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BMO MARTINGALES, A_p-CONDITION AND CLASSICAL OPERATORS

TESI DI DOTTORATO DI RICERCA

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Contents

Introduction

In this thesis we start studying the boundedness properties of various classical operators of harmonic analysis, in weighted Lebesgue and Orlicz spaces, in terms of the membership of the relevant weights to the so called A_p-classes. Subsequently, we describe both in the functional and probabilistic context the functions of Bounded Mean Oscillation functions (BMO) and their relations with A_p weights. We present some of our sharp results in the functional setting that we would like to extend to the BMO-Martingales space. We conclude describing an application in Mathematical Finance, due to Geiss [Ge].

The structure of thesis is the following. In the first chapter we describe the A_p and G_q classes and their properties. The A_p-class was introduced in [Mu1] in 1972, where the Hardy-Littlewood maximal operator is proved to be bounded in the Lebesgue space $L^p(\mathbb{R}^n, w)$, equipped with the measure $w(x)dx$, if and only if $w \in A_p$. The same role is played by the A_p-class in the study of boundedness of the the Hilbert transform [HMW] and of other singular integral operators [CFe].

The theory of weights arises often in various contexts of mathematical analysis including the theory of degenerate elliptic equations [FKS, Mo], the related nonlinear potential theory [HKM], and the theory of quasiregular maps [AIS].

One of the most useful results in the field, is probably, the selfimproving property of Muckenhoupt's weights. The surprising fact that the weights are more regular than they seem to be a priori was observed already by Muckenhoupt [Mu1]. The same phenomenon was studied by Gehring in [G] where he introduced the concept of reverse Hölder inequalities and proved that they improve themselves. Later Coifman and Fefferman [CFe] showed that Muckenhoupt's weights are exactly those weights which satisfy a reverse Hölder inequality. Since then reverse Hölder inequalities have had a wide number of applications in modern analysis.

In the second and third chapter we describe weak and strong inequalities for the Hardy-Littlewood Maximal Function in connections with A_p and G_q weights in the Lebesgue and Orlicz spaces.

In the fourth chapter we illustrate the boundedness properties of the singular operators through the A_p weights. In particular we obtain a sharp estimate for the Riesz Potential in terms of a precise power of the A_p-constant of a weight.

In the fifth chapter, we pass to study the BMO space, introduced in 1961 by John and Nirenberg [JN], and consider its connections with A_p-weights. In particular we show, in one dimension, sharp inequalities between some A_2-constants and BMO norm.

The BMO space is extremely important in various areas of Mathematics. In particular, in the last part of thesis we describe the probabilistic version of BMO and of the A_p-condition, through the BMO-martingale space, in which some of previous results continue to hold.

Finally, we give an informal introduction to the theory of Mathematical Finance with special emphasis on the BMO-martingale and show how this space is useful in the problem of pricing an option and describing the evolution of stock's price.

Chapter 1

Weighted Integral Inequalities

In this chapter we recall basic definitions and properties of A_p and G_q weights, respectively from Muckenhoupt [Mu1] and Gehring [G], and show some known results and our sharp results about the improvement of the integrability exponent.

1.1 The class of weights A_p and G_q

Let us assume that a weight ω is a non negative locally integrable function on $\mathbb{R}^n$ and we consider only cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes. The n-dimensional Lebesgue measure of a subset E of $\mathbb{R}^n$ will be denote by $|E|$, while we will set

$$\omega_Q = \fint_Q \omega(x)\, dx = \frac{1}{|Q|} \int_Q \omega(x)\, dx$$

to denote the mean value of ω over Q.

Definition 1.1. We say that ω satisfies the A_p**-condition** (briefly $\omega \in A_p(\mathbb{R}^n)$), $1 < p < \infty$, if there exists a constant $A \geq 1$ such that, for any cube $Q \subset \mathbb{R}^n$, one has

$$\fint_Q \omega(x)dx \left(\fint_Q \omega^{-\frac{1}{p-1}}(x)dx \right)^{p-1} \leq A. \tag{1.1}$$

We call the A_p**-constant** of ω as

$$A_p(\omega) = \sup_Q \fint_Q \omega(x)dx \left(\fint_Q \omega^{-\frac{1}{p-1}}(x)dx \right)^{p-1}, \tag{1.2}$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with edges parallel to the coordinate axes.

The A_p-class was introduced in 1972 by B. Muckenhoupt [Mu1] in connection with boundedness properties of the Hardy-Littlewood Maximal Operator M (see Chapter 2 for details) defined on the weighted space $L^p_{loc}(\mathbb{R}^n, \omega dx)$ by

$$Mf(x) = \sup_{x\in Q} \fint_Q |f(y)|dy. \tag{1.3}$$

One of the most prominent examples of a Muckenhoupt weights $\omega \in A_p$, $1 < p < \infty$, is given by $\omega(x) = |x|^\alpha$ when $-n < \alpha < n(p-1)$. Moreover, if $0<\delta<1$, then $\omega(x) = |x|^{-n(1-\delta)} \in A_1$; also $\omega(x) = |x|^{n(p-1)(1-\delta)} \in A_p$.

Now we report the specific definitions for A_p-class when $p = 1$ and $p = \infty$.

Definition 1.2. [Mu1] For $p = 1$, A_1**-class** consists of all weights ω such that the quantity

$$A_1(\omega) = \sup_Q \frac{\fint_Q \omega dx}{\operatorname{ess\,inf}_{x\in Q} \omega(x)} \tag{1.4}$$

is finite for every cube $Q \subset \mathbb{R}^n$ and we call it A_1**-constant** of ω.

Definition 1.3. [H] For $p = \infty$, A_∞**-class** consists of all weights ω such that the quantity

$$A_\infty(\omega) = \sup_Q \left(\fint_Q \omega dx\right)\left(\exp \fint_Q \log \frac{1}{\omega} dx\right) \tag{1.5}$$

is finite for every cube $Q \subset \mathbb{R}^n$ and we call it A_∞**-constant** of ω.

There is also a characterization that gives an equivalent definition of A_∞-class, namely

Proposition 1.1. *([Mu2], [CFe]) A locally integrable weight $\omega : \mathbb{R}^n \longrightarrow [0, +\infty)$ belongs to the A_∞-class iff there exist constants $0<\alpha\le 1 \le K$ so that*

$$\frac{|F|}{|Q|} \le K \left(\frac{\int_F wdx}{\int_Q wdx}\right)^\alpha \tag{1.6}$$

for each cube $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes and for each measurable set $F \subset Q$.

In Muckenhoupt's paper [Mu1] was proved the following result, also known "*backward propagation*" of the A_p condition: if a weight belongs to the class A_p , then it also belongs to the class $A_{p-\varepsilon}$ for some $\varepsilon > 0$. In other words, we have the inclusion

$$A_p \subset A_{p-\varepsilon}.$$

Two years later Coifman and Fefferman proved in [CFe] the following lemma.

Lemma 1.2. *[CFe] If $\omega \in A_p$, then $\omega \in A_{p-\varepsilon}$, where $\varepsilon \sim A_p(\omega)^{-\frac{1}{p-1}}$, and there exists a constant C such that*

$$A_{p-\varepsilon}(\omega) \leq C A_p(\omega).$$

Another important result, useful to illustrate the properties of A_p weights is the following

Theorem 1.3. *[W1] A locally integrable weight ω is in A_p, $p > 1$, if and only if there exists $1 < p_1 < p$ such that for every cube Q*

$$\left(\frac{|F|}{|Q|}\right)^{p_1} \leq A_{p_1}(\omega) \frac{\int_F \omega \, dx}{\int_Q \omega \, dx}$$

for every measurable subset F of Q.

Now, we pass to describe another important class of weights, the G_q-class, born almost simultaneously (1973), thanks to F.W. Gehring [G], in connection with local integrability properties of the gradient of quasiconformal mappings.

Definition 1.4. A weight v on the space $\mathbb{R}^n$ satisfies the **G_q-condition** if there exists a constant $G \geq 1$ such that, for all cubes $Q \subset \mathbb{R}^n$ as above, we have

$$\frac{\left(\fint_Q v^q(x)dx\right)^{\frac{1}{q}}}{\fint_Q v(x)dx} \leq G \tag{1.7}$$

and we refer to (1.7) as a "reverse" Hölder inequality. We call the **G_q-constant** of v as

$$G_q(v) = \sup_Q \left[\frac{\left(\fint_Q v^q dx\right)^{\frac{1}{q}}}{\fint_Q v dx}\right]^{q'} \tag{1.8}$$

where $q' = \frac{q}{q-1}$ is the Hölder exponent of q.

We consider the case $q = 1$ and $q = \infty$ and define G_1-class and G_∞-class.

Definition 1.5. G_1**-class** consists of all weights v such that $G_1(v)$**-constant**, defined by

$$G_1(v) = \sup_Q \left(\exp \fint_Q \frac{v}{v_Q} \log \frac{v}{v_Q} dx \right), \tag{1.9}$$

is finite, with $v_Q = \fint_Q v\, dx$.

Definition 1.6. G_∞**-class** consists of all weights v such that $G_\infty(v)$**-constant**, defined by

$$G_\infty(v) = \sup_Q \frac{\operatorname*{ess\,sup}_{x \in Q} v}{\fint_Q v\, dx} \tag{1.10}$$

is finite.

The following characterization gives an equivalent definition of G_1-class and it's similar to definition of A_∞ in (1.6):

Proposition 1.4. *A locally integrable weight $v : \mathbb{R}^n \longrightarrow [0, \infty)$ belongs to G_1-class iff there exist constants $0 < \beta \leq 1 \leq H$ so that*

$$\frac{\int_E w dx}{\int_Q w dx} \leq H \left(\frac{|E|}{|Q|} \right)^\beta \tag{1.11}$$

for each cube $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes and for each measurable set $E \subset Q$.

Gehring [G] showed the improvement of the integrability exponent in a reverse Hölder inequality also known "forward propagation" of G_q condition: that for any class G_q there exists an $\eta > 0$ for which we have the inclusion

$$G_q \subset G_{q+\eta}.$$

The last result has numerous applications in the theory of weighted spaces, quasiconformal mappings, and partial differential equations.

Now we report a result for G_q weights, similar to Theorem (1.3).

Theorem 1.5. *[Mi] A locally integrable weight v is in G_q, $q > 1$ if and only if there exists $q_1 > q$ such that for every cube Q*

$$\left(\frac{\int_E v dx}{\int_Q v dx}\right)^{q_1'} \leq G_{q_1}(v)\frac{|E|}{|Q|}$$

where $q_1' = \frac{q_1}{q_1-1}$, for every measurable subset E of Q.

The relationship between the Gehring and Muckenhoupt classes is studied in a number of papers. In [CFe] Coifman and Fefferman proved that each Gehring class is contained in a Muckenhoupt class and vice versa, i.e., we have inclusions of the form

$$A_p \subset G_q \qquad\qquad G_{q_1} \subset A_{p_1},$$

for some $1 \leq p, q, p_1, q_1 \leq \infty$.

We resume some of the properties of the classes A_p and G_q in the following proposition

Proposition 1.6. *[GR] Let ω a weight and let $1<p\ ,q<\infty$. Then*

1. $A_1 \subset A_p \subset A_q$, *for* $1 \leq p < q < \infty$.
2. $\omega \in A_p \Longrightarrow \omega^\alpha \in A_p$, $0 \leq \alpha \leq 1$.
3. $\omega \in A_p$ *if and only if* $\omega^{1-p'} \in A_{p'}$, *with* $p' = \frac{p}{p-1}$.
4. $\omega \in A_p$ *iff there exist* $u, v \in A_1$, *so that* $\omega = uv^{1-p}$.
5. $G_\infty \subset G_q \subset G_p$, *for* $1 < p \leq q < \infty$.

There are some limiting relations between constants A_1, A_∞, G_1 and G_∞ defined above, as the following theorems show.

Theorem 1.7. *[StWh] Let $\omega : \mathbb{R}^n \longrightarrow [0, +\infty)$ be a weight, then*

$$\omega \in G_q \iff \omega^q \in A_\infty.$$

Sbordone and Wik [SW] proved the following

$$A_\infty(\omega) = \lim_{p\to\infty} A_p(\omega), \tag{1.12}$$

and later Moscariello and Sbordone [MS] established an analogous result for G_q-weight as

$$G_1(v) = \lim_{q \to 1} G_q(v). \tag{1.13}$$

These formulas give a quantitative version of the equalities

$$A_\infty = \bigcup_{p>1} A_p = \bigcup_{q>1} G_q = G_1.$$

proved by Muckenhoupt in [Mu2].

A precise relation among A_∞ and G_1 constants in one dimension was due to R. Corporente [C] in the following theorem

Theorem 1.8. *[C] Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing homeomorphism onto such that h, h^{-1} are locally absolutely continuous. Then*

$$A_\infty(h') = G_1((h^{-1})'), \tag{1.14}$$

where h' is a first derivative of h.

1.2 Improvement of the integrability exponent

In this section we report some results about the so-called "sharp self-improvement of exponents" property of the A_p and G_q classes in one dimension.

Let us begin with some results about the improvement of the integrability exponent of a G_q-weight through the following Theorem

Theorem 1.9. *[DS] Let $q > 1$ and assume $v \in G_q$. Then $v \in G_r$ with $r \in [q, q_1)$ and q_1 is the unique solution of the equation*

$$\varphi(x) = 1 - B^q \, \frac{x-q}{x} \left(\frac{x}{x-1} \right)^q = 0$$

where B is such that

$$\left(\fint_I v^q dx \right)^{\frac{1}{q}} \leq B \fint_I v dx,$$

with $I \subset \mathbb{R}$. Moreover, for $q \leq q_0 < q_1$ we have

$$[G_{q_0}(v)]^{\frac{1}{q_0'}} \leq B^{\frac{1}{q'}} \left[\frac{q}{q_0 \varphi(q_0)} \right]^{\frac{1}{q}}.$$

The result is sharp.

Note that Theorem 1.9 also shows that the best integrability exponent of all non increasing functions in G_q is equal to the best integrability exponent of a power type function in G_q.

Theorem 1.9 was generalized to all functions in 1992 by Korenovskii [Ko1] in the following

Theorem 1.10. *[Ko1] Let $1 < q < \infty$ and let $f \in L^q(I)$. If $f \in G_q$, then $f \in G_p$ for $p \in [q, \beta)$ and β verifies the equation*

$$1 - M^q \frac{x-q}{x}\left(\frac{x}{x-1}\right)^q = 0$$

where M is such that

$$\left(\fint_I f^q dx\right)^{\frac{1}{q}} \leq M \fint_I f dx,$$

with $I \subset \mathbb{R}$.

In [Ko1] there is also a A_p version of the same result.

Theorem 1.11. *[Ko1] Let $p > 1$ and assume $\omega \in A_p$. Then $\omega \in A_s$ with $s \in [p, p_1)$ where p_1 is the unique solution of the equation*

$$\psi(x) = 1 - \frac{p-x}{p-1}(Ax)^{\frac{1}{p-1}} = 0$$

where A is such that

$$\fint_I \omega dx \left(\fint_I \omega^{-\frac{1}{p-1}} dx\right)^{p-1} \leq A,$$

with $I \subset \mathbb{R}$. Moreover, for $p_1 < p_0 \leq p$ we have

$$A_{p_0}(\omega) \leq A\left[\frac{p_0 - 1}{(p-1)\psi(p_0)}\right]^{p-1}.$$

It is worth noting that in the special case $p = q = 2$ we have explicit values of q_1 and p_1 and the above theorems enjoy a simpler presentation.

Theorem 1.12. *[Ko1] Let $\omega : \mathbb{R} \to [0, \infty[$ be a weight such that $A_2(\omega) = A < \infty$. Define for $1 + \sqrt{\frac{A-1}{A}} < s \leq 2$,*

$$\psi(s) = \frac{1}{s-1}[1 - A(2-s)s]. \tag{1.15}$$

Then $A_s(\omega) < \infty$ for any s in such a range and

$$A_s(\omega) \leq \frac{A}{\psi(s)}. \tag{1.16}$$

The result is sharp.

Theorem 1.13. *[DS] Let $v : \mathbb{R} \to [0, \infty[$ be a weight such that $G_2(v) = B < \infty$. Define for* $\;2 \le r < 1 + \sqrt{\frac{B}{B-1}}$,

$$\varphi(r) = r - G_r(v)(r-2)\frac{r^2}{(r-1)^2}. \tag{1.17}$$

Then $G_r(v) < \infty$ for any r in such a range and

$$G_r(v)^{\frac{2}{r'}} \le \frac{2B}{\varphi(r)}. \tag{1.18}$$

Various relations occurring among A_p and A_2 constants of weights and their powers are collected in the following

Lemma 1.14. *[S2] Let $\omega : \mathbb{R} \longrightarrow \mathbb{R}$ be a weight. For $p > 1$ we have*

$$[A_2(\omega^{\frac{1}{p-1}})]^{p-1} \le A_p(\omega)\ A_p(\omega^{-1}). \tag{1.19}$$

For $1 < p \le 2$ we have

$$A_p(\omega) \le [A_2(\omega^{\frac{1}{p-1}})]^{p-1}. \tag{1.20}$$

For $q > 1$ we have

$$A_2(\omega) \le A_q(\omega) A_q(\omega^{-1}). \tag{1.21}$$

Proof. For any interval $I \subset \mathbb{R}$, Hölder inequality implies

$$1 \le \fint_I \omega \fint_I \omega^{-1}$$

hence

$$\left[\fint_I \omega^{\frac{1}{p-1}} \fint_I \omega^{-\frac{1}{p-1}}\right]^{p-1} \le$$

$$\le \fint_I \omega \left(\fint_I \omega^{-\frac{1}{p-1}}\right)^{p-1} \cdot \fint_I \omega^{-1} \left(\fint_I \omega^{\frac{1}{p-1}}\right)^{p-1} \le A_p(\omega) A_p(\omega^{-1})$$

taking supremum with respect to all intervals I we obtain (1.19).

Fix an interval I and take p such that $1 < p \le 2$; then we have $1 \le \frac{1}{p-1}$ and Jensen inequality implies

$$\fint_I \omega \le \left(\fint_I \omega^{\frac{1}{p-1}}\right)^{p-1}$$

hence

$$\fint_I \omega \left(\fint_I \omega^{-\frac{1}{p-1}}\right)^{p-1} \le \left[\fint_I \omega^{-\frac{1}{p-1}} \cdot \fint_I \omega^{\frac{1}{p-1}}\right]^{p-1} \le \left[A_2(\omega^{\frac{1}{p-1}})\right]^{p-1}.$$

Taking supremum with respect to all intervals I we obtain (1.20).

If $q > 1$ assume

$$A_q(\omega)A_q(\omega^{-1}) = A < \infty.$$

Since that $A_q(w) = [A_p(w^{-\frac{1}{q-1}})]^{q-1}$ where $p = q/(q-1)$, we have

$$A_p(\omega^{\frac{1}{q-1}})A_p(\omega^{-\frac{1}{q-1}}) = A^{\frac{1}{q-1}}.$$

Replacing ω with $\omega^{\frac{1}{q-1}}$ in (1.19) we get

$$\left[A_2((\omega^{\frac{1}{q-1}})^{\frac{1}{p-1}})\right]^{p-1} \leq A_p(\omega^{\frac{1}{q-1}})\ A_p(\omega^{-\frac{1}{q-1}}).$$

But $(q-1)(p-1) = 1$, hence

$$A_2(\omega) \leq A$$

that is (1.21). □

Up to now we have been dealing with the self-improvement of exponents p and q in A_p and G_q classes.

Now we consider the problem of the exact G_q-class pertaining to all A_p-weights. This was solved for $p = 1$ in [BSW] and for $p > 1$ has been recently settled by Vasyunin [V], who found the exact range of exponents q so that a weight in the A_p-class belongs to the G_q-class. Let us first state the main result in [BSW].

Theorem 1.15. *[BSW] Let ω belong to the A_1-class with $A_1(\omega) = A$. Then, for every $1 \leq q < \frac{A}{A-1}$*

$$[G_q(\omega)]^{q-1} \leq \frac{1}{A^{q-1}(A+q-qA)}. \tag{1.22}$$

The constant on the right hand side as well as the upper bound of q cannot be improved. In fact, the weight $\omega(t) = \frac{t^{\frac{1}{(A-1)}}}{A}$ is an extremal, which gives equality in (1.22) and lies in L^q if and only if $q < \frac{A}{A-1}$.

