C}$. In our context then we have that:

$$f(z) = p(z, e^{e^z}) = e^{g(z)}, \tag{4.2}$$

where $g(z) \in \mathbb{C}[z]^E$. Let

$$p(x, y) = \sum_{n,m} \alpha_{n,m} x^n y^m,$$

where $\alpha_{n,m} \in \mathbb{Q}$. So we have $p(z, e^{e^z}) = \sum_{n,m} \alpha_{n,m} z^n e^{me^z} = e^{g(z)}$, where $g(z) \in \mathbb{C}[z]^E$ and by identity of polynomials, this is true if and only if the polynomial p is of the following form:

$$p(x, y) = \alpha y^k, \text{ for some } k \in \mathbb{N} \text{ and } \alpha \in \mathbb{Q}.$$

So, unless p is a polynomial in only the variable y, p has a solution of the form (w, e^{e^w}).

2. Now we need to assume Schanuel's Conjecture. Let w be a solution of $f(z) = 0$. We want to show that (w, e^{e^w}) is a generic point of the associated variety under the assumption that the complexes w and e^w are linearly independent over $\mathbb{Q}$. By Schanuel's Conjecture we have:

$$t.d.\mathbb{Q}(w, e^w, e^w, e^{e^w}) \geq 2.$$

Indeed, the transcendence degree is exactly 2 since w and e^{e^w} are algebraically dependent being solution of the polynomial $p(x, y)$. So the transcendence degree coincides with the dimension of the curve $p(x, y) = 0$. Then the point $(w, e^w, e^w e^{e^w})$ is generic.

□

Remark 4.3.1. *We have proved that the variety*

$$V = \begin{cases} p(z_1, w_2) = 0 \\ w_1 = z_2 \end{cases} \tag{4.3}$$

associated to the function $f(z)$ intersects the graph of exponentiation in a generic point, under the hypothesis that w and e^w are linearly independent. So, under this hypothesis we have that the strong exponential closure is satisfied relative to $f(z) = p(z, e^{e^z})$ for the exponential field $(\mathbb{C}, e^x)$. We, probably, can eliminate this hypothesis. We suppose now that w, e^w are linearly dependent, this means that

$$e^w = \frac{m}{n} w$$

where $m, n \in \mathbb{Z}$, $n \neq 0$. In particular w and e^w are algebraically dependent. We notice that w is necessarily transcendental over $\mathbb{Q}$. If not, then e^w is a root of a polynomial $q(w, y)$ in $\mathbb{Q}^{alg}[y]$, hence e^w is algebraic over $\mathbb{Q}$. This contradicts the Lindemann Weierstrass Theorem.

Now we suppose $m = n = 1$, hence $e^w = w$, and then $e^{e^w} = w$. So the polynomial $p(z, e^{e^z})$ reduces to a polynomial in one variable over $\mathbb{Q}$ evaluated at w, and this value is 0, so w is algebraic over $\mathbb{Q}$, which we showed is impossible (unless $p(x, y)$ is reducible).

Now, suppose $e^w = \frac{m}{n} w$, with $m \neq n$ we distinguishes two cases:

1. *If n divides m we can argue as in the previous case of $m = n = 1$, and we get again w algebraic over $\mathbb{Q}$ which we do not want. (Notice that this is the case for both positive and negative m).*

2. *If m and n are coprime, we have*

$$e^{ne^w} = w^m (\frac{m}{n})^m.$$

Let

$$q(x, y) = y^n - (\frac{m}{n})^m x^m.$$

In the previous theorem we assumed the polynomial $p(x, y)$ irreducible, so in particular $p(w, y)$ is irreducible over $Q(w)[y]$. We observe that if (w, e^{e^w}) is a solution of $p(x, y)$ then (w, e^{e^w}) is also a solution of $q(x, y)$. Hence $p(w, y)$ divides $q(w, y)$. But m, n are coprime and this implies that $q(x, y)$ is irreducible, hence the polynomial q is essentially p, then

$$p(w, y) = y^n - (\frac{m}{n})^m w^m.$$

In this particular case, we have that a solution w such that $e^w = \frac{m}{n} w$ will not be a generic solution of the 4.4.1, since the dimension of the variety and the dimension of the point are different, being 2 and 1 respectively. Probably using Nevanlinna Theory we can prove that there is a generic solution, by showing that not all solutions w of $f(z) = p(z, e^{e^z}) = 0$ are such that w and e^w are linearly dependent.

