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HESSIAN EQUATIONS, QUERMASSINTEGRALS AND SYMMETRIZATION

TESI DI DOTTORATO DI RICERCA

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Contents

Introduction

This thesis deals with the study of a class of fully nonlinear elliptic operators of second order, called k-Hessian operators. This study is motivated by several applications as for instance the mass transportation theory.

The classical mass transportation problem is due to Monge (1781) and it consists in finding an optimal map, which transports a material into a new construction minimizing the total cost. Denoted by X and Y initial and final configurations of the material, and by f^+, f^- the densities of the material respectively in X and Y, the total cost associated to the map T is

$$\int_X c(x, T(x)) f^+(x)\, dx,$$

where $c(x, y)$ is called the cost function and it represents the cost for the transport of a unit mass.

Under suitable assumptions, in [15] the author proves that, when the quadratic cost is considered, then the optimal transport map T is unique and it can be written as the gradient of a convex potential Φ:

$$T(x) = \nabla\Phi(x),$$

where the function Φ satisfies a particular Hessian equation

$$\det(D^2\Phi(x))f^-(\nabla\Phi(x)) = f^+(x).$$

Hence the optimal maps solve an Hessian equation. These equations appear in many optimization problems connected to the mass transportation problems. Some examples are the shape optimization problem in elasticity and the optimal transportation network problems.

Let $u \in C^2(\Omega)$, with Ω bounded, connected, open set of $\mathbb{R}^n$, and let $\lambda_1, \cdots, \lambda_n$, the eigenvalues of D^2u, (the Hessian matrix of u). The k-Hessian operator, with $1 \leq k \leq n$, is defined as follows

$$S_k(D^2(u)) = \sum_{i_1<i_2<\cdots<i_k} \lambda_{i_1} \cdot \lambda_{i_2} \cdots \lambda_{i_k}, \tag{1}$$

so, it is the sum of all $k \times k$ principal minors of the Hessian matrix D^2u and it is clear that for $k = 1$ and $k = n$, it reduces respectively, to the Laplacian and to the Monge-Ampère operator. Equations which involve Hessian operators are called Hessian equations.

It is known that S_1, i.e. the Laplacian operator is elliptic. This property is not in general true for $k > 1$, and the admissible functions for S_k, i.e. the class of those functions where S_k is elliptic, is the class of the k-convex functions. A function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is k-convex (strictly k-convex) in Ω if

$$S_j(D^2u) \geq 0 \ (> 0) \qquad \text{for } j = 1, \ldots, k.$$

The first part of the thesis is devoted to the study of integral functionals associated to the k-Hessian operators, which generalize the energy integrals. These integrals are called (p, k)-Hessian integrals and they are defined as

$$I_{p,k}[u, \Omega] = \int_\Omega S_k^{ij}(D^2u) u_i u_j |Du|^{p-k-1} \, dx \qquad 1 \leq p < \infty; \qquad k = 1, \cdots, n, \tag{2}$$

where $u_i = \dfrac{\partial u(x)}{\partial x_i}$, and $S_k^{ij} = \dfrac{\partial S_k(D^2u)}{\partial u_{ij}}$. For $k = 1$, the $(p, 1)$-Hessian integrals reduce to the classical Dirichlet energy integrals

$$I_{p,1}[u, \Omega] = \int_\Omega |Du|^p \, dx. \tag{3}$$

Since for the Dirichlet integrals the following Hardy-Sobolev inequality holds true, (see for instance [36], [62], [63])

$$\left(\int_\Omega \frac{u^q}{|x|^s} \, dx \right)^{\frac{p}{q}} \leq C \int_\Omega |Du|^p \, dx, \quad u \in W_0^{1,p}(\Omega) \tag{4}$$

where $1 < p < n$, $0 \leq s < p$ and $q \leq \dfrac{p(n-s)}{n-p}$ and the best constant C is known (see [73], [36]), we have investigated the question of finding a sharp Hardy-Sobolev type inequality for (p, k)-Hessian integrals, which extends (4).

In [85], [75], [31], a Sobolev type inequality for (p, k)-Hessian integrals is contained:

$$I_{p,k}[u, \Omega] \geq C(n, p, k, \Omega) \left[\int_\Omega |u|^q \, dx \right]^{\frac{p}{q}}, \tag{5}$$

for $q = \dfrac{pn}{n-k-p+1}$ and $1 < p < n-k+1$.

Here, we have obtained a more general inequality than (5), that is

$$I_{p,k}[u, \Omega] \geq C(n, p, k, q, s, \Omega) \left[\int_\Omega \frac{|u|^q}{|x|^s} \, dx \right]^{\frac{p}{q}} \tag{6}$$

for $1 < p < n-k+1$, $q \le \frac{p(n-s)}{n-k-p+1}$ and $0 \le s < p+k-1$, finding the best value of the constant(see [41]).

For the critical value $p = n-k+1$, one can obtain an Hardy type inequality with the best constant ([41])

$$I_{p,k}\left[u,\Omega\right] \ge \binom{n-1}{k-1}\left(\frac{n-k-p+1}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^{p+k-1}}\,dx. \tag{7}$$

The best constants appearing in (6) and in (7), as in the classical case, are not achieved. This immediately leads to an improvement of both inequalities. Indeed we get (see [41])

$$\begin{aligned} I_{p,k}\left[u,\Omega\right] \ge & \binom{n-1}{k-1}\left(\frac{n-k-p+1}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^{p+k-1}}\,dx+ \\ & + C_1 \int_\Omega \frac{|u|^p}{|x|^{p+k-1}} \left(\frac{1}{\log\left(\frac{C}{|x|}\right)}\right)^\gamma dx \end{aligned} \tag{8}$$

with $\gamma \ge 2$.

All the above results have been proved by means of suitable symmetrization arguments. We have considered a class of rearrangements which does not preserve the measure of the level sets of a function, but a suitable curvature measure. This measure is called quermassintegral. To define it, let us consider Ω a bounded, connected open set of $\mathbb{R}^n$, $n \ge 2$, with smooth boundary. Ω is $(k-1)$-convex if (see [75], [67], [69], [23], [78], [79])

$$H_j[\partial\Omega] = S_j(\kappa_1, \cdots, \kappa_n) \ge 0,$$

for $j = 1, \cdots, k-1$, where κ_i denote the i^{th}-mean curvature of $\partial\Omega$ and H_j is called j-mean curvature of Ω. The k^{th}-quermassintegral of Ω, $V_k(\Omega)$, is defined by (see for instance [75], [67], [69], [23])

$$V_k(\Omega) = \frac{1}{n\binom{n-1}{k-1}} \int_{\partial\Omega} H_{k-1}(\partial\Omega)\, d\mathcal{H}^{n-1}, \tag{9}$$

where $k = 1, \cdots, n$ and $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$.

If $u : \Omega \to]-\infty, 0]$ is a function with convex sublevel sets $\Omega_t = \{u < t\}$, and Ω is a bounded, convex open set of $\mathbb{R}^n$ with smooth boundary then the k-symmetrand of u, for $k = 0, \cdots, n-1$, is defined by (see [81], [75])

$$u_k^*(x) = \sup\left\{t \le 0 : \left(\frac{V_k(\Omega_t)}{\omega_n}\right)^{\frac{1}{n-k}} \le |x|, Du \ne 0 \text{ on } \Sigma_t\right\}, \quad x \in B_R, \tag{10}$$

where B_R is a ball, centered at the origin and of radius $R = \xi_k(\Omega)$, having hence, the same k-quermassintegral of Ω.

In the second part of the thesis, we have considered the eigenvalue problem associated to k-Hessian operator:

$$\begin{cases} S_k(D^2u) = \lambda(-u)^k \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{11}$$

In [85] and [44], it has been proved that if Ω is $(k-1)$-strictly convex, there exists a positive constant λ_k which depends only on n, k, and Ω, such that problem (4.1) admits a negative solution $u \in C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})$ for $\lambda = \lambda_k$.

λ_k is the eigenvalue of k-Hessian operator and u is its eigenfunction.

We have first proved a Faber-Krahn type inequality for λ_k ([42]). If Ω is bounded and strictly convex open set of $\mathbb{R}^n$, with $k = 1, \dots, n$, if $\lambda_k(\Omega)$ is the eigenvalue of (11) and $\lambda_k(\Omega^*_{k-1})$ the eigenvalue of the following problem

$$\begin{cases} S_k(D^2(v)) = \lambda(-v)^k & \text{in } \Omega^*_{k-1} \\ v = 0 & \text{on } \partial\Omega^*_{k-1} \end{cases} \tag{12}$$

where Ω^*_{k-1} is the ball centered at the origin with the same $(k-1)^{\text{th}}$-quermassintegral of Ω, then

$$\lambda_k(\Omega) \geq \lambda_k(\Omega^*_{k-1}). \tag{13}$$

This inequality is isoperimetric, i.e. the equality holds if and only if Ω is a ball.

Secondly, we proved a reverse Hölder inequality, also known as Payne-Rayner inequality, for eigenfunctions ([42]), that, for the first eigenfunctions of the Laplacian, reads as

$$\|u\|_r \leq K(r, q, n, \lambda_1)\|u\|_q \tag{14}$$

for $0 < q < r < +\infty$, $n \geq 2$, and where K is a suitable positive constant not depending on u, and K is sharp.

For eigenfunctions of k-Hessian operator, we have proved the following Payne-Rayner type inequality

$$\left(\int_0^{V_{k-1}(\Omega)} r^{\frac{n}{n-k+1}-1}(-\tilde{u}_{k-1})^p \, dr\right)^{\frac{1}{p}} \leq C(n, p, q, k, \lambda_k) \left(\int_0^{V_{k-1}(\Omega)} r^{\frac{n}{n-k+1}-1}(-\tilde{u}_{k-1})^q \, dr\right)^{\frac{1}{q}}, \tag{15}$$

where u is a solution of (11) with convex level sets. (15) holds as an equality if and only if Ω is a ball.

In the last part of the thesis, we have investigated some convexity properties of eigenfunctions of Hessian operators. In particular we have studied when it is possible to prove the convexity of level sets of the eigenfunctions.

For the Laplacian it has been proved that the first eigenfunction is log-convex (see [24], [55], [57], [58], [59]). For eigenfunctions of Hessian operators this question is still an open problem. A first result concerns the case when Ω is a ball. In this case, also the eigenfunctions of Hessian operators are log-convex. The main obstacle to extend this result to a general strictly convex set, is to use the Constant Rank Theorem (see for instance [24], [59]). This result is known for the Laplacian and it has been extended to fully nonlinear operators. It reads as (see [46])

Theorem 0.1. *Let Ω is a domain in $\mathbb{R}^n$, and $F = F(r, p, u, x)$ is a given function in $S^n \times \mathbb{R}^n \times \mathbb{R} \times \Omega$ and elliptic. Suppose*

(i) $F(A^{-1}, p, u, x)$ is local convex in (A, u, x) for each p fixed.

If $u \in C^{2,1}(\Omega)$ is a convex solution of

$$F(D^2u, Du, u, x) = 0,$$

then the rank of $D^2u(x)$ is constant in Ω.

The thesis is organized as follows: in the first chapter we introduce k-Hessian operators, k-convex functions and k-convex domains and we describe their mainly properties; moreover we recall some results concerning the existence of solutions to Dirichlet problems associated to k-Hessian operators. In the second chapter we define the rearrangements with respect to quermassintegrals, and we briefly recall their main properties used trought the thesis. In the third chapter we prove the Hardy-Sobolev type inequalities for (p, k)-Hessian integrals, their improvements and the optimality of the constants. In the fourth chapter we consider the eigenvalue problem associated to k-Hessian operators and we prove a Faber-Krahn inequality and a Payne-Rayner inequality for eigenfunctions. Finally we describe the open problem about the issue of log-convexity of eigenfunctions when the domain is assumed to be convex. In the last chapter we describe some possible applications involving Hessian equations.

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Chapter 1

Hessian operators

In this chapter we define k-Hessian operators, k-convex functions and k-convex domains in $\mathbb{R}^n$, and we recall some of their properties.

1.1 Symmetric functions

Let A be a $n \times n$ symmetric matrix and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ its eigenvalues. The k^{th}-elementary symmetric function is defined as follows (see for instance [23], [40], [61], [67], [75],[78],[79],[81], [69])

$$S_k(\lambda(A)) = \sum_{i_1<i_2<\cdots<i_k} \lambda_{i_1} \cdot \lambda_{i_2} \cdots \lambda_{i_k}. \tag{1.1}$$

Notice that $S_k(\lambda(A))$ is clearly, the sum of all $k \times k$ principal minors of the matrix A. Moreover we assume that $S_0(\lambda(A)) = 1$.

We can define the following set associated to S_k

$$\Gamma_k = \{\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n | S_j(\lambda) > 0 \forall j = 1, \cdots, k\},$$

in [40] it has been proved that Γ_k is an open, symmetric, convex cone of $\mathbb{R}^n$ with vertex at the origin. By definition it follows

$$\Gamma_n \subset \cdots \subset \Gamma_k \subset \cdots \subset \Gamma_1,$$

where $\Gamma_n = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n | \lambda_1 > 0, \cdots, \lambda_n > 0\}$, and Γ_1 is the half space $\{\lambda \in \mathbb{R}^n | \sum \lambda_i > 0\}$.

The following Proposition (see [61], [67],[23],[69],[66]) holds true

Proposition 1.1. *Let* $\lambda = (\lambda_1, \cdots, \lambda_n) \in \Gamma_k$*, with* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$*. Denoted by* $S_{k;i}(\lambda) = S_k(\lambda)|_{\lambda_i=0}$ *and by* $S_k^i(\lambda) = \frac{\partial}{\partial\lambda_i} S_k(\lambda)$*, then the following inequalities hold true*

(i) $S_k(\lambda + \eta) = S_k(\lambda) + \eta_i S_{k-1;i}(\lambda)$*, where* $\eta = (0, \cdots, 0, \eta_i, 0, \cdots, 0)$*;*

(ii) $S_k^i(\lambda) = S_{k-1;i} > 0$;

(iii) $\lambda_k \geq 0$ *and* $S_k(\lambda) \leq \binom{n}{k}\lambda_1 \cdots \lambda_k$;

(iv) $S_k^n(\lambda) \geq \cdots \geq S_k^1(\lambda)$;

(v) $(n-k)S_k(\lambda) = \sum_{i=1}^n S_{k;i}$

(vi) $\frac{1}{n}S_1(\lambda) \geq \cdots \geq \left[\binom{n}{k-1}^{-1} S_{k-1}(\lambda)\right]^{1/(k-1)} \geq \left[\binom{n}{k}^{-1} S_k(\lambda)\right]^{\frac{1}{k}}, \quad \forall k = 1, \ldots, n,$

and equality at any stage of (vi) implies $\lambda_1 = \cdots = \lambda_n$ *(see [61])*

The cone Γ_k may also be equivalently defined as the component $\{\lambda \in \mathbb{R}^n | S_k(\lambda) > 0\}$ containing the vector $(1, \cdots 1)$, and characterized as (see [23], [40], [61], [67],[78],[69])

$$\Gamma_k = \{\lambda \in \mathbb{R}^n | 0 < S_k(\lambda) \leq S_k(\lambda + \eta) \forall \eta \in \mathbb{R}^n, \eta_i \geq 0, i = 1, \cdots, n\}. \tag{1.2}$$

Moreover by (1.2) and (i) of Proposition 1.1, one can also prove that

$$\Gamma_k = \{\lambda \in \mathbb{R}^n | S_k(\lambda) > 0; S_k^r(\lambda) > 0 \text{ for } r = 1, \cdots, n\}. \tag{1.3}$$

We point out that S_k is an homogenous function of degree k, hence by Eulero theorem one gets

$$S_k(\lambda) = \frac{1}{k}\sum_{i=1}^n S_k^i(\lambda)\lambda_i. \tag{1.4}$$

Moreover S_k is invariant by rotation.

In order to prove a crucial property of S_k, we introduce the quotient of symmetric function defined as follows

$$\frac{S_k(\lambda)}{S_l(\lambda)}, \ \ l < k,$$

where $\lambda \in \overline{\Gamma_k}$ (see [61], [78],[23]).

The following result asserts that $\left(\frac{S_k(\lambda)}{S_l(\lambda)}\right)^{\frac{1}{k-l}}$, is superadditive (see [61]).

Theorem 1.1. *Let* m *and* k *be positive integers with* $m \leq k \leq n$*. If* λ *and* μ *are in* $\overline{\Gamma_k}$*, then*

$$\left(\frac{S_k(\lambda+\mu)}{S_{k-m}(\lambda+\mu)}\right)^{\frac{1}{m}} \geq \left(\frac{S_k(\lambda)}{S_{k-m}(\lambda)}\right)^{\frac{1}{m}} + \left(\frac{S_k(\mu)}{S_{k-m}(\mu)}\right)^{\frac{1}{m}}. \tag{1.5}$$

Proof. First, we prove (1.5) when $m = 1$ by induction on k. Indeed if $k = 1$, then inequality (1.5) is obvious. For $k > 1$, by definition of symmetric function we write

$$S_k(\lambda) = S_{k,i}(\lambda) + \lambda_i S_{k-1}(\lambda) - \lambda_i^2 S_{k-2;i}.$$

Summing on i and using (v) in Proposition 1.1 now gives

$$kS_k(\lambda) = S_1(\lambda)S_{k-1}(\lambda) - \sum_{i=1}^{n} \lambda_i^2 S_{k-2;i}(\lambda). \tag{1.6}$$

Now we define $g_k = S_k/S_{k-1}$ and $g_{k;i} = S_{k;i}/S_{k-1;i}$. Dividing by S_{k-1} in (1.6) and since $S_j(\lambda) = S_{j;i}(\lambda) + \lambda_i S_{k-1;i}(\lambda)$, we get

$$kg_k(\lambda) = S_1(\lambda) - \sum_i \frac{\lambda_i^2}{\lambda_i + g_{k-1,i}(\lambda)}.$$

Now we define

$$\varphi_k(\lambda, \mu) = g_k(\lambda + \mu) - g_k(\lambda) - g_k(\mu),$$

and we obtain

$$\varphi_k = \frac{1}{k} \sum_i \left(\frac{\lambda_i^2}{\lambda_i + g_{k-1,i}(\lambda)} + \frac{\mu_i^2}{\mu_i + g_{k-1,i}(\mu)} - \frac{(\lambda_i + \mu_i)^2}{\lambda_i + \mu_i + g_{k-1,i}(\lambda + \mu)} \right).$$

By the induction hypothesis and property (ii) in Proposition 1.1, $g_{k-1;i}$ is superadditive therefore we have

$$\begin{aligned} &\frac{\lambda_i^2}{\lambda_i + g_{k-1,i}(\lambda)} + \frac{\mu_i^2}{\mu_i + g_{k-1,i}(\mu)} - \frac{(\lambda_i + \mu_i)^2}{\lambda_i + \mu_i + g_{k-1,i}(\lambda + \mu)} \\ &\qquad \geq \frac{\lambda_i^2}{\lambda_i + g_{k-1,i}(\lambda)} + \frac{\mu_i^2}{\mu_i + g_{k-1,i}(\mu)} - \frac{(\lambda_i + \mu_i)^2}{\lambda_i + \mu_i + g_{k-1,i}(\lambda) + g_{k-1;i}(\mu)} \\ &\qquad = \frac{(\lambda_i g_{k-1;i}(\mu) - \mu_i g_{k-1;i}(\lambda))^2}{(\lambda_i g_{k-1;i}(\lambda))(\mu_i g_{k-1;i}(\mu))(\lambda_i + \mu_i + g_{k-1,i}(\lambda) + g_{k-1;i}(\mu))} \geq 0. \end{aligned}$$

It hence follows that $\varphi_k(\lambda + \mu) \geq 0$ and hence (1.5) is proved for $m = 1$ and any $k \geq 1$.

If $m > 1$, we can write

$$\left(\frac{S_k(\lambda)}{S_{k-m}(\lambda)} \right)^{\frac{1}{m}} = \left(\prod_{j=1}^{m} g_{k-j+1}(\lambda) \right)^{\frac{1}{m}};$$

by the case $m = 1$, each g_{k-j+1} is superadditive, hence since the product of superadditive function is superadditive also, the theorem is proved. □

The following Theorem proves an important property of S_k (see [61], [78], [23]).

Theorem 1.2. *Let $\lambda \in \Gamma_k$, then $S_k^{\frac{1}{k}}(\lambda)$ is concave.*

Proof. By definition of symmetric function, $S_k^{\frac{1}{k}}$ can be written as the geometric mean of $\frac{S_{j+1}}{S_j}$, for $j = 0, \cdots, k-1$. Since the product of superadditive functions is superadditive also, , by Theorem 1.1, $S_k^{\frac{1}{k}}$ is superadditive. Since $S_k^{\frac{1}{k}}$ is also homogenous of degree one, superadditivity implies the concavity and this conclude the proof of the Theorem. □

Remark 1.1. *We observe that since $\left(\frac{S_k}{S_l}\right)^{\frac{1}{k-l}}$, for $n \geq k > l \geq 0$, can be written as the geometric mean of $\frac{S_{j+1}}{S_j}$, for $j = l, \cdots, k-1$, repeating the same argument of the proof of Theorem 1.2, we obtain that also the quotient $\left(\frac{S_k}{S_l}\right)^{\frac{1}{k-l}}$ is concave.*

Whenever λ coincides to the vector of the eigenvalues of a matrix A, since by definition, S_k is the sum of all $k \times k$ principal minors of A, by definition of determinant by permutation, we get (see [61], [66], [81], [67])

$$S_k(A) = S_k(\lambda(A)) = \frac{1}{k!} \sum_{1 \leq i_1, \cdots, i_k \leq n} \delta_{i_1, \cdots, i_k}^{j_1, \cdots, j_k} a_{i_1 j_1} \cdots a_{i_k j_k}, \tag{1.7}$$

where $\delta_{i_1, \cdots, i_k}^{j_1, \cdots, j_k}$ is the generalized Kronecker delta; it is zero if $\{i_1, \cdots, i_k\} \neq \{j_1, \cdots, j_k\}$, equal to $+1$ (or -1) if $(i_1, \cdots, i_k)$ and $(j_1, \cdots, j_k)$ differ by an even (or odd) permutation.

By (1.7) we get

$$S_k^{ij}(A) = \frac{\partial}{\partial a_{ij}} S_k(A) = \frac{1}{(k-1)!} \sum_{1 \leq i, i_1, \cdots, i_{k-1} \leq n} \delta_{i, i_1, \cdots, i_{k-1}}^{j, j_1, \cdots, j_{k-1}} a_{i_1 j_1} \cdots a_{i_{k-1} j_{k-1}}, \tag{1.8}$$

and obviously

$$S_k(A) = \frac{1}{k} \sum_{i,j=1}^{n} S_k^{ij}(A) a_{ij}. \tag{1.9}$$

We observe that (1.8) and (1.9) are the analogous of respectively (ii) in Proposition 1.1 and of (1.4).

In particular if $A = D^2 u$, $u \in C^2(\Omega)$, moreover we have

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j} S_k^{ij}(D^2 u(x)) = 0. \tag{1.10}$$

Indeed deriving (1.8) and summing on j we get

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j} S_k^{ij}(D^2 u(x)) = \frac{1}{(k-1)!} \sum_{j=1}^{n} \sum_{1 \leq i, i_1, \cdots, i_{k-1} \leq n} \delta_{i, i_1, \cdots, i_{k-1}}^{j, j_1, \cdots, j_{k-1}} \frac{\partial}{\partial x_j} (u_{i_1 j_1} \cdots u_{i_{k-1} j_{k-1}})$$
$$= \frac{1}{(k-1)!} \sum_{j=1}^{n} \sum_{1 \leq i, i_1, \cdots, i_{k-1} \leq n} \delta_{i, i_1, \cdots, i_{k-1}}^{j, j_1, \cdots, j_{k-1}} (u_{i_1 j_1 j} \cdots u_{i_{k-1} j_{k-1}} + \cdots + u_{i_1 j_1} \cdots u_{i_{k-1} j_{k-1} j}),$$

where we assume that we can consider the third derivatives of u. In the other hand we can proceed by approximation and proceed as before. Now since the third derivatives of u, $u_{i_r j_r j}$ are symmetric in i_r and j while the Kronecker symbols are skew-symmetric in those indices, the sum over i_r and j vanishes.

The result (1.10) is contained in [66]. We observe that by (1.10) and (1.9) then $S_k(D^2u)$ is divergence free, i.e.

$$S_k(D^2u) = \frac{1}{k}\sum_{j=1}^{n}\left(S_k^{ij}(D^2u)u_i\right)_j . \tag{1.11}$$

1.2 k-Hessian operators and admissible functions

Let Ω be a bounded, connected, open set of $\mathbb{R}^n$. For a function $u \in C^2(\Omega)$ the k-Hessian operator, for $k = 1, 2, \ldots, n$, is defined by

$$F_k[u] = S_k(D^2u) = S_k(\lambda(D^2u)), \tag{1.12}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ denote the eigenvalues of the Hessian matrix of the second derivatives of u. Well known examples of k-Hessian operators are $F_1[u] = \Delta u$, the Laplacian operator and $F_n[u] = \det(D^2u)$, i. e. the Monge-Ampère operator. The equation

$$F_k[u] = f(x, u, Du) \tag{1.13}$$

is the k-Hessian equation, hence the 1-Hessian equation is the Poisson equation, while the n-equation is Monge-Ampère equation. It is known that $F_1[u] = \Delta u$ is always an elliptic operator, hence every function $u \in C^2(\Omega)$ is an admissible function for the Laplacian. This property is not in general true for $k > 1$.

Therefore, we recall the notion of ellipticity for a general second order partial differential operator on a domain Ω in $\mathbb{R}^n$ (see [45]) and after we will define the class of admissible function for k-Hessian operators.

Let us consider a general second order operator

$$F[u] = F(x, u, Du, D^2u),$$

where F is a real function on the set $\Gamma = \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$, ($\mathbb{R}^{n\times n}$ denotes the space of real symmetric $n \times n$ matrices) and $u \in C^2(\Omega)$.

$F(x, z, p, r)$ is fully nonlinear if it is nonlinear with respect to r. We observe that S_k is a linear operator only for $k = 1$, in the cases $k > 1$, it is a fully nonlinear partial differential operator of second order. Denoting by $\gamma = (x, z, p, r)$ the points of Γ, we have the following (see for instance [45], [23])

Definition 1.1. *F is elliptic (or degenerate elliptic) in a subset $\mathcal{U}$ of Γ if the matrix $[F_{ij}(\gamma)]$, where*

$$F_{ij}(\gamma) = \frac{\partial F}{\partial r_{ij}}(\gamma) \qquad i, j = 1, \ldots, n$$

is positive (or positive semi-definite) for all $\gamma \in \mathcal{U}$.

