On the Steady–State Navier–Stokes Equations in Two Dimensional Domains

Antonio Russo

Tesi di dottorato

XX Ciclo del Dottorato in Scienze Matematiche
Sede Amministrativa: Dipartimento di Matematica e Applicazioni "R. Caccioppoli" - Università di Napoli "Federico II'.
Consorziato con: Seconda Università degli Studidi Napoli, Università della Basilicata, Università di Salerno.

A nonna Elena

Contents

Introduzione

È ben nota l'importanza che riveste tanto nella Meccanica Razionale quanto nelle Scienze Applicate lo studio delle equazioni di Navier–Stokes[1] [22]

$$\left.\begin{aligned} \Delta \boldsymbol{u} - \mathcal{R}\boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla p = \boldsymbol{f} \\ \operatorname{div}\boldsymbol{u} = 0 \end{aligned}\right\} \quad \text{in } \Omega \tag{1}$$

nelle incognite $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ $(n = 2, 3)$ *atto di moto* e $p : \Omega \to \mathbb{R}$ *pressione idrodinamica*. Tale sistema governa il moto stazionario di un fluido viscoso incomprimibile, di *viscosità cinematica* ν, in una regione identificata con un dominio (aperto connesso) del piano o dello spazio Ω su cui agisce una forza assegnata di densità di volume $\boldsymbol{f}$; il parametro $\mathcal{R}$, noto come numero di Reynolds, è caratteristico del problema ed è definito dal rapporto

$$\mathcal{R} = \frac{lv}{\nu},$$

con l e v lunghezza e velocità "di riferimento" [22]. La $(1)_1$ traduce in termini differenziali la prima equazione di Eulero o del bilancio della quantità di moto, e la $(1)_2$ la condizione di incomprimibilità del mezzo. Una caratteristica delle equazioni (1) è che la pressione p interviene solo col suo gradiente e risulta determinata dalla conoscenza di $\boldsymbol{u}$, coerentemente con la classica impostazione lagrangiana della Meccanica nella quale un vincolo, in questo caso l'incomprimibilità del fluido, dà luogo ad un'incognita reazione vincolare (pressione idrodinamica) che appare nell'equazione del *bilancio tra le forze* e che "scompare" in quella delle potenze calcolate per velocità compatibili con il vincolo [4], [34]. Al sistema (1) va associata una condizione sul contorno $\partial\Omega$ che corrisponde alla "fisica" del problema che si vuole analizzare. Una classica richiesta, ragionevole per "non piccole" viscosità, assume che le particelle del fluido si incollino ai punti del bordo, ovvero che la velocità di una di esse aderente alla frontiera in un punto $\xi \in \partial\Omega$ assuma sempre la velocità di ξ. Tale condizione di "aderenza" si traduce formalmente nella seguente equazione da associare al sistema (1)

$$\boldsymbol{u} = \boldsymbol{a} \quad \text{su } \partial\Omega, \tag{2}$$

[1] La simbologia che useremo è specificata nel prossimo paragrafo.

con $\boldsymbol{a}$ campo vettoriale assegnato su $\partial\Omega$, soddisfacente, a norma della $(1)_1$, la condizione

$$\int_{\partial\Omega} \boldsymbol{a}\cdot\boldsymbol{n} = 0, \tag{3}$$

se Ω è limitato. Se, poi, la regione di moto è illimitata, ad esempio esterna ad un compatto di $\mathbb{R}^n$ o tanto estesa che una condizione del tipo (2) non è più controllabile a grande distanza, allora l'infinito diviene una frontiera fittizia sulla quale è naturale richiedere che $\boldsymbol{u}$ sia costante, ovvero che esista un assegnato vettore costante $\boldsymbol{u}_0$ tale che

$$\lim_{r\to+\infty} \boldsymbol{u}(x) = \boldsymbol{u}_0. \tag{4}$$

Se $\mathcal{R}$ è sufficientemente piccolo è del tutto ragionevole, almeno in un primo stadio, trascurare il termine non lineare $\mathcal{R}\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ pervenendo così al sistema di Stokes

$$\begin{aligned} \Delta\boldsymbol{u} - \nabla p &= \boldsymbol{f} \\ \operatorname{div}\boldsymbol{u} &= 0 \end{aligned} \quad \text{in } \Omega. \tag{5}$$

Naturalmente, alle equazioni (5) andranno associate le condizioni (2), (3) nei domini limitati e (2), (4) in quelli non limitati.

Esistenza ed unicità di una soluzione dei suddetti problemi ai limiti sono stati oggetto di una serie impressionante di ricerche a partire dalla scoperta delle equazioni avvenuta nel 1822 ad opera di C.L.M.H. Navier, innanzitutto per il sistema lineare (5). I primi risultati risentono dei mezzi analitici del tempo e sono confinati a regioni di forma particolare. Ad esempio, nell'esterno della sfera ed in assenza di forze di volume[2], nel 1851 G.G. Stokes determinò la soluzione esplicita delle equazioni (5) costante al bordo ed infinitesima all'infinito (cfr. [14], p. 245). Passando, poi, all'equivalente problema in due dimensioni, notò che, a differenza del precedente, esso non poteva ammettere alcuna soluzione. Questa osservazione, divenuta celebre nel seguito come *Paradosso di Stokes*, inaugurò un affascinante problema:

[2]Nel seguito di questa introduzione, per semplicità di esposizione, assumeremo $\boldsymbol{f} = \boldsymbol{0}$.

in un dominio esterno del piano Ω caratterizzare i dati al bordo $\boldsymbol{a}$ per i quali il sistema

$$\begin{aligned} \Delta \boldsymbol{u} - \nabla p = \boldsymbol{0} \quad & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 \quad & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{a} \quad & \text{su } \partial\Omega, \\ \lim_{r \to +\infty} \boldsymbol{u}(x) = \boldsymbol{u}_0 & \end{aligned} \tag{6}$$

ammette una soluzione. Tale questione è stata compiutamente affrontata e risolta, almeno da un punto di vista teorico, in ambito variazionale e per domini di classe C^2, da C.G. Simader e G.P. Galdi in un famoso lavoro del 1990 [18]. I risultati di [18] sono stati estesi al caso di domini di classe $C^{1,\alpha}$ $(\alpha > 0)$ e dati al bordo continui in [26] ed al caso di domini lipschitziani e dati al bordo negli spazi di Lebesgue in [39]. La condizione necessaria e sufficiente determinata in [18] affinché il sistema (6) ammetta una soluzione puó essere espressa dalla seguente relazione [26] (cfr. paragrafo 2.8)

$$\int_{\partial\Omega} (\boldsymbol{a} - \boldsymbol{u}_0) \cdot \boldsymbol{\psi} = 0, \tag{7}$$

per tutte le densità $\boldsymbol{\psi}$ dei potenziali idrodinamici di semplice strato costanti su $\complement\Omega$. Si osservi che la (7) ha valenza prevalentemete teorica, riuscendo applicabile soltanto quando siano note le densità $\boldsymbol{\psi}$. A quanto ci risulta, ciò è possibile solo nel caso in cui $\partial\Omega$ sia un'ellisse [26] (cfr. paragrafo 2.8).

Per il primi risultati di una certa completezza bisogna attendere la pubblicazione del lavoro di F.K.G. Odqvist del 1930 [32]. Utilizzando la teoria delle equazioni integrali di Fredholm e richiedendo che i dati siano sufficientemente regolari, Odqvist dimostra che il problema di Stokes interno ammette un'unica soluzione espressa da un potenziale di doppio strato e quello esterno tridimensionale una soluzione somma di un potenziale di doppio strato e di uno di semplice strato per ogni assegnazione del dato al bordo[3]. Passando poi al problema non lineare interno (1)–(2), un' applicazione dei precedenti risultati

[3] Una chiara esposizione dei risultati di Odqvist è riportata nel Capitole 3 della monografia di O.A. Ladyszenskaia [24].

e del teorema delle contrazioni di S. Banach consente ad Odqvist di dimostrare l'esistenza di una soluzione a patto che i dati abbiano una "norma hölderiana" sufficientemente piccola. Come osservato in [15] p.2, tali risultati erano del tutto coerenti sia dal punto di vista teorico che da quello applicativo, tenendo presente il carattere fortemente non lineare delle equazioni $(1)_1$ e il fatto che l'accordo della teoria di Navier–Stokes con gli esperimenti aveva luogo solo per piccoli numeri di Reynolds. Di conseguenza, notevole impressione suscitò tra gli esperti della disciplina la lettura del celeberrimo lavoro di J. Leray [25] del 1933. In esso, infatti, Leray dimostrò l'esistenza di una soluzione del problema di Navier–Stokes interno *per ogni numero di Reynolds* nella sola ipotesi di regolarità dei dati e di flusso nullo su ogni componente connessa $\partial\Omega_i$ di $\partial\Omega$:

$$\int_{\partial\Omega_i} \boldsymbol{a}\cdot\boldsymbol{n} = 0. \tag{8}$$

Osserviamo che per la $(1)_2$ la (8) è automaticamente soddisfatta se $\partial\Omega$ è connessa. Per quanto riguarda il problema esterno, Leray costruì con un metodo detto *dei domini invadenti*, sempre nell'ipotesi (8), una soluzione delle equazioni (1), (2) e (4) ad *integrale di Dirichlet finito*

$$\int_{\Omega} |\nabla\boldsymbol{u}|^2 < +\infty. \tag{9}$$

In tre dimensioni, Leray dimostrò che la (9) garantisce che $\boldsymbol{u}$ tende "in un certo modo" al vettore costante assegnato $\boldsymbol{u}_0$. In due dimensioni, invece la (9) non assicura alcun tipo di convergenza, potendo essere soddisfatta da funzioni divergenti all'infinito, come, ad esempio, $\log^\alpha r$, $\alpha \in (0, 1/2)$.

Limitandoci ai problemi lasciati aperti da Leray in dimensione due, quelli di maggiore spessore, a nostro avviso, sono i seguenti [4]

- (i) l'accertamento che la soluzione in domini esterni costruita da Leray soddisfi la condizione all'infinito (4) o almeno ammetta limite all'infinito.

[4] Per i problemi lasciati aperti da J. Leray in dimensione tre una dettagliata esposizione è contenuta nella monografia di G.P. Galdi [15]

- (ii) la rimozione o, almeno, l'indebolimento, dell'ipotesi (8);

Per quanto riguarda il problema (i) un fondamentale contributo fu portato nel 1974 da D. Gilbarg e H.F. Weinberger [19]. Essi dimostrarono che la soluzione di Leray[5] converge all'infinito ad un vettore costante $\boldsymbol{\kappa}$ nel senso della convergenza in media di ordine due:

$$\lim_{R\to+\infty}\int_0^{2\pi}|\boldsymbol{u}-\boldsymbol{\kappa}|^2(R,\theta)=0.$$

Successivamente è stato dimostrato in [17] che tale convergenza è uniforme. Se $\boldsymbol{\kappa}$ coincida o meno con $\boldsymbol{u}_0$ è attualmente un problema aperto. Il solo esempio noto di un problema risolto del tipo (i) si deve a G.P. Galdi [17]. Se $\partial\Omega$ è simmetrico rispetto agli assi coordinati, ovvero se

$$(\xi_1,\xi_2)\in\partial\Omega\ \Rightarrow(-\xi_1,\xi_2),(\xi_1,-\xi_2)\in\partial\Omega,$$

$\boldsymbol{u}_0=\boldsymbol{0}$ e

$$\begin{aligned}a_1(\xi_1,\xi_2)&=-a_1(-\xi_1,\xi_2)=a_1(\xi_1,-\xi_2),\\ a_2(\xi_1,\xi_2)&=a_2(-\xi_1,\xi_2)=-a_2(\xi_1,-\xi_2),\end{aligned}$$

allora il problema di Navier–Stokes esterno bidimensionale ammette almeno una soluzione. Se questa sia unica in qualche classe funzionale è un problema completamente aperto.

Un primo contributo al problema (ii) in domini limitati si deve a W. Borchers e K. Pileckas [3], G.P. Galdi [13] e L.I. Sazonov [40] , i quali, riprendendo un'idea originaria di R. Finn [9], dimostrarono che i risultati di Leray continuano a valere nell'ipotesi che i flussi

$$\Phi_i=\int_{\partial\Omega_i}\boldsymbol{a}\cdot\boldsymbol{n}$$

siano sufficientemente piccoli. Per quanto riguarda i domini esterni, recentemente abbiamo dimostrato [36] l'esistenza di una soluzione variazionale, convergente quindi all'infinito a norma dei risultati di [19], a patto che

$$\frac{\alpha\mathfrak{R}}{2\pi}\sum_{i=1}^{k}|\Phi_i|<1,$$

[5] Con tale locuzione intenderemo la soluzione costruita da Leray con il metodo dei domini invadenti

dove

$$\alpha = \sup_{\|\boldsymbol{w}\|_{D^{1,2}_\sigma(\mathbb{R}^2)}=1} \left| \int_{\mathbb{R}^2} (\log r)\, \mathrm{div}(\boldsymbol{w}\cdot\nabla\boldsymbol{w}) \right|.$$

Se, poi, Ω è simmetrico rispetto al suo centroide , ovvero se

$$(\xi_1,\xi_2)\in\partial\Omega \;\Rightarrow (-\xi_1,-\xi_2)\in\partial\Omega \tag{10}$$

e

$$\boldsymbol{a}(\xi_1,\xi_2) = -\boldsymbol{a}(-\xi_1,-\xi_2), \tag{11}$$

allora $\boldsymbol{u}$ converge uniformemente a $\mathbf{0}$ all'infinito.

Nel 1997 H. Fujita e H. Morimoto [12] portarono il seguente interessante contributo al problema (i) in domini limitati. Supposto che $\boldsymbol{a}$ ammetta la seguente decomposizione

$$\boldsymbol{a} = \mathcal{F}\boldsymbol{h} + \boldsymbol{\gamma},$$

con $\mathcal{F}\in\mathbb{R}$, $\boldsymbol{h}$ gradiente di una funzione armonica e $\boldsymbol{\gamma}\in W^{1/2,2}(\partial\Omega)$ soddisfacente la condizione

$$\int_{\partial\Omega} \boldsymbol{\gamma}\cdot\boldsymbol{n} = 0,$$

allora, a meno di un insieme numerabile $\mathcal{G}$ di valori di $\mathcal{F}$, esiste una costante positiva c_0 dipendente da $\mathcal{F}$, $\boldsymbol{h}$ ed Ω per cui, se

$$\|\boldsymbol{\gamma}\|_{W^{1/2,2}(\partial\Omega)} < c_0,$$

allora il sistema (1), (2) ammette una soluzione.

Osserviamo che in tutti i risultati su esposti, la formulazione dei problemi avviene in ambito variazionale e, quindi, i dati al bordo sono richiesti appartenere almeno ad opportuni spazi di traccia. I risultati di [12] sono stati estesi in un recente lavoro [37] a dati al bordo negli spazi di Lebesgue [6].

Lo scopo di questa tesi è quello di presentare, in forma ragionevolmente autosufficiente, quanto abbiamo appreso nel ciclo di dottorato

[6] In questo lavoro abbiamo anche esteso i risultati di Fujita–Morimoto nei domini esterni tridimensionali

sui problemi esposti ai punti (i), (ii) e il piccolo contributo fornito in questi anni alla loro comprensione.

Dopo aver esplicitato nel primo capitolo i principali simboli e gli strumenti matematici che useremo, nel capitolo 2 svilupperemo la teoria dell'esistenza ed unicità di soluzioni dei problemi interno ed esterno di Stokes, essenzialmente nell'ipotesi che i dati al bordo appartengano a qualche spazio $L^q(\partial\Omega)$ $(q > 1)$ e le forze allo spazio di Hardy $\mathcal{H}^1(\Omega)$. Tale studio, che terminerà con la dimostrazione del paradosso di Stokes, oltre ad essere propedeutico a quello del problema (non lineare) di Navier–Stokes, è di un certo interesse anche in virtù del fatto che le applicazioni più numerose della teoria riguardano i cosiddetti "moti lenti di un fluido viscoso" retti dal sistema lineare (5) (cfr., *e.g.*, [23]). Utilizzeremo la classica teoria delle equazioni integrali di Fredholm, seguendo l'impostazione sviluppata in [26], dove tuttavia particolare attenzione è riservata alle soluzioni classiche corrispondenti a dati al bordo continui e forze hölderiane.

Nel capitolo II esporremo la teoria esistenziale delle soluzioni del problema interno di Navier–Stokes nelle stesse ipotesi sui dati richieste nel problema lineare, con $q \geq 2$, dimostrando in conclusione i risultati del lavoro [37].

Nel terzo ed ultimo capitolo, tratteremo il ben più complesso problema esterno di Navier–Stokes, estendendo il risultato dimostrato in [36]. Precisamente, dimostremo che, se

$$\boldsymbol{a} \in L^2(\Omega)$$

e

$$\frac{\mathcal{R}}{2\pi}\sum_{i=1}^{m}|\Phi_i| < 1,$$

allora il sistema (1) ammette una soluzione

$$(\boldsymbol{u}, p) \in C^\infty(\Omega) \times C^\infty(\Omega)$$

che assume il dato al bordo $\boldsymbol{a}$ in un senso opportuno, che coincide con quello della convergenza non tangenziale e con quello classico, rispettivamente per $\boldsymbol{a} \in L^q(\partial\Omega)$ $(q > 2)$ e $\boldsymbol{a} \in C(\partial\Omega)$. Inoltre, esistono un vettore $\boldsymbol{\kappa}$ ed uno scalare p_0 tali che

$$\lim_{r\to+\infty} \boldsymbol{u}(x) = \boldsymbol{\kappa}, \quad \lim_{r\to+\infty} p(x) = p_0$$

uniformemente e, se valgono le (10), (11), allora $\boldsymbol{\kappa} = \boldsymbol{0}$.

Introduction

Well known is the great importance in Rational Mechanics as well as in applied sciences of the study of Navier–Stokes equations[7] [22]

$$\left.\begin{array}{r}\Delta \boldsymbol{u} - \mathcal{R}\boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla p = \boldsymbol{f} \\ \operatorname{div}\boldsymbol{u} = 0\end{array}\right. \quad \text{in } \Omega \tag{1}$$

in the unknowns $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ $(n = 2, 3)$ *kinetic field* and $p : \Omega \to \mathbb{R}^n$ *hydrodynamic pressure*. This system governs the steady motion of an incompressible viscous fluid with *kinematic viscosity* ν, in a region identified with a domain (open connected set) of the plane or of the space Ω, under the action of an assigned force whose volume density is f; the parameter $\mathcal{R}$, known as the Reynolds number, is characteristic of the problem and is defined by the ratio

$$\mathcal{R} = \frac{lv}{\nu}$$

where l are v are a "reference" length and a "reference" velocity respectively [22]. Equations $(1)_1$ express in differential terms the first Euler equation (the moment balance equation), while equation $(1)_2$ expresses the incompressibility condition for the fluid. A peculiar property of equations (1) is that the pressure p appears only through its gradient, and is determined from the knowledge of u, according to the classical lagrangian assessment of Mechanics, in which any constraint, in this case the incompressibility of the fluid, produces an unknown force of constraint (hydrodynamic pressure), which is present in the equation of the *balance of the forces* but "disappears" in the balance of their powers evaluated for velocities that are compatible with the constraint [4], [34]. To system (1) a condition on the boundary $\partial\Omega$, corresponding to the "phisics" of the particular problem considered, must be added. A classical requirement, which is reasonable for "not too small" viscosity, is that the boundary particles of the fluid are glued to the points of the bounding wall, *i.e.* that the velocity of any one of them which is in contact with this wall at a point $\xi \in \partial\Omega$ is

[7] The notation used here will be specified in the next Section.

always equal to the speed of ξ. Such "adherence" condition finds its formal expression in the following equation to be added to system (1):

$$\boldsymbol{u} = \boldsymbol{a} \quad \text{su } \partial\Omega, \tag{2}$$

where a is a prescribed vector field on $\partial\Omega$, satisfying, according to $(1)_2$, the condition

$$\int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{n} = 0 \tag{3}$$

if Ω is bounded. If the region occupied by the fluid in motion is instead unbounded, for instance external to a compact set of $\mathbb{R}^n$, or so wide that a condition of type (2) can be not experimentally checked at large distance, then the infinity becomes an additional, virtual boundary on which it is reasonable to require that $\boldsymbol{u}$ be constant, that is to say that an assigned constant vector $\boldsymbol{u}_0$ exist such that

$$\lim_{r \to +\infty} \boldsymbol{u}(x) = \boldsymbol{u}_0. \tag{4}$$

If $\mathcal{R}$ is sufficiently small, then it is quite reasonable — at least at a first stage — to neglect the nonlinear term $\mathcal{R}\boldsymbol{u} \cdot \nabla \boldsymbol{u}$, thus deriving the Stokes system

$$\begin{aligned} \Delta \boldsymbol{u} - \nabla p &= \boldsymbol{f} \\ \operatorname{div} \boldsymbol{u} &= 0 \end{aligned} \quad \text{in } \Omega. \tag{5}$$

Naturally, to equations (5) we must associate conditions (2), (3) in bounded domains and conditions (2), (4) in unbounded ones.

The question about the existence and the uniqueness of a solution to these boundary problems has been the object of many researches since the discovery of equations (1) in 1822, due to Navier, in particular for the linear system (5). The first results suffer the limits of the analytical tools of their time, and are confined to regions of particular shapes. For instance, in the external region to a ball, and in the absence of volume forces[8], Stokes determined in 1851 the explicit solution of equations (5) with velocity constant on the boundary and zero at infinity (see [14], p. 245). Turning then his attention to the

[8]In the sequel of this introduction, only for the sake of simplicity, we shall assume $\boldsymbol{f} = \boldsymbol{0}$.

equivalent problem in two dimensions, he observed that, opposite to the preceding one, it has no solution at all. This observation, which was thereafter known as the *Stokes Paradox*, opened a fascinating problem: in an external domain Ω of the plane, to characterize the boundary data a for which the system

$$\begin{aligned} \Delta \boldsymbol{u} - \nabla p = \boldsymbol{f} &\quad \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 &\quad \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{a} &\quad \text{su } \partial\Omega, \\ \lim_{r \to +\infty} \boldsymbol{u}(x) = \mathbf{0} & \end{aligned} \tag{6}$$

has a solution. This question has been tackled and, at least from a theoretical viewpoint, solved by C.G. Simader and G.P. Galdi in a celebrated paper of 1990 [18], in a variational framework and for regular domains. The results of [18] have been extended in [26] to the case of domains of class $C^{1,\alpha}$ ($\alpha > 0$) and continuous boundary data and in [39] to the case of Lipschitz domains and boundary data belonging to a Lebesgue space. The necessary and sufficient found in [18] in order that system (6) has a solution, can be expressed (for $\boldsymbol{f} = \mathbf{0}$) as follows [26] (see also Section 2.8)

$$\int_{\partial\Omega} (\boldsymbol{a} - \boldsymbol{u}_0) \cdot \boldsymbol{\psi} = 0, \tag{7}$$

for all densities $\boldsymbol{\psi}$ of the hydrodynamical simple layer potentials constant in $\complement\Omega$. Observe that relation (7) has a mainly theoretical meaning, as it may be applied only when the elements of $\mathcal{S}_q$ are known. As far as we are aware, this is possible only in the case in which $\partial\Omega$ is an ellipse [26] (see also Section 2.8).

The first reasonably complete results were published in the paper by F.K.G. Odqvist of 1930 [32]. Using the theory of integral equations of Fredholm, and requiring that the data be sufficiently regular, Odqvist proves that the internal Stokes problem has a unique solution expressed by a double layer potential and that the external Stokes problem has a solution which is the sum of a double layer potential and a simple layer potential for any prescribed boundary data[9]. Turning then to the nonlinear internal problem (1)–(3), an application of the previous results and of the contraction theorem of S. Banach allows Odqvist to prove the existence of a solution provided the data have a sufficiently small "Hölder norm". As pointed out in [15] p.2, these results were quite consistent from the theoretical viewpoint as well as in view of applications, bearing in mind the strongly nonlinear character of equations $(1)_1$ and the fact that only for small values of the Reynolds number there is agreement between experimental evidence and Navier–Stokes theory. As a consequence, a strong impact among the experts in the field had the celebrated paper by J. Leray [25] of 1933, where the existence of a solution to the internal Navier–Stokes problem was proved *for any Reynolds number* in the sole assumption of regular data and zero flux on any connected component $\partial\Omega_i$ of $\partial\Omega$:

$$\int_{\partial\Omega_i} \boldsymbol{a}\cdot\boldsymbol{n} = 0. \tag{8}$$

Observe that, for relation $(1)_2$, (8) is obviously satisfied if $\partial\Omega$ is connected. As far as the external problem is concerned, using a method called *of invading domains*, still under the hypothesis (8), Leray showed the existence of a solution to equations (1), (3) with a *finite Dirichlet integral*:

$$\int_{\Omega} |\nabla\boldsymbol{u}|^2 < +\infty. \tag{9}$$

In three dimensions, Leray proved that (9) assures that $\boldsymbol{u}$ "somehow" tends to the assigned constant vector $\boldsymbol{u}_0$. In two dimensions, instead, (9) cannot assure any type of convergence, as it can be satisfied by functions that diverge at infinity, such as, for instance, $\log^\alpha r$, $\alpha \in (0,1/2)$.

Confining ourselves to the problems left open by Leray in two dimensions, the ones of major moment, in our opinion, are the following[10]:

[9] A clear exposition of Odqvist's results can be found in Chapter 3 of the monograph by O.A. Ladyszenskaia [24].

[10] A detailed exposition of the problems left open by J. Leray in three dimensions is contained in the monograph by G.P. Galdi [15]

- (i) to ascertain that the solution in external domains constructed by Leray satisfies the condition at infinity (4) or at least has a limit at infinity;

- (ii) to drop, or at least to weaken, assumption (8).

As regards problem (i) a fundamental contribution was given in 1974 by D. Gilbarg e H.F. Weinberger [19]. They proved that Leray's solution[11] converges at infinity to a constant vector $\boldsymbol{\kappa}$ in the L^2 sense

$$\lim_{R\to+\infty}\int_0^{2\pi}|\boldsymbol{u}-\boldsymbol{\kappa}|^2(R,\theta)=0.$$

Subsequently it was proved in [17] that the above convergence is uniform. Whether $\boldsymbol{\kappa}$ coincides with $\boldsymbol{u}_0$ or not is still an open problem. The sole known example of a solved problem of type (i) is due to G.P. Galdi [17]. If $\partial\Omega$ is symmetric with respect to the coordinate axes, that is if

$$(\xi_1,\xi_2)\in\partial\Omega\ \Rightarrow(-\xi_1,\xi_2),(\xi_1,-\xi_2)\in\partial\Omega.$$

$\boldsymbol{f}=\boldsymbol{0}$, $\boldsymbol{u}_0=\boldsymbol{0}$ and

$$\begin{aligned}a_1(\xi_1,\xi_2)&=-a_1(-\xi_1,\xi_2)=a_1(\xi_1,-\xi_2),\\ a_2(\xi_1,\xi_2)&=a_2(-\xi_1,\xi_2)=-a_2(\xi_1,-\xi_2),\end{aligned}$$

then the two-dimensional external Navier–Stokes problem has at least one solution. Whether it is unique in some functional class is a quite open problem.

The first two contributions to the solution of problem (ii) in bounded domains are due to W. Borchers e K. Pileckas [3], G.P. Galdi [13] and L.I. Sazonov [40] who, by developing a previous idea of R. Finn [9], showed that Leray's results keep holding under the assumption that the fluxes

$$\Phi_i=\int_{\partial\Omega}\boldsymbol{a}\cdot\boldsymbol{n}$$

[11] By this expression we mean the solution constructed by the method of invading domains.

be sufficiently small. As regards external domains, we have recently proved [36] the existence of a solution of the Leray type, therefore converging at infinity according to the results of [19], [17], provided

$$\frac{\alpha\mathcal{R}}{2\pi}\sum_{i=1}^{k}|\Phi_i|<1,$$

where

$$\alpha=\sup_{\|\boldsymbol{w}\|_{D^{1,2}_{\sigma}(\mathbb{R}^2)}=1}\left|\int_{\mathbb{R}^2}(\log r)\,\mathrm{div}(\boldsymbol{w}\cdot\nabla\boldsymbol{w})\right|.$$

If in addition Ω is symmetric with respect to its centroid, *i.e.* if

$$(\xi_1,\xi_2)\in\partial\Omega\ \Rightarrow(-\xi_1,-\xi_2)\in\partial\Omega \tag{10}$$

and

$$\boldsymbol{a}(\xi_1,\xi_2)=-\boldsymbol{a}(-\xi_1,-\xi_2), \tag{11}$$

then $\boldsymbol{u}$ uniformly converges to $\mathbf{0}$ at infinity.

In 1997 H. Fujita e H. Morimoto [12] gave the following interesting contribution to the solution of problem (i) in bounded domains. If we assume that a can be written in the form

$$\boldsymbol{a}=\mathcal{F}\boldsymbol{h}+\boldsymbol{\gamma}$$

with $\mathcal{F}\in\mathbb{R}$ and $\boldsymbol{h}$ the gradient of an harmonic function and $\boldsymbol{\gamma}\in W^{1/2,2}(\partial\Omega)$ satisfying the condition

$$\int_{\partial\Omega}\boldsymbol{\gamma}\cdot\boldsymbol{n}=0,$$

then a countable set $\mathcal{G}$ of values of $\mathcal{F}$ exists such that if $\mathcal{F}\notin\mathcal{G}$, then one can find a positive constant c_0 depending on $\mathcal{F}$, $\boldsymbol{h}$ and Ω for which, if

$$\|\boldsymbol{\gamma}\|_{W^{1/2,2}(\partial\Omega)}<c_0,$$

then system (1)–(3) has a solution.

Observe that, for all the results here quoted, the problems are formulated in a variational framework, so that the boundary data are required to belong at least to suitable trace spaces. The results of [12] have been extended in a recent paper [37] to the case of boundary data belonging to Lebesgue spaces[12].

The aim of the present thesis is to present, in a sufficiently self-contained form, all we have learned during the four years of our Doctorate cycle about the problems described at items (i), (ii), as well as the little contribution we have been able to give to their understanding.

After explaining the most useful symbols and the mathematical tools we shall use throughout the whole work (second and third section of this chapter), in chapter 2 we shall develop the existence and uniqueness theory for solutions to internal and external Stokes problems, essentially under the assumption that the boundary data belong to some space $L^q(\partial\Omega)$ ($q > 1$) and the forces belong to the Hardy space $\mathcal{H}^1(\Omega)$. This study, which will end with the proof of the Stokes paradox, is somehow "preparatory" to that of the (nonlinear) Navier–Stokes problem, but has also an intrinsic interest because the most part of the applications of the theory are concerned with the so-called "slow motions of a viscous fluid", obviously described by the linear system (5) (cr., *e.g.*, [23]). We shall use the theory of integral equations of Fredholm, following the perspective developed in [26], where a particular emphasis is however reserved to classical solutions corresponding to continuous boundary data and Hölder continuous forces.

In chapter II we shall present the existence theory for solutions to the internal Navier–Stokes problem under the same assumptions on data as in the linear problem, with $q \geq 2$, and finally prove the results of paper [37]. In the third and last chapter, we shall treat the much more complex external Navier–Stokes problem, and give an extension of the result proved in [36]. Precisely, we shall show that, if

$$a \in L^q(\Omega),\ q \geq 2,$$

and

$$\frac{\mathcal{R}}{2\pi}\sum_{i=1}^{m}|\Phi_i| < 1,$$

[12]In this paper the results of i Fujita–Morimoto have been also extended to three–dimensional exterior domains

then system(1) has a solution

$$(\boldsymbol{u}, p) \in C^{\infty}(\Omega) \times C^{\infty}(\Omega),$$

which "takes" the boundary datum $\boldsymbol{a}$ in a suitable sense, which coincides with the one of the nontangential convergence and with the classical one for $\boldsymbol{a} \in L^q(\partial\Omega)$, $q > 1$, and $\boldsymbol{a} \in C(\partial\Omega)$ respectively. Furthermore, there exist a vector $\boldsymbol{\kappa}$ and a scalar p_0 such that

$$\lim_{r\to+\infty} \boldsymbol{u}(x) = \boldsymbol{\kappa}, \quad \lim_{r\to+\infty} p(x) = p_0,$$

uniformly and, if relations (10), (11) hold, then $\boldsymbol{\kappa} = \mathbf{0}$.

Chapter 1

Notation and mathematical tools

1.1 Notation

Throughout this thesis we shall adopt vector notation. $\mathbb{R}$ is the set of the real numbers, $\mathbb{N}$ is the set of the natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Italic light face letters, different from o, x, y, ξ and ζ denote scalars (in $\mathbb{R}$; o is the origin of the reference frame $(o, \{\boldsymbol{e}_1, \boldsymbol{e}_2\})$, with $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ orthonormal basis of $\mathbb{R}^2$ and we set $\boldsymbol{e}_3 = \boldsymbol{e}_1 \times \boldsymbol{e}_2$; italic bold–face lower case letters stand for vectors (in $\mathbb{R}^2$) and bold–face upper case letters stand for second order tensors; the identity map is denoted by $\mathcal{I}$, which is also used to denote the unit second–order tensor; x, y are points of $\mathbb{R}^2$ and ξ, ζ points on surfaces;

$$\boldsymbol{x} = x - o, \quad r = r(x) = |\boldsymbol{x}|, \quad \boldsymbol{x} = r\boldsymbol{e}_r, x \neq o;$$
$$\boldsymbol{a} \cdot \boldsymbol{A} = a_i A_{ij} \boldsymbol{e}_j, \quad \boldsymbol{A} \cdot \boldsymbol{a} = a_j A_{ij} \boldsymbol{e}_i, \quad \operatorname{tr} \boldsymbol{A} = A_{ii}, \ \boldsymbol{A} \cdot \boldsymbol{B} = A_{ij} B_{ij}$$

where a_i and A_{ij}, B_{ij} $(i, j = 1, 2)$ are the components of the vector $\boldsymbol{a}$ and of the second–order tensors $\boldsymbol{A}$, $\boldsymbol{B}$ in $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$, and the convention on repeated indexes is used; $\boldsymbol{a} \otimes \boldsymbol{b}$ is the second–order tensor with component $a_i b_j$; $\boldsymbol{A}^{\mathrm{T}}$ is the transpose of $\boldsymbol{A}$.

The greek letter Ω is reserved to denote a domain (open connected set) of $\mathbb{R}^2$; $\overline{\Omega}$ stands for its closure, $\partial\Omega$ its boundary and $\complement\Omega = \mathbb{R}^2 \setminus \Omega$;

$$S_R = \{x : |\boldsymbol{x}| < R\}, \quad T_R = S_{2R} \setminus S_R, \quad \Omega_R = \Omega \cap S_R.$$

Ω is bounded if there is a disk S_R such that $\Omega \subset S_R$; Ω is of class C^2 if , for every $\xi \in \partial\Omega$, there is a neighborhood of ξ (in $\partial\Omega$) with is the graph of a function of class C^2. The function $\delta = \delta(x)$ stands for the distance of x from $\partial\Omega$. We set

$$\Omega(t) = \{x \in \Omega : \delta(x) < t\}.$$

- Let Ω_i, $i = 0, 1, \dots, m$, be $m+1$ bounded domains ($m \in \mathbb{N}$) with connected boundaries and

  $$\Omega' = \bigcup_{i=1}^{m} \Omega_i; \quad \overline{\Omega}_i \subset \Omega_0,\ i \neq 0; \quad \overline{\Omega}_i \cap \overline{\Omega}_j = \varnothing,\ i \neq j \neq 0.$$

  In the sequel we shall consider the *bounded domain*

  $$\Omega = \Omega_0 \setminus \Omega' \tag{1.1.1}$$

  and the *exterior domain*

  $$\Omega = \mathbb{R}^2 \setminus \bigcup_{i=1}^{m} \overline{\Omega}_i. \tag{1.1.2}$$

  The unit normal vector $\boldsymbol{n}$ to $\partial\Omega$ is chosen interior with respect to Ω' and exterior with respect to Ω_0.

If Ω is exterior, S_{R_0} denotes a disk containing $\overline{\Omega'}$.

- We shall always assume that the above domains are of class C^2.

Let φ be a (scalar, vector or second order tensor) field in an exterior domain Ω and let f be a positive function in $(0, +\infty)$. As is customary,

$$\varphi = o(f) \Leftrightarrow \lim_{r \to +\infty} \frac{|\varphi(x)|}{f(r)} = 0$$

and

$$\varphi = O(f) \Leftrightarrow |\varphi(x)| \le cf(r), \quad \forall\, x \in \complement S_{R_0}.$$

Let $\varphi(x)$ be a k-time differentiable function in Ω; $\nabla\varphi$ is the vector with components $\partial_i\varphi = \partial\varphi/\partial x_i$ and we set

$$\nabla_i\varphi = \underbrace{\nabla \ldots \nabla}_{i-\text{times}}\varphi,\ i = 1, \ldots, k; \quad \nabla_1\varphi = \nabla\varphi, \quad \nabla_0\varphi = \varphi.$$

Let $k \in \mathbb{N}_0$. $C^k(\Omega)$ ($C(\Omega) = C^0(\Omega)$) is the linear space of functions φ such that $\nabla_i\varphi$ is continuous in Ω for all $i = 0, \ldots, k$; $C^k(\overline{\Omega})$ is the subspace of $C^k(\Omega)$ of functions φ such that $\nabla_i\varphi$ is bounded and uniformly continuous in Ω; it is a Banach space endowed with the norm

$$\|\varphi\|_{C^k(\overline{\Omega})} = \max_{i=0,\ldots k} \sup_{\Omega} |\nabla_i\varphi|;$$

$C^{k,\alpha}(\Omega)$, $\alpha \in (0,1]$, is the subspace of $C^k(\Omega)$ of functions φ such that $\nabla_i\varphi$ is (locally) Hölder continuous and $C^{k,\alpha}(\overline{\Omega})(\subset C^k(\overline{\Omega}))$ is the Banach space endowed with the norm

$$\|\varphi\|_{C^{k,\alpha}(\overline{\Omega})} = \max_{i=0,\ldots k} \sup_{x\neq y\in\Omega} \frac{|\nabla_i\varphi(x) - \nabla_i\varphi(y)|}{|x-y|^\alpha}$$

Analogous definitions are given for function defined over $\partial\Omega$.

The symbol $\mathfrak{R}$ denotes the three dimensional linear space of rigid motions $\boldsymbol{\varrho}$ on the plane

$$\boldsymbol{\varrho}(x) = \boldsymbol{\kappa} + \alpha\boldsymbol{e}_3 \times \boldsymbol{x},$$

for all constant vectors $\boldsymbol{k}$ and for all scalars α. Moreover, denoting by $\mathcal{A}$ a a subset of $\mathbb{R}^2$, $\mathfrak{R}_{\mathcal{A}}$ indicates the restrictions of the fields of $\mathfrak{R}$ to $\mathcal{A}$.

We set

$$C^\infty(\Omega) = \bigcap_{k\in\mathbb{N}_0} C^k(\Omega)$$

and we denote by $C_0^\infty(\Omega)$ the subset of C^∞ of functions φ such that $\nabla_k\varphi$ have a compact supports in Ω for all $k \in \mathbb{N}_0$.

Let $\boldsymbol{u}(x) = u_i(x)\boldsymbol{e}_i$, $\boldsymbol{S}(x) = S_{ij}(x)\boldsymbol{e}_i\otimes\boldsymbol{e}_j$ be a vector and a second–order tensor field in Ω, with $u_i, S_{ij} \in C^k(\Omega)$; $\nabla\boldsymbol{u}$ is the second–order tensor field with components $(\nabla \boldsymbol{u})_{ij} = \partial u_j/\partial x_i$, $\operatorname{div} \boldsymbol{u} = \operatorname{tr} \nabla \boldsymbol{u}$ and $\Delta \boldsymbol{u} = \operatorname{div} \nabla \boldsymbol{u}$; $\operatorname{div} \boldsymbol{S}$ is the vector field with components $\partial S_{ij}/\partial x_j$; accordingly, $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ is the vector with components $u_i \partial u_j/\partial x_i$. Analogous definitions are given for higher derivatives. We shall not distinguish in notation between scalar, vector or tensor function spaces; for instance, $\boldsymbol{u} \in C(\overline{\Omega})$ means $u_i \in C(\overline{\Omega})$, $i = 1, 2$ and $\|\boldsymbol{u}\|_{C(\overline{\Omega})} = \sup_\Omega |\boldsymbol{u}|$. We set

$$\hat{\nabla} \boldsymbol{u} = \tfrac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}}), \quad \tilde{\nabla} \boldsymbol{u} = \tfrac{1}{2}(\nabla \boldsymbol{u} - \nabla \boldsymbol{u}^{\mathrm{T}}).$$

The symbols c, c_0,... denote positive constants whose numerical value are not essential to our purposes and may have different value in a same line. When the dependence on a parameter λ has to be specified, we write $c(\lambda)$.

1.2 Some mathematical tools

In this section we recall the main properties of the function spaces we use in this work. We quickly premise some necessary tools of functional analysis [28].

If B a Banach space with norm $\|\cdot\|_B$, by B^* we denote its dual (the linear space of all linear and continuous functional from B to $\mathbb{R}$) endowed with topologies induced by the norm (strong topology)

$$\|u^*\|_{B^*} = \sup_{\|u\|_B=1} \langle u^*, u\rangle,$$

where $\langle u^*, u\rangle$ denotes the value of the functional u^* at u, by the family of seminorms $\{p_{u^*}\}_{u^*\in B^*}$ (weak topology)

$$p_{u^*}(u) = \langle u^*, u\rangle.$$

If the canonical immersion $B \to (B^*)^*$ is onto, then B is called reflexive. In such a case, if $A \subset B$ is bounded in the strong topology, then $\overline{A}$ is compact in the weak topology.

