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Exploring Representation Theory of Unitary Groups via Linear Optical Passive Devices

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Abstract. In this paper, we investigate some mathematical structures underlying the physics of linear optical passive (LOP) devices. We show, in particular, that with the class of LOP transformations on N optical modes one can associate a unitary representation of $\mathrm{U}(N)$ in the N-mode Fock space, representation which can be decomposed into irreducible sub-representations living in the subspaces characterized by a fixed number of photons. These (sub-)representations can be classified using the theory of representations of semi-simple Lie algebras. The remarkable case where $N = 3$ is studied in detail.

1. Introduction

Quantum Mechanics (QM) in finite-dimensional Hilbert spaces is considered nowadays as an extremely important subject, especially for the central role that it plays in Quantum Information Processing (QIP) and Quantum Computation (QC) — see, for instance [1, 2] and the rich bibliography therein. The most relevant mathematical objects attached to finite-dimensional Hilbert spaces are — from the point of view of QM, and, in particular, of QIP and QC — the associated unitary groups, and, more in general, the unitary representations acting in such spaces; in particular, *irreducible* unitary representations (i.u.r.'s) enjoy a special status, by virtue of the fact that every non-trivial vector in the carrier Hilbert space of a i.u.r. is a *cyclic* vector for such a representation [3].

Since many quantum-mechanical systems are characterized by *intrinsically* infinite-dimensional carrier Hilbert spaces, the problem of how to single out a finite-dimensional Hilbert space such that the evolution of the system can be confined in it, and, moreover, suitable for the applications (in particular, to QIP and QC) that one has in mind is of paramount importance. Fortunately, at least in principle, in the domain of quantum optical systems there is a simple and conceptually clear scheme for achieving this scope: to use states with a fixed total number of photons and linear optical passive (LOP) devices. This scheme plays, in fact, a central role in linear optical QC (see the review paper [4], and references therein).

Aim of the present paper is to describe some mathematical structures underlying the physics of LOP devices; in particular, we will study the unitary representations that are associated in a natural way with LOP transformations. We believe that this study, which further develops some ideas introduced in a previous paper [5] devoted to an algebraic approach to linear optical QC, may allow a deeper insight into the potential applications of LOP devices.

The paper is organized as follows. In Sect. 2, we show that with the class of LOP transformations on N optical modes is associated a unitary representation of the N-dimensional unitary group $\mathrm{U}(N)$ in the N-mode Fock space, representation which can be decomposed into sub-representations living in the subspaces characterized by a fixed number of photons; such (sub-)representations of $\mathrm{U}(N)$ will be called, for reasons that will be clarified later on, 'N-mode Jordan-Schwinger representations'. Next, in order to give a complete characterization of the Jordan-Schwinger representations, we will introduce, in Sect. 3, the so-called 'Jordan-Schwinger map'. This technical tool allows to apply in a straightforward way the theory of representations of semi-simple Lie algebras which will enable us to prove, in Sect. 4, the main result of the paper: the N-mode Jordan-Schwinger representations are i.u.r.'s that form, for $N = 2$, a maximal set of inequivalent i.u.r.'s, while, for $N \geq 3$, they form a special non-maximal set of inequivalent i.u.r.'s that can be characterized by means of their 'highest weights' (for the sake of simplicity, we will focus on the case where $N = 3$). Finally, in Sect. 5, conclusions are drawn.

2. LOP Transformations

The linear optical *passive* (LOP) transformations are defined as the class of linear transformations that act on a system of N optical modes *leaving unchanged the total number of photons* in the process (see e.g. [6]). In this section, we will discuss the mathematical framework underlying such important physical processes.

As it is well known, the fundamental mathematical object that describes a N-mode quantized e.m. field is the (N-mode) *Heisenberg-Weyl algebra* $\mathcal{W}(N)$, i.e. the $(2N+1)$-dimensional operator algebra generated by the basis elements $\{\widehat{a}_k, \widehat{a}_k^\dagger, \hat{\mathbb{I}}\}_{k=1}^N$ — where

$$\widehat{a}_k \;=\; \hat{\mathbb{I}} \otimes \cdots \otimes \hat{\mathbb{I}} \otimes \overbrace{\;\widehat{a}\;}^{k} \otimes \hat{\mathbb{I}} \otimes \cdots \otimes \hat{\mathbb{I}}\,, \tag{1}$$

with $\widehat{a}$ (resp. $\widehat{a}^\dagger$) denoting the standard annihilation (resp. creation) operator — which satisfy the canonical commutation relations:

$$[\widehat{a}_k, \widehat{a}_l^\dagger] = \delta_{kl}\,\hat{\mathbb{I}}\,, \quad [\widehat{a}_k, \widehat{a}_l] = [\widehat{a}_k^\dagger, \widehat{a}_l^\dagger] = 0\,, \quad k, l = 1, 2, \ldots, N\,. \tag{2}$$

The carrier Hilbert space of this operator algebra is the N*-mode* (bosonic) *Fock space* $\mathcal{H}_F^{(N)}$ endowed with the orthonormal basis $\{|n_1,\ldots,n_N\rangle\}_{n_1,\ldots,n_N=0}^{\infty}$, with

$$|n_1,\ldots,n_N\rangle \;=\; \Big(\prod_{k=1}^{N}\frac{1}{\sqrt{n_k!}}\,(\widehat{a}_k^{\dagger})^{n_k}\Big)|\mathbf{0}\rangle\,, \quad |\mathbf{0}\rangle\equiv|\overbrace{0,\ldots,0}^{N}\rangle\,. \tag{3}$$

