Advances in Just-Non-$\mathfrak{X}$ Groups and Minimal Non-$\mathfrak{X}$ Groups

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Abstract

Let $\mathfrak{X}$ be a class of groups. A group which does not belong to $\mathfrak{X}$ but all of whose proper quotients belong to $\mathfrak{X}$ is called Just-Non-$\mathfrak{X}$ group. A group which does not belong to $\mathfrak{X}$ but all of whose proper subgroups belong to $\mathfrak{X}$ is called Minimal-Non-$\mathfrak{X}$ group. Just-Non-$\mathfrak{X}$ groups and Minimal-Non-$\mathfrak{X}$ groups are correlated by structural results for different choices of the class $\mathfrak{X}$. Many authors investigated these groups and there is a long standing line of research in such a topic. Here some recent results have been shown in the context of the generalized FC-groups and in the context of the topological groups.

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Contents

1. The Problem in Literature

If $\mathfrak{X}$ is a class of groups, a group G which belongs to $\mathfrak{X}$ is said to be an $\mathfrak{X}$-*group.* A group G is said to be a *Just-Non-$\mathfrak{X}$ group*, or briefly a *JNX group* , if it is not an $\mathfrak{X}$-group but all of its proper quotients are $\mathfrak{X}$-groups. Of course, every simple group which is not an $\mathfrak{X}$-group is a JNX group, so the simple groups constitute an easy source of examples for Just-Non-$\mathfrak{X}$ groups. The structure of Just-Non-$\mathfrak{X}$ groups has already been studied for several choices of the class $\mathfrak{X}$, so there is a well developed theory about the topic (see [44]). Moreover the study of JNX groups has been investigated both in finite groups and infinite groups so that many techniques have general application (see [5, Section 2.3] and [44]).

H. Schunk was interested in studying JNX groups with respect to some problems of Local Theory of Finite Groups as [5, Theorem 2.3.7] and [20, Chapter 3] testify. JNX groups were called *groups of boundary* $\mathfrak{X}$ in the original works of H. Schunk and results of factorization as [5, Theorem 2.3.15, Propositions 2.3.16, 2.3.17, Theorems 2.3.20, 2.3.24, 2.4.12, Statements 6.5.10–6.5.19, Theorem 6.5.21, Corollary 6.5.22] were obtained. Further classical references can be found in [20, Chapters 6, 11].

Most of the times, the literature on JNX groups shows that their description overlaps either the results of H. Schunk, which we just mentioned, or a well-known splitting's theorem of I. Schur and H. Zassenhaus (see [20, 18.1, 18.2] or [68, 9.1.2]). In the context of locally finite groups, we may find generalizations of the results of H. Schunk [20, Chapters 6, 11], as described in [19, Chapter 6]. A variation of the results [20, Chapters 6, 11] has been recently given in the context of compact groups by [71]. Here, a compact group which is not a Lie group but all of its proper Hausdorff quotients are Lie groups has been investigated. This is the further proof that many techniques and methods have general application in topics concerning JNX groups.

On the other hand, the knowledge of JNX groups is often accompanied by the following notion, which is dual in a certain sense.

A group G is called *$\mathfrak{X}$-critical group*, or *Minimal-Non-$\mathfrak{X}$ group*, or briefly MNX group, if G is not an $\mathfrak{X}$-group but all of its proper subgroups are $\mathfrak{X}$-groups. There is a long standing line of research on MNX groups, as we can see in [68, Theorem 9.1.9, Exercise 9.1.11, Theorem 10.3.3] and [20, pp.59, 330, 402, 408, 480, 515, 525, 781]. Such a literature shows that the terminology and the notations are not uniform and some results can be found independently by means of different approaches (see [5, Theorem 6.4.4] and [67, Theorem 3.44]). For instance, we note that the terminology Minimal-Non-$\mathfrak{X}$ group is adopted by [50] and [67], while the terminology $\mathfrak{X}$-critical group is adopted by [4] and [20].

The reason why Just-Non-$\mathfrak{X}$ groups and Minimal-Non-$\mathfrak{X}$ groups are correlated is due to an unexpected symmetry in their structure as it is clear by comparing [44, Theorems 11.1, 11.2, 12.26, 12.30, 14.1, 14.2, 14.8, 14.10, 14.18, 14.19, 15.4, 15.5, 15.11, 16.21, 16.24, 16.28, 16.30, 16.31, 16.32, 16.33, 17.5, 17.7, 17.8, 17.9, Corollaries 12.27, 12.28, 12.29] with [68, Theorem 9.1.9, Exercise 9.1.11, Theorem 10.3.3].

For instance, if $\mathfrak{A}$ is the class of the abelian groups, Just-Non-$\mathfrak{A}$ groups have been completely described by M. F. Newman in [44, Theorems 11.1, 11.2]. He proved that a Just-Non-$\mathfrak{A}$ group is characterized to be a homomorphic image of a direct product of an extra-special group by a quasicyclic group. Minimal-Non-$\mathfrak{A}$ groups have been completely described by O. Yu. Schmidt in [5, p.268, l.4] (or in [68, Theorem 9.1.9]). It is interesting to point out the great symmetry which pervades the result of M. F. Newman and that of O. Yu. Schmidt. The Fitting subgroup $Fit(G)$ of a group G (see [68, p.133]) plays in the structure of a Just-Non-$\mathfrak{A}$ group the same role which is played by $G/Frat(G)$ in the structure of a Minimal-Non-$\mathfrak{A}$ group, where $Frat(G)$ denotes the Frattini subgroup of G (see [68, p.135]). We continue to find these analogies for many choices of $\mathfrak{X}$ and not only for $\mathfrak{X} = \mathfrak{A}$.

The importance of Just-Non-$\mathfrak{X}$ groups and Minimal-Non-$\mathfrak{X}$ groups becomes more relevant when we look at situations as in [5, l.2-18] or [50, Theorem 7.4.1]. Let $\mathfrak{P}$ be the class of polycyclic groups. For instance, [50, Theorem 7.4.1] states that a finitely generated group G, which is not a polycyclic group, has a suitable homomorphic image which is a Just-Non-$\mathfrak{P}$ group. This property holds for many choices of the class $\mathfrak{X}$ and not only for $\mathfrak{X} = \mathfrak{P}$. This shows that the knowledge of Just-Non-$\mathfrak{P}$ groups deals with the knowledge of all finitely generated groups.

Actually, many problems remain unsolved for JNX groups also for easy choices of $\mathfrak{X}$ as [47, Problem 9.74] and [44, Open Questions] show.

There are some cautionary observations which are necessary to note, in order to have an approach as in [44] to topological groups. The existence of a topology in a group does not allow us to speak in an usual way either of formations or of varieties of groups (see [5, 20, 35, 34, 38, 59, 58, 55, 56, 54, 53]). Literature on varieties of topological groups is relatively recent, as shown in [59, 58, 55, 56, 54, 53]. Unfortunately, most of the classical results of [44, Chapter 2] do not hold in the context of topological groups, because the Fitting subgroup cannot play the same role of the abstract case.

For these motivations, we have to move in the category of Hausdorff topological groups with corresponding morphisms (see [32, p.294]). In order to speak about quotients in a meaningful way in this category, we should refer to quotients modulo closed normal subgroups (see [32, Definition 1.9]). Many situations in this category show that we may not have any closed normal subgroup at all. Then it would seem reasonable to pick a category consisting of Hausdorff topological groups which have enough compact quotient groups to separate the points. In particular, the category of locally compact groups satisfies such requirements. They are well-known in literature and described for instance in [7, 26, 28, 33, 32, 60, 81, 80].

Our notation follows [5, 20, 32, 33, 38, 66, 67, 68, 88]. Except for these references, which are used almost everywhere in the present work, [1–8, 10–24, 27, 29–31, 38–52, 61–65, 67–70, 72–75, 77–79, 82–85] deal with the abstract case and are correlated to Sections 3, 4, 5. The remaining references deal with the topological case and are correlated to Sections 6, 7, 8.

2. Main Results

This Section is devoted to describe the main results which have been proved in the present PhD Thesis. Most of them can be found in [71, 74, 75]. Our approach seems to be innovative in the context of topological groups, as communicated in the referee reports of [71]. For this motivation, some results are still in progress and have not been illustrated here because we want to have uniformity in the exposition of the topic. Now we pass to list briefly the subjects of the following Sections.

Section 3 deals with $JNPC$ groups, which are JNX groups where $\mathfrak{X}$ is the class of the PC-groups, introduced in [22]. We find a good description of a $JNPC$ group with a unique minimal normal subgroup, solving partially some open questions in [44]. However, it seems to be still unknown the general structure of a $JNPC$ group.

Section 4 extends some of the results of previous Section 3 to $JNMC$ groups, which are a class of groups wider than the class of the $JNPC$ groups.

Section 5 deals with $MNPC$ groups, which can be considered as a dual class of groups with respect to that of $JNPC$ groups. These groups have been recently considered in [77]. $MNMC$ groups are also mentioned in Section 5, since their structure in periodic case is well known by [29, 63].

Section 6 and Section 7 point out the role of JNX groups in the context of topological groups. Compact JNL groups have been discussed in Section 6. Locally compact JNC groups have been discussed in Section 7. The definitions are a little bit technical and can be found respectively in Section 6 and Section 7. Roughly speaking, we will see in these two sections that the corresponding notions of MNL group and MNC group are delicate to formulate. Some examples will allow us to check when our definitions are meaningful.

Finally, Section 8 summarizes, both in the abstract case and in the topological case, some open questions which are still unsolved to the best of our knowledge. These questions are formulated here for the first time.

Our aim is to communicate most of the advances in JNX groups and MNX groups of the last years. This will be done looking at the methods and at the strategies, which have been adopted in literature. We hope that the colleagues who will read the present PhD thesis, or those who will investigate the topic in future, could have a valid instrument for going on. Indeed, only looking at the scientific production of the last ten years, it is interesting to note that JNX groups and MNX groups are constantly studied in Russian, Western European, Eastern European, Chinese, Arabic and Indian literature.

Furthermore, looking more deeply, one can see that a same result can be found with different approaches and methods. For this, the uniformity of treatment seems to be of equal interest of the main results.

3. $JNPC$ Groups

We recall that a group G is called *FC-group* if $G/C_G(\langle x\rangle^G)$ is a finite group for each element x of G. FC-groups are well known (see [67, 82]). Many generalizations of FC-groups have been obtained, looking at [6, 22, 23, 29, 41, 42, 52, 51, 64, 65, 73, 72].

A group G is called *PC-group*, or group with polycyclic-by-finite conjugacy classes, if $G/C_G(\langle x\rangle^G)$ is a polycyclic-by-finite group for each element x of G. An element x of a group G is called a *PC-element* of G if $G/C_G(\langle x\rangle^G)$ is a polycyclic-by-finite group. Of course, a group G is a PC-group if and only if each element of G is a PC-element of G. As noted in [52, Proposition B.2 a)], the set of all PC-elements of G forms a characteristic subgroup $PC(G)$ of G which is called the *PC-center* of G. Clearly, a PC-group extends the well known notion of FC-group. Note that the first generalization of the notion of FC-group seems to be due to Ya. D. Polovicky in [64, 65], where CC-groups have been introduced. Recall that a group G is called *CC-group*, or group with Chernikov conjugacy classes, if $G/C_G(\langle x\rangle^G)$ is a Chernikov group for each element x of G. See [67, Chapter 4] for details.

