Università degli Studi di Napoli FEDERICO II

Dottorato di ricerca in Scienze Matematiche

A.A. 2007–2008

NONLINEAR ELLIPTIC AND PARABOLIC EQUATIONS WITH MEASURE DATA

Tesi di Dottorato

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INTRODUCTION

This thesis is devoted to the study of boundary value problems for second order elliptic and parabolic equations having measure data. In order to explain the motivations of this study, let us begin by considering a class of Dirichlet problems for nonlinear elliptic equations of the type

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u)) = \mu & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{0.0.1}$$

where Ω is a bounded open set of $\mathbb{R}^N$, $N \geq 2$, $a : \Omega\times\mathbb{R}\times\mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function such that

$$a(x,s,\xi)\xi \geq \alpha\, |\xi|^p\,, \quad \alpha > 0, \tag{0.0.2}$$

$$|a(x,s,\xi)| \leq \left[|\xi|^{p-1} + |s|^{p-1} + a_0\,(x)\right], \quad a_0(x) \in L^{p'}(\Omega), \tag{0.0.3}$$

$$(a(x,s,\xi) - a(x,s,\eta), \xi - \eta) > 0, \quad \xi \neq \eta \tag{0.0.4}$$

a.e. $x \in \Omega, \forall s \in \mathbb{R}, \forall \xi,\, \eta \in \mathbb{R}^N$.

When the datum μ belongs to the dual space of $W_0^{1,p}(\Omega)$, the notion of weak solution is well-defined and the assumptions $(0.0.2) - (0.0.4)$ ensure both existence and uniqueness results, as established by the classical theory due to Leray and Lions ([80]; we refer to Chapter I for some basic results).

If the datum μ is a measure, the notion of weak solution is obviously not appropriate. Moreover, if we consider the notion of solution in the sense of distribution the classical counterexample due to Serrin ([104], see also [97], [2] shows that a "local" uniqueness result for Dirichlet problem does not hold.

These drawbacks force to find extra conditions on the distributional solutions in order to ensure both existence and uniqueness.

In the linear case, that is $p = 2$ and $a(x, \nabla u) = A(x)\nabla u$, where A is an uniformly elliptic matrix with $L^\infty(\Omega)$ coefficients, this problem has been studied by Stampacchia, who introduced and studied in [106] a notion defined by duality. This allowed him to prove both existence and uniqueness results. Such a solution satisfies the equation in distributional sense and moreover belongs to the Sobolev space $W_0^{1,q}(\Omega)$ with $q < \frac{N}{N-1}$. Stampacchia's framework can not be extended to nonlinear cases, except when $p = 2$ and the operator a is strongly monotone and Lipschitz continuous with respect to ∇u ([88]). The first existence results in nonlinear case are due to Boccardo and Gallouët. In [30] and [31] they proved the existence of a distributional solution to (0.0.1) which belongs to $W_0^{1,q}(\Omega)$, for $q < \frac{N(p-1)}{N-1}$ under the assumption $p > 2 - \frac{1}{N}$. Such a solution is found by a natural approximation method: the idea consists in fixing the solution as the limit of a sequence of solutions to (0.0.1) which, owing to the regularity of the right hand-side, are weak solutions. Such a solution is known as "Solution Obtained as Limit of Approximations" ([43], see also [47]). The assumption on p is motivated by the fact that, if $p \leq 2 - \frac{1}{N}$, then $\frac{N(p-1)}{N-1} \leq 1$.

Other equivalent notion of solutions have been introduced such as "entropy solution" in [48], [33], "renormalized solution" in [88], [87],[48]. These framework which concerns to measure in $L^1(\Omega)$ or in $L^1 + W^{-1,p'}$ allow to prove existence, uniqueness and continuity with the respect to the datum of the solutions with respect to μ. Finally the notion of renormalized solution has been extended to the case of a general measure in [48], where existence and partial uniqueness result have been proved (see also [69]).

We present all these solutions in Chapter III. We point out that, in spite of the different notion of solutions used in literature, all the existence results are obtained by constructing the solution u as the almost everywhere limit of the solutions u_n to problem (0.0.1) corresponding to smooth function f_n which converge to μ in the weak*-topology. This procedure can be semplified if a continuous dependence from the data result is available. Such a result holds changing (0.0.4) into the following

”strong monotonicity” conditions

$$(a(x,\xi)-a(x,\eta))\cdot(\xi-\eta)\geq\frac{\gamma}{2^{2-p}}\frac{|\xi-\eta|^2}{(|\xi|+|\eta|)^{2-p}},\qquad \xi\neq\eta, \tag{0.0.5}$$

if $1 < p < 2$, or

$$(a(x,\xi)-a(x,\eta))\cdot(\xi-\eta)\geq\frac{\gamma}{(p-1)2^{p-2}}|\xi-\eta|^p,\qquad \xi\neq\eta, \tag{0.0.6}$$

if $p \geq 2$.

Under such assumptions in [5] an existence result for “Solution Obtained as Limit of Approximations” to problem (0.0.1) is proved.

Now let us explain the bound on p, $1 < p < N$. If p is greater then N, then, by Sobolev embedding and duality arguments, the space of measures with bounded variation on Ω is a subset of $W^{-1,p'}(\Omega)$ therefore existence and uniqueness of a weak solution in $W_0^{1,p}(\Omega)$ is a consequence of theory of monotone operators. Furthermore, the case $p = N$ has been studied in [58], [65], [68].

These approaches have been extended in various directions: for example to nonlinear elliptic equations with lower order terms, nonlinear degenerate elliptic equations and nonlinear parabolic equations.

Let us consider the case of Dirichlet problem for nonlinear uniformly elliptic equations with lower order terms of the type

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u))-\operatorname{div}(\Phi(x,u))+H(x,\nabla u)=\mu & \text{in} \quad \Omega \\ u=0 & \text{on} \quad \partial\Omega, \end{cases} \tag{0.0.7}$$

where $a(x, s, \xi)$ is a Carathéodory function such that (0.0.2) – (0.0.4) hold, H: $\Omega \times \mathbb{R} \longrightarrow \mathbb{R}^N$ and $\Phi : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ are two Carathéodory functions such that

$$|H(x,\xi)|\leq b(x)\,|\xi|^{p-1},\quad b(x)\in L^N(\Omega), \tag{0.0.8}$$

$$|\Phi(x,s)|\leq c(x)\,|s|^{p-1},\quad c(x)\in L^{\frac{N}{p-1}}(\Omega). \tag{0.0.9}$$

and finally μ is a Radon measure with bounded variation on Ω.

Existence and uniqueness for such type of problems have been widely studied in literature. In the linear case, Stampacchia proved in [106] the existence and uniqueness of a solution by duality, if 0 is not in the spectrum of the operator, condition which is verified for example, if $\|c\|_{L^{\frac{N}{p-1}}(\Omega)}$ or $\|b\|_{L^N(\Omega)}$ is small enough. Existence results for problem (0.0.7) have been proved for example in [49] and [51] by using the classical symmetrization methods, in [20], [72], [73], [11] and [12] for renormalized solutions, in [28] for entropy solutions and in [5], [6] for SOLA

In Chapter III we present an existence result for SOLA's to (0.0.7) with $H = 0$. Precisely we consider the problem

$$\begin{cases} -\operatorname{div}\left(a\left(x,\nabla u\right)\right)-\operatorname{div}\left(\Phi\left(x,u\right)\right)=\mu & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{0.0.10}$$

where $a : (x, z) \in \Omega \times \mathbb{R}^N \to a\left(x, z\right) \in \mathbb{R}^N$ is a Carathéodory function satisfying:

$$a\left(x,\xi\right)\cdot\xi \geq \lambda\left|\xi\right|^p, \qquad \xi \in \mathbb{R}^N, \quad \lambda > 0, \tag{0.0.11}$$

$$\left|a\left(x,\xi\right)\right| \leq \Lambda\left|\xi\right|^{p-1}, \qquad \xi \in \mathbb{R}^N, \quad \Lambda > 0, \tag{0.0.12}$$

with $1 < p < N$, and

$$\left(a\left(x,\xi\right)-a\left(x,\eta\right)\right)\cdot\left(\xi-\eta\right) > 0, \qquad \xi \neq \eta, \tag{0.0.13}$$

for almost every $x \in \mathbb{R}^N$ and for every ξ, $\eta \in \mathbb{R}^N$. Furthermore $\Phi\left(x, s\right)$ satisfies assumption (0.0.9).

We present an existence result for SOLA contained in [56] and obtained by adapting the techniques used in [5] and [6]. The first step of such approach consists in proving some apriori estimates for the gradients of the weak solutions u_n to the approximated problems having regular data in terms of L^1-norm of the data. Such a proof is based on the choice of a suitable test function, built on the level sets of u_n, and a comparison result beetwen the sferically symmetric rearrangement of u_n and the solution to a suitable elliptic problem with symmetric data. The following step consists in showing that it is possible to pass to the limit in the approximated problems. Such a procedure is simplified by substituting the classical monotonicity assumption on a (0.0.2) with the strong monotonicity conditions (0.0.5) , (0.0.6) . Such existence result is already proved in [51]; however the approach used in [56] is different and simpler.

As far as uniqueness is concerned the presence of lower order terms does not allow us to use heavily the strong monotonicity conditions to get a continuity with respect to the data. However this can be obtained if we strenght the structural conditions of a, we assume that Φ is locally Lipschitz continuous and we impose further restrictions on the index p. In Chapter III we present two uniqueness results, proved in [56], when μ is not any more a measure but merely an L^1 function.

Other uniqueness results can be found in [37], [38] and [39]. These results are always in the context of "finite energy solutions"; this means that μ is taken in $L^m(\Omega)$ with $m \geq \frac{2N}{N+2}$. Finally the uniqueness of entropy solution has been obtained in [96] when Φ is locally Lipschitz continuous and has at most an exponential growth at infinity, while uniqueness results for renormalized solutions have been proved in [12] and in [18].

Chapter IV is devoted to the study of nonlinear elliptic problems which satisfy a more general ellipticity condition. More precisely we consider the following problem

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u)) + H(x,\nabla u) = \mu & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{0.0.14}$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function satisfying (0.0.4) and

$$a(x,s,\xi)\xi \geq \nu(x)\,|\xi|^p\,, \tag{0.0.15}$$

$$|a(x,s,\xi)| \leq \nu(x)\left[|\xi|^{p-1} + |s|^{p-1} + a_0(x)\right],\ a_0(x) \in L^{p'}(\nu), \tag{0.0.16}$$

a.e. for $x \in \Omega$, for every $s \in \mathbb{R}$, for every $\xi \in \mathbb{R}^N$. Moreover $\nu(x)$ is a nonnegative function satisfying

$$\nu(x) \in L^r(\Omega), r \geq 1, \tag{0.0.17}$$

$$v(x)^{-1} \in L^t(\Omega), \quad t \geq N/p, \quad 1 + 1/t < p < N(1 + 1/t). \tag{0.0.18}$$

Furthermore $H : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that

$$|H(x, \xi)| \leq b(x) |\nabla u|^{p-1}, \tag{0.0.19}$$

$$b(x) \in L^\tau(\Omega), \ \tau > \frac{p' \widetilde{p} t}{t - (t+1)(p' + \widetilde{p})}. \tag{0.0.20}$$

where $\widetilde{p}$ is defined by

$$\widetilde{p} = \frac{p^\#}{r'},$$

and

$$(p^\#)^{-1} = p^{-1}(1 + 1/t) - N^{-1}.$$

In the linear case, if μ is a Radon measure with bounded variation on Ω, the existence of a solution by duality method has been proved in [89]. In the nonlinear case, existence results for degenerate elliptic equations have been proved in [101] and in [16]. In the first paper the existence of "generalized solution" has been proved, while in the second one classical symmetrizated methods are used for operator with lower order terms.

In Chapter IV we prove an existence result contained in [54] for renormalized solutions to (0.0.14). This result is obtained by adapting the technique developed in [35] and used also in [18] for uniformly elliptic equations. The idea is to consider first a sequence of approximated problems having regular data. When the norm of b is small the operator is coercive, hence, by using $T_k(u)$ as test function in (0.0.14), we easily obtain an a priori estimate for the solutions to the approximated problem. When the norm of b is not small, we reduce in some sense the problem to a finite sequence of problems with norm of b small. We obtain again the apriori estimate which allows us to pass to the limit in the approximated problem. Finally, in Chapter V we study the existence of solutions to Cauchy-Dirichlet problems for nonlinear parabolic equations

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) = \mu & \text{in} \quad Q_T \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{0.0.21}$$

where Ω is a bounded open set of R^N, $N \geq 2$, Q_T is the cylinder $\Omega \times (0,T)$ and T is a real positive number. Furthermore $a(x,t,\xi) : \Omega \times (0,T) \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function such that

$$a(x,t,s,\xi)\xi \geq \alpha\,|\xi|^p\,, \quad \alpha > 0, \tag{0.0.22}$$

$$|a(x,t,s,\xi)| \leq \left[h(x,t) + |s|^{p-1} + |\xi|^{p-1}\right], \quad h(x,t) \in L^{p'}(Q_T), \tag{0.0.23}$$

$$(a(x,t,s,\xi) - a(x,t,s,\varrho), \xi - \varrho) > 0, \quad \xi \neq \varrho \tag{0.0.24}$$

for almost every $x \in \Omega$, $t \in (0,T)$ and for every $s \in \mathbb{R}$, ξ, $\varrho \in \mathbb{R}^N$ and μ is a Radon measure with bounded variation on Q_T. As in the elliptic case when $\mu \in L^{p'}(Q_T)$ and $u_0 \in L^2(\Omega)$ the problem admits a unique solution that lies in the space $C(0,T;L^2(\Omega))$ (see [81]). When the data are functions in $L^1(Q_T)$ or, more in general, measures we have to define a new notion of solution. The notion of SOLA, renormalized solution, entropy solution have been extended to the parabolic case. In [23] and [99] the existence of renormalized solutions and entropy solutions for nonlinear parabolic equations without lower order terms have been proved respectively; while in [95] a parabolic problem with a lower order term of the type $b(x,t)\,|\nabla u|^{p-1}$ is considered.
In Chapter V we also present an existence result, proved in [55], for renormalized solution to nonlinear parabolic problem of the type:

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) - \operatorname{div}\left(K(x,t,u)\right) = \mu & \text{in} \quad Q_T \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{0.0.25}$$

where $a(x,t,s,\xi) : \Omega \times (0,T) \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function such $(0.0.22) - (0.0.23)$ hold true.

Furthermore $K : \Omega \times (0,T) \times \mathbb{R} \longrightarrow \mathbb{R}^N$ is a Carathéodory function such

$$|K(x,t,\eta)| \leq c(x,t)\, |\eta|^{\gamma}\,, \tag{0.0.26}$$

where

$$c(x,t) \in \left(L^{\tau}(Q_T)\right)^N, \quad \tau > \frac{N+p}{p-1}, \tag{0.0.27}$$

$$\gamma = \frac{N+2}{N+p}(p-1), \tag{0.0.28}$$

$$\mu \in L^1(Q_T), \tag{0.0.29}$$

$$u_0 \in L^1(\Omega). \tag{0.0.30}$$

Such existence result is obtained by adapting the techniques used in [35] (see also [18]) in order to prove the apriori estimate, while we use the limit procedure introduced in [24] for passing to limit in the approximated problems.

Ringraziamenti Il primo ringraziamento è indirizzato alla prof. Anna Mercaldo, guida sicura nella mia formazione e crescita professionale. Mai sono venuti a mancare i suoi impagabili ed insostituibili suggerimenti.

Desidero inoltre ringraziare i prof Angelo Alvino e Vincenzo Ferone per i numerosi consigli che mi hanno fornito nella stesura di questa tesi.

Un particolare quanto sentito pensiero è rivolto ad Adamaria Perrotta e Bruno Volzone, amici sinceri e leali, meravigliosi compagni di avventura.

Ancora un grazie a Marta Chinnici, Assunta Tataranni, Flavia Ventriglia, la cui compagnia ha reso più gradevoli questi anni e a Filomena Feo per i suoi preziosi consigli. Desidero ringraziare gli amici con cui ho condiviso un'esperienza indimenticabile a Cortona: Roberto Argiolas, Sandra Martinez, Simone Cecchini, Maurizio Ceseri, Giulio Ciraolo.

Un particolare ringraziamento è rivolto alla mia cara amica Chiara Autieri con la quale ho condiviso la parte iniziale del percorso di studi.

Desidero infine ringraziare la mia famiglia, a cui dedico questa tesi, costante punto di riferimento, ineguagliabile sostegno e fonte di incoraggiamento nei miei studi. A mia madre, mio padre e mio fratello Mario il più grande dei pensieri, per ogni sorriso e per l'amore che sanno donarmi giorno dopo giorno.

CHAPTER I

ELLIPTIC EQUATIONS WITH DATA IN $W^{-1,P'}(\Omega)$

In this chapter we give a review of some classical existence results for weak solutions to Dirichlet problems concerning nonlinear elliptic operators ([81]). First of all, we refer to some classical results involving the so-called monotone and pseudo-monotone operators and then we show how these results can be applied to Dirichlet problems for nonlinear elliptic operators with lower order terms.

1.1 *Existence results for monotone and pseudo-monotone operators*

In this section we start with a few definitions and properties about monotone and pseudomonotone operators ([81]). From now on we will denote by V a reflexive and separable Banach space and by V' its dual space.

Definition 1.1.1 *An operator $A : V \longrightarrow V'$ is said to be monotone if it satisfies the following condition*

$$(A(u) - A(v), u - v)) \geq 0, \quad \forall\ u,\ v \in V.$$

Definition 1.1.2 *We say that $A : V \longrightarrow V'$ is hemicontinous operator if for every $u,\ v,\ w \in V$ the function*

$$\lambda \in \mathbb{R} \longrightarrow (A(u + \lambda v), w) \in \mathbb{R},$$

is continuous.

Obviously if A is a continuous operator then A is also hemicontinuous, but the contrary is not true in general. Neverthless, as showed by the following Lemma, hemicontinuity plus monotonicity and boundedness of an operator yields the continuity.

Lemma 1.1.3 *If A is bounded, hemicontinuous and monotone, then A is continuous from V to V' endowed with strongly and weakly topology respectively.*

A bounded, hemicontinuous and monotone operator is not enough to get an existence theorem. This result may be proved by assuming that the operator is coercive.

Definition 1.1.4 *An operator $A : V \longrightarrow V'$ is coercive if*

$$\lim_{\|v\| \longrightarrow \infty} \frac{(A(v), v)}{\|v\|} = +\infty.$$

Now we are able to prove a general existence result for monotone operators.

Theorem 1.1.5 *Let be $A : V \longrightarrow V'$a bounded, hemicontinuous, monotone and coercive operator. Then A is surjective that is, for every $f \in V'$ there exists $u \in V$ such that*

$$A(u) = f. \tag{1.1.1}$$

Proof. The idea is to built a solution of the equation (1.1.1) by constructing solutions of certain finite dimensional approximations to (1.1.1) and then passing to the limit. Let be w_1, w_2, ..., w_m a basis of V; for each $m \in \mathbb{N}$, there exists $u_m \in \{w_1, w_2, ..., w_m\}$ such that

$$(A(u_m), w_m) = (f, w_j), \quad 1 \leq j \leq m. \tag{1.1.2}$$

In fact, we observe that

$$(A(u_m), u_m) - (f, u_m) \geq (A(u_m), u_m) - c\,\|u_m\|\,.$$

This implies, thanks to the coercivity condition, that, for $\|u_m\|$ sufficiently large, $(A(u_m), u_m) - c\,\|u_m\| \geq 0$. On the other hand, by Lemma 1.1.3 , the function $v \longrightarrow$

$(A(v), v)$ is continuous on $\{w_1, w_2, ..., w_m\}$. Now we recall a well-known result: if $P : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is a continuous functions such that $P(x) \cdot x \geq 0$ when $|x| = r$, for some $r > 0$ then there exists a point $x \in B_r(0)$ such that $P(x) = 0$. This result, applied to the function $P(\eta) = (P_1(\eta), ..., P_m(\eta))$ where $P_j(\eta) = (A(\sum_{i=1}^{m} \eta_i w_i), w_j) - (f, w_j)$, $1 \leq j \leq m$, implies that there exists $u_m \in \{w_1, w_2, ..., w_m\}$ that solves (1.1.2). By (1.1.2), we get

$$(A(u_m), w_j) = (f, w_j) \leq \|f\|_{V'} \|u_m\|.$$

By coercivity, being A bounded, it follows that

$$\|u_m\|_V \leq C, \quad \|A(u_m)\|_{V'} \leq C,$$

which implies that, up to a subsequence,

$$\begin{aligned} u_m &\rightharpoonup u \quad \text{weakly in } V, \\ A(u_m) &\rightharpoonup \xi \quad \text{weakly in } V'. \end{aligned} \tag{1.1.3}$$

Passing to the limit, we get for any $1 \leq j \leq m$

$$(\xi, w_j) = (f, w_j), \tag{1.1.4}$$

that implies $\xi = f$. Moreover, by (1.1.2), we obtain

$$(A(u_m), u_m) = (f, u_m) \longrightarrow (f, u),$$

and by (1.1.4) we get

$$(A(u_m), u_m) \longrightarrow (\xi, u). \tag{1.1.5}$$

Hence the result is proved if we show that

$$\xi = A(u).$$

By monotonicity condition we have

$$(A(u_m) - A(v), u_m - v) \geq 0, \quad \forall v \in V.$$

Passing to the limit, by(1.1.3) and (1.1.5) , we obtain

$$(\xi - A(v), u - v) \geq 0, \quad \forall v \in V. \tag{1.1.6}$$

Let $w \in V$ and $t > 0$. Applying (1.1.6) , with $v = u + tw$, we get

$$(\xi - A(u + tw), w) \geq 0.$$

By hemicontinuity of the operator A, it follows that, for any $w \in V$, $(\xi - A(u), w) \geq 0$, which implies

$$(\xi - A(u), w) = 0, \quad \forall w \in V.$$

Hence $\xi = A(u)$ which completes the proof. ■

A very simple example of monotone operator to which it is possible to apply *Theorem* 1.1.5 is the so called p-Laplace operator $A(u) = \Delta_p u$, where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$. More generally assume that $\Omega \subset \mathbb{R}^n$ is an open bounded set, $1 < p < +\infty$ and $V = W_0^{1,p}(\Omega)$ and suppose also that $F : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a continuous monotone mapping which satisfies the following growth condition

$$|F(\xi)| \leq C(1 + |\xi|^{p-1}), \quad \forall\, \xi \in \mathbb{R}^N.$$

Then it is easy to verify that the operator

$$A : u \in W_0^{1,p}(\Omega) \longrightarrow -\operatorname{div}(F(\nabla u)) \in W^{-1,p'}(\Omega)$$

is bounded, hemicontinuous and monotone. So, by *Lemma 1.1.3.*, A is continuous from $W_0^{1,p}(\Omega)$ strongly to $W^{-1,p'}(\Omega)$. Furthermore if we assume that F satisfies the condition

$$F(\xi) \cdot \xi \geq \alpha\, |\xi|^p, \quad \forall\, \xi \in \mathbb{R}^N, \quad \alpha > 0,$$

then A is also a coercive operator. We deduce, thanks to *Theorem 1.1.5,* that for every $f \in W^{-1,p'}(\Omega)$ there exists $u \in W_0^{1,p}(\Omega)$ such that

$$-\operatorname{div}(F(\nabla u)) = f.$$

We refer to Section 1.2 for more details and examples.

We observe that the monotonicity assumption made in *Theorem* 1.1.5 is general not easy to test; such a condition can be replaced by a weaker one:

Definition 1.1.6 *An operator $A : V \longrightarrow V'$ is pseudo-monotone if*

(i) A is bounded,

(ii) if $u_j \rightharpoonup u$ weakly in V and $\liminf_{j \longrightarrow \infty}(A(u_j), u_j - v) \geq (A(u), u - v)\ \forall v \in V$.

The following Proposition establishes the relation between monotone and pseudo-monotone operators.

Proposition 1.1.7 *If A is a bounded, hemicontinuous and monotone operators then it is pseudo-monotone .*

By Theorem 1.1.5 and *Proposition* 1.1.9 it follows that

Theorem 1.1.8 *If $A : V \longrightarrow V'$ is a pseudo-monotone and coercive operator then, for every $f \in V'$ there exists at least a function $u \in V$ such that*

$$A(u) = f.$$

A very important example of pseudo-monotone operators to which Theorem 1.1.10 can be applied is the so called "operator of Calculus of Variation", whose definition is given below ([81])

Definition 1.1.9 *An operator $A : V \longrightarrow V'$ is said to be an operator of the Calculus of Variation type if it is bounded and it can be represented as*

$$A(v) = A(v, v),$$

where the operator $(u, v) \in V \times V \longrightarrow A(u, v) \in V'$ satisfies the following conditions

$$\begin{cases} \forall u \in V,\ v \in V \longrightarrow A(u, v) \in V' \text{ is bounded and hemicontinuous,} \\ (A(u) - A(v), u - v)) \geq 0, \end{cases} \tag{1.1.7}$$

$$\forall v \in V,\ u \in V \longrightarrow A(u,v) \in V' \text{ is bounded and hemicontinuous,} \tag{1.1.8}$$

$$\begin{cases} u_j \rightharpoonup u \text{ weakly in } V \text{ and if } (A(u_j,u_j) - A(u_j,u), u_j - u)) \longrightarrow 0 \\ \text{then } \forall v \in V,\ A(u_j,v) \rightharpoonup A(u,v) \text{ weakly in } V', \end{cases} \tag{1.1.9}$$

$$\begin{cases} u_j \rightharpoonup u \text{ weakly in } V \text{ and if } A(u_j,v) \rightharpoonup \psi \text{ weakly in } V' \\ \text{then } (A(u_j,v), u_j) \longrightarrow (\psi, u). \end{cases} \tag{1.1.10}$$

Proposition 1.1.10 *If A is an operator of Calculus of Variation then A is pseudo-monotone.*

By *Proposition* 1.1.12, *Proposition* 1.1.9 and *Remark* 1.1.7 it follows a general existence result.

Proposition 1.1.11 *Let be an operator $A : V \longrightarrow V'$ of Calculus of Variations type. Then, for any $f \in V'$, the equation $A(u) = f$ admits at least a solution.*

1.2 Applications to nonlinear elliptic equations

In this section we prove an existence result for a general class of pseudo-monotone operators: furthermore, we show how it is possible to get, thanks to the result proved in the previous section, the existence of a solution for operators involving lower order terms.

From now on we assume that Ω is a bounded open set of $\mathbb{R}^n$ and we consider a class of nonlinear problems of the type

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u)) + H(x,u,\nabla u) = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2.1}$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$, $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ are two Carathéodory functions such that

$$|a(x,s,\xi)| \le C\left[b(x) + |s|^{p-1} + |\xi|^{p-1}\right], \quad b(x) \in L^{p'}(\Omega),\ C > 0, \tag{1.2.2}$$

$$a(x,s,\xi)\cdot\xi \geq |\xi|^p, \tag{1.2.3}$$

$$(a(x,s,\xi)-a(x,s,\eta))\cdot(\xi-\eta) > 0, \quad \xi \neq \eta, \tag{1.2.4}$$

$$|H(x,s,\xi)| \leq C\left[b(x)+|s|^{p-1}+|\xi|^{p-1}\right], \quad C>0, \tag{1.2.5}$$

a.e. $x \in \Omega$, $\forall s \in \mathbb{R}$, $\forall \xi$, $\eta \in \mathbb{R}^N$ and

$$f \in W^{-1,p'}(\Omega).$$

Definition 1.2.1 *Let* $p > 1$*, then if* $f \in W^{-1,p'}(\Omega)$*, a function* $u \in W_0^{1,p}(\Omega)$ *is a weak solution of problem* (1.2.1) *if*

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla\varphi dx+\int_\Omega H(x,u,\nabla u)\varphi dx = < f,\varphi >_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}, \quad \forall\varphi \in W_0^{1,p}(\Omega). \tag{1.2.6}$$

If we denote by $B:(u,v)\in W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega) \longrightarrow B(u,v)\in\mathbb{R}$ the form defined by

$$B(u,v)=\int_\Omega a(x,u,\nabla u)\cdot\nabla v dx+\int_\Omega H(x,u,\nabla u)v dx,$$

then

$$\frac{B(u,v)}{\|v\|_{W_0^{1,p}(\Omega)}} \longrightarrow +\infty \quad \text{when } \|v\|_{W_0^{1,p}(\Omega)} \longrightarrow +\infty. \tag{1.2.7}$$

The form $v \longrightarrow B(u,v)$ is linear and continuous on $W_0^{1,p}(\Omega)$. So we can write

$$B(u,v)=(A(u),v),$$

where $A(u)=-\operatorname{div}(a(x,u,\nabla u))+H(x,u,\nabla u)$.

In order to prove the existence result given by Theorem 1.2.3 below, we recall a lemma which will be useful in the following ([81]). We omit the details for briefness.

Lemma 1.2.2 *Let us assume a satisfies the conditions* $(1.2.2)-(1.2.4)$. *Let*

$$u_j,\ u \in W_0^{1,p}(\Omega) \text{ such that } u_j \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega).$$

Put

$$F_j = (a(x,u,\nabla u_j) - a(x,u,\nabla u)) \cdot (\nabla u_j - \nabla u),$$

and suppose that

$$\int\limits_{\Omega} F_j(x)dx \longrightarrow 0.$$

Then, up to a subsequence,

$$\nabla u_j \longrightarrow \nabla u \quad a.e.\ in\ \Omega,$$

and

$$H(x,u_j,\nabla u_j) \rightharpoonup H(x,u,\nabla u) \quad weakly\ in\ L^{p'}(\Omega).$$

Theorem 1.2.3 *Let be* $\tilde{A} : u \in W_0^{1,p}(\Omega) \longrightarrow -\operatorname{div}(a(x,u,\nabla u)) \in W^{-1,p'}(\Omega)$ *and assume that conditions* (1.2.2) − (1.2.7) *hold. Then, if* $f \in W^{-1,p'}(\Omega)$ *then there exists* $u \in W_0^{1,p}(\Omega)$ *such that* $\tilde{A}(u) = f$ *in* $W^{-1,p'}(\Omega)$, *that is* (1.2.6) *holds true.*

Proof. We will prove that the operator

$$A(u) = -\operatorname{div}(a(x,u,\nabla u)) + H(x,u,\nabla u)$$

is an operator of Calculus of Variations type. Then the result follows from *Theorem* 1.1.12. Let us introduce the operator $A(u,v)$. Let be

$$A_1(u,v,w) = \int\limits_{\Omega} a(x,u,\nabla v) \cdot \nabla w dx,$$

$$A_2(u,w) = \int\limits_{\Omega} H(x,u,\nabla u) w dx.$$

The form $w \longrightarrow B_1(u,v,w) + B_2(u,w)$ is continuous on $W_0^{1,p}(\Omega)$. Hence

$$A_1(u,v,w) + A_2(u,w) = \tilde{A}(u,v,w) = (A(u,v),w), \quad A(u,v) \in W^{-1,p'}(\Omega).$$

So we have

$$A(u,u) = A(u).$$

Proof of (1.1.7), (1.1.8). By (1.2.4), we have

$$(A(u,u)-A(u,v),u-v)=(A_1(u,u,u-v)-A_1(u,v,u-v))\geq 0.$$

Moreover the function $v \longrightarrow A(u,v)$ is bounded and hemicontinuous from V to V'. Indeed for u, v_1, $v_2 \in W_0^{1,p}(\Omega)$ we have, for $\lambda \longrightarrow 0$,

$$a(x,u,\nabla(v_1+\lambda v_2)) \rightharpoonup a(x,u,\nabla v_1) \quad \textit{weakly} \text{ in } L^{p'}(\Omega),$$

$$H(x,u,\nabla(v_1+\lambda v_2)) \rightharpoonup H(x,u,\nabla v_1) \quad \textit{weakly} \text{ in } L^{p'}(\Omega),$$

hence for any $w \in W_0^{1,p}(\Omega)$ we have

$$\tilde{A}(x,v_1+\lambda v_2,w) \longrightarrow A(x,v_1,w) \quad if\ \lambda \longrightarrow 0,$$

and this proves (1.1.7). In a similar way we can prove (1.1.8).

Proof of (1.1.9). Using the notation of Lemma, we get

$$(A(u_j,u_j)-A(u_j,u),u_j-u)=\int_\Omega F(x)dx;$$

then if $u_j \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ and $(A(u_j,u_j)-A(u_j,u),u_j-u) \longrightarrow 0$, by Lemma 1.2.1, we get $H(x,u_j,\nabla u_j) \rightharpoonup H(x,u,\nabla u)$ weakly in $L^{p'}(\Omega)$; moreover, being

$$H(x,u_j,\nabla u_j) \rightharpoonup H(x,u,\nabla u) \quad \textit{weakly} \text{ in } L^{p'}(\Omega),$$

we have

$$\tilde{A}(u_j,v,w) \longrightarrow \tilde{A}(u,v,w) \quad \text{for any } w \in W_0^{1,p}(\Omega);$$

hence $A(u_j,v) \longrightarrow A(u,v)$ weakly in $W^{-1,p'}(\Omega)$.

Proof of (1.1.10). Let $u_j \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ and $A(u_j,v) \rightharpoonup \psi$ weakly in $W^{-1,p'}(\Omega)$. So $u_j \longrightarrow u$ strongly in $L^p(\Omega)$, hence by Carathéodory theorem

$$A_1(u_j,v,u_j) \longrightarrow A_1(u,v,u).$$

Moreover, being

$$|A_2(u_j,u_j-u)| \geq c\,\|u_j-u\|_{L^p},$$

it follows that

$$A_2(u_j, u_j - u) \longrightarrow 0. \tag{1.2.8}$$

But

$$A_2(u_j, u) = (A(u_j, v), u) - A_1(u_j, v, u) \longrightarrow (\psi, u) - A_1(u, v, u),$$

so by (1.2.8) we get

$$A_2(u_j, u_j) \longrightarrow (\psi, u) - A_1(u, v, u)$$

and finally

$$(A(u_j, v), u_j) = A_1(u_j, v, u_j) + A_2(u_j, u_j) \longrightarrow (\psi, u).$$

■

Remark 1.2.4 *The result just proved here take places into a wide literature about existence problems for elliptic equations with* $f \in W^{-1,p'}(\Omega)$. *If we consider the problem with the lower order term* $b(x)\,|\xi|^{\lambda}$*with* $0 \le \lambda \le p-1$, $b \in L^r(\Omega)$, $r > N$ *and* $-\operatorname{div}(c(x)\,|s|^{\gamma})$, *with* $0 \le \gamma \le p-1$, $c(x) \in L^{\sigma}(\Omega)$, $\sigma > \frac{N}{p-1}$ *the operator can fail to be coercive if the norm of* $\|b\|_{L^r}$ *and* $\|c\|_{L^\sigma}$ *are not small enough. The linear case has been studied by Stampacchia in [106]; he proved an existence result assuming that the norm of* $\|b\|_{L^r}$ *and* $\|c\|_{L^\sigma}$ *with and are sufficiently small or in particular the measure* $|\Omega|$ *is small enough. The nonlinear case has been studied in [50] in the case* $c = 0$ *and in [27] in the case* $b = 0$. *In [52] the effect of the two lower order terms are taking into account: the authors proved existence result without smallness hypotheses on the norm of* $\|b\|_{L^r}$ *and* $\|c\|_{L^\sigma}$ *except naturally in the case* $\lambda = \gamma = p-1$. *In this case , the existence result still hold only if the norm of* $\|b\|_{L^r}$ *or* $\|c\|_{L^\sigma}$ *is sufficiently small.*

CHAPTER II

NOTION OF SOLUTIONS

In this chapter we introduce some well-know notion of solutions for nonlinear elliptic problem whose data are L^1 function or Radon measure with bounded total variation. Let us consider the following Dirichlet problem:

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u)) = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.0.9}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, $1 < p < N$. Furthermore $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function such that

$$a(x,s,\xi)\xi \geq \alpha \left|\xi\right|^p, \quad \alpha > 0, \tag{2.0.10}$$

$$|a(x,s,\xi)| \leq \left[|\xi|^{p-1} + |s|^{p-1} + a_0\left(x\right)\right], \quad a_0(x) \in L^{p'}(\Omega), \tag{2.0.11}$$

$$(a(x,s,\xi) - a(x,s,\eta), \xi - \eta) > 0, \quad \xi \neq \eta, \tag{2.0.12}$$

$a.e. x \in \Omega, \forall s \in \mathbb{R}, \forall \xi, \eta \in \mathbb{R}^N$ and

$$f \text{ is a bounded Radon measure with bounded total variation.}$$

If $p > N$ then, by Sobolev embedding theorems and duality arguments, the space of measures with bounded variation on Ω is a subset of $W^{-1,p'}(\Omega)$ so that a natural notion of solution for problem (2.0.9) is that of weak solution. As pointed out in the previous chapter a function $u \in W_0^{1,p}(\Omega)$ is called a weak solution to (2.0.9) if

$$\int_\Omega a(x,u,\nabla u)\cdot \nabla\varphi = < f, \varphi >_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)}, \quad \forall \varphi \in W_0^{1,p}(\Omega).$$

The existence and uniqueness of weak solutions to (2.0.9) are consequence of the theory of monotone operators ([80], [81]; see also Chapter I) . However this framework can not be extended to the case $p \leq N$, since, as showed by the following simple example, we can not expect that the solution belongs to $W_0^{1,p}(\Omega)$.

