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ON THE L^p-SOLVABILITY OF THE DIRICHLET PROBLEM AND GENERALIZATIONS IN ORLICZ SPACES

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Introduction

In the late 1950s and early 1960s, the work of De Giorgi [DG] and Nash [N], and then Moser [Mos], initiated the study of regularity of solutions to divergence form elliptic equations with merely bounded measurable coefficients. Weak solutions in a domain Ω, a priori only in a Sobolev space $W^{1,2}_{loc}(\Omega)$, were shown to be Hölder continuous of some order depending just on ellipticity, and maximum principles and Harnack inequalities were established. The Dirichlet problem for such operators, with continuous data on the boundary, was established in [LSW]. This in turn paved the way for a more systematic and detailed study of the properties of the harmonic measures $d\omega_{\mathcal{L}}$ associated to $\mathcal{L} = \mathrm{div}(A\nabla)$ on a domain Ω. The classical properties of existence of non-tangential limits of solutions (*Fatou type theorems*) and comparison principles appeared in [CFMS], but owed a great deal to the earlier work of [HW2] on harmonic functions in Lipschitz domains.

The investigation into the solvability of L^p boundary value problems, in the sense of non-tangential convergence and L^p estimates on the non-tangential maximal function of solutions, really began with the study of harmonic functions in Lipschitz domains [D1], [D2]. In [D1], B. Dahlberg proved that, on any Lipschitz domain Ω, the harmonic measure, $d\omega_{\mathcal{L}}$, and the surface measure $d\sigma$ were mutually absolutely continuous, that $d\omega_{\mathcal{L}} \in \mathcal{A}_\infty(d\sigma)$ (the Muckenhoupt weight class A_∞). He proved that there exists a constant C such that for any radius r and every surface ball $\Delta_r \subset \partial\Omega$,

$$\left(\fint_{\Delta_r} k^2 d\sigma \right)^{\frac{1}{2}} \leqslant C \fint_{\Delta_r} k d\sigma, \tag{1}$$

where $d\omega_{\mathcal{L}} = kd\sigma$. The estimate (1) will imply the L^2 solvability of Dirichlet problem in the domain Ω. Until recently, most results proving solvability for those boundary value problems were carried out for operators $\mathcal{L} = \mathrm{div}(A\nabla)$ assuming the matrix A to be both real and symmetric. On the other hand there are a variety of reason to studying the non symmetric situation. These include the connections with non-divergence form equations, and the broader issue of obtaining estimates on elliptic measure in the absence of special L^2 identities which relate tangential and normal derivatives. In [KKPT] the study of non-symmetric divergence form operators with bounded measurable coefficients was initiated. In light of this we began to study the solvability of the Dirichlet problem for this class of operators when the boundary data varies in an Orlicz functional space L^Φ, extending the L^p situation.

We prove, in more than two dimensions, that the known condition (see [K], [KKPT])

$$\omega_{\mathcal{L}} \in B_q(d\sigma)$$

for the L^p solvability, is a necessary and sufficient condition also for the L^Φ-*solvability* of the Dirichlet problem, whenever L^Φ is in a suitable *class* of Orlicz space containing the Lebesgue space L^p.

Moreover, in dimension $n = 2$ we find a number of quantitative sharp results for the L^p Dirichlet problem. More precisely, assume that the elliptic operator $\mathcal{L} = \mathrm{div}(A(x)\nabla)$ is L^p- resolutive, $p > 1$, on the unit disc $\mathbb{D} \subset \mathbb{R}^2$. Then, there exists $\varepsilon > 0$ such that L is L^r-resolutive in the optimal range $p - \varepsilon < r \leqslant \infty$ (see after Theorem 1.4.1). We determine the precise value of ε in terms of p and of a natural "norm" of the harmonic measure $\omega_{\mathcal{L}}$. In planar case we study also the following problem: given two operators in our class, say $\mathcal{L}_0$ and $\mathcal{L}_1$, when the solvability of $\mathcal{L}_0$ guarantee solvability for the second operator $\mathcal{L}_1$? We will treat this subject for special couples of operators which are pull-back of the Laplacian via quasiconformal mappings and we will obtain simultaneous solvability results for this couple of operators.

Now, a few words about the organization of the thesis. It consists about seven chapters.

First chapter is devoted to introduce the formulation of the L^p Dirichlet problem, as well as definitions and known results. Then, in Chapters 2 and 3 we recall definitions and properties of Orlicz functional spaces and introduce the Hardy-Littlewood maximal operator together with some of its most interesting properties. Apart from the usual estimates for this operator we also obtain some new weighted inequalities, so that the results obtained in the course of the Chapter 3 seems to be of independent interest.

In Chapter 4 we consider a Young's function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the Δ_2 condition together with its complementary function and we give a necessary and sufficient condition for the L^Φ-*solvability* of the Dirichlet problem, where L^Φ is the Orlicz Space generated by the function Φ (see Section 4.1 for definitions). In last three chapters we confine ourself to the two dimensional case to obtain a number of *sharp quantitative results.* In Chapter 6 we consider sequences of operators and study the weak convergence of their harmonic measures. Finally in Chapter 7 we show a relation between the solvability in Orlicz context of Dirichlet and Neumann problem for a special class of operators.

Chapter 1

Definitions and backgrounds

In this chapter we introduce the formulation of the Dirichlet problem with boundary data in the Lebesgue space $L^p(d\sigma)$ and we report some of the known results.

1.1 The classical Dirichlet problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. For $K \geqslant 1$ we consider the class $\mathcal{E}(K)$ of measurable matrix field $A(x) = (a_{i,j}(x))_{i,j=1}^n \in L^\infty(\Omega, \mathbb{R}^n \times \mathbb{R}^n)$ verifying the uniform ellipticity condition:

$$\frac{|\xi|^2}{K} \leqslant \langle A(x)\xi, \xi\rangle \leqslant K\,|\xi|^2 \tag{1.1}$$

a.e $x \in \Omega$ and for all $\xi \in \mathbb{R}^n \setminus \{0\}$. The matrix A will not be assumed to be symmetric.

The space $W^{1,2}_{loc}(\Omega)$ denotes $\{f \in L^2_{loc}(\Omega) : \varphi f \in W^{1,2}(\Omega), \forall \varphi \in C_0^\infty(\Omega)\}$ where $W^{1,2}(\Omega)$ is the usual Sobolev space $\{f \in L^2(\Omega) : \int_\Omega |f|^2 + \int_\Omega |\nabla f|^2 < \infty\}$.

Consider the linear second order elliptic operator in divergence form

$$\mathcal{L} = \text{div}\,(A\nabla) = \frac{\partial}{\partial x_i} a_{i,j}(x) \frac{\partial}{\partial x_j}$$

(the repeated indices summation convention is used).

Definition 1.1.1. *A function $u \in W^{1,2}_{loc}(\Omega)$ is a solution to $\mathcal{L}u = 0$ in Ω if*

$$\int_\Omega a_{i,j}(x)\frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_j} = 0, \quad \forall \varphi \in C_0^\infty(\Omega). \tag{1.2}$$

Thanks to the pioneering work of [DG], [Mos] [N] and [LSW] we have local regularity, Harnack's principle, maximum principle, pointwise bounds for such solutions. It was observed firstly by Morrey [Mo] that the symmetry of the matrix A is not needed to get these results (see also [KKPT]). We report here some of these fundamental estimates. Here and below, we denote by $\fint_E f d\sigma$ the mean value of $f \in L^1(\partial\Omega)$ over the σ-measurable subset $E \subset \partial\Omega$. That is, $\fint_E f d\sigma = f_E = \frac{1}{\sigma(E)}\int_E f d\sigma$, where $\sigma(E) = \int_E d\sigma$.

Lemma 1.1.1. (Caccioppoli) *If $u \geqslant 0$ is an $\mathcal{L}$-subsolution in Ω (i.e. the integral in* (1.2) *is non-positive) and if $r > 0$ is such that $\overline{B_{2r}(X)} \subset \Omega$. Then,*

$$\fint_{B_r(X)} |\nabla u(z)|^2 dz \leqslant \frac{C_{K,n}}{r^2}\fint_{B_{2r}(X)} u^2(z)dz.$$

The interior regularity estimates are as follows. Here, $\mathrm{osc}_{B_r} u = \sup_{B_r} u - \inf_{B_r} u$ denotes the oscillation of u over the ball B_r.

Lemma 1.1.2. *If u is a nonnegative subsolution in Ω and $\overline{B_{2r}(X)} \subset \Omega$ then*

$$\sup_{B_r(X)} u \leqslant C_{K,p,n}\left(\fint_{B_{2r}(X)} u^p\right)^{\frac{1}{p}}, \qquad \forall p > 0$$

Lemma 1.1.3. (Interior Hölder Continuity) *If u is a solution to $\mathcal{L}$ in Ω, then*

$$osc_{B_r(X)} u \leqslant C_{K,n}\left(\frac{r}{R}\right)^\alpha \left(\fint_{B_R(X)} u^2\right)^{\frac{1}{2}}$$

for some $0 < \alpha < 1$, $\alpha = \alpha(K,n)$ and $0 < r < R < dist(X, \partial\Omega)$.

It is worth to point out here that the Hölder continuity rate of the solution only depends on the ellipticity of the operator.

Lemma 1.1.4. (Harnack's inequality) *Let $u \geqslant 0$ be a solution to the equation $\mathcal{L}u = 0$ in Ω, and assume that $r > 0$ is such that $B_{2r}(X) \subset \Omega$. Then,*

$$\sup_{B_r(X)} u \leqslant C_{K,n} \inf_{B_r(X)} u \tag{1.3}$$

Moreover, it holds

Lemma 1.1.5. *If u is a solution to $\mathcal{L}u = 0$ in Ω and $r > 0$ is such that $B_{2r}(X) \subset \Omega$, then there exists a $p > 2$, $p = p(K,n)$, such that*

$$\left(⨍_{B_r(X)} |\nabla u|^p dz\right)^{\frac{1}{p}} \leqslant C \left(⨍_{B_{2r}(X)} |\nabla u|^2 dz\right)^{\frac{1}{2}}. \tag{1.4}$$

Lemma 1.1.6. (Maximum principle) *If u is a solution to $\mathcal{L}u = 0$ in Ω, which is continuous in a neighborhood of $\partial\Omega$, then*

$$\sup_{\Omega} u \leqslant \sup_{\partial\Omega} u.$$

Now, let $f \in W_0^{1,2}(\Omega)^*$ (here $W_0^{1,2}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,2}(\Omega)$). By the Lax-Milgram lemma, there exists a unique $w \in W_0^{1,2}(\Omega)$ such that $\mathcal{L}w = f$ in Ω, in the sense that

$$\int_\Omega a_{i,j} \frac{\partial w}{\partial x_i} \frac{\partial \varphi}{\partial x_j} = \langle \varphi, f \rangle, \quad \forall \varphi \in C_0^\infty(\Omega).$$

Consider $g \in Lip_0(\mathbb{R}^n)$ such that $G_{|\partial\Omega} = g$, i.e. $\operatorname{supp} G$ is compact and

$$|G(x) - G(y)| \leqslant C|x - y|, \quad \forall x, y \in \mathbb{R}^n.$$

and let $f = \mathcal{L}G = \frac{\partial}{\partial x_i} a_{i,j}(x) \frac{\partial}{\partial x_j} G \in W_0^{1,2}(\Omega)^*$. Hence, there exists $w \in W_0^{1,2}(\Omega)$ which solve $\mathcal{L}w = \mathcal{L}G$ in the sense described above. Let $u = G - w$. Then, $u \in W^{1,2}(\Omega)$ and $\mathcal{L}u = 0$. Since $w \in W_0^{1,2}(\Omega)$, then $u_{|\partial\Omega}$ "$=$" g. Such u is called the *generalized solution* of the classical Dirichlet problem with data g.

It is worth to point out that u is well defined since if $G_1, G_2 \in Lip_0(\mathbb{R}^n)$, $G_{1|\partial\Omega} = G_{2|\partial\Omega} =$

g, then $G_1 - G_2 \in W_0^{1,2}(\Omega)$, and so $u_1 - u_2 \in W_0^{1,2}(\Omega)$, $\mathcal{L}(u_1 - u_2) = 0$ and hence $u_1 \equiv u_2$.

Suppose that for all $g \in Lip(\partial\Omega)$, the generalized solution $u \in C(\bar{\Omega})$, and consider now $f \in C(\partial\Omega)$. We find a sequence $g_j \in Lip(\partial\Omega)$, such that $g_j \to f$ uniformly on $\partial\Omega$. Denoting by u_j the corresponding solutions to the problems with data g_j, by Lemma 1.1.6 we have

$$\max_{\Omega} |u_j - u_k| \leqslant \max_{\Omega} |g_j - g_k|.$$

Thus $\{u_j\}$ converges uniformly in Ω to $u \in C(\bar{\Omega})$. Noting that for such solution a Caccioppoli inequality holds, for any $j \in \mathbb{N}$

$$⨍_{B_r(X)} |\nabla u_j|^2 dX \leqslant C(K, n) r^{-2} ⨍_{B_{2r}(X)} u_j^2$$

where $r > 0$ is such that $B_{2r} \subset \Omega$, we have $u \in W_{loc}^{1,2}(\Omega)$, $\mathcal{L}u = 0$ in Ω and $u_{|\partial\Omega} = f$. Another application of the maximum principle shows that u is independent of the choice of $\{g_j\}$ and hence is unique.

Definition 1.1.2. *A domain Ω is said to be regular for the operator $\mathcal{L}$ if for every boundary data $g \in Lip(\partial\Omega)$ the generalized solution of the classical Dirichlet problem $u \in C(\bar{\Omega})$.*

Theorem 1.1.7. *Ω is regular for $\mathcal{L}$ if and only if Ω is regular for the Laplacian $\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$.*

The notion of generalized solution, and of regular domain, come from the work of Littman, Stampacchia and Weinberger [LSW], which also proves the following *Wiener test* to characterize regular domains for our class of operators. It involves the notion of *capacity* that we now recall. If $E \subset B = \{|x| < 1\}$ is a closed set, then,

$$\operatorname{cap}(E) = \inf \int_B |\nabla \varphi|^2$$

where the infimum is taken over all $\varphi \in C_0^\infty(B)$, with $\varphi \geqslant 1$ on E.

Theorem 1.1.8. *Let $\Omega \subset\subset B = \{|x| < 1\}$. Then Ω is regular if and only if*

$$\int_0 cap({}^c\Omega \cup B_r(x))\frac{dr}{r^{n-1}} = +\infty$$

for all $x \in \partial\Omega$.

In particular any bounded Lipschitz domain Ω is regular.

We shall now recall a key notion of the theory, namely the '*harmonic measure*' associated with $\mathcal{L}$. To this effect let Ω be a regular domain in $\mathbb{R}^n$. Moreover, let $f \in C(\partial\Omega)$, $X \in \Omega$ and let us consider the linear functional

$$f \longrightarrow u(X) \tag{1.5}$$

on $C(\partial\Omega)$ where $u \in W^{1,2}_{loc}(\Omega) \cap C(\bar{\Omega})$ is the generalized solution of the classical Dirichlet problem (1.32). By the maximum principle, (1.5) is a bounded, positive continuous linear functional and $u \equiv 1$ if $f \equiv 1$. Therefore, by the Riesz representation theorem, there exists a family of regular Borel probability measures

$$\{\omega^X_{\mathcal{L}}\}_{X\in\Omega}$$

such that u represents as

$$u(X) = \int_{\partial\Omega} f(Q)d\omega^X_{\mathcal{L}}(Q) \tag{1.6}$$

This family of measures is called **$\mathcal{L}$-harmonic measure**. When no confusion arises, we will omit the reference to $\mathcal{L}$. Moreover, when $\Omega = B$, the unit ball in $\mathbb{R}^n$, we will simply denote by $\omega_{\mathcal{L}} = \omega^O_{\mathcal{L}}$ the harmonic measure of $\mathcal{L}$ in B evaluated at the origin O of the unit ball B. By abuse of notation we will sometimes refer to ω as the harmonic measure of $\mathcal{L}$ on Ω.

Next Lemma shows that the measures of the family ω^X, as X varies over Ω, are mutually absolutely continuous:

Lemma 1.1.9. *Let $E \subset \partial\Omega$ be a Borel set. Then $\omega^{X_0}(E) = 0$ if and only if $\omega^X(E) = 0$ for any $X \in \Omega$.*

Proof. By regularity of ω^{X_0} and ω^X it is enough to establish the claim for the compact subset of $\partial\Omega$. So, let $K \subset \partial\Omega$ be a compact set and suppose that $\omega^{X_0}(K) = 0$. Now, let $\varepsilon > 0$ be given. We can find an open set $U \supset K$ such that $\omega^{X_0}(U) < \varepsilon$. Let $g \in C(\partial\Omega)$, $0 \leqslant g \leqslant 1$, $g \equiv 1$ on K and let $u(X)$ be the generalized solution of the classical Dirichlet problem with data g. Clearly $\omega^X(K) \leqslant u(X)$. In fact by the non-negativity of g,

$$u(X) = \int_\Omega g d\omega^X \geqslant \int_K g d\omega^X = \omega^X(K) \tag{1.7}$$

Fix such an X and let $\Gamma \subset \Omega$ be a compact set containing X_0 and X. Hence we can recover Γ by a finite number m of balls $B_j = B(X_j, r_j) \subset \Omega$ $(J = 0, ..., m)$, such that $X_m = X$, $B(X_j, 2r_j) \subset \Omega$ and $B_{j-1} \cap B_j \neq \emptyset$, $j = 1, ..., m$. So, let $Y_j \in B_{j-1} \cap B_j$, $j = 1, ..., m$. We have, applying Lemma 1.1.4, $C = C(K, n)$,

$$u(X) = u(X_m) \leqslant Cu(Y_m) \leqslant C^2 u(Y_{m-1}) \leqslant ... \leqslant C^m u(Y_1) \leqslant C^{m+1} u(X_0).$$

Thus,

$$u(X) \leqslant C(K, n, X, X_0) u(X_0) \tag{1.8}$$

Therefore, by (1.7) and (1.8), it holds

$$\omega^X(K) \leqslant u(X) \leqslant Cu(X_0) = C \int_{\partial\Omega} g d\omega^{X_0} = C\omega^{X_0}(U) < C\varepsilon$$

and then $\omega^X(K) = 0$.

□

As we will see, for the purpose of solving boundary value problems, it is necessary to study the relationship between the harmonic measure $d\omega_{\mathcal{L}}$ and the surface measure $d\sigma$ for a given domain Ω. To this aim we need to introduce the Green's function and determine its relationship to harmonic measure. In [GW], Grüter and Widman made a systematic study of the Green's function, without assuming the symmetry of the matrix.

Theorem 1.1.10. [GW] *There exists a unique function $G : \Omega \times \Omega \to \mathbb{R} \cup \{+\infty\}$, $G \geqslant 0$, such that, for each $Y \in \Omega$ and $r > 0$,*

i) $G(\cdot, Y) \in W^{1,2}(\Omega \setminus B_r(Y)) \cap W_0^{1,1}(\Omega)$

ii) $\forall \varphi \in C_0^\infty(\Omega)$

$$\int a_{i,j}(X) \frac{\partial}{\partial X_i} G(X,Y) \frac{\partial}{\partial X_j} \varphi(X) dX = \varphi(Y)$$

(i.e. '$LG(\cdot, Y) = -\delta_Y$').

iii) $G(Y,X) = G^*(X,Y)$, *where G^* satisfies i) and ii) for A^*, the adjoint of A.*

iv) $G(X,Y) \leqslant C_K|X-Y|^{2-n}, \qquad \forall X, Y \in \Omega,$

v) $G(X,Y) \geqslant C_K|X-Y|^{2-n}, \qquad \forall X, Y \in \Omega,\ |X-Y| \leqslant \frac{1}{2}\ dist(Y, \partial\Omega)$

vi) $G(\cdot, Y) \in W_0^{1,p}(\Omega)$ *for all $1 \leqslant p \leqslant \frac{n}{n-1}$, uniformly in Y.*

vii) $G(X,Y) \leqslant C_K dist(Y, \partial\Omega)^\alpha |X-Y|^{2-n-\alpha}$, *where $\alpha = \alpha(K,n)$.*

viii) $|G(X,Y) - G(Z,Y)| \leqslant C_K(|X-Z|^\alpha)(|X-Y|^{2-n-\alpha} + |Z-Y|^{2-n-\alpha})$

Note that in dimension $n = 2$ the singularity in the bounds on the Green's function would be logarithmic.

In a smooth domain like the unit ball B, if the coefficients matrix $A \in C^\infty(\mathbb{R}^n)$, Green's theorem shows that $G \in C^\infty(\bar{B} \times \bar{B} \setminus \{(X,X) : X \in \bar{B}\})$. Green's formula then shows that

$$d\omega^{X_0}(Q) = A^*(Q)\nabla_Q G^*(Q,X_0) \cdot \overrightarrow{N}(Q) d\sigma$$

where $\overrightarrow{N}(Q)$ is the outward unit normal at $Q \in \partial B$. Moreover, by the Hopf maximum principle we have that

$$\left\langle A^*(Q)\nabla_Q G^*(Q,X_0), \overrightarrow{N}(Q) \right\rangle \geqslant \delta > 0$$

and hence $\log \left\langle A^*(Q)\nabla_Q G^*(Q,X_0), \overrightarrow{N}(Q) \right\rangle \in C^\infty(\partial B)$. Also since the generalized solution of the classical Dirichlet problem with data $g \in C(\partial B)$ is given by

$$u(X) = \int_{\partial B} A^*(Q)\nabla_Q G^*(Q,X_0) \cdot \overrightarrow{N}(Q) g(Q) d\sigma(Q)$$

it is obvious that the above expression still makes sense with $g \in L^p$, $1 \leqslant p \leqslant \infty$. Moreover, introducing the non-tangential approach regions

$$\Gamma_\beta(Q) = \{X \in B : |X - Q| \leqslant (1+\beta)\,\mathrm{dist}(X, \partial B)\} \tag{1.9}$$

($\beta > 0$) and, for any $Q \in \partial B$, the non tangential maximal function,

$$Nu(Q) = \sup_{X \in \Gamma_\beta(Q)} |u(X)| \tag{1.10}$$

one has (see [K] and reference therein contained),

$$Nu(Q) \leqslant C_\beta M(g)(Q) \tag{1.11}$$

where $Mg(Q) = \sup_{\Delta \ni Q} \frac{1}{\sigma(\Delta)} \int_\Delta |g(P)| d\sigma(P)$ is the Hardy-Littlewood maximal operator on

∂B. Because of the known estimate for M (see for example [Gc-RdF]), one has

$$\|Nu\|_{L^p(\partial B, d\sigma)} \leqslant C_{\beta,p} \|g\|_{L^p(\partial B, d\sigma)}, \qquad 1 < p \leqslant \infty \tag{1.12}$$

$$\sigma\{Nu > t\} \leqslant \frac{C_\beta}{t} \int_{\partial B} |g| d\sigma$$

This, combined with the fact that, for $g \in C(\partial B)$, $u \in C(\bar{B})$ allows one to conclude that, for $g \in L^p(\partial B, d\sigma)$, $1 \leqslant p \leqslant \infty$, u converges non-tangentially to g a.e. with respect to the measure $d\sigma$, i.e.

$$\lim_{X \to Y,\, X \in \Gamma_\beta} u(X) = u(Y)$$

for σ-almost every $Y \in \partial B$.

In general, to establish the relationship between the Green's function and harmonic measure is more delicate. This was carried out in [CFMS] (owing a great deal to the estimates in [HW2]) for symmetric elliptic operators $\mathcal{L}$. However, a careful inspection of the proofs of the results therein contained shows that all the estimates remain valid (with G replaced by G^* where appropriate) even in the non-symmetric case. This was observed by [KKPT]. We summarize these below.

1.2 Properties of harmonic measure

For the convenience of the reader we list here some of the most useful properties of the harmonic measure $\omega_{\mathcal{L}}$ for an operator $\mathcal{L}$ in our class. In any case we refer to [K] for more details.