As noted in Section 1, we may consider a *Just-Non-PC group*, or briefly a *$JNPC$ group*, as a Just-Non-$\mathfrak{X}$ group, where $\mathfrak{X}$ is the class of the PC-groups.

The following two lemmas recall properties of PC-groups which are described respectively in [22, Theorem 2.2] and [22, Lemma 2.4], so the proofs have been omitted.

Lemma 3.1. *Let G be a group. G is a PC-group if and only if $\langle X\rangle^G$ is a polycyclic-by-finite subgroup of G, where X is a finite subset of G.*

Lemma 3.1 can be also expressed by saying that a PC-group is a locally (normal and polycyclic-by-finite) group. See [67] for this terminology. It follows easily from Lemma 3.1 that in a group G, $\langle x\rangle^{PC(G)}$ is a polycyclic-by-finite group for each nontrivial PC-element x of G.

Lemma 3.2. *Quotients, subgroups and direct products of PC-groups are PC-groups.*

[22, Corollary 2.3, Lemma 2.4] give a weak closure by sections of PC-groups. However we know that finite extensions of FC-groups are FC-groups, but finite extensions of PC-groups can not be PC-groups. The following example is emblematic.

Example 3.3. Let G be the locally dihedral 2-group

$$G = D_{2^\infty} = \langle x\rangle \ltimes C_{2^\infty} = \langle x\rangle \ltimes P,$$

where x is an involution which acts on the quasicyclic 2-group P via $a^x = a^{-1}$, for each element $a \in P$. G is a finite extension of P by $\langle x\rangle$ and $G = \langle x\rangle^G$.

Clearly $\langle x \rangle^G$ is not a polycyclic-by-finite group so that G is not a PC-group thanks to Lemma 3.1. □

This fact is not expected because many closure properties of PC-groups come from closure properties of the class of all polycyclic-by-finite groups. Therefore Example 3.3 proves that a group which contains a normal PC-subgroup of finite index can not be a PC-group. On the other hand, a group G which contains a finite normal subgroup F whose quotient group G/F is a PC-group is certainly a PC-group. This is explained by the following statement.

Lemma 3.4. *If G is a $JNPC$ group, then G has no nontrivial polycyclic-by-finite normal subgroups.*

Proof. We claim that an extension of a polycyclic-by-finite group by a PC-group is likewise a PC-group. Assume that $G = HK$ is the product of a polycyclic-by-finite normal subgroup H of G by a PC-group K. It is enough to prove that G is a PC-group.

Let x be an element of G. Since G/H is a PC-group, $\langle xH \rangle^{G/H}$ is a polycyclic-by-finite group. Then $\langle x \rangle^K H/H$ is a polycyclic-by-finite group and so $\langle x \rangle^K$ is a polycyclic-by-finite group. Now $\langle x \rangle^G \leq H\langle x \rangle^K$, hence $\langle x \rangle^G$ is a polycyclic-by-finite group, which gives that G is a PC-group.

From the above argument, we have that a $JNPC$ group cannot contain nontrivial polycyclic-by-finite normal subgroups. The result follows. □

Another interesting fact is that a $JNPC$ group is subdirectly indecomposable.

Lemma 3.5. *If G is a $JNPC$ group, then every intersection of two nontrivial normal subgroups of G is nontrivial.*

Proof. Let H and K be two nontrivial normal subgroups of G. Suppose that $H \cap K$ is trivial. G is isomorphic to a subgroup of the direct product of G/H and G/K. But G/H and G/K are PC-groups, so Lemma 3.2 implies that G is a PC-group and this gives a contradiction. □

Theorem 3.6. *If G is a $JNPC$ group, then $Z(G)$ is trivial.*

Proof. If x is a nontrivial element of $Z(G)$, then $\langle x \rangle$ is a cyclic normal subgroup of G, against Lemma 3.4. This implies that $Z(G)$ is trivial. □

Unfortunately, the structure of PC-groups does not allow us to express a condition similar to [69, Proposition 2.2]. Recall that a *Just-Non-FC group*, or briefly a *$JNFC$ group*, is a Just-Non-$\mathfrak{X}$ group, where $\mathfrak{X}$ is the class of the FC-groups. Of course, a $JNFC$ group is a $JNPC$ group. Note that a $JNFC$ group G can not satisfy *max-n* as testified in [44], so it is clear that a $JNPC$ group can not satisfy *max-n*. In order to adapt [69, Proposition 2.2] and [44, Lemma 15.1], we recall the following notions. The *Hirsch-Plotkin radical* $HP(G)$ of a group G is defined to be the unique largest maximal normal locally nilpotent subgroup of G (see [67, §2, p.57-64] for details). Recall that a Just-Non-$\mathfrak{PF}$ group is a group which is not polycyclic-by-finite but all whose proper quotients are polycyclic-by-finite. This class of groups is actually the same of that of Just-Non-$\mathfrak{P}$ groups as we may see, comparing the classification in [69] and that in [44, Chapter 15].

Proposition 3.7. *Let G be a locally soluble $JNPC$ group. If the Hirsch-Plotkin radical of each proper quotient group of G satisfies max-ab, then G is a Just-Non-$\mathfrak{PF}$ group.*

Proof. [22, Theorem 3.2] implies that a locally soluble PC-group is hyperabelian, so that G has each proper quotient group which is hyperabelian. Now [67, Theorem 3.31] implies that each proper quotient group of G is polycyclic-by-finite. The result follows. □

Proposition 3.8. *Assume that G is a $JNPC$ group, H is a nontrivial normal subgroup of G, H satisfies max-n, $HP(G/H) = R/H$. If G/H is locally soluble and R/H satisfies max-ab, then G is a Just-Non-$\mathfrak{PF}$ group.*

Proof. [22, Theorem 3.2] implies that a locally soluble PC-group is hyperabelian, so that G/H has each proper quotient group which is hyperabelian. Now [22, Theorem 3.31] implies that G/H is polycyclic-by-finite. Since H satisfies *max-n* and G/H is a polycyclic-by-finite group, we may conclude that G satisfies *max-n*. It follows easily from Lemma 3.1 that a PC-group which satisfies *max-n* is a polycyclic-by-finite group. Thus each proper quotient group of G is a polycyclic-by-finite group and the result follows. □

Obviously each finitely generated $JNPC$ group is a Just-Non-$\mathfrak{PF}$ group. However an improvement of Propositions 3.7 and 3.8 can be furnished by means of [22, Lemmas 5.10 and 5.11] as follows.

Proposition 3.9. *Let G be a locally soluble $JNPC$ group and H be a normal subgroup of G. If each infinite subset of G/H contains a pair of elements which generate a polycyclic-by-finite subgroup, then G is a Just-Non-$\mathfrak{PF}$ group.*

Proof. Since G/H is locally soluble PC-group, it is hyperabelian from [22, Theorem 3.2]. Therefore we apply [22, Lemma 5.10] so that G/H is a polycyclic-by-finite group. Now the result follows. □

Remark 3.10. *From [69] we know that a $JNPC$ group which satisfies the conditions of Proposition 3.7 or Proposition 3.8 or Proposition 3.9 is completely classified.*

Using wreath products we are able to construct many $JNPC$ groups: this point of view was suggested at the first time by D. J. Robinson in [69] for Just-Non-$\mathfrak{PF}$ groups. This approach allows us to classify $JNFC$ groups, Just-Non-$\mathfrak{PF}$ groups and many other types of Just-Non-$\mathfrak{X}$ groups, where $\mathfrak{X}$ is a prescribed class of groups (see [44] for details).

A classical example of a $JNPC$ group is given by the group $G = C_\infty \ltimes \mathbb{Q}_p$, where C_∞ is infinite cyclic and $\mathbb{Q}_p$ is the additive group of rational numbers with denominator a power of p for a fixed prime p. G is a a Just-Non-$\mathfrak{PF}$ group with a unique minimal normal subgroup and it is in particular a $JNPC$ group with a unique minimal normal subgroup. Details can be found in [69]. Of course, each periodic $JNPC$ group is a $JNFC$ group, since the property to be an FC-group and the property to be a PC-group coincide in the periodic case. Note that the same example $G = C_\infty \ltimes \mathbb{Q}_p$ shows that there exists a nonperiodic $JNPC$ group which is a $JNFC$ group.

Remark 3.11. *Let G be a $JNPC$ group with max-n. If N is a normal nilpotent subgroup of G, then N is abelian.*

Proof. G will have each proper quotient which is a polycyclic-by-finite group and it is not a polycyclic-by-finite group. It is a Just-Non-$\mathfrak{PF}$ group and so the result follows from [69, (2.3)]. □

Lemma 3.12. *Let G be a group with minimal normal subgroup $A = Fit(G)$. Then A is either torsion-free abelian or p-elementary abelian for some prime p.*

Proof. Obviously, A is an abelian group. Denote with $T = T(G)$ the torsion subgroup of A. If T is trivial, then A is torsion-free and the result follows. Assume that T is nontrivial. Of course, T is characteristic in A and so normal in G. By the minimality of A, T must be equal to A. Then A is periodic. The Prüfer decomposition of A implies that A is the direct product of p-groups. Consider the socle $A[p]$ of A. This is nontrivial, since A is periodic. Again the minimality of A implies $A = A[p]$ and so A has finite exponent. Then A is p-elementary abelian and the result follows. □

Corollary 3.13. *Let G be a group. If A is either a maximal periodic or a maximal torsion-free abelian subgroup of G, then $C_G(A)$ is a maximal abelian subgroup of G.*

Proof. Of course, $A \leq C_G(A)$. Conversely, $A \geq C_G(A)$, because A is maximal abelian. Then the result follows with $A = C_G(A)$. □

Corollary 3.14. *Let G be a soluble group with minimal normal subgroup $A = Fit(G)$. Then $A = C_G(A)$.*

Proof. This is a well known fact which happens for the chief factors of a soluble group. A is in this situation. See [67]. □

The following result of D. J. Robinson will be useful (see [44, Theorem 4.5]).

Theorem 3.15. *Let G be a group with an abelian subgroup A satisfying the minimal condition on its G-invariant subgroups and let K be a normal subgroup of G satisfying the following conditions:*

(i) *$K \geq A$ and K/A is locally nilpotent;*

(ii) *the FC-hypercenter of $G/C_K(A)$ includes $K/C_K(A)$;*

(iii) *$A \cap Z(K)$ is trivial.*

Then G contains a free abelian subgroup X such that the index $|G : XA|$ is finite and $X \cap A = 1$ (nearly splitting of G on A). Moreover the complements of A in G fall into finitely many conjugacy classes.

The following notion can be useful in order to formulate our main results of classification of $JNPC$ groups with a unique minimal normal subgroup.

Definition 3.16. *In the situation of Lemma 3.12, we will say that a $JNPC$ group G with $Fit(G) = A$ has char$A$=0 if A is torsion-free. We will say that G has char$A$=p, for some prime p, if A is p-elementary abelian.*

The following two results classify a $JNPC$ group with a unique minimal normal subgroup. We may note that the Fitting subgroup plays a fundamental role in such a classification.