Example 2.0.5 *For $N \geq 2$ and $\Omega = B_1(0) = \left\{x \in \mathbb{R}^N : |x| < 1\right\}$, let σ_{N-1} is $(N-1)-$dimensional measure of $\partial B_1(0)$, and let be $\gamma = \frac{N-p}{p-1}$. Let us consider the function u defined by*

$$u(x) = \begin{cases} \frac{1}{\gamma}(|x|^{-\gamma} - 1) & if \quad 1 < p < N \\ -\log(|x|) & if \quad p = N, \end{cases}$$

belongs to $L^1_{loc}(\Omega)$ if and only if $\gamma < N$ that is $p > \frac{2N}{N+1}$.

Therefore the notion of weak solution does not fit the case when f is not an element of the dual space $W^{-1,p'}(\Omega)$. Moreover the classical counterexample due to Serrin shows that the solution in the sense of distribution is not unique.

Example 2.0.6 *Let be $\Omega = \left\{x \in \mathbb{R}^N : |x| < 1\right\}$ and $\lambda > 1$. Let us consider the problem*

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) = 0 & in \quad \Omega \\ u = x_1 & on \quad \partial\Omega, \end{cases} \tag{2.0.13}$$

where

$$a_{i,j} = (\lambda - 1)\,\frac{x_i x_j}{|x|^2} + \delta_{i,j},$$

It is easy to see that problem (2.0.13) *admits two distributional solutions* (*see [104], [2]*)

$$\bar{u}(x) = x_1\,|x|^{-\frac{n}{2}+\alpha}, \qquad \underline{u}(x) = x_1\,|x|^{-\frac{n}{2}-\alpha},$$

but $\underline{u}(x)$ seems to have the features of a pathological one to be rejected.

The previous argument imply that in order to get both existence and uniqueness results it is necessary to introduce new notions of solution. In this chapter we report the notions of entropy solution ([13]), SOLA ([43]), renormalized solution ([87] and [88]) and "generalized solution" ([100]).

2.1 Solution defined by duality method

In this section we deal with the notion of solution introduced by Stampacchia in [106] for linear operators. In that paper he defined a notion of solution by duality method and he proved an existence and uniqueness theorem. In particular he proved that the solution satisfies the equation (2.0.9) in the distributional sense and belongs to $W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$.

In order to explain the method introduced by Stampacchia, let us consider a bounded open set Ω of $\mathbb{R}^N$, with $N \geq 2$ and the linear elliptic problem

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1.1}$$

where $A(x) = (a_{i,j})$ is a matrix of coefficients belonging to $L^\infty(\Omega)$, satisfying the ellipticity condition

$$a_{i,j}(x)\xi_i\xi_j \geq \alpha\,|\xi|^2\,, \quad \forall \xi \in \mathbb{R}^2,\ \alpha > 0,$$

and f is a Radon measure with bounded variation on Ω.
If we define for every $u \in H_0^1(\Omega)$ the adjoint operator

$$L^*(u) = -\operatorname{div}(A^*(x)\nabla u),$$

where A^* denotes the transpose matrix of A, we consider the corresponding problem

$$\begin{cases} L^*u = f & \text{in} \quad \Omega \\ \quad u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1.2}$$

with $f \in W^{-1,p'}(\Omega)$ with $p' > N$. This problem admits a solution belonging to $C(\bar{\Omega})$ so, since $p' > N$,the operator

$$G_p^* : f \in W^{-1,p'}(\Omega) \longrightarrow u \in C(\bar{\Omega})$$

is well defined. This function G_p^* is linear and continuous so we can define the Green operator

$$G^* : \bigcup_{p'>N} W^{-1,p'}(\Omega) \longrightarrow C_0(\Omega),$$

with $G^* \Big|_{W^{-1,p'}(\Omega)} = G_p^*$.

Now we can give the definition of solution by duality to problem (2.1.1).

Definition 2.1.1 *A function $u \in L^1(\Omega)$ is a solution by duality to problem* (2.1.1) *if*

$$\int_\Omega ugdx = \int_\Omega G^*(g)df,$$

for every $g \in L^\infty(\Omega)$.

The following existence and uniqueness result hold true:

Theorem 2.1.2 *Let be f a measure with bounded variation on Ω. Then there exists a unique solution by duality to problem* (2.1.1) *. Morerover $u \in W_0^{1,q}(\Omega)$ with $q < \frac{N}{N-1}$.*

Stampacchia also proved in [106] that such a solution is Hölder continuous when $p > N$ but this continuity can be lost if $p \leq N$. In this case, the solution is continuous only if $f \in L(n,1)$ ([64], [1]). The continuity of the solution allows also to prove that the unique solution to problem (2.0.9) belongs to $L(\frac{N}{N-1}, \infty)$ and improves the result contained in [1] and in [106]. In [106] it is proven that $u \in H_0^{1,q}(\Omega)$ with $q < \frac{N}{N-1}$, while in [1] the author proved that $u \in L(\frac{N}{N-2}, \infty)$.

2.2 Solution Obtained as Limit of Approximations

Stampacchia's framework, based on a duality argument, cannot be extended to the case of a general nonlinear operator except when $p = 2$ (see [88]) when Stampacchia's ideas continue to work if the operator is strongly monotone and Lipschitz continuous with respect to ∇u. The first existence result in the nonlinear case is due to Boccardo and Gallouët. In [31] they proved the existence of a solution in the sense of distribution to problem (2.0.9) and they showed that such a solution belongs to the Sobolev space $W_0^{1,q}(\Omega)$ for every $1 < q < \frac{N(p-1)}{N-1}$. This solution is found by an approximating method which consists in finding a solution as limit of a sequence of solutions that are weak solution to (2.0.9) because of the regularity f the right hand-side.In particular they proved some apriori estimates which are the critical point in the passage under integral sign performed to define a solution known as Solution Obtained as Limit of approximation (SOLA) [43].

Definition 2.2.1 *A function* $u : \Omega \longrightarrow \mathbb{R}$ *is a SOLA to* (2.0.9) *if there exist two subsequences*

$$f_n \in L^\infty(\Omega) \quad f_n \longrightarrow f \quad \textit{strongly in } L^1(\Omega),$$

$$u_n \in W_0^{1,p}(\Omega) \quad \textit{and} \quad -\operatorname{div}(a(x, \nabla u_n)) = f_n \quad \textit{in } D'(\Omega),$$

$$u_n \longrightarrow u \quad \textit{a.e. in } \Omega.$$

The main result proved in [43] is the following existence result:

Theorem 2.2.2 *Let* $f \in L^1(\Omega)$, *then there exists a SOLA* u *of* (2.0.9) *which belongs to* $W_0^{1,q}(\Omega)$ *for every* $1 < q < \frac{N(p-1)}{N-1}$.

We point out that the procedure of passing to the limit under integral sign can be simplified by a result of continuity from the data. In order to obtain this result, hypothese (2.0.12) has to be replaced by more streghten conditions

$$(a(x,\xi) - a(x,\eta)) \cdot (\xi - \eta) \geq \alpha_1 |\xi - \eta|^p \quad if\, 2 \leq p \leq N, \tag{2.2.1}$$

$$(a(x,\xi) - a(x,\eta)) \cdot (\xi - \eta) \geq \alpha_1 \frac{|\xi - \eta|^2}{(1 + |\xi| + |\eta|)^{2-p}} \quad if\ 2 - \frac{1}{N} < p < 2, \tag{2.2.2}$$

where α, α_1 and β are positive constants.

Proposition 2.2.3 *Let assumption* (2.2.1) *or* (2.2.2) *hold and let be*

$$|a(x,\xi)| \leq A\, |\xi|^{p-1} \quad A > 0. \tag{2.2.3}$$

If f, g *are two regular functions and* u *and* v *are the solutions of*

$$-\operatorname{div}(a(x, \nabla u)) = f, \quad -\operatorname{div}(a(x, \nabla v)) = g,$$

then, for every $q < \frac{N(p-1)}{N-1}$

$$\|u-v\|_{W_0^{1,q}(\Omega)} \le \psi\left(\|f-g\|_{L^1(\Omega)}\right),$$

if $p \ge 2$, *where* ψ *is a positive function such that*

$$\lim_{s\longrightarrow 0^+} \psi(s) = 0,$$

and

$$\|u-v\|_{W_0^{1,q}(\Omega)} \le \Lambda\left(\|f\|_{L^1(\Omega)},\ \ \|g\|_{L^1(\Omega)},\ \ \|f-g\|_{L^1(\Omega)}\right),$$

if $1 < p < 2$, *where* Λ *is a function that tends to zero when* $\|f-g\|_{L^1(\Omega)}$ *tends to zero and all the other norms remain bounded.*

The previous result has been improved by Alvino and Mercaldo in [43]. In this paper the authors suggested a different and quick approach based on symmetrizzation methods (see [86] and [108]) which allows to prove apriori estimates for SOLAs to (2.0.9) and, under the stronger monotonicity assumption (2.2.1) , (2.2.2) , also a result of continuity from the data.

Proposition 2.2.4 *Let assumptions* (2.2.3) , (2.2.1) *or* (2.2.2) *hold and let be* u *and* v *two weak solutions to* (2.0.9) *with regular functions* f *and* g. *Then, for every* $q < \frac{N(p-1)}{N-1}$

$$\int_\Omega |\nabla(u-v)|^q\, dx \le C\,\|f-g\|_{L^1(\Omega)}^{\frac{q}{p-1}},$$

if $p \ge 2$,

$$\int_\Omega |\nabla(u-v)|^q\, dx \le C\left(\|f\|_{L^1(\Omega)}^{\frac{1}{p-1}} + \|f\|_{L^1(\Omega)}^{\frac{1}{p-1}}\right)\|f-g\|_{L^1(\Omega)}^{\frac{q}{p-1}},$$

if $1 < p < 2$, *where* C *depends on* N, p, $|\Omega|$, *and* q.

2.3 Entropy solution and renormalized solution

In this section we recall the equivalent notion of entropy solution ([48], [33]) and renormalized solution ([88], [87] and [48]) . In order to define this notions let us introduce the truncature operator. For a given constant $k > 0$ we define the function $T_k : \mathbb{R} \longrightarrow \mathbb{R}$ as

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \le k \\ k\text{sign}(s) & \text{if} \quad |s| > k. \end{cases} .$$

Now we want to give a sense to the derivative of a function $u \in W^{1,1}_{loc}(\Omega)$ generalizing the usual concept of weak derivative in $W^{1,1}_{loc}(\Omega)$.

Definition 2.3.1 *Let be $1 < p < \infty$ and let be u a measurable function defined on Ω which is almost everywhere finite and satisfies $T_k(u) \in W^{1,1}_{loc}(\Omega)$ for every $k > 0$. Then there exists ([13]) a measurable function $v : \Omega \to \mathbb{R}^N$ such that*

$$\nabla T_k(u) = v\chi_{\{|u|\le k\}} \quad a.e. \text{ in } \Omega, \quad for \;\; every\; k > 0. \tag{2.3.1}$$

We defined the gradient ∇u as this function v, and we denote $\nabla u = v$.

Remark 2.3.2 *We remark that the gradient defined in (2.3.1) is not the gradient used in the definition of Sobolev space, since it is possible that u does not belong to $L^1_{loc}(\Omega)$ or v does not belong to $(L^1_{loc}(\Omega))^N$. However, if v belongs to $(L^1_{loc}(\Omega))^N$, then u belongs to $W^{1,1}_{loc}(\Omega)$ and v is the distributional gradient of u. On the other hand, if u belongs to $L^1_{loc}(\Omega)$, the function v is not in general the distributional gradient of u. In fact, if Ω is the unit ball of $\mathbb{R}^N$ and $u(x) = \frac{x_1}{|x|^N}$ then $u \in L^q(\Omega)$, for every $q < \frac{N}{N-1}$ and*

$$\frac{\partial T_k(u)}{\partial x_1} = \left\{ \frac{1}{|x|^N} - N\frac{x_1^2}{|x|^{N+2}} \right\} \chi_{\{|u|\le k\}},$$

so

$$v_1 = \frac{1}{|x|^N} - N\frac{x_1^2}{|x|^{N+2}}$$

does not belong to $L^1_{loc}(\Omega)$*, which implies that* v *is not in* $\left(L^1_{loc}(\Omega)\right)^N$. *On the contrary, we have in the distributional sense*

$$\frac{\partial u}{\partial x_1} = \mathbf{pv}\left\{\frac{1}{|x|^N} - N\frac{x_1^2}{|x|^{N+2}}\right\} + \frac{1}{N}\sigma_{N-1}\delta_0,$$

where $\boldsymbol{pv}$ *denotes the principal value,* σ_{N-1} *the (N-1)- dimensional measure of the surface of the unit ball of* $\mathbb{R}^N$ *and* δ_0 *is the Dirac mass at the origin.*

Now we are able to introduce the definition of entropy solution.

Definition 2.3.3 *Let be* $f \in L^1(\Omega)$*. A measurable function* $u : \Omega \to \mathbb{R}$ *satisfying the condition* $T_k(u) \in W_0^{1,p}(\Omega)$ *for every* $k > 0$ *is an entropy solution to problem* (2.0.9) *if it results*

$$\int_\Omega a(x, \nabla u)\nabla T_k(u - \varphi)dx \le \int_\Omega T_k(u - \varphi)f dx, \tag{2.3.2}$$

for every $k > 0$ *and* $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

We underline that we did not assume that the entropy solution belongs to some Sobolev space but only that u is a measurable function.
Now it is possible to prove the following existence and uniqueness result (see [13] and [33]).

Theorem 2.3.4 *Let be* $f \in L^1(\Omega)$ *and let us assume conditions* $(2.0.10) - (2.0.12)$. *Then there exists a unique entropy solution to problem* (2.0.9).

An equivalent notion of solution to problem (2.0.9) is the so called renormalized solution; such a notion has been introduced in [87] and [88] for nonlinear elliptic equations when the datum $f \in L^1(\Omega)$.

Definition 2.3.5 *Let* $p > 1$ *and* $f \in L^1(\Omega)$*. A function* u *is a renormalized solution to* (2.0.9) *if it satisfies the following conditions:*

u is a measurable function, almost everywhere finite in Ω,

$$T_k(u) \text{ belongs to } W_0^{1,p}(\Omega), \text{ for every } k,$$

$$\frac{1}{n} \lim_{n \longrightarrow +\infty} \int_{\{n \leq |u| \leq 2n\}} a(x, \nabla u) \cdot \nabla u dx = 0,$$

$$\int_\Omega h(u) a(x, \nabla u) \nabla v dx + \int_\Omega h'(u) a(x, \nabla u) \nabla u \; v dx = \int_\Omega f h(u) v dx,$$

for every $h \in W^{1,\infty}(\mathbb{R})$ with compact support in $\mathbb{R}$ and $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Thanks to this notion we are able to get an existence and uniqueness result:

Theorem 2.3.6 *Let be $f \in L^1(\Omega)$ and let us assume conditions* (2.0.10) − (2.0.12) *hold. Then there exists a unique renormalized solution to problem* (2.0.9) .

This theorem has been improved in [48]; in this work the authors extended the definition of renormalized solution to the general case where f is a Radon measure with bounded variation on Ω.

Such definition needs the notion of p-capacity which we brifly recall here. The p-capacity $cap_p(K, \Omega)$ of a compact set $K \subset \Omega$ with respect to Ω is

$$cap_p(K, \Omega) = \inf \left\{ \int_\Omega |\nabla \varphi|^p : \varphi \in C_c^\infty(\Omega), \quad \varphi \geq \chi_K \right\},$$

where χ_K is the characteristic function of k. If $U \subseteq \Omega$ is an open set, we denote by

$$cap_p(U, \Omega) = \sup \left\{ cap_p(K, \Omega) : K \text{ compact}, \; K \subseteq \Omega \right\}.$$

Finally, the p-capacity of any subset $B \subseteq \Omega$ is defined by

$$cap_p(B, \Omega) = \sup \left\{ cap_p(U, \Omega) : B \text{ open}, \; B \subseteq \Omega \right\}.$$

We denote by $M_b(\Omega)$ the space of all Radon measure with bounded variation on Ω and by $C_b^0(\Omega)$ the space of all bounded and continuous functions on Ω so $\int_\Omega \varphi d\mu$ is well defined for $\varphi \in C_b^0(\Omega)$. Finally we denote by μ^+, μ^- and $|\mu|$ the positive, negative part and total variation of measure μ in $M_b(\Omega)$. We denote by $M_0(\Omega)$ as the set of all measures μ in $M_b(\Omega)$ which are absolutely continuous with respect to the p-capacity, i.e. which satisfy $\mu(B) = 0$ for every Borel set $B \subseteq \Omega$ such that $cap_p(B, \Omega) = 0$. We denote by $M_s(\Omega)$ as the set of all the measures μ in $M_b(\Omega)$ which are singular with respect to the p-capacity, i.e. which are concentrated in a set $E \subset \Omega$ such that $cap_p(E, \Omega) = 0$. The following proposition shows an important decomposition result (see [66] for the proof).

Proposition 2.3.7 *For every Radon measure with bounded variation on Ω there exists an unique pair of measures (μ_0, μ_s) with $\mu_0 \in M_0(\Omega)$ and $\mu_s \in M_s(\Omega)$ such that $\mu = \mu_0 + \mu_s$.*

The measures μ_0 and μ_s are called the absolutely continuous part and the singular part of μ with respect to the p-capacity. For what concerns μ_0 the following decomposition result holds ([33]):

Proposition 2.3.8 *Let μ_0 be a a Radon measure with bounded variation on Ω. Then μ_0 belongs to $M_0(\Omega)$ if it belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$. Thus if μ_0 belongs to $M_0(\Omega)$, there exists f in $L^1(\Omega)$ and $g \in \left(L^{p'}(\Omega)\right)^N$ such that*

$$\mu_0 = f - \operatorname{div}(g)$$

in the sense of distributions; moreover

$$\int_\Omega v d\mu_0 = \int_\Omega f v dx + \int_\Omega g \cdot \nabla v dx, \quad \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

Now we are able to introduce the notion of renormalized solution to problem (2.0.9) when f is a Radon measure with bounded variation on Ω.

Definition 2.3.9 *Let be μ a Radon measure with bounded variation on Ω and assume that a satisfies* $(2.0.10) - (2.0.12)$ *. A function u is a renormalized solution to problem* (2.0.9) *if it satisfies the following conditions:*

u is a measurable function, almost everywhere finite,

$$T_k(u) \text{ belongs to } W_0^{1,p}(\Omega), \text{ for every } k,$$

for every $\varphi \in C_b^0(\Omega)$ *we have*

$$\lim_n \frac{1}{n} \int_{\{n \le u < 2n\}} a(x, \nabla u) \cdot \nabla u \varphi dx = \int_\Omega \varphi d\mu_s^+, \tag{2.3.3}$$

and

$$\lim_n \frac{1}{n} \int_{\{-2n \le u < -n\}} a(x, \nabla u) \cdot \nabla u \varphi dx = \int_\Omega \varphi d\mu_s^-, \tag{2.3.4}$$

for every $h \in W^{1,\infty}(\mathbb{R})$ *with compact support in* $\mathbb{R}$ *we have*

$$\int_\Omega h(u) a(x, \nabla u) \nabla \varphi dx + \int_\Omega h'(u) a(x, \nabla u) \nabla u \varphi dx = \int_\Omega \varphi h(u) d\mu_0, \tag{2.3.5}$$

for every $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *such that* $h(u)\varphi$ *belongs to* $W_0^{1,p}(\Omega)$.

Remark 2.3.10 *We observe that conditions* (2.3.3) *and* (2.3.4) *are equivalent to say that the sequences* $\frac{1}{n} a(x, \nabla u) \cdot \nabla u \chi_{\{n \le u < 2n\}}$ *and* $\frac{1}{n} a(x, \nabla u) \cdot \nabla u \chi_{\{-2n \le u < -n\}}$ *converge to* μ_s^+ *and* μ_s^- *respectively in the narrow topology. In other words, the measures* μ_s^+ *and* μ_s^- *can be obtained through the energy of the solution.*

The main result contained in [48] is the following existence result:

Theorem 2.3.11 *Let be* μ *a Radon measure with bounded variation on* Ω *and assume that a satisfies* (2.0.10)−(2.0.12) *. Then there exists a renormalized solution to* (2.0.9) *.*

We observe that the Radon measure with bounded variation on Ω is not the general datum for problem (2.0.9) . Indeed, there exists elements in $W^{-1,p'}(\Omega)$ which are not measures, such that the data

$$\mu - \operatorname{div}(F), \tag{2.3.6}$$

$F \in \left(L^{p'}(\Omega)\right)^N$ can be considered. However, the new term $-\operatorname{div}(F)$, does not give an additional difficulty because of its "regularity" and the Theorem 2.3.11 still hold with the same proof whenever Definition 2.3.9 is modified as follows. The condition(2.3.3) (similar consideration apply to (2.3.4)) have to be replaced by

$$E(t, s) = \frac{1}{t-s} \int_{\{s \le u \le t\}} a(x, \nabla u) \cdot \nabla u \varphi dx,$$

and it is possible to prove that a subsequence $E(t_n, s_n)$, for some conveniently chosen t_n e s_n, converges to

$$\int_{\Omega} \varphi d\mu_s^+.$$

Solution 2.3.12 ***Remark 2.3.13*** *The result contained in this chapter can be extended to nonlinear elliptic Neumann problems in [3], [15], [62], [41], [63], [98].*

CHAPTER III

UNIFORMLY ELLIPTIC EQUATIONS WITH LOWER ORDER TERMS

In this chapter we consider a class of nonlinear elliptic problems of the type

$$\begin{cases} -\operatorname{div}(a(x,\nabla u)) - \operatorname{div}(\Phi(x,u)) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.0.7}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, $1 < p < N$, $a : (x,z) \in \Omega \times \mathbb{R}^N \to a(x,z) = (a_i(x,z)) \in \mathbb{R}^N$ is a Carathéodory function satisfying:

$$a(x,\xi) \cdot \xi \geq \lambda |\xi|^p, \qquad \xi \in \mathbb{R}^N, \quad \lambda > 0, \tag{3.0.8}$$

$$|a(x,\xi)| \leq \Lambda |\xi|^{p-1}, \qquad \xi \in \mathbb{R}^N, \quad \Lambda > 0, \tag{3.0.9}$$

with $1 < p < N$, and

$$(a(x,\xi) - a(x,\eta)) \cdot (\xi - \eta) > 0, \qquad \xi \neq \eta, \tag{3.0.10}$$

for almost every $x \in \mathbb{R}^N$ and for every ξ, $\eta \in \mathbb{R}^N$.

Furthermore

$$|\Phi(x,s)| \leq c(x)\,|s|^{p-1} \tag{3.0.11}$$

where $c(x) \in L^{\frac{N}{p-1}}(\Omega)$, $c(x) \geq 0$ a. e. in Ω.

Finally

$$f \text{ is a } L^1(\Omega) \text{ function or a Radon measure with bounded total variation.}$$

In this Chapter we prove the existence of a SOLA to problem (3.0.7) if f is a Radon measure with bounded variation on Ω and some uniqueness results when f is a L^1- function. Such results are contained in [56]. Stampacchia in [106] studied problem (3.0.7) in the linear case under the assumption of smallness of $\|c\|_{L^{\frac{N}{p-1}}}$. Furthermore many authors have proved existence results for problem (3.0.7) in the nonlinear case (see [28] and [12]; see also [52], [72] and [73]). In [56] we prove the same result with a different and quick approach. The same approach has been followed in [5] for the nonlinear problem (3.0.7) with $\Phi = 0$ and in [6] when the lower order term is of the type $b(x)\,|\nabla u|^{p-1}$. It is based on the choice of a timely test function and a comparison result between the solution to problem (3.0.7) with regular data and the solution of a suitable spherically symmetric problem (see [17]). This comparison result has been obtained by classical symmetrizzation methods introduced by Talenti and Maz'ja (see [108] and [86]).

For what concerns the uniqueness results, the first attempt for regular data is due to Trudinger ([109]) . Then, the case $f \in H^{-1}(\Omega)$ has been studied in [10] when $\Phi = 0$ and in [33] when Φ is a Lipschitz continuous function. Other uniqueness results can be found in [37], [38] and [39],. These results are always in the context of finite energy solutions; with respect to the datum f, this means that f is taken in $L^m(\Omega)$ with $m \geq \frac{2N}{N+2}$.

On the other hand, assuming that $f \in L^1(\Omega)$, the uniqueness of entropy solution has been obtained in [94] when Φ is locally Lipschitz continuous and has at most an exponential growth at infinity.

Other uniqueness results have been also proved for renormalized solutions in [12] and in [19], (see also [70], [74]).

In this chapter we prove uniqueness results of SOLA's when $f \in L^1(\Omega)$ and Φ is locally Lipschitz continuous with respect to the second variable.

The uniqueness can be proved if we get an estimate of the gradient of the difference of two solutions in term of the L^1- norm of the difference of data. Unfortunately, the assumptions of existence are not enought to have this result even if $\Phi = 0$ and f is regular. To overcome this difficulty we have to modify the classical monotonicity hypotheses on the structure of the operator.

3.1 Definitions and preliminary results

In this section we recall the definition of rearrangementes and some properties which will be used throughout.

Let us consider a measurable function $u : \Omega \to \mathbb{R}$, where Ω is a measurable subset of $\mathbb{R}^N$. We denote by μ the distribution function of u

$$\mu(t) = \left|\{x \in \Omega : |u(x)| \geq t\}\right|, \qquad t \geq 0,$$

and by u^* the decreasing rearrangement of u

$$u^*(s) = \sup\{t \geq 0 : \mu(t) > s\}, \qquad s \in (0, |\Omega|).$$

The increasing rearrangement u_* of u is defined as

$$u_*(s) = u^*(|\Omega| - s), \qquad s \in (0, |\Omega|).$$

If ω_N denotes the measure of unit ball of $\mathbb{R}^N$ and $\Omega^\#$ the ball of $\mathbb{R}^N$ centered in the origin such that $|\Omega| = \left|\Omega^\#\right|$, the sferically decreasing and the sferically increasing rearrangements of u are

$$u^\#(x) = u^*\left(\omega_N |x|^N\right), \qquad u_\#(x) = u_\#\left(\omega_N |x|^N\right), \qquad x \in \Omega^\#.$$

If $1 < q < +\infty$ and $1 \leq r \leq +\infty$, the Lorentz space $L^{q,r}(\Omega)$ is the class of function u such that

$$\|u\|^*_{q,r} = \left(\int_0^{+\infty} \left[u^*(s)\, s^{\frac{1}{q}}\right]^r \frac{ds}{s}\right)^{\frac{1}{r}} < +\infty, \tag{3.1.1}$$

$$\|u\|_{q,\infty}^{*} = \sup_{s>0} u^{*}(s)\, s^{\frac{1}{q}} < +\infty. \tag{3.1.2}$$

We remark that $L^{q,q}(\Omega) = L^{q}(\Omega)$, and $L^{q,\infty}(\Omega)$ is the Marcinkiewicz space $L^{q}-weak$; moreover, if Ω is bounded, the following embeddings hold (see [76], [91])

$$L^{q,r_1}(\Omega) \subset L^{q,r_2}(\Omega), \qquad r_1 < r_2,$$

$L^{q_1,r}(\Omega) \subset L^{q}(\Omega)$, $q < q_1$.

Here we just recall some inequalities which will be useful later down (for an exhaustive treatment of rearrangements see [40] and [77]). Here we just recall the Hardy-Littlewood inequality

$$\int_{\Omega^{\#}} u^{\#}(x)\, v_{\#}(x)\, dx \leq \int_{\Omega} |u(x)\, v(x)|\, dx \leq \int_{\Omega^{\#}} u^{\#}(x)\, v^{\#}(x)\, dx, \tag{3.1.3}$$

with u, v measurable functions (see [75]), and a Sobolev-type inequality (see [8])

$$\int_{\Omega^{\#}} |x|^{N\alpha - p} \left[u^{\#}(x)\right]^{p} dx \leq \omega_N^{-\alpha} \left(\frac{p}{N-p+N\alpha}\right)^{p} \int_{\Omega} \left[\mu(|u(x)|)\right]^{\alpha} |\nabla u|^{p}\, dx, \tag{3.1.4}$$

where $u \in W_0^{1,p}(\Omega)$, $p < N$, $\alpha > 0$ and μ is the distribution function of u. Finally we recall a comparison result between the solution of the nonlinear elliptic problem (3.0.7) with regular datum and the solution of a suitable problem with radially symmetric datum. The result, contained in [17], is

$$u^{*}(s) \leq 2^{p-1} \int_{s}^{|\Omega|} \frac{1}{(N\omega_N^{\frac{1}{N}})^{p'} t^{p'\left(1-\frac{1}{N}\right)}} \left(\int_0^t f^{*}(\tau)\, d\tau\right)^{\frac{1}{p-1}} \exp\left(\int_s^t \frac{C(r)^{\frac{1}{p-1}}}{N\omega_N^{\frac{1}{N}} r^{1-\frac{1}{N}}} dr\right) dt, \tag{3.1.5}$$

for $a.e.$ $s \in (0, |\Omega|]$. Here $C(r)$ is defined as

$$\int_{|u|>t} c(x)^{p'}\, dx = \int_0^{\mu(t)} C(s)^{p'}\, ds;$$

the following lemma shows that $C(r)$ can be obtained as weak limit of functions having the same rearrangement of $c(r)$

Lemma 3.1.1 *If $f \in L^p(\Omega)$, $p \geq 1$, there exists a sequence $\{f_k\}$ of functions such that $f_k^* = f^*$ and*

$$f_k \rightharpoonup F \qquad \text{in } L^p(0, |\Omega|)\,, \text{ if } p > 1,$$

and

$$\lim_{k\to+\infty} \int_0^{|\Omega|} f_k(s)\, g(s)\, ds = \int_0^{|\Omega|} F(s)\, g(s)\, ds, \qquad g \in BV([0, |\Omega|])\,, \text{ if } p = 1.$$

As conseguence of the previous result any Lebesgue or Lorentz norm of $C(r)$ can be estimated from above with the same norm of $c(r)$; this implies that $C(r)$ and $c(r)$ have the same sommability and so (3.1.5) becomes

$$u^*(s) \leq K \|f\|_{L^1}^{\frac{1}{p-1}} s^{-\frac{N-p}{N(p-1)}}, \tag{3.1.6}$$

where K is a constant depending on $|\Omega|$, N, p, $\|c\|_{L^{p'}}$.
We explicitelly remark that analogous inequalities have been proved in the linear case in [108], [86].

3.2 A priori estimates

In this section we prove a priori estimates for weak solutions to problem (3.0.7) in terms of the L^1- norm of the datum. From now on, we assume that

$$p > 2 - \frac{1}{N}. \tag{3.2.1}$$

We underline that this condition is set only to avoid technicalities: the result, in fact, can be proved also in the case $1 < p \leq 2 - \frac{1}{N}$.

Theorem 3.2.1 *Under assumptions (3.0.8) - (3.0.11), if u is a weak solution to problem (3.0.7) with $f \in C^\infty(\Omega)$, then we have:*

$$\left(\int_\Omega |\nabla u|^q\, dx\right)^{\frac{1}{q}} \leq K \|f\|_{L^1}^{\frac{1}{p-1}}, \tag{3.2.2}$$

where $q < \frac{N(p-1)}{N-1}$ and K is a positive constant depending on N, p, q, λ, $|\Omega|$, $\|c\|_{L^{\frac{N}{p-1}}}$.

Proof. Under hypotheses (3.0.8) - (3.0.11) there exists a unique weak solution $u \in W_0^{1,p}(\Omega)$ (see [81]).

So, let be μ the distribution function of u; we define:

$$\varphi(x) = \operatorname{sign}[u(x)] \int_0^{|u(x)|} [\mu(t)]^\alpha, \qquad \alpha > 0. \tag{3.2.3}$$

We observe that φ is a valid test function because $\varphi \in W_0^{1,p}(\Omega)$. Using the definition of weak solution to problem (3.0.7) we have:

$$\int_\Omega [a(x, \nabla u) \cdot \nabla u] [\mu(|u(x)|)]^\alpha dx + \int_\Omega \Phi(x,u) \cdot \nabla u \ [\mu(|u(x)|)]^\alpha dx = \int_\Omega f\varphi dx. \tag{3.2.4}$$

From (3.0.8):

$$\int_\Omega [a(x, \nabla u) \cdot \nabla u] [\mu(|u(x)|)]^\alpha dx \geq \lambda \int_\Omega |\nabla u|^p [\mu(|u(x)|)]^\alpha dx. \tag{3.2.5}$$

On the other hand, by (3.0.11) and by Hölder inequality we have:

$$\int_\Omega \Phi(x,u) \cdot \nabla u \ [\mu(|u(x)|)]^\alpha dx \leq \|c\|_{L^{\frac{N}{p-1}}} \left(\int_\Omega |\nabla u|^p [\mu(|u(x)|)]^\alpha dx \right)^{\frac{1}{p}} \times$$
$$\times \left(\int_\Omega |u|^{\frac{Np}{N-p}} [\mu(|u(x)|)]^{\frac{N\alpha}{N-p}} dx \right)^{\frac{1}{p'} \frac{N-p}{N}}. \tag{3.2.6}$$

By coarea formula and by (3.1.6), (3.2.6) becomes:

$$\int_\Omega \Phi(x,u) \cdot \nabla u \ [\mu(|u(x)|)]^\alpha dx \leq K \|f\|_{L^1} \left(\int_\Omega |\nabla u|^p [\mu(|u(x)|)]^\alpha dx \right)^{\frac{1}{p}} \times$$
$$\times \left(\int_0^{|\Omega|} s^{-p' + \frac{N\alpha}{N-p}} ds \right)^{\frac{1}{p'} \frac{N-p}{N}}, \tag{3.2.7}$$

where $K = K\left(|\Omega|, \ N, \ p, \ \|c\|_{L^{\frac{N}{p-1}}}\right)$; for the rest of the paper K is a constant which can vary from line to line.

Assuming that

$$\alpha > \frac{N-p}{N(p-1)}, \tag{3.2.8}$$

the last integral in (3.2.7) is finite, so we obtain:

$$\int_\Omega \Phi(x,u) \cdot \nabla u [\mu(|u(x)|)]^\alpha dx \leq K \|f\|_{L^1} \left(\int_\Omega |\nabla u|^p [\mu(|u(x)|)]^\alpha dx \right)^{\frac{1}{p}}. \tag{3.2.9}$$

From (3.2.4), (3.2.5), (3.2.9) we get:

$$\lambda \int_{\Omega} |\nabla u|^p \left[\mu\left(|u(x)|\right)\right]^\alpha dx \le K \|f\|_{L^1} \left(\int_{\Omega} |\nabla u|^p \left[\mu\left(|u(x)|\right)\right]^\alpha dx \right)^{\frac{1}{p}} + \int_{\Omega} |f| \, |\varphi| \, dx. \tag{3.2.10}$$

Now we evaluate the $L^\infty-$ norm of φ; since α satisfies condition (3.2.8) we have:

$$\sup_{\Omega} |\varphi(x)| = \int_0^{+\infty} [\mu(t)]^\alpha dt = \alpha \int_0^{|\Omega|} s^{\alpha-1} u^*(s) \, ds.$$

By Hölder inequality, (3.2.8) and by Sobolev-type inequality (3.1.4) we obtain:

$$\sup_{\Omega} |\varphi(x)| \le \left(\int_0^{|\Omega|} s^{\alpha - \frac{(N-1)p'}{N}} ds \right)^{\frac{1}{p'}} \left(\int_0^{|\Omega|} s^{\alpha - \frac{p}{N}} [u^*(s)]^p \, ds \right)^{\frac{1}{p}} \le$$

$$\le K \left(\int_{\Omega} |\nabla u|^p \left[\mu\left(|u(x)|\right)\right]^\alpha dx \right)^{\frac{1}{p}}. \tag{3.2.11}$$

Coming back to (3.2.10), by (3.2.11) we have:

$$\lambda \int_{\Omega} |\nabla u|^p \left[\mu\left(|u(x)|\right)\right]^\alpha dx \le K \|f\|_{L^1} \left(\int_{\Omega} |\nabla u|^p \left[\mu\left(|u(x)|\right)\right]^\alpha dx \right)^{\frac{1}{p}}. \tag{3.2.12}$$

So

$$\int_{\Omega} |\nabla u|^p \left[\mu\left(|u(x)|\right)\right]^\alpha dx \le K \|f\|_{L^1}^{p'}, \tag{3.2.13}$$

where $K = K\left(|\Omega|, \ N, \ p, \ \|c\|_{L^{\frac{N}{p-1}}}, \ \lambda\right)$.

From (3.2.13) we can deduce a priori estimates for the gradient of weak solutions to problem (3.0.7).