Let B' be another Banach space and let

$$\mathcal{K} : B \to B' \tag{1.2.1}$$

be a linear operator from B to B'. Recall that operator

$$\mathcal{K}^* : B'^* \to B^* \tag{1.2.2}$$

defined by the reciprocity relation

$$\langle \mathcal{K}^*[u'^*], u\rangle = \langle u'^*, \mathcal{K}[u]\rangle, \quad \forall\, u \in B,\ \forall\, u'^* \in B'^*,$$

is called the adjoint of $\mathcal{K}$.

The operator (1.2.1) is completely continuous if it is contiguous and $\overline{\mathcal{K}(A)}$ is a compact subset of B' for every bounded subset A of B, *i.e*, if for every bounded sequence $\{u_k\}_{k\in\mathbb{N}}$ of B from $\{\mathcal{K}[u_k]\}_{k\in\mathbb{N}}$ we can extract a subsequence which converges in B'. By a well–known theorem $\mathcal{K}$ is completely continuous if and only if $\mathcal{K}^*$ is completely continuous.

Consider the operator

$$\mathcal{I} + \mu\mathcal{K} : B \to B \tag{1.2.3}$$

for $\mu \in \mathbb{R}$. The following result is known as *Fredholm alternative* [28].

Lemma 1.2.1 *If $\mathcal{K}$ is linear and completely continuous, then there is a countable subset G of $\mathbb{R}$ such that the map (1.2.3) is invertible for all $\mu \notin G$. Moreover, if $\mu \in G$, then*

$$\dim \operatorname{Ker} \big(\mathcal{I} + \mu\mathcal{K}\big) = \dim \operatorname{Ker} \big(\mathcal{I} + \mu\mathcal{K}^*\big) \in \mathbb{N}_0$$

and the equation

$$u' = \big(\mathcal{I} + \mu\mathcal{K}\big)[u] \quad \Big[resp.\ u'^* = \big(\mathcal{I} + \mu\mathcal{K}^*\big)[u^*]\Big]$$

has a solution if and only if

$$\langle u^*, u' \rangle = 0 \quad \Big[resp.\ \langle u'^*, u \rangle = 0\Big]$$

for all $u^* \in \operatorname{Ker} \big(\mathcal{I} + \mu\mathcal{K}^*\big)$ $\Big[$*resp.* $u \in \operatorname{Ker} \big(\mathcal{I} + \mu\mathcal{K}\big)\Big]$.

Consider a map

$$\mathcal{T} : B \to B.$$

A fixed point of $\mathcal{T}$ is a point $\bar{u}$ of B such that

$$\bar{u} = \mathcal{T}[\bar{u}].$$

The following results are classical (see, *e.g.*, [8]).

Lemma 1.2.2 *If $\mathcal{T}$ is a contraction, i.e., there is $\mu \in (0, 1)$ such that*

$$\|\mathcal{T}[u]\|_B \le \mu \|u\|_B,$$

then $\mathcal{T}$ has a unique fixed point in B.

We shall use a simple consequence of Lemma 1.2.2.

Lemma 1.2.3 *Let*

$$\mathcal{T}[u] = u_0 + \mathcal{W}[u],$$

with

$$\|\mathcal{W}[u]\|_B \leq c_0 \|u\|_B^2.$$

If

$$\|u_0\|_B < \frac{1}{4c_0},$$

then $\mathcal{T}$ has a unique fixed point in the ball

$$S = \left\{ u \in B : \|u\|_B < \frac{1}{2c_0} \right\}.$$

Proof - If $u \in S$ then

$$\|\mathcal{T}[u]\|_B \leq \frac{1}{4c_0} + c_0 \|u\|_B^2 \leq \frac{1}{2c_0}$$

so that $\mathcal{T}$ maps the Banach space $\bar{S}$ into itself. Since

$$\|\mathcal{T}[u_1 - u_2]\|_B \leq c_0 \|u_1 - u_2\|_B^2 \leq \frac{1}{2} \|u_1 - u_2\|_B,$$

$\mathcal{T}$ is a contraction on S, where by Lemma 1.2.2 it has a unique fixed point. □

Lemma 1.2.4 . *Let $\mathcal{T}$ be a completely continuous map from B into itself. If the set*

$$\{u \in B : u = \mu \mathcal{T}[u],\ \mu \in [0, 1]\}$$

is bounded, then $\mathcal{T}$ has a fixed point.

- $L^q(\Omega)$ (Lebesgue's spaces) and $W_0^{k,q}(\Omega)$ (Sobolev's spaces), with $k \in \mathbb{N}$ and $q \in (1, +\infty)$ are the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{L^q(\Omega)} = \left\{ \int_\Omega |u|^q \right\}^{1/q},$$

$$\|u\|_{W^{1,q}(\Omega)} = \left\{ \|u\|_{L^q(\Omega)}^q + \sum_{i=1}^{k} \|\nabla_i u\|_{L^q(\Omega)}^q \right\}^{1/q},$$

respectively.

We denote by $L^1_{\text{loc}}(\overline{\Omega})$ the linear space of all $u \in L^1(\Omega')$ for all measurable subset Ω' of Ω and by $L^1_{\text{loc}}(\Omega)$ the linear space of all $u \in L^1(K)$ for all compact K contained in Ω.

$L^\infty(\Omega)$ is the space of all $u \in L^1_{\text{loc}}(\overline{\Omega})$ such that $|u(x)| \leq c$ for some positive constant c and almost all $x \in \Omega$. $L^\infty(\Omega)$ is a Banach space with norm

$$\|u\|_{L^\infty(\Omega)} = \inf\{c : |u(x)| \leq c, \text{ a.e. on } \Omega\}.$$

$L^{q'}(\Omega)$ is the dual space of $L^q(\Omega)$ $(q > 1)$, where

$$\frac{1}{q} + \frac{1}{q'} = 1, \tag{1.2.4}$$

while $L^\infty(\Omega)$ is the dual space of $L^1(\Omega)$. Therefore, for $q > 1$, $L^q(\Omega)$ is reflexive so that, if $\{u_k\}_{k\in\mathbb{N}}$ is a bounded sequence in $L^q(\Omega)$, $(q > 1)$, then there is $u \in L^q(\Omega)$ such that along a subsequence $\{u_{k'}\}$

$$\lim_{k'\to+\infty} \int_\Omega u_{k'} v = \int_\Omega uv, \tag{1.2.5}$$

for all $v \in L^{q'}(\Omega)$.

- $W^{k,q}(\Omega)$ is the completion of the space

$$\{u \in C^k(\Omega) : \|u\|_{W^{k,q}(\Omega)} < +\infty\}$$

with respect to the norm $\|u\|_{W^{k,q}(\Omega)}$. Analogous definitions are given for the Lebesgue and Sobolev spaces on $\partial\Omega$ [28].

Let B and B' be two Banach space. The symbol $B \hookrightarrow B'$ means that $B \subset B'$ and $\mathcal{I} : x \in B \to x \in B'$ is continuous. Moreover, $B \overset{c}{\hookrightarrow} B'$ means that $\mathcal{I}$ is also compact.

We recall the classical embedding theorems [21], [28].

Lemma 1.2.5 *If $k \in \mathbb{N}$ and $q \in [1, +\infty)$, then*[1]

$$W^{k,q}(\Omega) \hookrightarrow L^{q^*}(\Omega), \quad q < 2,\ q^* = 2q/(2 - kq) \in [1, +\infty);$$
$$W^{1,2}(\Omega) \hookrightarrow L^{q}(\Omega), \quad q \in [2, +\infty).$$

Moreover, if Ω is bounded, then

$$W^{k,q}(\Omega) \hookrightarrow C^{h,\mu}(\overline{\Omega}), \quad kq > 2,\ h = [k - 2/q], \quad \mu = k - h - 2/q,\ \text{for } k - 2/q \notin \mathbb{N};$$
$$W^{k,q}(\Omega) \hookrightarrow C^{k-1-2/q,\mu}(\overline{\Omega}), \quad kq > 2, \mu \in (0,1) \quad \text{for } k - 2/q \in \mathbb{N}.$$

Lemma 1.2.6 *Let Ω be a bounded domain. Then*

$$W^{1,q}(\Omega) \overset{c}{\hookrightarrow} L^{\bar{q}}(\Omega), \quad \bar{q} < 2q/(2 - q);$$
$$W^{1,2}(\Omega) \overset{c}{\hookrightarrow} L^{q}(\Omega), \quad q \in [1, +\infty),$$
$$C^{0,\mu}(\overline{\Omega}) \overset{c}{\hookrightarrow} C(\overline{\Omega}), \quad \mu > 0.$$

The space $W^{1-1/q,q}(\partial\Omega)$ $(q > 1)$ is the set of all $u \in L^q(\partial\Omega)$ such that the seminorm

$$[u]^q_{1-1/q} = \int_{\partial\Omega} \left\{ \int_{\partial\Omega} \frac{u(\xi) - u(\zeta)|^q}{|\xi - \zeta|^q} \, \mathrm{d}s_\zeta \right\} \mathrm{d}s_\xi$$

is finite. Endowed with the norm

$$\|u\|_{W^{1-1/q,q}(\partial\Omega)} = \left\{ \|u\|^p_{L^q(\partial\Omega)} + [u]^q_{1-1/q} \right\}^{1/q},$$

$W^{1-1/q,q}(\partial\Omega)$ is a Banach space. Let $L^q_{\mathrm{div}}(\Omega)$ and $L^q_{\mathrm{div},0}$ be the completion of $C_0^\infty(\overline{\Omega})$ and $C_0^\infty(\Omega)$ respectively with respect to the norm

$$\|\boldsymbol{u}\|_{L^q_{\mathrm{div}}(\Omega)} = \|\boldsymbol{u}\|_{L^q(\Omega)} + \|\operatorname{div} \boldsymbol{u}\|_{L^q(\Omega)}.$$

[1] As is customary, if $h \in \mathbb{R}$, then $[h]$ means the smallest integer such that $[h] < h$ and $\{h\} = h - [h]$.

Lemma 1.2.7 *Let Ω be a bounded domain. The classical trace operators*

$$\varphi \in C(\overline{\Omega}) \to \operatorname{tr}_{|\partial\Omega}\varphi \in C(\partial\Omega),$$
$$\boldsymbol{u} \in C(\overline{\Omega}) \to \operatorname{tr}_{|\partial\Omega}(\boldsymbol{u}\cdot\boldsymbol{n}) \in C(\partial\Omega)$$

extend uniquely to bounded onto maps

$$\tau[\varphi] : W^{1,q}(\Omega) \to W^{1-1/q,q}(\partial\Omega) \hookrightarrow L^{q/(2-q)}(\partial\Omega),$$
$$\tau_n[\varphi] : L^q_{\mathrm{div}}(\Omega) \to [W^{1-1/q',q'}(\Omega)]^*$$

for every $q \in (1, +\infty)$. Moreover $\operatorname{Ker}\tau = W^{1,q}_0(\Omega)$, $\operatorname{Ker}\tau_n = L^q_{\mathrm{div},0}(\Omega)$[2] *and the generalized divergence theorem holds*

$$\int_\Omega \boldsymbol{u}\cdot\nabla\varphi = \langle \tau_n[\boldsymbol{u}\cdot\boldsymbol{n}], \tau[\varphi]\rangle - \int_\Omega \varphi \operatorname{div}\boldsymbol{u}, \qquad (1.2.6)$$

for all $\varphi \in W^{1,q'}(\Omega)$ and $\boldsymbol{u} \in L^q_{\mathrm{div}}(\Omega)$.

- Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ and let

$$\mathcal{G} = \{\phi \in C^\infty_0(\mathbb{R}^2) : \ \operatorname{supp}\phi \subset S_1(o), \ \|\nabla\phi\|_{L^\infty(\mathbb{R}^2)} < 1\}.$$

Set

$$u^\star(x) = \sup_{t>0}\sup_{\phi\in\mathcal{G}}\left|\int_{\mathbb{R}^2} \frac{1}{t^3}\phi\left(\frac{x-y}{t}\right) u(y)\mathrm{d}a_y\right|.$$

We say that u belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ if

$$u^\star \in L^1(\mathbb{R}^2).$$

$\mathcal{H}^1(\mathbb{R}^2)$ is a Banach space with the norm defined by

$$\|u\|_{\mathcal{H}^1(\mathbb{R}^2)} = \|u^\star\|_{L^1(\mathbb{R}^2)}.$$

If $u \in \mathcal{H}^1(\mathbb{R}^2)$, then necessarily

$$\int_{\mathbb{R}^2} u = 0$$

[2]If $\mathcal{T}$ is an operator between the spaces B and B', $\operatorname{Ker}\mathcal{T} = \{u \in B : \mathcal{T}[u] = 0\}$.

and the set

$$\left\{u \in C_0^\infty(\mathbb{R}^2) : \int_{\mathbb{R}^2} u = 0\right\}$$

is dense in $\mathcal{H}^1(\mathbb{R}^2)$ [44].

The Hardy space on Ω, $\mathcal{H}^1(\Omega)$, if defined as the set of all $f \in L^1(\Omega)$ such that function

$$\tilde{f}(x) = \begin{cases} f(x), & x \in \Omega, \\ 0, & x \notin \Omega \end{cases}$$

belongs to $\mathcal{H}^1(\mathbb{R}^2)$.

Let $\boldsymbol{u} \in L^1_{\rm loc}(\Omega)$. We say that $\boldsymbol{u}$ is weakly divergence free if

$$\int_\Omega \boldsymbol{u} \cdot \nabla \phi = 0, \quad \forall \varphi \in C_0^\infty(\Omega).$$

If V is a linear subspace of $L^1_{\rm loc}(\Omega)$, by V_σ we denote of all weakly divergence free vector field of V. Of course, if $\boldsymbol{u} \in W^{1,q}_{\sigma,\rm loc}(\Omega)$, then $\operatorname{div} \boldsymbol{u} = 0$ almost everywhere in Ω.

- $D_0^{1,q}(\Omega)$ and $D^{1,q}(\Omega)$ are the completion of $C_0^\infty(\Omega)$ and $C_0^\infty(\overline{\Omega})$ respectively with respect to the semi–norm $\|\nabla u\|_{L^q(\Omega)}$. A function in $D_0^{1,q}(\Omega)$ has zero trace on $\partial\Omega$.

Te following result is proved in [5].

Lemma 1.2.8 *If* $\boldsymbol{u} \in D^{1,2}_\sigma(\mathbb{R}^2)$, *then* $\operatorname{div}(\boldsymbol{u} \cdot \nabla \boldsymbol{u}) \in \mathcal{H}^1(\mathbb{R}^2)$.

Let B a bounded measurable subset of $\mathbb{R}^2$. For $u \in L^1(B)$, we set

$$u_B = \frac{1}{|B|} \int_B u,$$

where $|B|$ denotes the measure of B.

A proof of the following results can be find in [13].

Lemma 1.2.9 *Let Ω be a bounded domain. There is a positive constant $c(\Omega, q)$ such that*

$$\int_\Omega |u - u_\Omega|^q \leq c \int_\Omega |\nabla u|^q, \tag{1.2.7}$$

for all $u \in D^{1,q}(\Omega)$, and

$$\int_\Omega |u|^q \leq c \int_\Omega |\nabla u|^q, \tag{1.2.8}$$

for all $u \in D_0^{1,q}(\Omega)$, $q \in (1, +\infty)$.

The constant c in (1.2.7) is invariant under dilatation; in particular, if Ω is the shell T_R and $q = 2$, then

$$\int_{T_R} |u - u_{T_R}|^2 \leq c_0 R^2 \int_{T_R} |\nabla u|^2, \tag{1.2.9}$$

with c_0 independent of R.

By (1.2.8) if Ω is bounded, then the spaces $D_0^{1,q}(\Omega)$ and $W_0^{1,q}(\Omega)$ are equivalent.

Lemma 1.2.10 *Let Ω be a bounded domain. There is a positive constant $c_\sigma(\Omega, q)$ such that*

$$\int_\Omega |u|^q \leq c_\sigma \int_\Omega |\nabla u|^2,$$

for all $u \in W_0^{1,2}(\Omega)$, $q \in (1, +\infty)$.

Lemma 1.2.11 *Let Ω be a bounded domain. There is a positive constant $c(\Omega, q)$ such that*

$$\int_\Omega \frac{|u|^q}{\delta^q} \leq c \int_\Omega |\nabla u|^q,$$

for all $u \in W_0^{1,q}(\Omega)$, $q \in (1, +\infty)$.

Let (r,θ) be a polar coordinate system in $\mathbb{R}^2$. Set

$$\bar{u}(r) = \int_0^{2\pi} u(r,\theta).$$

Lemma 1.2.12 *If $\partial_\theta u(r,\theta) \in L^2(0,2\pi)$, then*

$$\int_0^{2\pi} |u-\bar{u}|^2(r,\theta) \le \int_0^{2\pi} |\partial_\theta u|^2(r,\theta), \tag{1.2.10}$$

Lemma 1.2.13 *Let Ω be a bounded domain. If $f \in W_0^{k-1,q}(\Omega)$, with $k \in \mathbb{N}$, $q \in (1,+\infty)$ and $f_\Omega = 0$, then the problem*

$$\operatorname{div} \boldsymbol{u} = f \quad \text{in } \Omega \tag{1.2.11}$$

admits a solution $\boldsymbol{u} \in W_0^{k,q}(\Omega)$ and

$$\|\boldsymbol{u}\|_{W^{k,q}(\Omega)} \le c\|f\|_{W^{k-1,q}(\Omega)},$$

with $c(\Omega,q)$ invariant under dilatations.

Lemma 1.2.14 *Let Ω be a bounded or an exterior domain. If $\boldsymbol{u} \in L^q(\Omega)$ and*

$$\int_\Omega \boldsymbol{u}\cdot\boldsymbol{\varphi} = 0,$$

for all $\boldsymbol{\varphi} \in D_{\sigma,0}^{1,q'}(\Omega)$, then there is a unique $Q \in L^q(\Omega)$ such that

$$\int_\Omega \boldsymbol{u}\cdot\boldsymbol{\phi} = \int_\Omega Q \operatorname{div} \boldsymbol{\phi},$$

for all for all $\boldsymbol{\phi} \in D_0^{1,q'}(\Omega)$.

We shall also use the elementary arithmetic–geometric mean inequality

$$\pm\, 2ab \le \alpha|a|^2 + \frac{|b|^2}{\alpha}, \tag{1.2.12}$$

for all $a,b \in \mathbb{R}$ and for all positive α.

Chapter 2

Steady Stokes flow in bounded and exterior domains

2.1 The Navier–Stokes equations

The stationary flows of a viscous fluid $\mathcal{F}$ filling a region identified with a domain Ω of $\mathbb{R}^2$ are given by the solutions of the steady–state Navier–Stokes equations

$$\begin{aligned} \Delta \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nabla p = \boldsymbol{f} \quad & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 \quad & \text{in } \Omega, \end{aligned} \tag{2.1.1}$$

where

- $\boldsymbol{u} : \Omega \to \mathbb{R}^2$ is the (unknown) kinetic field,
- $p : \Omega \to \mathbb{R}$ is the (unknown) pressure field,
- $\boldsymbol{f} : \Omega \to \mathbb{R}^2$ is the (assigned) body force field,

and for simplicity we set the Reynolds number equal to one.

Very clear expositions of the basic physical facts concerning equations (2.1.1) can be found in [14], [15], [22].

Unless otherwise explicitly stated, from now on we shall consider

▶ the bounded and exterior domains defined by (1.1.1), (1.1.2) respectively with boundary of class C^2. ◁

The boundary value problems associated to system (2.1.1) in a bounded and in an exterior domain will be called *interior* and *exterior* respectively.

The stress tensor field associated to $(\boldsymbol{u}, p)$ is the second order tensor field

$$\boldsymbol{T}(\boldsymbol{u}, p) = 2\hat{\nabla}\boldsymbol{u} - p\mathcal{I},$$

while the traction field on $\partial\Omega$ is the vector field[1]

$$\boldsymbol{T}(\boldsymbol{u}, p)\boldsymbol{n} = 2\hat{\nabla}\boldsymbol{u}\cdot\boldsymbol{n} - p\boldsymbol{n}.$$

With this notation (2.1.1) can be written in the *dynamical form* [22]

$$\begin{aligned} &\dot{\boldsymbol{u}} = \operatorname{div}\boldsymbol{T}(\boldsymbol{u}, p) - \boldsymbol{f} \\ &\operatorname{div}\boldsymbol{u} = 0 \end{aligned} \qquad \text{in } \Omega, \tag{2.1.2}$$

where the so called *inertial term*

$$\dot{\boldsymbol{u}} = \boldsymbol{u}\cdot\nabla\boldsymbol{u}$$

is the *molecular derivative* of $\boldsymbol{u}$ [22].

In the exterior problem we can identify Ω' with an "obstacle" $\mathcal{B}$ moving in the fluid. In this case the net force exerted by $\mathcal{F}$ on $\mathcal{B}$ is the vector

$$\boldsymbol{s} = \int_{\partial\Omega}\boldsymbol{T}(\boldsymbol{u}, p)\boldsymbol{n} \tag{2.1.3}$$

while the total force exerted by the environment on $\mathcal{B}$ is the vector

$$\boldsymbol{\tau} = \int_{\partial\Omega}\boldsymbol{T}(\boldsymbol{u}, p)\boldsymbol{n} + \int_{\Omega}\boldsymbol{f} \tag{2.1.4}$$

where, of course, we have assumed that the integrals have a meaning.

[1]Recall that $\boldsymbol{n}$ denotes the unit normal exterior to $\partial\Omega$ (with respect to Ω') for Ω bounded, and interior to Ω (with respect to Ω') for Ω exterior.

▶ Let $\boldsymbol{f} \in [D_0^{1,q'}(\Omega)]^*$. A *weak solution* (variational solution for $q = 2$)[2] of the Navier–Stokes equations is a pair

$$(\boldsymbol{u}, p) \in W^{1,q}_{\sigma,\mathrm{loc}}(\Omega) \times L^q_{\mathrm{loc}}(\Omega) \tag{2.1.5}$$

which satisfies the relation

$$\int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{\phi} + \int_\Omega \boldsymbol{u} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{\phi} + \langle \boldsymbol{f}, \boldsymbol{\phi} \rangle = \int_\Omega p \operatorname{div} \boldsymbol{\phi} \tag{2.1.6}$$

for all $\boldsymbol{\phi} \in C_0^\infty(\Omega)$. ◁

▶ Let $\boldsymbol{f} \in C^{0,\mu}_{\mathrm{loc}}(\Omega)$, $\mu \in (0, 1]$. A *classical solution* of the Navier–Stokes equations is a pair

$$(\boldsymbol{u}, p) \in C^2(\Omega) \times C^1(\Omega) \tag{2.1.7}$$

which satisfies pointwise equations (2.1.1). ◁

[2]In the sequel, we shall not give a particular emphasis to weak solutions of the Stokes and Navier–Stokes problems. It will appear clear from the context as existence of a weak solution can be derived by our results under suitable assumptions on $\boldsymbol{f}$ and $\boldsymbol{a}$. On the other hand, a complete treatment of the variational theory is performed in [14], [15] (see also [24], [45]).

2.2 The Navier–Stokes problem

Let $\boldsymbol{a}$ a field on $\partial\Omega$ which denotes the velocity field of the boundary of $\partial\Omega$. The classical *adherence boundary condition* required in a motion of $\mathcal{F}$ consists in assuming that *a particles* of $\mathcal{F}$ which lies in a point ξ has the same velocity of ξ, *i.e.* the velocity field of $\mathcal{F}$ at the boundary coincides with $\boldsymbol{a}$. Formally, the boundary value problem associated to the stationary Navier–Stokes equations consists in finding a solution to the equations

$$\begin{aligned} \Delta\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla p = \boldsymbol{f} \quad & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 \quad & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{a} \quad & \text{on } \partial\Omega. \end{aligned} \tag{2.2.1}$$

If Ω is unbounded we also require that

$$\lim_{r\to+\infty} \boldsymbol{u}(x) = \boldsymbol{u}_0, \tag{2.2.2}$$

where $\boldsymbol{u}_0$ is an assigned constant vector.

► If $\boldsymbol{f} \in [D_0^{1,q'}(\Omega)]^*$ and $\boldsymbol{a} \in W^{1-1/q,q}(\partial\Omega)$, a weak solution (variational solution for $q = 2$) of the Navier–Stokes problem (2.2.1) is a pair (2.1.5) such that $\operatorname{tr}\boldsymbol{u}_{\partial\Omega} = \boldsymbol{a}$. If $\boldsymbol{f} \in C^{0,\mu}_{\mathrm{loc}}(\Omega)$ and $\boldsymbol{a} \in C(\partial\Omega)$, a classical solution of system (2.2.1) is a pair (2.1.7) such that $\boldsymbol{u}$ satisfies pointwise $(2.2.1)_3$. Analogous definitions are given for problem (2.2.1)–(2.2.2). ◁

► Let $\boldsymbol{f} = \boldsymbol{0}$. Taking the curl operator in $(2.2.1)_1$ we see that the function

$$\omega = \partial_1 u_2 - \partial_2 u_1 \tag{2.2.3}$$

satisfies the equation

$$\Delta\omega - \operatorname{div}(\omega\boldsymbol{u}) = 0. \tag{2.2.4}$$

Moreover, taking the div operator in $(2.2.1)_1$ one notes that the pressure field is a solution of the Poisson equation

$$\Delta p + \operatorname{div}(\boldsymbol{u}\cdot\nabla\boldsymbol{u}) = 0. \tag{2.2.5}$$

Multiplying $(2.2.1)_1$ scalarly by $\boldsymbol{u}$ and taking into account that

$$\begin{aligned}\boldsymbol{u}\cdot\Delta\boldsymbol{u} &= \tfrac{1}{2}\Delta|\boldsymbol{u}|^2 - |\nabla\boldsymbol{u}|^2,\\ |\nabla\boldsymbol{u}|^2 &= \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \omega\boldsymbol{e}_3\times\boldsymbol{u},\end{aligned}$$

it is simple to see that the *head pressure* function

$$\Pi = p + \tfrac{1}{2}|\boldsymbol{u}|^2 \tag{2.2.6}$$

is a solution of the equation

$$\Delta\Pi - \operatorname{div}(\Pi\boldsymbol{u}) = \omega^2. \tag{2.2.7}$$

◁

2.3 The Stokes equations

By formally setting $\boldsymbol{u}\cdot\nabla\boldsymbol{u} = \mathbf{0}$ in (2.1.1) , we have the Stokes equations

$$\begin{aligned} \Delta\boldsymbol{u} - \nabla p = \boldsymbol{f} \qquad &\text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 \qquad &\text{in } \Omega. \end{aligned} \tag{2.3.1}$$

Of course, the definition of regular and weak solutions of (2.3.1) are the ones we gave for the Navier–Stokes equations with the nonlinear term equal to zero.

The pressure field p in systems (2.1.1) and (2.3.1) are defined within an additive arbitrary constant. As usual, if Ω is bounded, we normalize $p(\in L^1(\Omega))$, by setting

$$\int_\Omega p = 0. \tag{2.3.2}$$

The equations

$$\begin{aligned} \Delta\boldsymbol{u} &= \nabla p, \\ \operatorname{div}\boldsymbol{u} &= 0 \end{aligned} \tag{2.3.3}$$

admits the fundamental solutions (see [24] Ch. 3) $(\boldsymbol{U}(x-y), \boldsymbol{q}(x-y))$ defined by

$$\begin{aligned} U_{ij}(x-y) &= -\frac{1}{4\pi}\left[\delta_{ij}\log\frac{1}{|x-y|} + \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2},\right] \\ q_i(x-y) &= \frac{1}{2\pi}\frac{\partial}{\partial x_j}\log\frac{1}{|x-y|}. \end{aligned} \tag{2.3.4}$$

Let $\boldsymbol{f} \in C_0^\infty(\Omega)$, let $(\boldsymbol{u}, p)$ be a regular solution of equations (2.3.1) and set

$$\boldsymbol{T}'(\boldsymbol{U}(x-y), \boldsymbol{q}(x-y)) = p\mathcal{I} + \nabla_y\boldsymbol{U} + \nabla_y\boldsymbol{U}^{\mathrm{T}}.$$

Following [24] Ch. 3, we can prove the following representation formula of $\boldsymbol{u}$ in a bounded domain

$$\begin{aligned}
\boldsymbol{u}(x) &= \int_{\Omega} \boldsymbol{U}(x-y)\cdot \boldsymbol{f}(y)\,\mathrm{da}_y - \int_{\partial\Omega} \boldsymbol{U}(x-\xi)\cdot[\boldsymbol{T}(\boldsymbol{u},p)\boldsymbol{n}](\xi)\,\mathrm{ds}_\xi \\
&\quad + \int_{\partial\Omega} \boldsymbol{u}(\xi)\cdot \boldsymbol{T}'(\boldsymbol{U}(x-\xi),\boldsymbol{q}(x-\xi))\cdot \boldsymbol{n}(\xi)\,\mathrm{ds}_\xi, \\
p(x) &= \int_{\Omega} \boldsymbol{q}(x-y)\cdot \boldsymbol{f}(y)\,\mathrm{da}_y - \int_{\partial\Omega} \boldsymbol{q}(x-\xi)\cdot[\boldsymbol{T}(\boldsymbol{u},p)\boldsymbol{n}](\xi)\,\mathrm{ds}_\xi \\
&\quad - 2\int_{\partial\Omega} \boldsymbol{u}(\xi)\cdot \nabla_\xi \boldsymbol{q}(x-\xi)\cdot \boldsymbol{n}(\xi)\,\mathrm{ds}_\xi.
\end{aligned} \tag{2.3.5}$$

Then, starting from (2.3.5) and repeating a classical argument (see, *e.g.*, [7] p. 120), we see that if Ω is an exterior domain and $\boldsymbol{u} = o(r)$, then there is a constant vector $\boldsymbol{u}_0$ and a constant scalar p_0 such that

$$\begin{aligned}
\boldsymbol{u}(x) &= \boldsymbol{u}_0 + \int_{\Omega} \boldsymbol{U}(x-y)\cdot \boldsymbol{f}(y)\,\mathrm{da}_y + \int_{\partial\Omega} \boldsymbol{U}(x-\xi)\cdot[\boldsymbol{T}(\boldsymbol{u},p)\boldsymbol{n}](\xi)\,\mathrm{ds}_\xi, \\
&\quad - \int_{\partial\Omega} \boldsymbol{u}(\xi)\cdot \boldsymbol{T}'(\boldsymbol{U}(x-\xi),\boldsymbol{q}(x-\xi))\cdot \boldsymbol{n}(\xi)\,\mathrm{ds}_\xi, \\
p(x) &= p_0 + \int_{\Omega} \boldsymbol{q}(x-y)\cdot \boldsymbol{f}(y)\,\mathrm{da}_y + \int_{\partial\Omega} \boldsymbol{q}(x-\xi)\cdot[\boldsymbol{T}(\boldsymbol{u},p)\boldsymbol{n}](\xi)\,\mathrm{ds}_\xi \\
&\quad + 2\int_{\partial\Omega} \boldsymbol{u}(\xi)\cdot \nabla_\xi \boldsymbol{q}(x-\xi)\cdot \boldsymbol{n}(\xi)\,\mathrm{ds}_\xi.
\end{aligned} \tag{2.3.6}$$

Hence it easily follows

Lemma 2.3.1 *Let* $(\boldsymbol{u}, p)$ *be a regular solution of system* (2.3.1) *in an exterior domain. If* $\boldsymbol{f} \in C_0^\infty(\Omega)$ *and*

$$\boldsymbol{u} = o(r), \tag{2.3.7}$$

then there are a constant vector $\boldsymbol{u}_0$ *and a scalar* p_0 *such that* $(\boldsymbol{u}, p)$ *admits the following representation*

$$\begin{aligned}
\boldsymbol{u}(x) &= \boldsymbol{u}_0 + \boldsymbol{U}(x)\cdot\boldsymbol{\tau} + \boldsymbol{\omega}(x) \\
p(x) &= p_0 + \boldsymbol{q}(x)\cdot\boldsymbol{\tau} + \alpha(x)
\end{aligned} \tag{2.3.8}$$

with $\boldsymbol{\tau}$ *defined in* (2.1.4) *and* $\boldsymbol{\omega}$*,* α *satisfy*

$$\nabla_k \boldsymbol{\omega} = O(r^{1-k-n}), \quad \nabla_k \alpha = O(r^{-k-n}),$$

for all $k \in \mathbb{N}_0$*.*

Note that if

$$\boldsymbol{u} = o(\log r), \tag{2.3.9}$$

then (2.3.8) implies that

$$\boldsymbol{\tau} = \boldsymbol{0}. \tag{2.3.10}$$

From Lemma 2.3.1 we have

Theorem 2.3.1 *Let* $(\boldsymbol{u}, p)$ *be a regular solution of system* (2.3.1) *in an exterior domain. If* $\boldsymbol{u}$ *is constant on* $\partial\Omega$ *and vanishes at infinity, then then* $\boldsymbol{u} = \boldsymbol{0}$ *in* Ω*.*

PROOF - Multiply $(2.3.3)_1$ scalarly by $\boldsymbol{u}$. Then, making use of the identities

$$\boldsymbol{u} \cdot \operatorname{div} \boldsymbol{T}(\boldsymbol{u}, p) = \operatorname{div}[\boldsymbol{u} \cdot \boldsymbol{T}(\boldsymbol{u}, p)] - 2|\hat{\nabla}\boldsymbol{u}|^2,$$

integrating over Ω_R and taking into account (2.3.10) and that $\boldsymbol{u}$ is constant on $\partial\Omega$ (say $\boldsymbol{c}$), we get

$$\begin{aligned} 2\int_{\Omega_R} |\hat{\nabla}\boldsymbol{u}|^2 &= -\boldsymbol{c} \cdot \int_{\partial\Omega} \boldsymbol{T}(\boldsymbol{u}, p)\boldsymbol{n} + \int_{\partial S_R} \boldsymbol{u} \cdot \boldsymbol{T}(\boldsymbol{u}, p)\boldsymbol{e}_R \\ &= \int_{\partial S_R} \boldsymbol{u} \cdot \boldsymbol{T}(\boldsymbol{u}, p)\boldsymbol{e}_R. \end{aligned} \tag{2.3.11}$$

By Lemma 2.3.1 and (2.3.9)

$$\boldsymbol{u} = O(r^{-1}, \quad p = p_0 + O(r^{-2}), \quad \nabla\boldsymbol{u} = O(r^{-2})$$

so that, since

$$\int_{\partial S_R} p\boldsymbol{u} \cdot \boldsymbol{e}_R = \int_{\partial S_R} (p - p_0)\boldsymbol{u} \cdot \boldsymbol{e}_R,$$

it holds

$$\int_{\partial S_R} \boldsymbol{u} \cdot \boldsymbol{T}(\boldsymbol{u}, p)\boldsymbol{e}_R = O(R^{-1}).$$

Then letting $R \to +\infty$ in (2.3.11) we see that $\nabla \boldsymbol{u} = \boldsymbol{0}$ in Ω. Hence it follows that $\boldsymbol{u} = \boldsymbol{0}$ in Ω, taking into account that Ω is connected and $\boldsymbol{u}$ vanishes on $\partial\Omega$. □

We shall need also the following well–known uniqueness results which follows from the the well–known relation

$$2\int_{\Omega} |\hat{\nabla}\boldsymbol{u}|^2 = \int_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{T}(\boldsymbol{u}, p)\boldsymbol{n}$$

and (2.3.11) respectively in bounded and exterior domains.

Lemma 2.3.2 *If $(\boldsymbol{u}, p)$ is a regular solution of system (2.3.3) in a bounded domain vanishing on $\partial\Omega$, then $\boldsymbol{u} = \boldsymbol{0}$ and p is a constant.*

Lemma 2.3.3 *If $(\boldsymbol{u}, p)$ is a regular solution of system (2.3.3) in a bounded domain such that $\boldsymbol{T}(\boldsymbol{u}, p)\boldsymbol{n} = \boldsymbol{0}$ on $\partial\Omega$, then $p = 0$ and $\boldsymbol{u}$ is a rigid motion.*

Lemma 2.3.4 *If $(\boldsymbol{u}, p)$ is a regular solution of system (2.3.3) in an exterior domain such that $\boldsymbol{T}(\boldsymbol{u}, p)\boldsymbol{n} = \boldsymbol{0}$ on $\partial\Omega$ and $\boldsymbol{u} \cdot \boldsymbol{T}(\boldsymbol{u}, p) = o(r^{-1})$, then $p = 0$ and $\boldsymbol{u}$ is constant vector*

• *Remark 2.3.1*

Note that if $(\boldsymbol{u}, p)$ is a variational solution of system (2.3.3) then we can use the generalized divergence theorem (1.2.6) to see that (2.3.5) and (2.3.8) and their consequences also hold (with $\boldsymbol{f}$ vanishing outside a bounded set for Ω exterior) for weak solutions of equations (2.3.1). Moreover, if $(\boldsymbol{u}, p)$ is a variational solution, we have

$$2\int_{\Omega} |\hat{\nabla}\boldsymbol{u}|^2 = \langle \boldsymbol{T}(\boldsymbol{u}, p)\boldsymbol{n}, \boldsymbol{u}\rangle$$

for bounded domains and

$$2\int_{\Omega} |\hat{\nabla}\boldsymbol{u}|^2 = -\langle \boldsymbol{T}(\boldsymbol{u}, p)\boldsymbol{n}, \boldsymbol{u}\rangle + \int_{\partial S_R} \boldsymbol{u} \cdot T(\boldsymbol{u}, p)\boldsymbol{e}_R$$

for exterior domains. Hence it follows that Theorem 2.3.1 and Lemmas 2.3.2, 2.3.3, 2.3.4 retain their validity for variational solutions. ◇

2.4 The Stokes volume potential

The pair

$$\begin{aligned}\mathcal{V}[\boldsymbol{f}](x) &= \int_\Omega \boldsymbol{U}(x-y)\cdot\boldsymbol{f}(y)\,\mathrm{da}_y\\ \mathcal{P}[\boldsymbol{f}](x) &= \int_\Omega \boldsymbol{q}(x-y)\cdot\boldsymbol{f}(y)\,\mathrm{da}_y\end{aligned}\tag{2.4.1}$$

is known as *Stokes' volume potential* with density $\boldsymbol{f}$. It is a solution of equations (2.3.1) in a sense we specify below.

The integral transforms (2.4.1) enjoy several important properties (see [27], [28], [44]). We recall here the ones we shall use more frequently in the sequel. If Ω is a bounded domain, then

(i) $\boldsymbol{f}\in C^{0,\mu}(\Omega)\Rightarrow(\mathcal{V}[\boldsymbol{f}],\mathcal{P}[\boldsymbol{f}])\in C^{2,\mu}(\Omega)\times C^{1,\mu}(\Omega)$;

(ii) $\boldsymbol{f}\in L^q(\Omega),\ q\in(1,+\infty)\Rightarrow(\mathcal{V}[\boldsymbol{f}],\mathcal{P}[\boldsymbol{f}])\in W^{2,q}(\Omega)\times W^{1,q}\Omega)$;

(iii) $\boldsymbol{f}\in[W^{1,q'}(\Omega)]^*,\ q\in(1,+\infty)\Rightarrow(\mathcal{V}[\boldsymbol{f}],\mathcal{P}[\boldsymbol{f}])\in W^{1,q}(\Omega)\times L^q(\Omega)$.

If Ω is a bounded or an exterior domain, then

(iv) $\boldsymbol{f}\in\mathcal{H}^1(\Omega)\Rightarrow(\mathcal{V}[\boldsymbol{f}],\mathcal{P}[\boldsymbol{f}])\in[D^{2,1}(\mathbb{R}^2)\cap D^{2,1}(\mathbb{R}^2)]\times D^{1,1}(\mathbb{R}^2)$.

Note that, since for $\varphi\in C_0^\infty(\mathbb{R}^2)$

$$\varphi(x_1,x_2)=\int_\infty^{x_1}\int_\infty^{x_2}\partial^2_{\xi_1\xi_2}\varphi(\xi_1,\xi_2)$$

it holds

$$\|\varphi\|_{L^\infty(\mathbb{R}^2)}\le\|\varphi\|_{D^{2,1}(\mathbb{R}^2)}.$$

Since $D^{2,1}(\mathbb{R}^2)$ is the closure of $C_0^\infty(\mathbb{R}^2)$ with respect to $\|\varphi\|_{D^{2,1}(\mathbb{R}^2)}$, denoting by $C_0(\mathbb{R}^2)$ the completion of $C_0^\infty(\mathbb{R}^2)$ with respect to $\|\varphi\|_{L^\infty(\mathbb{R}^2)}$, we have that

$$D^{2,1}(\mathbb{R}^2)\hookrightarrow C_0(\mathbb{R}^2).$$

Recall that a function in $C_0(\mathbb{R}^2)$ tends to zero uniformly at infinity (see, *e.g.*, [35]). Therefore, in the case (iv) the field $\mathcal{V}[\boldsymbol{f}]$ is continuous and tends to zero uniformly at infinity.

As a simple consequence of the above properties, we can say that $(\mathcal{V}[\boldsymbol{f}], \mathcal{P}[\boldsymbol{f}])$ satisfies (2.3.1) pointwise in Ω if $\boldsymbol{f}$ is Hölder continuous in Ω; pointwise almost everywhere if $\boldsymbol{f} \in \mathcal{H}^1(\Omega)$ and (2.1.5) (with $\boldsymbol{u} \cdot \nabla \boldsymbol{u} = \boldsymbol{0}$) if $\boldsymbol{f} \in [W^{1,q'}(\Omega)]^*$.