F is uniformly elliptic in $\mathcal{U}$, if there exist two positive constants Λ and λ such that

$$\lambda I \leq [F_{ij}(\gamma)] \leq \Lambda I, \qquad \forall \gamma \in \mathcal{U}$$

where I is the unit matrix.

If F is elliptic (degenerate elliptic, uniformly elliptic) in the whole set Γ, then we simply say that F is elliptic (degenerate elliptic, uniformly elliptic) in Ω.

Definition 1.2. *If $u \in C^2(\Omega)$ and F is elliptic (degenerate elliptic, uniformly elliptic) in the whole set $\mathcal{U} = \{(x, u(x), Du(x), D^2u(x)), x \in \Omega\}$, then we say F is elliptic (degenerate elliptic, uniformly elliptic) with respect to u in Ω, or that u is an admissible solution to F.*

The previous definition extends the usual definition of ellipticity known for linear operators, to the class of nonlinear operators. We observe that by previous definition the admissible solution of Monge-Ampère equation are convex function.

Definition 1.3 (**k-convex function**). *Let Ω be a bounded connected open set of $\mathbb{R}^n$. A function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is k-convex (strictly k-convex) in Ω if*

$$S_j(D^2u) \geq 0 \ (> 0) \quad \text{for } j = 1, \ldots, k.$$

We denote the class of k-convex function in Ω by $\overline{\Phi}^2_k(\Omega)$ and that one of k-strictly convex by $\Phi^2_k(\Omega)$.

We observe that $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is k-convex function if and only if $\lambda(D^2u) \in \overline{\Gamma_k}$.

Remark 1.2. *By definition we have that if u is k-convex, then u is j-convex $1 \leq j < k$. We observe that a function u is 1-convex if and only if it is subharmonic, and u is n-convex if and only if it is convex. This implies that k-convex functions are subharmonic, $\overline{\Phi}^2_k(\Omega) \subset \overline{\Phi}^2_1(\Omega)$ and convex functions are k-convex for $1 \leq k < n$, $\overline{\Phi}^2_n \subset \overline{\Phi}^2_k(\Omega)$.*

By (1.3), we can prove the following result

Theorem 1.3. *Let $1 < k \leq n$; then S_k is elliptic (degenerate) with respect to u if and only if u is strictly k-convex (k-convex).*

Proof. By definition 1.1, S_k is elliptic with respect to u if the matrix $[S_k^{ij}(D^2u)]$, given in (1.8), is positive definite; hence if its eigenvalues are positive. By (1.3), it is sufficient to prove that the eigenvalue of $[S_k^{ij}]$ are $S_k^i(\lambda(D^2u))$.

Indeed if λ_i denote the i^{th} eigenvalue of D^2u, since D^2u is a symmetric matrix we have

$$\operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n) = C^T D^2 u C,$$

where C is an orthogonal matrix.

Hence we have

$$\lambda_r = c_{ir} u_{ij} c_{jr}$$
$$u_{ij} = c_{ir} \lambda_r c_{jr}.$$

Then we have

$$\frac{\partial S_k}{\partial u_{ij}} = c_{ir} \frac{\partial S_k}{\partial \lambda_r} c_{jr}$$
$$\frac{\partial S_k}{\partial \lambda_r} = c_{ir} \frac{\partial S_k}{\partial u_{ij}} c_{jr},$$

hence

$$\operatorname{diag}(S_{k,1}, \cdots, S_{k,n}) = C^T S_k^{ij} C.$$

This complete the proof of the theorem. □

Now we recall same properties of k-convex functions (see for instance [78], [79]).

First we observe that by (vi) of Proposition 1.1 we get the following result

Lemma 1.1. *Let Ω be a bounded, connected open set of $\mathbb{R}^n$. Let $u \in \Phi_k^2(\Omega)$ with zero boundary value. Then u is negative in Ω.*

Proof. It is sufficiently to observe that by (vi) of Proposition 1.1 u is subharmonic and by the assumptions, u vanishes on the boundary. Hence by the classical comparison principle ([45]), we have the claim. □

In the next we denote by $\overline{\Phi}_k^{2,0}$ and $\overline{\Phi}_k^{2,0}$ respectively the set of k-convex and strictly k-convex functions which are zero on the boundary of Ω.

One may ask when composing k-convex (strictly) functions we get that the composite function is also k-convex (strictly). The following result solves this problem (see [78], [79]).

Proposition 1.2. *Let Ω be a bounded connected open set of $\mathbb{R}^n$. Let $u_1, \cdots, u_m \in \overline{\Phi}_k^2(\Omega)$ ($\Phi_k^2(\Omega)$) and f be a convex, nondecreasing function in $\mathbb{R}^m$. Then the composite function $w = f(u_1, \cdots, u_m) \in \overline{\Phi}_k^2(\Omega)$ ($\Phi_k^2(\Omega)$) also.*

Proof. By simple calculation we have

$$w_{ij} = \sum_{p=1}^{m} \left[\frac{\partial f}{\partial u_p}(u_p)_{ij} + \sum_{q=1}^{m} \frac{\partial^2 f}{\partial u_p \partial u_q}(u_p)_i (u_q)_j \right],$$

and it is clear that $D^2 w \in \overline{\Gamma_k}$ since

- $\overline{\Gamma_k}$ is a convex cone;
- $\dfrac{\partial f}{\partial u_p} \geq 0 \quad \forall p = 1, \cdots, m;$
- $\left[\dfrac{\partial^2 f}{\partial u_p \partial u_q} \right] \geq 0.$

□

In order to conclude this section we give a weak definition of k-convex functions (see for instance [78]) which we will need to define the weak solutions of the Hessian equations.

Definition 1.4. *Let Ω be a bounded connected open set of $\mathbb{R}^n$ and let $u \in C^0(\Omega)$. u is weak k-convex (strictly) if there exists a sequence $\{u_m\}$, u_m is k-convex (strictly) for all $m \in \mathbb{N}$, such that in any open set Ω', $\Omega' \subset \Omega$, u_m converges uniformly to u.*

We denote the class of weak k-convex functions in Ω, by $\overline{\Phi}_k^0(\Omega)$ and that one weak strictly k-convex by $\Phi_k^0(\Omega)$. We observe that $\overline{\Phi}_k^2(\Omega) \subset \overline{\Phi}_k^0(\Omega)$.

Remark 1.3. *We point out that Proposition 1.2 holds true also for the functions of $\overline{\Phi}_k^0(\Omega)$ since one can proceed by approximation and to use Proposition 1.2 for the sequence u_m.*

If $u \in \overline{\Phi}^0_k(\Omega)$ let

$$u_\epsilon(x) = \epsilon^{-n} \int \rho\left(\frac{x-y}{\epsilon}\right) u(y)\, dy, \qquad \text{for } 0 < \epsilon < \text{dist}(x, \partial\Omega),$$

where ρ is the usual mollifier, i.e. $\rho \in C^\infty_0(\mathbb{R}^n)$, $\rho(x) > 0$ for $|x| < 1$, $\rho(x) = 0$ for $|x| \geq 1$, $\int_{\mathbb{R}^n} \rho\, dx = 1$.

The following Proposition holds true (see for instance [78], [79]),

Proposition 1.3. *$u_\epsilon \in C^\infty(\Omega') \cap \overline{\Phi}^2_k(\Omega')$ for any $\Omega' \subset \Omega$ satisfying dist$(\Omega', \Omega) \geq \epsilon$. Moreover, as $\epsilon \searrow 0$, $u_\epsilon \searrow u$ in Ω'.*

Finally, we can give a further criterion to define weak k-convex function (see [78]):

Lemma 1.2. *A function $u \in C^0(\Omega)$ is weak k-convex in Ω if and only if its restriction to any subset $\Omega' \subset \Omega$ is the limit of a monotone decreasing sequence in $\overline{\Phi}^2_k(\Omega')$.*

The assertion of the previous Lemma is a direct consequence of the definition of weak k-convex functions and of the Proposition 1.3.

1.3 k-convex domains

Let Ω be a bounded, connected and open set of $\mathbb{R}^n$ with boundary $\partial\Omega \in C^{2,\alpha}$, i.e., in any neighborhood of a point $x_0 \in \partial\Omega$, it coincides with the graphic of a function $\phi \in C^{2,\alpha}$. Next by simplicity, we say that $\partial\Omega$ is smooth when we assume $\partial\Omega \in C^{2,\alpha}$.

Let us suppose that $\partial\Omega$ has the $n-1$ principal curvatures oriented so that convex domains have non negative curvatures. We denote by $\nu(x)$ the outward normal at the point $x \in \partial\Omega$, then it is known that $\kappa_1(x), \cdots, \kappa_{n-1}(x)$ are the eigenvalues of $D\nu(x)$. Using 1-Hessian operator and n-Hessian operator on $\kappa_1(x), \cdots, \kappa_{n-1}(x)$, we can define H and K, respectively the mean curvature and the Gauss curvature of $\partial\Omega$ as

$$H = \frac{S_1(\kappa_1, \cdots, \kappa_{n-1})}{n-1}$$

$$K = S_{n-1}(\kappa_1, \cdots, \kappa_{n-1}).$$

This suggest us the following general definition

Definition 1.5. *Let Ω be a bounded, connected and open set of $\mathbb{R}^n$ with smooth boundary. The m^{th} mean curvature of $\partial\Omega$ is defined by*

$$H_m[\partial\Omega] = S_m(k_1, k_2, \cdots, k_{n-1}), \tag{1.14}$$

where $m = 0, \cdots, n-1$ and where we assume $H_0[\partial\Omega] = S_0(k_1, \cdots, k_{n-1}) = 1$.

By definition, we observe that Ω is convex if and only if $H_{n-1}[\partial\Omega] = K \geq 0$. By properties of symmetric function, this is equivalent to have $H_m[\partial\Omega] \geq 0$ for all $m = 1, \cdots n-1$. This property can be generalized by the following definition (see for instance [67], [79], [75], [23], [69])

Definition 1.6. *Let Ω be a bounded, connected and open set of $\mathbb{R}^n$ with smooth boundary. Ω is said to be k-convex (strictly k-convex) if*

$$H_j[\partial\Omega] \geq 0 \ (> 0), \qquad j = 1, \cdots, k.$$

We observe that Ω is $(n-1)$-convex if and only if it is convex. Moreover it is clear that if Ω is k-convex then Ω is j-convex for all $j = 1, \cdots, k$. In particular if Ω is convex then Ω is k-convex for all $k = 1, \cdots, n$. The reverse assertion is not in general true. There are counterexamples of sets which are k-convex but are not convex. In $\mathbb{R}^3$ it suffices to take a set with nonnegative mean curvature H but with negative Gauss curvature.

Let us consider the curve $\alpha : x = f(z)$, where f is a nonnegative function such that $f \in C^2([a,b])$, $f(a) = f(b) = 0$ and $\lim_{z\to a^+} f'(z) = -\lim_{z\to b^-} f'(z)$. Let us rotate α with respect to the axis z and we denote by Ω the set obtained. We observe that, by assumptions on f, Ω is smooth. It is simple to compute the mean curvatures of $\partial\Omega$ which are the following

$$\kappa_1(z) = -\frac{f''(z)}{(1+f'(z)^2)^{\frac{3}{2}}} \tag{1.15}$$

$$\kappa_2(z) = \frac{1}{f(z)(1+f'(z)^2)^{\frac{1}{2}}}. \tag{1.16}$$

Hence we get

$$H(z) = \frac{1}{2}\frac{1+f'^2(z) - f(z)f''(z)}{f(z)(1+f'(z)^2)^{\frac{3}{2}}}$$

$$K(z) = -\frac{f''(z)}{f(z)(1+f'(z)^2)^3}.$$

Hence we get that Ω is 1-convex if and only if H is positive, i.e.

$$f''(z) \leq \frac{1+f'(z)^2)}{f(z)},$$

and we observe that it suffices that f'' is positive at same point of Ω to have that $K < 0$, i.e. Ω is not convex.

If we take for example the function f whose graphic is defined by the following polar equation:

$$\rho(\theta) = \cos(2\theta) + a \qquad \theta \in [0, \pi], \quad a > 1,$$

and we rotate f with respect to the axis z, then by (1.15), the mean curvatures of $\partial\Omega$ are

$$\kappa_1(\theta) = \frac{-\rho\rho'' + \rho'^2 + \rho^2}{(\rho'^2 + \rho^2)}^{3/2}$$

$$\kappa_2(\theta) = \frac{\rho \sin\theta - \rho' \cos(\theta)}{\rho \sin\theta \,(\rho'^2 + \rho^2)^{1/2}}.$$

We observe that κ_2 is always positive and since

$$a - 1 \leq \left(\rho'^2 + \rho^2\right)^{1/2} \leq 2(a + 1),$$

we have

$$\kappa_2(\theta) \geq \frac{1}{(\rho'^2 + \rho^2)^{1/2}} \geq \frac{1}{2(a + 1)},$$

moreover

$$\kappa_2 \geq \begin{cases} \dfrac{(a-1)(a-5)}{8(a+1)^3} & \text{if } a > 5; \\ \dfrac{(a-1)(a-5)}{(a-1)^3} & \text{if } 1 < a < 5 \end{cases}.$$

We observe that

$$K(\theta) = \kappa_1(\theta)\kappa_2(\theta),$$

which for $a < 5$ is not positive in each point since for example in $\theta = \frac{\pi}{2}$ is negative.

Instead H is

$$H(\theta) = \kappa_1(\theta) + \kappa_2(\theta) \geq \frac{1}{2(a+1)} + \frac{(a-1)(a-5)}{(a-1)^3} > 0 \Leftrightarrow a < \frac{5 - \sqrt{52}}{3} \cup a > \frac{5 + \sqrt{52}}{3}.$$

Hence if we take $a \in (\frac{5 + \sqrt{52}}{3}, 5)$, we get that Ω is 1-convex but is not convex (see picture).

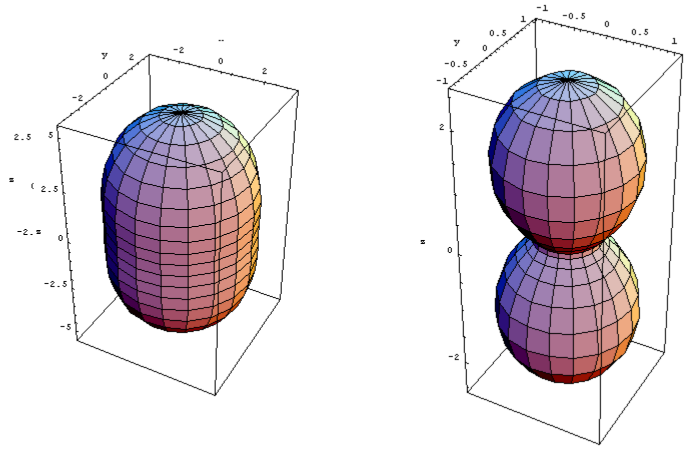


Figure 1.1: The first picture represents Ω with $a \in \left((5+\sqrt{52})/3, 5\right)$, the second one with $a = 1,5$.

Let Ω be a bounded, $(k-1)$-convex, open set of $\mathbb{R}^n$ and let u, k-convex; let us consider

$$\Omega_t = \{x \in \Omega : u(x) < t\}$$

and

$$\Sigma_t = \partial\Omega_t = \{x \in \Omega : u(x) = t\}.$$

Then the mean curvatures of Σ_t, when Σ_t is not degenerate, are (see [75])

$$H_{k-1}(\Sigma_t) = \left[D\frac{Du}{|Du|}\right]_{k-1}, \quad \text{if } Du \neq 0. \tag{1.17}$$

Moreover the following pointwise equality holds true ([75])

$$S_k^{ij}(D^2u)u_iu_j = |Du|^{k+1}\left[D\frac{Du}{|Du|}\right]_{k-1}, \quad \text{if } Du \neq 0, \tag{1.18}$$

where S_k^{ij} is defined in (1.8).

Finally we recall the following Reilly identity which holds true for non-degenerate Σ_t (see for instance [65])

$$\frac{d}{dt}\int_{\Sigma_t} H_{k-1}(\Sigma_t)d\mathcal{H}^{n-1} = k\int_{\Sigma_t}\frac{H_k(\Sigma_t)}{|Du|}d\mathcal{H}^{n-1}. \tag{1.19}$$

Then one may ask what is the connection between k-convex sets and k-convex functions. The following result gives an answer to this question

Theorem 1.4. *Let Ω be a bounded, connected and open set of $\mathbb{R}^n$ and let $u \in \overline{\Phi}_k^2(\Omega)$. Then the level sets of u are $(k-1)$-convex.*

Proof. Without loss of generality we can prove the result for the level set $\Omega_0 = \{x \in \Omega | u(x) \leq 0\}$.

First we assume that u is strictly k-convex (if not we may consider $u_\epsilon(x) = u(x) + \epsilon\frac{\|x\|^2}{2}$, which is strictly k-convex and there pass to the limit).

We denote by ν the inner normal at $\partial\Omega_0$. Since u is k-convex by Hopf's lemma, we have $u_\nu < 0$ on $\partial\Omega_0$.

For any point $x_0 \in \partial\Omega_0$, up to translations and rotations of the coordinates axes, we may assume that x_0 is the origin and that the first $(n-1)$ directions of the axes are the principal ones. Then

$$x_n = \rho(x') = \tfrac{1}{2}\sum_{i=1}^{n-1} \kappa_1 x_i^2 + o(|x'|^2),$$

where $x' = (x_1, \cdots, x_{n-1})$.

Differentiating the boundary condition $u(x', \rho(x')) = 0$, we get

$$u_{ij}(0) + u_n\rho_{ij}(0) = 0, \quad 1 \leq j < n,$$

$$u_{ij}(0) = -u_n\kappa_i\delta_{ij} = |u_n|\kappa_i\delta_{ij}.$$

Hence we obtain

$$S_k^{nn}(D^2u) = |u_n|^{k-1}S_k(\kappa),$$

this complete the proof. □

1.4 The Dirichlet problem: Hessian equations

In this section we consider the following Dirichlet problem for Hessian operator

$$\begin{cases} S_k(D^2u) = f(x) \text{ in } \Omega \\ u = \phi(x) \text{ on } \partial\Omega \end{cases} \tag{1.20}$$

We may ask when there exist admissible solutions of (**??**) and what are the assumptions on f and on Ω in order to have the existence of a solution.

A first result which solves the existence problem is the following (see for instance [23]).

Theorem 1.5. *Let $k = 1, \cdots, n$ and let $\Omega \subset \mathbb{R}^n$ be $(k-1)$-strictly convex. If $f \in C^\infty(\overline{\Omega})$, with $f > 0$ in $\overline{\Omega}$, and let $\phi \in C^\infty(\partial\Omega)$, then there exists a unique solutions $u \in C^\infty(\overline{\Omega}) \cap \Phi_k^2(\Omega)$ of (1.20).*

Remark 1.4. *We observe that by Theorem 1.3, admissible solutions are strictly k-convex functions; this implies that the assumption on the set Ω is necessary by Proposition 1.4. Moreover the strictly k-convexity of the solution, implies assumption on the positivity of f.*

The proof of Theorem 1.5, is based on the continuity method. Therefore, we briefly describe the idea of this method. Let $u_0 \in C^\infty(\overline{\Omega})$ be a strictly k-convex subsolution of (4.24), i.e. $u_0 = \phi$ on $\partial\Omega$ and

$$S_k(D^2 u_0) \geq f(x) \text{ in } \Omega.$$

For each t, $0 \leq t \leq 1$, we wish to find a strictly k-convex solution, u_t in $C^{2,\alpha}(\overline{\Omega})$ of

$$\begin{cases} S_k(D^2 u_t) = tf + (1-t) S_k(D^2 u_0) \text{ in } \Omega \\ u_t = \phi \text{ on } \partial\Omega \end{cases} \tag{1.21}$$

The space $C^{2,\alpha}$, $0 < \alpha < 1$, consists of functions $u \in C^2(\overline{\Omega})$ whose second derivatives satisfy a Hölder condition with a fixed exponent α in $\overline{\Omega}$; the corresponding norm is

$$\begin{aligned} |u|_{C^{2,\alpha}} = \max_{\overline{\Omega}} |u| + \sum_i \max_{\overline{\Omega}} |u_i| + \sum_{i,j} \max_{\overline{\Omega}} |u_{ij}| \\ + \sum_{i,j} \sup_{x \neq y \in \overline{\Omega}} \frac{|u_{ij}(x) - u_{ij}(y)|}{|x-y|^\alpha}. \end{aligned}$$

For $t = 0$ we have a solution u_0. Using the implicit function theorem, one finds that the set of t for which (1.21) has a solution is open. If one can establish the $C^{2,\alpha}$ a priori estimate

$$|u_t|_{C^{2,\alpha}} \leq C,$$

where C is independent of t, it follows that the set of such t is also closed, and hence it is the whole unit interval. The function u_1 is then our desired solution of (1.20).

The principal steps are the a priori estimate and the existence of a subsolution. In [23], the authors obtain a $C^{2,\alpha}$ a priori estimate for solution of (1.20) using the existence of a subsolution and the comparison principle. The construction of a subsolution strongly depends by the assumption on Ω.

If one assumes that the subsolution exists a priori, it is not necessary suppose the geometric condition on Ω. In this case we have the following theorem contained in [51].

Theorem 1.6. *Let Ω be a bounded, connected, open set of $\mathbb{R}^n$, with smooth boundary. Let us consider the following Dirichlet problem*

$$\begin{cases} S_k(D^2u) = \eta(x, u, Du) \text{ in } \Omega \\ u = \phi \text{ on } \partial\Omega, \end{cases} \tag{1.22}$$

for $k = 2, \cdots, n$. Then there exists a unique solution u in $C^\infty(\Omega)$ of (1.22), provided that there exists a strict subsolution and that the function $\psi = \eta^{\frac{1}{k}}$ satisfies the following condition

$$\psi(x, z, p) \text{ is convex in } p\ \forall (x, z) \in \overline{\Omega} \times \mathbb{R};$$

$$\inf_p \psi > 0 \qquad \inf_p \frac{\partial \psi}{\partial z} > 0 \qquad \sup_p \frac{|D_x \psi|}{1 + |p|} < \infty,$$

where a strict subsolution v to the problem (1.22) is a strictly k-convex function such that

$$\begin{cases} S_k(D^2v) \geq \psi(x, v, Dv) + \delta \text{ in } \Omega \\ v = \phi \text{ on } \partial\Omega, \end{cases}$$

In the previous section we defined weak k-convex functions. One may ask if there exists a definition of weak solution of (1.20) and whenever it exists.

In this direction we have the following (see [78], [77])

Definition 1.7. *Let $p \geq 1$, and let $f \in L^p(\Omega)$. A function $u \in C^0(\Omega)$ is called an admissible weak solution of (1.20) if there exists a sequence $\{u_m\} \subset C^2(\Omega)$ of k-convex functions such that*

$$u_m \to u \qquad \text{in } C^0(\Omega)$$

$$F(D^2u_m) \to f \qquad \text{in } L^1_{loc}(\Omega).$$

We observe in particular that an admissible weak solution is a weak k-convex function, since u_m are k-convex function.

By definition it is clear that the existence of weak solution of (1.20) can be proved by approximation, using Theorem 1.5 for the following approximation problems

$$\begin{cases} S_k(D^2u_m) = f_m \text{ in } \Omega \\ u_m = \phi_m \text{ on } \partial\Omega, \end{cases} \tag{1.23}$$

where $\phi_m, f_m \in C^\infty(\overline{\Omega})$ satisfy $\phi_m \to \phi$ in $C^0(\overline{\Omega})$, and $f_m \to f$ in $L^p(\Omega)$ and $f_m > 0$ in Ω.

It is obvious that for the existence of a weak solution also a comparison result is needed. This result is contained in [77] and it is the following

Theorem 1.7. *Let $u, v \in C^0(\overline{\Omega}) \cup C^2(\Omega)$, satisfy $u \leq v$ on $\partial\Omega$, u is k-convex in Ω and the differential inequalities,*

$$S_k(D^2u) \geq \psi_1, \qquad S_k(D^2v) \leq \psi_2,$$

whenever the function $w = v - u$, is k-convex, where $\psi_1, \psi_2 \geq 0$, $\psi_1, \psi_2 \in L^P(\Omega)$, for some $p \geq 1$. Then we have the estimates,

$$\|(u-v)^+\|_{L^q(\Omega)} \leq C(diam(\Omega))^{2+\frac{n}{q}-\frac{n}{kp}} \|\psi_2^{\frac{1}{k}} - \psi_1^{\frac{1}{k}}\|_{L^{kp}(\Omega)}, \tag{1.24}$$

for $1 \leq q \leq \infty$ and

$$\frac{2}{n} + \frac{1}{q} - \frac{1}{kp} \geq 0 \quad (>0) \text{ if } p > 1 \quad (p = 1), \tag{1.25}$$

where C is a constant depending on k, n, p, q.

By definition 1.7 and by Theorems 1.24 and 1.5, one can prove the following existence and uniqueness result (see for instance [77])

Theorem 1.8. *Let Ω be a strictly $(k-1)$-convex set of $\mathbb{R}^n$, $k = 1, \cdots, n$. Let $\phi \in C^0(\overline{\Omega})$ and $f \geq 0, \in L^p(\Omega)$, for $p > \frac{n}{2k}$. Then there exists a unique admissible weak solution $u \in C^0(\overline{\Omega})$ of (1.20) in Ω.*

Chapter 2

Symmetrization for quermassintegrals

In this chapter we define rearrangements with respect to integrals depending on curvatures known as quermassintegral, and we describe their main properties.

2.1 Quermassintegrals and rearrangements

Let Ω be a bounded, connected open set of $\mathbb{R}^n$, $n \geq 2$, and assume Ω $(k-1)$-convex. The k^{th} quermassintegral of Ω, $V_k(\Omega)$, is defined by (see for instance [75], [67], [69], [23])

$$V_k(\Omega) = \frac{1}{n\binom{n-1}{k-1}} \int_{\partial\Omega} H_{k-1}(\partial\Omega)\, d\mathcal{H}^{n-1}, \tag{2.1}$$

where $k = 1, \cdots, n$, $H_{k-1}(\partial\Omega)$ is the $(k-1)^{\text{th}}$ mean curvature of $\partial\Omega$ defined in (1.14) and where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$.

Moreover we assume $V_0(\Omega) = |\Omega|$, the usual Lebesgue measure of Ω.