Theorem 3.17. *Let G be a soluble $JNPC$ group. If $A = Fit(G)$ is minimal normal in G of char$A = 0$ and G/A is locally nilpotent, then*

(i) *A is torsion-free abelian;*

(ii) *$A = C_G(A)$ is the unique minimal normal subgroup of G;*

(iii) *G contains a free abelian subgroup X such that $|G : XA|$ is finite and $X \cap A$ is trivial (nearly splitting of G on A). If G splits over A, the complements of A fall into finitely many conjugacy classes.*

Proof. (i). Follows by Lemma 3.12.
(ii). By Corollary 3.14, $A = C_G(A)$. Since A is a minimal normal subgroup of G, A is obviously the unique minimal normal subgroup of G. The result follows.
(iii). By the previous steps (i) and (ii), A is an abelian subgroup of G which satisfies the minimal condition on its G-invariant subgroups. G/A is a locally nilpotent PC-group such that $G/C_G(A) = G/A$. Now Theorem 3.6 implies that $Z(G) \cap A$ is trivial. We may apply Theorem 3.15 so that (iii) is proved. □

Theorem 3.18. *Let G be a soluble $JNPC$ group. If $A = Fit(G)$ is minimal normal in G of char$A = p$ for some prime p and G/A is locally nilpotent,*

(i) *A is p-elementary abelian;*

(ii) *$A = C_G(A)$ is the unique minimal normal subgroup of G;*

(iii) *G contains a free abelian subgroup X such that $|G : XA|$ is finite and $X \cap A$ is trivial (nearly splitting of G on A). If G splits over A, the complements of A fall into finitely many conjugacy classes.*

Proof. A similar argument as in Theorem 3.17 can be applied.□

We want to inform the reader that further improvements of Theorems 3.17 and 3.18 have been given in [74] and [75].

4. $JNMC$ Groups

A *minimax group* is a group which has a series of finite length whose factors satisfy either the maximal condition or the minimal condition on subgroups. The maximal condition on subgroups is often denoted with *max* and the minimal condition on subgroups is often denoted with *min*. Thus *minimax* is a finiteness property which generalizes both *max* and *min*. It is easy to verify that the class of minimax groups is closed with respect to homomorphic images, subgroups and extensions. A group G is said to be *soluble minimax* if it has a characteristic series of finite length whose factors are abelian minimax groups. Abelian minimax groups are well-known: an abelian group is minimax if and only if it is an extension of a group with *max* by a group with *min* [67, Lemma 10.31] and consequently soluble minimax groups are well-known [67, Sections 10.3 and 10.4].

A group G is said to be an MC*-group*, or group with (soluble minimax)-by-finite conjugacy classes, if $G/C_G(\langle x\rangle^G)$ is a finite extension of a soluble minimax group, for each element x of G. MC-groups have been introduced in [40] and studied in [41]. Following [67], a finite extension of a soluble minimax group is called a (*soluble minimax*)-*by-finite* group. The class of the (soluble minimax)-by-finite groups is denoted by $\mathfrak{S}_2\mathfrak{F}$ (see [19, 67]).

In a group G, an element x of G is said to be an MC*-element* of G if $G/C_G(\langle x\rangle^G)$ is a finite extension of a soluble minimax group. Obviously, a group G is an MC-group if and only if its elements are all MC-elements and it is clear that each FC-group is an MC-group. The set of all MC-elements of a group G forms a characteristic subgroup $MC(G)$ of G as noted in [52, Propostion B.2, a)]. $MC(G)$ is called the MC*-center* of G and contains the center $Z(G)$ of G.

In this Section, we are interested in studying Just-Non-$\mathfrak{X}$ groups, where $\mathfrak{X}$ is the class of the MC-groups. These groups are called *Just-Non-*MC *groups*, or briefly $JNMC$ *groups*. Of course, $JNPC$ groups, $JNFC$ groups and the groups studied in [44, Section 16] are $JNMC$ groups.

We have already noted in Section 3 that there are correlations between $JNPC$ groups and the groups which have been studied in [69]. The same happens when we study $JNMC$ groups: it will be useful to consider the groups which have been introduced in [27].

We will proceed as in Section 3, listing some general properties of MC-groups and then stating our main results. The next two lemmas can be found in [41] so that their proofs have been omitted.

Lemma 4.1. *Let G be a group. If X is a finite subset of MC-elements of G, then $\langle X\rangle^G$ is a (soluble minimax)-by-finite subgroup of G.*

If G is an MC-group, then Lemma 4.1 can be expressed by saying that G is a locally (normal and (soluble minimax)-by-finite) group.

Lemma 4.2. *The class of MC-groups is closed with respect to forming homomorphic images, subgroups and direct products of its members.*

Note that a *Just-Non-*$\mathfrak{S}_2\mathfrak{F}$ group is a Just-Non-X group, where $\mathfrak{X} = \mathfrak{S}_2\mathfrak{F}$.

Proposition 4.3. *Let G be a $JNMC$ group. If each proper quotient of G is finitely generated, then G is a Just-Non-*$\mathfrak{P}\mathfrak{F}$ *group.*

Proof. A finitely generated (soluble minimax)-by-finite group is a polycyclic-by-finite group. See [50, Chapter 5]. Then G will be a group which is not polycyclic-by-finite but all whose proper quotients are polycyclic-by-finite. The result follows from this fact. □

As we have seen in the previous section, it is an interesting fact that a $JNMC$ group is subdirectly indecomposable.

Lemma 4.4. *If G is a $JNMC$ group, then every intersection of two nontrivial normal subgroups of G is nontrivial.*

Proof. Let H and K be two nontrivial normal subgroups of G. Suppose that $H \cap K$ is trivial. G is isomorphic to a subgroup of the direct product of G/H and G/K. But G/H and G/K are MC-groups, so Lemma 4.2 implies that G is an MC-group and this contradicts the fact that G is a $JNMC$ group. □

Lemma 4.5. *If G is a $JNMC$ group, then G has no nontrivial (soluble minimax)-by-finite normal subgroups.*

Proof. We claim that an extension of a (soluble minimax)-by-finite group by a MC-group is likewise a MC-group. Assume that $G = HK$ is the product of a (soluble minimax)-by-finite normal subgroup H of G by a MC-group K. It is enough to prove that G is a MC-group.

Let x be an element of G. Since G/H is a MC-group, $\langle xH\rangle^{G/H}$ is a (soluble minimax)-by-finite group. Then $\langle x\rangle^K H/H$ is a (soluble minimax)-by-finite group and so $\langle x\rangle^K$ is a (soluble minimax)-by-finite group. Now $\langle x\rangle^G \le H\langle x\rangle^K$, hence $\langle x\rangle^G$ is a (soluble minimax)-by-finite group, which gives that G is a MC-group.

From the above argument, we have that a $JNMC$ group cannot contain nontrivial (soluble minimax)-by-finite normal subgroups. The result follows. □

Theorem 4.6. *If G is a $JNMC$ group, then $Z(G)$ is trivial.*

Proof. If $Z(G)$ is nontrivial, then an element x of $Z(G)$ implies that $\langle x\rangle$ goes against Lemma 4.5. The result follows. □

Remark 4.7. *Let G be a $JNMC$ group such that each quotient of G is an FC-group. If N is a normal nilpotent subgroup of G, then N is abelian.*

Remark 4.7 points out a well-known fact for $JNFC$ groups. Of course, a group in the situation of Remark 4.7 cannot be an FC-group. Then it must be a $JNFC$ group. The role of the nilpotent subgroups which is described is one of the main results in [21]. This is the main point which allows to prove that the Fitting subgroup of a $JNFC$ group is abelian. As the expert reader may note, we were not able to prove in an elementary way that the Fitting subgroup of a $JNPC$ group is abelian. Unfortunately, the same is true for a $JNMC$ group. But, it is elementary the proof of the following fact, which we repeat in order to note that it is of general interest.

Lemma 4.8. *Let G be a group with minimal normal subgroup $A = Fit(G)$. Then A is either torsion-free abelian or p-elementary abelian for some prime p.*

Proof. Obviously, A is an abelian group. Denote with $T = T(G)$ the torsion subgroup of A. If T is trivial, then A is torsion-free and the result follows. Assume that T is nontrivial. Of course, T is characteristic in A and so normal in G. By the minimality of A, T must be equal to A. Then A is periodic. The Prüfer decomposition of A implies that A is the direct product of p-groups. Consider the socle $A[p]$ of A. This is nontrivial, since A is periodic. Again the minimality of A implies $A = A[p]$ and so A has finite exponent. Then A is p-elementary abelian and the result follows. □

Lemma 4.9. *Let G be a group. If A is either a maximal periodic or a maximal torsion-free abelian subgroup of G, then $C_G(A)$ is a maximal abelian subgroup of G.*

Proof. Of course, $A \leq C_G(A)$. Conversely, $A \geq C_G(A)$, because A is maximal abelian. Then the result follows with $A = C_G(A)$. □

Corollary 4.10. *Let G be a soluble group with minimal normal subgroup $A = Fit(G)$. Then $A = C_G(A)$.*

Proof. This is a well known fact which happens for the chief factors of a soluble group. A is in this situation. See [67]. □

We need of the following definition in order to formulate the main results of the present section.

Definition 4.11. *In the situation of Lemma 4.8, we will say that a $JNMC$ group G with $Fit(G) = A$ has char$A$=0 if A is torsion-free. We will say that G has char$A$=p, for some prime p, if A is p-elementary abelian.*

The following two results classify a $JNMC$ group with a unique minimal normal subgroup.

Theorem 4.12. *Let G be a soluble $JNMC$ group. If $A = Fit(G)$ is minimal normal in G of char$A = 0$ and G/A is locally nilpotent, then*

(i) *A is torsion-free abelian;*

(ii) *$A = C_G(A)$ is the unique minimal normal subgroup of G;*

(iii) *G contains a free abelian subgroup X such that $|G : XA|$ is finite and $X \cap A$ is trivial (nearly splitting of G on A). If G splits over A, the complements of A fall into finitely many conjugacy classes.*

Proof. (i). Follows by Proposition 4.8.
(ii). By Corollary 4.10, $A = C_G(A)$. Since A is a minimal normal subgroup of G, A is obviously the unique minimal normal subgroup of G. The result follows.
(iii). By the previous steps (i) and (ii), A is an abelian subgroup of G which satisfies the minimal condition on its G-invariant subgroups. G/A is a locally nilpotent MC-group such that $G/C_G(A) = G/A$. Now Theorem 4.6 implies that $Z(G) \cap A$ is trivial. We may apply Theorem 3.15 so that (iii) is proved. □

Theorem 4.13. *Let G be a soluble $JNMC$ group and p be a prime. If $A = Fit(G)$ is minimal normal in G of char$A = p$ and G/A is locally nilpotent, then*

(i) *A is p-elementary abelian;*

(ii) *$A = C_G(A)$ is the unique minimal normal subgroup of G;*

(iii) *G contains a free abelian subgroup X such that $|G : XA|$ is finite and $X \cap A$ is trivial (nearly splitting of G on A). If G splits over A, the complements of A fall into finitely many conjugacy classes.*

Proof. We may apply an argument as in Theorem 4.12. □

Note that a periodic MC-group is a CC-group in the sense of [64]. Therefore the following result is obvious.