► Let Ω be bounded. If $\boldsymbol{f} = \operatorname{div} \boldsymbol{F}$, with $\boldsymbol{F} \in L^q(\Omega)$, then $\boldsymbol{f} \in [D^{1,q'}(\Omega)]^*$ and from (III) it follows that $\mathcal{V}[\boldsymbol{f}] \in W^{1,q}(\Omega)$ and $\tau\big[\mathcal{V}[\boldsymbol{f}]\big] \in L^{q/(2-q)}(\partial\Omega)$. ◁

2.5 The Stokes layer potentials

Let $\partial\Omega$ be a closed curve of class C^2. In the next three sections we shall only consider the bounded domain defined by the point interior to $\partial\Omega$, and the exterior domain

$$\Omega^- = \mathbb{R}^2 \setminus \overline{\Omega^+}.$$

This will allows us to use in a more simple way the methods of layer potentials to study the boundary value problems associated to the Stokes equations. Then, in Section 2.12 we shall observe as the results we derive can be extended to the general cases (1.1.1) and (1.1.2).

Let $\boldsymbol{\psi}, \boldsymbol{\varphi} \in L^1(\partial\Omega)$. The *Stokes simple and double layer potentials* are the pair defined respectively by [24]

$$\begin{aligned} \boldsymbol{v}[\boldsymbol{\psi}](x) &= -\int_{\partial\Omega} \boldsymbol{U}(x-\zeta)\cdot\boldsymbol{\psi}(\zeta)\,\mathrm{d}s_\zeta, \\ P[\boldsymbol{\psi}](x) &= -\int_{\partial\Omega} \boldsymbol{q}(x-\zeta)\cdot\boldsymbol{\psi}(\zeta)\,\mathrm{d}s_\zeta \end{aligned} \tag{2.5.1}$$

and

$$\begin{aligned} \boldsymbol{w}[\boldsymbol{\varphi}](x) &= \int_{\partial\Omega} \boldsymbol{\varphi}(\zeta)\cdot\boldsymbol{T}'(\boldsymbol{U},\boldsymbol{q})(x-\zeta)\cdot\boldsymbol{n}(\zeta)\,\mathrm{d}s_\zeta, \\ \varpi[\boldsymbol{\varphi}](x) &= -2\,\mathrm{div}\int_{\partial\Omega} (\boldsymbol{q}(x-\zeta)\cdot\boldsymbol{\psi}(\zeta))\boldsymbol{n}(\zeta)\,\mathrm{d}s_\zeta. \end{aligned} \tag{2.5.2}$$

They are analytical solutions to system (2.3.3) in $\mathbb{R}^2 \setminus \partial\Omega$. Moreover, in virtue of the expression of the fundamental solution (2.3.4)

$$\boldsymbol{v}[\boldsymbol{\psi}](x) = O(\log r), \quad \nabla_k \boldsymbol{v}[\boldsymbol{\psi}](x),\ \nabla_{k-1} P[\boldsymbol{\psi}](x) = O(r^{-k}), \tag{2.5.3}$$

for all $k \in \mathbb{N}$,

$$\nabla_k \boldsymbol{v}[\boldsymbol{\psi}](x),\ \nabla_k P[\boldsymbol{\psi}](x) = O(r^{-1-k}) \ \Leftrightarrow\ \int_{\partial\Omega} \boldsymbol{\psi} = \boldsymbol{0}, \tag{2.5.4}$$

for all $k \in \mathbb{N}_0$, and

$$\nabla_k \boldsymbol{w}[\boldsymbol{\varphi}](x) = O(r^{-1-k}), \quad \nabla_k \varpi[\boldsymbol{\varphi}](x) = O(r^{-k-2}) \tag{2.5.5}$$

for all $k \in \mathbb{N}_0$.

It is a classical result (see [27] Lemma 33) that the trace of $\boldsymbol{v}[\boldsymbol{\psi}]$ on $\partial\Omega$ is a field defined almost everywhere on $\partial\Omega$ by the limit

$$\mathcal{S}[\boldsymbol{\psi}](\xi) = \lim_{t\to 0^+} \boldsymbol{v}[\boldsymbol{\psi}](\xi \pm t\boldsymbol{n}), \tag{2.5.6}$$

exists, for almost all $\xi \in \partial\Omega$. If $\boldsymbol{\psi}$ is continuous, then (2.5.6) holds everywhere on $\partial\Omega$.

A simple computation shows that [24]

$$\boldsymbol{K}(x,\zeta) = \boldsymbol{T}'(\boldsymbol{U},\boldsymbol{q})(x-\zeta)\cdot\boldsymbol{n}(\zeta) = -\frac{1}{\pi}\frac{(x-\zeta)\otimes(x-\zeta)[(x-\zeta)\cdot\boldsymbol{n}(\zeta)]}{|x-\zeta|^4}.$$

Since $\boldsymbol{n}$ is of class C^1, the *kernel* $\boldsymbol{K}(x,\zeta)$ is bounded on $\partial\Omega$. Hence it follows that the trace of $\boldsymbol{w}[\boldsymbol{\varphi}]$ exists on both sides of $\partial\Omega$ and suffers a weak discontinuity (see [27] Ch. II):

$$\mathcal{W}^{\pm}[\boldsymbol{\varphi}] = \lim_{t\to 0^+} \boldsymbol{w}[\boldsymbol{\varphi}](\xi \mp t\boldsymbol{n}) = (\pm\tfrac{1}{2}\mathcal{I} + \mathcal{K})[\boldsymbol{\varphi}](\xi), \tag{2.5.7}$$

for almost all $\xi \in \partial\Omega$, where we set

$$\mathcal{K}[\boldsymbol{\varphi}](\xi) = \int_{\partial\Omega} \boldsymbol{\varphi}(\zeta)\cdot\boldsymbol{T}'(\boldsymbol{U},\boldsymbol{q})(\xi-\zeta)\cdot\boldsymbol{n}(\zeta)\,\mathrm{ds}_\zeta.$$

The trace of the traction of the simple layer potential is given by

$$\begin{aligned}\mathcal{T}^{\pm}[\boldsymbol{\psi}] &= \lim\nolimits_{t\to 0^+}[\boldsymbol{T}(\boldsymbol{v}[\boldsymbol{\psi}], P[\boldsymbol{\psi}])\boldsymbol{n}](\xi \pm t\boldsymbol{n}) \\ &= (\pm\tfrac{1}{2}\mathcal{I} - \mathcal{K}^*)[\boldsymbol{\psi}](\xi),\end{aligned} \tag{2.5.8}$$

where $\mathcal{K}^*$ is the adjoint of $\mathcal{K}$ defined by

$$\mathcal{K}^*[\boldsymbol{\psi}](\xi) = -\frac{1}{\pi}\int_{\partial\Omega} \frac{(\xi-\zeta)\otimes[(\xi-\zeta)\cdot\boldsymbol{\psi}(\zeta)][(\xi-\zeta)\cdot\boldsymbol{n}(\xi)]}{|\xi-\zeta|^4}\,\mathrm{ds}_\zeta.$$

From (2.5.6) and (2.5.8) it follows the *jump relations*

$$\mathcal{W}^{+}[\boldsymbol{\varphi}] - \mathcal{W}^{-}[\boldsymbol{\varphi}] = \boldsymbol{\varphi} \tag{2.5.9}$$

and

$$\mathcal{T}^{+}[\boldsymbol{\psi}] - \mathcal{T}^{-}[\boldsymbol{\psi}] = \boldsymbol{\psi}. \tag{2.5.10}$$

In the sequel we recall some classical properties of the Stokes layer potentials that are a consequence of the following well–known result (see [27], Ch. II).

Lemma 2.5.1 *The operators* $\mathcal{K}$, $\mathcal{K}^*$ *map* $L^q(\partial\Omega)$, $q > 1$, *into* $C^{0,\mu}(\partial\Omega)$ *and* $C^{0,\mu}(\partial\Omega)$ *into* $C^{1,\mu}(\partial\Omega)$, *for every* $\mu \in (0,1)$.

Then, by lemma 1.2.6 $\mathcal{K}, \mathcal{K}^*$ are compact from $L^q(\partial\Omega)$ into itself and to the equation (say)

$$\boldsymbol{a} = (\tfrac{1}{2}\mathcal{I} + \mathcal{K})[\boldsymbol{\varphi}] \tag{2.5.11}$$

we can apply the Fredholm alternative (Lemma 1.2.1) to say that

- (i) either the homogeneous equation

$$(\tfrac{1}{2}\mathcal{I} + \mathcal{K})[\boldsymbol{\varphi}] = \mathbf{0}$$

has only the null solution and (2.5.11) is uniquely solvable for every $\boldsymbol{a} \in L^q(\partial\Omega)$;

- (ii) or

$$\dim \operatorname{Ker}\left(\tfrac{1}{2}\boldsymbol{\varphi} + \mathcal{K}[\boldsymbol{\varphi}]\right) = \dim \operatorname{Ker}\left(\tfrac{1}{2}\boldsymbol{\psi} + \mathcal{K}^*[\boldsymbol{\psi}]\right) \in \mathbb{N}$$

and (2.5.11) is solvable if and only if

$$\int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{\psi} = 0, \tag{2.5.12}$$

for all $\boldsymbol{\psi} \in \operatorname{Ker}\left(\tfrac{1}{2}\boldsymbol{\psi} + \mathcal{K}^*[\boldsymbol{\psi}]\right)$. Of course, the same results hold for the equation $\boldsymbol{a} = -\frac{1}{2}\boldsymbol{\varphi} + \mathcal{K}[\boldsymbol{\varphi}]$.

Moreover, from Lemma 2.5.1 it follows

- (j) if $\boldsymbol{a} \in C^{k,\mu}(\partial\Omega)$ ($k = 0, 1, \mu \in (0,1)$) satisfies (2.5.11), then $\boldsymbol{\varphi} \in C^{k,\mu}(\partial\Omega)$;

- (jj) if $\boldsymbol{a} \in W^{1-1/q,q}(\partial\Omega)$, $q \in (1, +\infty)$, satisfies (2.5.11), then $\boldsymbol{\varphi} \in W^{1-1/q,q}(\partial\Omega)$;

- (jjj) if $\boldsymbol{a} \in W^{1,q}(\partial\Omega)$, $q \in (1, +\infty)$, satisfies (2.5.11), then $\boldsymbol{\varphi} \in W^{1,q}(\partial\Omega)$.

Clearly,

$$\mathrm{Ker}\,\left(\tfrac{1}{2}\boldsymbol{\varphi}+\mathcal{K}[\boldsymbol{\varphi}]\right)\subset C^{1,\mu}(\partial\Omega),$$

for every $\mu\in(0,1)$. Then from the classical Lyapounov–Tauber theorem (see [27], Theorem 15.V) it follows

$$\lim_{t\to 0^+}[\boldsymbol{T}(\boldsymbol{w}[\boldsymbol{\varphi}],\varpi[\boldsymbol{\varphi}])\boldsymbol{n}](\xi+t\boldsymbol{n})=\lim_{t\to 0^+}[\boldsymbol{T}(\boldsymbol{w}[\boldsymbol{\varphi}],\varpi[\boldsymbol{\varphi}])\boldsymbol{n}](\xi-t\boldsymbol{n}) \tag{2.5.13}$$

i.e, the traction of the double layer potential with a regular density is continuous across $\partial\Omega$.

2.6 The Stokes problem in bounded domains

Let $\boldsymbol{a} \in L^q(\partial\Omega)$ and let $\boldsymbol{f} \in \mathcal{H}^1(\Omega^+)$. The interior Stokes problem is to find a solution to the equations

$$\begin{aligned} \Delta \boldsymbol{u} - \nabla p = \boldsymbol{f} \quad & \text{in } \Omega^+, \\ \operatorname{div} \boldsymbol{u} = 0 \quad & \text{in } \Omega^+, \\ \boldsymbol{u} = \boldsymbol{a} \quad & \text{on } \partial\Omega. \end{aligned} \tag{2.6.1}$$

Following a classical procedure [24], [26], we look for a solution of (2.6.1) expressed by

$$\begin{aligned} \boldsymbol{u}(x) &= \boldsymbol{w}[\boldsymbol{\varphi}] + \mathcal{V}[\boldsymbol{f}], \\ p(x) &= \varpi[\boldsymbol{\varphi}] + \mathcal{P}[\boldsymbol{f}] \end{aligned} \tag{2.6.2}$$

in the unknown field $\boldsymbol{\varphi} \in L^q(\partial\Omega)$, for some $\boldsymbol{\psi} \in \operatorname{Ker} \mathcal{T}^-$. Then by (2.5.7), we have to solve the functional equation

$$\boldsymbol{\alpha} = \boldsymbol{a} - \mathcal{V}[\boldsymbol{f}]_{|\partial\Omega} = (\tfrac{1}{2}\mathcal{I} + \mathcal{K})[\boldsymbol{\varphi}] = \mathcal{W}^+[\boldsymbol{\varphi}] \tag{2.6.3}$$

in $L^q(\partial\Omega)$. Therefore, in virtue of Lemma 2.5.1 and Fredholm's alternative the only thing we have to do is to determine the linear space $\operatorname{Ker} \mathcal{T}^-$. Taking into account that

$$\boldsymbol{v}[\boldsymbol{n}] = -\int_{\partial\Omega} \boldsymbol{U}(x-\zeta)\cdot\boldsymbol{n}(\zeta)\,\mathrm{ds}_\zeta = -\int_{\Omega^+} \operatorname{div} \boldsymbol{U}(x-y)\,\mathrm{da}_y = \boldsymbol{0},$$

we have that $\boldsymbol{v}[\boldsymbol{n}] = \boldsymbol{0}$ and $P[\boldsymbol{n}] = c$ in Ω^+. On the other hand, since

$$\int_{\partial\Omega} \boldsymbol{n} = \boldsymbol{0},$$

by (2.5.5) Theorem 2.3.1 implies that $\boldsymbol{v}[\boldsymbol{n}] = \boldsymbol{0}$ and $P[\boldsymbol{n}] = 0$ in Ω^- so that from (2.5.10) it follows that $\boldsymbol{n} \in \operatorname{Ker} \mathcal{T}^-$. If $\boldsymbol{\psi}(\neq c\boldsymbol{n}) \in \operatorname{Ker} \mathcal{T}^-$, taking into account that by (2.5.10)

$$\int_{\partial\Omega} \boldsymbol{\psi} = \int_{\partial\Omega} \mathcal{T}^+[\boldsymbol{\psi}] = \boldsymbol{0},$$

Lemma 2.3.4 implies that $\boldsymbol{v}[\boldsymbol{\psi}] = \mathbf{0}$ in $\mathbb{R}^2$, $P[\boldsymbol{\psi}] = 0$ in Ω^- and $P[\boldsymbol{\psi}] = \text{constant}$ in Ω^+ so that

$$\operatorname{Ker} \mathcal{T}^+ = \operatorname{sp}\{\boldsymbol{n}\}. \tag{2.6.4}$$

Hence it follows

Theorem 2.6.1 *If $\boldsymbol{f} \in \mathcal{H}^1(\Omega^+)$ and $\boldsymbol{a} \in L^q(\partial\Omega)$ $(q > 1)$ satisfies*

$$\int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{n} = 0, \tag{2.6.5}$$

then system (2.6.1) has a solution given by (2.6.2) with $\boldsymbol{\varphi} \in L^q(\partial\Omega)$; $\boldsymbol{u}$ is continuous in Ω^+, satisfies $(2.6.1)_{1,2}$ almost everywhere in Ω,

$$\lim_{t\to 0^+} \boldsymbol{u}(\xi - t\boldsymbol{n}(\xi)) = \boldsymbol{a}(\xi), \tag{2.6.6}$$

for almost all $\xi \in \partial\Omega$ and

$$\|\boldsymbol{u}\|_{L^{2q}(\Omega^+)} \le c\left\{\|\boldsymbol{a}\|_{L^q(\partial\Omega)} + \|\boldsymbol{f}\|_{\mathcal{H}^1(\Omega^+)}\right\}. \tag{2.6.7}$$

If $\boldsymbol{a} \in C(\partial\Omega)$, then $\boldsymbol{u} \in C(\overline{\Omega})$, (3.3.1) holds everywhere on $\partial\Omega$ and there is a positive constant depending only on $\partial\Omega$ such that

$$\|\boldsymbol{u}\|_{C(\overline{\Omega^+})} \le c\left\{\|\boldsymbol{a}\|_{C(\partial\Omega)} + \|\boldsymbol{f}\|_{\mathcal{H}^1(\Omega^+)}\right\}. \tag{2.6.8}$$

From (2.6.3), (j)–(jjj) of Section 2.5 and (ı),(ıı), (ııı) of Section 2.4 it follows that more regular is $\boldsymbol{\alpha}$, then more regular is the corresponding solution $(\boldsymbol{u}, p)$. In particular, assuming for simplicity for $\boldsymbol{f} = \mathbf{0}$, we can state

Theorem 2.6.2 *If $\boldsymbol{a} \in W^{1-1/q,q}(\partial\Omega)$ $(q > 1)$ satisfies (2.6.5), then system (2.6.1) has a unique weak solution and*

$$\|\boldsymbol{u}\|_{W^{1,q}(\Omega^+)} + \|p\|_{L^q(\partial\Omega)} \le c\|\boldsymbol{a}\|_{W^{1-1/q,q}(\partial\Omega} \tag{2.6.9}$$

Moreover, if $\boldsymbol{a} \in C^{k,\mu}(\partial\Omega)$ $(k = 0, 1)$, then

$$\begin{aligned} \|\boldsymbol{u}\|_{C^{0,\mu}(\overline{\Omega^+})} &\le c\|\boldsymbol{a}\|_{C^{0,\mu}(\partial\Omega)}, \\ \|\boldsymbol{u}\|_{C^{1,\mu}(\overline{\Omega^+})} + \|p\|_{C^{0,\mu}(\overline{\Omega^+})} &\le c\|\boldsymbol{a}\|_{C^{1,\mu}(\partial\Omega)}. \end{aligned} \tag{2.6.10}$$

Proof - We have only to prove uniqueness which is trivial for $q \ge 2$ (see Remark 2.3.1). Le $\boldsymbol{u} \in W^{1,q}_{\sigma,0}(\Omega)$ $(q < 2)$ be a weak solution to equations (2.3.3) and let $(\boldsymbol{z}, Q) \in W^{1,q'}(\Omega) \times L^{q'}(\Omega)$ be weak solution of the system

$$\begin{aligned} \Delta \boldsymbol{z} - \nabla Q = \boldsymbol{\phi} \quad & \text{in } \Omega^+, \\ \operatorname{div} \boldsymbol{z} = 0 \quad & \text{in } \Omega^+, \\ \boldsymbol{z} = \boldsymbol{0} \quad & \text{in } \partial\Omega. \end{aligned} \tag{2.6.11}$$

with $\boldsymbol{\phi} \in C_0^\infty(\Omega)$. Since

$$\begin{aligned} \boldsymbol{z} \cdot \operatorname{div} \boldsymbol{T}(\boldsymbol{u}, p) &= \operatorname{div}[\boldsymbol{z} \cdot \boldsymbol{T}(\boldsymbol{u}, p)] - \operatorname{div}[\boldsymbol{u} \cdot \boldsymbol{T}(\boldsymbol{z}, Q)] \\ &\quad + \boldsymbol{u} \cdot \operatorname{div} \boldsymbol{T}(\boldsymbol{z}, Q), \end{aligned} \tag{2.6.12}$$

integrating on Ω^+, using the generalized divergence theorem (1.2.6) and taking into account that $\boldsymbol{u} = \boldsymbol{z} = \boldsymbol{0}$ on $\partial\Omega$, we get

$$\int_{\Omega^+} \boldsymbol{u} \cdot \boldsymbol{\phi} = 0$$

for all $\boldsymbol{\phi} \in C_0^\infty(\Omega^+)$. Hence it follows $\boldsymbol{u} = \boldsymbol{0}$ in Ω^+. □

Let $(\boldsymbol{u}, p)$ be the solution in Theorem 2.6.1. Let $\boldsymbol{a}_k \in C^1(\partial\Omega)$ be a sequence which converges to $\boldsymbol{a}$ strongly in $L^q(\partial\Omega)$ and such that

$$\int_{\partial\Omega} \boldsymbol{a}_k \cdot \boldsymbol{n} = 0.$$

Let $(\boldsymbol{u}_k, p_k)$ be the solution of system (2.6.1) with data $(\boldsymbol{a}_k, \boldsymbol{f})$. The regularity of $(\boldsymbol{u}_k, p_k)$ and $(\boldsymbol{z}, Q)$ allow us to use (2.6.12) and integrate on Ω^+ to get

$$\int_{\Omega^+} \boldsymbol{u}_k \cdot \boldsymbol{\phi} = \int_{\partial\Omega} \boldsymbol{a}_k \cdot \boldsymbol{T}(\boldsymbol{z}, Q)\boldsymbol{n} + \int_{\Omega^+} \boldsymbol{f} \cdot \boldsymbol{z}.$$

Hence, obviously,

$$\begin{aligned} \int_{\Omega^+} (\boldsymbol{u}_k - \boldsymbol{u}) \cdot \boldsymbol{\phi} + \int_{\Omega^+} \boldsymbol{u} \cdot \boldsymbol{\phi} &= \int_{\partial\Omega} (\boldsymbol{a}_k - \boldsymbol{a}) \cdot \boldsymbol{T}(\boldsymbol{z}, Q)\boldsymbol{n} \\ &\quad + \int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{T}(\boldsymbol{z}, Q)\boldsymbol{n} + \int_{\Omega^+} \boldsymbol{f} \cdot \boldsymbol{z}. \end{aligned} \tag{2.6.13}$$

In virtue of (2.6.7) the sequence $\{\boldsymbol{u}_k\}_{k\in\mathbb{N}}$ converges strongly to $\boldsymbol{u} \in L^{2q}(\Omega^+)$. Therefore, letting $k \to +\infty$ in (2.6.13) implies that $\boldsymbol{u}$ satisfies the relation

$$\int_{\Omega^+} \boldsymbol{u} \cdot \boldsymbol{\phi} = \int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{T}(\boldsymbol{z}, Q)\boldsymbol{n} + \int_{\Omega^+} \boldsymbol{f} \cdot \boldsymbol{z}. \tag{2.6.14}$$

According to [33], [39] we shall call *very weak solution* of system (2.6.2) a field $\boldsymbol{u} \in L^q(\Omega^+)$ which satisfies (2.6.14) for all $\boldsymbol{\phi} \in C_0^\infty(\Omega^+)$, with $(\boldsymbol{z}, Q)$ solution of system (2.6.11). Then, we can state the following uniqueness theorem.

Theorem 2.6.3 *If $\boldsymbol{f} \in \mathcal{H}^1(\Omega^+)$ and $\boldsymbol{a} \in L^q(\partial\Omega)$ $(q > 1)$ satisfies* (2.6.5)*, then system* (2.6.2) *has a a unique very weak solution.*

2.7 The Stokes problem in exterior domains

Let $\boldsymbol{a} \in L^q(\partial\Omega)$, $q > 1$, and let $\boldsymbol{f} \in \mathcal{H}^1(\Omega^-)$. The exterior Stokes problem is to find a solution to the equations

$$\begin{aligned} \Delta\boldsymbol{u} - \nabla p = \boldsymbol{f} &\quad \text{in } \Omega^-, \\ \operatorname{div}\boldsymbol{u} = 0 &\quad \text{in } \Omega^-, \\ \boldsymbol{u} = \boldsymbol{a} &\quad \text{on } \partial\Omega, \\ \lim_{r\to+\infty} \boldsymbol{u}(x) = \boldsymbol{u}_0, & \end{aligned} \tag{2.7.1}$$

where $\boldsymbol{u}_0$ is an assigned constant vector.

Let us first determine the kernel of the operator

$$\mathcal{W}^-[\boldsymbol{\varphi}] = (-\tfrac{1}{2}\mathcal{I} + \mathcal{K})[\boldsymbol{\varphi}].$$

If $\boldsymbol{\varphi} \in \operatorname{Ker}\mathcal{W}^-$, then by uniqueness $\boldsymbol{w}[\boldsymbol{\varphi}] = \boldsymbol{0}$ and $\varpi[\boldsymbol{\varphi}] = 0$ in Ω^- so that $\boldsymbol{T}(\boldsymbol{w}[\boldsymbol{\varphi}], \varpi[\boldsymbol{\varphi}])\boldsymbol{n} = \boldsymbol{0}$. Hence, taking into account (2.5.13) and Lemma 2.3.3, it follows that $\varpi[\boldsymbol{\varphi}] = 0$ and $\boldsymbol{w}[\boldsymbol{\varphi}]$ is a rigid field in Ω^+. Then from the jump conditions (2.5.9) it follows that necessarily $\boldsymbol{\varphi} \in \mathfrak{R}_{\partial\Omega}$. On the other hand, from $(2.3.5)_1$ and (2.5.7) we easily see that any field of $\mathfrak{R}_{\partial\Omega}$ lies in $\operatorname{Ker}\mathcal{W}^-$. Therefore,

$$\dim\mathcal{W}^- = \dim\mathcal{T}^+ = \dim\mathfrak{R}_{\partial\Omega} = 3. \tag{2.7.2}$$

If $\boldsymbol{\psi} \in \mathcal{T}^+$, then by Lemma 2.3.3 $\boldsymbol{v}[\boldsymbol{\psi}]$ is a rigid field in Ω^+ and

$$\operatorname{Ker}\mathcal{T}^+ = \{\boldsymbol{\psi} : \mathcal{S}[\boldsymbol{\psi}] \in \mathfrak{R}_{\partial\Omega^+},\ P[\boldsymbol{\psi}] = 0 \text{ in } \Omega^+\}$$

Moreover, by the jump conditions (2.5.10)

$$\mathcal{T}^-[\boldsymbol{\psi}] = -\boldsymbol{\psi}. \tag{2.7.3}$$

Let

$$\mathfrak{C} = \{\boldsymbol{\psi} : \mathcal{S}[\boldsymbol{\psi}] = \text{constant vector},\ P[\boldsymbol{\psi}] = 0 \text{ in } \Omega^+\}.$$

If $\boldsymbol{\psi}(\neq \mathbf{0}) \in \mathfrak{C}$, then $\boldsymbol{\psi}_{\partial\Omega} \neq \mathbf{0}$, or else $(\boldsymbol{v}[\boldsymbol{\psi}], P[\boldsymbol{\psi}])$ satisfies the hypotheses of Theorem 2.3.1 so that $\boldsymbol{v}[\boldsymbol{\psi}] = \mathbf{0}$, $P[\boldsymbol{\psi}] = 0$ in Ω^- and, as a consequence $\boldsymbol{\psi} = \mathbf{0}$. If $\{\boldsymbol{\psi}_i\}_{i=1,2,3} \subset \mathfrak{C} \setminus \{\mathbf{0}\}$, then the system $\{\int_{\partial\Omega} \boldsymbol{\psi}_i\}_{i=1,2,3}$ of $\mathbb{R}^2$ is linearly dependent so that there are three nonzero scalars α_i such that $\alpha_i \int_{\partial\Omega} \boldsymbol{\psi}_i = \mathbf{0}$. Therefore, the pair $(\boldsymbol{v}[\boldsymbol{\psi}], P[\boldsymbol{\psi}])$, with $\boldsymbol{\psi} = \alpha_i \boldsymbol{\psi}_i$ satisfies the hypotheses of Theorem 2.3.1. Hence $\alpha_i \boldsymbol{\psi}_i = \mathbf{0}$ and, as a consequence,

$$\dim \mathfrak{C} = 2.$$

There is a density $\boldsymbol{\psi} \in \operatorname{Ker} \mathcal{T}^+$ such that $\mathcal{S}[\boldsymbol{\psi}]$ is not a pure translation, or else we have that $\dim \operatorname{Ker} \mathcal{T}^+ = \dim \mathfrak{C} = 2$. By adding to $\boldsymbol{\psi}$ a field in $\boldsymbol{\psi}' \in \mathfrak{C}$ such that

$$\int_{\partial\Omega} (\boldsymbol{\psi} + \boldsymbol{\psi}') = \mathbf{0},$$

we see that $\mathcal{S}[\boldsymbol{\psi}'']$ is (nonconstant) rigid field, with $\boldsymbol{\psi}''_{\partial\Omega} = \mathbf{0}$. Therefore, we can define the space

$$\dim \mathfrak{L} = \{\boldsymbol{\psi} : \mathcal{S}[\boldsymbol{\psi}](\xi) \in \mathfrak{R}_{\partial\Omega},\ P[\boldsymbol{\psi}] = 0,\ \text{in } \Omega^+,\ \boldsymbol{\psi}_{\partial\Omega} = \mathbf{0}\}$$

and write

$$\operatorname{Ker} \mathcal{T}^+ = \mathfrak{C} \oplus \mathfrak{L}.$$

• *Remark 2.7.1*

It is not excluded that $\mathcal{S}[\boldsymbol{\psi}] = \mathbf{0}$ for some nonzero $\boldsymbol{\psi} \in \mathfrak{C}$. Indeed, if $\Omega^+ = S_R$, we know that [26] (see also section 2.8) the simple layer potentials

$$\boldsymbol{v}[\boldsymbol{e}_i](x) = \frac{1}{4\pi}\left[\int_{\partial S_R} \log|x-\zeta|\boldsymbol{e}_i \, \mathrm{d}s_\zeta - \int_{\partial S_R} \frac{(x-\zeta)(x_i-\zeta_i)}{|x-\zeta|^2} \, \mathrm{d}s_\zeta\right],$$

$i = 1, 2$, are constant in S_R and

$$\boldsymbol{v}[\boldsymbol{e}_1](0) = \frac{\boldsymbol{e}_1}{4\pi}\left[\int_{\partial S_R} \log|\boldsymbol{\zeta}| - \int_{\partial S_R} \frac{|\zeta_1|^2}{|\boldsymbol{\zeta}|^2}\right] = \frac{R}{2}\Big(\log R - \frac{1}{2}\Big),$$

$$\boldsymbol{v}[\boldsymbol{e}_2](0) = \frac{\boldsymbol{e}_2}{4\pi}\left[\int_{\partial S_R} \log|\boldsymbol{\zeta}| - \int_{\partial S_R} \frac{|\zeta_2|^2}{|\boldsymbol{\zeta}|^2}\right] = \frac{R}{2}\Big(\log R - \frac{1}{2}\Big).$$

Then for the disk $\Omega^{+} = S_{\sqrt{e}}$ we have that $\mathcal{S}[\boldsymbol{e}_i] = \mathbf{0}$, $i = 1, 2$ [39]. $\diamondsuit$

Let

$$\mathfrak{M} = \{\boldsymbol{\psi} \in \mathfrak{C} : \mathcal{S}[\boldsymbol{\psi}] = \mathbf{0}\}$$

and set

$$\ell = \dim \mathfrak{M} \leq 2.$$

• *Remark 2.7.2*

Let $\boldsymbol{\psi}(\neq \mathbf{0}) \in \mathfrak{L}$. Bearing in mind that

$$\mathcal{S}[\boldsymbol{\psi}] = \boldsymbol{\kappa} + \alpha \boldsymbol{e}_3 \times \boldsymbol{\xi}, \qquad \int_{\partial\Omega} \boldsymbol{\psi} = \mathbf{0},$$

with $\boldsymbol{\kappa}$ and $\alpha(\neq 0)$ constants, an integration by parts (2.5.5) and (2.7.3) yield

$$2\int_{\Omega} |\hat{\nabla} \boldsymbol{v}[\boldsymbol{\psi}]|^2 = -\int_{\partial\Omega} \mathcal{S}[\boldsymbol{\psi}] \cdot \mathcal{T}^{-}[\boldsymbol{\psi}] = \alpha \boldsymbol{e}_3 \cdot \int_{\Omega} \boldsymbol{\xi} \times \boldsymbol{\psi}.$$

Hence it follows that

$$\int_{\Omega} \boldsymbol{\xi} \times \boldsymbol{\psi} \neq \mathbf{0}.$$

$\diamondsuit$

We are now in a position to prove the following

Theorem 2.7.1 *If $\boldsymbol{a} \in L^q(\partial\Omega)$ $(q > 1)$ and $\boldsymbol{f} \in \mathcal{H}^1(\Omega^{-})$, then system* (2.7.1)$_{1,2,3}$ *has a solution expressed by*

$$\begin{aligned} \boldsymbol{u}(x) &= \boldsymbol{w}[\boldsymbol{\varphi}] + \boldsymbol{v}[\boldsymbol{\psi}] + \mathcal{V}[\boldsymbol{f}] + \ell\boldsymbol{\kappa}, \\ p(x) &= \varpi[\boldsymbol{\varphi}] + P[\boldsymbol{\psi}] + \mathcal{P}[\boldsymbol{f}], \end{aligned} \tag{2.7.4}$$

with $\boldsymbol{\varphi} \in L^q(\Omega)$, $\boldsymbol{\psi} \in \operatorname{Ker} \mathcal{T}^{+}$ and $\boldsymbol{\kappa}$ constant vector defined by

$$\int_{\partial\Omega} (\boldsymbol{a} - \ell\boldsymbol{\kappa}) \cdot \boldsymbol{\psi}' + \int_{\Omega^{-}} \boldsymbol{f} \cdot \boldsymbol{v}[\boldsymbol{\psi}'] = 0, \quad \boldsymbol{\psi}' \in \mathfrak{M}. \tag{2.7.5}$$

$\boldsymbol{u}$ is continuous in Ω^{-}, and (3.3.1) *holds almost everywhere on $\partial\Omega$.*

Proof - Consider the functional equation

$$\boldsymbol{a} - \mathcal{S}[\boldsymbol{\psi}] - \mathcal{V}[\boldsymbol{f}]_{|\partial\Omega} - \ell\boldsymbol{\kappa} = \big(-\tfrac{1}{2}\mathcal{I} + \mathcal{K}\big)[\boldsymbol{\varphi}] = \mathcal{W}^-[\boldsymbol{\varphi}]. \qquad (2.7.6)$$

In virtue of Fredholm alternative we need to find $\boldsymbol{\psi} \in \operatorname{Ker}\mathcal{T}^+$ such that

$$\int_{\partial\Omega} \big[\boldsymbol{a} - \mathcal{V}[\boldsymbol{f}]_{|\partial\Omega} - \ell\boldsymbol{\kappa}\big] \cdot \boldsymbol{\psi}' = \int_{\partial\Omega} \mathcal{S}[\boldsymbol{\psi}] \cdot \boldsymbol{\psi}' \qquad (2.7.7)$$

for all $\boldsymbol{\psi}' \in \operatorname{Ker}\mathcal{T}^+$. Of course, to show this it is sufficient to prove that the homogeneous system

$$\int_{\partial\Omega} \mathcal{S}[\boldsymbol{\psi}] \cdot \boldsymbol{\psi}' = 0, \quad \forall\, \boldsymbol{\psi}' \in \operatorname{Ker}\mathcal{T}^+, \qquad (2.7.8)$$

has only the trivial solution. Let $\ell = 0$. Choosing $\boldsymbol{\psi}' \in \mathfrak{C}$ in (2.7.8), we have

$$\int_{\partial\Omega} \mathcal{S}[\boldsymbol{\psi}] \cdot \boldsymbol{\psi}' = \int_{\partial\Omega} \mathcal{S}[\boldsymbol{\psi}'] \cdot \boldsymbol{\psi} = \boldsymbol{c} \cdot \int_{\partial\Omega} \boldsymbol{\psi} = 0, \qquad (2.7.9)$$

for all constant vectors $\boldsymbol{c}$. Hence

$$\int_{\partial\Omega} \boldsymbol{\psi} = \mathbf{0}. \qquad (2.7.10)$$

Now, we a choose $\boldsymbol{\psi}' = \boldsymbol{\psi}$ in (2.7.8) and in virtue of (2.7.3) we get

$$\int_{\partial\Omega} \mathcal{S}[\boldsymbol{\psi}] \cdot \mathcal{T}^-[\boldsymbol{\psi}] = 0. \qquad (2.7.11)$$

By $(2.7.10)_1$ and (2.5.5) we can integrate over Ω to get

$$2\int_{\Omega} |\hat{\nabla}\boldsymbol{v}[\boldsymbol{\psi}]|^2 = -\int_{\partial\Omega} \mathcal{S}[\boldsymbol{\psi}] \cdot \mathcal{T}^-[\boldsymbol{\psi}] = 0. \qquad (2.7.12)$$

Hence it follows that $\mathcal{T}^-[\boldsymbol{\psi}] = \mathbf{0}$ and by (2.7.3) that $\boldsymbol{\psi} = \mathbf{0}$.

If $\ell = 2$, then (2.7.5) assures that (2.7.7) is satisfied for all $\boldsymbol{\psi}' \in \mathfrak{C}$ so that $\boldsymbol{\psi}$ lies necessarily in $\mathfrak{L}$. Therefore, choosing $\boldsymbol{\psi}' = \boldsymbol{\psi}$ in (2.7.8), we have once again (2.7.12). Hence the desired conclusion follows.

If $\ell = 1$, then (2.7.5) implies that (2.7.7) is satisfied for all densities in $\mathfrak{M}$ and $\boldsymbol{\psi} = \boldsymbol{\psi}'' + \boldsymbol{\psi}_0 \in (\mathfrak{C} \setminus \mathfrak{M}) \oplus \mathfrak{L}$. Since

$$\int_{\partial\Omega} \boldsymbol{\psi}_0 = \mathbf{0},$$

choosing first $\boldsymbol{\psi} = \boldsymbol{\psi}''$, then $\boldsymbol{\psi} = \boldsymbol{\psi}_0$ in (2.7.8) and repeating the above argument, we have that $\boldsymbol{\psi} = \mathbf{0}$. The theorem is completely proved. □

In general

$$\boldsymbol{u}(x) = O(\log r).$$

However, starting from (2.7.13) we can construct a solution of system $(2.7.1)_{1,2,3}$ which converges to a constant vector at infinity as follows. Let $\bar{\boldsymbol{\psi}} \in \mathfrak{C}$ be such that

$$\int_{\partial\Omega} (\boldsymbol{\psi} + \bar{\boldsymbol{\psi}}) = \mathbf{0}.$$

Then, setting

$$\boldsymbol{u}_0 = \ell\boldsymbol{\kappa} - \mathcal{S}[\bar{\boldsymbol{\psi}}],$$

we have

Theorem 2.7.2 *If $\boldsymbol{a} \in L^q(\partial\Omega)$ $(q > 1)$ and $\boldsymbol{f} \in \mathcal{H}^1(\Omega^-)$, then system $(2.7.1)_{1,2,3}$ has a solution expressed by*

$$\begin{aligned} \boldsymbol{u}(x) &= \boldsymbol{w}[\boldsymbol{\varphi}] + \boldsymbol{v}[\boldsymbol{\psi}] + \mathcal{V}[\boldsymbol{f}] + \boldsymbol{\gamma}, \\ p(x) &= \varpi[\boldsymbol{\varphi}] + P[\boldsymbol{\psi}] + \mathcal{P}[\boldsymbol{f}]. \end{aligned} \tag{2.7.13}$$

with $\boldsymbol{\varphi} \in L^q(\Omega)$, $\boldsymbol{\psi} \in \operatorname{Ker} \mathcal{T}^+$ such that

$$\int_{\partial\Omega} \boldsymbol{\psi} = \mathbf{0},$$

and $\boldsymbol{\gamma}$ constant vector defined by

$$\int_{\partial\Omega} (\boldsymbol{a} - \boldsymbol{\gamma}) \cdot \boldsymbol{\psi}' + \int_{\Omega^-} \boldsymbol{f} \cdot \boldsymbol{v}[\boldsymbol{\psi}'] = 0, \quad \boldsymbol{\psi}' \in \mathfrak{C}. \tag{2.7.14}$$

$\boldsymbol{u}$ is continuous in Ω^-, and (3.3.1) holds almost everywhere on $\partial\Omega$.

▶ Of course, the above solutions enjoys (locally) all the regularity properties stated for bounded domains. ◁

2.8 Uniqueness and Stokes' paradox

From (2.7.5) it follows that if $\boldsymbol{\phi} \in C_0^\infty(\Omega^-)$ is chosen such that

$$\int_{\Omega^-} \boldsymbol{\phi} = \mathbf{0}, \quad \int_{\Omega^-} \boldsymbol{\phi} \cdot \boldsymbol{v}[\boldsymbol{\psi}'] = 0, \quad \boldsymbol{\psi}' \in \mathfrak{C}, \tag{2.8.1}$$

then the system

$$\begin{aligned} \Delta \boldsymbol{z} - \nabla Q = \boldsymbol{\phi} \quad & \text{in } \Omega^-, \\ \operatorname{div} \boldsymbol{z} = 0 \quad & \text{in } \Omega^-, \\ \boldsymbol{z} = \mathbf{0} \quad & \text{on } \partial\Omega, \\ \lim_{r \to +\infty} \boldsymbol{z}(x) = \mathbf{0} \quad & \end{aligned} \tag{2.8.2}$$

has a regular solution vanishing at infinity.