In (2.1), the constant, is chosen so that the k^{th} quermassintegral of a ball B_R, of radius $R > 0$, is

$$V_k(B_R) = \omega_n R^{n-k}.$$

Observe that for $k = 1$ we have

$$V_1(\Omega) = \frac{1}{n}\mathcal{H}^{n-1}(\partial\Omega),$$

hence V_1 is proportional to the perimeter of Ω. However, in general, V_k is the mean of m dimensional area of projections of Ω onto m dimensional subspaces of $\mathbb{R}^n$.

When Ω is convex, the following Steiner formula holds true (see for instance [75], [67], [69], [81])

$$V_0(\Omega + tB) = |\Omega + tB| = \sum_{i=0}^{n} \binom{n}{i} t^i V_i(\Omega), \tag{2.2}$$

where B is the unit ball of $\mathbb{R}^n$ centered at the origin, and the set $\Omega + tB$ is the Minkowski sum of the sets Ω and tB, defined by $\Omega + tB = \{x + ty, x \in \Omega, y \in B\}$.

In [69], it has been proved that the quermassintegrals of convex sets have a monotonicity property, i.e., if Ω_1, Ω_2, are two convex sets of $\mathbb{R}^n$, such that $\Omega_1 \subset \Omega_2$, then

$$V_k(\Omega_1) \leq V_k(\Omega_2), \quad \forall k = 0, \cdots, n.$$

This property is not in general, true whenever Ω_1 and Ω_2 are only $(k-1)$-convex, $k = 0, \cdots, n-1$. In order to prove this fact, we give the following example. Let us consider the surface of the torus of $\mathbb{R}^3$; the torus is obtained by a rotation with respect to the axis z of the circle of radius r, centered at the point $(R, 0)$, contained in the plan x, z (see picture).

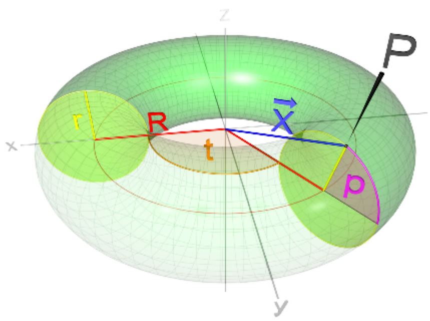


Figure 2.1: R is the distance from the origin to the center of the circle of radius r, and t e p are the angles of the rotation.

The parametric equations of the surface are,

$$\varphi(p, t) = ((R + r\cos p)\cos t, (R + r\cos p)\sin t, r\sin p) \qquad t, p \in [0, 2\pi], \quad r < R. \tag{2.3}$$

We can compute the principal curvatures of the torus which are

$$\kappa_1 = \frac{1}{r}$$

$$\kappa_2 = \frac{\cos p}{R + r\cos p}.$$

Hence the Gauss and the mean curvature are respectively

$$K = \frac{\cos p}{r(R + r\cos p)}$$

$$H = \frac{1}{2}\frac{R + 2r\cos p}{r(R + r\cos p)}.$$

We observe that K is not positive while, since $R > r$, we have $H > 0$.

Let us consider the ball B_ρ centered at the origin and of radius $\rho = R + r + \epsilon$. It is clear that $T \subset B_\rho$, where T denotes the set having the torus as its boundary.

Computing the 2-quermassintegral, that is $V_2(T) = \frac{1}{6}\int_{[0,2\pi]^2} 2H\,dp\,dt$, it results

$$\begin{aligned} V_2(T) &= \frac{1}{6}\iint_{[0,2\pi]^2}(R + 2r\cos p)\,dp\,dt \\ &= \frac{2}{3}\pi^2 R. \end{aligned}$$

$$V_2(B_\rho) = \frac{4}{3}\pi(R + r + \epsilon).$$

Hence we get

$$V_2(T) - V_2(B_\rho) = \frac{2}{3}\pi^2 R - \frac{4}{3}\pi(R + r + \epsilon) = \frac{2}{3}\pi[(\pi - 2)R - 2(r + \epsilon)].$$

So, choosing $R > 3r$, we obtain the claim.

Now let us consider

$$\Omega_t = \{x \in \Omega : u(x) < t\} \qquad \Sigma_t = \partial\Omega_t = \{x \in \Omega : u(x) = t\},$$

where u is a function which level sets are $(k-1)$-convex. The k^{th}-mean radius of Ω, for $k = 0, \cdots, n-1$, is (see for instance [75], [67], [69], [23])

$$\xi_k(\Omega) = \left(\frac{V_k(\Omega)}{\omega_n}\right)^{\frac{1}{n-k}}. \tag{2.4}$$

In particular for $k = 0$ and $k = 1$ we have

$$\begin{aligned} \xi_0(\Omega) &= \left(\frac{|\Omega|}{\omega_n}\right)^{\frac{1}{n}} \\ \xi_1(\Omega) &= \left(\frac{\mathcal{H}^{n-1}(\partial\Omega)}{n\omega_n}\right)^{\frac{1}{n-1}}. \end{aligned}$$

We observe that in the case Ω is a ball B_R of radius $R > 0$ we get

$$\xi_k(B_R) = R \qquad \forall k = 0, \cdots, n-1,$$

which in part justifies the name of mean radius.

By monotonicity of quermassintegrals on convex sets, we have

$$\xi_k(\Omega_1) \leq \xi_k(\Omega_2),$$

whenever $\Omega_1 \subset \Omega_2$, Ω_1, Ω_2 convex.

Moreover in [81], [75] it has been proved the following isoperimetric inequality

Theorem 2.1. *Let Ω be a bounded, convex set of $\mathbb{R}^n$, then the following isoperimetric inequality holds true*

$$\xi_m(\Omega) \geq \xi_l(\Omega) \qquad \text{for } 0 \leq l \leq m \leq n-1, \tag{2.5}$$

equality holding if and only if Ω is a ball.

Clearly

$$V_k(\Omega) \geq \omega_n \left(\frac{V_{k-1(\Omega)}}{\omega_n} \right)^{\frac{n-k}{n-k+1}},$$

which means that among convex sets Ω with fixed $V_{k-1}(\Omega)$, the balls minimize $V_k(\Omega)$.

We observe that for $k = 1$ (2.5) reduces to the classical isoperimetric inequality (see [9], [69])

$$\frac{1}{n\omega_n^{\frac{1}{n}}} \mathcal{H}^{n-1}(\partial\Omega) \geq |\Omega|^{1-\frac{1}{n}}$$

Recently in [52] the authors prove that (2.5), holds true even if Ω is not convex, but it is only a k-convex starshaped set.

It remains an open problem to prove that (2.5) holds true in general when Ω is only k-convex, for $1 \leq k < n$. Let Ω be a bounded, convex, open set of $\mathbb{R}^n$, with smooth boundary; let u be a non positive measurable function which vanishes on $\partial\Omega$ with convex, smooth level sets. If

$$\Omega_t = \{x \in \Omega : u(x) < t\},$$

then, for $k = 0, \cdots, n-1$, we can define

$$\lambda_k : (-\infty, 0] \to [0, +\infty), \qquad \lambda_k(t) = V_k(\Omega_t), \tag{2.6}$$

and $V_k(\emptyset) = 0$.

By definition of quermassintegrals, for $k = 0$, λ_0 reduces to distribution

$$\lambda_0(t) = |\{x \in \Omega : u(x) < t\}|,$$

while $\lambda_1(t) = \dfrac{1}{n}\lambda(t)$, where $\lambda(t)$ is the perimeter of Ω_t.

We observe that the support of λ_k is $(\inf_\Omega u, 0]$ and its range is $[0, V_k(\Omega))$. Moreover we have

Proposition 2.1. $\forall k = 0, \ldots, n-1, \quad \lambda_k$ *is an non-decreasing function.*

Proof. The proof is a consequence of the monotonicity property of quermassintegrals. Indeed let $t_1 \leq t_2 \leq 0$; clearly $\Omega_{t_1} \subset \Omega_{t_2}$. By definition we have $\lambda_k(\Omega_{t_1}) = V_k(\Omega_{t_1})$ and $\lambda_k(\Omega_{t_2}) = V_k(\Omega_{t_2})$. Since Ω_{t_1} and Ω_{t_2} are convex, then $V_k(\Omega_{t_1}) \leq V_k(\Omega_{t_2})$ and the claim follows. □

Let us define the following class of functions

Definition 2.1. *Let Ω be a convex set of $\mathbb{R}^n$. We denote by $\mathcal{A}(\Omega)$ the subset of functions $u \in C^\infty(\Omega) \cup C^{1,1}(\overline{\Omega})$ such that $u = 0$ on the boundary and u has convex sublevel sets $\Omega_t = \{u < t\}$.*

Remark 2.1. *We observe that if $u \in \mathcal{A}$ u is nonpositive.*

Remark 2.2. *We point out that if $u \in \mathcal{A}$, λ_k is monotone, but it can be seen that the monotonicity property still holds true whenever the convexity of the sublevel sets is replaced by the k-convexity.*

Now we can give the notion of the rearrangements by quermassintegrals.

Definition 2.2. *Let $u : \Omega \to]-\infty, 0]$ be a measurable function which vanishes on $\partial\Omega$ and with convex, smooth level sets. Then the k^{th} increasing rearrangement of u with respect to the k^{th} quermassintegral of its level sets, is*

$$\tilde{u}_k(s) = \sup\{t \leq 0 : \lambda_k(t) \leq s\}, \qquad k = 0, \cdots, n-1, \tag{2.7}$$

where λ_k is defined in (2.6).

For $k = 0$, u_0 is the usual Schwartz rearrangement

By definition we have

- $\tilde{u}_k$ has supported in $(0, V_k(\Omega)]$;
- $\tilde{u}_k$ is a non positive function and its range is $(\inf_\Omega u, 0]$;
- $\tilde{u}_k(\lambda_k(t)) \geq t$, $\forall t \in (\inf_\Omega u, 0]$ and equality holding if u is smooth;
- $\tilde{u}_k$ is a non-decreasing function.

In a similar way we can define the k^{th} decreasing rearrangement by

$$\underline{u}_k(s) = \tilde{u}_k(V_k(\Omega) - s)$$

Moreover we have

- $\underline{u}_k$ has supported in $(0, V_k(\Omega)]$;
- $\underline{u}_k$ is a non positive function and its range is $(\inf_\Omega u, 0]$;
- $\underline{u}_k(\lambda_k(t)) \leq t$, $\forall t \in (\inf_\Omega u, 0]$ and equality holding if u is smooth;
- $\underline{u}_k$ is a non-increasing function.

Let Ω be a bounded, convex, open set of $\mathbb{R}^n$ with smooth boundary and let $u \in \mathcal{A}(\Omega)$. Now, for $k = 0, \cdots, n-1$, we can define the k-symmetrand of u, (see [81], [75])

$$u_k^*(x) = \sup\{t \leq 0 : \xi_k(\Omega_t) \leq |x|, Du \neq 0 \text{ on } \Sigma_t\}, \quad x \in B_R \tag{2.8}$$

where B_R is the ball, centered at the origin with radius $R = \xi_k(\Omega)$, having hence, the same k-quermassintegral of Ω.

By definition we have

- u_k^* is radially symmetric nondecreasing along the radius.
- $u_k^* < 0$ in B_R.
- $V_k(\Omega_t) = V_k(\Omega_t^*)$, where $\Omega_t^* = \{u_k^* < t\}$ are ball centered at the origin with radius $R_t = \xi_k(\Omega_t)$.

Denoted by $\rho_k(r) = u_k^*(|x|)$, where $r = |x|$, we have (see [81], [75])

- ρ_k id defined in $[0, \xi_k(\Omega)]$;
- $\rho_k(0) = \min_{\overline{\Omega}} u$ and $\rho_k(\xi_k(\Omega)) = 0$;

- ρ_k is a non-decreasing function in $[0, \xi_k(\Omega)]$.

Moreover we have (see [81], [75])

Proposition 2.2. *The function $\rho_k \in C^{0,1}([0,R])$ and moreover*

$$0 \le \rho_k'(r) \le \sup_{\Omega} |Du| = M, \qquad a.e.\ in[0,R]. \tag{2.9}$$

Proof. Let $t \le 0$; by (1.19), and by definition of k-quermassintegral, we have

$$\begin{aligned}
\frac{d}{dt}V_k(\Omega_t) &= \binom{n}{k}^{-1} \int_{\Sigma_t} \frac{H_k(\Sigma_t)}{|Du|} d\mathcal{H}^{n-1} \\
&\ge \binom{n}{k}^{-1} \frac{1}{M} \int_{\Sigma_t} H_k(\Sigma_t) d\mathcal{H}^{n-1} \\
&= \frac{n-k}{M} V_{k+1}(\Omega_t).
\end{aligned}$$

Hence we obtain

$$\left(\frac{d}{dt}V_k(\Omega_t)\right) \frac{1}{(n-k)V_{k+1}(\Omega_t)} \ge \frac{1}{M}.$$

By definition of k-mean radius, by monotonicity property of quermassintegral and by isoperimetric inequality (2.5) we have

$$\begin{aligned}
\left(\frac{d}{dt}\xi_k(\Omega_t)\right) &= \frac{1}{n-k} \left(\frac{V_k(\Omega_t)}{\omega_n}\right)^{\frac{1}{n-k}-1} \frac{1}{\omega_n} \frac{d}{dt} V_k(\Omega_t) \\
&\ge \frac{1}{M} \left(\frac{V_k(\Omega_t)}{\omega_n}\right)^{-\frac{n-k-1}{n-k}} \frac{V_{k+1}(\Omega_t)}{\omega_n} \ge \frac{1}{M}.
\end{aligned}$$

Hence we have obtained

$$\left(\frac{d}{dt}\xi_k(\Omega_t)\right) \ge \frac{1}{M}. \tag{2.10}$$

Finally by definition of k-symmetrand of u, we have

$$\left(\frac{d}{dt}\xi_k(\Omega_t)\right) = \frac{1}{\frac{d}{dt}u_k^*(\xi_k(\Omega_t))},$$

ssubstituting in (3.39), we get

$$\rho'(r) \le M, \qquad \text{a. e. } r \in [0,R].$$

□

2.2 Some inequalities about rearrangements

In this section we prove some inequalities about rearrangements with respect to quermassintegrals.

Let Ω be a bounded, convex open set of $\mathbb{R}^n$ with smooth boundary and let u belonging $\mathcal{A}(\Omega)$.

By definition of u_k^* and by (2.5), it follows

$$|u < t| \leq |u_k^* < t|, \tag{2.11}$$

this implies

$$\|u\|_{L^p(\Omega)} \leq \|u_k^*\|_{L^p(B_R)}, \quad p \geq 1, \tag{2.12}$$

for every $p \geq 1$, where B_R, is the ball centered at the origin and with radius $\xi_k(\Omega)$.

Hence u and its k-symmetrand, are not equimeasurable functions.

The following extension of Hardy-Littlewood inequality holds.

Theorem 2.2. *Let Ω be a bounded, convex open set of $\mathbb{R}^n$, with smooth boundary, and let $f, g \in \mathcal{A}(\Omega)$. The following Hardy-Littlewood inequality holds true*

$$\int_\Omega f g \, dx \leq \int_{B_R} f_k^* g_k^* \, dx, \tag{2.13}$$

where B_R is the ball centered at the origin and of radius $\xi_k(\Omega)$, $k = 1, \cdots, n$.

Proof. (2.2) is an immediate consequence of the classical Hardy-Littlewood inequality and (2.5). We have

$$\int_\Omega f g \, dx \leq \int_{\Omega^\#} f^\# g^\# \, dx, \leq \int_{B_R} f_k^* g_k^* \, dx,$$

where $\Omega^\#$ is the ball having the same measure of Ω. The first inequality is the classical Hardy-Littlewood inequality. The second one holds true by the fact that $\Omega^\#$ is contained in B_R for all k and by

$$0 \geq f^\#(x) \geq f_k^*(x),$$

where we assume that the rearrangements are zero outside its domains. □

Now we define the following functionals integral called (p, k)-Hessian integrals (see [75], [81]).

Definition 2.3. *Let Ω be a bounded, convex open set of $\mathbb{R}^n$ with smooth boundary, and let $u \in \mathcal{A}(\Omega)$. The (p,k)-Hessian integral, for $1 \geq p < \infty$ and $k = 1, \cdots, n$, is defined by*

$$I_{p,k}\left[u, \Omega\right] = \int_m^0 dt \int_{\Sigma_t} H_{k-1}(\Sigma_t) \left|Du\right|^{p-1} \, d\mathcal{H}^{n-1}, \tag{2.14}$$

where $m = \min_\Omega u$ and $\Sigma_t = \partial\Omega_t$, for $t \in (m, 0)$. When $p = k+1$, $(k+1,k)$-Hessian integrals are called Hessian integrals and they are denoted by $I_k[u,\Omega] = \int_\Omega S_k(D^2u)(-u)\, dx$.

By (1.17), (1.18), and by coarea formula, (p,k)-Hessian integral can be rewritten as follows

$$I_{p,k}\left[u, \Omega\right] = \int_\Omega S_k^{ij}(D^2u)u_iu_j \left|Du\right|^{p-k-1} \, dx. \tag{2.15}$$

For a radially symmetric function the (p,k)-Hessian integrals can be written ([81], [75])

$$I_{p,k}\left[u^*_{k-1}, B_R\right] = n\binom{n-1}{k-1}\omega_n \int_0^R f^p\left(\omega_n r^{n-k+1}\right) r^{n-k} \, dr \tag{2.16}$$

where $f\left(\omega_n \left|x\right|^{n-k+1}\right) = \left|\nabla u^*_{k-1}(x)\right|$, and $R = \xi_{k-1}(\Omega)$.

We observe that for $k = 1$, since $S_1^{ij} = \delta_{ij}$, $(p,1)$-Hessian integrals are generalizations of the classical Dirichlet energy integrals

$$I_{p,1}\left[u, \Omega\right] = \int_\Omega \left|Du\right|^p \, dx. \tag{2.17}$$

The following result gives an extension of the classical Polya-Szegö principle (see [75], [81]),

Theorem 2.3. *Let Ω be a bounded, convex, open set of $\mathbb{R}^n$ with smooth boundary, and let $u \in \mathcal{A}(\Omega)$. Then for any $p \geq 1$,*

$$I_{p,k}\left[u, \Omega\right] \geq I_{p,k}\left[u^*_{k-1}, B_R\right], \tag{2.18}$$

where B_R is the ball centered at the origin with radius $\xi_{k-1}(\Omega)$.

Proof. For a.e. $t \in (m, 0)$, by Hölder inequality we have

$$\begin{aligned} I_{p,k}[u,\Omega] &= \int_{\Sigma_t} H_{k-1}(\Sigma_t)|Du|^{p-1} \, d\mathcal{H}^{n-1} \\ &\geq \left(\int_{\Sigma_t} H_{k-1}(\Sigma_t) \, d\mathcal{H}^{n-1}\right)^p \left(\int_{\Sigma_t} \frac{H_{k-1}(\Sigma_t)}{|Du|} \, d\mathcal{H}^{n-1}\right)^{1-p}. \end{aligned} \tag{2.19}$$

By Reilly's formula (1.19), isoperimetric inequality (2.5) and by definition of $(k-1)$-symmetrand of u, by (2.19) we have

$$\begin{aligned}
&\int_{\Sigma_t} H_{k-1}(\Sigma_t)|Du|^{p-1}\,d\mathcal{H}^{n-1} \\
&\geq \left[k\binom{n}{k}V_k(\Omega_t)\right]^p \left[\binom{n}{k-1}\frac{d}{dt}V_{k-1}(\Omega_t)\right]^{1-p} \\
&\geq \left[k\binom{n}{k}\omega_n^{\frac{1}{n-k+1}}V_{k-1}^{\frac{n-k}{n-k+1}}(\Omega_t)\right]^p \left[\binom{n}{k-1}\frac{d}{dt}V_{k-1}(\Omega_t)\right]^{1-p} \\
&= \int_{\Sigma_t} H_{k-1}(u^*_{k-1}=t)|Du^*_{k-1}|^{p-1}\,d\mathcal{H}^{n-1}.
\end{aligned} \tag{2.20}$$

Inequality (2.18) then follows from definition of (p,k)-Hessian integrals. □

Let Ω be a bounded, convex, open set of $\mathbb{R}^n$ with smooth boundary. Let f, g be non positive measurable functions vanishing on $\partial\Omega$ with smooth, convex sublevel sets. Let us define the following relation

$$f \prec_{k-1} g \Leftrightarrow \begin{cases} \int_0^{V_{k-1}(\Omega)}(-\tilde{f}_{k-1})\,ds = \int_0^{V_{k-1}(\Omega)}(-\tilde{g}_{k-1})\,ds \\ \int_0^t(-\tilde{f}_{k-1})\,ds \leq \int_0^t(-\tilde{g}_{k-1})\,ds \qquad t\in[0,V_{k-1}(\Omega)], \end{cases}$$

and denote by $K_{k-1}(f)$ the set

$$K_{k-1}(f) = \{\varphi : \varphi \prec_{k-1} f\}.$$

$K_{k-1}(f)$ is the set of those functions with $(k-1)$-rearrangements dominated by the $(k-1)$-rearrangement of f.

The following result shows some properties of the set $K_{k-1}(f)$.

Theorem 2.4. *Let Ω be a bounded, convex, open set of $\mathbb{R}^n$ with smooth boundary. Let f, g be nonpositive measurable functions vanishing on $\partial\Omega$ with smooth, convex sublevel sets. Then the following statements are equivalent*

(i) $\int_0^t(-\tilde{f}_{k-1})\,ds \leq \int_0^t(-\tilde{g}_{k-1})\,ds \qquad t\in[0,V_{k-1}(\Omega)]$,

(ii) for all $\varphi \in L^\infty_+(0,V_{k-1}(\Omega))$*, decreasing ,* $\int_0^{V_{k-1}(\Omega)}\left(-\tilde{f}_{k-1}\right)\varphi\,ds \leq \int_0^{V_{k-1}(\Omega)}(-\tilde{g}_{k-1})\,\varphi\,ds$

(iii) $\int_0^{V_{k-1}(\Omega)} F\left(-\tilde{f}_{k-1}\right)\,ds \leq \int_0^{V_{k-1}(\Omega)} F(-\tilde{g}_{k-1})\,dx$*, for all convex, non-negative Lipschitz function F such that $F(0)=0$.*

Proof. Here we follow [6].

First claim: (i)⇒ (ii); This assertion follows directly by the chain rule

$$\begin{aligned}\int_0^{V_{k-1}(\Omega)}(-\tilde f_{k-1})\varphi\,ds &= -\int_0^{V_{k-1}(\Omega)}\left(\int_0^t(-\tilde f_{k-1})(s)\,ds\right)d\varphi(t)+\varphi(V_{k-1}(\Omega))\int_0^{V_{k-1}(\Omega)}(-\tilde f_{k-1})(t)\,dt.\\ &\le -\int_0^{V_{k-1}(\Omega)}\left(\int_0^t(-\tilde g_{k-1})(s)\,ds\right)d\varphi(t)+\varphi(V_{k-1}(\Omega))\int_0^{V_{k-1}(\Omega)}(-\tilde g_{k-1})(t)\,dt.\\ &= \int_0^{V_{k-1}(\Omega)}(-\tilde g_{k-1})\varphi\,ds\end{aligned}$$

Second claim:(ii) ⇒ (iii); It is enough to prove the claim for C^1 convex functions F.

Let us put

$$\varphi = F'(-\tilde f_{k-1}),$$

then φ satisfies assumptions in (ii) so

$$\int_0^{V_{k-1}(\Omega)}(-\tilde f_{k-1})(F'(-\tilde f_{k-1}))\,ds \le \int_0^{V_{k-1}(\Omega)}(-\tilde g_{k-1})F'(-\tilde f_{k-1})\,ds,$$

that is

$$\int_0^{V_{k-1}(\Omega)}F'(-\tilde f_{k-1})\left((-\tilde g_{k-1})-(-\tilde f_{k-1})\right)ds \ge 0. \tag{2.21}$$

By convexity

$$\int_0^{V_{k-1}(\Omega)}F(-(\tilde g_{k-1}))-F(-(\tilde f_{k-1}))\,ds \ge \int_0^{V_{k-1}(\Omega)}F'(-\tilde f_{k-1})\left((-\tilde g_{k-1})-(-\tilde f_{k-1})\right)ds \ge 0,$$

we obtain the claim.

Third claim: (iii)⇒ (i); To prove this assertion we argue by contradiction: assume (iii) and assume that (i) does not hold. Then, let $[\underline{r},\overline{r}]$ be a maximal interval such that $\int_0^t\left((-\tilde f_{k-1})-(\tilde g_{k-1})\right)ds>0$. Notice that by (iii), we have $\underline{r}>0$ and $\overline{r}<V_{k-1}(\Omega)$ and so $\forall t\in[\underline{r},\overline{r}]$

$$\int_0^{\underline{r}}\left((-\tilde f_{k-1})-(\tilde g_{k-1})\right)ds=0 \quad \int_0^{\overline{r}}\left((-\tilde f_{k-1})-(\tilde g_{k-1})\right)ds=0 \qquad 0\le\underline{r}<\overline{r}\le V_{k-1}(\Omega).$$

Then we can find $r_1\in]\underline{r},\overline{r}[$, such that

$$-\tilde f_{k-1}(r_1) > -\tilde g_{k-1}(r_1).$$

Taking $F(t)=(t-(-\tilde g_{k-1}(r_1)))^+$, we get

$$\begin{aligned}\int_0^{V_{k-1}(\Omega)}F(-\tilde f_{k-1})\,ds &= \int_0^{V_{k-1}(\Omega)}\left((-\tilde f_{k-1}(r))-(-\tilde g_{k-1}(r_1))\right)^+ds \ge \int_0^{r_1}\left((-\tilde f_{k-1}(r))-(-\tilde g_{k-1}(r_1))\right)\\ &> \int_0^{r_1}\left((-\tilde g_{k-1}(r))-(-\tilde g_{k-1}(r_1))\right)ds = \int_0^{V_{k-1}(\Omega)}F(-\tilde g_{k-1})\,ds,\end{aligned}$$

and this is a contradiction. □

An immediate consequence of the previous result is:

Proposition 2.3. *Let Ω be a bounded, convex, open set of $\mathbb{R}^n$ with smooth boundary. Let f, g be non positive measurable functions vanishing on $\partial\Omega$ with smooth, convex sublevel sets. Let $h \in L^\infty_+(0, V_{k-1}(\Omega))$, increasing. If $hf \prec_{k-1} hg$, then*

$$\int_0^{V_{k-1}(\Omega)} h(t)F(-\tilde{f}_{k-1}(t))\,dt \le \int_0^{V_{k-1}(\Omega)} h(t)F(-\tilde{g}_{k-1}(t))\,dt,$$

for all convex, non-negative Lipschitz function F such that $F(0) = 0$.