For any fixed $q < \frac{N(p-1)}{N-1}$,we choose α such that $q < \frac{p}{1+\alpha}$. By Hardy-Littlewood inequality (3.1.3) we get

$$\| |\nabla u| \|_{L^q}^p \le K \|\nabla u\|_{\frac{p}{1+\alpha},p}^p,$$

with $\alpha > \frac{N-p}{N(p-1)}$. So by (3.2.13)

$$\| |\nabla u| \|_{L^q}^p \le K \int_{\Omega} |\nabla u|^p \left[\mu\left(|u(x)|\right)\right]^\alpha dx \le K \|f\|_{L^1}^{p'}.$$

■

3.3 Existence result

In this section we prove the existence of a SOLA to problem (3.0.7). As pointed out before, the SOLA is obtained as limit of approximations and the starting point of this procedure is the estimate (3.2.2). The convergence of the gradients of approximated solutions is obtained in a easily way since we prove a result which gives an estimate of the difference of the gradients in terms of the difference of two solutions. To this aim we substitute assumptions (3.0.8) and (3.0.10) by conditions

$$(a(x,\xi)-a(x,\eta))\cdot(\xi-\eta)\geq\beta\frac{|\xi-\eta|^2}{(|\xi|+|\eta|)^{2-p}},\qquad \xi\neq\eta, \tag{3.3.1}$$

if $1<p<2$, or

$$(a(x,\xi)-a(x,\eta))\cdot(\xi-\eta)\geq\beta\,|\xi-\eta|^p\,,\qquad \xi\neq\eta, \tag{3.3.2}$$

if $p\geq 2$. In order to avoid some technicalities, we suppose that $c(x)\in L^{\infty}(\Omega)$, but the same result is still valid if $c(x)\in L^{\frac{N}{p-1}}(\Omega)$ under the hypotheses given in Remark 3.2.

Now we prove the following result

Proposition 3.3.1 *Let us assume (3.0.9), (3.0.11) with $c(x)\in L^{\infty}(\Omega)$, (3.3.1), (3.3.2). Let be u, v weak solutions to problem (3.0.7) with data f, $g\in C^{\infty}(\Omega)$ respectively, q such that $q<\frac{N(p-1)}{N-1}$ and $m\leq q^*=\frac{Nq}{N-q}$.*
If $p\geq 2$, and

$$|\Phi(x,u)-\Phi(x,v)|\leq K\,(|u|+|v|)^{p-2}\,|u-v|\,, \tag{3.3.3}$$

then we have

$$\||\nabla(u-v)|\|_{L^q}\leq K\,\|u-v\|_{L^m}^{\frac{1}{p-1}}\left[\|f\|_{L^1}^{\frac{p-2}{(p-1)^2}}+\|g\|_{L^1}^{\frac{p-2}{(p-1)^2}}+\|f\|_{L^1}^{\frac{1}{p}}+\|g\|_{L^1}^{\frac{1}{p}}\right]. \tag{3.3.4}$$

If $1<p<2$ and

$$|\Phi(x,u)-\Phi(x,v)|\leq K\,|u-v|^{p-1}\,, \tag{3.3.5}$$

then we have:

$$\left\| |\nabla (u-v)| \right\|_{L^q} \le K \left\| u-v \right\|_{L^m}^{\frac{\sigma}{2}} \left(\|f\|_{L^1} + \|g\|_{L^1} \right)^{\frac{2-\sigma}{2(p-1)}} + \left\| u-v \right\|_{L^m}^{\frac{1}{2}} \left(\|f\|_{L^1} + \|g\|_{L^1} \right)^{\frac{p}{2(p-1)}}, \tag{3.3.6}$$

with σ *a suitable constant such that* $0<\sigma<\min\left\{p-1, \frac{m}{q'}\right\}$. *The constant* K *depends on* N, β, p, q, $|\Omega|$, $\|c\|_{L^\infty}$.

Proof.

Let be u and v the weak solutions to problem (3.0.7) with data f, $g \in C^\infty(\Omega)$ respectively. Denoted by μ the distribution function of $|u-v|$, let us consider the test function

$$\varphi(x) = \operatorname{sign}(u-v) \int_0^{|u-v|(x)} [\mu(t)]^\alpha \, dt,$$

with $\alpha > 0$. Taking φ in (3.0.7) with data f and g, and subtracting get

$$\int_\Omega \left[\left(a(x,\nabla u) - a(x,\nabla v) \right) \cdot \nabla (u-v) \right] \left[\mu\left(|u-v|(x) \right) \right]^\alpha dx =$$

$$= \int_\Omega \left[\Phi(x,u) - \Phi(x,v) \right] \cdot \nabla (u-v) \left[\mu\left(|u-v|(x) \right) \right]^\alpha dx + \int_\Omega (f-g)\, \varphi dx. \tag{3.3.7}$$

Case $p \ge 2$.

From (3.3.2) the left-hand side of (3.3.7) satisfies

$$\int_\Omega \left[\left(a(x,\nabla u) - a(x,\nabla v) \right) \cdot \nabla (u-v) \right] \left[\mu\left(|u-v|(x) \right) \right]^\alpha dx \ge$$

$$\ge \beta \int_\Omega |\nabla (u-v)|^p \left[\mu\left(|u-v|(x) \right) \right]^\alpha dx. \tag{3.3.8}$$

Now we evaluate the right-hand side. From (3.3.3) and Young inequality we deduce

$$\int_\Omega \left[\Phi(x,u) - \Phi(x,v) \right] \cdot \nabla (u-v) \left[\mu\left(|u-v|(x) \right) \right]^\alpha dx \le$$

$$\le K \int_\Omega |\nabla (u-v)|^p \left[\mu\left(|u-v|(x) \right) \right]^\alpha dx +$$

$$+ K \|c\|_{L^\infty} \int_\Omega \left[|u| + |v| \right]^{(p-2)p'} |u-v|^{p'} \left[\mu\left(|u-v|(x) \right) \right]^\alpha dx. \tag{3.3.9}$$

Now we get an estimate of the last integral in (3.3.9). From (3.1.3), Hölder inequality and (3.1.6) we have

$$\int_\Omega [|u|+|v|]^{(p-2)p'} |u-v|^{p'} [\mu(|u-v|(x))]^\alpha dx \leq$$

$$\leq K \|u-v\|^{p'}_{\frac{Np}{N\alpha-p+N},p} \left[\|f\|_{L^1}^{\frac{p(p-2)}{(p-1)^2}} + \|g\|_{L^1}^{\frac{p(p-2)}{(p-1)^2}}\right] \left(\int_0^{|\Omega|} s^{\alpha+\frac{p}{N(p-2)}-\frac{p(N-p)}{N(p-1)}} ds\right)^{\frac{p-2}{p-1}}. \tag{3.3.10}$$

Since $\alpha > \frac{N-p}{N(p-1)}$, the last integral in (3.3.10) is finite. By (3.3.10), inequality (3.3.9) becomes

$$\int_\Omega [\Phi(x,u)-\Phi(x,v)] \cdot \nabla(u-v) [\mu(|u-v|(x))]^\alpha dx \leq$$

$$\leq K \int_\Omega |\nabla(u-v)|^p [\mu(|u-v|(x))]^\alpha dx +$$

$$+K \|u-v\|^{p'}_{\frac{Np}{N\alpha-p+N},p} \left[\|f\|_{L^1}^{\frac{p(p-2)}{(p-1)^2}} + \|g\|_{L^1}^{\frac{p(p-2)}{(p-1)^2}}\right], \tag{3.3.11}$$

Let us consider the last term in (3.3.7)

$$\int_\Omega |f-g| |\varphi| dx \leq \|f-g\|_{L^1} \|\varphi\|_{L^\infty}. \tag{3.3.12}$$

Since α satisfies (3.2.8), then

$$\sup_\Omega |\varphi(x)| = \int_0^{+\infty} [\mu(t)]^\alpha dt = \alpha \int_0^{|\Omega|} s^{\alpha-1} (u-v)^*(s) ds = \alpha \|u-v\|_{\frac{1}{\alpha},1}. \tag{3.3.13}$$

Fixed $m < q^*$, we choose α in such a way that

$$\frac{Np}{N\alpha-p+N} < m < q^*. \tag{3.3.14}$$

Hence it results that $\frac{1}{\alpha} < m$. So using (3.3.8), (3.3.11) - (3.3.13) and condition (3.3.14), by (3.3.7) we obtain

$$\int_\Omega |\nabla(u-v)|^p [\mu(|u-v|(x))]^\alpha dx \leq$$

$$\leq K \|u-v\|_{L^m}^{\frac{p}{p-1}} \left[\|f\|_{L^1}^{\frac{p(p-2)}{(p-1)^2}} + \|g\|_{L^1}^{\frac{p(p-2)}{(p-1)^2}} + \|f-g\|_{L^1}\right]. \tag{3.3.15}$$

From Hardy-Littlewood inequality, and choosing once again α so that $q < \frac{p}{1+\alpha}$,we get

$$\int_\Omega |\nabla (u-v)|^p \left[\mu\left(|u-v|(x)\right)\right]^\alpha dx \geq K \left\||\nabla (u-v)|\right\|^p_{\frac{p}{1+\alpha},p} \geq K \left\||\nabla (u-v)|\right\|^p_{L^q}.$$

Therefore, by (3.3.15) and by last inequality we obtain (3.3.4).

Case $1 < p < 2$.

Let us consider the function

$$G(x) = \frac{|\nabla (u-v)|^{\frac{2}{p}}}{\left(|\nabla u| + |\nabla v|\right)^{\frac{2-p}{p}}}.$$

Coming back to (3.3.7), now we consider the hypotheses (3.3.1) on a and the assumption (3.3.5) on Φ,we get:

$$\int_\Omega G(x)^p \left[\mu\left(|u-v|(x)\right)\right]^\alpha dx \leq K \int_\Omega |c(x)| \left|\nabla (u-v)\right| |u-v|^{p-1} \left[\mu\left(|u-v|(x)\right)\right]^\alpha dx + \\ + \int_\Omega |f-g| \, |\varphi| \, dx. \tag{3.3.16}$$

Fixed $0 < \sigma < p-1$, we set

$$I = \int_\Omega |c(x)| \left|\nabla (u-v)\right| |u-v|^{p-1-\sigma} |u-v|^\sigma \left[\mu\left(|u-v|(x)\right)\right]^\alpha dx.$$

Let be $m < q^*$, we choose α such that $\frac{1}{\alpha} < m < q^*$. Now let σ be $0 < \sigma < \min\left\{p-1, \frac{m}{q'}\right\}$. Denoted by θ the positive number such that $\frac{1}{\theta} = 1 - \frac{\sigma}{m} - \frac{1}{q}$, we can apply Hölder inequality and have

$$I \leq \|c\|_{L^\infty} \|u-v\|^\sigma_{L^m} \left\||\nabla u| + |\nabla v|\right\|_{L^q} \left(\int_\Omega |u-v|^{(p-1-\sigma)\theta} \left[\mu\left(|u-v|(x)\right)\right]^{\alpha\theta} dx\right)^{\frac{1}{\theta}}.$$

From the Hardy-Littlewood inequality, the comparison result (3.1.6) applied to u^* and v^*, and by a priori estimate (3.2.2) on the gradient of u and v, we finally have:

$$I \leq K \|u-v\|^\sigma_{L^m} \left(\|f\|_{L^1} + \|g\|_{L^1}\right)^{\frac{p-\sigma}{p-1}}. \tag{3.3.17}$$

By (3.3.16) and (3.3.17) , we get

$$\int_\Omega G(x)^p \left[\mu(|u-v|)(x))\right]^\alpha dx \leq K \|u-v\|^\sigma_{L^m} \left(\|f\|_{L^1} + \|g\|_{L^1}\right)^{\frac{p-\sigma}{p-1}} + \int_\Omega |(f-g)| \, |\varphi| \, dx.$$

The term $\int_\Omega |(f-g)|\,|\varphi|\,dx$ is evolved as in the case $p \geq 2$. So we obtain:

$$\int_\Omega G(x)^p\,[\mu(|u-v|)(x))]^\alpha\,dx \leq K\,\|u-v\|_{L^m}^\sigma\,(\|f\|_{L^1}+\|g\|_{L^1})^{\frac{p-\sigma}{p-1}}+$$

$$+K\,\|u-v\|_{L^m}\,\|f-g\|_{L^1}\,. \tag{3.3.18}$$

Now we estimate from below the left-hand side of (3.3.18) by proceeding as in the proof of Theorem 3.2.1. Indeed, by Hardy-Littlewood inequality:

$$\int_\Omega G(x)^p\,[\mu(|u-v|)(x))]^\alpha\,dx \geq \|G\|^p_{\frac{p}{1+\alpha},p}\,.$$

If we choose once again α in way that $q < \frac{p}{1+\alpha}$, we have:

$$\|G\|^p_{L^q} \leq K\,\|u-v\|_{L^m}^\sigma\,(\|f\|_{L^1}+\|g\|_{L^1})^{\frac{p-\sigma}{p-1}}+$$

$$+\,K\,\|u-v\|_{L^m}\,\|f-g\|_{L^1}\,. \tag{3.3.19}$$

Since

$$\int_\Omega |\nabla(u-v)|^q\,dx \leq K\,\|G\|_{L^q}^{qp/2}\left(\|f\|_{L^1}^{\frac{1}{p-1}}+\|g\|_{L^1}^{\frac{1}{p-1}}\right)^{q(1-\frac{p}{2})},$$

then

$$\|\nabla(u-v)\|_{L^q} \leq K\,\|G\|_{L^q}^{p/2}\left(\|f\|_{L^1}^{\frac{1}{p-1}}+\|g\|_{L^1}^{\frac{1}{p-1}}\right)^{1-\frac{p}{2}}. \tag{3.3.20}$$

From (3.3.19) and (3.3.20) we have:

$$\||\nabla\,(u-v)|\|_{L^q} \leq K\,\|u-v\|_{L^m}^{\frac{\sigma}{2}}\,(\|f\|_{L^1}+\|g\|_{L^1})^{\frac{2-\sigma}{2(p-1)}}+\|u-v\|_{L^m}^{\frac{1}{2}}\,(\|f\|_{L^1}+\|g\|_{L^1})^{\frac{p}{2(p-1)}}\,,$$

where $K = K(N,\,p,\,q,\,\beta,\,|\Omega|\,,\,\|c\|_{L^\infty})$. ■

Remark 3.3.2 *The result of the Proposition* 3.3.1 *is still valid when* $c(x)$ *belongs to* $L^{\frac{N}{p-1}}$*: the only difference from our case is that* σ *has to be choosen as*

$$0 < \sigma < \min\left\{\frac{m}{q'},\ \frac{\alpha-\frac{1}{p^*}}{\frac{1}{m}-\frac{N-p}{N(p-1)}}\right\}.$$

Now we are ready to prove the following result:

Theorem 3.3.3 *Let* $p > 2-\frac{1}{N}$ *and let us assume conditions* (3.0.8) , *(3.0.11), (3.3.1), (3.3.2). If* $f \in L^1(\Omega)$, *or, more in general, if* f *is a Radon measure with bounded total variation, there exists at least a SOLA for problem* (3.0.7) . *Moreover such solution belongs to* $W_0^{1,q}(\Omega)$ *for every* q *which satisfies*

$$1 < q < \frac{N(p-1)}{N-1}.$$

Proof . Let be $\{f_n\}_{n\in\mathbb{N}}$ a sequence of functions in $C^\infty(\Omega)$ which *weakly*$-*$ converges to f in the sense of measures. By well-known results there exists a solution $u_n \in W_0^{1,p}(\Omega)$ to problem (3.0.7) with datum f_n (see [52], [81]) ; so we have:

$$\int_\Omega (a(x, \nabla u_n) \cdot \nabla\varphi) dx = \int_\Omega \Phi(x, u_n) \cdot \nabla\varphi dx + \int_\Omega f_n \varphi dx, \qquad \forall \varphi \in C_0^\infty(\Omega). \quad (3.3.21)$$

The a priori estimate (3.2.2) ensurse that the sequence $\{u_n\}$ is bounded in $W_0^{1,q}(\Omega)$, $\forall$ $q < \frac{N(p-1)}{N-1}$ and so it is compact in L^m for every $m < q^*$. By Proposition 3.3.1, we can deduce that there exists a subsequence, denoted again with $\{u_n\}$, which converges in $W_0^{1,q}(\Omega)$ to a function v.

Now we pass to the limit in (3.3.21) . Let us consider each term in (3.3.21) . Obviously we have

$$\lim_{n\to+\infty} \int_\Omega f_n \varphi dx = \int_\Omega \varphi df \qquad \forall \varphi \in C_0^\infty(\Omega). \quad (3.3.22)$$

For what concerns the terms on the left side, using growth conditons (3.0.9) on a and (3.0.11) on Φ and the properties of Nemytskii operators (see [78]) we arrive at

$$a(x, \nabla u_n) \overset{(L^1)^N}{\to} a(x, \nabla u), \quad (3.3.23)$$

and

$$\Phi(x, u_n) \overset{(L^1)^N}{\to} \Phi(x, u). \quad (3.3.24)$$

(see also **[6]** for a detailed discussion). Therefore, thanks to (3.3.22) , (3.3.23) , (3.3.24) , we can pass to limit in (3.3.21) ; this ensures that u is a SOLA. ■

3.4 Uniqueness results

The previous result is not enough to get the continuity with respect to data. To this aim, we first need an estimate of rearrangements of the difference of two solutions in term of L^1 norm of the difference of data. To get this result, we have to strength hypotheses on the structure of the operator; in particular choose as model of the lower order term $\Phi(x,u) = c(x)\,|u|^{\gamma}$; it satisfies conditions (3.4.2) or (3.4.6).

The range of variability of γ, specified in the following propositions, includes the classical case $\gamma = p-1$ under more restrictive assumptions on the value of p.

Lemma 3.4.1 *Let $p \geq 2$, let us assume condition* (3.0.9) *and*

$$(a(x,\xi) - a(x,\eta)) \cdot (\xi - \eta) \geq \delta(1 + |\xi| + |\eta|)^{p-2}\,|\xi - \eta|^2\,, \quad \delta > 0, \quad \xi,\eta \in \mathbb{R}^N, \tag{3.4.1}$$

$$|\Phi_s(x,s)| \leq c(x)\,|s|^{\gamma-1}\,, \tag{3.4.2}$$

with

$$c(x) > 0, \quad c(x) \in L^r(\Omega), \quad r > N \tag{3.4.3}$$

and

$$1 \leq \gamma < \frac{N-1}{N-p} - \frac{N(p-1)}{r(N-p)}. \tag{3.4.4}$$

If u and v are solutions to problem (3.0.7) *with data f and g in $L^1(\Omega)$, then*

$$(u-v)^*(s) \leq K\,\|f-g\|_{L^1}\, s^{-\frac{N-2}{N}}, \tag{3.4.5}$$

where K is a constant which depends on N, δ, $|\Omega|$, p, $\|f\|_{L^1}$, $\|g\|_{L^1}$, r.

Lemma 3.4.2 *Let $2 - \frac{1}{N} < p < 2$. Let us assume conditions* (3.0.9), (3.3.1) *and*

$$|\Phi_s(x,s)| \leq c(x)\,(1 + |s|)^{\gamma-1}\,, \tag{3.4.6}$$

with

$$c(x) > 0, \quad c(x) \in L^r(\Omega), \quad r > \frac{2N(p-1)}{1+Np-2N} \tag{3.4.7}$$

and

$$\gamma < \frac{(N-1)(p-1)}{N-p} - \frac{N(p-1)}{r(N-p)}. \tag{3.4.8}$$

If u and v are solutions to problem (3.0.7) *with data f and g in $L^1(\Omega)$, then*

$$(u-v)^*(s) \le K \|f-g\|_{L^1} s^{-\frac{N-2}{N}-(2-p)\zeta}, \tag{3.4.9}$$

for some $\zeta > \frac{N-1}{N(p-1)}$. The constant K depends on N, β, $|\Omega|$, p, $\|f\|_{L^1}$, $\|g\|_{L^1}$, r, q.

Proof of Lemma 3.4.1.

Set $w = u - v$ and $h = f - g$. For every fixed t, k, positive constants, we consider the function

$$\Psi = \begin{cases} k \text{ sign} w & \text{if } |w| > t+k \\ (w-t) \text{ sign} w & \text{if } t < |w| \le t+k \\ 0 & \text{otherwise.} \end{cases}$$

Using Ψ as test function in equation (3.0.7) with f and g as data, subtracting and dividing by k, we have

$$\frac{1}{k} \int\limits_{t<|w|\le t+k} [a(x,\nabla u) - a(x,\nabla v)] \cdot \nabla w dx = \frac{1}{k} \int\limits_{t<|w|\le t+k} [\Phi(x,u) - \Phi(x,v)] \cdot \nabla w dx +$$

$$+ \int\limits_{|w|>t+k} h \; signw \; dx + \frac{1}{k} \int\limits_{t<|w|\le t+k} h \left[(w-t) \; signw\right] dx. \tag{3.4.10}$$

We set

$$\nu(x) = (1 + |\nabla u| + |\nabla v|)^{p-2}. \tag{3.4.11}$$

By using assumptions (3.4.1), (3.4.2) and the definition of $\nu(x)$, (3.4.10) becomes

$$\frac{\delta}{k} \int\limits_{t<|w|\le t+k} \nu(x) |\nabla w|^2 dx \le \frac{(t+k)}{k} \int\limits_{t<|w|\le t+k} c(x)(|u|+|v|)^{\gamma-1} |\nabla w| dx +$$

$$+ \int\limits_{|w|>t+k} h \; signw \; dx + \frac{1}{k} \int\limits_{t<|w|\le t+k} h \left[(w-t) \; signw\right] dx.$$

Now, since $\nu(x) \geq 1$, applying Hölder inequality and letting k go to zero in the previous inequality, we obtain

$$-\frac{d}{dt}\int_{|w|>t} \nu(x)\left|\nabla w\right|^2 dx \leq \frac{t}{\delta}\left(-\frac{d}{dt}\int_{|w|>t} \nu(x)\left|\nabla w\right|^2 dx\right)^{\frac{1}{2}} \times \left[\left(-\frac{d}{dt}\int_{|w|>t} c(x)^2(|u|+|v|)^{2(\gamma-1)}dx\right)^{\frac{1}{2}}\right] + + \frac{1}{\delta}\int_{|w|>t} |h|\, dx. \tag{3.4.12}$$

If μ denotes the distribution function of w, proceeding as in [7], it is possible to define a function T such that

$$T(\mu(t))\left|\mu'(t)\right| = -\frac{d}{dt}\left(\int_{|w|>t} c(x)^2(|u|+|v|)^{2(\gamma-1)}dx\right). \tag{3.4.13}$$

The function defined in (3.4.13) is a weak limit of functions having the same rearrangement of $c(x)^2(|u|+|v|)^{2(\gamma-1)}$.

By Hardy-Littlewood inequality and by the definitions of T, we obtain

$$-\frac{d}{dt}\int_{|w|>t} \nu(x)\left|\nabla w\right|^2 dx \leq \frac{t}{\delta}\left|\mu'(t)\right|^{\frac{1}{2}}\left[T(\mu(t))\right]^{\frac{1}{2}} \times \times\left(-\frac{d}{dt}\int_{|w|>t} \nu(x)\left|\nabla w\right|^2 dx\right)^{\frac{1}{2}} + \frac{1}{\delta}\int_0^{\mu(t)} h^*(s)ds. \tag{3.4.14}$$

On the other hand, denoted by $k_N = \omega_N^{1/N}N$, by isoperimetric and Schwarz inequalities, since $\nu(x) \geq 1$, it follows (see [107])

$$k_N\mu(t)^{1-\frac{1}{N}} \leq -\frac{d}{dt}\int_{|w|>t} |\nabla w|\, dx \leq \left(-\frac{d}{dt}\int_{|w|>t} \nu(x)\left|\nabla w\right|^2 dx\right)^{\frac{1}{2}}\left|\mu'(t)\right|^{\frac{1}{2}}, \tag{3.4.15}$$

that is

$$1 \leq k_N^{-1}\mu(t)^{\frac{1}{N}-1}\left(-\frac{d}{dt}\int_{|w|>t} \nu(x)\left|\nabla w\right|^2 dx\right)^{\frac{1}{2}}\left|\mu'(t)\right|^{\frac{1}{2}}. \tag{3.4.16}$$

By (3.4.14) , (3.4.16) we obtain

$$-\frac{d}{dt}\int_{|w|>t} \nu(x)\left|\nabla w\right|^2 dx \leq \frac{t}{\delta}\left(-\frac{d}{dt}\int_{|w|>t} \nu(x)\left|\nabla w\right|^2 dx\right)^{\frac{1}{2}}\left|\mu'(t)\right|^{\frac{1}{2}}\left[T(\mu(t))\right]^{\frac{1}{2}} + +\frac{1}{\delta}k_N^{-1}\mu(t)^{\frac{1}{N}-1}\left(-\frac{d}{dt}\int_{|w|>t} \nu(x)\left|\nabla w\right|^2 dx\right)^{\frac{1}{2}}\left|\mu'(t)\right|^{\frac{1}{2}}\int_0^{\mu(t)} h^*(s)ds. \tag{3.4.17}$$

Using the (3.4.15) , (3.4.17) becomes

$$1 \leq k_N^{-1}\frac{t}{\delta}\mu(t)^{\frac{1}{N}-1}\left|\mu'(t)\right|\left[T(\mu(t))\right]^{\frac{1}{2}}+\frac{1}{\delta}k_N^{-2}\mu(t)^{2(\frac{1}{N}-1)}\left|\mu'(t)\right|\int_0^{\mu(t)}h^*(s)ds.$$

Integrating the previous inequality between 0 and t, and using the definition of $w^*(s)$, we have

$$w^*(s) \leq K\int_s^{|\Omega|}w^*(t)t^{\frac{1}{N}-1}T(t)^{\frac{1}{2}}dt+K\int_s^{|\Omega|}t^{2(\frac{1}{N}-1)}\left(\int_0^t h^*(\tau)d\tau\right)dt.$$

Now we apply the Gronwall's Lemma. we get

$$w^*(s) \leq K\int_s^{|\Omega|}\left\{t^{2(\frac{1}{N}-1)}\left(\int_0^t h^*(\tau)d\tau\right)\left(\exp\int_s^t \tau^{\frac{1}{N}-1}T(\tau)^{\frac{1}{2}}d\tau\right)\right\}dt. \qquad (3.4.18)$$

Now we have to impose the right conditions on γ which ensure that the following integral

$$\int_0^{|\Omega|}\tau^{\frac{1}{N}-1}T(\tau)^{\frac{1}{2}}dr < +\infty$$

This happens if the function T belongs to some space L^{ϑ} with $\vartheta > \frac{N}{2}$. But we have already observed that T has the same sommability of $c(x)^2\left(|u|+|v|\right)^{2(\gamma-1)}$, so $\frac{1}{\vartheta}=\frac{2(\gamma-1)}{q^*}+\frac{2}{r}$. Hence we have to impose that

$$\vartheta > \frac{N}{2} \quad \text{and} \quad q^* < \frac{N(p-1)}{N-p}.$$

This conditions are verified if we choose γ as in (3.4.4) . Then, the (3.4.18) becomes

$$\left(u-v\right)^*(s) \leq K\int_s^{|\Omega|}t^{2(\frac{1}{N}-1)}\left(\int_0^t h^*(\tau)d\tau\right)dt \leq K\left\|f-g\right\|_{L^1}s^{-\frac{N-2}{N}},$$

where K depends on N, δ, $|\Omega|$, p, $\|f\|_{L^1,}$ $\|g\|_{L^1,}$ r. ■

Proof of Lemma 3.4.2.

The proof of Lemma 3.4.2 is similar to that of Lemma 3.4.1. Let us consider (3.4.10) and (3.4.11) . By (3.3.1) , (3.4.6) , we get

$$\frac{\beta}{k}\int\limits_{t<|w|\leq t+k}\left|\nabla w\right|^2\left(|\nabla u|+|\nabla v|\right)^{p-2}dx \leq \frac{(t+k)}{k}\int\limits_{t<|w|\leq t+k}c(x)(1+|u|+|v|)^{\gamma-1}\left|\nabla w\right|dx+$$

$$+\int\limits_{|w|>t+k} h\ signw\ dx + \frac{1}{k}\int\limits_{t<|w|\le t+k} h\left[(w-t)\ signw\right]dx.$$

Since $p<2$, we have $(|\nabla u|+|\nabla v|)^{\frac{2-p}{2}} \le (1+|\nabla u|+|\nabla v|)^{\frac{2-p}{2}}$, and so

$$\frac{\beta}{k}\int\limits_{t<|w|\le t+k} |\nabla w|^2 (|\nabla u|+|\nabla v|)^{p-2}dx \le$$

$$\le \frac{(t+k)}{k}\int\limits_{t<|w|\le t+k} c(x)\frac{(1+|u|+|v|)^{\gamma-1}|\nabla w|}{(|\nabla u|+|\nabla v|)^{\frac{2-p}{2}}}(1+|\nabla u|+|\nabla v|)^{\frac{2-p}{2}}dx+$$

$$+\int\limits_{|w|>t+k} h\ signw\ dx + \frac{1}{k}\int\limits_{t<|w|\le t+k} h\left[(w-t)\ signw\right]dx. \qquad (3.4.19)$$

By Hölder inequality in (3.4.19) and letting k go to 0, we have

$$-\frac{d}{dt}\int_{|w|>t}|\nabla w|^2 (|\nabla u|+|\nabla v|)^{p-2}dx \le Kt\left(-\frac{d}{dt}\int_{|w|>t} c(x)^2\frac{(1+|u|+|v|)^{2(\gamma-1)}}{\nu}dx\right)^{\frac{1}{2}}\times$$

$$\times\left(-\frac{d}{dt}\int_{|w|>t}|\nabla w|^2 (|\nabla u|+|\nabla v|)^{p-2}dx\right)^{\frac{1}{2}} + K\int_{|w|>t}|h|\,dx. \qquad (3.4.20)$$

Arguing as in the previous proof, we introduce two functions $\bar{T}$ and $\bar{\nu}$ such that

$$\bar{T}(\mu(t))\,|\mu'(t)| = -\frac{d}{dt}\int\limits_{|w|>t} c(x)^2\frac{(1+|u|+|v|)^{2(\gamma-1)}}{\nu}dt, \qquad (3.4.21)$$

$$\bar{\nu}(\mu(t))\,|\mu'(t)| = -\frac{d}{dt}\int\limits_{|w|>t}\frac{1}{\nu}dt. \qquad (3.4.22)$$

From (3.4.15) , we obtain

$$k_N\mu(t)^{1-\frac{1}{N}} \le \left(-\frac{d}{dt}\int\limits_{|w|>t}|\nabla w|^2 (|\nabla u|+|\nabla v|)^{p-2}dx\right)^{\frac{1}{2}}\bar{\nu}(\mu(t))^{\frac{1}{2}}\,|\mu'(t)|^{\frac{1}{2}}. \qquad (3.4.23)$$

Therefore, by (3.4.23) the (3.4.20) becomes

$$-\frac{d}{dt}\int_{|w|>t}|\nabla w|^2 (|\nabla u|+|\nabla v|)^{p-2}dx \le Kt\bar{T}(\mu(t))^{\frac{1}{2}}\,|\mu'(t)|^{\frac{1}{2}}\times$$

$$\times\left(-\frac{d}{dt}\int_{|w|>t}|\nabla w|^2 (|\nabla u|+|\nabla v|)^{p-2}dx\right)^{\frac{1}{2}}+$$

$$+K\mu(t)^{\frac{1}{N}-1}\bar{\nu}(\mu(t))^{\frac{1}{2}}\left|\mu'(t)\right|^{\frac{1}{2}}\times$$

$$\times\left(-\frac{d}{dt}\int_{|w|>t}|\nabla w|^2\left(|\nabla u|+|\nabla v|\right)^{p-2}dx\right)^{\frac{1}{2}}\int_0^{\mu(t)}h^*(\tau)d\tau \tag{3.4.24}$$

Proceeding as in the case $p \geq 2$, we obtain

$$w^*(s)\leq K\int_s^{|\Omega|}w^*(t)t^{\frac{1}{N}-1}\bar{T}(t)^{\frac{1}{2}}\bar{\nu}(t)^{\frac{1}{2}}dt+K\int_s^{|\Omega|}t^{2(\frac{1}{N}-1)}\bar{\nu}(t)\left(\int_0^t h^*(\tau)d\tau\right)dt,$$

then, by Gronwall's Lemma and integration by parts

$$w^*(s)\leq K\int_s^{|\Omega|}\left\{t^{2(\frac{1}{N}-1)}\bar{\nu}(t)\left(\int_0^t h^*(\tau)d\tau\right)\left(\exp\int_s^t\tau^{\frac{1}{N}-1}\bar{T}(\tau)^{\frac{1}{2}}\bar{\nu}(\tau)^{\frac{1}{2}}d\tau\right)\right\}dt. \tag{3.4.25}$$

Now, we want to impose the right conditions on γ which ensure that

$$\int_0^{|\Omega|}\tau^{\frac{1}{N}-1}\bar{T}(\tau)^{\frac{1}{2}}\bar{\nu}(\tau)^{\frac{1}{2}}d\tau<+\infty.$$

$\bar{T}$ has the same sommability of $c(x)^2\frac{(1+|u|+|v|)^{2(\gamma-1)}}{\nu(x)}$ and $\bar{\nu}$ has the same sommability of $\frac{1}{\nu(x)}$; recalling the expression of ν and the estimate (3.2.2), we deduce that $\left(\bar{T}\bar{\nu}\right)^{\frac{1}{2}}$ belongs to $L^{\vartheta}(\Omega)$ with $\frac{1}{\vartheta}=\frac{\gamma-1}{q^*}+\frac{2-p}{q}+\frac{1}{r}$. So the integral is finite for every γ such that

$$\vartheta>N,\quad q^*<\frac{N(p-1)}{N-p}\quad\text{and}\quad q<\frac{N(p-1)}{N-1},$$

which holds true if condition (3.4.8) is satisfied. Coming back to (3.4.25), we arrive at

$$w^*(s)\leq K\left\|h\right\|_{L^1}\int_s^{|\Omega|}t^{2(\frac{1}{N}-1)}\bar{\nu}(t)dt.$$

Using the sommability of $\bar{\nu}(t)$ and the Hölder inequality, we have

$$w^*(s)\leq K\left\|h\right\|_{L^1}\left\|\bar{\nu}\right\|_{L^{\frac{q}{2-p}}}\left[\int_s^{|\Omega|}t^{2(\frac{1}{N}-1)\frac{q}{q-2+p}}dt\right]^{\frac{q-2+p}{q}},$$

and then

$$(u-v)^*(s)\leq K\left\|f-g\right\|_{L^1}s^{-\frac{N-2}{N}-(2-p)\varsigma},$$

for some $\varsigma>\frac{N-1}{N(p-1)}$; here K depends on N, β, $|\Omega|$, p, $\|f\|_{L^1}$, $\|g\|_{L^1}$, r, q. ■

Thanks to Lemma 3.4.1 and 3.4.2, now we are able to prove the following uniqueness results:

Theorem 3.4.3 *Let $p \geq 2$, let us assume hypotheses of Lemma 3.4.1. Then the problem* (3.0.7) *has a unique SOLA.*

Theorem 3.4.4 *Let $2 - \frac{1}{N} < p < 2$, let us assume hypotheses of Lemma 3.4.2. Then the problem* (3.0.7) *has a unique SOLA.*

Proof of Theorem 3.1.2. From (3.4.5) and (3.3.4) we get:

$$\|\nabla(u-v)\|_{L^q} \leq K \left(\int_0^{|\Omega|} (u-v)^{*\,m} ds \right)^{\frac{1}{m(p-1)}} \leq$$

$$\leq K \|f-g\|_{L^1}^{\frac{1}{p-1}} \times \left(\int_0^{|\Omega|} s^{-\frac{N-2}{N}m} ds \right)^{\frac{1}{m(p-1)}}.$$

The last inequality holds for every m such that $\frac{Np}{N\alpha - p + N} < m < q^*$. If we choose

$$\alpha > \frac{N-2}{N},$$

then

$$\left(\int_0^{|\Omega|} s^{-\frac{N-2}{N}m} ds \right)^{\frac{1}{m(p-1)}}$$

is finite, and so we get

$$\|\nabla(u-v)\|_{L^q} \leq K \|f-g\|_{L^1}^{\frac{1}{p-1}},$$

which ensures the uniqueness of the SOLA. ■

Proof of Theorem 3.1.3. From (3.4.9) and (3.3.6) we obtain that:

$$\|\nabla(u-v)\|_{L^q} \leq K \|u-v\|_{L^{q^*}}^{\frac{1}{2}} + K \|u-v\|_{L^{q^*}}^{\frac{\sigma}{2}} \leq$$

$$\leq K \|f-g\|_{L^1}^{\frac{1}{2}} \left(\int_0^{|\Omega|} s^{-\frac{N-2}{N}q^* - \frac{(2-p)q^*}{q}} ds \right)^{\frac{1}{2q^*}} + K \|f-g\|_{L^1}^{\frac{\sigma}{2}} \left(\int_0^{|\Omega|} s^{-\frac{N-2}{N}q^* - \frac{(2-p)q^*}{q}} ds \right)^{\frac{\sigma}{2q^*}}.$$

If γ satisfies condition (3.4.8) the last integral is sommable, so we get:

$$\|\nabla(u-v)\|_{L^q} \leq K \|f-g\|_{L^1}^{\frac{1}{2}} + K \|f-g\|_{L^1}^{\frac{\sigma}{2}}.$$

■

CHAPTER IV

NON-UNIFORMLY ELLIPTIC EQUATIONS

In this chapter we prove a class of Dirichlet problem for degenerate elliptic equations of the type

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u)) + H(x,\nabla u) = \mu & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4.0.26}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, $1 < p < N$, $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function such that

$$a(x,s,\xi)\xi \geq \nu(x)\left|\xi\right|^p, \tag{4.0.27}$$

$$|a(x,s,\xi)| \leq \nu(x)\left[|\xi|^{p-1} + |s|^{p-1} + a_0(x)\right], \ a_0(x) \in L^{p'}(\nu), \tag{4.0.28}$$

for almost every $x \in \Omega$, for every s$\in \mathbb{R}$, for every $\xi \in \mathbb{R}^N$, where $\nu(x)$ is a nonnegative function verifying

$$\nu(x) \in L^r(\Omega), r \geq 1, \tag{4.0.29}$$

$$v(x)^{-1} \in L^t(\Omega), \quad t \geq N/p, \quad 1 + 1/t < p < N(1 + 1/t). \tag{4.0.30}$$

Morover we assume that a is monotone, that is

$$(a(x,s,\xi) - a(x,s,\eta), \xi - \eta) > 0, \tag{4.0.31}$$

for a.e $x \in \Omega$, for every s$\in \mathbb{R}$, for every $\xi, \eta \in \mathbb{R}^N, \xi \neq \eta$.