In order to state the result from [V], we fix $p > 1$ and $\delta > 1$ and denote by $x = x(p,\delta)$ the positive solution to the equation

$$(1-x)(1-x/p)^{-p} = \frac{1}{\delta}.$$

Then $0 < x \leq 1$ and we put

$$p^* = p^*(p,\delta) = \frac{p-x}{x(p-1)}$$

we have the following

Theorem 1.16. *[V] Suppose that a weight ω belongs to A_p and let $A = A_p(\omega)$. Then ω belongs to G_q for each $1 \le q < p^*(p, A)$. The bound for q is optimal.*

We report a result contained in [S1] that gives a simple proof of previous theorem in a special case.

Theorem 1.17. *[S2] Suppose that a non-decreasing weight $\omega : [a, b] \to [0, \infty)$ belongs to A_2 and $A = A_2(\omega)$. Then for $1 \le q < \sqrt{\frac{A}{A-1}}$, ω^{-1} belongs to G_q and for any $[c, d] \subset [a, b]$*

$$\left(\fint_c^d \omega^{-q}\, dx \right)^{1/q} \le \frac{q}{A - q^2(A-1)} \fint_c^d \omega^{-1}\, dx. \tag{1.23}$$

The result is sharp.

Another point of view concerns the improvement of power exponents pertaining to A_2 weights. Namely, assume that the weights ω belongs to A_2 and set $A = A_2(\omega)$. Then, it is easy to check that

$$A_2(\omega^\theta) \le A^\theta \qquad \text{for } 0 \le \theta \le 1.$$

In the next Theorem we describe the so called optimal "self-improvement of exponents" property of the A_2 class.

Theorem 1.18. *[AS] Assume $A_2(\omega) = A < \infty$, then for $1 \le \tau < \sqrt{\frac{A}{A-1}}$ we have $\omega^\tau \in A_2$ and*

$$A_2(\omega^\tau)^{\frac{1}{2\tau}} \le \frac{\tau A}{A - \tau^2(A-1)}. \tag{1.24}$$

The upper bound on τ cannot be improved.

Proof. Let us recall that the exact continuation of Muckenhoupt condition A_2 in one dimension ([Ko1], [S2], [V]) reads as follows: for $1 + \sqrt{\frac{A-1}{A}} < s \le 2$

$$A_s(\omega) \le \frac{A}{\psi(s)} \tag{1.25}$$

with

$$\psi(s) = \frac{1}{s-1}[1 - As(2-s)]. \tag{1.26}$$

From definition of $A_s(\omega)$, we deduce, for any interval $I \subset \mathbb{R}$,

$$\fint_I \omega^{-\frac{1}{s-1}} \leq \left[\frac{1}{\fint_I \omega} \cdot \frac{A}{\psi(s)} \right]^{1/(s-1)} \tag{1.27}$$

and also, taking into account that $A = A_2(\omega) = A_2(\omega^{-1})$ we deduce that

$$\fint_I \omega^{\frac{1}{s-1}} \leq \left[\frac{1}{\fint_I \omega^{-1}} \cdot \frac{A}{\psi(s)} \right]^{1/(s-1)} . \tag{1.28}$$

Multiplying (1.27) and (1.28) and using the Hölder inequality in the form

$$1 \leq \fint_I \omega(x)dx \fint_I \omega^{-1}(x)dx,$$

we obtain

$$\fint_I \omega^{\frac{1}{s-1}}(x)dx \fint_I \omega^{-\frac{1}{s-1}}(x)dx \leq \left[\frac{A}{\psi(s)} \right]^{2/(s-1)} .$$

Hence, for $1 + \sqrt{\frac{A-1}{A}} < s \leq 2$ we have

$$A_2(\omega^{\frac{1}{s-1}}) \leq \left[\frac{A}{\psi(s)} \right]^{2/(s-1)} .$$

If we set $\tau = \frac{1}{s-1}$ we obtain immediately, for the range $1 < \tau < \sqrt{\frac{A}{A-1}}$,

$$[A_2(\omega^\tau)]^{1/2\tau} \leq \frac{A}{\varphi(\tau)}$$

where $\varphi(\tau) = \tau \left[1 - A(1 - \frac{1}{\tau^2}) \right]$ which coincides with (1.24).
The optimality is seen by mean of power functions. Namely, choose $\omega(x) = |x|^r$ with $0 < r < 1$, then we have

$$A_2(|x|^r) = \frac{1}{1 - r^2}$$

and $A_2(|x|^{r\tau}) = \dfrac{1}{1 - \tau^2 r^2} < \infty$ if and only if $1 < \tau < \sqrt{\frac{A}{A-1}} = \dfrac{1}{r}$. $\square$

Chapter 2

The Hardy-Littlewood Maximal Operator

In this chapter we start to define the weak and strong type inequalities. A weighted inequality for an operator is a boundedness result from some L^p space to some L^q space when at least one of those spaces is taken with respect to a measure different from Lebesgue measure. Many times the measures are absolutely continuous with respect to Lebesgue measure and the densities are called weights; that is, if $d\mu(x) = \omega(x)dx$, ω is the weight.

A question of considerable interest in harmonic analysis is, "What types of weights ω have the property that T is bounded on $L^p(\omega)$?" where $1 < p < \infty$, and T is an operator which is bounded on the (unweighted) space L^p. Here, $L^p(\omega) = L^p(\mathbb{R}^n, \omega)$ denotes the weighted Lebesgue space, that is as usual, the Banach space of all measurable functions f on $\mathbb{R}^n$ with finite norm, defined by

$$||f||_{L^p(\omega)} = (\int_{\mathbb{R}^n} |f(x)|^p \omega(x)\ dx)^{1/p},$$

where a weight ω is supposed to be a non-negative locally integrable function.

Typically T is the Hardy-Littlewood maximal operator, singular integral operators, or various related operators of interest in harmonic analysis.

In this chapter we will describe the Hardy-Littlewood maximal operator, first introduced by Hardy and Littlewood [HL] in one dimensional case for the purpose of the application to Complex Analysis.

It is a classical tool in harmonic analysis but recently it has been successfully used in studying Sobolev functions and partial differential equations.

We will focus attention on weighted norm inequalities for Hardy-Littlewood type maximal operators. The importance of the maximal operator stems from the fact that it plays a very significant role in the estimate of different operators in analysis.

We will consider weak type $(1,1)$ inequalities satisfied by several types of Hardy-Littlewood maximal operators. As is well known, weak type $(1,1)$ inequalities satisfied by Hardy-Littlewood maximal operators are keys to prove their strong type (p,p) boundedness via Marcinkiewiczs interpolation theorem.

2.1 Hardy-Littlewood maximal function

Let f be a locally integrable function on $\mathbb{R}^n$. Then, we have different definitions for the H-L Maximal function, as following

Definition 2.1. Let $B_r = B(0,r)$ be the Euclidean ball of radius r centered at the origin. The **Hardy-Littlewood maximal function** M^*f is defined by

$$M^*f(x) = \sup_{r>0} \fint_{B_r} |f(x-y)|dy. \tag{2.1}$$

This can equal to $+\infty$.

We can define the Hardy-Littlewood maximal function in another way, with cubes in place of balls, namely

Definition 2.2. If Q_r is the cube $[-r,r]^n$, we define the **centered Hardy-Littlewood maximal function** as

$$M^{'}f(x) = \sup_{r>0} \frac{1}{(2r)^n} \int_{Q_r} |f(x-y)|dy. \tag{2.2}$$

When $n=1$, M^* and $M^{'}$ coincide, while if $n>1$, then there exist constants c_n and C_n, depending only on n, such that

$$c_n M^{'}f(x) \leq M^*f(x) \leq C_n M^{'}f(x). \tag{2.3}$$

Because of inequality (2.3), the two operators M^* and $M^{'}$ are essentially interchangeable, and one can choose that more appropriate, depending on the circumstances. In fact, we can also give a more general definition for the H-L maximal function:

Definition 2.3. Let f be a locally integrable function on $\mathbb{R}^n$. The **non-centered Hardy-Littlewood maximal function** is defined by:

$$Mf(x) = \sup_{x \in Q} \fint_Q |f(y)|dy, \tag{2.4}$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to coordinate axes and containing x.

Again, M^* is pointwise equivalent to M, so in the following we call M the **Hardy-Littlewood maximal operator** and Mf the Hardy-Littlewood maximal function.

Note that the maximal operator M is sublinear and homogeneous, that is,

$$M(f+g) \leq Mf + Mg \qquad \textit{and} \qquad M(\lambda f) = \lambda(Mf), \quad \forall \lambda \geq 0.$$

The Hardy-Littlewood maximal operator appears in many places but some of its most notable uses are in the proofs of the Lebesgue differentiation theorem and Fatou's theorem and in the theory of singular integral operators.

2.2 Weak and strong type inequalities for the H-L maximal operator

We begin recalling the definition of weak and strong type inequalities.

Definition 2.4. Let (X, μ) and (Y, ν) be measure spaces, and let T be an operator from $L^p(X, \mu)$ into the space of measurable functions from Y to $\mathbb{C}$. We say that T is **weak** (p, q), $q < \infty$, if

$$\nu(\{y \in Y : |Tf(y)| > \lambda\}) \leq \left(\frac{C\|f\|_p}{\lambda} \right)^q,$$

and we say that it is weak (p, ∞) if it is a bounded operator from $L^p(X, \mu)$ to $L^\infty(Y, \nu)$.

We say that T is **strong** (p, q) if it is bounded from $L^p(X, \mu)$ to $L^q(Y, \nu)$. If T is strong (p, q) then it is weak (p, q), in fact if we let $E_\lambda = \{y \in Y : |Tf(y)| > \lambda\}$, then

$$\nu(E_\lambda) = \int_{E_\lambda} d\nu \leq \int_{E_\lambda} \left| \frac{Tf(x)}{\lambda} \right|^q d\nu \leq \frac{\|Tf\|_q^q}{\lambda^q} \leq \left(\frac{C\|f\|_p}{\lambda} \right)^q.$$

When $(X,\mu) = (Y,\nu)$ and T is the identity, the weak (p,p) inequality is the classical Chebyshev inequality.

Definition 2.5. Let (X,μ) be a measure space and let $f : X \longrightarrow \mathbb{C}$ be a measurable function. We call the function $a_f : (0,\infty) \longrightarrow [0,\infty]$, given by

$$a_f(x) = \mu(\{x \in X : |f(x)| > \lambda\}),$$

the **distribution function** of f associated with μ.

Proposition 2.1. *Let $\phi : [0,\infty) \longrightarrow [0,\infty)$ be differentiable, increasing and such that $\phi(0) = 0$. Then*

$$\int_X \phi(|f(x)|)d\mu = \int_0^\infty \phi'(\lambda)a_f(\lambda)d\lambda.$$

To prove the previous equality it is enough to observe that the left-hand side is equivalent to

$$\int_X \int_0^{|f(x)|} \phi'(\lambda)d\lambda d\mu$$

and change the order of integration. If, in particular, $\phi(\lambda) = \lambda^p$ then

$$\|f\|_p^p = p\int_0^\infty \lambda^{p-1}a_f(\lambda)d\lambda. \tag{2.5}$$

Since weak inequalities measure the size of the distribution function, representation (2.5) of the L^p norm is ideal for proving the following interpolation theorem, which will let us deduce L^p boundedness from weak inequalities. It applies to a larger class of operators than linear ones (note that maximal operators are not linear): an operator T from a vector space of measurable functions to measurable functions is sublinear if

$$|T(f_0 + f_1)| \leq |Tf_0| + |Tf_1|,$$

and

$$|T(\lambda f)| = |\lambda||Tf|, \quad \forall\lambda \in \mathbb{C}.$$

Theorem 2.2. ***(Marcinkiewicz Interpolation)*** *Let (X,μ) and (Y,ν) be measure spaces, $1 \leq p_0 < p_1 \leq \infty$, and let T be a sublinear operator from $L^{p_0}(X,\mu) + L^{p_1}(X,\mu)$ to the measurable functions on Y, that is weak (p_0,p_0) and weak (p_1,p_1). Then T is strong (p,p) for $p_0 < p < p_1$.*

The following statements are central to the utility of the Hardy-Littlewood maximal operator.

1. For $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, $M(f)$ is finite almost everywhere.

2. If $f \in L^1(\mathbb{R}^n)$, then there exists a positive constant c such that, for all $\alpha > 0$;
$$|\{x|Mf(x) > \alpha\}| \leq \frac{c}{\alpha}\int_{\mathbb{R}^n} |f|.$$
This property is called a weak-type bound and establishes that M is weak $(1,1)$; it can be proved using the Vitali covering lemma.

3. If $f \in L^p(\mathbb{R}^n)$, $1 < p \leq \infty$, $M(f) \in L^p(\mathbb{R}^n)$ and
$$||Mf||_{L^p} \leq C_p||f||_{L^p}$$
where C is a constant depends only on p.

Property (3) says the operator $M : f \longrightarrow Mf$ is bounded on $L^p(\mathbb{R}^n)$ (strong (p,p)). It is clearly true when $p = \infty$, since we cannot take an average of a bounded function and obtain a value larger than the largest value of the function.

It is worth noting (3) does not hold for $p = 1$. This can be easily proved by calculating $M\chi$ where χ is the characteristic function of the unit ball centered at the origin.

M is not bounded in $L^1(\mathbb{R}^n)$, in fact for $f \geq 0$, Mf is not in L^1 unless $f(x) = 0$ for a.e. x, since $Mf(x) \geq C|x|^{-n}$ for large x, with $C > 0$ if $f \neq 0$.

We have the following very important Theorem about local integrability of maximal operator.

Theorem 2.3. *(Hardy-Littlewood Maximal Theorem) Let f be an integrable function supported in a cube $Q \subset \mathbb{R}^n$. Then $Mf \in L^1(Q)$ if and only if $f \log f \in L^1(Q)$.*

The properties (2) and (3) are contained in this theorem

Theorem 2.4. *The operator M is weak $(1,1)$ and strong (p,p), $1 < p \leq \infty$.*

Moreover, let us note that, from definition, follows

$$\|Mf\|_\infty \leq \|f\|_\infty,$$

so by the Marcinkiewicz interpolation theorem, to prove Theorem 2.4, it will be enough to prove that M is weak $(1,1)$.

Now, we show a proof when $n = 1$ but before we need the following one-dimensional covering lemma.

Lemma 2.5. *Let $\{I_\alpha\}_{\alpha\in A}$ be a collection of intervals in $\mathbb{R}$ and let K be a compact set contained in their union. Then there exists a finite subcollection $\{T_j\}$ such that*

$$K \subset \bigcup_j I_j \qquad \textit{and} \qquad \sum_j \chi_{I_j}(x) \leq 2, \quad x \in \mathbb{R}.$$

Proof. of Theorem (2.4) (n=1)

Let $E_\lambda = \{x \in \mathbb{R} : Mf(x) > \lambda\}$. If $x \in E_\lambda$ then there exists an interval I_x centered at x such that

$$\frac{1}{|I_x|}\int_{I_x} |f| > \lambda.$$

Let $K \subset E_\lambda$ be compact. Then $K \subset \bigcup I_x$, so by lemma 2.5 there exists a finite collection $\{I_j\}$ of intervals such that $K \subset \bigcup_j I_j$ and $\sum_j \chi_{I_j} \leq 2$. Hence,

$$|K| \leq \sum_j |I_j| \leq \sum_j \frac{1}{\lambda}\int_{I_j} |f| \leq \frac{1}{\lambda}\int_{\mathbb{R}} \sum_j \chi_{I_j}|f| \leq \frac{2}{\lambda}\|f\|_1.$$

Since the previous inequality holds for every compact $K \subset E_\lambda$, the weak $(1,1)$ inequality for M follows immediately. $\square$

Note that Lemma 2.5 is not valid in dimensions greater than 1. Theorem 2.4 can be proved in $\mathbb{R}^n$ using dyadic maximal function but we don't investigate it here.

2.3 Weighted norm inequalities for the H-L maximal operator

A very interesting question in harmonic analysis is what type of weights ω have the property that an operator T is bounded in $L^p(\omega)$ with $1<p<\infty$ and where T is bounded in $L^p(\mathbb{R}^n)$.

We start giving a generalization of the maximal function. Let μ be a positive Borel measure on $\mathbb{R}^n$, finite on compact sets and satisfying the following doubling condition:

$$\mu(2Q) \leq C\mu(Q) \tag{2.6}$$

for every cube $Q \subset \mathbb{R}^n$, with $C > 0$ independent of Q. We say that μ is a doubling measure.

Definition 2.6. Let μ as above, $d\mu = \omega(x)dx$, and let $f \in L^1_{loc}(\mathbb{R}^n)$. The **weighted Hardy-Littlewood maximal function** is defined by:

$$M_\omega f(x) = \sup_{x\in Q} \frac{1}{\omega(Q)} \int_Q |f(y)|\omega(y)dy, \tag{2.7}$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to coordinate axes containing x.

The analogous of the property (3), in Section 2.2, for weighted maximal operators is the following

Theorem 2.6. *Let μ a doubling measure in $\mathbb{R}^n$ such that $d\mu = \omega(x)dx$, then for every p, with $1 < p < \infty$, there is a constant $C_p > 0$ such that for every $f \in L^p(\omega)$, we have*

$$\left(\int_{\mathbb{R}^n} (M_\omega f(x))^p \omega(x)dx\right)^{\frac{1}{p}} \leq C_p \left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx\right)^{\frac{1}{p}}.$$

Also the Theorem 2.3 can be extended to M_ω for a doubling measure μ such that $d\mu = \omega(x)dx$.

It is fundamental the fact that M_ω is bounded in $L^p(w)$ for every $1 < p < \infty$, if the weight ω is doubling. In particular M_ω is bounded if ω is a A_∞ weight.

It was B. Muckenhoupt who made the key discovery, by studying the weighted inequality for the maximal function.

In particular, it is known that Muckenhoupt's A_p-condition is a necessary and sufficient condition for boundedness in the case of the Hardy-Littlewood maximal operator or singular integral operators, as proved in the following Muckenhoupt's theorem [Mu1].

Theorem 2.7. *[Mu1] Let $1 < p < \infty$, then M is a bounded operator in $L^p(\omega)$ if and only if $\omega \in A_p$.*

We have also a weak-type of Theorem 2.7.

Theorem 2.8. *[Bu1] For $1 \leq p < \infty$, the weak (p,p) inequality*

$$\omega(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}) \leq \frac{c(p,n)}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx$$

holds if and only if $\omega \in A_p$.

The following Theorem due to Perez is a generalization of the previous results.

Theorem 2.9. *[P] The following statements are equivalent.*

1. *For every $1 < p < \infty$, and whenever $\omega \in A_p$*

$$M : L^p(\omega) \longrightarrow L^p(\omega)$$

2. *For every $1 < p < \infty$, and whenever $\omega \in A_\infty$*

$$M_\omega : L^p(\omega) \longrightarrow L^p(\omega).$$

Some years later Buckley in [Bu1] proved a result which shows how the operator norms specifically depend from the A_p-constant of ω.

Theorem 2.10. *[Bu1] If $\omega \in A_p$, $1 \leq p < \infty$, then*

$$\|Mf\|^p_{L^p(\omega)} \leq C(p) A_p(\omega)^{p'} \|f\|^p_{L^p(\omega)}$$

where p' is the conjugate exponent of p and C is a positive constant depending only on p. The power p' for $A_p(\omega)$ is the smallest possible, and hence the best one, since $A_p(\omega) \geq 1$.

Before to see the proof of the Theorem 2.10 we need of a preliminary Lemma.

Lemma 2.11. *[Bu1] If $f \in L^p(\omega)$ and $f_{Q_k} \geq \alpha > 0$ for each of the disjoint cubes $\{Q_k\}$, then*

$$\sum_k \omega(Q_k) \leq A_p(\omega) \left(\frac{\|f\|_{L^p(\omega)}}{\alpha} \right)^p.$$

Proof. (**of Theorem 2.10**) First, we show that for $1 \leq p < \infty$,

$$\omega(\{Mf > \alpha\}) \leq c(p,n) A_p(\omega) \left(\frac{\|f\|_{L^p(\omega)}}{\alpha} \right)^p. \tag{2.8}$$

where $c(p,n)$ is a positive constant, depending only on $p \geq 1$ and $n \in \mathbb{N}$.

Without loss of generality, we assume that $f(x) \geq 0$ and that $\|f\|_{L^p(\omega)} = 1$. Suppose that $Mf(x) > \alpha > 0$ so that $f_{Q_k} \geq \alpha$ for some cube Q_k centered at x. Let $E_r = \{x : |x| < r, Mf(x) > \alpha\}$. The Besicovich covering Lemma [Be] tells us that E_r can be covered by the union of N_n collections of disjoint cubes, on each of which the mean value of f is at least α. Choose the collection $\{Q_k\}$, whose union has maximal ω-measure. Thus,

$$\omega(E_r) \leq N_n w \left(\bigcup_k Q_k \right) \leq \frac{c(p,n) A_p(\omega)}{\alpha^p},$$

by Lemma 2.11. Letting $r \longrightarrow \infty$, we get (2.8).