Open Problem: Has $f(z) = p(z, e^{e^z})$ infinitely many solutions in $\mathbb{C}$?

4.4 Solutions of exponential polynomials over a Zilber's field

In this section we investigate the set of solutions of exponential polynomials over a Zilber's field.

We consider the problem studied over $\mathbb{C}$ in the previous section, but relative to a Zilber's field. We want to study when $p(z, e^{e^z}) = 0$ has a solution in a Zilber's field K, where $p(x, y) \in K[x, y]$.

It is necessary to consider the following variety $V \subseteq K^2 \times (K^2)^*$

$$V = \begin{cases} p(z_1, w_2) = 0 \\ w_1 = z_2 \end{cases} \tag{4.4}$$

We observe that the variety (4.4) has dimension 2. In order to show that there is a solution of $f(z) = p(z, e^{e^z}) = 0$ in K, it is enough to show that the variety V is normal and free, since then from Zilber's axioms the variety V intersects the graph of exponentiation in a generic point. We prove the following result:

Theorem 4.4.1. *The variety* $V \subseteq K^2 \times (K^2)^*$ *defined by*

$$V = \begin{cases} p(z_1, w_2) = 0 \\ w_1 = z_2 \end{cases}$$

is normal and free unless $p(u, v) = \alpha v^h$ *for some* $h \in \mathbb{N}$ *and* $\alpha \in K$.

Proof: We first show that the variety is normal, i.e. for every $(z_1, z_2, w_1, w_2) \in V$, and for any $k \leq 2$, we have to prove that $t.d.\mathbb{Q}(z'_1, \ldots, z'_k, w'_1, \ldots, w'_k) \geq k$.

For $k = 1$ we have

$$z' = mz_1 + nz_2$$

and

$$w' = w_1^m \cdot w_2^n,$$

for some $m, n \in \mathbb{Z}$. Suppose by contradiction that $t.d.\mathbb{Q}(mz_1 + nz_2, w_1^m \cdot w_2^n) = 0$, this means that there exist two polynomials $q(x), s(x) \in \mathbb{Q}[x]$ such that

$$q(mz_1 + nz_2) = 0, \text{ and } s(w_1^m \cdot w_2^n) = 0, \tag{4.5}$$

and this implies that there are polynomials $q'(x, y), s'(x, y) \in \mathbb{Q}[x, y]$ such that

$$q'(mz_1, nz_2) = 0, \text{ and } s'(w_1^m, w_2^n) = 0. \tag{4.6}$$

But $(z_1, z_2, w_1, w_2) \in V$, so (4.6) implies that z_1, z_2 are algebraically dependent over $\mathbb{Q}$, and w_1, w_2 are algebraically dependent over $\mathbb{Q}$, and this cannot happen since dimension of V is 2.

For $k = 2$ we have

$$z_1' = mz_1 + nz_2, \; z_2' = sz_1 + tz_2$$

and

$$w_1' = w_1^m \cdot w_2^n, \; w_2' = w_1^s \cdot w_2^t.$$

Suppose by contradiction that

$$t.d.\mathbb{Q}(z_1', z_2', w_1', w_2') = t.d.\mathbb{Q}(mz_1 + nz_2, sz_1 + tz_2, w_1^m \cdot w_2^n, w_1^s \cdot w_2^t) = 1.$$

Fixed m, n greater or equal than s, t respectively, and without loss of generality, this means that there exists a polynomial $q(x, y, z) \in \mathbb{Q}(z_2')[x, y, z]$ such that,

$$q(mz_1 + nz_2, w_1^m \cdot w_2^n, w_1^s \cdot w_2^t) = 0. \tag{4.7}$$

This implies that there is a polynomial $q'(x,y,z,t) \in \mathbb{Q}[x,y,z,t]$ such that

$$q'(mz_1, nz_2, w_1^m, w_2^n) = 0. \tag{4.8}$$

but the point $(z_1, z_2, w_1, w_2) \in V$, so (4.8) is in contradiction with the dimension of V.