A generic LOP device is usually depicted as $2N$-port, namely, as a 'black box' with N inputs and N outputs — $N\geq 2$ — respectively corresponding to the field operators $\{\widehat{a}_k\}_{k=1}^{N}$ and $\{\widehat{b}_k\}_{k=1}^{N}$. Thus, it is described by a linear transformation on $\mathrm{span}\{\widehat{a}_1,\ldots,\widehat{a}_N\}$ mapping the basis element $\widehat{a}_k$ into the operator $\widehat{b}_k$, $k=1,\ldots,N$; the property of photon-number conservation is expressed by the following condition:

$$\sum_{k=1}^{N}\widehat{a}_k^{\dagger}\widehat{a}_k \;=\; \sum_{k=1}^{N}\widehat{b}_k^{\dagger}\widehat{b}_k\,. \tag{4}$$

A simple calculation shows that this condition is sufficient to guarantee that the canonical commutation relations (2) still hold for the operators $\{\widehat{b}_k,\widehat{b}_k^{\dagger},\hat{\mathbb{I}}\}_{k=1}^{N}$, which then form another basis for the algebra $\mathcal{W}(N)$, and can indeed be interpreted as the field operators of the output modes. Precisely, denoting by $\underline{U}$ the $N\times N$ matrix representing the LOP transformation

$$\widehat{b}_k \;=\; \sum_{l=1}^{N}\underline{U}_{kl}\,\widehat{a}_l\,, \tag{5}$$

from condition (4) one can easily prove that $\underline{U}\,\underline{U}^{\dagger}=\underline{U}^{\dagger}\,\underline{U}=\underline{\mathrm{Id}}$, where $\underline{\mathrm{Id}}$ is the identity matrix. Hence, with any LOP transformation it can be naturally associated a *unitary matrix* $\underline{U}$; conversely, any unitary matrix defines a LOP transformation. Thus, there is a one-to-one correspondence between LOP $2N$-ports and the elements of the group $\mathrm{U}(N)$.

On the other hand, formula (5) implies also that the output field operators $\{\widehat{b}_k,\widehat{b}_k^{\dagger}\}_{k=1}^{N}$, together with the identity operator $\hat{\mathbb{I}}$, form another *irreducible* set of generators of the algebra $\mathcal{W}(N)$. Hence, according to the Stone-von Neumann theorem [7] on canonical commutation relations, they must be unitarily equivalent to the input field operators $\{\widehat{a}_k,\widehat{a}_k^{\dagger}\}_{k=1}^{N}$, i.e.

$$\widehat{b}_k \;=\; \widehat{U}^{\dagger}\,\widehat{a}_k\,\widehat{U}\,, \quad k=1,\ldots,N\,, \tag{6}$$

where $\widehat{U}$ is a suitable unitary operator in $\mathcal{H}_F^{(N)}$, uniquely defined up to an arbitrary phase factor; therefore: *with any LOP transformation, identified by a unitary matrix* $\underline{U}$ *and* (5), *one can associate a unitary operator* $\widehat{U}$ *— satisfying relation* (6) *— which is unique up to a phase factor.*

Let us highlight an important property of every unitary operator $\widehat{U}$ satisfying relation (6). To this aim, let us decompose the N-mode Fock space $\mathcal{H}_F^{(N)}$ as the orthogonal sum of all subspaces with a fixed number of photons:

$$\mathcal{H}_F^{(N)} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n^{(N)}, \quad \mathcal{H}_n^{(N)} := \mathrm{span}\left\{|n_1, \ldots, n_N\rangle \in \mathcal{H}_F^{(N)} \ : \ n_1 + \ldots + n_N = n\right\}; \tag{7}$$

notice that from a simple calculation one finds that the dimension of the n-photon subspace is given by

$$\dim(\mathcal{H}_n^{(N)}) \ = \ \frac{(n+N-1)!}{n!\,(N-1)!}\,. \tag{8}$$

Denoting by $\widehat{\mathrm{P}}_n^{(N)}$ the orthogonal projector onto the n-photon subspace $\mathcal{H}_n^{(N)}$, with $n = 0, 1, 2, \ldots$, and by $\widehat{n}^{(N)} := \sum_{k=1}^{N} \widehat{n}_k$, $\widehat{n}_k \equiv \widehat{a}_k^\dagger \widehat{a}_k$, the *total number of photons operator*, we have the following spectral decomposition:

$$\widehat{n}^{(N)} \ = \ \sum_{n=0}^{\infty} n\, \widehat{\mathrm{P}}_n^{(N)}\,. \tag{9}$$

Using the definition of $\widehat{n}^{(N)}$ and condition (4), one can easily check that the unitary operator U *commutes* with $\widehat{n}^{(N)}$:

$$\widehat{U}\widehat{n}^{(N)} \ = \ \widehat{U}\,\widehat{n}^{(N)}\,\widehat{U}^\dagger\,\widehat{U} \ = \ \sum_{k=1}^{N}\left(\widehat{U}\,\widehat{b}_k^\dagger \widehat{b}_k\,\widehat{U}^\dagger\right)\widehat{U} \ = \ \sum_{k=1}^{N} \widehat{a}_k^\dagger \widehat{a}_k\,\widehat{U} \ = \ \widehat{n}^{(N)}\,\widehat{U}\,. \tag{10}$$

From the spectral decomposition (9) it follows that *the commutation relation* (10) *is equivalent to the fact that the n-photon subspace* $\mathcal{H}_n^{(N)}$ *is invariant with respect to the unitary operator* $\widehat{U}$, i.e. $\widehat{U}\mathcal{H}_n^{(N)} = \mathcal{H}_n^{(N)}$, $n = 0, 1, \ldots$; in particular, we have:

$$\widehat{U}|\mathbf{0}\rangle = e^{i\phi(\widehat{U})}|\mathbf{0}\rangle\,, \quad \text{for some} \quad \phi(\widehat{U}) \in [0, 2\pi[\quad (\text{hence: } \widehat{U}^\dagger\,|\mathbf{0}\rangle = e^{-i\phi(\widehat{U})}\,|\mathbf{0}\rangle)\,. \tag{11}$$