Suppose now that $p > 1$, if $\omega \in A_p$ then $\omega \in A_{p-\varepsilon}$ where $\varepsilon \sim A_p(\omega)^{1-p'}$, see Lemma 1.2, and trivially $w \in A_{p+\varepsilon}$, with norm no larger than $A_p(\omega)$. Applying the Marcinkiewicz Interpolation Theorem to the corresponding weak-type results at $p - \varepsilon$ and $p + \varepsilon$, we get the strong type result we require with the indicated bound for the operator norm.

To see that the power $A_p(\omega)^{p'}$ is best possible, we give an example for $\mathbb{R}$ (a similar example works in $\mathbb{R}^n$ for any n). Let $\omega(x) = |x|^{(p-1)(1-\delta)}$, so that $A_p(\omega) \sim \frac{1}{\delta^{p-1}}$. Now, $f(x) = |x|^{-1+\delta} \chi_{[0,1]} \in L^p(\omega)$. It is easy to see that $Mf \geq \frac{f}{\delta}$ and so

$$\frac{\|Mf\|^p_{L^p(\omega)}}{\|f\|^p_{L^p(\omega)}} \geq C\delta^{-p} \sim A_p(\omega)^{p'}.$$

□

This result is sharp in the sense that $A_p(\omega)^{\frac{p'}{p}}$ cannot be replaced by $\varphi(A_p(\omega)^{\frac{p'}{p}})$ for any positive non decreasing function φ growing slower than $t^{\frac{p'}{p}}$.

Now we report similar results of weighted integral inequalities for the Maximal Operator by mean of Gehring condition (see Chapter 1).

Theorem 2.12. *[P] Let $1 < p < \infty$. Then*

$$\begin{cases} M_\omega : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n) \\ M_\omega : L^{p'}(\omega^{p'}) \longrightarrow L^{p'}(\omega^{p'}) \end{cases}$$

if and only if

$$\begin{cases} \omega \in G_{p'} \\ M : L^{p'}(\mathbb{R}^n) \longrightarrow L^{p'}(\mathbb{R}^n) \\ M_{\omega^{p'}} : L^p(\omega^{p'}) \longrightarrow L^p(\omega^{p'}). \end{cases}$$

2.4 Our results: sharp weak-type inequalities for the H-L maximal operator on weighted Lebesgue space

In this section we will find the best constant in the corresponding weak estimates on $L^p(\mathbb{R}, dx)$ for the weighted maximal function M_v, under the assumptions $G_q(v) < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

Let us to consider the Mf, (uncentred) maximal function of $f \in L^1_{loc}(\mathbb{R}, \omega dx)$.

Our aim is to prove the following sharp result in one dimension.

Theorem 2.13. *[A1] The best constant in the theorem 2.8 is* $c(p,1) = 2$*.*

In the special case $\omega(x) = 1$ we reobtain a sharp result due to [Ber],[GM] and [GK]. In order to prove theorem (2.13), we actually prove the following weighted double weak type inequality, in the same spirit of [GK] .

Theorem 2.14. *[A1] Let* ω *be a* A_p *weight on* $\mathbb{R}$*,* $1 \leq p < \infty$*, and assume that* $f : \mathbb{R} \to [0, \infty)$ *belongs to* $L^p_{loc}(\mathbb{R}, \omega)$*. Then we have,* $\forall \lambda > 0$*:*

$$\int_{\{Mf>\lambda\}} \omega(x)dx \;+\; \int_{\{f>\lambda\}} \omega(x)dx \leq \tag{2.9}$$

$$\leq \frac{2A_p(\omega) - 1}{\lambda^p} \int_{\{Mf>\lambda\}} f^p \omega(x)dx \;+\; \frac{1}{\lambda^p} \int_{\{f>\lambda\}} f^p \omega(x)dx.$$

The inequality is sharp.

Proof. Fix $\lambda > 0$ and set

$$E_\lambda = \{x : Mf(x) > \lambda\}$$

we may assume that $\int_{\{f>\lambda\}} \omega(x)dx = \omega(\{f > \lambda\}) < \infty$. For any $x \in E_\lambda$ there exists an interval I_x containing x such that

$$\fint_I f(y)dy > \lambda. \tag{2.10}$$

By Lindelöf's theorem there exists a countable subcollection $\{I_j\}$ of $\{I_x\}$ such that

$$\bigcup_{j\in\mathbb{N}} I_j = \bigcup_{x\in\mathbb{E}_\lambda} I_x.$$

As in [GK], define for $m \in \mathbb{N}$

$$F^m = \bigcup_{j=1}^{m} I_j.$$

By lemma 4.4 in [Ga] there exist two subcollections of $\{I_j : j = 1, 2, ..., m\}$, S_1 and S_2 such that

$$F^m = (\bigcup_{j\in S_1} I_j) \cup (\bigcup_{j\in S_2} I_j) = F_1 \cup F_2$$

and $I_k \cap I_j = \emptyset$ if $j, k \in S_1$ or if $j, k \in S_2$. For $i = 1, 2$ we have, by (2.10)

$$\omega(F_i) = \sum_{j\in S_i} \omega(I_j) < \frac{1}{\lambda^p} \sum_{j\in S_i} \omega(I_j) \left(\fint_{I_j} f(x)dx\right)^p. \tag{2.11}$$

Using Hölder inequality we have

$$\omega(F_i) \leq \frac{1}{\lambda^p} \sum_{j\in S_i} \frac{\omega(I_j)}{|I_j|^p} \int_{I_j} f^p(x)\omega dx \left[\int_{I_j} \omega^{-\frac{1}{p-1}} dx\right]^{p-1}$$

$$= \frac{1}{\lambda^p} \sum_{j\in S_i} \frac{\omega(I_j)}{|I_j|} \left[\fint_{I_j} \omega^{-\frac{1}{p-1}} dx\right]^{p-1} \int_{I_j} f^p \omega dx$$

$$\leq \frac{1}{\lambda^p} A_p(\omega) \sum_{j\in S_i} \int_{I_j} f^p \omega dx$$

by the assumption $A_p(\omega) < \infty$. Since the intervals I_j for $j \in S_i$ are pairwise disjoint we deduce

$$\omega(F_i) \leq \frac{A_p(\omega)}{\lambda^p} \int_{F_i} f^p \omega dx \qquad \text{for } i = 1, 2. \tag{2.12}$$

Therefore

$$\omega(F^m)+\omega(F_1\cap F_2)=\omega(F_1)+\omega(F_2)\leq \tag{2.13}$$

$$\leq \frac{A_p(\omega)}{\lambda^p}\int_{F_1} f^p\omega dx+\frac{A_p(\omega)}{\lambda^p}\int_{F_2} f^p\omega dx=$$

$$=\frac{A_p(\omega)}{\lambda^p}\int_{F^n} f^p\omega dx+\frac{A_p(\omega)}{\lambda^p}\int_{F_1\cap F_2} f^p\omega dx.$$

By mean of the inequality [GK]

$$\frac{1}{\lambda^p}\int_{F_1\cap F_2} f^p\omega dx+\omega(\{f>\lambda\})\leq \frac{1}{\lambda^p}\int_{\{f>\lambda\}} f^p\omega dx+\omega(F_1\cap F_2) \tag{2.14}$$

and, using (2.13) and (2.14), we deduce that

$$\omega(F^m)+\omega(\{f>\lambda\})\leq \frac{A_p(\omega)}{\lambda^p}\int_{F^m} f^p\omega dx+\frac{1}{\lambda^p}\int_{\{f>\lambda\}} f^p\omega dx+\frac{A_p(\omega)-1}{\lambda^p}\int_{E_\lambda} f^p\omega dx.$$

Since F^m is an increasing sequence of measurable sets whose union is E_λ, we get (2.9) letting $m\to\infty$. Since inequality (2.9) reduces for $\omega=1$ to inequality (2.1) in [GK], it is sharp. □

Theorem (2.13) is a simple consequence of Theorem (2.14) as for as we note that $\{f>\lambda\}\subset\{Mf>\lambda\}$.

The counter part of Theorem (2.14) in the setting of weighted maximal operator acting on unweighted space is the following Theorem (2.15). Our aim is to prove the following sharp form of weak type property of M_v on $L^p(\mathbb{R})$, where M_v is defined as in 2.7.

Theorem 2.15. *[A1] Let $1\leq p,q\leq\infty$, if $G_q(v)<\infty$ and $f\in L^p(\mathbb{R})$, $\frac{1}{p}+\frac{1}{q}=1$, then, for any $\lambda>0$ we have*

$$|\{M_vf>\lambda\}\leq \frac{2G_q(v)}{\lambda^p}\|f\|^p_{L^p(\mathbb{R})}. \tag{2.15}$$

The inequality is sharp.

Similarly as before, we first prove the following double weak type inequality.

Theorem 2.16. *[A1] Let $G_q(v) < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

(2.16)
$$|\{M_v f > \lambda\}| + |\{f > \lambda\}| \leq \frac{2G_q(v) - 1}{\lambda^p} \int_{\{M_v f > \lambda\}} f^p(x)dx + \frac{1}{\lambda^p} \int_{\{f > \lambda\}} f^p(x)dx.$$

Proof. Fix $\lambda > 0$ and set

$$E_\lambda = \{x : M_v f(x) > \lambda\}.$$

For any $x \in E_\lambda$ there exists an interval I_x containing x such that

$$\frac{1}{v(I_x)} \int_{I_x} f(y)v(y)dy > \lambda.$$

By Lindelöf's theorem, there exists a countable subcollection $\{I_j\}$ of $\{I_x\}$ such that

$$\bigcup_{j \in \mathbb{N}} I_j = \bigcup_{x \in E_\lambda} I_x.$$

Define for $m \in \mathbb{N}$

$$F^m = \bigcup_{J=1}^{m} I_j.$$

Then, there exist two subcollections of $I_j : j = 1, 2, ..., m$, S_1 and S_2 such that

$$F^m = (\bigcup_{j \in S_1} I_j) \cup (\bigcup_{j \in S_2} I_j) = F_1 \cup F_2$$

and $I_k \cap I_j = \emptyset$ if $j, k \in S_1$ or if $j, k \in S_2$. For $i = 1, 2$ we have, applying Hölder's inequality

$$|F_i| = \sum_{j \in S_i} |I_j| < \frac{1}{\lambda^p} \sum_{j \in S_i} |I_j| \left(\frac{1}{v(I_j)} \int_{I_j} f v dx \right)^p \leq$$

$$\leq \frac{1}{\lambda^p} \sum_{j \in S_i} \frac{|I_j|}{v(I_j)^p} \int_{I_j} |f|^p dx \left(\int_{I_j} |v|^q dx \right)^{\frac{p}{q}} =$$

$$= \frac{1}{\lambda^p} \sum_{j \in S_i} \left[\frac{\left(⨍_{I_j} |v|^q dx \right)^{\frac{1}{q}}}{⨍_{I_j} v dx} \right]^p \int_{I_j} |f|^p dx \leq \frac{G_q(v)}{\lambda^p} \int_{F_i} f^p dx.$$

Hence, we obtain

$$|F_i| \leq \frac{G_q(v)}{\lambda^p} \int_{F_i} f^p dx.$$

Therefore

$$\begin{aligned}(2.17) \qquad |F^m| + |F_1 \cap F_2| = |F_1| + |F_2| \leq \\ \leq \frac{G_q(v)}{\lambda^p} \int_{F_1} f^p dx + \frac{G_q(v)}{\lambda^p} \int_{F_2} f^p dx = \\ = \frac{G_q(v)}{\lambda^p} \int_{F^m} f^p dx + \frac{G_q(v)}{\lambda^p} \int_{F_1 \cap F_2} f^p dx.\end{aligned}$$

By mean of the inequality [GK]

$$(2.18) \qquad \frac{1}{\lambda^p} \int_{F_1 \cap F_2} f^p dx + |\{f > \lambda\}| \leq \frac{1}{\lambda^p} \int_{\{f > \lambda\}} f^p dx + |F_1 \cap F_2|,$$

and, using (2.17) and (2.18), we deduce

$$|F^m| + |F_1 \cap F_2| + \frac{1}{\lambda^p} \int_{F_1 \cap F_2} f^p dx + |\{f > \lambda\}| \leq$$

$$\leq \frac{G_q(v)}{\lambda^p} \int_{F^m} f^p dx + \frac{G_q(v)}{\lambda^p} \int_{F_1 \cap F_2} f^p dx + \frac{1}{\lambda^p} \int_{\{f > \lambda\}} f^p dx + |F_1 \cap F_2|.$$

Therefore

$$|F^m| + |\{f > \lambda\}| \leq \frac{2G_q(v) - 1}{\lambda^p} \int_{F^m} f^p(x) dx + \frac{1}{\lambda^p} \int_{\{f > \lambda\}} f^p dx.$$

Since F^m is an increasing sequence of measurable sets whose union is E_λ, we get (2.16) letting $m \longrightarrow \infty$. □

Thanks our results and Theorem (1.12) and Theorem (1.13) mentioned in Chapter 1, we deduce the exact continuation of the weak type properties for M and M_v.

Proposition 2.17. *Let $A_2(\omega) = A < \infty$; then for $1 + \sqrt{\frac{A-1}{A}} < s \leq 2$ we have*

$$\int_{\{Mf > \lambda\}} \omega \leq \frac{2A}{\lambda^s \psi(s)} \|f\|^s_{L^s(\mathbb{R}, \omega)}$$

where $\psi(s)$ is as (1.15) in Theorem 1.12.

Proposition 2.18. *Let $G_2(v) = B < \infty$; then for $2 \leq r < 1 + \sqrt{\frac{B}{B-1}}$ we have*

$$|\{M_v f > \lambda\}| \leq \frac{2}{\lambda^r} \left[\frac{2B}{\varphi(r)}\right]^{\frac{r'}{2}} \|f\|^r_{L^r(\mathbb{R})}$$

where $\varphi(r)$ is as (1.17) in Theorem 1.13.

Chapter 3

Sharp estimates for the weighted Maximal Operator in Orlicz spaces

In this chapter we prove an integral inequality for the weighted maximal function in Orlicz spaces, generalizing a previous result due to C. Perez [P], with precise evaluation of the norm in terms of the G_q-constant of the weight, in the same spirit of works by Buckley [Bu1] and Capone-Fiorenza [CF].

3.1 Introduction

Let us start to recall some definitions and notations.

A ***Young function*** is a convex function $\Phi : [0, \infty) \longrightarrow [0, \infty)$, increasing on $[0, \infty)$ and satisfying

$$\lim_{t\to 0} \frac{\Phi(t)}{t} = 0, \qquad \lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty.$$

Moreover, Φ has a derivative φ which is nondecreasing, nonnegative, such that $\varphi(0_+) = 0$ and $\varphi(\infty) = \infty$ and so that

$$\Phi(t) = \int_0^t \varphi(x)dx$$

and we can take φ to be right-continuous. The Young function complementary to Φ is given by

$$\Psi(t) = \sup_s \{st - \Phi(s)\} = \int_0^t \psi(x)dx$$

where $\psi(x) = \inf\{s : \varphi(s) \geq x\}$. These functions verify the Young's inequality

$$ab \leq \Phi(a) + \Psi(b) \qquad \forall\, a, b > 0.$$

More in general, if a Young function Φ is not necessarily convex we have an Orlicz function. The **Orlicz space**, $L^{\Phi}(\Omega)$ consists of all measurable functions f on Ω such that

$$\int_{\Omega} \Phi\left(\frac{|f|}{\lambda}\right) dx < \infty \quad \text{for some } \lambda > 0.$$

$L^{\Phi}(\Omega)$ is a complete linear metric space with respect the following distance function:

$$dist_{\Phi(f,g)} = \inf\left\{\lambda > 0 : \int_{\Omega} \Phi\left(\frac{|f-g|}{\lambda}\right) dx \leq \lambda\right\}.$$

If Φ is a Young function, $L^{\Phi}(\Omega)$ can be equipped with the Luxemburg norm

$$\|f\|_{L^{\Phi}} = \inf\left\{\lambda > 0 : \int_{\Omega} \Phi\left(\frac{|f|}{\lambda}\right) dx \leq 1\right\},$$

and becomes a Banach space.

If we put $\Phi(t) = t^p$, $0 < p < \infty$ then the space $L^{\Phi}(\Omega)$ coincides with the usual Lebesgue space $L^p(\Omega)$. Note that $L^p(\Omega)$ is a Banach space only when $p \geq 1$.

In the same way we can define the weighted Orlicz class as the set of all functions f for which

$$\int_{\Omega} \Phi\left(\frac{|f|}{\lambda}\right) \omega dx < \infty \quad \text{for some } \lambda > 0,$$

where ω is a non negative measurable function on the space $\mathbb{R}^n$. As before $L^{\Phi}(\Omega, \omega)$ denotes the **weighted Orlicz space**.

In the following we are going to report some common important properties and results about Young functions.

Definition 3.1. Let Φ be a Young function. Φ satisfies the Δ_2-condition ($\Phi \in \Delta_2$) if there is $c > 0$ such that

$$\Phi(2t) \leq c\Phi(t), \qquad \forall t \geq 0. \tag{3.1}$$

Note that if $\forall\ \Phi, \Psi \in L^{\Phi}(\Omega)$ complementary functions we have that both verify Δ_2 - condition, $L^{\Phi}(\Omega)$ is a reflexive Orlicz space.

Now we report a result about the equivalence between a growth condition and Δ_2-condition.

Theorem 3.1. *[KR] Let Φ be a Young function, then*

$$\Phi \in \Delta_2 \Longleftrightarrow p\Phi(t) \leq t\Phi'(t) \leq q\Phi(t) \qquad \forall t > 0 \tag{3.2}$$

with $1 < p \leq q$.

We give an example of growth exponents of a Young function.

Example 3.1. The Young function

$$\Phi(t) = \begin{cases} t^2 & t \in [0,1] \\ e^{2(t-1)} & t \in [1,2] \\ \frac{e^2}{16}t^4 & t \in [2,+\infty[\end{cases}$$

verifies the following growth condition

$$2\Phi(t) \leq t\Phi'(t) \leq 4\Phi(t), \ \forall t > 0.$$

In many applications (calculus of variations, interpolation etc.) it is useful to assume that a Young function is, in a certain sense between two powers t^p and t^q.

3.2 Preliminary results

In the Chapter 2 we have seen that one reason why the class of Muckenhoupt weights is important for analysis is that the maximal operator is continuous in weighted L^p-spaces, if and only if the weight function is a Muckenhoupt weight.

In [CF] the boundedness properties of the Hardy-Littlewood maximal operator M was generalized to weighted Orlicz spaces $L^\Phi(\omega dx)$ with a precise dependence of the constant c in the inequality

$$\int_{\mathbb{R}^n} \Phi(Mf)\omega \, dx \leq c \int_{\mathbb{R}^n} \Phi(|f|) \, dx$$

in terms of growth conditions on Φ and $A_p(\omega)$-constant. This extended a previous weighted maximal Theorem between Orlicz spaces due to Kerman-Torchinsky [KT].

Now, we want to give an analogous theorem related to the weighted maximal operator

$$M_v g(x) = \sup_{x \in Q} \frac{1}{\int_Q v(y)\, dy} \int_Q |g(y)| v(y)\, dy$$

over the unweighted Orlicz space $L^\Psi(dx)$ as the following

$$\int_{\mathbb{R}^n} \Psi(M_v g)\, dx \leq c \int_{\mathbb{R}^n} \Psi(g)\, dx$$

generalizing a previous result due to C. Perez [P] concerning the case $\Psi(t) = t^p$.

Our study is motivated by the fact that recently, there has been new interest in computing norms of some classical operators between weighted spaces, how we will see in next sections.

To estimate parameter in terms of the G_q constant involved, we will follow a technique of reducing the problem to the one dimensional case via non increasing rearrangement in the spirit of the following result [DS].

Lemma 3.2. *[A2] If $n = 1$ and $v^* : [a, b] \longrightarrow [0, \infty)$ is a non increasing function such that $G_q(v^*) < \infty$. Then, there exists $\eta \geq c_1 [G_q(v^*)]^{1-q}$ such that*

$$G_{q+\eta}(v^*) \leq c_o\, G_q(v^*) \tag{3.3}$$

with c_o and c_1 depending only q.