Corollary 4.14. *If G is a periodic $JNMC$ group, then it is a group described in [44, Section 16].*

5. $MNPC$ Groups

The present Section deals with a dual problem which has been investigated in Section 3 and Section 4. Many results have been obtained on Minimal-Non-$\mathfrak{X}$ groups, for various classes of groups $\mathfrak{X}$. Here we are interested in studying Minimal-Non-$\mathfrak{X}$ groups, where $\mathfrak{X}$ is either the class of the PC-groups (respectively, the class of the MC-groups). Such groups will be called *$MNPC$ groups* (respectively, *$MNMC$ groups*). Both $MNPC$ groups and $MNMC$ groups are generalizations of the notion of Minimal-Non-FC group (or briefly *$MNFC$ group*), where the class of the FC-groups is involved. Another useful notion, generalizing that of $MNFC$ group, is the notion of Minimal-Non-CC group (or briefly *$MNCC$ group*), where the class of the CC-groups is involved.

In [7] and [8] (see also [82, Section 8]) $MNFC$ groups have been classified when they have a nontrivial finite or abelian factor group. $MNFC$ groups are finite cyclic extensions of divisible p-groups of finite rank. J. Otál and J. Peña proved in [63] that there are no $MNCC$ groups which have a nontrivial finite or abelian factor group. We study $MNPC$ groups having a nontrivial finite factor group and we prove that they are finite cyclic extensions of divisible groups of finite rank. Contrary to the FC-case, these groups are not necessary periodic. Note that the imposition of the condition of having a nontrivial finite factor group is to avoid Tarski groups, that is infinite nonabelian groups whose proper subgroups are finite.

Finitely generated $MNPC$ groups will be analyzed separately. Since among them there are Tarski groups as the example of A. Yu. Ol'šhanskiĭ shows [62], we cannot obtain a good description of them. However in Theorem 5.1 we give some conditions that these groups have to satisfy.

We can find the definitions of PC-center and PC-hypercenter in [6, 48, 52, 51, 72, 75]. These definitions extend some well-known situations of the FC-case (see [67, Section 4.3]).

Theorem 5.1. *Let G be a finitely generated $MNPC$ group. Then:*

(i) *G has no nontrivial locally graded factor groups. In particular, G is a perfect group and has no proper subgroups of finite index.*

(ii) *The center, the hypercenter, the PC-center and the PC-hypercenter of G coincide.*

(iii) *$G/Frat(G)$ is an infinite simple group and for all x in $G \setminus Frat(G)$ we have that $G = \langle x\rangle^G$.*

Note that the simplicity of $G/Frat(G)$ has been proved for Minimal-Non-$\mathfrak{X}$ groups, when $\mathfrak{X}$ is the class of nilpotent groups [61], finite-by-nilpotent groups [84], locally finite-by-nilpotent groups [18] and torsion-by-nilpotent groups [83].

We will characterize $MNPC$ groups with a nontrivial finite factor group in Theorem 5.2. The following notion will be useful for stating the next result.

A group is said to have *finite* (*Prüfer*) *rank* n if every finitely generated subgroup can be generated by n elements and n is the least such integer.

Theorem 5.2. *Let G be a group and suppose that G^*, the finite residual of G, is a proper subgroup of G. Then G is an $MNPC$ group if, and only if, the following conditions hold:*

(i) *there exists $x \in G$ such that $G = \langle G^*, x\rangle$. Moreover, G^* is nontrivial and there is a prime p and a positive integer n such that $x^{p^n} \in G^*$;*

(ii) *G^* is either a q-group for a suitable prime q, or a torsion-free group. Furthermore, G^* is a divisible abelian group of finite rank;*

(iii) *$G' = G^*$;*

(iv) *if N is a proper G-admissible subgroup of G^*, then N is a finitely generated group;*

(v) *if H is a proper normal subgroup of G, then HG^* is a proper abelian subgroup of G. In particular, H is an abelian group.*

Clearly, every periodic $MNPC$ group is an $MNFC$ group. But in view of the results of V. V. Belyaev and N. Sesekin [82, Theorem 8.11], one can deduce from Theorem 5.2 that the converse holds for groups having a nontrivial finite factor group. So that we have the following consequence.

Corollary 5.3. *Let G be a group having a nontrivial finite factor group. Then G is a periodic $MNPC$ group if and only if G is an $MNFC$ group.*

Since abelian q-groups of finite rank (q prime) are the direct product of finitely many quasicyclic groups or finite cyclic groups, they satisfy the minimal condition on subgroups. So that one can deduce the following result still from Theorem 5.2.

Corollary 5.4. *Let G be a group having a nontrivial finite factor group. If G is a periodic $MNPC$ group, then G is a Chernikov group.*

Let K be a copy of the group C_{2^∞} and consider the locally dihedral 2-group G which is the semidirect product of K by a cyclic group $\langle x\rangle$ of order 2 such that x inverts each element of K. We have $\langle x\rangle^G = \langle x\rangle [K, x] = \langle x\rangle K = G$ so that G is not a PC-group as it is not a polycyclic-by-finite group. We deduce that G' is not a polycyclic-by-finite group, so that $G' = K$. By construction, each proper subgroup of G is either abelian or finite. Then each proper subgroup of G is a PC-group. G is an $MNPC$ group having the nontrivial finite factor G/K of order 2. A first property in the finitely generated case is the following.

Lemma 5.5. *Let G be a finitely generated $MNPC$ group. Then G has no proper subgroups of finite index.*

Proof. Suppose that H is a proper subgroup of G of finite index. Then H is a finitely generated PC-group. So by Lemma 3.1 H, and therefore G, is a polycyclic-by-finite group. Hence G is a PC-group, which is a contradiction. □

Since finitely generated locally graded groups, in particular polycyclic-by-finite groups, have proper subgroups of finite index, we can deduce the following result.

Corollary 5.6. *Let G be an $MNPC$ group. Then G is a locally graded group if and only if G is not a finitely generated group.*

The next result is a first step in order to prove Theorem 5.1.

Lemma 5.7. *Let G be a finitely generated minimal $MNPC$ group. Then G has no nontrivial locally graded factor groups. In particular, G is a perfect group.*

Proof. Let N be a normal subgroup of G such that G/N is a nontrivial locally graded group. Since G/N is finitely generated, it has a proper normal subgroup H/N such that $(G/N)/(H/N)$ is finite. Thus H is a proper subgroup of G of finite index, which is a contradiction by Lemma 5.5. □

Since a finitely generated PC-group is a polycyclic-by-finite group, we can deduce the following result.

Corollary 5.8. *A finitely generated $MNPC$ group has no nontrivial factor groups which are PC-groups.*

Proof of Theorem 5.1. (i). Follows from Corollary 5.6 and Lemma 5.7.

(ii). Since G is a perfect group, the center of $G/Z(G)$ is trivial. So that $Z(G)$ is the hypercenter of G. Now let x be a PC-element of G, then $G/C_G(\langle x\rangle^G)$ is a polycyclic-by-finite group. We deduce from Lemma 5.7 that $G = C_G(\langle x\rangle^G)$. Thus $x \in Z(G)$, hence $Z(G)$ is the PC-center of G. Clearly, Corollary 5.8 gives that $\overline{G} = G/PC(G)$ is an $MNPC$ group. Thus as before, if $\overline{x}$ is a PC-element of $\overline{G}$ then $\overline{x}$ belongs to the center of $\overline{G}$. Now $\overline{G} = G/Z(G)$, so $Z(\overline{G})$ is trivial. Thus the set of PC-elements of $\overline{G}$ is trivial and therefore the PC-center of G is the PC-hypercenter of G.

(iii). Since G is a finitely generated group, $Frat(G)$ is a proper subgroup, so $G/Frat(G)$ is an infinite group. Suppose that $G/Frat(G)$ is not a simple group and let N be a normal subgroup of G such that $Frat(G) \lneq N \lneq G$. Therefore there is a maximal subgroup M of G such that $N \nleq M$. It follows that $G = MN$ and therefore $G/N \simeq M/M \cap N$. Since M is a proper subgroup of G, it is a PC-group. We deduce that G/N is a PC-group, which is a contradiction by Corollary 5.8. Therefore $G/Frat(G)$ is a simple group. Let x be an element of $G \setminus Frat(G)$. Since $G/Frat(G)$ is a simple group, $\langle x\rangle^G Frat(G) = G$ and this gives that $G = \langle x\rangle^G$. □

In order to prove Theorem 5.2, we adapt the proof of Belyaev and Sesekin [82, Theorem 8.11] to the case of PC-groups.

Lemma 5.9. *Let $G = HK$ be the product of a normal polycyclic-by-finite subgroup H by a PC-subgroup K. Then G is a PC-group.*

Proof. See the proof of Lemma 3.4. $\square$

Lemma 5.10. *Let H be a subgroup of a PC-group G. If H has no proper subgroups of finite index, then it is contained in $Z(G)$.*

Proof. Since G is a PC-group, $G/Z(G)$ is a residually polycyclic-by-finite group and therefore it is a residually finite group. So $HZ(G)/Z(G)$ is a residually finite group which has no proper subgroups of finite index, hence it is trivial and therefore $H \leq Z(G)$. $\square$

Lemma 5.11. *Let H be a normal subgroup of finite index in a $MNPC$ group G. Then G/H is a cyclic group of p-power order, where p is a prime.*

Proof. First, note that in view of Theorem 5.1, G is not a finitely generated group. So that every finitely generated subgroup of G is a proper subgroup and therefore it is a polycyclic-by-finite group by Lemma 3.1. Since G is not a PC-group, there exists a nontrivial element x of G such that $\langle x\rangle^G$ is not a polycyclic-by-finite group. Assume that $\langle H, x\rangle$ is a proper subgroup of G. Then $\langle H, x\rangle$ is a PC-group so that $\langle x\rangle^H$ is a polycyclic-by-finite group. Since G/H is a finite group, $G = HF$, where F is a finitely generated subgroup of G. So that

$$\langle x\rangle^G = \langle x\rangle^{HF} = \left(\langle x\rangle^H\right)^F = K^F$$

where $K = \langle x\rangle^H$. But both K and F are finitely generated groups, so $\langle K, F\rangle$ is also a finitely generated group and therefore it is a polycyclic-by-finite group. It follows that K^F is a polycyclic-by-finite group and this gives the contradiction that $\langle x\rangle^G$ is a polycyclic-by-finite group. Thus $G = \langle H, x\rangle$ and G/H is a cyclic group, as claimed.