Furthermore $H : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that

$$|H(x,\xi)| \leq b(x)\left|\xi\right|^{p-1} + b_0(x), \tag{4.0.32}$$

with

$$b_0(x) \in L^1(\Omega), \tag{4.0.33}$$

$$b(x) \in L^{\tau}(\Omega),\ \tau > \frac{p'\widetilde{p}t}{t-(t+1)(p'+\widetilde{p})}, \tag{4.0.34}$$

where $\widetilde{p}$ is defined by

$$\widetilde{p} = \frac{p^{\#}}{r'}, \tag{4.0.35}$$

and let $p^{\#}$ denote the number $(p^{\#})^{-1} = p^{-1}(1+1/t) - N^{-1}$.

The existence of a distributional solution for non-uniformly elliptic problem has also been proved by Rakotoson in [101] when $H = 0$ and by Betta, Del Vecchio and Posteraro in [16] by using classical symmetrization methods. Regularity results for such a solution are proved in [42].

Here we prove an existence result for renormalized solutions contained in [55]. The main features in studying this problem are the non-uniformly ellipticity condition and the fact that the operator is not coercive. The proof of our result consists of several steps. Firstly we introduce the "approximated problems", then we prove an apriori estimate for the gradients of its solutions. This estimate can be easily done when b is small enough since in this case the operator is coercive. When the norm of b is not small ,we use Bottaro-Marina technique ([35]) which consists in decomposing b in a finite sum of terms having norm small enough. Finally we pass to the limit in the approximated problems by using a stability result which is an extention of a result proved in [48] (see also [72]). The same approach has been used in [20] where a is an uniformly elliptic operator (see also [72], [73] for uniformly elliptic operator with two lower order terms).

4.1 Definitions and first properties

In this section we recall some properties of the measure ([48]), a few properties of weighetd Sobolev spaces ([89]) and, finally, the definition of renormalized solution to degenerate nonlinear elliptic equations with measure data ([48]).

4.1.1 Weighted Sobolev spaces

Let be Ω a bounded domain in $\mathbb{R}^N$, $N \geq 2$, and let $\nu(x)$ be a nonnegative function on Ω such that $\nu(x) \in L^r(\Omega)$, $r \geq 1$, $v(x)^{-1} \in L^t(\Omega)$, $t \geq N$. We denote by $L^p(\nu)$, $p > 1$, the space of measurable functions u such that

$$\|u\|_{L^p(\nu)} = \left(\int_\Omega |u|^p \, \nu(x) dx \right)^{1/p} < \infty, \tag{4.1.1}$$

and by $W^{1,p}(\nu)$ the completation of the space $C^1(\overline{\Omega})$ with respect to the norm

$$\|u\|_{W^{1,p}(\nu)} = \|u\|_{L^p(\nu)} + \| |\nabla u| \|_{L^p(\nu)} \,. \tag{4.1.2}$$

Moreover we denote by $W_0^{1,p}(\nu)$ the closure of $C_0^1\left(\overline{\Omega}\right)$ in $W^{1,p}(\nu)$ and by $W^{-1,p'}(1/\nu)$ the dual space of $W_0^{1,p}(\nu)$. The elements of the dual space $W^{-1,p'}(1/\nu)$ are described by the following Proposition ([89]).

Proposition 4.1.1 *If T is a continuous linear functional on $W_0^{1,p}(\nu)$ then there exist (N+1) functions f_0, f_1,..., f_N on $\overline{\Omega}$ such that*

$$f_0 \in L^{p'}(\nu^{-1}), \quad f_j \in L^{p'}(\nu^{-1}), \quad 1 \leq j \leq N \tag{4.1.3}$$

and

$$T(u) = \int_\Omega (f_0 u \nu^{2/p-1} + f_j u_{x_j} \nu^{2/p-1}) dx, \quad for \ u \in W_0^{1,p}(\nu). \tag{4.1.4}$$

Conversely, any set of functions f_0, f_1,..., f_N satisfying (4.1.3) *defines a continuous linear functional by means of* (4.1.4) *on $W_0^{1,p}(\nu)$.*

Now we recall some Sobolev-type inequalities which will be used in the following ([89]).

Proposition 4.1.2 *Let ν be a nonnegative function on Ω such that*

$$\nu \in L^r(\Omega), \ r \geq 1, \ \nu^{-1} \in L^t(\Omega), \ t \geq N, \tag{4.1.5}$$

Let p be a real number such that

$$t \geq N/p, \ 1 + 1/t < p < N(1 + 1/t). \tag{4.1.6}$$

Then there exists a continuous linear imbedding of $W_0^{1,p}(\nu)$ *in* $L^{p^\#}(\Omega)$ *and there exists a constant* $C_0 > 0$ *depending on* N, p, ν, t, *such that*

$$\|u\|_{L^{p^\#}(\Omega)} \leq C_0 \, \||\nabla u|\|_{L^p(\nu)}, \qquad \forall u \in W_0^{1,p}(\nu). \tag{4.1.7}$$

Proposition 4.1.3 *Let* ν *be a nonnegative function on* Ω *such that* (4.1.5) *holds true and let* p *a real number such that* (4.1.6) *holds true. Let* $\widetilde{p}$ *denote the number*

$$\widetilde{p} = \frac{p^\#}{r'}.$$

Then there exists a continuous linear imbedding of $W_0^{1,p}(\nu)$ *in* $L^{\widetilde{p}}(\nu)$ *and a constant* $C_0' > 0$ *depending on* N, p, ν, t, *such that*

$$\|u\|_{L^{\widetilde{p}}(\nu)} \leq C \, \||\nabla u|\|_{L^p(\nu)}, \quad \forall u \in W_0^{1,p}(\nu). \tag{4.1.8}$$

Finally we recall a Poincarè-type inequality for weighted Sobolev spaces:

Proposition 4.1.4 *Let* ν *be a nonnegative function on* Ω *such that* (4.1.5) *holds true and let* p *a real number such that* (4.1.6) *holds true. Then* $W_0^{1,p}(\nu)$ *is continuously imbedded in* $L^p(\nu)$.

4.1.2 Decomposition of measure*s*

Now we recall the definition of (p, ν)-capacity ([86]), which extends the notion of p-capacity given in Chapter II.

The (p,ν)-capacity $cap_{p,\nu}(K,\Omega)$ of a compact set $K \subset \Omega$ with respect to Ω is defined by

$$cap_{p,\nu}(K,\Omega) = \inf\left\{ \int_\Omega \nu(x)\,|\nabla\varphi|^p : \varphi \in C_0^\infty(\Omega),\, \varphi \geq \chi_K \right\},$$

where χ_K denotes the characteristic function of K . If $U \subseteq \Omega$ is a open set, we denote

$$cap_{p,\nu}(U,\Omega) = \sup\left\{ cap_{p,\nu}(K,\Omega) : K \ compact, K \subseteq U \right\},$$

while the (p,ν)-capacity of any subset $B \subseteq \Omega$ is defined as

$$cap_{p,\nu}(B,\Omega) = \inf\left\{ cap_{p,\nu}(U,\Omega) : U \ open, B \subseteq U \right\}.$$

We denote by $M_{0,\nu}(\Omega)$ the set of all measures μ in $M_b(\Omega)$ which are absolutely continuous with respect to the (p,ν)-capacity, that is $\mu(B) = 0$ for every Borel set $B \subseteq \Omega$ such that $cap_{p,\nu}(B,\Omega) = 0$. We define $M_{s,\nu}(\Omega)$ the set of all measures μ in $M_b(\Omega)$ which are singular with respect to the (p,ν)-capacity that is which are concentrated in a set $E \subset \Omega$ such that $cap_{p,\nu}(E,\Omega) = 0$.
The following proposition is analogous to Proposition 2.3.7 and it is a consequence of the decomposition result proved in [66].

Proposition 4.1.5 *For every measure in $M_b(\Omega)$ there exists an unique pair of measures (μ_0, μ_s) with $\mu_0 \in M_{0,\nu}(\Omega)$ and $\mu_s \in M_{s,\nu}(\Omega)$, such that $\mu = \mu_0 + \mu_s$.*

The measures μ_0 and μ_s are the absolutely continuous part and the singular part of μ with respect to the (p,ν)-capacity. Moreover, by adapting the proof of the result proved in [33], we obtain the following properties which states that

Proposition 4.1.6 *Let μ_0 be a measure in $M_b(\Omega)$. Then μ_0 belongs to $M_{0,\nu}(\Omega)$ if and only if it belongs to $L^1(\Omega) + W^{-1,p'}(1/\nu)$. Then there exists $f \in L^1(\Omega)$ and $g \in (L^{p'}(1/\nu^{p-1}))^N$ such that*

$$\mu_0 = f - \operatorname{div}(g), \tag{4.1.9}$$

in the sense of distributions.

By Proposition 4.1.5 and 4.1.6 we get

Proposition 4.1.7 *Every measure μ in $M_b(\Omega)$ can be decomposed as follows*

$$\mu = \mu_0 + \mu_s = f - \operatorname{div}(g) + \mu_s^+ - \mu_s^-, \tag{4.1.10}$$

where μ_0 is a measure in $M_{0,\nu}(\Omega)$. It can be written as $f - \operatorname{div}(g)$, with $f \in L^1(\Omega)$ and $g \in (L^{p'}(1/\nu^{p-1}))^N$, μ_s^+, μ_s^- are two nonnegative measures in $M_{s,\nu}(\Omega)$, which are concentrated in two disjoint subset E^+, E^- of zero (p,ν)-capacity.

4.1.3 A technical result

In this section we prove a generalization of a result proved in [13] (see also [20]) which allows us to obtain an a priori estimates for the gradients of the solutions. Here we denote by

$$\text{meas}_\nu E = \int_E \nu(x)dx,$$

for any measurable set $E \subseteq \mathbb{R}^N$.

We will use the weigthed Lorentz spaces $L^{r,\infty}(\nu)$, $0 < r \le \infty$ which is the set of measurable functions such that

$$\|u\|_{L^{r,\infty}(\nu)} = \sup_{t>0} t \text{ meas}_\nu \{x \in \Omega : |u| > t\}^{1/r} < +\infty.$$

The main result of this section is the following:

Lemma 4.1.8 *Assume that Ω is an open subset of $\mathbb{R}^N$ with finite Lebesgue measure and let ν be a function such that* (4.1.5) *and* (4.1.6) *hold true. Let u be a measurable function such that $T_k(u) \in W_0^{1,p}(\nu)$ with $k > 0$ and such that*

$$\int_\Omega \nu(x) |\nabla T_k(u)|^p \le Mk + L, \tag{4.1.11}$$

where M and L are given constants. Then $|u|^{p-1} \in L^{\widetilde{p}/p,\infty}(\nu), |\nabla u|^{p-1} \in L^{\frac{p'\widetilde{p}}{p'+\widetilde{p}},\infty}(\nu)$ *and*

$$\left\| |u|^{p-1} \right\|_{L^{\widetilde{p}/p,\infty}(\nu)} \leq C(N,p)\left[M + |\Omega|^{1/\widetilde{p}} L^{1/p'}\right], \tag{4.1.12}$$

$$\left\| |\nabla u|^{p-1} \right\|_{L^{p'\widetilde{p}/(p'+\widetilde{p}),\infty}(\nu)} \leq C(N,p)\left[M + |\Omega|^{1/\widetilde{p}} L^{1/p'}\right], \tag{4.1.13}$$

where $C(N,p)$ is a constant depending only on N and p .

Remark 4.1.9 *Under the assumption of Lemma 4.1.8, the functions $|u|^{p-1}$ and $|\nabla u|^{p-1}$ belong to the weighted Lorentz spaces $L^{\widetilde{p}/p,\infty}(\nu)$ and $L^{\frac{p'\widetilde{p}}{p'+\widetilde{p}},\infty}(\nu)$ respectively. By classical results ([89]) such spaces are imbedded in the weigthed spaces $L^{\gamma}(\nu)$, $\gamma < \frac{\widetilde{p}}{p}$ and $L^{q}(\nu)$, $q < \frac{p'\widetilde{p}}{p'+\widetilde{p}}$. Actually we will just use estimates of $|u|^{p-1}$ and $|\nabla u|^{p-1}$ in $L^{\gamma}(\nu)$, and $L^{q}(\nu)$ respectively.*
Moreover we observe that, if ν is a constant then $\frac{\widetilde{p}}{p} = \frac{N}{N-p}$ and $\frac{p'\widetilde{p}}{p'+\widetilde{p}} = \frac{N}{N-1}$.

Proof.

We begin by proving (4.1.12). For any $h_0 > 0$, we have

$$\begin{aligned} \left\| |u|^{p-1} \right\|_{L^{\widetilde{p}/p,\infty}(\nu)} &= \sup_{h>0} h \operatorname{meas}_{\nu}\left\{x \in \Omega : |u|^{p-1} > h\right\}^{p/\widetilde{p}} \leq \\ &\leq \sup_{0<h\leq h_0} h \operatorname{meas}_{\nu}\left\{x \in \Omega : |u|^{p-1} > h\right\}^{p/\widetilde{p}} \\ &+ \sup_{h\geq h_0} h \operatorname{meas}_{\nu}\left\{x \in \Omega : |u|^{p-1} > h\right\}^{p/\widetilde{p}} \\ &\leq h_0 |\Omega|^{p/\widetilde{p}} + \sup_{h\geq h_0} h \operatorname{meas}_{\nu}\left\{x \in \Omega : |u|^{p-1} > h\right\}^{p/\widetilde{p}}. \end{aligned} \tag{4.1.14}$$

Now we observe that

$$\begin{aligned} h^{\widetilde{p}} \operatorname{meas}_{\nu}\{x\epsilon\Omega : |u| > h\} &\leq \int_{\Omega} T_h^{\widetilde{p}}(u)\nu(x)dx \leq \\ &\leq C(N,p)\left(\int_{\Omega} |\nabla T_h(u)|^p v(x)dx\right)^{\widetilde{p}/p} \\ &\leq C(N,p)(Mh+L)^{\widetilde{p}/p}, \end{aligned}$$

that is for every h > 0

$$h^{\widetilde{p}/p-1}\ \mathrm{meas}_\nu \left\{x\in\Omega : |u|^{p-1} > h\right\} \leq C(N,p)(Mh^{1/p-1}+L)^{\widetilde{p}/p}.$$

Then we obtain

$$\mathrm{meas}_\nu \left\{x\in\Omega : |u|^{p-1} > h\right\} \leq C(N,p)(Mh^{-1}+Lh^{-p'})^{\widetilde{p}/p},$$

that is

$$h\ \mathrm{meas}_\nu \left\{x\in\Omega : |u|^{p-1} > h\right\}^{p/\widetilde{p}} \leq C(N,p)(M+Lh^{-p'}). \qquad (4.1.15)$$

By (4.1.15) , inequality (4.1.14) becomes

$$\left\| |u|^{p-1} \right\|_{L^{\widetilde{p}/p,\infty}(\nu)} \leq h_0 \left|\Omega\right|^{p/\widetilde{p}} + C(N,p)(M+Lh^{1-p'}).$$

Taking $h_0 = \frac{L^{(p-1)/p}}{|\Omega|^{(p-1)/\widetilde{p}}}$, we have (4.1.12) .

Now we estimate the $|\nabla u|^{p-1}$

For every $\mu > 0$ and every $k > 0$, we have

$$\mu^p\ \mathrm{meas}_\nu \{x\in\Omega : |\nabla u| > \mu \text{ and } |u|\leq k\} \leq \int_{|u|\leq k} |\nabla u|^p\, v(x)dx \qquad (4.1.16)$$

$$= \int_\Omega |\nabla T_k(u)|^p\, v(x)dx \leq (Mk+L),$$

$$meas_\nu \left\{x\in\Omega : |\nabla u|^{p-1} > \mu \text{ and } |u| > k\right\} \leq \frac{(Mk+L)^{\widetilde{p}/p}}{k^{\widetilde{p}/p}}. \qquad (4.1.17)$$

By (4.1.16) , (4.1.17) we have

$$\mathrm{meas}_\nu \left\{x\in\Omega : |\nabla u|^{p-1} > \mu\right\} \leq \frac{(Mk+L)}{\mu^{p'}} + C(N,p)\frac{(Mk+L)^{\widetilde{p}/p}}{k^{\widetilde{p}}}. \qquad (4.1.18)$$

Now if we write $k = a+b$ with $a > 0$ and $b > 0$, (4.1.18) becomes

$$\mathrm{meas}_\nu \left\{x\in\Omega : |\nabla u|^{p-1} > \mu\right\} \leq \frac{Ma}{\mu^{p'}} + \frac{Mb}{\mu^{p'}} + \frac{M}{\mu^{p'}} + \qquad (4.1.19)$$

$$+ C(N,p)2^{\widetilde{p}/p}M^{\widetilde{p}/p}(a+b)^{\frac{\widetilde{p}}{p}-\widetilde{p}} +$$

$$+ C(N,p)2^{\widetilde{p}/p}L^{\widetilde{p}/p}(a+b)^{-\widetilde{p}},$$

for every $\mu > 0 \quad a > 0, b > 0$. If we observe that $\frac{\widetilde{p}}{p} - \widetilde{p} = \widetilde{p}\left(\frac{1}{p} - 1\right) = -\frac{\widetilde{p}}{p'} < 0$ we deduce

$$(a+b)^{\frac{\widetilde{p}}{p}-\widetilde{p}} \leq a^{-\frac{\widetilde{p}}{p'}},\ (a+b)^{-\widetilde{p}} \leq b^{-\widetilde{p}}.$$

By $(4.1.18), (4.1.19)$, we obtain

$$\begin{aligned} \text{meas}_\nu \left\{x \in \Omega : |\nabla u|^{p-1} > \mu\right\} &\leq \frac{Ma}{\mu^{p'}} + \frac{Mb}{\mu^{p'}} + \frac{M}{\mu^{p'}} + \\ &+ C(N,p)2^{\widetilde{p}/p} M^{\widetilde{p}/p} a^{-\frac{\widetilde{p}}{p'}} + C(N,p)2^{\widetilde{p}/p} L^{\widetilde{p}/p} b^{-\widetilde{p}} \\ &\leq C(N,p)\left[\frac{Ma}{\mu^{p'}} + a^{-\widetilde{p}/p'} M^{\widetilde{p}/p} + \frac{Mb}{\mu^{p'}} + b^{-\widetilde{p}} L^{\widetilde{p}/p}\right] \\ &+ C(N,p)\frac{L}{\mu^{p'}}. \end{aligned}$$

If we take $a = M^{\frac{\widetilde{p}-p}{p+\widetilde{p}(p-1)}} \mu^{\frac{p^2}{(p-1)[p+\widetilde{p}(p-1)]}}$ and $b = \left(\frac{\mu^{p'} L^{\widetilde{p}/p}}{M}\right)^{\frac{1}{1+\widetilde{p}}}$, we obtain

$$\begin{aligned} \text{meas}_\nu \left\{x \in \Omega : |\nabla u|^{p-1} > \mu\right\} &\leq C(N,p)\left[\left(\frac{M}{\mu}\right)^{(\widetilde{p}p)/(p+\widetilde{p}(p-1))}\right] + \\ &+ C(N,p)\left[\left(\frac{ML^{1/p}}{\mu^{p'}}\right)^{\widetilde{p}/(\widetilde{p}+1)} + \frac{L}{\mu^{p'}}\right]. \end{aligned}$$

Since $\frac{\widetilde{p}p}{p+\widetilde{p}(p-1)} = \frac{p'+\widetilde{p}}{p'\widetilde{p}}$, we have

$$\begin{aligned} \text{meas}_\nu \left\{x \in \Omega : |\nabla u|^{p-1} > \mu\right\}^{\frac{p'+\widetilde{p}}{p'\widetilde{p}}} &\leq C(N,p)\left[\frac{M}{\mu} + M^{\frac{p'+\widetilde{p}}{p'(\widetilde{p}+1)}} L^{\frac{p'+\widetilde{p}}{p'p(\widetilde{p}+1)}} \mu^{-\frac{p'+\widetilde{p}}{(\widetilde{p}+1)}}\right] \\ &+ C(N,p)\left[L^{\frac{p'+\widetilde{p}}{p'\widetilde{p}}} \mu^{-\frac{p'+\widetilde{p}}{\widetilde{p}}}\right], \end{aligned}$$

or equivantely

$$\begin{aligned} \mu \ \text{meas}_\nu \left\{x \epsilon \Omega : |\nabla u|^{p-1} > \mu\right\}^{\frac{p'+\widetilde{p}}{p'\widetilde{p}}} &\leq C(N,p)\left[M + M^{\frac{p'+\widetilde{p}}{p'(\widetilde{p}+1)}} \left(L^{\frac{p'+\widetilde{p}}{p'\widetilde{p}}} / \mu^{\frac{p'}{\widetilde{p}}}\right)^{\frac{\widetilde{p}}{p(\widetilde{p}+1)}}\right] + \\ &C(N,p)\left(L/\mu^{\frac{p'^2}{p'+\widetilde{p}}}\right)^{\frac{p'+\widetilde{p}}{p'\widetilde{p}}}. \end{aligned}$$

By Young inequality we have

$$\mu \text{ meas}_\nu \left\{x \in \Omega : |\nabla u|^{p-1} > \mu\right\}^{\frac{p'+\widetilde{p}}{p'\widetilde{p}}} \leq C(N,p)\left[M + \left(\frac{p'+\widetilde{p}}{p'(\widetilde{p}+1)}\right) M\right] \tag{4.1.20}$$
$$+ C(N,p)\left[\left(\frac{\widetilde{p}}{p(\widetilde{p}+1)}\right)\left(\frac{L}{\mu^{p'^2/(p'+\widetilde{p})}}\right)^{\frac{p'+\widetilde{p}}{p'\widetilde{p}}}\right]$$
$$+ C(N,p)\left[\left(\frac{L}{\mu^{p'^2/(p'+\widetilde{p})}}\right)^{\frac{p'+\widetilde{p}}{p'\widetilde{p}}}\right]$$
$$\leq C(N,p)\left[M + \frac{L^{\frac{p'+\widetilde{p}}{p'\widetilde{p}}}}{\mu^{p'/\widetilde{p}}}\right].$$

By (4.1.20) we deduce that

$$\sup_{\mu>0} \mu \text{ meas}_\nu \left\{x \in \Omega : |\nabla u|^{p-1} > \mu\right\}^{\frac{p'+\widetilde{p}}{p'\widetilde{p}}} \leq$$

$$\sup_{0<\mu\leq\mu_0} \mu \text{ meas}_\nu \left\{x \in \Omega : |\nabla u|^{p-1} > \mu\right\}^{\frac{p'+\widetilde{p}}{p'\widetilde{p}}} +$$
$$+ \sup_{\mu\geq\mu_0} \mu \text{ meas}_\nu \left\{x \in \Omega : |\nabla u|^{p-1} > \mu\right\}^{\frac{p'+\widetilde{p}}{p'\widetilde{p}}}$$
$$\leq \mu_0 \, |\Omega|^{\frac{p'+\widetilde{p}}{p'\widetilde{p}}} + \sup_{\mu\geq\mu_0} c\left[M + \frac{L^{\frac{p'+\widetilde{p}}{p'\widetilde{p}}}}{\mu^{p'/\widetilde{p}}}\right] \leq$$
$$c\left[\mu_0 \, |\Omega|^{\frac{p'+\widetilde{p}}{p'\widetilde{p}}} + M + \frac{L^{\frac{p'+\widetilde{p}}{p'\widetilde{p}}}}{\mu^{p'/\widetilde{p}}}\right].$$

By choosing $\mu_0 = \left(\frac{L}{|\Omega|}\right)^{1/p'}$ we have (4.1.13). ■

4.1.4 Definition of renormalized solution

Now we recall the definition of renormalized solution to problem (4.0.26), which is an extension of the Definition 2.3.9 given in Chapter II..

Definition 4.1.10 *We say that u is a renormalized solution of* (4.0.26) *if it satisfies the following conditions*

$$\begin{cases} u \text{ is measurable on } \Omega, \text{ almost everywhere finite,} \\ T_k(u) \in W_0^{1,p}(\nu), \; k > 0 \end{cases}$$

$$|u|^{p-1} \in L^{\gamma}(\nu), \; \gamma < \frac{\widetilde{p}}{p}; \tag{4.1.21}$$

$$|\nabla u|^{p-1} \in L^{q}(\nu), \; q < \frac{p'\widetilde{p}}{p'+\widetilde{p}}; \tag{4.1.22}$$

$$\lim_{n\to\infty} \frac{1}{n} \int_{n\le u<2n} a(x,u,\nabla u)\nabla u\varphi = \int_{\Omega} \varphi d\mu_s^{+}; \tag{4.1.23}$$

$$\lim_{n\to\infty} \frac{1}{n} \int_{-2n\le u<n} a(x,u,\nabla u)\nabla u\varphi = \int_{\Omega} \varphi d\mu_s^{-}; \tag{4.1.24}$$

for every $\varphi \in C_b^0(\Omega)$ *and*

$$\int_{\Omega} a(x,u,\nabla u)\nabla u h'(u)v + \int_{\Omega} a(x,u,\nabla u)\nabla v h(u) + \int_{\Omega} H(x,\nabla u)h(u)v \tag{4.1.25}$$

$$= \int_{\Omega} fh(u)v + \int_{\Omega} g\nabla u h'(u)v + \int_{\Omega} g\nabla v h(u);$$

for every $v \in W^{1,p}(\nu) \cap L^{\infty}(\Omega)$ *and for every* $h \in W^{1,\infty}(\mathbb{R})$ *with compact support in* $\mathbb{R}$*, which are such that* $h(u)v \in W_0^{1,p}(\nu)$.

4.2 Existence result

In this section we prove the existence of a renormalized solution to problem (4.0.26). The proof of our result consists of several steps. Firstly we introduce the "approximated problems" and we prove an apriori estimate for the gradients of its solutions. Finally we pass to the limit in the approximated problems by using a stability result.

Theorem 4.2.1 *Assume that* (4.0.27) – (4.0.35) *hold. Then there exists at least a renormalized solution to problem* (4.0.26).

4.2.1 Approximated problems

According to (4.1.10) the bounded Radon measure μ can be decomposed in

$$\mu = f - \operatorname{div}(g) + \mu_s^+ - \mu_s^-,$$

where $f \in L^1(\Omega)$, $g \in (L^{p'}(1/\nu^{p-1}))^N$, μ_s^+, μ_s^- two nonnegative measures in $M_{s,\nu}(\Omega)$ which are concentrated in two disjont subsets E^+ and E^- of zero (p,ν)-capacity. The measure μ can be approximated by a sequence μ_ε that is

$$\mu_\varepsilon = f_\varepsilon - \operatorname{div}(g) + \lambda_\varepsilon^+ - \lambda_\varepsilon^-,$$

where

$$\begin{cases} f_\varepsilon & \text{is a sequence of functions in } L^1(\Omega) \\ & \text{that converges to } f \text{ in } L^1(\Omega) \text{ weakly,} \end{cases} \tag{4.2.1}$$

$$\begin{cases} \lambda_\varepsilon^+ & \text{is a sequence of nonnegative functions in } L^\alpha(\Omega) \\ & \text{with } \alpha = \frac{Npt}{pt(N+1)-N(t+1)} \text{ that converges to } \mu_s^+ \\ & \text{in the narrow topology of measures,} \end{cases} \tag{4.2.2}$$

$$\begin{cases} \lambda_\varepsilon^- & \text{is a sequence of nonnegative functions in } L^\alpha(\Omega) \\ & \text{with } \alpha = \frac{Npt}{pt(N+1)-N(t+1)} \text{that converges to } \mu_s^- \\ & \text{in the narrow topology of measures.} \end{cases} \tag{4.2.3}$$

We observe that $\mu_\varepsilon \in W^{-1,p'}(1/\nu)$ by Propositions 4.1.7.

Let us denote by

$$H_\varepsilon(x,\xi) = T_{1/\varepsilon}(H(x,\xi)). \tag{4.2.4}$$

By (4.0.32) we have

$$|H_\varepsilon(x,\xi)| \leq |H(x,\xi)| \leq b(x)\,|\nabla u|^{p-1} + b_0(x),$$

$$|H_\varepsilon(x,\xi)| \leq \frac{1}{\varepsilon}. \tag{4.2.5}$$

Let us consider the following approximated problem:

$$\begin{cases} -\operatorname{div}(a(x,u_\varepsilon,\nabla u_\varepsilon)) + H_\varepsilon(x,\nabla u_\varepsilon) = \mu_\varepsilon & \text{in} \quad \Omega \\ u_\varepsilon = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{4.2.6}$$

A function u_ε is a weak solution to such a problem if it satisfies the following conditions:

$$\begin{cases} u_\varepsilon \in W_0^{1,p}(\nu), \\ \int\limits_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\nabla v + \int\limits_\Omega H_\varepsilon(x,\nabla u_\varepsilon)v = \int\limits_\Omega f_\varepsilon v + \int\limits_\Omega g\nabla v + \\ \qquad\qquad \int\limits_\Omega \lambda_\varepsilon^+ v + \int\limits_\Omega \lambda_\varepsilon^- v, \quad \forall v \in W_0^{1,p}(\nu). \end{cases} \tag{4.2.7}$$

The existence of a weak solution u_ε to problem (4.2.6) is obtained by adapting the proof of classical results proved by Leray and Lions ([80]). Moreover it is easy to verify that a weak solution to problem (4.2.6) is also a renormalized solution to problem (4.2.6).

4.2.2 A priori estimates

Now we prove a priori estimates in the weighted spaces $L^\gamma(\nu)$, $\gamma < \frac{\widetilde{p}}{p}$ for the sequence $|u_\varepsilon|^{p-1}$ and a priori estimate in the space $L^q(\nu)$, $q < \frac{p'\widetilde{p}}{p'+\widetilde{p}}$ for the sequence of the gradients $|\nabla u_\varepsilon|^{p-1}$. The proof is divided in several steps. We begin by considering the case where $\|b\|_{L^r(\Omega)}$ is small enough for the sake of semplicity. Then, in the general case, we adapt the technique of Bottaro and Marina ([35]).

Theorem 4.2.2 *Under the hypotheses of Theorem* 3.1 *every solution u_ε to the problem* (4.2.7) *satisfies*

$$\left\| |u_\varepsilon|^{p-1} \right\|_{L^\gamma(\nu)} \le c, \tag{4.2.8}$$

$$\left\| |\nabla u_\varepsilon|^{p-1} \right\|_{L^q(\nu)} \le c, \tag{4.2.9}$$

where $\gamma < \frac{\widetilde{p}}{p}$ *and* $q < \frac{p'\widetilde{p}}{p'+\widetilde{p}}$, *c is a positive constant which depends only on* p, $|\Omega|$, $N, \|b(x)\|_{L^{\tau}(\Omega)}$, $\sup_{\varepsilon} \|f_{\varepsilon}\|_{L^1(\Omega)}, \sup_{\varepsilon} (\lambda_{\varepsilon}^{+}(\Omega) + \lambda_{\varepsilon}^{-}(\Omega))$, *and on the rearrangement of* $b(x)$.

Proof. *The case where* $\|b(x)\|_{L^{\tau}(\Omega)}$ *is small*

Using $T_k(u_\varepsilon)$, $k > 0$, as a test function in (4.2.7),we obtain

$$\int_{\Omega} a(x, u_\varepsilon, \nabla u_\varepsilon) \nabla T_k(u_\varepsilon) + \int_{\Omega} H_\varepsilon(x, \nabla u_\varepsilon) T_k(u_\varepsilon) \tag{4.2.10}$$
$$= \int_{\Omega} f_\varepsilon T_k(u_\varepsilon) + \int_{\Omega} g \nabla T_k(u_\varepsilon) + \int_{\Omega} \lambda_\varepsilon^{+} T_k(u_\varepsilon) - \int_{\Omega} \lambda_\varepsilon^{-} T_k(u_\varepsilon).$$

Now we estimate the single integrals in equality (4.2.10) . From the ellipticity condition (4.0.27) we obtain

$$\int_{\Omega} a(x, u_\varepsilon, \nabla u_\varepsilon) \nabla T_k(u_\varepsilon) \geq \int_{|u_\varepsilon| \leq k} a(x, u_\varepsilon, \nabla u_\varepsilon) \nabla u_\varepsilon \tag{4.2.11}$$
$$\geq \int_{|u_\varepsilon| \leq k} \nu(x) |\nabla u_\varepsilon|^p = \int_{\Omega} \nu(x) |\nabla T_k(u_\varepsilon)|^p.$$

By (4.0.32) and by Hölder inequality we have

$$\left| \int_{\Omega} H_\varepsilon(x, \nabla u) T_k(u_\varepsilon) \right| \leq \int_{\Omega} b(x) |\nabla u_\varepsilon|^{p-1} T_k(u_\varepsilon) + \int_{\Omega} b_0(x) T_k(u_\varepsilon) \leq \tag{4.2.12}$$
$$\leq k \int_{\Omega} b(x) |\nabla u_\varepsilon|^{p-1} + k \int_{\Omega} b_0(x)$$
$$\leq k \|b(x)\|_{L^{\tau}(\Omega)} \left\|\nu^{-1}(x)\right\|_{L^t(\Omega)}^{1/q} \left\| |\nabla u_\varepsilon|^{p-1} \right\|_{L^q(\nu)}$$
$$+ k \|b_0\|_{L^1(\Omega)},$$

Moreover it results:

$$\int_{\Omega} f_\varepsilon T_k(u_\varepsilon) \leq k \|f_\varepsilon\|_{L^1(\Omega)}, \tag{4.2.13}$$

$$\left| \int_{\Omega} \lambda_n^{+} T_k(u_\varepsilon) \right| \leq k \int_{\Omega} \lambda_n^{+}, \tag{4.2.14}$$

$$\left|\int_\Omega \lambda_n^- T_k\left(u_\varepsilon\right)\right| \leq k\int_\Omega \lambda_n^- . \tag{4.2.15}$$

Now we estimate the last integral in (4.2.10) . From Young inequality we obtain

$$\left|\int_\Omega g\nabla T_k\left(u_\varepsilon\right)\right| \leq \frac{1}{p}\int_\Omega \nu(x)\left|\nabla T_k\left(u_\varepsilon\right)\right|^p + \frac{1}{p'}\int_\Omega g^{p'}\nu^{-1/(p-1)}. \tag{4.2.16}$$

By definition of the convergence of measures in the narrow topology, we have

$$\sup_\varepsilon \|f_\varepsilon\|_{L^1(\Omega)} + \sup_\varepsilon\left(\lambda_\varepsilon^+(\Omega) + \lambda_\varepsilon^-(\Omega)\right) < +\infty. \tag{4.2.17}$$

By (4.2.10) – (4.2.17) we obtain

$$\begin{aligned}\int_\Omega \nu(x)\left|\nabla T_k(u_\varepsilon)\right|^p \leq k\ p'\left[\ \|b(x)\|_{L^\tau(\Omega)}\left\|\nu^{-1}(x)\right\|_{L^t(\Omega)}^{1/q}\ \right]\left\|\left|\nabla u_\varepsilon\right|^{p-1}\right\|_{L^q(\Omega)} \\ + k\ p'\|b_0(x)\|_{L^1(\Omega)} + p'\sup_\varepsilon\|f_\varepsilon\|_{L^1(\Omega)} + \\ + k\ p'\sup_\varepsilon\left(\lambda_\varepsilon^+(\Omega) + \lambda_\varepsilon^-(\Omega)\right) + \|g\|_{L^{p'}(1/\nu^{p-1})}^{p'} .\end{aligned} \tag{4.2.18}$$

Now we define

$$\begin{aligned}M = p'\left[\|b(x)\|_{L^\tau(\Omega)}\left\|\nu^{-1}(x)\right\|_{L^t(\Omega)}^{1/q}\left\|\left|\nabla u_\varepsilon\right|^{p-1}\right\|_{L^q(\Omega)}\right] \\ + p^{'}\left[\|b_0(x)\|_{L^1(\Omega)} + \sup_\varepsilon\|f_\varepsilon\|_{L^1(\Omega)} + \sup_\varepsilon\left(\lambda_\varepsilon^+(\Omega) + \lambda_\varepsilon^-(\Omega)\right)\right],\end{aligned} \tag{4.2.19}$$

$$M^* = p'\ \left[\|b_0(x)\|_{L^1(\Omega)} + \sup_\varepsilon\|f_\varepsilon\|_{L^1(\Omega)} + \sup_\varepsilon\left(\lambda_\varepsilon^+(\Omega) + \lambda_\varepsilon^-(\Omega)\right)\right], \tag{4.2.20}$$

$$L = \|g\|_{L^{p'}(1/\nu^{p-1})}^{p'} . \tag{4.2.21}$$

Therefore (4.2.18) becomes

$$\int_\Omega \nu\left(x\right)\left|\nabla T_k\left(u_\varepsilon\right)\right|^p \leq Mk + L. \tag{4.2.22}$$

By Lemma 4.1.8 we obtain

$$\left\|\left|\nabla u_\varepsilon\right|\right\|_{L^q(\nu)}^{p-1} \leq c\left(N,p\right)\left[M + |\Omega|^{\frac{1}{p}}\,L^{\frac{1}{p'}}\right]. \tag{4.2.23}$$

We have

$$\left\| |\nabla u_\varepsilon|^{p-1} \right\|_{L^q(\nu)} \leq c\,(N,p)\, \|b(x)\|_{L^\tau(\Omega)} \left\| \nu^{-1}(x) \right\|_{L^t(\Omega)}^{\frac{1}{q}} \left\| |\nabla u_\varepsilon|^{p-1} \right\|_{L^q(\nu)} + \\ + c\,(N,p) \left[M^* + |\Omega|^{\frac{1}{p}} L^{\frac{1}{p'}} \right].$$

If the norm of the coefficient b is small enough, i.e.

$$c(N,p)\, \|b(x)\|_{L^\tau(\Omega)} \left\| \nu^{-1}(x) \right\|_{L^t(\Omega)}^{\frac{1}{q}} < 1,$$

we obtain

$$\left\| |\nabla u_\varepsilon|^{p-1} \right\|_{L^q(\nu)} \leq \frac{c\,(N,p) \left[M^* + |\Omega|^{\frac{1}{p}} L^{\frac{1}{p'}} \right]}{1 - c(N,p)\, \|b(x)\|_{L^\tau(\Omega)} \left\| \nu^{-1}(x) \right\|_{L^t(\Omega)}^{\frac{1}{q}}},$$

which is the a priori estimate (4.2.9) .