2.3 Applications to Hessian equations: comparison results

In this section we will show as symmetrization techniques can be used to prove comparison results. In particular we will see that the rearranged solution to certain Dirichlet problem involving Hessian operators, can be estimated by the solution to the conveniently symmetrized problem. Results of this type are well known for a large class of partial differential operators, as for example the Laplacian and the p-Laplacian (see for instance [9],[5],[6].[7] [72]).

Let Ω be a bounded, strictly convex open set of $\mathbb{R}^n$ with smooth boundary. Let u be the solution to problem

$$\begin{cases} S_k(D^2u) = f(x) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{2.22}$$

such that its level sets are convex, and let v the solution to

$$\begin{cases} S_k(D^2v) = f^\# \text{ in } \Omega^*_{k-1} \\ v = 0 \text{ on } \partial\Omega^*_{k-1}, \end{cases} \tag{2.23}$$

where Ω^*_k is the ball centered at the origin having the same k^{th}-quermassintegral of Ω and $f^\#$ is the Schwarz symmetrand of f.

We observe that at least in the case $k = 1$ and $k = n$, there exists a function u having convex sublevel sets (see [24], [59], [68]).

The following comparison result, extends the classical theorem in [72].

Theorem 2.5. *Let us denote by u^*_{k-1} the $(k-1)$-symmetrand of u. Then*

$$v \le u^*_{k-1}.$$

Proof. The proof use standard tools of rearrangements (see for instance [73], [9], [75],[81], [14] [72]). However we give the proof for completeness.

We first observe that the solution to (2.23) can be written explicitly. Indeed since Ω_k^* is a ball, the solution v is radially symmetric. Hence we have $v(x) = v(|x|) = \varphi(r)$, where $|x| = r$ and so we obtain

$$\frac{\partial}{\partial x_i} v = v_i = \varphi' \frac{x_i}{r},$$
$$\frac{\partial^2}{\partial x_i x_j} v = v_{ij} = \varphi'' \frac{x_i x_j}{r^2} + \varphi' \left(\frac{r^2 \delta_{ij} x_i x_j}{r^3} \right),$$

for $i, j = 1, \cdots, n$, and where δ_{ij} denotes the usual Kronecker's symbol.

Since S_k is invariant under rotations (see section 2.1), we can consider only the points of the form $x = (r, 0, \cdots, 0)$ where $D^2 v$ is diagonal and we get

$$D^2 v = \operatorname{diag}(\varphi'', \frac{\varphi'}{r}, \cdots, \frac{\varphi'}{r}).$$

Hence we have

$$\begin{aligned} S_k(D^2 v) &= \binom{n-1}{k-1} \varphi'' \left(\frac{\varphi'}{r} \right)^{k-1} + \binom{n-1}{k} \left(\frac{\varphi'}{r} \right)^k \\ &= \binom{n-1}{k-1} r^{1-n} \left(\frac{r^{n-k}}{k} \varphi' \right)'. \end{aligned}$$

Finally we obtain that φ solves the following one-dimensional problem

$$\begin{cases} \binom{n-1}{k-1} r^{1-n} \left(\frac{r^{n-k}}{k} \varphi' \right)' = f^0(x) \text{ in } (0, \xi_{k-1}(\Omega)) \\ \varphi'(0) = \varphi(\xi_{k-1}(\Omega)) = 0, \end{cases} \tag{2.24}$$

where $f^0(r) = f^\#(|x|)$.

Integrating explicitly we get

$$v(x) = -\left(\frac{n}{\binom{n}{k}} \right)^{\frac{1}{k}} \int_{|x|}^{\xi_{k-1}(\Omega)} \left(r^{-n+k} \int_0^r f^0(s) s^{n-1}\, ds \right)^{\frac{1}{k}} dr.$$

Now let Ω_t be the sub-level set of u, i.e. $\Omega_t = \{x \in \Omega : u(x) < t\}\}$ for $\min_\Omega m \le t < 0$. Let us integrate equation in (2.24) on Ω_t. We obtain

$$\int_{\Omega_t} S_k(D^2 u)\, dx = \int_{\Omega_t} f\, dx \tag{2.25}$$

The following equality holds true (see for instance [66])

$$\int_{\Omega_t} S_k(D^2 u)\, dx = \frac{1}{k} \int_{\Sigma_t} |Du|^k H_{k-1}(\Sigma_t)\, d\mathcal{H}^{n-1}. \tag{2.26}$$

By (1.19), Hölder inequality and by (2.26) we get

$$\begin{aligned}\int_{\Omega_t} S_k(D^2u)\,dx &= \frac{1}{k}\int_{\Sigma_t} |Du|^k H_{k-1}(\Sigma_t)\,d\mathcal{H}^{n-1} \\ &\geq \frac{1}{k}\frac{\left(\int_{\Sigma_t} H_{k-1}(\Sigma_t)\,d\mathcal{H}^{n-1}\right)^{k+1}}{\left(\int_{\Sigma_t} |Du|^{-1}H_{k-1}(\Sigma_t)\,d\mathcal{H}^{n-1}\right)^{k}} \\ &= \frac{1}{k\binom{n}{k-1}^k}\frac{\left(\int_{\Sigma_t} H_{k-1}(\Sigma_t)\,d\mathcal{H}^{n-1}\right)^{k+1}}{\left(\frac{d}{dt}V_{k-1}(\Omega_t)\right)^k}\end{aligned}$$

where Σ_t denotes the boundary of Ω_t, i.e. the t level set of u.

By definition of V_{k-1} we have

$$\int_{\Sigma_t} H_{k-1}(\Sigma_t)\,d\mathcal{H}^{n-1} = n\binom{n-1}{k-1}V_k(\Omega_t) = n\omega_n\binom{n-1}{k-1}(\xi_k)^{n-k}$$

so we can write

$$\int_{\Omega_t} S_k(D^2u)\,dx \geq \frac{1}{k}\frac{\left[n\omega_n\binom{n-1}{k-1}\right]^{k+1}[\xi_k(\Omega_t)]^{(n-k)(k+1)}}{\binom{n}{k-1}^k\left(\frac{d}{dt}V_{k-1}(\Omega_t)\right)^k}$$

By definition of ξ_k we have

$$\frac{d}{dt}V_{k-1}(\Omega_t) = \omega_n(n-k+1)\left(\xi_{k-1}(\Omega_t)\right)^{n-k}\frac{d}{dt}\left(\xi_{k-1}(\Omega_t)\right);$$

thus using the isoperimetric inequality for the mean radius, which we can use since the level sets of u are convex, we get

$$\int_{\Omega_t} S_k(D^2u)\,dx \geq \frac{1}{k}\frac{\left[n\binom{n-1}{n-k}\right]^{k+1}\omega_n\left[\xi_{k-1}(\Omega_t)\right]^{(n-k)}}{\binom{n}{k-1}^k(n-k+1)^k\left(\frac{d}{dt}\xi_{n-k+1}(\Omega_t)\right)^k}.$$

Now setting $r = \xi_{k-1}(\Omega_t)$ and observing that

$$u^*_{k-1}\left(\xi_{k-1}(\Omega_t)\right)' = \left(\frac{d}{dt}\xi_{k-1}(\Omega_t)\right)^{-1},$$

we finally obtain

$$\int_{\Omega_t} S_k(D^2u)\,dx \geq \omega_n\binom{n}{k}r^{n-k}\left(u^*_{k-1}(r)'\right)^k. \tag{2.27}$$

Let us come back to (2.25). For the integral in the right hand side of (2.25), by Hardy-Littlewood inequality, we obtain

$$\int_{\Omega_t} f\,dx \leq \int_{\Omega_t} f^\#\,dx \leq n\omega_n\int_0^{\xi_{k-1}(\Omega_t)} f^0(s)s^{n-1}\,ds. \tag{2.28}$$

Now by (2.25), (2.27) and (2.28), we get

$$\left(u_{k-1}^*(r)'\right)^k \leq n\binom{n}{k}^{-1} r^{-(n-k)} \int_0^r f^0(s)s^{n-1}\,ds \tag{2.29}$$

where $r \in [0, \xi_{k-1}(\Omega_t)]$. Integrating between r and $R = \xi_{k-1}(\Omega)$, we obtain the assert. $\square$

Recently it has been proved a comparison result for the solutions to more general problems involving hessian operators where f can depends also on u (see for instance [14].). We recall it in the chapter 5 since it needs to define the eigenvalue problem associated to Hessian operators.

Chapter 3

Hardy-Sobolev type inequalities for (p, k)-Hessian Integrals

In this chapter, using symmetrization for quermassintegrals, we prove Hardy-Sobolev type inequalities for (p, k)-Hessian integrals which extend the classical results, known for Sobolev functions (see for instance [75], [62], [63], [36] and [11]).

3.1 Hardy-Sobolev inequalities for (p, k)-Hessian Integrals

The classical Sobolev inequality, also called Sobolev embedding theorem, for weakly differentiable functions is (see for instance [45], [11], [73], [62], [63], [1])

Theorem 3.1. *Let Ω be a bounded, open set of $\mathbb{R}^n$. Then there exists a positive constant $C = C(n, p)$ such that for any $u \in W_0^{1,p}(\Omega)$,*

$$\|u\|_{np/n-p} \leq C(n,p)\|Du\|_p \quad \text{for } p < n; \tag{3.1}$$

$$\sup_{\Omega} |u| \leq C(n,p)|\Omega|^{\frac{1}{n-1/p}}\|Du\|_p, \ \text{for } p > n, \tag{3.2}$$

which mean

$$W_0^{1,p}(\Omega) \subset L^{np/n-p}(\Omega) \ \text{if } p < n;$$

$$W_0^{1,p}(\Omega) \subset C^0(\overline{\Omega}) \ \text{if } p > n.$$

An important question concerning inequality (3.1), was to find the smallest constant which is admissible in (3.1). The answer to this question, has been given in [73], where the author, using symmetrization argument, has been proved that if $\Omega = \mathbb{R}^n$, the best value of the constant C is

$$C = \pi^{-\frac{1}{2}} n^{-\frac{1}{p}} \left(\frac{p-1}{n-p} \right)^{1-1/p} \left\{ \frac{\Gamma(1+n/2)\Gamma(n)}{\Gamma(n/p)\Gamma(1+n-n/p)} \right\}^{1/n}, \qquad 1 < p < n. \tag{3.3}$$

An extension of the Sobolev inequality (3.1) is the following (see for instance [36], [62], [63])

Theorem 3.2. *Let Ω be a bounded, open set of $\mathbb{R}^n$. Then there exists a positive constant $C = C(n,p,q,s,\Omega)$ such that for any $u \in W_0^{1,p}(\Omega)$,*

$$\left(\int_\Omega \frac{u^q}{|x|^s} \, dx \right)^{\frac{p}{q}} \leq C \int_\Omega |Du|^p \, dx, \tag{3.4}$$

where $1 < p < n$, $0 \leq s < p$ and $q \leq \dfrac{p(n-s)}{n-p}$.

Inequality (3.4) is called Hardy-Sobolev type inequality and it is an extension of (3.1) which we get taking $s = 0$.

In the chapter 3, we have defined (p,k)-Hessian integrals as an extension of the energy integrals. It is hence clear what is now our goal. We deal to obtain sharp Hardy-Sobolev type inequality for (p,k)-hessian integrals, which extend (3.4) obtained taking $k = 1$.

First results in this direction are contained for instance in [85], [75], [31], where essentially it has been proved a Sobolev type inequality for (p,k)-hessian integrals which extend (3.1); the main result is

Theorem 3.3. *Let Ω be a bounded, convex, open set of $\mathbb{R}^n$ with smooth boundary and let be $u \in \mathcal{A}(\Omega)$, $(k = 1, \ldots, n)$. If $1 < p < n-k+1$ then exists a constant C depending only from n, k, p, Ω, such that*

$$I_{p,k}[u,\Omega] \geq C(n,p,k,\Omega) \left[\int_\Omega |u|^q \, dx \right]^{\frac{p}{q}} \tag{3.5}$$

for $q = \dfrac{pn}{n-k-p+1}$.

In the paper [41] a more general result, which extend (3.4), is given. It is the following

Theorem 3.4. *Let Ω be a bounded, convex, open set of $\mathbb{R}^n$ with smooth boundary and let be $u \in \mathcal{A}(\Omega)$, $(k = 1, \ldots, n)$. If $1 < p < n-k+1$ then exists a constant C depending* *only from n, k, p, s, Ω, such that*

$$I_{p,k}\left[u,\Omega\right] \geq C\left(n,p,k,q,s,\Omega\right)\left[\int_{\Omega}\frac{|u|^{q}}{|x|^{s}}\,dx\right]^{\frac{p}{q}} \tag{3.6}$$

for $q \leq \dfrac{p\,(n-s)}{n-k-p+1}$ and $0 \leq s < p+k-1$. The constant C is given by

$$C = \binom{n-1}{k-1}\left(\frac{p+k-s-1}{p}\right)^{\frac{p}{q}-1}\frac{\pi^{-\frac{p}{2}}}{n-s}\quad\left(\frac{p-1}{n-p-k+1}\right)^{p-1}$$
$$\left[\frac{\Gamma\left(1+\frac{p(n-s)}{2(p+k-s-1)}\right)\Gamma\left(\frac{p(n-s)}{p+k-s-1}\right)}{\Gamma\left(\frac{n-s}{p+k-s-1}\right)\Gamma\left(1+\frac{(p-1)(n-s)}{p+k-s-1}\right)}\right]^{\frac{p+k-s-1}{n-s}} \tag{3.7}$$

Proof. Let us consider the $(k-1)$-symmetrand of u, u^*_{k-1}. Denoted by $R = \xi_{k-1}(\Omega)$, by Hardy-Littlewood inequality (2.2) we have:

$$\int_{\Omega}\frac{|u|^{q}}{|x|^{s}}\,dx \leq \int_{B_R}\frac{\left|u^*_{k-1}\right|^{q}}{|x|^{s}}\,dx. \tag{3.8}$$

Then Polya-Szegö principle for Hessian Integrals (2.18) and (3.8) allows us to prove inequality (3.6) only for the $(k-1)$-symmetrand of u, i.e. u^*_{k-1}. Now writing

$$u^*_{k-1}(x) = u^*_{k-1}(|x|) = \rho(r), \qquad r = |x|,$$

we have, by (2.16):

$$I_{p,k}\left[u^*_{k-1}(|x|), B_R\right] = n\omega_n\binom{n-1}{k-1}\int_0^R(\rho'(r))^p r^{n-k}\,dr \tag{3.9}$$

In order to prove (3.6), we have to show that the following one-dimensional inequality holds true

$$n\omega_n\binom{n-1}{k-1}\int_0^R(\rho'(r))^p r^{n-k}\,dr \geq n\omega_n C\left[\int_0^R|\rho|^q\,r^{n-1-s}\,dr\right]^{\frac{p}{q}} \tag{3.10}$$

Let $s < p+k-1$ and let us make the change of variable $t = r^{p/(p+k-s-1)}$. Then the inequality (3.10) becames

$$\int_0^{R^{p/(p+k-s-1)}}(\rho'(t))^p t^{d-1}\,dt \geq \overline{C}\left[\int_0^{R^{p/(p+k-s-1)}}|\rho(t)|^q t^{d-1}\,dt\right]^{\frac{p}{q}} \tag{3.11}$$

where

$$d = \frac{p\,(n-s)}{p+k-s-1} \qquad \text{and} \tag{3.12}$$

$$\overline{C} = C\binom{n-1}{k-1}^{-1}\left(\frac{p}{p+k-s-1}\right)^{\frac{p}{q}+p-1}. \tag{3.13}$$

Now, extending to zero the function ρ in $\mathbb{R}^+$ inequality (5.1), and then (5.1), follows immediately from Lemma 2 in [73]. The constant in such Lemma is sharp and a straightforward computation gives the constant in (3.7). We finally observe that after the change of variables, the inequality (5.1) can be viewed as a Sobolev embedding theorem in the fractional dimension d in (3.12) and we must require

$$p < d \quad \text{i.e.} \quad p < n-k+1.$$

□

Remark 3.1. *We observe that in particular cases inequality (3.1) reads as follows:*

(i) If we take $s = 0$ we obtain the Sobolev inequality for (p,k)-Hessian Integrals (3.5).

(ii) If we take $k = 1$ then we have

$$I_{p,1}\left[u,\Omega\right] = \int_\Omega |Du|^p \, dx$$

hence the inequality (3.1) becames the well known Hardy-Sobolev inequality 3.4([62], [11], [63], [39]) and if we take also $s = 0$ we have the classical Sobolev inequality with the best constant (see [73]).

For inequality (3.1), as the classical case for $s = p$, the exponent, $s = p + k - 1$, is critical. In this case as for Lebesgue's integrals, we obtain a different inequality from 3.6, which is the Hardy inequality for (p,k)-Hessian integrals (see [41])

Theorem 3.5. *Let Ω be a bounded, convex open set of $\mathbb{R}^n$ with smooth boundary and let $u \in A_{k-1}(\Omega)$, $k = 1, \ldots, n$. If $1 < p < n-k+1$ the following inequality holds:*

$$I_{p,k}\left[u,\Omega\right] \geq \binom{n-1}{k-1}\left(\frac{n-k-p+1}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^{p+k-1}}\, dx \tag{3.14}$$

Before proving the inequality (3.14), we need the following Lemma about the Hardy type inequalities in one dimension that we find in [11].

Lemma 3.1. *Let ψ be a nonnegative measurable function on $(0,\infty)$ and suppose $-\infty < \lambda < 1$ and $1 < d \leq \infty$. Then the following inequality holds:*

$$\left\{\int_0^\infty \left(t^{1-\lambda}\int_t^\infty \psi(s)\frac{1}{s}\,ds\right)^d \frac{1}{t}\,dt\right\}^{\frac{1}{d}} \leq \frac{1}{1-\lambda}\left\{\int_0^\infty \left(t^{1-\lambda}\psi(t)\right)^d \frac{1}{t}\,dt\right\}^{\frac{1}{d}} \tag{3.15}$$

Now we proceed with the proof of (3.14).

Proof of Theorem 3.5. We will see that the inequality (3.14) is an immediate consequence of the previous Lemma. Indeed, we can rewrite (3.14) as one-dimensional inequality.

Let us consider $u^*_{k-1}(x)$, the $(k-1)$-symmetrand of u. Using the same arguments of the proof of Theorem 3.4, we again can prove inequality (3.14) only for $u^*_{k-1}(x)$. Writing

$$u^*_{k-1}(x) = u^*_{k-1}(|x|) = \rho(r),$$

we must prove the following one dimensional inequality:

$$\begin{aligned} I_{p,k}\left[u, B_R\right] &= n\omega_n \binom{n-1}{k-1} \int_0^R (\rho')^p r^{n-k}\,dr \\ &\geq \binom{n-1}{k-1}\left(\frac{n-k-p+1}{p}\right)^p \int_0^R |\rho|^p\, r^{n-k-p}\,dr. \end{aligned} \tag{3.16}$$

Then taking

$$\begin{aligned} \psi(s) &= \rho'(s)s \\ \lambda &= \frac{2p+k-n-1}{p} < 1 \\ d &= p \end{aligned}$$

and extending the function ρ to zero for $r > R$, by inequality (3.15) we have (3.16).

□

3.2 Optimality of constants

In this section we prove that inequality (3.1) and (3.14) are sharp.

Now we show the optimality of the constant in (3.14). A similar argument can be repeated for (3.1). For all $\epsilon > 0$ we define the following functions:

$$u_\epsilon\left(x\right) = u_\epsilon\left(|x|\right) = \left(\frac{1}{R^q+\epsilon^q}\right)^{\frac{n-k+1}{p}} - \left(\frac{1}{r^q+\epsilon^q}\right)^{\frac{n-k+1}{p}}$$

where $q = \frac{n-k-p+1}{n-k+1}$, $\quad R = \xi_{n-k+1}(\Omega)$ and $r = |x|$.

We prove that

$$\lim_{\epsilon \to 0} \frac{I_{p,k}\left[u_\epsilon, \Omega\right]}{\int_\Omega \frac{|u_\epsilon|^p}{|x|^{p+k-1}}\, dx} = \binom{n-1}{k-1} \left(\frac{n-k-p+1}{p}\right)^p. \tag{3.17}$$

First we prove that $u_\epsilon \in \mathcal{A}(\Omega)$. We observe that

- $u_\epsilon \in C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega}), \quad \forall \epsilon > 0$
- $u_\epsilon \le 0 \quad$ in Ω and $u_\epsilon = 0 \quad$ on $\partial\Omega$

Moreover, since u_ϵ is radially symmetric, its level sets are balls and hence they are convex, hence $u_\epsilon \in \mathcal{A}$. Now we consider the limit (3.17). We have:

$$\begin{aligned} I_{p,k}\left[u_\epsilon, B_R\right] =& n\omega_n \binom{n-1}{k-1} \int_0^R (u_\epsilon')^p r^{n-k}\, dr \\ =& n\omega_n \binom{n-1}{k-1} \left(\frac{n-k-p+1}{p}\right)^p \int_0^{R/\epsilon} \left(\frac{1}{r^{\frac{n-k-p+1}{n-k+1}}+1}\right)^{n-k+p+1} r^{n-k-\frac{p^2}{n-k+1}}\, dr \end{aligned}$$

On the other hand we have:

$$\int_\Omega \frac{|u_\epsilon|^p}{|x|^{p+k-1}}\, dx = n\omega_n \int_0^{R/\epsilon} \left[\left(\frac{1}{r^{\frac{n-k-p+1}{n-k+1}}+1}\right)^{\frac{n-k+1}{p}} - \left(\frac{1}{\left(\frac{R}{\epsilon}\right)^{\frac{n-k-p+1}{n-k+1}}+1}\right)^{\frac{n-k+1}{p}}\right]^p r^{n-k-p}\, dr. \tag{3.18}$$

Setting $\psi(r) = \dfrac{1}{r^{\frac{n-k-p+1}{n-k+1}}+1}$, the limit (3.17) can be written as follows:

$$\begin{aligned} &\lim_{\epsilon \to 0} \frac{I_{p,k}\left[u_\epsilon, \Omega\right]}{\int_\Omega \frac{|u_\epsilon|^p}{|x|^{p+k-1}}\, dx} \\ &= \lim_{s \to \infty} \binom{n-1}{k-1} \left(\frac{n-k-p+1}{p}\right)^p \frac{\displaystyle\int_0^s \psi(r)^{n-k+p+1} r^{n-k-\frac{p^2}{n-k+1}}\, dr}{\displaystyle\int_0^s \left[\psi(r)^{\frac{n-k+1}{p}} - \psi(s)^{\frac{n-k+1}{p}}\right]^p r^{n-k-p}\, dr} \\ &= \lim_{s \to \infty} \binom{n-1}{k-1} \left(\frac{n-k-p+1}{p}\right)^p \frac{\displaystyle\int_0^s \psi(r)^{n-k+p+1} r^{n-k-\frac{p^2}{n-k+1}}\, dr}{\displaystyle\psi(s)^{n-k+1} \int_0^s \left[\frac{\psi(r)}{\psi(s)}^{\frac{n-k+1}{p}} - 1\right]^p r^{n-k-p}\, dr} \end{aligned} \tag{3.19}$$

Now we set

$$\begin{aligned} \frac{1}{\sigma} &= \frac{\psi(r)}{\psi(s)} \\ r &= \left[\frac{\sigma}{\psi(s)} - 1\right]^{\frac{n-k+1}{n-k-p+1}} \end{aligned}$$

Hence the integral in the denominator of (3.19) becomes:

$$\psi(s)^{n-k+1}\int_0^s \left[\frac{\psi(r)}{\psi(s)}^{\frac{n-k+1}{p}}-1\right]^p r^{n-k-p}\,dr =$$

$$= \left(\frac{n-k+1}{n-k-p+1}\right)\psi(s)^{n-k}\int_{\psi(s)}^1 \left[\left(\frac{1}{\sigma}\right)^{\frac{n-k+1}{p}}-1\right]^p \left(\frac{\sigma}{\psi(s)}-1\right)^{n-k} d\sigma$$

$$= \left(\frac{n-k+1}{n-k-p+1}\right)\sum_{j=0}^{n-k}(-1)^{n-k-j}\psi(s)^{n-k-j}\binom{n-k}{j}\int_{\psi(s)}^1 \left[\left(\frac{1}{\sigma}\right)^{\frac{n-k+1}{p}}-1\right]^p \sigma^j\,d\sigma. \tag{3.20}$$

Now we observe that:

$$\lim_{s\to\infty}\psi(s)^{n-k-j}\quad\int_{\psi(s)}^1 \left[\left(\frac{1}{\sigma}\right)^{\frac{n-k+1}{p}}-1\right]^p \sigma^j\,d\sigma = \frac{1}{n-k-j}\qquad \text{if } j\neq n-k, \tag{3.21}$$

and

$$\lim_{s\to\infty}\frac{\displaystyle\int_{1/t_s}^1 \left[\left(\frac{1}{\sigma}\right)^{\frac{n-k+1}{p}}-1\right]^p \sigma^{n-k}\,d\sigma}{-\log\psi(s)} = 1 \tag{3.22}$$

Using (3.19), (3.20),(3.21), (3.22), we have,

$$\lim_{s\to\infty}\binom{n-1}{k-1}\left(\frac{n-k-p+1}{p}\right)^p \frac{\displaystyle\int_0^s \psi(r)^{n-k+p+1} r^{n-k-\frac{p^2}{n-k+1}}\,dr}{\psi(s)^{n-k+1}\displaystyle\int_0^s \left[\frac{\psi(r)}{\psi(s)}^{\frac{n-k+1}{p}}-1\right]^p r^{n-k-p}\,dr} =$$

$$= \binom{n-1}{k-1}\left(\frac{n-k-p+1}{p}\right)^p \lim_{s\to\infty}\frac{\displaystyle\int_0^s \psi(r)^{n-k+p+1} r^{n-k-\frac{p^2}{n-k+1}}\,dr}{\left(\frac{n-k+1}{n-k-p+1}\right)(-\log\psi(s))}$$

$$= \binom{n-1}{k-1}\left(\frac{n-k-p+1}{p}\right)^p.$$

Hence (3.17) holds and $\binom{n-1}{k-1}\left(\frac{n-k-p+1}{p}\right)^p$ is the best constant in the inequality (3.14).

3.3 Improved Hardy inequality for (p,k)-Hessian Integrals

In the previous section we have proved the Hardy inequality for (p,k)-Hessian Integrals and we have given the best value of the constant.

Since the best value of the constant, is not achieved, in this section we deal to improve Hardy inequality (3.5), by adding on the right-hand side of (3.14) a second term involving the singular weight $\left(\frac{1}{\log\left(\frac{1}{|x|}\right)}\right)^{\gamma}$, analogously to the classical energy integrals (see [16], [29], [84]). Moreover we will prove that the optimal value of γ is $\gamma = 2$.