Now we prove that the variety is also free. Suppose that there exist $m_1, m_2 \in \mathbb{Z}$ and $b \in K$ such that

$$V \subseteq \{m_1 z_1 + m_2 z_2 = b\}$$

or there exist $m_1, m_2 \in \mathbb{Z}$ and $d \in K$ with $d \neq 0$ with

$$V \subseteq \{w_1^{m_1} \cdot w_2^{m_2} = d\}.$$

In the first case, from $p(z_1, w_2) = 0$ it follows that $m_1 = 0$ or $m_2 = 0$, hence $m_2 z_2 = b$ or $m_1 z_1 = b$. In both cases we a get a contradiction with the dimension of the variety, e. g. if $m_1 = 0$, we have $z_1 = \frac{b}{m_1}$ so, $p(\frac{b}{m_1}, w_2) = 0$, hence the dimension of the variety is 1, and this is a contradiction.

In the second case, from $p(z_1, w_2) = 0$ it follows that $m_1 = 0$ or $m_2 = 0$ and again we get a contradiction with the dimension of the variety. Indeed, if $m_2 = 0$ we have $w_1^{m_1} = d$. Moreover, $z_2 = w_1$ and so $z_2^{m_1} = w_1^{m_1}$, that is $z_2^{m_1} = d$. But K is an algebraically closed field, so let z_2 be one of the $m_1 th$-root of d, hence we have that the dimension of the variety is 1, and this is a contradiction.

The variety V is free unless z_1 does not occur in the polynomial p, in which case we have that the polynomial p has no solution, and the polynomial $p(u,v) \in K[u,v]$ is necessarily equal to αv^h for some $h \in \mathbb{N}$.

□

We now consider a different case, let $p(z,u,v) \in K[z,u,v]$. We want to answer the following question:

Does there exist an element $\alpha \in K$ such that $p(\alpha, e^{\alpha}, e^{e^{\alpha}}) = 0$?

In answering this question we proceed as before appealing to the axioms introduced by Zilber.

Theorem 4.4.2. *The variety* $V \subseteq K^2 \times (K^2)^*$ *defined by*

$$V = \begin{cases} p(z_1, z_2, w_2) = 0 \\ w_1 = z_2 \end{cases}$$

is normal and free, unless $p(z, u, v) = \delta u^h v^t$ *for some* $h, t \in \mathbb{N}$ *and* $\delta \in K$.

Proof: We first prove that V is normal, i.e. for any $k \leq 2$, we have to prove that $t.d.\mathbb{Q}(z'_1, \ldots, z'_k, w'_1, \ldots, w'_k) \geq k$. Suppose by contradiction that

$$t.d.\mathbb{Q}(z'_1, \ldots, z'_k, w'_1, \ldots, w'_k) < k.$$

For $k = 1$, suppose that

$$t.d.\mathbb{Q}(z'_1, w'_1) = 0$$

so, we have that there exist two polynomials $q(x), s(x) \in \mathbb{Q}[x]$ such that

$$q(mz_1 + nz_2) = 0, \text{ and } s(w_1^m \cdot w_2^n) = 0.$$

This implies that

$$q'(mz_1, nz_2) = 0, \text{ and } s'(w_1^m, w_2^n) = 0, \tag{4.9}$$

where $q'(x, y), s'(x, y) \in \mathbb{Q}[x, y]$. But $(z_1, z_2, w_1, w_2) \in V$, so (4.9) implies that z_1, z_2 are algebraically dependent over $\mathbb{Q}$ and w_1, w_2 are algebraically dependent over $\mathbb{Q}$, and this cannot happen since dimension of V is 2.

Now we suppose that for $k = 2$ $t.d.\mathbb{Q}(z'_1, z'_2, w'_1, w'_2) = 1$, and we proceed as in the previous theorem getting a contradiction.

For the freeness we proceed as in the previous theorem. The variety V is free unless z_1 does not occur in the polynomial p. This implies that the polynomial p does not have any zero, and the polynomial $p(z,u,v) \in K[z,u,v]$ is necessarily equal to $\delta u^h v^k$ for some $h, k \in \mathbb{N}$.