Proof. By [DS], if $q_1 > q$ is the unique solution to the equation

$$\varphi(x) = 1 - \frac{x-q}{x} (\frac{x}{x-1})^q [G_q(v^*)]^{q-1} = 0$$

and if q_2 is any number in the interval (q, q_1), we have

$$[G_{q_2}(v^*)]^{\frac{1}{q_2'}} \leq \left[\frac{q}{q_2 \varphi(q_2)} \right]^{1/q} [G_q(v^*)]^{\frac{1}{q'}}.$$

The equation $\varphi(q_1) = 0$ can be rewritten as

$$\frac{(q_1 - q)}{q_1} \left(\frac{q_1}{q_1 - 1} \right)^q [G_q(v^*)]^{q-1} = 1.$$

Set now $\eta = q_2 - q$; since $\eta < q_1 - q$, then we have

$$\frac{\eta}{q_1} \left(\frac{q_1}{q_1 - 1} \right)^q [G_q(v^*)]^{q-1} < 1,$$

and therefore

$$\eta < q_1 \left(\frac{q_1 - 1}{q_1}\right)^q [G_q(v^*)]^{1-q}.$$

Defining $\widetilde{c_1} = q_1 \left(\frac{q_1-1}{q_1}\right)^q$, for $0 < \eta_o < \widetilde{c_1}[G_q(v^*)]^{1-q}$, we have

$$[G_{q+\eta_o}(v^*)]^{1-\frac{1}{q+\eta}} \leq \left[\frac{q}{(q+\eta_o)\varphi(q+\eta_o)}\right]^{1/q} [G_q(v^*)]^{\frac{1}{q'}}.$$

The result follows choosing $\eta < \widetilde{c_1}[G_q(v^*)]^{1-q}$ such that $\eta > \frac{1}{2}\widetilde{c_1}[G_q(v^*)]^{1-q} = c_1[G_q(v^*)]^{1-q}$ and $c_o = \left[\frac{q}{(q+\eta_o)\varphi(q+\eta_o)}\right]^{1/q}$. □

Lemma 3.3. *[A2] Let $G_q(v) < \infty$ and let Q be a fixed cube in $\mathbb{R}^n$. Then the non increasing rearrangement $v^*(t)$ of v on Q belongs to G_q on the interval $[0, |Q|]$ and we have*

$$G_q(v^*) \leq c(n, q)\, G_q(v)$$

for some constant $c(n, q) > 0$.

Proof. By Theorem 4.1 in [S3] (for $\varphi(t) = t^q$), we have

$$G_q(v^*) \leq [(2^n + 1)3^{qn}]^{\frac{1}{q-1}}\, G_q(v). \tag{3.4}$$

□

Lemma 3.4. *[A2] If $G_q(v) < \infty$, then there exists a constant $\eta \geq c(n, q)[G_q(v)]^{1-q}$ such that*

$$G_{q+\eta}(v) \leq c(n, q)\, G_q(v). \tag{3.5}$$

Proof. Let us fix Q a cube in $\mathbb{R}^n$. According to Lemma 3.3, if v^* is the non increasing rearrangement of v on Q, then there exists $c(n, q)$ such that

$$G_q(v^*) \leq c(n, q) G_q(v). \tag{3.6}$$

Then, using Lemma 3.2 and equation (3.6), we obtain

$$G_{q+\eta}(v^*) \leq c_2(n, q)\, G_q(v),$$

where $c_2(n, q)$ is positive constant, depending on n, q.

In particular, for a positive constant

$$\left(\fint_0^{|Q|} v^*(t)^{q+\eta}\, dt\right)^{\frac{1}{q+\eta}} \leq c'(n, q, \eta)[G_q(v)]^{\frac{q+\eta-1}{q+\eta}} \fint_0^{|Q|} v^*(t)\, dt.$$

By the familiar properties of the rearrangements, and then passing to sup with respect to the cube Q, we deduce

$$(3.7) \qquad \left(\fint_Q v^{q+\eta}\,dx\right)^{\frac{1}{q+\eta}} \leq c'(n,q,\eta)[G_q(v)]^{\frac{q+\eta-1}{q+\eta}}\fint_Q v,$$

that is equivalent to (3.5). □

3.3 Our main theorem

We prove the main Theorem.

Theorem 3.5. *[A2] If $\Psi : [0,\infty) \longrightarrow [0,\infty)$ is a convex increasing function such that*

$$r\Psi(t) \leq t\Psi'(t) \leq s\Psi(t) \qquad \forall\, t > 0$$

with $1 < r \leq s$, and if v is a weight verifying the $G_{\frac{r}{r-1}}$ condition, then

$$\int_{\mathbb{R}^n} \Psi(M_v f)\,dx \leq c(n,r,s)[G_{\frac{r}{r-1}}(v)]^{\frac{r}{r-1}}\int_{\mathbb{R}^n}\Psi(|f|)\,dx$$

for any $f \in L^1(\mathbb{R}^n)$.

Lemma 3.6. *If Ψ verifies the conditions of Theorem 3.5, then there exists a positive constant c such that $\Psi(\lambda t) \leq c\,max\{\lambda^r,\lambda^s\}\Psi(t)$, for all $\lambda, t > 0$.*

Proof. See [FK]. □

Now, we prove Theorem 3.5 as follows:

Proof. Let f such that $\Psi(|f|) \in L^1(\mathbb{R}^n)$, otherwise there is nothing to prove. Put

$$[f]_\lambda = \begin{cases} f & \text{if} \quad |f| > \lambda \\ 0 & \text{if} \quad |f| \leq \lambda \end{cases} \qquad , \qquad [f]^\lambda = \begin{cases} 0 & \text{if} \quad |f| > \lambda \\ f & \text{if} \quad |f| \leq \lambda \end{cases}$$

so that $f = [f]_\lambda + [f]^\lambda$. Since $G_{\frac{r}{r-1}}(v) < \infty$, by Lemma 3.4, there exists $\eta > 0$ such that $G_{r_0}(v) < \infty$ with $r_0 = \frac{r}{r-1} - \eta$. If $s_0 = \frac{r}{r-1} + \eta$ we have easily also $G_{s_0}(v) < \infty$ and therefore by theorem (2.15) we have

$$\int_{\mathbb{R}^n} \Psi(M_v f)\,dx = \int_0^\infty \Psi'(\lambda)|\{M_v(f) > \lambda\}|d\lambda \leq$$

$$\leq \int_0^\infty \Psi'(\lambda)|\{M_v(f_\lambda) > \frac{\lambda}{2}\}|d\lambda +$$

$$+ \int_0^\infty \Psi'(\lambda)|\{M_v(f^\lambda) > \frac{\lambda}{2}\}|d\lambda \leq$$

$$\leq c(n)2^{r_0} G_{\frac{r_0}{r_0-1}}(v) \int_0^\infty \frac{\Psi'(\lambda)}{\lambda^{r_0}} \left(\int_{\mathbb{R}^n} |f_\lambda|^{r_0} \, dx \right) +$$

$$+ c(n)2^{s_0} G_{\frac{s_0}{s_0-1}}(v) \int_0^\infty \frac{\Psi'(\lambda)}{\lambda^{s_0}} \left(\int_{\mathbb{R}^n} |f^\lambda|^{s_0} \, dx \right) =$$

$$= c(n)2^{r_0} G_{\frac{r_0}{r_0-1}}(v) \int_0^\infty \frac{\Psi'(\lambda)}{\lambda^{r_0}} \left(\int_{|f|>\lambda} |f|^{r_0} \, dx d\lambda \right) +$$

$$+ c(n)2^{s_0} G_{\frac{s_0}{s_0-1}}(v) \int_0^\infty \frac{\Psi'(\lambda)}{\lambda^{s_0}} \left(\int_{|f|\leq\lambda} |f|^{s_0} \, dx d\lambda \right) =$$

$$= c(n)2^{r_0} G_{\frac{r_0}{r_0-1}}(v) \int_{\mathbb{R}^n} |f(x)|^{r_0} \left(\int_0^{|f(x)|} \frac{\Psi'(\lambda)}{\lambda^{r_0}} d\lambda \right) \, dx +$$

$$+ c(n)2^{s_0} G_{\frac{s_0}{s_0-1}}(v) \int_{\mathbb{R}^n} |f(x)|^{s_0} \left(\int_{|f(x)|}^\infty \frac{\Psi'(\lambda)}{\lambda^{s_0}} d\lambda \right) \, dx.$$

Using Lemma 3.6, we have

$$\int_0^{|f(x)|} \frac{\Psi'(\lambda)}{\lambda^{r_0}} d\lambda \leq cq \int_0^{|f(x)|} \frac{1}{\lambda^{r_0+1}} \left(\frac{\lambda}{|f(x)|} \right)^{p-\frac{\eta}{2}} \Psi(|f(x)|) d\lambda = \frac{2qc}{\eta} \frac{\Psi(|f(x)|)}{|f(x)|^{r_0}}.$$

In the same way, we have

$$\int_{|f(x)|}^\infty \frac{\Psi'(\lambda)}{\lambda^{s_0}} d\lambda \leq \frac{2qc}{\eta} \frac{\Psi(|f(x)|)}{|f(x)|^{s_0}}.$$

Hence,

$$\int_{\mathbb{R}^n} \Psi(M_v f) \, dx \leq \left(c(n) \frac{2qc}{\eta} \int_{\mathbb{R}^n} \Psi(|f(x)|) \, dx \right) \left(2^{r_0} G_{\frac{r_0}{r_0-1}}(v) + 2^{s_0} G_{\frac{s_0}{s_0-1}}(v) \right).$$

Using Lemma 3.4 for $q = r/(r-1)$

$$\frac{G_{\frac{r_0}{r_0-1}}(v)}{\eta} \sim [G_{\frac{r}{r-1}}(v)]^{\frac{r}{r-1}}.$$

Therefore,

$$\int_{\mathbb{R}^n} \Psi(M_v f) \, dx \leq c(n,r) [G_{\frac{r}{r-1}}(v)]^{\frac{r}{r-1}}. \int_{\mathbb{R}^n} \Psi(|f(x)| \, dx.$$

$\square$

Chapter 4

Sharp estimates for some operators on weighted Lebesgue spaces

For a long time it has been of interest to find, for a given operator T, sharp bounds for the operator norms $||T||_{L^p(\omega)}$ in terms of the A_p-constant of the weight ω. The aim is on controlling the operator norm by a suitable power of the A_p-constant of ω. One seeks to prove an estimate of the form

$$||Tf||_{L^p(\omega)} \leq CA_p^r(\omega)||f||_{L^p(\omega)}$$

for a suitable r where the constant C is independent of f or ω. Since $A_p(\omega) \geq 1$ it is desirable to find estimates with $r = r(p)$ as small as possible.

To obtain the sharp estimates is of course significantly more difficult than to just prove continuity. Such estimates have applications in PDE; see for example in the case of the Hilbert transform, the work by Fefferman, Kenig and Pipher [FKP]. More recently, Volberg and Petermichl [AIS], [PV] solved a famous problem related to quasiregular maps through a sharp bound for the weighted Beurling operator.

In this chapter we illustrate some result of optimal estimate for the growth of the norm of the operators as Hilbert transform, Riesz transform, Beurling-Alhfors, Calderon-Zygmund, Potential Riesz, in terms of the A_p-constant. In particular, we will exhibited the optimal exponent for the A_p-constant in a weighted inequality for the fractional integral operator I_α, also called Riesz potential, of any order $\alpha \in (0, n)$.

4.1 Good-λ methods

Good-λ inequalities, brought to Harmonic Analysis, provide a powerful tool to prove boundedness results for operators or at least comparisons of two operators. A typical good-λ inequality for two non-negative functions F and G is as follows: for every $0 < \delta < 1$, there exists $\gamma = \gamma(\delta)$ and for every $\omega \in A_\infty$,

$$\omega\{x : F(x) > 2\lambda, G(x) \leq \gamma\lambda\} \leq C\delta\omega\{x : F(x) > \lambda\}.$$

The usual approach for proving such an estimate consists in first deriving a local version of it with respect to the underlying doubling measure, and then passing to the weighted measure using that $\omega \in A_\infty$.

Weighted good-λ estimates encode a lot of information about F and G, since they give a comparison of the ω-measure of the level sets of both functions. One gets, for instance, that for every $0 < p < \infty$ and all $\omega \in A_\infty$ then $||F||_{L^p(\omega)}$ is controlled by $||G||_{L^p(\omega)}$. The same inequality holds with $L^{p,\infty}$ in place of L^p or with some other function spaces. Thus, the size of F is controlled by that of G.

In applications, one tries to control a specific operator T to be studied by a maximal one M whose properties are known by setting $F = Tf$ and $G = Mf$.

4.2 Calderón- Zygmund operator

Calderón-Zygmund operators have been thoroughly studied since the 50s. They are singular integral operators associated with a kernel satisfying certain size and smoothness conditions. Let us recall some concepts.

Let $K(x, y)$ be a locally integrable function defined off the diagonal $x = y$ in $\mathbb{R}^n \times \mathbb{R}^n$, which satisfies the size estimate

$$|K(x, y)| \leq \frac{c}{|x - y|^n} \tag{4.1}$$

and, for some $\varepsilon > 0$, the regularity condition

$$(4.2) \qquad |K(x,y) - K(z,y)| + |K(y,x) - K(y,z)| \leq c\frac{|x-z|^{\varepsilon}}{|x-y|^{n+\varepsilon}},$$

whenever $2|x-z| < |x-y|$. A continuous linear operator $T : C_0^{\infty}(\mathbb{R}^n) \longrightarrow L^1_{loc}(\mathbb{R}^n)$ is a Calderón-Zygmund operator if it extends to a bounded operator on $L^2(\mathbb{R}^n)$, and there is a kernel K satisfying (4.1) and (4.2) such that

$$T(x) = \int_{\mathbb{R}^n} K(x,y)\, f(y)dy$$

for any $f \in C_0^{\infty}(\mathbb{R}^n)$ and $x \notin supp(f)$.

If T is a Calderón-Zygmund operator with smooth kernel, in particular it is already bounded on (unweighted) L^2, it was shown in [Co] and in [CFe].

In the same papers it is provided that if T is any Calderón-Zygmund operator with standard kernel and if M is the Hardy-Maximal operator, then for any $0 < p < \infty$ and $\omega \in A_{\infty}$, there is a constant C depending on p and ω such that,

$$\int_{\mathbb{R}^n} |Tf(x)|^p \omega(x)\, dx \leq C \int_{\mathbb{R}^n} Mf(x)^p \omega(x)\, dx$$

for any function f such that the left-hand side is finite.

This means that T is controlled by the Hardy-Littlewood maximal function M in $L^p(\omega)$ for all $0 < p < \infty$ and $w \in A_{\infty}$ and therefore T is bounded on $L^p(\omega)$ if M is bounded on $L^p(\omega)$, which by Muckenhoupt's theorem means $\omega \in A_p$. In particular, the range of unweighted L^p boundedness of T, that is the set of p for which T is strong-type (p,p), is $(1,\infty)$.

Moreover, in [LOP], authors proved the following A_1 bound for the Calderón-Zygmund operator

$$||T||_{L^p(\omega)} \leq c(n,p)A_1(\omega),$$

where c is a positive constant depending only on p and the dimension and where A_1 stands for A_1-constant of Muckenhoupt of ω.

4.3 Riesz Trasform

We recall that if $f \in L^p(\mathbb{R}^n)$, $1<p<\infty$, then for $i=1,...,n$ the Riesz transforms are defined by

$$R_i(f)(x) = \lim_{\varepsilon \to 0} d_n \int_{|y| \geq \varepsilon} \frac{y_i}{|y|^{n+1}} f(x-y)\, dy$$

with $d_n = \Gamma((n+1)/2)\pi^{(n+1)/2}$. It is known that the multiplier for this operator is given by $d_n x_i/|x|$, which is formally the Fourier transform of the kernel of $R_i(f)$.

The symbol R_i stands for the i^{th} direction Riesz transform on $\mathbb{R}^n$ and is defined by its Fourier multiplier as follows:

$$\widehat{R_i} f(x) = i\frac{x_i}{|x|}\widehat{f}(x).$$

On R^n, it is well-known that the classical Riesz transforms R_i , $1 \leq i \leq n$, are bounded on $L^p(\mathbb{R}^n, dx)$ for $1 < p < \infty$ and are of weak-type $(1,1)$ with respect to dx. As a consequence of the weighted theory for classical Calderón-Zygmund operators, the Riesz transforms are also bounded on $L^p(\mathbb{R}^n, \omega(x)dx)$ for all $\omega \in A_p$, $1<p<\infty$, and are of weak-type $(1,1)$ with respect to $\omega(x)dx$ for $\omega \in A_1$. Furthermore, it can be shown that the A_p condition on the weight is necessary for the weighted L^p boundedness of the Riesz transforms.

Recently, Petermichl [P2] established the best possible bound on the norm of the Riesz transforms in the weighted Lebesgue space $L^p(\mathbb{R}^n, \omega)$ in terms of the A_p-constant of the weight ω, for $1<p<\infty$. The result is contained in the following theorem

Theorem 4.1. *[P2] There exists a constant c so that for all weights $\omega \in A_2$ the Riesz transforms as operators in weighted space $R_i : L^2(\mathbb{R}^n, \omega) \longrightarrow L^2(\mathbb{R}^n, \omega)$ have operator norm $||R_i||_{L^2(\omega)} \leq cA_2(\omega)$. This result is sharp.*

4.4 Hilbert Trasform

In mathematics and in signal processing, the Hilbert transform is a linear operator which takes a function, $f(x)$, to another function, $H(f)(x)$, with the same domain. The Hilbert transform is named after David Hilbert, who first introduced the operator in order to solve a special case of the Riemann-Hilbert problem for holomorphic functions. It is a basic tool in Fourier analysis, and provides a concrete means for realizing the conjugate of a given function or

Fourier series. Furthermore, in harmonic analysis, it is an example of a singular integral operator, and of a Fourier multiplier. The Hilbert transform is also important in the field of signal processing where it is used to derive the analytic representation of a signal $f(x)$.

The Hilbert transform was originally defined for periodic functions, or equivalently for functions on the circle, in which case it is given by convolution with the Hilbert kernel. More commonly, however, the Hilbert transform refers to a convolution with the Cauchy kernel, for functions defined on the real line R (the boundary of the upper half-plane).

Definition 4.1. The Hilbert transform is defined on $\mathbb{R}$ is defined as follows:

$$Hf(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{f(y)}{x-y} \stackrel{def}{=} \lim_{\varepsilon \to 0} \int_{|x-t|>\varepsilon} \frac{f(y)}{x-y}\, dy \tag{4.3}$$

when the integral exists (because of the possible singularity at $x = y$, the integral is to be considered as a Cauchy principal value).

We recall that the Hf is zero if f is a constant, while the Hilbert transform of a real function is a real function.

Boundedness

Before to describe the weighted case, we recall the boundedness properties of H in the classical Lebesgue space. If $1 < p < \infty$, then the Hilbert transform on $L^p(\mathbb{R})$ is a bounded linear operator, meaning that there exists a constant C_p such that

$$||Hf||_p \leq C_p ||f||_p,$$

for all $f \in L^p(\mathbb{R})$. This theorem is due to Riesz (1928) [R] (see also [Ti]). The best constant C_p is given by

$$C_p = \begin{cases} tang\frac{\pi}{2p}, & for \quad 1 < p \leq 2 \\ cot\frac{\pi}{2p} & for \quad 2 < p < \infty. \end{cases}$$

This result is due to Pichorides [Pi].

The weighted case is now well known through the famous Helson-Szëgo Theorem [HS] and the Hunt-Muckenhoupt-Wheeden Theorem [HMW].

The first one states that the Hilbert transform H is bounded in the weighted space $L^2(\omega)$ if and only if ω is of the form $\omega = \exp\{u + Hv\}$ with $u, v \in L^\infty$

and $||u||_\infty < \pi/2$. This condition must be equivalent to A_2-condition but there is not a direct proof of the equivalence.

The Hunt - Muckenhoupt - Wheeden theorem states that the Hilbert transform H is bounded in $L^p(\omega)$ if and only if $\omega \in A_p$. More precisely,

Theorem 4.2. *[HMW] Let $1 < p < \infty$. For the inequality*

$$\int_{-\infty}^{+\infty} |(Hf)(x)|^p \omega(x)\, dx \leq c \int_{-\infty}^{+\infty} |f(x)|^p \omega(x)\, dx,$$

where the positive constant c does not depend on $f \in L^p(\mathbb{R}, \omega)$, it's necessary and sufficient that $\omega \in A_p$.

Buckley [Bu2] also showed that the Hilbert transform is bounded on $L^p(w)$ with an operator norm which is at most a multiple of $A_p^\alpha(w)$, where $max\{1, p'/p\} \leq \alpha$. In particular, for $p = 2$ he showed that the dependence on $A_2(w)$ was at least linear, and at most quadratic.

Petermichl and Pott [PP] very elegantly showed $\alpha \leq 3/2$ in the following theorem

Theorem 4.3. *[PP] $H : L^2(\mathbb{R}, \omega) \longrightarrow L^2(\mathbb{R}, \omega)$ has operator norm $||H||_{L^2(\omega)} \leq cA_2(\omega)^{3/2}$.*

Later, Petermichl [P1] improved this estimate in the following theorem.