Thus, $e^{i\phi(\widehat{U})}$ can be regarded as the phase factor which is left undetermined in relation (6); as a consequence, if we set once and for all $e^{i\phi(\widehat{U})} = 1$, then, *there is a unique unitary operator* $\widehat{U}$ *associated with a given unitary* $N \times N$ *matrix* $\underline{U}$ *by means of relations* (5) *and* (6) *such that*

$$\widehat{U}|\mathbf{0}\rangle = |\mathbf{0}\rangle\,. \tag{12}$$

Therefore, by this association, we can define the map

$$\widehat{\Upsilon}^{(N)} : \mathrm{U}(N) \ni \underline{U} \ \mapsto \ \widehat{\Upsilon}^{(N)}(\underline{U}) \in \mathcal{U}(\mathcal{H}_F^{(N)})\,, \tag{13}$$

where $\mathcal{U}(\mathcal{H}_F^{(N)})$ is the unitary group of the N-mode Fock space, i.e. the group of all unitary operators in $\mathcal{H}_F^{(N)}$.

Observe, now, that the map $\widehat{\Upsilon}^{(N)}$ satisfies the following property:

$$\widehat{\Upsilon}^{(N)}(\underline{U}\,\underline{U}') \ = \ \widehat{\Upsilon}^{(N)}(\underline{U})\,\widehat{\Upsilon}^{(N)}(\underline{U}')\,, \quad \forall\, \underline{U}, \underline{U}' \in \mathrm{U}(N)\,; \tag{14}$$

indeed, given any couple of unitary $N \times N$ matrices $\underline{U}, \underline{U}'$, we have:

$$\begin{aligned} \widehat{\Upsilon}^{(N)}(\underline{U}')^\dagger\, \widehat{\Upsilon}^{(N)}(\underline{U})^\dagger\, \widehat{a}_k \widehat{\Upsilon}^{(N)}(\underline{U})\, \widehat{\Upsilon}^{(N)}(\underline{U}') &= \sum_{l,m=1}^{N} \underline{U}_{kl}\, \underline{U}'_{lm}\, \widehat{a}_m \\ &= \widehat{\Upsilon}^{(N)}(\underline{U}\,\underline{U}')^\dagger\, \widehat{a}_k \widehat{\Upsilon}^{(N)}(\underline{U}\,\underline{U}'), \end{aligned} \tag{15}$$

for every $k \in \{1, \ldots, N\}$, and

$$\widehat{\Upsilon}^{(N)}(\underline{U})\, \widehat{\Upsilon}^{(N)}(\underline{U}')\, |\mathbf{0}\rangle = |\mathbf{0}\rangle\,. \tag{16}$$

By property (14), the map $\widehat{\Upsilon}^{(N)}$ is a homomorphism of the group $\mathrm{U}(N)$ into the unitary group of the N-mode Fock space, namely, $\widehat{\Upsilon}^{(N)}$ *is a unitary representation of* $\mathrm{U}(N)$ *in* $\mathcal{H}_F^{(N)}$.

Notice that we have obtained this result using only the powerful theorem of Stone and von Neumann on canonical commutation relations; up to this point, we did not exploit the explicit form of the operator $\widehat{U} = \widehat{\Upsilon}^{(N)}(\underline{U})$, actually we did not even exhibit such form. In order to obtain the expression of the unitary operator $\widehat{\Upsilon}^{(N)}(\underline{U})$, one can first observe that

$$[\widehat{\Upsilon}^{(N)}(\underline{U}), \widehat{\mathrm{P}}_n^{(N)}] = 0\,, \quad \forall\, \underline{U} \in \mathrm{U}(N)\,, \quad \forall\, n \in \{0, 1, \ldots\}\,; \tag{17}$$

then, one can write the following decomposition:

$$\widehat{\Upsilon}^{(N)}(\underline{U}) = \sum_{n=0}^{\infty} \widehat{\Upsilon}_n^{(N)}(\underline{U})\,, \quad \widehat{\Upsilon}_n^{(N)}(\underline{U}) := \widehat{\Upsilon}^{(N)}(\underline{U})\, \widehat{\mathrm{P}}_n^{(N)} = \widehat{\mathrm{P}}_n^{(N)}\, \widehat{\Upsilon}^{(N)}(\underline{U})\, \widehat{\mathrm{P}}_n^{(N)}\,. \tag{18}$$

Hence, the representation $\widehat{\Upsilon}^{(N)}$ is not irreducible and can be decomposed as the orthogonal sum of finite-dimensional sub-representations. Precisely, let us define the finite-dimensional unitary representation $\Upsilon_n^{(N)}$ of $\mathrm{U}(N)$ in $\mathcal{H}_n^{(N)}$ by

$$\Upsilon_n^{(N)}(\underline{U})|\psi\rangle = \widehat{\Upsilon}_n^{(N)}(\underline{U})|\psi\rangle\,, \quad \forall\, \underline{U} \in \mathrm{U}(N)\,, \quad \forall\, |\psi\rangle \in \mathcal{H}_n^{(N)}\,; \tag{19}$$

then, we have:

$$\widehat{\Upsilon}^{(N)} = \bigoplus_{n=0}^{\infty} \Upsilon_n^{(N)}\,. \tag{20}$$

The representation $\Upsilon_0^{(N)}$ is, obviously, just the trivial representation of $\mathrm{U}(N)$.