It remains to prove that G/H has p-power order, where p is a prime. Let $|G : H| = mn$ for suitable integers $m, n > 1$ such that $(m, n) = 1$. Then $\langle H, x^m\rangle$ and $\langle H, x^n\rangle$ are proper subgroups of G so that they are PC-groups. So both $\langle x^m\rangle^H$ and $\langle x^n\rangle^H$ are polycyclic-by-finite groups. But for every positive integer i we have that

$$\langle x^i\rangle^G = \langle x^i\rangle^{\langle x,H\rangle} = \langle x^i\rangle^{\langle x\rangle H} = \left(\langle x^i\rangle^{\langle x\rangle}\right)^H = \langle x^i\rangle^H$$

so that both $\langle x^m\rangle^G$ and $\langle x^n\rangle^G$ are polycyclic-by-finite groups and therefore $\langle x^m\rangle^G\langle x^n\rangle^G$ is a polycyclic-by-finite group, too. But $(m, n) = 1$ so that

$$\langle x\rangle^G = \langle x^m\rangle^G\langle x^n\rangle^G$$

and therefore $\langle x\rangle^G$ is a polycyclic-by-finite group, which is a contradiction. Then G/H has p-power order, where p is a suitable prime, as claimed. □

Proof of Theorem 5.2. (i). Let H_1 and H_2 be two normal subgroups of finite index in G. Then $|G : H_1 \cap H_2|$ is also a finite number which is divisible by both $|G : H_1|$ and $|G : H_2|$. We deduce that the prime p of Lemma 5.11 is the same for all normal subgroups of finite index in G. On the other hand, if $|G : H_1| = p^n$ and $|G : H_2| = p^m$, where $n \leq m$, then $H_2 \leq H_1$ because the finite cyclic group $G/(H_1 \cap H_2)$ has a unique subgroup of each index. Thus the set of normal subgroups of G of finite index is a chain and therefore it is finite. It follows that G/G^* is a finite group and Lemma 5.11 gives that there is x in G and a prime p and a positive integer n such that $G = \langle G^*, x\rangle$ and $x^{p^n} \in G^*$. Clearly G^* is a nontrivial group.

(ii). By Lemma 5.10, G^* is an abelian group and G^* is divisible because it has no proper subgroups of finite index.

From (i), we have that $G = \langle G^*, x\rangle$. We claim that $\langle H, x\rangle$ is a proper subgroup of G, whenever H is a G-admissible proper subgroup of G^*. If $G = \langle H, x\rangle$, then G/H, and therefore G^*/H, is a nontrivial cyclic group. So G^*/H , and therefore G^*, has a proper subgroup of finite index, which is a contradiction. Now, let T be the torsion subgroup of G^*. It is well known [25, Section 19] that the torsion subgroup of a divisible abelian subgroup is divisible, so that $G^* = T \times S$, where S is a torsion-free G-admissible divisible subgroup of G^*. If both T and S are proper subgroups of G^*, then both $\langle T, x\rangle$ and $\langle S, x\rangle$ are proper subgroups of G by the previous argument. We deduce by Lemma 5.10, that x centralizes both T and S, so that G is an abelian group, which is a contradiction. It follows that either $G^* = T$ or $G^* = S$, that is, either G^* is a periodic group or a torsion-free group.

We know from [25, Section 19] that T (respectively, S) is the direct product of quasicyclic groups (respectively, copies of the additive group $\mathbb{Q}$ of the rational numbers). Since G is not abelian, $G^* \not\leq Z(G)$. So there exists a subgroup K of G^* such that K is isomorphic to a quasicyclic group C_{q^∞} for some prime q or K is isomorphic to $\mathbb{Q}$ and $K \not\leq Z(G)$. By Lemma 3.2 we deduce that $G = \langle K, x\rangle$. Now $N = K^G$, then

$$N = K^{\langle x\rangle} = KK^x \dots K^{x^{p^n-1}}.$$

This implies that N is either an abelian q-group or a torsion-free abelian group. Since $K, K^x, \dots, K^{x^{p^n-1}}$ have rank 1, N has finite rank, because it is a product of finitely many subgroups of finite rank [67, Lemma 1.44].

On a hand G/N is a cyclic group; on another hand G cannot have subgroups of arbitrary finite index. This implies that G/N is a finite group, from which we deduce that $N = G^*$. Therefore, the result follows.

(iii). Suppose that G' is a proper subgroup of G^*. Then as before $H = \langle G', x\rangle$ is proper in G so that $\langle x\rangle^H$ is a polycyclic-by-finite group. But $H' = [G', \langle x\rangle]$ since G' is an abelian group, so H' is a polycyclic-by-finite group. By Lemma 3.1, we deduce that G/H' is an $MNPC$ group, and since a factor group of a divisible abelian group is also divisible, there is no loss of generality if we suppose that $H' = 1$. It follows that G' is a central subgroup of G, from which we deduce that

$$[G^*, x] = [(G^*)^{p^n}, x] = [G^*, x]^{p^n} = [G^*, x^{p^n}] = 1.$$

This gives that G is an abelian group, which is a contradiction. Therefore $G' = G^*$, as claimed.

(iv) Let N be a proper G-admissible subgroup of G^*. Then, as before, $\langle N, x\rangle$ is a proper subgroup of G, so that $\langle x\rangle^N$, and therefore $[N, \langle x\rangle]$, is a polycyclic-by-finite group. Hence it suffices to prove that $N/[N, \langle x\rangle]$ is a finitely generated group. Since a section of a group of finite rank is of finite rank [67, Lemma 1.44] , there is no loss of generality if we assume that $[N, \langle x\rangle] = 1$, so that N is a central subgroup of G. Now let r be an integer and

$$G^* = G' = [G^*, \langle x\rangle] = \langle [a, x^r] : a \in G^*\rangle.$$

Since N is central we deduce that each element of N is of the form $[a, x]$, where $a \in G^*$. We have $[a, x]^{p^n} = [a, x^{p^n}] = 1$, so that N is of finite exponent. Therefore N is an abelian p-group of finite rank and of finite exponent, hence N is a finite group, as required.

(v) Let H be a proper normal subgroup of G^*. Assume that $G = HG^*$. Then $H/H \cap G^*$ is finite. Note that since H is a proper subgroup of G, $H \cap G^*$ is a proper subgroup of G^*. We deduce by (iv) that $H \cap G^*$ is a polycyclic group and this gives that H is a polycyclic-by-finite group. It follows that G is an extension of a polycyclic-by-finite group by an abelian group. Therefore G is a PC-group by Lemma 3.1, which is a contradiction, so that HG^* is a proper subgroup of G. Since G/G^* is a cyclic group of order p^n, there is a positive integer $i \leq n$ such that $HG^* = \langle G^*, x^{p^i}\rangle$. By Lemma 3.2, we deduce that HG^*, and therefore H, is an abelian group.

Conversely, suppose that G is a group which satisfies conditions (i)-(v). If G is a PC-group, then Lemma 3.2 gives that G is an abelian group and therefore the condition (iii) gives that G^* is a trivial group which contradicts (i). Let H be a proper subgroup of G. If HG^* is a proper subgroup of G, then by the condition (v) we have that HG^*, and therefore H, is an abelian group. So that H is a PC-group. Now assume that $G = HG^*$. Since G/G^* is a group of order p^n, H^{p^n} is a subgroup of G^*. Clearly H^{p^n} is a proper G-admissible subgroup of G^*. We deduce by (iv) that H^{p^n} is a polycyclic group. Now G is of finite rank because G^* is of finite rank and G/G^* is a cyclic group. It follows that H/H^{p^n} is a metabelian group of finite rank and of finite exponent, hence it is a finite group. So H is a polycyclic-by-finite group and therefore it is a PC-group. We conclude that G is an $MNPC$ group, as claimed. □

6. Compact JNL Groups

The present Section deals with topological groups and with the property ro be a Lie group. A compact group G is called a *compact Lie group* if it has a faithful finite dimensional representation (see [32, Definition 2.41]). If N is a normal closed subgroup of a compact group G, it is possible to consider the homogeneous space G modulo N which is a compact group with the quotient topology induced by N. We will refer always to this sense of quotients in the present Section.

If G is a topological group, let $\mathcal{N}(G)$ denote the set of all normal subgroups of G such that $N \in \mathcal{N}(G)$ if and only if G/N is a Lie group. Then $G \in \mathcal{N}(G)$; further $\{1\} \in \mathcal{N}(G)$ if and only if G is a Lie group. If $N \in \mathcal{N}(G)$ and M is a closed normal subgroup of G such that $N \leq M$, then $M \in \mathcal{N}(G)$. If G is a compact group, then $\mathcal{N}(G)$ converges to 1 and the natural morphism $G \rightarrow \lim_{N \in \mathcal{N}(G)} G/N$ is an isomorphism of compact groups (see [32, p.17-23]). The connected component of the identity will be denoted G_0 (see [32, p.23]). A group G is said to be a *compact Just-Non-Lie group*, or briefly a *compact JNL group*, if G is a compact non-Lie group such that all closed normal subgroups $N \neq \{1\}$ are contained in $\mathcal{N}(G)$.

Following the previous definition of compact *JNL* group, we may investigate without ambiguity those compact groups which are non-Lie groups, but are rich of Lie-quotients. Under this point of view, our aims are close to the aims of [44], then we follow the classical approach of studying groups which have a prescribed property but whose proper quotient groups do not have it.

Recall that a topological group G has *no small subgroups*, respectively, *no small normal subgroups* if there is a neighborhood U of the identity such that for every subgroup, respectively, normal subgroup H of G if H is contained in U then H is trivial.

A compact Lie group G is characterized to satisfying one of the equivalent conditions of the following lemma. This has been recalled for convenience of the reader (see [32, 33]).

Lemma 6.1. *Let G be a compact group and $\mathbb{K}$ denote the field of real numbers or the field of complex numbers. The following statements are equivalent:*

(a) *G has a faithful finite dimensional representation.*

(b) *G has a faithful finite dimensional orthogonal (or unitary) representation.*

(c) *G is isomorphic as topological group to a (compact) group of orthogonal (or unitary) matrices.*

(d) *G is isomorphic as topological group to a closed subgroup of the multiplicative group of some Banach algebra A over $\mathbb{K}$.*

(e) *There is a Banach algebra A over $\mathbb{K}$ and an injective morphism from G into the multiplicative group A^{-1} of A.*

(f) *G has no small subgroups.*

(g) *G has no small normal subgroups.*

Proof. See [32, Corollary 2.40].□

A first characterization of compact abelian *JNL* groups can be obtained thanks to [32, Proposition 2.42]. In the next statement we will introduce the *character* $\widehat{G}$ *of a group* G (see [32]).

Proposition 6.2. *Let G be a compact abelian group. G is a compact JNL group if and only if $\widehat{G}$ is not a finitely generated abelian group but each $\widehat{G/N}$ is finitely generated abelian, where $N \neq \{1\}$ is a closed subgroup of G.*

Proof. A compact abelian Lie group is characterized by [32, Proposition 6.42] only as that group which is a character group of a finitely generated abelian group. This means that a compact abelian group G is a compact Lie group if and only if $\widehat{G}$ is an abelian group which is finitely generated. By negation, G is not a compact Lie group if and only if $\widehat{G}$ is not a finitely generated abelian group. Since all closed subgroups $N \neq \{1\}$ are contained in $\mathcal{N}(G)$, $\widehat{G/N}$ has to be finitely generated abelian by [32, Proposition 2.42]. Now the result follows. □

Example 6.3. Let p be a prime number. The group $\mathbb{Z}_p$ of p-adic integers (see [32, Example 1.28]) is a torsion-free abelian compact *JNL* group, as its character group $\widehat{\mathbb{Z}}_p = \mathbb{Z}(p^\infty)$ is the Prüfer group. $\mathbb{Z}_p$ is not a Lie group but the proper quotients of $\mathbb{Z}_p$ are discrete cyclic of p-power order and these are Lie groups. We note that $\mathbb{Z}_p$ has a nonsingleton closed normal abelian subgroup $A = p\mathbb{Z}_p$ of index p and $\mathbb{Z}_p$ does not split over A. This implies that a compact *JNL* group can not split over a normal closed nonsingleton subgroup whose index is finite.