General case

Now we want to decompose the term $b(x)$ in a finite sum of terms. We will decompose the term $|\nabla u_\varepsilon|^{p-1}$ in a sum of term of type

$$|\nabla u_\varepsilon|^{p-1}\, \chi_{\{m_{i+1} < |u_\varepsilon| < m_i\}}.$$

The values of the costant m_i depend on m but not their number. I will vary between 0 and $I \leq I^{\,*}$ which is independent of N.

Now we define a new set Z_N so that the measure of the set $\{x \in \Omega : m_{i+1} < |u_\varepsilon| < m_i\}$ will be continuous respect to the parameter m for m_i given. Since Ω is a bounded open set, $|\Omega|$ is finite and the set of constant c such that

$$|\{x \in \Omega : |u_\varepsilon(x)| = c\}| > 0$$

is at most countable.

Let be Z_N^c the union of all those sets and $Z_N = \Omega \backslash Z_N^c$ the union of all the sets such that $|\{x \in \Omega : |u_\varepsilon(x)| = c\}| = 0$. Since

$$\nabla u_\varepsilon = 0 \quad a.e. \text{ on } \{x \in \Omega : |u_\varepsilon(x)| = c\},$$

we obtain that

$$\nabla u_\varepsilon = 0 \quad a.e. \text{ on } Z_n^c.$$

With this choice of m_i we know that for m_i fixed and $0 < m < m_i$ the function

$$m \longrightarrow |Z_n \cap \{m < |u_\varepsilon| < m_i\}|$$

is continuous.

First step:

Define for $m > 0$ the function that is

$$S_m(s) = \begin{cases} 0 & |s| \leq m \\ (|s| - m)\ sign(s) & |s| > m. \end{cases} \tag{4.2.24}$$

Using $T_k(S_m(u_\varepsilon))$, with m to be specified, as a test function in $(4.2.7)$, we obtain

$$\int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon) \nabla T_k(S_m(u_\varepsilon)) + \int_\Omega H_\varepsilon(x, \nabla u_\varepsilon) T_k(S_m(u_\varepsilon)) =$$

$$= \int_\Omega f_\varepsilon T_k(S_m(u_\varepsilon)) + \int_\Omega g \nabla T_k(S_m(u_\varepsilon)) + \int_\Omega \lambda_n^+ T_k(S_m(u_\varepsilon)) \tag{4.2.25}$$
$$- \int_\Omega \lambda_n^- T_k(S_m(u_\varepsilon)).$$

Now we estimate the singular integrals in $(4.2.25)$.

By elliptic conditions $(4.0.27)$, we have

$$\int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon) \nabla T_k(S_m(u_\varepsilon)) \geq \int_\Omega \nu(x) |\nabla T_k(S_m(u_\varepsilon))|^p; \tag{4.2.26}$$

Moreover

$$\int_\Omega f_\varepsilon T_k(S_m(u_\varepsilon)) \leq k \|f_\varepsilon\|_{L^1(\Omega)}, \tag{4.2.27}$$

$$\left| \int_\Omega \lambda_n^+ T_k(S_m(u_\varepsilon)) \right| \leq k \int_\Omega \lambda_n^+, \tag{4.2.28}$$

$$\left| \int_\Omega \lambda_n^- T_k(S_m(u_\varepsilon)) \right| \leq k \int_\Omega \lambda_n^-, \tag{4.2.29}$$

By Young inequality we obtain

$$\left|\int_\Omega g\nabla T_k\left(S_m\left(u_\varepsilon\right)\right)\right| \leq \frac{1}{p}\int_\Omega \nu(x)\left|\nabla T_k\left(S_m\left(u_\varepsilon\right)\right)\right|^p + \frac{1}{p'}\int_\Omega g^{p'}\nu^{-1/(p-1)}. \qquad (4.2.30)$$

Using the definition of $S_m(u_\varepsilon)$ and that $\nabla u_\varepsilon = \nabla S_m(u_\varepsilon)$ a.e. in $Z_\varepsilon = \Omega - Z_\varepsilon^c$ we have by (4.0.32)

$$\left|\int_\Omega H_\varepsilon(x,\nabla u_\varepsilon)T_k\left(S_m\left(u_\varepsilon\right)\right)\right| \leq \int_\Omega \left|b(x)\left|\nabla u_\varepsilon\right|^{p-1}T_k\left(S_m\left(u_\varepsilon\right)\right)\right| + \int_\Omega \left|b_0(x)T_k\left(S_m\left(u_\varepsilon\right)\right)\right| \qquad (4.2.31)$$

$$\leq \left\|b(x)\right\|_{L^\tau(Z_\varepsilon\cap\{|u_\varepsilon|>m\})}\left\|\nu^{-1}(x)\right\|^{\frac{1}{q}}_{L^t(Z_\varepsilon\cap\{|u_\varepsilon|>m\})}\times$$
$$\times\left\|\left|\nabla\left(S_m\left(u_\varepsilon\right)\right)\right|^{p-1}\right\|_{L^q(\nu)} + k\left\|b_0\right\|_{L^1(\Omega)}.$$

By (4.2.26) – (4.2.31) we have for $k>0$

$$\int_\Omega \nu\left(x\right)\left|\nabla T_k\left(S_m\left(u_\varepsilon\right)\right)\right|^p \leq M_1 k + L, \qquad (4.2.32)$$

where

$$M_1 = p'\left\|b(x)\right\|_{L^\tau(Z_\varepsilon\cap\{|u_\varepsilon|>m\})}\left\|\nu^{-1}(x)\right\|^{\frac{1}{q}}_{L^t(Z_\varepsilon\cap\{|u_\varepsilon|>m\})}\times \qquad (4.2.33)$$
$$\times\left\|\left|\nabla\left(S_m\left(u_\varepsilon\right)\right)\right|^{p-1}\right\|_{L^q(\nu)} + M^*,$$

and where $M^*$and L are defined by (4.2.20) – (4.2.21).

By Lemma 4.1.8 we have

$$\left\|\left|\nabla\left(S_m\left(u_\varepsilon\right)\right)\right|^{p-1}\right\|_{L^q(\nu)} \leq c\left(N,p\right)\left\|b(x)\right\|_{L^\tau(|Z_\varepsilon\cap\{|u_\varepsilon|>m\}|)}\left\|\nu^{-1}(x)\right\|^{\frac{1}{q}}_{L^t(\Omega)}\times$$
$$\times\left\|\left|\nabla\left(S_m\left(u_\varepsilon\right)\right)\right|^{p-1}\right\|_{L^q(\nu)} + c\left(N,p\right)\left[M^* + |\Omega|^{\frac{1}{p}}L^{\frac{1}{p'}}\right].$$

Now we observe that

$$\left\|b(x)\right\|_{L^1(Z_\varepsilon\cap\{|u_\varepsilon|>m\})} = \int_{|Z_\varepsilon\cap\{|u_\varepsilon|>m\}|} b\left(x\right) \leq \int_0^{|Z_\varepsilon\cap\{|u_\varepsilon|>m\}|} b^*(t). \qquad (4.2.34)$$

If

$$c(N,p)\left\|b(x)\right\|_{L^\tau(|Z_\varepsilon\cap\{|u_\varepsilon|>m\}|)}\left\|\nu^{-1}(x)\right\|^{\frac{1}{q}}_{L^t(\Omega)} \leq \frac{1}{2}, \qquad (4.2.35)$$

then we choose $m = m_1 = 0$. If (4.2.35) is not true, we choose $m = m_1 > 0$ such that

$$c(N,p)\left(\int_0^{|Z_\varepsilon\cap\{|u_\varepsilon|>m_1\}|} b^*(t)^\tau dt\right)^{\frac{1}{\tau}}\left(\int_\Omega \nu^{-t}\right)^{\frac{1}{tq}} = \frac{1}{2}.$$

The function $m \longrightarrow |Z_\varepsilon \cap \{|u_\varepsilon| > m\}|$ is continuous, decreasing and goes to 0 when m goes to ∞ . Now if we define δ by

$$c(N,p)\left(\int_0^{\delta} b^*(t)^\tau dt\right)^{\frac{1}{\tau}}\left(\int_\Omega \nu^{-t}\right)^{\frac{1}{tq}} = \frac{1}{2}, \tag{4.2.36}$$

we have $|Z_\varepsilon \cap \{|u_\varepsilon| > m_1\}| = \delta$. With this choise of m, we obtain

$$\left\||\nabla\left(S_m\left(u_\varepsilon\right)\right)|^{p-1}\right\|_{L^q(\nu)} \leq 2\; c\left(N,p\right)\left[M^* + |\Omega|^{\frac{1}{\bar{p}}}\, L^{\frac{1}{p'}}\right]. \tag{4.2.37}$$

Second step:

Define for $0 \leq m < m_1$ the function

$$S_{m,m_1}\left(u_\varepsilon\right) = \begin{cases} m_1 - m & u_\varepsilon > m_1 \\ u_\varepsilon - m & m \leq u_\varepsilon \leq m_1 \\ 0 & -m \leq u_\varepsilon \leq m \\ u_\varepsilon + m & -m_1 \leq u_\varepsilon \leq -m \\ m - m_1 & u_\varepsilon < -m_1 \end{cases} \tag{4.2.38}$$

Using the test function $T_k\left(S_{m,m_1}\left(u_\varepsilon\right)\right)$ with m to be specified later we have

$$\int_\Omega a\left(x,u_\varepsilon,\nabla u_\varepsilon\right)\nabla T_k\left(S_{m,m_1}\left(u_\varepsilon\right)\right) + \int_\Omega H_\varepsilon(x,\nabla u_\varepsilon) T_k\left(S_{m,m_1}\left(u_\varepsilon\right)\right) = \tag{4.2.39}$$

$$\int_\Omega f_\varepsilon T_k\left(S_{m,m_1}\left(u_\varepsilon\right)\right) + \int_\Omega g\nabla T_k\left(S_{m,m_1}\left(u_\varepsilon\right)\right) + \int_\Omega \lambda_n^+ T_k\left(S_{m,m_1}\left(u_\varepsilon\right)\right) - \int_\Omega \lambda_n^- T_k\left(S_{m,m_1}\left(u_\varepsilon\right)\right).$$

Now as in the previous step we have

$$\int_\Omega a\left(x,u_\varepsilon,\nabla u_\varepsilon\right)\nabla T_k\left(S_{m,m_1}\left(u_\varepsilon\right)\right) \geq \int_\Omega \nu\left(x\right)\left|\nabla T_k\left(S_{m,m_1}\left(u_\varepsilon\right)\right)\right|^p .$$

$$\int_\Omega f_\varepsilon T_k\left(S_{m,m_1}\left(u_\varepsilon\right)\right) \leq k\left\|f_\varepsilon\right\|_{L^1(\Omega)}, \tag{4.2.40}$$

$$\left|\int_\Omega \lambda_n^+ T_k\left(S_{m,m_1}\left(u_\varepsilon\right)\right)\right| \leq k\int_\Omega \lambda_n^+, \tag{4.2.41}$$

$$\left|\int_\Omega \lambda_n^- T_k\left(S_{m,m_1}\left(u_\varepsilon\right)\right)\right| \leq k\int_\Omega \lambda_n^-. \tag{4.2.42}$$

By Young inequality we have

$$\int_\Omega g\nabla T_k\left(S_{m,m_1}\left(u_\varepsilon\right)\right) \leq \frac{1}{p}\int_\Omega \nu(x)\left|\nabla T_k S_{m,m_1}\left(u_\varepsilon\right)\right|^p + \frac{1}{p'}\int_\Omega g^{p'}\nu^{-1/(p-1)}. \tag{4.2.43}$$

Since $S_{m,m_1}\left(u_\varepsilon\right) = 0$ when $\left|u_\varepsilon\right| \leq m$, we obtain

$$\begin{aligned}\left|\int\limits_\Omega H_\varepsilon(x,\nabla u_\varepsilon) T_k\left(S_{m,m_1}\left(u_\varepsilon\right)\right)\right| &\leq k\int_{m<|u_\varepsilon|<m_1} b\left(x\right)\left|\nabla u_\varepsilon\right|^{p-1} + \\ &+ k\int_{|u_\varepsilon|\geq m_1} b\left(x\right)\left|\nabla u_\varepsilon\right|^{p-1} + k\int\limits_\Omega b_0(x) \\ &\leq k\left\|b\left(x\right)\right\|_{L^\tau(Z_\varepsilon\cap\{m<|u_\varepsilon|<m_1\})}\left\|\nu^{-1}\right\|_{L^t(\Omega)}^{\frac{1}{q}}\left\|\left|\nabla\left(S_{m,m_1}\left(u_\varepsilon\right)\right)\right|^{p-1}\right\|_{L^q(v)} \\ &+ k\left\|b\left(x\right)\right\|_{L^\tau(\Omega)}\left\|\nu^{-1}\right\|_{L^t(\Omega)}^{\frac{1}{q}}\left\|\left|\nabla\left(S_{m_1}\left(u_\varepsilon\right)\right)\right|^{p-1}\right\|_{L^q(v)} + k\left\|b_0(x)\right\|_{L^1(\Omega)}.\end{aligned} \tag{4.2.44}$$

Combining $(4.2.39), (4.2.44)$ we have for $k > 0$

$$\int\limits_\Omega \nu\left(x\right)\left|\nabla T_k\left(S_{m,m_1}\left(u_\varepsilon\right)\right)\right|^p \leq M_2 k + L, \tag{4.2.45}$$

where

$$\begin{aligned}M_2 &= p'\left\|b(x)\right\|_{L^\tau(Z_\varepsilon\cap\{m_1<|u_\varepsilon|<m\})}\left\|\nu^{-1}(x)\right\|_{L^t(\Omega)}^{\frac{1}{q}}\left\|\left|\nabla\left(S_{m,m_1}\left(u_\varepsilon\right)\right)\right|^{p-1}\right\|_{L^q(\nu)} + \\ &+ \left\|b\left(x\right)\right\|_{L^\tau(\Omega)}\left\|\nu^{-1}\right\|_{L^t(\Omega)}^{\frac{1}{q}}\left\|\left|\nabla\left(S_{m_1}\left(u_\varepsilon\right)\right)\right|^{p-1}\right\|_{L^q(v)} + M^*,\end{aligned} \tag{4.2.46}$$

where M^* and L are defined in $(4.2.20), (4.2.21)$.

By $(4.2.9)$ we have

$$\begin{aligned}&\left\|\left|\nabla\left(S_{m,m_1}\left(u_\varepsilon\right)\right)\right|^{p-1}\right\|_{L^q(\nu)} \\ &\leq c\left(N,p\right)\left\|b(x)\right\|_{L^\tau(|Z_\varepsilon\cap\{m<|u_\varepsilon|<m_1\}|)}\left\|\nu^{-1}(x)\right\|_{L^t(\Omega)}^{\frac{1}{q}}\left\|\left|\nabla\left(S_{m,m_1}\left(u_\varepsilon\right)\right)\right|^{p-1}\right\|_{L^q(\nu)} \\ &+ c(N,p)\left[\left\|b\left(x\right)\right\|_{L^\tau(\Omega)}\left\|\nu^{-1}\right\|_{L^t(\Omega)}^{\frac{1}{q}}\left\|\left|\nabla\left(S_{m_1}\left(u_\varepsilon\right)\right)\right|^{p-1}\right\|_{L^q(v)}\right] \\ &+ c(N,p)\left[M^* + |\Omega|^{\frac{1}{p}} L^{\frac{1}{p'}}\right].\end{aligned}$$

If

$$c\,(N,p)\,\|b(x)\|_{L^{\tau}(|Z_{\varepsilon}\cap\{m<|u_{\varepsilon}|<m_1\}|)}\,\left\|\nu^{-1}(x)\right\|_{L^{t}(\Omega)}^{\frac{1}{q}}\leq\frac{1}{2}, \tag{4.2.47}$$

then we choose $m = m_2 = 0$. If (4.2.47) does not hold, we choose $m = m_2 > 0$ so that

$$c\,(N,p)\,\|b(x)\|_{L^{\tau}(|Z_{\varepsilon}\cap\{m<|u_{\varepsilon}|<m_1\}|)}\,\left\|\nu^{-1}(x)\right\|_{L^{t}(\Omega)}^{\frac{1}{q}}=\frac{1}{2}.$$

The function $m \longrightarrow |Z_\varepsilon \cap \{|u_\varepsilon| > m\}|$ is continuous, decreasing and goes to 0 when m goes to ∞ . Now if we define δ by

$$c(N,p)\left(\int_0^{\delta} b^*(t)^{\tau}dt\right)^{\frac{1}{\tau}}\left(\int_{\Omega}\nu^{-t}\right)^{\frac{1}{tq}}=\frac{1}{2}, \tag{4.2.48}$$

we have $|Z_\varepsilon \cap \{m < |u_\varepsilon| < m_1\}| = \delta$. With this choice of m we obtain

$$\begin{aligned}\left\||\nabla\left(S_{m,m_1}\left(u_{\varepsilon}\right)\right)|^{p-1}\right\|_{L^{q}(\nu)}\leq 2\;c(N,p)&\left[\|b\left(x\right)\|_{L^{\tau}(\Omega)}\left\|\nu^{-1}\right\|_{L^{t}(\Omega)}^{\frac{1}{q}}\right]\times\\ &\times\left\||\nabla\left(S_{m_1}\left(u_{\varepsilon}\right)\right)|^{p-1}\right\|_{L^{q}(v)}+\\ &+2c(N,p)\left[M^{*}+|\Omega|^{\frac{1}{p}}\,L^{\frac{1}{p'}}\right].\end{aligned} \tag{4.2.49}$$

Third step:

Now we define for $0 \leq m < m_2$ the function $S_{m,m_2} : \mathbb{R} \longrightarrow \mathbb{R}$

$$S_{m,m_2}\left(u_{\varepsilon}\right)=\begin{cases} m_2-m & u_{\varepsilon}>m_2\\ u_{\varepsilon}-m & m\leq u_{\varepsilon}\leq m_2\\ 0 & -m\leq u_{\varepsilon}\leq m\\ u_{\varepsilon}+m & -m_2\leq u_{\varepsilon}\leq -m\\ m-m_2 & u_{\varepsilon}<-m_2.\end{cases} \tag{4.2.50}$$

As in the previous step we use $T_k\left(S_{m,m_2}\left(u_\varepsilon\right)\right)$ with m to be specified later as test function in the (4.2.7)

$$\int_{\Omega}\nu\left(x\right)|\nabla T_k\left(S_{m,m_1}\left(u_{\varepsilon}\right)\right)|^{p}\leq M_3k+L,$$

where

$$M_3 = p' \|b(x)\|_{L^\tau(Z_\varepsilon \cap \{m_1<|u_\varepsilon|<m\})} \left\|\nu^{-1}(x)\right\|^{\frac{1}{q}}_{L^t(\Omega)} \left\||\nabla\left(S_{m,m_2}\left(u_\varepsilon\right)\right)|^{p-1}\right\|_{L^q(\nu)} +$$
$$+ \|b\left(x\right)\|_{L^\tau(\Omega)} \left\|\nu^{-1}\right\|^{\frac{1}{q}}_{L^t(\Omega)} \left\|\left|\nabla\left(S_{m_{2,1}}\left(u_\varepsilon\right)\right)\right|^{p-1}\right\|_{L^q(v)}$$
$$+ \|b\left(x\right)\|_{L^\tau(\Omega)} \left\|\nu^{-1}\right\|^{\frac{1}{q}}_{L^t(\Omega)} \left\||\nabla\left(S_{m_1}\left(u_\varepsilon\right)\right)|^{p-1}\right\|_{L^q(v)} + M^*,$$

where M^* and L are defined in $(4.2.20), (4.2.21)$.

As before we use Lemma 4.1.8, we choose $m = m_3 = 0$ in the case

$$c\left(N,p\right) \|b(x)\|_{L^\tau(|Z_\varepsilon \cap \{m<|u_\varepsilon|<m_2\}|)} \left\|\nu^{-1}(x)\right\|^{\frac{1}{q}}_{L^t(\Omega)} \leq \frac{1}{2} \tag{4.2.51}$$

and if (4.2.51) does not hold, we choose $m = m_3 > 0$ so that

$$c\left(N,p\right) \|b(x)\|_{L^\tau(|Z_\varepsilon \cap \{m<|u_\varepsilon|<m_2\}|)} \left\|\nu^{-1}(x)\right\|^{\frac{1}{q}}_{L^t(\Omega)} = \frac{1}{2}. \tag{4.2.52}$$

Now as in the previous step we observe that m_3 depends on n and

$$|Z_\varepsilon \cap \{m < |u_\varepsilon| < m_2\}| = \delta.$$

With this choice of m we obtain

$$\begin{aligned} \left\||\nabla\left(S_{m,m_1}\left(u_\varepsilon\right)\right)|^{p-1}\right\|_{L^q(\nu)} \leq{}& 2\ c(N,p) \left[\|b\left(x\right)\|_{L^\tau(\Omega)} \left\|\nu^{-1}\right\|^{\frac{1}{q}}_{L^t(\Omega)}\right] \\ & \left\||\nabla\left(S_{m_2,m_1}\left(u_\varepsilon\right)\right)|^{p-1}\right\|_{L^q(v)} + \\ & + 2\ c(N,p) \left[\|b\left(x\right)\|_{L^\tau(\Omega)} \left\|\nu^{-1}\right\|^{\frac{1}{q}}_{L^t(\Omega)}\right] \\ & \left\||\nabla\left(S_{m_1}\left(u_\varepsilon\right)\right)|^{p-1}\right\|_{L^q(v)} + \\ & + 2\ c(N,p) \left[M^* + |\Omega|^{\frac{1}{p}}\, L^{\frac{1}{p'}}\right]. \end{aligned} \tag{4.2.53}$$

Final step

We repeat the procedure until $i = I$. If

$$c\left(N,p\right) \|b(x)\|_{L^\tau(|Z_\varepsilon \cap \{m<|u_\varepsilon|<m_{I-1}\}|)} \left\|\nu^{-1}(x)\right\|^{\frac{1}{q}}_{L^t(\Omega)} \leq \frac{1}{2},$$

then we choose $m_I = 0$. Now we want to estimate I. We observe

$$|\Omega| \geq |Z_N| \geq |Z_N \cap \{|u_\varepsilon| > m_1\}| + |Z_N \cap \{m_2 < |u_\varepsilon| < m_1\}| +$$
$$+ |Z_N \cap \{m_3 < |u_\varepsilon| < m_2\}| + ... + + |Z_N \cap \{m_{I-1} < |u_\varepsilon| < m_{I-2}\}| .$$

By $(4.2.36), (4.2.48)$, we know that

$$|Z_N \cap \{|u_\varepsilon| > m_1\}| = |Z_N \cap \{m_2 < |u_\varepsilon| < m_1\}| =$$
$$= |Z_N \cap \{m_3 < |u_\varepsilon| < m_2\}| = ... = |Z_N \cap \{m_{I-1} < |u_\varepsilon| < m_{I-2}\}| = \delta,$$

with δ that does not depend on N. We realize that

$$|\Omega| \geq (I-1)\delta,$$

$$I \leq I^*, \tag{4.2.54}$$

where $I^* = 1 + \left[\dfrac{|\Omega|}{\delta}\right]$. From (4.2.54) we deduce that I depends only on $b(x)$ through the definition of δ but it does not depend on N.

Now we define $m_0 = +\infty$, $S_{m_1,m_0} = S_{m_1}$

$$\begin{cases} X_i = \left\| \left| \nabla \left(S_{m_i,m_{i-1}} (u_\varepsilon) \right) \right|^{p-1} \right\|_{L^q(\nu)} & 1 \leq i \leq I \\ a = 2C(N,p) \left\| b(x) \right\|_{L^\tau(\Omega)} \left\| \nu^{-1} \right\|_{L^t(\Omega)}^{\frac{1}{q}} \\ b = 2\ C(N,p) \left[M^* + |\Omega|^{\frac{1}{p}} L^{\frac{1}{p'}} \right]. \end{cases}$$

By $(4.2.37), (4.2.49), (4.2.53)$, we deduce

$$\begin{array}{lll} X_1 \leq b & X_2 \leq aX_1 + b & X_3 \leq aX_2 + aX_1 + b \\ X_I \leq aX_{I-1} + ... + aX_1 + b & for\ I \leq I^* & \end{array} .$$

By induction we have

$$X_i \leq (a+1)^{i-1} b, \qquad \text{for } 1 \leq i \leq I,$$

$$|\nabla u_\varepsilon|^{p-1} = \sum_{i=1}^{I} |\nabla u_\varepsilon|^{p-1} \chi_{\{m_i<|u_\varepsilon|<m_{i-1}\}} = \sum_{i=1}^{I} \left|\nabla S_{m_i,m_{i-1}}(u_\varepsilon)\right|^{p-1}. \tag{4.2.55}$$

By (4.2.55) we have

$$\begin{aligned} \left\| |\nabla u_\varepsilon|^{p-1} \right\|_{L^q(\nu)} &\leq \sum_{i=1}^{I} \left\| |\nabla S_{m_i,m_{i-1}}(u_\varepsilon)|^{p-1} \right\|_{L^q(\nu)} \leq \sum_{i=1}^{I} X_i \leq \\ &\leq \sum_{i=1}^{I} (a+1)^{i-1} b = b\left[\frac{(a+1)^I - 1}{a}\right] \leq \frac{b}{a}\left((a+1)^{I^*} - 1\right). \end{aligned} \tag{4.2.56}$$

In the same way we obtain the estimate for $|u_\varepsilon|^{p-1}$. ■

4.2.3 A stability result

In this section we prove a stability result which unsures that the solution u_ε converges almost everywhere to renormalized solution u.The result is obtained by adapting the technique developed in [48] and used also in [72]. A new proof of this stability result, which does not need the strong convergence of the truncations of the solutions in the energy space, has been proved in [84]

In order to prove the stability result, we consider the nonlinear problem

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u) = \mu_\varepsilon & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4.2.57}$$

where ε is a sequence of positive numbers that converges to zero, $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function such that $(4.0.27), (4.0.28)$ are satisfied and μ_ε is a Radon measure with bounded total variation in Ω.

Since μ_ε and μ are measures in $M_b(\Omega)$, they can be decomposed as in the Proposition 4.1.7, as follows

$$\mu_\varepsilon = f_\varepsilon - \operatorname{div}(g_\varepsilon) + \lambda_\varepsilon^+ - \lambda_\varepsilon^-, \tag{4.2.58}$$

$$\mu = f - \operatorname{div}(g) + \mu_s^+ - \mu_s^-, \tag{4.2.59}$$

where sequences f_ε, λ_ε^+, λ_ε^- satisfy $(4.2.1)-(4.2.3)$ and g_ε is a sequence such that

$$\begin{cases} g_\varepsilon \text{ is a sequence of functions in } \left(L^{p'}(1/\nu^{p-1})\right)^N \\ g_\varepsilon \to g \ \left(L^{p'}(1/\nu^{p-1})\right)^N \text{ strongly.} \end{cases} \tag{4.2.60}$$

The following stability result holds true:

Theorem 4.2.3 *Assume that* $(4.0.27)$, $(4.0.28)$, $(4.2.1)-(4.2.3)$, $(4.2.60)$ *hold. Let* u_ε *be a renormalized solution of* $(4.2.57)$. *Then* u_ε *converges almost everywhere to renormalized solution* u *to the problem*

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u) = \mu & in \quad \Omega \\ u = 0 & on \quad \partial\Omega. \end{cases} \tag{4.2.61}$$

According to Proposition 4.1.7 λ_ε^+ and λ_ε^- can be decomposed in the following way

$$\lambda_\varepsilon^+ = \lambda_{\varepsilon,0}^+ + \lambda_{\varepsilon,s}^+,$$

$$\lambda_\varepsilon^- = \lambda_{\varepsilon,0}^- + \lambda_{\varepsilon,s}^-,$$

with $\lambda_{\varepsilon,0}^+$, $\lambda_{\varepsilon,0}^- \in M_{0,v}(\Omega), \lambda_{\varepsilon,0}^+$, $\lambda_{\varepsilon,0}^- \geq 0$ and $\lambda_{\varepsilon,s}^+$, $\lambda_{\varepsilon,s}^- \in M_{s,v}(\Omega), \lambda_{\varepsilon,s}^+$, $\lambda_{\varepsilon,s}^- \geq 0$. On the other hand μ_ε can be decomposed as follows

$$\mu_\varepsilon = \mu_{\varepsilon,0} + \mu_{\varepsilon,s} = \mu_{\varepsilon,0} + \mu_{\varepsilon,s}^+ + \mu_{\varepsilon,s}^-,$$

where $\mu_{\varepsilon,0} \in M_{0,v}(\Omega)$ and $\mu_{\varepsilon,s}^+, \mu_{\varepsilon,s}^-$ are two nonnegative measure in $M_{s,v}(\Omega)$ which are concentrated on two disjoint subset E_ε^+ and E_ε^- of (p,ν)-capacity. As in [48] we have

$$0 \leq \mu_{\varepsilon,s}^+ \leq \lambda_{\varepsilon,0}^+ \qquad 0 \leq \mu_{\varepsilon,s}^- \leq \lambda_{\varepsilon,0}^-. \tag{4.2.62}$$

Sketch of the proof.

First step

By using the same tecnique used in the previous, we obtain a priori estimate for ∇u_ε thanks to which we have that $u_\varepsilon \to u$ a.e. Now we want to prove that u is a solution of (4.2.61) in the sense of distribution.

We know that u_ε is a renormalized solution of the problem (4.2.57) and so it is also a solution in the sense of distribution that is

$$\int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon)\nabla\varphi = \int_\Omega \varphi d\mu_\varepsilon, \forall\varphi \in C_0^\infty(\Omega). \tag{4.2.63}$$

Otherwise by a result contained in [97] we have

$$a(x, u_\varepsilon, \nabla u_\varepsilon) \longrightarrow a(x, u, \nabla u) \text{ in } L^1(\Omega) \text{ strongly} \tag{4.2.64}$$

and by passing to the limit in (4.2.63), we obtain that u is a distributional solution of (4.2.61).

Second step

In this step we adapt the proof given in [72]. We prove that

$$\limsup_{n\longrightarrow\infty}\limsup_{\varepsilon\to 0}\frac{1}{n}\int_{\{n<u_\varepsilon<2n\}} a(x, u_\varepsilon, \nabla u_\varepsilon)\nabla u_\varepsilon\varphi \leq \int_\Omega \varphi d\mu_s^+, \tag{4.2.65}$$

$$\limsup_{n\longrightarrow\infty}\limsup_{\varepsilon\to 0}\frac{1}{n}\int_{\{-2n<u_\varepsilon<n\}} a(x, u_\varepsilon, \nabla u_\varepsilon)\nabla u_\varepsilon\varphi \leq \int_\Omega \varphi d\mu_s^-, \tag{4.2.66}$$

$\forall\varphi \in C^1\left(\overline{\Omega}\right)$ with $\varphi \geq 0$.

Now we define for $n \geq 1$ $s_n : \mathbb{R} \longrightarrow \mathbb{R}$ and $h_\eta : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$s_n(r) = \frac{T_{2n}(r) - T_n(r)}{n}, \tag{4.2.67}$$

$$h_\eta(r) = 1 - |s_\eta(r)|. \tag{4.2.68}$$

If we take $h_\eta(u_\varepsilon)s_n(u_\varepsilon^+)\varphi$ as test function and with $\eta \longrightarrow \infty$ we have

$$\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)s_n(u_\varepsilon^+)\nabla\varphi + \frac{1}{n}\int_{\{n<u_\varepsilon<2n\}} a(x,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon\varphi$$
$$= \int_\Omega f_\varepsilon s_n(u_\varepsilon^+)\varphi + \int_\Omega g_\varepsilon s_n(u_\varepsilon^+)\nabla\varphi + \frac{1}{n}\int_{\{n<u_\varepsilon<2n\}} g_\varepsilon\nabla u_\varepsilon\varphi +$$
$$\int_\Omega s_n(u_\varepsilon^+)\varphi d\lambda_{\varepsilon,0}^+ - \int_\Omega s_n(u_\varepsilon^+)\varphi d\lambda_{\varepsilon,0}^- + \int_\Omega \varphi d\mu_{\varepsilon,s}^+,$$

$\forall\varphi \in C^1(\overline{\Omega})$ non negative.

Since $s_n(u_\varepsilon^+)$ is bounded and $s_n(u_\varepsilon^+) \longrightarrow s_n(u^+)$ $a.e$, by Lebesgue convergence theorem we have

$$\lim_{n\longrightarrow\infty}\lim_{\varepsilon\longrightarrow 0}\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)s_n(u_\varepsilon^+)\nabla\varphi = \lim_{n\longrightarrow\infty}\int_\Omega a(x,u,\nabla u)s_n(u^+)\nabla\varphi = 0 \quad (4.2.69)$$

and by (4.2.62)

$$\lim_{n\longrightarrow\infty}\lim_{\varepsilon\longrightarrow 0}\int_\Omega f_\varepsilon s_n(u_\varepsilon^+)\varphi = \lim_{n\longrightarrow\infty}\int_\Omega f s_n(u^+)\varphi = 0. \quad (4.2.70)$$

By Hölder inequality we have

$$\frac{1}{n}\int_{\{n<u_\varepsilon<2n\}} g_\varepsilon\nabla u_\varepsilon\varphi \leq \|\varphi\|_{L^\infty(\Omega)}\|g_\varepsilon\|_{L^{p'}(1/\nu^{p-1})} \times \frac{1}{n}\left(\int_{\{n<u_\varepsilon<2n\}}\nu\,|\nabla u_\varepsilon|^p\right)^{1/p}.$$

Since $g_\varepsilon \to g$ $\left(L^{p'}(1/\nu^{p-1})\right)^N$ strongly,.we have

$$\lim_{n\longrightarrow\infty}\limsup_{\varepsilon\longrightarrow 0}\frac{1}{n}\int_{\{n<u_\varepsilon<2n\}} g_\varepsilon\nabla u_\varepsilon\varphi = 0, \quad (4.2.71)$$

and finally by (4.2.62)

$$\int_\Omega s_n(u_\varepsilon^+)\varphi d\lambda_{\varepsilon,0}^+ + \int_\Omega \varphi d\mu_{\varepsilon,s}^+ \leq \int_\Omega \varphi d\lambda_\varepsilon^+. \quad (4.2.72)$$

By $(4.2.69) - (4.2.72)$ we obtain $\forall\varphi \in C^1(\overline{\Omega})$

$$\frac{1}{n}\int_{\{n<u_\varepsilon<2n\}} a(x,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon\varphi \leq \omega(\varepsilon,n) + \int_\Omega \varphi d\lambda_\varepsilon^+, \quad (4.2.73)$$

where ω is a function such that $\lim_{n\longrightarrow\infty}\limsup_{\varepsilon\longrightarrow 0}\omega(\varepsilon,n)=0$. Since λ_ε^+ converges to μ_s^+ in the narrow topology we have (4.2.65).