Here and below we will restrict our attention to the case when Ω is the unit ball $B \subset \mathbb{R}^n$. Since our class of operators is invariant under bi-Lipschitzian transformations of $\mathbb{R}^n$, the following results extend immediately to Ω bounded Lipschitz domain. *

*In particular our argument depend only on certain geometric properties of B characterizing a special class of domains, the so called Non-tangentially accessible domains, N.T.A. (see for example [JK] for more details) useful in the study of regularity of free boundaries

i) Let $Q \in \partial B$ and let $A_r(Q) = (1-r)Q$. Then, there exists a positive constant M such that

$$\omega^X(\Delta_r(Q)) \geqslant M.$$

for any $X \in B_{r/2}(A_r(Q))$.

ii) For any point $X \in B \setminus B_{r/2}(A_r(Q))$ it holds

$$r^{n-2}G(X, A_r(Q)) \leqslant M\omega^X(\Delta_{2r}(Q))$$

iii) For any point $X \in B \setminus B_{2r}(Q)$ it holds

$$\omega^X(\Delta_r(Q)) \leqslant Mr^{n-2}G(X, A_r(Q)).$$

Hence, by *ii*), *iii*) and the Harnack's inequality, we have

iv) For $X \in B \setminus B_{2r}(Q)$,

$$\omega^X(\Delta_r(Q)) \simeq r^{n-2}G(X, A_r(Q)). \tag{1.13}$$

In particular, for any $X \in B \setminus B_{4r}(Q)$,

$$\omega^X(\Delta_{2r}(Q)) \leqslant M\omega^X(\Delta_r(Q)). \tag{1.14}$$

Condition (1.14) is called *doubling condition* of the harmonic measure. In Chapter 5 we will also investigate about this property (see Section 5.3).

Here and below, for any $Q \in \partial B$ we assume $T_r(Q) = B_r(Q) \cap B$.

v) **(Comparison Principle)** If u, v are nonnegative solutions in $T_{2r}(Q)$, continuous in $\overline{T_{2r}(Q)}$ and vanishing on $\Delta_{2r}(Q)$, then there exists a constant $M > 0$ such that for any

$X \in T_r(Q)$ it holds

$$M^{-1}\frac{u(A_r(Q))}{v(A_r(Q))} \leqslant \frac{u(X)}{v(X)} \leqslant M\frac{u(A_r(Q))}{v(A_r(Q))}$$

vi) Let $Q, Q_0 \in \partial B$, $\Delta = \Delta_r(Q)$ and let $\Delta' = \Delta_s(Q_0) \subset \Delta$. Then,

$$\omega^{A_r(Q)}(\Delta') \simeq \frac{\omega^X(\Delta')}{\omega^X(\Delta)} \tag{1.15}$$

for any $X \in B \setminus T_{2r}(Q)$.

Definition 1.2.1. *The Radon-Nykodym derivative of* ω^X *with respect to* ω*, i.e. the function*

$$K(X,Q) = \frac{d\omega^X}{d\omega}(Q).$$

is defined to be the Kernel function of $\mathcal{L}$*.*

Note that by the mutual absolute continuity of ω^X and ω (see Theorem 1.1.9) K is well defined. Moreover, by the *doubling property* of ω (1.14) and the Lebesgue differentiation theorem for doubling measures, for a.e. $Q \in \partial B$ with respect to ω,

$$K(X,Q) = \lim_{\Delta' \downarrow Q} \frac{\omega^X(\Delta')}{\omega(\Delta')}. \tag{1.16}$$

A priori K is defined for ω-a.e. $Q \in \partial B$. However it is shown that the limit in (1.16) exists for σ-a.e. Q and that K is Hölder continuous with respect to Q. In particular it satisfies the following two estimates

i) Let us fix a point Q_0 on the boundary of B and assume $A = A_r(Q_0)$, $\Delta_j = \Delta_{2^j r}(Q_0)$ and $R_j = \Delta_j \setminus \Delta_{j-1}, j \geqslant 0$. Then,

$$\operatorname*{ess\,sup}_{Q \in R_j} K(A,Q) \leqslant M\frac{2^{-j\alpha}}{\omega(\Delta_j)}. \tag{1.17}$$

where $\alpha = \alpha(K, n) > 0$. In particular, on $\Delta_{j+1} \setminus \Delta_j$ we have

$$K(X,Q) \leqslant C \frac{2^{-j\alpha}}{\omega(\Delta_{j+1})}. \tag{1.18}$$

ii) For any $X \in B$,

$$|K(X,Q_1) - K(X,Q_2)| \leqslant C_X |Q_1 - Q_2|^{\alpha}$$

where α is a positive constant depending on $\mathcal{L}$.

For $f \in L^1(d\omega)$ define $u(X) = \int_{\partial B} f d\omega^X = \int_{\partial B} f(Q)K(X,Q)d\omega(Q)$, so that $Lu = 0$ in B. More generally, if ν is finite, signed, Borel measure on B since $K(X,\cdot)$ is continuous, we can define $u(X) = \int_{\partial B} K(X,Q)d\nu(Q)$, which is again a solution to $Lu = 0$. For $f \in L^1(d\omega)$, we let

$$M_\omega f = \sup_{\Delta \ni Q} \frac{1}{\omega(\Delta)} \int_\Delta |f| d\omega$$

and for ν a finite Borel measure, with total variation $|\nu|$,

$$M_\omega(\nu) = \sup_{\Delta \ni Q} \frac{1}{\omega(\Delta)} |\nu|(\Delta)$$

denote the Hardy-Littlewood maximal operator associated to ω on ∂B (see Chapter 3 for more details). Since ω verifies the doubling condition we have

Theorem 1.2.1. *The following estimates are true:*

i) $\omega\{Q \in \partial B : M_\omega(\nu)(Q) > t\} \leqslant \frac{M}{t}|\nu|(\partial B)$.

ii) $\|M_\omega f\|_{L^p(d\omega)} \leqslant M_p \|f\|_{L^p(d\omega)}, \quad 1 < p \leqslant \infty$.

Lemma 1.2.2. *Let ν be a finite Borel measure on ∂B and let*

$$u(X) = \int_{\partial B} K(X,Q) d\nu.$$

Then

i) For any point $P \in \partial B$,

$$Nu(P) \leqslant C_\alpha M_\omega(\nu)(P).$$

ii) If in additional $\nu \geqslant 0$ *then*

$$M_\omega(\nu)(P) \leqslant C_\alpha Nu(P)$$

Proof. Let us start by proving i). To this aim, let $P \in \partial B$, $X \in \Gamma_\beta(P)$ and let $r = |X - P| \simeq \operatorname{dist}(X, \partial B)$. Moreover, assume $\Delta_j = \Delta_{2^j r}(P)$. Then,

$$u(X) = \sum_{j=0}^{\infty} \int_{\Delta_{j+1} \setminus \Delta_j} K(X,Q) d\nu(Q) + \int_{\Delta_r(P)} K(X,Q) d\nu(Q)$$

Now, by (1.18) we have

$$\sum_{j=0}^{\infty} \int_{\Delta_{j+1} \setminus \Delta_j} K(X,Q) d\nu(Q) \leqslant C_\alpha M_\omega(\nu)(P).$$

On the other hand, by (1.16) and (1.15), for any $Q \in \Delta_r(P)$,

$$K(X,Q) \simeq \frac{1}{\omega(\Delta_r(P))}.$$

Then

$$\int_{\Delta_r(P)} K(X,Q) d\nu(Q) \leqslant C_\alpha M_\omega(\nu)(P),$$

so that i) follows.

To prove ii), let $\nu \geqslant 0$. Then

$$u(X) \geqslant \int_{\Delta_r(P)} K(X,Q) d\nu(Q) \simeq \frac{1}{\omega(\Delta_r(P))} \int_{\Delta_r(P)} d\nu,$$

so that observing that $r > 0$ is arbitrary, the thesis is completely proved. $\square$

1.3 A brief review of the real variable theory of weights

To explain some of the known results on the L^p- solvability we need to recall some facts about the real variable theory of weights, which are the key ingredients in a number of important papers [D1], [D2], [CFMS], [FKP], [K].

Definition 1.3.1. *A function $w : \partial B \to \mathbb{R}$ will be called a **weight** if it is positive and if $w \in L^1(\partial B, d\sigma)$, σ being the surface measure on ∂B.*

Let μ be any non negative, Borel measure on ∂B satisfying the doubling condition

$$\mu(\Delta_{2r}(Q)) \leqslant C\mu(\Delta_r(Q)) \tag{1.19}$$

where Q is a point on $\partial B, \Delta_r(Q)=$ $\mathrm{B}_r(Q) \cap \partial\mathrm{B}$, $B_r(Q)$ the ball of $\mathbb{R}^n$ with center Q and radius r (for example $\mu = \omega$, the harmonic measure associated to any elliptic operator $\mathcal{L}$, or $\mu = \sigma$).

Let us now introduce some definitions about the A_∞- class of measures on $\partial\mathbb{B}$.

Definition 1.3.2. *Let ν be another non-negative measure on ∂B. Then ν belongs to $A_\infty(d\mu)$, if there exist constants $0 < \beta \leqslant 1 \leqslant H < \infty$ so that*

$$\frac{\nu(E)}{\nu(\Delta)} \leqslant H \left(\frac{\mu(E)}{\mu(\Delta)}\right)^\beta, \tag{1.20}$$

for any spherical ball $\Delta \subset \partial B$ and any measurable set $E \subset \Delta$.

Condition (1.20) implies that ν is absolutely continuous with respect to μ. For this reason the Radon Nikodym derivative $k = d\nu/d\mu$ is viewed as a weight, which we will call an A_∞-weight. We then sometimes will write $k \in A_\infty$.

Moreover (1.20) is a 'scale invariant' version of absolute continuity, which unlike ordinary absolute continuity, defines an equivalence relationship (see [Gc-RdF], [K], [Go], [R] for more details).

Definition 1.3.3. *Let μ and ν be as before. We say that the measure μ supported on ∂B belongs to the Gehring class $B_q(d\nu)$ (and we will write $\mu \in B_q(d\nu)$), $q > 1$, if $d\mu$ is absolutely continuous with resect to $d\nu$, i.e. $d\mu = kd\nu$, and the Radon-Nikodym derivative $k = \frac{d\nu}{d\mu} \in L^q(d\nu)$ and verifies the "reverse Hölder inequality"*

$$\left(\frac{1}{\nu(\Delta_r(Q))} \int_{\Delta_r(Q)} k^q d\nu \right)^{\frac{1}{q}} \leqslant C \left(\frac{1}{\nu(\Delta_r(Q))} \int_{\Delta_r(Q)} kd\nu \right)$$

for all surface ball $\Delta_r(Q)$.

It is well known that A_∞ is the union of Gehring classes B_q:

$$A_\infty = \cup_{q>1} B_q \tag{1.21}$$

Definition 1.3.4. *For any A_∞ measure ν on ∂B we define*

$$\tilde{B}_1(\nu) = \inf \left\{ \frac{H}{\beta} : 0 < \beta \leqslant 1 \leqslant H \textit{ and condition (1.20) holds} \right\}. \tag{1.22}$$

If we switch the role of the measures σ and ν on ∂B in (1.20) are preserved the properties of the weights supported (see [CF]).

Theorem 1.3.1. *The measure ν supported on ∂B belongs to A_∞ with respect to σ if and only if there exist constants $0 < \alpha \leqslant 1 \leqslant M$ such that*

$$\frac{\sigma(F)}{\sigma(\Delta)} \leqslant M \left(\frac{\nu(F)}{\nu(\Delta)} \right)^\alpha, \tag{1.23}$$

for any spherical ball $\Delta \subset \partial B$ and for any measurable set $F \subset \Delta$.

It is therefore natural to associate to weight ν a constant defined as

$$\tilde{A}_\infty(\nu) = \inf \left\{ \frac{M}{\alpha} : 0 < \alpha \leqslant 1 \leqslant M \text{ and condition (1.23) holds} \right\}. \tag{1.24}$$

We emphasize explicitly that a measure ν belongs to A_∞ if and only if $\tilde{A}_\infty(\nu) < \infty$ or, equivalently, $\tilde{B}_1(\nu) < \infty$. That is why we will call (1.22) and (1.24) A_∞- constants of ν.

Remark 1.3.1. If $n = 2$ and ω is defined by $\frac{d\omega}{d\sigma} = \sigma^\alpha$ with $\alpha \in (-1, 0]$, then $\omega \in A_\infty$ and $\tilde{B}_1(\omega) = \frac{1}{\alpha+1}$.

The main properties of this class of measures are summarized in what follows.

Theorem 1.3.2. *The following properties hold:*

(i) $\nu \in A_\infty(d\mu)$ if and only if, given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that if $E \subset \Delta_r(Q)$, $\Delta_r(Q)$ any surface ball, then

$$\frac{\mu(E)}{\mu(\Delta_r(Q))} < \delta \Rightarrow \frac{\nu(E)}{\nu(\Delta_r(Q))} < \epsilon.$$

(ii) $\nu \in A_\infty(d\mu)$ if and only if there exist $C > 0$, $\eta > 0$, $\vartheta > 0$, such that $\forall E \subset \Delta_r(Q)$, we have

$$\frac{\mu(E)}{\mu(\Delta_r(Q))} \leqslant C \left(\frac{\nu(E)}{\nu(\Delta_r(Q))}\right)^\vartheta$$

and

$$\frac{\nu(E)}{\nu(\Delta_r(Q))} \leqslant C \left(\frac{\mu(E)}{\mu(\Delta_r(Q))}\right)^\eta.$$

(iii) If $\nu \in B_q(d\mu)$, $q > 1$, then there exists $\epsilon > 0$ such that $\nu \in B_{q+\epsilon}(d\mu)$.

(iv) $\nu \in B_q(d\mu)$ if and only if

$$M_\nu f = \sup_{Q\in\Delta} \fint_\Delta |f| d\nu,$$

verifies

$$\|M_\nu f\|_{L^p(d\mu)} \leqslant C\|f\|_{L^p(d\mu)}, \quad \frac{1}{p} + \frac{1}{q} = 1$$

Definition 1.3.5. *For $1 < q < \infty$, let $\nu \in B_q(d\mu)$, and let k be as above. We define B_q-*

constant of the measure ν with respect to μ, the quantities

$$B_{q,\mu}(\nu) = \sup_{\Delta} \left[\frac{\left(\frac{1}{\mu(\Delta)} \int_{\Delta} k^q d\mu \right)^{\frac{1}{q}}}{\frac{1}{\mu(\Delta)} \int_{\Delta} k d\mu} \right]^p, \quad \frac{1}{q} + \frac{1}{p} = 1, \tag{1.25}$$

where the supremum is taken over all the surface ball $\Delta \subset \partial B$.

We report here the following sharp result on higher integrability for B_q weights (see [S], Theorem 2.1), which represents the quantitative form of the "self improvement" property of Gehring classes which is optimal in one dimension.

Theorem 1.3.3. **[S]** *Let $q > 1$ and assume that $\omega : [a, b] \subset \mathbb{R} \to [0, +\infty[$ satisfies the condition*

$$B_q(\omega) = B_{q,dx}(\omega) = B < \infty. \tag{1.26}$$

Let $q_1 > q$ be the unique solution to the equation:

$$\varphi(y) = 1 - B^{q-1} \frac{y - q}{y} \left(\frac{y}{y - 1} \right)^q = 0. \tag{1.27}$$

Then, for $q \leqslant \theta < q_1$,

$$[B_\theta(\omega)]^{\frac{1}{\theta'}} \leqslant B^{\frac{1}{q'}} \left[\frac{q}{\theta \varphi(\theta)} \right]^{\frac{1}{q}} \tag{1.28}$$

($\frac{1}{\theta} + \frac{1}{\theta'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$). *The result is sharp, because there exists ω satisfying (1.26) not belonging to $L^{q_1}_{loc}([a, b])$*

The class of B_q weights arises in connection with the Muckenhoupt weights, namely the space A_p.

Definition 1.3.6. *Let $1 < p < \infty$. We say that the measure ω belongs to the Muckenhoupt class A_p if ω is absolutely continuous with respect to σ and the Radon-Nikodym derivative $k = \frac{d\omega}{d\sigma}$ satisfies the condition*

$$A_p(\omega) = \sup_{\Delta} \left(\fint_{\Delta} k d\sigma \right) \left(\fint_{\Delta} k^{\frac{1}{1-p}} d\sigma \right)^{p-1} < \infty \tag{1.29}$$

where the supremum is taken over all surface ball $\Delta \subset \partial B$.

The constant $A_p(\omega) \geqslant 1$ is named A_p- constant of ω. It is well known that $A_\infty = \cup_{p>1} A_p$, $\nu \in B_q(d\mu)$ if and only if $\mu \in A_p(d\mu)$, $\frac{1}{p} + \frac{1}{q} = 1$. see [Gc-RdF].

From the definition one can easily see that if $p > r$ then $A_p \subset A_r$ and $A_\infty = \cup_r A_r$. It can also be proved that if $\omega \in A_p$ then $d\omega$ is a doubling measure (see [St1] Chapter 5, 1.5). However, the converse is not true, as the function $w(x) = |x|^\alpha$ is doubling if $-n < \alpha$ but it is in A_p only if in addition $\alpha < n(p-1)$. An even better example is given by the totally singular doubling measure $d\mu = \Pi_{k=1}^\infty [1 + a\cos(3^k 2\pi x)]dx$ where $-1 < \alpha < 1$ (see [St1], Chapter 1 or [Zy], Chapter 5 for more details).

The following theorem provides a link between the classes A_p and B_q.

Following the proof of Theorem 1 in [C], one can see that it holds:

Proposition 1.3.4. *Let w be a weight on ∂B such that the measure $d\mu = wd\sigma$ is doubling. Let $d\nu = zd\mu$, $z > 0$ on ∂B and $z \in L^1(d\mu)$. If there exist $0 < \gamma \leqslant 1$ and $C > 0$ such that*

$$\frac{\mu(E)}{\mu(\Delta)} \leqslant C \left(\frac{\nu(E)}{\nu(\Delta)} \right)^\gamma, \quad \forall \Delta, \quad \forall E \subset \Delta \tag{1.30}$$

then there exist $\delta > 0$, $K > 0$, such that

$$\left(\frac{1}{\mu(\Delta)} \int_\Delta z^{1+\delta} d\mu \right)^{\frac{1}{1+\delta}} \leqslant K \frac{1}{\mu(\Delta)} \int_\Delta z d\mu, \quad \forall \Delta. \tag{1.31}$$

Moreover the constants K and δ in (1.31) are dependent only upon the constants C and γ in (1.30) and upon the constant in the doubling condition of μ.

For more details we refer the reader to the papers B. Muckenhoupt [M], R. R. Coifman and C. Fefferman [CF], A. P. Calderón [C] where the theory of A_∞ weights is extensively studied.

1.4 The L^p-Dirichlet problem.

In the previous sections we described a series of results on the Dirichlet problem for general second order elliptic, divergence form operators with bounded measurable coefficients. Also, we pointed out (in the comments after Theorem 1.1.10, in particular inequalities (1.11) and (1.12)) how, in the case when the coefficients are smooth, further results are possible. We will now isolate several particularly interesting questions, which are well understood for smooth coefficients, and formulate them for general operators $\mathcal{L}$, in more general context. Deciding to what extent these facts remain valid in this situation has been the subject of intense investigation in the last twentyfive years.

Hence, let us consider the classical Dirichlet boundary value problem:

$$\begin{cases} \mathcal{L}u = 0 \quad \text{in } \mathbb{B} \\ u_{|\partial\mathbb{B}} = f \end{cases} \tag{1.32}$$

where

$$\mathcal{L} = \operatorname{div}(A(x)\nabla\,) \tag{1.33}$$

is an elliptic operator with coefficient matrix $A \in \mathcal{E}(K)$.

Definition 1.4.1. *For $1 < p < \infty$, Problem (1.32) is called L^p-* solvable *and the operator (1.33) is said L^p-* resolutive*, if there exists a constant $C_p > 0$ for which the following holds:*

For any $f \in C(\partial\mathbb{B})$ the unique solution $u \in W^{1,2}_{loc}(\mathbb{B}) \cap C(\bar{\mathbb{B}})$ to (1.32) satisfies the uniform estimate

$$\|Nu\|_{L^p(\partial\mathbb{B},d\sigma)} \leqslant C_p \, \|f\|_{L^p(\partial\mathbb{B},d\sigma)} \,. \tag{1.34}$$

Note that one can similarly define the Dirichlet problem in $L^p(d\mu)$ where μ is a general measure on ∂B.

Now, by Lemma 1.2.2 and Theorem 1.3.2, *iv)* we are in position to recall the following key result:

Theorem 1.4.1. **[K]** *Let $1 < p < \infty$, $q = \frac{p}{p-1}$. The following are equivalent:*

i) The Dirichlet problem (1.32) is L^p-solvable;

ii) The $\mathcal{L}$-harmonic measure ω is absolutely continuous with respect to σ, and the Radon-Nykodym derivative $k = \frac{d\omega}{d\sigma} \in L^q(d\sigma)$ with

$$\left(\frac{1}{\sigma(\Delta)}\int_{\Delta} k^q\right)^{\frac{1}{q}} \leqslant C\left(\frac{1}{\sigma(\Delta)}\int_{\Delta} k\right), \tag{1.35}$$

for any surface ball $\Delta \subset \partial B$.

The maximum principle and interpolation show that if $\mathcal{L}$ is L^p-resolutive, then it is also L^r-resolutive in the range $p \leqslant r \leqslant \infty$. Moreover (see Lemma 1.3.2, *iii*)) shows that if $\mathcal{L}$ is L^p-resolutive, then there exists $\varepsilon > 0$ such that $\mathcal{L}$ is also $L^{p-\varepsilon}$-resolutive.

Remark 1.4.1. Suppose that we have two operators $\mathcal{L}_0$ and $\mathcal{L}_1$ whose respective coefficient matrices A_0 and A_1 coincide on a neighborhood of ∂B. Then if $\mathcal{L}_0$ is L^p-resolutive then also $\mathcal{L}_1$ is L^p-resolutive (see for example [FKP]). Thus we see that the L^p-solvability is a property that depends only on the behavior of the coefficients of $\mathcal{L}$ near the boundary ∂B.

Theorem 1.4.2. *Let $A \in \mathcal{E}(K)$ and suppose $\mathcal{L} = \operatorname{div}(A\nabla)$ be L^p- resolutive. Then, for any $f \in L^p(\partial B, d\sigma)$ there exists a unique $u \in W^{1,2}_{loc}(B) \cap C(\bar{B})$ such that*

i) $\mathcal{L}u = 0$ in B,

ii) $Nu \in L^p(\partial B, d\sigma)$

iii) u converges non-tangentially to f for σ-almost any $P \in \partial B$.