□

Remark 4.4.1. *The two previous results show that an exponential polynomial over a Zilber's field K has always a solution in K unless it is of a particular form. From Theorem 4.4.2 it follows that the function $f(z) = p(z, e^z, e^{e^z})$ has always a solution in K unless $f(z) = \delta e^{H(z)}$, where $H(z) \in K[z]^E$ and $\delta \in K$.*

This characterization reminds a result of Henson and Rubel for $(\mathbb{C}, exp)$ which we will recall in the next section. Their result was the starting point to characterize those exponential polynomials over a Zilber's field K which have no solutions in K.

We notice that also in the case of the polynomial $p(z, e^z)$ analyzed by Marker in [9], the function $f(z) = p(z, e^z)$ has always a zero in a Zilber's field, since the variety defined by $p(z,u) = 0$ is normal and free, unless $p(u,v) = \alpha v^h$ for some $h \in \mathbb{N}$.

4.4.1 Characterization of exponential polynomials in a Zilber's field

Until now we have been interested in solutions of exponential polynomials in a Zilber's field.

In this section we switch to consider the opposite question, i.e. we want to give a necessary and sufficient condition in order to characterize when an exponential polynomial has no solutions in a Zilber's field.

For the complex exponential field such characterization exists and it is due to Henson and Rubel in [12]:

Let $F(z_1, \ldots, z_n) \in \mathbb{C}[z_1, \ldots, z_n]^E$

$$F(z_1, \ldots, z_n) \text{ has no roots in } \mathbb{C} \text{ iff } F(z_1, \ldots, z_n) = e^{G(z_1, \ldots, z_n)},$$

where $G(z_1, \ldots, z_n) \in \mathbb{C}[z_1, \ldots, z_n]^E$.

In the proof Henson and Rubel use Nevanlinna Theory and, moreover, in their paper they claim that no direct algebraic approach can be used instead of Nevanlinna Theory. As we will see, if Zilber's Conjecture is true, we give an alternative proof of their result using only algebraic methods.

In the last two theorems of previous section we proved that in some special cases (with at most two iterations of exponentials) Henson and Rubel's result holds for a Zilber's field, but we want a general result.

Our first attempt of the proof was based essentially on Zilber's axioms, examining when the variety defined by the polynomial in consideration was normal and free. But this approach has revealed unsatisfactory for a general polynomial.

Recall the following definition (see also Chapter 1).

Definition 4.4.2. *Let $F(\overline{z}) \in K[\overline{z}]^E$. We say that $F(\overline{z})$ is prime if the quotient $\frac{K[\overline{z}]^E}{\langle F \rangle^E}$ is a domain. In fact it is an E-domain.*

Remark 4.4.3. *If we consider the augmentation map from $K[\overline{z}]^E$ into to the polynomial ring $K[\overline{z}]$ we have that the augmentation ideal I (which we recall is an E-ideal) is prime, since the quotient $\frac{K[\overline{z}]^E}{I}$ is isomorphic to $K[\overline{z}]$ which is a domain. (In the classical case we have that I is also a maximal ideal).*

Now we state our main result:

Theorem 4.4.3. *Let $F(z_1,\ldots,z_n) \in K[z_1,\ldots,z_n]^E$ where K is a Zilber's field and F is prime and irreducible. Then*

$$F(z_1,\ldots,z_n) \text{ has no roots in } K \text{ iff } F(z_1,\ldots,z_n) = e^{H(z_1,\ldots,z_n)},$$

where $H(z_1,\ldots,z_n) \in K[z_1,\ldots,z_n]^E$

Proof: ($\Leftarrow$) obvious.

We first prove the following lemma which will be useful for the proof of the other implication. In the sequel (R, D, E) will denote a characteristic 0 domain R with a partial exponential function E defined on R whose domain is D.