Third step

Now we want to prove that u is a renormalized solution. As in [48] we consider for $\delta>0$ two cut-off functions ψ_δ^+ and $\psi_\delta^-\in C_0^\infty(\Omega)$ such that

$$0\le\psi_\delta^+\le 1\qquad 0\le\psi_\delta^-\le 1\qquad \sup p\left(\psi_\delta^+\right)\cap\sup p\left(\psi_\delta^-\right)=\emptyset, \tag{4.2.74}$$

$$\lim_{\delta\longrightarrow 0}\int_\Omega\left|\nabla\psi_\delta^+\right|^p=\lim_{\delta\longrightarrow 0}\int_\Omega\left|\nabla\psi_\delta^+\right|^p=0,$$

$$\lim_{\delta\longrightarrow 0}\int_\Omega\psi_\delta^+d\mu_s^+=\lim_{\delta\longrightarrow 0}\int_\Omega\psi_\delta^-d\mu_s^-=0,$$

$$\lim_{\delta\longrightarrow 0}\int_\Omega(1-\psi_\delta^+)d\mu_s^+=\lim_{\delta\longrightarrow 0}\int_\Omega(1-\psi_\delta^-)d\mu_s^-=0,$$

$$\lim_{\delta\longrightarrow 0}\lim_{\varepsilon\longrightarrow 0}\int_\Omega\psi_\delta^-d\lambda_\varepsilon^+=\lim_{\delta\longrightarrow 0}\lim_{\varepsilon\longrightarrow 0}\int_\Omega\psi_\delta^+d\lambda_\varepsilon^-=0, \tag{4.2.75}$$

$$\lim_{\delta\longrightarrow 0}\lim_{\varepsilon\longrightarrow 0}\int_\Omega(1-\psi_\delta^+)d\lambda_\varepsilon^+=\lim_{\delta\longrightarrow 0}\lim_{\varepsilon\longrightarrow 0}\int_\Omega(1-\psi_\delta^+)d\lambda_\varepsilon^+=0. \tag{4.2.76}$$

Using $h_n(u_\varepsilon)h(u)v(1-\psi_\delta^+-\psi_\delta^-)$ as test function where h_n is defined in (4.2.68) we have

$$\begin{aligned}
&\int_\Omega h_n'(u_\varepsilon)h(u)v(1-\psi_\delta^+-\psi_\delta^-)\left[a(x,u_\varepsilon,\nabla u_\varepsilon)\right]\nabla u_\varepsilon+ \\
&\int_\Omega h_n(u_\varepsilon)h'(u)v(1-\psi_\delta^+-\psi_\delta^-)\left[a(x,u_\varepsilon,\nabla u_\varepsilon)\right]\nabla u+ \\
&\int_\Omega h_n(u_\varepsilon)h(u)(1-\psi_\delta^+-\psi_\delta^-)\left[a(x,u_\varepsilon,\nabla u_\varepsilon)\right]\nabla v+ \\
&\int_\Omega h_n(u_\varepsilon)h(u)v\left[a(x,u_\varepsilon,\nabla u_\varepsilon)\right]\nabla(1-\psi_\delta^+-\psi_\delta^-) \\
&\int_\Omega f_\varepsilon h_n(u_\varepsilon)h(u)v(1-\psi_\delta^+-\psi_\delta^-)+\int_\Omega g_\varepsilon\nabla\left[h_n(u_\varepsilon)h(u)v(1-\psi_\delta^+-\psi_\delta^-)\right]+ \\
&\int_\Omega h_n(u_\varepsilon)h(u)v(1-\psi_\delta^+-\psi_\delta^-)d\lambda_{\varepsilon,0}^++\int_\Omega h_n(u_\varepsilon)h(u)v(1-\psi_\delta^+-\psi_\delta^-)d\lambda_{\varepsilon,0}^-.
\end{aligned} \tag{4.2.77}$$

Now we pass to the limit in (4.2.77) for $\varepsilon \longrightarrow 0, n \longrightarrow \infty$ and $\delta \longrightarrow 0$.

Since $(1-\psi_\delta^+ - \psi_\delta^-)a(x,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon$ is positive, letting $\delta \longrightarrow 0$ we have

$$\lim_{\delta\longrightarrow 0}\lim_{n\longrightarrow\infty}\sup\lim_{\varepsilon\longrightarrow 0}\sup\frac{1}{n}\int_{\{n<|u_\varepsilon|<2n\}}|h(u)v|\,(1-\psi_\delta^+ - \psi_\delta^-)a(x,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon = 0. \quad (4.2.78)$$

By [97] we know that $T_{2n}(u_\varepsilon) \longrightarrow T_{2n}(u)$ a.e so we have

$$a(x,T_{2n}(u_\varepsilon),\nabla T_{2n}(u_\varepsilon)) \longrightarrow a(x,T_{2n}(u),\nabla T_{2n}(u)).$$

Since $|h_n(u_\varepsilon)| \le 1$ and $h_n(u_\varepsilon) \longrightarrow h_n(u)$ a.e, replacing $a(x,u,\nabla u)$ by $a(x,T_M(u),\nabla T_M(u))$ and letting $n \longrightarrow \infty$, $\delta \longrightarrow 0$ we have

$$\lim_{\delta\longrightarrow 0}\lim_{n\longrightarrow\infty}\lim_{\varepsilon\longrightarrow 0}\int_\Omega h_n(u_\varepsilon)h^{'}(u)v(1-\psi_\delta^+ - \psi_\delta^-)\left[a(x,u_\varepsilon,\nabla u_\varepsilon)\right]\nabla u \quad (4.2.79)$$
$$= \int_\Omega h^{'}(u)v\left(a(x,u,\nabla u)\right)\nabla u.$$

In the same way

$$\lim_{\delta\longrightarrow 0}\lim_{n\longrightarrow\infty}\lim_{\varepsilon\longrightarrow 0}\int_\Omega h_n(u_\varepsilon)h(u)(1-\psi_\delta^+ - \psi_\delta^-)\left(a(x,u_\varepsilon,\nabla u_\varepsilon)\right)\nabla v \quad (4.2.80)$$
$$= \int_\Omega h(u)\left(a(x,u,\nabla u)\right)\nabla v,$$

$$\lim_{\delta\longrightarrow 0}\lim_{n\longrightarrow\infty}\lim_{\varepsilon\longrightarrow 0}\int_\Omega h_n(u_\varepsilon)h(u)v\left[a(x,u_\varepsilon,\nabla u_\varepsilon)\right]\nabla(1-\psi_\delta^+ - \psi_\delta^-) = 0. \quad (4.2.81)$$

Using the point-wise convergence of u_ε and the definition of h_n and the definition of $\psi_\delta^+, \psi_\delta^-$ we obtain

$$\lim_{\delta\longrightarrow 0}\lim_{n\longrightarrow\infty}\lim_{\varepsilon\longrightarrow 0}\int_\Omega f_\varepsilon h_n(u_\varepsilon)h(u)v(1-\psi_\delta^+ - \psi_\delta^-) = \int_\Omega f h(u)v. \quad (4.2.82)$$

To conclude that u is a renormalized solution we have to prove

$$\lim_{n\to\infty}\frac{1}{n}\int_{n<u<2n} a(x,u,\nabla u)\nabla u\varphi = \int_\Omega \varphi d\mu_s^+, \quad (4.2.83)$$

$$\lim_{n\to\infty}\frac{1}{n}\int_{-2n<u<-n} a(x,u,\nabla u)\nabla u\varphi = \int_\Omega \varphi d\mu_s^-, \tag{4.2.84}$$

$\forall\varphi \in C_b^0(\Omega)$.

By the pointwise convergence of u_ε and by Fatou Lemma it follows

$$\limsup_{n\to\infty}\frac{1}{n}\int_{n<u<2n} a(x,u,\nabla u)\nabla u\varphi \leq \int_\Omega \varphi d\mu_s^+, \tag{4.2.85}$$

$$\lim_{n\to\infty}\frac{1}{n}\int_{-2n<u<-n} a(x,u,\nabla u)\nabla u\varphi \leq \int_\Omega \varphi d\mu_s^-, \tag{4.2.86}$$

$\forall\varphi \in C^1(\overline{\Omega})$ with $\varphi \geq 0$.

Since u is a solution in the sense of distribution, taking $h_n(u)\psi$ as test function where h_n is defined in (4.2.68) and $\psi \in C_0^\infty(\Omega)$, letting $\eta \to \infty$, $\delta \to 0$ we obtain

$$\liminf_{n\to\infty}\frac{1}{n}\int_{n<u<2n} a(x,u,\nabla u)\nabla u\varphi \geq \int_\Omega \varphi d\mu_s^+, \tag{4.2.87}$$

$\forall\varphi \in C^1(\overline{\Omega})$ with $\varphi \geq 0$.

If we take $\varphi(1-\psi_\delta^+)\psi_\delta^-$ as test function we have (4.2.84)

Fourth step

Now we want to prove that $T_k(u_\varepsilon)$ converges to $T_k(u)$ strongly in $H_0^1(\Omega)$ for any $k>0$. By (4.0.27) , the point-wise convergence of $u_\varepsilon, \nabla u_\varepsilon$ we have

$$\int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u) \leq \liminf_{\varepsilon\longrightarrow 0}\int_\Omega a(x,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon))\nabla T_k(u_\varepsilon).$$

If we take $h_n(u_\varepsilon)T_k(u_\varepsilon)$ as test function and letting $n\to\infty, \varepsilon\to 0$ we have

$$\limsup_{\varepsilon\to 0}\int_\Omega a(x,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon))\nabla T_k(u_\varepsilon) \leq \int_\Omega a(x,u,\nabla u)\nabla T_k(u), \tag{4.2.88}$$

so we conclude that

$$\lim_{\varepsilon \to 0} \int_\Omega a(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) \nabla T_k(u_\varepsilon) \leq \int_\Omega a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u). \tag{4.2.89}$$

By (4.2.89) we have

$$\lim_{\varepsilon \to 0} \int_\Omega a(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) \nabla T_k(u_\varepsilon) = \int_\Omega a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u). \tag{4.2.90}$$

Because of (4.0.27) and the point-wise convergence of $\nabla T_k(u_\varepsilon)$ [101] $\nabla T_k(u_\varepsilon) \to \nabla T_k(u)$ in $(L^p(\Omega))^N$ strongly. ■

4.2.4 Passage to the limit

The weak solution u_ε to approximated problem (4.2.6) satisfies

$$\begin{cases} -\operatorname{div}(a(x, u_\varepsilon, \nabla u_\epsilon)) = \Phi_\varepsilon - \operatorname{div}(g) & \text{in} \quad \Omega \\ u_\varepsilon \in W_0^{1,p}(\nu) & \text{on} \quad \partial\Omega, \end{cases} \tag{4.2.91}$$

where $\Phi_\varepsilon = f_\varepsilon - H_\varepsilon(x, \nabla u_\varepsilon) + \lambda_\varepsilon^+ - \lambda_\varepsilon^-$.

Using $T_k(u_\varepsilon)$ as test function in (4.2.91) we have for some $\widetilde{M}$ and $\widetilde{L}$

$$\int_\Omega v(x) \left|\nabla T_k(u_\varepsilon)\right|^p \leq \widetilde{M} k + \widetilde{L}, \tag{4.2.92}$$

for every $k > 0$ and every $\varepsilon > 0$.

We know that there exists a function u such that u_ε converges to u a.e., ∇u exists a.e., $T_k(u) \in W_0^{1,p}(\nu)$ and ∇u_ε converges to ∇u almost everywhere [101] By Fatou Lemma and by (4.2.92) we deduce that

$$\int_\Omega v(x) \left|\nabla T_k(u)\right|^p \leq \widetilde{M} k + \widetilde{L}. \tag{4.2.93}$$

From (4.2.93) and by Lemma 4.1.8 we deduce that $|\nabla u|^{p-1} \in L^q(\nu)$ and $|u|^{p-1} \in L^\gamma(\nu)$. By a result proved in [101] we know that $a(x, u_\varepsilon, \nabla u_\varepsilon) \longrightarrow a(x, u, \nabla u)$ in $L^1(\Omega)$ strongly.

Therefore $b(x)\,|\nabla u_\varepsilon|^{p-1} \longrightarrow b(x)\,|\nabla u|^{p-1}$ almost everywhere in Ω and for every measurable set $E \subset \Omega$

$$\int_E |b(x)| \left|\nabla u_\varepsilon^{p-1}\right| \leq c\, \|b(x)\|_{L^\tau(E)} \left\|v^{-1}\right\|_{L^t(E)}^{\frac{1}{q}}.$$

By Vitali-Lebesgue theorem we have $b(x)\,|\nabla u_\varepsilon|^{p-1} \longrightarrow b(x)\,|\nabla u|^{p-1}$ in $L^1(\Omega)$ strongly. Thanks to these results the weak solution u_ε of

$$\begin{cases} -\operatorname{div}(a(x, u_\varepsilon, \nabla u_\varepsilon)) = f_\varepsilon - H_\varepsilon(x, \nabla u_\varepsilon) - \operatorname{div}(g) + \lambda_\varepsilon^+ - \lambda_\varepsilon^- \\ u_\varepsilon \in W_0^{1,p}(\nu) \end{cases} \tag{4.2.94}$$

is also a renormalized solution to (4.2.94) and the stability results (Theorem 4.2.3) asserts that u is also a renormalized solution to the problem

$$\begin{cases} -\operatorname{div}(a(x, u, \nabla u)) + H(x, \nabla u) = f - \operatorname{div}(g) + \mu_s^+ - \mu_s^- & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which proves the theorem.

CHAPTER V

PARABOLIC EQUATIONS

This chapter is devoted to the study of nonlinear parabolic problems when tha data are L^1-function or more in general a Radon measure with bounded variation. In the first section we recall the definition of functional spaces which are essential to study of parabolic problems and the different notion of solutions. In Section II we prove the existence of a renormalized solution for a class of nonlinear parabolic equations having lower order terms.

5.0.5 Spaces of functions and notion of solutions

In this section we recall some feature about spaces of functions with values in a Banach space. Let be V a real Banach space, for $1 \leq p < \infty$, let us denote by $L^p((a,b);V)$ the space of measurable functions $u : [a,b] \to V$ such that

$$\|u\|_{L^p((a,b);V)} = \left(\int_a^b \|u\|_V^p \, dt \right)^{\frac{1}{p}} < \infty,$$

and $L^\infty((a,b);V)$ the space of measurable functions such that:

$$\|u\|_{L^\infty((a,b);V)} = \operatorname*{ess\,sup}_{[a,b]} \|u\|_V < \infty.$$

Let us recall that, for $1 \leq p \leq \infty$, $L^p((a,b);V)$ is a Banach space. Moreover if for $1 \leq p < \infty$ and V', the dual space of V, is separable, then the dual space of $L^p((a,b);V)$ can be identified with $L^{p'}(a,b;V')$.

For our purpose V will mainly be either the Lebesgue space $L^p(\Omega)$ or the Sobolev space $W_0^{1,p}(\Omega)$, with $1 \leq p < \infty$. We denote by $L^p((a,b);W_0^{1,p}(\Omega))$ the space of all functions $u : \Omega \times [a,b] \to \mathbb{R}$ which belong to $L^p(\Omega \times (a,b))$ and such that $\nabla u =$

$(u_{x_1}, ..., u_{x_N})$ belongs to $(L^p(\Omega \times (a,b))^N$ (often, for simplicity, we will indicate this space only by $L^p(\Omega \times (a,b))$. Moreover,

$$\left(\int_a^b |\nabla u|^p \; dxdt \right)^{\frac{1}{p}}$$

defines an equivalent norm by Poincaré's inequality.

Given a function in $L^p((a,b);V)$ it is possible to define a time derivative of u in the space of vector valued distributions $\mathcal{D}'((a,b);V)$ which is the space of linear continuous functions from $C_0^\infty(a,b)$ into V. In fact, the definition is the following:

$$\langle u_t, \psi \rangle = -\int_a^b u \, \psi_t \; dt \,, \qquad \forall \;\; \psi \in C_0^\infty(a,b),$$

where the equality is meant in V. In the following, we will also use sometimes the notation $\frac{\partial u}{\partial t}$ instead of u_t. We recall the following classical embedding result (see [46] for the proof)

Theorem 5.0.4 *Let H be an Hilbert space such that:*

$$V \underset{dense}{\hookrightarrow} H \hookrightarrow V' \,.$$

Let $u \in L^p((a,b);V)$ be such that u_t, defined as above in the distributional sense, belongs to $L^{p'}((a,b);V')$. Then u belongs to $C([a,b];H)$.

This result also allows us to deduce, for functions u and v enjoying these properties, the integration by parts formula:

$$\int_a^b \langle v, u_t \rangle \; dt + \int_a^b \langle u, v_t \rangle \; dt = (u(b), v(b)) - (u(a), v(a)) \,, \tag{5.0.95}$$

where $\langle \cdot, \cdot \rangle$ is the duality between V and V' and $(\cdot, \cdot)$ the scalar product in H. Notice that (5.0.95) makes sense thanks to Theorem 5.0.4. Its proof relies on the fact that $C_0^\infty((a,b);V)$ is dense in the space of functions $u \in L^p((a,b);V)$ such that $u_t \in L^{p'}((a,b);V')$ endowed with the norm $\|u\| = \|u\|_{L^p((a,b);V)} + \|u_t\|_{L^{p'}((a,b);V')}$, together with the fact that (5.0.95) is true for $u,\ v \in C_0^\infty((a,b);V)$ by the theory of integration and derivation in Banach spaces.

Now we recall some further results that will be very useful in what follows. The first one contains a generalization of the integration by parts formula (5.0.95) where the time derivative of a function is less regular, and its proof can be found in [61].

Lemma 5.0.5 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous piecewise C^1 function such that $f(0) = 0$ and f' is zero away from a compact set of $\mathbb{R}$. Let us denote $F(s) = \int_0^s f(r)dr$. If $u \in L^p((0,T);W_0^{1,p}(\Omega))$ is such that $u_t \in L^{p'}((0,T);W^{-1,p'}(\Omega)) + L^1(Q_T)$ and if $\psi \in C^\infty(\overline{Q})$, then we have*

$$\int_0^T \langle u_t, f(u)\psi\rangle \; dt = \int\limits_\Omega F(u(T))\psi(T)\; dx - \int\limits_\Omega F(u(0))\psi(0)\; dx - \iint\limits_{Q_T} \psi_t\; F(u)\; dxdt.$$

Now we state an embedding theorem, well-known Gagliardo-Nirenberg embedding theorem, that will play a central role in our work ([90]).

Theorem 5.0.6 (Gagliardo-Nirenberg) *Let v be a function in $W_0^{1,q}(\Omega) \cap L^\rho(\Omega)$ with $q \geq 1$, $\rho \geq 1$. Then there exists a positive constant C, depending on N, q and ρ, such that*

$$\|v\|_{L^\gamma(\Omega)} \leq C\|\nabla v\|^\theta_{(L(\Omega))^N}\|v\|^{1-\theta}_{L^\rho(\Omega)}\,,$$

for every θ and γ satisfying

$$0 \leq \theta \leq 1, \quad 1 \leq \gamma \leq +\infty, \quad \frac{1}{\gamma} = \theta\left(\frac{1}{q} - \frac{1}{N}\right) + \frac{1-\theta}{\rho}\,.$$

An immediate consequence of the previous result is the following embedding result:

Corollary 5.0.7 *Let $v \in L^q((0,T);L^q(\Omega)) \cap L^\infty((0,T);L^\rho(\Omega))$, with $q \geq 1$, $\rho \geq 1$. Then $v \in L^\sigma(\Omega)$ with $\sigma = q\frac{N+\rho}{N}$ and*

$$\iint\limits_{Q_T} |v|^\sigma\; dxdt \leq C\|v\|^{\frac{\rho q}{N}}_{L^\infty(0,T;L^\rho(\Omega))}\iint\limits_{Q_T}|\nabla v|^q\; dxdt\,.$$

Now we give some basic result about nonlinear parabolic problems and we introduce the relative notion of solutions.

Let be Ω a bounded open set of R^N, $N \geq 2$, Q_T is the cylinder $\Omega \times (0,T)$, where T is a real positive number. Let us consider the nonlinear parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) = f & \text{in} \quad Q_T \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{5.0.96}$$

where $a(x,t,s,\xi) : \Omega \times (0,T) \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function such that

$$a(x,t,s,\xi)\xi \geq \alpha\, |\xi|^p\,, \quad \alpha > 0, \tag{5.0.97}$$

$$|a(x,t,s,\xi)| \leq \nu \left[h(x,t) + |s|^{p-1} + |\xi|^{p-1}\right], \quad \nu > 0,\ h(x,t) \in L^{p'}(Q_T), \tag{5.0.98}$$

$$(a(x,t,s,\xi) - a(x,t,s,\varrho), \xi - \varrho) > 0, \quad \xi \neq \varrho \tag{5.0.99}$$

for almost every $x \in \Omega$, $t \in (0,T)$ and for every $s \in \mathbb{R}$, ξ, $\varrho \in \mathbb{R}^N$.

If $f \in L^{p'}(Q_T)$ and $u_0 \in L^2(\Omega)$ then problem (5.0.96) admits a unique solution $u \in C(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega)$ in the weak sense, that is

$$-\int_\Omega u_0\varphi(0)dx - \int_0^T < \varphi_t, u > dt + \iint_{Q_T} a(x,t,u,\nabla u)\cdot\nabla\varphi dxdt = \int_0^T < f,\varphi >_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)} dt,$$

for all $\varphi \in L^p(0,T;W_0^{1,p}(\Omega)$ and $\varphi_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)$ such that $\varphi(T) = 0$ ([81]).

When the datum is not in the dual space it is not possible to use a variational framework ([81]). For this reason in [45] (see also [30]) the following notion of solution have been introduced :

Definition 5.0.8 *Let be f a bounded Borel measure. A function u is a solution to* (5.0.96) *in the sense of distribution if*

$$u \in L^1(0,T;W_0^{1,1}(\Omega)),$$

$$a(x,t,u,\nabla u) \in L^1(Q_T),$$

and u satisfies the equation (5.0.96) *in the following weak sense:*

$$-\iint_{Q_T} u\frac{\partial\varphi}{\partial t}dxdt + \iint_{Q_T} a(x,t,u,\nabla u)\cdot\nabla\varphi dxdt = \iint_{Q_T}\varphi df,$$

for every $\varphi \in C^\infty(\bar{Q}_T)$ such that $\varphi = 0$ in a neighborhood of $\partial\Omega \times (0,T) \cup (\Omega \times \{T\})$.

The authors proved the existence of at least a solution of (5.0.96) in $L^q(0,T;W_0^{1,q}(\Omega))$, with $q < \frac{N(p-1)+p}{N+1}$ and p real number such that $p > \frac{2N+1}{N+1}$. Existence results have been also given in [30] where the sommability of the solution with respect to the space and time is more precise. In particular the solution belongs to the space $L^r(0,T;W_0^{1,q}(\Omega))$, where r and q are two real number such that $1 \le q < \min\left\{\frac{N(p-1)}{N-1},\ p\right\}$, $1 \le r \le p$ and $\frac{N(p-2)+p}{r} + \frac{N}{q} > N+1$. Other existence result for this kind of problem have also been proved in [95] where a lower order term is considered. Unfortunately, as in the elliptic case, the sommability is not enough to get uniqueness result. These difficulties are overcame in the linear case [106] by duality method. To be more explicit let us consider the following linear parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(M(x,t)\nabla u)) = f & \text{in} \quad Q_T \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ u(x,0) = 0 & \text{in} \quad \Omega, \end{cases} \tag{5.0.100}$$

where $f \in L^1(Q_T)$ and M is a matrix with bounded, measurable coefficients satisfying the condition (5.0.97) with $p = 2$. Let be

$$\begin{cases} \frac{\partial w}{\partial t} - \operatorname{div}(M^*(x,t)\nabla w)) = g & \text{in} \quad Q_T \\ w(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ w(x,0) = 0 & \text{in} \quad \Omega, \end{cases}$$

the adjoint problem where M^* is the adjoint matrix and $g \in L^\infty(Q_T)$. A function u is said to be a "solution by duality" if

$$\iint_{Q_T} ugdxdt = \iint_{Q_T} fwdxdt.$$

When $f \in L^1(Q_T)$ e $u_0 \in L^1(\Omega)$ problem (5.0.100) admits a unique solution by duality. In the nonlinear case, with $f \in L^1(Q_T)$, the problem is not well defined and uniqueness is not always guaranteed so the equivalent notion of entropy and renormalized solutions have been introduced [99] and [23], [61].

Definition 5.0.9 *Let be $f \in L^1(Q_T)$ and $u_0 \in L^1(\Omega)$. A function $u \in C\left([0,T]\,; L^1(\Omega)\right)$ is said to be an entropy solution if*

$$T_k(u) \in L^p((0,T); W_0^{1,p}(\Omega)), \quad k > 0$$

and

$$\int_\Omega \Theta_k(u-\phi)(T)dx - \int_\Omega \Theta_k(u_0-\phi)(T)dx + \int_0^T < \phi_t, T_k(u-\varphi) > dt +$$

$$\int_0^T \int_\Omega a(x,t,u,\nabla u)\nabla T_k(u-\phi)dxdt \le \int_0^T \int_\Omega f T_k(u-\phi)dxdt,$$

for all $k > 0$ and $\phi \in L^p((0,T); W_0^{1,p}(\Omega)) \cap L^\infty(Q_T) \cap C\left([0,T]\,; L^1(\Omega)\right)$ such that $\phi_t \in L^{p'}((0,T); W^{-1,p'}(\Omega)) + L^1(\Omega)$, where $\Theta_k(s) = \int_0^s T_k(\sigma)d\sigma$.

Definition 5.0.10 *A real function u defined in Q_T is a renormalized solution of* (5.0.96) *if it satisfies the following conditions:*

$$u\text{: } Q_T \longrightarrow \bar{\mathbb{R}} \text{ is a measurable function on } Q_T \text{ and } u \in L^\infty((0,T); L^1(\Omega)),$$

$$T_k(u) \in L^p((0,T); W_0^{1,p}(\Omega)), \text{ for any } k > 0,$$

$$\frac{1}{n}\lim_{n\to+\infty} \int_{\{n\le|u|\le 2n\}} |\nabla u|^p = 0,$$

and if for every function $S \in W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has a compact support

$$\frac{\partial S(u)}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)S'(u)) + S''(u)a(x,t,u,\nabla u)\nabla u + = fS'(u) \quad \text{in } D'(\Omega),$$

and

$$S(u)(t=0) = S(u_0) \quad \text{in } \Omega.$$

5.1 Existence results for operators with lower order terms

In this section we prove existence results for nonlinear parabolic problems with lower order terms. In particular we consider a nonlinear parabolic problem which can be formally written as

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) + \operatorname{div}(K(x,t,u)) = f & \text{in} \quad Q_T \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{5.1.1}$$

where $a(x,t,s,\xi) : \Omega \times (0,T) \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$, $K(x,t,s) : \Omega \times (0,T) \times \mathbb{R} \longrightarrow \mathbb{R}^N$ are Carathéodory functions satisfying (5.0.97) , (5.0.99),

$$|a(x,t,s,\xi)| \leq \nu \left[h(x,t) + |\xi|^{p-1} \right], \quad \nu > 0,\ h(x,t) \in L^{p'}(Q_T), \tag{5.1.2}$$

$$|K(x,t,\eta)| \leq c(x,t)\, |\eta|^{\gamma}, \tag{5.1.3}$$

$$c(x,t) \in (L^{\tau}(Q_T))^N, \quad \tau > \frac{N+p}{p-1}, \tag{5.1.4}$$

$$\gamma = \frac{N+2}{N+p}(p-1), \tag{5.1.5}$$

for almost every $x \in \Omega$, $t \in (0,T)$, for every $s \in \mathbb{R}$ and for every $\xi \in \mathbb{R}^N$.
Moreover

$$f \in L^1(Q_T), \tag{5.1.6}$$

$$u_0 \in L^1(\Omega). \tag{5.1.7}$$

Under these assumptions, the above problem does not admit, in general, a weak solution since the field $a(x,t,u,\nabla u)$ and $K(x,t,u)$ do not belong to $\left(L^{p'}(\Omega)\right)^N$. To overcome this difficulty we refer to the notion of renormalized solution (see [30], [25], [48]). The existence of a renormalized solution to problem (5.1.1) with $c(x,t) = 0$ has been proved in [22], [23] where $a(x,t,s,\xi)$ is indipendent of s and in [25] where a lower order term $\operatorname{div}(\Phi(u))$, with Φ continuous function in $\mathbb{R}^N$ is considered. Here we prove the existence of a renormalized solution for problem (5.1.1) ; this result is contained in [55]. The proof consists of several steps. First of all we introduce the approximated problem, then we prove an apriori estimate for the gradient of its solution following an idea contained in [77]. The estimate can be easily obtained if we consider a subcylinder $\Omega \times (0,t)$, $t \in (0,T)$. For this reason we decompose the entire cylinder Q_T into a finite number of subcylinder. Finally we pass to the limit using the same procedure followed in [25].

Definition 5.1.1 *A real function u defined in Q_T is a renormalized solution of* (5.1.1) *if it satisfies the following conditions:*

$$u \in L^\infty((0,T);L^1(\Omega)), \tag{5.1.8}$$

$$T_k(u) \in L^p((0,T);W_0^{1,p}(\Omega)), \ \textit{for any } k>0, \tag{5.1.9}$$

$$\lim_{n\to+\infty} \int_{\{n\le|u|\le n+1\}} a(x,t,u,\nabla u)\nabla u = 0, \tag{5.1.10}$$

and if for every function $S \in W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has a compact support

$$\frac{\partial S(u)}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)S'(u)) + S''(u)a(x,t,u,\nabla u)\nabla u +$$

$$+\operatorname{div}(K(x,t,u)S'(u)) - S''(u)K(x,t,u)\nabla u + H(x,t,\nabla u)S'(u) = fS'(u) \quad \textit{in } D'(\Omega) \tag{5.1.11}$$

and

$$S(u)(t=0) = S(u_0) \quad \textit{in } \Omega. \tag{5.1.12}$$

We observe that this equation can be formally obtained through pointwise multiplication of (5.1.1) by $S'(u)$ and all terms in (5.1.11) have a meaning since $T_k(u) \in L^p((0,T);W_0^{1,p}(\Omega))$, for any $k>0$ and S' has a compact support. In particular, there exists $M>0$ such that *supp*$S' \subset [-M,M]$ and

$$\iint_{Q_T} K(x,t,u)S'(u)\nabla u = \iint_{Q_T} K(x,t,T_M(u))S'(u)\nabla T_M(u),$$

and such integral is well defined thanks to the assumptions $(5.1.3)-(5.1.14)$ and the fact that $T_k(u) \in L^p((0,T);W_0^{1,p}(\Omega))$ and $S \in W^{2,\infty}(\mathbb{R})$.

In this section we also study the following nonlinear parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div} a(x,t,u,\nabla u) + H(x,t,u) = f & \text{in} \quad Q_T \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{5.1.13}$$

$a(x,t,s,\xi) : \Omega \times (0,T) \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function satisfying $(5.0.97)$, $(5.0.99)$ and $(5.1.2)$, $H(x,t,\xi) : \Omega \times (0,T) \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function such that

$$|H(x,t,\eta)| \leq b(x,t) \left|\nabla u\right|^{\delta}, \tag{5.1.14}$$

$$b(x,t) \in L^r(Q_T), \quad \text{with } r > \frac{p(N+1)-N}{(p-1)(N+1)}, \tag{5.1.15}$$

$$\delta \leq \frac{p(N+1)-N}{N+2}. \tag{5.1.16}$$

Finally f and u_0 are two functions satisfying $(5.1.6)$, $(5.1.7)$.

In [77], under the assumption $p \geq 2 - \frac{1}{N+1}$, Porzio proved the existence of a solution in the sense of distribution to problem (5.1.13) which belongs to $L^m((0,T);W_0^{1,m}(\Omega))$ for $m < \frac{p(N+1)-N}{N+1}$. Let us explicitly remark that the assumption on p assures that $\frac{p(N+1)-N}{N+1} > 1$. To overcome this assumption on p we refer again to the notion of renormalized solution.

Definition 5.1.2 *A real function u defined in Q_T is a renormalized solution of* (5.1.13) *if it satisfyies the following conditions* $(5.1.8)-(5.1.10)$ *and if for every function $S \in W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has a compact support*

$$\frac{\partial S(u)}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)S'(u)) + S''(u)a(x,t,u,\nabla u)\nabla u+$$
$$+H(x,t,\nabla u)S'(u) = fS'(u) \quad \text{in } D'(\Omega) \tag{5.1.17}$$

and

$$S(u)(t=0) = S(u_0) \quad \text{in } \Omega. \tag{5.1.18}$$

In this section we prove the existence of a renormalized solution respectively to problem (5.1.1) and (5.1.13); such result is contained in [55]. We think that the existence of a renormalized solution could be shown also for a nonlinear parabolic problem involving both lower order term -div $K(x,t,u)$ and $H(x,t,u)$. This question is analized in [71].

In order to prove the existence results, we prove a technical lemma (we follow the same method used in [13]), that yields two estimates for $|u_\varepsilon|^{p-1}$ and $|\nabla u_\varepsilon|^{p-1}$ in the Lorentz spaces $L^{\frac{p(N+1)-N}{N(p-1)},\infty}(Q_T)$ and $L^{\frac{p(N+1)-N}{(N+1)(p-1)},\infty}(Q_T)$ respectively. Moreover by imbedding theorems, these apriori bounds imply two estimates in the Lebesgue spaces $L^m(Q_T)$ and $L^s(Q_T)$ with $m < \frac{p(N+1)-N}{N(p-1)}$ and $s < \frac{p(N+1)-N}{(N+1)(p-1)}$.

Lemma 5.1.3 *Assume that Ω is an open set of $\mathbb{R}^N$ of finite measure and $1 < p < +\infty$. Let be u a measurable function satisfying $T_k(u) \in L^p((0,T); W_0^{1,p}(\Omega)) \cap L^\infty((0,T); L^2(\Omega))$for every positive k and such that*

$$\sup_{t\in(0,T)} \int_\Omega |T_k(u)|^2 + \iint_{Q_T} |\nabla T_k(u)|^p \leq Mk, \quad \forall k > 0, \tag{5.1.19}$$

where M is a positive constant. Then

$$\left\| |u|^{p-1} \right\|_{L^{\frac{p(N+1)-N}{N(p-1)},\infty}(Q_T)} \leq CM^{\left(\frac{p}{N}+1\right)\frac{N}{N+p'}} |Q_T|^{\frac{1}{p'}\frac{N}{N+p'}}, \tag{5.1.20}$$

$$\left\| |\nabla u|^{p-1} \right\|_{L^{\frac{p(N+1)-N}{(N+1)(p-1)},\infty}(Q_T)} \leq CM^{\frac{(N+2)(p-1)}{p(N+1)-N}}, \tag{5.1.21}$$

where C is a constant which depends only on N and p.

Proof. By Gagliardo - Niremberg and by (5.1.19) we have

$$k^{\frac{p(N+2)}{N}} meas\{(x,t)\in Q_T : |u| > k\} \leq C \sup_{t\in(0,T)} \left(\int_\Omega |T_k(u)|^2\right)^{\frac{p}{N}} \tag{5.1.22}$$

$$\times \iint_{Q_T} |\nabla T_k(u)|^p \leq C\,(Mk)^{\frac{p}{N}+1}, \tag{5.1.23}$$

that is

$$meas\left\{(x,t)\in Q_T : |u|^{(p-1)} > k\right\} \leq CM^{\frac{p}{N}+1} k^{-\frac{N+p'}{N}(p-1)}. \tag{5.1.24}$$

By (5.1.24) we deduce that $|u|^{(p-1)} \in L^{\frac{p(N+1)-N}{N(p-1)},\infty}(Q_T)$. Furthermore, by (5.1.24) we get

$$\left\| |u|^{p-1} \right\|_{L^{\frac{p(N+1)-N}{N(p-1)},\infty}(Q_T)} = \sup_{k>0} k \left[meas \left\{ (x,t) \in Q_T : |u|^{(p-1)} > k \right\} \right]^{\frac{N}{N+p'}} \leq$$

$$\leq \sup_{0<k<k_0} k \; meas \left\{ (x,t) \in Q_T : |u|^{(p-1)} > k \right\}^{\frac{N}{N+p'}} +$$

$$+ \sup_{k>k_0} k \; meas \left\{ (x,t) \in Q_T : |u|^{(p-1)} > k \right\}^{\frac{N}{N+p'}} \leq$$

$$\leq k_0 \, |Q_T|^{\frac{N}{N+p'}} + C \left[M^{\frac{p}{N}+1} k_0^{-\frac{N+p'}{N}(p-1)} \right]^{\frac{N}{N+p'}}.$$

Taking $k_0 = \frac{M^{\left(\frac{p}{N}+1\right)\frac{N}{N+p'}}}{|Q_T|^{\frac{N}{p}\frac{1}{N+p'}}}$ we have (5.1.20).