The main result is the following (see [41])

Theorem 3.6. *Let Ω be a bounded, convex, open set of $\mathbb{R}^n$ with smooth boundary. Let C be a positive constant such that $C \geq \sup_{\Omega}\left(|x|\, e^{\frac{2}{p}}\right)$ and $1 < p < n-k+1$. Then the following statements hold:*

i) there exists a constant $C_1 > 0$ depending on n, p, k, C such that

$$\begin{aligned} I_{p,k}[u,\Omega] \geq &\binom{n-1}{k-1}\left(\frac{n-k-p+1}{p}\right)^p \int_{\Omega} \frac{|u|^p}{|x|^{p+k-1}}\, dx + \\ &+ C_1 \int_{\Omega} \frac{|u|^p}{|x|^{p+k-1}} \left(\frac{1}{\log\left(\frac{C}{|x|}\right)}\right)^{\gamma} dx \end{aligned} \tag{3.23}$$

for every functions $u \in \mathcal{A}(\Omega)$, if and only if $\gamma \geq 2$.

ii) For $2 \leq p < n-k+1$ there exists a constant $C_2 > 0$ depending on n, p, q, C, Ω such that for any $u \in \mathcal{A}(\Omega)$, we have

$$\begin{aligned} I_{p,k}[u,\Omega] \geq &\binom{n-1}{k-1}\left(\frac{n-k-p+1}{p}\right)^p \int_{\Omega} \frac{|u|^p}{|x|^{p+k-1}}\, dx + \\ &+ C_1 \int_{\Omega} \frac{|u|^p}{|x|^{p+k-1}} \left(\frac{1}{\log\left(\frac{C}{|x|}\right)}\right)^{2} dx + C_2 \left[\int_{\Omega} \frac{|u|^q}{|x|^{\beta}}\, dx\right]^{\frac{p}{q}} \end{aligned} \tag{3.24}$$

where $1 < q < \dfrac{p(n-\beta)}{n-k-p+1}$ and $0 \leq \beta < p+k-1$.

This theorem extends the results in [29] and [16], which hold for energy integrals.

Proof. We organize the proof in the following manner: first we prove the validity of inequalities (3.23) and (3.24) and finally we show the optimality of $\gamma = 2$.

Let $u \in \mathcal{A}(\Omega)$, since the singular weight $(\log(\frac{C}{|x|}))^{\gamma}$ is a decreasing function with respect to $r = |x|$ under our assumption on C, we can prove both inequalities, (3.23) and (3.24), only for the $(k-1)$-symmetrand of u, i.e. $u^*_{k-1}(x)$ by Hardy-Littlewood inequality for rearrangements(see [9], [11]) and Polya-Szegö principle for Hessian Integral (2.18).

Writing

$$u_{k-1}^*(|x|) = \rho(r),$$

and setting $d = n-k+1$, the inequalities (3.23) and (3.24) that we have to prove become respectively the following one-dimensional inequalities

$$\begin{aligned} I_{p,k}\left[u_{k-1}^*(|x|), B_R\right] =& \binom{n-1}{k-1} \int_0^R (\rho'(r))^p r^{d-1}\, dr \\ \geq& \binom{n-1}{k-1} \left(\frac{n-k-p+1}{p}\right)^p \int_0^R \frac{|\rho|^p}{r^p} r^{d-1}\, dr + \\ &+ C_1 \int_0^R \frac{|\rho|^p}{r^p (\log(\frac{C}{r}))^\gamma} r^{d-1}\, dr \end{aligned} \tag{3.25}$$

$$\begin{aligned} I_{p,k}\left[u_{k-1}^*(|x|), B_R\right] =& \binom{n-1}{k-1} \int_0^R (\rho'(r))^p r^{d-1}\, dr \\ \geq& \binom{n-1}{k-1} \left(\frac{n-k-p+1}{p}\right)^p \int_0^R \frac{|\rho|^p}{r^p} r^{d-1}\, dr + \\ &+ C_1 \int_0^R \frac{|\rho|^p}{r^p \log(\frac{C}{r})^2} r^{d-1}\, dr + \\ &+ C_2 \left[\int_0^R \frac{|\rho|^q}{r^{\beta-k+1}} r^{d-1\, dr}\right]^{\frac{p}{q}}. \end{aligned} \tag{3.26}$$

We observe that (3.25) and (3.26) follow from Theorem 1.1 in [29] where the dimension n is replaced by d.

Now we shall prove the optimality.

We suppose that $1 < p < n-k+1$ and $0 \leq \gamma < 2$. We observe that in the case $\gamma = 0$ the optimality holds true since $\beta_{n,p,k} = \binom{n-1}{k-1}(\frac{n-k-p+1}{p})^p$ is the best constant in the Hardy inequality (3.5). Hence we can suppose $0 < \gamma < 2$.

Now we define the following functional

$$J_\gamma(u) = \frac{I_{p,k}\left[u, \Omega\right] - \binom{n-1}{k-1} \left(\frac{n-k-p+1}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^{p+k-1}}\, dx}{\int_\Omega \frac{|u|^p}{|x|^{p+k-1}} \left(\frac{1}{\log\left(\frac{C}{|x|}\right)}\right)^\gamma dx}. \tag{3.27}$$

To prove optimality of $\gamma = 2$ we shall prove that

$$\inf_{\substack{u \in \mathcal{A}(\Omega) \\ u \neq 0}} J_\gamma(u) = 0. \tag{3.28}$$

To prove (3.28), we will construct a sequence $u_\epsilon \in A_{k-1}(\Omega) \quad \forall \epsilon > 0$ such that

$$\lim_{\epsilon \to 0} J_\gamma(u_\epsilon) = 0. \tag{3.29}$$

Now, without loss of generality, we suppose that Ω is the unit ball $B_1(0)$, we set $\alpha = \frac{n-k-p+1}{p}$, and for every $\epsilon > 0$, we define the following function

$$\tilde{u}_\epsilon = \begin{cases} -\dfrac{2}{e\alpha\epsilon^{2\alpha}\log(\frac{1}{\epsilon})} & 0 \le r \le \epsilon^2 \\ -\dfrac{2}{e\alpha\epsilon^{2\alpha}\log(\frac{1}{\epsilon})} + \dfrac{\log(\frac{r}{\epsilon^2})}{r^{\alpha\log(\frac{1}{\epsilon})}} & \epsilon^2 \le r \le \epsilon^2 e^{\frac{1}{\alpha}} \\ -\dfrac{\log(\frac{r}{\epsilon^2})}{r^{\alpha\log(\frac{1}{\epsilon})}} & \epsilon^2 e^{\frac{1}{\alpha}} \le r \le \epsilon \\ -\dfrac{\log(r)}{r^{\alpha\log(\frac{1}{\epsilon})}} & \epsilon \le r \le 1, \end{cases} \tag{3.30}$$

where $r = |x|$.

We observe that this function is negative, increasing, continuous, has convex level sets and it is null on the boundary of $B_1(0)$ but it is not a smooth function, hence $\tilde{u}_\epsilon \notin \mathcal{A}(B_1(0))$.

We moreover observe that if we compute $J_\gamma(\tilde{u}_\epsilon)$ we obtain that (3.29) holds true. In fact using the same arguments of [29]we have:

$$\int_{B_1} \frac{|\tilde{u}_\epsilon|^p}{|x|^{p+k-1}}\, dx \ge \frac{2n\omega_n}{p+1}\log(\frac{1}{\epsilon}) \tag{3.31}$$

Moreover we have

$$\begin{aligned} I_{p,k} &= n\omega_n \binom{n-1}{k-1} \int_0^1 (\frac{\partial \tilde{u}_\epsilon}{\partial r})^p r^{n-k}\, dr \\ &= \frac{2n\omega_n \binom{n-1}{k-1}\alpha^p}{p+1}\log(\frac{1}{\epsilon}) + O(\frac{1}{\log(\frac{1}{\epsilon})}). \end{aligned} \tag{3.32}$$

Hence by (3.31) and (3.32) we obtain that the numerator of $J_\gamma(\tilde{u}_\epsilon)$ is:

$$I_{p,k}\left[u,\Omega\right] - \binom{n-1}{k-1}\left(\frac{n-k-p+1}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^{p+k-1}} \le O(\frac{1}{\log(\frac{1}{\epsilon})}). \tag{3.33}$$

Now we consider the denominator of $J_\gamma(\tilde{u}_\epsilon)$ and we have

$$\int_\Omega \frac{|u|^p}{|x|^{p+k-1}} \left(\frac{1}{\log\left(\frac{C}{|x|}\right)}\right)^\gamma dx \ge \tilde{C}(\log(\frac{1}{\epsilon}))^{1-\gamma}, \tag{3.34}$$

where $\tilde{C}$ is a positive constant depending only by n, p.

Finally, by (3.34) and (3.33) and since $0 < \gamma < 2$ we have

$$J_\gamma(\tilde{u}_\epsilon) \to 0 \quad \text{for} \quad \epsilon \to 0, \tag{3.35}$$

hence the claim is proved.

Now in order, to have a sequence $u_\epsilon \in \mathcal{A}(B_1(0))$ that fulfils (3.29), we regularize $\tilde{u}_\epsilon$. But before, to have that the regularization of $\tilde{u}_\epsilon$ converges to $\tilde{u}_\epsilon$ uniformly in $B_1(0)$, we extend $\tilde{u}_\epsilon(r)$ up to $r = 1 + \epsilon$ as follows

$$\overline{u}_\epsilon = \begin{cases} \tilde{u}_\epsilon(r) & 0 \le r \le 1 \\ \frac{1}{\log(\frac{1}{\epsilon})} r - \log(\frac{1}{\epsilon}) & 1 \le r \le 1 + \epsilon. \end{cases} \tag{3.36}$$

Now we define the regularization of $\overline{u}_\epsilon$ as follows

$$\overline{u}_{\epsilon,h}(x) = h^{-n} \int_{B_{1+\epsilon}(0)} \rho(\frac{x-y}{h}) \overline{u}_\epsilon(y)\, dy, \tag{3.37}$$

where $h > 0$ and such that $h < \text{dist}(x, \partial B_{1+\epsilon}(0))$ and ρ is the usual mollifier.

Now we prove that $\overline{u}_{\epsilon,h} \in A_{k-1}(B_1(0))$. First of all we need to have $\overline{u}_{\epsilon,h}(x) = 0$ on $\partial B_1(0)$. Hence we define the following function

$$v_{\epsilon,h}(x) = \overline{u}_{\epsilon,h}(x) - \overline{C}, \tag{3.38}$$

where $\overline{C} = \overline{u}_{\epsilon,h}(x) \mid_{\partial B_1(0)}$.

Using the property of $\tilde{u}_\epsilon$, it is not difficult to prove the following statements:

- $v_{\epsilon,h}$ is a radially function on $B_1(0)$ and its level sets are convex.
- $v_{\epsilon,h}$ is increasing with respect to $r = |x|$ on $B_1(0)$, for h sufficiently small.
- $v_{\epsilon,h} = 0$ on $\partial B_1(0)$
- $v_{\epsilon,h}$ is a negative function on $B_1(0)$.

Hence $v_{\epsilon,h} \in \mathcal{A}(B_1(0))$.

Finally we shall prove that

$$J_\gamma(v_{\epsilon,h}) \to 0 \quad \text{for } \epsilon \to 0. \tag{3.39}$$

Since $v_{\epsilon,h}$ is a radially function and setting $v_{\epsilon,h}(|x|) = \varphi(r)$, we have

$$\begin{aligned} J_\gamma(u) &= \frac{I_{p,k}\left[u,\Omega\right] - \binom{n-1}{k-1}\left(\frac{n-k-p+1}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^{p+k-1}}\,dx}{\int_\Omega \frac{|u|^p}{|x|^{p+k-1}}\left(\frac{1}{\log\left(\frac{C}{|x|}\right)}\right)^\gamma dx} \\ &= \frac{\binom{n-1}{k-1}\int_0^1 (\varphi'(r))^p r^{n-k}\,dr - \binom{n-1}{k-1}\left(\frac{n-k-p+1}{p}\right)^p \int_0^1 |\varphi(r)|^p r^{n-p-k}\,dr}{\int_0^1 \frac{|\varphi(r)|^p}{(\log(\frac{C}{r}))^\gamma} r^{n-p-k}\,dr}. \end{aligned} \tag{3.40}$$

By the following

$$v_{\epsilon,h} \to \tilde{u}_\epsilon \quad \text{for } \epsilon \to 0 \qquad \text{uniformly on } B_1(0);$$

$$Dv_{\epsilon,h} = (Dv_\epsilon)_h \to D\tilde{u}_\epsilon \quad \text{for } \epsilon \to 0 \qquad \text{uniformly on } B_1(0),$$

we have

$$J_\gamma(v_{\epsilon,h}) \to J_\gamma(\tilde{u}_\epsilon) \qquad \text{for } h \to 0. \tag{3.41}$$

Finally, by (3.35) and (5.8), we have

$$J_\gamma(v_{\epsilon,h}) \to 0 \qquad \text{for } \epsilon \to 0,$$

hence the optimality is proved.

This completes the proof of the theorem. □

Chapter 4

Eigenvalue problems for k-Hessian operators

In this chapter we consider the eigenvalue problems for Hessian operators and using symmetrization for quermassintegrals, we prove a Faber-Krahn inequality for the first eigenvalue and a Payne-Rayner inequality for eigenfunctions.

4.1 Payne-Rayner type inequalities for eigenfunctions

Let us consider the eigenvalue problem associated to k-Hessian operator

$$\begin{cases} S_k(D^2u) = \lambda(-u)^k \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{4.1}$$

In [85] and [44], it has been proved the following result

Theorem 4.1. *If Ω is $(k-1)$ strictly convex, there exists a positive constant λ_k which depends only on n, k, and Ω, such that problem (4.1) admits a negative solution $u \in C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega}) \cap \Phi_k^2$ for $\lambda = \lambda_k$ moreover*

(i) if $(\mu, \psi) \in [0, \infty) \times \left(C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})\right)$, is another solution of (4.1), then $\lambda_k = \mu$ and $\psi = \alpha u$, for some positive constant α, i.e. λ_k is simple;

(ii) if $\Omega_1 \subset \Omega_2$ then $\lambda_k(\Omega_1) \geq \lambda_k(\Omega_2)$.

In [85], it has also been given the following variational characterization of $\lambda_k(\Omega)$

$$\lambda_k = \min_{\substack{u\in\Psi_0^k(\Omega)\\ u\neq 0}} \frac{\int_\Omega(-u)S_k(D^2u), dx}{\int_\Omega(-u)^{k+1}, dx}. \tag{4.2}$$

We refer to λ_k as the eigenvalue of k-Hessian operator and to u as eigenfunction of k-Hessian operator.

For $k = 1$ and $k = n$ we obtain respectively the first eigenvalue of the Laplacian operator, λ_1, and the eigenvalue of Monge-Ampère operator, λ_n. Various inequalities have been given for λ_1 and λ_n and their corresponding eigenfunctions. For example in [30] it has been shown that the following isoperimetric inequality, named reverse Hölder inequality or *Payne-Rayner inequality*, holds true for the first eigenfunction of the laplacian operator

$$\|u\|_r \leq K(r, q, n, \lambda_1)\|u\|_q \tag{4.3}$$

for $0 < q < r < +\infty$, $n \geq 2$, and where K is a suitable positive constant and K is sharp.

A similar result holds true also for the first eigenfunction of p-laplacian operator under suitable assumptions (see [5], [2]).

In the case $k = n$, i.e. the Monge-Ampère operator, in dimension $n = 2$, in [13] it has been proven a similar inequality, where in the right hand side of (4.3) does not appear $\|u\|_q$, but $\|u^\star\|_q$, where $u^\star$ is the rearrangement of u with respect to the perimeter of its level lines. The aim of this section is to obtain a Payne-Rayner type inequality for eigenfunctions of k-Hessian operator, which will extend (4.3), obtained putting $k = 1$ and $k = n$ respectively.

To prove this result we argue as in [5], [13], [2]. It is essentially based on a comparison result between u and a suitable eigenfunction v of k-Hessian operator with Ω replaced by a ball B, centered at the origin and such that the corresponding eigenvalue is equal to λ_k. Moreover this result is a consequence of an another comparison result between solutions and subsolutions of a one-dimensional eigenvalue problem of the following type

$$\begin{cases} -(|\varphi'|^{\gamma-2}\varphi' s^\beta)' = \mu s^\alpha |\varphi|^{\gamma-2}\varphi & s \in (0, b) \\ \varphi(0) = \varphi'(b) = 0 \end{cases} \tag{4.4}$$

In the next sections, we give general results about the first eigenvalue of (1.9) and we prove the comparison result which allows us to prove finally, the Payne-Rayner inequality.

4.1.1 The one-dimensional case

In this section we give some general results about the first eigenvalue of the following one-dimensional problem

$$\begin{cases} -(|\varphi'|^{\gamma-2}\varphi' s^{\beta})' = \mu s^{\alpha}|\varphi|^{\gamma-2}\varphi & s \in (0,b) \\ \varphi(0) = \varphi'(b) = 0 \end{cases} \tag{4.5}$$

where:

- $b \in \mathbb{R}$, $b > 0$;
- $\gamma = \dfrac{k+1}{k}$;
- $\beta = -(\gamma - 1)\left(\dfrac{n}{n-k+1} - 1\right)$;
- $\alpha = -\gamma\left(\dfrac{n-k}{n-k+1}\right)$.

Let E be a bounded and measurable subset of $\mathbb{R}^n$, $n \geq 2$, let $c, d \in L^1(E)$ be a non negative functions and let be $1 \leq p \leq +\infty$. We put

$$L^p(E,c) = \{u : E \to \mathbb{R},\ \text{measurable},\ \|u\|_{L^p(E,c)} = \int_E c(x)|u(x)|^p\,dx < +\infty\}.$$

We denote by $W_0^{1,p}(E,c,d)$ the completion of the space $C_0^{\infty}(E)$ with respect to the following norm

$$\|u\|_{W_0^{1,p}(E,c,d)} = \|u\|_{L^p(E,c)} + \|Du\|_{L^p(E,d)}$$

If the case $c(x) = x^{\alpha}$ and $d(x) = x^{\beta}$ the following theorem (see [63]) is useful to establish whenever the embedding of $W_0^{1,\gamma}((0,b), s^{\alpha}, s^{\beta})$ in the weighted space $L^{\gamma}((0,b), s^{\alpha})$ is compact.

Theorem 4.2. *Let be p, q, such that $1 \leq p \leq q < +\infty$. Let be $\alpha, \beta \in \mathbb{R}$, $0 < b < +\infty$ and $p' = \frac{p}{p-1}$. Then*

$$\left(\int_0^b |u|^q x^{\alpha}\,dx\right)^{\frac{1}{q}} \leq C\left(\int_0^b |u'|^p x^{\beta}\,dx\right)^{\frac{1}{p}} \tag{4.6}$$

if and only if

(i) $\alpha \geq \beta\dfrac{p}{q} - \dfrac{q}{p'} - 1$, *if* $\beta \neq p - 1$;

(ii) $\alpha > -1$, *if* $\beta = p - 1$.

Now we can give the following result:

Theorem 4.3. *Let b, γ, α, β be as previous. Then for every $k = 1, \dots, n$, exists μ_1, the first eigenvalue of (4.5), and it is positive and simple. Moreover every positive eigenfunction of (4.5) associated with μ_1 is strictly increasing in $(0, b)$.*

Proof. We argue as in [5], [13]. First we prove the existence of μ_1 and of the corresponding eigenfunction.

In order to prove our claim, it is enough to prove that the following eigenvalue problem

$$\begin{cases} -(|\phi'|^{\gamma-2}\phi'\sigma(s))' = \lambda\tau(s)|\phi|^{\gamma-2}\phi & s \in (0, 2b) \\ \phi(0) = \phi(2b) = 0 \end{cases} \tag{4.7}$$

where

$$\sigma(s) = \begin{cases} s^\beta \text{ if } s \in (0, b); \\ (2b - s)^\beta \text{ if } s \in (b, 2b); \end{cases} \qquad \tau(s) = \begin{cases} s^\alpha \text{ if } s \in (0, b); \\ (2b - s)^\alpha \text{ if } s \in (b, 2b), \end{cases}$$

admits a unique positive solution up to a multiplicative constant, which is symmetric with respect to the line $s = b$ and strictly increasing in $(0, b)$.

Indeed the first eigenvalue μ_1 of the problem (4.5), can be found as minimum of the following Rayleigh quotient

$$\mu_1 = \min_{\substack{\varphi \in W^{1,\gamma}((0,b), s^\alpha, s^\beta) \\ \varphi(0)=0 \\ \varphi \neq 0}} \frac{\int_0^b |\varphi'|^\gamma s^\beta \, ds}{\int_0^b |\varphi|^\gamma s^\alpha \, ds}, \tag{4.8}$$

else the first eigenvalue ν_1 of the problem (4.7), can be found as minimum of the following Rayleigh quotient

$$\nu_1 = \min_{\substack{\phi \in W_0^{1,\gamma}((0,2b), \tau(s), \sigma(s)) \\ \phi \neq 0}} \frac{\int_0^{2b} |\phi'|^\gamma \sigma(s) \, ds}{\int_0^{2b} |\phi|^\gamma \tau(s) \, ds}. \tag{4.9}$$

We observe that by symmetry, we can extend in $(0, 2b)$ any function φ which realizes the minimum in (4.8), and it can be used as test function in (4.9). Conversely, if ϕ minimizes (4.9), its restriction to $(0, b)$, can be used as test function in (4.8). Hence, now we prove the claim for the problem (4.7).

The existence of a first eigenfunction and of the first eigenvalue ν_1, can be proven by standard tools of the calculus of variations. It is enough to observe that, since $\beta < \gamma - 1$ and $\alpha > \beta - \gamma$, by our assumptions, the embedding of $W_0^{1,\gamma}((0, b), s^\alpha, s^\beta)$ in the space $L^\gamma((0, b), s^\alpha)$ is compact by Theorem 4.2.

Now we prove that every positive eigenfunctions of (4.7) associated with the first eigenvalue is strictly increasing in $(0, b)$.

We observe that if ϕ_1 minimize (4.9) then $|\phi_1|$ does also, i.e. is an eigenfunction.

By Harnack's inequality (see [74])it follows

$$\sup_I |\phi_1| \leq C \inf_I |\phi_1| \tag{4.10}$$

for every $I \subset (0, b)$.

By (4.10) it follows then $|\phi_1| > 0$ or $|\phi_1| \equiv 0$ in $(0, b)$. Hence we obtain that ϕ_1 has constant sign in $(0, b)$.

Now we suppose that ϕ_1 is positive and we prove that ϕ_1 is strictly increasing in $(0, b)$. The symmetry of the problem (4.7), ensure that a positive solution is symmetric with respect to the line $s = b$. Since ϕ_1 is a positive eigenfunction of (4.5) then ϕ_1 is a distributional solution to

$$\begin{cases} -(|\phi_1'|^{\gamma-2}\phi_1'\sigma(s))' = \nu_1\tau(s)(\phi_1)^{\gamma-1}\phi_1 & s \in (0, b) \\ \phi_1(0) = \phi_1(2b) = 0. \end{cases} \tag{4.11}$$

Hence it belongs to $C^1_{\text{loc}}(0, 2b)$ (see for instance [35]). So we can integrate the equation between s and b where $s < b$ and we have

$$-\int_s^b (|\phi_1'|^{\gamma-2}\phi_1'\sigma(s))'\,ds = \nu_1 \int_s^b \tau(s)|\phi_1|^{\gamma-2}\phi_1\,ds.$$

Since $\phi_1 > 0$, then we obtain

$$|\phi_1'|^{\gamma-2}\phi_1'\sigma(s) = \nu_1 \int_s^b \tau(s)\phi_1^{\gamma-1}\,ds$$

Hence $\phi_1' > 0$ in $(0, b)$, i.e. ϕ_1 is strictly increasing in $(0, b)$.

Finally now we prove that λ_1 is simple, i.e. all the other associated eigenfunctions are scalar multiples of each other. Arguing as in [10] and [56] the simplicity of λ_1 is a consequence of the convexity with respect to ϕ^γ of the following functional

$$\int_0^{2b} |\phi'|^\gamma \sigma(s)\,ds,$$

and this complete the proof. □

4.1.2 Faber-Krahn and reverse Hölder inequalities

In this section we will prove the Faber-Krahn inequality for the first eigenvalue of Hessian operators.

Let us consider the eigenvalue problem for Hessian operator (4.1). The first eigenvalue satisfies the following comparison result

Theorem 4.4. *Let Ω be a bounded strictly convex open set of $\mathbb{R}^n$, with $k = 1, \dots, n$. Let $\lambda_k(\Omega)$ be the eigenvalue of (4.1) and $\lambda_k(\Omega^*_{k-1})$ the eigenvalue of the following problem*

$$\begin{cases} S_k(D^2(v)) = \lambda(-v)^k & \text{in } \Omega^*_{k-1} \\ v = 0 & \text{on } \partial\Omega^*_{k-1} \end{cases} \tag{4.12}$$

where Ω^*_{k-1} is the ball centered at the origin and such that $V_{k-1}(\Omega) = V_{k-1}(\Omega^*_{k-1})$. Then the following inequality holds:

$$\lambda_k(\Omega) \geq \lambda_k(\Omega^*_{k-1}), \tag{4.13}$$

equality holding whenever Ω is a ball.

Proof. Let $u \in \Phi^2_k(\Omega)$ be an eigenfunction associated to $\lambda_k(\Omega)$ such that its level lines are convex. An eigenfunction that satisfies this assumption exists at least for $k = 1$ and $k = 2$ (see for instance [59], [58], [55], [24]).