The representation $\Upsilon_1^{(N)}$ plays, instead, a special role. In fact, setting $\widehat{U} \equiv \widehat{\Upsilon}^{(N)}(\underline{U})$, we have:

$$\langle\mathbf{0}|\widehat{a}_k \widehat{U} \widehat{a}_l^\dagger|\mathbf{0}\rangle = \langle\mathbf{0}|\widehat{U}(\widehat{U}^\dagger \widehat{a}_k \widehat{U})\widehat{a}_l^\dagger|\mathbf{0}\rangle = \langle\mathbf{0}|\widehat{b}_k \widehat{a}_l^\dagger|\mathbf{0}\rangle = \sum_{m=1}^{N} \underline{U}_{km}\langle\mathbf{0}|\widehat{a}_m \widehat{a}_l^\dagger|\mathbf{0}\rangle\,, \tag{21}$$

where we have used the fact that $\widehat{U}^\dagger|\mathbf{0}\rangle = |\mathbf{0}\rangle$ and $\widehat{U}^\dagger \widehat{a}_k \widehat{U} = \widehat{b}_k$; then, from relation (21) it follows that

$$\langle\mathbf{0}|\widehat{a}_k\, \Upsilon_1^{(N)}(\underline{U})\widehat{a}_l^\dagger|\mathbf{0}\rangle = \langle\mathbf{0}|\widehat{a}_k \widehat{\Upsilon}^{(N)}(\underline{U})\widehat{a}_l^\dagger|\mathbf{0}\rangle = \underline{U}_{kl}\,, \quad k, l \in \{1, \ldots, N\}\,. \tag{22}$$

Thus, $\Upsilon_1^{(N)}$ turns out to be a realization of the *fundamental representation* of $\mathrm{U}(N)$ in the one-photon subspace of $\mathcal{H}_F^{(N)}$. The remaining representations $\{\Upsilon_n^{(N)}\}_{n\geq 2}$ can be determined, in principle, using the same technique, but obtaining for the matrix elements, in general, cumbersome expressions having the form of a homogeneous polynomial of degree n (for $\Upsilon_n^{(N)}(\underline{U})$) in the variables $\{\underline{U}_{kl}\}_{k,l=1,\ldots,N}$. It is then natural to ask the following questions:

1. Is there a way for expressing the representations $\{\Upsilon_n^{(N)}\}_{n\geq 2}$ of $\mathrm{U}(N)$ by a compact formula?
2. Are the representations $\{\Upsilon_n^{(N)}\}_{n\geq 2}$ irreducible, and do they form, together with the trivial and the fundamental representations, *a maximal set of inequivalent irreducible unitary representations* of $\mathrm{U}(N)$?

It will be shown in the following that there exists a mathematical tool that allows one to give a straightforward answer to such questions, namely, the so-called 'Jordan-Schwinger map' (shortly, J-S map).

3. The Jordan-Schwinger Map

The general formulation of the J-S map [8, 9, 10] gives a simple procedure allowing one to obtain the so called *bosonic realization* of a Lie algebra. Consider the operator realization of the $\mathrm{gl}(N)$ algebra generated by the basis elements

$$\widehat{d}_{kl} := \widehat{a}_k^\dagger \widehat{a}_l \,, \quad k,l = 1,\ldots N \quad (\text{hence: } [\widehat{d}_{kl}, \widehat{d}_{pq}] = \delta_{lp}\widehat{d}_{kq} - \delta_{qk}\widehat{d}_{pl})\,. \tag{23}$$

Next, let $\mathcal{A}$ be an M-dimensional matrix Lie algebra, and let $\{\underline{A}^{(m)}\}_{m=1}^M$ be a basis of $\mathcal{A}$ consisting of, say, $N\times N$ matrices ($M \leq N^2$). Then, one can define the linear operators

$$\widehat{A}^{(m)} := \sum_{k,l=1}^{N} \underline{A}_{kl}^{(m)} \widehat{d}_{kl}\,, \quad m = 1,\ldots,M\,, \tag{24}$$

acting in the N-mode Fock space $\mathcal{H}_F^{(N)}$. The linear operators $\{\widehat{A}^{(m)}\}_{m=1}^M$ form a basis of the M-dimensional 'bosonic realization' $\widehat{\mathcal{A}}$ of the matrix Lie algebra $\mathcal{A}$ since, as the reader may verify using the commutation relations in (23), the operators $\{\widehat{A}^{(m)}\}_{m=1}^M$ preserve the commutation rules of the basis matrices $\{\underline{A}^{(m)}\}_{m=1}^M$:

$$[\widehat{A}^{(m)}, \widehat{A}^{(r)}] = \sum_{k,l=1}^{N} [\underline{A}^{(m)}, \underline{A}^{(r)}]_{kl} \widehat{d}_{kl}\,, \quad m,r = 1,\ldots,M\,. \tag{25}$$

The one-to-one correspondence $\underline{A}^{(m)} \mapsto \widehat{A}^{(m)}$, $m = 1,\ldots,M$ — *extended by linearity* — is the J-S map associated with the matrix Lie algebra $\mathcal{A}$: $\mathsf{JS}_{\mathcal{A}} : \mathcal{A} \to \widehat{\mathcal{A}}$. In the special case of the Lie algebra $\mathrm{u}(N)$ of $\mathrm{U}(N)$, i.e. the matrix algebra consisting of all antihermitian $N\times N$ matrices, we will set: $\breve{\Upsilon}^{(N)}(\underline{A}) \equiv \mathsf{JS}_{\mathrm{u}(N)}(\underline{A})$.

At this point, one can prove (see ref. [5]) that the following important relation holds:

$$\widehat{\Upsilon}^{(N)}(\exp(\underline{A})) \;=\; \exp(\check{\Upsilon}^{(N)}(\underline{A}))\,, \quad \forall\, \underline{A} \in \mathrm{u}(N)\,. \tag{26}$$

We stress that this formula holds *unambiguously*; namely, recalling that the exponential map for unitary groups is surjective but not injective, nevertheless we have: $\exp(\underline{A}) = \exp(\underline{A}') \Rightarrow \exp\big(\check{\Upsilon}^{(N)}(\underline{A})\big) = \exp\big(\check{\Upsilon}^{(N)}(\underline{A}')\big)$.