Now we prove the estimate involving the gradient of u. For every $\lambda > 0$ and for every $k > 0$ we have

$$meas \{(x,t) \in Q_T : |\nabla u| > \lambda\} \leq meas \{(x,t) \in Q_T : |\nabla u| > \lambda \text{ and } |u| \leq k\} +$$
$$meas \{(x,t) \in Q_T : |\nabla u| > \lambda \text{ and } |u| > k\}.$$

By (5.1.19) we know that

$$\lambda^p meas \{(x,t) \in Q_T : |\nabla u| > \lambda \text{ and } |u| \leq k\} \leq \iint_{|u_\varepsilon| \leq k} \lambda^p \leq \iint_{Q_T} |\nabla T_k(u)|^p \leq Mk,$$

that is

$$meas \left\{ (x,t) \in Q_T : |\nabla u|^{(p-1)} > \lambda \text{ and } |u| \leq k \right\} \leq \frac{Mk}{\lambda^{p'}}. \tag{5.1.25}$$

Thanks to (5.1.23) we obtain

$$meas \left\{ (x,t) \in Q_T : |\nabla u|^{(p-1)} > \lambda \text{ and } |u| > k \right\} \leq C M^{\frac{p}{N}+1} k^{-\frac{p(N+1)-N}{N}}. \tag{5.1.26}$$

From (5.1.25), (5.1.26) we deduce that

$$meas \left\{ (x,t) \in Q_T : |\nabla u|^{(p-1)} > \lambda \right\} \leq \frac{Mk}{\lambda^{p'}} + C M^{\frac{p}{N}+1} k^{-\frac{p(N+1)-N}{N}}. \tag{5.1.27}$$

If we take $k = M^{\frac{1}{N+1}} \lambda^{\frac{N}{(N+1)(p-1)}}$ (5.1.27) becomes

$$meas\left\{(x,t) \in Q_T : |\nabla u|^{(p-1)} > \lambda\right\} \le C \frac{M^{\frac{N+2}{N+1}}}{\lambda^{\frac{p(N+1)-N}{(N+1)(p-1)}}}. \tag{5.1.28}$$

By (5.1.28) we have

$$\left\| |\nabla u|^{p-1} \right\|_{L^{\frac{p(N+1)-N}{(N+1)(p-1)},\infty}(Q_T)} = \sup_{\lambda>0} \lambda \left[meas\left\{(x,t) \in Q_T : |\nabla u|^{(p-1)} > \lambda\right\}\right]^{\frac{(N+1)(p-1)}{p(N+1)-N}} \tag{5.1.29}$$

$$\le \sup_{0<\lambda<\lambda_0} \lambda \; meas\left\{(x,t) \in Q_T : |\nabla u|^{(p-1)} > \lambda\right\}^{\frac{(N+1)(p-1)}{p(N+1)-N}} +$$

$$+ \sup_{\lambda>\lambda_0} \lambda \; meas\left\{(x,t) \in Q_T : |\nabla u|^{(p-1)} > \lambda\right\}^{\frac{(N+1)(p-1)}{p(N+1)-N}} \le$$

$$\le \lambda_0 \, |Q_T|^{\frac{(N+1)(p-1)}{p(N+1)-N}} + CM^{\frac{(N+2)(p-1)}{p(N+1)-N}}.$$

If we choose $\lambda_0 = M^{\frac{(N+2)(p-1)}{p(N+1)-N}} |Q_T|^{-\frac{(N+1)(p-1)}{p(N+1)-N}}$ we have (5.1.21). ∎

5.1.1 Existence result for problem (5.1.1)

The main result of this section is the following existence result which proof is contained in [55].

Theorem 5.1.4 *Under the assumptions* (5.0.97), (5.0.99), (5.1.2), (5.1.3) – (5.1.7) *there exists at least renormalized solution to problem* (5.1.1).

Proof

The proof consists of several steps. In the first step we consider the approximated problem. In the second step we obtain apriori estimates for the solutions and its gradient. In Step 3 we followed the idea contained in [25] and we show that the limit of the solution of the approximated problem belongs to $L^\infty((0,T); L^1(\Omega))$. In Step 4 we define a time regularization of $T_k(u)$. In Step 5 we prove a lemma that is essential in order to develop in Step 6 the monotonicity method. In Step 7 we show that the limit of the solution of the approximated problem satisfyies conditions (5.1.10), (5.1.11), (5.1.12) .

Step 1. For $\varepsilon > 0$ let us consider the following approximated problem

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t} - \operatorname{div}(a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)) + \operatorname{div}\left(K_\varepsilon(x,t,u_\varepsilon)\right) = f_\varepsilon & \text{in} \quad Q_T \\ u_\varepsilon(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ u_\varepsilon(x,0) = (u_0)_\varepsilon(x) & \text{in} \quad \Omega, \end{cases} \tag{5.1.30}$$

where

$$a_\varepsilon(x,t,s,\xi) = a(x,t,T_{\frac{1}{\varepsilon}}(s),\xi) \text{ a.e. in } Q_T, \ s \in \mathbb{R}, \ \xi \in \mathbb{R}^N, \tag{5.1.31}$$

$$\begin{gathered} f_\varepsilon \text{ is a sequence of function in } L^{p'}(Q_T) \\ \text{such that } f_\varepsilon \to f \text{ a.e. and strongly in } L^1(Q_T), \end{gathered} \tag{5.1.32}$$

$$\begin{gathered} (u_0)_\varepsilon \text{ is a sequences of function in } L^2(\Omega) \text{ such} \\ \text{that } (u_0)_\varepsilon \to u_0 \text{ a.e. and strongly in } L^1(\Omega), \end{gathered} \tag{5.1.33}$$

and

$$\begin{gathered} |K_\varepsilon(x,t,\eta)| \le |K(x,t,\eta)| \le c(x,t)\,|\eta|^\gamma \\ \text{and } |K_\varepsilon(x,t,\eta)| \le c(x,t)\left(\tfrac{1}{\varepsilon}\right)^\gamma . \end{gathered} \tag{5.1.34}$$

By well-known result (e. g. [81]) there exists at least a weak solution to (5.1.30) which belongs to $L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$.

Step 2 If we take $T_k(u_\varepsilon)$ as test function in (5.1.30) and we integrate between $(0,t)$ where $t \in (0,t_1)$ is arbitrary fixed and $t_1 \in (0,T)$ will be choosen later, using condition (5.1.34) we have

$$\iint_{Qt} (u_\varepsilon)_t T_k(u_\varepsilon) + \iint_{Qt} a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla T_k(u_\varepsilon) \le$$

$$\iint_{Qt} c(x,t)\ |u_\varepsilon|^\gamma\,|\nabla T_k(u_\varepsilon)| + \iint_{Qt} f_\varepsilon T_k(u_\varepsilon). \tag{5.1.35}$$

On the other hand, if we denote by

$$\psi_k(s) = \int_0^s T_k(\sigma)d\sigma = \begin{cases} \frac{r^2}{2} & \text{if} \quad |r| \leq k \\ k\,|r| - \frac{k^2}{2} & \text{if} \quad |r| \geq k \end{cases}, \tag{5.1.36}$$

we have

$$\iint\limits_{Qt} (u_\varepsilon)_t T_k(u_\varepsilon) = \int_\Omega \psi_k(u_\varepsilon(t)) - \int_\Omega \psi_k\left((u_0)_\varepsilon\right). \tag{5.1.37}$$

Moreover it results

$$\frac{1}{2}\left|T_k(s)\right|^2 \leq \psi_k(s) \leq k\,|s|\,, \quad \forall k > 0. \tag{5.1.38}$$

Using (5.1.38) , by(5.1.37) we get

$$\iint\limits_{Qt} (u_\varepsilon)_t T_k(u_\varepsilon) \geq \frac{1}{2}\int_\Omega \left|T_k(u_\varepsilon)\right|^2 - k\int_\Omega \left|(u_0)_\varepsilon\right|. \tag{5.1.39}$$

By (5.1.35), (5.1.39) , (5.0.97) we obtain

$$\frac{1}{2}\int_\Omega \left|T_k(u_\varepsilon)\right|^2 + \alpha \iint\limits_{Qt} \left|\nabla T_k(u_\varepsilon)\right|^p \leq \iint\limits_{Qt} c(x,t)\ \ |u_\varepsilon|^\gamma \left|\nabla T_k(u_\varepsilon)\right| +$$

$$+k\int_\Omega \left|(u_0)_\varepsilon\right| + k\iint\limits_{Qt} f_\varepsilon. \tag{5.1.40}$$

If we take the supremum for $t \in (0, t_1)$ and we define

$$M = \left\|u_0\right\|_{L^1(\Omega)} + \sup_\varepsilon \left\|f_\varepsilon\right\|_{L^1(Q_T)}, \tag{5.1.41}$$

inequality (5.1.40) becomes

$$\frac{1}{2}\sup_{t\in(0,t_1)}\int_\Omega \left|T_k(u_\varepsilon)\right|^2 + \alpha \iint\limits_{Q_{t_1}} \left|\nabla T_k(u_\varepsilon)\right|^p \leq Mk + \iint\limits_{Qt_1} c(x,t)\ \ |u_\varepsilon|^\gamma \left|\nabla T_k(u_\varepsilon)\right|. \tag{5.1.42}$$

By Gagliardo-Niremberg and Young inequalities we have

$$\iint\limits_{Qt_1} c(x,t)\ \ |u_\varepsilon|^\gamma \left|\nabla T_k(u_\varepsilon)\right| \leq C\frac{\gamma}{N+2}\left\|c(x,t)\right\|_{L^\tau(Q_{t_1})} \sup_{t\in(0,t_1)}\int_\Omega \left|T_k(u_\varepsilon)\right|^2 +$$

$$+C\frac{N+2-\gamma}{N+2}\left\|c(x,t)\right\|_{L^\tau(Q_{t_1})}\left(\iint\limits_{Q_{t_1}} \left|\nabla T_k(u_\varepsilon)\right|^p\right)^{\left(\frac{1}{p}+\frac{N\gamma}{(N+2)p}\right)\frac{N+2}{N+2-\gamma}}. \tag{5.1.43}$$

Since $\gamma = \frac{(N+2)}{N+p}(p-1)$, using (5.1.42) and (5.1.43) we obtain

$$\frac{1}{2}\sup_{t\in(0,t_1)}\int_\Omega |T_k(u_\varepsilon)|^2+\alpha\iint\limits_{Q_{t_1}} |\nabla T_k(u_\varepsilon)|^p \leq Mk+C\frac{\gamma}{N+2}\left\|c(x,t)\right\|_{L^\tau(Q_{t_1})}\sup_{t\in(0,t_1)}\int_\Omega |T_k(u_\varepsilon)|^2$$

$$+C\frac{N+2-\gamma}{N+2}\left\|c(x,t)\right\|_{L^\tau(Q_{t_1})}\iint\limits_{Q_{t_1}} |\nabla T_k(u_\varepsilon)|^p\,,$$

that is equivalent to

$$\left[\frac{1}{2}-C\frac{\gamma}{N+2}\left\|c(x,t)\right\|_{L^\tau(Q_{t_1})}\right]\sup_{t\in(0,t_1)}\int_\Omega |T_k(u_\varepsilon)|^2+$$

$$+\left[\alpha-\frac{N+2-\gamma}{N+2}\left\|c(x,t)\right\|_{L^\tau(Q_{t_1})}\right]\iint\limits_{Q_{t_1}} |\nabla T_k(u_\varepsilon)|^p \leq Mk.$$

If we choose t_1 such that

$$\frac{1}{2}-C\frac{\gamma}{N+2}\left\|c(x,t)\right\|_{L^\tau(Q_{t_1})} \tag{5.1.44}$$

and

$$\alpha-\frac{N+2-\gamma}{N+2}\left\|c(x,t)\right\|_{L^\tau(Q_{t_1})} \tag{5.1.45}$$

are positive. Let us denote by C the minimum between (5.1.44) and (5.1.45), we have

$$\sup_{t\in(0,t_1)}\int_\Omega |T_k(u_\varepsilon)|^2+\iint\limits_{Q_{t_1}} |\nabla T_k(u_\varepsilon)|^p \leq CMk. \tag{5.1.46}$$

The last inequality allows us to deduce apriori bounds for the solutions u_ε and its gradient ∇u_ε. In fact, by Lemma 5.1.3 we obtain apriori estimates for u_ε and ∇u_ε on the cylinder Q_{t_1} in term of M. Here we use the same technique used in [77]. If we consider a partition of the entire interval $[0,T]$ into a finite number of intervals $[0,t_1]$, $[t_1,t_2]$, ..., $[t_{n-1},T]$ such that for each interval $[t_{i-1},t_i]$ the condition (5.1.45) holds we deduce that

$$|u_\varepsilon|^{p-1}\in L^m(Q_T) \quad \text{with } m<\frac{p(N+1)-N}{N(p-1)}, \tag{5.1.47}$$

and

$$|\nabla u_\varepsilon|^{p-1}\in L^s(Q_T) \quad \text{with } s<\frac{p(N+1)-N}{(N+1)(p-1)}. \tag{5.1.48}$$

Step 3. Now we proceed as in [25]; we report the proof for sake of completeness. By (5.1.46) it follows that

$$T_k(u_\varepsilon) \text{ is bounded in } L^p((0,T);W_0^{1,p}(\Omega)), \tag{5.1.49}$$

and

$$T_k(u_\varepsilon\ ) \text{ is bounded in } L^\infty((0,T);L^1(\Omega)), \tag{5.1.50}$$

indipendently of ε for any $k \geq 0$ so there exists a subsequence still denoted by u_ε

$$T_k(u_\varepsilon) \rightharpoonup T_k(u) \quad \text{in } L^p((0,T);W_0^p(\Omega)). \tag{5.1.51}$$

Moreover, proceeding as in [23], [26] is possible to prove that for any $S \in W^{2,\infty}(\mathbb{R})$ with S' compact the term

$$\frac{\partial S(u_\varepsilon)}{\partial t} \text{ is bounded in } L^1(Q_T) + L^{p'}((0,T);W^{-1,p'}(\Omega)), \tag{5.1.52}$$

indipendently of ε. In fact, by pointwise moltiplication of $S'(u_\varepsilon)$ in the equation (5.1.30) we have

$$\frac{\partial S(u_\varepsilon)}{\partial t} - \operatorname{div}(a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)S'(u_\varepsilon)) + S''(u_\varepsilon)a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon +$$

$$+\operatorname{div}(K_\varepsilon(x,t,u_\varepsilon)S'(u_\varepsilon)) - S''(u_\varepsilon)K_\varepsilon(x,t,u_\varepsilon)\nabla u_\varepsilon = f_\varepsilon S'(u_\varepsilon) \quad \mathit{in}\ D'(\Omega). \tag{5.1.53}$$

Now each term in (5.1.53) is taking into account. Because of (5.1.2) , (5.1.31) , and (5.1.50) the term $-\operatorname{div}(a(x,t,u_\varepsilon,\nabla u_\varepsilon)S'(u_\varepsilon)) + S''(u_\varepsilon)a(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon + f_\varepsilon S'(u_\varepsilon)$ is bounded in $L^1(Q_T) + L^{p'}((0,T);W^{-1,p'}(\Omega))$ indipendently of ε. If we recall that $S'(u_\varepsilon)$ has a compact support contained in $[-k,k]$, by (5.1.34) it follows that for $0 < \varepsilon < \frac{1}{k}$

$$\left| \iint_{Q_T} K_\varepsilon(x,t,u_\varepsilon)^{p'} S'(u_\varepsilon)^{p'} \right| \leq \iint_{Q_T} c(x,t)^{p'} \ \left| T_{\frac{1}{\varepsilon}}(u_\varepsilon) \right|^{p'\gamma} |S'(u_\varepsilon)|^{p'} =$$

$$\iint_{|u_\varepsilon| \leq k} c(x,t)^{p'} \ |T_k(u_\varepsilon)|^{p'\gamma} \, |S'(u_\varepsilon)|^{p'} \,,$$

Furthermore, by Hölder and Gagliardo- Niremberg inequality, it results

$$\iint_{|u_\varepsilon|\le k} c(x,t)^{p'} \ |T_k(u_\varepsilon)|^{p'\gamma} |S'(u_\varepsilon)|^{p'} \le$$

$$\|c(x,t)\|^{p'}_{L^\tau(Q_T)} \left[\left(\sup_{t\in(0,T)} \int_\Omega |T_k(u_\varepsilon)|^2 \right)^{\frac{p}{N}} + \iint_{Q_T} |\nabla T_k(u_\varepsilon)|^p \right] \le c_k$$

where c_k is a constant indipendently of ε which will vary from line to line.

In the same way, by (5.1.34) we have

$$\left| \iint_{Q_T} (S''(u_\varepsilon)K_\varepsilon(x,t,u_\varepsilon)\nabla u_\varepsilon)^{p'} \right| \le \iint_{Q_T} S''(u_\varepsilon)^{p'} |c(x,t)|^{p'} \ \left| T_{\frac{1}{\varepsilon}}(u_\varepsilon) \right|^{\gamma p'} |\nabla u_\varepsilon|^{p'} . \tag{5.1.54}$$

Furthermore, for $0<\varepsilon<\frac{1}{k}$, by Hölder and Gagliardo-Niremberg inequality we deduce that

$$\iint_{Q_T} S''(u_\varepsilon)^{p'} |c(x,t)|^{p'} \ \left| T_{\frac{1}{\varepsilon}}(u_\varepsilon) \right|^{\gamma p'} |\nabla u_\varepsilon|^{p'} = \iint_{Q_T} S''(u_\varepsilon)^{p'} |c(x,t)|^{p'} \ |T_k(u_\varepsilon)|^{\gamma p'} |\nabla T_k(u_\varepsilon)|^{p'} \le c_k.$$

Now we want to prove an estimate which will be useful to prove (5.1.10). For any integer $n\ge 1$ let us consider the function

$$\theta_n(r) = T_{n+1}(r) - T_n(r) = \begin{cases} 0 & \text{if} \quad |r|\le n \\ (|r|-n)\ \text{sign}(r) & \text{if} \quad n\le |r| \le n+1 \\ \text{sign} r & \text{if} \quad |r|\ge n+1. \end{cases} \tag{5.1.55}$$

We observe that $\|T_{n+1}(r)-T_n(r)\|_{L^\infty(\mathbb{R})} \le 1$ for any $n\ge 1$ and for any r, $T_{n+1}(r) - T_n(r) \longrightarrow 0$ when $n\longrightarrow +\infty$. Using $\theta_n(u_\varepsilon)$ as test function in (5.1.30), by (5.1.34) and Young inequality we get

$$\int_\Omega \tilde{\theta}_n(u_\varepsilon)(T) + \iint_{Qt} a_\varepsilon(x,t,u,\nabla u_\varepsilon)\nabla\theta_n(u_\varepsilon) \le$$

$$\iint_{Q_t} c(x,t) \left| T_{\frac{1}{\varepsilon}}(u_\varepsilon) \right|^\gamma |\nabla \theta_n(u_\varepsilon)| + \int_\Omega \tilde{\theta}_n(u_0)_\varepsilon + \iint_{Q_t} f_\varepsilon \theta_n(u_\varepsilon), \tag{5.1.56}$$

for almost $t \in (0,T)$, where

$$\tilde{\theta}_n(r) = \int_0^r \theta_n(s)ds.$$

Since $\theta_n \geq 0$ and for $\varepsilon < \frac{1}{n+1}$

$$a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla\theta_n(u_\varepsilon) = a(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla\theta_n(u_\varepsilon) \text{ a.e. in } Q_t,$$

inequality (5.1.56) implies that

$$\iint_{Q_t} a(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla\theta_n(u_\varepsilon) \leq \iint_{Q_t} c(x,t)\left|T_{\frac{1}{\varepsilon}}(u_\varepsilon)\right|^\gamma |\nabla\theta_n(u_\varepsilon)| +$$
$$+\int_\Omega \tilde{\theta}_n(u_0)_\varepsilon + \iint_{Q_t} f_\varepsilon\theta_n(u_\varepsilon), \tag{5.1.57}$$

a.e. $t \in (0,T)$, for $\varepsilon < \frac{1}{n+1}$. On the other hand, the boundedness of $T_k(u_\varepsilon)$ (5.1.49), (5.1.52) and the apriori estimate of u_ε, in the the Lorentz spaces imply that there exists a subsequence, still denoted by u_ε, such that

$$u_\varepsilon \to u \text{ a.e. in } Q_T, \tag{5.1.58}$$

where u is a measurable function defined on Q_T (we follow the same procedure used in [23], [26]). Furthermore, by definition of θ_n, we get

$$\theta_n(u_\varepsilon) \rightharpoonup \theta_n(u) \text{ weakly in } L^p((0,T);W_0^p(\Omega)). \tag{5.1.59}$$

Since $a_\varepsilon(x,t,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon)$ is bounded in $\left(L^{p'}(Q_T)\right)^N$ indipendently of ε for $0 < \varepsilon < \frac{1}{k}$, it follows that

$$a_\varepsilon(x,t,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon)) \rightharpoonup \sigma_k \text{ weakly in } \left(L^{p'}(Q_T)\right)^N, \tag{5.1.60}$$

when ε goes to zero for any $k > 0$ and $n \geq 1$ and σ_k belongs to $\left(L^{p'}(Q_T)\right)^N$for any $k > 0$.

Let us prove that u belongs to $L^\infty(0,T;L^1(\Omega))$. If we take $T_k(u_\varepsilon)$ as test function in (5.1.30) , by (5.1.34) we have

$$\int_\Omega \psi_k(u_\varepsilon)(t) + \iint_{Qt} a_\varepsilon(x,t,u,\nabla u_\varepsilon)\nabla T_k(u_\varepsilon) \leq \iint_{Qt} |c(x,t)|\left|T_{\frac{1}{\varepsilon}}(u_\varepsilon)\right|^\gamma |\nabla T_k(u_\varepsilon)|$$

$$+\int_\Omega \psi_k(u_0)_\varepsilon + \iint_{Q_t} f_\varepsilon T_k(u_\varepsilon) \tag{5.1.61}$$

for almost $t \in (0,T)$ and $0 < \varepsilon < \frac{1}{k}$. By Hölder and Gagliardo-Niremberg inequality we have

$$\iint_{Qt} |c(x,t)| \left|T_{\frac{1}{\varepsilon}}(u_\varepsilon)\right|^\gamma |\nabla T_k(u_\varepsilon)| \leq \|c(x,t)\|_{L^\tau(Q_t)} \left(\sup_{t\in(0,T)} \int_\Omega |T_k(u_\varepsilon)|^2\right)^{\frac{p-1}{N+p}} \tag{5.1.62}$$

$$\times \| |\nabla T_k(u_\varepsilon)| \|_{L^p(Q_t)}^{\frac{p(N+1)}{N+p}} \leq c_k. \tag{5.1.63}$$

Howevere, by (5.1.38) it follows that

$$\int_\Omega \psi_k(u_0)_\varepsilon + \iint_{Q_t} f_\varepsilon T_k(u_\varepsilon) \leq \int_\Omega |(u_0)_\varepsilon| + k \iint_{Q_t} f_\varepsilon. \tag{5.1.64}$$

Using (5.1.62) , (5.1.64) in (5.1.61) , we have

$$\int_\Omega \psi_k(u_\varepsilon)(t) \leq c_k + \int_\Omega |(u_0)_\varepsilon| + k \iint_{Q_t} f_\varepsilon.$$

Finally, by (5.1.32) , (5.1.33) it is possible to pass to the lim inf in the previous inequality as ε goes to 0 and to obtain

$$\int_\Omega \psi_k(u)(t) \leq k \left[\|f\|_{L^1(Q_T)} + \|u_0\|_{L^1(Q_T)}\right] + c_k.$$

Thanks to the definition of ψ_k, the last inequality becomes

$$k \int_\Omega |u(x,t)| \leq \frac{3}{2} k^2 |\Omega| + k \left[\|f\|_{L^1(Q_T)} + \|u_0\|_{L^1(Q_T)}\right] + c_k,$$

for almost any $t \in (0,T)$, which shows that $u \in L^\infty(0,T; L^1(\Omega))$.

Now, coming back to (5.1.57), for $0 < \varepsilon < \frac{1}{n+1}$ we have

$$\iint_{Q_T} a(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla\theta_n(u_\varepsilon) \leq \iint_{n\leq|u_\varepsilon|\leq n+1} c(x,t) |T_{n+1}(u_\varepsilon)|^\gamma |\nabla u_\varepsilon| +$$

$$+ \int_\Omega \tilde{\theta}_n(u_0)_\varepsilon + \iint_{Q_T} f_\varepsilon \theta_n(u_\varepsilon).$$

Using the weakly convergence of $T_k(u_\varepsilon)$ and the pointwise convergence of u_ε it follows that

$$\overline{\lim_{\varepsilon\to 0}}\iint\limits_{Q_T} a(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla\theta_n(u_\varepsilon) \le \overline{\lim_{\varepsilon\to 0}}\iint_{n\le|u_\varepsilon|\le n+1} c(x,t)\left|T_{n+1}(u_\varepsilon)\right|^\gamma \left|\nabla\theta_n(u_\varepsilon)\right| +$$

$$+\int_\Omega \tilde{\theta}_n(u_0) + \iint\limits_{Q_T} f\theta_n(u). \tag{5.1.65}$$

On the other hand, since $\nabla\theta_n(u_\varepsilon) = \chi_{\{n\le|u_\varepsilon|\le n+1\}}\nabla u_\varepsilon$ a.e. in Q_T, by Young inequality

$$\iint_{n\le|u_\varepsilon|\le n+1} c(x,t)\left|T_{\frac{1}{\varepsilon}}(u_\varepsilon)\right|^\gamma \nabla u_\varepsilon \le \frac{\alpha^{-\frac{p'}{p}}}{p'}\iint_{n\le|u_\varepsilon|\le n+1} c(x,t)^{p'}\left|T_{n+1}(u_\varepsilon)\right|^{p'\gamma} +$$

$$\frac{\alpha}{p}\iint\limits_{Q_T}\left|\nabla\theta_n(u_\varepsilon)\right|^p. \tag{5.1.66}$$

Using (5.1.66) in (5.1.65), the weak convergence of $\theta_n(u_\varepsilon)$ and (**??**) imply that

$$\frac{\alpha}{p'}\iint\limits_{Q_T}\left|\nabla\theta_n(u)\right|^p \le \iint\limits_{Q_T} f\theta_n(u) + \int_\Omega \tilde{\theta}_n(u_0) + \frac{\alpha^{-\frac{p'}{p}}}{p'}\iint_{n\le|u_\varepsilon|\le n+1} c(x,t)\left|u_\varepsilon\right|^\gamma \nabla\theta_n(u_\varepsilon). \tag{5.1.67}$$

The last inequality, together with the assumptions (5.1.6) , (5.1.7) , show that $\theta_n(u)$ is bounded in $L^p((0,T);W_0^p(\Omega))$ indipendently of n. Thanks to the pointwise convergence of $\theta_n(u)$ to 0 when $n\to+\infty$, $\theta_n(u)$ goes to zero weakly in $L^p((0,T);W_0^p(\Omega))$ as $n\to+\infty$. As a consequence

$$\lim_{n\to+\infty}\iint\limits_{Q_T} f\theta_n(u) = 0$$

and

$$\lim_{n\to+\infty}\iint_{n\le|u_\varepsilon|\le n+1} c(x,t)^{p'}\left|u\right|^{p'\gamma} = 0,$$

when $n\to+\infty$. Moreover $\tilde{\theta}_n(u_0)\to 0$ a.e in Ω when $n\to+\infty$ and $\left|\tilde{\theta}_n(u_0)\right| \le |u_0|$ a.e. in Ω. Since $u_0\in L^1(\Omega)$, by Lebesgue's convergence Theorem we obtain for $n\to+\infty$

$$\int_\Omega \tilde{\theta}_n(u_0)\to 0.$$

Therefore,

$$\lim_{n\to+\infty}\iint_{Q_T} |\nabla\theta_n(u)|^p = 0$$

Finally, passing to the limit as $n \to +\infty$ in (5.1.65) and (5.1.67) we get

$$\lim_{n\to+\infty}\overline{\lim_{\varepsilon\to 0}}\iint_{n\leq|u_\varepsilon|\leq n+1} a(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon = 0, \tag{5.1.68}$$

and

$$\theta_n(u) \to 0 \text{ strongly in } L^p((0,T);W_0^{1,p}(\Omega)) \tag{5.1.69}$$

as $n \to +\infty$.

Step 4 In this step we introduce a time reguralitazion of the $T_k(u)$ for $k > 0$ in order to performe the monotonicity method. This kind of regularization has been introduced at the first time by R. Landes in [43] and can be defined as follows. Let be v_0^μ a sequence of functions defined on Ω such that

$$v_0^\mu \in L^\infty(\Omega)\cap W_0^{1,p}(\Omega) \text{ for all } \mu > 0, \tag{5.1.70}$$

$$\|v_0^\mu\|_{L^\infty(\Omega)} \leq k \quad \forall\mu > 0, \tag{5.1.71}$$

$$v_0^\mu \to T_k(u_0) \quad \text{a.e. in } \Omega \text{ and } \frac{1}{\mu}\|\nabla v_0^\mu\|^p_{L^p(\Omega)} \to 0, \text{ as } \mu \text{ tends to } +\infty. \tag{5.1.72}$$

Existence of such subsequence (v_0^μ) is easy to establish [40]. For fixed $k \geq 0$ and $\mu > 0$, the monotone problem

$$\begin{cases} \frac{\partial(T_k(u))_\mu}{\partial t} + \mu((T_k(u))_\mu - T_k(u)) = 0 & \text{in} \quad D'(Q_T) \\ (T_k(u))_\mu(t=0) = v_0^\mu & \text{in} \quad \Omega. \end{cases} \tag{5.1.73}$$

admits a unique solution $(T_k(u))_\mu \in L^\infty(Q_T)\cap L^p((0,T);W_0^{1,p}(\Omega))$. We observe that

$$\frac{\partial(T_k(u))_\mu}{\partial t} \in L^p((0,T);W_0^{1,p}(\Omega)). \tag{5.1.74}$$

The behavior of $(T_k(u))_\mu$ as $\mu \to \infty$ has been proved in [43], [40], [33]. Here we just recall that from (5.1.70) – (5.1.73), it follows that

$$\begin{gathered}(T_k(u))_\mu \to T_k(u) \text{ a.e. in } Q_{T,} \text{ in } L^\infty(Q_T) \text{ weakly-* and} \\ \text{strongly in } L^p((0,T); W_0^{1,p}(\Omega)) \text{ as } \mu \to +\infty,\end{gathered} \tag{5.1.75}$$

$$\|(T_k(u))_\mu\|_{L^\infty(Q_T)} \leq \max\left(\|T_k(u)\|_{L^\infty(Q_T)}, \|v_0^\mu\|_{L^\infty(\Omega)}\right) \leq k, \tag{5.1.76}$$

for any $\mu > 0$ and any $k \geq 0$. This definition of $(T_k(u))_\mu$ allows us to prove the following lemma whose proof can be found in [25]

Lemma 5.1.5 *Let $k \geq 0$ be fixed. Let S be an increasing $C^\infty(\mathbb{R})-$ function such that $S(r) = r$ for $|r| \leq k$ and $\sup p S'$ is compact. Then*

$$\underline{\lim}_{\mu\to+\infty} \lim_{\varepsilon\to 0} \int_0^T \int_0^t < \frac{\partial (S(u_\varepsilon)}{\partial t}, (T_k(u_\varepsilon) - (T_k(u))_\mu) > \geq 0,$$

where $< .,. >$ denotes the duality pairing between $L^1(\Omega) + W^{-1,p'}(\Omega)$ and $L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$.

Step 5. In this step we prove a lemma which is the critical point in the development of the monotonicity method .

Lemma 5.1.6 *The subsequence of u_ε satisfies for any $k \geq 0$*

$$\overline{\lim_{\varepsilon\to 0}} \int_0^T \int_0^t \int_\Omega a(x,t,u_\varepsilon, \nabla T_k(u_\varepsilon))\nabla T_k(u_\varepsilon) \leq \int_0^T \int_0^t \int_\Omega \sigma_k \nabla T_k(u),$$

where σ_k is defined in (5.1.60).

Proof

Let be S_n a sequence of increasing $C^\infty(\mathbb{R})-$function such that

$$S_n(r) = r \quad \text{for } |r| \leq n, \tag{5.1.77}$$

$$suppS_n' \subset [-(n+1), (n+1)], \tag{5.1.78}$$

$$\|S_n''\|_{L^\infty(\mathbb{R})} \leq 1, \tag{5.1.79}$$

for any $n \geq 1$. By pointwise multiplication of $S_n^{'}(u_\varepsilon)$ in we have

$$\frac{\partial S_n(u_\varepsilon)}{\partial t} - \operatorname{div}(a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)S_n'(u_\varepsilon)) + S_n''(u_\varepsilon)a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon +$$

$$+\operatorname{div}(K_\varepsilon(x,t,u_\varepsilon)S_n'(u_\varepsilon)) - S_n''(u_\varepsilon)K_\varepsilon(x,t,u_\varepsilon)\nabla u_\varepsilon = f_\varepsilon S_n'(u_\varepsilon) \quad \text{in } D'(\Omega).$$

We observe that $\frac{\partial S_n(u_\varepsilon)}{\partial t} \in L^1(Q_T) + L^{p'}(0,T); W^{-1,p'}(\Omega))$.

For $k \geq 0$, let us consider

$$W_\mu^\varepsilon = T_k(u_\varepsilon) - (T_k(u_\varepsilon))_\mu, \tag{5.1.80}$$

where $(T_k(u))_\mu$ has been defined in (5.1.73) . If we integrate over $(0,t)$ and $(0,T)$ we have

$$\int_0^T\int_0^t < \frac{\partial(S(u_\varepsilon)}{\partial t}, W_\mu^\varepsilon > + \int_0^T\int_0^t\int_\Omega a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)S_n'(u_\varepsilon)\nabla W_\mu^\varepsilon +$$

$$+\int_0^T\int_0^t\int_\Omega S_n''(u_\varepsilon)a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon W_\mu^\varepsilon - \int_0^T\int_0^t\int_\Omega K_\varepsilon(x,t,u_\varepsilon)S_n'(u_\varepsilon)\nabla W_\mu^\varepsilon$$

$$-\int_0^T\int_0^t\int_\Omega S_n''(u_\varepsilon)K_\varepsilon(x,t,u_\varepsilon)\nabla u_\varepsilon W_\mu^\varepsilon = \int_0^T\int_0^t\int_\Omega f_\varepsilon S_n'(u_\varepsilon)W_\mu^\varepsilon. \tag{5.1.81}$$

Now we pass to the limit in (5.1.81) as ε tends to 0, μ tends to $+\infty$ and then n tends to $+\infty$ for k real number fixed. In particular we want to prove that for any fixed $k \geq 0$

$$\underline{\lim}_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_0^T\int_0^t < \frac{\partial(S(u_\varepsilon)}{\partial t}, W_\mu^\varepsilon > \geq 0 \quad \text{for any } n \geq k, \tag{5.1.82}$$

$$\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_0^T\int_0^t\int_\Omega K_\varepsilon(x,t,u_\varepsilon)S_n'(u_\varepsilon)\nabla W_\mu^\varepsilon = 0 \quad \text{for any } n \geq 1, \tag{5.1.83}$$

$$\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_0^T\int_0^t\int_\Omega S_n''(u_\varepsilon)K_\varepsilon(x,t,u_\varepsilon)\nabla u_\varepsilon W_\mu^\varepsilon = 0 \quad \text{for any } n \geq 1, \tag{5.1.84}$$

$$\lim_{n\to+\infty}\overline{\lim_{\mu\to+\infty}}\ \overline{\lim_{\varepsilon\to 0}}\left|\int_0^T\int_0^t\int_\Omega S_n''(u_\varepsilon)a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon W_\mu^\varepsilon\right| = 0, \tag{5.1.85}$$

$$\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_0^T\int_0^t\int_\Omega f_\varepsilon S_n'(u_\varepsilon)W_\mu^\varepsilon = 0 \quad \text{for any } n\geq 1. \tag{5.1.86}$$

The proof of (5.1.82) can be easily obtained by appling Lemma 5.1.5 to the function S_n for any fixed $n\geq k$. Let us recall the main properties of W_μ^ε. In view of (5.1.58), (5.1.80), (5.1.51), for any fixed $\mu>0$

$$W_\mu^\varepsilon \rightharpoonup T_k(u_\varepsilon)-(T_k(u_\varepsilon))_\mu \text{ weakly in } L^p((0,T);W_0^{1,p}(\Omega)),$$

as ε goes to 0. Then by (5.1.75), (5.1.76) we have

$$\left\|W_\mu^\varepsilon\right\|_{L^\infty(Q_T)}\leq 2k, \quad \text{for any } \varepsilon>0 \text{ and for any } \mu>0. \tag{5.1.87}$$

From (5.1.80), (5.1.87) we deduce that for fixed $\mu>0$

$$W_\mu^\varepsilon \rightharpoonup T_k(u)-(T_k(u))_\mu \text{ a.e. in } Q_T \text{ and in } L^\infty(Q_T) \text{ weakly-*}, \tag{5.1.88}$$

when $\varepsilon\to 0$.