Theorem 4.4. *[P1] There exists a constant c so that for all weights $\omega \in A_2$ the Hilbert transform as an operator in weighted space $H : L^2(\mathbb{R}, \omega) \longrightarrow L^2(\mathbb{R}, \omega)$ has operator norm $||H||_{L^2(\omega)} \leq cA_2(\omega)$ and this result is sharp.*

This theorem immediately implies the following more general version:

Theorem 4.5. *[GR] For $2 \leq p < \infty$ there exists $c(p)$ only depending on p so that $H : L^p(\mathbb{R}, \omega) \longrightarrow L^p(\mathbb{R}, \omega)$ has operator norm $||H||_{L^2(\omega)} \leq cA_p(\omega)$.*

4.5 Riesz Potenzial : a sharp estimate

Recall that a weight ω, namely a locally integrable nonnegative function in $\mathbb{R}^n$, belongs to the class A_p for some $p \in (1, \infty)$ if the quantity

$$A_p(\omega) = \sup_B \left(\frac{1}{|B|} \int_B \omega(x)\, dx \right) \left(\frac{1}{|B|} \int_B \omega(x)^{\frac{1}{1-p}}\, dx \right)^{p-1} \tag{4.4}$$

is finite. Here, B denotes any ball in $\mathbb{R}^n$ and $|B|$ stands for its Lebesgue measure.

Up to now we have seen that boundedness properties of various classical operators of harmonic analysis in weighted Lebesgue spaces can be characterized in terms of the A_p-weights. Now, we address a parallel issue for the Riesz Potential I_α, of any order $\alpha \in (0, n)$. The Riesz potential of a function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy \qquad \text{for } x \in \mathbb{R}^n. \tag{4.5}$$

A very classical result states that I_α is bounded from $L^p(\mathbb{R}^n)$ into $L^{\frac{np}{n-\alpha p}}(\mathbb{R}^n)$ if $1 < p < \frac{n}{\alpha}$. Weighted inequalities for fractional integrals can be characterized in terms of the A_p condition. As shown in [MW], a necessary and sufficient condition for the boundedness of I_α from $L^p(\mathbb{R}^n, \omega)$ into $L^{\frac{np}{n-\alpha p}}(\mathbb{R}^n, \omega)$ is that $\omega^{\frac{np}{n-\alpha p}} \in A_{1+\frac{q}{p'}}$, where $p' = \frac{p}{p-1}$. (See also [Ma] for an alternate treatment of weighted inequalities for potentials in terms of capacities.)

If ω is merely in A_p for some $p \in (1, \frac{n}{\alpha})$, then there still exists a power $\overline{p}$, larger than p and depending on $A_p(\omega)$, such that I_α is bounded from $L^p(B, \omega)$ into $L^{\overline{p}}(B, \omega)$ for any ball $B \subset \mathbb{R}^n$ [FKS].

Our result provides sharp quantitative information on this statement, and reads as follows.

Theorem 4.6. *[ACiS] Let $n \geq 2$, $\alpha \in (0, n)$ and $1 < p < \frac{n}{\alpha}$. Let $\omega \in A_p$. Then, there exist positive constants $k = k(\alpha, p, n)$ and $C = C(\alpha, p, n)$ such that if*

$$p - kA_p(\omega)^{\frac{1}{1-p}} < q < p \tag{4.6}$$

then

$$\left(\frac{1}{\int_{B_R} \omega(x) dx} \int_{B_R} |I_\alpha f(x)|^{\frac{nqp}{nq-\alpha p}} \omega(x)\, dx \right)^{\frac{nq-\alpha p}{nqp}} \leq \tag{4.7}$$

$$\leq CR^\alpha A_p(\omega)^{\frac{nq-\alpha}{nq(p-1)}} \left(\frac{1}{\int_{B_R} \omega(x) dx} \int_{B_R} |f(x)|^p \omega(x)\, dx \right)^{\frac{1}{p}}$$

for any ball $B_R \subset \mathbb{R}^n$ of radius R and every function $f \in L^p(B_R, \omega)$ (continued by 0 outside B_R). Moreover, the exponent $\frac{nq-\alpha}{nq(p-1)}$ for $A_p(\omega)$ is sharp.

Our approach to Theorem 4.6 is related to an argument from [CFr]. In fact, its proof relies upon a combination of an estimate for $I_\alpha f$ in terms of Mf appearing in [He] and the Buckley's result (see Theorem 2.10), and exploits a property of A_p weights established in [CFe]. It is however interesting that keeping track of the exact dependence of the quantities involved in this argument can lead to the sharp bound (4.7).

Proof. (**of the Theorem** (4.6))

Given any $\varepsilon > 0$, define

$$I_\alpha^\varepsilon f(x) = \int_{\{|x-y|\geq\varepsilon\}\bigcap B_R} f(y)|x-y|^{\alpha-n}dy \qquad \text{for } x \in B_R. \tag{4.8}$$

A constant $C_1 = C_1(\alpha, p, n)$ exists such that

$$|I_\alpha f(x) - I_\alpha^\varepsilon f(x)| \leq C_1\varepsilon^\alpha Mf(x) \qquad \text{for } x \in B_R \tag{4.9}$$

see [He]. On the other hand, by Hölder's inequality,

(4.10)

$$|I_\alpha^\varepsilon f(x)| \leq ||f||_{L^p(B_R,\omega)} \left(\int_{\{|x-y|\geq\varepsilon\}\bigcap B_R} |x-y|^{(\alpha-n)p'}\omega(y)^{\frac{1}{1-p}}\, dy\right)^{\frac{1}{p'}} \qquad \text{for } x \in B_R.$$

Owing to a result from [CFe], a constant k as in the statement exists such that, if q fulfills (4.6), then $\omega \in A_q$ and

$$A_q(\omega) \leq C_2 A_p(\omega)^{\frac{1}{1-p}} \tag{4.11}$$

for some constant $C_2 = C_2(p, n)$. For any such q, another application of Hölder's inequality to the integral on the right-hand side of (4.10) yields

(4.12)

$$|I_\alpha^\varepsilon f(x)| \leq C_3||f||_{L^p(B_R,\omega)}\varepsilon^{\alpha-\frac{nq}{p}} \left(\int_{B_R} \omega(y)^{\frac{1}{1-q}}\, dy\right)^{\frac{q-1}{p}} \qquad \text{for } x \in B_R,$$

for some constant $C_3 = C_3(\alpha, p, n)$. Combining (4.9) and (4.12), and choosing

$$\varepsilon = \left(\frac{Mf(x)}{||f||_{L^p(B_R,\omega)} \left(\int_{B_R} \omega(y)^{\frac{1}{1-q}}\, dy\right)^{\frac{q-1}{p}}}\right)^{-\frac{p}{nq}}$$

entail that

(4.13)

$$I_\alpha f(x) \leq C_4(Mf(x))^{1-\frac{\alpha p}{nq}}||f||^{\frac{\alpha p}{nq}}_{L^p(B_R,\omega)} \left(\int_{B_R} \omega(y)^{\frac{1}{1-q}}\, dy\right)^{\frac{\alpha}{nq'}} \qquad \text{for } x \in B_R,$$

for some constant $C_4 = C_4(\alpha, p, n)$. From (4.13) and (4.5) one infers that

(4.14)

$$||I_\alpha f||_{L^{\frac{nqp}{nq-\alpha p}}(B_R,\omega)} \le C_4 ||Mf(x)||_{L^p(B_R,\omega)}^{\frac{nq-\alpha p}{nq}} ||f||_{L^p(B_R,\omega)}^{\frac{\alpha p}{nq}} \left(\int_{B_R} \omega(y)^{\frac{1}{1-q}}\, dy \right)^{\frac{\alpha}{nq'}}$$
$$\le C_5 A_p(\omega)^{\frac{nq-\alpha p}{nq(p-1)}} ||f||_{L^p(B_R,\omega)} \left(\int_{B_R} \omega(y)^{\frac{1}{1-q}}\, dy \right)^{\frac{\alpha}{nq'}},$$

for some constant $C_5 = C_5(\alpha, p, n)$, whence, by the definition of $A_q(\omega)$, one obtains that

$$\left(\frac{1}{\int_{B_R} \omega(x)\, dx} \int_{B_R} |I_\alpha f(x)|^{\frac{nqp}{nq-\alpha p}} \omega(x)\, dx \right)^{\frac{nq-\alpha p}{nqp}} \tag{4.15}$$
$$\le C_6 R^\alpha A_p(\omega)^{\frac{nq-\alpha p}{nq(p-1)}} A_q(\omega)^{\frac{\alpha}{nq}} \left(\frac{1}{\int_{B_R} \omega(x)\, dx} \int_{B_R} |f(x)|^p \omega(x)\, dx \right)^{1/p},$$

for some constant $C_6 = C_6(\alpha, p, n)$. Hence, inequality (4.7) follows, owing to (4.11).

In order to prove that the exponent $\frac{nq-\alpha p}{nq(p-1)}$ in (4.7) is sharp, consider a ball centered at 0 of any radius R, and functions f and weights ω having the form $f(x) = \phi(w_n|x|^n)$ and $\omega(x) = \psi(w_n|x|^n)$ for $x \in \mathbb{R}^n$, for some functions $\phi, \psi : [0, \infty) \longrightarrow [0, \infty)$, with ϕ vanishing outside $[0, w_n R^n)$. Here, w_n denotes the Lebesgue measure of the unit ball. One has

$$I_\alpha f(x) \ge 2^{\alpha-n} \int_{\{|y|<|x|\}} \frac{f(y)}{|x|^{n-\alpha}} dy = 2^{\alpha-n} |x|^{\alpha-n} \int_0^{w_n|x|^n} \phi(r)\, dr. \tag{4.16}$$

Thus, on setting $w_n R^n = t$, we get

(4.17)

$$\sup_f \frac{\left(\frac{1}{\int_{B_R} \omega(x)\, dx} \int_{B_R} |I_\alpha f(x)|^{\frac{nqp}{nq-\alpha p}} \omega(x)\, dx \right)^{\frac{nq-\alpha p}{nqp}}}{\left(\frac{1}{\int_{B_R} \omega(x)\, dx} \int_{B_R} |f(x)|^p \omega(x)\, dx \right)^{1/p}}$$
$$\ge C_7 \sup_\phi \frac{\left(\int_0^t \psi(s) ds\right)^{\frac{\alpha}{nq}} \left(\int_0^t \left(\int_0^s \phi(r) dr\right)^{\frac{nqp}{nq-\alpha p}} s^{\frac{(\alpha-n)qp}{nq-\alpha p}} \psi(s) ds \right)^{\frac{nq-\alpha p}{nqp}}}{\left(\int_0^t \phi(s)^p \psi(s) ds \right)^{1/p}}$$
$$\ge C_7 \left(\int_0^t \psi(s) ds \right)^{\frac{\alpha}{nq}} \sup_{0<\tau<t} \left(\int_\tau^t r^{\frac{(\alpha-n)qp}{nq-\alpha p}} \psi(r) dr \right)^{\frac{nq-\alpha p}{npq}} \left(\int_0^\tau \psi(r)^{\frac{1}{1-p}} dr \right)^{\frac{1}{p'}}$$

for some constant $C_7 = C_7(\alpha, n)$, where the last inequality holds by a classical characterization of one-dimensional Hardy-type inequalities – see e.g. [Ma, Theorem 1.3.1/1].

Now, choose $\psi(s) = s^{(p-1)(1-\delta)}$ for $s > 0$, with $\delta \in (0,1)$. It is then well known that $A_p(\omega) \approx \delta^{1-p}$ as $\delta \to 0^+$, up to multiplicative constants depending only on p and n [Bu1]. Next, take $q = p - a\delta$, for sufficiently small a depending on k, in such a way that condition (4.6) is fulfilled. Computations show that, with these choices of ψ and q, the rightmost side of (4.17) $\approx \delta^{\frac{\alpha}{nq}-1}$ as $\delta \to 0^+$, up to multiplicative constants depending only on α, p and n. Since $A_p(\omega)^{\frac{nq-\alpha}{nq(p-1)}} \approx \delta^{\frac{\alpha}{nq}-1}$ as well, the sharpness of the exponent $\frac{nq-\alpha}{nq(p-1)}$ follows.

$\square$

Chapter 5

BMO-Space

In this chapter we will illustrate the space of functions of bounded mean oscillation (BMO space), introduced by John and Nirenberg (1961), and extremely important in various areas of analysis including harmonic analysis, PDEs and function theory.

In Harmonic analysis, this space plays the same role in the theory of Hardy spaces that the space L^∞ of bounded functions plays in the theory of L^p-spaces.

5.1 Definitions and notations

We begin with some notations. If $\Omega \subset \mathbb{R}^n$ is any measurable set of finite positive measure $|\Omega|$ and f is an integrable function, let us recall that $f_\Omega = \frac{1}{|\Omega|}\int_\Omega f dx = ⨍_\Omega f dx$ indicates the integral mean of f over Ω.

Definition 5.1. A function f is said to be a **BMO function** if it is in $L^1_{loc}(\mathbb{R}^n)$ and the "mean oscillation" is finite, that is

$$⨍_Q |f - f_Q|\, dx < \infty. \tag{5.1}$$

Here Q is a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes.

The supremum is called the **BMO norm** of f and is denoted by $||f||_*$:

$$\|f\|_* = \sup_Q ⨍_Q |f - f_Q|\, dx. \tag{5.2}$$

Every bounded (measurable) function is in BMO and

$$||f||_* \le \inf\{||f-a||_\infty : a \ \ constant\}.$$

Clearly, constant function have zero BMO norm.

Note that BMO space is a Banach space.

Moreover, Fefferman (1971) showed that the BMO space is dual to H^1, the Hardy space with $p=1$.

It's important to note that BMO is not equal to L^∞ but the inclusion holds $L^\infty \subset BMO$ and we have

$$\|f\|_* \le 2\|f\|_\infty.$$

A typical example of a function that is in BMO but not in L^∞ is $\log|x|$. Now we sketch the proof.

Let $I=(a,b)\subset\mathbb{R}$. We show for an appropriate choice of C_I,

$$\fint_I |\log|x| - C_I|\ dx \le 1, \tag{5.3}$$

which in turn implies that $\|\log|\cdot|\|_* \le 2$.

In order to prove (5.3) we consider three cases:

i) $0<a<b$

ii) $-b<a<b$

iii) the rest

In the case **i)**, we pick $C_I=\log b$ and note that

$$\int_I |\log|x| - \log b|\ dx = \int_{(a,b)} (\log b - \log x)\ dx =$$

$$= \int_{(a,b)} \log b\ dx - \int_{(a,b)} \log x\ dx = (b-a) - a(\log b - \log a).$$

Therefore,

$$\fint_I |\log|x| - \log b|\ dx = 1 - a\frac{\log b - \log a}{b-a},$$

and (5.3) follows since $0<a<b$.

In the case **ii)** we may restrict ourselves to $-b<a<0<b$. Again pick $C_I=\log b$ and note that

$$\int_I |\log|x| - \log b|\ dx = \int_{(a,-a)} |\log|x| - \log b|\ dx + \int_{(-a,b)} (\log b - \log x)\ dx = W+K.$$

From the above computation we have

$$K = (b + a) + a(\log b - \log(-a)).$$

To compute W we observe that the integrand is an even function, so

$$W = 2 \lim_{\varepsilon \to 0^+} \int_{(\varepsilon, -a)} (\log b - \log x)\ dx = 2(-a \log b + a \log(-a) - a).$$

Thus

$$W + K = (b - a) + a(\log b - \log(-a))$$

and so

$$⨍_I |\log|x| - \log b|\ dx = 1 - (-a)\frac{(\log b - \log(-a))}{b + a}\frac{b + a}{b - a}.$$

Since $-b < a < 0 < b$ also in this case (5.3) follows.

The remaining cases can be reduced to either i) or ii) since we are dealing with an even function.

Now we give an example of function that does not belong to BMO.

Example 5.1. Let us show that the function $g(x) = \text{sign (x)} \log \frac{1}{|x|}$ does not belong to $BMO([-1, 1])$. Indeed, for $0 < h < 1$ and $I \equiv [-h, h]$ we have $g_I = 0$ and

$$⨍_I |g(y) - g_I|\ dy = \frac{1}{2h} \int_{-h}^{h} \left|\log \frac{1}{|x|}\right| dx = \frac{1}{h} \int_0^h \log \frac{1}{x} dx = 1 + \log \frac{1}{h} \xrightarrow{h \to 0} \infty.$$

This example shows that if the absolute value of a function belongs to the BMO-class, this does not imply that the function itself is a BMO-function.

We shall give a result which provide many example of BMO functions.

Theorem 5.1. *[GR] If ω is an A_1 weight, then $\log \omega \in BMO$ with a norm depending only on the $A_1(\omega)$.*

5.2 Estimates of rearrangements of the BMO-functions

The aim of the present section is to show that if a function f is in BMO, then its non-increasing rearrangement f^* is also in BMO. The importance of the equimeasurable rearrangements of functions comes from the fact that in certain cases they preserve the properties of the original functions and in the same time have a simpler form. Let us give the definitions.

Definition 5.2. The **non-increasing rearrangement** of the function f is a non-increasing function f^* such that it is equimeasurable with $|f|$, i.e., for all $y > 0$ they have the same distribution function (see Definition 2.5)

$$a_{f^*}(y) = |\{x \in [0, |E|] : f^*(x) > y\}| = |\{t \in E : f^*(t) > y\}| = a_f(y)$$

for any measurable set $E \subset \mathbb{R}^n$.

This property does not define the non-increasing rearrangement uniquely: it can take different values at points of discontinuity (the set of such points is at most countable). For definiteness let us assume in addition that the function f^* is continuous from the left on $(0, |E|]$. The relation between the distribution function and the non-increasing rearrangement is given by the following equality:

$$f^*(x) = \inf\{y > 0 : a_f(y) < x\}, \;\; 0 < x < |E|.$$

This formula shows that in a certain sense the non-increasing rearrangement is the inverse function to the distribution function.

An equivalent definition of the non-increasing rearrangement can be written in the following way:

$$f^*(x) = \sup_{D \subset E,\, |D|=x} \inf_{y \in D} |f(y)|, \qquad 0 < x < |E|.$$

Sometimes instead of the non-increasing rearrangement it is more convenient to use the **non-decreasing rearrangement**. For the function f, measurable on the set $E \subset R^n$, the non-decreasing rearrangement is defined via the following equality:

$$f_*(x) = \inf_{D \subset E,\, |D|=x} \sup_{y \in D} |f(y)|, \qquad 0 < x < |E|.$$

The function f_* is non-negative, it is equimeasurable with $|f|$ on E and it is non-decreasing on $[0, |E|)$. The connection between the non-increasing and non-decreasing rearrangements is given by the equality

$$f_*(x) = f^*(|E| - x)$$

which holds true at every point of continuity, i.e. almost everywhere on $(0, |E|)$.

The equimeasurability of functions f^*, f_* and $|f|$ implies that

$$\int_0^{|E|} \varphi(f^*(u))\, du = \int_0^{|E|} \varphi(f_*(u))\, du = \int_E \varphi(|f(x)|)\, dx.$$

The most important properties of the equimeasurable rearrangements f^* and f_* follow directly from their definition and consist in the identities:

$$\sup_{D\subset E,\, |D|=x} \int_D |f(y)|\, dy = \int_0^x f^*(u)\, du, \qquad 0 < x < |E|$$

$$\inf_{D\subset E,\, |D|=x} \int_D |f(y)|\, dy = \int_0^x f_*(u)\, du, \qquad 0 < x < |E|.$$

Often it is useful to consider the following functions

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(u)\, du, \quad f_{**}(t) = \frac{1}{t}\int_0^t f_*(u)\, du, \ \ t > 0.$$

Theorem 5.2. *[BDS] Let $f \in BMO(\mathbb{R}^n)$, then*

$$f^{**}(x) - f^*(x) \le 2^{n+4}\|f\|_*, \qquad 0 < x < \infty.$$

In particular, from Theorem 5.2 it follows that the rearrangement operator is bounded in BMO.

The following Theorem shows that the non-increasing rearrangement f^* of a BMO-function f is also a BMO-function.

Theorem 5.3. *($(n = 1)$, [GRo]; $(n \ge 1)$, [BDS]) Let $f \in BMO(\mathbb{R}^n)$. Then $f^* \in BMO([0,\infty))$ and*

$$\|f^*\|_* \le C\|f\|_* ,$$

where the constant C depends only on the dimension n of the space (one can take $C = 2^{n+5}$).