Let $\boldsymbol{a} \in L^q(\partial\Omega)$ and let $\boldsymbol{a}_k \in C^1(\partial\Omega)$ be a sequence which converges to $\boldsymbol{a}$ strongly in $L^q(\partial\Omega)$. Let $(\boldsymbol{u}_k, p_k)$ be the solution of system $(2.7.1)_{1,2,3,}$ with data $(\boldsymbol{a}_k, \boldsymbol{f})$, expressed by (2.7.13). The regularity of $(\boldsymbol{u}_k, p_k)$ and $(\boldsymbol{z}, Q)$ allow us to use (2.6.12) and integrate on Ω_R^- to get

$$\begin{aligned} \int_{\Omega_R^-} \boldsymbol{u}_k \cdot \boldsymbol{\phi} = & - \int_{\partial\Omega} \boldsymbol{a}_k \cdot \boldsymbol{T}(\boldsymbol{z}, Q)\boldsymbol{n} + \int_{\Omega_R^-} \boldsymbol{f} \cdot \boldsymbol{z} \\ & + \int_{\partial S_R} [\boldsymbol{u}_k \cdot \boldsymbol{T}(\boldsymbol{z}, Q) - \boldsymbol{z} \cdot \boldsymbol{T}(\boldsymbol{u}_k, p_k)] \cdot \boldsymbol{n}. \end{aligned} \tag{2.8.3}$$

In virtue of the behavior at infinity of $(\boldsymbol{u}_k, p_k)$, $(\boldsymbol{z}, Q)$ and their derivatives, letting $R \to +\infty$ in (2.8.3), we have

$$\int_{\Omega} \boldsymbol{u}_k \cdot \boldsymbol{\phi} = - \int_{\partial\Omega} \boldsymbol{a}_k \cdot \boldsymbol{T}(\boldsymbol{z}, Q)\boldsymbol{n} + \int_{\Omega^-} \boldsymbol{f} \cdot \boldsymbol{z}. \tag{2.8.4}$$

By the expression of $\boldsymbol{u}_k - \boldsymbol{u}$ it is not difficult to see that for R sufficiently large, there is a positive constant $c(R)$ such that

$$\|\boldsymbol{u}_k - \boldsymbol{u}\|_{L^q(\Omega_R^-)} \le c(R) \|\boldsymbol{a}_k - \boldsymbol{a}\|_{L^q(\partial\Omega)}.$$

Therefore, if $\operatorname{supp} \boldsymbol{\phi} \subset \Omega_R$, we can let $k \to +\infty$ in (2.8.4) to have

$$\int_{\Omega^-} \boldsymbol{u} \cdot \boldsymbol{\phi} = - \int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{T}(\boldsymbol{z}, Q)\boldsymbol{n} + \int_{\Omega^-} \boldsymbol{f} \cdot \boldsymbol{z}. \tag{2.8.5}$$

The arbitrariness of R allows us to say that $\boldsymbol{u}$ meets (2.8.5) for all $\boldsymbol{\phi} \in C_0^\infty(\Omega)$ satisfying (2.8.1), with $(\boldsymbol{z}, Q)$ solution to equations (2.8.2). Therefore, calling *very weak solution* of system $(2.7.1)_{1,2,3,}$ a field $\boldsymbol{u} \in L^q_{\text{loc}}(\overline{\Omega})$ which satisfies relation (2.8.5), we see that if $\boldsymbol{u}$ is a very weak solution corresponding to zero data, then

$$\int_{\Omega^-} \boldsymbol{u} \cdot \boldsymbol{\phi} = 0$$

for all $\boldsymbol{\phi} \in C_0^\infty(\overline{\Omega^-})$ satisfying (2.8.1). Hence it follows that $(\boldsymbol{u}, p)$ belongs to the linear space

$$\mathfrak{F} = \{(\boldsymbol{v}[\boldsymbol{\psi}'] - \mathcal{S}[\boldsymbol{\psi}'], P[\boldsymbol{\psi}']),\ \boldsymbol{\psi}' \in \mathfrak{C}\}.$$

Theorem 2.8.1 *If $\boldsymbol{a} \in L^q(\partial\Omega)$ $(q > 1)$ and $\boldsymbol{f} \in \mathcal{H}^1(\overline{\Omega^-})$, then system $(2.7.1)_{1,2,3}$ has a unique very weak solution expressed by (2.7.13) modulo a pair in $\mathfrak{F}$ and a constant pressure.*

Let us pass to consider problem (2.7.1). If we choose $\boldsymbol{a}$, $\boldsymbol{f}$ and $\boldsymbol{u}_0$ such that

$$\int_{\partial\Omega} (\boldsymbol{a} - \boldsymbol{u}_0) \cdot \boldsymbol{\psi}' + \int_{\Omega^-} \boldsymbol{f} \cdot \boldsymbol{v}[\boldsymbol{\psi}'] = 0, \quad \boldsymbol{\psi}' \in \mathfrak{C}, \tag{2.8.6}$$

then the pair (2.7.13), with $\boldsymbol{\gamma} = \boldsymbol{u}_0$, is a solution of system (2.7.1). Of course, $(\boldsymbol{u} - \boldsymbol{u}_0, p)$ is a solution to the Stokes equations, taking the value $\boldsymbol{a} - \boldsymbol{u}_0$ on $\partial\Omega$ and vanishing at infinity. Repeating the steps from (2.8.3) to (2.8.5) where now $(\boldsymbol{z}, Q)$ is the solution of equations (2.6.11) given by (2.7.13), with $\boldsymbol{\phi}$ arbitrary field in $C_0^\infty(\Omega)$, it is not difficult to see that $\boldsymbol{u}$ satisfies the relation

$$\int_{\Omega^-} \boldsymbol{u} \cdot \boldsymbol{\phi} = -\int_{\partial\Omega} (\boldsymbol{a} - \boldsymbol{u}_0) \cdot \boldsymbol{T}(\boldsymbol{z}, Q)\boldsymbol{n} + \int_{\Omega^-} \boldsymbol{f} \cdot \boldsymbol{z}. \tag{2.8.7}$$

As we did above, we call *very weak solution* of system (2.6.1) a field $\boldsymbol{u} \in L^q_{\text{loc}}(\overline{\Omega^-})$ which satisfies (2.8.7) for all $\boldsymbol{\phi} \in C_0^\infty(\overline{\Omega^-})$ and $\boldsymbol{z} = o(r)$ solution of equations (2.6.11). Let $\boldsymbol{u}$ be a very weak solution of problem (2.6.1). Choosing $(\boldsymbol{z},Q)\in\mathfrak{F}$ and taking into account that $\boldsymbol{T}(\boldsymbol{z},Q)\boldsymbol{n}=-\boldsymbol{\psi}'$, from (2.8.7) it follows that $\boldsymbol{a}$, $\boldsymbol{f}$ and $\boldsymbol{u}_0$ must necessarily satisfy (2.8.6). As a consequence, we have the well–known *Stokes paradox* of viscous hydrodynamics for very weak solutions [39].

Theorem 2.8.2 *If $\boldsymbol{a}\in L^q(\partial\Omega)$ $(q>1)$ and $\boldsymbol{f}\in\mathcal{H}^1(\Omega^-)$, then system (2.7.1) has a unique very weak solution modulo a constant pressure if and only if $\boldsymbol{a}$, $\boldsymbol{f}$ and $\boldsymbol{u}_0$ satisfy the compatibility condition (2.8.6).*

▶ We aim now at deriving the expression of the fields in $\operatorname{Ker}\mathcal{T}^+$ when $\partial\Omega$ is an ellipse. To this end we follow [26].

We know that for every rigid field $\boldsymbol{\varrho}$ on $\partial\Omega$ [24]

$$\int_{\partial\Omega}(\xi-\zeta)\cdot\boldsymbol{n}(\zeta)\frac{(\xi-\zeta)\otimes(\xi-\zeta)}{|\xi-\zeta|^4}\boldsymbol{\varrho}(\zeta)\mathrm{d}a_\zeta=-\frac{\pi}{4}\boldsymbol{\varrho}(\xi). \tag{2.8.8}$$

Let $\partial\Omega$ be the ellipse of equation

$$\boldsymbol{\xi}\cdot\boldsymbol{A}\cdot\boldsymbol{\xi}=1.$$

Since

$$\boldsymbol{n}(\xi)=\frac{\boldsymbol{A}\boldsymbol{\xi}}{|\boldsymbol{A}\boldsymbol{\xi}|},$$

we have

$$\begin{aligned}(\xi-\zeta)\cdot\boldsymbol{n}(\xi)&=\frac{(\xi-\zeta)\cdot\boldsymbol{A}\boldsymbol{\xi}}{|\boldsymbol{A}\boldsymbol{\xi}|}=\frac{1}{|\boldsymbol{A}\boldsymbol{\xi}|}(\boldsymbol{\xi}\cdot\boldsymbol{A}\boldsymbol{\xi}-\boldsymbol{\zeta}\cdot\boldsymbol{A}\boldsymbol{\xi})=\\&=\frac{1}{|\boldsymbol{A}\boldsymbol{\xi}|}(\boldsymbol{\zeta}\cdot\boldsymbol{A}\boldsymbol{\zeta}-\boldsymbol{\xi}\cdot\boldsymbol{A}\boldsymbol{\zeta})=-\frac{|\boldsymbol{A}\boldsymbol{\zeta}|}{|\boldsymbol{A}\boldsymbol{\xi}|}\frac{(\xi-\zeta)\cdot\boldsymbol{A}\boldsymbol{\zeta}}{|\boldsymbol{A}\boldsymbol{\zeta}|}=\\&=-\frac{|\boldsymbol{A}\boldsymbol{\zeta}|}{|\boldsymbol{A}\boldsymbol{\xi}|}(\xi-\zeta)\cdot\boldsymbol{n}(\zeta).\end{aligned}$$

Therefore, (2.8.8) implies

$$-\int_{\partial\Omega}(\xi-\zeta)\cdot\boldsymbol{n}(\xi)\frac{(\xi-\zeta)\otimes(\xi-\zeta)}{|\xi-\zeta|^4}\frac{\boldsymbol{\varrho}(\zeta)}{|\boldsymbol{A}\cdot\boldsymbol{\zeta}|}\mathrm{d}a_\zeta=-\frac{\pi}{4}\frac{\boldsymbol{\varrho}(\xi)}{|\boldsymbol{A}\cdot\boldsymbol{\xi}|}. \tag{2.8.9}$$

Hence, taking into account that

$$\frac{1}{|\boldsymbol{A}\cdot\boldsymbol{\xi}|}=\boldsymbol{\xi}\cdot\boldsymbol{n}(\xi),$$

it follows that

$$\operatorname{Ker}\mathcal{T}^{+} = \operatorname{sp}\{(\boldsymbol{\xi}\cdot\boldsymbol{n}(\xi))(\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\times\boldsymbol{\xi})\}.$$

Therefore, coupling this information with Theorem 2.8.2, we have

Theorem 2.8.3 *Let $\partial\Omega$ be an ellipse. If $\boldsymbol{a}\in L^q(\partial\Omega)$ $(q>1)$ and $\boldsymbol{f}\in\mathcal{H}^1(\Omega^-)$, then system (2.6.1) has a unique very weak solution modulo a constant pressure if and only if*

$$\int_{\partial\Omega}(\boldsymbol{a}-\boldsymbol{u}_0)\cdot(\boldsymbol{\xi}\cdot\boldsymbol{n})\boldsymbol{e}_i + \int_{\Omega}\boldsymbol{f}\cdot\boldsymbol{v}[(\boldsymbol{\xi}\cdot\boldsymbol{n})\boldsymbol{e}_i] = 0, \quad i=1,2. \tag{2.8.10}$$

In particular, if $\partial\Omega$ is a disk, then the problem

$$\begin{aligned} \Delta\boldsymbol{u}-\nabla p &= \boldsymbol{0} && \text{in } \Omega^-,\\ \operatorname{div}\boldsymbol{u} &= 0 && \text{in } \Omega^-,\\ \boldsymbol{u} &= \boldsymbol{a} && \text{on } \partial\Omega,\\ \lim_{r\to+\infty}\boldsymbol{u}(x) &= \boldsymbol{u}_0 \end{aligned}$$

has a solution if and only if

$$u_{0i} = \frac{1}{2|\Omega^+|}\int_{\partial\Omega}\boldsymbol{a}\cdot(\boldsymbol{\xi}\cdot\boldsymbol{n})\boldsymbol{e}_i \tag{2.8.11}$$

◁

► If $(\boldsymbol{u},p)$ is a solution to equations (2.3.3) which converges at infinity to a constant vector $\boldsymbol{u}_0$, then from (2.8.11) it follows that

$$\boldsymbol{u}_0 = \frac{1}{2\pi}\int_0^{2\pi}\boldsymbol{u}(R,\theta),$$

for all $S_R\supset\Omega^+$. ◁

► If $\boldsymbol{a}\in C(\partial\Omega)$, $\boldsymbol{f}\in\mathcal{H}^1(\Omega^-)$ and $\boldsymbol{u}_0$ satisfy (2.8.6), then

$$|\boldsymbol{u}_0| \le c\big\{\|\boldsymbol{a}\|_{C(\partial\Omega)}+\|\boldsymbol{f}\|_{\mathcal{H}^1(\Omega^-)}\big\}.$$

On the other hand, we have

$$\|\boldsymbol{u}-\boldsymbol{u}_0\|_{C(\overline{\Omega})} \le c\big\{\|\boldsymbol{a}-\boldsymbol{u}_0\|_{C(\partial\Omega)}+\|\boldsymbol{f}\|_{\mathcal{H}^1(\Omega^-)}\big\}.$$

Therefore, putting together the above estimates, we get

$$\|\boldsymbol{u}\|_{C(\overline{\Omega})} \le c\big\{\|\boldsymbol{a}\|_{C(\partial\Omega)}+\|\boldsymbol{f}\|_{\mathcal{H}^1(\Omega^-)}\big\}.$$

◁

2.9 Self–propelled Stokes flow

Let $\boldsymbol{a}$, $\boldsymbol{\varrho}$ be an assigned field on $\partial\Omega$ and a rigid motion respectively. A self–propelled motion of Ω^+ is a solution of the following equations [16]

$$
\begin{aligned}
\Delta \boldsymbol{u} - \nabla Q &= \mathbf{0} && \text{in } \Omega^-,\\
\operatorname{div} \boldsymbol{u} &= 0 && \text{in } \Omega^-,\\
\boldsymbol{u} &= \boldsymbol{a} + \boldsymbol{\varrho} && \text{on } \partial\Omega,\\
\lim_{r\to+\infty} \boldsymbol{u}(x) &= \mathbf{0}, &&\\
\int_{\partial\Omega} \boldsymbol{T}(\boldsymbol{u},p)\boldsymbol{n} &= \mathbf{0}, &&\\
\int_{\partial\Omega} \boldsymbol{\xi} \times \boldsymbol{T}(\boldsymbol{u},p)\boldsymbol{n} &= \mathbf{0}. &&
\end{aligned}
\tag{2.9.1}
$$

Let $(\boldsymbol{u}, Q)$ be a solution of equations $(2.7.1)_{1,2,3}$ and let

$$\boldsymbol{\psi} \in \mathfrak{C} \oplus \mathfrak{L}$$

be such that (see Remark 2.7.2)

$$
\begin{aligned}
&\int_{\partial\Omega} [\boldsymbol{T}(\boldsymbol{u},p)\boldsymbol{n} + \boldsymbol{\psi}] = \mathbf{0},\\
&\int_{\partial\Omega} \boldsymbol{\xi} \times [\boldsymbol{T}(\boldsymbol{u},p)\boldsymbol{n} + \boldsymbol{\psi}] = \mathbf{0}.
\end{aligned}
$$

Then it is readily seen that the pair

$$
\begin{aligned}
&\boldsymbol{u}'(x) = \boldsymbol{u}(x) + \boldsymbol{v}[\boldsymbol{\psi}],\\
&p'(x) = p + P[\boldsymbol{\psi}].
\end{aligned}
\tag{2.9.2}
$$

is a very weak solution of system (2.9.1) if and only if

$$\int_{\partial\Omega} (\boldsymbol{a} + \boldsymbol{\varrho}) \cdot \boldsymbol{\psi} = \mathbf{0}, \quad \forall\, \boldsymbol{\psi} \in \mathfrak{C} \oplus \mathfrak{L}. \tag{2.9.3}$$

Therefore, we can state the following

Theorem 2.9.1 $\boldsymbol{a} \in L^q(\partial\Omega)$ $(q > 1)$. *Then system* (2.9.1) *has a unique very weak solution modulo a constant pressure, expressed by* (2.9.2), *if and only if* $\boldsymbol{a}$ *and* $\boldsymbol{\varrho}$ *satisfy* (2.9.3).

• *Remark 2.9.1*

If $\partial\Omega$ is an ellipse and $\boldsymbol{\varrho} = \boldsymbol{\kappa} + \alpha \boldsymbol{e}_3 \times \boldsymbol{x}$, then (2.9.3) reads

$$\kappa_i = -\frac{1}{2|\Omega^+|} \int_{\partial\Omega} (\boldsymbol{\xi} \cdot \boldsymbol{n}) \boldsymbol{a} \cdot \boldsymbol{e}_i \;, \quad i = 1, 2,$$
$$\alpha = -\frac{1}{h} \int_{\partial\Omega} (\boldsymbol{\xi} \cdot \boldsymbol{n}) \boldsymbol{a} \cdot (\boldsymbol{e}_3 \times \boldsymbol{\xi}),$$

with

$$h = 4 \int_{\Omega^+} |\boldsymbol{x}|^2.$$

◇

2.10 The Neumann problem

By the layer potentials approach we can also treat the classical Neumann problem for the Stokes system in bounded and exterior domains. To this end we follow the argument in [42].

Let us start by considering the system

$$\begin{aligned} \Delta \boldsymbol{u} - \nabla Q &= \boldsymbol{f} && \text{in } \Omega^+, \\ \operatorname{div} \boldsymbol{u} &= 0 && \text{in } \Omega^+, \\ \boldsymbol{T}(\boldsymbol{u}, p)\boldsymbol{n} &= \boldsymbol{s} && \text{on } \partial\Omega, \end{aligned} \tag{2.10.1}$$

where $\boldsymbol{s}$ is an assigned field in $L^q(\partial\Omega)$ and $\boldsymbol{f} \in L^{2q/(q+1)}(\Omega^+)$ $(q > 1)$. Looking for a solution expressed by a single layer potential, we are led to consider the Fredholm equation in $L^q(\partial\Omega)$

$$\mathcal{T}^+[\boldsymbol{\psi}] = (\tfrac{1}{2}\mathcal{I} - \mathcal{K}^*)[\boldsymbol{\psi}] = \boldsymbol{s} - \boldsymbol{T}(\mathcal{V}[\boldsymbol{f}], \mathcal{P}[\boldsymbol{f}])\boldsymbol{n}_{|\partial\Omega}. \tag{2.10.2}$$

If $\boldsymbol{\varphi} \in \operatorname{Ker} \mathcal{W}^-$, then $\boldsymbol{w}[\boldsymbol{\varphi}]$ is a regular solution to the equations (2.3.3) vanishing at infinity. Taking into account the behavior at infinity of a double layer potential, by uniqueness we have that $\boldsymbol{w}[\boldsymbol{\varphi}] = \boldsymbol{0}$ and $\varpi[\boldsymbol{\varphi}] = 0$ in Ω and by (2.5.13) $\boldsymbol{T}(\boldsymbol{w}[\boldsymbol{\varphi}], [\varpi[\boldsymbol{\varphi}])\boldsymbol{n} = \boldsymbol{0}$ on $\partial\Omega$. Therefore, (2.5.13) and Lemma 2.3.3 assure that $\boldsymbol{w}[\boldsymbol{\varphi}]$ is a rigid displacement in Ω^+ and by the jump conditions (2.5.9) necessarily $\boldsymbol{\varphi} \in \mathfrak{R}_{\partial\Omega}$. On the other hand, it is simple to see that every $\boldsymbol{\varphi} \in \mathfrak{R}_{\partial\Omega}$ belongs to $\operatorname{Ker} \mathcal{W}^+$ so that

$$\operatorname{Ker} \mathcal{W}^+ = \mathcal{R}_{\partial\Omega}.$$

Therefore we have

Theorem 2.10.1 *If $\boldsymbol{s} \in L^q(\partial\Omega)$ and $\boldsymbol{f} \in L^{2q/(q+1)}(\Omega^+)$ $(q > 1)$ satisfies*

$$\begin{aligned} \int_{\partial\Omega} \boldsymbol{s} &= \int_{\Omega^+} \boldsymbol{f}, \\ \int_{\partial\Omega} \boldsymbol{x} \times \boldsymbol{s} &= \int_{\Omega^+} \boldsymbol{x} \times \boldsymbol{f}, \end{aligned} \tag{2.10.3}$$

then system (2.10.1) has a unique solution modulo a rigid motion, expressed by

$$\begin{aligned} \boldsymbol{u}(x) &= \boldsymbol{v}[\boldsymbol{\psi}] + \mathcal{V}[\boldsymbol{f}], \\ p(x) &= P[\boldsymbol{\psi}] + \mathcal{P}[\boldsymbol{f}], \end{aligned} \tag{2.10.4}$$

for some $\boldsymbol{\psi} \in L^q(\partial\Omega)$. *If* $\boldsymbol{f} \in L^s(\Omega^+)$ $(s > 2)$, *then*

$$\lim_{t\to 0^+} [\boldsymbol{T}(\boldsymbol{u}, p)\boldsymbol{n}](\xi - t\boldsymbol{n}(\xi)) = \boldsymbol{s}(\xi), \tag{2.10.5}$$

for almost all $\xi \in \partial\Omega$. *Moreover, if* $\boldsymbol{s} \in C(\partial\Omega)$, *then* (2.10.5) *is satisfied everywhere on* $\partial\Omega$.

Proof - Existence is a simple consequence of Fredholm's alternative and (2.10.3); (2.10.5) follows from the properties of the single layer potential and the continuity of $\nabla\mathcal{V}[\boldsymbol{f}]$, $\mathcal{P}[\boldsymbol{f}]$ under the hypothesis $\boldsymbol{f} \in L^s(\Omega^+)$ $(s > 2)$. Since $\boldsymbol{u} \in W^{1,2q}(\Omega^+)$, uniqueness is a consequence of Lemma 2.3.3. □

Let us pass to consider the Neumann problem in Ω^-:

$$\begin{aligned} \Delta\boldsymbol{u} - \nabla p = \boldsymbol{f} \quad & \text{in } \Omega^-, \\ \operatorname{div}\boldsymbol{u} = 0 \quad & \text{in } \Omega^-, \\ \boldsymbol{T}(\boldsymbol{u}, p)\boldsymbol{n} = \boldsymbol{s} \quad & \text{on } \partial\Omega, \\ \lim_{r\to+\infty} \boldsymbol{u}(x) = \boldsymbol{0}. & \end{aligned} \tag{2.10.6}$$

The following theorem holds.

Theorem 2.10.2 *Let* $\boldsymbol{s} \in L^q(\partial\Omega)$ *and let* $\boldsymbol{f} \in L^{2q/(q+1)}(\Omega^-)\cap\mathcal{H}^1(\Omega^-)$ $(q > 1)$. *If*

$$\int_{\partial\Omega} \boldsymbol{s} = \boldsymbol{0}, \tag{2.10.7}$$

then system (2.10.6) *has a unique solution expressed by*

$$\begin{aligned} \boldsymbol{u}(x) &= \boldsymbol{v}[\boldsymbol{\psi}] + \boldsymbol{w}[\boldsymbol{\varphi}] + \mathcal{V}[\boldsymbol{f}], \\ p(x) &= P[\boldsymbol{\psi}] + \varpi[\boldsymbol{\varphi}] + \mathcal{P}[\boldsymbol{f}], \end{aligned} \tag{2.10.8}$$

with $\boldsymbol{\psi} \in L^q(\partial\Omega)$ *and* $\boldsymbol{\varphi} \in \operatorname{Ker}\mathcal{W}^+$. *If* $\boldsymbol{f} \in L^s(\Omega^-)$ $(s > 2)$, *then* (2.10.5) *holds and if* $\boldsymbol{s} \in C(\partial\Omega)$, *then* (2.10.5) *is satisfied everywhere on* $\partial\Omega$.

Proof - We know that $\operatorname{Ker}\mathcal{T}^- = \operatorname{sp}\{\boldsymbol{n}\}$ so that $\dim\operatorname{Ker}\mathcal{W}^+ = 1$. If $\boldsymbol{\varphi} \in \operatorname{Ker}\mathcal{W}^+$, then by uniqueness $\boldsymbol{w}[\boldsymbol{\varphi}] = \boldsymbol{0}$ and (2.5.9) implies

$$\mathcal{W}^-[\boldsymbol{\varphi}] = -\boldsymbol{\varphi}. \tag{2.10.9}$$

Therefore by Fredholm alternative, the equation

$$\mathcal{T}^-[\psi] = -(\tfrac{1}{2}\mathcal{I} + \mathcal{K}^*)[\psi] = \boldsymbol{s} - \boldsymbol{T}(\mathcal{V}[\boldsymbol{f}], \mathcal{P}[\boldsymbol{f}])\boldsymbol{n}_{|\partial\Omega} - \boldsymbol{T}(\boldsymbol{w}[\boldsymbol{\varphi}], \varpi[\boldsymbol{\varphi}])\boldsymbol{n}$$

has a solution if and only if

$$\int_{\partial\Omega} [\boldsymbol{s} - \boldsymbol{T}(\mathcal{V}[\boldsymbol{f}], \mathcal{P}[\boldsymbol{f}])\boldsymbol{n}_{|\partial\Omega} - \boldsymbol{T}(\boldsymbol{w}[\boldsymbol{\varphi}], \varpi[\boldsymbol{\varphi}])\boldsymbol{n}] \cdot \boldsymbol{\varphi}' = 0$$

for all $\boldsymbol{\varphi}' \in \operatorname{Ker} \mathcal{W}^+$, or equivalently if and only if the equation

$$\int_{\partial\Omega} [\boldsymbol{T}(\boldsymbol{w}[\boldsymbol{\varphi}], \varpi[\boldsymbol{\varphi}])\boldsymbol{n}] \cdot \boldsymbol{\varphi}' = 0, \quad \forall \boldsymbol{\varphi} \in \operatorname{Ker} \mathcal{W}^+, \tag{2.10.10}$$

has only the null solution. To show this, choose $\boldsymbol{\varphi}' = \boldsymbol{\varphi}$ in (2.10.10) and note that by (2.10.9) an integration over Ω^- gives

$$2\int_{\Omega^-} |\hat{\nabla} \boldsymbol{w}[\boldsymbol{\varphi}]|^2 = -\int_{\partial\Omega} \mathcal{W}^+[\boldsymbol{\varphi}] \cdot \boldsymbol{T}(\boldsymbol{w}[\boldsymbol{\varphi}], \varpi[\boldsymbol{\varphi}])\boldsymbol{n} = 0. \tag{2.10.11}$$

Hence it easily follows that $\boldsymbol{\varphi} = \boldsymbol{0}$. Condition (2.10.7) guarantees that $(2.10.6)_4$ is satisfied. Uniqueness and the second part of the theorem are proved by the argument we used in the proof of Theorem 2.10.1.

□

• *Remark 2.10.1*

Note that if $\boldsymbol{\varphi}$ is a (nonzero) density of $\operatorname{Ker} \mathcal{W}^+$, then necessarily

$$\int_{\partial\Omega} \boldsymbol{\varphi} \cdot \boldsymbol{n} \neq 0.$$

Indeed, if

$$\int_{\partial\Omega} \boldsymbol{\varphi} \cdot \boldsymbol{n} = 0,$$

since $\boldsymbol{T}(\boldsymbol{w}[\boldsymbol{\varphi}], \varpi[\boldsymbol{\varphi}])\boldsymbol{n} = c\boldsymbol{n}$, (2.10.9) implies that

$$\int_{\partial\Omega} \mathcal{W}^+[\boldsymbol{\varphi}] \cdot \boldsymbol{T}(\boldsymbol{w}[\boldsymbol{\varphi}], \varpi[\boldsymbol{\varphi}])\boldsymbol{n} = 0.$$

Hence by (2.10.11) it follows $\boldsymbol{\varphi} = \boldsymbol{0}$. ♢

2.11 Solutions of the Stokes problem expressed by simple layer potentials

Thanks to the results of the foregoing sections, we can make use of a classical argument in potential theory, to show that the solution of the Stokes problems we found above can be expressed by simple layer potentials plus volume potentials. We shall assume for simplicity $\boldsymbol{f} = \mathbf{0}$. The extension to the general case presents is achieved with few changes.

Let $(\boldsymbol{u}, p)$ be the solution of the Stokes problem in Ω^+ expressed by (2.6.2) and corresponding to

$$\boldsymbol{a} \in W^{1,q}(\partial\Omega), \tag{2.11.1}$$

with $q > 1$, and $\boldsymbol{a}$ satisfying

$$\int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{n} = 0. \tag{2.11.2}$$

In vitue of Theorem 2.10.1 and Lemma 2.3.3, the solution of the Neumann problem with boundary datum

$$\boldsymbol{s} = \boldsymbol{T}(\boldsymbol{u}, p)\boldsymbol{n} \in L^q(\partial\Omega) \tag{2.11.3}$$

is given by

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{v}[\boldsymbol{\psi}] + \boldsymbol{\varrho}, \\ p &= P[\boldsymbol{\psi}], \end{aligned} \tag{2.11.4}$$

for some $\boldsymbol{\varrho} \in \mathfrak{R}$. Now, by the results of Section 2.7 the field $\boldsymbol{\varrho}$ can be expressed in Ω^+ by

$$\boldsymbol{\varrho} = \boldsymbol{v}[\boldsymbol{\psi}'] + \gamma\boldsymbol{\kappa},$$

with $\boldsymbol{\psi}' \in \operatorname{Ker} \mathcal{T}^+$ and $\boldsymbol{\kappa}$ constant vector, and (2.11.4) takes the form

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{v}[\boldsymbol{\psi} + \boldsymbol{\psi}'] + \gamma\boldsymbol{\kappa}, \\ p &= P[\boldsymbol{\psi} + \boldsymbol{\psi}']. \end{aligned}$$

Consider now the solution (2.7.13) under hypotheses (2.11.1), (2.11.2). The solution of the Neumann problem with boundary datum (2.11.3) is expressed by (2.10.8). Then by Lemma 2.3.4

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{v}[\boldsymbol{\psi}] + \boldsymbol{w}[\boldsymbol{\varphi}] + \boldsymbol{\kappa}, \\ p &= P[\boldsymbol{\psi}] + \varpi[\boldsymbol{\varphi}], \end{aligned} \tag{2.11.5}$$

with $\boldsymbol{\psi} \in L^q(\partial\Omega)$, $\boldsymbol{\varphi} \in \operatorname{Ker} \mathcal{W}^-$ and $\boldsymbol{\kappa}$ constant vector. Since

$$\boldsymbol{a} = \mathcal{S}[\boldsymbol{\psi}] - \boldsymbol{\varphi} + \boldsymbol{\kappa},$$

taking into account that $\boldsymbol{v}[\boldsymbol{\psi}]$ is divergence free in $\mathbb{R}^2$, by (2.11.2) we have

$$\int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{n} = -\int_{\partial\Omega} \boldsymbol{\varphi} \cdot \boldsymbol{n} = 0.$$

Hence by Remark 2.10.1 it follows that $\boldsymbol{\varphi} = \boldsymbol{0}$ and (2.11.5) takes the form

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{v}[\boldsymbol{\psi}] + \boldsymbol{\kappa}, \\ p &= P[\boldsymbol{\psi}]. \end{aligned}$$

2.12 Some remarks on the Stokes problem in domains with nonconnected boundaries

We assumed $\partial\Omega$ connected only to make the exposition more simple and alleviate notation. Indeed, with few (sometimes laborious) changes we can prove the results of the foregoing sections for the Stokes problem in the bounded domain (1.1.1) and in the exterior domains (1.1.2).

As far as the Stokes problem in domain (1.1.1) is concerned, retaining the definitions and the notation we used, we can prove that

$$\dim \operatorname{Ker} \mathcal{T}^- = 3k+1$$

and

$$\operatorname{Ker} \mathcal{T}^- = \{\boldsymbol{\psi} : \mathcal{S}[\boldsymbol{\psi}] \in \mathfrak{R}_{\partial\Omega_i}, P[\boldsymbol{\psi}] = 0 \text{ in } \Omega_i\} \oplus \operatorname{sp}\{\boldsymbol{n}\}.$$

With this information at hand, we can show that, under the hypotheses on $\boldsymbol{a}$ and $\boldsymbol{f}$ required in Theorem 2.6.1, system (2.6.1) in the domain (1.1.1) have a solution expressed by (2.7.13), with $\boldsymbol{\psi} \in \operatorname{Ker} \mathcal{T}^- \setminus \operatorname{sp}\{\boldsymbol{n}\}$.

All the properties stated in Section 2.6 hold unchanged.

Passing to consider problem (2.7.1) in the domain (1.1.2), one shows that

$$\dim \operatorname{Ker} \mathcal{T}^- = 3k$$

and

$$\operatorname{Ker} \mathcal{T}^- = \{\boldsymbol{\psi} : \mathcal{S}[\boldsymbol{\psi}] \in \mathfrak{R}_{\partial\Omega_i}, P[\boldsymbol{\psi}] = 0 \text{ in } \Omega_i\}.$$

Moreover,

$$\dim \mathfrak{C} = 2.$$

Once again we can follow the Fredholm alternative to show that the results in section 2.7 and 2.8 hold unchanged.

Analogous results hold for the Neumann problem in (1.1.1), (1.1.2) [42].

• *Remark 2.12.1*

If we consider only solutions of the exterior Stokes problem in the spaces $D^{1,q}(\Omega)$, then we easily rediscover the results in [18]. The solutions $\boldsymbol{h} \in \mathcal{S}_q$ (see the introduction) of system (2.3.3) in $D^{1,q}(\Omega)$, which G.P. Galdi and C.G. Simader called *exceptional*, are here the simple layer potential $\boldsymbol{v}[\boldsymbol{\psi}]$ with $\psi \in \mathfrak{C}$. Of course, $\mathcal{S}_q = \{\mathbf{0}\}$ for $q \in (1, 2]$.

◊

• *Remark 2.12.2*

Let

$$\boldsymbol{\sigma}(x) = -\sum_{1=1}^{m} \frac{x - x_i}{|x - x_i|^2} \int_{\partial\Omega_i} \boldsymbol{a} \cdot \boldsymbol{n}$$

and note that

$$\int_{\partial\Omega_i} \frac{(\xi - x_i) \cdot \boldsymbol{n}(\xi)}{|\xi - x_i|^2} = -1.$$

In view of the application of the above results to the existence theorem for the Navier–Stokes problem, let us denote by $(\boldsymbol{v}, p_s)$ the solution to the Stokes problem with boundary datum

$$\boldsymbol{\alpha} = \boldsymbol{a} - \boldsymbol{\sigma}.$$

Then the pair

$$(\boldsymbol{u}_s = \boldsymbol{v} + \boldsymbol{\sigma}, p_s) \tag{2.12.1}$$

is a very weak solution to system (2.6.1) such that

$$\int_{\partial\Omega_i} \boldsymbol{v} \cdot \boldsymbol{n} = 0, \quad i = 1, \ldots, m. \tag{2.12.2}$$

◊

Chapter 3

Steady Navier–Stokes flow in bounded domains

3.1 Existence of a very weak solution

- Throughout this chapter we shall consider the bounded domain by Ω the domain

$$\Omega_0 \setminus \overline{\Omega'}, \quad \Omega' = \bigcup_{i=1}^{m} \Omega_i.$$

we assume to be of class C^2.

Let us recall that the (interior) Navier–Stokes problem is to find a solution of the system

$$\begin{aligned} \Delta \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nabla p = \boldsymbol{f} \quad & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 \quad & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{a} \quad & \text{on } \partial\Omega. \end{aligned} \tag{3.1.1}$$

Let $(\boldsymbol{u}_s, p_s)$ be the very weak solution of the Stokes problem corresponding to data $(\boldsymbol{a}, \boldsymbol{f})$ given by (2.12.1) with

$$\boldsymbol{a} \in L^2(\partial\Omega), \quad \boldsymbol{f} \in \mathcal{H}^1(\Omega) \tag{3.1.2}$$

and

$$\int_{\partial\Omega} \boldsymbol{a}\cdot\boldsymbol{n} = 0. \tag{3.1.3}$$

The volume potential

$$\mathcal{C}[\boldsymbol{u}] = \mathcal{V}[\boldsymbol{u}\cdot\nabla\boldsymbol{u}]$$

maps boundedly $L^4_\sigma(\Omega)$ into $W^{1,2}_\sigma(\Omega)$. If $\{\boldsymbol{u}_k\}_{k\in\mathbb{N}}$ is a bounded sequence in $L^4(\Omega)$, then $\{\mathcal{C}[\boldsymbol{u}_k]\}_{k\in\mathbb{N}}$ is bounded in $W^{1,2}(\Omega)$. By Lemma 1.2.5 from $\{\mathcal{C}[\boldsymbol{u}_k]\}_{k\in\mathbb{N}}$ we can extract a subsequence which converges strongly in $L^4(\Omega)$ so that $\mathcal{C}$ is completely continuous form $L^4_\sigma(\Omega)$ into itself. Let $(\mathcal{Y}[\boldsymbol{u}], \mathcal{Q}[u])$ be the solution of the Stokes problem with boundary datum $-\operatorname{tr}_{|\partial\Omega}\mathcal{C}[\boldsymbol{u}] \in W^{1/2,2}(\partial\Omega)$ and zero body force. Since by the estimates about solutions of the Stokes problem and the trace theorem

$$\|\mathcal{Y}[\boldsymbol{u}]\|_{W^{1,2}(\Omega)} \le c\|\operatorname{tr}_{|\partial\Omega}\mathcal{C}[\boldsymbol{u}]\|_{W^{1/2,2}(\partial\Omega)} \le \|\mathcal{C}[\boldsymbol{u}]\|_{W^{1,2}(\Omega)},$$

we have that also the operator $\mathcal{Y}$ is completely continuous from $L^4_\sigma(\Omega)$ into itself. Therefore, the operator

$$\mathcal{N}[\boldsymbol{u}] = (\mathcal{Y}+\mathcal{C})[\boldsymbol{u}]$$

maps $L^4_\sigma(\Omega)$ into $W^{1,2}_{\sigma,0}(\Omega)$ and is completely continuous from $L^4_\sigma(\Omega)$ into itself. Now, it is natural to look for a solution of system (3.1.1) as a fixed point of the equation

$$\boldsymbol{u}' = \boldsymbol{u}_s + \mathcal{N}[\boldsymbol{u}]. \tag{3.1.4}$$

The pressure field associated to (the fixed point) $\boldsymbol{u}$ will be the field

$$p = p_s + \mathcal{Q}[\boldsymbol{u}] + \mathcal{P}[\boldsymbol{u}\cdot\nabla\boldsymbol{u}]. \tag{3.1.5}$$

- We call the pair $(\boldsymbol{u}, p)$ *very weak solution* of the Navier Stokes problem (3.1.1).

To do this we appeal to Lemma 1.2.4. A solution of the functional equation

$$\boldsymbol{u} = \lambda(\boldsymbol{u}_s + \mathcal{N}[\boldsymbol{u}]), \tag{3.1.6}$$

for $\lambda \in [0,1]$, satisfies the equations

$$\begin{aligned} \Delta \boldsymbol{w} - \lambda(\boldsymbol{u}_s + \boldsymbol{w}) \cdot \nabla(\boldsymbol{u}_s + \boldsymbol{w}) - \nabla Q = \mathbf{0} \quad & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{w} = 0 \quad & \text{in } \Omega, \\ \boldsymbol{w} = \mathbf{0} \quad & \text{on } \partial\Omega, \end{aligned} \tag{3.1.7}$$

where we set $\boldsymbol{w} = \mathcal{N}[\boldsymbol{u}] \in W^{1,2}_{\sigma,0}(\Omega)$, for some pressure field $Q \in L^2(\Omega)$. The field $\boldsymbol{w}$ is a variational solution of system (3.1.7), *i.e*,

$$\int_\Omega \nabla \boldsymbol{w} \cdot \nabla \boldsymbol{\phi} = \lambda \int_\Omega (\boldsymbol{u}_s + \boldsymbol{w}) \cdot \nabla \boldsymbol{\phi} \cdot (\boldsymbol{u}_s + \boldsymbol{w}) \tag{3.1.8}$$

for all $\boldsymbol{\phi} \in W^{1,2}_{\sigma,0}(\Omega)$.