Lemma 4.4.4. *Let (R, D, E) be a partial E-domain where D=dom(E), $\mathbb{Q} \subseteq R$ and suppose (R, D, E) satisfies Schanuel's Conjecture. Let S be a domain extending R. There is a subset D_1 of S with $D_1 = D \oplus \mathbb{Q} \cdot t$ for some $t \in S$ such that (S, D_1, E_1) is a partial E-domain with E_1 extending E, satisfying (SC), and moreover (S, D_1, E_1) is a strong extension of (R, D, E).*

Proof: We want to prove that there exists an element ω in some algebraically closed field extending S, which is transcendental over S and which we will use to extend the domain of E in S. For this purpose we consider the following infinite set Σ of formulas in the language of $\mathcal{L}_{\mathcal{S}} \cup \{\omega_r\}_{r\in\mathbb{Q}}$ where ω_r are new constant symbols:

$$\Sigma = Diag(S) \cup \Gamma \cup \Delta \tag{4.10}$$

where $\Gamma = \{F(\omega_1) \neq 0 : F(x) \in S[x]\}$, and $\Delta = \{\omega_{r_1} \cdot \omega_{r_2} = \omega_{r_1+r_2} : r_1, r_2 \in \mathbb{Q}\}$.

We want to find a model of this theory. We use the Compactness Theorem in order to find a such model. Let $\Sigma' \subseteq \Sigma$ and Σ' be finite. Let $F_1,\ldots,F_k$ be the polynomials of Γ which appear in Σ', i. e. $F_i(\omega_1) \neq 0$ for $i = 1,\ldots,k$, and let $\omega_{r_1} \cdot \omega_{r_2} = \omega_{r_1+r_2}$ where $r_1, r_2 \in T \subseteq \mathbb{Q}$ with T be finite subset of Δ contained in Σ'.

Let $r_1, \ldots, r_h$ be the rationals which appear in the finite subset T of $\mathbb{Q}$. Let

$$r_1 = \frac{p_1}{q_1}, \ldots, r_h = \frac{p_h}{q_h},$$

and let $N = q_1 \cdot \ldots \cdot q_h$. We have $r_1 = \frac{p'_1}{N}, \ldots r_h = \frac{p'_h}{N}$.

We can choose an element $\alpha \in S^{alg}$ such that $F_1(\alpha) \neq 0, \ldots, F_k(\alpha) \neq 0$. Define $\omega_{\frac{1}{N}}$ be one of the Nth roots of α, i. e. $\omega_{\frac{1}{N}} = \sqrt[N]{\alpha}$ and $\omega_{\frac{p'_i}{N}} = \sqrt[N]{\alpha^{p'_i}}$, for $i = 1, \ldots, h$. So, S^{alg} is a model of Σ'. By compactness there exists a model of Σ in which there is an element ω that is transcendental over S.

Let $D_1 = D \oplus \mathbb{Q} \cdot t$, where t is not in R. If $\gamma \in D_1$, then $\gamma = d + rt$, where d and r are uniquely determined and $d \in D$, $r \in \mathbb{Q}$. We extend E to D_1 in the freest possible way as follows

$$E_1(\gamma) = E(d) \cdot \omega^r,$$

where $\omega = E_1(t)$. E_1 extends the morphism E, and satisfies the axioms of exponentiation.

Now we want to prove that the extension is strong. We observe that in general, if $A \subseteq R \subseteq S$ and A finite, $\delta_R(A) \leq \delta_S(A)$, and for all $A \subseteq D$ we have that $dim_S(A) \leq dim_R(A)$. So, in order to prove that the extension is strong it is enough to show that $dim_S(A) \geq dim_R(A)$, for all finite $A \subseteq D$, that is there is no finite $Y \supseteq A$, with $Y \subseteq S$ such that $\delta_S(Y) \leq \delta_R(A)$. For this purpose, let $B = \{b_1, \ldots, b_k\} \subseteq D_1$. We can write uniquely $b_i = a'_i + r_i \cdot t$ for $i = 1, \ldots, k$, where $a'_i \in D$ and $r_i \in \mathbb{Q}$. We want to prove that if

$$\delta_R(A) = t.d.(A, E(A)) - l.d(A) = L.$$

then

$$\delta_S(A, b_1, \ldots, b_k) = t.d.(A, b_1, \ldots, b_k, E_1(A), E_1(b_1), \ldots, E_1(b_k)) - l.d(A, b_1, \ldots, b_k) \geq L.$$

If $r_i = 0$ for all $i = 1, \ldots, k$ then the dimensions are the same. If $r_i \neq 0$ for some i, without loss of generality we can analyze the case of a single $r \neq 0$, that is we consider