Proof. Let us start by the proof of the existence. Let $\mathcal{L}$ be L^p resolutive and let $f \in L^p(\partial B, d\sigma)$. Moreover, let $\{f_j\}_{j\in\mathbb{N}}$ be a sequence of functions such that:

$$f_j \in C(\partial B), \ \forall j \in \mathbb{N}$$

$$\|f_j - f\|_{L^p(\partial B, d\sigma)} \to 0, \qquad \text{as } j \to \infty$$

and let $u_j \in W^{1,2}_{loc}(B) \cap C(\bar{B})$ be the corresponding solution (i.e. such that $\mathcal{L}u_j = 0$ in B and $u_{j|\partial B} = f_j$, for any $j \in \mathbb{N}$). By the linearity of $\mathcal{L}$, we have $\mathcal{L}(u_k - u_j) = 0$ in B, and $u_k - u_j = f_k - f_j$ on ∂B. Hence,

$$\|N(u_k - u_j)\|_{L^p(\partial B, d\sigma)} \leqslant C\|f_k - f_j\|_{L^p(\partial B, d\sigma)} \to 0 \qquad \text{as } j, k \to \infty$$

Unless to consider a subsequence, we can suppose

$$\|N(u_k - u_j)\|_{L^p(\partial B, d\sigma)} \leqslant \frac{1}{2^j} \qquad \forall j \in \mathbb{N},\ \forall k \geqslant j.$$

and so we have

$$\left[\int_{\partial B} \left(\sum_{j\in\mathbb{N}} \sup_{X\in\Gamma_\beta(Q)} |u_{j+1}(X) - u_j(X)|\right)^p d\sigma(Q)\right]^{\frac{1}{p}} \leqslant \sum_{j\in\mathbb{N}} \|N(u_{j+1} - u_j)\|_{L^p(\partial B, d\sigma)} \leqslant 1 \quad (1.36)$$

Hence, for σ- almost every $Q \in \partial B$ the first series in (1.36) is finite, i.e. there exists $E_0 \subseteq \partial B$ such that $\sigma(E_0) = 0$ and

$$\sum_{j\in\mathbb{N}} \sup_{X\in\Gamma_\beta(Q)} |u_{j+1}(X) - u_j(X)| < \infty, \qquad \forall Q \in \partial B \setminus E_0$$

i.e. the series

$$\sum_{j\in\mathbb{N}} (u_{j+1}(X) - u_j(X)) \qquad (1.37)$$

is totally convergent in $\Gamma_\beta(Q), \forall Q \in \partial B \setminus E_0$. Let now $0 < r < 1$, B_r be the ball with radius r and concentric with the unit ball B. Observing that $\bar{B}_r$ is compact and that $\Gamma_\beta(Q)$ are open set recovering $\bar{B}_r$, we have that there exists a finite number of points $Q_1, Q_2, Q_3, ... \in \partial B$

such that

$$\bar{B}_r \subseteq \Gamma_\beta(Q_1) \cup \Gamma_\beta(Q_2) \cup \Gamma_\beta(Q_3) \cup ...$$

By $\sigma(E_0) = 0$, we have that $\partial B \setminus E_0$ is a subset dense of ∂B, we can suppose that $Q_1, Q_2, Q_3, ... \in \partial B \setminus E_0$. Hence the series (1.37) is totally, and so uniformly, convergent in $\bar{B}_r$, $\forall 0 < r < 1$. So

$$\sum_{j\in\mathbb{N}} (u_{j+1}(X) - u_j(X)) \text{ is locally uniformly convergent in } B. \tag{1.38}$$

Suppose now

$$u(X) = u_1(X) + \sum_{j=1}^{\infty} (u_{j+1}(X) - u_j(X)) = \lim_j u_j(X).$$

By $\mathcal{L}u_j = 0$ and by (1.38) we obtain $\mathcal{L}u = 0$ in B. To prove $iii)$, let us assume

$$N_r v(Q) = \sup_{X \in \bar{B}_r \cap \Gamma_\beta(Q)} |v(X)|,$$

for any $Q \in \partial B$. Obviously $N_r v(Q) \leqslant Nv(Q)$, $\lim_{r\to 1^-} N_r v(Q) = Nv(Q)$, and N_r increases with r. Let us also observe that

$$\lim_k |u_k(X) - u_j(X)| = |u(X) - u_j(X)|$$

uniformly in $\bar{B}_r$ $(0 < r < 1)$. Moreover, for any $X \in \bar{B}_r \cap \Gamma_\beta(Q)$ we have $|u_k(X) - u_j(X)| \leqslant N_r(u_k - u_j)(Q)$. Then,

$$|u(X) - u_j(X)| = \lim_k |u_k(X) - u_j(X)| \leqslant \liminf_k N_r(u_k - u_j)(Q)$$

for any $X \in \bar{B}_r \cap \Gamma_\beta(Q)$, and so

$$N_r(u - u_j)(Q) \leqslant \liminf_k N_r(u_k - u_j)(Q) \leqslant \liminf_k N(u_k - u_j)(Q).$$

Hence, using the Fatou's lemma

$$\|N_r(u-u_j)\|_p \leqslant \liminf_k \|N(u_k-u_j)\|_p \leqslant C\lim_k \|f_k-f_j\|_p = C\|f-f_j\|_p.$$

On the other hand

$$\lim_{r\to 1^-} N_r(u-u_j) = N(u-u_j)$$

increasing with respect to r, and then, by Beppo Levi's theorem

$$\|N(u-u_j)\|_p = \lim_{r\to 1^-} \|N_r(u-u_j)\|_p \leqslant C\|f-f_j\|_p \to 0, \quad \text{as } j\to\infty$$

So, unless to consider a subsequence again, there exists $E_0 \subset \partial B$, $\sigma(E_0)=0$ such that

$$\lim_j N(u-u_j)(Q) = 0 \qquad \text{and} \qquad \lim_j f_j(Q) = f(Q)$$

for any $Q \in \partial B \setminus E_0$. Now, observing that

$$|u(X)-f(Q)| \leqslant |u(X)-u_j(X)| + |u_j(X)-f_j(Q)| + |f_j(Q)-f(Q)|$$

for any $X \in B, Q \in \partial B$, we obtain the thesis. In fact, 1)$\forall Q \in \partial B \setminus E_0$, $\forall \varepsilon > 0, \exists \nu \in \mathbb{N}$ such that $j > \nu$ implies

$$|u(X)-f(Q)| \leqslant 2\varepsilon + |u_j(X)-f_j(Q)|, \qquad \text{for any } X \in \Gamma_\beta(Q).$$

Moreover, by the assumption, for any $j \in \mathbb{N}$,

$$\lim_{X\to Q} u_j(X) = u_j(Q) = f_j(Q),$$

and then

$$\lim_{X\to Q, X\in\Gamma_\beta(Q)} u(X) = f(Q),$$

for any $Q \in \partial B \setminus E_0$, i.e. the non tangential convergence of u to f $\sigma-$a.e.

To show uniqueness, let $\mathcal{L}$ be L^p- solvable, $Nu \in L^p(\partial B)$ and that u converges non-tangentially to 0 σ-a.e. Let us show that $u(0) = 0$. To this aim, let $G(Y)$ denote the Green's function for $\mathcal{L}$ with pole at the origin, and let $\delta(X) = dist(X, \partial B)$. Moreover let $\phi_j \in C_0^\infty(B)$ be a sequence of functions such that:

1) $\phi_j \equiv 1$ on $\{\delta(X) \geqslant \frac{1}{j}\}$;

2) $\mathrm{supp}\phi_j \subset \{\delta(X) > \frac{1}{2j}\}$;

3) $|\nabla \phi_j| \leqslant C_j$;

4) $0 \leqslant \phi_j \leqslant 1$

and let $R_j = \{X : \frac{1}{2j} \leqslant \delta(X) \leqslant \frac{1}{j}\}$. Using Theorem 1.1.10, $ii)$, we see that

$$\begin{aligned} u(0) = u(0)\phi_k(0) = \int_B a_{i,j}(Y)\frac{\partial G(Y)}{\partial Y_i}\frac{\partial}{\partial Y_j}(u\phi_k)(Y)dY = \\ = \int_B a_{i,j}(Y)\frac{\partial G}{\partial Y_i}(Y)\frac{\partial u}{\partial Y_j}\phi_k(Y)(Y)dY + \\ + \int_B a_{i,j}(Y)\frac{\partial G}{\partial Y_i}(Y)\frac{\partial \phi_k}{\partial Y_j}(Y)(u)(Y)dY. \end{aligned}$$

We first estimate last integral in last equality. In order to do so, we use Lemma 1.1.1 applied to G on balls of size $\frac{1}{k}$, Lemma 1.1.2 applied to $|u|$ and (1.13), to conclude that

$$\left| \int_B a_{i,j}(Y)\frac{\partial G}{\partial Y_i}(Y)\frac{\partial \phi_k}{\partial Y_j}(Y)(u)(Y)dY \right| \leqslant C\int_{\partial B} m(Q)\cdot Nu_{\frac{1}{k}}(Q)d\sigma(Q),$$

where $m(Q) = \sup_{\Delta \ni Q} \frac{\omega(\Delta)}{\sigma(\Delta)}$, and

$$Nu_{\frac{1}{k}}(Q) = \sup_{X\in\Gamma_\alpha(Q), X\in B_{\frac{1}{k}}(Q)} |u(X)|.$$

Note that, since $\mathcal{L}$ is L^p- resolutive, then by Theorem 1.4.1, $ii)$ we have $m(Q) \in L^q(\partial B, d\sigma)$,

$\frac{1}{p} + \frac{1}{q} = 1$. Also, our assumption on u implies that $\|Nu_{\frac{1}{k}}\|_{L^p} \to 0$ as $k \to \infty$, and hence

$$\int_B a_{i,j}(Y) \frac{\partial G}{\partial Y_i}(Y) \frac{\partial \phi_k}{\partial Y_j}(Y)(u)(Y) dY \to 0.$$

In a similar way, applying Lemma 1.1.1 also to u on balls of size $\frac{1}{k}$, and integrating by parts

$$\int_B a_{i,j}(Y) \frac{\partial G}{\partial Y_i}(Y) \frac{\partial u}{\partial Y_j} \phi_k(Y)(Y) dY = - \int a_{i,j}(Y) G(Y) \frac{\partial u}{\partial Y_j} \frac{\partial \phi_k}{\partial Y_i}(Y) dY.$$

So the statement is completely proved. $\square$

Chapter 2

Young functions and Orlicz spaces

The Orlicz functional spaces represent one of the most immediate generalization of the Lebesgue spaces L^p, $(1 \leqslant p \leqslant \infty)$. They are a class of Banach spaces of measurable functions which play a primary role in many areas of mathematical analysis. We will collect here some definitions and results related to it, many of which are contained in [KR], [RR].

2.1 Orlicz spaces

A *Young's function* is a convex function of the type $\Phi : [0, +\infty) \to [0, +\infty)$ such that

$$\Phi(t) = \int_0^t \varphi(s) ds,$$

where $\varphi : [0, \infty[\to \mathbb{R}$ is nondecreasing, right-continuous and such that

$$\varphi(s) > 0 \quad \forall s > 0, \quad \varphi(0) = 0, \quad \lim_{s \to \infty} \varphi(s) = +\infty.$$

For example, the functions $\Phi_1(t) = \frac{t^p}{p}$ $(p > 1)$ and $\Phi_2(t) = e^{t^2} - 1$ are Young functions. The Young's function $\Psi(t)$, *complementary to* $\Phi(t)$, is defined as

$$\Psi(t) = \sup_{s>0}\{st - \Phi(s)\} = \int_0^t \varphi^{-1}(s) ds,$$

where

$$\varphi^{-1}(s) = \sup\{u : \varphi(u) \leqslant s\} \tag{2.1}$$

is the inverse generalized of φ. Note that whenever φ is continuous and strictly increasing than (2.1) coincide with the classical inverse function of φ.

EXAMPLE 2.1.1. As we already pointed out, the function $\Phi_1(t) = \frac{t^p}{p}$, $p > 1$, is a Young function. We shall compute the complementary function to it. Clearly, $\varphi_1(t) = t^{p-1}$ and $\varphi_1^{-1}(t) = s^{q-1}$, $\frac{1}{p} + \frac{1}{q} = 1$, and $\Psi_1(t) = \int_0^t \varphi_1^{-1}(s)ds = \frac{t^q}{q}$.

As a second example, we shall compute the Young function complementary to the Young function $\Phi_2(t) = e^t - t - 1, t \geqslant 0$. For this function we have that $\varphi_2(t) = e^t - 1$, from which it follows that $\varphi_2^{-1}(s) = \log(s+1)$, $s \geqslant 0$ and

$$\Psi_2(t) = \int_0^t \varphi_2^{-1}(s)ds = (1+t)\log(1+t) - t.$$

We note that it is impossible in many cases to find an explicit formula for the complementary Young function. For example, if $\Phi(t) = e^{t^2} - 1$, then $\varphi(t) = 2te^{t^2}$ and we cannot express $\varphi^{-1}(s)$ in the explicit form.

Sometimes we will consider *Orlicz functions*, i.e. continuously increasing functions $\Phi : [0,\infty) \to [0,\infty)$ verifying

$$\Phi(0) = 0, \qquad \lim_{t\to\infty} \Phi(t) = \infty.$$

From now on, Ω will denote a bounded domain in $\mathbb{R}^n$. If μ is a measure supported on Ω, the Orlicz Space $L^\Phi(d\mu) = L^\Phi(\Omega, d\mu)$ consists of all measurable functions on Ω for which there exists $K > 0$ such that

$$\int_\Omega \Phi\left(\frac{|f|}{K}\right) d\mu < \infty.$$

$L^\Phi(\Omega)$ is a complete linear metric space with respect to the following distance function:

$$\mathrm{dist}_\Phi(f,g) = \inf\left\{K > 0 \ \int_\Omega \Phi\left(\frac{|f-g|}{K}\right) d\mu \leqslant K\right\}.$$

If Φ is a Young function, L^Φ becomes a Banach space when equipped by the *Luxemburg norm*

$$\|f\|_{L^\Phi} = \inf\left\{k > 0;\ \int_\Omega \Phi\left(\frac{|f|}{k}\right) d\mu \leqslant \Phi(1)\right\} \tag{2.2}$$

It is easy to see that if we let $\Phi(t) = \frac{t^p}{p}$, $1 \leqslant p < \infty$ then the norm defined in (2.2) is equivalent to the classical L^p-norm

$$\|f\|_p = \left(\int_\Omega |f|^p d\mu\right)^{\frac{1}{p}}$$

so that in this case the space $L^\Phi(\Omega)$ coincide with the usual Lebesgue space $L^p(\Omega)$. Another important example is the exponential class defined with the Orlicz function $\Phi(t) = e^t - 1$. A pair of Young complementary function (Φ, Ψ) are also called *Hölder conjugate couple.* In fact the following *Hölder's inequality* holds,

$$\left|\int_\Omega \langle f, g\rangle\right| \ \leqslant\ C_{(\Phi,\Psi)}\, \|f\|_\Phi\, \|g\|_\Psi$$

for any $f \in L^\Phi(\Omega)$ and any $g \in L^\Psi(\Omega)$.

To define the dual space of L^Φ, we will need the following *doubling* property

Definition 2.1.1. *We say that a Young's function $\Phi(t)$ satisfies the Δ_2-condition (we will write $\Phi \in \Delta_2$) if there exists a constant $l > 0$ such that*

$$\Phi(2t) \leqslant l\Phi(t), \quad \forall t \geqslant 0. \tag{2.3}$$

When $\Phi \in \Delta_2$, the smallest constant l such that (2.3) is true, i.e.

$$l = \sup_{t>0} \frac{\Phi(2t)}{\Phi(t)}$$

is greater or equal than 2. In the sequel we will call it the *doubling constant* of Φ.

Let us explicitly observe that the Δ_2-condition (2.3) is equivalent to the more general property:

$$\forall A>0,\ \exists B>0\ :\ \Phi(At)\leqslant B\Phi(t),\quad \forall t\geqslant 0. \tag{2.4}$$

and

$$\forall B>0,\ \exists A>0\ :\ \Phi(At)\leqslant B\Phi(t),\quad \forall t\geqslant 0. \tag{2.5}$$

Theorem 2.1.1. *Let Φ, Ψ be complementary Young functions with $\Phi \in \Delta_2$. Then every bounded linear functional defined on $L^\Phi(\Omega)$ is uniquely represented by a function $g \in L^\Psi(\Omega)$ as*

$$f \to \langle f, g\rangle$$

Without a doubling condition the dual of $L^\Phi(\Omega)$ does not have a nice description. If we consider the complementary functions

$$\Phi(t) = t\log^{\frac{1}{\alpha}}(e+t) \quad \Psi(t) = e^{t^\alpha} - 1$$

with $\alpha > 0$, we find that the dual to $L\log^{\frac{1}{\alpha}} L(\Omega)$ is the exponential class $Exp_\alpha(\Omega) = L^\Psi(\Omega)$, but not conversely.

Theorem 2.1.2. *Let Φ be an Orlicz function (not necessarily a Young function) satisfying the Δ_2 condition. Then the space $C_0^\infty(\Omega)$ is dense in the metric space $L^\Phi(\Omega)$.*

In particular, if also the complementary $\Psi(t)$ obey the Δ_2 condition, the Banach spaces L^Φ and L^Ψ are mutually dual. We shall require that Ψ satisfies Δ_2 too. In this case we have that $\Phi(t)$ and $\Psi(t)$ are *essentially equal* to $t\varphi(t)$ and $t\varphi^{-1}(t)$ respectively, for all $t \geqslant 0$, and then φ and φ^{-1} also satisfy the Δ_2 condition. Moreover, for the inverse functions of Φ and Ψ, we have:

$$t \leqslant \Phi^{-1}(t)\Psi^{-1}(t) \leqslant 2t, \qquad \forall t \geqslant 0. \tag{2.6}$$

Sometimes, when the complementary Young functions Φ and Ψ verify the Δ_2 condition, we will write for short $\Phi \in \nabla_2$.

If $\Phi \in \Delta_2$, it is an easy computation to show that the following inequality holds true

$$\Phi(t) \leqslant t\varphi(t) \leqslant l\Phi(t), \qquad \forall t \geqslant 0. \tag{2.7}$$

Inequality (2.7) implies ([GIS], pag. 692) that

$$\Phi(\lambda t) \leqslant \lambda^l \Phi(t), \quad \forall t \geqslant 0, \quad \forall \lambda \geqslant 1. \tag{2.8}$$

Let us observe that respectively from (2.7), (2.8) and (2.7) again, we have:

$$\lambda t\varphi(\lambda t) \leqslant l\Phi(\lambda t) \leqslant l\lambda^l \Phi(t) \leqslant l\lambda^l t\varphi(t) \qquad \forall t \geqslant 0, \quad \forall \lambda \geqslant 1,$$

from which,

$$\varphi(\lambda t) \leqslant l\lambda^{l-1}\varphi(t), \quad \forall t \geqslant 0, \quad \forall \lambda \geqslant 1. \tag{2.9}$$

We note that, if $\Phi(t)$ is a Young function and $\Psi(t)$ is its complementary function, the Young's inequality holds:

$$st \leqslant \Phi(s) + \Psi(t), \qquad \forall s, t \geqslant 0 \tag{2.10}$$

and, from Lagrange theorem and the monotonicity of φ, the following inequality

$$|\Phi(|A|) - \Phi(|B|)| \leqslant \varphi(|A| + |B|)|A - B| \tag{2.11}$$

holds for all A, $B \in \mathbb{R}^n$. Moreover, by the convexity of Φ and by the Δ_2 condition, it holds

$$\Phi(a + b) \leqslant \frac{l}{2}(\Phi(a) + \Phi(b)), \qquad \forall a, b > 0. \tag{2.12}$$

Here below we just recall some properties of a Young function Φ :

i) If $\Phi \in \Delta_2$, the inverse function Φ^{-1} of Φ verifies

$$\Phi^{-1}(a + b) \leqslant \Phi^{-1}(a) + \Phi^{-1}(b), \quad \forall a, b \geqslant 0. \tag{2.13}$$

Moreover Φ^{-1} verifies (2.3) with Φ replaced by Φ^{-1} and $l = 2$.

ii) Let Ψ be the complementary function of Φ. We have,

$$\Psi\left(\frac{\Phi(t)}{t}\right) \leqslant \Phi(t), \quad \forall t > 0. \tag{2.14}$$

iii) If $\Phi \in \Delta_2$ the following inequality holds true

$$\Phi^{-1}(\lambda t) \leqslant \lambda \Phi^{-1}(t), \quad \forall t \geqslant 0, \quad \forall \lambda \geqslant 1. \tag{2.15}$$

iv) For any open cube Q_0 in $\mathbb{R}^n$ and F measurable, we have (see e.g. [Zi]):

$$\int_{Q_0} \Phi(|F|)dx = \int_0^\infty \varphi(t)\,|\{x \in Q_0 : |F(x)| > t\}|\,dt. \tag{2.16}$$

v) If the function φ is convex, clearly the inverse function φ^{-1} of φ is concave and so the complementary function Ψ of Φ verifies (2.3).

The following proposition relates a Young function satisfies Δ_2-condition with power-like functions.

Proposition 2.1.3. **[KR]** *Let Φ, Ψ be complementary Young functions, then*

$$\Phi, \Psi \in \Delta_2 \Longleftrightarrow \exists\, p, q, 1 < p \leqslant q < \infty : p\Phi(t) \leqslant t\varphi(t) \leqslant q\Phi(t) \quad \forall t > 0. \tag{2.17}$$

Definition 2.1.2. *We say that the Young function Φ verifies the Δ' condition (and we will write $\Phi(t) \in \Delta'$) if it is* submultiplicative, *i.e. if there exist a positive constant C such that*

$$\Phi(st) \leqslant C\Phi(s)\Phi(t) \qquad \forall\, s, t \geqslant 0. \tag{2.18}$$

It is quite simply to prove that if a Young function $\Phi \in \Delta'$ then it also satisfies the Δ_2 condition.

EXAMPLE 2.1.2. If $\Phi_1(t) = \frac{t^p}{p}$ $(p > 1)$, then obviously $\Phi \in \Delta'$. A second example of Young function which satisfy the Δ' condition is given by $\Phi_2(t) = t^p(|\log t| + 1)$, $p > 1$, $t > 0$. In fact an easy computation shows that $\Phi_2(st) \leqslant \Phi_2(s)\Phi_2(t)$ for any $s, t > 0$.

If we consider the function $\Phi_3(t) = \frac{t^2}{\log(e+t)}$, then it is an easy to show that $\Phi_3 \in \Delta_2$ but $\Phi_3 \notin \Delta'$.

Let w be a weight and let Φ, Ψ be complementary Young functions verifying Δ_2-condition. We say that $w \in A_\Phi -$ class if there exists $A \geqslant 1$ such that

$$\forall \varepsilon > 0, \quad \left(\fint_I \varepsilon w dx\right) \varphi \left(\fint_I \varphi^{-1}\left(\frac{1}{\varepsilon w}\right) dx\right) \leqslant A \tag{2.19}$$

for all bounded intervals I in $\mathbb{R}$, where $\Phi' = \varphi$. Whenever (2.19) holds, we will write for short $w \in A_\Phi$.

The A_Φ-class of weights was introduced by Kerman and Torchinsky in [KT] and it extend the definition of Muckenhoupt A_p weight to the framework of the Orlicz spaces. We recall indeed (see [M]) that a weight w belongs to A_p-class, $1 < p < \infty$, if there exists $A \geqslant 1$ such that

$$\fint_I w dx \left(\fint_I w^{-\frac{1}{p-1}} dx\right)^{p-1} \leqslant A, \tag{2.20}$$

for all bounded intervals I in $\mathbb{R}$.

We are going to characterize those weights for which a weighted inequality of strong type for the Hilbert transform holds. Recall that the Hilbert transform in $\mathbb{R}$ is given by

$$Hf(y) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{|x-y|>\varepsilon} \frac{f(x)}{y-x} dx, \qquad y \in \mathbb{R}. \tag{2.21}$$

Theorem 2.1.4. [KK] *Let w be a weight on $\mathbb{R}^n$ and let Φ be a Young function verifying the Δ_2 condition together with its complementary function. Then the inequality*

$$\int_{\mathbb{R}} \Psi(Hf) w(y) dy \leqslant C \int_{\mathbb{R}} \Psi(|f|) w(y) dy$$

holds for all $f \in L^\Phi$ if and only if $w \in A_\Phi$.

For more details see [KK].