Let us prove (5.1.83). For any fixed $n\geq 1$ and $0<\varepsilon<\frac{1}{n+1}$ it results

$$S_n'(u_\varepsilon)K_\varepsilon(x,t,u_\varepsilon)\nabla W_\mu^\varepsilon = S_n'(u_\varepsilon)K_\varepsilon(x,t,T_{n+1}(u_\varepsilon))\nabla W_\mu^\varepsilon \text{ a.e. in } Q_T,$$

since $suppS'\subset[-(n+1),n+1]$. On the other hand,

$$S_n'(u_\varepsilon)K_\varepsilon(x,t,T_{n+1}(u_\varepsilon)) \longrightarrow S_n'(u)K(x,t,T_{n+1}(u)) \text{ a.e. in } Q_T,$$

and

$$|S_n'(u_\varepsilon)K_\varepsilon(x,t,T_{n+1}(u_\varepsilon))|\leq c(x,t)(n+1)^\gamma \text{ for } n\geq 1.$$

By (5.1.77) and the strongly convergence of $(T_k(u_\varepsilon))_\mu$ in $L^p((0,T);W_0^{1,p}(\Omega))$ we obtain (5.1.83)

Proof (5.1.84). For any fixed $n\geq 1$ and $0<\varepsilon<\frac{1}{n+1}$

$$S_n''(u_\varepsilon)K_\varepsilon(x,t,u_\varepsilon)\nabla u_\varepsilon W_\mu^\varepsilon = S_n''(u_\varepsilon)K_\varepsilon(x,t,T_{n+1}(u_\varepsilon))\nabla T_{n+1}(u_\varepsilon)W_\mu^\varepsilon \text{ a.e. in } Q_T,$$

as in the previous step it is possible to pass to the limit for $\varepsilon \longrightarrow 0$ since, by (5.1.87), (5.1.58), (5.1.88)

$$S_n''(u_\varepsilon)K_\varepsilon(x,t,T_{n+1}(u_\varepsilon))W_\mu^\varepsilon \longrightarrow S_n''(u)K(x,t,T_{n+1}(u))W_\mu \text{ a.e. in } Q_T,$$

and

$$|S_n''(u)K(x,t,T_{n+1}(u))W_\mu| \leq 2k\,|c(x,t)|\,(n+1)^\gamma.$$

Finally by (5.1.77) we obtain (5.1.84).

Let us prove (5.1.85). Due to (5.1.77), (5.1.78) $suppS' \subset [-(n+1),-n] \cup [n,n+1]$ for any $n \geq 1$, as a consequence

$$\left|\int_0^T\int_0^t\int_\Omega S_n''(u_\varepsilon)a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon W_\mu^\varepsilon\right| \leq$$

$$T\left\|S_n''(u_\varepsilon)\right\|_{L^\infty(\mathbb{R})}\left\|W_\mu^\varepsilon\right\|_{L^\infty(Q_T)}\int_{\{n\leq|u_\varepsilon|\leq n+1\}} a(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon,$$

for any $n \geq 1$, any $\varepsilon \leq \frac{1}{n+1}$ and any $\mu > 0$. The above inequality together with (5.1.79) and (5.1.87) make it possible to obtain

$$\overline{\lim_{\mu\to+\infty}}\,\overline{\lim_{\varepsilon\to 0}}\left|\int_0^T\int_0^t\int_\Omega S_n''(u_\varepsilon)a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon W_\mu^\varepsilon\right| \leq C\overline{\lim_{\varepsilon\to 0}}\left|\int_{\{n\leq|u_\varepsilon|\leq n+1\}} a(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon\right|,$$

for any $n \geq 1$, where C is indipendently of ε.

By (5.1.68) it is possible to pass to the limit as n tends to $+\infty$ and to establish (5.1.85).

Proof of (5.1.86). By (5.1.32), the pointwise convergence of u_ε and W_μ^ε and its boundness it is possible to pass to the limit for $\varepsilon \longrightarrow 0$ for any $\mu > 0$ and any $n \geq 1$

$$\lim_{\varepsilon\to 0}\int_0^T\int_0^t\int_\Omega f_\varepsilon S_n'(u_\varepsilon)W_\mu^\varepsilon = \int_0^T\int_0^t\int_\Omega fS_n'(u)(T_k(u)-T_k(u))_\mu).$$

Now for fixed $n \geq 1$, using (5.1.76) it possible to pass to the limit as μ tends to $+\infty$ in the above equality.

Now we turn back to the proof of Lemma 5.2.4. Due to (5.1.82) – (5.1.86) we can to pass to the limit-sup when ε tends to zero, then to the limit-sup when μ tends to $+\infty$ and to the limit as n tends to $+\infty$ in (5.1.81) . Using the definiton of W_μ^ε we deduce that for any $k \geq 0$

$$\lim_{n\to+\infty} \overline{\lim_{\mu\to+\infty}} \overline{\lim_{\mu\to+\infty}} \int_0^T \int_0^t \int_\Omega S_n^{'}(u_\varepsilon) a_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)(\nabla T_k(u_\varepsilon) - \nabla(T_k(u))_\mu) \leq 0.$$

Since $S_n^{\prime}(u_\varepsilon)a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla T_k(u_\varepsilon) = a(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla(T_k(u_\varepsilon)$ for $k \leq \frac{1}{\varepsilon}$ and $k \leq n$, using the properties of $S_n^{\prime}$ the above inequality implies that for $k \leq n$

$$\overline{\lim_{\varepsilon\to 0}} \int_0^T \int_0^t \int_\Omega a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla T_k(u_\varepsilon) \leq$$

$$\lim_{n\to+\infty} \overline{\lim_{\mu\to+\infty}} \overline{\lim_{\varepsilon\to 0}} \int_0^T \int_0^t \int_\Omega (S_n^{'}(u_\varepsilon) a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)(\nabla T_k(u_\varepsilon))_\mu \tag{5.1.89}$$

On the other hand, for $\varepsilon \leq \frac{1}{n+1}$

$$S_n^{'}(u_\varepsilon)a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon) = S_n^{'}(u_\varepsilon)a(x,t,T_{n+1}(u_\varepsilon),\nabla T_{n+1}(u_\varepsilon)) \text{ a.e. in } Q_T.$$

Furthermore by (5.1.60) it follows that for fixed $n \geq 1$

$$S_n^{'}(u_\varepsilon)a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon) \to S_n^{'}(u_\varepsilon)\sigma_{n+1} \text{ weakly in } L^{p^{'}}(Q_T)$$

when ε tends to 0. Finally, using the strong convergence of $(T_k(u))_\mu$ to $T_k(u)$ in $L^p(0,T;\ W_0^{1,p}(\Omega))$ as μ tends to $+\infty$, we get

$$\lim_{\mu\to+\infty} \lim_{\varepsilon\to 0} \int_0^T \int_0^t \int_\Omega (S_n^{'}(u_\varepsilon) a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)(\nabla T_k(u_\varepsilon))_\mu = \int_0^T \int_0^t \int_\Omega S_n^{'}(u_\varepsilon)\sigma_{n+1}\nabla T_k(u) \tag{5.1.90}$$

as soon as $k \leq n$ since (5.1.77) implies that $S_n^{'}(r) = 1$ for$|r| \leq n$. Now for $k \leq n$ we have

$$a(x,t,T_{n+1}(u_\varepsilon),\nabla T_{n+1}(u_\varepsilon))\chi_{\{|u_\varepsilon|\leq k\}} = a(x,t,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon))\chi_{\{|u_\varepsilon|\leq k\}} \text{ a.e. in } Q_T,$$

which implies that , by (5.1.58) , (5.1.60) , passing to the limit when ε tends to 0,

$$\sigma_{n+1}\chi_{\{|u|\le k\}} = \sigma_k\chi_{\{|u|\le k\}} \text{ a.e. in } Q_T - \{|u| = k\} \text{ for } k \le n. \tag{5.1.91}$$

Finally, by (5.1.91) and (5.1.60) we have for $k \le n$,

$$\sigma_{n+1}\nabla T_k(u) = \sigma_k\nabla T_k(u) \text{ a.e. in } Q_T.$$

Recalling (5.1.89) , (5.1.90) the proof of lemma is complete.

Step 6. In this step we prove that u satisfies the equation (5.1.11) . First of all we prove that the weak limit σ_k of $a(x,t,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon))$ can be identified with $a(x,t,T_k(u),\nabla T_k(u))$. In order to prove this result we recall the following lemma:

Lemma 5.1.7 *The subsequence satisfies the following condition for any* $k \ge 0$

$$\lim_{\varepsilon\to 0}\int_0^T\int_0^t\int_\Omega [a(x,t,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon)) - a(x,t,T_k(u_\varepsilon),\nabla T_k(u))]\times[\nabla T_k(u_\varepsilon) - \nabla T_k(u)] = 0. \tag{5.1.92}$$

Proof

Let $k \ge 0$ be fixed. By (5.0.97) we have

$$\lim_{\varepsilon\to 0}\int_0^T\int_0^t\int_\Omega [a(x,t,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon)) - a(x,t,T_k(u_\varepsilon),\nabla T_k(u))]\times[\nabla T_k(u_\varepsilon) - \nabla T_k(u)] \ge 0. \tag{5.1.93}$$

Furthermore, by (5.1.2) (5.1.58) we have

$$a(x,t,T_k(u_\varepsilon),\nabla T_k(u)) \longrightarrow a(x,t,T_k(u),\nabla T_k(u)) \text{ a.e. in } Q_T,$$

and

$$|a(x,t,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon))| \le \nu\left[h(x,t) + |\nabla T_k(u_\varepsilon)|^{p-1}\right] \text{ a.e. in } Q_T,$$

uniformly with respect to ε. As a consequence

$$a(x,t,T_k(u_\varepsilon),\nabla T_k(u)) \longrightarrow a(x,t,T_k(u),\nabla T_k(u)) \text{ strongly in } (L^{p'}(Q_T))^N. \tag{5.1.94}$$

Finally, by Lemma 5.2.4, (5.1.58) , (5.1.60) and (5.1.94) make it possible to pass to the limit-sup as ε tends to 0 in (5.1.93) and we have (5.1.92) .

Lemma 5.1.8 *For fixed $k \geq 0$, we have*

$$\sigma_k = a(x,t,T_k(u),\nabla T_k(u)) \text{ a.e. in } Q_T, \tag{5.1.95}$$

and as ε tends to 0

$$a(x,t,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon))\nabla T_k(u_\varepsilon) \rightharpoonup a(x,t,T_k(u),\nabla T_k(u))\nabla T_k(u) \tag{5.1.96}$$

weakly in $L^1((0,T)\times\Omega)$.

Proof. We observe that for any $k > 0$, any $0 < \varepsilon < \frac{1}{k}$ and any $\xi \in \mathbb{R}^N$

$$a_\varepsilon(x,t,T_k(u_\varepsilon),\xi) = a(x,t,T_k(u_\varepsilon),\xi) = a_{\frac{1}{k}}(x,t,T_k(u_\varepsilon),\xi) \text{ a.e. in } Q_T.$$

By (5.1.51), (5.1.92)

$$\lim_{\varepsilon\to 0}\int_0^T\int_0^t\int_\Omega a_{\frac{1}{k}}(x,t,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon))\nabla T_k(u_\varepsilon) = \int_0^T\int_0^t\int_\Omega \sigma_k \nabla T_k(u). \tag{5.1.97}$$

Since, for fixed $k > 0$, the function $a_{\frac{1}{k}}(x,t,s,\xi)$ is continuous and bounded with respect to s, the usual Minty's argument applies in view of (5.1.51), (5.1.60), and (5.1.97). It follows that (5.1.95) holds true (the case $k = 0$ being trivial). In order to prove (5.1.96), by (5.0.97) and (5.1.92) give that for any $k \geq 0$ and any $T' < T$

$$[a(x,t,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon)) - a(x,t,T_k(u_\varepsilon),\nabla T_k(u))] \times [\nabla T_k(u_\varepsilon) - \nabla T_k(u)] \longrightarrow 0$$

strongly in $L^1((0,T)\times\Omega)$ as ε tends to 0. Moreover by (5.1.51), (5.1.60), (5.1.94), and (5.1.95) we have

$$a(x,t,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon))\nabla T_k(u) \rightharpoonup a(x,t,T_k(u),\nabla T_k(u))\nabla T_k(u) \text{ weakly in } L^1(Q_T),$$

and

$$a(x,t,T_k(u_\varepsilon),\nabla T_k(u))\nabla T_k(u) \longrightarrow a(x,t,T_k(u),\nabla T_k(u))\nabla T_k(u) \text{ strongly in } L^1(Q_T),$$

as ε tends to 0.

Using the above convergence result in (5.1.97) shows that for any $k \geq 0$ and any $T' < T$

$$a(x,t,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon))\nabla T_k(u_\varepsilon) \rightharpoonup a(x,t,T_k(u),\nabla T_k(u))\nabla T_k(u) \tag{5.1.98}$$

weakly in $L^1((0,T')\times\Omega)$ as ε tends to 0.

In order to extend the functions $a(x,t,s,\xi)$, f on a time interval $(0,\bar{T})$ with $\bar{T}>T$ in such a way that (5.0.97) , (5.0.99) , 5.1.2 – (5.1.7) hold true with $\bar{T}$ in place of T, we can show that the convergence result (5.1.98) is still valid in $L^1(Q_T)-$ weak, namely that (5.1.98) holds true.

Now we prove that u satisfies (5.1.10) . To this end we remark that for any fixed $n\geq 0$ we have

$$\iint\limits_{\{n\leq|u_\varepsilon|\leq n+1\}} a(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon = \iint\limits_{Q_T} a(x,t,u_\varepsilon,\nabla u_\varepsilon)(\nabla T_{n+1}(u_\varepsilon)-\nabla T_n(u_\varepsilon)) =$$

$$= \iint\limits_{Q_T} a(x,t,T_{n+1}(u_\varepsilon),\nabla T_{n+1}(u_\varepsilon))(\nabla T_{n+1}(u_\varepsilon)-\nabla T_n(u_\varepsilon)) +$$

$$- \iint\limits_{Q_T} a(x,t,T_{n+1}(u_\varepsilon),\nabla T_{n+1}(u_\varepsilon))\nabla T_n(u_\varepsilon).$$

According to (5.1.96) , one is at liberty to pass to the limit as ε tends to 0 for fixed $n\geq 0$ and to obtain

$$\lim_{\varepsilon\to 0} \iint\limits_{\{n\leq|u_\varepsilon|\leq n+1\}} a(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon = \iint\limits_{Q_T} a(x,t,T_{n+1}(u),\nabla T_{n+1}(u))\nabla T_{n+1}(u)$$

$$- \iint\limits_{Q_T} a(x,t,T_n(u),\nabla T_n(u))\nabla T_n(u) = \iint\limits_{\{n\leq|u|\leq n+1\}} a(x,t,u,\nabla u)\nabla u. \tag{5.1.99}$$

Taking the limit as n tends to $+\infty$ in (5.1.99) and using the estimate (5.1.68) show that u satisfies (5.1.10) . Our aim is to prove that u satisfies (5.1.11) and (5.1.12) .Now we want to prove that u satisfies the equation (5.1.11) . Let be S a function in $W^{2,\infty}(\mathbb{R})$ such that $suppS'\subset[-k,k]$ where k is a real positive number. In the following we show how it is possible to pass to the limit in (5.1.53). Since $u_\varepsilon\longrightarrow u$ $a.e.$ in Q_T and in $L^\infty(Q_T)$ weak-*, using the boundness of $S(u_\varepsilon)$ it follows that $\frac{\partial S(u_\varepsilon)}{\partial t}\longrightarrow\frac{\partial S(u)}{\partial t}$ in $D'(\Omega)$. We observe that the term $a_\varepsilon(x,t,u_{\varepsilon,}\nabla u_\varepsilon)$ can be identified with $a_\varepsilon(x,t,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon))$ for $\varepsilon\leq\frac{1}{k}$, so using the pointwise convergence of

$u_\varepsilon \longrightarrow u$ in Q_T, the weakly convergence of $T_k(u_\varepsilon\ ) \rightharpoonup T_k(u) \quad$ in $L^p((0,T);W_0^p(\Omega))$ we get

$$a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)S'(u_\varepsilon) \rightharpoonup a(x,t,T_k(u_\varepsilon),\nabla T_k(u))S'(u) \quad \text{in } L^{p'}(Q_T),$$

and

$$S''(u_\varepsilon)a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon \rightharpoonup S''(u)a(x,t,T_k(u),\nabla T_k(u))\nabla T_k(u) \quad \text{in } L^1(Q_T).$$

Furthermore, since

$$K_\varepsilon(x,t,u_\varepsilon)S'(u_\varepsilon) = K_\varepsilon(x,t,T_k(u_\varepsilon))S'(u_\varepsilon) \text{ a.e. in } Q_T,$$

by (5.1.34)

$$|K_\varepsilon(x,t,T_k(u_\varepsilon))S'(u_\varepsilon)| \leq |c(x,t)|\, k^\gamma,$$

it follows that

$$K_\varepsilon(x,t,T_k(u_\varepsilon))S'(u_\varepsilon) \ \rightarrow K(x,t,T_k(u))S'(u) \text{ strongly in } L^{p'}(Q_T).$$

In a similar way, it results

$$S''(u_\varepsilon)K_\varepsilon(x,t,u_\varepsilon)\nabla u_\varepsilon = S''(T_k(u_\varepsilon))K_\varepsilon(x,t,T_k(u_\varepsilon))\nabla T_k(u_\varepsilon) \text{ a.e. in } Q_T,$$

and

$$S''(u_\varepsilon)K_\varepsilon(x,t,u_\varepsilon) \rightarrow S''(u)K(x,t,u) \text{ a.e. in } Q_T,$$

so, using the weakly convergence of $T_k(u_\varepsilon)$ in $L^p((0,T);W_0^p(\Omega))$ it is possible to prove that

$$S''(u_\varepsilon)K(x,t,u_\varepsilon)\nabla u_\varepsilon \rightarrow S''(u)K(x,t,u)\nabla u \text{ in } L^1(Q_T).$$

Finally by (5.1.32) we deduce that

$$f_\varepsilon S'(u_\varepsilon) \longrightarrow fS'(u) \quad \text{ in } L^1(Q_T).$$

It remains to prove that $S(u)(t=0) = S(u_0)$ in Ω. By (5.1.53) the term $\frac{\partial S(u_\varepsilon)}{\partial t}$ is bounded in $L^1(Q_T) + L^{p'}((0,T);W^{-1,p'}(\Omega))$ so by Aubin's type lemma it follows that $S(u_\varepsilon)$ belongs to $C^0\left([0,T]\,;W^{-1,s}(\Omega)\right)$ where $s<\inf(p',\frac{N}{N-1})$ and $S(u_\varepsilon)(t=0)=S(u_0)_\varepsilon$ converges to $S(u)(t=0)$ strongly in $W^{-1,s}(\Omega)$. Using (5.1.33) and the boundeness of S we have

$$S(u_0)_\varepsilon \longrightarrow S(u_0) \quad \text{strongly in } L^d(\Omega),\ d<\infty,$$

and then $S(u_0)=S(u)(t=0)$.

■

5.1.2 Existence result for problem (5.1.13)

In this section we prove the existence of a renormalized solution to problem (5.1.13). The result contained in [55] is the following:

Theorem 5.1.9 *Under the hypotheses* (5.0.97), (5.0.99), (5.1.2), (5.1.6), (5.1.7), (5.1.14) – (5.1.16) *there exists a renormalized solution of* (5.1.13) .

Proof

Here we follow the same technique used in the previous section. We divide the proof into several steps.

Proof. *Step 1* Let us consider the approximated problem

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t}-\operatorname{div}(a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon))+H_\varepsilon(x,t,\nabla u_\varepsilon)=f_\varepsilon & \text{in} \quad Q_T \\ u_\varepsilon(x,t)=0 & \text{on} \quad \partial\Omega\times(0,T) \\ u_\varepsilon(x,0)=\left(u_0\right)_\varepsilon(x) & \text{in} \quad Q_T, \end{cases} \tag{5.1.100}$$

where

$$H_\varepsilon(x,t,\eta)=T_{\frac{1}{\varepsilon}}(x,t,\eta), \tag{5.1.101}$$

$$\left|H_\varepsilon(x,t,\eta)\right|\le b(x,t)\left|\eta\right|^\delta,\ \text{and}\ \left|H_\varepsilon(x,t,\eta)\right|\le\varepsilon, \tag{5.1.102}$$

and $a_\varepsilon(x,t,\nabla u_\varepsilon)$, f_ε and $(u_0)_\varepsilon$ have been defined in (5.1.31), (5.1.32), (5.1.33). Under the assumptions (5.1.32), (5.1.33), (5.1.102), (5.1.101) the problem (5.1.100) admits a unique solution $u_\varepsilon\in L^\infty(0,T;L^2(\Omega))\cap L^p(0,T;W_0^{1,p}(\Omega))$.

Step 2 In this step we obtain the apriori estimates for the solution u_ε and its gradient ∇u_ε. To this end, let us consider $T_k(u_\varepsilon)$ as test function in (5.1.100) and we integrate between $(0,t)$, where $t \in (0,t_1)$ and $t_1 \in (0,T)$ will be choosen later, by (5.1.102) we have

$$\iint\limits_{Q_t} (u_\varepsilon)_t T_k(u_\varepsilon) + \iint\limits_{Qt} a(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla T_k(u_\varepsilon)$$

$$\leq \iint\limits_{Qt} b(x,t)\ \left|\nabla T_k(u_\varepsilon)\right|^\delta T_k(u_\varepsilon) + \iint\limits_{Q_t} f_\varepsilon T_k(u_\varepsilon). \tag{5.1.103}$$

Using (5.0.97), (5.1.39) and Hölder inequality we have

$$\frac{1}{2}\int_\Omega \left|T_k(u_\varepsilon)\right|^2 + \alpha \iint\limits_{Q_t} \left|\nabla T_k(u_\varepsilon)\right|^p \leq k\int_\Omega \left|(u_0)_\varepsilon\right|^2 +$$

$$+k\left\|b\right\|_{L^r(Q_t)} \left\|\left|\nabla T_k(u_\varepsilon)\right|^{p-1}\right\|_{L^s(Q_t)}^{\frac{\delta}{p-1}} + k\left\|f_\varepsilon\right\|_{L^1(Q_t)}. \tag{5.1.104}$$

Taking the supremum for $t \in (0,t_1)$ inequality (5.1.104) becomes

$$\frac{1}{2}\sup_{t\in(0,t_1)}\int_\Omega \left|T_k(u_\varepsilon)\right|^2 + \alpha \iint\limits_{Q_{t_1}} \left|\nabla T_k(u_\varepsilon)\right|^p \leq M_1 k, \tag{5.1.105}$$

where

$$M_1 = \left\|b\right\|_{L^r(Q_{t_1})} \left\|\left|\nabla T_k(u_\varepsilon)\right|^{p-1}\right\|_{L^s(Q_{t_1})}^{\frac{\delta}{p-1}} + \sup_\varepsilon \left\|f_\varepsilon\right\|_{L^1(Q_T)} + \left\|u_0\right\|_{L^1(\Omega)},$$

for $s < \frac{p(N+1)-N}{(N+1)(p-1)}$. By Lemma 5.1.3 we get

$$\left\|\left|\nabla u_\varepsilon\right|^{p-1}\right\|_{L^s(Q_T)} \leq C\left[\sup_\varepsilon \left\|f_\varepsilon\right\|_{L^1(Q_T)} + \left\|u_0\right\|_{L^1(\Omega)}\right]^{\frac{(N+2)(p-1)}{p(N+1)-N}}$$

$$+C\left(\left\|b\right\|_{L^r(Q_{t_1})}\right)^{\frac{(N+2)(p-1)}{p(N+1)-N}} \left(\iint_{Q_{t_1}} \left|\nabla T_k(u_\varepsilon)\right|^{(p-1)s} + 1\right)^{\frac{\delta}{s}\frac{(N+2)}{p(N+1)-N}},$$

for some constant C. Since $\delta \leq \frac{p(N+1)-N}{N+2}$ it results

$$\left\|\left|\nabla u_\varepsilon\right|^{p-1}\right\|_{L^s(Q_T)} \leq C\left[\sup_\varepsilon \left\|f_\varepsilon\right\|_{L^1(Q_T)} + \left\|u_0\right\|_{L^1(\Omega)}\right]^{\frac{(N+2)(p-1)}{p(N+1)-N}} +$$

$$+C\left(\|b\|_{L^r(Q_{t_1})}\right)^{\frac{(N+2)(p-1)}{p(N+1)-N}}\left(\iint_{Q_{t_1}}|\nabla T_k(u_\varepsilon)|^{(p-1)s}+1\right)^{\frac{1}{s}}. \qquad (5.1.106)$$

If we choose t_1 such that

$$1-C\,\|b\|_{L^r(Q_{t_1})}^{\frac{(N+2)(p-1)}{p(N+1)-N}}>0, \qquad (5.1.107)$$

inequality (5.1.106) becomes

$$\left\||\nabla u_\varepsilon|^{p-1}\right\|_{L^s(Q_{t_1})}\le C,\quad s<\frac{p(N+1)-N}{(N+1)(p-1)},$$

for some constant C. Our aim is to obtain two apriori estimates for u_ε and its gradient on the entire cylinder. The technique that here we follow is the same method used in the previous section: we consider a partition of the entire interval $[0,T]$ into a finite number of intervals $[0,t_1]$,..., $[t_{n-1},T]$ and for each of them we assume that a condition like (5.1.107) holds. In this way we obtain the apriori bounds (5.1.47) . The estimate for the solution u_ε is a natural consequence of the last inequality. In fact, by Lemma 5.1.2, we know that

$$\left\||u|^{p-1}\right\|_{L^m(Q_{t_1})}\le CM_1^{\left(\frac{p}{N}+1\right)\frac{N}{N+p'}}\,|Q_T|^{\frac{1}{p'}\frac{N}{N+p'}},$$

where M_1 is now indipendetly on ε.

Step 3 Now we proceed as in Step 3 of the first section (see also [25]) . By (5.1.105) $T_k(u_\varepsilon)$ is bounded indipendently of ε for any positive k, so (5.1.51) hold.
Moreover, if we multiplicate by $S'(u_\varepsilon)$ in the equation (5.1.100) , for any $S\in W^{2,\infty}(\mathbb{R})$ such that S' is compact we have

$$\frac{\partial S(u_\varepsilon)}{\partial t}-\operatorname{div}(a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)S'(u_\varepsilon))+S''(u_\varepsilon)a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon+$$

$$+S'(u_\varepsilon)H_\varepsilon(x,t,\nabla u_\varepsilon)=f_\varepsilon S'(u_\varepsilon)\quad\text{in } D'(\Omega). \qquad (5.1.108)$$

As in the first section we observe that $\frac{\partial S(u_\varepsilon)}{\partial t}$ is bounded in $L^1(Q_T)+L^{p'}((0,T);W^{-1,p'}(\Omega))$ indipendently of ε. In fact, the term $S'(u_\varepsilon)H_\varepsilon(x,t,\nabla u_\varepsilon)$ is bounded in $L^{p'}(Q_T)$ indipendently of ε since by

$$\iint\limits_{Q_T} |S'(u_\varepsilon)H_\varepsilon(x,t,\nabla T_k(u_\varepsilon))|^{p'} \leq \|b(x,t)\|_{L^r(Q_T)} \, \||\nabla T_k(u_\varepsilon)|\|_{L^p(Q_T)}^{\delta p'} \leq c_k.$$

Let us prove that $u \in L^\infty((0,T);L^1(\Omega))$. If we take $T_k(u_\varepsilon)$ as test function in (5.1.100) , by (5.1.102) ,we have

$$\int_\Omega \psi_k(u_\varepsilon)(t) + \iint_{Q_t} a_\varepsilon(x,t,u,\nabla u_\varepsilon)\nabla T_k(u_\varepsilon) \leq \iint_{Q_t} b(x,t)\,|\nabla u_\varepsilon|^\delta\, T_k(u_\varepsilon)$$
$$+ \int_\Omega \psi_k(u_0)_\varepsilon + \iint_{Q_t} f_\varepsilon T_k(u_\varepsilon), \tag{5.1.109}$$

for almost $t \in (0,T)$ and $0 < \varepsilon < \frac{1}{k}$, where $\psi_k(s)$ is defined by (5.1.36) . If we take the lim inf in the previous inequality, by (5.1.47) we have

$$\int_\Omega \psi_k(u_\varepsilon)(t) \leq c_k + \int_\Omega |(u_0)_\varepsilon| + k \iint_{Q_t} f_\varepsilon,$$

which implies (5.1.8) .

The next step is to prove that $\theta_n(u) \to 0$ strongly in $L^p((0,T);W_0^{1,p}(\Omega))$. To this end let us Let us consider $\theta_n(u_\varepsilon)$ where $\theta_n(u_\varepsilon)$ is defined by (5.1.55) . Arguing as in the first section, there exists a subsequence, still denoted by u_ε, such that (5.1.58) hold. Furthermore, (5.1.59) , (5.1.60) are valid too. If we take $\theta_n(u_\varepsilon)$ as test function in (5.1.100) , for $\varepsilon < \frac{1}{n+1}$, by (5.1.102) we have

$$\int_\Omega \tilde{\theta}_n(u_\varepsilon)(T) + \iint_{Qt} a_\varepsilon(x,t,u,\nabla u_\varepsilon)\nabla \theta_n(u_\varepsilon) \leq$$
$$+ \iint_{Q_t} b(x,t)\,|\nabla u_\varepsilon|^\delta\, \theta_n(u_\varepsilon) + \int_\Omega \tilde{\theta}_n(u_0)_\varepsilon + \iint_{Q_t} f_\varepsilon \theta_n(u_\varepsilon). \tag{5.1.110}$$

We observe that, by Hölder and Young inequality

$$\iint_{Q_T} b(x,t)\left|\nabla u_\varepsilon\right|^\delta \theta_n(u_\varepsilon) = \iint_{|u_\varepsilon|\le n+1} b(x,t)\left|\nabla \theta_n(u_\varepsilon)\right|^\delta \theta_n(u_\varepsilon) + \tag{5.1.111}$$

$$+\iint_{|u_\varepsilon|>n+1} b(x,t)\left|\nabla u_\varepsilon\right|^\delta \theta_n(u_\varepsilon) \le \frac{\alpha}{p}\iint_{Q_T}|\nabla\theta_n(u_\varepsilon)|^p +$$

$$\frac{\alpha^{-\frac{p'}{p}}}{p'}\left(\iint_{Q_T}(b(x,t)\theta_n(u_\varepsilon))^{(\frac{p}{\delta})'}\right)^{\frac{1}{(\frac{p}{\delta})'}} +$$

$$+\left\||\nabla u_\varepsilon|^{p-1}\right\|_{L^s(Q_T)}\left(\iint_{|u_\varepsilon|>n+1} b(x,t)^r\right)^{1/r}.$$

By (5.1.110) and (5.1.111) we have

$$(\alpha-\frac{\alpha}{p})\iint_{Q_T}|\nabla\theta_n(u_\varepsilon)|^p \le \frac{\alpha^{-\frac{p'}{p}}}{p'}\left(\iint_{Q_T}(b(x,t)\theta_n(u_\varepsilon))^{(\frac{p}{\delta})'}\right)^{\frac{1}{(\frac{p}{\delta})'}} +$$

$$c_k\left(\iint_{|u_\varepsilon|>n+1} b(x,t)^r\right)^{1/r} + \int_\Omega \tilde{\theta}_n(u_0)_\varepsilon + \iint_{Q_T} f_\varepsilon\theta_n(u_\varepsilon).$$

Finally, letting $\varepsilon\to 0$ and $n\to+\infty$ we have

$$\lim_{n\to+\infty}\iint_{Q_T}|\nabla\theta_n(u)|^p = 0.$$

As in the first section, this result imply that (5.1.68) and (5.1.69) hold.

Step 4 In this step we prove a lemma which is useful to develop the monotonicity method.

■

Lemma 5.1.10 *The subsequence of u_ε satisfies for any $k\ge 0$*

$$\overline{\lim_{\varepsilon\to 0}}\int_0^T\int_0^t\int_\Omega a(x,t,u_\varepsilon,\nabla T_k(u_\varepsilon))\nabla T_k(u_\varepsilon) \le \int_0^T\int_0^t\int_\Omega \sigma_k\nabla T_k(u),$$

where σ_k is defined in (5.1.60).

Proof. Let be S_n a sequence of increasing $C^\infty(\mathbb{R})-$function such that (5.1.77) $-$ (5.1.79) hold for any $n \geq 1$. By pointwise multiplication of $S_n^{'}(u_\varepsilon)$ in we have

$$\frac{\partial S_n(u_\varepsilon)}{\partial t} - \operatorname{div}(a(x,t,u_\varepsilon,\nabla u_\varepsilon)S_n'(u_\varepsilon)) + S_n''(u_\varepsilon)a(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon +$$

$$+H_\varepsilon(x,t,\nabla u_\varepsilon)S_n'(u_\varepsilon)) = f_\varepsilon S_n'(u_\varepsilon) \quad \text{in } D'(\Omega).$$

For $k \geq 0$, let us consider $W_\mu^\varepsilon = T_k(u_\varepsilon) - (T_k(u_\varepsilon))_\mu$, as test function, where $(T_k(u))_\mu$ has been defined in (5.1.73) and we integrate over $(0,t)$ and $(0,T)$

$$\int_0^T\int_0^t < \frac{\partial(S(u_\varepsilon)}{\partial t}, W_\mu^\varepsilon > + \int_0^T\int_0^t\int_\Omega a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)S_n'(u_\varepsilon)\nabla W_\mu^\varepsilon +$$

$$+\int_0^T\int_0^t\int_\Omega S_n''(u_\varepsilon)a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon W_\mu^\varepsilon - \int_0^T\int_0^t\int_\Omega H_\varepsilon(x,t,\nabla u_\varepsilon)S_n'(u_\varepsilon)W_\mu^\varepsilon =$$

$$= \int_0^T\int_0^t\int_\Omega f_\varepsilon S_n'(u_\varepsilon)W_\mu^\varepsilon. \tag{5.1.112}$$

Thanks to (5.1.82) , (5.1.85) , (5.1.86) proved in the previous section we only have to prove that

$$\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_0^T\int_0^t\int_\Omega H_\varepsilon(x,t,\nabla u_\varepsilon)S_n'(u_\varepsilon)W_\mu^\varepsilon = 0 \quad \text{for any } n \geq 1. \tag{5.1.113}$$

This result can be easily obtained if we observe that for $n \geq 1$

$$H_\varepsilon(x,t,\nabla u_\varepsilon)S_n'(u_\varepsilon)W_\mu^\varepsilon = H_\varepsilon(x,t,\nabla T_{n+1}(u_\varepsilon))S_n'(u_\varepsilon)W_\mu^\varepsilon \text{ a.e. in } Q_T,$$

since $suppS' \subset [-(n+1),(n+1)]$. Furthermore, by (5.1.88) and (5.1.58) we have

$$S_n'(u_\varepsilon)W_\mu^\varepsilon \to S_n'(u)W_\mu \text{ a.e. in } Q_T, \tag{5.1.114}$$

which implies, thanks to the boundness character of $S_n'(u_\varepsilon)W_\mu^\varepsilon$, that

$$S_n'(u_\varepsilon)W_\mu^\varepsilon \to S_n'(u)W_\mu \text{ strongly in } L^{p'}(Q_T). \tag{5.1.115}$$

On the other hand, by (5.1.51)

$$H_\varepsilon(x,t,\nabla T_{n+1}(u_\varepsilon)) \rightharpoonup H(x,t,\nabla T_{n+1}(u)) \text{ in } L^p((0,T);W_0^{1,p}(\Omega)). \tag{5.1.116}$$

Finally, by (5.1.115) , (5.1.116) we obtain (5.1.113) .

Step 5 It remains to prove that u satisfies the equation (5.1.11) . To this end let's go back to (5.1.108) . The scheme that we use to pass to the limit in the previous equation is the same used in (5.1.53) except for the lower order term. Using the boundness of S, the condition (5.1.101) , the pointwise convergence of u_ε we have

$$S'(u_\varepsilon) \to S'(u) \text{ strongly in } L^p(Q_T). \tag{5.1.117}$$

On the other hand, for $\varepsilon \leq \frac{1}{k}$,

$$H_\varepsilon(x,t,\nabla u_\varepsilon) = H_\varepsilon(x,t,\nabla T_k(u_\varepsilon)) \text{ a.e. in } Q_T,$$

and

$$H_\varepsilon(x,t,\nabla T_k(u_\varepsilon)) \rightharpoonup H(x,t,\nabla T_k(u)) \text{ in } L^p((0,T);W_0^{1,p}(\Omega)). \tag{5.1.118}$$

By (5.1.118) , (5.1.117) we obtain

$$H_\varepsilon(x,t,\nabla u_\varepsilon)S'(u_\varepsilon) \longrightarrow H(x,t,\nabla u)S'(u) \quad \text{in } L^1(Q_T).$$

■

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