Let

$$\Phi_i = \int_{\partial\Omega_1} \boldsymbol{a} \cdot \boldsymbol{n}.$$

Let us show that, if

$$\Phi = \frac{1}{4\pi} \sum_{i=1}^{m} |\Phi_i| \left(\max_{\partial\Omega} \log|x - x_i| - \min_{\partial\Omega} \log|x - x_i| \right) < 1, \tag{3.1.9}$$

then all the variational solutions of system (3.1.7) are uniformly bounded in $W^{1,2}_0(\Omega)$, *i.e.*, there is a positive constant c independent of $(\boldsymbol{w}, Q)$ such that

$$\int_\Omega |\nabla \boldsymbol{w}|^2 \leq c. \tag{3.1.10}$$

To show (3.1.10) we follow a classical argument of J. Leray [25]. It (3.1.10) is not true, then there is a sequence of variational solutions $\{\boldsymbol{w}_k\}_{k\in\mathbb{N}}$ of (3.1.7) and a sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to+\infty} J_k^2 = \lim_{k\to+\infty} \int_\Omega |\nabla \boldsymbol{w}_k|^2 = +\infty, \quad \lim_{k\to+\infty} \lambda_k = \lambda_0 \in [0,1].$$

Then, from (3.1.7) it follows that the field

$$\boldsymbol{w}'_k = \frac{\boldsymbol{w}_k}{J_k}$$

satisfies the relation

$$\frac{1}{J_k}\int_\Omega \nabla \boldsymbol{w}'_k \cdot \nabla \boldsymbol{\phi} = \lambda_k \int_\Omega \boldsymbol{w}'_k \cdot \nabla \boldsymbol{\phi} \cdot \boldsymbol{w}'_k + \frac{\lambda_k}{J_k^2}\int_\Omega \boldsymbol{u}_s \cdot \nabla \boldsymbol{\phi} \cdot \boldsymbol{u}_s + \frac{\lambda_k}{J_k}\int_\Omega (\boldsymbol{w}'_k \cdot \nabla \boldsymbol{\phi} \cdot \boldsymbol{u}_s + \boldsymbol{u}_s \cdot \nabla \boldsymbol{\phi} \cdot \boldsymbol{w}'_k). \tag{3.1.11}$$

Since

$$\int_\Omega |\nabla \boldsymbol{w}'_k|^2 = 1,$$

by Lemma 1.2.5 from $\{\boldsymbol{w}'_k\}_{k\in\mathbb{N}}$ we can extract a subsequence, still denoted by the same symbol, which converges strongly in $L^4(\Omega)$ and weakly in $W^{1.2}(\Omega)$ to a field $\boldsymbol{w}' \in W^{1,2}_{\sigma,0}(\Omega)$:

$$\begin{aligned} &\lim_{k\to+\infty}\int_\Omega |\boldsymbol{w}'_k - \boldsymbol{w}'|^4 = 0, \\ &\lim_{k\to+\infty}\int_\Omega \nabla \boldsymbol{w}'_k \cdot \nabla \boldsymbol{\phi} = \int_\Omega \nabla \boldsymbol{w}' \cdot \nabla \boldsymbol{\phi}. \end{aligned} \tag{3.1.12}$$

By Hölder's inequality and Lemma 1.2.10

$$\begin{aligned} &\left|\int_\Omega \nabla \boldsymbol{w}'_k \cdot \nabla \boldsymbol{\phi}\right| \le \|\nabla \boldsymbol{w}'_k\|_{L^2(\Omega)}\|\nabla \boldsymbol{\phi}\|_{L^2(\Omega)} \le \|\nabla \boldsymbol{\phi}\|_{L^2(\Omega)} \\ &\left|\int_\Omega \boldsymbol{u}_s \cdot \nabla \boldsymbol{\phi} \cdot \boldsymbol{u}_s\right| \le \|\nabla \boldsymbol{\phi}\|_{L^2(\Omega)}\|\boldsymbol{u}_s\|^2_{L^4(\Omega)}, \\ &\left|\int_\Omega \boldsymbol{u}_s \cdot \nabla \boldsymbol{\phi} \cdot \boldsymbol{w}'_k\right| \le \|\nabla \boldsymbol{\phi}\|_{L^2(\Omega)}\|\boldsymbol{w}'_k\|_{L^4(\Omega)}\|\boldsymbol{u}_s\|_{L^4(\Omega)} \\ &\qquad \le c_\sigma \|\nabla \boldsymbol{\phi}\|_{L^2(\Omega)}\|\boldsymbol{u}_s\|_{L^4(\Omega)}\|\nabla \boldsymbol{w}'_k\|_{L^2(\Omega)} \le c_\sigma \|\nabla \boldsymbol{\phi}\|_{L^2(\Omega)}\|\boldsymbol{u}_s\|_{L^4(\Omega)}, \\ &\left|\int_\Omega \boldsymbol{w}'_k \cdot \nabla \boldsymbol{\phi} \cdot \boldsymbol{u}_s\right| \le c_\sigma \|\nabla \boldsymbol{\phi}\|_{L^2(\Omega)}\|\boldsymbol{u}_s\|_{L^4(\Omega)}. \end{aligned}$$

Moreover, since

$$\int_\Omega (\boldsymbol{w}'_k \cdot \nabla \boldsymbol{\phi} \cdot \boldsymbol{w}'_k - \boldsymbol{w}' \cdot \nabla \boldsymbol{\phi} \cdot \boldsymbol{w}') = \int_\Omega (\boldsymbol{w}'_k - \boldsymbol{w}') \cdot \nabla \boldsymbol{\phi} \cdot \boldsymbol{w}'_k + \int_\Omega \boldsymbol{w}' \cdot \nabla \boldsymbol{\phi} \cdot (\boldsymbol{w}'_k - \boldsymbol{w}')$$

and

$$\left|\int_\Omega (\boldsymbol{w}'_k - \boldsymbol{w}') \cdot \nabla \boldsymbol{\phi} \cdot \boldsymbol{w}'_k\right| \le \|\nabla \boldsymbol{w}'_k\|_{L^2(\Omega)} \|\boldsymbol{w}'_k - \boldsymbol{w}'\|_{L^4(\Omega)} \|\boldsymbol{w}'_k\|_{L^4(\Omega)},$$
$$\left|\int_\Omega \boldsymbol{w}' \cdot \nabla \boldsymbol{\phi} \cdot (\boldsymbol{w}'_k - \boldsymbol{w}')\right| \le \|\nabla \boldsymbol{w}'_k\|_{L^2(\Omega)} \|\boldsymbol{w}'_k - \boldsymbol{w}'\|_{L^4(\Omega)} \|\boldsymbol{w}'\|_{L^4(\Omega)},$$

taking into account (3.1.12), we can let $k \to +\infty$ in (3.1.11) to have

$$\lim_{k\to+\infty} \int_\Omega \boldsymbol{w}'_k \cdot \nabla \boldsymbol{\phi} \cdot \boldsymbol{w}'_k = \int_\Omega \boldsymbol{w}' \cdot \nabla \boldsymbol{\phi} \cdot \boldsymbol{w}' = \int_\Omega \boldsymbol{w}' \cdot \nabla \boldsymbol{w}' \cdot \boldsymbol{\phi} = 0,$$

for all $\boldsymbol{\phi} \in W^{1,2}_{\sigma,0}(\Omega)$. Then by Lemma 1.2.14, there is a field $Q' \in L^q(\Omega)$, $q < 2$, such that

$$\int_\Omega \boldsymbol{w}' \cdot \nabla \boldsymbol{w}' \cdot \boldsymbol{\phi} = \int_\Omega Q' \operatorname{div} \boldsymbol{\phi}, \quad \forall\, \boldsymbol{\phi} \in W^{1,2}_0(\Omega).$$

so that the pair $(\boldsymbol{w}, Q')$ is a variational solution of the Euler equations

$$\begin{aligned} \lambda_0 \boldsymbol{w}' \cdot \nabla \boldsymbol{w}' + \nabla Q' &= \boldsymbol{0} \quad \text{in } \Omega, \\ \operatorname{div} \boldsymbol{w}' &= 0 \quad \text{in } \Omega, \\ \boldsymbol{w}' &= \boldsymbol{0} \quad \text{on } \partial\Omega. \end{aligned} \tag{3.1.13}$$

The following Lemma holds [1].

Lemma 3.1.1 *If $(\boldsymbol{w}', Q') \in W^{1,2}_{\sigma,0}(\Omega) \times L^q(\Omega)$, $q < 2$, is a variational solution of system (3.1.13), then Q' is a constant Q'_i (say) on every $\partial\Omega_i$.*

Proof - Let $\boldsymbol{w}' \in C^\infty_0(\Omega)$ and let ξ_0 be a point of the connected component $\partial\Omega_i$ of $\partial\Omega$. Choose a new coordinate system (x_1, x_2) in which the points of a neighborhood $\mathfrak{I}$ of ξ_0 in $\overline{\Omega}$ are expressed by

$$A = \{(x_1, x_2) : |x_1| < \epsilon,\ x_2 \in (h(x_1), h(x_1) + \alpha)\},$$

for some positive ϵ and α, with $h(x_1)$ function of class C^2 and the x_2–axis pointing along the inner normal to $\partial\Omega'$. By Hardy's inequality

$$\int_A \frac{|\boldsymbol{w}'|^2}{|x_2 - h(x_1)|^2} \le 4 \int_A |\nabla \boldsymbol{w}'|^2$$

we get

$$\begin{aligned}\int_A \frac{|\boldsymbol{w}'\cdot\nabla\boldsymbol{w}'|}{|x_2-h(x_1)|} &\le \left\{\int_A \frac{|\boldsymbol{w}'|^2}{|x_2-h(x_1)|^2}\int_A|\nabla\boldsymbol{w}'|^2\right\}^{1/2}\\ &\le 2\int_A|\nabla\boldsymbol{w}'|^2.\end{aligned} \tag{3.1.14}$$

Writing (3.1.14) for a sequence $\{\boldsymbol{w}'_k\}_{k\in\mathbb{N}}\in C_0^\infty(\Omega)$ which converges to $\boldsymbol{w}'$ in $W_0^{1,2}(\Omega)$ and letting $k\to+\infty$ we see that (3.1.14) holds for $\boldsymbol{w}'\in W_{\sigma,0}^{1,2}(\Omega)$. Then, taking into account $(3.1.13)_1$, we have

$$\int_A|\nabla Q'| = o(\alpha). \tag{3.1.15}$$

If $\phi\in C_0^\infty(\Sigma)$, then

$$\begin{aligned}\int_A Q'\nabla_{x_1}\phi &= \int_{|x_1|<\epsilon}\int_0^\alpha Q'(x_1,h(x_1)+x_2)\nabla_{x_1}\phi(x_1)\\ &= \int_{|x_1|<\epsilon}\int_0^\alpha \nabla_{x_1}\big[Q'(x_1,h(x_1)+x_2)\phi(x_1)\big]\\ &= \int_{|x_1|<\epsilon}\int_0^\alpha \phi(x_1)\big[\nabla_{x_1}Q'(x_1,h(x_1)+x_2\\ &+\int_{|x_1|<\epsilon}\int_0^\alpha \nabla_{x_1}h(x_1)\partial_{x_2}Q'(x_1,h(x_1)+x_2)\big].\end{aligned}$$

Hence, dividing both sides by α, letting $\alpha\to 0$ and taking into account (3.1.15), it follows that

$$\int_{|x_1|<\epsilon} Q'\nabla_{x_1}\phi = 0, \quad \forall\,\phi\in C_0^\infty(V),$$

and, as a consequence, that Q' is constant on $\partial\Omega_i$, we denote by Q'_i. Note that, in general $Q'_i\neq Q'_j$ for $i\neq j$. □

Choosing now $\boldsymbol{\phi}=\boldsymbol{w}_k$ in (3.1.11), we have

$$1 = \lambda_0\int_\Omega \boldsymbol{w}'_k\cdot\nabla\boldsymbol{w}'_k\cdot\boldsymbol{u}_s + \frac{1}{J_k}\int_\Omega \boldsymbol{u}_s\cdot\nabla\boldsymbol{w}'_k\cdot\boldsymbol{u}_s. \tag{3.1.16}$$

Since

$$\left|\int_\Omega \boldsymbol{u}_s \cdot \nabla \boldsymbol{w}'_k \cdot \boldsymbol{u}_s\right| \le \|\boldsymbol{u}_s\|^2_{L^4(\Omega)} \|\nabla \boldsymbol{w}'_k\|_{L^2(\Omega)} \le \|\boldsymbol{u}_s\|^2_{L^4(\Omega)},$$

and

$$\int_\Omega (\boldsymbol{w}'_k \cdot \nabla \boldsymbol{w}'_k \cdot \boldsymbol{u}_s - \boldsymbol{w}' \cdot \nabla \boldsymbol{w}' \cdot \boldsymbol{u}_s) =$$
$$= \int_\Omega [(\boldsymbol{w}'_k - \boldsymbol{w}') \cdot \nabla \boldsymbol{w}'_k \cdot \boldsymbol{u}_s + \boldsymbol{w}' \cdot \nabla(\boldsymbol{w}'_k - \boldsymbol{w}') \cdot \boldsymbol{u}_s,$$

$$\left|\int_\Omega [(\boldsymbol{w}'_k - \boldsymbol{w}') \cdot \nabla \boldsymbol{w}'_k \cdot \boldsymbol{u}_s\right| \le \|\boldsymbol{w}'_k - \boldsymbol{w}'\|_{L^4(\Omega)} \|\boldsymbol{u}_s\|_{L^4(\Omega)},$$
$$\left|\int_\Omega [\boldsymbol{w}' \cdot \nabla(\boldsymbol{w}'_k - \boldsymbol{w}') \cdot \boldsymbol{u}_s\right| \le \|\nabla(\boldsymbol{w}'_k - \boldsymbol{w}')\|_{L^2(\Omega)} \|\boldsymbol{w}'\|_{L^4(\Omega)} \|\boldsymbol{u}_s\|_{L^4(\Omega)},$$

we can let $k \to +\infty$ in (3.1.16) to get

$$1 = \lambda_0 \int_\Omega \boldsymbol{w}' \cdot \nabla \boldsymbol{w}' \cdot \boldsymbol{u}_s. \tag{3.1.17}$$

In virtue of $(3.1.13)_1$, Lemma 3.1.1 and (2.12.2) we have

$$\lambda_0 \int_\Omega \boldsymbol{w}' \cdot \nabla \boldsymbol{w}' \cdot \boldsymbol{v} = -\int_\Omega \boldsymbol{v} \cdot \nabla Q' = \sum_{i=1}^m Q'_i \int_{\partial\Omega} \boldsymbol{v} \cdot \boldsymbol{n} = 0.$$

Then

$$\begin{aligned} 1 = \lambda_0 \int_\Omega \boldsymbol{w}' \cdot \nabla \boldsymbol{w}' \cdot \boldsymbol{\sigma} &= -\lambda_0 \sum_{i=1}^m \frac{\Phi_i}{2\pi} \int_\Omega \frac{\boldsymbol{w}' \cdot \nabla \boldsymbol{w}' \cdot (x - x_i)}{|x - x_i|^2} \\ &= \lambda_0 \sum_{i=1}^m \frac{\Phi_i}{2\pi} \int_\Omega (\log|x - x_i|) \nabla \boldsymbol{w}' \cdot \nabla \boldsymbol{w}'^{\mathrm{T}}. \end{aligned} \tag{3.1.18}$$

Since an easy computation shows that

$$\int_\Omega \nabla \boldsymbol{w}' \cdot \nabla \boldsymbol{w}'^{\mathrm{T}} = \int_\Omega (|\hat{\nabla} \boldsymbol{w}'|^2 - |\tilde{\nabla} \boldsymbol{w}'|^2)$$

and by the first Korn's inequality [22]

$$2\int_\Omega |\hat{\nabla}\boldsymbol{w}'|^2 = 2|\tilde{\nabla}\boldsymbol{w}'|^2 = \int_\Omega |\nabla \boldsymbol{w}'|^2,$$

(3.1.18) implies

$$1 - \Phi \leq 1 - \lambda_0\Phi < 0. \tag{3.1.19}$$

Since (3.1.19) contradicts our assumption (3.1.9), we see that (3.1.10) holds true. Then we can use Lemma 1.2.4 to get

Theorem 3.1.1 *Let Ω be a bounded domain of class C^2. Let $\boldsymbol{a} \in L^2(\partial\Omega)$ satisfies (3.1.3) and let $\boldsymbol{f} \in \mathcal{H}^1(\Omega)$. If (3.1.9) holds, then system (3.1.1) has a solution $(\boldsymbol{u}, p)$, with $\boldsymbol{u}$ fixed point of equation (3.1.4).*

As far as the regularity properties of $(\boldsymbol{u}, p)$ are concerned, we note that:

▶ From (3.1.7), written with $\lambda = 1$, it follows that $\boldsymbol{w}$ is a variational solution of the Stokes equations

$$\begin{aligned} \Delta\boldsymbol{w} - \operatorname{div}\boldsymbol{F} - \nabla Q = \mathbf{0} \quad &\text{in } C,\\ \operatorname{div}\boldsymbol{w} = 0 \quad &\text{in } C,\\ \boldsymbol{w} = \boldsymbol{\omega} \quad &\text{on } \partial C, \end{aligned} \tag{3.1.20}$$

where C is any disk such that $\overline{C} \subset \Omega$, $\boldsymbol{\omega} = \operatorname{tr}_{|\partial C}\boldsymbol{w} \in W^{1/2,2}(\partial S_R)$ and $\boldsymbol{F} = (\boldsymbol{u}_s + \boldsymbol{w}) \otimes (\boldsymbol{u}_s + \boldsymbol{w})$. Since $\boldsymbol{u}_s \in W^{2,1}_{\rm loc}(\Omega)$ and $\boldsymbol{w} \cdot \nabla\boldsymbol{w} \in L^q(\Omega)$, $q < 2$, we see that $\operatorname{div}\boldsymbol{F} \in L^q(C)$, $q < 2$ so that $\boldsymbol{w}$ can be written as sum of layer potentials over ∂C and the volume potential $\mathcal{V}[\operatorname{div}\boldsymbol{F}] \in W^{2,1}(C)$. Hence by the arbitrariness of C it follows that $\boldsymbol{u} \in W^{2,1}_{\rm loc}(\Omega)$. Analogously, one can show that $p \in W^{1,1}_{\rm loc}(\Omega)$ so that $(\boldsymbol{u}, p)$ satisfies equations $(3.1.1)_{1,2}$ almost everywhere in Ω. From what we said it is clear that $(\boldsymbol{u}, p)$ enjoys the same regularity properties as the solution $(\boldsymbol{u}_s, p_s)$ of the Stokes problem. In particular, for $\boldsymbol{f} = \mathbf{0}$ (say), $(\boldsymbol{u}, p) \in C^\infty(\Omega) \cap C^\infty(\Omega)$ and

- if $\boldsymbol{a} \in W^{1-1/q,q}(\partial\Omega)$, then

$$(\boldsymbol{u}, p) \in W^{1,q}(\Omega) \times L^q(\Omega).$$

- if $\boldsymbol{a} \in W^{1,q}(\partial\Omega)$, then

$$(\boldsymbol{u}, p) \in W^{1,2q}(\Omega) \cap .$$

- if $\boldsymbol{a} \in C(\partial\Omega)$, then

$$(\boldsymbol{u}, p) \in C^2(\Omega) \cap C(\overline{\Omega}).$$

then $(\boldsymbol{u}, p) \in C^\infty(\Omega) \times C^\infty(\Omega)$. ◁

▶ The boundary condition is assumed in the following way

$$\lim_{t\to 0^+} \boldsymbol{u}_s(\xi - t\boldsymbol{n}) = \boldsymbol{a}(\xi),$$

for almost all $\xi \in \partial\Omega$, and $\boldsymbol{w}_{|\partial\Omega} = \boldsymbol{0}$ in the sense of the trace in the Sobolev space $W_0^{1,2}(\Omega)$. If $\boldsymbol{a} \in L^q(\partial\Omega)$, $q > 2$, then $\boldsymbol{w} \in W_0^{1,t}(\Omega)$ for some $t > 2$ so that by Lemma 1.2.5 $\boldsymbol{w} \in C(\overline{\Omega})$ and

$$\lim_{t\to 0^+} \boldsymbol{u}(\xi - t\boldsymbol{n}) = \boldsymbol{a}(\xi), \tag{3.1.21}$$

for almost all $\xi \in \partial\Omega$. Moreover, if $\boldsymbol{a} \in C(\partial\Omega)$, then (3.1.21) holds for all $\xi \in \partial\Omega$.

◁

• *Remark 3.1.1*

Existence of a solution of system (3.1.1) under the only hypothesis (3.1.3) is easily established, provided $\|\boldsymbol{u}_s\|_{L^q(\Omega)}$ ($q > 2$) is suitably small, as follows. Using the above notation, let us observe that the operator $\mathcal{C}$ maps boundedly $L^q_\sigma(\Omega)$ into $W^{1,q/2}_\sigma(\Omega)(\supset L^q_\sigma(\Omega))$ and by standard estimates we see that there is a positive constant c_0, depending on Ω, such that

$$\|\mathcal{N}[\boldsymbol{u}]\|_{L^q(\Omega)} \leq c_0 \|\boldsymbol{u}\|^2_{L^q(\Omega)}.$$

Therefore, in virtue of Lemma 1.2.3, the map (3.1.4) is a contraction in the ball

$$\left\{\boldsymbol{u} \in L^q_\sigma(\Omega) : \|\boldsymbol{u}\|_{L^q(\Omega)}\right\} < \frac{1}{2c_0}$$

provided

$$\|\boldsymbol{u}_s\|_{L^q(\Omega)} < \frac{1}{4c_0}, \tag{3.1.22}$$

where it has a fixed point, which is (with the pressure field (3.1.5)) a solution of system (3.1.1). Observe that, if

$$\boldsymbol{a} \in L^{q/2}(\partial\Omega), \quad \boldsymbol{f} \in \mathcal{H}^1(\Omega),$$

in virtue of the estimate (2.6.7), then there is a constant c_ℓ (depending on Ω) such that

$$\|\boldsymbol{u}\|_{L^q(\Omega)} \leq c_\ell \left\{\|\boldsymbol{a}\|_{L^{q/2}(\partial\Omega)} + \|\boldsymbol{f}\|_{\mathcal{H}^1(\Omega)}\right\},$$

and (3.1.22) is satisfied for

$$\|\boldsymbol{a}\|_{L^{q/2}(\partial\Omega)} + \|\boldsymbol{f}\|_{\mathcal{H}^1(\Omega)} < \frac{1}{4c_0c_\ell}.$$

$\diamondsuit$

3.2 Uniqueness of a very weak solution

We aim now at discussing the important problem of uniqueness of the very weak solution of the Navier–Stokes problem whose existence is guaranteed by Theorem 3.1.1. We want to select a uniqueness class, determined under restrictions only on the data and where the solutions enjoys the same properties as the ones specified in Theorem 3.1.1.

Let $(\boldsymbol{u}, p)$, $(\boldsymbol{u}+\boldsymbol{w}, p+Q) \in L^4_\sigma(\Omega)$ are two solutions of system (3.1.1) corresponding to $\boldsymbol{a}$ and $\boldsymbol{f}$, with $\boldsymbol{u}$ fixed point of equation (3.1.4). Then the pair $(\boldsymbol{w}, Q)$ satisfies the equations

$$\begin{aligned} \Delta\boldsymbol{w} - (\boldsymbol{u}+\boldsymbol{w})\cdot\nabla\boldsymbol{w} - \boldsymbol{w}\cdot\nabla\boldsymbol{u} - \nabla Q = \boldsymbol{0} \quad & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{w} = 0 \quad & \text{in } \Omega, \\ \boldsymbol{w} = \boldsymbol{0} \quad & \text{on } \partial\Omega. \end{aligned} \tag{3.2.1}$$

In virtue of the properties of solutions of the Stokes problem, $\boldsymbol{w} \in W^{1,2}_{\sigma,0}(\Omega)$ and $Q \in L^2(\Omega)$. Let $\delta(x)$ be the distance of x from $\partial\Omega$ and let

$$g(x) = \begin{cases} 0, & \delta(x) < \delta_0, \\ 1, & \delta(x) > 2\delta_0, \\ \delta_0^{-1}(\delta(x)-\delta_0), & \delta_0 \le \delta(x) \le 2\delta_0, \end{cases} \tag{3.2.2}$$

with δ_0 small positive number. Then an integration by parts gives

$$\begin{aligned} \int_\Omega g|\nabla\boldsymbol{w}|^2 = \frac{1}{2}\int_\Omega |\boldsymbol{w}|^2(\boldsymbol{u}+\boldsymbol{w})\cdot\nabla g + \int_\Omega p\boldsymbol{w}\cdot\nabla g \\ - \int_\Omega \nabla g\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{w} + \int_\Omega (\boldsymbol{u}\cdot\boldsymbol{w})\boldsymbol{w}\cdot\nabla g \\ + \int_\Omega g\boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{u}. \end{aligned} \tag{3.2.3}$$

Set

$$T_{\delta_0} = \{x \in \Omega : \delta_0 < \delta(x) < 2\delta_0\}.$$

By Hölder's inequality, the properties of the function g and Lemma 1.2.11, we have

$$\left|\int_\Omega |\boldsymbol{w}|^2\boldsymbol{w}\cdot\nabla g\right| \le \left\{\int_{T_{\delta_0}} |\boldsymbol{w}|^4 \int_\Omega \frac{|\boldsymbol{w}|^2}{\delta^2}\right\}^{1/2} \le c\|\boldsymbol{w}\|^2_{L^4(T_{\delta_0})}\|\nabla\boldsymbol{w}\|_{L^2(\Omega)},$$

$$\left|\int_\Omega p\boldsymbol{w}\cdot\nabla g\right| \le \left\{\int_{T_{\delta_0}} p^2 \int_\Omega \frac{|\boldsymbol{w}|^2}{\delta^2}\right\}^{1/2} \le c\|p\|_{L^2(T_{\delta_0})}\|\nabla\boldsymbol{w}\|_{L^2(\Omega)},$$

$$\begin{aligned}\left|\int_\Omega \nabla g\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{w}\right| &\le \left\{\int_{T_{\delta_0}} |\nabla\boldsymbol{w}|^2 \int_\Omega \frac{|\boldsymbol{w}|^2}{\delta^2}\right\}^{1/2}\\ &\le c\|\nabla\boldsymbol{w}\|_{L^2(T_{\delta_0})}\|\nabla\boldsymbol{w}\|_{L^2(\Omega)},\end{aligned}$$

$$\begin{aligned}\left|\int_\Omega (\boldsymbol{u}\cdot\boldsymbol{w})\boldsymbol{w}\cdot\nabla g\right| &\le \left\{\int_{T_{\delta_0}} |\boldsymbol{u}|^4 \int_{T_{\delta_0}} |\boldsymbol{w}|^4\right\}^{1/4} \left\{\int_\Omega \frac{|\boldsymbol{w}|^2}{\delta^2}\right\}^{1/2}\\ &\le c\|\boldsymbol{u}\|_{L^4(T_{\delta_0})}\|\boldsymbol{w}\|_{L^4(T_{\delta_0})}\|\nabla\boldsymbol{w}\|_{L^2(\Omega)}.\end{aligned}$$

Therefore, letting $\delta_0 \to 0$ in (3.2.3) yields

$$\int_\Omega |\nabla\boldsymbol{w}|^2 = \int_\Omega \boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{u}. \tag{3.2.4}$$

By Lemma 1.2.10

$$\|\boldsymbol{w}\|_{L^4(\Omega)} \le c_\sigma\|\nabla\boldsymbol{w}\|_{L^2(\Omega)}, \tag{3.2.5}$$

so that

$$\begin{aligned}\left|\int_\Omega \boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{u}\right| &\le \|\boldsymbol{w}\|_{L^4(\Omega)}\|\boldsymbol{u}\|_{L^4(\Omega)}\|\nabla\boldsymbol{w}\|_{L^2(\Omega)}\\ &\le c_\sigma\|\boldsymbol{u}\|_{L^4(\Omega)}\|\nabla\boldsymbol{w}\|^2_{L^2(\Omega)}\end{aligned}$$

and (3.2.4) yields

$$(1 - c_\sigma\|\boldsymbol{u}\|_{L^4(\Omega)})\int_\Omega |\nabla\boldsymbol{w}|^2 \le 0. \tag{3.2.6}$$

Therefore, if

$$c_\sigma\|\boldsymbol{u}\|_{L^4(\Omega)} < 1 \tag{3.2.7}$$

then $\boldsymbol{w} = 0$. Let us look for a condition on $\boldsymbol{a}$ and $\boldsymbol{f}$ assuring that (3.2.7) is satisfied. To this end consider system (3.1.7). Multiplying scalarly $(3.1.7)_1$ by $g\boldsymbol{w}$, where g is the function (3.2.2) and proceeding as we did to get (3.2.4), we arrive at

$$\int_\Omega |\nabla \boldsymbol{w}|^2 = \int_\Omega (\boldsymbol{w} + \boldsymbol{u}_s) \cdot \nabla \boldsymbol{w} \cdot \boldsymbol{u}_s, \tag{3.2.8}$$

Since

$$\left| \int_\Omega \boldsymbol{w} \cdot \nabla \boldsymbol{w} \cdot \boldsymbol{u}_s \right| \leq c_\sigma \|\boldsymbol{u}_s\|_{L^4(\Omega)} \|\nabla \boldsymbol{w}\|^2_{L^2(\Omega)},$$

$$\left| \int_\Omega \boldsymbol{u}_s \cdot \nabla \boldsymbol{w} \cdot \boldsymbol{u}_s \right| \leq \|\boldsymbol{u}_s\|^2_{L^4(\Omega)} \|\nabla \boldsymbol{w}\|_{L^2(\Omega)},$$

from (3.2.6) it follows

$$(1 - c_\sigma \|\boldsymbol{u}_s\|_{L^4(\Omega)}) \|\nabla \boldsymbol{w}\|_{L^2(\Omega)} \leq \|\boldsymbol{u}_s\|^2_{L^4(\Omega)}. \tag{3.2.9}$$

Since by Minkowski's inequality

$$\|\boldsymbol{u}\|_{L^4(\Omega)} \leq \|\boldsymbol{u}_s\|_{L^4(\Omega)} + \|\boldsymbol{w}\|_{L^4(\Omega)},$$

from (3.2.5), (3.2.9) and (3.2.6) it follows

Theorem 3.2.1 *If the solution $\boldsymbol{u}_s$ of the Stokes problem corresponding to $\boldsymbol{a} \in L^2(\partial\Omega)$ and $\boldsymbol{f} \in \mathcal{H}^1(\Omega)$ satisfy*

$$c_\sigma \|\boldsymbol{u}_s\|_{L^4(\Omega)} < 1, \tag{3.2.10}$$

and

$$c_\sigma \|\boldsymbol{u}_s\|_{L^4(\Omega)} + \frac{c_\sigma^2 \|\boldsymbol{u}_s\|^2_{L^4(\Omega)}}{(1 - c_\sigma \|\boldsymbol{u}_s\|_{L^4(\Omega)})} < 1, \tag{3.2.11}$$

where c_σ is the constant appearing in (3.2.5), then system (3.1.1) has a unique very weak solution[1].

[1] Recall the pressure field is normalized by (2.3.2).

3.3 The Amick theorem for very weak solutions

As we said in the introduction, an outstanding open problem in the theory of the steady Navier–Stokes equations is to prove (or disprove) existence of a solution of system (3.1.1) in domains with nonconnected boundaries under the only (necessary) assumption

$$\int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{n} = 0. \tag{3.3.1}$$

To the best of our knowledge the only result which makes use only of (3.3.1) is due to C.J. Amick (1984) [1]. He proved existence of a variational solution under suitable hypotheses of symmetry on the domain and the data we are specifying. The purpose of this section is to extend Amick's theorem to very weak solutions [38].

Let $\partial\Omega$ be symmetric with respect to the x_1–axis, *i.e.*,

$$(\xi_1, \xi_2) \in \partial\Omega \Longrightarrow (\xi_1, -\xi_2) \in \partial\Omega$$

and

$$\{x_2 = 0\} \cap \partial\Omega_i \neq \varnothing, \quad i = 0, 1, \dots, m.$$

Let $\boldsymbol{a} \in L^2(\Omega)$ and $\boldsymbol{f} \in \mathcal{H}^1(\Omega)$ be symmetric, *i.e.*,

$$\begin{aligned} a_1(\xi_1, \xi_2) &= a_1(\xi_1, -\xi_2), \\ a_2(\xi_1, \xi_2) &= -a_2(\xi_1, -\xi_2) \end{aligned}$$

for almost all $\xi \in \partial\Omega$ and

$$\begin{aligned} f_1(x_1, x_2) &= f_1(x_1, -x_2), \\ f_2(x_1, x_2) &= -f_2(x_1, -x_2) \end{aligned}$$

for almost all $x \in \Omega$. We say that a solution $(\boldsymbol{u}, p)$ of system (3.1.1) is *symmetric* if

$$\begin{aligned} u_1(x_1, x_2) &= u_1(x_1, -x_2), \\ u_2(x_1, x_2) &= -u_2(x_1, -x_2), \\ p(x_1, x_2) &= p(x_1, -x_2). \end{aligned}$$

The linear subspace of $W^{1,2}(\Omega)$ (say) of symmetric functions are closed and, as a consequence, Banach spaces. The proof of Theorem 3.1.1 goes unchanged in these spaces until (3.1.17). Of course, if we are able to show that the constant values of the pressure on $\partial\Omega_i$ are all the same, then (3.1.17) leads to a contradiction and so to the desired existence of a *symmetric solution* $(\boldsymbol{u}, p)$.

We premise the following lemma.

Lemma 3.3.1 *Let* $(\boldsymbol{w}', Q') \in W^{1,2}_{\sigma,0}(\Omega) \times L^q(\Omega)$ $(q < 2)$ *be a solution of system* (3.1.13). *Then* $Q' \in W^{2,1}(\Omega) \cap C(\overline{\Omega})$.

PROOF - Recall that Q' is a solution of the Poisson equation

$$\Delta Q' + \lambda_0 \operatorname{div}(\boldsymbol{w} \cdot \nabla \boldsymbol{w}) = 0 \tag{3.3.2}$$

constant on every $\partial\Omega_i$. The volume potential

$$\mathcal{Q}'[\boldsymbol{w}] = -\frac{\lambda_0}{2\pi} \int_{\partial\Omega} (\log|x-y|) \operatorname{div}(\boldsymbol{w} \cdot \nabla \boldsymbol{w})(y) \, \mathrm{da}_y$$

belongs in $W^{2,1}(\mathbb{R}^2)$ so that, in particular, it is a continuous function in $\mathbb{R}^2$. Then Q' can be expressed as $\mathcal{Q}[\boldsymbol{w}]$ plus harmonic layer potentials with regular densities. Hence it follows that $Q' \in W^{2,1}(\Omega) \cap C(\overline{\Omega})$. $\square$

Near the points where $\partial\Omega$ intersects the x_1–axis, the curves $\partial\Omega_i$ and $\partial\Omega_j$ (say) can be expressed as the graphs of two functions ϕ and ψ of class C^2, $\{\varphi(x_2), x_2) : x_2 \in (-\delta, \delta\}$ and $\{\psi(x_2), x_2(: x_2 \in (-\delta, \delta)\}$ and $\{\psi(x_2), x_2) : x_2 \in (-\delta, \delta\}$. Let

$$A = \{(x_1, x_2) : x_2 \in (-\delta_0, \delta_0),\ x_1 \in (\varphi(x_2), \psi(x_2)\},$$

The equation

$$\partial_1 Q + \frac{\lambda_0}{2} \partial_1 {w_1'}^2 = -\lambda_0 w_2' \partial_2 w_1' \tag{3.3.3}$$

is satisfied almost everywhere in Ω. On the other hand by Gagliardo's theorem [28] $\partial_1 Q$ is summable on every $\{x_2 = c\} \cap \Omega$. Therefore, integrating (3.3.3) on A, we have

$$\int_{-\delta_0}^{\delta_0} \mathrm{d}x_2 \int_{\varphi(x_2)}^{\psi(x_2)} \partial_1 (Q + \frac{\lambda_0}{2} {w_1'}^2)(x_1, x_2) \mathrm{d}x_2 = -\lambda_0 \int_A w_2' \partial_2 w_1'.$$

Since $Q + \frac{1}{2}u_1^2 \in W^{1,q}(\Omega)$, $q < 2$, by Fubini's theorem the function

$$\partial_1 (Q + \frac{\lambda_0}{2} {w'_1}^2)(x_1, x_2)$$

is summable for almost all $x_2 \in (-\delta, \delta)$. Therefore, we have

$$\begin{aligned} \int_{-\delta_0}^{\delta_0} [Q(\psi(x_2), x_2) - Q(\varphi(x_2), x_2)] \mathrm{d}x_2 &= 2\delta_0 (Q_j - Q_i) \\ &= -\lambda_0 \int_A w'_2 \partial_2 w'_1. \end{aligned} \tag{3.3.4}$$

Since by Lemma 1.2.11

$$\frac{1}{\delta_0^2} \int_A w_2^2 \le c \int_A \frac{w_2^2}{x_2^2} \le c \int_A |\nabla \boldsymbol{w}|^2,$$

from (3.3.4) it follows

$$\begin{aligned} |Q_j - Q_i| &\le \frac{c}{\delta_0} \left| \int_A w_2 \partial_2 w_1 \right| \le c \left\{ \int_A \frac{w_2^2}{x_2^2} \int_A |\nabla \boldsymbol{w}|^2 \right\}^{1/2} \\ &\le c \int_A |\nabla \boldsymbol{w}|^2. \end{aligned}$$

Then, letting $\delta_0 \to 0$, we see that $Q_i = Q_j$. Hence it follows

Theorem 3.3.1 *Let Ω be symmetric with respect to the x_1–axis and let $\{x_2 = 0\} \cap \partial\Omega_i \neq \varnothing$, $i = 0, 1, \dots, m$.. If $\boldsymbol{a} \in L^2(\partial\Omega)$, $\boldsymbol{f} \in \mathcal{H}^1(\Omega)$ are symmetric and $\boldsymbol{a}$ satisfy (3.3.1), then system (3.1.1) has a very weak solution $(\boldsymbol{u}, p)$. Moreover, if $\boldsymbol{u}_s$ satisfies (3.2.10) and (3.2.11), then $(\boldsymbol{u}, p)$ is unique.*

3.4 A mixed problem

Let Ω be the domain of $\mathbb{R}^2$ with boundary

$$\partial\Omega = \Gamma \cup \Sigma \tag{3.4.1}$$

where Γ is the union of Lipschitz curves and Σ is the union of a finite number of segments of the x_1-axis (see figure 3.4.1).

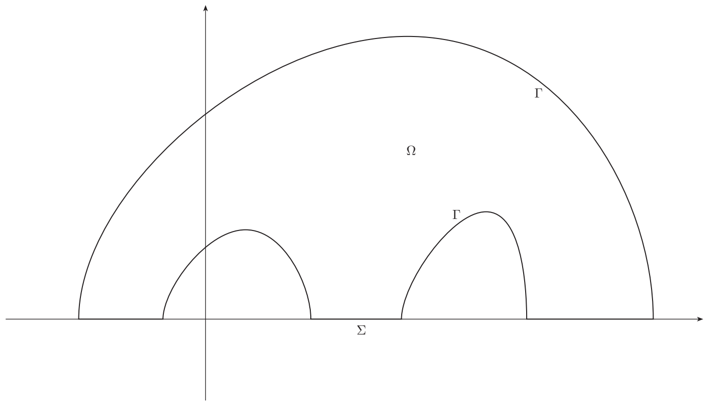

Figure 3.4.1: *the domain Omega.*

In this section we shall consider the mixed boundary value problem

$$\begin{aligned} \Delta \boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla p = \boldsymbol{f} \quad & \text{in } \Omega,\\ \operatorname{div}\boldsymbol{u} = 0 \quad & \text{in } \Omega,\\ \boldsymbol{u} = \boldsymbol{a} \quad & \text{on } \Gamma,\\ u_2 = 0 \quad & \text{on } \Sigma,\\ \partial_1 u_2 + \partial_2 u_1 = 0 \quad & \text{on } \Sigma. \end{aligned} \tag{3.4.2}$$

Note that $(3.4.2)_{4,5}$ require respectively that the normal component of the velocity and the tangential component of the stress vanish on $\partial\Sigma$ (slip conditions). The field $\boldsymbol{a}$ must satisfy the compatibility condition

$$\int_\Gamma \boldsymbol{a}\cdot\boldsymbol{n} = 0, \tag{3.4.3}$$

where $\boldsymbol{n}$ denotes the outward unit normal to $\partial\Omega$.

Let $\mathcal{M}$ be the mirror transformation $(x_1, x_2) \to (x_1, -x_2)$ and let $\tilde{\Omega}$ be the bounded domain with boundary

$$\overline{\Gamma} \cup \mathcal{M}(\Gamma) \tag{3.4.4}$$

(see figure 3.4.2).

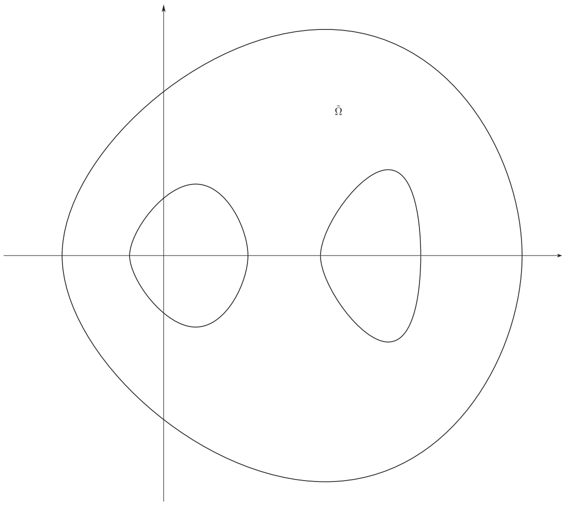

Figure 3.4.2: *the symmetric domain* $\tilde{\Omega}$.

Denote by $\tilde{\boldsymbol{a}} = (\tilde{a}_1, \tilde{a}_2)$ and $\tilde{\boldsymbol{f}} = (\tilde{f}_1, \tilde{f}_2)$ the extension of $\boldsymbol{a}$ and $\boldsymbol{f}$ in $\tilde{\Omega}$ defined respectively by

$$\tilde{f}_1(x_1, -x_2) = f_1(x_1, x_2), \quad \tilde{f}_2(x_1, -x_2) = -f_2(x_1, x_2)$$

and

$$\tilde{a}_1(\xi_1, -\xi_2) = a_1(\xi_1, \xi_2), \quad \tilde{a}_2(\xi_1, -\xi_2) = -a_2(\xi_1, \xi_2).$$

As a simple consequence of Theorem 3.3.1, we have

Theorem 3.4.1 *If* Ω *is of class* C^2, $\boldsymbol{a} \in L^2(\partial\Omega)$ *satisfies* (3.4.3) *and* $\boldsymbol{f} \in \mathcal{H}^1(\Omega)$, *then system* (3.4.2) *has a solution.*

Proof - Since $\tilde{\Omega}$, $\tilde{\boldsymbol{a}}$ and $\tilde{\boldsymbol{f}}$ satisfy the hypotheses of Theorem 3.3.1, system (3.1.1) has a a symmetric solution $(\tilde{\boldsymbol{u}}, \tilde{p})$. Of course, the restriction of $(\tilde{\boldsymbol{u}}, \tilde{p})$ to Ω satisfies $(3.4.2)_{1,2,3}$ and by symmetry $(3.4.2)_{4,5}$..