$b_1 = a'_1, \ldots, b_m = a'_m$, and $b_i = \alpha + rt$, with $\alpha \in D$ and $r \neq 0$. So we have

$$\delta_S(A, b_1, \ldots, b_k) =$$

$$t.d.(A, a'_1, \ldots, a'_m, \alpha + rt, E_1(A), E_1(a'_1), \ldots, E_1(a'_m), E_1(\alpha + rt)) +$$

$$-l.d(A, a'_1, \ldots, a'_m, \alpha + rt) =$$

$$t.d(A, a'_1, \ldots, a'_m, \alpha + rt, E(A), E(a'_1), \ldots, E(a'_m), E(\alpha) \cdot \omega^r) - l.d(A, a'_1, \ldots, a'_m, \alpha + rt),$$

and we want to show that this is greater or equal than L. Indeed,

$$l.d.(A, a'_1, \ldots, a'_m, \alpha + rt) = l.d(A, a'_1, \ldots, a'_m) + 1,$$

and also

$$t.d(A, a'_1, \ldots, a'_m, \alpha + rt, E(A), E(a'_1), \ldots, E(a'_m), E(\alpha) \cdot \omega^r) =$$

$$t.d(A, a'_1, \ldots, a'_m, \alpha + rt, E(A), E(a'_1), \ldots, E(a'_m)) + 1$$

from the transcendence of ω over S. So we have that the dimensions coincide, that is S is a strong extension of R. Moreover, we have that $\delta_S(Y) \geq 0$ for all finite $Y \subseteq S$, hence we have that Schanuel's Conjecture is true in (S, D_1, E_1).

□

We can generalize this result to a particular case, that is the case of group rings. Using a similar proof we can show the following result.

Proposition 4.4.5. *Let (R, A, E) be a partial E-domain, satisfying Schanuel's Conjecture. Then $(R_1, A \oplus B, E_1)$, where B is a divisible group and $R_1 = R[t^B]$ and E_1 is a natural extension of E, is a strong extension of R and satisfies Schanuel's Conjecture.*

Now we have all the ingredients to prove Theorem 4.4.3:

Proof of Theorem 4.4.3 ($\Rightarrow$) Suppose by contradiction that

$$F(z_1, \ldots, z_n) \neq e^{H(z_1, \ldots, z_n)},$$

for all $H(z_1, \ldots, z_n) \in K[z_1, \ldots, z_n]^E$.

We will prove that the polynomial F has a zero in K. By the construction of the exponential polynomial ring (see 1.1.3) we have that $F(z_1, \ldots, z_n) \in R_{k+1} = R_k[t^{B_k}]$ for some $k \in \mathbb{N}$, so $F = \sum_{n=1}^{N} a_n t^{b_n}$, where $a_n \in R_k$. We will prove that

$$\frac{K[z_1, \ldots, z_n]^E}{\langle F \rangle^E} = \frac{\bigcup R_h}{\langle F \rangle^E}$$

is isomorphic to the limit of a certain group rings. We will show that

$$\frac{\bigcup_h R_h}{\langle F \rangle^E} = \bigcup_h \left(\frac{R_h}{\langle F \rangle^E \cap R_h}\right),$$

and we will prove that the quotients

$$\frac{R_h}{\langle F \rangle^E \cap R_h}$$

can be viewed as group rings.

We need to distinguish the following two cases:

Case 1. The linear dimension of $(b_1, \ldots, b_N)_{\mathbb{Q}} > 1$. In this case we may assume $b_N \notin (b_1, \ldots, b_{N-1})_{\mathbb{Q}}$. We want to construct the quotient, so for this purpose we impose

$$F = \sum_{n=1}^{N} a_n t^{b_n} = 0,$$

and we have

$$t^{b_N} = -\frac{1}{a_N} \sum_{n=1}^{N-1} a_n t^{b_n}.$$

We observe that if $F \in R_{k+1}$ and F is irreducible then $\langle F\rangle^E \subseteq \bigcup_{h>k} R_h$. We have to construct

$$\frac{R_h}{\langle F\rangle^E \cap R_h},$$

for all $h \in \mathbb{N}$. Denote

$$\frac{R_h}{\langle F\rangle^E \cap R_h} = \widetilde{R}_h.$$

If $h \leq k$ we have $\widetilde{R}_h = R_h$, since $\langle F\rangle^E \cap R_h = \{0\}$.