By (4.2), we get

$$\lambda_k(\Omega) = \min_{\substack{v \in \Psi^k_0(\Omega) \\ v \neq 0}} \frac{\int_\Omega (-v) S_k(\lambda(D^2 v))\, dx}{\int_\Omega (-v)^{k+1}\, dx} = \frac{\int_\Omega (-u) S_k(\lambda(D^2 u))\, dx}{\int_\Omega (-u)^{k+1}} dx. \tag{4.14}$$

By (2.26, (2.14), coarea formula, Polya-Szegö for Hessian integrals (2.18) and Hardy-Littlewood inequality (2.2) we obtain

$$\begin{aligned} \frac{\int_\Omega (-u) S_k(\lambda(D^2 u)), dx}{\int_\Omega (-u)^{k+1}, dx} &= \frac{\int_m^0 dt \int_{\Sigma_t} H_{k-1}(\Sigma_t)\, |Du|^k\ d\mathcal{H}^{n-1}}{\int_\Omega (-u)^{k+1}, dx} \\ &\geq \binom{n-1}{k-1} \frac{\int_0^R ((u^*_{k-1})')^k r^{n-k}\, dr}{\int_0^R (-u^*_{k-1})^{k+1} r^{n-1}, dr} \\ &\geq \min_{\substack{w \in C^{0,1}\Omega^*_{k-1} \\ w \neq 0 \\ w' \geq 0}} \frac{\binom{n-1}{k-1} \int_0^R (w')^k r^{n-k}\, dr}{\int_0^R r^{n-1} (-w)^{k+1}\, dr} = \eta_1(\Omega^*_{k-1}) \end{aligned} \tag{4.15}$$

where $m = \min_\Omega u$, $R = \xi_{k-1}(\Omega)$, and η_1 is the first eigenvalue of the following one-dimensional problem

$$\begin{cases} \binom{n-1}{k-1} r^{-n+1} (\frac{r^{n-k}}{k} (w')^k)' = \eta(-w)^k & r \in (0, R) \\ w(R) = w'(0) = 0 \end{cases} \tag{4.16}$$

Denoting by w_1 the eigenfunction associated to η_1, then w_1 is unique up to positive scalar multiplication, and w_1 is a weak solution of (4.16). If we consider the eigenfunction

v_1 associated to the eigenvalue $\lambda_k(\Omega^*_{k-1})$, by (4.2), we observe that also v_1 is a classical solution of (4.16) with respect to $\eta_1(\Omega^*_{k-1}) = \lambda_k(\Omega^*_{k-1})$.

By simplicity of η_1, we obtain that $\eta_1 = \lambda_k(\Omega^*_{k-1})$ and w_1 coincides with v_1 up to multiplicative positive constant. Then we get

$$\lambda_k(\Omega) \geq \lambda_k(\Omega^*_{k-1}). \tag{4.17}$$

If $\lambda_k(\Omega) = \lambda_k(\Omega^*_{k-1})$, one can easily check that the level sets of u, $\{u \leq t\}$ are balls and that the quantities

$$H_{k-1}(\Sigma_t)\, |Du|^k\,,$$

are constant on $\{u = t\}$. This obviously implies that $|Du|$ is constant on $\{u = t\}$. Since $|Du| \neq 0$ we can conclude as in [7] □

Now we can prove a Payne-Rayner type inequality for eigenfunctions of the Hessian operators in a strictly convex set $\Omega \subset \mathbb{R}^n$.

Before giving the main result we quote the following lemma

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded, strictly convex open set. Let u be a solution to (4.1) with convex level sets. Then the following inequality holds:*

$$\tilde{u}'_{k-1}(s) \leq \frac{(\lambda_k n)^{\frac{1}{k}}}{(n-k+1)^{\frac{k+1}{k}}} \frac{s^{-\frac{(k+1)(n-k)}{k(n-k+1)}}}{\binom{n}{k}^{\frac{1}{k}} \omega_n^{\frac{2}{n-k+1}}} \left(\int_0^s t^{\frac{n}{n-k+1}-1} (-\tilde{u}_{k-1})^k \, dt \right)^{\frac{1}{k}} \tag{4.18}$$

where $s \in [0, V_{n-k+1}(\Omega)]$ and $\tilde{u}_{k-1} = u^*_{k-1}\left(\left(\frac{s}{\omega_n}\right)^{\frac{1}{n-k+1}} \right)$

Proof. We first observe that the assumption of the existence of a solution to (4.1) with convex level sets, is satisfied at least for $k = 1$ and $k = 2$ (see for instance [59], [58], [55], [24]).

(4.18) follows by standard tools about rearrangements (see for instance [9], [72], [14]). In the next we argue as in [14]. Let Ω_t be the t sublevel set of u, i.e. $\Omega_t = \{x \in \Omega : u(x) < t\}\}$ for $\min_\Omega \leq t < 0$ }.

Let us integrate equation in (4.24) on Ω_t. Since u is an eigenfunction of S_k, we obtain

$$\int_{\Omega_t} S_k(D^2u)\, dx = \lambda_k \int_{\Omega_t} (-u)^k \, dx \tag{4.19}$$

By (2.26), Hölder inequality and by (1.19) we get

$$\int_{\Omega_t} S_k(D^2u)\,dx = \frac{1}{k}\int_{\Sigma_t} |Du|^k H_{k-1}(\Sigma_t)\,d\mathcal{H}^{n-1}$$
$$\geq \frac{1}{k}\frac{\left(\int_{\Sigma_t} H_{k-1}(\Sigma_t)\,d\mathcal{H}^{n-1}\right)^{k+1}}{\left(\int_{\Sigma_t} |Du|^{-1}H_{k-1}(\Sigma_t)\,d\mathcal{H}^{n-1}\right)^{k}}$$
$$= \frac{1}{k\binom{n}{k-1}^k}\frac{\left(\int_{\Sigma_t} H_{k-1}(\Sigma_t)\,d\mathcal{H}^{n-1}\right)^{k+1}}{\left(\frac{d}{dt}V_{k-1}(\Omega_t)\right)^{k}}$$

where Σ_t denotes the boundary of Ω_t, i.e. the t level set of u.

By definition of V_{k-1} we have

$$\int_{\Sigma_t} H_{k-1}(\Sigma_t)\,d\mathcal{H}^{n-1} = n\binom{n-1}{n-k}V_k(\Omega_t) = n\omega_n\binom{n-1}{n-k}(\xi_k)^{n-k}$$

so we can write

$$\int_{\Omega_t} S_k(D^2u)\,dx \geq \frac{1}{k}\frac{\left[n\omega_n\binom{n-1}{k-1}\right]^{k+1}[\xi_k(\Omega_t)]^{(n-k)(k+1)}}{\binom{n}{k-1}^k\left(\frac{d}{dt}V_{k-1}(\Omega_t)\right)^k}$$

By definition of ξ_k we have

$$\frac{d}{dt}V_{k-1}(\Omega_t) = \omega_n(n-k+1)\left(\xi_{k-1}(\Omega_t)\right)^{n-k}\frac{d}{dt}\left(\xi_{k-1}(\Omega_t)\right);$$

thus recalling the isoperimetric inequality (2.5), we get

$$\int_{\Omega_t} S_k(D^2u)\,dx \geq \frac{1}{k}\frac{\left[n\binom{n-1}{n-k}\right]^{k+1}\omega_n\,[\xi_{k-1}(\Omega_t)]^{(n-k)}}{\binom{n}{k-1}^k\left(\frac{d}{dt}\xi_{k-1}(\Omega_t)\right)^k}.$$

Now setting $r = \xi_{k-1}(\Omega_t)$ and observing that

$$u^*_{k-1}\left(\xi_{k-1}(\Omega_t)\right)' = \left(\frac{d}{dt}\xi_{k-1}(\Omega_t)\right)^{-1}$$

we finally obtain

$$\int_{\Omega_t} S_k(D^2u)\,dx \geq \omega_n\binom{n}{k}r^{n-k}\left(u^*_{k-1}(r)'\right)^k.$$

Let us come back to (4.19). For the integral in the right hand side of (4.19), by Hardy-Littlewood inequality (2.2), we obtain

$$\int_{\Omega_t} |u|^k\,dx \leq n\omega_n\int_0^{\xi_{k-1}(\Omega_t)}(-u^*_{k-1}(t))^k t^{n-1}\,dt \tag{4.20}$$

Now by (4.19), (4.1.2) and (4.20) we get

$$\left(u_{k-1}^*(r)'\right)^k \leq \binom{n}{k}^{-1} r^{-(n-k)}\lambda_k n \int_0^r (-u_{k-1}^*(t))^k t^{n-1}\, dt \tag{4.21}$$

with $r \in [0, \xi_{k-1}(\Omega_t)]$.

Now we make a change of variables putting $s = \omega_n r^{n-k+1}$. Denoting by $\tilde{u}_{k-1}(s) = u_{k-1}^*\left(\left(\frac{s}{\omega_n}\right)^{\frac{1}{n-k+1}}\right)$ and substituting in (4.21) we get

$$\tilde{u}_{k-1}'(s) \leq \frac{(\lambda_k n)^{\frac{1}{k}}}{(n-k+1)^{\frac{k+1}{k}}} \frac{s^{-\frac{(k+1)(n-k)}{k(n-k+1)}}}{\binom{n}{k}^{\frac{1}{k}} \omega_n^{\frac{2}{n-k+1}}} \left(\int_0^s t^{\frac{n}{n-k+1}-1}(-\tilde{u}_{k-1})^k\, dt\right)^{\frac{1}{k}} \tag{4.22}$$

where $s \in [0, V_{k-1}(\Omega)]$. □

Remark 4.1. *Inequality (4.18) for $k = 1$ and $k = n$ becomes respectively the results of [72] and of [13].*

Remark 4.2. *If we consider the solution v of the following problem*

$$\begin{cases} S_k(D^2(v)) = \lambda(-v)^k & \text{in } \Omega_{k-1}^* \\ v = 0 & \text{on } \partial\Omega_{k-1}^* \end{cases} \tag{4.23}$$

then (4.18) becomes an equality and v coincides with its $(k-1)$-symmetrand.

Let us consider a fixed eigenfunction u with convex level sets, to the following problem:

$$\begin{cases} S_k(D^2(u)) = \lambda(-u)^k & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.24}$$

where $\Omega \subset \mathbb{R}^n$ be a smooth, bounded, strictly convex open set. Let B_R be the ball centered at the origin such that $\lambda_k(\Omega) = \lambda_k(B_R)$. A straightforward calculation shows that

$$B = \left\{x \in \mathbb{R}^n : |x| < \left(\frac{\kappa_1}{\lambda_k(\Omega)}\right)^{\frac{1}{2k}}\right\},$$

where κ_1 is the first eigenvalue of the problem (4.24) in the unit ball.

Let v_q be the eigenfunction corresponding to $\lambda_k(B_R)$ such that

$$\int_0^{V_{k-1}(\Omega)} r^{\frac{n}{n-k+1}-1}(-\tilde{u}_{k-1})^q\, dr = \int_0^{V_{k-1}(B_R)} r^{\frac{n}{n-k+1}-1}(-\tilde{v}_{k-1})^q\, dr \quad 0 < q < +\infty \tag{4.25}$$

and let v_∞ be the eigenfunction corresponding to $\lambda_1(B_R)$ having the same L^∞ norm as u.

The following comparison result holds.

Theorem 4.5. *Let u and v_q be defined as above then the following statements hold:*

(i) if $0 < q < +\infty$ *then*

$$\int_0^s r^{\frac{n}{n-k+1}-1}(-\tilde{u}_{k-1})^q \, dr \le \int_0^s r^{\frac{n}{n-k+1}-1}(-\tilde{v}_{k-1})^q \, dr \tag{4.26}$$

where $s \in [0, V_{k-1}(B_R)]$*;*

(ii) if $q = +\infty$ *then*

$$-\tilde{u}_{k-1}(s) \ge -v_\infty(s). \tag{4.27}$$

Proof. We put

$$U(s) = \int_0^s r^{\frac{n}{n-k+1}-1}(-\tilde{u}_{k-1})^k \, dr, \quad s \in [0, V_{k-1}(\Omega)] \tag{4.28}$$

$$V(s) = \int_0^s r^{\frac{n}{n-k+1}-1}(-\tilde{v}_{k-1})^k \, dr, \quad s \in [0, V_{k-1}(B_R)]. \tag{4.29}$$

We observe that U and V solve the following one-dimensional problems

$$\begin{cases} -\left[\left(\dfrac{U'(s)}{s^{\frac{n}{n-k+1}-1}}\right)^{\frac{1}{k}}\right]' \le \dfrac{(n\lambda_k)^{\frac{1}{k}}}{(n-k+1)^{\frac{k+1}{k}}\binom{n}{k}\omega_n^{\frac{2}{n-k+1}}} s^{-\frac{k+1}{k}\left(\frac{n-k}{n-k+1}\right)} U^{\frac{1}{k}}(s) \quad \text{q.o. in } [0, V_{k-1}(\Omega)] \\ U(0) = U'(V_{k-1}(\Omega)) = 0 \end{cases} \tag{4.30}$$

$$\begin{cases} -\left[\left(\dfrac{V'(s)}{s^{\frac{n}{n-k+1}-1}}\right)^{\frac{1}{k}}\right]' = \dfrac{(n\lambda_k)^{\frac{1}{k}}}{(n-k+1)^{\frac{k+1}{k}}\binom{n}{k}\omega_n^{\frac{2}{n-k+1}}} s^{-\frac{k+1}{k}\left(\frac{n-k}{n-k+1}\right)} V^{\frac{1}{k}}(s) \quad \text{q.o. in } [0, V_{k-1}(B_R)] \\ V(0) = V'(V_{k-1}(B_R)) = 0 \end{cases} \tag{4.31}$$

We start to consider the case $q = k$. First of all we observe that by Faber-Krahn inequality (4.13), we have $V_{k-1}(B_R) \le V_{n-k+1}(\Omega)$ and hence, by our assumptions, we get

$$U(V_{k-1}(B_R)) \le V(V_{k-1}(B_R)).$$

We will prove that $U(s) \le V(s)$ in $(0, V_{k-1}(B_R))$.

Let us suppose, ab absurdo, that there is a positive maximum of $W(s) = U(s) - V(s)$ in $[0, V_{k-1}(B_R)]$. Then there exists $s_1 \in [0, V_{k-1}(B_R)]$ such that $W(s_1) \ge 0$ and $W'(s_1) = U'(s_1) - V'(s_1) = 0$.

Let us define the following function

$$Z(s) = \begin{cases} U(s) & s \in [0, s_1]; \\ V(s) + W(s_1) & s \in [s_1, V_{k-1}(B_R)]. \end{cases}$$

Since the function $\left(\frac{Z'(s)}{s^{\frac{n}{n-k+1}-1}}\right)^{\frac{1}{k}}$ is absolutely continuous in $[\epsilon, V_{k-1}(B_R)]$, $\forall \epsilon > 0$, we have

$$\left(\frac{Z'(s)}{s^{\frac{n}{n-k+1}-1}}\right)^{\frac{1}{k}} = \begin{cases} \left(\frac{U'(s)}{s^{\frac{n}{n-k+1}-1}}\right)^{\frac{1}{k}} & s \in [0, s_1]; \\ \left(\frac{V'(s)}{s^{\frac{n}{n-k+1}-1}}\right)^{\frac{1}{k}} & s \in [s_1, V_{k-1}(B_R)]. \end{cases}$$

Hence by (4.30) and (4.31), Z solves the following:

$$\begin{cases} -\left[\left(\frac{Z'(s)}{s^{\frac{n}{n-k+1}-1}}\right)^{\frac{1}{k}}\right]' \le \frac{(n\lambda_k)^{\frac{1}{k}}}{(n-k+1)^{\frac{k+1}{k}}\binom{n}{k}\omega_n^{\frac{2}{n-k+1}}} s^{-\frac{k+1}{k}\left(\frac{n-k}{n-k+1}\right)} Z^{\frac{1}{k}}(s) \quad \text{q.o. in } [0, V_{k-1}(B_R)] \\ Z(0) = Z'(V_{k-1}(B_R)) = 0 \end{cases} \tag{4.32}$$

Now we claim that

$$\lim_{\epsilon \to 0^+} \left(\frac{Z'(\epsilon)}{\epsilon^{\frac{n}{n-k+1}-1}}\right)^{\frac{1}{k}} Z(\epsilon) = 0; \tag{4.33}$$

we will show (4.33) at the end of the proof.

Multiplying both sides of (4.32) by $Z(s)$, integrating by parts and using (4.33), we get

$$\int_0^{V_{k-1}(B_R)} \frac{(Z'(s))^{\frac{k+1}{k}}}{\left(s^{\frac{n}{n-k+1}-1}\right)^{\frac{1}{k}}} ds \le \mu \int_0^{V_{k-1}(B_R)} Z(s)^{\frac{k+1}{k}} s^{-\frac{k+1}{k}\frac{n-k}{n-k+1}} ds,$$

where

$$\mu = \frac{(n\lambda_k)^{\frac{1}{k}}}{(n-k+1)^{\frac{k+1}{k}}\binom{n}{k}\omega_n^{\frac{2}{n-k+1}}}.$$

Hence we obtain

$$\frac{\displaystyle\int_0^{V_{k-1}(B_R)} \frac{(Z'(s))^{\frac{k+1}{k}}}{\left(s^{\frac{n}{n-k+1}-1}\right)^{\frac{1}{k}}} ds}{\displaystyle\int_0^{V_{k-1}(B_R)} Z(s)^{\frac{k+1}{k}} s^{-\frac{k+1}{k}\frac{n-k}{n-k+1}} ds} \le \mu$$

This implies, by simplicity of μ (see Theorem 4.3), that Z is proportional to V. Hence by its definition, it follows that $Z(s)$ coincides with V. So we have a contradiction, since we have supposed that $W(s_1) > 0$.

In order to conclude the proof for $q = k$, we have to prove (4.33). First of all, we observe that by Theorem 4.2, the embedding

$$W^{1,\frac{k+1}{k}}((0,V_{k-1}(B_R)),s^{-\frac{k+1}{k}\frac{n-k}{n-k+1}},s^{-\frac{1}{k}(\frac{n}{n-k+1}-1)})\hookrightarrow L^{\frac{k+1}{k}}((0,V_{k-1}(B_R)),s^{-\frac{k+1}{k}\frac{n-k}{n-k+1}}),$$

is compact.

By absolutely continuity of $\left(\frac{Z'(s)}{s^{\frac{n}{n-k+1}-1}}\right)^{\frac{1}{k}}$, and by (4.32) we have

$$\left(\frac{Z'(\epsilon)}{\epsilon^{\frac{n}{n-k+1}-1}}\right)^{\frac{1}{k}} Z(\epsilon)\le \mu^{k+1}\left(\int_\epsilon^{V_{k-1}(B_R)} s^\alpha Z^{\frac{1}{k}}\,ds\right)\left[\int_0^\epsilon s^\beta\left(\int_s^{V_{k-1}(B_R)} t^\alpha Z^{\frac{1}{k}}\,dt\right)^k ds\right],$$

where $\alpha=-\frac{k+1}{k}\frac{n-k}{n-k+1}$ and $\beta=\frac{n}{n-k+1}-1$.

By Hölder inequality and since $Z\in L^{\frac{k+1}{k}}((0,V_{k-1}(B_R)),s^\alpha)$, we have

$$\left(\frac{Z'(\epsilon)}{\epsilon^{\frac{n}{n-k+1}-1}}\right)^{\frac{1}{k}} Z(\epsilon)\le \mu^{k+1}\|Z\|^{\frac{k+1}{k}}\left(J(\epsilon)\right)^{\frac{k}{k+1}}\left[\int_0^\epsilon s^\beta\left(J(s)\right)^{\frac{k^2}{k+1}}\,ds\right] \tag{4.34}$$

where

$$J(x)=\int_x^b t^\alpha\,dt,\qquad b=V_{n-k+1}(B_R).$$

To calculate $J(x)$, we have to consider separately the cases $k>\frac{n}{2}$, $k<\frac{n}{2}$, $k=\frac{n}{2}$. Indeed we have

$$J(x)=\begin{cases} C(n,k)\left(b^{\frac{2k-n}{k(n-k+1)}}-x^{\frac{2k-n}{k(n-k+1)}}\right) & \text{if } k>\frac{n}{2};\\ |C(n,k)|\left(x^{\frac{2k-n}{k(n-k+1)}}-b^{\frac{2k-n}{k(n-k+1)}}\right) & \text{if } k<\frac{n}{2};\\ \log\left(\frac{b}{x}\right) & \text{if } k=\frac{n}{2},\end{cases}$$

where $C(n,k)=\left(\frac{2k-n}{k(n-k+1)}\right)^{-1}$.

Now we come back to (4.34). For $k>\frac{n}{2}$ we have

$$\left(\frac{Z'(\epsilon)}{\epsilon^{\frac{n}{n-k+1}-1}}\right)^{\frac{1}{k}} Z(\epsilon)\le \mu^{k+1}\|Z\|^{\frac{k+1}{k}}C(n,k)b^{\frac{2k-n}{k(n-k+1)}}\epsilon^{\frac{n}{n-k+1}},$$

hence letting ϵ to 0 we have (4.33).

If $k<\frac{n}{2}$ we get

$$\left(\frac{Z'(\epsilon)}{\epsilon^{\frac{n}{n-k+1}-1}}\right)^{\frac{1}{k}} Z(\epsilon)\le \mu^{k+1}\|Z\|^{\frac{k+1}{k}}|C(n,k)|C_1(n,k)\epsilon^{\frac{2k}{n-k+1}},$$

where $C_1=\frac{2k^2+n}{(k+1)(n-k+1)}>0$. Hence again letting ϵ to *zero* we have (4.33).

Finally for $k=\frac{n}{2}$ we obtain

$$\left(\frac{Z'(\epsilon)}{\epsilon^{\frac{n-2}{n+2}}}\right)^{\frac{2}{n}} Z(\epsilon)\le \mu^{\frac{n+2}{2}}\|Z\|^{\frac{n+2}{n}}\left(\log\left(\frac{b}{\epsilon}\right)\right)^{\frac{n}{n+2}}\int_0^\epsilon s^{\frac{n-2}{n+2}}\left(\log\left(\frac{b}{s}\right)\right)^{\frac{n^2}{2(n+2)}}\,ds.$$

Then by Hopital theorem we have

$$\lim_{\epsilon\to 0}\left(\log\left(\frac{b}{\epsilon}\right)\right)^{\frac{n}{n+2}}\int_0^{\epsilon} s^{\frac{n-2}{n+2}}\left(\log\left(\frac{b}{s}\right)\right)^{\frac{n^2}{2(n+2)}}ds=\lim_{\epsilon\to 0}\left(\epsilon^{\frac{4n}{(n+2)^2}}\log\left(\frac{b}{\epsilon}\right)\right)^{\frac{n+2}{2}}=0.$$

This conclude the proof of (i) for $q=k$.

Let us consider the general case $0<q<+\infty$. We define

$$\begin{aligned} U(s) &= \int_0^s r^{\frac{n}{n-k+1}-1}(-\tilde{u}_{k-1})^q\,dr, \quad s\in[0,V_{k-1}(\Omega)] \\ V(s) &= \int_0^s r^{\frac{n}{n-k+1}-1}(-\tilde{v}_{k-1})^q\,dr, \quad s\in[0,V_{k-1}(B_R)]. \end{aligned}$$

By definition we have $U_q(0)=V_q(0)=0$, and by our assumptions, $U_q(V_{k-1}(B_R))\le V_q(V_{k-1}(B_R))$.

If, ab absurdo, $U_q\not\leq V_q$ then there exists $s_1\in(0,V_{k-1}(B_R))$, such that the function $W(s)=U(s)-V(s)$ has a positive maximum in s_1 and hence $U_q'(s_1)=V_q'(s_1)$. This implies that, by definition, $U'(s_1)=V'(s_1)$ and hence, if we argue as before, we obtain again $U(s)\le V(s)$ for $s\in(0,s_1]$.

By (4.30) and (4.31) then we get

$$\left(\left(\frac{U'(s)}{s^{\frac{n}{n-k+1}-1}}\right)^{\frac{1}{k}}\right)'\le\left(\left(\frac{V'(s)}{s^{\frac{n}{n-k+1}-1}}\right)^{\frac{1}{k}}\right)',$$

hence integrating we obtain that $(-\tilde{u}_{k-1})\le(-\tilde{v}_{k-1})$ in $(0,s_1]$, which implies that $U_q(s_1)\le V_q(s_1)$. This is a contradiction and hence the proof of the part (i) of the Theorem is completed.

Now we prove the part (ii) of the theorem.

If $V_{k-1}(B_R)=V_{k-1}(\Omega)$, it follows that $\lambda_k(\Omega)=\lambda_k(\Omega^*_{k-1})$; then $\Omega=\Omega^*_{k-1}$ up to translation, and the assertion is obvious.

Let us suppose that $V_{k-1}(B_R)<V_{k-1}(\Omega)$. Let us define the following

$$s_1=\inf E=\inf\{s\in(0,V_{k-1}(B_R)):|\tilde{u}_{k-1}(t)|\ge\tilde{v}^{\infty}_{k-1}(t),\forall t\in(s,V_{k-1}(B_R))\}.$$

By our assumptions $E\neq\emptyset$ since $|\tilde{u}_{k-1}(V_{k-1}(B_R))|>\tilde{v}^{\infty}_{k-1}(V_{k-1}(B_R))=0$.

By definition moreover we have $\tilde{u}_{k-1}(s_1)=\tilde{v}^{\infty}_{k-1}(s_1)$. We will show that $s_1=0$. If ab absurd, we suppose that $s_1>0$, proceeding as before, we obtain that $U(s)\le V(s)$, $s\in[0,s_1]$, hence by (4.30) and (4.31) then we get

$$\left(\left(\frac{U'(s)}{s^{\frac{n}{n-k+1}-1}}\right)^{\frac{1}{k}}\right)'\le\left(\left(\frac{V'(s)}{s^{\frac{n}{n-k+1}-1}}\right)^{\frac{1}{k}}\right)',$$

hence we obtain that $(-\tilde{u}_{k-1}) \leq (-\tilde{v}_{k-1})$ a.e. $(0, s_1]$, this is a contradiction and hence the proof of the theorem is completed. □

A consequence the Theorem 4.5 is the following *reverse Hölder inequality*,

Theorem 4.6. *Let Ω be a bounded, smooth, strictly convex, open set of $\mathbb{R}^n$. Let u be an eigenfunction of k-Hessian operator in Ω with convex level sets. Then if $0 < q < p \leq +\infty$ the following reverse inequality holds true*

$$\left(\int_0^{V_{k-1}(\Omega)} r^{\frac{n}{n-k+1}-1}(-\tilde{u}_{k-1})^p \, dr\right)^{\frac{1}{p}} \leq C(n,p,q,k,\lambda_k)\left(\int_0^{V_{k-1}(\Omega)} r^{\frac{n}{n-k+1}-1}(-\tilde{u}_{k-1})^q \, dr\right)^{\frac{1}{q}}, \tag{4.35}$$

and equality holds whenever Ω is a ball.

Proof. Let us choose v_q as in Theorem 4.5, and let us extend v_q to zero in $(V_{k-1}(B_R), V_{k-1}(\Omega))$.