2.2 Zygmund spaces

The Zygmund spaces, denoted by $L^p \log^\alpha L(\Omega)$, correspond to the Orlicz function $\Phi(t) = t^p \log^\alpha(a+t)$ with $1 \leqslant p < \infty$, $\alpha \in \mathbb{R}$ and suitable large constant a.
The defining function $\Phi(t) = t^p \log^\alpha(e+t)$, $1 \leqslant p < \infty$ is a Young function when $\alpha \geqslant 1-p$ and there we have the following estimate

$$\|f\|_{L^p \log^{-1} L} \;\leqslant\; \|f\|_p \;\leqslant\; \|f\|_{L^p \log L}$$

and

$$\|f\|_{L^p \log L} \;\leqslant\; \left[\int |f|^p log\left(e + \frac{|f|}{\|f\|_p}\right)\right]^{\frac{1}{p}} \;\leqslant\; 2\,\|f\|_{L^p \log L}.$$

The non-linear functional

$$[[f]]_{p,\alpha} = \left[\int_{\mathbb{R}^n} |f|^p \log^\alpha\left(e + \frac{|f|}{\|f\|_p}\right)\right]^{\frac{1}{p}}, \qquad p \geqslant 1 \quad \text{and} \quad \alpha \geqslant 0,$$

is equivalent to the Luxemburg norm, given at (2.2), and the following estimates are true

$$\|f\|_{L^p \log^{-1} L} \;\leqslant\; \|f\|_{L^p} \;\leqslant\; \|f\|_{L^p \log^\alpha L} \;\leqslant\; [[f]]_{p,\alpha} \;\leqslant\; 2\|f\|_{L^p \log^\alpha L} \tag{2.22}$$

Whenever $a, b > 1$ and $\alpha, \beta \in \mathbb{R}$ are coupled by the relationships

$$\frac{1}{c} = \frac{1}{a} + \frac{1}{b}, \quad \frac{\gamma}{c} = \frac{\alpha}{a} + \frac{\beta}{b}$$

the following Hölder-type inequalities holds

$$\|fg\|_{L^c \log^\gamma L} \;\leqslant\; C\|f\|_{L^a \log^\alpha L} \cdot \|g\|_{L^b \log^\beta L},$$

where C is a positive constant depending on α, β, a and b. Hölder's inequality for Zygmund spaces takes the form

$$\|\varphi_1 ... \varphi_k\|_{L^p \log^\alpha L} \leqslant c\,\|\varphi_1\|_{L^{p_1} \log^{\alpha_1} L} \;\cdots\; \|\varphi_k\|_{L^{p_k} \log^{\alpha_k} L}$$

where $p_1, p_2, ..., p_k > 1$; $\alpha_1, \alpha_2, ..., \alpha_k \in \mathbb{R}$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + ... + \frac{1}{p_k}$, $\frac{\alpha}{p} = \frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2} + ... + \frac{\alpha_k}{p_k}$. The constant here does not depend on the functions $\varphi_i \in L^{p_i} \log^{\alpha_i} L$.

If we take as Hölder conjugate couple $\Phi(t) = t \log(e + t)$ and $\Psi(t) = e^t - 1$ defining the Zygmund and exponential classes, respectively, we have the following estimate

$$\left| \int_\Omega \langle f, g \rangle \right| \leqslant 4\,\|f\|_{L \log L}\,\|g\|_{Exp}.$$

In view of the same homogeneities on each side we can assume Luxemburg norm equal 1. From the definition of these norms we find

$$\int_\Omega |f| \log(e + |f|) = \log(e + 1)$$

and

$$\int_\Omega (e^{|g|} - 1) = e - 1$$

Then we have the elementary inequality

$$|f||g| \leqslant |f| \log(1 + |f|) + e^{|g|} - 1 \tag{2.23}$$

to conclude that $\int_\Omega |f||g| \leqslant 4$ as desired.

Thus $Exp(\Omega)$ is the dual space to the Zygmund space $L \log L(\Omega)$. $L \log L$ and Exp have traditionally be regarded as more general Orlicz spaces.

2.3 Indices of Orlicz spaces

The aim of this section is to introduce the so called *indices* of a Young function and to establish connections between those indices and the growth conditions on Young functions.

Definition 2.3.1. *Let*

$$h(s) = \sup_{t>0} \frac{\Phi^{-1}(t)}{\Phi^{-1}(st)}. \tag{2.24}$$

The upper and lower Boyd indices ρ and θ of L^Φ are

$$\rho = \inf_{0<s<1} -\frac{logh(s)}{logs} = \lim_{s\to 0^+} -\frac{\log h(s)}{\log s} \tag{2.25}$$

and

$$\theta = \inf_{1<s<\infty} -\frac{\log h(s)}{\log s} = \lim_{s\to \infty} -\frac{\log h(s)}{\log s} \tag{2.26}$$

respectively.

The right wing equalities in (2.25) and (2.26) follow from known properties of subadditive functions (since $\log h$ enjoys this property).

It is easy to see that in the case of Lebesgue spaces $L^\Phi = L^p$ one has $\rho = \theta = p^{-1}$. We list here some properties of these indices we will make use of below (see [KT] and reference therein contained):

Proposition 2.3.1. *Let Φ and Ψ be Young complementary functions, both verifying the Δ_2 condition. The following properties hold:*

i) $0 < \theta \leqslant \rho < 1$*;*

ii) *given a fixed $0 < r < \rho^{-1}$, there exists an s_0, with $0 < s_0 < 1$, such that*

$$\Phi(st) \leqslant \left(\frac{s}{s_0}\right)^r \Phi(t), \tag{2.27}$$

for all $t > 0$, $0 < s < 1$;

iii) *For any s such that $0 < s < 1$ we have $h(s) \geqslant s^{-\rho}$; so, for any fixed $0 < s < 1$ there is a $t > 0$ such that*

$$\frac{\Phi^{-1}(t)}{\Phi^{-1}(st)} > \frac{s^{-\rho}}{2}. \tag{2.28}$$

Furthermore, under the above hypotheses, the following holds true:

Proposition 2.3.2. *Let Φ and Ψ be as in Proposition 2.3.1 and let Φ_δ, $\delta > 0$ be such that*

$$\varphi_\delta^{-1}(t) = \left(\varphi^{-1}(t)\right)^{1+\delta}.$$

Then, the upper index ρ' of L^{Φ_δ} is greater than the upper index ρ of L^Φ.

For the proof of last proposition we remaind the reader to [KT], Lemma 2. Moreover, following the proof of the cited Lemma, it is possible to compute exactly the upper index of L^{Φ_δ}, that is $\rho' = \frac{1+\delta\rho-\theta_\Psi}{1+\delta\rho}$, where θ_Ψ is the lower Boyd index of L^Ψ.

For a complete analysis of these properties we refer to [Bo], [MO] and to the results obtained in [KT].

The definition of the Boyd indices is very simple, nevertheless, a particular computation might be extremely difficult. Now we want to give an effective method of establishing their values. In fact it is possible to estimate the Boyd indices of a Young function Φ in terms of the growth condition

$$p\Phi(t) \leqslant t\varphi(t) \leqslant q\Phi(t),$$

$t \geqslant 0$, giving birth to the *Simonenko indices* (see [Si]).

Definition 2.3.2. *Let Φ be a Young function, and let us consider the best p and q such that*

$$p\Phi(t) \leqslant t\varphi(t) \leqslant q\Phi(t) \qquad \forall t > 0.$$

holds. We will assume

$$p(\Phi) = \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)} \qquad \text{and} \qquad q(\Phi) = \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)}. \tag{2.29}$$

The numbers $p(\Phi)$ and $q(\Phi)$ are called lower Simonenko and upper Simonenko index of Φ respectively.

In the same way we can define the *Simonenko indices* of the complementary function Ψ, $p(\Psi)$ and $q(\Psi)$.

The following property, useful in the sequel, is contained in [RR]

Proposition 2.3.3. *Let $\Phi, \Psi \in \Delta_2$. Then, for any s such that $0 < s \leqslant 1$ and for any $t > 0$,*

$$\frac{\Phi^{-1}(st)}{\Phi^{-1}(t)} \leqslant s^{q(\Phi)^{-1}}. \tag{2.30}$$

It is known ([KR], Theorem 5.1) that

$$\Phi, \Psi \in \Delta_2 \qquad \Longleftrightarrow \qquad 1 < p(\Phi) \leqslant q(\Phi) < \infty. \tag{2.31}$$

Setting

$$h_\Phi(\lambda) = \sup_{t>0} \frac{\Phi(\lambda t)}{\Phi(t)}, \quad \lambda > 0 \tag{2.32}$$

the numbers

$$\underline{\alpha}(\Phi) = \lim_{\lambda \to 0^+} \frac{\log h_\Phi(\lambda)}{\log \lambda} = \sup_{0<\lambda<1} \frac{\log h_\Phi(\lambda)}{\log \lambda} \tag{2.33}$$

and

$$\overline{\alpha}(\Phi) = \lim_{\lambda \to \infty} \frac{\log h_\Phi(\lambda)}{\log \lambda} = \inf_{1<\lambda<\infty} \frac{\log h_\Phi(\lambda)}{\log \lambda} \tag{2.34}$$

are called the fundamental lower index of Φ and the fundamental upper index of Φ, respectively. The numbers $\underline{\alpha}(\Phi)$ and $\overline{\alpha}(\Phi)$ are reciprocals of the Boyd indices ρ and θ respectively (see Boyd [Bo]). Moreover, $\Phi \in \Delta_2$ if and only if $\overline{\alpha} < \infty$ (see [KK]). Always $1 \leqslant \underline{\alpha} \leqslant \overline{\alpha}$ and it is $\overline{\alpha} > 1$ if and only if the complementary function Ψ satisfies the Δ_2 condition. The couples $\underline{\alpha}(\Psi)$ and $\overline{\alpha}(\Phi)$, and $\overline{\alpha}(\Psi)$ and $\underline{\alpha}(\Phi)$ behave similarly as conjugate exponents of power functions (see e.g. [Bo]), namely $\underline{\alpha}(\Psi) = \overline{\alpha}(\Phi)/(\overline{\alpha}(\Phi) - 1)$ and $\overline{\alpha}(\Psi) = \underline{\alpha}(\Phi)/(\underline{\alpha}(\Phi) - 1)$.

As observed above the definition of the Boyd (and then of the fundamental) indices does not often represent an efficient tool for computation. The following theorem give an answer in this direction

Theorem 2.3.4. **[FiK]** *Let Φ be a Young function. If there exist*

$$r_0 = \lim_{t\to 0} \frac{t\varphi(t)}{\Phi(t)} \qquad \textit{and} \qquad r_\infty = \lim_{t\to\infty} \frac{t\varphi(t)}{\Phi(t)},$$

then,

$$\underline{\alpha}(\Phi) = \min\{r_0, r_\infty\} \qquad \textit{and} \qquad \overline{\alpha}(\Phi) = \max\{r_0, r_\infty\}$$

Example 2.3.1. Let us consider the Young function

$$\Phi_1(t) = t^p \log^\alpha(e+t),$$

$1 < p < \infty$ and $\alpha \geqslant 0$. Applying Theorem 2.3.4 one can easily compute $\underline{\alpha}(\Phi) = \overline{\alpha}(\Phi) = p$ whenever $\alpha > 1$ and $\underline{\alpha}(\Phi) = p$, $\overline{\alpha}(\Phi) = p + \alpha$, if $\alpha = 1$.

Let

$$\Phi_2(t) = \begin{cases} t^2 & \text{if} \quad 0 \leqslant t < 1, \\ 2t - 1 & \text{if} \quad 1 \leqslant t < 2, \\ t^2/2 + 1 & \text{if} \quad 2 \leqslant t \end{cases}$$

Then, Φ_2 is a Young function and an easy computation gives $p(\Phi_2) = \frac{4}{3}$ and $q(\Phi_2) = 2$. On the other hand by Theorem 2.3.4, $\underline{\alpha}(\Phi_2) = 2$. This shows that, in general, $\underline{\alpha}(\Phi_2) \neq p(\Phi_2)$.

As another example, let $r \geqslant 1$, $s > 0$ and let

$$\Phi_3(t) = \begin{cases} 0 & \text{if} \quad t = 0 \\ t^r \exp\left(\sqrt{1 + s\log^+ t}\right) & \text{if} \quad t > 0 \end{cases}$$

(see Talenti [T]). Then, simply by applying Theorem 2.3.4 we obtain $\underline{\alpha}(\Phi_3) = \overline{\alpha}(\Phi_3) = r$. Moreover, as to the Simonenko indices we have $p(\Phi_3) = r$ and $q(\Phi_3) = r + \frac{s}{2}$.

Now, let $\Phi, \Psi \in \Delta_2$. By (2.29) we have, for any $t > 0$,

$$\frac{p(\Phi)}{t} \leqslant \frac{\varphi(t)}{\Phi(t)} \leqslant \frac{q(\Phi)}{t}$$

Let us fix $\delta > 0$. By integrating over the interval $[\delta, t]$, last inequality implies

$$\log\left(\frac{t}{\delta}\right)^p \leqslant \log\frac{\Phi(t)}{\Phi(\delta)} \leqslant \log\left(\frac{t}{\delta}\right)^q.$$

Hence,

$$\frac{\Phi(\delta)}{\delta^p}t^p \leqslant \Phi(t) \leqslant \frac{\Phi(\delta)}{\delta^q}t^q$$

so that

$$p\frac{\Phi(\delta)}{\delta^p}t^{p-1} \leqslant \varphi(t) \leqslant q\frac{\Phi(\delta)}{\delta^q}t^{q-1}, \quad \forall t > \delta. \tag{2.35}$$

So we have

$$\forall\,\delta > 0,\ \exists\, c_1,\, c_2 > 0 : \quad c_1 t^{p-1} \leqslant \varphi(t) \leqslant c_2 t^{q-1}. \quad \forall t > \delta. \tag{2.36}$$

We have the following connections between growth condition on Φ and fundamental indices.

Lemma 2.3.5. **[FiK]** *Let Φ be a Young function satisfying the growth condition $p\Phi(t) \leqslant t\varphi(t) \leqslant q\Phi(t),\ \ \forall t \geqslant 0$. Then we have*

$$p \leqslant p(\Phi) \leqslant \underline{\alpha}(\Phi) \leqslant \overline{\alpha}(\Phi) \leqslant q(\Phi) \leqslant q. \tag{2.37}$$

Theorem 2.3.6. **[KT]** *Let Φ and Ψ be a couple of Young function both satisfying the Δ_2 condition and w a weight on $\mathbb{R}^n$. The following conditions are equivalent*

(i) w verifies the A_Φ condition (2.19)

(ii) $w \in A_p$ where $p = \underline{\alpha}(\Phi)$.

Chapter 3

Maximal operator and weighted inequality

The weighted norm inequalities have become one of the most dynamically developing parts of harmonic analysis since the early 70's and the pioneering result by B. Muckenhoupt [M]. Solutions of many important problems have been closely linked with weight problems. The mentioned paper by B. Muckenhoupt triggerred a flood of results on weighted inequalities and related topics; in this paper it was shown among others that the one weight norm inequality for the (unweighted) maximal operator is true iff the weight satisfies the A_p condition (see Definition 1.3.6). The A_p weights provide an extraordinary beautiful answer to a number of challenging problems which had arisen already in the 30's in connection with fundamental results due to G. H. Hardy and J.E. Littlewood. Theorems on boundedness of weighted maximal operator and of the Hilbert transform followed very soon (see R. A. Hunt, B. Muckenhoupt and R. L. Wheeden [HMW], R. R. Coifman and C. Fefferman [CF]).

This chapter is intended to study weighted norm inequalities for the Hardy-Littlewood maximal operator together with some of its most immediate and interesting properties. Apart from the usual estimates for this operator we also obtain some new weighted inequalities. In particular we will obtain a weighted integral inequality in the Orlicz context and we will give a new characterization of the Gehring class of weight in connection with a special class of Orlicz functional spaces (see Theorem 3.2.1). Moreover, in Section 3.3 we will also study the boundedness of the Hardy Littlewood maximal operator in the variable exponent spaces $W^{1,p(\cdot)}$, extending a result due to J. Kinnunen and P. Lindqvist, known in the classical setting. Hence the results obtained in the course of the chapter seems to be of independent interest.

In next chapters will be clear the importance of those weight problems. It stems not only from the theory of functions itself, but it is also clear from the numerous applications to our boundary value problems and imbedding theorems (see for example Theorem 4.1.1).

3.1 Maximal Operator on Lebesgue space

Let $B \subset \mathbb{R}^n$ be the unit ball and let f be a locally integrable function on ∂B. For $Q \in \partial B$ we define

$$Mf(Q) = \sup_{\Delta \ni Q} \fint_{\Delta} |f(Y)| dY$$

where the supremum is taken over all surface ball Δ containing Q.

Mf will be called the Hardy-Littlewood maximal function of f and the operator M sending f to Mf, the Hardy-Littlewood maximal operator.

Theorem 3.1.1. *The Hardy-Littlewood maximal operator is of "strong type" (p, p), i.e., there exists a positive constant C such that for any function $f \in L^p(\partial B, d\sigma)$, it holds*

$$\|Mf\|_{L^p} \leqslant C\|f\|_{L^p}.$$

Moreover M is of "weak type" $(1, 1)$, i.e. for any $t > 0$, there exists a constant $C > 0$ such that

$$|\{Q \in \partial B : Mf(Q) > t\}| \leqslant \frac{C}{t} \int_{\partial B} f d\sigma$$

We will consider also a more general version of the Hardy Littlewood operator. Let ν be a positive measure. For any $f \in L^1(d\nu)$ consider the operator defined by

$$M_\nu(f)(Q) = \sup_{\Delta \ni Q} \frac{1}{\nu(\Delta)} \int_\Delta |f|\, d\nu.$$

It holds the following

Theorem 3.1.2. *A doubling measure ν belongs to the Gehring class $B_q(d\mu)$ iff the weighted Hardy-Littlewood maximal operator M_ν verifies*

$$\|M_\nu f\|_{L^p(d\mu)} \leqslant C \|f\|_{L^p(d\mu)}, \qquad \frac{1}{q} + \frac{1}{p} = 1. \tag{3.1}$$

Combining Lemma 2.2 of [MS] and Theorem 2.5 of [Bu] with slight modification, we have:

Proposition 3.1.3. *Let $1 < q < \infty$, and let $\nu \in B_q(d\mu)$. Then:*

$$\|M_\nu f\|_{L^p(d\mu)} \leqslant C(n,p)^{\frac{1}{p}} \left[B_{q,\mu}(\nu)\right]^{\frac{q}{p}} \|f\|_{L^p(d\mu)}, \qquad \frac{1}{q} + \frac{1}{p} = 1, \tag{3.2}$$

and so, for all $\lambda > 0$,

$$\mu\left(\{M_\nu f > \lambda\}\right) \leqslant C(n,p) \frac{[B_{q,\mu}(\nu)]^q}{\lambda^p} \int_\Omega |f|^p d\mu. \tag{3.3}$$

Finally, we want just recall the following version of the Marcinkiewicz theorem (cfr. [StW]). Here and below, if v is a weight on ∂B and A is a σ-measurable set, we will write $v(A) = \int_A v d\sigma$.

Theorem 3.1.4. *Let T be a sublinear operator, and let v be a weight on ∂B. Suppose that T is simultaneously of restricted weak-types (p_1, p_1) and (p_2, p_2), $1 < p_1 < p_2 < \infty$, with respect to the measure $dv = v d\sigma$, i.e.*

$$\int_{\{T\chi_E > \lambda\}} dv \leqslant \frac{C}{\lambda^{p_i}} v(E), \quad i = 1, 2 \tag{3.4}$$

E measurable subset of ∂B, C independent on E and on the positive constant λ. Then T is also of 'strong type' (r,r), for all $p_1 < r < p_2$, that is

$$\|Tf\|_{L^r(dv)} \leqslant K \|f\|_{L^r(dv)} \tag{3.5}$$

K independent on f.

If $T = M_w$, then the restricted weak type can be characterized as follows:

Proposition 3.1.5. *Let w, v be weights on ∂B, and let the measure dv be doubling. The weighted Hardy-Littlewood maximal operator M_w is of restricted weak-type (p,p) with respect to dv, i.e.*

$$\int_{\{M_w\chi_E > \lambda\}} dv \leqslant \frac{C}{\lambda^p} v(E), \quad 1 \leqslant p < \infty \tag{3.6}$$

with C independent on E and on the positive constant λ, iff there exists $K > 0$ such that for all Δ, and for all measurable $E \subset \Delta$,

$$\frac{w(E)}{w(\Delta)} \leqslant K \left(\frac{v(E)}{v(\Delta)}\right)^{\frac{1}{p}} \tag{3.7}$$

Proof. .

(3.6) $\Longrightarrow$ (3.7)

Observing that, by the definition of the operator M_w, if $E \subset \Delta$

$$M_w\chi_E(P) \geqslant \frac{\chi_\Delta(P)}{w(\Delta)} \int_\Delta \chi_E(Q) w d\sigma(Q) = \frac{w(E)}{w(\Delta)} \chi_\Delta(P),$$

results from (3.6)

$$v(\Delta) \leqslant \int_{\left\{M_w\chi_E > \frac{w(E)}{w(\Delta)}\right\}} dv \leqslant C v(E) \left(\frac{w(\Delta)}{w(E)}\right)^p$$

that is (3.7).

(3.7) $\Longrightarrow$ (3.6)

We have

$$M_v\chi_E(P) = \sup_{\Delta \ni P} \frac{1}{v(\Delta)} \int_\Delta \chi_E(Q) v(Q) d\sigma(Q) = \sup_{\Delta \ni P} \frac{v(E \cap \Delta)}{v(\Delta)}.$$

and analogously for M_w. Then, by (3.7)

$$(M_w \chi_E)^p \leqslant K^p M_v \chi_E,$$

so that $\{M_w \chi_E > \lambda\} \subseteq \left\{M_v \chi_E > \frac{\lambda^p}{K^p}\right\}$. Now, the measure dv doubling implies that the operator M_v is of weak-type $(1,1)$ with respect to dv; in particular,

$$\int_{\{M_v \chi_E > \lambda\}} dv(Q) \leqslant \frac{C_1}{\lambda} v(E)$$

and then (3.6) follows with $C = C_1 K^p$. □

3.2 Maximal operator on Orlicz spaces

The following result gives necessary and sufficient conditions to ensure the boundedness of the weighted maximal operator on Orlicz functional spaces.

Theorem 3.2.1. **[Z2]** *Let w, v be weights on ∂B, such that the measures $dv = v d\sigma$ and $dw = w d\sigma$ are doubling, and let $\Phi(t) = \int_0^t \varphi(s) ds$ be a Young's function which, together with its complementary function $\Psi(t)$, satisfies the Δ_2 condition. Then, the following are equivalent:*

i) *There exists a constant $C > 0$, independent on f, such that:*

$$\int_{\partial B} \Phi(M_w f) v d\sigma \leqslant C \int_{\partial B} \Phi(|f|) v d\sigma;$$

ii) *$w \in B_\Phi(dv)$, that is:*

$$\left(\frac{1}{w(\Delta)} \int_\Delta \epsilon v d\sigma \right) \varphi \left(\frac{1}{w(\Delta)} \int_\Delta \varphi^{-1} \left(\frac{w}{\epsilon v} \right) w d\sigma \right) \leqslant K \tag{3.8}$$

for all surface balls Δ and for all $\epsilon > 0$;

iii) *$w \in B_{q_0}(dv)$, where $\frac{1}{p_0}+\frac{1}{q_0}=1$, p_0^{-1} upper index of L^Φ.*

Proof. *i)* $\Rightarrow$ *ii)*

Let us consider:

$$\|\chi_\Delta\|_{L^\Phi(\varepsilon dv)} = \inf\left\{k>0 : \int_{\partial B} \Phi\left(\frac{\chi_\Delta}{k}\right)\varepsilon v d\sigma \leqslant 1\right\} = \frac{1}{\Phi^{-1}\left(\frac{1}{\varepsilon v(\Delta)}\right)} \tag{3.9}$$

and

$$T_\varepsilon := \left\|\frac{w\chi_\Delta}{\varepsilon v}\right\|_{L^\Psi(\varepsilon dv)} = \inf\left\{k>0 : \int_{\partial B} \Psi\left(\frac{w\chi_\Delta}{k\varepsilon v}\right)\varepsilon v d\sigma \leqslant 1\right\} \tag{3.10}$$

We can immediately observe that $T_\varepsilon > 0$, unless $\sigma(\Delta)=0$, which we exclude. Indeed, $T_\varepsilon = 0$ implies that the function $\frac{w\chi_\Delta}{\varepsilon v}$ is zero dv-a.e., but $w, v > 0$ implies $\sigma(\Delta) = 0$. On the other hand, the converse of the Hölder's inequality implies the existence of a nonnegative function f, supported by Δ, with norm $\|f\|_{L^\Phi(\varepsilon dv)} = 1$ and such that $\int_\Delta f w d\sigma = \int_{\partial B} f \frac{w\chi_\Delta}{\varepsilon v}\varepsilon dv = T_\varepsilon$ and then $M_w f(P) \geqslant \frac{T_\varepsilon}{w(\Delta)}$, $\forall P \in \Delta$; this implies, by *i)*, $T_\varepsilon < \infty$.