5.3 The distance in BMO to L^∞

We have seen that BMO functions are not necessarily bounded, so it's natural to study the distance between $f \in BMO$ to L^∞, defined as follows

$$dist_{BMO}(f, L^\infty) = \inf_{g\in L^\infty} ||f - g||_*. \tag{5.4}$$

Beginning to consider again the function $\log|x|$. Fix $(0, b) = I \subset \mathbb{R}$ and consider those $x \in I$ where $\log|x|$ is large, i.e., consider the set

$$E_\lambda = \{x \in I : |\log|x| - C_I| > \lambda\}, \qquad \lambda > 0,$$

where $C_I = (\log|\cdot|)_I$. We are interested in E_λ for large values of λ. We can write E_λ as the sum of two sets:

$$E_\lambda = \{x \in I : x > e^{\lambda + C_I}\} \cup \{x \in I : x < e^{-\lambda + C_I}\}.$$

If λ is large the first set is empty and so for λ big enough we get:

$$|E_\lambda| \leq |\{x \in I : x < e^{-\lambda + C_I}\}| = e^{-\lambda} e^{C_I}.$$

Now by Jensen inequality

$$e^{C_I} \leq \fint_I e^{\log x}\, dx = \frac{|I|}{2}$$

and consequently

$$|E_\lambda| \leq \frac{|I|}{2} e^{-\lambda}.$$

The remarkable fact is that a similar estimate holds for arbitrary $f \in BMO$ and $I \subset \mathbb{R}$. The bounds for the distance (5.4) are expressed in terms of constants in the following theorem, due to John and Nirenberg

Theorem 5.4. *[JN] There exist constants C_1, C_2, depending only on the dimension n, such that for every $f \in BMO(\mathbb{R}^n)$ and every cube $Q \subset \mathbb{R}^n$*

$$|\{x \in Q : |f(x) - f_Q| > \lambda\}| \leq C_1\, |Q|\, e^{-\left(\frac{C_2 \lambda}{\|f\|_*}\right)}, \qquad \lambda > 0. \tag{5.5}$$

In this theorem, the authors showed that the distribution function, corresponding to a function of bounded mean oscillation, is exponentially decreasing.

Remark 5.1. *In terms of equimeasurable rearrangements the inequality (5.5) can be rewritten in the following form:*

$$(f - f_Q)^*(x) = \frac{\|f\|_*}{C_2} \log \frac{C_1 |Q|}{x}, \qquad 0 < x \leq |Q|. \tag{5.6}$$

So, if $f \in BMO$, then its equimeasurable rearrangement do not grow faster than the logarithmic function as the argument tends to zero.

Remark 5.2. *In a certain sense the John-Nirenberg theorem is invertible. Namely, if f is a locally summable on $\mathbb{R}^n$ function such that for any cube $Q \subset \mathbb{R}^n$*

$$|\{x \in Q : |f(x) - f_Q| > \lambda\}| \leq C_1 |Q|\, e^{-C_2 \lambda}, \qquad \lambda > 0 \tag{5.7}$$

where the constants C_1 and C_2 do not depend on Q, then we want to prove that $f \in BMO(\mathbb{R}^n)$.

Indeed, let us rewrite (5.7) *in the form*

$$(f - f_Q)^*(x) \leq \frac{1}{C_2} \log \frac{C_1 |Q|}{x}, \qquad 0 < x \leq |Q|.$$

Then

$$\frac{1}{|Q|} \int_Q |f(x) - f_Q|\, dx = \frac{1}{|Q|} \int_0^{|Q|} (f - f_Q)^*(y) dy \leq \frac{1}{C_2} \frac{1}{|Q|} \int_0^{|Q|} \log \frac{C_1 |Q|}{y}\, dy =$$

$$= \frac{1}{C_2} \int_0^1 \log \frac{C_1}{u}\, du = \frac{1}{C_2}(1 + \log C_1).$$

Taking the supremum over all cubes $Q \subset \mathbb{R}^n$, we obtain

$$\|f\|_* \leq \frac{1}{C_2}(1 + \log C_1).$$

The John-Nirenberg theorem implies the following

Corollary 5.5. *If $f \in BMO(\mathbb{R}^n)$, then $f \in L^p_{loc}(\mathbb{R}^n)$, for any $p < \infty$.*

Proof. It is enough to prove that $(f - f_Q) \in L^p(Q)$ for any cube $Q \in \mathbb{R}^n$. The John-Nirenberg inequality in the form 5.6 yields

$$\int_Q |f - f_Q|^p dx = \int_0^{|Q|} (f - f_Q)^*(t)\, dt \leq$$

$$\leq \left(\frac{\|f\|_*}{C_2}\right)^p \int_0^{|Q|} \log^p \left(\frac{C_1 |Q|}{t}\right)\, dt =$$

$$= \left(\frac{\|f\|_*}{C_2}\right)^p |Q| C_1 \int_0^{\frac{1}{C_1}} \log^p \left(\frac{1}{u}\right)\, du < \infty.$$

□

Corollary 5.6. *Let $f \in L^1_{loc}(\mathbb{R}^n)$ verify* (5.5)*, then for $\lambda > \frac{\|f\|_*}{C_2}$ and for any cube Q,*

$$⨍_Q e^{\frac{|f(x) - f_Q|}{\lambda}} dx \leq \frac{C_1}{(C_2 \frac{\lambda}{\|f\|_*}) - 1}.$$

5.4 A precise interplay between BMO space and A_2-class.

The interplay between BMO and A_2 is well known: f belongs to BMO if and only if there exists $\lambda \geq 1$ such that $\omega = e^{f/\lambda}$ belongs to A_2.

In the following we will try to illustrate this subject giving optimal bounds, in the particular case $n = 1$.
For $f \in BMO$, consider the set

$$I_f = \{\lambda > 0 \ : \ A_2(e^{f/\lambda}) < \infty\}. \tag{5.8}$$

As observed in [GJ] this set is non empty. Note that the "openess" property of A_2-class, namely:

if $\omega \in A_2$, then automatically $\omega^\tau \in A_2$ for some $\tau > 1$

(see Chapter 1) implies that I_f does not have a minimum. In [GJ] the infimum of I_f was introduced

$$\varepsilon(f) = \inf I_f \tag{5.9}$$

and recognized to give the right upper and lower bounds for the distance of f to L^∞ (defined in (5.4)), through the following theorem.

Theorem 5.7. *([GJ]) If $f \in L^1_{loc}(\mathbb{R}^n)$ then*

$$c_1 \, dist_{BMO}(f, L^\infty) \ \leq \ \varepsilon(f) \ \leq \ c_2 \, dist_{BMO}(f, L^\infty)$$

where c_1 and c_2 are constants depending only on the dimension.

Our aim is to establish more precise estimates for $\varepsilon(f)$ in the one-dimensional case. For example, we will show that in Theorem 5.7 we can take $c_2 = e/2$.

We will also obtain another expression for the functional $\varepsilon(f)$ in which A_2-constants appear.

We will rely on the following version of John-Nirenberg inequality due to A. Korenovskii ([Ko2], [Ko3]).

Theorem 5.8. *Let $f \in BMO(\mathbb{R})$. Then for any interval I and for any $\lambda > 0$*

$$\frac{1}{|I|}|\{x \in I \ : \ |f(x) - f_I| > \lambda\}| \ \leq \ e^{1+\frac{2}{e}} \, exp\left(\frac{-2\lambda}{e\|f\|_*}\right).$$

The constant $(2/e)$ in the exponent cannot be increased.

Proposition 5.9. *[AS] The function f belongs to BMO if and only if I_f is a non empty set. If we define*

$$\varepsilon(f) = \inf I_f$$

then

$$I_f = (\varepsilon(f), \infty) \tag{5.10}$$

and

$$\varepsilon(f) \leq \frac{e}{2} \|f\|_*. \tag{5.11}$$

Proof. Condition $A_2(e^{f/\lambda}) < \infty$ is equivalent to

$$s(f,\lambda) = \sup_I ⨍_I e^{\frac{|f-f_I|}{\lambda}} \, dx \ < \infty \tag{5.12}$$

where the supremum is taken with respect to all intervals $I \subset \mathbb{R}$. Actually, for $\lambda > 0$ the following inequalities hold

$$\frac{1}{2}\, s(f,\lambda) \leq A_2(e^{f/\lambda}) \leq s(f,\lambda)^2 \tag{5.13}$$

(see Ch. IV, Corollary 2.18 in [GR]).

Then it is obvious that

$$\lambda_0 \in I_f, \quad \lambda_1 > \lambda_0 \implies \lambda_1 \in I_f.$$

Moreover, due to Theorem 1.18, the set I_f does not contain its infimum $\varepsilon(f)$. This means that (5.10) holds true. To establish (5.11) we repeat a standard argument [GR] invoking Theorem 5.8

$$⨍_I e^{|f-f_I|/\lambda} = \int_0^\infty \frac{e^{t/\lambda}}{\lambda} |\{x \in I : |f(x) - f_I| > t\}| \, dt \leq$$

$$\leq \int_0^\infty \frac{e^{t/\lambda}}{\lambda} \, e^{(1\,+\,2/e)} \, e^{-(2/e\|f\|_*)\,t} \, |I| \, dt =$$

$$= |I| \, \frac{e^{(1\,+\,2/e)}}{\lambda} \int_0^\infty e^{(\frac{1}{\lambda}\,-\,2/e\|f\|_*)\,t} \, dt = |I| \, \frac{e^{(1\,+\,2/e)}}{\lambda} \left(\frac{2}{e\|f\|_*} - \frac{1}{\lambda} \right)^{-1}$$

if $\lambda > \frac{e}{2}\|f\|_*$. $\square$

Corollary 5.10. *[AS] For any $f \in BMO(\mathbb{R})$*

$$\varepsilon(f) \leq \frac{e}{2} \, dist_{BMO}(f, L^\infty). \tag{5.14}$$

Proof. It is easy to check that for $g \in L^\infty$

$$\varepsilon(f) = \varepsilon(f - g).$$

Then, using (5.11) we obtain

$$\varepsilon(f) \le \frac{e}{2}\|f - g\|_*$$

for any $g \in L^\infty$. This immediately implies (5.14). $\square$

More precisely, we obtain another representation for $\varepsilon(f)$.

Theorem 5.11. *[AS] For any $f \in BMO$*

$$\varepsilon(f) \; = \inf\left\{\lambda \sqrt{\frac{A_2(e^{f/\lambda}) - 1}{A_2(e^{f/\lambda})}} \;:\; \lambda \in I_f\right\}. \tag{5.15}$$

Proof. By Theorem 1.18 we deduce that, if $A_2(e^f) = A < \infty$, then

$$\varepsilon(f) \; \le \; \sqrt{\frac{A-1}{A}}. \tag{5.16}$$

In fact, for $\omega = e^f$ and $\lambda > \sqrt{\frac{A-1}{A}}$, we deduce $A_2(\omega^{\frac{1}{\lambda}}) < \infty$. Hence the inclusion

$$\left(\sqrt{\frac{A-1}{A}}, \infty\right) \;\; \subset I_f$$

holds and this implies (5.16).

Moreover, by applying this observation with f/λ in place of f and using the following property of the functional $\varepsilon(f)$:

$$\varepsilon(\mu f) = \mu\, \varepsilon(f) \qquad\qquad for \quad \mu > 0,$$

we deduce, for $\lambda \in I_f$

$$\frac{1}{\lambda}\varepsilon(f) \le \sqrt{\frac{A_2(e^{f/\lambda}) - 1}{A_2(e^{f/\lambda})}},$$

hence

$$\varepsilon(f) \le \inf\left\{\lambda\sqrt{\frac{A_2(e^{f/\lambda}) - 1}{A_2(e^{f/\lambda})}} : \quad \lambda \in I_f\right\}.$$

To get the inequality (5.15) it is sufficient to observe that

$$\inf\left\{\lambda\sqrt{\frac{A_2(e^{f/\lambda}) - 1}{A_2(e^{f/\lambda})}} : \quad \lambda \in I_f\right\} \le \inf\{\lambda : \lambda \in I_f\} = \varepsilon(f).$$

$\square$

Corollary 5.12. *[AS] For any $f \in BMO$, we have*

$$0 \le \varepsilon(f) \le 1; \tag{5.17}$$

moreover

$$\varepsilon(f) < 1 \tag{5.18}$$

if and only if

$$A_2(e^f) < \infty.$$

Proof. Let us introduce, as in [GJ] and in [To], for $f \in BMO$

$$p(f) = \inf\{p > 1 : A_p(e^{\pm f}) < \infty\},$$

then by Lemma 1.4 in [GJ] one has

$$p(f) = \varepsilon(f) + 1 \le 2.$$

Hence (5.17) holds true.
From (5.16) we deduce that if $A_2(e^f) < \infty$, then

$$\varepsilon(f) \le \sqrt{\frac{A-1}{A}} < 1.$$

Conversely, if $\varepsilon(f) < 1$, there exists $\lambda_0 < 1$ such that

$$A_2(e^{f/\lambda_0}) < \infty.$$

In view of (5.12), (5.13) we obtain

$$s(f, \lambda_0) < \infty$$

and therefore $s(f, 1) < \infty$, which in turns implies $A_2(e^f) < \infty$. $\square$

5.5 Explicit bounds for the norm of composition operators acting on $BMO(\mathbb{R})$

In this section we improve a recent result of Gotoh [Go] who establishes a precise relation among constants in the P. W. Jones [Jo] Theorem about homeomorphisms of the line preserving BMO. We give also an explicit bound for the distance to L^∞ after composition (see [ACS]).

Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing homeomorphism. In the recent paper [Go], the relation between the norm of the operator

$$U : f \in BMO \longrightarrow f \circ h^{-1} \in BMO$$

and the A_∞-constants α, K of $\omega = h'$ according to Proposition 1.1 , was determined. The following Theorem gives an important relation between A_∞ and $BMO(\mathbb{R})$ that we need in the following.

Theorem 5.13. *[Jo] The following conditions are equivalent*

i) *There exists $c \geq 1$ such that*

$$\|f \circ h^{-1}\|_* \leq c\|f\|_*$$

for any $f \in BMO(\mathbb{R})$;

ii) $h' \in A_\infty$;

iii) $(h^{-1})' \in A_\infty$.

Theorem 5.14. *[Go] Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing homeomorphism, if h' verifies*

$$\frac{|I|}{|J|} \leq K \left(\frac{\int_I h' \, dx}{\int_J h' \, dx} \right)^\alpha$$

for any interval $J \subset \mathbb{R}$ and for each measurable set $I \subset J$, where $K \geq 1 \geq \alpha > 0$, then

$$\|f \circ h^{-1}\|_* \leq C \frac{K}{\alpha} \tag{5.19}$$

where $C > 0$ is some universal constant.

In the following Theorem we give an expression for the constant C in (5.19).

Theorem 5.15. *[ACS] Let h be an increasing homeomorphism from $\mathbb{R}$ into itself and assume that $\omega = h'$ verifies the A_∞ condition:*

$$\frac{\int_E \omega \, dx}{\int_I \omega \, dx} \leq K \left(\frac{|E|}{|I|} \right)^\alpha \tag{5.20}$$

for any interval $I \subset \mathbb{R}$ and for each measurable set $E \subset I$, where $K \geq 1 \geq \alpha > 0$. Then

$$\|f \circ h^{-1}\|_* \leq \frac{K}{\alpha} e^{2+\frac{2}{e}} \|f\|_* \tag{5.21}$$

for any $f \in BMO(\mathbb{R})$.

Proof. Following [Go], we fix the interval I and set $I' = h(I)$. It is worth noting that assumption (5.20) for $\omega = h'$ reads as

$$\frac{|h(E)|}{|h(I)|} \leq K \left(\frac{|E|}{|I|} \right)^{\alpha} \tag{5.22}$$

for E measurable, $E \subset I$. Fix $f \in BMO$ and set $g = f \circ h^{-1}$. By the John-Nirenberg Theorem, see Theorem 5.8, if we define for $t > 0$

$$E_t = \{x \in I : |f(x) - f_I| > t\}$$

we have

$$\frac{|E_t|}{|I|} \leq e^{1+\frac{2}{e}} \cdot e^{-\frac{2t}{e\|f\|_*}}. \tag{5.23}$$

On the other hand, let I' be an interval of $\mathbb{R}$, if we set

$$\mu(t) = |\{y \in I' : |g(y) - f_I| > t\}|$$

we have, by (5.22) and (5.23),

$$\mu(t) = |h(E_t)| \leq |h(I)| \cdot K \left(e^{1+\frac{2}{e}} \cdot e^{-\frac{2t}{e\|f\|_*}} \right)^{\alpha}. \tag{5.24}$$

By well known inequalities and identities from measure theory:

$$⨍_{I'} |g - g_{I'}| \leq 2 ⨍_{I'} |g - f_I| = \frac{2}{|I'|} \int_0^{\infty} \mu(t)\, dt \tag{5.25}$$

and by the simple calculations induced by (5.24)

$$\int_0^{\infty} \mu(t)\, dt \leq |I'| \cdot K e^{(1+\frac{2}{e})\alpha} \frac{e}{2\alpha} \|f\|_*$$

we arrive at the estimate

$$⨍_{I'} |g - g_{I'}| \leq \frac{K}{\alpha} e^{(2+\frac{2}{e})} \|f\|_*.$$

Taking supremum with respect to the intervals, we obtain (5.21). □

Now our aim is to give an explicit bound for the distance to L^{∞} after composition. Let us begin with the following Lemma which is in the same spirit as Theorem 2.7 in [JN1].

Lemma 5.16. *[ACS] Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be a homeomorphism such that $(h^{-1})' \in A_p$, $1 < p < \infty$. Let ω be a weight on $\mathbb{R}$ and set $A_2(\omega) = A$; then, for $0 \leq \sigma < \frac{1}{p}\sqrt{\frac{A}{A-1}}$ we have*

$$A_2(\omega^\sigma \circ h^{-1})^{\frac{1}{2}} \leq \left[A_p(h^{-1})'\right]^{\frac{1}{p}} \left[\frac{\sigma p A}{A - \sigma^2 p^2 (A-1)}\right]^\sigma . \tag{5.26}$$

The inequality is sharp.