Hence by previous theorem we get $u^q \in K_{k-1}(v^q)$. By Proposition 2.3 and Theorem 2.4, we obtain

$$\left(\int_0^{V_{k-1}(\Omega)} r^{\frac{n}{n-k+1}-1}(-\tilde{u}_{k-1})^p \, dr\right)^{\frac{1}{p}} \leq C(p,q,k,n,\lambda_k)\left(\int_0^{V_{k-1}(\Omega)} r^{\frac{n}{n-k+1}-1}(-\tilde{u}_{k-1})^q \, dr\right)^{\frac{1}{q}}$$

where

$$C(p,q,k,n,\lambda_k) = \frac{\left(\int_0^{V_{k-1}(\Omega)} r^{\frac{n}{n-k+1}-1}(-\tilde{v}_{k-1})^p \, dr\right)^{\frac{1}{p}}}{\left(\int_0^{V_{k-1}(\Omega)} r^{\frac{n}{n-k+1}-1}(-\tilde{v}_{k-1})^q \, dr\right)^{\frac{1}{q}}}.$$

□

Remark 4.3. *In [5] it is proved that if u is an eigenfunction of p-Laplacian operator then*

$$\|u\|_p \leq (constant)\|u\|_q, \qquad 0 < q < p \leq +\infty.$$

In the case of Hessian operator we can only deduce by (4.35) and (2.12) that

$$\|u\|_p \leq (constant)\|u^*_{k-1}\|_q, \qquad 0 < q < p \leq +\infty,$$

since the rearrangements with respect to the quermassintegral does not preserve the L_q-norms.

4.2 An open problem: convexity properties of eigenfunctions

The main assumption to prove the Payne-Rayner inequality for eigenfunctions of k-hessian operators, is the convexity of its level sets. This property has been proved for the Laplacian while, for the Monge-Ampère operator is obvious since the admissible functions are convex. For S_k, with $k = 2, \cdots, n-1$, this question is again an open problem. Actually we are studying this problem and we have seen that, just for $k = 2$, it is very difficult to extend the result well known for the Laplacian to 2-Hessian operator.

In the linear case it is known that the first eigenfunction of the Laplacian with zero boundary value, is strictly log-convex and hence that its level sets are convex. This result was proved in [58] using a maximum principle named Korewaar's concavity maximum principle successively improved by Kennington in [57], which is the following

Theorem 4.7. *Let Ω be a bounded convex domain in $\mathbb{R}^n$, and let $w \in C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$ be a solution of the elliptic equation*

$$a_{ij}(Dw)w_{ij} = b(x, w, Dw) \quad \text{in } \Omega, \tag{4.36}$$

where b is nondecreasing in w and jointly convex in (x, w) and

$$a_{ij} \in C^{1+\alpha}(\mathbb{R}^n),$$
$$b \in C^{\alpha}(\Omega) \times C^{1+\alpha}(\Omega \times \mathbb{R}^n).$$

Then for each $t \in (0, 1)$ the concavity function of w,

$$C_t(x, y) = w(tx + (1-t)y) - tw(x) - (1-t)w(y), \tag{4.37}$$

cannot have a negative interior minimum on $\Omega \times \Omega$.

We explicitly observe that Korewaar's concavity maximum principle seems to be strongly related on the linearity of the operator with respect to the second derivatives of w. In [68], this technique has been used to prove the same result for eigenfunctions of the p-Laplacian. Indeed linearizing the p-Laplacian operator, the author obtain an operator of the type (4.36). In our case by linearizing the k-Hessian operator, using the divergence form, some dependencies on the second derivatives of the solution appear, hence it cannot be used directly.

In [24], Caffarelli and Friedman proved the same result in dimension $n = 2$ using different techniques strongly based on the Rank Theorem and on the Deformation method. These techniques have also been used by Korewaar and Lewis in [59] to prove a similar result for a class of quasilinear elliptic partial differential equations. Recently these techniques have been used for geometric fully nonlinear partial differential equations associated to the Christoffel-Minkowski problem (see for instance [28], [49], [48] and [50]). Then, we have tried to prove that also the eigenfunctions of S_2 are strictly *log*-convex using the Caffarelli Friedman techniques. So we need a suitable version of the Rank Theorem, to ensure that the strictly log convexity of eigenfunctions is true whenever Ω is the unitary ball and so we can conclude by means of the classical continuity method. We will see that the very crucial step is to ensure that a Constant rank Theorem holds.

4.2.1 Convexity properties in a ball

It is interesting to observe that we can prove this result whenever Ω is a ball only using the symmetry of the problem.

The following result holds:

Proposition 4.1. *Let B_R be a ball centered at the origin with radius $R > 0$. Let be $u \in C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})$ be the eigenfunction of the 2-Hessian operator in B_R. Then $w = \log(-u)$ is strictly concave in B_R and hence $-w$ is strictly convex in B_R.*

Proof. Since u is a radially function we set

$$u(x) = \varphi(|x|) = \varphi(r),$$

where $r = |x|$, $r \in [0, R]$, $\varphi(r) < 0$ $\forall r \in (0, R)$ and φ is an increasing function in $(0, R)$, i.e.

$$\varphi'(r) > 0 \quad \forall r \in (0, R)$$
$$\varphi'(0) = \varphi(R) = 0.$$

Now we set $w = \log(-\varphi(r))$ $\forall r \in (0, R)$. We shall show that w is strictly concave in $(0, R)$.

Since w is a smooth function, it is sufficient to prove that $w'' < 0$ in $(0, R)$.

It is simple to check that w solves the following O.D.E.

$$\frac{d}{dr}(r^{n-2}w'^2) = \frac{2r^{n-1}}{n-1}\left[\lambda - \frac{n-1}{r}w'^3\right].$$

Let us integrate the equation (4.2.1) between 0 and r, we have

$$w'^2(r)r^{n-2} = \frac{2\lambda}{n(n-1)}r^n - 2\int_0^r \rho^{n-2}w'^3 d\rho$$

Dividing by r^{n-2} we have

$$w'^2(r) = \frac{2\lambda}{n(n-1)}r^2 - \frac{2}{r^{n-2}}\int_0^r \rho^{n-2}w'^3 d\rho. \tag{4.38}$$

By (4.38) and since $w' < 0$ in $(0, R)$, we have

$$w' = -\sqrt{\frac{2\lambda}{n(n-1)}r^2 - \frac{2}{r^{n-2}}\int_0^r \rho^{n-2}w'^3 d\rho}. \tag{4.39}$$

Now we compute w''

$$w''(r) = -\frac{\frac{2\lambda r}{n(n-1)} - w'^3(r) + \frac{n-2}{r^{n-1}}\int_0^r \rho^{n-2}w'^3 d\rho}{\sqrt{\frac{2\lambda}{n(n-1)}r^2 - \frac{2}{r^{n-2}}\int_0^r \rho^{n-2}w'^3 d\rho}}. \tag{4.40}$$

Now we will study the sign of w'' near the origin . Rewriting (4.40) as follows

$$w''(r) = -\frac{\frac{2\lambda r}{n(n-1)} + \frac{1}{r^{n-1}}\int_0^r \rho^{n-2}[(n-2)w'^3(\rho) - (n-1)w'^3(r)]d\rho}{r\sqrt{\frac{2\lambda}{n(n-1)} - \frac{2}{r^n}\int_0^r \rho^{n-2}w'^3 d\rho}}$$

we have

$$w''(r) = -\frac{\frac{2\lambda}{n(n-1)}}{\sqrt{\frac{2\lambda}{n(n-1)} - \frac{2}{r^n}\int_0^r \rho^{n-2}w'^3 d\rho}} - \frac{1}{r^n}\frac{\int_0^r \rho^{n-2}[(n-2)w'^3(\rho) - (n-1)w'^3(r)]d\rho}{\sqrt{\frac{2\lambda}{n(n-1)} - \frac{2}{r^n}\int_0^r \rho^{n-2}w'^3 d\rho}}. \tag{4.41}$$

It results

$$\lim_{r\to 0}\frac{1}{r^n}\int_0^r \rho^{n-2}w'^3 d\rho = \lim_{r\to 0}\frac{r^{n-2}w'^3(r)}{nr^{n-1}} = \frac{1}{n}\lim_{r\to 0}\frac{w'^3(r)}{r} \tag{4.42}$$

Since $w'(0) = 0$ we have,

$$\lim_{r\to 0}\frac{3w'^2(r)w''(r)}{n} = \frac{3}{n}w'^2(0)w''(0) = 0. \tag{4.43}$$

The conclusion is that the first term in (4.42) for $r \to 0$ converges to $-\sqrt{\frac{2\lambda}{n(n-1)}} < 0$ and the second term converges to zero. Consequently by continuity, we can deduce that $w''(0) = -\sqrt{\frac{2\lambda}{n(n-1)}} < 0$ and it is negative in a neighborhood of the origin. Now we prove that w'' cannot change sign in $(0, R)$ and hence that it remains negative in $(0, R)$, that is $w''(s) < 0$ in $(0, R)$.

Let

$$\overline{r} = \sup E = \sup\{\rho : w''(r) < 0 \ \forall r \in (0, \rho)\},$$

where $E \neq \emptyset$ since in E is contained at least a neighborhood of the origin. By definition of $\overline{r}$, we have that $0 < \overline{r} \leq R$ and $w''(r) < 0$ in $(0, \overline{r})$. Now we prove that $\overline{r} = R$. Indeed by absurd, if we suppose that $\overline{r} < R$, we can compute $w''(\overline{r})$ and by definition of $\overline{r}$ we have $w''(\overline{r}) = 0$. Moreover it holds

$$w''(\overline{r}) = \frac{-\frac{2\lambda}{n(n-1)} - \frac{1}{\overline{r}^n}\int_0^{\overline{r}} \rho^{n-2}[(n-2)w'^3(\rho) - (n-1)w'^3(\overline{r})]d\rho}{\sqrt{\frac{2\lambda}{n(n-1)} - \frac{2}{\overline{r}^n}\int_0^{\overline{r}} \rho^{n-2} w'^3 d\rho}}, \tag{4.44}$$

Hence we study the sign of

$$-\frac{2\lambda}{n(n-1)} - \frac{1}{\overline{r}^n}\int_0^{\overline{r}} \rho^{n-2}[(n-2)w'^3(\rho) - (n-1)w'^3(\overline{r})]d\rho.$$

The first term is negative. Since $w' < 0$ and by definition of $\overline{r}$ we have

$$(n-1)[w'^3(\rho) - w'^3(\overline{r})] > 0.$$

So we have proved that $w''(\overline{r}) < 0$ and this is a contradiction. Hence $\overline{r} = R$ consequently w is strictly concave in $(0, R)$ and the proposition is proven. □

4.2.2 Convexity properties of eigenfunctions

At the first time let us assume that a Constant rank theorem holds true and let us check if we can prove the log-convexity of eigenfunctions of S_2. We first have to see what equation $w = -\log(-u)$ solves. We have the following

Proposition 4.2. *Let u be an eigenfunction of (4.1) and let us consider the function $w = -\log(-u)$. Then w is a solution of the following equation:*

$$S_2(D^2 w) - \sum_{i,j=1}^{n} a_{ij}(Dw) w_{ij} = \lambda \tag{4.45}$$

where

$$a_{ij} = \begin{cases} |Dw|^2 - w_i^2 \ \text{if } i = j \\ -w_i w_j \ \text{if } i \neq j. \end{cases} \tag{4.46}$$

Proof. We observe that by Proposition 1.2 w is a strictly 2-convex function. Since $S_2(D^2w) = \sum_{\substack{i,j=1 \\ i<j}}^{n} (w_{ii}w_{jj} - w_{ij}^2)$, substituting the first and second derivatives of w we have

$$
\begin{aligned}
S_2(D^2w) &= \sum_{\substack{i,j=1 \\ i<j}}^{n} \left(\frac{u_i^2}{u^2} - \frac{u_{ii}}{u}\right)\left(\frac{u_j^2}{u^2} - \frac{u_{jj}}{u}\right) - \left(\frac{u_iu_j}{u^2} - \frac{u_{ij}}{u}\right) \\
&= \frac{1}{u^2}\sum_{\substack{i,j=1 \\ i<j}}^{n} (u_{ii}u_{jj} - u_{ij}^2) - \frac{1}{u^3}\sum_{\substack{i,j=1 \\ i<j}}^{n} (u_i^2u_{jj} + u_j^2u_{ii} - 2u_iu_ju_{ij}) \\
&= \lambda + \sum_{\substack{i,j=1 \\ i<j}}^{n} w_{ii}w_j^2 - 2\sum_{\substack{i,j=1 \\ i<j}}^{n} w_iw_jw_{ij} \\
&= \lambda + \sum_{i,j=1}^{n} a_{ij}(Dw)w_{ij}.
\end{aligned}
$$

□

The following result asserts that equation (4.45) remains elliptic

Lemma 4.2. *Let u be an eigenfunction of (4.1) and let us consider the function $w = -\log(-u)$. Equation (4.45) is elliptic with respect to $w = -\log(-u)$ where u is an eigenfunction of S_2.*

Proof. In the next we set $F(D^2w, Dw) = S_2(D^2w) - \sum_{i,j=1}^{n} a_{ij}(Dw)w_{ij}$.

By (1.11) we have

$$F = \sum_{i=1}^{n}(S_2^{ij} - a_{ij}(Dw))w_{ij},$$

moreover, by the definition of w, since $S_2(D^2w) = \sum_{\substack{i,j=1 \\ i<j}}^{n}(w_{ii}w_{jj} - w_{ij}^2)$ we obtain

$$
\begin{aligned}
F^{ii} = \frac{\partial F}{\partial w_{ii}} = S_2^{ii} - a_{ii} &= \sum_{\substack{j=1 \\ j\neq i}}^{n}(w_{jj} - a_{ii}) \\
&= \sum_{\substack{j=1 \\ j\neq i}}^{n}\left(\frac{u_j^2}{u^2} - \frac{u_{jj}}{u}\right) - \sum_{\substack{j=1 \\ j\neq i}}^{n}\frac{u_j^2}{u^2} = \frac{1}{|u|}S_2^{ii}(D^2u).
\end{aligned}
$$

For $i \neq j$ we have

$$F^{ij} = \frac{\partial F}{\partial w_{ij}} = S_2^{ij} - a_{ij} = -w_{ij} - a_{ij} = -\frac{u_iu_j}{u^2} + \frac{u_{ij}}{u} + \frac{u_iu_j}{u^2} = \frac{1}{|u|}S_2^{ij}(D^2u).$$

Hence

$$[F^{ij}] = \frac{1}{|u|}[S_2^{ij}], \tag{4.47}$$

and since u is strictly 2-convex, F is elliptic. □

Before prove the convexity of w, we also need the following Lemma.

Lemma 4.3. *Let Ω be a bounded, strictly convex open set of $\mathbb{R}^n$, with the boundary $\partial\Omega \in C^{2,\alpha}$. Let u be eigenfunction of the following problem*

$$\begin{cases} S_2(D^2u) = \lambda u^2 & x \in \Omega \\ u = 0 & on\ \partial\Omega, \end{cases} \tag{4.48}$$

then $w = -\log(-u)$ is strictly convex in some neighborhood of $\partial\Omega$.

Proof. The proof is analogous to the one contained in [28]. By property (vi) of Proposition 1.1 we have $\triangle u \geq \overline{C}S_2((D^2u)) > 0$, where $\overline{C}$ is a positive constant. Then by Hopf Lemma, $\frac{\partial u}{\partial \nu} > 0$, where ν denotes the outward unit normal at $\partial\Omega$. Setting $g(t) = -\log(-t)$, $t < 0$, we have $g' > 0$, $g'' > 0$ and $w = g(u)$.

Fix a point $\overline{x} \in \Omega$ with small distance d to $\partial\Omega$, and let $x_0 \in \partial\Omega$ such that $|\overline{x} - x_0| = d$. Since $u_\nu > 0$ we have

$$g'(u(\overline{x})) = -\frac{1}{u(\overline{x})} \sim \frac{c}{d} \tag{4.49}$$

$$g''(u(\overline{x})) \sim \frac{c_1}{d^2}, \tag{4.50}$$

where c, c_1 are two positive constants and where $\sim \dfrac{A}{d^\alpha}$ means $= \dfrac{A + o(1)}{d^\alpha}$ with $o(1) \to 0$ if $d \to 0$.

We denote by ν the direction of the normal to $\partial\Omega$ at x_0 and by τ_i the i^{th} tangential direction at x_0 for $i = 1, \dots, n-1$. Then the following statements hold true

$$u_{\tau_i}(x_0) = 0$$

$$u_{\tau_i}(\overline{x}) \sim O(d).$$

Now we compute the second derivatives of w in the directions τ_i and ν at $\overline{x}$. By (4.49) and (4.50) we have

$$w_{\nu\nu} = g''u_\nu^2 + g'u_{\nu\nu} \sim \frac{c_1}{d^2}u_\nu^2 + O\left(\frac{1}{d}\right)u_{\nu\nu} \sim \frac{\overline{c}}{d^2} \tag{4.51}$$

$$w_{\nu\tau_i} = g''u_\nu u_{\tau_i} + g'u_{\nu\tau_i} \sim O\left(\frac{1}{d}\right) \tag{4.52}$$

$$w_{\tau_i\tau_j} = g''u_{\tau_i}u_{\tau_j} + g'u_{\tau_i\tau_j} \sim O\left(\frac{1}{d}\right) \text{ for } i \neq j. \tag{4.53}$$

Finally since $u_\nu = k_i u_{\tau_i\tau_i}$ on $\partial\Omega$, where k_i is the curvature and since $k_i > 0$ by strict convexity of Ω, we have

$$w_{\tau_i\tau_i} = g'' u_{\tau_i}^2 + g' u_{\tau_i\tau_i} \sim \frac{1}{d^2} u_{\tau_i}^2 + \frac{c}{d}\frac{u_\nu}{k} \sim \frac{c_2}{d}, \tag{4.54}$$

where c_2 is a positive constant.

It follows that the Hessian matrix of w is positive definite at $\overline{x}$ provided d is small enough. The Lemma is proven. □

Now, if we assume that a Constant Rank Theorem holds, we can prove the following main result

Theorem 4.8. *Let Ω be an open, bounded, strictly convex set of $\mathbb{R}^n$ with boundary $\partial\Omega \in C^{2,\alpha}$. Let $u \in C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})$ be an eigenfunction of*

$$\begin{cases} S_2(D^2 u) = \lambda u^2 & x \in \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.55}$$

then $w = -\log(-u)$ is strictly convex in Ω; consequently the level lines of u are strictly convex.

Proof. We use the continuity method as in [59], [24]. For $0 \le t < 1$, let Ω_t be a family of bounded, open and strictly convex sets in $\mathbb{R}^n$ with smooth boundary such that $\Omega_0 = B_1$, the unit ball, $\Omega_1 = \Omega$ and $\partial\Omega_t \to \partial\Omega_s$ as $t \to s$ in the sense of Hausdorff distance, whenever $0 \le s \le 1$. Hence we consider a continuous deformation, i.e., a function that deform B_1 continuously into Ω by the family (Ω_t) where the sets Ω_t have the same properties of Ω.

Let u_t be the eigenfunction of S_2 in Ω_t, such that $\|u_t\|_{C^2(\Omega_t)} = 1$ and we denote $w_t = -\log(-u_t)$.

By properties of deformation, we have that $u_t \to u_s$ locally in C^2 for $t \to s$, $\forall t, s \in [0,1]$

Since u_0 is the eigenfunction of 2-Hessian operator in the unit ball, by Proposition 4.1, w_0 is strictly convex in B_1. Since the deformation is continuous, for t near $t = 0$, w_t remains strictly convex, i.e. there exists $t_0 > 0$ such that w_t is strictly convex for any $t \in [0, t_0)$.

We define

$$\bar{t} = \sup \overline{E} = \sup\{t : w_i \text{ is strictly convex } \forall i \in [0,t]\}.$$

We observe that $\overline{E} \neq \emptyset$, moreover $0 < \bar{t} \leq 1$.

We shall prove that $\bar{t} = 1$. Indeed if $\bar{t} < 1$ then there exists $\delta > \bar{t}$ such that $D^2 w_\delta$ is semi-definite positive but not positive definite in Ω_δ. By Proposition 2 and Constant Rank Theorem then $D^2 w_\delta$ has constant rank in Ω_δ and $\text{rank} D^2 w_\delta < n$. By Lemma 4.3, w_δ is strictly convex in some neighborhood of $\partial\Omega_\delta$ where there are also some internal point of Ω_δ. Hence at this point we have $\text{rank} D^2 w_\delta = n$ and also $\text{rank} D^2 w_\delta < n$. This is a contradiction. Hence also $D^2 w_1 = D^2 w$ is positive definite and w is strictly convex in Ω.

This completes the proof of the Theorem. □

We observe that we have used a Constant Rank theorem for equation (4.45) and not for the eigenvalue problem. In the next section we exploit if this result holds true for (4.45).

4.2.3 The Constant Rank Theorem

The classical Constant Rank Theorem for fully nonlinear elliptic equations is contained in [28] and it is the following

Theorem 4.9. *Let $\Psi \subset \mathbb{R}^n$ be an open cone, denote $Sym(n) = \{n \times n$ real symmetric matrices$\}$, and set*

$$\overline{\Psi} = \{A \in Sym(n) : \lambda(A) \in \Psi\}.$$

We assume $g \in C^2(\Psi)$ symmetric and such that

$$g_{\lambda_i}(\lambda) = \frac{\partial g}{\partial \lambda_i}(\lambda) > 0 \qquad \forall i = 1, \ldots, n, \quad \forall \lambda \in \Psi,$$

and extend it to $F : \overline{\Psi} \to \mathbb{R}$ by $F(A) = g(\lambda(A))$. Moreover we define $\overline{F}(A) = F(A^{-1})$ whenever $A^{-1} \in \overline{\Psi}$, and we assume $\overline{F}$ is locally convex. Under those conditions, assume u is a C^3 convex solution of the following equation in a domain Ω in $\mathbb{R}^n$:

$$F(D^2 u(x)) = f(x, u(x), Du(x)) \quad \forall x \in \Omega,$$

for some $f \in C^{1,1}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$. If $f(x, u, p)$ is concave in $\Omega \times \mathbb{R}$ for any fixed $p \in \mathbb{R}^n$, then the Hessian matrix $(D^2 u)$ has constant rank in Ω.

Proof. The proof of this result is very technique hence we give just an idea.

Let x_0 be a point of Ω such that D^2w has minimal rank at x_0 and we set $\text{rank}(D^2w(x_0)) = l,\ 1 \leq l \leq n$. If $l = n$ the theorem is true. Suppose that $l < n$. Hence we have

$$S_l(D^2w(x_0)) = C > 0$$
$$S_{l+1}(D^2w(x_0)) = 0.$$

We will prove that the rank of D^2w is l in each point of Ω.

Now we pick an open neighborhood $\mathcal{O}$ of x_0, such that for any $z \in \mathcal{O}$ we have

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$$
$$\lambda_1 \geq \lambda_2 \geq \lambda_l \geq C_0 > 0$$

where λ_i is the i^{th} eigenvalue of D^2w and C_0 is a positive constant depending only on $\|w\|_{C^3}$, $\|b\|_{C^2}$.

Let $G = \{1, 2, \ldots, l\}$ and $B = \{l+1, \ldots, n\}$ the "good "and "bad "sets of indices respectively in the sense that λ_i, with $i \in G$, are the good eigenvalues of D^2w. We set

$$\phi(x) = S_{l+1}(D^2w(x)), \qquad \forall x \in \Omega,$$

We observe that ϕ is defined in Ω, $\phi \geq 0$ in Ω since D^2w is semi-definite positive and $\phi(x_0) = 0$.

To prove the Theorem we will show that $\phi = 0\ \forall x \in \Omega$. Using the same notation in [24], for two function $h(y)$, $k(y)$, defined in an open set $\mathcal{O} \subset \mathbb{R}^n$, we say that $h(y) \lesssim k(y)$ if there exist two positive constants $c_1, c_2 > 0$, depending only on $\|w\|_{C^3}$, $\|b\|_{C^2}$ and C_0, such that

$$(h - k)(y) \leq (c_1|\nabla\phi| + c_2\phi)(y). \tag{4.56}$$

If (4.56) holds true $\forall y \in \mathcal{O}$, we write $h \lesssim k$ and we also write $h(y) \sim k(y)$ if $h(y) \lesssim k(y)$ and $k(y) \lesssim h(y)$.

The main step of the proof is to prove

$$\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \lesssim 0. \tag{4.57}$$

To get (4.57), we use strongly that $\overline{F}(A) = F(A^{-1})$ whenever $A^{-1} \in \overline{\Psi}$, is locally convex and that $f(x, u, p)$ is concave in $\Omega \times \mathbb{R}$ for any fixed $p \in \mathbb{R}^n$.

Since $\phi \geq 0 \in \Omega$ and $\phi(x_0) = 0$, then by (4.57), it follows, from the strong minimum principle (see [45]) that $\phi = 0\ \forall z \in \mathcal{O}$. Hence we conclude that $\phi = 0$ locally. Thus, the set $E = \{x \in \Omega :\ \text{rank}D^2w = l\}$ is open in Ω. But E is also closed, so by connectivity of Ω, $E = \Omega$. Hence $\phi(x) = 0\ \forall x \in \Omega$ and the rank of D^2w is constant.

The theorem is proven. $\square$

We observe that this theorem cannot be used in our case since in the equation (4.45), we have also the dependence by the gradient.

Recently a more general Constant Rank Theorem has been proved (see [46])

Theorem 4.10. *Let Ω is a domain in $\mathbb{R}^n$, and $F = F(r, p, u, x)$ is a given function in $S^n \times \mathbb{R}^n \times \mathbb{R} \times \Omega$ and elliptic. Suppose*

(i) $F(A^{-1}, p, u, x)$ is local convex in (A, u, x) for each p fixed.

If $u \in C^{2,1}(\Omega)$ is a convex solution of

$$F(D^2u, Du, u, x) = 0,$$

then the rank of $D^2u(x)$ is constant in Ω.

The assumption (i) is necessary to have that for any $i \in B$, where B, G are the same of the one defined above,

$$\begin{aligned}\sum_{\alpha,\beta,\gamma,\eta\in G} F^{\alpha,\beta,\gamma\eta}u_{i\alpha\beta}u_{i\gamma\eta} + 2\sum_{\alpha,\beta\in G} F^{\alpha\beta}\sum_{j\in G}\frac{1}{\lambda_j}u_{ij\alpha}u_{ij\beta} + \sum_{\alpha,\beta\in G} F^{\alpha\beta,u}u_{i\alpha\beta}u_i + \\ + 2\sum_{\alpha\beta\in G} F^{\alpha\beta,x_i}u_{i\alpha\beta} + F^{u,u}u_i^2 + 2F^{u,x_i}u_i + F^{x_ix_i} \geq 0.\end{aligned} \tag{4.58}$$

Since Theorem 4.10 holds for a class of operators which includes that in (4.45), we have only to check if condition (4.58) is satisfied. Let us denote by $F(Dw, D^2w)$ the following operator,

$$F(Dw, D^2w) = S_2(D^2w) - \sum_{i,j=1}^{n} a_{ij}(Dw)w_{ij}$$

where $a_{ij}(Dw)$ are

$$a_{ij} = \begin{cases} |Dw|^2 - w_i^2 \text{ if } i = j \\ -w_iw_j \text{ if } i \neq j. \end{cases} \tag{4.59}$$

Let us assume that D^2w is diagonal then

$$F(p, \lambda(D^2w)) = \sum_{1\leq i<j\leq n}$$

We can also assume that D^2w is positive definite else we can reduce to the set G.