□

• *Remark 3.4.1*

If $\boldsymbol{a}$ and $\boldsymbol{f}$ are more regular, then more regular is the corresponding solution. In particular, if $\boldsymbol{a} \in C(\partial\Gamma)$ and $a_2 = 0$ at the end points of the segment of Σ, then $\boldsymbol{u}$ is continuous in $\overline{\Omega}$. For more complete regularity results concerning the solution and its uniqueness, we quote [38]. ♢

3.5 A maximum modulus estimate

There is another approach to prove the uniform estimate (3.1.10) assuring existence of a solution of system (3.1.1) which comes back again to the classical paper of J. Leray [25], based on the use of a suitable cut–off function on the boundary (see [11], [13] Ch. VIII, in the variational context). A value of this method is to furnish a priori estimate for the solution, like a maximum modulus theorem, as derived in [43] for $\boldsymbol{f} = \mathbf{0}$ and regular $\boldsymbol{a}$ under assumption (8) (in the introduction):

$$\|\boldsymbol{u}\|_{C(\overline{\Omega})} \le c\big\|\boldsymbol{a}\|_{C(\partial\Omega)} + \|\boldsymbol{a}\|^2_{C(\partial\Omega)}\big\} \tag{3.5.1}$$

In this section, we show as we can apply this argument in the more general context of boundary data $\boldsymbol{a} \in L^2(\partial\Omega)$ and under hypothesis (3.1.9) on the flux.

Let $(\boldsymbol{u}_s = \boldsymbol{v} + \boldsymbol{\sigma}, p_s)$ be the solution of the Stokes system corresponding to $\boldsymbol{f} = \mathbf{0}$ and $\boldsymbol{a} \in L^2(\partial\Omega)$ satisfying (3.1.3) (see Remark 2.12.2). Following [14], denote by γ the stream function of $\boldsymbol{v}$ defined in Ω by the line integral

$$\gamma(x) = \int_{x_0}^{x} (v_1 \mathrm{d}x_2 - v_2 \mathrm{d}x_1),$$

with x_0 fixed point of Ω. Since

$$\int_{\partial\Omega_i} \boldsymbol{v} \cdot \boldsymbol{n} = 0,$$

γ is singlevalued and it holds

$$\boldsymbol{v} = \operatorname{curl} \boldsymbol{\gamma} \quad \text{in } \Omega, \tag{3.5.2}$$

with $\boldsymbol{\gamma} = \gamma \boldsymbol{e}_3$. Of course, $\boldsymbol{\gamma}$ is defined within an additive constant we can choose in such a way that by (1.2.7), (2.6.7) and Lemma 1.2.5

$$\|\boldsymbol{\gamma}\|_{L^\infty(\Omega)} \le c\|\boldsymbol{\gamma}\|_{W^{1,4}(\Omega)} \le c\|\boldsymbol{v}\|_{L^4(\Omega)} \le c\|\boldsymbol{a}\|_{L^2(\partial\Omega)}. \tag{3.5.3}$$

Let δ_0 be a (small) positive number and let w be a C^∞ function in $\mathbb{R}$, vanishing in $(-\infty, 0]$ and equal to 1 in $[1, +\infty)$. For $0 < \delta_0 \ll 1$, the function

$$g_{\delta_0}(x) = w\Big(\alpha^{-1}\Big(\log\log\frac{1}{\delta(x)} - \log\log\frac{1}{\delta_0}\Big)\Big) \tag{3.5.4}$$

vanishes in $\Omega \setminus \Omega(\delta_0)$, is equal to 1 in $\Omega(\delta_0^{e^{-\alpha}})$ and

$$\nabla g_{\delta_0}(x) = -\frac{w'}{\delta(x)\log\delta(x)}\nabla\delta(x).$$

Of course

$$\lim_{\delta_0\to 0}\sup|\delta(x)\nabla g_{\delta_0}(x)| = 0. \tag{3.5.5}$$

Since $\partial\Omega$ is of class C^2, then g_δ is of class C^2 in $\mathbb{R}^2$ and

$$\nabla g_{\delta_0} = 0 \quad \text{in } T_{\delta_0} = \Omega(\delta_0)\setminus\Omega(\delta_0^{e^{-\alpha}}),$$

The field

$$\boldsymbol{h} = \mathrm{curl}(g_{\delta_0}\boldsymbol{\gamma}) + \boldsymbol{\sigma}, \tag{3.5.6}$$

is equal to $\boldsymbol{u}_s$ in $\Omega(\delta_0)$ is a solution of the equations

$$\begin{aligned} \Delta\boldsymbol{h} - \nabla Q' = \boldsymbol{\phi} \quad &\text{in } \Omega,\\ \mathrm{div}\,\boldsymbol{w} = 0 \quad &\text{in } \Omega, \end{aligned} \tag{3.5.7}$$

for some pressure field Q', with $\boldsymbol{\phi} \in C_0^2(T(\delta_0))$. Moreover, $\boldsymbol{h}$ takes the value $\boldsymbol{a}$ on the boundary in the sense of (3.1.21) for almost all $\xi \in \partial\Omega$.

Let $VMO(\mathbb{R}^2)$[2] be the space obtained by completing $C_0^\infty(\mathbb{R}^2)$ with respect to the seminorm [6]

$$\sup_{\{x\in\mathbb{R}^2, R>0\}}\frac{1}{|S_R(x)|}\int_{S_R(x)}|\varphi - \varphi_{S_R(x)}|,$$

with $S_R(x) = \{y\,|y-x| < R\}$. It is well–known that

$$[VMO(\mathbb{R}^2)]^* = \mathcal{H}^1(\mathbb{R}^2).$$

Therefore, since

$$[W^{1,2}(\mathbb{R}^2) \hookrightarrow VMO(\mathbb{R}^2),$$

it holds

$$\mathcal{H}^1(\mathbb{R}^2) \hookrightarrow [W^{1,2}(\mathbb{R}^2)]^*.$$

[2]This is the acronym of *Vanishing Mean Oscillation*.

In particular, if Ω is a bounded domain and $\boldsymbol{f} \in \mathcal{H}^1(\Omega)$, then

$$|\langle \boldsymbol{f}, \boldsymbol{w}\rangle| \leq c\|\boldsymbol{f}\|_{\mathcal{H}^1(\Omega)}\|\nabla \boldsymbol{w}\|_{L^2(\Omega)} \tag{3.5.8}$$

for all $\boldsymbol{w} \in D_0^{1,2}(\Omega)$.

Let $\boldsymbol{f} \in \mathcal{H}^1(\Omega)$ and let $\boldsymbol{w} \in W_{\sigma,0}^{1,2}(\Omega)$ be a variational solution of the equations

$$\begin{aligned} \Delta \boldsymbol{w} - (\boldsymbol{h} + \boldsymbol{w}) \cdot \nabla(\boldsymbol{h} + \boldsymbol{w}) - \nabla Q - \boldsymbol{\phi} = \boldsymbol{f} \quad & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{w} = 0 \quad & \text{in } \Omega, \\ \boldsymbol{w} = \boldsymbol{0} \quad & \text{on } \partial\Omega. \end{aligned} \tag{3.5.9}$$

Then, the usual integration by parts gives

$$\begin{aligned} \int_\Omega |\nabla \boldsymbol{w}|^2 = \int_\Omega \boldsymbol{w} \cdot \nabla \boldsymbol{w} \cdot \boldsymbol{h} + \int_\Omega \boldsymbol{h} \cdot \nabla \boldsymbol{w} \cdot \boldsymbol{h} \\ - \int_\Omega \boldsymbol{\phi} \cdot \boldsymbol{w} - \langle \boldsymbol{f}, \boldsymbol{w}\rangle. \end{aligned} \tag{3.5.10}$$

By proceeding as we did in the proof Theorem 3.1.1, we see that

$$\begin{aligned} &\left|\int_\Omega \boldsymbol{w} \cdot \nabla \boldsymbol{w} \cdot \boldsymbol{\sigma}\right| \leq \Phi \int_\Omega |\nabla \boldsymbol{w}|^2 \\ &\left|\int_\Omega \boldsymbol{h} \cdot \nabla \boldsymbol{w} \cdot \boldsymbol{h}\right| \leq \|\boldsymbol{h}\|_{L^4(\Omega)}^2 \|\nabla \boldsymbol{w}\|_{L^2(\Omega)} \\ &\left|\int_\Omega \boldsymbol{\phi} \cdot \boldsymbol{w}\right| \leq c\|\boldsymbol{\phi}\|_{L^2(\Omega)}\|\nabla \boldsymbol{w}\|_{L^2(\Omega)} \leq c\|\boldsymbol{h}\|_{L^4(\Omega)}\|\nabla \boldsymbol{w}\|_{L^2(\Omega)}. \end{aligned}$$

Therefore, (3.5.10) and (3.5.8) implies

$$\begin{aligned} (1-\Phi)\int_\Omega |\nabla \boldsymbol{w}|^2 \leq c\left\{\|\boldsymbol{h}\|_{L^4(\Omega)}\| + \|\boldsymbol{h}\|_{L^4(\Omega)}^2 + \|\boldsymbol{f}\|_{\mathcal{H}^1(\Omega)}\right\}\|\nabla \boldsymbol{w}\|_{L^2(\Omega)} \\ + \left|\int_\Omega \boldsymbol{w} \cdot \nabla \boldsymbol{w} \cdot \operatorname{curl}(g_{\delta_0}\boldsymbol{\gamma})\right|. \end{aligned} \tag{3.5.11}$$

It remains to majorize the last integral in (3.5.11). To do this note that by (3.5.3)

$$\left|\int_{\Omega} \boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot\operatorname{curl}(g_{\delta_0}\boldsymbol{\gamma})\right| \le c\int_{\Omega(2\delta_0)} |\boldsymbol{w}||\nabla\boldsymbol{w}||\boldsymbol{v}| + c\int_{T(\delta_0)} |\nabla g_{\delta_0}||\boldsymbol{w}||\nabla\boldsymbol{w}| = \mathcal{J}_1 + \mathcal{J}_2.$$

Now, by Hölder's inequality (3.5.3) and Lemmas 1.2.10, 1.2.11

$$\mathcal{J}_1 \le \|\boldsymbol{v}\|_{L^4(\Omega(2\delta_0))}\|\boldsymbol{w}\|_{L^4(\Omega)}\|\nabla\boldsymbol{w}\|_{L^2(\Omega)} \le c\|\boldsymbol{v}\|_{L^4(\Omega(2\delta_0))}\|\nabla\boldsymbol{w}\|^2_{L^2(\Omega)}$$

and

$$\begin{aligned}\mathcal{J}_2 &\le c\sup|\delta(x)\nabla g_{\delta_0}(x)|\int_{T(\delta_0)} \delta_0^{-1}|\boldsymbol{w}||\nabla\boldsymbol{w}| \\ &\le c\sup|\delta(x)\nabla g_{\delta_0}(x)|\left\{\int_{\Omega}\frac{|\boldsymbol{w}|^2}{\delta_0^2}\int_{\Omega}|\nabla\boldsymbol{w}|^2\right\}^{1/2} \\ &\le c\sup|\delta(x)\nabla g_{\delta_0}(x)|\int_{\Omega}|\nabla\boldsymbol{w}|^2\end{aligned}$$

Taking into account (3.5.5) and

$$\lim_{\delta_0\to 0}\|\boldsymbol{v}\|_{L^4(\Omega(2\delta_0))} = 0,$$

we can choose δ_0 such that

$$\|\boldsymbol{v}\|_{L^4(\Omega(2\delta_0))} + \sup|\delta(x)\nabla g_{\delta_0}(x)| < 1 - \Phi.$$

Therefore, (3.5.11) yields

$$\|\nabla\boldsymbol{w}\|_{L^2(\Omega)} \le c(\Omega,\delta_0)\left\{\|\boldsymbol{h}\|_{L^4(\Omega)} + \|\boldsymbol{h}\|^2_{L^4(\Omega)} + \|\boldsymbol{f}\|_{\mathcal{H}^1(\Omega)}\right\}. \quad (3.5.12)$$

From (3.5.12) it follows

Theorem 3.5.1 *Let Ω be a bounded domain of class C^2, let $\boldsymbol{f} \in \mathcal{H}^1(\Omega)$ and let $\boldsymbol{a} \in L^2(\partial\Omega)$ satisfy*

$$\int_{\partial\Omega} \boldsymbol{a}\cdot\boldsymbol{n} = 0.$$

If (3.1.9) *holds, then system* (3.1.1) *has a very weak solution expressed by* $\boldsymbol{w}+\boldsymbol{h}$*, with* $\boldsymbol{h}$ *given by* (3.5.6) *and* $\boldsymbol{w} \in W^{1,2}_{\sigma,0}(\Omega)$ *variational solution of equations* (3.5.9)*. Moreover,*

$$\|\boldsymbol{u}\|_{L^4(\Omega)} \leq c\big\{\|\boldsymbol{a}\|_{L^2(\partial\Omega)} + \|\boldsymbol{a}\|^2_{L^2(\partial\Omega)} + \|\boldsymbol{f}\|_{\mathcal{H}^1(\Omega)}\big\} \tag{3.5.13}$$

and if $\boldsymbol{a} \in C(\partial\Omega)$*, then*

$$\|\boldsymbol{u}\|_{C(\overline{\Omega})} \leq c\big\{\|\boldsymbol{a}\|_{C(\partial\Omega)} + \|\boldsymbol{a}\|^2_{C(\partial\Omega)} + \|\boldsymbol{f}\|_{\mathcal{H}^1(\Omega)}\big\}, \tag{3.5.14}$$

with c *independent of* $\boldsymbol{a}$*.*

Proof - Existence and estimate (3.5.13) follows from (3.5.12). If $\boldsymbol{a} \in C(\partial\Omega)$, then $\boldsymbol{h} \in C(\overline{\Omega})$ and (3.5.21) follow from the results about Stokes equations. □

Now, we aim at proving Theorem 3.3.1 by the above method. To this end we follows an argument of H. Morimoto [29]. Assume that Ω and $\boldsymbol{a}$ satisfy the symmetric assumptions in Theorem 3.3.1. Let $\boldsymbol{h}$ be a symmetric divergence free extension of $\boldsymbol{a}$ in Ω, expressed (say) by (3.5.6), and set

$$\Omega_+ = \{x \in \Omega : x_2 \geq 0\}.$$

Since Ω_+ is simple connected, we can write

$$\boldsymbol{h} = \operatorname{curl}\boldsymbol{\varphi} \quad \text{in } \Omega_+,$$

where $\boldsymbol{\varphi} \in C^\infty(\Omega_+) \cap W^{1,4}(\Omega_+)$ enjoys the same properties as the field $\boldsymbol{\gamma}$ in (3.5.2). Of course,

$$\boldsymbol{z} = \operatorname{curl}(g_{\delta_0}\boldsymbol{\varphi}) \tag{3.5.15}$$

with g_{δ_0} defined by (3.5.6), is a divergence free extension of

$$\tilde{\boldsymbol{h}} = \operatorname{tr}_{|\partial\Omega_+}\boldsymbol{h},$$

and

$$\begin{aligned} &\tilde{\boldsymbol{h}} = \boldsymbol{a} \quad \text{on } \partial\Omega \cap \partial\Omega_+,\\ &h_2 = 0, \quad \text{on } \{x_2 = 0\} \cap \Omega. \end{aligned} \tag{3.5.16}$$

Let

$$\Omega_1^+(\delta_0) = \{x \in \Omega_+ : \text{dist}\,(x, \partial\Omega \cap \partial\Omega_+) < \delta_0\},$$
$$\Omega_0^+(\delta_0) = \{x \in \Omega_+ : \text{dist}\,(x, \{x_2 = 0\} \cap \Omega) < \delta_0\}.$$

Let $\boldsymbol{w} \in W^{1,2}_{\sigma,0}(\Omega)$ be symmetric and consider the integral

$$\int_{\Omega^+(\delta_0)} \boldsymbol{w} \cdot \nabla \boldsymbol{w} \cdot \boldsymbol{z} = \int_{\Omega_1^+(\delta_0)} \boldsymbol{w} \cdot \nabla \boldsymbol{w} \cdot \boldsymbol{z} + \int_{\Omega_0^+(\delta_0)} \boldsymbol{w} \cdot \nabla \boldsymbol{w} \cdot \boldsymbol{z} = \mathcal{J}_1 + \mathcal{J}_2.$$

Since $\boldsymbol{w}$ is zero on $\partial\Omega$, we can follows the argument we used in the proof of (3.5.12) to see that

$$|\mathcal{J}_1| \le c_1(\delta_0) \int_{\Omega_1^+} |\nabla \boldsymbol{w}|^2,$$

with $c_1(\delta_0) \underset{\delta_0 \to 0}{\longrightarrow} 0$. From the well–known vector identity

$$\boldsymbol{w} \cdot \nabla \boldsymbol{w} = \nabla |\boldsymbol{w}|^2 + \boldsymbol{\omega} \times \boldsymbol{w}$$

and the boundary properties of $\boldsymbol{z}$, it is not difficult to see that “near to” $\{x_2 = 0\}$

$$\nabla \delta(x) = \boldsymbol{e}_2,$$

$$\begin{aligned}\mathcal{J}_2 &= \int_{\Omega_0^+(\delta_0)} \boldsymbol{\omega} \times \boldsymbol{w} \cdot \boldsymbol{z} = \int_{\Omega_0^+(\delta_0)} \omega \big[w_2 \partial_2 (g_{\delta_0} \varphi_1) + w_1 \partial_1 (g_{\delta_0} \varphi_2)\big] \\ &= \int_{\Omega_0^+(\delta_0)} g_{\delta_0} \omega \big[w_2 \partial_2 \varphi_1 + w_1 \partial_1 \varphi_2\big] + \int_{\Omega_0^+(\delta_0)} \varphi_1 \omega w_2 \partial_2 g_{\delta_0} = \mathcal{J}_1' + \mathcal{J}_2'.\end{aligned}$$

It is evident that

$$|\mathcal{J}_1'| \le c_1'(\delta_0) \int_{\Omega^+} |\nabla \boldsymbol{w}|^2,$$

with $c_1'(\delta_0) \underset{\delta_0 \to 0}{\longrightarrow}$. Moreover, by Lemma 1.2.11 and (3.5.5)

$$\begin{aligned}|\mathcal{J}_2'| &\le c \int_{\Omega_0^+(\delta_0)} |\omega w_2| \partial_2 g_{\delta_0} \le c \sup |x_2 \nabla g_{\delta_0}(x_2)| \int_{\Omega_0^+(\delta_0)} \frac{|\omega w_2|}{x_2} \\ &\le c_2'(\delta_0) \int_{\Omega^+} |\nabla \boldsymbol{w}|^2,\end{aligned}$$

with $c_2'(\delta_0) \underset{\delta_0\to 0}{\to} 0$. Collecting the above results, we see that

$$\left|\int_{\Omega_0^+} \boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{z}\right| \le c(\delta_0)\int_{\Omega_0^+}|\nabla\boldsymbol{w}|^2, \tag{3.5.17}$$

with $c(\delta_0) \underset{\delta_0\to 0}{\to} 0$.

Now, it is evident that the field

$$\tilde{z}(x_1,x_2) = \begin{cases}(z_1(x_1,x_2), z_2(x_1,x_2)), & (x_1,x_2)\in\Omega^+,\\ (z_1(x_1,x_2), -z_2(x_1,-x_2)), & (x_1,x_2)\in\Omega\setminus\Omega^+\end{cases} \tag{3.5.18}$$

is a symmetric extension field of $\boldsymbol{a}$ in Ω such that

$$\int_\Omega |\boldsymbol{v}\cdot\nabla\boldsymbol{w}\cdot\tilde{\boldsymbol{z}}| \le c(\delta)\int_\Omega|\nabla\boldsymbol{w}|^2. \tag{3.5.19}$$

for all symmetric field $\boldsymbol{w}\in W^{1,2}_{\sigma,0}(\Omega)$, with

$$\lim_{\delta_0\to 0} c(\delta_0) = 0.$$

Starting from (3.5.19) and reasoning as we did in the proof of Theorem 3.5.1, we have

Theorem 3.5.2 *If Ω, $\boldsymbol{f}$ and $\boldsymbol{a}$ satisfy the hypotheses of Theorem 3.3.1 then system (3.1.1) has a very weak solution. Moreover,*

$$\|\boldsymbol{u}\|_{L^4(\Omega)} \le c\big\{\|\boldsymbol{a}\|_{L^2(\partial\Omega)} + \|\boldsymbol{a}\|^2_{L^2(\partial\Omega)} + \|\boldsymbol{f}\|_{\mathcal{H}^1(\Omega)}\big\}, \tag{3.5.20}$$

and if $\boldsymbol{a}\in C(\partial\Omega)$, then

$$\|\boldsymbol{u}\|_{C(\overline{\Omega})} \le c\big\{\|\boldsymbol{a}\|_{C(\partial\Omega)} + \|\boldsymbol{a}\|^2_{C(\partial\Omega)} + \|\boldsymbol{f}\|_{\mathcal{H}^1(\Omega)}\big\}, \tag{3.5.21}$$

with c independent of $\boldsymbol{a}$.

3.6 The Fujita–Morimoto approach

A contribution to problem (ii) stated in the introduction was given by H. Fujita and H. Morimoto [12] (1997) (see also [31]). They assume $\boldsymbol{f} \in [D_0^{1,q'}(\Omega)]^*$ and consider a boundary value expressed by

$$\boldsymbol{a} = \mathcal{F}\boldsymbol{h} + \boldsymbol{\gamma} \tag{3.6.1}$$

where

$$\boldsymbol{h} = \nabla\beta, \tag{3.6.2}$$

with $\mathcal{F} \in \mathbb{R}$, $\beta \in W^{2,2}(\Omega)$ harmonic function and $\boldsymbol{\gamma} \in W^{1/2,2}(\partial\Omega)$ satisfying

$$\int_{\partial\Omega} \boldsymbol{\gamma} \cdot \boldsymbol{n} = 0. \tag{3.6.3}$$

They prove that there is a countable subset $\mathcal{G}$ of $\mathbb{R}$ such that if $\mathcal{F} \notin \mathcal{G}$ and $\|\boldsymbol{\gamma}\|_{W^{1/2,2}(\partial\Omega)} + \|\boldsymbol{f}\|_{[D_0^{1,q'}(\Omega)]^*}$ is sufficiently small, then system (3.1.1) has a weak solution. Moreover, H. Morimoto [30] proved that if

$$\beta = \log|x|$$

and Ω is the annulus

$$\Omega = \{x : R < |\boldsymbol{x}| < 2R\},$$

then $\mathcal{G} = \varnothing$. Accordingly, in such a case under hypothesis (3.6.1) and modulo a *smallness* of $\boldsymbol{\gamma}$, we have existence of a solution of system (3.1.1) for every *outflow*

$$\int_{\partial S_R} \boldsymbol{a} \cdot \boldsymbol{n}.$$

The purpose of this section is to show that this result continues to hold under more weak hypotheses on the data [37].

Let β be a harmonic function in Ω such that

$$\nabla\beta \in L^q(\Omega), \quad \operatorname{tr} \nabla\beta_{|\partial\Omega} \in L^{q/2}(\partial\Omega) \tag{3.6.4}$$

The following theorem holds [37]

Theorem 3.6.1 *Let Ω be a bounded domain of class C^2, let $\boldsymbol{f} \in \mathcal{H}^1(\Omega)$ and let $\boldsymbol{a}$ be given by (3.6.1) with $\boldsymbol{\gamma} \in L^{q/2}(\partial\Omega)$, $q > 2$, and β harmonic function in Ω satisfying (3.6.4). There is a countable subset $\mathcal{G}$ of $\mathbb{R}$ such that if $\mathcal{F} \notin \mathcal{G}$ and $\|\boldsymbol{\gamma}\|_{L^{q/2}(\partial\Omega)} + \|\boldsymbol{f}\|_{\mathcal{H}^1(\Omega)}$ is sufficiently small, then system (3.1.1) has a solution.*

Proof - Let us first look for a solution of the problem

$$\begin{aligned} \Delta \boldsymbol{v} - \mathcal{F}\boldsymbol{v}\cdot\nabla\boldsymbol{h} - \mathcal{F}\boldsymbol{h}\cdot\nabla\boldsymbol{v} - \nabla Q = \boldsymbol{0} \quad & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{v} = 0 \quad & \text{in } \Omega, \\ \boldsymbol{v} = \boldsymbol{0} \quad & \text{on } \partial\Omega. \end{aligned} \tag{3.6.5}$$

Since, for $\boldsymbol{v} \in L^q_\sigma(\Omega)$, $|\boldsymbol{h}||\boldsymbol{v}| \in L^{q/2}(\Omega)$, the operator

$$\mathcal{J}[\boldsymbol{v}] = \mathcal{V}[\boldsymbol{v}\cdot\nabla\boldsymbol{h} + \boldsymbol{h}\cdot\nabla\boldsymbol{v}]$$

maps $L^q_\sigma(\Omega)$ into $W^{1,q/2}_\sigma(\Omega)$. Since $q > 2$, by Lemma 1.2.6 $\mathcal{J}$ is completely continuous from $L^q_\sigma(\Omega)$ into itself. Let $\mathcal{A}[\boldsymbol{v}]$ be the solution of the Stokes problem in Ω with zero body force and boundary datum $-\operatorname{tr}_{|\partial\Omega}\mathcal{J}[\boldsymbol{v}] \in W^{1-2/q,2/q}(\partial\Omega)$. By the estimates on weak solutions of the Stokes system and the trace theorem we have

$$\|\mathcal{A}[\boldsymbol{u}]\|_{W^{1,q/2}(\Omega)} \le c\|\operatorname{tr}_{|\partial\Omega}\mathcal{J}[\boldsymbol{v}]\|_{W^{1-2/q,q/2}(\partial\Omega)} \le c\|\mathcal{J}[\boldsymbol{v}]\|_{W^{1,q/2}(\Omega)},$$

so that also $\mathcal{A}$ is completely continuous from $L^q_\sigma(\Omega)$ into itself. By Lemma 1.2.1 the functional equations

$$\mathcal{L}[\boldsymbol{v}] = \big(\mathcal{I} - \mathcal{F}\mathcal{J} - \mathcal{F}\mathcal{A}\big)[\boldsymbol{v}] = \boldsymbol{\phi}$$

has a unique solution for all $\mathcal{F}$ modulo a countable subset $\mathcal{G}$ of $\mathbb{R}$. Therefore, for every $\mathcal{F} \notin \mathcal{G}$ the operator $\mathcal{L}$ is invertible.

Consider now the operator $\mathcal{N}$ from $L^q_\sigma(\Omega)$ into $W^{1,q/2}_{\sigma,0}(\Omega)$, defined at the beginning of Section 3.1 and denote by $\boldsymbol{u}_\gamma$ the solution of the Stokes problem with data $(\boldsymbol{f},\boldsymbol{\gamma})$. Taking into account that the $\mathcal{L}$ is invertible for $\mathcal{F} \notin \mathcal{G}$ we can apply the operator $\overset{-1}{\mathcal{L}}$ to the equation

$$\mathcal{L}[\boldsymbol{v}] = \boldsymbol{u}_\gamma + \mathcal{N}[\boldsymbol{z}] \tag{3.6.6}$$

to get

$$\boldsymbol{v} = \overset{-1}{\mathcal{L}}[\boldsymbol{u}_\gamma] + \overset{-1}{\mathcal{L}}\big[\mathcal{N}[\boldsymbol{z}]\big]. \tag{3.6.7}$$

Since

$$\|\overset{-1}{\mathcal{L}}\big[\mathcal{N}[\boldsymbol{z}]\big]\|_{L^q(\Omega)} \le c_0\|\boldsymbol{z}\|^2_{L^q(\Omega)},$$

if

$$\|\overset{-1}{\mathcal{L}}[\boldsymbol{u}_\gamma]\|_{L^q(\Omega)} < \frac{1}{4c_0},$$

or equivalently if $\|\boldsymbol{\gamma}\|_{L^{q/2}(\partial\Omega)} + \|\boldsymbol{f}\|_{\mathcal{H}^1(\Omega)}$ is sufficiently small, then by Lemma 1.2.3 the map $\overset{-1}{\mathcal{L}}\big[\mathcal{N}[\boldsymbol{z}]\big]$ is a contraction in the ball

$$\mathcal{B} = \left\{\boldsymbol{z} \in L^q_\sigma(\Omega) : \|\boldsymbol{z}\| < \frac{1}{2c_0}\right\}.$$

Then equation (3.6.7) has a unique fixed point in in $\mathcal{B}$

$$\mathcal{L}[\boldsymbol{v}] = \boldsymbol{u}_\gamma + \mathcal{N}[\boldsymbol{v}]. \tag{3.6.8}$$

Taking the Stokes operator in (3.6.8) we see that $\boldsymbol{v}$ satisfies the equations

$$\begin{aligned} \Delta\boldsymbol{v} - \mathcal{F}\boldsymbol{v}\cdot\nabla\boldsymbol{h} - \mathcal{F}\boldsymbol{h}\cdot\nabla\boldsymbol{v} - \boldsymbol{v}\cdot\nabla\boldsymbol{v} - \nabla p = \boldsymbol{f} \quad & \text{in } \Omega,\\ \operatorname{div}\boldsymbol{v} = 0 \quad & \text{in } \Omega,\\ \boldsymbol{v} = \boldsymbol{\gamma} \quad & \text{on } \partial\Omega \end{aligned} \tag{3.6.9}$$

for a suitable pressure field p. Moreover, noting that

$$\Delta\boldsymbol{h} - \boldsymbol{h}\cdot\nabla\boldsymbol{h} = -\nabla\beta\cdot\nabla\nabla\beta = -\tfrac{1}{2}\nabla|\boldsymbol{h}|^2,$$

the pair $(\boldsymbol{h}, -\frac{1}{2}\nabla|\boldsymbol{h}|^2)$ is a solution of the Navier Stokes problem with zero body force and boundary datum $\boldsymbol{h}$. Hence it follows that the pair

$$(\mathcal{F}\boldsymbol{h} + \boldsymbol{v}, p - \tfrac{1}{2}\mathcal{F}\nabla|\boldsymbol{h}|^2)$$

gives the desired solution of our problem. □

▶ If $\boldsymbol{h} \in L^{2q/(2-q)}(\Omega)$, then the solution $\boldsymbol{v}$ in (3.6.5) belongs to $W^{1,2}_{\sigma,0}(\Omega)$. Therefore, multiplying $(3.6.5)_1$ scalarly by $g\boldsymbol{v}$, where g is the function (3.2.2), and reasoning as we did to prove (3.2.4), we arrive at

$$\int_\Omega |\nabla \boldsymbol{v}|^2 = \mathcal{F} \int_\Omega \boldsymbol{v} \cdot \nabla \boldsymbol{v} \cdot \boldsymbol{h}. \tag{3.6.10}$$

Taking into account that

$$\|\boldsymbol{v}\|_{L^q(\Omega)} \le c' \|\nabla \boldsymbol{v}\|_{L^2(\Omega)},$$

by Hölder's inequality we have

$$\begin{aligned} \left| \int_\Omega \boldsymbol{v} \cdot \nabla \boldsymbol{v} \cdot \boldsymbol{h} \right| &\le \|\boldsymbol{v}\|_{L^q(\Omega)} \|\boldsymbol{h}\|_{L^{2q/(q-2)}(\Omega)} \|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \\ &\le c' \|\boldsymbol{h}\|_{L^{2q/(q-2)}(\Omega)} \|\nabla \boldsymbol{v}\|^2_{L^2(\Omega)}. \end{aligned}$$

Then, (3.6.10) yields

$$\big(1 - |\mathcal{F}| c' \|\boldsymbol{h}\|_{L^{2q/(q-2)}(\Omega)}\big) \int_\Omega |\nabla \boldsymbol{v}|^2 \le 0.$$

As a consequence, if in Theorem 3.6.1 $\boldsymbol{h} \in L^{2q/(2-q)}(\Omega)$, then the elements of $\mathcal{G}$ does not belong to the interval $(-c_0, c_0)$, with

$$c_0 = \frac{1}{c' \|\boldsymbol{h}\|_{L^{2q/(q-2)}(\Omega)}}.$$

◁

Chapter 4

Steady Navier–Stokes flow in exterior domains

4.1 D–solutions of the exterior Navier–Stokes problem

We consider now the Navier–Stokes system

$$\begin{aligned} \Delta \boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla p = \boldsymbol{f} \quad &\text{in } \Omega,\\ \operatorname{div}\boldsymbol{u} = 0 \quad &\text{in } \Omega,\\ \boldsymbol{u} = \boldsymbol{a} \quad &\text{on } \partial\Omega, \end{aligned} \tag{4.1.1}$$

$$\lim_{r\to+\infty} \boldsymbol{u}(x) = \boldsymbol{u}_0 \tag{4.1.2}$$

in the exterior domain

$$\Omega = \mathbb{R}^2 \setminus \overline{\Omega'}, \quad \Omega' = \bigcup_{i=1}^{m} \Omega_i.$$

Definition 4.1.1 *A (variational) solution* $(\boldsymbol{u}, p) \in W^{1,2}_{\text{loc}} \times L^2_{\text{loc}}(\Omega)$ *of equations* (4.1.1)$_{1,2}$ *such that*

$$\int_{\complement S_{R_0}} |\nabla \boldsymbol{u}|^2$$

is called D–solution.

At least for $\boldsymbol{u}_0 = \mathbf{0}$ the existence of a solution of system (4.1.1)–(4.1.2) is an open problem [17]. If $\boldsymbol{u}_0 \neq \mathbf{0}$ by a theorem of D.R. Smith and R. Finn [10] (see also [15], Ch. X) we know that if $\boldsymbol{f} = \mathbf{0}$ and an hölderian norm of $\|\boldsymbol{a} - \boldsymbol{u}_0\|$ is sufficiently small, then a regular solution certainly exists. It is not known whether this results continues to holds for general data or at least for data obeying the Fujita–Morimoto decomposition (3.6.1).

As we said in the introduction, the conditions

$$\Phi_i = \int_{\partial\Omega_i} \boldsymbol{a} \cdot \boldsymbol{n} = 0, \quad i = 1, \dots, m,$$

guarantee (for $\boldsymbol{f} = \mathbf{0}$ say) the existence of a D–solution which converges uniformly to constant vector $\boldsymbol{\kappa}$. While in the linear case (exterior Stokes problem) we know the relation which must be satisfied by $\boldsymbol{a}$ and $\boldsymbol{\kappa}$ to assure existence of a solution (Stokes paradox), in the problem we are considering we know no relation between $\boldsymbol{\kappa}$ and $\boldsymbol{u}_0$.

The main purpose of this chapter is to present an improvement of the results of [36] where is showed that existence of a D–solution holds provided the "fluxes" $|\Phi_i|$ are sufficiently small. Moreover, for symmetric domains and data we prove this solution converges to zero at infinity uniformly.

4.2 Asymptotic behavior of D–solutions

In this section we deal with the asymptotic properties of a D–solution by essentially follow the scheme outlined in [20] (see also [15] and [17]). To this end we shall need the following lemmas we prove for the sake of completeness.

Let (r,θ) be a polar coordinate system in $\mathbb{R}^2$ and let $\varphi \in L^1_{\rm loc}(\Omega)$. Set

$$\bar\varphi(r) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(r,\theta).$$

and

$$\boldsymbol{e}_r = \frac{\boldsymbol{x}}{r} = (\cos\theta, \sin\theta), \quad \boldsymbol{e}_\theta = (-\sin\theta, \cos\theta).$$

Lemma 4.2.1 *Let $\varphi \in C^\infty(\complement S_{R_0})$. If $\nabla\varphi, \Delta\varphi \in L^2(\complement S_{R_0})$, then $\nabla_2\varphi \in L^2(\complement S_{R_0})$.*

Proof - Since

$$\begin{aligned} g^2\partial_{ii}\varphi\partial_{jj}\varphi &= \partial_i(g^2\partial_i\varphi\partial_{ijj}\varphi) - g^2\partial_i\varphi\partial_{ijj}\varphi - 2g\partial_i\varphi\partial_{jj}\varphi\partial_i g \\ &= \partial_i(g^2\partial_i\varphi\partial_{ijj}\varphi) - \partial_j(g^2\partial_i\varphi\partial_{ij}\varphi) - 2g\partial_i\varphi\partial_{jj}\varphi\partial_i g \\ &+ g^2\partial_{ij}\varphi\partial_{ij}\varphi + 2g\partial_i\varphi\partial_{ij}\varphi\partial_j g, \end{aligned}$$

where g is a smooth function vanishing in $\complement S_{2R}$, equal to one in S_R and such that $|\nabla g| \le cR^{-1}$, $R \gg R_0$, an integration on $\complement S_{R_0}$ yields

$$\int_{\complement S_{R_0}} |g\nabla_2\varphi|^2 = \int_{\complement S_{R_0}} |g\Delta\varphi|^2 + 2\int_{\complement S_{R_0}} g(\Delta\varphi\nabla\varphi - \nabla\varphi\cdot\nabla_2\varphi)\cdot\nabla g. \tag{4.2.1}$$

By (1.2.12)

$$2g|\nabla\varphi\cdot\nabla_2\varphi\cdot\nabla g| \le \alpha|g\nabla_2\varphi|^2 + \alpha^{-1}|\nabla\varphi|^2|\nabla g|^2,$$

for all positive α. Then, choosing $\alpha < 1$, (4.2.1) implies

$$(1-\alpha)\int_{\complement S_{R_0}} |g\nabla_2\varphi|^2 \le \int_{\complement S_{R_0}} |g\Delta_2\varphi|^2 + \int_{T_R} \big(|\nabla\varphi|^2 + |\Delta\varphi||\nabla\varphi|\big). \tag{4.2.2}$$

If $\nabla\varphi$ and $\Delta\varphi \in L^2(\complement S_{R_0})$, then $|\Delta\varphi||\nabla\varphi| \in L^1(\complement S_{R_0})$ and we can let $R \to +\infty$ in (4.2.2) to get $\nabla_2\varphi \in L^2(\complement S_{R_0})$. □

Lemma 4.2.2 *If $\varphi \in D^{1,2}(\complement S_{R_0})$, then*

$$\int_{\complement S_{R_0}} \frac{\varphi^2}{r^2 \log^2 r} \leq 4 \int_{\complement S_{R_0}} |\nabla \varphi|^2 + \frac{2}{\log R_0} \int_0^{2\pi} \varphi^2(R_0, \theta) \tag{4.2.3}$$

and

$$\int_0^{2\pi} \varphi^2(R, \theta) = o(\log R). \tag{4.2.4}$$

Proof - By the density of $C^1(\complement S_{R_0})$ in $D^{1,2}(\complement S_{R_0})$ it is sufficient to show (4.2.3) and (4.2.4) for smooth functions. By the basic calculus

$$\int_{\complement S_{R_0} \cap S_R} \frac{\varphi^2}{r^2 \log^2 r} = \int_0^{2\pi} \int_{R_0}^R \frac{\varphi^2}{r \log^2 r} = -\int_0^{2\pi} \int_{R_0}^R \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{\varphi^2}{\log r} \right)$$
$$+2 \int_0^{2\pi} \int_{R_0}^R \frac{\varphi \partial_r \varphi}{\log r} \leq \frac{1}{\log R_0} \int_0^{2\pi} \varphi^2(R_0, \theta) + 2 \int_0^{2\pi} \int_{R_0}^R \frac{\varphi \partial_r \varphi}{\log r}. \tag{4.2.5}$$

Since by (1.2.12)

$$2 \frac{\varphi \partial_r \varphi}{\log r} \leq \frac{\varphi^2}{2r \log^2 r} + 2r |\partial_r u|^2$$

(4.2.5) implies

$$\int_{\complement S_{R_0} \cap S_R} \frac{\varphi^2}{r^2 \log^2 r} \leq 4 \int_{\complement S_{R_0} \cap S_R} |\partial_r \varphi|^2 + \frac{2}{\log R_0} \int_0^{2\pi} \varphi^2(R_0, \theta).$$

Hence, letting $R \to +\infty$, (4.2.3) follows.