From now the symbol $\mathbb{Z}_p$ will denote always the group of Example 6.3. In the category of compact groups, we may extend a Lie group by another Lie group and we will obtain again a Lie group. This follows easily from Lemma 6.1.

Lemma 6.4. *Let G be a compact JNL group.*

(i) *G does not have a closed normal nonsingleton Lie subgroup.*

(ii) *G does not have a finite normal nonsingleton subgroup.*

(iii) *If G is abelian, then G is torsion-free.*

Proof. (i). If N is a closed normal nonsingleton Lie subgroup, then $N \in \mathcal{N}(G)$ and so G/N is a Lie group. But then G, as an extension of a Lie group by a Lie group is a Lie group, contrary to the definition of compact

JNL group. Finally, (i) implies (ii) and (ii) implies (iii). □

Proposition 6.5. *Let G be a compact JNL group. If M is a nonsingleton closed normal subgroup of G, then G contains a subgroup which is properly smaller than M.*

Proof. The set $\mathcal{N}(G)$ of all normal subgroups such that G/N is a Lie group is a nontrivial filterbasis intersecting in $\{1\}$ while not containing $\{1\}$. If M is a nonsingleton closed normal subgroup, and G is a compact *JNL* group, then $M \in \mathcal{N}(G)$. Now there is an $N \in \mathcal{N}(G)$ such that $M \not\subseteq N$. Then $M \cap N \in \mathcal{N}(G)$ is properly smaller than M. □

Proposition 6.5 implies that a compact *JNL* group with nonsingleton center contains always a subgroup which is properly smaller than the center. The following notion will be used in the next statement.

Let G be a compact group and N be a closed normal subgroup of G. We will say that N has *no closed normal subgroups which are core-free in* G if for each closed normal subgroup M of N, then the subgroup $M_G \neq \{1\}$, where $M_G = \bigcap_{g \in G} M^g$ is the core of M in G. This terminology is standard for abstract groups and can be found for instance in [68].

Proposition 6.6. *Let G be a compact JNL group.*

(i) *If $Z(G) \neq \{1\}$, then $Z(G)$ is a compact JNL group. Furthermore G is a finite central extension of $\mathbb{Z}_p$.*

(ii) *Assume that $N \neq \{1\}$ is a closed normal subgroup of G. If N has no closed normal subgroups which are core-free in G, then N is a compact JNL group.*

Proof. (i). By (i) of Lemma 6.4, $Z(G)$ is not a Lie group. Let $N \neq \{1\}$ be a closed subgroup of $Z(G)$. Then N is a closed normal subgroup of G and thus G/N is a Lie group. In particular, $Z(G)/N \leq G/N$ is a Lie group. We have proved that $Z(G)$ is an abelian *JNL* group. Since G is profinite, $G/Z(G)$ must be finite. In particular, G/N and $Z(G)/N$ are finite. Therefore $Z(G)$ is isomorphic to $\mathbb{Z}_p$ so that G is a finite central extension of $\mathbb{Z}_p$.
(ii). Let $M \neq \{1\}$ be a closed normal subgroup of N. Then $M_G \neq \{1\}$ and G/M_G is a Lie group. Thus the closed subgroup N/M_G is a Lie group, too. But then $N/M \simeq (N/M_G)/(M/M_G)$ is a Lie group as well. □

Proposition 6.7. *Let G be a compact JNL group.*

(i) *If G is abelian, then $G \simeq \mathbb{Z}_p$ for some prime number p.*

(ii) *If G is nilpotent, then G is abelian.*

Proof. (i). By duality $A = \widehat{G}$ is an abelian group which is not finitely generated, bu in which every proper subgroup is finitely generated. We write A

additively. By (iii) of Lemma 6.4, G is torsion-free and so A is divisible (see [32, Corollary 8.5]). Thus $A \simeq \mathbb{Q}^{(I)} \oplus \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty)^{(I_p)}$, where I and I_p are suitable sets and $\mathbb{P}$ denotes the set of all prime numbers (see [32, Theorem A1.42]).
Suppose that $I \neq \varnothing$. Let $a \neq 0$, $a \in \mathbb{Q}^{(I)}$. Then $\frac{1}{2^\infty}\mathbb{Z}\cdot a$ is a proper subgroup of A that is not finitely generated. This is a contradiction. Thus A is a divisible torsion group.
Write $A_p = \mathbb{Z}(p^\infty)^{(I_p)}$ for its p-primary components. Since A is nonzero, at least one A_p is nonzero. Let $A^{(p)}$ denote the sum $\bigoplus_{q\neq p} A_q$ of all q-primary components A_q for q prime number which is distinct by p. Suppose $A^{(p)} \neq \{0\}$. Then $A_p \neq A$ and thus A_p is finitely generated contradicting the fact that a Prüfer group is not finitely generated. Thus $A = \mathbb{Z}(p^\infty)^{(I_p)}$. Let $j \in I_p$, then $A \simeq \mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p^\infty)^{(I_p\setminus\{j\})}$. If the second summand were nonzero, then $\mathbb{Z}(p^\infty)$ would have to be finitely generated, which is not. Thus $I_p = \{j\}$ and A is a Prüfer group. This causes its dual G to be a group $\mathbb{Z}_p$ of p-adic integers so that the result follows.

(ii). If we can prove that G is abelian, whenever G is nilpotent of class at most 2, then we are done, because the second center $Z_2(G)$ has class of nilpotence at most 2 and would have class 2 if G is nonabelian.
Thus we may assume that G is nilpotent of class at most 2 without loss of generality. Then $[G, G] \leq Z(G)$, and

$$[g_1 Z(G), g_2 Z(G)] = \{[g_1, g_2]\},$$

where $g_1, g_2 \in G$, so that the bihomomorphic function

$$b : G \times G \to Z(G)$$

factors through a bihomomorphic function

$$B : G/Z(G) \times G/Z(G) \to Z(G).$$

Now $Z(G)$ is normal in G and since $G \neq \{1\}$ we have $Z(G) \neq \{1\}$ as G is nilpotent. Since G is a compact *JNL* group, $G/Z(G)$ is a Lie group. Therefore $Z(G)$ is open in G and $G/Z(G)$ is finite. Now $[G, G] = B(G/Z(G) \times G/Z(G))$ is a union of compact Lie subgroups $\bigcup_{g\in G} B(G/Z(G), gZ(G))$. On the other hand, $Z(G) \simeq \mathbb{Z}_p$ by (i) of Proposition 6.6 and (i) above. In particular, $Z(G)$ does not contain any nonsingleton Lie subgroups. Hence $B(G/Z(G), gZ(G)) = \{1\}$ for each $g \in G$ and thus $[G, G] = 1$.□

Example 6.8. The compact topological ring $\mathbb{Z}_p$ of p-adic integers contains the multiplicative group E of $p-1$ roots of unity. So we can form $G = \mathbb{Z}_p \rtimes E$ with E acting on $\mathbb{Z}_p$ by multiplication. Then G is a profinite centerfree metabelian group for each $p > 2$ (see [66] and [88]). Every nonsingleton normal subgroup of G contains one of the form $p^k\mathbb{Z}_p \times \{1\}$ and thus is contained in $\mathcal{N}(G)$. Therefore G is a compact *JNL* group. We note that this example illustrates that solubility of compact *JNL* groups does not imply commutativity.

Proposition 6.7 proves that a compact abelian *JNL* group is totally disconnected. However, this conclusion holds more generally, as the following result shows.

Theorem 6.9. *A compact JNL group G is totally disconnected.*

Proof. Assume that G is a compact *JNL* group and $G_0 \neq \{1\}$. We shall derive a contradiction.

(a). Since G is a compact *JNL* group, G/G_0 is a Lie group and thus, as a totally disconnected group, is finite.

(b). We will denote with S the commutator subgroup $[G_0, G_0]$ of G_0 and with A the identity component $Z(G_0)_0$ of $Z(G_0)$. Both of these subgroups are characteristic subgroups of G_0. Set $\Delta = S \cap A$. We claim that $\Delta = \{1\}$.

Suppose that $\Delta \neq \{1\}$. Then G/Δ is a Lie group. In particular, S/Δ is a Lie group, whence $S/Z(S)$ is a Lie group, since $\Delta \leq Z(S)$. The factor group $S/Z(S)$ is of the form $\prod_{j \in J} S_j$ for a family of centerfree compact connected simple Lie groups (see [32, Theorem 9.24]), and thus J is finite. Then [32, Theorem 9.19] allows us to conclude that Δ is finite. From (ii) of Lemma 6.4 we have a contradiction. Therefore $\Delta = \{1\}$ and thus we have a direct product decomposition

$$G = S \times A$$

(see [32, Theorem 9.24]).

(c). Suppose that $S \neq 1$. Then G/S, and therefore G_0/S, is a Lie group. Hence $A \simeq G_0/S$ is a Lie group. Then (i) of Lemma 6.4 implies that $A = \{1\}$ and therefore $G_0 = S = \prod_{j \in J} S_j$. Also, G_0 is centerfree. By Lee's Theorem [32, Theorem 9.41] there is a finite group F such that $G = F \ltimes S$. Since the factors S_j are simple, the action of F induces a permutation group on J. But F is finite, then there is a finite subset I of J which is invariant under this action. Then $\prod_{j \in I} S_j$ is a nonsingleton normal subgroup of G and is a Lie group as a finite product of Lie groups. Now (i) of Lemma 6.4 implies that $S = \{1\}$, and thus we know that $G_0 = A$ is abelian.

(d). The factor group $\Gamma = G/A$ acts as a finite group of automorphisms on A, and every Γ-invariant nonsingleton subgroup B of A is normal in G. Then G/B is a Lie group and thus A/B is a finite dimensional torus (see [32, 33] for details). By Pontryagin duality (see [32, Theorem 1.37]), Γ acts as a finite automorphism group on the character group $\widehat{A}$ and

> (∗) every proper Γ-invariant subgroup P of $\widehat{A}$ is finitely generated free.

We write $\widehat{A}$ additively. Let $R = \mathbb{Z}[\Gamma]$ be the integral group ring: this makes naturally $\widehat{A}$ into an R-module. We claim that rank $\widehat{A} < \infty$, where rank $\widehat{A}$ denotes the rank of the torsion-free abelian group A (see [32, Appendix 1]).

If $|\Gamma| = n$ is a positive integer, then rank $R = n$, and the R-submodule $\langle P \rangle_R$ generated by an arbitrary subgroup P of $\widehat{A}$ satisfies the condition

$$\text{rank } \langle P \rangle_R \leq n \cdot \text{ rank } P.$$

As a consequence, if we suppose that $\widehat{A}$ has infinite rank, then we can construct a proper R-submodule of $\widehat{A}$ of infinite rank in the following way. We take infinitely many distinct elements $\widehat{a_1}, \widehat{a_2}, \ldots$ in $\widehat{A}$ and consider

$$\langle \widehat{a_1} \rangle \times \langle \widehat{a_2} \rangle \times \ldots.$$

This is against $(*)$.
Thus rank $\widehat{A} < \infty$ and we conclude that $\widehat{A}$ is finitely generated free. Then $\widehat{A}$ is a finite dimensional torus. But, G would be a Lie group and this cannot be. This final contradiction proves the result. □

Since a totally disconnected compact Lie group is finite (see [32, Exercise 2.8, (ii)]), Theorem 6.9 shows that a compact *JNL* group is not finite but has each proper quotient which is finite. Conversely if G is a totally disconnected compact group, which is not finite but all whose proper quotients are finite, then G is obviously a compact *JNL* group.