Now we claim that there exists a constant K_1 such that for all Δ and for all $\varepsilon > 0$

$$\|\chi_\Delta\|_{L^\Phi(\varepsilon dv)}\, T_\varepsilon \leqslant K_1 w(\Delta). \tag{3.11}$$

Indeed, with the same f as before, we have

$$\frac{T_\varepsilon}{w(\Delta)}\chi_\Delta(Q) \leqslant M_w f(Q) \qquad \forall Q \in \partial B. \tag{3.12}$$

Being $\Phi(t)$ an increasing function, yielding *i)* and integrating we have:

$$\int_\Delta \Phi\left(\frac{T_\varepsilon}{w(\Delta)}\right)\varepsilon v d\sigma \leqslant \int_{\partial B}\Phi\left(M_w f\right)\varepsilon v d\sigma \leqslant C\int_{\partial B}\Phi\left(|f|\right)\varepsilon v d\sigma \leqslant C \tag{3.13}$$

that is

$$\frac{T_\varepsilon}{w(\Delta)} \leqslant \Phi^{-1}\left(\frac{C}{\varepsilon v(\Delta)}\right)$$

Let us choose

$$K_1 = h(C^{-1}) = \sup_{t>0} \frac{\Phi^{-1}(t)}{\Phi^{-1}(C^{-1}t)}.$$

Taking $t = \frac{C}{\varepsilon v(\Delta)}$, (3.11) follows.

Now, since $t\varphi^{-1}(t) \leqslant q(\Psi)\Psi(t)$ $\forall t > 0$, we have:

$$\int_{\partial B} \frac{w\chi_\Delta}{T_\varepsilon} \varphi^{-1}\left(\frac{w\chi_\Delta}{v\varepsilon T_\varepsilon}\right) d\sigma \leqslant q(\Psi) \int_{\partial B} \Psi\left(\frac{w\chi_\Delta}{v\varepsilon T_\varepsilon}\right) \varepsilon v d\sigma \leqslant q(\Psi)$$

and then, by (3.11) we have

$$\frac{\|\chi_\Delta\|_{L^\Phi(\varepsilon dv)}}{K_1 w(\Delta)} \int_\Delta \varphi^{-1}\left(\frac{w}{\varepsilon v} \frac{\|\chi_\Delta\|_{L^\Phi(\varepsilon dv)}}{K_1 w(\Delta)}\right) w d\sigma \leqslant q(\Psi) \tag{3.14}$$

Now, let us consider the function of ε:

$$\theta(\varepsilon) = \frac{\|\chi_\Delta\|_{L^\Phi(\varepsilon dv)}}{K_1 \varepsilon w(\Delta)}$$

Let us remark that, from (2.6), it follows that $\theta(\varepsilon)$ is essentially equal to the function

$$\theta_1(\varepsilon) = \frac{v(\Delta)}{K_1 w(\Delta)} \Psi^{-1}\left(\frac{1}{\varepsilon v(\Delta)}\right)$$

and hence, $\lim_{\varepsilon\to o^+} \theta(\varepsilon) = +\infty$, and $\lim_{\varepsilon\to+\infty} \theta(\varepsilon) = 0$. Moreover $\theta(\varepsilon)$ is continuous, and so there exists $\varepsilon > 0$ such that $\theta(\varepsilon) = 1$, essentially equal to $\left[v(\Delta)\Psi\left(\frac{K_1 w(\Delta)}{v(\Delta)}\right)\right]^{-1}$.

Now, applying these results to (3.14) we obtain

$$\int_\Delta \varphi^{-1}\left(\frac{w}{v}\right) w d\sigma \leqslant K_2 q(\Psi) v(\Delta) \Psi\left(\frac{K_1 w(\Delta)}{v(\Delta)}\right) \leqslant K_2 \frac{q(\Psi)}{p(\Psi)} K_1 w(\Delta) \varphi^{-1}\left(\frac{K_1 w(\Delta)}{v(\Delta)}\right). \tag{3.15}$$

Then,

$$\varphi\left(\frac{1}{w(\Delta)} \int_\Delta \varphi^{-1}\left(\frac{w}{v}\right) w d\sigma\right) \leqslant \varphi(At) \tag{3.16}$$

by assuming $A = K_2 \frac{q(\Psi)}{p(\Psi)} K_1$ and $t = \varphi^{-1}\left(\frac{K_1 w(\Delta)}{v(\Delta)}\right)$. Now, from the generalized Δ_2 condition for Φ, let $B>0$ such that $\Phi(At)\leqslant B\Phi(t)$, $t>0$. Then we have

$$\varphi(At)\leqslant\frac{q(\Phi)}{At}\Phi(At)\leqslant\frac{q(\Phi)B}{At}\Phi(t)\leqslant\frac{q(\Phi)B}{p(\Phi)At}t\varphi(t)=\frac{q(\Phi)}{p(\Phi)}\frac{p(\Psi)}{q(\Psi)}\frac{B}{K_1K_2}\frac{w(\Delta)}{v(\Delta)}$$

from which the assertion *ii*) follows for $\varepsilon=1$ with $K=\frac{q(\Phi)}{p(\Phi)}\frac{p(\Psi)}{q(\Psi)}\frac{B}{K_1K_2}$. The same proof applies to εv; for the constant K depends only on C, and so the assertion is proved in the general case.

Now to prove that *ii*) implies *iii*) we need some preliminary results:

Lemma 3.2.2. *Let Φ, p_0, w, and v be as in Theorem 3.2.1. Then, $w\in B_\Phi(dv)$ implies that the weighted Hardy-Littlewood maximal operator M_w is bounded from $L^r(dv)$ to itself, for all $r>p_0$.*

Proof. By the interpolation criterion (Theorem 3.1.4) it is enough to prove that M_w is of restricted weak-type (p_0,p_0) that is (3.7) with $p=p_0$. We have, by duality between L^Φ and L^Ψ:

$$\frac{w(E)}{w(\Delta)}=\frac{1}{w(\Delta)}\int_{\partial B}\chi_E\left(\frac{\chi_E w}{\varepsilon v}\right)\varepsilon dv\leqslant\frac{1}{w(\Delta)}\|\chi_E\|_{L^\Phi(\varepsilon dv)}\left\|\frac{w\chi_E}{\varepsilon v}\right\|_{L^\Psi(\varepsilon dv)}.$$

We claim that

$$\left\|\frac{w\chi_E}{\varepsilon v}\right\|_{L^\Psi(\varepsilon dv)}\leqslant C_1 w(\Delta)\Phi^{-1}\left(\frac{1}{\varepsilon v(\Delta)}\right)\tag{3.17}$$

Indeed, let us observe that (3.8) is equivalent to

$$\int_\Delta\varphi^{-1}\left(\frac{w}{\epsilon v}\right)wd\sigma\leqslant w(\Delta)\varphi^{-1}\left(\frac{Kw(\Delta)}{\epsilon v(\Delta)}\right)\tag{3.18}$$

Observing that $t\varphi^{-1}(t)\geqslant\Psi(t)$, we have:

$$\left\|\frac{w\chi_E}{\varepsilon v}\right\|_{L^\Psi(\varepsilon dv)}\leqslant\inf\left\{k>0|\int_\Delta\Psi\left(\frac{w}{k\varepsilon v}\right)\varepsilon vd\sigma\leqslant 1\right\}$$

$$\leqslant\inf\left\{k>0|\int_\Delta\frac{w}{k}\varphi^{-1}\left(\frac{w}{k\varepsilon v}\right)d\sigma\leqslant 1\right\},$$

and from (3.18)

$$\left\|\frac{w\chi_E}{\varepsilon v}\right\|_{L^{\Psi}(\varepsilon dv)} \leqslant \inf\left\{k>0 | \frac{1}{k} w(\Delta)\varphi^{-1}\left(\frac{Kw(\Delta)}{k\varepsilon v(\Delta)}\right)\leqslant 1\right\}$$

$$\leqslant \inf\left\{k>0 | \frac{K}{\varepsilon v(\Delta)} \leqslant \Phi\left(\frac{k}{w(\Delta)}\right)\right\}$$

$$= w(\Delta)\Phi^{-1}\left(\frac{K}{\varepsilon v(\Delta)}\right) \leqslant C_1 w(\Delta)\Phi^{-1}\left(\frac{1}{\varepsilon v(\Delta)}\right)$$

(where $C_1 = h(K^{-1})$), i.e. the (3.17). So we obtain

$$\frac{w(E)}{w(\Delta)} \leqslant C_1 \frac{\Phi^{-1}\left(\frac{1}{\varepsilon v(\Delta)}\right)}{\Phi^{-1}\left(\frac{1}{\varepsilon v(E)}\right)}. \tag{3.19}$$

Then the statement follows from Proposition 2.3.1, (iii), taking $s = \frac{v(E)}{v(\Delta)} < 1$ and $\varepsilon = \frac{1}{tv(E)}$. □

Lemma 3.2.3. *Let Φ, p_0, w and v be as in Theorem 3.2.1, and let Φ_δ, $\delta > 0$ be such that*

$$\varphi_\delta^{-1}(t) = \left(\varphi^{-1}(t)\right)^{1+\delta}$$

Then, $w \in B_\Phi(dv)$ implies $w \in B_{\Phi_\delta}(dv)$, for small δ.

Proof. Suppose $w \in B_\Phi(dv)$. We want to prove that $w \in B_{\Phi_\delta}(dv)$ for some $\delta > 0$, i.e.

$$\left(\frac{1}{w(\Delta)}\int_\Delta \epsilon v d\sigma\right)\varphi_\delta\left(\frac{1}{w(\Delta)}\int_\Delta \varphi_\delta^{-1}\left(\frac{w}{\epsilon v}\right) w d\sigma\right) \leqslant K \tag{3.20}$$

for all surface balls Δ and for all $\epsilon > 0$. For this purpose, it is sufficient to prove, for $z_\varepsilon = \varphi^{-1}\left(\frac{w}{\varepsilon v}\right)$,

$$\left(\frac{1}{w(\Delta)}\int_\Delta z_\varepsilon^{1+\delta} dw\right)^{\frac{1}{1+\delta}} \leqslant C\frac{1}{w(\Delta)}\int_\Delta z_\varepsilon dw \quad \forall\varepsilon>0,\ \forall\Delta. \tag{3.21}$$

Indeed, if (3.21) holds true, by $\varphi_\delta(t) = \varphi(t^{\frac{1}{1+\delta}})$, we have

$$\left(\frac{1}{w(\Delta)}\int_\Delta \varphi_\delta^{-1}\left(\frac{w}{\varepsilon v}\right)dw\right)^{\frac{1}{1+\delta}} \leqslant C\frac{1}{w(\Delta)}\int_\Delta \varphi^{-1}\left(\frac{w}{\varepsilon v}\right)dw$$

and than,

$$\begin{aligned}\varphi_\delta\left(\frac{1}{w(\Delta)}\int_\Delta \varphi_\delta^{-1}\left(\frac{w}{\varepsilon v}\right)dw\right) &\leqslant \varphi\left(C\frac{1}{w(\Delta)}\int_\Delta \varphi^{-1}\left(\frac{w}{\varepsilon v}\right)dw\right)\\ &\leqslant C'\varphi\left(\frac{1}{w(\Delta)}\int_\Delta \varphi^{-1}\left(\frac{w}{\varepsilon v}\right)dw\right) \leqslant K\frac{w(\Delta)}{\varepsilon v(\Delta)}\end{aligned}$$

i.e. (3.20). So let us prove (3.21). To begin, we prove (3.21) for $\varepsilon = 1$. Denote $z = z_1 = \varphi^{-1}(\frac{w}{v})$ and set

$$\tilde{\varphi}(t) = \frac{1}{\varphi^{-1}(t^{-1})} \qquad \left(\tilde{\varphi}^{-1}(t) = \frac{1}{\varphi(t^{-1})}\right).$$

Now, for $\varepsilon = 1$ the inequality

$$\left(\frac{1}{w(\Delta)}\int_\Delta v d\sigma\right)\varphi\left(\frac{1}{w(\Delta)}\int_\Delta \varphi^{-1}\left(\frac{w}{v}\right)w d\sigma\right) \leqslant K \tag{3.22}$$

becomes, also by the Δ_2-condition for φ^{-1}

$$\frac{1}{w(\Delta)}\int_\Delta z w d\sigma \leqslant K'\varphi^{-1}\left(\left(\frac{1}{w(\Delta)}\int_\Delta \frac{1}{\varphi(z)}w d\sigma\right)^{-1}\right) \tag{3.23}$$

that is

$$\left(\frac{1}{w(\Delta)}\int_\Delta z w d\sigma\right)\tilde{\varphi}\left(\frac{1}{w(\Delta)}\int_\Delta \tilde{\varphi}^{-1}\left(\frac{1}{z}\right)w d\sigma\right) \leqslant K', \quad \forall\Delta. \tag{3.24}$$

Now, we claim that (3.24) implies that for all Δ there exists $\lambda = \lambda_\Delta$ such that

$$\left\|\frac{\chi_\Delta}{\lambda z}\right\|_{L^{\tilde{\Psi}}(\lambda z dw)} \leqslant C w(\Delta)\tilde{\Phi}^{-1}\left(\frac{1}{\lambda(zw)(\Delta)}\right) \tag{3.25}$$

(where $\lambda z dw$ stays for the measure $\lambda z w d\sigma$). To prove (3.25) let us observe that $\tilde{\varphi}$ and $\tilde{\varphi}^{-1}$

satisfy both the Δ_2-condition, and then $\tilde{\Phi}$ and $\tilde{\Psi}$ obey Δ_2 too. Then, from (3.24)

$$\left(\frac{1}{w(\Delta)}\int_\Delta \tilde{\varphi}^{-1}\left(\frac{1}{z}\right) wd\sigma\right) \leqslant \tilde{\varphi}^{-1}\left(K'\frac{w(\Delta)}{(zw)(\Delta)}\right) \leqslant C'\tilde{\varphi}^{-1}\left(\frac{w(\Delta)}{(zw)(\Delta)}\right), \tag{3.26}$$

for all Δ. Now, let $\lambda = \lambda_\Delta$ such that

$$\frac{1}{\lambda} = (zw)(\Delta)\tilde{\Psi}\left(\frac{w(\Delta)}{(zw)(\Delta)}\right). \tag{3.27}$$

We have

$$\frac{1}{\lambda} \sim w(\Delta)\tilde{\varphi}^{-1}\left(\frac{w(\Delta)}{(zw)(\Delta)}\right)$$

that is

$$\tilde{\varphi}^{-1}\left(\frac{1}{\lambda w(\Delta)}\right) \sim \frac{w(\Delta)}{(zw)(\Delta)}$$

and then

$$\tilde{\Phi}\left(\frac{1}{\lambda w(\Delta)}\right)\lambda \sim \frac{1}{(zw)(\Delta)}.$$

So,

$$\frac{1}{\lambda w(\Delta)} \sim \tilde{\Phi}^{-1}\left(\frac{1}{\lambda (zw)(\Delta)}\right). \tag{3.28}$$

We have, by (3.26),

$$\int_\Delta \frac{\tilde{\Psi}\left(\frac{w(\Delta)}{(zw)(\Delta)}\right)}{\frac{w(\Delta)}{(zw)(\Delta)}\tilde{\varphi}^{-1}\left(\frac{w(\Delta)}{(zw)(\Delta)}\right)}\tilde{\varphi}^{-1}\left(\frac{1}{z}\right)\lambda wd\sigma \leqslant C'$$

then,

$$\int_\Delta \frac{1}{q(\tilde{\Psi})}\left(\frac{1}{z}\tilde{\varphi}^{-1}\left(\frac{1}{z}\right)\right)\lambda zwd\sigma \leqslant C'.$$

Hence,

$$\int_\Delta \frac{1}{C'}\frac{p(\tilde{\Psi})}{q(\tilde{\Psi})}\left(\frac{1}{z}\right)\lambda zwd\sigma \leqslant 1.$$

Then, by (2.8), there exists $C > 0$ such that

$$\int_{\Delta} \tilde{\Psi}\left(\frac{C}{z}\right) \lambda z w d\sigma \leqslant 1$$

and so, by (3.28), (3.25) follows.

Now, $\forall E \subset \Delta$ measurable set, by the equality

$$\|\chi_E\|_{L^{\tilde{\Phi}}(\lambda z dw)} = \frac{1}{\tilde{\Phi}^{-1}\left(\frac{1}{(\lambda z w)(E)}\right)}, \tag{3.29}$$

and by (3.25), we have

$$\frac{w(E)}{w(\Delta)} \leqslant C \frac{\tilde{\Phi}^{-1}\left(\frac{1}{\lambda(zw)(\Delta)}\right)}{\tilde{\Phi}^{-1}\left(\frac{1}{\lambda(zw)(E)}\right)}$$

and then, by (2.30) with $s = \frac{(zw)(E)}{(zw)(\Delta)}$, we obtain

$$\frac{w(E)}{w(\Delta)} \leqslant C\left(\frac{(zw)(E)}{(zw)(\Delta)}\right)^{q(\tilde{\Phi})^{-1}}, \qquad \forall \Delta, \quad \forall E \subset \Delta. \tag{3.30}$$

Now, observing that $w \in B_{\Phi}(dv)$ implies that $z_{\varepsilon} \in L^1(dw)$ for all $\varepsilon > 0$, by Proposition 1.3.4 applied to $d\mu = w d\sigma$ and $d\nu = zw d\sigma$, we have that there exist $\delta > 0$ and $K > 0$ such that

$$\left(\frac{1}{w(\Delta)} \int_{\Delta} z^{1+\delta} dw\right)^{\frac{1}{1+\delta}} \leqslant K\left(\frac{1}{w(\Delta)} \int_{\Delta} z dw\right) \tag{3.31}$$

for any Δ, that is (3.21) with $\varepsilon = 1$. Now, let us observe that the constant K' in (3.24) is independent on ε; then, the constants in (3.30) are independent on ε too. So, by the last assertion in Theorem 1.3.4, the constants δ and K in (3.31) are independent on ε, and then the proof holds true also in the general case. □

Now, we can conclude that $ii) \Longrightarrow iii)$. Indeed, we have that $w \in B_{\Phi}(dv)$ implies $w \in B_{\Phi_\delta}(dv)$ (Lemma 3.2.3); then the maximal operator M_w is bounded from $L^r(dv)$ to itself, for all $r > \frac{1}{\rho'}$, ρ' upper index of L^{Φ_δ} (Lemma 3.2.2). In particular, (Proposition 2.3.2), M_w is bounded from $L^{p_0}(dv)$ to itself, that is $w \in B_{q_0}(dv)$.

Finally, $iii) \Longrightarrow i)$:

Let $w \in B_{q_0}(dv)$, $\frac{1}{p_0}+\frac{1}{q_0}=1$. Hence, by Theorem 1.3.2, (iv), there exists an $\varepsilon>0$ such that $w \in B_{q_0+\varepsilon}(dv)$. Assume $m=q_0+\varepsilon$; from Proposition 3.1.3, we have, f measurable,

$$\begin{aligned}\int_{\partial B}\Phi(M_w f)vd\sigma &= \int_{\partial B}\left(\int_0^{M_w f}\varphi(s)ds\right)dv\\ &\leqslant q(\Phi)\int_{\partial B}\left(\int_0^{M_w f}\frac{\Phi(s)}{s}ds\right)dv\\ &= q(\Phi)\int_0^{+\infty}\left(\int_{\{P\in\partial B: M_w f(P)>s\}}dv\right)\frac{\Phi(s)}{s}ds\end{aligned}$$

$$\leqslant 2^{m'}c(n,m)q(\Phi)B\int_0^{+\infty}\left(\frac{1}{s}\right)^{m'}\left(\int_{\left\{|f|>\frac{s}{2}\right\}}|f|^{m'}dv\right)\frac{\Phi(s)ds}{s},$$

where $B=B_{m,v}(w)^m$. Then, by Fubini's theorem

$$\int_{\partial B}\Phi(M_w f)vd\sigma \leqslant c(n,m)q(\Phi)BC'\int_{\partial B}\left(\int_0^1\frac{\Phi(|f|s)}{s^{m'}}\frac{ds}{s}\right)vd\sigma$$

where C' is the doubling constant of $\Phi(t)$. Let now $m_0'=\frac{m'+p_0}{2}$. So $m_0'<p_0$, and then, as already mentioned, there is an s_0, $0<s_0<1$ such that $\Phi(st)\leqslant\left(\frac{s}{s_0}\right)^{m_0'}\Phi(t)$, $t>0$, $0<s<1$. Then,

$$\begin{aligned}\int_{\partial B}\Phi(M_w f)vd\sigma &\leqslant \frac{c(n,m)q(\Phi)BC'}{s_0^{m_0'}}\int_{\partial B}\Phi(|f|)\left(\int_0^1 s^{m_0'-(m'+1)}ds\right)vd\sigma\\ &= \frac{c(n,m)q(\Phi)BC'}{(m_0'-m')s_0^{m_0'}}\int_{\partial B}\Phi(|f|)vd\sigma\end{aligned}$$

This completes the proof of Theorem 3.2.1. □

3.3 Maximal operator on variable exponent spaces

Let Ω be an open subset of $\mathbb{R}^n$ (often we will assume Ω be connected). For a measurable function $p : \Omega \to [1, +\infty]$, $L^{p(\cdot)}(\Omega)$ is defined to be the set of all measurable functions $f : \Omega \to \mathbb{R}$ such that for some $\lambda > 0$,

$$\varrho\,(p(.), \Omega,\ f/\lambda) = \int_{\Omega \setminus \Omega_{p(\cdot),\infty}} |f(x)/\lambda|^{p(x)}\, dx + \|f/\lambda\|_{\infty, \Omega_{p(\cdot),\infty}} < \infty,$$

where $\Omega_{p(\cdot),\infty} = \{x \in \Omega : p(x) = \infty\}$.

The set $L^{p(\cdot)}(\Omega)$ becomes a Banach function space when equipped with the norm

$$\|f\|_{p(\cdot)} := \inf \Big\{ \lambda > 0 : \varrho\,(p(.), \Omega,\ f/\lambda) \leqslant 1 \Big\}.$$

These spaces are referred to as variable Lebesgue spaces or, more simply, as variable L^p spaces.

They have been studied for long time, but only quite recently their important applications have been found for example in the fluid dynamics, elasticity, and in particular in the study of properties of electrorheological fluids (see for instance [Zh], [Ru]).

For more information on their basic properties, see Kováčik and Rákosník [KoR] or Harjulehto and Hästö [HaHä]; for applications see [CUFN], [Di], [AM] and the references they contain.

In this section we will investigate about the boundedness properties of the Maximal operator between those spaces. To this aim, let us define $\Phi(\Omega)$ to be the set of all measurable functions $p\colon \Omega \to [1, \infty]$ and

$$p_- = \operatorname{ess\,inf}_{x\in\Omega}\ p(x), \quad p_+ = \operatorname{ess\,sup}_{x\in\Omega}\ p(x).$$

In [PRu], M. Pick and M. Ružička proved that for the maximal operator

$$\mathcal{M}f(x) = \sup_{B \ni x} \frac{1}{|B|} \int_{B \cap \Omega} |f(y)| dy$$

(where the supremum is taken over all balls B which contain x and for which $|B \cap \Omega| > 0$) to be bounded on the space $L^{p(\cdot)}(\Omega)$, where Ω is a bounded domain of $\mathbb{R}^n$, the uniform local continuity condition

$$|p(x) - p(y)| \leqslant \frac{C}{-\ln |x-y|} \tag{3.32}$$

is "close" to be a necessary condition, in the sense that there are also counter-examples where it is shown that (3.32) is not necessary (see for instance [Le]).

In [Di] it is proved that the condition is sufficient.

The following result was shown by D. Cruz-Uribe, A. Fiorenza and C. J. Neugebauer in [CUFN] (see also A. Nekvinda [Ne] and C. Capone, D. Cruz-Uribe and A. Fiorenza [CCF]).

Theorem 3.3.1. *Given an open set $\Omega \subset \mathbb{R}^n$, let $p \in \Phi(\Omega)$, $1 < p_- \leqslant p_+ < \infty$, satisfying the following conditions*

$$|p(x) - p(y)| \leqslant \frac{C}{-\log |x-y|}, \quad x, y \in \Omega, \quad |x-y| < \frac{1}{2} \tag{3.33}$$

and

$$|p(x) - p(y)| \leqslant \frac{C}{\log(e + |x|)}, \quad x, y \in \Omega, \quad |y| \geqslant |x|. \tag{3.34}$$

Then the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded on $L^{p(\cdot)}(\Omega)$.