Proof. We will use Theorem 1.18 which describes the so called optimal "self-improvement of exponents" property of the A_2 class. Let σ to be determined later and set

$$L = \fint_I (\omega \circ h^{-1}(x))^\sigma \, dx \fint_I \frac{1}{(\omega \circ h^{-1}(x))^\sigma} dx.$$

We make the change of variables $t = h^{-1}(x)$, $h^{-1}(I) = J$ in the first integral:

$$\frac{1}{|I|}\int_I \omega^\sigma \circ h^{-1}(x) \, dx \; = \; \frac{1}{|I|}\int_J \frac{\omega^\sigma(t)}{(h^{-1})'(h(t))} \, dt \leq$$

by Hölder's inequality

$$\leq \left(\frac{1}{|I|}\int_J \omega^{\sigma p}(t) dt\right)^{\frac{1}{p}} \left(\frac{1}{|I|}\int_J \frac{1}{[(h^{-1})'(h(t))]^{p'}}\right)^{\frac{1}{p'}} .$$

We change back to the x variable into the last integral, obtaining

$$\frac{1}{|I|}\int_J \frac{1}{[(h^{-1})'(h(t))]^{p'}} \, dt = \frac{1}{|I|}\int_I [(h^{-1})'(x)]^{1-p'} dx$$

hence, taking into account that $\dfrac{|I|}{|J|} = \fint_I (h^{-1})'$,

$$\frac{1}{|I|}\int_I \omega^\sigma \circ h^{-1}(x) \, dx \leq \left[\frac{|J|}{|I|}\right]^{\frac{1}{p}} \left(\fint_J \omega^{\sigma p}(t) dt\right)^{\frac{1}{p}} \left(\fint_I (h^{-1})'^{(1-p')}(x) \, dx\right)^{\frac{1}{p'}} \leq$$

$$\leq \left(\fint_J \omega^{\sigma p}(t) dt\right)^{\frac{1}{p}} \left[A_p(h^{-1})'\right]^{\frac{1}{p}} .$$

Similarly, the second factor in L can be majorized as follows

$$\fint_I (\omega^\sigma \circ h^{-1}(x))^{-\sigma} \, dx \leq \left(\fint_J \omega^{-\sigma p}(t) dt\right)^{\frac{1}{p}} \left[A_p(h^{-1})'\right]^{\frac{1}{p}}$$

and hence

$$L \leq \left[\fint_J \omega^{\sigma p} \fint_J \omega^{-\sigma p}\right]^{\frac{1}{p}} \left[A_p(h^{-1})'\right]^{\frac{2}{p}} .$$

Taking supremum with respect to J, we obtain

$$L \le [A_2(\omega^{\sigma p})]^{\frac{1}{p}} \left[A_p((h^{-1})')\right]^{\frac{2}{p}}$$

and, finally, taking supremum with respect to I on L

$$A_2(\omega^{\sigma} \circ h^{-1}) \le [A_2(\omega^{\sigma p})]^{\frac{1}{p}} \left[A_p((h^{-1})')\right]^{\frac{2}{p}}.$$

We now choose σ. From Theorem 1.18 it follows that, if $\tau = \sigma p < \sqrt{\frac{A}{A-1}}$, then $A_2(\omega^{\sigma p}) < \infty$. Then, we choose $\sigma < \frac{1}{p}\sqrt{\frac{A}{A-1}}$ and (1.24) gives

$$[A_2(\omega^{\sigma p})]^{\frac{1}{p}} \le \left[\frac{\sigma p A}{A - \sigma^2 p^2 (A-1)}\right]^{2\sigma}.$$

It remains to show that the inequality (5.26) is sharp. This is a consequence of the choice $h(t) = t$ which reduces (5.26) to the form

$$A_2(\omega^{\sigma p})^{1/\sigma p} \le \frac{\sigma p A}{A - \sigma^2 p^2 (A-1)}.$$

which agrees with the sharp implication in Theorem 1.18. $\square$

Let us now consider the functional $\varepsilon(f) = \inf I_f$ where $f \in BMO$ and I_f is defined by (5.8). From Proposition 5.9 and Theorem 5.7 we know that f belongs to BMO if and only if I_f is not empty and that $\varepsilon(f)$ is equivalent to the distance functional

$$dist_{BMO}(f, L^{\infty}) = \inf_{g \in L^{\infty}} \|f - g\|_*.$$

Let us prove the following:

Theorem 5.17. *[ACS] Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing homeomorphism such that $(h^{-1})'$ belongs to the A_p-class. Then for any $f \in BMO(\mathbb{R})$*

$$\varepsilon(f \circ h^{-1}) \le p\, \varepsilon(f). \tag{5.27}$$

Moreover, there exists an equivalent norm $\|\cdot\|_*'$ on BMO such that

$$dist'_{BMO}(f \circ h^{-1}, L^{\infty}) \le p\, dist'_{BMO}(f, L^{\infty}). \tag{5.28}$$

Proof. Fix $\lambda \in I_f$ and set $A_\lambda = A_2(e^{f/\lambda})$. Let us prove that

$$\varepsilon(f \circ h^{-1}) \le \lambda\, p \sqrt{\frac{A_\lambda - 1}{A_\lambda}}. \tag{5.29}$$

By previous lemma, with $\omega = e^{f/\lambda}$ we deduce that for $0 \le \sigma < \frac{1}{p}\sqrt{\frac{A_\lambda}{A_\lambda - 1}}$ one has

$$A_2\left(e^{\dfrac{\sigma f \circ h^{-1}}{\lambda}}\right) < \infty.$$

In other words, for $\mu > \lambda\ p\sqrt{\frac{A_\lambda - 1}{A_\lambda}}$, μ belongs to the set $I_{f \circ h^{-1}}$ and this immediately implies (5.29).

Let us recall that actually (see Theorem 5.11)

$$\varepsilon(f) = \inf\{\lambda\sqrt{\frac{A_\lambda - 1}{A_\lambda}} : \lambda \in I_f\}.$$

Then by (5.29) we get (5.27).

Let us note that if h is an increasing homeomorphism such that $(h^{-1})' \in A_1$ and also $h' \in A_1$, then inequality (5.27) reduces to the optimal identity

$$\varepsilon(f \circ h^{-1}) = \varepsilon(f)$$

for any $f \in BMO$. In this sense our result is sharp. In fact we benefit of the coupled inequality to (5.27)

$$\varepsilon(g \circ h) \le p\, \varepsilon(g)$$

for any $g \in BMO$ and for any $p > 1$. Passing to the limit in both inequalities we obtain the stated identity.

Now let us observe that, since $(h^{-1})'$ belongs to A_p, in particular it belongs to A_∞ and then by Theorem 5.13 there exists $c > 0$ such that

$$\|f \circ h^{-1}\|_* \le c\, \|f\|_* \tag{5.30}$$

for any $f \in BMO$. Now it is a routine matter to see that

$$dist_{BMO}(f \circ h^{-1}, L^\infty) \le c\, dist_{BMO}(f, L^\infty) \tag{5.31}$$

with the same constant c than in (5.30), for any $f \in BMO$. To this end, we note that for any $f, g \in BMO$ (5.30) implies that

$$\|f \circ h^{-1} - g \circ h^{-1}\|_* \le c\, \|f - g\|_*. \tag{5.32}$$

If we restrict ourselves to $g \in L^\infty$ by (5.32) we deduce

$$dist_{BMO}(f \circ h^{-1}, L^\infty) \le \|f \circ h^{-1} - g \circ h^{-1}\|_*. \tag{5.33}$$

In view of (5.32),(5.33) we conclude with (5.31). By means of Theorem 5.7 and Theorem 5.8 we deduce the inequality

$$\varepsilon(f \circ h^{-1}) \leq c\, k_2 \, \frac{e}{2} \, \varepsilon(f)$$

which is largely less precise than (5.27).

To prove (5.28) remember ([Ga], p. 258) that, if H denotes the Hilbert transform:

$$Hg(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{g(y)}{x-y} \, dy,$$

and $\varphi \in BMO$, then $\varphi = f + Hg + \alpha$ with $f \in L^\infty$, $g \in L^\infty$ and α constant, and

$$\|\varphi\|'_* = \inf\{\|f\|_\infty + \|g\|_\infty : \varphi = f + Hg + \alpha\} \tag{5.34}$$

defines a norm on BMO equivalent to $\|\varphi\|_*$. Now if we set

$$dist'_{BMO}(\varphi, L^\infty) = \inf_{\psi \in L^\infty} \|\varphi - \psi\|'_*$$

the identity

$$dist'_{BMO}(\varphi, L^\infty) = \frac{\pi}{2} \varepsilon(\varphi) \tag{5.35}$$

holds for any $\varphi \in BMO$ ([Ga], Corollary 6.6). If we equipe BMO with the norm (5.34) in view of (5.27), (5.28) holds. $\square$

Chapter 6

BMO-Martingale and probabilistic A_p-condition

The theory of martingales is a powerful tool for studying properties of stopping times which are of great importance in risk theory, mathematical finance, statistical sequential analysis etc. We will illustrate some of classical results and some recent developments in the theory of martingales and its applications.

The origin of martingale lies in the history of games of chance. Martingale referred to a class of betting strategies that was popular in 18th century France.

The concept of martingale in probability theory was introduced by Paul Pierre Lvy, and much of the original development of the theory was done by Joseph Leo Doob.

The Martingale was long considered to be a necessary condition for an efficient asset market, one in which the information contained in past prices is instantly, fully and perpetually reflected in the asset's current price.

This chapter is articulated in this way: in the first section we recall some classical definitions of Theory of Probability; in the second section we define martingale as a process stochastic. Subsequently, we introduce the BMO-martingale space and its relationships with a probabilistic version of A_p-condition. Finally, we conclude with an application of BMO-martingales in Mathematical Finance.

6.1 Preliminary definitions

Definition 6.1. In probability theory, the **sample space** or universal sample space, Ω of an experiment or random trial is the nonempty set of all possible outcomes or states of nature and are often given the symbol ω. For example, if the experiment is tossing a coin, the sample space is the set {head, tail}. For tossing a single six-sided die, the sample space is {1, 2, 3, 4, 5, 6}. Any subset of the sample space is usually called an **event**. More precisely, an event is a set of outcomes to which a probability is assigned.

Definition 6.2. A **σ-algebra** $\mathcal{F}$ over a set Ω is a nonempty collection of subsets of Ω that is closed under complementation and countable unions of its members. It is a Boolean algebra, completed to include countably infinite operations.

Formally, a σ-algebra $\mathcal{F}$ is characterized by the following properties:

(i) $\emptyset \in \mathcal{F}$

(ii) $F \in \mathcal{F} \Longrightarrow F^C \in \mathcal{F}$, where $F^C = \Omega \backslash F$

(iii) $A_n \in \mathcal{F}, n \in \mathbb{N} \Longrightarrow A := \bigcup\limits_{n \in \mathbb{N}} A_n \in \mathcal{F}$.

Elements of the σ-algebra are called **measurable sets**. An ordered pair $(\Omega, \mathcal{F})$, where Ω is a set and $\mathcal{F}$ is a σ-algebra over Ω, is called a measurable space. If U is an arbitrary family of subsets of Ω then we can form a special σ-algebra from U, called the σ-algebra generated by U. We denote it by $\sigma(U)$ and it is the smallest σ-algebra over Ω that contains U.

Definition 6.3. The **probability measure** P is a function $P : \mathcal{F} \to [0, 1]$, that assigns to each event a probability between 0 and 1. It must satisfy the probability axioms:

(a) $P(\emptyset) = 0,\ P(\Omega) = 1$,

(b) if $A_n \in \mathcal{F}, n \in \mathbb{N}$, with $A_n \cap A_m = \emptyset, n \neq m$ then $P(\bigcup\limits_{n \in \mathbb{N}} A_n) = \sum\limits_{n \in \mathbb{N}} P(A_n)$.

Because P is a function defined on $\mathcal{F}$ and not on Ω, the set of events is not required to be the complete power set of the sample space; that is, not every set of outcomes is necessarily an event.

The triple $(\Omega, \mathcal{F}, P)$ is called **probability space**. The subsets I of Ω which belong to $\mathcal{F}$ are called $\mathcal{F}$-measurable sets. In a probability context these sets are called events and we use the interpretation

$$P(I) = \text{“the probability that the event I occurs”}.$$

In particular, if $P(I) = 1$ we say that “I occurs with probability 1”, or “almost surely (a.s.)”.

Definition 6.4. If $(\Omega, \mathcal{F}, P)$ is a given probability space, then the function $Y : \Omega \longrightarrow \mathbb{R}^n$ is called **$\mathcal{F}$-measurable** if

$$Y^{-1}(U) := \{w \in \Omega; Y(w) \in U\} \in \mathcal{F},$$

for every open sets $U \in \mathbb{R}^n$.

Definition 6.5. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Then a **random variable** X is formally defined as a $\mathcal{F}$-measurable function,

$$X : \Omega \longrightarrow \mathbb{R}^n \qquad\qquad X : w \rightarrow X(w).$$

An interpretation of this is that the preimage of the “well-behaved” subsets of $\mathbb{R}^n$ are events (elements of $\mathcal{F}$), and hence are assigned a probability by P.

Definition 6.6. In probability theory the **expected value** (or mathematical expectation, or mean) of a discrete random variable is the sum of the probability of each possible outcome of the experiment multiplied by the outcome value (or payoff). In general, if X is a random variable defined on a probability space $(\Omega, \mathcal{F}, P)$, then the expected value of X (denoted $E(X)$ or sometimes or $\langle X \rangle$) is defined as

$$E(X) = \int_\Omega X dP.$$

Note that not all random variables have an expected value, since the integral may not exist.

Definition 6.7. In probability theory, a **conditional expectation** (also known as conditional expected value) is the expected value of a real random variable with respect to a conditional probability distribution.

Thus, if X is a random variable, and A is an event whose probability is not 0, then the conditional probability distribution of X given A assigns a probability $P(X = x \mid A)$ to the interval $(-\infty, x]$, and we have a conditional probability distribution, which may have a first moment, called $E(X|A)$, the conditional expectation of X given the event A.

The conditional expectation of X given random variable Y, denoted by $E(X|Y)$, is another random variable, obtained essentially by averaging the random variable X down to the granularity of the random variable Y. The expected value has the property of monotonicity and linearity.

Definition 6.8. Given a complete probability space $(\Omega, \mathcal{F}, P)$, a **filtration** $\mathcal{F}_t,\ t \geq 0$, is an increasing sequence of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$ such that

- $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$
- $\mathcal{F}_t = \bigcap\limits_{u>t} \mathcal{F}_u$ for all $t \geq 0$.

A $\sigma-$algebra represents the information known until "t".

Now, we are able to define a stochastic process, which is the counterpart to a deterministic process. In fact, there is not a unique way of how the process might evolve under time (as is the case, for example, for solutions of an ordinary differential equation); in a stochastic or random process there is some indeterminacy in its future evolution, described by probability distributions. This means that even if the initial condition is known, there are many possibilities the process might go to, but some paths are more probable and others less.

Familiar examples of processes modeled as stochastic time series include stock market and exchange rate fluctuations, signals such as speech, audio and video, medical data, blood pressure or temperature, and random movement such as Brownian motion or random walks. We give a more rigorous definition.

Definition 6.9. Given a probability space $(\Omega, \mathcal{F}, P)$, a **stochastic process** is a collection of $\mathbb{R}^n$-valued random variables indexed by a set T, denoted by $\{X_t\}_{t \in T}$. It can be expressed by the form

$$X = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in T}, (X_t)_{t \in T}, P)$$

where $\mathcal{F}_t$ is the filtration and X_t is $\mathcal{F}_t$-measurable.

Usually $T \subset \mathbb{R}^+$ and $t \in T$ is seen as "time".

6.2 Martingales

The theory of martingales plays a very important an useful role in the study of stochastic processes. A martingale is a stochastic process (i.e., a sequence of random variables) such that the conditional expected value of an observation at some time t, given all the observations up to some earlier time s, is equal to the observation at that earlier time s. Precise definitions are given below.

Definition 6.10. A **discrete-time martingale** is a discrete-time stochastic process, i.e. a sequence of random variable $M_1, M_2, M_3, \ldots$ that satisfies for all t

$$E(|M_t|) < +\infty;$$

$$E(M_{(t+1)}|M_1, \ldots, M_t) = M_t$$

i.e., the conditional expected value of the next observation, given all the past observations, is equal to the last observation.

Definition 6.11. Similarly, a **continuous-time martingale** with respect to the stochastic process X_t is a stochastic process M_t such that for all t

$$E(|M_t|) < +\infty;$$

$$E(M_t|\{X_\tau, \tau \leq s\}) = M_s, \qquad \forall s \leq t.$$

Definition 6.12. In full generality, a stochastic process M is a **martingale** with respect to a filtration $\mathcal{F}_t$ and probability measure P if

1. M is adapted to the filtration, i.e. each M_t is $\mathcal{F}_t$-measurable for all t;

2. M_t is integrable for all t, i.e. $E(|M_t|) < +\infty$, for all t;

3. $E(M_{t+1}|\mathcal{F}_t) = M_t$ for all t.

A related notion is that of a super or sub-martingale. If, in the definition of a martingale, we replace the equality in 3. by an inequality we get super or sub-martingales.
For a **sub-martingale** we demand the relation for every t,

$$M_t \leq E(M_{t+1}|\mathcal{F}_t),$$

while for a **super-martingale** the relation is for every t

$$M_t \geq E(M_{t+1}|\mathcal{F}_t).$$

Definition 6.13. An adapted process $X = (X_t, \mathcal{F}_t)$ is said to be a **semi-martingale** if X_t can be written as $M_t + A_t$, where M is a local martingale and A is a stochastic process that is locally of bounded variation.

Semimartingales are good integrators, forming the largest class of processes with respect to which the Ito integral (see definition (6.15))can be defined. Examples of semimartingales are all continuously differentiable processes, Brownian motion and Poisson processes. Note that sub-martingales and super-martingales are semimartingales.

The most important (continuous) stochastic process which is a martingale is Brownian motion, which gets its name from the botanist Robert Brown (1828). In fact, he observed how particles of pollen suspended in water moved erratically on a microscopic scalefirst moving in one direction and then zig zagging in another. The motion was caused by water molecules randomly buffeting the particle of pollen.

Later the one-dimensional Brownian motion was used by Louis Bachelier around 1900 in finance for modeling random behavior that evolves over time, as fluctuations in an asset's price. A first rigorous proof of its (mathematical) existence was given by Norbert Wiener in 1921 and for this reason it is called Brownian motion o Wiener process.

Definition 6.14. Let $(W_t, \mathcal{F}_t)_{t\in T}$ be an $\mathbb{R}$-valued continuous stochastic process on $(\Omega, \mathcal{F}, P)$. Then $(W_t, \mathcal{F}_t)_{t\in T}$ is called a **standard Brownian motion** if

- $W_0 = 0$
- W_t is almost surely continuous
- W_t has independent increments with distribution $W_t - W_s \sim N(0, t-s)$ (for $0 = s < t$).

$N(\mu, \sigma^2)$ denotes the normal distribution with expected value μ and variance σ^2.

Now we give a definition of stochastic integral, born to define a new integral for stochastic processes. It extends the methods of calculus to stochastic processes such as Brownian motion and has important applications in mathematical finance and stochastic differential equations.

Definition 6.15. The Ito stochastic integral can be written in this way:

$$I_t = \int_0^t H_s dW_s,$$

where W is a Brownian motion or, more generally, a semimartingale and H is a locally bounded predictable process. The paths of Brownian motion fail to satisfy the requirements to be able to apply the standard techniques of calculus because it is not differentiable at any point and has infinite variation over every time interval. Then, the integral cannot be defined in the usual way as Riemann-Stieltjes integral but the integral can be defined as long as the integrand H is adapted, which means that its value at time t can only depend on information available up until this time.

Definition 6.16. Let $(\Omega, \mathcal{F})$ be a measurable space and $(\mathcal{F}_t)_{t\in T}$ be a filtration. Then, an $\mathcal{F}-$measurable random variable $\tau : \Omega \longrightarrow T \cup \{+\infty\}$ is said to be a **stopping time** with respect to $\mathcal{F}_t$, if for all $t \in T$

$$\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t.$$

A stopping time may actually take the value ∞ on a nonempty subset of Ω. Moreover, τ is a **finite stopping time** if

$$P(\tau(\omega) < +\infty) = 1.$$

The idea behind the definition of a stopping time is that the decision to stop at time t can be based only on the information available up to that time.

An example in real life might be the time at which a gambler leaves the gambling table, which might be a function of his previous winnings (for example, he might leave only when he goes broke), but he can't choose to go or stay based on the outcome of games that haven't been played yet.

The following theorem called Doob's optional stopping theorem (or optional sampling theorem) is one of central facts in the theory of martingales sequences. It says that, under certain conditions, the expected value of a martingale at a stopping time is equal to its initial value.

Theorem 6.1. ***Doob's optional stopping theorem***
Let $M_t : t \geq 0$ be a sequence of random variables defined on a probability space $\Omega, \mathcal{F}, \mathbb{P}$, which is a martingale sequence with respect to the filtration $\mathcal{F}_t$ and $0 \leq \tau_1 \leq \tau_2 \leq C$ be two bounded stopping times. Then

$$E(|M_{\tau_2}| \mid \mathcal{F}_{\tau_1}) = M_{\tau_1}.$$

Stopping times are frequently used to generalize certain properties of stochastic processes to situations in which the required property is satisfied in only a local sense.

Definition 6.17. Let $\{M_t\}$ be a martingale and τ be a stopping time adapted a filtration $\{\mathcal{F}_t\}$ and let $t \wedge \tau$ denote $\min\{t, \tau\}$; note that $t \wedge \tau$ is also a stopping time. Then, the process $M_t^\tau = M_{t\wedge\tau}$ is a martingale, and we say that $\{M_t^\tau\}$ is the martingale $\{M_t\}$ stopped at t.

Definition 6.18. A martingale M_t is said **local martingale** if there exists a sequence of stopping times $\tau_n : \Omega \longrightarrow [0, \infty)$ increasing to infinity, such that the stopped martingale $M_{t\wedge\tau_n}$ is a martingale for each n. Such a sequence (τ_n) of stopping times is called **fundamental sequence**.

Definition 6.19. A stopping time τ is **predictable** if it is equal to the limit of an increasing sequence of stopping times τ_n satisfying $\tau_n < \tau$ whenever $\tau > 0$.

We assume that all local martingales with respect to the filtration $\mathcal{F}_t$ are continuous. It's important to observe that the following properties are equivalent [ESY]:

1. any local martingale is continuous,

2. any stopping time is predictable,

3. for every stopping time τ and every $\mathcal{F}_t$-measurable random variable U, there exists a continuous local martingale M with $M_\tau = U$ a.s..

6.3 BMO-martingale and probabilistic A_p condition

In the Chapter 5 we have define the BMO space in the analytic context; now we will see it in the probabilistic setting and show some of exciting results about BMO in the theory of exponential local martingales.

Definition 6.20. Let M be a uniformly integrable $\mathcal{F}_t$-martingale satisfying $M_0 = 0$. For $1 \leq p < \infty$ set

$$\|M\|_{BMO_p} = \sup_{\tau} \|E[|M_\infty - M_\tau|^p|\mathcal{F}_\tau]^{\frac{1}{p}}\|_\infty$$

where the supremum is taken over all stopping times τ. The normed linear space $\{M : ||M||_{BMO_p} < \infty\}$ with norm $||M||_{BMO_p}$ is denoted by BMO$_p$.