By the basic calculus and Schwarz's inequality

$$\left| \int_0^{2\pi} \varphi^2(R, \theta) \right| \leq \left| \int_0^{2\pi} \varphi^2(R_0, \theta) \right| + \left| \int_0^{2\pi} \int_{R_0}^R \partial_r \varphi \right|^2$$
$$\leq c + \int_0^{2\pi} \int_{R_0}^R \frac{1}{r} \int_0^{2\pi} \int_{R_0}^R |\partial_r \varphi|^2 r \leq c + (\log R) \int_{\complement S_R} |\nabla \varphi|^2.$$

Hence (4.2.4) follows □

Lemma 4.2.3 *Let $\varphi \in D^{1,q}(\mathbb{R}^2)$, with $q > 2$. Then*

$$\varphi(x) \le c\left\{\int_0^{2\pi} |\varphi|(|x|,\theta) + \|\nabla\varphi\|_{L^q(S_1(x))}\right\}, \tag{4.2.6}$$

Proof - We essentially follow [17], Lemma 3.10. Let us start from the well–known representation formula

$$\varphi(x) = \int_0^{2\pi} \varphi(r',\theta') + \frac{1}{2\pi}\int_{S_\rho(x)} \frac{(x-y)\cdot\nabla\varphi(y)}{|x-y|^2}\mathrm{d}a_y.$$

with $x = (r,\theta)$ and (ρ',θ') is a polar coordinate system with origin at x and $\rho \in [0,1]$. Since by Hölder's inequality for $q > 2$

$$\left|\int_{S_\rho(x)} \frac{(x-y)\cdot\nabla\varphi(y)}{|x-y|^2}\mathrm{d}a_y\right| \le \|\nabla\varphi\|_{L^q(S_\rho)}\left\{\int_{S_\rho(x)} \frac{\mathrm{d}a_y}{|x-y|^{q'}}\right\}^{1/q'}$$

and a simple computation shows that

$$\int_{S_\rho(x)} \frac{\mathrm{d}a_y}{|x-y|^{q'}} = \int_0^{2\pi}\int_0^\rho \frac{1}{t^{q'-1}} < +\infty.$$

Then we have

$$|\varphi(x)| \le \int_0^{2\pi} |\varphi(r',\theta')| + c\|\nabla\varphi\|_{L^q(S_1(x))}. \tag{4.2.7}$$

Multiplying (4.2.7) by r' and integrating over $r' \in [0,1]$, we have

$$|\varphi(x)| \le c\{\|\varphi\|_{L^1(S_1(x))} + \|\nabla\varphi\|_{L^q(S_1(x))}\}. \tag{4.2.8}$$

On the other hand, for almost all $\theta' \in (0,2\pi)$,

$$\varphi(\rho,\theta') = \varphi(x) + \int_0^\rho \partial_t\varphi(t,\theta')\mathrm{d}t.$$

Hence

$$\int_0^{2\pi} |\varphi(\rho,\theta')| \le 2\pi|\varphi(x)| + \int_0^{2\pi}\int_0^\rho |\partial_t\varphi(t,\theta')|. \tag{4.2.9}$$

Since

$$\int_0^{2\pi}\int_0^{\rho}|\partial_t\varphi(t,\theta')| \le c\left\{\int_0^{2\pi}\int_0^1 \frac{1}{t^{q'-1}}\right\}^{1/q'}\|\nabla\varphi\|_{L^q(S_1(x))},$$

(4.2.9) yields

$$\int_0^{2\pi}|\varphi(\rho,\theta')| \le c\left\{|\varphi(x)| + \|\nabla\varphi\|_{L^q(S_1(x))}\right\}.$$

Hence, multiplying by ρ and integrating over $\rho \in [0,1]$ and $\theta \in [0,2\pi]$, it follows

$$\|\varphi\|_{L^1(S_1(x))} \le c\left\{\int_0^{2\pi}|\varphi(|x|,\theta)| + \|\nabla\varphi\|_{L^q(S_1(x))}\right\}. \tag{4.2.10}$$

Putting together (4.2.8), (4.2.10), we get (4.2.6). □

Lemma 4.2.4 *Let* $\boldsymbol{w} \in W^{1,2}_\sigma(S_R \setminus S_\rho)$, $\rho < R$. *Then*

$$\left|\int_{S_R\setminus S_\rho}\boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot\frac{\boldsymbol{e}_r}{r}\right| \le \int_{S_R}|\nabla\boldsymbol{w}|^2. \tag{4.2.11}$$

Proof - We follows [20]. Note that, since

$$\partial_1 w_1 = -\partial_2 w_2$$

and

$$\partial_\theta w_i = -r\sin\theta\partial_1 w_i + r\cos\theta\partial_2 w_i, \quad i = 1,2,$$

it holds

$$\begin{aligned}\boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{e}_r &= \cos\theta w_1\partial_1 w_1 + \cos\theta w_2\partial_2 w_1\\ &+ \sin\theta w_1\partial_1 w_2 + \sin\theta w_2\partial_2 w_2\\ &= -\sin\theta w_2\partial_1 w_1 + \cos\theta w_2\partial_2 w_1\\ &- \cos\theta w_1\partial_2 w_2 + \sin\theta w_1\partial_1 w_2 = \frac{1}{r}(w_2\partial_\theta w_1 - w_1\partial_\theta w_2),\end{aligned}$$

almost everywhere in S_R. Then, taking into account that

$$\int_0^{2\pi} \partial_\theta w_2 = \int_0^{2\pi} \partial_\theta w_1 = 0,$$

for almost all $\theta \in (0, 2\pi)$, we have

$$\int_{S_R} \boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot\frac{\boldsymbol{e}_r}{r} = \int_0^R \frac{1}{r}\int_0^{2\pi} \big[(w_2-\bar{w}_2)\partial_\theta w_1-(w_1-\bar{w}_1)\partial_\theta w_2\big]. \quad (4.2.12)$$

Since by Lemma 1.2.10

$$\int_0^{2\pi} |w_i - \bar{w}_i|^2 \leq \int_0^{2\pi} |\partial_\theta w_i|^2.$$

using (1.2.12) and Schwarz's inequality, we get

$$\begin{aligned}
\left|\int_0^{2\pi}(w_1 - \bar{w}_1)\partial_\theta w_2\right| &\leq \left\{\int_0^{2\pi} |w_1 - \bar{w}_1|^2 \int_0^{2\pi} |\partial_\theta w_2|^2\right\}^{1/2} \\
&\leq \left\{\int_0^{2\pi} |\partial_\theta w_1|^2 \int_0^{2\pi} |\partial_\theta w_2|^2\right\}^{1/2} \leq \frac{1}{2}\int_0^{2\pi} |\partial_\theta w_1|^2 + \frac{1}{2}\int_0^{2\pi} |\partial_\theta w_2|^2, \\
\left|\int_0^{2\pi}(w_2 - \bar{w}_2)\partial_\theta w_1\right| &\leq \frac{1}{2}\int_0^{2\pi} |\partial_\theta w_1|^2 + \frac{1}{2}\int_0^{2\pi} |\partial_\theta w_2|^2,
\end{aligned}$$

for almost all $r \in (\rho, R)$. Therefore, (4.2.4) follows from (4.2.12), bearing in mind that

$$\int_0^{2\pi} |\partial_\theta \boldsymbol{w}| \leq \int_0^{2\pi} |\nabla \boldsymbol{w}|^2 r^2.$$

□

Recall that we set

$$\omega = \partial_1 u_2 - \partial_2 u_1$$

and

$$\Phi = p + \tfrac{1}{2}|\boldsymbol{u}|^2.$$

Lemma 4.2.5 *If $\boldsymbol{f}$ has a compact support and $\boldsymbol{u}$ is a D–solution, then*

$$\int_{\complement S_{R_0}} |\nabla\omega|^2 < +\infty \tag{4.2.13}$$

and

$$\lim_{R\to+\infty} \max_{\theta\in[0,2\pi]} \Pi(R,\theta)$$

exists.

Proof - Let w be a nonnegative regular function in $\mathbb{R}$, equal to zero in $(-\infty, 0]$ and to 1 in $[1, +\infty)$. Let

$$g(x) = w(\delta^{-1}(\log\log R - \log\log r)) \tag{4.2.14}$$

where $R \gg R_0$ and δ is a small positive number. It is easy to see that

$$g(x) = \begin{cases} 1, & x \in S_{R^{e^{-\delta}}}, \\ 0, & x \notin S_R. \end{cases}$$

Moreover,

$$\nabla g = -\frac{w'\boldsymbol{e}_r}{\delta r \log r}.$$

Let $h(\omega)$ be the function defined by

$$h(\omega) = \begin{cases} \omega^2, & |\omega| \le \omega_0, \\ \omega_0(2|\omega| - \omega_0), & |\omega| > \omega_0, \end{cases}$$

where ω_0 is a positive constant such that

$$\omega_0 > \max_{\partial S_{R_0}} |\omega|.$$

Assume that supp $\boldsymbol{f} \subset S_{R_0}$, multiply the equation

$$\Delta\omega - \boldsymbol{u}\cdot\nabla\omega = 0 \quad \text{in } \complement S_{R_0} \tag{4.2.15}$$

by gh' and integrate over $\complement S_{R_0}$. Then, using the identities

$$\begin{aligned} &\nabla h = h'\nabla\omega \\ &gh'\Delta\omega = \operatorname{div}(gh'\nabla\omega) - gh''|\nabla\omega|^2 - h'\nabla\omega\cdot\nabla g \\ &\qquad\quad = \operatorname{div}(g\nabla h - h\nabla g) - gh''|\nabla\omega|^2 + h\Delta g \\ &gh'\nabla\omega\cdot\boldsymbol{u} = g\nabla h\cdot\boldsymbol{u} = \operatorname{div}(gh\boldsymbol{u}) - h\boldsymbol{u}\cdot\nabla g, \end{aligned}$$

we have

$$\int_{\complement S_{R_0}} g h''|\nabla\omega|^2 = \int_{\complement S_{R_0}} h(\Delta g + \boldsymbol{u}\cdot\nabla g) + \tfrac{1}{2}\int_{\partial S_{R_0}} (\partial_n\omega^2 - \omega^2\boldsymbol{u}\cdot\boldsymbol{n}). \tag{4.2.16}$$

Since

$$h(\omega) \le \min\{\omega^2, 2\omega_0|\omega|\},$$

taking into account the properties of g, we have

$$\left|\int_{\complement S_{R_0}} h\Delta g\right| \le \int_{\complement S_{R_\delta}} \omega^2$$

$$\left|\int_{\complement S_{R_0}} h\boldsymbol{u}\cdot\nabla g\right| \le c\omega_0\int_{\complement S_{R_\delta}} \frac{|\boldsymbol{u}||\omega|}{r\log r} \le c\omega_0\left\{\int_{\complement S_{R_\delta}} \frac{|\boldsymbol{u}|^2}{r^2\log^2 r} + \int_{\complement S_{R_\delta}} \omega^2\right\}.$$

Therefore, from (4.2.16) it follows

$$\int_{\substack{\complement S_{R_0}\cap S_{R_\delta}\\(\omega\le\omega_0)}} |\nabla\omega|^2 \le c\omega_0\left\{\int_{\complement S_{R_\delta}} \frac{|\boldsymbol{u}|^2}{r^2\log^2 r} + \int_{\complement S_{R_\delta}} \omega^2\right\} + \tfrac{1}{2}\int_{\partial S_{R_0}} |\partial_n\omega^2 - \tfrac{1}{2}\omega^2\boldsymbol{u}\cdot\boldsymbol{n}|. \tag{4.2.17}$$

Hence, letting $R\to\infty$ and taking into account that $\omega, (r\log r)^{-1}\boldsymbol{u} \in L^2(\Omega)$,

$$\int_{\substack{\complement S_{R_0}\\(\omega\le\omega_0)}} |\nabla\omega|^2 \le \tfrac{1}{2}\int_{\partial S_{R_0}} |\partial_n\omega^2 - \omega^2\boldsymbol{u}\cdot\boldsymbol{n}|. \tag{4.2.18}$$

Taking into account that c is independent of ω_0, we can let $\omega_0\to+\infty$ to get

$$\int_{\complement S_{R_0}} |\nabla\omega|^2 \le \tfrac{1}{2}\int_{\partial S_{R_0}} |\partial_n\omega^2 - \tfrac{1}{2}\omega^2\boldsymbol{u}\cdot\boldsymbol{n}|$$

and so (4.2.13).

From (2.2.7) it follows that $\Pi \in C^\infty(\complement S_{R_0})$ satisfies the *elliptic inequality*

$$\Delta \Pi - \boldsymbol{u} \cdot \nabla \Pi \geq 0.$$

Then by the classical *maximum principle* (see, *e.g.*, [27]) the function

$$\max_{\theta \in [0, 2\pi]} \Pi(R, \theta)$$

is monotone for $R > R_0$ and, as a consequence, has a limit for $R \to +\infty$. □

Let us collect now the main summability and asymptotic properties of a D–solutions.

Lemma 4.2.6 *If* $\boldsymbol{f}$ *has a compact support and* $(\boldsymbol{u}, p)$ *is a* D*–solution, then*

$$\begin{aligned} &\nabla_k \boldsymbol{u} \in L^q(\complement S_{R_0}), \\ &\nabla_{k+1} p \in L^q(\complement S_{R_0}), \end{aligned} \tag{4.2.19}$$

for every $k \in \mathbb{N}$ *and* $q \geq 2$*. Moreover,* p *admits the following decomposition at a large distance*

$$p(x) = \sum_{i=1}^{4} p_i(x), \tag{4.2.20}$$

for all $|\boldsymbol{x}| > R_0$*, with*

$$\begin{aligned} &p_1 \in D^{2,1}(\complement S_{R_0}) \cap D^{1,2}(\complement S_{R_0}), \\ &p_i \in D^{1,q}(\complement S_{R_0}) \cap L^{2q/(2-q)}(\complement S_{R_0}), \ q \in (1,2), \quad i = 2, 3, \\ &\nabla_k p_2(x) = O(r^{-1-k}). \end{aligned}$$

In particular,

$$p \in D^{1,2}(\complement S_{R_0})$$

and, if

$$\int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{n} = 0, \tag{4.2.21}$$

then

$$p \in D^{2,1}(\complement S_{R_0})$$

Proof - Since

$$
\begin{aligned}
\Delta u_1 &= \partial_{11}u_1 + \partial_{22}u_1 = -\partial_{21}u_2 + \partial_{22}u_1 = -\partial_2(\partial_1 u_2 - \partial_2 u_1) = -\partial_2\omega \\
\Delta u_2 &= \partial_{11}u_2 + \partial_{22}u_2 = \partial_{11}u_2 - \partial_{12}u_1 = \partial_1\omega,
\end{aligned} \tag{4.2.22}
$$

(4.2.13) implies that

$$\Delta \boldsymbol{u} \in L^2(\complement S_{R_0})$$

so that by Lemma 4.2.1

$$\nabla \boldsymbol{u} \in W^{1,2}(\complement S_{R_0}) \tag{4.2.23}$$

and, in particular, by Lemma 1.2.5 $\nabla \boldsymbol{u} \in W^{1,q}(\complement S_{R_0})$. Hence by taking into account Lemmas 4.2.2 – 4.2.3, it follows

$$\boldsymbol{u} = o(\sqrt{\log r}). \tag{4.2.24}$$

Since $\boldsymbol{u} \in C^\infty(\complement S_{R_0})$, from (4.2.15) we see that a derivative $\omega_j = \partial_j\omega$ is a solution of the equation

$$\Delta\omega_j - \boldsymbol{u}\cdot\nabla\omega_j - \partial_j\boldsymbol{u}\cdot\nabla\omega = 0. \tag{4.2.25}$$

Mulyiply (4.2.25) by $g^2\omega_j$, where g is the function (4.2.14). Then, by the identities

$$
\begin{aligned}
&g^2\omega_j\Delta\omega_j = \operatorname{div}(g^2\omega_j\nabla\omega_j) - g^2|\nabla\omega_j|^2 - 2g\omega_j\nabla\omega_j\cdot\nabla g, \\
&g^2\omega_j\boldsymbol{u}\cdot\nabla\omega_j = \tfrac{1}{2}\operatorname{div}(g^2\omega_j^2\boldsymbol{u}) - g\omega_j^2\boldsymbol{u}\cdot\nabla g, \\
&g^2\omega_j\partial_j\boldsymbol{u}\cdot\nabla\omega = \operatorname{div}(g^2\omega\omega_j\partial_j\boldsymbol{u}) - g^2\omega\nabla\omega_j\cdot\partial_j\boldsymbol{u} - 2g\omega\omega_j\partial_j\boldsymbol{u}\cdot\nabla g
\end{aligned}
$$

and the divergence theorem, an integration over $\complement S_{R_0}$ gives

$$
\begin{aligned}
\int_{\complement S_{R_0}} |g\nabla\omega_j|^2 = &-2\int_{\complement S_{R_0}} g(2\omega_j\nabla\omega_j + \omega\omega_j\partial_j\boldsymbol{u})\cdot\nabla g \\
&- \int_{\complement S_{R_0}} g\omega_j^2\boldsymbol{u} - \int_{\complement S_{R_0}} g^2\omega\nabla\omega_j\cdot\partial_j\boldsymbol{u} + J,
\end{aligned}
$$

where J groups the boundary integrals on ∂S_{R_0}. Hence, in virtue of (4.2.24) and the inequalities

$$\begin{aligned}
\left|\int_{\complement S_{R_0}} g\omega_j \nabla\omega_j\cdot\nabla g\right| &\le \alpha\int_{\complement S_{R_0}}|g\nabla\omega_j|^2 + c\int_{\complement S_{R_\delta}}\omega_j^2,\\
\left|\int_{\complement S_{R_0}} g\omega_j^2\boldsymbol{u}\cdot\nabla g\right| &\le c\int_{\complement S_R}\frac{|\boldsymbol{u}||\omega_j|^2}{r\log r}\le c\int_{\complement S_{R_\delta}}|\omega_j|^2,\\
\left|\int_{\complement S_{R_0}} g\omega\omega_j\partial_j\boldsymbol{u}\cdot\nabla g\right| &\le c\left\{\int_{\complement S_{R_\delta}}|\nabla\boldsymbol{u}|^4\int_{\complement S_{R_\delta}}\omega^2\right\}^{1/2},\\
\left|\int_{\complement S_{R_0}} g^2\omega\nabla\omega_j\cdot\partial_j\boldsymbol{u}\right| &\le \alpha\int_{\complement S_{R_0}}|g\nabla\omega_j|^2 + c\int_{\complement S_{R_0}}|\nabla\boldsymbol{u}|^4,
\end{aligned}$$

choosing α suitably small, it follows

$$\int_{\complement S_{R_0}}|g\nabla\omega_j|^2\le c \tag{4.2.26}$$

Therefore, letting $R\to+\infty$ in (4.2.26) implies that $\nabla\omega_j\in L^2(\complement S_{R_0})$. Iterating this argument we arrive at $(4.2.19)_1$; $(4.2.19)_2$ is a consequence of $(4.2.19)_2$ and equations $(4.1.1)_1$.

From (4.2.23), (4.2.24) and equations $(4.1.1)_1$ we have

$$\int_{\complement S_{R_0}}\frac{|\nabla p|^2}{\log r}<+\infty. \tag{4.2.27}$$

Write

$$\boldsymbol{u}=\boldsymbol{w}+\boldsymbol{\gamma},$$

where

$$\boldsymbol{\gamma}(x)=\frac{\boldsymbol{e}_r}{r}\int_{\partial\Omega}\boldsymbol{a}\cdot\boldsymbol{n}.$$

Of course

$$\int_{\partial S_{R_0}}\boldsymbol{w}\cdot\boldsymbol{n}=0. \tag{4.2.28}$$

Consider the equation

$$\Delta p+\operatorname{div}(\boldsymbol{u}\cdot\nabla\boldsymbol{u})=0\quad\text{in }\complement S_{R_0}. \tag{4.2.29}$$

Let ϕ be a function of class C^∞ in $\mathbb{R}^2$, vanishing on $S_{\bar{R}}$ and equal to 1 outside $S_{2\bar{R}}$, $\bar{R} \gg R_0$. By (4.2.28) Lemma 1.2.13 assures that the equation

$$\operatorname{div} \boldsymbol{h} + \operatorname{div}(\phi \boldsymbol{w}) = 0 \quad \text{in } T_{\bar{R}}$$

has a solution $\boldsymbol{h} \in C_0^\infty(T_{\bar{R}})$. Multiply (4.2.29) by ϕ. Then the function $Q = \phi^2 p$ is a solution of the equation

$$\Delta Q = \operatorname{div} \boldsymbol{\nu} + \varphi \quad \text{in } \mathbb{R}^2, \tag{4.2.30}$$

with

$$\boldsymbol{\nu} = -(\phi \boldsymbol{w} + \boldsymbol{h}) \cdot \nabla(\phi \boldsymbol{w} + \boldsymbol{h}) - 2\phi^2 \boldsymbol{w} \cdot \nabla \boldsymbol{\gamma} + \phi^2 \boldsymbol{\gamma} \cdot \nabla \boldsymbol{\gamma} = \sum_{i=1}^{3} \boldsymbol{\nu}_i$$

and $\varphi \in C_0^\infty(T_{\bar{R}})$.

Taking into account (4.2.27), by uniqueness Q admits the representation by the volume potential

$$\begin{aligned} Q(x) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \Big[\sum_{i=1}^{3} \operatorname{div} \boldsymbol{\nu}_i(y) + \varphi(y) \Big] \log|x-y| \mathrm{d}a_y \\ &= \sum_{i=1}^{4} Q_i(x). \end{aligned} \tag{4.2.31}$$

Since by Lemma (1.2.8) $\operatorname{div} \boldsymbol{\nu}_1 \in \mathcal{H}^1(\mathbb{R}^2)$, we have that

$$Q_1 \in D^{2,1}(\mathbb{R}^2).$$

Now, taking into account that $\nabla \boldsymbol{\gamma}(x) = O(r^{-2})$ and (4.2.24), it holds

$$\lim_{R \to +\infty} \int_{\partial S_R} \log|x-y| (\boldsymbol{w} \cdot \nabla \boldsymbol{\gamma} \cdot \boldsymbol{e}_r)(\zeta) \, \mathrm{d}s_\zeta = 0$$

and

$$\lim_{R \to +\infty} \int_{\partial S_R} \log|x-y| (\boldsymbol{\gamma} \cdot \nabla \boldsymbol{\gamma} \cdot \boldsymbol{e}_r)(\zeta) \, \mathrm{d}s_\zeta = 0$$

so that

$$Q_i(x) = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{(x-y) \cdot (g \boldsymbol{\nu}_i)(y)}{|x-y|^2} \mathrm{d}a_y, \quad i = 2, 3,$$

Hence, noting that $g\boldsymbol{\nu}_i \in L^q(\mathbb{R}^2)$, for all $q > 1$, by well–known results about integral transforms ([27], Ch. VI) it follows that

$$Q_i(x) \in D^{1,q}(\mathbb{R}^3) \cap L^{2q/(2-q)}(\mathbb{R}^3),\ q \in (1,2), \quad i = 2,3.$$

Therefore, there is a constant Q_0 such that

$$\lim_{r\to+\infty} \sum_{1=1}^{2} Q_i(x) = Q_0.$$

uniformly. On the other hand,

$$\int_\Omega \varphi = 0,$$

otherwise we have a contradiction with (4.2.27). Hence

$$\nabla_k Q_2 = O(r^{-k-1}),$$

for $k \in \mathbb{N}_0$ and (4.2.20) is proved, bearing in mind that $p = Q$ at large distance. The last part of the theorem is evident. □

▶ The second part of Lemma 4.2.6 seems to be new. The hypothesis that $\boldsymbol{f}$ has a compact support can be relaxed by requiring, for instance, that $\boldsymbol{u} = \mathbf{0}$ on $\partial\Omega$ or $\operatorname{div}(\phi^2 \boldsymbol{f}) \in \mathcal{H}^1(\mathbb{R}^2)$ for some $\bar{R} > R_0$. ◁

▶ Lef $\boldsymbol{f}$ have a compact support. Since $\Delta \boldsymbol{u} \in L^2(\complement S_{R_0})$ and $\nabla p \in L^2(\complement S_{R_0})$, then

$$\boldsymbol{w} \cdot \nabla \boldsymbol{w} \in L^2(\complement S_{R_0}).$$

◁

Lemma 4.2.7 *If* $\boldsymbol{f}$ *has a compact support and* $\boldsymbol{u}$ *is a* D*–solution, then*

$$\int_{\complement S_{R_0}} r|\nabla\omega|^2 < +\infty. \tag{4.2.32}$$

PROOF - From (4.2.16), written with $h = \omega^2$ and g replaced by rg, we get

$$\int_{\complement S_{R_0}} rg|\nabla\omega|^2 = \int_{\complement S_{R_0}} \omega^2(\Delta g + \boldsymbol{u}\cdot\nabla g) + J' \tag{4.2.33}$$

with J' boundary integral on ∂S_{R_0}. Hence, taking into account that by the properties of the function g and (4.2.24)

$$|\Delta g + \boldsymbol{u}\cdot\nabla g| \le c,$$

(4.2.32) follows by letting $R \to +\infty$ in (4.2.33). □

Lemma 4.2.8 *If $\boldsymbol{f}$ has a compact support and $(\boldsymbol{u}, p)$ is a D–solution, then*

$$\lim_{r\to+\infty} \nabla_k p(x) = 0, \tag{4.2.34}$$

uniformly, for all $k \in \mathbb{N}_0$,

$$\lim_{r\to+\infty} \nabla_k \boldsymbol{u}(x) = \boldsymbol{0}, \tag{4.2.35}$$

uniformly, for all $k \in \mathbb{N}$, and

$$\boldsymbol{u} = o(\sqrt{\log r}). \tag{4.2.36}$$

If $\boldsymbol{a}$ satisfies (4.2.21), then

$$\lim_{r\to+\infty} \int_0^{2\pi} r|\nabla p|(r,\theta) = 0. \tag{4.2.37}$$

Moreover, if $\boldsymbol{u} \in L^\infty(\complement S_{R_0})$, then there is a constant vector $\boldsymbol{\kappa}$ such that

$$\lim_{r\to+\infty} \boldsymbol{u}(x) = \boldsymbol{\kappa}, \tag{4.2.38}$$

uniformly.

PROOF - Relations (4.2.34), (4.2.35), for $k \in \mathbb{N}$, and (4.2.36) are a simple consequence of the above lemmas, while (4.2.34) for $k = 0$ is proved by noting that (4.2.20) implies that p has a constant limit at infinity and, since p is defined modulo an additive constant, we can always choose this constant such that

$$\lim_{r\to+\infty} p(x) = 0. \tag{4.2.39}$$

To prove (4.2.37), it is sufficient we note that by (4.2.34) for large R we have

$$\begin{aligned}\int_0^{2\pi} |\nabla p(R,\theta)| &= \int_0^{2\pi} \left| \int_R^{+\infty} \partial_r \nabla p(r,\theta) \right| \le \int_0^{2\pi} \int_R^{+\infty} |\nabla_2 p(r,\theta)| \\ &\le \frac{1}{R} \int_0^{2\pi} \int_R^{+\infty} |\nabla_2 p(r,\theta)| r \le \frac{1}{R} \int_{\complement S_R} |\nabla_2 p|.\end{aligned}$$

In virtue of Lemma 4.2.5, (4.2.39) implies that

$$\lim_{R\to+\infty} \left\{ \max_{\theta\in[0,2\pi]} |\boldsymbol{u}(R,\theta)| \right\} = \ell \in [0,+\infty]. \tag{4.2.40}$$

Therefore, if $\boldsymbol{u} \in L^\infty(\complement S_{R_0})$, then $\ell \in [0,+\infty)$ and, if $\ell = 0$, then

$$\lim_{r\to+\infty} \boldsymbol{u}(x) = \boldsymbol{0}$$

uniformly. Let us show now that

$$\ell = \lim_{R\to+\infty} |\bar{\boldsymbol{u}}(R)|. \tag{4.2.41}$$

From the trace theorem[1]

$$\begin{aligned}\int_0^{2\pi} |\boldsymbol{u} - \bar{\boldsymbol{u}}|^2(R,\theta) &\le \int_0^{2\pi} |\boldsymbol{u} - \boldsymbol{u}_{T_R}|^2 \\ &\le \frac{c}{R^2} \int_{T_R} |\boldsymbol{u}|^2 + c\int_{T_R} |\nabla \boldsymbol{u}|^2 \le c\int_{T_R} |\nabla \boldsymbol{u}|^2.\end{aligned}$$

[1] We use here the inequality

$$\int_{\mathcal{B}} |\varphi - \varphi_{\mathcal{B}}|^q \le \int_{\mathcal{B}} |\varphi - \alpha|^q$$

for all $\alpha \in \mathbb{R}$ which is easily proved by minimizing the integral at the right hand side in $\mathbb{R}$.

Hence it follows that

$$\lim_{R\to+\infty}\int_0^{2\pi}|\boldsymbol{u}-\bar{\boldsymbol{u}}|^2(R,\theta)=0. \tag{4.2.42}$$

By (4.2.40) for every sequence $\{R_k\}_{k\in\mathbb{N}}$, there is a corresponding sequence $\{\theta_k\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to+\infty}|\boldsymbol{u}(R_k,\theta_k)|=\ell. \tag{4.2.43}$$

From the integral theorem of the mean there is $R_k\in(2^k,2^{k+1})$ such that

$$\log 2\int_0^{2\pi}|\partial_\theta\boldsymbol{u}|^2(R_k,\theta)=\int_{2^k}^{2^{k+1}}\frac{1}{r}\int_0^{2\pi}|\partial_\theta\boldsymbol{u}|^2\le\int_{S_{2^{k+1}}\setminus S_{2^k}}|\nabla\boldsymbol{u}|^2.$$

Since $\boldsymbol{u}$ is a D–solution,

$$\lim_{k\to+\infty}\int_{S_{2^{k+1}}\setminus S_{2^k}}|\nabla\boldsymbol{u}|^2=0.$$

Therefore,

$$\lim_{k\to+\infty}\int_0^{2\pi}|\partial_\theta\boldsymbol{u}|^2(R_k,\theta)=0. \tag{4.2.44}$$

By the basic calculus

$$|\boldsymbol{u}(R_k,\theta_k)-\boldsymbol{u}(R_k,\theta)|^2\le\int_0^{2\pi}|\partial_\theta\boldsymbol{u}|^2(R_k,\theta).$$

Then, taking into account (4.2.43) and the inequality

$$\big||\boldsymbol{u}(R_k,\theta)|-\ell\big|^2\le 2|\boldsymbol{u}(R_k,\theta_k)-\ell|^2+2|\boldsymbol{u}(R_k,\theta_k)-\boldsymbol{u}(R_k,\theta)|^2$$

we have

$$\lim_{k\to+\infty}|\boldsymbol{u}(R_k,\theta)|=\ell, \tag{4.2.45}$$

uniformly in θ. Since by Lemma 1.2.10

$$\int_0^{2\pi}|\boldsymbol{u}(R_k,\theta)-\bar{\boldsymbol{u}}(R_k)|^2\le\int_0^{2\pi}|\partial_\theta\boldsymbol{u}|^2(R_k,\theta),$$

we have

$$|\boldsymbol{u}(R_k,\theta)-\bar{\boldsymbol{u}}(R_k)|^2 \le c\int_0^{2\pi} |\partial_\theta \boldsymbol{u}|^2(R_k,\theta).$$

Therefore, taking into account (4.2.45), we get

$$\lim_{k\to+\infty} |\bar{\boldsymbol{u}}(R_k)| = \ell. \tag{4.2.46}$$

For every $R\in(2^k,2^{k+1})$ by Schwarz's inequality it holds

$$\begin{aligned}|\bar{\boldsymbol{u}}(R)-\bar{\boldsymbol{u}}(R_k)|^2 &= \frac{1}{4\pi^2}\left|\int_{R_k}^{R}\int_0^{2\pi}\partial_r \boldsymbol{u}\right|^2 \\ &\le \frac{1}{2\pi}\left\{\int_{R_k}^{R}\frac{1}{r}\right\}\left\{\int_{\complement S_{R_k}}|\nabla \boldsymbol{u}|^2\right\} \le c\int_{\complement S_{R_k}}|\nabla \boldsymbol{u}|^2.\end{aligned}$$

Hence, letting $k\to+\infty$ and bearing in mind (4.2.46), (4.2.41) follows.

If $\ell>0$, then there is a positive R_0 such that

$$|\bar{\boldsymbol{u}}| > \ell/2, \quad \forall\, r > R_0. \tag{4.2.47}$$

Let $\psi(r)$ be the argument of the vector $\bar{\boldsymbol{u}}(r)$, *i.e.*,

$$\begin{aligned}\bar{u}_1(r) &= |\bar{u}(r)|\cos\psi(r),\\ \bar{u}_2(r) &= |\bar{u}(r)|\sin\psi(r).\end{aligned}$$

By a simple computation we have

$$\psi' = \frac{\bar{u}_1\bar{u}_2' - \bar{u}_2\bar{u}_2'}{|\bar{\boldsymbol{u}}|^2}.$$

Now, from

$$\begin{aligned}\boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{e}_\theta &= -\sin\theta w_1\partial_1 w_1 - \sin\theta w_2\partial_2 w_1 + \cos\theta w_1\partial_1 w_2\\ &\quad + \cos\theta w_2\partial_2 w_2 = -\cos\theta w_2\partial_1 w_1 - \sin\theta w_2\partial_2 w_1\\ &\quad + \cos\theta w_1\partial_1 w_2 + \sin\theta w_1\partial_2 w_2 = r(w_1\partial_r w_2 - w_2\partial_r w_1),\end{aligned}$$

and (4.2.22), we get

$$\partial_r\omega + u_1\partial_r u_2 - u_2\partial_r u_1 + \frac{1}{r}\partial_\theta p = 0.$$

Hence, taking the average over θ in (4.1.1)$_1$ and dividing by $|\bar{\boldsymbol{u}}|^2$, it follows

$$2\pi\psi'(r) = \frac{1}{|\bar{\boldsymbol{u}}|^2}\int_0^{2\pi}[(u_2-\bar{u}_2)\partial_r u_1](r,\theta) - \frac{1}{|\bar{\boldsymbol{u}}|^2}\int_0^{2\pi}[\partial_r\omega + (u_1-\bar{u}_1)\partial_r u_2](r,\theta), \tag{4.2.48}$$

for all $r > R_0$. Integrating over (ρ, R) and taking into account (4.2.47), (4.2.48) yields

$$|\psi(R)-\psi(\rho)| \le c\left\{\int_\rho^R\int_0^{2\pi}|(u_2-\bar{u}_2)\partial_r u_1| + c\int_\rho^R\int_0^{2\pi}|(u_1-\bar{u}_1)\partial_r u_2| + \int_\rho^R\int_0^{2\pi}|\partial_r\omega|\right\}. \tag{4.2.49}$$

By Lemma 1.2.10 and Schwarz's inequality

$$\begin{aligned}\int_\rho^R\int_0^{2\pi}|(u_2-\bar{u}_2)\partial_r u_1| &\le \left\{\int_\rho^R\frac{1}{r}\int_0^{2\pi}|u_2-\bar{u}_2|^2\int_\rho^R r\int_0^{2\pi}|\partial_r u_1|^2\right\}^{1/2}\\ &\le \int_{S_R\setminus S_\rho}|\nabla\boldsymbol{u}|^2,\\ \int_\rho^R\int_0^{2\pi}|(u_1-\bar{u}_1)\partial_r u_2| &\le \int_{S_R\setminus S_\rho}|\nabla\boldsymbol{u}|^2.\end{aligned}$$

Also,

$$\int_\rho^R\int_0^{2\pi}|\partial_r\omega| \le c\left\{\int_\rho^R\frac{1}{r^2}\right\}^{1/2}\int_{S_R\setminus S_\rho} r|\nabla\omega|^2 \le c\int_{S_R\setminus S_\rho} r|\nabla\omega|^2.$$

Then, letting $\rho \to +\infty$ in (4.2.49) and taking into account Lemma 4.2.7, we see that $\psi(r)$ converges to a constant $\psi_0 \in [0, 2\pi]$ and

$$\lim_{R\to+\infty}\frac{1}{2\pi}\int_0^{2\pi}\boldsymbol{u}(R,\theta) = \boldsymbol{\kappa}, \tag{4.2.50}$$

with

$$\boldsymbol{\kappa} = (\ell\cos\psi_0, \ell\sin\psi_0).$$

Now, (4.2.42) and (4.2.50) implies that

$$\lim_{R\to+\infty}\int_0^{2\pi}|\boldsymbol{u}-\boldsymbol{\kappa}|(R,\theta)=0. \tag{4.2.51}$$

By Lemma 4.2.3

$$|\boldsymbol{u}(x)-\boldsymbol{\kappa}|\le c\left\{\int_0^{2\pi}|\boldsymbol{u}-\boldsymbol{\kappa}|(|x|,\theta)+\|\nabla\boldsymbol{u}\|_{L^q(S_1(x))}\right\},$$

for some $q>2$. Therefore, (4.2.51) yields (4.2.38). □

• *Remark 4.2.1*

In virtue of classical results of C. J. Amick [2], if $\boldsymbol{a}$ and $\boldsymbol{f}$ are zero, then a D–solution $\boldsymbol{u}$ bounded in $\complement S_{R_0}$. Then by Lemma 4.2.8 $\boldsymbol{u}$ converges uniformly to a constant vector at infinity. ♢

4.3 Existence of a D–solution

Let $\boldsymbol{a} \in L^2(\partial\Omega)$ and let

$$\boldsymbol{u}_s = \boldsymbol{v} + \boldsymbol{\sigma}$$

be the very weak solution of the Stokes problem corresponding to $\boldsymbol{f} = \boldsymbol{0}$ and $\boldsymbol{a}$, defined in Remark 2.12.2. Recall that

$$\int_{\partial\Omega_i} \boldsymbol{v} \cdot \boldsymbol{n} = 0, \quad i = 1, \dots, m. \tag{4.3.1}$$

Let g be a regular function, vanishing outside the disk $S_{2\bar{R}}$ and equal to one in $S_{\bar{R}}$, with $\bar{R} \gg R_0$. By Lemma 1.2.13 the problem

$$\operatorname{div} \boldsymbol{\zeta} + \operatorname{div}(g\boldsymbol{v}) = 0 \quad \text{in } T_{\bar{R}}$$

has a solution $\boldsymbol{\zeta} \in W_0^{2,2}(T_R)$. The field

$$\boldsymbol{h} = \boldsymbol{z} + \boldsymbol{\sigma}, \tag{4.3.2}$$

with

$$\boldsymbol{z} = \begin{cases} \boldsymbol{v}, & \text{in } \Omega_{\bar{R}}, \\ \boldsymbol{\zeta} + g\boldsymbol{v}, & \text{in } T_{\bar{R}}, \\ \boldsymbol{0}, & \text{in } \complement S_{\bar{R}} \end{cases} . \tag{4.3.3}$$

is a divergence free extension of $\boldsymbol{a}$ in Ω and $\boldsymbol{z}$ has compact support in Ω. Note that, in particular,

$$\boldsymbol{h} \in L^4_\sigma(\Omega). \tag{4.3.4}$$

Let

$$\Phi = \frac{1}{2\pi} \sum_{i=1}^{m} |\Phi_i|,$$

where

$$\Phi_i = \int_{\partial\Omega_i} \boldsymbol{a} \cdot \boldsymbol{n}$$

and x_i is a fixed point in Ω_i.

The following theorem holds.

Theorem 4.3.1 *Let* $\boldsymbol{f} \in [D_0^{1,2}(\Omega)]^*$, *let* $\boldsymbol{a} \in L^2(\partial\Omega)$. *If*

$$\Phi < 1, \tag{4.3.5}$$

then system (4.1.1) *has a solution*

$$(\boldsymbol{u}, p) \in [D^{1,2}(\complement S_{R_0}) \cap W^{2,1}_{\text{loc}}(\Omega)] \times [D^{1,2}(\complement S_{R_0}) \cap W^{1,1}_{\text{loc}}(\Omega)]. \tag{4.3.6}$$

and if

$$\int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{n} = 0,$$

then

$$p \in D^{2,1}(\complement S_{R_0}). \tag{4.3.7}$$

If $\boldsymbol{a} \in L^q(\partial\Omega)$, $q > 2$, *then*

$$\lim_{t \to 0^+} \boldsymbol{u}_s(\xi + t\boldsymbol{n}) = \boldsymbol{a}(\xi),$$

for almost all $\xi \in \partial\Omega$, *and if* $\boldsymbol{a} \in C(\partial\Omega)$, *then* $\boldsymbol{u} \in C_{\text{loc}}(\overline{\Omega})$. *Moreover, if* $\boldsymbol{f}$ *has a compact support, then*

$$\lim_{r \to +\infty} p(x) = 0, \tag{4.3.8}$$

uniformly, and there is a constant vector $\boldsymbol{\kappa}$ *such that*

$$\lim_{r \to +\infty} \boldsymbol{u}(x) = \boldsymbol{\kappa}, \tag{4.3.9}$$

uniformly.

Proof - We look for a solution of system (4.1.1) expressed by

$$\boldsymbol{u} = \boldsymbol{h} + \boldsymbol{w},$$

with $\boldsymbol{h}$ given by (4.3.2) and $\boldsymbol{w} \in D^{1,2}_{\sigma,0}(\Omega)$ variational solution of equations

$$\begin{aligned} \Delta\boldsymbol{w} - (\boldsymbol{h} + \boldsymbol{w}) \cdot \nabla(\boldsymbol{h} + \boldsymbol{w}) - \nabla Q + \Delta\boldsymbol{h} = \boldsymbol{f} \quad & \text{in } \Omega. \\ \operatorname{div} \boldsymbol{w} = 0 \quad & \text{in } \Omega. \\ \boldsymbol{w} = \boldsymbol{0} \quad & \text{on } \partial\Omega. \end{aligned} \tag{4.3.10}$$

for some pressure field $Q \in L^2_{\text{loc}}(\overline{\Omega})$.