Corollary 6.10. *Assume that G is a compact JNL group. If M is a nonsingleton closed normal nilpotent subgroup of G, then M is isomorphic to $\mathbb{Z}_p$ for a suitable prime p.*

Proof. Theorem 6.9 implies that G is totally disconnected, then G/M is a finite group. We have also that M/N is finite, where $N \neq \{1\}$ is a closed normal subgroup of M. If M is a Lie subgroup of G, then G is a Lie group, against the definition of compact *JNL* group. Then M cannot be a Lie group such that all its closed normal subgroups $N \neq \{1\}$ are contained in $\mathcal{N}(M)$. It follows that M is a compact *JNL* group. The remainder of the proof follows from Proposition 6.7. □

In the situation of Corollary 6.10 one can conclude that the smallest closed subgroup F containing all nilpotent normal subgroups of G is abelian. Moreover if F is nonsingleton, then it is isomorphic to some $\mathbb{Z}_p$, where p is a suitable prime. This circumstance is analogous to [44, Theorems 10.5, 10.9, 10.10], where the Fitting subgroup is involved.

Corollary 6.11. *Assume that G is a compact JNL group and A is a nonsingelton closed abelian normal subgroup of G. If $p(A)$ is a prime which does not divide $|G/A|$, then G splits over A, that is, G is the semidirect product of $\mathbb{Z}_p$ and a finite group which is isomorphic to G/A.*

Proof. By Theorem 6.9 and Corollary 6.10 there exists a prime $p = p(A)$

such that A is a nonsigleton open abelian normal subgroup of G which is isomorphic to $\mathbb{Z}_p$. Since Theorem 6.9 implies that G is totally disconnected, we may apply directly [36, Satz III] and the result follows. □

In the situation of Corollary 6.11, [36, Satz III] shows that all the complements of A are conjugated via inner automorphisms of G. This circumstance is not new, because it can be found in most of the classical results on Just-Non-$\mathfrak{X}$ groups, where $\mathfrak{X}$ is a given class of groups. We are referring to situations as in [44, Theorems 11.1, 11.2, 12.26, 12.30, 14.1, 14.2, 14.8, 14.10, 14.18, 14.19, 15.4, 15.5, 15.11, 16.21, 16.24, 16.28, 16.30, 16.31, 16.32, 16.33, 17.5, 17.7, 17.8, 17.9] and [44, Corollaries 12.27, 12.28, 12.29]. Note that also Theorems 3.17-3.18 of Section 3 and Theorems 4.12-4.13 of Section 4 deal with the same circumstance.

Clearly, a profinite group is a compact *JNL* group if and only if every nonsingleton closed normal subgroup is open. However the consideration of the Nottingham Group can be useful to visualize profinite groups which are neither compact nor Lie groups but all whose proper quotients are Lie groups.

Example 6.12. Let p be a prime number. Following [88, p.66–67], $\mathbb{F}_p[t]$ denotes the formal power series algebra over the field with p elements $\mathbb{F}_p$ in an indeterminate t. Write A for the group of (continuous) automorphisms of $\mathbb{F}_p[t]$ and for each integer $n \geq 1$ let J_n be the kernel of the homomorphism from A to the automorphism group of $\mathbb{F}_p[t]/(t^{n+1})$, where (t^{n+1}) is the ideal generated by t^{n+1}. From [88, p.66–67], it is known that J_1 coincides with the inverse limit of J_1/J_n for $n \geq 1$ and each J_1/J_n is a finite group of order p^n. Thus J_1 is a pro-p group. Moreover J_1 is centerfree, profinite, pronilpotent, is not a Lie group, has each nonsingleton closed normal subgroup which is not a Lie group. Here J_1 has each proper quotient which is a Lie group, but J_1 is not a compact group since it is not a strictly projective limit of compact Lie groups [32, Corollary 2.43]. □

From [88, Propositions 2.2.2, 2.3.2, 2.4.3], the problem to classify all profinite (respectively prosoluble, respectively pronilpotent) groups, which are not Lie groups but all whose proper quotients are Lie groups, can be reduced to the corresponding problem for pro-p-groups (p is a prime). However, we appear to know nothing about a complete classification for such groups.

Coming back to compact *JNL* groups, a rich example of soluble compact *JNL* group which allows us to visualize the situation in the nonabelian case is the following.

Example 6.13 (Hofmann-Russo Group). We take a prime number $p \neq 2$ and let $A = \mathbb{Z}_p^2$ be the free $\mathbb{Z}_p$-module of rank 2. Every closed (additive) subgroup of A is obviously a free $\mathbb{Z}_p$-module of rank at most 2. A $\mathbb{Z}_p$-submodule of rank 2 is an open subgroup of A, and, equivalently, has finite index in A. A $\mathbb{Z}_p$-submodule of rank 1 of A is of the form $\mathbb{Z}_p \cdot (a, b)$ for each $a, b \in \mathbb{Z}_p$.

Let $R \subseteq \mathbb{Z}_p \setminus p\mathbb{Z}$ denote the multiplicative group of $(p-1)$-th roots of unity.

Let Γ denote a group of automorphisms of A with the matrix representations

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \qquad \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$$

where $a, b \in R$

We note that Γ is a group of monomial matrices and it is isomorphic to a semidirect product of the diagonal subgroup of R^2 by the cyclic group of order 2. In particular, $|\Gamma| = 2(p-1)^2$.

Now let

$$G = \Gamma \ltimes A$$

denote the semidirect product with respect to the natural action of Γ on A. We will see that

$$(**) \qquad G \text{ is a compact } JNL \text{ group.}$$

Let N be a nonsingleton closed normal subgroup of G. We must show that N has finite index in G. Since it suffices to show that the normal subgroup $N \cap (A \times \{1\})$ has finite index in G, we may assume that $N = B \times \{1\}$, where B is a Γ-invariant $\mathbb{Z}_p$-submodule of A. We must show that rank B=2, for then B is open in A; therefore A/B is finite. If rank B=0, then $N = \{1\}$ and this is not relevant for the definition of compact *JNL* group. Assume that rank B=1. We will get to a contradiction. Now $B = \mathbb{Z}_p \cdot (a, b)$ for suitable elements $a, b, \in \mathbb{Z}_p$, not both of which are zero. Since B is Γ-invariant, for each $\gamma \in \Gamma$ there is a nonzero $\lambda = \lambda_\gamma \in \mathbb{Z}_p$ such that $\gamma(a, b) = \lambda \cdot (a, b)$. If $b = 0$, then $a \neq 0$ and we let

$$\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

whence $(\lambda_\tau a, 0) = \lambda_\tau(a, 0) = \tau(a, 0) = (0, a)$, which is impossible. Likewise $a = 0$ is impossible, and so $a \neq 0 \neq b$. Then $(\lambda_\tau a, \lambda_\tau b) = \lambda_\tau(a, b) = \tau(a, b) = (b, a)$, and so $\lambda_\tau = b/a = a/b$. We conclude that $(a + b)(a - b) = a^2 - b^2 = 0$, then either $a = b$ or $a = -b$. We set

$$\alpha = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$$

for some $1 \neq r \in R$. The existence of such r is due to the fact that $p \neq 2$. Then, in the first case, $\lambda_\alpha a, \lambda_\alpha a) = \lambda_\alpha(a, b) = \alpha(a, b) = (ra, b)$. We first conclude $\lambda_\alpha = 1$, then $a = ra$ and so $r = u$, a contradiction. In the second case, we obtain $\lambda_\alpha a = ra$, then $-\lambda_\alpha a = \lambda_\alpha b = b = -a$. So again we get $\lambda_\alpha = 1$ and $r = 1$. This final contradiction proves $(**)$.

From Proposition 6.7, a compact abelian *JNL* group is isomorphic to $\mathbb{Z}_p$, but in our case $A \not\cong \mathbb{Z}_p$. Then the construction of our group G shows that

(i) *G is a soluble compact JNL group with a nonsingleton abelian normal closed subgroup $A \times \{1\}$ such that A has finite index in G and is not a compact JNL group;*

(ii) *G is a compact JNL group which is profinite and soluble of derived length 3. Moreover G'' is abelian and $G'' \not\simeq \mathbb{Z}_p$.*

(iii) *G is centerfree.*

(iv) *By taking the direct product of finitely many finite cyclic groups and G, we may construct a centerfree soluble compact JNL group of arbitrary derived length.* □

The group $\mathbb{Z}_p$ has many nonclosed proper nonsingleton subgroups. We might wish to replace E by the full group of units $\mathbb{Z}_p \setminus p\mathbb{Z}_p$ of $\mathbb{Z}_p$ in Example 6.8. The result is a more complicated metabelian profinite group, but also one that is not a compact *JNL* group. Still, observations like the following can be made.

Proposition 6.14. *Let G be a compact JNL group. There is a descending sequence*

$$G = G_1 \geq G_2 \geq G_3 \geq \ldots \geq \{1\}$$

of closed normal subgroups of G converging to 1 such that G_n/G_{n+1} is a finite product of simple groups or groups of prime order, for each positive integer $n \geq 1$. In particular, G is a second countable and thus metric profinite group.

Proof. The totally disconnected compact *JNL* group G cannot be finite, since it is not a Lie group. Then it has a descending family of compact normal subgroups

$$G = G_1 \geq G_2 \geq G_3 \geq \ldots$$

converging to 1, such that each factor group G_n/G_{n+1} is a finite product of simple groups or groups of prime order, for each positive integer $n \geq 1$ (see [32, Theorem 9.91]). If G_{n+1}, then $G_n \neq \{1\}$, and thus $G_n \in \mathcal{N}(G)$. Hence G/G_n is a Lie group and thus is finite since G is totally disconnected. □

Proposition 6.15. *Let G be a compact JNL group and A be a nonsingleton closed central subgroup of G. Then G splits over A.*

Proof. Let p be a prime. From Proposition 6.6, G is a finite central extension of $Z(G) \simeq \mathbb{Z}_p$. From (i) of Proposition 6.7, $A \simeq \mathbb{Z}_p$. We deduce that G is a finite central extension of A. But, A is torsion-free normal and $|G/A|$ is finite. Then G splits on A as claimed.□

A simple consideration can be done in order to have a deep knowledge of compact *JNL* groups without nonsingleton torsion-free normal subgroups. As it is shown in [32, Theorem 9.23], centerfree compact groups can be easily described in terms of Mayer-Vietoris sequences.

Proposition 6.16. *Let G be a compact JNL group without nonsingleton torsion-free normal subgroups. Then G is centerfree.*

Proof. Assume that $Z(G) \neq \{1\}$ and let $z \in Z(G)$ with $z \neq 1$. The subgroup $\langle z \rangle$ cannot be neither finite from (ii) of Lemma 6.4 nor torsion-free from the hypothesis. Then $z = 1$ and we obtain a contradiction. The result is proved.$\square$

7. Locally Compact JNC Groups

A *Just-Non-Compact group*, or briefly a *JNC group*, is a Hausdorff topological group which is not a compact group but all of whose proper Hausdorff quotients are compact groups. All simple groups (that is, groups without proper nonsingleton closed normal subgroups) are JNC groups by default. So one should concentrate on the nonsimple groups in the class. If G is discrete, then G is a JNC group if and only if G is an infinite group all of whose proper quotients are finite groups: this is another topic of interest to the algebraic theory of groups, and we assume from here on out that a JNC group is a nondiscrete nonsimple Hausdorff topological group.