In particular, for $p = 1$, we have

$$\|M\|_{BMO} = \sup_{\tau} \|E[|M_\infty - M_\tau||\mathcal{F}_\tau]\|_\infty,$$

and when $M \in BMO$, M is said BMO**-martingale**.

Note that, if $M \in BMO$ and τ is a stopping time, then $M^\tau \in BMO$ and $||M^\tau||_{BMO} \leq ||M||_{BMO}$. It can be shown that for any $p, q \in [1, \infty)$ we have $BMO_p = BMO_q$ (see [Ka]). Therefore we will often omit the index and simply write BMO for the set of BMO martingales.

Moreover, the following result holds

Corollary 6.2. *[Ka] Let $1 < p < \infty$. There is a positive constant C_p depending only on p, such that for any uniformly integrable martingale M*

$$\|M\|_{BMO} \le \|M\|_{BMO_p} \le C_p \|M\|_{BMO}.$$

Now, let L_∞ be the class of all bounded martingales and let H_∞ be the class of all martingales M such that $\langle M\rangle_\infty$ is bounded. Since $||M||_{BMO} \le 2||M||_\infty$ and $||M||_{BMO_2} \le ||\langle M\rangle_\infty||_\infty^{1/2}$, these two classes L_∞ and H_∞ are contained in BMO.

The following Theorem is the John-Nirenberg inequality in BMO-martingales space.

Theorem 6.3. *[Ka] (John-Nirenberg inequality) If $\|M\|_{BMO} < \frac{1}{4}$, then for any stopping time τ*

$$E[\exp(|M_\infty - M_\tau|)|\mathcal{F}_\tau] \le \frac{1}{1 - 4\|M\|_{BMO}}. \tag{6.1}$$

The next inequality, which is also called the John-Nirenberg inequality, was given by Garsia [Gar] for discrete parameter martingales and by Meyer [Me] for general martingales.

Theorem 6.4. *If $\|M\|_{BMO_2} < 1$, then for every stopping time τ*

$$E[\exp(\langle M\rangle_\infty - \langle M\rangle_\tau)|\mathcal{F}_\tau] \le \frac{1}{1 - \|M\|_{BMO_2}^2}. \tag{6.2}$$

The following Remark shows the connection between BMO-functions and BMO-martingales.

Remark 6.1. *[Ka] Let $D = \{z : |z| < 1\}$ be the unit disc in the complex plane, ∂D its boundary and $m(d\theta)$ the normalized Lebesgue measure on ∂D. An integrable real valued function f is in $BMO(\mathbb{R})$ if there exists a positive constant C such that for all intervals $I \subset \partial D$,*

$$\frac{1}{m(I)} \int_I |f - f_I| \, m(d\theta) \le C,$$

where $f_I = \frac{1}{m(I)} \int_I f \, dm$ and the smallest constant with the previous property is denoted by $\|f\|_*$ is the BMO-norm of a function. Now, let

$$h(z) = \int_0^{2\pi} f(t) P(r, \theta - t) \, m(dt) \quad (z = re^{i\theta} \in D),$$

where $P(r,\eta) = \frac{1-r^2}{1-2r\cos(\eta)+r^2}$ *is the Poisson kernel. Then* h *is the harmonic function in* D *with boundary function* f*. Let now* $B = B(B_t, \mathcal{F}_t)$ *be the complex Brownian motion starting at 0 and let* $\tau = \inf\{t : |B_t| = 1\}$*. The process* $(h(B_{t\wedge\tau}), \mathcal{F}_{t\wedge\tau})$ *is a uniformly integrable martingale. In particular, if* f *is in* BMO*, then the process* $h(B^\tau)$ *is a* BMO*-martingale and there are constants* $C_1, C_2 > 0$*, independent of* f*, such that*

$$C_1\|f\|_* \leq \|h(B^\tau)\|_{BMO} \leq C_2\|f\|_{BMO}.$$

Conversely, if X *is a uniformly integrable martingale adapted to the filtration* $(\mathcal{F}_{t\wedge\tau})$*, then there is a unique Borel measurable function* f *defined on* ∂D *such that* $f(B_\tau) = E[X_\infty|\sigma(B_\tau)]$*. Let us consider the mapping* $J : X \longrightarrow f$*. Then there is a constant* C *such that*

$$\|J(X)\|_* \leq C\|X\|_{BMO}$$

for all BMO*-martingales* X *adapted to the filtration* $(\mathcal{F}_{t\wedge\tau})$*. The family of all real-valued* BMO*-functions on* ∂D *is identified in this way with the family of all* BMO*-martingales* X *which have* X_∞ *measurable with respect to* $\sigma(B_\tau)$*.*

Now, we give the definition of Exponential martingale.

Definition 6.21. For a continuous local martingale M, with quadratic variation $\langle M\rangle$, we define **exponential local martingale** $\mathcal{E}(M)$ as

$$\mathcal{E}(M)_t := \exp\left(M_t - \frac{1}{2}\langle M\rangle_t\right) \quad 0 \leq t < \infty.$$

with $\mathcal{E}(M)_0 = 1$ and where $\langle M\rangle_t$ denotes the increasing process associated with M_t. Moreover, $\mathcal{E}(M)$ solves the "stochastic differential equation" (SDE)

$$d\mathcal{E}(M)_t = \mathcal{E}(M)_t dM_t, \qquad t \geq 0,$$

with initial condition $\mathcal{E}(M)_0 = 1$.

It plays an essential role in various questions concerning the absolutely continuity of probability laws of stochastic processes.

We observe that $\mathcal{E}(M)$ is a local martingale, it is a martingale if and only if $E[\mathcal{E}(M)_t] = 1$ for all $t > 0$. In fact, generally, we have $E[\mathcal{E}(M)_t] \leq 1$ for every t.

However, $\mathcal{E}(M)$ is not always a uniformly integrable martingales and verify this is often difficult.

The following Theorem gives a sufficient condition to have a uniformly integrable martingale. This fact is very useful when we have to solve mathematical finance problems.

Theorem 6.5. *[Ka] Let M be a martingale in BMO, then the stochastic exponential $\mathcal{E}(M)$ is an uniformly integrable martingale.*

Now, we give an analogous definition of A_p-condition in probabilistic setting.

Definition 6.22. Let $1 < p < \infty$. We say that the stochastic exponential $\mathcal{E}(M)$ satisfies **A_p-condition** if

$$\sup_{\tau} \|E[\{\mathcal{E}(M)_\tau/\mathcal{E}(M)_\infty\}^{1/(p-1)}|\mathcal{F}_\tau]\|_\infty < \infty,$$

where the supremum is taken over all stopping times τ. Particularly, if $p = 1$

$$\sup_{\tau} \|\mathcal{E}(M)_\tau/\mathcal{E}(M)_\infty\|_\infty < \infty.$$

We observe that the previous A_p-condition is a probabilistic version of the Muckenhoupt's condition described in the first chapter.

The next result is similar to Lemma (1.2).

Lemma 6.6. *[Ka] Let $1 < p < \infty$. If $\mathcal{E}(M)$ satisfies A_p condition, then it satisfies also $A_{p-\varepsilon}$ for some ε with $0 < \varepsilon < p - 1$.*

In the next Theorem it is shown the connection between A_p and BMO-martingale.

Theorem 6.7. *[Ka] The following conditions are equivalent.*

(a) *$M \in BMO$.*

(b) *$\mathcal{E}(M)$ satisfies A_p-condition for some $p \geq 1$.*

(c) $\sup_{\tau} \left\| E[\log^+ \frac{\mathcal{E}(M)_\tau}{\mathcal{E}(M)_\infty}|\mathcal{F}_\tau] \right\|_\infty < \infty.$

6.4 The distance in BMO to L_∞

In this section we see comparable upper and lower bounds for the distance in BMO to L_∞ (class of all bounded martingales) in probabilistic setting.

For $M \in BMO$-martingales, let $a(M)$ be the infimum of the set of $a > 0$ for which

$$\sup_{\tau} \|E[\exp(a|M_\infty - M_\tau|)|\mathcal{F}_\tau]\|_\infty < \infty\},$$

where the supremum is taken over all the stopping times τ and let $dist_{BMO}(\ ,\ )$ be the distance on the space BMO deduced from the norm $\|\cdot\|_{BMO}$, by usual procedure.

Then, there is a very beautiful relation between $a(M)$ and $dist_{BMO}(M, L_\infty)$:

Theorem 6.8. *([Va], [E]) Let $M \in BMO$ be a martingale, then we have*

$$\frac{dist_{BMO}(M, L_\infty)}{4} \le a(M) \le \frac{dist_{BMO}(M, L_\infty)}{4}. \tag{6.3}$$

Note that Theorem 6.8 is the probabilistic version of Theorem 5.7 where $a(M)$ "plays the role" of $\varepsilon(f)$.

This result was originally obtained in 1978 by J. Garnett and P. Jones ([GJ]) in classic analysis. In 1980, N. Th. Varopoulo established the probabilistic version for Brownian martingales and in 1985 M. Emery proved it for continuous martingales.

Dellacherie, Meyer and Yor proved in [DMY] that L_∞ is neither closed nor dense in BMO whenever $BMO \neq L_\infty$. In the classical setting Garnett and Jones [Ga] proved for locally integrable function f on $\mathbb{R}^n$ that

$$f \in BMO - closure \text{ of } L_\infty \Longleftrightarrow e^f, e^{-f} \in A_p, \ \ \forall p > 1.$$

Now we report a probabilistic analogue of this result. For a uniformly integrable martingale M, let

$$p(M) = \inf\{p > 1 : E[\exp(M_\infty)|\mathcal{F}], E[\exp(-M_\infty)|\mathcal{F}] \in A_p\}. \tag{6.4}$$

From Hölder inequality it follows that $E[\exp(M_\infty)|\mathcal{F}]$ satisfies A_p, for all $p > p(M)$.

Lemma 6.9. *[Ka] If $p(M) < \infty$, then $p(M) \le 2$, $M \in BMO$ and*

$$p(M) - 1 = a(M).$$

Theorem 6.10. *[Ka] Let $M \in BMO$, then M belongs to the BMO-closure of $L_\infty \iff E[\exp(M_\infty)|\mathcal{F}],\ E[\exp(-M_\infty)|\mathcal{F}] \in A_p$, for all p.*

6.5 Sharp estimates on the norm of the martingale transform

In the previous chapters, we have investigated some sharp estimates for singular integrals, which are one of the main subject of Harmonic Analysis. Now we briefly see the martingale transforms, which serve as a good analog of singular integrals. The discrete (martingale) approach very often gives the technical key to the continuous (singular operator) case. In its turn, for example, the weighted estimate of the Hilbert transform with matrix weight is equivalent to finding important regularity properties of vector (=multivariate) stationary stochastic processes.

Let $(M_t)_{t\geq 0}$ be a sequence, denoted by M, of a real integrable functions on a probability space $(\Omega, \mathcal{F}, P)$ and $(Y_t)_{t\geq 0}$ its difference sequence : $M_t = \sum_{k=0}^{t} Y_k$, $t \geq 0$. Then, if for all $t \geq 1$ the expectation of a product of Y_t and $\varphi(Y_0, ..., Y_{t-1})$ is zero for all real bounded continuous functions φ on $\mathbb{R}^n$, then M is a martingale. We can define the **martingale transform** M' of M as $M'_t = \sum_{k=0}^{t} \varepsilon_k Y_k$, where $(\varepsilon_t)_{t\geq 0}$ is a sequence of numbers $\varepsilon_t \in \{-1, 1\}$.

Notice that M' is also a martingale. Moreover, it is important to note that M_t may be sums of independent random variables with mean zero. But the independence of the increments may be destroyed and M'_t will in general no longer have the independent increments property.

The boundedness of singular integral operators in $L^2(\omega)$ for $\omega \in A_2$ has been known for a long time, by the Hunt-Muckenhaupt-Wheeden Theorem and in the Chapter 4 we have seen some of results of optimal estimate for the growth of the norm of some operators.

Now, we show some result, see [Wt] and [DPV] for martingale transforms, but before, we need to recall some their notations.

We consider the standard *dyadic lattice* as the family of intervals $\mathcal{L}$:=

$\{[m2^n, (m+1)2^n;\ m, n \in \mathbb{Z}\}$. Each interval $I \subset \mathbb{R}$ gives to its *Haar function*, denote by h_I:

$$h_I := \frac{\chi_{I_l} - \chi_{I_r}}{\sqrt{|I|}}$$

where I_l, I_r denote the left and right children of I respectively, and χ_E stands for the characteristic function of the set E. Denote by $\mathcal{L}(I)$ the set of all dyadic subintervals of the interval I, including I itself.

Hence, we consider the martingale transform T_σ as the operator defined by

$$T_\sigma f := \sum_{J \in \mathcal{L}} \sigma_I \langle f, h_J \rangle h_J,$$

where σ_I assumes the values $+1$ and -1 only.

One of the problem is to find an estimate on the norm $||T_\sigma f||$.

For $p = 2$, Wittwer [Wt] showed that the the martingale transform is bounded linearly in $A_2(\omega)$, Muckenhoupt constant of a weight ω.

Theorem 6.11. *[Wt]*

$$||T_\sigma f||_{L^2(\omega)} \leq c A_2(\omega) ||f||_{L^2(\omega)}$$

for any $\omega \in A_2$ and $f \in L^2(\omega)$.

Subsequently, Dragičević, Petermichl and Volberg [DPV], generalized the Burkholder's sharp estimate [Bur], obtained a sharp L^p bounds for martingale transform.

Theorem 6.12. *[DPV] For every σ and every $1 < p < \infty$,*

$$||T_\sigma f||_p \leq p^* - 1.$$

The motivation of their work was to search for L^p estimates for the Ahlfors-Beurling operator T.

6.6 Mathematical Finance: the Black-Scholes model

In this section we give briefly, thanks to Geiss's paper, some ideas about the possible applications in Mathematical Finance of previous results, without giving precise definitions and theorems. The field of Mathematical Finance has undergone a remarkable development since the seminal papers by F. Black and M. Scholes [BS] and R. Merton [M], in which the famous *Black-Scholes Option Pricing Formula* was derived.

In 1997 the Nobel prize in Economics was awarded to R. Merton and M. Scholes for this achievement, thus also honoring the late F. Black.

The idea of developing a *formula* for the price of an option goes back as far as 1900, when L. Bachelier firstly had the innovative idea of using a stochastic process as a model for pricing an option.

In this chapter we will consider the so-called European call option.

Definition 6.23. An **European call option** is generally defined as a contract between two parties in which one has the right but not the obligation to buy (call) one unit of the underlying stock at a fixed time T (time horizon) and at a fixed price K (the strike price). Moreover, the buyer pays a price for this right.

This determines the value C_T of the option (payoff) at time T as a function of the (unknown) value S_T of the stock at time T, namely

$$C_T = (S_T - K)_+.$$

We note that the option is worthless (at time T) if $S_T \leq K$, and is worth the difference between S_T and K if $S_T > K$.

In 1965, P.R. Samuelson, Nobel prize winner, proposed as a model for a stock price process the **geometric Brownian motion** with parameters $\mu \in \mathbb{R}$ (*drift*) and $\sigma^2 > 0$ (*volatility*) obeying the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{6.5}$$

where $(W_t)_{0 \leq t \leq T}$ is a standard Brownian motion starting at $W_0 = 0$.

Here, μ is the average growth rate and volatility is the variable that determines the magnitude of random changes in short-term interest rates. In other words, to greater the volatility corresponds a greater range in which interest rates are likely to fluctuate.

Using Ito's formula one finds that the solution to (6.5) is given by the process

$$S_t = S_0 \exp\left[(\mu - \frac{\sigma^2}{2})t + \sigma W_t\right] \tag{6.6}$$

which is called geometric Brownian motion and describes the price evolution of a stock. Hence, S_t follows a log-normal process i.e.

$$ln\ S_t \sim N\left[ln\ S_0 + (\mu - \frac{\sigma^2}{2})t, \sigma^2 t\right],$$

while $\ln S_t/S_0$ follows normal distibution with expectation $(\mu - \frac{\sigma^2}{2})t$ and variance $\sigma^2 t$.

This model is today one of the standard reference model to describe the price evolution of a stock; although promoted by Samuelson, it now is often called the **Black-Scholes model**. The mathematical structure of the problem and its connections to martingale theory were subsequently worked out and clarified by J.M. Harrison and D.M. Kreps; a detailed account can be found in Harrison/Pliska (1981).

6.7 The approximation of stochastic integrals and weighted BMO

The seller, with the received payment for option, can realize an replicating portfolio in order to minimize the risk. A market in which is possible to replicate an option is said to be a **complete market**. Completeness is a rather delicate property which typically gets lost if one considers even minor modifications of a basic complete model. For instance, in the classical Black-Scholes model, the geometric brownian motion becomes incomplete if the volatility is influenced by a second stochastic factor or if one adds a jump component to the model.

However, this portfolio will be construct entirely self financing and thus deterministic (non stochastic) and it will have the same payoff as the call option at expiration and therefore, by the fundamental theorem of finance, the portfolio value must equal the call option value. An economically very reasonable assumption on a financial market consists of requiring that there are no arbitrage opportunities, i.e. opportunity for risk-free profits. It states that two equivalent goods in the same competitive market must have the same price. The **principle of no arbitrage** is a concept of central importance to the theory, which allows to determinate a unique option price in the Black-Scholes model.

Let us consider a typical situation in financial mathematics, where we assume a semimartingale $S = (S_t)_{0 \leq t \leq T}$ where $S_t > 0$ stands for the price of a risky asset at time t, and a random variable $f(S_t)$ (the function $f : (0, \infty) \longrightarrow [0, \infty)$ is a Borel measurable) describing the pay-off of an european option. Assume that

$$f(S_t) = v_0 + \int_0^T H_u dS_u,$$

where v_0 is the option's price and $(H_u)_{0 \leq u \leq T}$ is a predictable process (also satisfying an appropriate regularity condition).

The Ito stochastic integral represents the payoff of the continuous-time trading strategy consisting of holding an amount H_t of the stock at time t. In this situation, the condition that H is adapted corresponds to the necessary restriction that the trading strategy can only make use of the available information at any time and implies that the stochastic integral will not diverge when calculated as a limit of Riemann sums.

From a financial point of view, this formula states that seller can replicate the option $f(S_t)$ at an initial cost given by the constant v_0 and subsequently trading in the stock S as prescribed by the predictable trading strategy H.

The problem is the that dynamic replication assumes continuous asset price movements, but real asset prices can move discontinuously (markets are not always open), destroying the possibility of accurate replication and providing a meaningful likelihood of bankruptcy for any uncovered option seller who does not have unlimited capital.

Hence, in practice one has to replaced the continuously adjusted hedging portfolio by a discretely adjusted one. Keeping the initial value v_0 and trading at time knots $\tau = (t_i)_{i=1}^n$ this yields to the hedging error

$$Err(\tau) := \int_0^T H_u dS_u - \sum_{i=1}^{n} v_{i-1}(S_{t_i} - S_{t_{i-1}}),$$

where the $\mathcal{F}_{t_{i-1}}$-measurable v_{i-1} describe the positions in the discretely adjusted portfolio. If one wishes to trade n times only, then it is naturally to ask for $0 = t_0 < ... < t_n = T$ such that the error $\mathrm{Err}(\tau)$ becomes minimal in some sense.

In 2005 S. Geiss [Ge] combined two objects rather different at first glance: spaces of stochastic processes having weighted bounded mean oscillation (weighted BMO) and the approximation of certain stochastic integrals, driven by the geometric Brownian motion, by integrals over piece-wise constant integrands.

Recall, for reader's convenience, the definition of weighted BMO, as follows

Definition 6.24. [Ge] Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{X_t\}_{t\in T}$ be an adapted stochastic process, with $X_0 = 0$ and $\{\Phi_t\}_{t\in T}$ be a geometrical Brownian motion with $\Phi_t > 0$, $\forall t \in T$, then

$$||X||_{BMO}^{\Phi} = \sup_{\tau} \left\| E\left[\frac{|X_T - X_\tau|^p}{\Phi_\tau^p} | \mathcal{F}_\tau \right] \right\|_\infty^{\frac{1}{p}}$$

where the supremum is taken over all stopping times τ.

Usually, the approximation error is measured with respect to L^2, but this approach has some drawbacks: the resulting distributional tail estimates are rather weak.

Then, one would use L^p spaces with $2 < p < \infty$, but spaces of weighted BMO are more appropriate because in general, estimates with respect to BMO-spaces imply L^p estimates and secondly, by a weighted John-Nirenberg type Theorem, one can obtain significant better tail-estimates.

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