At step $h = k$ we isolate $(b_N)_{\mathbb{Q}}$, so we split off $(b_N)_{\mathbb{Q}}$ as a summand of B_k, that is

$$B_k = \widetilde{B_k} \oplus (b_N)_{\mathbb{Q}},$$

and we can write

$$R_{k+1} = R_k[t^{\widetilde{B_k} \oplus (b_N)_{\mathbb{Q}}}] \cong R_k[t^{\widetilde{B_k}}][t^{(b_N)_{\mathbb{Q}}}].$$

We break the step $h = k$ into two new steps. In the first step we define

$$E_{k+\frac{1}{2}} : R_k \to \widetilde{\widetilde{R}}_{k+1} = R_k[t^{\widetilde{B}_k}],$$

and $E_{k+\frac{1}{2}}$ is defined as usual (see 1.1.3). In the second step we want to extend $E_{k+\frac{1}{2}}$ to $(b_N)_{\mathbb{Q}}$, using $F = 0$. Let $\alpha \in (b_N)_{\mathbb{Q}}$, so $\alpha = \frac{r}{s} \cdot b_N$ where $r, s \in \mathbb{Z}$, and $s \neq 0$. We define $\widetilde{E}_k(\frac{r}{s} \cdot b_N)$ in the only possible way determined by $F = 0$, that is

$$\widetilde{E}_k(\frac{r}{s} \cdot b_N) = t^{\frac{r}{s} \cdot b_N} = (-\frac{1}{a_N} \sum_{n=1}^{N-1} a_n E_{k+\frac{1}{2}}(b_n))^{\frac{r}{s}}.$$

Let $\theta = -\frac{1}{a_N} \sum_{n=1}^{N-1} a_n E_{k+\frac{1}{2}}(b_n)$, then $\widetilde{E}_k(\frac{r}{s} \cdot b_N) = \theta^{\frac{r}{s}}$. For each s we have finitely many choices for $\theta^{\frac{1}{s}}$, and we need a uniform way of choosing an sth root of θ for all s. In order to do this we use Konig's Lemma. It is not difficult to prove that in the algebraic closure of the field of fractions of R_{k+1} the law of exponentiation holds. So, we have

$$\widetilde{E}_k : \widetilde{\widetilde{R}}_{k+1} \to \widetilde{R}_{k+1} = \widetilde{\widetilde{R}}_{k+1}[\theta^{\frac{r}{s}} : r, s \in \mathbb{Z}].$$

For the next steps we continue as in the usual construction of the E-polynomial ring case, that is at step $k+2$ we define

$$\widetilde{R}_{k+2} = \widetilde{R}_{k+1}[t^{B_{k+1}}].$$

So, we have

$$\bigcup_h \widetilde{R}_h = \bigcup_h \left(\frac{R_h}{\langle F \rangle \cap R_h}\right).$$

From Lemma 4.4.5 at each step the partial E-domains $\widetilde{R}_{h+1}$ is a strong extension of $\widetilde{R}_h$ for all h, and satisfies Schanuel's Conjecture. So, we have that the algebraic closure of the limit of $\widetilde{R}_h$'s is a strong extension of K, (see also Lemma 4.2.10). It is left to show that the limit of $\widetilde{R}_h$'s is isomorphic to the quotient $\frac{K[\overline{z}]^E}{\langle F \rangle^E}$, that is

$$\bigcup_{h>k} \left(\frac{R_h}{\langle F \rangle^E \cap R_h}\right) \cong \frac{\bigcup R_h}{\langle F \rangle^E},$$

where the isomorphism is the natural one,

$$\varphi : \frac{K[\overline{z}]^E}{\langle F \rangle^E} \longrightarrow \bigcup_{h>k} \left(\frac{R_h}{\langle F \rangle^E \cap R_h}\right)$$

$$p(\overline{z}) + \langle F \rangle^E \longmapsto p(\overline{z}) + \langle F \rangle^E \cap R_h.$$

It is very easy to prove that this is an E-morphism of E-rings. Moreover, φ is an isomorphism. It is trivial to prove that φ is surjective. It is left to show that

$$\ker \varphi = \langle F \rangle^E.$$

We have two cases:

1) If $p(\overline{z}) \in \langle F \rangle^E$ then $p(\overline{z}) \in \langle F \rangle^E \cap R_h$, for all h except finitely many.