Let $\mathcal{N}(G)$ denote the set of proper nonsingleton normal subgroups of a topological group G. Since an extension of a compact group by a compact group is a compact group, the following is clear.

Remark 7.1. *If G is a JNC group, then all members $N \in \mathcal{N}(G)$ are noncompact groups.*

This allows us to look for the normal subgroups of a topological group. A topological group is called *almost connected* if G/G_0 is a compact group, where G_0 has been already introduced at page 26. So we have as follows.

Lemma 7.2. *A JNC group G is either totally disconnected or almost connected with noncompact identity component.*

Proof. If $G_0 = \{1\}$, then G is totally disconnected. If $G_0 = G$, then G is connected and thus almost connected. If $G_0 \in \mathcal{N}(G)$, then G_0 is noncompact by Remark 7.1 and so G/G_0 is a compact group. □

We know that all locally compact almost connected groups are pro-Lie groups by Yamabe's Theorem [86, 87].

Theorem 7.3. *Let G be a JNC pro-Lie group. Then G is a Lie group and is a semidirect product $G_0 \rtimes F$ of the connected Lie group G_0 by the finite group F.*

Proof. Assume that G is not a Lie group. Then $\mathcal{N}(G)$ contains a filterbasis $\mathcal{F}$ of normal subgroups such that G/N is a compact Lie group and $G = \lim_{N\in\mathcal{F}} G/N$. Now each G/N is a compact group, since G is a JNC group. Hence G is a compact group as projective limit of compact groups. This contradicts our assumptions on G so that G is a Lie group.

By Lemma 7.2, G is either totally disconnected or almost connected. Since a totally disconnected Lie group is discrete, this case is not allowed by the definition of a JNC group. Hence G is an almost connected group which means that G has only finitely many connected components. By Dong Hoon Lee's Supplement Theorem [32, Theorem 6.74], there is a finite group F such that

$G = G_0F$ and $G_0 \cap F$ is normal in G. Since $G_0 \cap F$ is a finite group, Remark 7.1 implies that $G_0 \cap F = \{1\}$ so that the result follows. □

Recall that every Lie group has a unique largest compact normal subgroup $C(G)$ (see [32]). Then we may deduce as follows.

Lemma 7.4. *Let G be a JNC Lie group. Then the nilradical $N(G)$, that is the largest connected normal nilpotent subgroup, is either a singleton or a vector group.*

Proof. Assume that $N(G) \neq \{1\}$. A connected nilpotent Lie group is simply connected if and only if it has no compact subgroups. A compact subgroup of a nilpotent Lie group is central. By definitions, $N(G)$ is contained in $C(G)$. Since $C(G) = \{1\}$, it follows that $N(G)$ is simply connected. Note that $N(G)/N(G)'$ is never a compact group. Therefore, if $N(G)$ is nonsingleton, then $N(G)' \in \mathcal{N}(G)$, that is $N(G)' = \{1\}$, and so $N(G)$ is an abelian group. Follows that $N(G)$ is isomorphic to $\mathbb{R}^n$ for some integer $n \geq 1$. □

Recall that G_0 is reductive if $N(G)$ is singleton (see [32, p.38]).

Proposition 7.5. *Let G be a JNC Lie group such that $N(G) \neq \{1\}$. Then there exists some integer $n \geq 1$ such that $G \simeq \mathbb{R}^n \rtimes K$ with a compact Lie group K operating irreducible and effectively on $\mathbb{R}^n$.*

Proof. From Lemma 7.4, we know that there exists some integer $n \geq 1$ such that $N(G) \simeq \mathbb{R}^n$. By the definition of JNC group, $G/N(G)$ is a compact group. Then the Vector Group Splitting Theorem [32, Theorem 11.15] implies that $G \simeq \mathbb{R}^n \rtimes K$ for a maximal compact subgroup K of G. Since the representations of a compact group on a finite dimensional vector space are completely reducible, a K-invariant nonsingleton proper vector subgroup V of $N(G)$ would be in $\mathcal{N}(G)$. But, G/V would contain the noncompact subgroup $N(G)/V$. So K acts irreducibly on $N(G)$. The set of elements of K leaving all of $N(G)$ elementwise fixed is $D = K \cap Z(N(G), G)$, where $Z(N(G), G)$ denotes the centralizer of $N(G)$ in G. The normalizer of D in G contains K and $N(G)$. Thus D is normal in G and fails to meet $N(G)$, whence G/D fails to be compact. Hence $D = \{1\}$, that is, K acts effectively on $N(G)$. The result follows. □

The assertion that K acts effectively and the compactness of K means that K may be identified with a closed subgroup of the orthogonal group $O(n)$ of $\mathbb{R}^n$ with respect to a suitable positive definite inner product on $\mathbb{R}^n$. Conversely, every closed subgroup K of $O(n)$, acting irreducibly on $\mathbb{R}^n$ give arise to a JNC group $\mathbb{R}^n \rtimes K$.

Note that the case that K is singleton is not excluded; the singleton group acts irreducibly and effectively on $\mathbb{R}$, and so $G = \mathbb{R}$ is included in the class of JNC groups, classified in Proposition 7.5.

Proposition 7.6. *Let G be a JNC Lie group such that $N(G) = \{1\}$. Then there exists some integer $k \geq 1$ such that $G \simeq S^k \rtimes F$ for a centerfree simple connected noncompact Lie group S and a finite group F permuting the k factors transitively and effectively, that is, no element of F commutes with all elements of G_0.*

Proof. The identity component $Z(G_0)_0$ of the center of G_0 is contained in $N(G)$, and so is trivial. Hence $Z(G_0)$ is discrete and G_0 is semisimple. Since $C(G_0)$ is trivial, G_0 has no compact factors. Since the center $Z(G_0)$ is characteristic, it is normal in G. Since $G_0/Z(G_0)$ is not a compact group as G_0 has no compact factors, we conclude that $Z(G_0) = \{1\}$. This means that G_0 is a centerfree group. Therefore, $G_0 \simeq S_1 \times S_2 \ldots \times S_k$ for simple adjoint noncompact connected Lie groups S_j, where $j \in \{1, 2, \ldots, k\}$. Thus

$$G \simeq (S_1 \times S_2 \ldots \times S_k) \rtimes F$$

for a finite group F by Theorem 7.3. The finite group F permutes the set of factors $\{S_1, S_2 \ldots, S_k\}$. If there is more than one orbit, then a partial product $N = S_{j_1} \times S_{j_2} \ldots \times S_{j_p}$ extended over a proper orbit give a nonsingleton proper normal subgroup such that G_0/N is not a compact group. This is not possible. If F permutes the factors transitively, then they are all isomorphic. The result follows. $\Box$

We know that the automorphism group of S^k in Proposition 7.6; there are many ways how F can act as a group of automorphisms of S^k. But, however it acts, it must not leave a proper nonsingleton normal subgroup of S^k invariant.

Conversely, if S is a centerfree simple connected noncompact Lie group and F is a finite subgroup of the Lie group of automorphism of S^k for some $k \geq 1$ permuting the factors transitively, then $S^k \rtimes F$ is a locally compact JNC group.

In essence, the results of the present Section say that we know explicitly the what the locally compact JNC groups are, if they are not totally disconnected. In the direction of totally disconnected groups, we know only that locally compact JNC groups are not prodiscrete. The examples we constructed above can be mimicked to some extent by using the irreducible and faithful representations of certain compact groups over finite dimensional vector spaces over nonarchimedean locally compact fields. However, this seems to be a wide field.

8. Some Open Questions

The present Section deals with some open questions in the context topological groups when we want to investigate JNX groups and MNX groups. There is literature in the abstract case as we mentioned in Section 1 of the present paper. It seems reasonable that a line of research as in [20, Chapters 6, 11] could be open in the context of topological groups, considering varieties of topological groups in the sense of [35, 34, 59, 58, 55, 56, 54, 53, 60, 80]. To the best of our knowledge, a systematic study in such a direction of research has been still not given. Looking at similar situations for abstract groups, we may formulate some open questions. For doing this, we need of the following notions.

Definitions 8.1. *Let Ω be a class of topological groups and $\mathfrak{V}(\Omega)$ be a variety of Hausdorff groups generated by Ω. Let G be a topological group in $\mathfrak{V}(\Omega)$, $N \neq \{1\}$ be a normal closed subgroup of G, $M \neq G$ be a normal closed subgroup of G. Define*

$$\mathcal{M}_{\mathfrak{V}(\Omega)}(G) = \{N \triangleleft G \ : \ G/N \in \mathfrak{V}(\Omega)\}$$

and

$$\widehat{\mathcal{M}}_{\mathfrak{V}(\Omega)}(G) = \{M \triangleleft G \ : \ M \in \mathfrak{V}(\Omega)\}.$$

(i) *A locally compact Just-Non-$\mathfrak{V}(\Omega)$ group G, or briefly a locally compact JNV group, is a locally compact group G which is not in $\mathfrak{V}(\Omega)$ such that all its closed normal subgroups $N \neq \{1\}$ are in $\mathcal{M}_{\mathfrak{V}(\Omega)}(G)$.*

(ii) *A locally compact Minimal-Non-$\mathfrak{V}(\Omega)$ group G, or briefly a locally compact MNV group, is a locally compact group G which is not in $\mathfrak{V}(\Omega)$ such that all its closed normal subgroups $M \neq G$ are in $\widehat{\mathcal{M}}_{\mathfrak{V}(\Omega)}(G)$.*

Assume that G is a locally compact group. If G is a simple group not belonging to $\mathfrak{V}(\Omega)$, that is, if G is a group not belonging to $\mathfrak{V}(\Omega)$ and without proper nonsingleton closed normal subgroups, then G is both a locally compact JNV groups and a locally compact MNV group by default. Therefore, an easy source of examples of groups satisfying Definition 8.1 is given.

Problem 8.2.

(i) *What is the structure of a locally compact JNV group?*

(ii) *What is the structure of a locally compact MNV group?*

It seems reasonable that a description as in Section 7 could be obtained for locally compact JNV groups. The results of Section 6 show that this is possible for compact groups when the variety of Lie groups is considered. To the best of our knowledge, a systematic study in such a direction of research is still open. Of course, the nilradical $N(G)$ of a topological group G should be substituted by an appropriate subgroup which should testify the verbal structure of G; the marginal structure of G; the properties of the variety with respect to forming suitable products, quotients and subgroups; the topological closure properties of the variety. The terminology which we have just mentioned follows [35, 34, 59, 58, 55, 56, 54, 53, 60, 80].

On the other hand, the literature in abstract groups seems to suggest that also a satisfactory description for locally compact MNV groups can be obtained. To the best of our knowledge, a systematic study in such a direction of research is still open.

Finally, the role of the Frattini subgroup and that of the Fitting subgroup suggest to formulate the following open questions, both in abstract case and in topological case.

Problem 8.3.

(i) *Let $\mathfrak{X}$ be a class of groups. Is it possible to introduce a categorical approach to JNX groups and MNX groups?*

(ii) *Is it possible to do the same of the previous step (i) in the category of Hausdorff topological groups?*

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List of Symbols

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