Let $\{R_k\}_{k\in\mathbb{N}}$ be a positive, increasing and unbounded sequence in $(0,+\infty)$, with $R_1 > R_0$. Consider the problem

$$\begin{aligned} \Delta \boldsymbol{w} - (\boldsymbol{h}+\boldsymbol{w})\cdot\nabla(\boldsymbol{h}+\boldsymbol{w}) - \nabla Q + \Delta \boldsymbol{h} = \boldsymbol{f} \ \text{ in } \Omega_{R_k}, \\ \operatorname{div} \boldsymbol{w} = 0 \ \text{ in } \Omega_{R_k}, \\ \boldsymbol{w} = \boldsymbol{0} \ \text{ on } \partial\Omega_{R_k}. \end{aligned} \tag{4.3.11}$$

By Theorem 3.1.1 assumption (4.3.25) assures that system (4.5.5) has a variational solution $(\boldsymbol{w}_k, Q_k) \in W^{1,2}_{\sigma,0}(\Omega_{R_k}) \times L^2(\Omega_{R_k})$, *i.e*, the field $\boldsymbol{w}_k$ satisfies the relation

$$\begin{aligned} \int_\Omega \nabla \boldsymbol{w}_k \cdot \nabla \boldsymbol{\phi} = \int_\Omega (\boldsymbol{h}+\boldsymbol{w}_k)\cdot\nabla\boldsymbol{\phi}\cdot(\boldsymbol{h}+\boldsymbol{w}_k) \\ - \int_\Omega \nabla \boldsymbol{h}\cdot\nabla\boldsymbol{\phi} - \langle \boldsymbol{f}, \boldsymbol{w}\rangle, \end{aligned} \tag{4.3.12}$$

for all $\boldsymbol{\phi} \in W^{1,2}_{\sigma,0}(\Omega_{R_k})$, where we extended $\boldsymbol{w}_k$ to the whole of Ω by setting $\boldsymbol{w}_k = \boldsymbol{0}$ in $\complement S_{R_k}$. Let us show that the sequence $\{\boldsymbol{w}_k\}_{k\in\mathbb{N}}$ is uniformly bounded in $D^{1,2}(\Omega)$, *i.e.*, there is a positive constant c such that

$$J_k^2 = \int_\Omega |\nabla \boldsymbol{w}_k|^2 \le c \tag{4.3.13}$$

for all $k \in \mathbb{N}$. Indeed, if (4.3.13) is not true, following [25], a subsequence exists, still denoted by the same symbol, such that

$$\lim_{k\to+\infty} J_k = +\infty.$$

The field

$$\boldsymbol{w}'_k = \frac{\boldsymbol{w}_k}{J_k}$$

is uniformly bounded in $D^{1,2}_{\sigma,0}(\Omega)$, because

$$\int_\Omega |\nabla \boldsymbol{w}'_k|^2 = 1, \tag{4.3.14}$$

and by (4.3.12)

$$
\begin{aligned}
\frac{1}{J_k}\int_\Omega \nabla \boldsymbol{w}'_k \cdot \nabla \boldsymbol{\phi} = &\int_\Omega \boldsymbol{w}'_k \cdot \nabla \boldsymbol{\phi} \cdot \boldsymbol{w}'_k + \frac{1}{J_k^2}\int_\Omega \boldsymbol{h} \cdot \nabla \boldsymbol{\phi} \cdot \boldsymbol{h} \\
&+ \frac{1}{J_k}\int_\Omega (\boldsymbol{h} \cdot \nabla \boldsymbol{\phi} \cdot \boldsymbol{w}'_k + \boldsymbol{w}'_k \cdot \nabla \boldsymbol{\phi} \cdot \boldsymbol{h} \\
&- \frac{1}{J_k^2}\int_\Omega \nabla \boldsymbol{h} \cdot \nabla \boldsymbol{\phi}) - \frac{1}{J_k^2}\langle \boldsymbol{f}, \boldsymbol{\phi}\rangle.
\end{aligned}
\tag{4.3.15}
$$

In virtue of (4.3.14) by Lemma 1.2.6 we can extract a sequence from $\{\boldsymbol{w}'_k\}_{k\in\mathbb{N}}$, we denote by the same symbol, which converges weakly in $D^{1,2}(\Omega)$ and strongly in $L^q_{\text{loc}}(\overline{\Omega})$, $q \in (1, +\infty)$ to a field $\boldsymbol{w}' \in D^{1,2}_{\sigma,0}(\Omega)$ such that

$$\int_\Omega |\nabla \boldsymbol{w}'|^2 \le 1.$$

Choose $\boldsymbol{\phi} \in C^\infty_{\sigma,0}(\Omega)$ in (4.3.15) and let $k \to +\infty$. By what we said above, proceeding as we did in the proof of Theorem 3.1.1, we see that $\boldsymbol{w}'$ satisfies the Euler equations

$$
\begin{aligned}
\boldsymbol{w}' \cdot \nabla \boldsymbol{w}' + \nabla Q' = \boldsymbol{0} \quad &\text{in } \Omega, \\
\operatorname{div} \boldsymbol{w} = 0 \quad &\text{in } \Omega, \\
\boldsymbol{w} = \boldsymbol{0} \quad &\text{on } \partial\Omega,
\end{aligned}
\tag{4.3.16}
$$

for some pressure field $Q' \in W^{1,2}(\overline{\Omega})$. From Lemma 3.1.1 it follows that Q' is a constant Q'_i on every $\partial\Omega_i$.

Choosing $\boldsymbol{\phi} = \boldsymbol{w}_k$ in (4.3.15), we have

$$
\begin{aligned}
1 = &\int_\Omega \boldsymbol{w}'_k \cdot \nabla \boldsymbol{w}'_k \cdot \boldsymbol{\sigma} + \int_\Omega \boldsymbol{w}'_k \cdot \nabla \boldsymbol{w}'_k \cdot \boldsymbol{z} \\
&+ \frac{1}{J_k}\int_\Omega [\nabla \boldsymbol{w}'_k \cdot (\boldsymbol{h} \otimes \boldsymbol{h} - \nabla \boldsymbol{h})] - \frac{1}{J_k}\langle \boldsymbol{f}, \boldsymbol{w}'_k \rangle.
\end{aligned}
\tag{4.3.17}
$$

Now,

$$\int_\Omega \boldsymbol{w}'_k \cdot \nabla \boldsymbol{w}'_k \cdot \boldsymbol{\sigma} = \sum_{i=1}^m \frac{\Phi_i}{2\pi}\int_\Omega \boldsymbol{w}'_k \cdot \nabla \boldsymbol{w}'_k \cdot \frac{x - x_i}{|x - x_i|^2}.$$

Extending $\boldsymbol{w}'_k$ in Ω' by setting $\boldsymbol{w}'_k = \mathbf{0}$ in Ω', by Lemma 4.2.4 we have

$$\left|\int_\Omega \boldsymbol{w}'_k \cdot \nabla \boldsymbol{w}'_k \cdot \frac{x - x_i}{|x - x_i|^2}\right| = \left|\int_{\mathbb{R}^2} \boldsymbol{w}'_k \cdot \nabla \boldsymbol{w}'_k \cdot \frac{x - x_i}{|x - x_i|^2}\right| \leq \int_\Omega |\nabla \boldsymbol{w}'_k|^2 = 1$$

so that

$$\left|\int_\Omega \boldsymbol{w}'_k \cdot \nabla \boldsymbol{w}'_k \cdot \boldsymbol{\sigma}\right| \leq \Phi.$$

Therefore, (4.3.17) implies

$$\begin{aligned} 1 - \Phi \leq \int_\Omega & \boldsymbol{w}'_k \cdot \nabla \boldsymbol{w}'_k \cdot \boldsymbol{z} \\ & + \frac{1}{J_k} \int_\Omega [\nabla \boldsymbol{w}'_k \cdot (\boldsymbol{h} \otimes \boldsymbol{h} - \nabla \boldsymbol{h})] - \frac{1}{J_k} \langle \boldsymbol{f}, \boldsymbol{w}'_k \rangle. \end{aligned} \tag{4.3.18}$$

Taking into account that $\boldsymbol{z}$ has a compact support and $\boldsymbol{\sigma} = O(r^{-1})$, we get

$$\begin{aligned} \left|\int_\Omega (\boldsymbol{z} + \boldsymbol{\sigma}) \cdot \nabla \boldsymbol{w}'_k \cdot \boldsymbol{z}\right| & \leq \|\nabla \boldsymbol{w}'_k\|_{L^2(\Omega)} \|\boldsymbol{z} + \boldsymbol{\sigma}\|^2_{L^2(T_{\bar{R}})} \|\boldsymbol{z}\|^2_{L^4(\Omega)} \\ & \leq c \|\nabla \boldsymbol{w}'_k\|^2_{L^4(\Omega)} \leq c, \\ \left|\int_\Omega \boldsymbol{\sigma} \cdot \nabla \boldsymbol{w}'_k \cdot \boldsymbol{\sigma}\right| \leq c \int_\Omega \frac{|\nabla \boldsymbol{w}'_k|}{r^2} & \leq c \left\{\int_\Omega \frac{1}{r^4} \int_\Omega |\nabla \boldsymbol{w}'_k|^2\right\}^{1/2} \leq c, \\ \left|\int_\Omega \nabla \boldsymbol{w}'_k \cdot \nabla \boldsymbol{h}\right| & \leq \|\nabla \boldsymbol{w}'_k\|_{L^2(\Omega)} \|\nabla \boldsymbol{h}\|_{L^2(\Omega)} \leq c, \\ |\langle \boldsymbol{f}, \boldsymbol{w} \rangle| \leq \|\nabla \boldsymbol{w}'_k\|_{L^2(\Omega)} & \|\boldsymbol{f}\|_{[D_0^{1,2}(\Omega)]^*} \leq c \end{aligned} \tag{4.3.19}$$

Therefore, we can let $k \to +\infty$ in (4.3.18) and use (4.3.1), (4.3.16) to get

$$\begin{aligned} (1 - \Phi) \leq \int_\Omega \boldsymbol{w}' \cdot \nabla \boldsymbol{w}' \cdot \boldsymbol{z} & = - \int_\Omega \boldsymbol{z} \cdot \nabla Q' \\ & = \sum_{i=1}^m Q'_i \int_{\partial \Omega_i} \boldsymbol{v} \cdot \boldsymbol{n} = 0. \end{aligned} \tag{4.3.20}$$

Since (4.3.20) contradicts (4.3.25), we see that (4.3.13) holds true.

Starting from (4.3.13), let us construct a sequence which converges to a variational solution of system (4.5.5). To this end we follow [41]. By Lemma 1.2.6 we can extract from $\{\boldsymbol{w}_k\}_{k\in\mathbb{N}}$ a subsequence which converges strongly in $L^q(\Omega_{R_1})$, $q \in (1, +\infty)$. Of course, this subsequence still satisfies (4.3.13) so that from it we can extract a subsequence which converges strongly in $L^q(\Omega_{R_2})$, $q \in (1, +\infty)$. Proceeding in this way, we construct a sequence of subsequence of $\{\boldsymbol{w}_k\}_{k\in\mathbb{N}}$, we can write as lines of a matrix, such that the k^{th} line converges strongly in $L^q(\Omega_{R_j})$, $j = 1, \dots k$. Therefore, taking the diagonal of this matrix, we have a sequence $\{\tilde{\boldsymbol{w}}_k\}$ which converges strongly in $L^q(\Omega_{R_k})$ for all $k \in \mathbb{N}$ and weakly in $D^{1,2}(\Omega)$ to a field $\tilde{\boldsymbol{w}} \in D^{1,2}_{\sigma,0}(\Omega)$. Therefore, for all $\boldsymbol{\phi} \in C^{\infty}_{\sigma,0}(\Omega)$,

$$\lim_{k\to+\infty} \int_{\Omega} \nabla\tilde{\boldsymbol{w}}_k \cdot \nabla\boldsymbol{\phi} = \int_{\Omega} \nabla\tilde{\boldsymbol{w}} \cdot \nabla\boldsymbol{\phi}$$

and, proceeding as we did in the proof of Theorem 3.1.1, we get

$$\lim_{k\to+\infty} \int_{\Omega} (\boldsymbol{h} + \tilde{\boldsymbol{w}}_k) \cdot (\nabla\boldsymbol{\phi}) \cdot (\boldsymbol{h} + \tilde{\boldsymbol{w}}_k) = \int_{\Omega} (\boldsymbol{h} + \tilde{\boldsymbol{w}}) \cdot (\nabla\boldsymbol{\phi}) \cdot (\boldsymbol{h} + \tilde{\boldsymbol{w}}).$$

Hence $\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{h} \in D^{1,2}(\complement S_{R_0})$ is a variational solution of system (4.5.5) and (4.3.6), (4.3.7), (4.3.8) have been proved in Lemma 4.3.1. The regularity properties follows from the analogous ones we proved for bounded domains.

By Lemma 4.3.1 to prove (4.3.9) it is sufficient to show that $\boldsymbol{u}$ is bounded in a neighborhood of the infinity. To this end we follow [19]. If we repeat the proof of Lemma 4.2.5 with a the function ηg where g is given by (4.2.14) and η is a regular function such that

$$\eta(r) = \begin{cases} 0, & r < R_k, \\ 1, & r > R_0, \end{cases}$$

we see that

$$\int_{S_{R_k^{e-\delta}} \setminus S_{R_0}} |\nabla\omega_k|^2 \le c, \tag{4.3.21}$$

uniformly on k. Making use of (4.3.21) and proceeding as we did to prove (4.2.20), with ηg instead of ϕ, we have in particular that

$$\|p_k\|_{L^\infty(S_{3R/4}\setminus S_{R_0})} \le c \tag{4.3.22}$$

uniformly on k. By the integral theorem of the mean, there is $\rho \in (R_k/2, 3R_k/4)$ such that

$$\begin{aligned}\log(3/2)\int_0^{2\rho}|\partial_\theta \boldsymbol{u}_k|^2(\rho,\theta) &= \int_{R_k/2}^{3R_k/4}\frac{1}{r}\int_0^{2\pi}|\partial_\theta \boldsymbol{u}_k|^2 \\ &\le \int_{S_{3R_k/4}\setminus S_{R_k/2}}|\nabla \boldsymbol{u}_k|^2 \le c.\end{aligned} \tag{4.3.23}$$

Integrating the identity

$$\boldsymbol{u}(\rho,\theta)-\boldsymbol{u}(\rho,\alpha)=\int_\alpha^\theta \partial_\theta \boldsymbol{u}(\rho,\vartheta)$$

on $\alpha \in (0,2\pi)$ and using Schwarz's inequality and (4.3.23), we have

$$|\boldsymbol{u}(\rho,\theta)-\bar{\boldsymbol{u}}(\rho)| \le c\int_0^{2\pi}|\partial_\theta \boldsymbol{u}_k|^2 \le c.$$

Therefore, since for $R > R_k/8$,

$$\begin{aligned}|\bar{\boldsymbol{u}}(R)| = |\bar{\boldsymbol{u}}(R)-\bar{\boldsymbol{\sigma}}(R_k)| &= |\bar{\boldsymbol{u}}(R)-\boldsymbol{u}(R_k)| = \left|\int_R^{R_k}\int_0^{2\pi}\partial_r \boldsymbol{u}\right| \\ &\le \frac{1}{2\pi}\left\{\int_R^{R_k} r\int_0^{2\pi}|\partial_r \boldsymbol{u}|^2\right\}^{1/2}\left\{\int_R^{R_k}\frac{1}{r}\right\}^{1/2} \\ &\le \left(\frac{\log 8}{2\pi}\right)^{\frac12}\left\{\int_{\Omega_{R_k}}|\nabla \boldsymbol{w}_k|^2\right\}^{1/2} \le c,\end{aligned}$$

it follows that for every k there is $\rho \in (R_k/2, 3R_k/4)$ such that

$$|\boldsymbol{u}(\rho,\theta)| \le c \tag{4.3.24}$$

uniformly in k. The *head pressure* field

$$\Pi_k = p_k + \tfrac12|\boldsymbol{u}_k|^2$$

satisfies the inequality

$$\Delta \Pi_k - \boldsymbol{u}_k\cdot\nabla \Pi_k = \omega_k^2 \ge 0,$$

then by the classical *maximum principle* (see, *e.g.*, [27])

$$\max_{S_{R_k/2}} \Pi_k \le \max_{S_\rho \setminus S_{R_0}} \Pi_k \le \max_{\partial S_\rho} \Pi_k + \max_{\partial S_{R_0}} \Pi_k \le c$$

uniformly in k. Then, bearing in mind (4.3.22), we see that there is a positive constant c independent of k such that

$$|\boldsymbol{u}_k(x)| \le c$$

for all $x \in S_{R_k/2} \setminus S_{R_0}$. Hence the desired result follows at once. □

If Ω is Lipschitz[2] and $\boldsymbol{a} \in W^{1/2,2}(\partial\Omega)$, then we construct the divergence free extension of $\boldsymbol{h}$ starting from the variational solution of the Stokes problem and, by literally repeating the above argument, we can prove the following theorem [36].

Theorem 4.3.2 *Let Ω be an exterior Lipschitz domain of $\mathbb{R}^2$. If $\boldsymbol{a} \in W^{1/2,2}(\partial\Omega)$, $\boldsymbol{f} \in [D_0^{1,2}(\Omega)]^*$ and*

$$\Phi < 1, \tag{4.3.25}$$

then system (4.1.1) has a variational solution $\boldsymbol{u} \in D^{1,2}(\Omega)$.

[2]This means that for all ξ is $\partial\Omega$ there is a neighboorhood of ξ (on $\partial\Omega$) which is a graph of a Lipschitz continuous function.

4.4 Existence of polar symmetric solutions

Let Ω is symmetric with respect to o, *i.e.*,

$$x \in \Omega \Rightarrow -x \in \Omega.$$

By a polar symmetric function we mean a scalar field φ in in Ω (or on $\partial\Omega$) such that

$$\varphi(-x) = -\varphi(x),$$

almost everywhere. It is clear that if $H(\subset L^1_{\rm loc}(\Omega))$ is a Banach space, the set of all polar symmetric functions of H is a closed subspace of H. Therefore, if $\boldsymbol{a}$ and $\boldsymbol{f}$ are polar symmetric then the reasoning we used in the proof of Theorem 4.3.1 yields existence of a polar symmetric solution $\boldsymbol{u}$; the associated pressure field p is an even function of x: $p(x) = p(-x)$. Since

$$\int_0^{2\pi} \boldsymbol{u}(R, \theta) = \boldsymbol{0},$$

for all $R > R_0$, by (1.2.9)

$$\int_{T_R} |\boldsymbol{u}|^2 = \int_{T_R} |\boldsymbol{u} - \boldsymbol{u}_{T_R}|^2 \le c \int_{T_R} |\nabla \boldsymbol{u}|^2. \tag{4.4.1}$$

Coupling (4.4.1) with the trace theorem gives

$$\int_0^{2\pi} |\boldsymbol{u}|^2(R, \theta) \le \frac{c}{R^2} \int_{T_R} |\boldsymbol{u}|^2 + \int_{T_R} |\nabla \boldsymbol{u}|^2 \le c \int_{T_R} |\nabla \boldsymbol{u}|^2.$$

Therefore, if $\boldsymbol{u}$ is a D–solution, then

$$\lim_{r \to +\infty} \int_0^{2\pi} |\boldsymbol{u}|^2 = 0 \tag{4.4.2}$$

and we can state

Theorem 4.4.1 *Let Ω be polar symmetric and let $(\boldsymbol{u}, p)$ be the solution of Theorem (4.3.2). If*

$$\boldsymbol{a}(\xi) = -\boldsymbol{a}(-\xi),$$

for almost all $\xi \in \partial\Omega$ *and*

$$\boldsymbol{f}(x) = -\boldsymbol{f}(-x),$$

for almost all $x \in \Omega$, *then*

$$\lim_{r\to+\infty} \boldsymbol{u}(x) = \boldsymbol{0}$$

uniformly.

▶ The above theorem is a slight improvement of a results of G.P. Galdi [17] and, as far as we are aware, it is the only case where we know the constant vector $\boldsymbol{u}_0$ to which the D–solution $\boldsymbol{u}$ (in Theorem 4.3.1) converges. Note that if $\boldsymbol{a} \in W^{1/2,2}(\partial\Omega)$, then we can assume Ω to be only Lipschitz [36]. ◁

• *Remark 4.4.1*

Let Ω be polar symmetric and let $\boldsymbol{\psi} \in \mathfrak{C}$. Since

$$\begin{aligned}\mathcal{S}[\boldsymbol{\psi}](-\xi) &= \int_{\partial\Omega} \boldsymbol{U}(-\xi-\zeta)\boldsymbol{\psi}(\zeta)\,\mathrm{d}a_\zeta = \int_{\partial\Omega} \boldsymbol{U}(-\xi+\zeta)\boldsymbol{\psi}(-\zeta)\,\mathrm{d}a_\zeta \\ &= \int_{\partial\Omega} \boldsymbol{U}(\xi-\zeta)\boldsymbol{\psi}(-\zeta)\,\mathrm{d}a_\zeta = \mathcal{S}[\boldsymbol{\psi}](\xi) = \int_{\partial\Omega} \boldsymbol{U}(\xi-\zeta)\boldsymbol{\psi}(\zeta)\,\mathrm{d}a_\zeta,\end{aligned}$$

we have

$$\int_{\partial\Omega} \boldsymbol{U}(\xi-\zeta)[\boldsymbol{\psi}(\zeta)-\boldsymbol{\psi}(-\zeta)]\,\mathrm{d}a_\zeta = \boldsymbol{0} \tag{4.4.3}$$

for all $\xi \in \partial\Omega$. If $\dim \mathfrak{M}_0 = 0$, then (4.4.3) yields

$$\boldsymbol{\psi}(\zeta) = \boldsymbol{\psi}(-\zeta)$$

for all $\zeta \in \partial\Omega$ so that $\boldsymbol{\psi}$ is an even function of ζ. If $\dim \mathfrak{M}_0 \neq 0$, then (4.4.3) implies that $\boldsymbol{\psi}(\zeta) - \boldsymbol{\psi}(-\zeta) = \bar{\boldsymbol{\psi}}(\zeta) \in \mathfrak{M}_0$. If $\boldsymbol{\psi}$ were not an even function, then we should have

$$\bar{\boldsymbol{\psi}}(-\zeta) = \boldsymbol{\psi}(-\zeta) - \boldsymbol{\psi}(\zeta) = -\bar{\boldsymbol{\psi}}(\zeta),$$

so that $\bar{\boldsymbol{\psi}}(\zeta)$ should be an odd function of ζ. Hence

$$\int_{\partial\Omega} \bar{\boldsymbol{\psi}} = \mathbf{0}.$$

Since this is absurd, we conclude that $\boldsymbol{\psi}(\zeta)$ is an even function of ζ. Therefore, the data $\boldsymbol{a}$ and $\boldsymbol{f}$ in Theorem 4.4.1 satisfy

$$\int_{\partial\Omega} \boldsymbol{a}\cdot\boldsymbol{\psi} = \int_{\partial\Omega} \boldsymbol{f}\cdot\boldsymbol{\psi} = 0$$

so that in our particular case we have the compatibility condition for the existence of a solution of the Stokes problem in an exterior domain

$$\int_{\partial\Omega} \boldsymbol{a}\cdot\boldsymbol{\psi} - \int_{\partial\Omega} \boldsymbol{f}\cdot\boldsymbol{\psi} = \boldsymbol{u}_0\cdot\int_{\partial\Omega} \boldsymbol{\psi}. \tag{4.4.4}$$

It is then reasonably to ask whether a condition like (4.4.4) is necessary for the existence of a solution of system (4.1.1), (4.1.2). For "small"

$$\boldsymbol{u}_0 \neq \mathbf{0} \tag{4.4.5}$$

this is excluded by the results in [10]. Hence it follows that in the nonlinear case and under assumption (4.4.5) the Stokes paradox (in general) does not hold. ♢

• *Remark 4.4.2*

Note that if $\boldsymbol{u}_0 = \boldsymbol{e}_1$ and $\boldsymbol{a}$, $\boldsymbol{f}$ are zero, then the Leray method yields to a bounded sequence (in $D^{1,2}(\Omega)$) of solutions $\{\boldsymbol{u}_k\}$ of the equations

$$\begin{aligned} \Delta\boldsymbol{u}_k - \mathcal{R}\boldsymbol{u}_k\cdot\nabla\boldsymbol{u}_k - \nabla Q_k &= \mathbf{0} \quad \text{in } \Omega_{R_k},\\ \operatorname{div}\boldsymbol{u}_k &= 0 \quad \text{in } \Omega_{R_k},\\ \boldsymbol{u}_k &= \mathbf{0} \quad \text{on } \partial\Omega,\\ \boldsymbol{u}_k &= \boldsymbol{e}_1 \quad \text{on } \partial\Omega_{R_k}, \end{aligned}$$

where $\mathcal{R}$ is the Reynolds number. Then $\boldsymbol{u}_k$ converges uniformly in every compact of Ω to a classical solution of the system

$$\begin{aligned} \Delta\boldsymbol{u} - \mathcal{R}\boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla Q &= \mathbf{0} \quad \text{in } \Omega,\\ \operatorname{div}\boldsymbol{u} &= 0 \quad \text{in } \Omega,\\ \boldsymbol{u} &= \mathbf{0} \quad \text{on } \partial\Omega \end{aligned} \tag{4.4.6}$$

and there is a *mysterious vector* $\boldsymbol{\kappa}$ such that

$$\lim_{r\to+\infty} \boldsymbol{u}(x) = \boldsymbol{\kappa}$$

uniformly.

- Is the vector $\boldsymbol{\kappa}$ nonzero?

This interesting problem was posed and solved by C.J. Amick [2] for regular domains (of class C^3, say) symmetric with respect to the x_1–axis (see also [17]). For general regular domains or for less regular symmetric domain (Lipschitz, say) it is open. A simple consequence of Amick's results is that the Stokes paradox is typical of the linear problem. Indeed, at least for symmetric domains

$$\boldsymbol{\kappa} = \mathbf{0} \;\Leftrightarrow\; \mathcal{R} = 0.$$

$\diamondsuit$

4.5 Existence of symmetric solutions

In this section we aim at extending in some sense a classical result of C. J. Amick [1][3] to exterior domains.

Let us use the notations and definitions of Section 3.3. Accordingly, Ω_i are m simply connected domains, symmetric with respect to the x_1–axis, $\partial\Omega_i \cap \{x_2 = 0\} = \varnothing$, for every i, and $\boldsymbol{a}$ is symmetric fields. Assume, as is always possible, that $o \in \Omega'$.

If $\boldsymbol{\alpha} \in L^2(\partial\Omega)$ is symmetric and satisfies

$$\int_{\partial\Omega} \boldsymbol{\alpha} \cdot \boldsymbol{n} = 0,$$

then by H. Morimoto's argument outlined in Section 3.5, we arrive at constructing a divergence free symmetric extension of $\boldsymbol{\alpha}$ expressed by

$$\boldsymbol{z} = \operatorname{curl}(g_\delta \boldsymbol{\varphi}), \tag{4.5.1}$$

where g_δ is the function defined by (3.5.4) and the estimate

$$\left| \int_\Omega \boldsymbol{w} \cdot \nabla \boldsymbol{w} \cdot \boldsymbol{z} \right| \leq c(\delta_0) \int_\Omega |\nabla \boldsymbol{w}|^2, \tag{4.5.2}$$

holds for all $\boldsymbol{w} \in D^{1,2}_{\sigma,0}(\Omega)$, with

$$\lim_{\delta_0 \to 0} c(\delta_0) = 0.$$

Let (4.5.1) be the extension of $\boldsymbol{a} - \boldsymbol{\sigma}_0$ in Ω, with

$$\boldsymbol{\sigma}_0(x) = -\frac{\boldsymbol{e}_r}{2\pi|\boldsymbol{x}|} \int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{n}.$$

Of course, the field

$$\boldsymbol{h} = \boldsymbol{z} + \boldsymbol{\sigma}_0$$

is a divergence free symmetric extension of $\boldsymbol{a}$ in Ω.

The following theorem holds.

[3] see Section 3.3

Theorem 4.5.1 *Let Ω be an exterior domain of $\mathbb{R}^2$ of class C^2, symmetric with respect to the x_1–axis. If $\boldsymbol{f} \in [D_0^{1,2}(\Omega)]^*$ and $\boldsymbol{a} \in L^2(\partial\Omega)$ are symmetric and*

$$\left| \int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{n} \right| < 2\pi, \tag{4.5.3}$$

then system (4.1.1) has a D–solution solution $(\boldsymbol{u}, p)$ and there is a scalar $\alpha \in [0,1]$ such that

$$\boldsymbol{\kappa} = \alpha \boldsymbol{e}_1. \tag{4.5.4}$$

Proof - Let $\{R_k\}_{k\in\mathbb{N}}$ be a positive, increasing and unbounded sequence in $(0, +\infty)$, with $R_1 > R_0$. By Theorem (3.3.1) the system

$$\begin{aligned} \Delta \boldsymbol{w} - (\boldsymbol{h} + \boldsymbol{w}) \cdot \nabla(\boldsymbol{h} + \boldsymbol{w}) - \nabla Q + \Delta \boldsymbol{h} = \boldsymbol{f} \quad &\text{in } \Omega_{R_k} \\ \operatorname{div} \boldsymbol{w} = 0 \quad &\text{in } \Omega_{R_k}, \\ \boldsymbol{w} = \boldsymbol{0} \quad &\text{on } \partial\Omega_{R_k}, \end{aligned} \tag{4.5.5}$$

has a solution $\boldsymbol{w}_k \in D^{1,2}_{\sigma,0}(\Omega)$. Therefore, we can repeat *ad litteram* the proof of Theorem 4.3.1 from (4.5.5) to (4.3.18) to get

$$\begin{aligned} 1 - \frac{1}{2\pi} \left| \int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{n} \right| \leq \int_{\Omega} \boldsymbol{w}'_k \cdot \nabla \boldsymbol{w}'_k \cdot \boldsymbol{z} \\ + \frac{1}{J_k} \int_{\Omega} [\nabla \boldsymbol{w}'_k \cdot (\boldsymbol{h} \otimes \boldsymbol{h} - \nabla \boldsymbol{h})] - \frac{1}{J_k} \langle \boldsymbol{f}, \boldsymbol{w}'_k \rangle. \end{aligned}$$

Hence, taking into account (4.5.2) and choosing δ_0 such that

$$1 - \frac{1}{2\pi} \left| \int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{n} \right| - c(\delta_0) = c_0(\delta_0) > 0,$$

it follows

$$c_0(\delta_0) \leq \frac{1}{J_k} \int_{\Omega} [\nabla \boldsymbol{w}'_k \cdot (\boldsymbol{h} \otimes \boldsymbol{h} - \nabla \boldsymbol{h})] - \frac{1}{J_k} \langle \boldsymbol{f}, \boldsymbol{w}'_k \rangle. \tag{4.5.6}$$

Then, using (4.3.19) we are allowed to let $k \to +\infty$ in (4.5.6) to have

$$c_0(\delta_0) = 0$$

and, as a consequence,

$$\left|\int_{\partial\Omega} \boldsymbol{a}\cdot\boldsymbol{n}\right| \geq 2\pi.$$

Since this contradicts hypothesis (4.5.3), the argument used in the last part of the proof of Theorem 4.3.1 yields existence of a symmetric solution. Finally, (4.5.4) follows from the fact that

$$\int_0^{2\pi} u_2(R,\theta) = 0$$

for large R and Lemma 4.1 of [17]. □

• *Remark 4.5.1*

Let

$$\mathbb{R}^2_+ = \{x \in \mathbb{R}^2 : x_2 > 0\}.$$

In the unbounded domain

$$\Omega = \mathbb{R}^2_+ \cap \complement\overline{\Omega'}$$

where Ω' is symmetric with respect to the x_1–axis, consider the mixed problem

$$\begin{aligned} \Delta\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla p = \boldsymbol{0} \quad & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 \quad & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{a} \quad & \text{on } \Gamma, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad & \text{on } \Sigma, \\ \boldsymbol{t}\cdot\boldsymbol{T}(\boldsymbol{u},p)\boldsymbol{n} = 0 \quad & \text{on } \Sigma, \end{aligned} \tag{4.5.7}$$

where

$$\begin{aligned} \Gamma &= \partial\Omega' \cap \mathbb{R}^2_+, \\ \Sigma &= \partial\Omega \setminus \Gamma \end{aligned}$$

By reasoning as we did to prove Theorem 3.4.1, it is not difficult to show that if $\boldsymbol{a} \in L^2(\Gamma)$ and

$$\left|\int_{\partial\Omega} \boldsymbol{a}\cdot\boldsymbol{n}\right| < 4\pi,$$

then system (4.5.7) has a solution $(\boldsymbol{u}, p)$ such that

$$\int_{\Omega\setminus S_{R_0}} |\nabla \boldsymbol{u}|^2 < +\infty$$

there is a scalar $\alpha \in [0,1]$ such that

$$\lim_{r\to+\infty} \boldsymbol{u}(x) = \alpha \boldsymbol{e}_1$$

uniformly. $\diamondsuit$

It is not difficult to see that if $m = 2k+1$, $k \in \mathbb{N}_0$, $\partial\Omega_i$ is symmetric with respect to both the coordinate axes:

$$(\xi_1, \xi_2) \in \partial\Omega \ \Rightarrow (-\xi_1, \xi_2), (\xi_1, -\xi_2) \in \partial\Omega,$$

$o \in \Omega_k$ and

$$\begin{aligned} a_1(x_1,x_2) &= -a_1(-x_1,x_2) = a_1(x_1,-x_2),\\ a_2(x_1,x_2) &= a_2(-x_1,x_2) = -a_2(x_1,-x_2),\\ f_1(\xi_1,\xi_2) &= -f_1(-\xi_1,\xi_2) = f_1(\xi_1,-\xi_2),\\ f_2(\xi_1,\xi_2) &= f_2(-\xi_1,\xi_2) = -f_2(\xi_1,-\xi_2), \end{aligned} \tag{4.5.8}$$

then we can repeat the argument in the proof of Theorem 4.5.1 to have

Theorem 4.5.2 *Let Ω be an exterior domain of $\mathbb{R}^2$ of class C^2, symmetric with respect to the coordinate axes. If $\boldsymbol{f} \in [D_0^{1,2}(\Omega)]^*$, $\boldsymbol{a} \in L^2(\partial\Omega)$ satisfy (4.5.8) and (4.5.3) holds, then system (4.1.1) has a D–solution solution $(\boldsymbol{u}, p)$ and*

$$\lim_{r\to+\infty} \boldsymbol{u}(x) = \boldsymbol{0},$$

uniformly.

• *Remark 4.5.2*

Let

$$\mathbb{Q} = \{x \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}.$$

and let $\partial\Omega'$ be connected.

Consider system (4.5.7) in the unbounded domain

$$\Omega = \mathbb{Q} \cap \complement\overline{\Omega}',$$

where Ω' is symmetric with respect to the coordinate axes and

$$\Gamma = \partial\Omega' \cap \mathbb{Q},$$
$$\Sigma = \partial\Omega \setminus \Gamma.$$

Once again, by reasoning as we did to prove Theorem 3.4.1 (with minor modification), it is not difficult to show that if $\boldsymbol{a} \in L^2(\Gamma)$ and

$$\left| \int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{n} \right| < 8\pi,$$

then (4.5.7) has a solution $(\boldsymbol{u}, p)$ such that

$$\int_{\Omega \setminus S_{R_0}} |\nabla \boldsymbol{u}|^2 < +\infty$$

and

$$\lim_{t \to +\infty} \boldsymbol{u}(x) = \boldsymbol{0}$$

uniformly. $\diamondsuit$

Bibliography

[1] C.J. Amick: Existence of solutions to the nonhomogeneous steady Navier–Stokes equations, *Indiana Univ. math. J.* **33** (1984), 817–830. 73, 82, 131

[2] C.J. Amick: On Leray's problem of steady Navier–Stokes flow past a body in the plane, *Acta Math.* **161** (1984), 71–130. 118, 130

[3] W. Borchers and K. Pileckas: Note on the flux problem for stationary Navier–Stokes equations in domains with multiply connected boundary, *Acta App. Math.* **37** (1994), 21–30. 5, 13

[4] B. Carbonaro and G. Starita, The principle of virtual velocities, *Quaderni Mat.* **1** (1997), 3–95. 1, 9

[5] R.R. Coifman, J.L. Lions, Y. Meier and S. Semmes: Compensated compactness and Hardy spaces , *J. Math. Pures App.* IX Sér. 72, 247–286 (1993). 27

[6] R.R. Coifman and G. Weiss: Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.* **83** (1977), pp. 569645. 89

[7] R. Dautray and J.L. Lions, *Mathematical analysis and numerical methods for science and technology*, vol IV, Springer–Verlag (1990). 37

[8] L.C. Evans: *Partial differential equations*, AMS (1998). 22

[9] R. Finn: On the steady-state solutions of the Navier–Stokes equations. III, *Acta Math.* **105** (1961), 197–244. 5, 13

[10] R. Finn and D.R. Smith: On the stationary solutions of the Navier–Stokes equations in two dimensions *Arch. Ration. Mech. Anal.* **25** (1967), 26–39. 100, 129

[11] H. Fujita: On the existence and regularity of the steady–state solutions of the Navier–Stokes equation, *J. Fac. Sci. Univ. Tokyo*, **9** (1961), 59–102. 88

[12] H. Fujita and H. Morimoto: A remark on the existence of the Navier-Stokes flow with non-vanishing outflow condition, *GAKUTO Internat. Ser. Math. Sci. Appl.* **10** (1997), 53–61. 6, 14, 15, 95

[13] G.P. Galdi: On the existence of steady motions of a viscous flow with non–homogeneous conditions, *Le Matematiche* **66** (1991), 503–524. 5, 13, 27, 88

[14] G.P. Galdi: *An Introduction to the Mathematical Theory of the Navier–Stokes Equations*, vol. I, revised edition, Springer Tracts in Natural Philosophy (ed. C. Truesdell) **38**, Springer–Verlag (1998). 2, 10, 31, 33, 88

[15] G.P. Galdi: *An Introduction to the Mathematical Theory of the Navier–Stokes Equations*, vol. II revised edition, Springer Tracts in Natural Philosophy (ed. C. Truesdell) **39**, Springer–Verlag (1998) 4, 12, 31, 33, 100, 101

[16] G.P. Galdi: On the steady self–propelled motion of a body in a viscous incompressible fluid, *Arch. Rational Mech. Anal.* **148** (1999), 53–88.

[17] G.P.Galdi: Stationary Navier-Stokes problem in a two-dimensional exterior domain, in *Stationary partial differential equations* Vol. I, 71–155, Handb. Differ. Equ., North-Holland (2004). 59

[18] G.P. Galdi and C. G. Simader, Existence, uniqueness and L^q estimates for the Stokes problem in an exterior domain, *Arch. Rational Mech. Anal.* **112** (1990), 291–318. 5, 13, 14, 100, 101, 103, 128, 130, 133

[19] D. Gilbarg and H. Weinberger: Asymptotic properties of Leray's solutions of the stationary two–dimensional Navier–Stokes equations, *Russian Math. Surveys* **29** (1974), 109–123. 3, 11, 67

[20] D. Gilbarg and H.F. Weinberger: Asymptotic properties of steady plane solutions of the Navier–Stokes equations with bounded Dirichlet integral, *Ann. Scuola Norm. Sup. Pisa* (4) **5** (1978), 301–404 5, 13, 14, 124

[21] E. Giusti: *Metodi Diretti nel Calcolo delle Variazioni*, Unione Matematica Italiana (1994). 101, 104

[22] M. E. Gurtin: *An introduction to continuum mechanics*, Pergamon Press (1981) 24

[23] J. Happel and H. Brenner: *Low Reynolds Number Hydrodynamics*, (1983) Martin Nijhoff Publishers. 1, 9, 31, 32, 76

[24] O.A. Ladyzhenskaia: *The Mathematical theory of viscous incompressible fluid*, Gordon and Breach (1969). 7, 15

[25] J. Leray: Étude de diverses équations intégrales non linéaire et de quelques problèmes que pose l'hydrodynamique, *J. Math. Pures Appl.* **12** (1933), 1–82. 3, 12, 33, 36, 42, 43, 46, 57

[26] P., Maremonti, R. Russo, R. and G. Starita : On the Stokes equations: the boundary value problem. *Quad. Mat.* **4** (1999), 69–140. 4, 12, 71, 88, 121

[27] C. Miranda : *Partial differential equations of elliptic type*. Springer–Verlag, 1970. 3, 7, 11, 15, 46, 51, 57

[28] C. Miranda: *Istituzioni di analisi funzionale lineare*. Unione Matematica Italiana, Oderisi Gubbio Editrice (1978). 40, 43, 45, 108, 112, 126

[29] H. Morimoto: A remark on the existence of a 2–D steady Navier–Stokes flow in bounded symmetric domain under general outflow condition, . *J. Math. Fluid Mech.* **9** (2007), 411–418. 21, 22, 24, 40, 83

[30] H. Morimoto: General outflow condition for Navier–Stokes flow, *Recent topics on mathematical theory of viscous incompressible fluid*,ed. H. Kozono - Y. Shibata, Kinokuniya, Tokyo, Lecture Notes in Num. Appl. Anal. **16** (1998), 209–224. 92

95

[31] H. Morimoto and S. Ukai: Perturbation of the NavierStokes flow in an annular domain with the non-vanishing outflow condition, *J. Math. Sci., Univ. Tokyo* **3** (1996), 7382. 95

[32] F.K.G. Odqvist: Über die randwertaufgaben der hydrodynamik zäher flüssigkeiten, *Math. Z.* **32** (1930), 329–375. 3, 11

[33] J. Necas: *Les méthodes directes en théorie des équations élliptiques*, Masson-Paris and Academie-Prague (1967). 49

[34] S. Rionero, *Lezioni di Meccanica Razionale*, Liguori editore, Napoli (1973). 1, 9

[35] W. Rudin, *Real and Complex analysis*, McGraw-Hill (1966). 40

[36] A. Russo: A note on the exterior two–dimensional steady-state Navier–Stokes problem, *J. Math. Fluid. Mech.* (2007) online. 5, 7, 14, 15, 100, 126, 128

[37] A. Russo and G. Starita: On the existence of steady-state solutions to the Navier–Stokes system for large fluxes, *Ann. Scuola Nor. Sup. Pisa* Ser. V **7** (2008), 171–180. 6, 7, 15, 95, 96

[38] A. Russo and G. Starita: A mixed problem for the steady Navier–Stokes equations, *Math. Comp. Moleling* doi.10.1016/j.m.c.m.2007.12.026 82, 87

[39] R. Russo: On the existence of solutions to the stationary Navier–Stokes equations, *Ricerche Mat.* **52** (2003), 285–348. 3, 11, 49, 52, 57

[40] L.I. Sazonov: On the existence of a stationary symmetric solution of the two–dimensional fluid flow problem, *Math. Zametki* **54** (1963), 138–141 (in Russian); English transl.: *Math. Notes* **54** (1963), 1280–1284.

[41] H. Sohr: *Navier–Stokes equations. An elementary functional analytic approach*, Birkhäuser (2001). 5, 13

[42] G. Starita and A. Tartaglione: On the Neumann problem for the Stokes system, *Math. Meth. Mod. Appl. Sci.*, **12** (2002), 813-834 124

[43] V.A. Solonnikov: On an estimate for the maximum modulus of the solution of a stationary problem for Navier-Stokes equations, *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov* (POMI) **249** (1997), 294–302; english transl.: *J. Math. Sci.* (New York) **101** (2000), 3563–3569. 61, 66

[44] E. Stein: *Harmonic analysis: real–variables methods, orthogonality and oscillatory integrals*, Princeton University Press (1993). 88

[45] R. Temam: *Navier–Stokes equations*, North–Holland (1977). 27, 40

33