Conditions (3.33) and (3.34) are the so called log-Hölder continuity conditions one locally and one at infinity. Let us observe in fact that (3.34) is the natural analogue of (3.33) at infinity. It implies that there is some number p_∞ such that $p(x) \to p_\infty$ as $|x| \to \infty$, and this limit holds uniformly in all directions.

In the same spirit of the definition of variable L^p spaces, we can consider the *variable Sobolev space* $W^{1,p(\cdot)}(\Omega)$ consisting of all functions $f : \Omega \to \mathbb{R}$ such that $Df \in L^{p(\cdot)}(\Omega)$

endowed with the norm

$$\|f\|_{1,p(\cdot)} = \|f\|_{p(\cdot)} + \|Df\|_{p(\cdot)}$$

where Df is the weak gradient of f. It is easy to see that if $p(x) = p$ is constant, then $W^{1,p(\cdot)}$ equals $W^{1,p}$.

Now, for a locally integrable function $f : \Omega \to [-\infty, +\infty]$ let us consider the local Hardy-Littlewood maximal function $\mathcal{M}_\Omega f : \Omega \to [0, \infty]$ as

$$\mathcal{M}_\Omega f(x) = \sup \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy$$

where the supremum is taken over all the balls *centered* at x with radius $0 < r < dist(x, \partial\Omega)$. In other words, all open balls centered at x and contained in Ω are admissible. In the case $\Omega = \mathbb{R}^n$ we simply write M. Obviously the uncentred maximal operator $\mathcal{M}$ is larger than the local one $\mathcal{M}_\Omega$. Due to J. Kinnunen and P. Lindqvist [KiLi] is the following known result about the boundedness of $\mathcal{M}_\Omega$ in the Sobolev space $W^{1,p}$.

Theorem 3.3.2. *Let $1 < p \leqslant \infty$. If $u \in W^{1,p}$, then $\mathcal{M}_\Omega u \in W^{1,p}(\Omega)$ and*

$$|D\mathcal{M}_\Omega u(x)| \leqslant c\mathcal{M}_\Omega |Du|(x), \tag{3.35}$$

for almost every $x \in \Omega$.

For related results see also [Ki], [HajOn], [KiSa].

The last part of this chapter will be devoted to extend this result to the context of variable Sobolev spaces. More precisely we shall prove the following:

Theorem 3.3.3. **[Z3]** *Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $p \in \Phi(\Omega)$ be such that $1 < p_- \leqslant p_+ < \infty$ and (3.33) holds true. Moreover let the maximal operator $\mathcal{M}_\Omega$ be bounded on $L^{p(\cdot)}$. Then $u \in W^{1,p(\cdot)}(\Omega)$ implies $\mathcal{M}_\Omega u \in W^{1,p(\cdot)}(\Omega)$ and (3.35) holds true.*

Let us notice that the main tool in order to prove Theorem 3.3.3 is a recent result contained in [CUF] about the convergence of approximate identities in variable L^p spaces.

This in particular gives criteria for smooth functions to be dense in the variable Sobolev spaces.

In order to prove Theorem 3.3.3 we need some preliminary results, which may be of independent interest.
Firstly, let us recall a recent result contained in [CUF].

We begin by recalling the definition of approximate identities. Let φ be an integrable function defined on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} \varphi dx = 1$. For each $t > 0$, define the function $\varphi_t(x) = \frac{1}{t^n}\varphi(\frac{x}{t})$. Note that by a change of variables, $\|\varphi_t\|_1 = \|\varphi\|_1$. Define the radial majorant of φ to be the function $\tilde{\varphi} = \sup_{|y|\geqslant|x|} |\varphi(y)|$. If $\tilde{\varphi}$ is integrable, we will say that the sequence $\{\varphi_t\}$ is a *potential-type approximate identity.* This is the case for example of the bounded functions φ of compact support.

Theorem 3.3.4. *Given a set Ω and $p(\cdot) \in \Phi(\Omega)$, let φ be such that $\{\varphi_t\}$ is a potential-type approximate identity.*

Then for all $f \in L^{p(\cdot)}(\Omega)$, $\{\varphi_t * f\}$ converges to f pointwise almost everywhere.

We want also to recall the following result on the density of smooth functions in the variable Sobolev spaces [CUF]:

Theorem 3.3.5. *Given an open set Ω, let $p(\cdot) \in \Phi(\Omega)$ be such that $p_+ < \infty$ and (3.33) holds. Then for $k \geqslant 1$, the set*

$$C^\infty \cap W^{k,p(\cdot)}(\Omega)$$

is dense in $W^{k,p(\cdot)}$.

Lemma 3.3.6. *Let $0 < |\Omega| < \infty$. If $p(\cdot), q(\cdot) \in \Phi(\Omega)$ are such that $p(x) \leqslant q(x)$, a. e. $x \in \Omega$, then*

$$\|f\|_{p(\cdot),\Omega} \leqslant (1 + |\Omega|)\|f\|_{q(\cdot),\Omega}. \tag{3.36}$$

See [KoR] for more details.

Lemma 3.3.7. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $p(\cdot) \in \Phi(\Omega)$ be such that $p_+ < +\infty$. If $f_j \rightharpoonup f$ and $g_j \rightharpoonup g$ weakly in $L^{p(\cdot)}(\Omega)$ and $f_j(x) \leqslant g_j(x)$, j=1,2,... a.e. in Ω, then $f(x) \leqslant g(x)$ a.e. in Ω.*

Proof. Let $p'(\cdot) \in \Phi(\Omega)$ the conjugate exponent function of $p(\cdot)$, i.e. such that $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ a.e. $x \in \Omega$. By the hypothesis $p_+ < +\infty$ we have that the dual space of $L^{p(\cdot)}$ is $L^{p'(\cdot)}$ and that C_0^∞ is dense in $L^{p'(\cdot)}$ (see [KoR] Corollary 2.7 and Theorem 2.11). So we have:

$$f_j \rightharpoonup f \ \text{ weakly in } L^{p(\cdot)} \Rightarrow \int_\Omega f_j(x)h(x)dx \to \int_\Omega f(x)h(x)dx, \quad \forall h \in C_0^\infty(\Omega).$$

Analogously

$$g_j \rightharpoonup g \ \text{ weakly in } L^{p(\cdot)} \Rightarrow \int_\Omega g_j(x)h(x)dx \to \int_\Omega g(x)h(x)dx, \quad \forall h \in C_0^\infty(\Omega).$$

Moreover, if in particular $h(x) \geqslant 0$, we have

$$\int_\Omega f_j(x)h(x)dx \leqslant \int_\Omega g_j(x)h(x)dx,$$

and so by passing to the limit,

$$\int_\Omega f(x)h(x)dx \leqslant \int_\Omega g(x)h(x)dx. \tag{3.37}$$

Now, let $\varphi(x) \in C_0^\infty(\mathbb{R}^n), \varphi(x) \geqslant 0$ so that (φ_t) is a potential type approximate identity. Let $\delta > 0$, and let us fix $y \in \Omega_\delta = \{y \in \Omega : \operatorname{dist}(y, \partial\Omega) > \delta\}$. One can easily observe that if supp $\varphi \subseteq \overline{B}(0, R)$, then supp $\varphi_t(y-\cdot) \subseteq \overline{B}(y, tR)$, and so $t \in (0, \frac{\delta}{R})$ implies supp $\varphi_t(y-\cdot) \subset\subset \Omega$. Next, by choosing in (3.37) $h(x) = \varphi_t(y-x)$ we have,

$$\int_\Omega f(x)h(x)dx = \int_{\mathbb{R}^n} f(x)\varphi_t(y-x)dx = f * \varphi_t(y). \tag{3.38}$$

Analogously,

$$\int_\Omega g(x)h(x)dx = \int_{\mathbb{R}^n} g(x)\varphi_t(y-x)dx = g * \varphi_t(y). \tag{3.39}$$

Then, by passing to the limit $t \to 0$, by (3.37), (3.38), (3.39), and Theorem 3.3.4, we obtain

$$f(y) \leqslant g(y) \quad a.e.\ y \in \Omega_\delta.$$

By the arbitrary choice of $\delta > 0$, the statement follows .

□

Lemma 3.3.8. *If* $f, g \in W^{1,p(\cdot)}(\Omega)$*, then* $f^+ = \max\{f, 0\}$, $f^- = \min\{f, 0\}$, $|f|$, $\max\{f, g\}$, $\in W^{1,p(\cdot)}(\Omega)$.

Proof. Let us just observe that we can write $h(x) = \max\{f, g\}(x) = ((f-g)^+ + g)(x)$. Then, by Lemma 7.6 in [GT] the statement easily follows. □

Note that, in particular,

$$Dh = \begin{cases} Df & if\ f \geqslant g \\ Dg & if\ f \leqslant g \end{cases}$$

and

$$D|f| = \begin{cases} Df & if\ f > 0 \\ 0 & if\ f = 0 \\ -Df & if\ f < 0 \end{cases}$$

so that $|D|f|| = |Df|$.

Proposition 3.3.9. *Let* $0 < t < 1$, $x \in \Omega$ *and*

$$u_t(x) = \fint_{B(x,t\delta(x))} u(y)dy, \tag{3.40}$$

where $\delta(x) = \mathrm{dist}(x, \partial\Omega)$*. Moreover, let* $p \in \Phi(\Omega)$ *be as in the statement of Theorem 3.3.3. Then if* $u \in W^{1,p(\cdot)}(\Omega)$*, we get* $u_t \in W^{1,p(\cdot)}(\Omega)$ *and*

$$|Du_t(x)| \leqslant 2\mathcal{M}_\Omega|Du|(x), \quad 0 < t < 1 \tag{3.41}$$

for almost every $x \in \Omega$.

Proof. The proof falls naturally into two parts. The first step concerns the case $u \in C^\infty \cap W^{1,p(\cdot)}(\Omega)$ since our assumptions imply that $C^\infty \cap W^{1,p(\cdot)}(\Omega)$ is dense in $W^{1,p(\cdot)}(\Omega)$. The idea of the proof of this step goes back at least as far as [KiLi], but for the convenience of the reader we repeat the relevant material.

Let $0 < t < 1$ be fixed. Thanks to Rademacher's theorem the function δ is differentiable a.e. in Ω. Moreover, $|D\delta(x)| = 1$ for a.e. $x \in \Omega$. The Leibnitz rule gives

$$D_i u_t(x) = D_i \left(\frac{1}{\omega(t\delta(x))^n} \right) \int_{B(x,t\delta(x))} u(y)dy$$

$$+ \left(\frac{1}{\omega(t\delta(x))^n} \right) \Big(\int_{B(x,t\delta(x))} D_i u(y)dy + t \int_{\partial B(x,t\delta(x))} u(y) d\mathcal{H}^{n-1}(y) D_i\delta(x) \Big),$$

$i = 1, ..., n$, for a.e. $x \in \Omega$. Thus

$$Du_t(x) = n \left(\frac{D\delta(x)}{\delta(x)} \right) \Big(\fint_{\partial B(x,t\delta(x))} u(y) d\mathcal{H}^{n-1}(y) - \fint_{B(x,t\delta(x))} u(y)dy \Big) \tag{3.42}$$

$$+ \fint_{B(x,t\delta(x))} Du(y)dy,$$

for a.e. $x \in \Omega$.

Now, our aim is to estimate the difference between the two integrals in Identity (3.42). To this end, let us suppose that $B(x,R) \subset \Omega$. By the Green's formula we get

$$\fint_{\partial B(x,R)} u(y) d\mathcal{H}^{n-1}(y) - \fint_{B(x,R)} u(y)dy = \frac{1}{n} \fint_{B(x,R)} Du(y)(y-x)dy.$$

Moreover

$$\Big| \fint_{B(x,R)} Du(y)(y-x)dy \Big| \leqslant R \fint_{B(x,R)} |Du(y)|dy \leqslant R\mathcal{M}_\Omega |Du|(x).$$

So we obtain

$$\Big| \fint_{\partial B(x,R)} u(y) d\mathcal{H}^{n-1}(y) - \fint_{B(x,R)} u(y)dy \Big| \leqslant \frac{R}{n} \mathcal{M}_\Omega |Du|(x). \tag{3.43}$$

Now we multiply (in the sense of the scalar product) bought sides of the vector Identity (3.42) with an arbitrary unit vector $e = (e_1, ..., e_n)$. By Schwarz inequality and taking into account Identity (3.43) with $R = t\delta(x)$ we have

$$\begin{aligned}
&|eDu_t(x)| \\
&\leqslant n\frac{|eD\delta(x)|}{\delta(x)}\frac{t\delta(x)}{n}\mathcal{M}_\Omega|Du|(x) + \left|\fint_{B(x,R)} eDu(y)dy\right| \\
&\leqslant (t+1)\mathcal{M}_\Omega|Du|(x)
\end{aligned}$$

for almost every $x \in \Omega$.

Since $0 < t < 1$ and e is arbitrary, (3.41) is proved for smooth functions.

The second step concerns the case $u \in W^{1,p(\cdot)}(\Omega)$. By density arguments, there is a sequence φ_j of functions in in $C^\infty \cap W^{1,p(\cdot)}(\Omega)$ such that $\varphi_j \to u$ in $W^{1,p(\cdot)}(\Omega)$. Fix $0 < t < 1$.We can see that

$$u_t(x) = \fint_{B(x,t\delta(x))} \lim_j \varphi_j(y)dy.$$

Since $\varphi_j \to u$ in $L^{p(\cdot)}(\Omega)$ and $p(x) \geqslant p_- > 1$ a.e. then, by Lemma 3.3.6, $\varphi_j \to u$ in $L^1(B(x, t\delta(x)))$. Thus

$$u_t(x) = \lim_j (\varphi_j)_t(x),$$

pointwise in Ω. Now

$$|(\varphi_j)_t(x)| \leqslant \fint_{B(x,t\delta(x))} |\varphi_j(y)|dy \leqslant \mathcal{M}_\Omega\varphi_j(x)$$

$j = 1, 2...$ for all $x \in \Omega$. Since the estimate (3.41) is true for C^∞ functions we have

$$|D(\varphi_j)_t(x)| \leqslant 2\mathcal{M}_\Omega|D\varphi_j|(x) \tag{3.44}$$

$j = 1, 2...$ for a.e. $x \in \Omega$.

These inequalities and Theorem 3.3.1 imply that

$$\begin{aligned}
&\|(\varphi_j)_t\|_{1,p(\cdot),\Omega}\\
&= \|(\varphi_j)_t\|_{p(\cdot),\Omega} + \|D(\varphi_j)_t\|_{p(\cdot),\Omega}\\
&\leqslant \|\mathcal{M}_\Omega(\varphi_j)\|_{p(\cdot),\Omega} + 2\|\mathcal{M}_\Omega|D\varphi_j|\|_{p(\cdot),\Omega}\\
&\leqslant c(n,p(\cdot),\Omega)\big(\|\varphi_j\|_{p(\cdot),\Omega} + \|D\varphi_j\|_{p(\cdot),\Omega}\big)\\
&= c(n,p(\cdot),\Omega)\|\varphi_j\|_{1,p(\cdot),\Omega} < \infty.
\end{aligned}$$

Thus $(\varphi_j)_t$ is a bounded sequence in $W^{1,p(\cdot)}(\Omega)$ that converges pointwise to u_t. Moreover in our assumption is $1 < p_-$ and $p_+ < +\infty$, so we have that $W^{1,p(\cdot)}$ is reflexive (see [KoR], Theorem 3.1). Then by the weak compacteness of $W^{1,p(\cdot)}(\Omega)$ there exists a subsequence of $(\varphi_j)_t$ (we will omit the explicit passage to it) which converges weakly in $W^{1,p(\cdot)}(\Omega)$. In particular we have $u_t \in W^{1,p(\cdot)}(\Omega)$ and

$$D(\varphi_j)_t \rightharpoonup Du_t \quad \text{weakly in } L^{p(\cdot)}(\Omega). \tag{3.45}$$

Moreover, by the sublinearity of the maximal function we obtain

$$|\mathcal{M}_\Omega|D\varphi_j|(x) - \mathcal{M}_\Omega|Du|(x)| \leqslant |\mathcal{M}_\Omega(|D\varphi_j| - |Du|)(x)|$$

for every $x \in \Omega$. Using again Theorem 3.3.1 we get

$$\begin{aligned}
&\|\mathcal{M}_\Omega|D\varphi_j|(x) - \mathcal{M}_\Omega|Du|(x)\|_{p(\cdot),\Omega}\\
&\leqslant \|\mathcal{M}_\Omega(|D\varphi_j| - |Du|)(x)\|_{p(\cdot),\Omega}\\
&\leqslant c(n,p(\cdot),\Omega)\||D\varphi_j| - |Du|\|_{p(\cdot),\Omega}.
\end{aligned}$$

Thus

$$\mathcal{M}_\Omega|D\varphi_j| \to \mathcal{M}_\Omega|Du| \;\text{ strongly in } L^{p(\cdot)}(\Omega). \tag{3.46}$$

By (3.45), (3.46) and (3.44), to complete the proof we have to apply Lemma 3.3.7 using strong convergence instead of weak convergence for one of the sequences.

The rest of this section is devoted to the proof of Theorem 3.3.3 (see [Z3]).

Proof. (of Theorem 3.3.3) Let $u \in W^{1,p(\cdot)}(\Omega)$. Let us observe that, by Lemma 3.3.8, $|u| \in W^{1,p(\cdot)}(\Omega)$.

The idea is to observe that the maximal function $\mathcal{M}_\Omega$ can be expressed as the supremum of a suitable increasing sequencen ([KiLi]). Let t_j, $j = 1, 2, ...$ be a sequence of rational numbers such that $0 < t_j < 1$ and let us denote $u_j = |u|_{t_j}$. From Lemma 3.3.8 and Proposition 3.3.9 $u_j \in W^{1,p(\cdot)}(\Omega)$ and

$$|Du_j(x)| \leqslant 2\mathcal{M}_\Omega|D|u||(x) = 2\mathcal{M}_\Omega|Du|(x)$$

$j = 1, 2...$ for a.e. $x \in \Omega$.

Thus let us define $v_k : \Omega \to [-\infty, +\infty]$ as

$$v_k(x) = \max_{1 \leqslant j \leqslant k} u_j(x),$$

$k = 1, 2, ...$ for a.e. $x \in \Omega$.

Let us observe that for a locally integrable function f the function

$$\fint_{B(x, r\delta(x))} |f(y)| dy$$

is continuous with respect to r, $0 < r < 1$. Indeed the integral is a function absolutely continuous with respect to the integration set and then, for all fixed $x \in \Omega$, $\int_{B(x,r\delta(x))} |f(y)| dy$ is continuous with respect to the radius r. So the same holds true for the mean value $\fint_{B(x,r\delta(x))} |f(y)| dy$. Then the supremum taken on the set (0,1) is the same as the one taken on a subset dense. In particular we have (3.40)

$$\mathcal{M}_\Omega(x) = \sup_k v_k(x).$$

Moreover, by Lemma 3.3.8, $v_k \in W^{1,p(\cdot)}(\Omega)$. So v_k is an increasing sequence converging pointwise to $\mathcal{M}_\Omega u$ and

$$|Dv_k(x)| = |D \max_{1 \leqslant j \leqslant k} u_j(x)| \leqslant \max_{1 \leqslant j \leqslant k} |Du_j(x)| \leqslant 2\mathcal{M}_\Omega |Du|(x)$$

$k = 1, 2...$ for a.e. $x \in \Omega$.

Since $v_k(x) \leqslant \mathcal{M}_\Omega u(x)$ $k = 1, 2...$ for a.e. $x \in \Omega$ we obtain

$$\begin{aligned}
&\|v_k\|_{1,p(\cdot),\Omega} \\
&= \|v_k\|_{p(\cdot),\Omega} + \|Dv_k\|_{p(\cdot),\Omega} \\
&\leqslant \|\mathcal{M}_\Omega(u)\|_{p(\cdot),\Omega} + 2\|\mathcal{M}_\Omega |Du|\|_{p(\cdot),\Omega} \\
&\leqslant c(n, p(\cdot), \Omega)\big(\|u\|_{p(\cdot),\Omega} + \|Du\|_{p(\cdot),\Omega}\big) \\
&= c(n, p(\cdot), \Omega)\|u\|_{1,p(\cdot),\Omega}.
\end{aligned}$$

Since the weak compactness it follows $\mathcal{M}_\Omega(u) \in W^{1,p(\cdot)}(\Omega)$ with $v_k \to \mathcal{M}_\Omega(u)$ and $Dv_k \to D\mathcal{M}_\Omega(u)$ weakly in $L^{p(\cdot)}(\Omega)$. Therefore from Lemma 3.3.7

$$|D\mathcal{M}_\Omega(u)(x)| \leqslant 2\mathcal{M}_\Omega |Du|(x).$$

$\square$

Remark 3.3.1. Let us observe that in case $\Omega = \mathbb{R}^n$, the proof can be obtained more easily by considering a slight modification of the ones contained in [Ki].

3.4 Some remarks

Let us say a few words about the inequality (1.34) for the nontangential maximal operator N, as defined by (1.10). The heart of the matter is the weighted norm inequality for the

Hardy-Littlewood (unweighted) maximal function of $f \in L^1_{loc}(\mathbb{R})$

$$Mf(x) = \sup_{I \ni x} ⨍_I |f(y)|dy \tag{3.47}$$

Here we have confined ourselves to the one dimensional case, not only for the sake of simplicity but also because it suffices to express out ideas in reasonable generality. A well known result of B. Muckenhoupt asserts that for a measure ω in $\mathbb{R}$ we have

$$\int_{\mathbb{R}} Mf(x)^p w(x)dx \leqslant c_p \int_{\mathbb{R}} |f(y)|^p w(x)dx \tag{3.48}$$

if and only if $A_p(w) < \infty$. S. Buckley [Bu] noted that

$$c_p = c(p,n)A_p(w)^{\frac{p}{p-1}} \tag{3.49}$$

This constant exhibits the best possible dependence on $A_p(\omega)$.

Here we deal with somewhat dual situation where we have the weighted maximal function of $g \in L^1_{loc}(\mathbb{R})$ defined by

$$M_v g(t) = \sup_{J \ni t} \frac{1}{\int_J v(\tau)d\tau} \int_J |g(\tau)|v(\tau)d\tau \tag{3.50}$$

where $v \in A_\infty$ on $\mathbb{R}$. As already mentioned (see Lemma 1.2.2) the following estimate

$$Nu(\sigma) \leqslant c\ M_\omega g(\sigma), \qquad \text{a.e. } \sigma \in \partial\mathbb{D} \tag{3.51}$$

with ω being the harmonic measure associated with an operator $\mathcal{L}$ and its coefficients matrix $A \in \mathcal{E}(K)$, while u is the solution to

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \mathbb{D} \\ u_{|\partial\mathbb{D}} = g \in C(\partial\mathbb{D}) \end{cases} \tag{3.52}$$

This problem involves a weighted maximal operator which satisfies the norm inequality

$$\int_{\mathbb{R}} M_v g(t)^q dt \leqslant c \int_{\mathbb{R}} |g(t)|^q dt, \tag{3.53}$$

We wish to give a short proof of such an inequality in case $q = 2$. Let us begin with the following

Lemma 3.4.1. *Let f, $g, \in L^1_{loc}(\mathbb{R})$ be coupled by the relation $g(t) = f(h^{-1}(t))$, for almost every $t \in \mathbb{R}$, where $h : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism. Then*

$$M_{(h^{-1})'} g(t) = Mf(h^{-1}(t)) \tag{3.54}$$

for a.e. $t \in \mathbb{R}$.