Relation (26) allows to decompose the unitary representation $\widehat{\Upsilon}^{(N)}$ into irreducible sub-representations (that, with $\mathrm{U}(N)$ being compact, must be finite-dimensional). In fact, the classification of the irreducible representations of $\mathrm{U}(N) = \mathrm{U}(1) \times \mathrm{SU}(N)$ can be trivially reduced to the classification of the irreducible representations of $\mathrm{SU}(N)$; hence, since $\mathrm{SU}(N)$ is simply connected, to the classification of the irreducible representations of the complexified Lie algebra $\mathrm{su}(N)_{\mathbb{C}}$, task that can be accomplished by virtue of the well known theory of representations of semi-simple Lie algebras. As it will be shown in the next section, by virtue of the J-S map, this theory translates beautifully from its abstract setting into the language of the N-mode Fock space $\mathcal{H}_F^{(N)}$ and of LOP transformations on this space. In particular, one can prove the following fact. The finite-dimensional sub-representations of $\widehat{\Upsilon}^{(N)}$ that appear in the decomposition formula (20) are actually *irreducible*. Moreover — differently from the case of U(2), case where the sequence $\{\Upsilon_n^{(2)}\}_{n=0}^{\infty}$ is a maximal set of inequivalent irreducible unitary representations (see ref. [5]) — the irreducible representations $\{\Upsilon_n^{(N)}\}_{n=0}^{\infty}$ of $\mathrm{U}(N)$, for $N \geq 3$, are mutually inequivalent but do not form a maximal set.

We will call the representations of $\mathrm{U}(N)$, with $N = 2, 3, \ldots$, implementable by means of LOP devices acting in the subspaces $\{\mathcal{H}_n^{(N)}\}_{n=0}^{\infty}$ (characterized by a fixed number of photons) of the N-mode Fock space, i.e. the unitary representations $\{\Upsilon_n^{(N)}\}_{n=0}^{\infty}$ — which will be shown, in the next section, to be irreducible — *N-mode Jordan-Schwinger representations.*

4. Characterizing the 3-Mode Jordan-Schwinger Representations

Let us now focus on the J-S representations of U(3): first, as anticipated, $N = 3$ is the simplest case where the J-S representations do not form a maximal set of inequivalent irreducible unitary representations; second, it contains all the basic features of the general case $N \geq 3$; third, the case of three optical modes (with a small number of photons) corresponds to a feasible experimental setup.

We will consider the basis of the complex Lie algebra $\mathrm{su}(3)_{\mathbb{C}}$ formed by the eight 3×3 matrices $\{\sigma_z^{1,2}, \sigma_z^{2,3}, \sigma_\pm^{1,2}, \sigma_\pm^{2,3}, \sigma_\pm^{1,3}\}$, where:

$$\sigma_z^{1,2} \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \sigma_z^{2,3} \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \sigma_+^{1,2} \;=\; \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \sigma_-^{1,2\dagger}$$

$$\sigma_+^{2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \sigma_-^{2,3\dagger}, \quad \sigma_+^{1,3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \sigma_-^{1,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \sigma_+^{1,3\dagger}. \tag{27}$$

Notice that the linear span $\boldsymbol{\sigma}^{1,2}$ of $\{\sigma_z^{1,2}, \sigma_+^{1,2}, \sigma_-^{1,2}\}$ is a subalgebra of $\mathrm{su}(3)_{\mathbb{C}}$ isomorphic to $\mathrm{su}(2)_{\mathbb{C}}$ (as it can be seen by ignoring the third row and the third column in each matrix). Similarly, $\boldsymbol{\sigma}^{2,3} := \mathrm{span}\{\sigma_z^{2,3}, \sigma_+^{2,3}, \sigma_-^{2,3}\}$ is another copy of $\mathrm{su}(2)_{\mathbb{C}}$ embedded in $\mathrm{su}(3)_{\mathbb{C}}$. Thus, we have the following commutation relations:

$$\left[\sigma_z^{1,2}, \sigma_\pm^{1,2}\right] = \pm 2\sigma_\pm^{1,2}, \quad \left[\sigma_z^{2,3}, \sigma_\pm^{2,3}\right] = \pm 2\sigma_\pm^{2,3}, \quad \left[\sigma_+^{1,2}, \sigma_-^{1,2}\right] = \sigma_z^{1,2}, \quad \left[\sigma_+^{2,3}, \sigma_-^{2,3}\right] = \sigma_z^{2,3}.$$

Next, we list the commutation relations linking the two subalgebras $\boldsymbol{\sigma}^{1,2}$ and $\boldsymbol{\sigma}^{2,3}$, namely,

$$\left[\sigma_z^{1,2}, \sigma_\pm^{2,3}\right] = \mp\sigma_\pm^{1,2}, \quad \left[\sigma_z^{2,3}, \sigma_\pm^{1,2}\right] = \mp\sigma_\pm^{1,2}, \quad \left[\sigma_\pm^{1,2}, \sigma_\pm^{2,3}\right] = \pm\sigma_\pm^{1,3}, \quad \left[\sigma_\pm^{1,2}, \sigma_\mp^{2,3}\right] = 0.$$

Besides, we have:

$$\left[\sigma_z^{1,2}, \sigma_\pm^{1,3}\right] = \pm\sigma_\pm^{1,3}, \; \left[\sigma_z^{2,3}, \sigma_\pm^{1,3}\right] = \pm\sigma_\pm^{1,3}, \; \left[\sigma_\pm^{1,2}, \sigma_\mp^{1,3}\right] = \mp\sigma_\mp^{2,3}, \; \left[\sigma_\pm^{2,3}, \sigma_\mp^{1,3}\right] = \pm\sigma_\mp^{1,2}.$$

Notice that the element of the algebra that one could naturally denote by $\sigma_z^{1,3}$ does not appear in the chosen basis of generators since it can be written as the sum of the basis elements $\sigma_z^{1,2}$ and $\sigma_z^{2,3}$; indeed: $\left[\sigma_+^{1,3}, \sigma_-^{1,3}\right] = \sigma_z^{1,2} + \sigma_z^{2,3}$. All the remaining matrix commutation relations involving the generators (27) vanish.