2) If $p(\overline{z}) \notin \langle F \rangle^E$ then $p(\overline{z}) \notin \langle F \rangle^E \cap R_h$, for all h, that is $\varphi(p(\overline{z}) + \langle F \rangle^E) \neq \langle F \rangle^E \cap R_h$.

So $\ker \varphi = \langle F \rangle^E$.

We can conclude that the algebraic closure of $\frac{K[\overline{z}]^E}{\langle F\rangle^E}$ denoted by K^* is a strong extension of K, and since F has a zero in K^* then it has to have a zero also in K, but this is a contradiction with our assumption, and the proof is completed.

Case 2. The linear dimension of $(b_1, \ldots, b_N)_{\mathbb{Q}} = 1$. Without loss of generality we can assume that $b_i \in (b_1)_{\mathbb{Q}}$ where $i = 2, \ldots N$, so we can write

$$F = a_1 t^{b_1} + a_2 t^{b_1(\frac{s_2}{r_2})} + \ldots + a_N t^{b_1(\frac{s_N}{r_N})},$$

where $s_i, r_i \in \mathbb{Z}$, with $i = 2, \ldots, N$. Let $r = l.c.m.(r_2, \ldots, r_N)$, we have

$$F = a_1(t^{\frac{b_1}{r}})^r + a_2(t^{\frac{b_1}{r}})^{s_2} + \ldots + a_N(t^{\frac{b_1}{r}})^{s_N}.$$

Using the same notations as in case 1, we have that $\widetilde{R}_h = R_h$ if $h \leq k$, since $\langle F\rangle^E \cap R_h = \{0\}$.

At step $h = k$ we can split off $(b_1)_{\mathbb{Q}}$ as summand of B_k, that is

$$B_k = \widetilde{B}_k \oplus (b_1)_{\mathbb{Q}}.$$

So we can write

$$R_{k+1} = R_k[t^{\widetilde{B_k} \oplus (b_1)_{\mathbb{Q}}}] \cong R_k[t^{\widetilde{B_k}}][t^{(b_1)_{\mathbb{Q}}}],$$

and we introduce a new stage. As a consequence also the exponential map E_k will be got into two stages. Let $\widetilde{\widetilde{R}}_{k+1} = R_k[t^{\widetilde{B}_k}]$, and

$$E_{k+\frac{1}{2}} : R_k \to \widetilde{\widetilde{R}}_{k+1} = R_k[t^{\widetilde{B}_k}],$$

where $E_{k+\frac{1}{2}}$ is defined as in the previous case. We now want to extend $E_{k+\frac{1}{2}}$ to $(b_1)_{\mathbb{Q}}$. Let $\alpha \in (b_1)_{\mathbb{Q}}$, so $\alpha = \frac{m}{n} b_1$, where $m, n \in \mathbb{Z}$ and $n \neq 0$. We define $\widetilde{E}_k(\frac{m}{n} \cdot b_1)$ in the only possible way determined by $F = 0$, that is

$$\widetilde{E}_k(\frac{m}{n} \cdot b_1) = t^{\frac{m}{n} \cdot b_1}.$$

Let $\omega = t^{b_1}$ then $\widetilde{E}_k(\frac{m}{n} \cdot b_1) = \omega^{\frac{m}{n}}$. For each n we have finitely many choices for $\omega^{\frac{1}{n}}$, and we need a uniform way of choosing an nth root of ω for all n. And also in this case we use Konig's Lemma. It is not difficult to prove that in the algebraic closure of R_{k+1} the law of exponentiation holds.

We have added a root of F, so we have obtained the quotient of R_{k+1} over $\langle F \rangle$. For the next steps we continue as in the previous case, at each step the extension we get is a strong extension of the previous partial E-ring and satisfies Schanuel's Conjecture using again Lemma 4.4.5. This completes the proof.

□

Remark 4.4.6. *In the proof of Theorem 4.4.3 we used only purely algebraic methods, so if Zilber's Conjecture is true our methods would give an alternative proof of Henson and Rubel's result for* $\mathbb{C}$ *without using Nevanlinna Theory.*

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