Proof. We first prove the inequality $\leqslant$ in (3.54). Fix $t \in \mathbb{R}$ to show that for any interval (a, b) containing t we have

$$\frac{1}{\int_a^b (h^{-1})'(\tau)d\tau} \int_a^b f(h^{-1}(\tau))(h^{-1})'(\tau)d\tau \leqslant Mf(h^{-1}(t)). \tag{3.55}$$

Inequality (3.55) follows by the change of variables: $h^{-1}(\tau) = \sigma$:

$$\frac{1}{\int_a^b (h^{-1})'(\tau)d\tau} \int_a^b f(h^{-1}(\tau))(h^{-1})'(\tau)d\tau = \frac{1}{h^{-1}(b) - h^{-1}(a)} \int_{h^{-1}(a)}^{h^{-1}(b)} f(\sigma)d\sigma, \tag{3.56}$$

where its should be noted that $h^{-1}(a) < h^{-1}(t) < h^{-1}(b)$.

The opposite inequality is proved similarly. Indeed, fix t and consider any interval $(c, d) \ni h^{-1}(t)$. In order to prove that

$$\frac{1}{d-c} \int_c^d f(\sigma)d\sigma \leqslant \sup_{(a,b)\ni t} \frac{\int_a^b f(h^{-1}(\tau))(h^{-1})'(\tau)d\tau}{\int_a^b (h^{-1})'(\tau)d\tau} \tag{3.57}$$

we perform the change of variables $h(\sigma) = \tau$,

$$\frac{1}{d-c}\int_c^d f(\sigma)d\sigma = \frac{1}{d-c}\int_{h(c)}^{h(d)} f(h^{-1}(\tau))(h^{-1})'(\tau)d\tau. \tag{3.58}$$

Introducing the end points $a = h(c)$, $b = h(d)$, we see that $t \in (a, b)$ and

$$\frac{1}{d-c}\int_c^d f(\sigma)d\sigma \leqslant \frac{1}{\int_a^b (h^{-1})'(\tau)d\tau}\int_a^b f(h^{-1}(\tau))(h^{-1})'(\tau)d\tau. \tag{3.59}$$

This completes the proof of the Lemma. $\square$

Corollary 3.4.2. *Let $h : \mathbb{R} \to \mathbb{R}$ be an increasing homeomorphism. Define two weights on $\mathbb{R}$:*

$$w(t) = h'(t) \qquad v(s) = (h^{-1})'(s).$$

Assume that there exists a constant $c_0 > 1$ such that

$$\int_{\mathbb{R}} M_v g(t)^2 dt \leqslant c_0 \int_{\mathbb{R}} g(t)^2 dt \tag{3.60}$$

for any $g \in L^2(\mathbb{R}, dx)$. Then, the inequality

$$\int_{\mathbb{R}} Mf(x)^2 w(x)dx \leqslant c_0 \int_{\mathbb{R}} f(x)^2 w(x)dx, \tag{3.61}$$

holds for any $f \in L^2(\mathbb{R}, w(x)dx)$. Conversely, (3.61) yields (3.60).

Proof. Assume (3.61) holds, for any $f \in L^2(\mathbb{R}, wdx)$. For $g \in L^2(\mathbb{R}, dx)$, we set $f(x) = g(h(x))$ and compute

$$\int f(x)^2 w(x)dx = \int f(x)^2 h'(x)dx = \int f(h^{-1}(t))^2 dt = \int g(t)^2 dt. \tag{3.62}$$

Similarly, in view of (3.54)

$$\int Mf(x)^2 w(x)dx = \int Mf(x)^2 h'(x)dx = \int Mf(h^{-1}(t))^2 dt = \int M_v g(t)^2 dt. \tag{3.63}$$

Now, from (3.61), (3.62) and (3.63) we deduce that

$$\int_{\mathbb{R}} M_v g(t)^2 dt \leqslant c_0 \int_{\mathbb{R}} g(t)^2 dt. \tag{3.64}$$

Similarly the reader may verify that (3.60) for $g \in L^2(\mathbb{R}, dx)$ implies (3.61) for $f \in L^2(\mathbb{R}, w(x)dx)$.

□

Chapter 4

On the Dirichlet problem with Orlicz boundary data

Let us consider a Young's function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the Δ_2 condition together with its complementary function Ψ, and let us consider the Dirichlet problem for a second order elliptic operator in divergence form $\mathcal{L} = \operatorname{div}(A\nabla u)$:

$$\begin{cases} \mathcal{L}u = 0 & \text{in } B \\ u_{|\partial B} = f, \end{cases}$$

where $A \in \mathcal{E}(K)$ and B is the unit ball of $\mathbb{R}^n$. In this chapter we give a necessary and sufficient condition for the L^Φ-solvability of the problem, where L^Φ is the Orlicz Space generated by the function Φ.

4.1 The L^Φ-solvability

Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a Young's function that satisfies the Δ_2 -condition together with its complementary function Ψ. The Dirichlet problem (1.32) is said to be L^Φ-**solvable** if for any $f \in C^0(\partial B)$ there exists a unique solution $u \in W^{1,2}_{loc}(B) \cap C^0(\bar{B})$ to (1.32) which satisfies the uniform estimate

$$\int_{\partial B} \Phi(Nu)d\sigma \leqslant C \int_{\partial B} \Phi(|f|)d\sigma \tag{4.1}$$

Let us observe that for $L^\Phi = L^p$ the integral inequality (4.1) corresponds to the norm inequalities (1.34). We will show that condition (*ii*) of Theorem 1.4.1 is a necessary and sufficient condition also for the L^Φ-solvability of the problem (1.32), where Φ is a given Young's function such that the upper index of L^Φ is p_0^{-1} (see Definition 2.3.1).

As a corollary of Theorem 3.2.1 we have the following extension of Theorem 1.4.1. Namely:

Theorem 4.1.1. **[Z2]** *Let B be the unit ball of $\mathbb{R}^n$ and let $\Phi(t) = \int_0^t \varphi(\tau)d\tau$ be a Young's function that satisfies the Δ_2 -condition together with its complementary function $\Psi(s) = \int_0^s \varphi^{-1}(\tau)d\tau$, and let p_0^{-1} be the upper index of the Orlicz Space $L^\Phi(\partial B, d\sigma)$. Then the following are equivalent:*

i) *The Dirichlet problem (1.32) is L^Φ-solvable.*

ii) *The $\mathcal{L}$-harmonic measure ω is absolutely continuous with respect to σ, and $k = \frac{d\omega}{d\sigma} \in B_\Phi(d\sigma)$, that is:*

$$\left(\frac{1}{k(\Delta)}\int_\Delta \epsilon d\sigma\right)\varphi\left(\frac{1}{k(\Delta)}\int_\Delta \varphi^{-1}\left(\frac{k}{\epsilon}\right)kd\sigma\right) \leqslant K \tag{4.2}$$

for all surface balls Δ and for all $\epsilon > 0$.

iii) *The $\mathcal{L}$-harmonic measure ω belongs to $B_{q_0}(d\sigma)$, $\frac{1}{p_0} + \frac{1}{q_0} = 1$, i.e. ω is absolutely continuous with respect to σ, and $k = \frac{d\omega}{d\sigma} \in L^{q_0}(d\sigma)$, with*

$$\left(\frac{1}{\sigma(\Delta)}\int_\Delta k^{q_0}\right)^{\frac{1}{q_0}} \leqslant C\left(\frac{1}{\sigma(\Delta)}\int_\Delta k\right), \qquad \forall\Delta.$$

Remark 4.1.1. It is worth to point out that in case $\Phi(t) = t^p$, $1 < p < 1$, condition *ii)* is exactly the *reverse Hölder condition iii)*. Hence, last theorem extends Theorem 1.4.1 in the sense that we obtain that condition $\omega_{\mathcal{L}} \in B_{q_0}$ characterize solvability of the Dirichlet problem also when the boundary data is in a suitable *class* of Orlicz space, containing L^{q_0}, and this class is identified by the upper Boyd index $\rho = \frac{1}{p_0}$.

Proof. The equivalence $(ii) \Leftrightarrow (iii)$ follows directly by Theorem 3.2.1 applied to the weight functions $v(x) = 1$ and $w(x) = k(x)$. Moreover $(iii) \Rightarrow (i)$. In fact, using Theorem 3.2.1 again, we have that there exists a constant $C > 0$, such that for any $f \in C(\partial B)$:

$$\int_{\partial B} \Phi(M_k f) d\sigma \leqslant C \int_{\partial B} \Phi(|f|) d\sigma. \tag{4.3}$$

Hence, by the pointwise estimates

$$Nu(P) \leqslant C_\beta M_\omega f(P), \quad \forall P \in \partial B,$$

contained in Lemma 1.2.2 the L^Φ- solvability directly follows.

To prove $(i) \Rightarrow (iii)$ we firstly need to show that L^Φ-solvability of the Problem (1.32) implies that the harmonic measure ω is absolutely continuous with respect to the surface measure σ. To this aim, let us observe that ω and σ are Borel positive regular probability measures on ∂B.

Now, let K be a compact set on ∂B. By the regularity of σ it holds

$$\forall \varepsilon > 0, \ \exists \text{ an open set } A \subset \partial B \text{ such that } K \subset A \text{ and } \sigma(A \setminus K) < \varepsilon.$$

Moreover, by the Uryson Lemma

$$\exists f \in C^0(\partial B) \text{ such that: } 0 \leqslant f \leqslant 1 \text{ on } \partial B, \ f = 1 \ \text{ on } K, \ f = 0 \text{ on } \partial B \setminus A.$$

For such function f we have (for any point on ∂B)

$$M_\omega f \geqslant \frac{1}{\omega(\partial B)} \int_{\partial B} f d\omega = \int_A f d\omega \geqslant \int_K f d\omega = \omega(K),$$

so that

$$\int_{\partial B} \Phi(M_\omega f) d\sigma \geqslant \Phi(\omega(K))\sigma(\partial B). \tag{4.4}$$

Moreover,

$$\int_{\partial B} \Phi(|f|) d\sigma = \int_A \Phi(f) d\sigma \leqslant \Phi(1)\sigma(A) < \Phi(1)(\sigma(K) + \varepsilon). \tag{4.5}$$

Hence, by (4.4), (4.5) and by i), we have

$$\Phi(\omega(K)) \leqslant \frac{c\Phi(1)}{\sigma(\partial B)}(\sigma(K) + \varepsilon))$$

and so, by the arbitrarity of ε, it holds

$$\omega(K) \leqslant \Phi^{-1}(c'\sigma(K)). \tag{4.6}$$

Now, by the regularity of ω and of σ, (4.6) holds for any Borel set and then for any Lebesgue set E on ∂B. In fact,

$$\begin{aligned}
\omega(E) &= \sup\{\omega(K) \mid K \text{ compact set, } K \subset E\} \\
&\leqslant \Phi^{-1}(c' \sup\{\sigma(K) \mid K \text{ compact set, } K \subset E\}) \\
&= \Phi^{-1}(c'\sigma(E)).
\end{aligned}$$

Hence the absolute continuity of ω with respect to σ,

$$\forall \varepsilon > 0, \exists \delta > 0 : \sigma(E) < \delta \Rightarrow \omega(E) < \varepsilon \tag{4.7}$$

holds true. Indeed one can choose in (4.7), $\delta = \frac{\Phi(\varepsilon)}{c'}$.

Now to complete our proof we need to show that $\omega \in B_{q_0}$. Let us start by observing that the weighted maximal operator M_ω is pointwise subadditive. In fact, let $f, g \in L^1(d\omega)$. Obviously we have

$$\int_\Delta |f+g|k d\sigma \leqslant \int_\Delta |f|k d\sigma + \int_\Delta |g|k d\sigma$$

for any surface ball $\Delta \subset \partial B$, so that

$$M_\omega(f+g)(P) \leqslant M_\omega f(P) + M_\omega g(P)$$

for any $P \in \partial B$. Hence, for any $f \in L^\Phi(d\sigma)$, let us consider f^+ and f^- be the positive and negative part of f respectively, i.e.

$$f^+ = \begin{cases} f & if \;\; f \geqslant 0 \\ 0 & if \;\; f \leqslant 0 \end{cases}$$

$$f^- = \begin{cases} 0 & if \;\; f \geqslant 0 \\ -f & if \;\; f \leqslant 0 \end{cases}$$

so that

$$f = f^+ - f^-.$$

Obviously $f \in L^\Phi(d\sigma)$ implies that also f^+ and f^- are in $L^\Phi(d\sigma)$. Then, let u_1, u_2 be the solution of the L^Φ- problem with boundary data f^+ and f^- respectively. By the subadditivity of M_ω and by the second part of Lemma 1.2.2, we have

$$M_\omega f \leqslant M_\omega f^+ + M_\omega f^- \leqslant C_\beta(Nu_1 + Nu_2).$$

On the other hand, by the Δ_2 condition and the convexity of Φ, by (2.12) and by the assumption of L^Φ-solvability, we obtain

$$\begin{aligned}\int_{\partial B}\Phi(M_\omega f)d\sigma &\leqslant \int_{\partial B}\Phi(C_\beta(Nu_1+Nu_2))d\sigma\\ &\leqslant C(\beta,\Phi)\left(\int_{\partial B}\Phi(Nu_1)d\sigma+\int_{\partial B}\Phi(Nu_2)d\sigma\right)\\ &\leqslant C(\beta,\Phi)\left(\int_{\partial B}\Phi(f^+)d\sigma+\int_{\partial B}\Phi(f^-)d\sigma\right)\\ &\leqslant C(\beta,\Phi)\left(\int_{\partial B}\Phi(|f|)d\sigma\right)\end{aligned}$$

Hence, we obtain that for any $f\in L^\Phi(d\sigma)$:

$$\int_{\partial B}\Phi(M_k f)d\sigma \leqslant C\int_{\partial B}\Phi(|f|)d\sigma. \tag{4.8}$$

Using again Theorem 3.2.1 the thesis follows. □

It is worth to point out that in general the integral inequality (4.1) is *stronger* than the norm inequality (1.34). Indeed it holds the following

Proposition 4.1.2. *Let $\Phi\in\Delta_2$ be a Young function. Then*

$$\int_{\partial B}\Phi(M_\omega f)dx \leqslant C\int_{\partial B}\Phi(|f|)dx \tag{4.9}$$

implies

$$\|M_\omega f\|_{L^\Phi}\leqslant K\|f\|_{L^\Phi(dx)} \tag{4.10}$$

Proof. Assume (4.9). Then, in particular, for any $h>0$,

$$\int_{\partial B}\Phi\left(\frac{M_\omega f}{h}\right)dx\leqslant C\int_{\partial B}\Phi\left(\frac{|f|}{h}\right)dx.$$

Let

$$k_f=\|f\|_{L^\Phi}=\inf\left\{k:\int_{\partial B}\Phi\left(\frac{|f|}{k}\right)dx\leqslant 1\right\},$$

and let $B = \frac{1}{C}$. There exists $A > 0$ such that $\Phi(At) \leqslant B\Phi(t)$. Then, for any $h = \frac{k}{A} > \frac{k_f}{A}$ we have

$$\begin{aligned}
&\int_{\partial B} \Phi\left(\frac{M_\omega f}{h}\right) dx = \int_{\partial B} \Phi\left(M_\omega \frac{f}{h}\right) dx \leqslant \\
&\leqslant C \int_{\partial B} \Phi\left(\frac{|f|}{h}\right) dx \\
&\leqslant CB \int_{\partial B} \Phi\left(\frac{|f|}{k}\right) dx \\
&\leqslant 1.
\end{aligned}$$

Hence, the infimum of h such that $\int_{\partial B} \Phi\left(\frac{M_\omega f}{h}\right) dx \leqslant 1$, verifies $\inf h \leqslant \frac{k_f}{A} = \frac{1}{A}\|f\|_{L^\Phi}$, i.e.

$$\|M_\omega f\|_{L^\Phi} \leqslant \frac{1}{A}\|f\|_{L^\Phi(dx)}$$

as we claimed. □

Note that from the 'openness' of the condition $\omega \in B_q(d\sigma)$ (Theorem 1.3.2, iv)), it follows that the L^Φ-solvability implies also the L^{Φ_δ}-solvability of the Problem (1.32), with suitable $\delta > 0$, and the upper index of L^{Φ_δ} is bigger than the one of L^Φ.

Moreover, we observe that, in the case $\Phi(t) = tlg^\alpha(e+t)$, $0 \leqslant \alpha < 1$, Theorem 1.4.1 does not hold (see Section 4.3).

In what follows we will show same examples of Young function Φ such that Theorem 4.1.1 can be applied.

4.2 Examples

Let

$$\Phi(t) = t^p \log^\alpha(e+t) \tag{4.11}$$

Obviously $\Phi(t)$ verifies the Δ_2 condition. Moreover, Φ is a Young function

i) when $p > 2$ and $\alpha > 2 - p$. In particular, choosing $\alpha < 0$, we have $\Phi(t) < t^p$. This implies $L^p \subset L^\Phi$,

ii) when $p > \frac{3}{2}$ and $\alpha > 3 - 2p$ so that it can be $\alpha < 0$, and hence $\Phi(t) < t^p$. This implies again $L^p \subset L^\Phi$.

Once that $\Phi_{p,\alpha}$ is a Young function, using [FiK] we have $\rho < 1$ (or equivalently $\underline{\alpha}(\Phi) > 1$) so that Φ and its complementary function Ψ verify the Δ_2 condition. Hence, Theorem 4.1.1 can be applied for example when

$$\Phi(t) = t^p \log^\alpha(e + t), \qquad \forall p > \frac{3}{2}, \alpha > 3 - 2p.$$

Note that thanks to [FiK] we can exactly compute the upper index ρ of Φ. Indeed by easy computation we have

$$\lim_{t \to 0} \frac{t\varphi(t)}{\Phi(t)} = p$$

Analogously,

$$\lim_{t \to \infty} \frac{t\varphi(t)}{\Phi(t)} = p$$

Hence the upper and lower Boyd index of Φ are $\rho = \vartheta = \frac{1}{p}$.

4.3 The $L(lgL)^\alpha$-unsolvability. A counterexample

In this section we will show, with an example, the unsolvability of the Dirichlet problem with $L(lgL)^\alpha$ boundary data. To be more precise, we will show that in case $f \in L(lgL)^\alpha$, $0 \leqslant \alpha < 1$, an analogue inequality to (1.34)

$$\|Nu\|_{L^1(\partial B, d\sigma)} \leqslant C \, \|f\|_{L(lgL)^\alpha(\partial B, d\sigma)}$$

does not hold.

EXAMPLE 4.3.1. Let B be the unit ball in $\mathbb{R}^2$, $f \in L(lgL)^{\alpha}(\partial B, d\sigma)$ and let u be the solution to the Dirichlet problem for the Laplacian

$$\begin{cases} \Delta u = 0 \quad \text{in } B \\ u_{|\partial B} = f \end{cases} \tag{4.12}$$

We recall that a solution of the problem (4.12) is given by the Poisson integral formula.

$$u(\varrho\cos\vartheta, \varrho\sin\vartheta) = \frac{1-\varrho^2}{2\pi}\int_{-\pi}^{\pi}\frac{f(\cos s\sin s)}{1+\varrho^2-2\varrho\cos(\vartheta-s)}ds \tag{4.13}$$

Let now consider the function $\Phi : [0,+\infty[\to [0,+\infty[$ defined by

$$\Phi = \Phi(t) = tlg^{\alpha}(e+t).$$

When $\alpha \geqslant 0$, Φ is a convex, continuous, strictly increasing function with $\Phi(0) = 0, \lim_{t\to\infty}\Phi(t) = +\infty$.

For small $\delta > 0$, let $\gamma = \gamma(\delta)$ be the (unique) positive solution to the equation:

$$\delta\gamma lg^{\alpha}(e+\gamma) = lg^{\alpha}(e+1) = \Phi(1) \tag{4.14}$$

(Note that for $\alpha = 0$, $\gamma = \frac{1}{\delta}$) and, for $Q \in \partial B$, $Q = (\cos s, \sin s)$ define:

$$f_\delta = f_\delta(Q) = f_\delta(s) = \begin{cases} \gamma & \text{if } 0 \leqslant s \leqslant \delta \\ 0 & \text{elsewhere} \end{cases}$$

These functions f_δ, $\delta > 0$ belong to $L(lgL)^{\alpha}(\partial B, d\sigma)$ with unit norm. Indeed, for $\lambda > 0$, we have

$$\int_{\partial B}\Phi(\lambda f(Q))d\sigma(Q) = \int_0^{\delta}\lambda\gamma lg^{\alpha}(e+\lambda\gamma)ds = \delta\lambda\gamma lg^{\alpha}(e+\lambda\gamma) \gtreqless \Phi(1) \Leftrightarrow$$

$$\lambda \gtreqless 1$$

(the equivalence follows by (4.14), being Φ a strictly increasing function) and then, recalling that, by definition:

$$\|f_\delta\|_{L(lgL)^\alpha(\partial B)} = \inf\left\{\frac{1}{\lambda} : \int_{\partial B} \Phi(\lambda f(Q))d\sigma(Q) \leqslant \Phi(1)\right\},$$

we have

$$\|f_\delta\|_{L(lgL)^\alpha(\partial B)} = 1, \quad \forall\delta. \tag{4.15}$$

If for any δ, $u_\delta = P[f_\delta]$ is the Poisson solution to the problem (4.12) with $f = f_\delta$, we have:

$$Nu_\delta(W) \geqslant \sup_{0\leqslant\varrho<1} u(\varrho W), \ \ \forall W = (\cos\vartheta, \sin\vartheta) \in \partial B,$$

that is:

$$Nu_\delta(W) \geqslant \frac{1}{2\pi} \sup_{0\leqslant\varrho<1} \int_0^\delta \frac{1-\varrho^2}{1+\varrho^2-2\varrho\cos(\vartheta-s)}\gamma ds,$$

and, if $\delta \leqslant \vartheta \leqslant \frac{\pi}{2}$,

$$Nu_\delta(W) \geqslant \frac{\delta\gamma}{2\pi\sin\vartheta} \geqslant \frac{\delta\gamma}{2\pi\vartheta}.$$

Hence, for the norms we have:

$$2\pi\|Nu_\delta\|_{L^1(\partial B)} \geqslant \delta\gamma \int_\delta^{\frac{\pi}{2}} \frac{1}{\vartheta} d\vartheta = \delta\gamma[\lg\vartheta]_\delta^{\frac{\pi}{2}} \geqslant \delta\gamma \lg\frac{1}{\delta}. \tag{4.16}$$

On the other hand, by (4.14) it follows that:

$$\delta\gamma = \frac{\Phi(1)}{\lg^\alpha(\gamma+e)},$$

and then

$$\lg^\alpha(\gamma+e) = \frac{\Phi(1)}{\delta\gamma}.$$

Consequently:

$$\lg\lg^\alpha(\gamma+e) = \lg\left(\frac{\Phi(1)}{\delta\gamma}\right)$$

that is

$$\lg\left(\frac{1}{\delta}\right) = \lg\gamma + \alpha\lg\lg(\gamma+e) - \lg(\Phi(1));$$

we also obtain that, when $\delta \to 0^+$,

$$\gamma\lg^{\alpha}(\gamma+e) \to +\infty$$

and so $\gamma \to +\infty$ too. By the above considerations and by (4.16), we have:

$$\frac{2\pi}{\Phi(1)}\lim_{\delta\to 0^+}\|Nu_\delta\|_{L^1(\partial B)} \geqslant$$

$$\frac{1}{\Phi(1)}\lim_{\delta\to 0^+}\left(\delta\gamma\lg\frac{1}{\delta}\right) =$$

$$\lim_{\gamma\to+\infty}\frac{\lg\gamma + \alpha\lg\lg(\gamma+e) - \lg(\Phi(1))}{\lg^{\alpha}(\gamma+e)} = \begin{cases} +\infty & \text{if} \quad 0 \leqslant \alpha < 1 \\ 1 & \text{if} \quad \alpha = 1 \\ 0 & \text{if} \quad \alpha > 1 \end{cases}$$

Hence, by the last inequality and by (4.15), follows that, if $0 \leqslant \alpha < 1$, then there is no constant $C > 0$ satisfying:

$$\|Nu\|_{L^1(\partial B, d\sigma)} \leqslant C\,\|f\|_{L(\lg L)^{\alpha}(\partial B, d\sigma)}$$

with $u = P[f]$, for any $f \in L^{\infty}(\partial B)$ ($\subset L(\lg L)^{\alpha}(\partial B, d\sigma)$).