For a given $p \in \mathbb{R}^n$, we have to prove that

$$\sum_{i,j} F_{\lambda_i\lambda_j}v_iv_j + 2\sum_i \frac{F_{\lambda_i}}{\lambda_i}v_i^2 \leq 0, \qquad \forall v \in \mathbb{R}^n. \tag{4.60}$$

The derivatives of F are

$$F_{\lambda_i} = \sum_{j \neq i} \lambda_j - a_{ii}$$

$$F_{\lambda_i \lambda_j} = \begin{cases} 1 \text{ if } i \neq j \\ 0 \text{ if } i = j. \end{cases}$$

Let us now to replace in (4.60),

$$\begin{aligned} \sum_i \sum_{j\neq i} v_i v_j + 2 \sum_i \sum_{j \neq i} \frac{\lambda_j}{\lambda_i} v_i^2 - 2 \sum_i \frac{v_i^2}{\lambda_i} a_{ii} &= \sum_i \sum_{j \neq i} \left(v_i v_j + 2 \frac{\lambda_j}{\lambda_i} v_i^2 \right) - 2 \sum_i \frac{v_i^2}{\lambda_i} a_{ii} \\ &\geq \sum_i \sum_{j \neq i} \left(-|v_i v_j| + 2 \frac{\lambda_j}{\lambda_i} v_i^2 \right) - 2 \sum_i \frac{v_i^2}{\lambda_i} a_{ii} \end{aligned}$$

By Young's inequality we get

$$\begin{aligned} \sum_i \sum_{j\neq i} v_i v_j + 2 \sum_i \sum_{j \neq i} \frac{\lambda_j}{\lambda_i} v_i^2 - 2 \sum_i \frac{v_i^2}{\lambda_i} a_{ii} &\geq \sum_i \sum_{j \neq i} \frac{\lambda_j}{\lambda_i} v_i^2 - 2 \sum_i \frac{v_i^2}{\lambda_i} a_{ii} \\ &= \sum_i \frac{v_i^2}{\lambda_i} \left(\sum_{j \neq i} \lambda_j - 2 a_{ii} \right) \geq 0 \Leftrightarrow \lambda_{\min} \geq \frac{2 a_{\max}}{n-1}. \end{aligned}$$

Hence (4.60) holds true if we require

$$\lambda_{\min} \geq \frac{2 a_{\max}}{n-1}, \tag{4.61}$$

where $\lambda_{\min}$ is the smallest positive eigenvalue of $D^2 w$. But this condition is not admissible for our function w. Indeed if w does not satisfy (4.61), we can rescaling considering $\frac{w}{\mu}$, for a positive constant μ. But replacing in (4.45) we do not obtain a condition on μ. Hence we can not procede in this way, and in particular since (4.60) is not satisfied we can not use the Theorem 4.10.

Assumption (4.60) is necessary to prove a Constant Rank Theorem, hence we have proved to change the function w to obtain a function which satisfies (4.60).

If we take $w = f(u)$, where u is an eigenfunction of S_2 and f is increasing and convex, we get that w satisfies the following equation

$$S_2(D^2 w) - \frac{f''(u)}{f'^2(u)} \sum_{i,j} a_{ij}(Dw) w_{ij} = \lambda u^2 f'^2(u), \tag{4.62}$$

where a_{ij} are the same in (4.59). Hence taking for example

$$f = (-u)^\alpha, \quad \alpha \neq 0,$$

we get

$$S_2(D^2 w) - \frac{(\alpha - 1)}{\alpha w} \sum_{i,j} a_{ij}(Dw) w_{ij} = \lambda \alpha^2 w^2, \tag{4.63}$$

which does not satisfies condition (i) in Theorem 4.10, hence we can not have a α-convexity of u. We obtain the same result also taking

$$f(u) = (-\log(-u))^{\alpha},$$

$$f(u) = \log(-\log(-u)),$$

where we suppose $-u < 1$ which is not a restriction since the first eigenvalue of S_2 is simple.

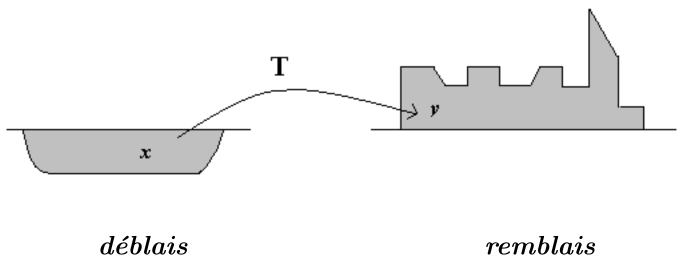
It remains hence a open problem to prove that for (4.45) holds true the Constant Rank Theorem.

Chapter 5

Applications

5.1 Mass transfer problem

Mass transportation theory goes back to the french geometer Gaspard Monge in 1781, when he published his famous work, *Mémoire sur la théorie des déblais et des remblais*; a *déblai* is an amount of material extracted from the earth of a mine while, a *remblai*, is a material put into a new construction. The problem considered by Monge was the following: assume to have a certain amount of soil to extract from the ground and assume that this amount has to be transported somewhere to be incorporated in a construction (see picture).



The place where the material should be extracted, and the one where it should be transported to, are known. Since the transport is costly, the main scope is to minimize the total cost. Monge assumed that the transport cost for the unit mass along a certain distance was given by the product of the mass by the distance.

A simple mathematical formulation of this problem is the following (see for instance [27], [64], [21], [87]):

let $f^-, f^+ : \mathbb{R}^2 \to \mathbb{R}$ two positive measurable functions (f^+ is the *déblai* and f^- the *remblai*), such that

$$\int_{\mathbb{R}^2} f^+(x)\,dx = \int_{\mathbb{R}^2} f^-(y)\,dy. \tag{5.1}$$

The problem consists in finding a transport function T such that

$$\int_{t^{-1}(B)} f^+(x)\,dx = \int_B f^-(y)\,dy, \qquad \forall B \in \mathcal{B}(\mathbb{R}^2), \tag{5.2}$$

where $\mathcal{B}(\mathbb{R}^2)$ is the Borel σ-algebra of $\mathbb{R}^2$, and T minimizes

$$\int_{\mathbb{R}^2} c(x, t(x)) f^+(x)\,dx,$$

where $c(x, t(x))$ is the cost function. The cost function considered by Monge was $c(x, y) = |y - x|$.

In order to have a more general formulation of the Monge problem, we give some preliminary notion ([87], [88], [64], [21], [37]).

Definition 5.1. *A Polish space is a complete, separable metric space, equipped with its Borel σ-algebra.*

Let X be a Polish space. We denote by $M_+(X)$ the space of finite Borel measures on X, i.e. the space of measures with finite mass, and by $\int f(x)\,d\mu$, the integral of a measurable function with respect to the measure μ.

Let X, Y, two Polish spaces. If μ is a Borel measure on X, and T is a Borel map between X and Y, then we denote by $T_\#\mu$ the image measure (or push-forward) of μ by T and it is a Borel measure on Y defined by

$$(T_\sharp\mu)\,[B] = \mu[T^{-1}(B)], \qquad \forall B \in \mathcal{B}(Y).$$

It is clear by definition that the push-forward is mass-preserving, i.e.

$$\|\mu\|_{M_+(X)} = \|T_\#\mu\|_{M_+(Y)},$$

where $\|\mu\|, \|T_\#\mu\|$ denote the total variation of the measure μ and of $T_\#(\mu)$ respectively.

Let $f^+ \in M_+(X)$, $f^- \in M_+(Y)$, i.e. the measure generated by the integrals of the functions f^-, f^+ which are belong to $M_+(X)$ and $M_+(Y)$ respectively. Moreover we assume

$$\|f^+\|_{M_+(X)} = \|f^-\|_{M_+(Y)}. \tag{5.3}$$

A Borel function $T : X \to Y$, is an admissible transport map if $T_\sharp f^+ = f^-$, i.e. condition (5.2). We observe that (5.3) is necessary to have the existence of an admissible transport map.

Let $c : X \times Y \to [0, +\infty)$ a lower semi-continuous function which is the cost for the transport of a unit mass; then the total cost of the transport map is

$$\mathcal{C}(T) = \int_X c(x, T(x)) df^+(x). \tag{5.4}$$

The Monge problem consists in finding T, admissible transport map which minimizes (5.4).

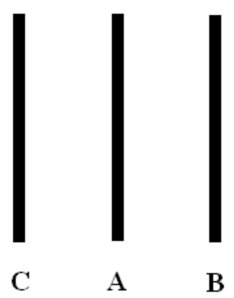
In general the problem does not admit a solution. To see this, we give some examples (see [87], [21], [64]).

Example 1: Let us take the measures $f^+ = \delta_x$, then for all function $T : X \to Y$, $T_\# f^+ = \delta_{T^{-1}(x)}$. It is clear that taking $f^- = \dfrac{\delta_{y_1} + \delta_{y_2}}{2}$, $y_1, y_2 \in Y$, there is no map T which transports f^+ into f^-, and so the Monge problem in this case is meaningless.

Example 2: Consider now

$$f^+ = \mathcal{H}^1 \llcorner A \text{ and } f^- = \frac{1}{2}\mathcal{H}^1 \llcorner B + \frac{1}{2}\mathcal{H}^1 \llcorner C,$$

where A, B, C are the segments below



If we consider the Euclidean cost, then the class of admissible transport maps is not empty but the minimum in the Monge problem is not attained.Indeed if the distance between the lines is L and the height is H, it can be seen that the infimum of the cost is $H \cdot L$ while every transport map has a cost strictly grater than $H \cdot L$.

To get around this facts, Kantorovich gave a relaxed formulation of the Monge problem ([27], [21], [87], [88], [64]).

The Monge-Kantorovich problem is

$$\min\left\{\int_{X\times Y} c(x,y)\,d\gamma(x,y), \quad \gamma \text{ admissible plan}\right\}, \tag{5.5}$$

where $\gamma \in M_+(X\times Y)$ is an admissible transport plan if f^+ and f^-, are its marginals, i.e. $f^+(B_1) = \gamma(B_1 \times Y)$ and $f^-(B_2) = \gamma(X \times B_2)$, for every Borel sets $B_1 \subset X$ and $B_2 \subset Y$.

Hence in Monge-Kantorovich problem one can find optimal admissible transport plan and not optimal transport map. The relation between transport plan and transport map is that a transport map T corresponds to the following transport plan,

$$\gamma_T = (Id \times T)_{\#} f^+.$$

Theorem 5.1. *There exists an optimal transport plan γ_{opt}.*

It is simple to prove that $\mathcal{C}(T) = \mathcal{C}(\gamma_T)$ and moreover if f^+ is nonatomic, we have

$$\inf\{\mathcal{C}(T), \text{ T transport map }\} = \min\{\mathcal{C}(\gamma), \gamma \text{ transport plan }\}$$

The advantage of this formulation is that by Theorem 5.1, an optimal admissible transport plan always exists. Then in order to solve the Monge problem one needs to prove that to the optimal transport plan corresponds a transport map.

When the cost is quadratic, $X = Y = \mathbb{R}^d$, and f^+f^- are two bounded, positive measurable functions with compact support in $\mathbb{R}^d$, such that

$$\int_{\mathbb{R}^d} f^+(x)\,dx = \int_{\mathbb{R}^d} f^-(x)\,dx = 1,$$

we have the following existence theorem of optimal transport map (see [15], [38], [27], [64])

Theorem 5.2. *There is a unique optimal transport map T defined on the support of f^+ satisfying*

$$\int_{T^{-1}(A)} f^+(x)\,dx = \int_A f^-(x)\,dx, \qquad \forall \text{ Borel set } A.$$

The map T is characterized as the unique application of this class which can be written as the gradient of a convex potential Φ:

$$T(x) = \nabla\Phi(x).$$

If moreover f^+ and f^- are strictly positive and Hölder continuous on their support, which we assume to be strictly convex, then the potential Φ has Hölder continuous derivatives up to the second order and satisfies in the classical sense the Monge-Ampère equation:

$$det(D^2\Phi(x))f^-(\nabla\Phi(x)) = f^+(x), \tag{5.6}$$

hence T solves

$$det(DT(x))f^-(T(x)) = f^+(x). \tag{5.7}$$

The previous result asserts hence, that under suitable assumptions, the optimal admissible transport map is the unique solution of a Monge-Ampère equation which is a n-Hessian equation. Hence in this case, the problem to find optimal transport map reduces to solve a Hessian equation.

It is interesting to see as the mass transportation theory can be used to give a proof of the classical isoperimetric inequality (see for instance [9], [11], [37]).

Theorem 5.3. *Let $E \subset \mathbb{R}^n$, having the same measure of the unit ball $B \subset \mathbb{R}^n$ (having finite perimeter). Then $P(E) \geq P(B)$, the equality holds true if and only if $E = B$ up to translations.*

Proof. The following proof is due to Gromov (see for instance [64], [88]).

Let us consider the transport problem with $X = Y = \mathbb{R}^n$, $f^+ = \chi_E$ and $f^- = \chi_B$ and let us consider the euclidean quadratic cost, $c(x, y) = |x - y|^2$. By Theorem 5.2, there exists an optimal transport map $T : \mathbb{R}^n \to \mathbb{R}^n$, such that $T(x) = D\varphi(x)$, φ is the potential, $\varphi : \mathbb{R}^n \to \mathbb{R}$, φ convex. Moreover T satisfies

$$\det(DT(x))f^-(T(x)) = f^+(x).$$

Since

$$f^+(x) = f^-(T(x)) = 1, \quad f^+\text{-a.e. } x,$$

then

$$\det(DT(x)) \equiv 1,$$

respect to f^+.

Hence, denoted by $\lambda_1, \ldots, \lambda_n$ the eigenvalues of $D^2(\varphi(x))$, since φ is convex, $\lambda_i \geq 0$, for $i = 1, \ldots, n$. Then by statement (vi) in Proposition 1.1, we get

$$1 \equiv \det(D^2\varphi(x)) = \big(\det(D^2\varphi(x)\big)^{\frac{1}{n}} \leq \frac{\Delta(\varphi(x))}{n}, \tag{5.8}$$

That is

$$\mathrm{div}(T) = \Delta\varphi \geq n.$$

Hence

$$\begin{aligned} P(B) = n|B| = n\int_B f^-(y)\,dy = n\int_{\mathbb{R}^n} f^+(x)\,dx \leq \int_{\mathbb{R}^n} f^+(x)\mathrm{div}(T(x))\,dx \\ = -\int_{\mathbb{R}^n} T(x)d(Df^+) \leq \int_{\mathbb{R}^n} |T(x)|d(|Df^+|) \leq \int_{\mathbb{R}^n} d(|Df^+|) = P(E), \end{aligned}$$

where we have used the fact that $|T(x)| \leq 1$ since $T(x) \subset B$ and where by $d(|Df^+|)$ we are denoting the total variation of Df^+ in $\mathbb{R}^n$. □

The mass transportation theory plays an important role in several areas of ma thematics. In this section we describe some optimization problems related to the mass transportation theory.

Optimal network

We give a model for the description of an urban transportation network and we describe the problem which consists in finding the design of the network which has the best transportation performances (see for instance [21], [19]).

Suppose that the following objects are given:

- Ω is a compact smooth domain of $\mathbb{R}^n$, $n \geq 2$, and in the model it represents the urban area we are dealing with;
- f^+ is a probability measure on Ω which represents the density of residents in the urban area Ω;
- f^- is a probability measure on Ω which represents the density of services in the urban area Ω.

The unknown of the problem is the transportation network Σ that has to be designed in an optimal way to transport the residents f^+ into the services f^-. We assume that Σ is a closed connected 1-dimensional subset of Ω, with total length bounded by a given constant L.

To find the optimal transportation network, we have to introduce a cost functional. Let us consider the following two functions

- $A : [0, +\infty[\to [0, +\infty[$, continuous and nondecreasing, such that $A(0) = 0$; A represents the cost for residents of travelling by their own means. In particular

$A(t)$ represents the cost to cover a length t by one's own means (walking, car fuel,...).

- $B : [0, +\infty[\to [0, +\infty[$, lower semi-continuous and nondecreasing; B represents the cost for residents of travelling by using the network. In particular $B(t)$ represents the cost to cover a length t by using the transportation network (tickets, time consumption,...).

If we define

$$c_\Sigma(x, y) = \inf \left\{ A\left(\mathcal{H}^1(\Gamma \setminus \Sigma)\right) + B\left(\mathcal{H}^1(\Gamma \cap \Sigma)\right) : \Gamma \text{ connects } x \text{ to } y \right\},$$

the optimal transportation network problem consists in minimizing the Monge-Kanorovich cost (5.5), where c is c_Σ, on the admissible sets Σ which are simply closed connected sets with $\mathcal{H}^1(\Sigma) \leq L$. About the existence of the optimal transportation network, we have the following result

Theorem 5.4. *For all $f^+, f^-, \Omega, L, A, B$ as above, there exists at least an optimal transportation network which solves the Monge-Kantorovitch problem (5.5).*

Shape optimization problems in elasticity

The problem consists in finding the elastic structure (seen as a distribution of a given amount of elastic material) which, for a given force field $f \in \mathbb{R}^n$, and for a given total mass, gives the best resistance in terms of minimal compliance ([19], [20], [21]).

The unknown mass distribution is then a nonnegative measure which may vary in the class of admissible choices with total mass m prescribed, and with support possibly constrained in a given design region D.

The optimization criterium is the elastic compliance $\mathcal{C}(\mu)$. For any $n \times n$ matrix z, we denote by $z^* = sym(z)$, the symmetric part of z and by $e(u)$ the strain tensor$e(u)$ =sym(Du), where $u : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function with compact support which is given and which represents a smooth displacement. We denote by $j(Du)$ the energy density associated to u:

$$j(z) = \beta |z^*|^2 + \frac{\alpha}{2} |\mathrm{tr} z^*|^2,$$

with α, β are the so called Lamé constants. This is the case when the material is homogeneous, isotropic and linearly elastic.

For a given mass distribution μ, the stored elastic energy of a smooth displacement u is

$$J(\mu, u) = \int j(Du)\, d\mu,$$

so that the total energy associated to μ and relative to a smooth displacement u is

$$E(\mu, u) = J(\mu, u) - \langle f, u \rangle,$$

where the last term represents the work of the force field.

In order to take into account possibly prescribed Dirichlet boundary conditions, we consider a closed subset Σ of $\mathbb{R}^n$ and we impose that the admissible displacements vanish on Σ.

Now we may define the energy of a measure μ as the infimum

$$\mathcal{E}(\mu) = \inf \{E(\mu, u) : u \text{ smooth, } u = 0 \text{ on } \Sigma\},$$

and the compliance is

$$\mathcal{C}(\mu) = -\mathcal{E}(\mu).$$

Finally, the optimization problem is

$$\min \left(\mathcal{C}(\mu) : \mu \in M^+(\mathbb{R}^n), \text{spt}\mu \subset D, \int d\mu = m \right). \tag{5.9}$$

Now we deal to prove that the shape optimization problem has an equivalent formulation in terms of the Monge-Kantorovich mass transport problem.

Let $\Omega \subset \mathbb{R}^n$, let us introduce the usual geodesic distance defined on $\overline{\Omega} \times \overline{\Omega}$ by

$$\begin{aligned} d_\Omega(x, y) = &\min \left\{ \int_0^1 |\gamma'(t)|\, dt : \gamma \in Lip([0, 1]; \overline{\Omega}), \gamma(0) = x, \gamma(1) = y \right\} = \\ &= \sup\{|\varphi(x) - \varphi(y)| : |D\varphi| \leq 1 \text{ on } \Omega\} \end{aligned} \tag{5.10}$$

Let us denote by $Lip_1(\Omega, d_\Omega)$ the class of functions defined in Ω that are Lipschitz continuous with respect to the distance d_Ω.

Let us consider the mass transport problem associated to the cost function d_Ω. Hence given $f^+, f^- \in \mathcal{M}^+(\overline{\Omega})$, such that $f^+(\overline{\Omega}) = f^-(\overline{\Omega}))$, we define

$$\Phi(f^+, f^-) = \min \left\{ \int d_\Omega(x, y)\, d\gamma(x, y), \gamma \text{ transport plan} \right\}$$

The next result makes the link between Φ and the following quantity (see [22])

$$I(f, \mathcal{U}, \Omega) = \inf\{ \int \rho^0(\lambda) : \lambda \in \mathcal{M}(\mathbb{R}^n; \mathbb{R}^{n \times n}), \text{spt}\lambda \subset \Omega, f + \text{div}\lambda \in \mathcal{U}^0\}, \tag{5.11}$$

where

- $\mathcal{U} = \{u \in \mathcal{D}(\mathbb{R}^n; \mathbb{R}^n)\}$;
- $\mathcal{U}^0 = \{T \in \mathcal{D}'(\mathbb{R}^n; \mathbb{R}^n)\}$,
- j is a convex, positively p-homogeneous, with $p > 1$, which when we consider the shape optimization problem, reduces to the elastic energy,
- ρ is a convex, positively 1-homogeneous function such that $j(z) = \frac{1}{p}\rho(z)^p$ and ρ^0 is the polar function associated to ρ.

Theorem 5.5. *The following facts hold:*

(i) If two measures $f^+, f^- \in \mathcal{M}^+(\overline{\Omega})$ are such that $f^+(\overline{\Omega}) = f^-(\overline{\Omega}))$, then setting $f = f^+ - f^-$ we have

$$\Phi(f^+, f^-) = I(f, \mathcal{U}, \Omega).$$

(ii) Let $f \in \mathcal{M}^+(\overline{\Omega})$ and let $c = \int df$. We have

$$I(f, \mathcal{U}, \Omega) = \Phi(f^+, f^-).$$

Finally the relation between shape optimization problem in the scalar case and Monge-Kantorovich problem follows by the fact that in [22] it has been proved that the Compliance functional is strongly related to $I(f, \mathcal{U}, D)$ as the following Theorem asserts

Theorem 5.6. *Assume that $I(f, \mathcal{U}, D)$ is finite. Then the following facts hold*

(i) The mass optimization problem (5.9) admits a solution μ and we have

$$\mathcal{C}(\mu) = \frac{(I(f, \mathcal{U}, D))^{p'}}{p' m^{1/p-1}}.$$

(ii) If μ is a solution of (5.9), then one has

$$I(f, \mathcal{U}, D) = \min\{\int \rho^0(\sigma)\, d\mu : spt\mu \subset D, \int d\mu = m,$$
$$\sigma \in L^1_\mu(\mathbb{R}^n; \mathbb{R}^{n \times n}), f + div(\sigma\mu) \in \mathcal{U}^0\},$$

where σ verifies

$$\rho^0(\sigma) = \frac{I(f, \mathcal{U}, D)}{m}, \quad \mu - \textit{almost everywhere}.$$

(iii) Conversely, if λ is a solution of (5.11), then $\mu = \dfrac{m}{I(f,\mathcal{U},D)}\rho^0(\lambda)$ *is optimal for (5.9).*

About the existence of the shape optimization problem in elasticity we have,

Theorem 5.7. *The mass optimization problem (5.9), admits at least a solution.*

5.2 The obstacle problem

The classical obstacle problem consists in studying the properties of minimizers of the Dirichlet integral

$$J(u) = \int_\Omega |\nabla u|^2 \, dx,$$

in a domain $\Omega \subset \mathbb{R}^n$, among all those configurations u with prescribed boundary values, $u|_{\partial\Omega=0}$ (or in general $u|_{\partial\Omega} = f$), and constrained to remain, in Ω, above a prescribed function φ which represents the obstacle (see for instance [25], [26], [32]).

More precisely, in the Hilbert space $H_0^1(\Omega)$ of all functions u with square integrable gradient, define K to be the closed convex set

$$K = \{u \in H_0^1(\Omega), u \geq \varphi\},$$

where Ω is a smooth domain of $\mathbb{R}^n$ and φ is a smooth function on $\overline{\Omega}$ with $\varphi|_{\partial\Omega} < 0$. The obstacle problem consists in minimizing $J(u)$ on K. The minimizers are called solutions of the obstacle problem.

This problem is motivated by the description of the equilibrium position of a membrane (the graph of u) that is fixed to a rigid support and pushed from below by an obstacle φ. Ω is the reference configuration of the unspanned membrane. The equilibrium position u of the membrane is going to be the minimizer of the elastic energy, which remains above φ.

It is known (see for instance [32], [25], [26]) that the solution u of the obstacle problem is the smallest sub-solution of the following problem

$$\begin{cases} -\ \Delta u \geq 0 & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \\ \quad u \geq \varphi & \text{in } \Omega. \end{cases}$$

The set $\Lambda = \{x \in \Omega : u(x) = \varphi(x)\}$, is called the coincidence set and, as well as u, it is an unknown of the problem and it is often named also free boundary set. Another question related to the obstacle problem is to study the regularity of Λ.

Actually, an analogous obstacle problem has been formulated for Monge-Ampère equation (see for instance [60], [70]). In this case the problem consists in finding the greatest viscosity sub-solution of

$$\begin{cases} \det(D^2u) & \geq & 1 & \text{in } \Omega \\ u & = & 0 & \text{on } \partial\Omega \\ u & \leq & \varphi & \text{in } \Omega \\ u & & \text{convex} & \text{in } \Omega \end{cases}$$

Clearly u has to be convex, to have admissible solution of elliptic Monge-Ampère equation. Since u is convex and vanishes on the boundary, it is negative in Ω and so it has to remain below the obstacle φ. Than the obstacle φ has to be positive on $\partial\Omega$, and negative somewhere in Ω.

About the existence of the solution of the obstacle problem we have the following result (see [60], [70])

Theorem 5.8. *Let Ω a bounded, convex domain of $\mathbb{R}^n$ with smooth boundary and let $\varphi \in C^{2,\alpha}(\overline{\Omega})$ such that $\varphi > 0$ on $\partial\Omega$, and $\varphi(x_0) < 0$ for some $x_0 \in \Omega$. Then there exists the greatest viscosity subsolution of (5.2) which is smaller than φ in Ω and it is in $C^{1,1}\overline{\Omega}$.*

The proof of the existence of the solution of the obstacle problem is a consequence of the Perron's method for Monge-Ampère operator.

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