Now, let $\boldsymbol{\varrho}$ be a representation of the complex Lie algebra $\mathrm{su}(3)_{\mathbb{C}}$ in a finite-dimensional (complex) Hilbert space $V \cong \mathbb{C}^K$; by means of the exponential map from su(3) onto SU(3), one can associate with $\boldsymbol{\varrho}$ a unitary representation R of SU(3) in V: $R(\exp(\sigma))\, v = \exp(\boldsymbol{\varrho}(\sigma))\, v$, for any $\sigma \in \mathrm{su}(3)$ (with su(3) regarded as a real form of $\mathrm{su}(3)_{\mathbb{C}}$) and any $v \in V$. The group representation R is irreducible if and only if the corresponding algebra representation $\boldsymbol{\varrho}$ is. According to results from the theory of representations of semi-simple Lie algebras [11], the fact whether the representation $\boldsymbol{\varrho}$ is irreducible or not can be decided by determining the associated 'weights' and 'roots'. We recall that a pair of complex numbers $\mu = (\mu_1, \mu_2)$ is called a *weight* for $\boldsymbol{\varrho}$ if there exists a nonzero vector $v \in V$ such that

$$\boldsymbol{\varrho}\,(\sigma_z^{1,2})v \;=\; \mu_1\, v\,, \qquad \boldsymbol{\varrho}\,(\sigma_z^{2,3})v \;=\; \mu_2\, v\,, \tag{28}$$

where the common nontrivial eigenvector v of $\boldsymbol{\varrho}\,(\sigma_z^{1,2})$ and $\boldsymbol{\varrho}\,(\sigma_z^{2,3})$ is called a *weight vector* for $\boldsymbol{\varrho}$. Similarly, a pair $\alpha = (\alpha_1, \alpha_2) \in \mathbb{C}^2$ is called a *root* if

- $|\alpha_1| + |\alpha_2| \neq 0$, i.e. α_1 and α_2 are not both zero;
- there exists a nonzero element $\sigma \in \mathrm{su}(3)_{\mathbb{C}}$ such that

$$[\sigma_z^{1,2}, \sigma] \;=\; \alpha_1\sigma\,, \qquad [\sigma_z^{2,3}, \sigma] \;=\; \alpha_2\sigma\,, \tag{29}$$

  where the element σ, which is nothing but a weight vector for the adjoint representation of $\mathrm{su}(3)_{\mathbb{C}}$, is called a *root vector* corresponding to the root α.

The commutation relations involving the generators (27) reveal what the roots for $\mathrm{su}(3)_{\mathbb{C}}$ are. There are six roots that we list below together with the corresponding root vectors:

α	σ
$(2,-1)$	$\sigma_+^{1,2}$
$(-1,2)$	$\sigma_+^{2,3}$
$(1,1)$	$\sigma_+^{1,3}$
$(-2,1)$	$\sigma_-^{1,2}$
$(1,-2)$	$\sigma_-^{2,3}$
$(-1,-1)$	$\sigma_-^{1,3}$

It will be convenient to single out the two roots $\dot{\alpha}=(2,-1)$ and $\ddot{\alpha}=(-1,2)$; they are called *positive simple roots* and have the property that all other roots can be expressed as linear combinations of $\dot{\alpha}$ and $\ddot{\alpha}$ with integer coefficients, which are (for each root) either all greater than or equal to zero, or all less than or equal to zero, as the reader may easily check. By means of the positive simple roots we can introduce a *partial ordering* relation among the weights. We will say that a weight μ is *higher* than another weight $\tilde{\mu}$ (in symbols, $\mu \succ \tilde{\mu}$) if $\mu-\tilde{\mu}$ can be written in the form

$$\mu-\tilde{\mu} \;=\; \dot{c}_{\mu,\tilde{\mu}}\dot{\alpha}+\ddot{c}_{\mu,\tilde{\mu}}\ddot{\alpha}\,, \tag{30}$$

with $\dot{c}_{\mu,\tilde{\mu}}\geq 0$ and $\ddot{c}_{\mu,\tilde{\mu}}\geq 0$.

At this point, a complete characterization of the finite-dimensional representations of $\mathrm{su}(3)_{\mathbb{C}}$ is provided by the well known *Theorem of the Highest Weight*:

1. every finite-dimensional representation of $\mathrm{su}(3)_{\mathbb{C}}$ has at least one weight, and it is irreducible if and only if it admits a *unique* highest weight;
2. two irreducible representations of $\mathrm{su}(3)_{\mathbb{C}}$ have the same highest weight if and only if they are equivalent;
3. for every irreducible representation of $\mathrm{su}(3)_{\mathbb{C}}$, its highest weight is of the form $\mu=(\mu_1,\mu_2)$ with μ_1 and μ_2 being non-negative integers; conversely, if μ_1 and μ_2 are non-negative integers, then there exists an irreducible representation of $\mathrm{su}(3)_{\mathbb{C}}$ (unique up to equivalence) with highest weight $\mu=(\mu_1,\mu_2)$.

Moreover, we recall that from the Weyl character formula one can derive the following relation for the dimension of the vector space V_{μ_1,μ_2} of the representation of $\mathrm{su}(3)_{\mathbb{C}}$ associated with the highest weight $\mu=(\mu_1,\mu_2)$:

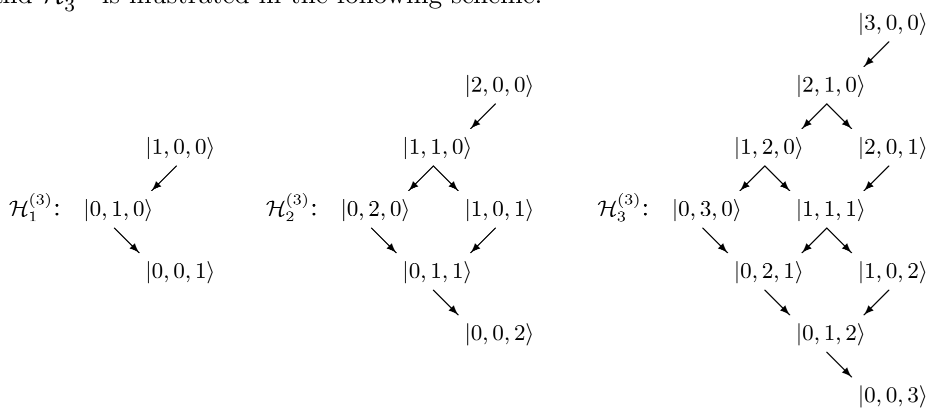
$$\dim(V_{\mu_1,\mu_2}) \;=\; \frac{1}{2}(\mu_1+1)(\mu_2+1)(\mu_1+\mu_2+2)\,. \tag{31}$$

Let us now apply the above results to the characterization of the J-S representations of U(3). First, let ϱ_n be the representation of su(3) in $\mathcal{H}_n^{(3)}$ defined by: $\varrho_n(\underline{A})\,\psi \;=\; \check{\Upsilon}^{(3)}(\underline{A})\,\psi \;=\; \sum_{k,l=1}^{3}\underline{A}_{kl}\widehat{d}_{kl}\,\psi$, with $\underline{A}\in\mathrm{su}(3)$, $\psi\in\mathcal{H}_n^{(3)}$. Thus, ϱ_n is the finite-dimensional sub-representation of $\check{\Upsilon}^{(3)}$ naturally associated, via the exponential map, with the finite-dimensional unitary representation $\Upsilon_n^{(3)}$ of U(3) in

$\mathcal{H}_n^{(3)}$: $\Upsilon_n^{(3)}(e^{i\phi}\exp(\underline{A})) = e^{in\phi}\exp(\varrho_n(\underline{A}))$, with $e^{i\phi} \in \mathrm{U}(1)$, $\underline{A} \in \mathrm{su}(3)$. Hence, the group representation $\Upsilon_n^{(3)}$ is irreducible if and only if the algebra representation ϱ_n is, or equivalently, if and only if the corresponding *complexified* representation $\boldsymbol{\varrho}_n$ of $\mathrm{su}(3)_{\mathbb{C}}$ is irreducible ($\boldsymbol{\varrho}_n(\underline{A} + i\underline{A}') = \varrho_n(\underline{A}) + i\varrho_n(\underline{A}')$, $\underline{A}, \underline{A}' \in \mathrm{su}(3)$). Next, observe that $\check{\Upsilon}^{(3)}(\sigma_z^{1,2}) = \widehat{a}_1^\dagger\widehat{a}_1 - \widehat{a}_2^\dagger\widehat{a}_2 = \widehat{n}_1 - \widehat{n}_2$, $\check{\Upsilon}^{(3)}(\sigma_z^{2,3}) = \widehat{a}_2^\dagger\widehat{a}_2 - \widehat{a}_3^\dagger\widehat{a}_3 = \widehat{n}_2 - \widehat{n}_3$. Then, for the finite-dimensional representation $\boldsymbol{\varrho}_n$ of $\mathrm{su}(3)_{\mathbb{C}}$ in $\mathcal{H}_n^{(3)}$ we have:

$$\begin{aligned} \boldsymbol{\varrho}_n(\sigma_z^{1,2})\,|n_1,n_2,n_3\rangle &= (n_1 - n_2)\,|n_1,n_2,n_3\rangle\,, \qquad n_1+n_2+n_3 = n\,, \quad (32)\\ \boldsymbol{\varrho}_n(\sigma_z^{2,3})\,|n_1,n_2,n_3\rangle &= (n_2 - n_3)\,|n_1,n_2,n_3\rangle\,, \qquad n_1+n_2+n_3 = n\,; \end{aligned}$$

i.e. $|n_1,n_2,n_3\rangle \in \mathcal{H}_n^{(3)}$ is a weight vector for $\boldsymbol{\varrho}_n$ with weight $(n_1 - n_2, n_2 - n_3)$. It is easy to show that the weight $\mu_n \equiv (n, 0)$, corresponding to the state $|n,0,0\rangle$, is higher than any other weight for $\boldsymbol{\varrho}_n$. Indeed, the difference between the weights $\mu, \tilde{\mu}$ corresponding to a couple of nonzero vectors $|n_1,n_2,n_3\rangle, |\tilde{n}_1,\tilde{n}_2,\tilde{n}_3\rangle \in \mathcal{H}_n^{(3)}$ is of the form $\mu - \tilde{\mu} = (\Delta n_1 - \Delta n_2, \Delta n_2 - \Delta n_3)$, with $\Delta n_j = n_j - \tilde{n}_j$. Thus, if we set $(\Delta n_1 - \Delta n_2, \Delta n_2 - \Delta n_3) = (2\dot{c}_{\mu,\tilde{\mu}} - \ddot{c}_{\mu,\tilde{\mu}}, -\dot{c}_{\mu,\tilde{\mu}} + 2\ddot{c}_{\mu,\tilde{\mu}})$ — recall relation (30) — taking into account that $\sum_{j=1}^3 \Delta n_j = 0$, we find: $\dot{c}_{\mu,\tilde{\mu}} = \Delta n_1$, $\ddot{c}_{\mu,\tilde{\mu}} = -\Delta n_3$. Therefore, for $|n_1,n_2,n_3\rangle = |n,0,0\rangle$ and $\mu = \mu_n$, the integers $\dot{c}_{\mu,\tilde{\mu}}$ and $\ddot{c}_{\mu,\tilde{\mu}}$ are always non-negative, hence, $\mu_n \succ \tilde{\mu}$; this proves that μ_n is the (unique) highest weight for $\boldsymbol{\varrho}_n$, which is then an irreducible representation of $\mathrm{su}(3)_{\mathbb{C}}$. The situation for the case of the one-photon, two-photon and three-photon subspaces $\mathcal{H}_1^{(3)}$, $\mathcal{H}_2^{(3)}$ and $\mathcal{H}_3^{(3)}$ is illustrated in the following scheme:



In this scheme the arrows go from *higher* to *lower* weights. Furthermore, each arrow ↙ corresponds to an application of the operator of $\widehat{a}_2^\dagger\widehat{a}_1$ in the subspace $\mathcal{H}_1^{(3)}$ (resp. $\mathcal{H}_2^{(3)}$, $\mathcal{H}_3^{(3)}$), i.e. to the action of $\boldsymbol{\varrho}_1(\sigma_-^{1,2})$ (resp. $\boldsymbol{\varrho}_2(\sigma_-^{1,2})$, $\boldsymbol{\varrho}_3(\sigma_-^{1,2})$), while each arrow ↘ corresponds to an application of the operator of $\widehat{a}_3^\dagger\widehat{a}_2$ in the subspace $\mathcal{H}_1^{(3)}$ (resp. $\mathcal{H}_2^{(3)}$, $\mathcal{H}_3^{(3)}$), i.e. to the action of $\boldsymbol{\varrho}_1(\sigma_-^{2,3})$ (resp. $\boldsymbol{\varrho}_2(\sigma_-^{2,3})$, $\boldsymbol{\varrho}_3(\sigma_-^{2,3})$). In general, for $n \geq 1$, one can easily show that by suitable applications of the operators $\{\boldsymbol{\varrho}_n(\sigma_\pm^{1,2})\}$, $\{\boldsymbol{\varrho}_n(\sigma_\pm^{2,3})\}$ one can generate from every vector of the standard basis $\{|n_1,n_2,n_3\rangle\}_{n_1+n_2+n_3=n}$ of $\mathcal{H}_n^{(3)}$ any other element of this basis.

5. Conclusions

In this paper, we have shown, by means of the theorem of Stone and von Neumann on canonical commutation relations, that with the class of LOP transformations on N optical modes, $N \geq 2$, is associated a unitary representation of the group $\mathrm{U}(N)$ in the N-mode Fock space $\mathcal{H}_F^{(N)}$, representation which has been denoted by $\widehat{\Upsilon}^{(N)}$. This representation can be decomposed as an orthogonal sum of the sub-representations $\{\Upsilon_n^{(N)}\}_{n\in\{0\}\cup\mathbb{N}}$, where the representation $\Upsilon_n^{(N)}$ lives in the n-photon subspace $\mathcal{H}_n^{(N)}$. By virtue of the J-S map, one can express the representation $\widehat{\Upsilon}^{(N)}$ by a compact formula and, applying results from the theory of representations of semi-simple Lie algebras, characterize the 'N-mode J-S representations' $\{\Upsilon_n^{(N)}\}_{n\in\{0\}\cup\mathbb{N}}$. It turns out that, only in the special case where $N = 2$, the J-S representations form a *maximal* set of inequivalent irreducible unitary representations, while, for $N \geq 3$, the N-mode J-S representations form a certain non-maximal set of inequivalent irreducible unitary representations that can be characterized by means of the associated 'highest weights'. In the case where $N = 3$ — the case which, for the sake of simplicity, has been studied in detail — we can do a simple check of the results obtained, comparing formula (8) with formula (31), formulae that should both give the dimension of the vector space where the irreducible representation $\Upsilon_n^{(N)}$ acts; in fact, we get:

$$\dim(V_{n,0}) \;=\; \frac{1}{2}\,(n+1)(n+2) \;=\; \frac{(n+2)!}{2(n!)} \;=\; \dim(\mathcal{H}_n^{(3)})\,. \tag{33}$$

We believe that these results may pave the way to an intriguing 'experimental exploration', via LOP devices, of the representations of unitary groups and to interesting applications to QIP and QC.

Bibliography

[1] M. Nielsen, I. Chuang, *Quantum Information and Quantum Computation*, Cambridge University Press, 2000.

[2] A. Galindo, M. A. Martin-Delgado, Rev. Mod. Phys. **74**, 347 (2002).

[3] S. A. Gaal, *Linear Analysis and Representation Theory*, Springer-Verlag, 1973.

[4] P. Kok *et al.*, quant-ph/0512071.

[5] P. Aniello, R. Coen Cagli, J. Opt. B: Quantum Semiclass. Opt. **7**, S711 (2005).

[6] U. Leonhardt, Rept. Prog. Phys. **66**, 1207 (2003).

[7] M. H. Stone, Proc. Nat. Acad. Scie. U.S.A. **16**, 172 (1930);
J. von Neumann, Math. Ann. **104**, 570 (1931);
see also: J. Dixmier, Comp. Math. **13**, 263 (1958).

[8] P. Jordan, Z. Phys. **94**, 531 (1935);
J. Schwinger, *On angular momentum*, in: *Quantum Theory of Angular Momentum*, L. C. Biedenharn and H. Van Dam, eds., Academic Press, 1965.

[9] V. I. Man'ko, G. Marmo, P. Vitale, and F. Zaccaria, Int. J. Mod. Phys. **A9**, 5541 (1994).

[10] A. Wunsche, J. Opt. B: Quantum Semiclass. Opt. **2**, 73 (2000).

[11] V. S. Varadarajan, *Lie Gorups, Lie Algebras and Their Representations*, Springer-Verlag, 1984;
Z. X. Wan, *Lie Algebras*, Pergamon Press, 1975.