4.4 On the case $\alpha = 1$

In this section we establish a maximal inequality which could be useful in view of sufficient conditions for the $L(\lg L)$-solvability of the Dirichlet problem (1.32). It corresponds to a limit case, as $q \to \infty$, of the B_q-condition. To this purpose, we recall the main result of B.

Muckenhoupt [M] about a weighted maximal function. Given a measure m on an interval J, define:

$$M_m f(x) = \sup_{y \in J} \frac{\int_x^y |f(t)| dm(t)}{\int_x^y dm(t)}$$

where the quotient is to be taken as 0 if the numerator and the denominator are both ∞. It holds the following:

Theorem 4.4.1. *Let m be a Borel measure on an interval J which is 0 on sets consisting of single points. Let $U(x)$ and $V(x)$ be nonnegative functions on J, assume that $1 \leqslant p < \infty$, $0 \leqslant a < \infty$ and given $f(x)$ on J let E_a be the subset of J where $M_m(f) > a$. Then, there is a constant, B, independent on f and a such that*

$$\int_{E_a} U(x) dm(x) \leqslant B a^{-p} \int_J |f(x)|^p V(x) dm(x) \tag{4.17}$$

iff there is a constant K such that for any subinterval I of J,

$$\left[\int_I U(x) dm(x) \right] \left[\int_I [V(x)]^{\frac{-1}{p-1}} dm(x) \right]^{p-1} \leqslant K [m(I)]^p. \tag{4.18}$$

In particular we have the next

Corollary 4.4.2. *Let $v > 0$ be a weight on J. Then, if $1 \leqslant p < \infty$ and $0 \leqslant a < \infty$, there is a constant B such that*

$$|\{M_v f > a\}| \leqslant \frac{B}{a^p} \int_J |f(x)|^p dx \tag{4.19}$$

for all f iff there is a constant K such that for every subinterval I of J,

$$|I| \left[\int_I v^{\frac{1}{p-1}} v dx \right]^{p-1} \leqslant K \left[\int_I v dx) \right]^p. \tag{4.20}$$

(Note that if v is a weight on the interval J we can choose in the last theorem $U(x) = V(x) = \frac{1}{v(x)}$ and so $U dm = V dm = dx$, $m = v dx$).

Hence, we have (4.19), $p = 1$ iff there is a constant K such that:

$$ess \sup_{x \in I} \leqslant K \frac{1}{|I|} \int_I v dx$$

for all subinterval I of J (that is the well known Gehring condition $v \in G_\infty(dx)$).

With trivial changes the results hold also for maximal operator where the interval J is taken with both extremal points variable and so we obtain the following result

Theorem 4.4.3. **[Z1]** *Let B be the unit circle in $\mathbb{R}^2$ and let v be a weight, $v \in G_\infty$. Then the weighted Hardy-Littlewood maximal operator*

$$M_v f(x) = \sup_{\Delta \ni x} \frac{1}{\int_\Delta v dx} \int_\Delta |f| v dx$$

is such that

$$M_v : f \in LlgL(dx) \to M_v f \in L^1(dx) \tag{4.21}$$

Proof. We preliminary observe that the following equality, which is an immediate consequence of Fubini's theorem, holds:

$$\int_{\partial B} |f| dx = \int_0^\infty |\{|f| > t\}| dt$$

If $v \in G_\infty(dx)$ there exists a constant $C > 0$ (independent on f) such that, $x > 0$,

$$|\{M_v f > x\}| \leqslant \frac{C}{x} \int_{\partial B} |f| dx$$

for any function $f \in L^1(dx)$. Now, fix $x > 0$,and define $f = g_x + h_x$ where

$$g_x = \begin{cases} f & \text{if } |f| > \frac{x}{2} \\ 0 & \text{elsewhere} \end{cases}$$

It is $h_x = f - g_x$ and then $\|h_x\|_{L^\infty} \leqslant \frac{x}{2}$, so $\|M_v h_x\|_{L^\infty} \leqslant \frac{x}{2}$; by the subadditivity of the maximal operator M_v, $M_v f \leqslant M_v g_x + \frac{x}{2}$, and then $\{M_v f > x\} \subset \{M_v f > \frac{x}{2}\}$. Applying

Muckenoupt's result to g_x we have

$$|\{M_v f > x\}| \leqslant |\{M_v f > \frac{x}{2}\}| \leqslant \frac{C}{x}\int_0^\infty |\{|g_x| > t\}|dt$$

Observe that

$$|\{|g_x| > t\}| = \begin{cases} |\{|f| > t\}| & \text{if } t > \frac{x}{2} \\ |\{|f| > \frac{x}{2}\}| & \text{elsewhere} \end{cases}$$

and then

$$|\{M_v f > x\}| \leqslant \frac{C}{x}\int_0^{\frac{x}{2}} |\{|g_x| > t\}|dt + \frac{C}{x}\int_{\frac{x}{2}}^\infty |\{|g_x| > t\}|dt =$$

$$\frac{C}{x}\int_0^{\frac{x}{2}} |\{|f| > \frac{x}{2}\}|dt + \frac{C}{x}\int_{\frac{x}{2}}^\infty |\{|f| > t\}|dt =$$

$$\frac{C}{x}\frac{x}{2}|\{|f| > \frac{x}{2}\}| + \frac{C}{x}\int_{\frac{x}{2}}^\infty |\{|f| > t\}|dt$$

Obviously, we have

$$\int_{\partial B} M_v f dx = \int_0^2 |\{M_v f > x\}|dx + \int_2^\infty |\{M_v f > x\}|dx.$$

The first term of the last is bounded by

$$\int_0^2 |\{M_v f > x\}|dx = \int_0^2 \int_{\{M_v f > x\}} d\vartheta dx \leqslant 2|\partial B|$$

and for the second one we have

$$\int_2^\infty |\{M_v f > x\}|dx \leqslant \int_2^\infty C|\{|f| > \frac{x}{2}\}|dx + C\int_2^\infty \frac{1}{x}\int_{\frac{x}{2}}^\infty |\{|f| > t\}|dtdx$$

But

$$\int_2^\infty C\left|\{|f| > \frac{x}{2}\}\right| dx \leqslant C\int_0^\infty \left|\{|f| > \frac{x}{2}\}\right| dx = C\|f\|_{L^1(\partial B)}$$

and

$$\int_2^\infty \frac{1}{x}\int_{\frac{x}{2}}^\infty |\{|f|>t\}|\,dtdx = \int_1^\infty \frac{1}{x}\int_x^\infty |\{|f|>t\}|\,dtdx =$$

$$\int_1^\infty \frac{1}{x}\left[\int_x^\infty \left(\int_{\{|f|>t\}} d\vartheta\right) dt\right] dx =$$

by Fubini's theorem

$$\int_1^\infty \frac{1}{x}\left[\int_{\{|f|>x\}}\int_x^{|f(e^{i\vartheta})|} dtd\vartheta\right] dx \leqslant$$

$$\leqslant \int_1^\infty \frac{1}{x}\left[\int_{\{|f|>x\}}\int_0^{|f(e^{i\vartheta})|} dtd\vartheta\right] dx =$$

By integrating

$$= \int_1^\infty \frac{1}{x}\left[\int_{\{|f|>x\}} |f(e^{i\vartheta})| d\vartheta\right] dx =$$

by Fubini's theorem again

$$= \int_{\{|f|>1\}} |f(e^{i\vartheta})| \int_1^{|f(e^{i\vartheta})|} \frac{1}{x} dxd\vartheta =$$

and by integrating

$$= \int_{\{|f|>1\}} |f(e^{i\vartheta})| \lg |f(e^{i\vartheta})| d\vartheta = \int_{\partial B} |f| \lg^+ |f| d\vartheta < \infty,$$

that is

$$\|M_v f\|_{L^1(dx)} \leqslant 2|\partial B| + C\,\|f\|_{L^1(dx)} + C\int_{\partial B} |f| \lg^+ |f| d\vartheta.$$

This completes our proof.

$\square$

Chapter 5

L^p-solvability in dimension $n = 2$. Sharp results

In this chapter we will concentring our attention to the case $n = 2$. This will give us the possibility to obtain a number of *quantitative* results, many of which sharp.

So, let us denote by $\mathbb{D}$ the unit disc in $\mathbb{R}^2$ and assume that the elliptic operator $\mathcal{L} = \operatorname{div}(A(x)\nabla)$ is L^p- resolutive, $p > 1$ on $\mathbb{D}$. One of our results is Proposition 5.4.1 in which we explore the "self improving" property of Gehring classes ($\omega \in B_q \Rightarrow \omega \in B_{q+\varepsilon}$), see Theorem 1.3.3. Thanks to Theorem 1.4.1, we are able to answer in the same vain the L^p- solvability question; L^p- solvability $\Rightarrow$ $L^{p-\eta}$- solvability). The point to make here is that we found the supremum of such η in terms of the B_q constant of ω. For the sake of readibility we formulate here the following particular case of Proposition 5.4.1

Theorem 5.0.4. *Assume that Problem (1.32) is L^2- solvable and set $B = B_2(\omega)$. Then this problem is also L^r- solvable, whenever*

$$r > 1 + \sqrt{\frac{B-1}{B}}. \tag{5.1}$$

This lower bound for r in terms of B is best possible.

It is worth pointing out that $1 + \sqrt{\frac{B-1}{B}} \to 1$ as $B \to 1$, which is the case of the Laplacian.

Indeed, let $\mathcal{L} = \Delta$ be the Laplace operator. We have $\omega_{\mathcal{L}} = \frac{1}{2\pi} d\sigma$ and then $B = 1$. So we re-obtain the known fact (see for example [F]) that the Laplacian "tends" to have its L^p-Dirichlet problem solvable in the full range $1 \leqslant p < \infty$.

From now on we will denote by $\mathcal{E}^s(K)$ the subclass of $\mathcal{E}(K)$ of symmetric matrices. In Section 5.3 we will specify the precise doubling property of ω as a function of the ellipticity constant K, at least for harmonic measures ω relative to the operators $\mathcal{L} = \operatorname{div}(\mathcal{A}(x)\nabla\,)$ on the half plane $\mathbb{R}^2_+$ with $\mathcal{A} \in \mathcal{E}^s(K)$ verifying $\det \mathcal{A} = 1$ a.e. We notice that in this case ω turns out to be equal to $\frac{dh}{1+h^2}$ for certain homeomorphism $h : \mathbb{R} \to \mathbb{R}$ with

$$D_2(dh) \leqslant e^{C(K-1)}, \tag{5.2}$$

where C turns out to be an absolute constant, Theorem 5.3.1.

Let us denote by $\mathcal{E}_1(K)$ the subclass of $\mathcal{E}^s(K)$ of symmetric matrix functions satisfying the condition

$$\det \mathcal{A}(x) = 1 \quad a.e.\ x \in \mathbb{D}.$$

The restriction to coefficient matrices $\mathcal{A} \in \mathcal{E}_1(K)$ poses any loss of generality. For this we recall [IS] that, if $u \in W^{1,2}_{loc}$ solves $\operatorname{div}(A(x)\nabla u\,) = 0$ for some $A \in \mathcal{E}^s(K)$ then there is a correction $\mathcal{A} \in \mathcal{E}_1(K)$ such that $\operatorname{div}(\mathcal{A}\nabla u) = 0$.

Now let $\mathcal{L}_0 = \operatorname{div}(A_0\nabla\,)$ and $\mathcal{L}_1 = \operatorname{div}(A_1\nabla\,)$ be two (elliptic) operators and suppose that we know that the Dirichlet problem is solvable for the first operator $\mathcal{L}_0$. A natural question arises as to whether one can easily verify that the second operator has its Dirichlet problem solvable as well?

For example, consider the two operators $\mathcal{L}_0 = \Delta$ and $\mathcal{L}_1 = \operatorname{div}(\mathcal{A}_1\nabla\,)$ with $\mathcal{A}_1 \in \mathcal{E}_1(K)$. It is well known that $\mathcal{L}_0$ is L^2- resolutive and there are a number of interesting results in order $\mathcal{L}_1$ to be L^q- resolutive, many of which require the coefficients of $\mathcal{L}_1$ to be uniformly close to those of Δ (i.e. δ_{ij}) as we approach the boundary of $\mathbb{D}$ (see e.g. [D3], [FKP], [F]).

Here we present a different point of view: an elliptic operator is thought of as a perturbation of the Laplacian after a suitable change of variables.

Actually, *all* matrices in $\mathcal{E}_1(K)$ generate *pull-back* of Laplacian via K- quasiconformal mappings. More precisely, let $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F = (\alpha, \beta)$ be K- *quasiconformal*; that is, F is a homeomorphism of class $W^{1,2}_{loc}(\mathbb{R}^2; \mathbb{R}^2)$ such that

$$|DF(x)|^2 \leqslant \left(K + \frac{1}{K}\right) J_F(x) \qquad \text{a.e.}. \tag{5.3}$$

Here $|DF(x)|$ stands for the Hilbert-Schmidt norm of the differential matrix $DF(x)$ and $J_F(x)$ for the Jacobian determinant of F. Then, with $\mathbb{R}^2_+$ denoting the half-plane $x_2 > 0$, we have $F(\mathbb{R}^2_+) = \mathbb{R}^2_+$ and $F(\mathbb{R}) = \mathbb{R}$. Moreover, if u satisfies $\Delta u = 0$, then $v = u \circ F$ is a solution to $\mathcal{L}v = \operatorname{div}(\mathcal{A}\nabla v) = 0$ where $\mathcal{A} = \mathcal{A}(x_1, x_2)$ is given by

$$\mathcal{A} = \frac{1}{J_F} \begin{pmatrix} \beta_{x_1}^2 + \beta_{x_2}^2 & -\alpha_{x_1}\beta_{x_1} - \alpha_{x_2}\beta_{x_2} \\ \\ -\alpha_{x_1}\beta_{x_1} - \alpha_{x_2}\beta_{x_2} & \alpha_{x_1}^2 + \alpha_{x_2}^2 \end{pmatrix} \tag{5.4}$$

and verifies (1.1), see [IS]. Hence $\mathcal{L} = \Delta_F$ is the *pull-back* under F of the Laplacian. It is well known that $\mathcal{A}$ belongs to $\mathcal{E}_1(K)$, see [IS] e.g.

In Section 5.2 we will prove the following theorem which enlights a quantitative version of the solvability for couples of special elliptic operators $\mathcal{L}_0$ and $\mathcal{L}_1$. It reveals a kind of duality between the associated harmonic measures $\omega_{\mathcal{L}_0}$ and $\omega_{\mathcal{L}_1}$, expressed by equality of the respective A_∞- constants (for the definitions of the constants $\tilde{A}_\infty(\nu)$ or $\tilde{B}_1(\nu)$ see Section 1.3).

Theorem 5.0.5. *Let $F : \mathbb{D} \to \mathbb{D}$ be a K- quasiconformal mapping. Then, the operator*

$$\mathcal{L}_0 = \Delta_F \tag{5.5}$$

is resolutive if and only if

$$\mathcal{L}_1 = \Delta_{F^{-1}} \tag{5.6}$$

is resolutive. Actually, for the harmonic measures $\omega_{\mathcal{L}_0}$ and $\omega_{\mathcal{L}_1}$ we have

$$\tilde{A}_\infty(\omega_{\mathcal{L}_0}) = \tilde{B}_1(\omega_{\mathcal{L}_1}) \quad \text{and} \quad \tilde{A}_\infty(\omega_{\mathcal{L}_1}) = \tilde{B}_1(\omega_{\mathcal{L}_0}). \tag{5.7}$$

Moreover, if $\mathcal{L}_0$ is L^p- resolutive for a $p > 1$ and let $B = B_q(\omega_{\mathcal{L}_0}) < \infty$ with $q = \frac{p}{p-1}$, then $\mathcal{L}_1$ is L^r- resolutive for

$$r > \frac{p-x}{p(1-x)} \tag{5.8}$$

where $x = x(B,p) \in (0,1)$ is the unique solution to the equation

$$\left(1 - \frac{x}{p}\right)^p = B(1-x). \tag{5.9}$$

The result is sharp.

We point out that under the definitions (5.5) and (5.6) it is not really meaningful to speak of the "distance" between $\mathcal{L}_0$ and $\mathcal{L}_1$. Indeed the underlying domains of operators $\mathcal{L}_0$ and $\mathcal{L}_1$ are $\mathbb{D}$ and $F(\mathbb{D})$ respectively. Nevertheless, even after composition with most natural map F, the coefficient matrix $\mathcal{A}_1 \circ F$ is not close to $\mathcal{A}_0$ in the sense of any natural distance between coefficients as it is shown in Example 5.2.1.

In Section 5.2 we also show that the solvability of the problem for a matrix $\mathcal{A}$ in our class is equivalent, up to a rotation of $\frac{\pi}{2}$ of the unit disc $\mathbb{D}$, to the solvability of the problem for the inverse matrix $\mathcal{A}^{-1}$ (see Theorem 5.2.2). In this case, the integrability exponent for solvability is the same.

As a corollary of Theorem 5.0.5 we obtain the following

Theorem 5.0.6. *Let $F : \mathbb{D} \to \mathbb{D}$ be K- quasiconformal, $\mathcal{L}_0 = \Delta_F$, $\mathcal{L}_1 = \Delta_{F^{-1}}$, and the operator $\mathcal{L}_0$ be L^2- resolutive. Then also the operator $\mathcal{L}_1$ is L^2- resolutive, provided $B_2(\omega_{\mathcal{L}_0}) < \frac{4}{3}$.*

Another sharp L^p- solvability result pertains to *Serrin's type operator*

$$\mathcal{L} = \operatorname{div}(\mathcal{A}(x)\nabla\ )$$

where $\mathcal{A}(x) \in \mathcal{E}_1(K)$ takes the form

$$\mathcal{A}(x) = \frac{I}{K} + \left(K - \frac{1}{K}\right)\frac{x \otimes x}{|x|^2} \qquad \text{for } x = (x_1, x_2) \in \mathbb{R}^2_+,$$

for some $K \geqslant 1$, as in [IS]. We notice that the Radon-Nikodym derivative $k = \frac{d\omega_{\mathcal{L}}}{dx}$ of the associated harmonic measure $\omega_{\mathcal{L}}$ belongs to the Gehring class B_q if and only if $1 < q < \frac{K}{K-1}$.

5.1 Quasiconformal mappings and Beltrami equations

For the convenience of the reader, we recall basic feature of quasiconformal mappings which are relevant to our results.

Let Ω_1 and Ω_2 be planar domains and $F : \Omega_1 \to \Omega_2$ be a homeomorphism. F is said to be quasiconformal if:

i) F belongs to Sobolev class $W^{1,2}_{loc}(\Omega_1)$,

ii) F satisfies the *complex Beltrami equation*:

$$\overline{\partial}F(z) = \mu(z)\partial F(z), \qquad \text{where } \|\mu\|_{L^\infty} < 1. \tag{5.10}$$

Here we have used the Cauchy-Rieman derivatives $\overline{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$ and $\partial = \frac{1}{2}(\partial_x - i\partial_y)$ with respect to the complex variable $z = x + iy$.

The function μ is called the *Beltrami coefficient* or *complex dilatation* of F. It determines F, unique up to a (post) composition with a conformal transformation.

Expressing the directional derivatives $\partial_\alpha F(z)$ in terms of $\overline{\partial}F$ and ∂F, (5.10) is equivalent to the distortion inequality

$$\max_\alpha |\partial_\alpha F| \leqslant K \min_\alpha |\partial_\alpha F|. \tag{5.11}$$

Here the smallest possible choice of the constant K is

$$K = \frac{1 + \|\mu\|_{L^\infty}}{1 - \|\mu\|_{L^\infty}} \in [1, \infty).$$

Obviously, the condition $\|\mu\|_{L^\infty} < 1$ is equivalent to $K < \infty$. When $\mu = 0$, or equivalently $K = 1$, we obtain the usual Cauchy-Riemann system, and F is conformal in Ω_1 in the classical sense, i. e. it is an analytical one-to-one map. That is why the constant K in (5.11) gives us the degree of nonconformality of F. Traditionally we refer to such F as K-quasiconformal mappings.

Next, we us define a 2×2 measurable matrix function

$$\mathcal{A}(x) = \left[\frac{{}^tDF(x)DF(x)}{J_F(x)}\right]^{-1} \tag{5.12}$$

where DF stands for the Jacobian matrix of F and $J_F(x) = \det DF(x)$ its Jacobian determinant which is almost everywhere positive. A simple computation shows that $\det \mathcal{A} = 1$, $\mathcal{A}$ is symmetric, and, using (5.11), we can prove the uniform ellipticity

$$\frac{1}{K}|\xi|^2 \leqslant \langle \mathcal{A}(x)\xi, \xi\rangle \leqslant K|\xi|^2, \quad \xi \in \mathbb{R}^2. \tag{5.13}$$

An important point here is that a converse statement is also true. More precisely, given *any* measurable symmetric matrix $\mathcal{A}(x)$ on the unit disc $\mathbb{D} \subset \mathbb{R}^2$ with $\det \mathcal{A} = 1$ that satisfies (5.13), there exists a K-quasiconformal mapping $F : \mathbb{D} \to \mathbb{D}$ for which (5.12) holds a.e. $x \in \mathbb{D}$. For this we recall the *measurable Riemann mapping* theorem, see the seminal work of Morrey [Mo], and [IM] for most recent account.

Theorem 5.1.1. *Let μ be a measurable function defined in $\mathbb{D} \subseteq \mathbb{C}$ such that $\|\mu\|_{L^\infty} < 1$. Then there is a K-quasiconformal mapping $g : \mathbb{D} \to \mathbb{C}$ whose Beltrami coefficient equals μ almost everywhere. Moreover, every $W^{1,2}_{loc}(\mathbb{D}, \mathbb{C})$ solution F to the Beltrami equation takes the form*

$$F(z) = H(g(z))$$

where $H : g(\mathbb{D}) \to \mathbb{C}$ is a holomorphic function.

We shall confront this result with the classical *Riemann mapping theorem*

Theorem 5.1.2. *Let Ω be a proper simply connected open subset of $\mathbb{C}$. Then, there exists a conformal mapping $\eta : \Omega \to \mathbb{D}$.*

In particular, combining Theorem 5.1.1 and Theorem 5.1.2, we see that for every μ as before there exists a K quasiconformal $F : \mathbb{D} \to \mathbb{D}$, $K = \frac{1+\|\mu\|_{L^\infty}}{1-\|\mu\|_{L^\infty}} \in [1, \infty)$, such that

$$\frac{\overline{\partial} F(z)}{\partial F(z)} = \mu(z), \quad \text{a.e. } z \in \mathbb{D}, \tag{5.14}$$

In addition to that we have uniqueness of homeomorphic solutions $F : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ which are *"normalized"* by fixing the values $F(0, 0) = (0, 0)$ and $F(1, 0) = (1, 0)$.

Theorem 5.1.3. *Given a 2×2 matrix $A(x) = [a_{ij}(x)]$ defined on the unit disc, such that $a_{ij} = a_{ji}$, $\det A = 1$ and (**??**) holds. Then there exists a unique normalized K-quasiconformal mapping $F : \mathbb{D} \to \mathbb{D}$ satisfying (5.12).*

In fact equation (5.12) is equivalent to a complex Beltrami equation with

$$\mu(x) = \frac{a_{22} - a_{11} - 2ia_{12}}{a_{22} + a_{11} + 2}, \qquad \|\mu\|_{L^\infty} < 1.$$

For more details see [IM].

5.2 Some cases of simultaneous solvability for two different operators

In this section we consider two elliptic operators which arise as pull-back of the Laplacian: $\mathcal{L}_0 = \Delta_F$ and $\mathcal{L}_1 = \Delta_{F^{-1}}$. We prove the solvability of Dirichlet problem for $\mathcal{L}_1$ knowing it for $\mathcal{L}_0$ (and conversely). Moreover, we show that the A_∞- constants of the respective harmonic measures $\omega_{\mathcal{L}_0}$ and $\omega_{\mathcal{L}_1}$ agree in a suitable way (Theorem 5.0.5).

Let $\mathcal{L}_0$ and $\mathcal{L}_1$ be operators as in Theorem 5.0.5. As we already mentioned, it is not meaningful to speak about the "distance" between $\mathcal{L}_0$ and $\mathcal{L}_1$. Indeed, the operators $\mathcal{L}_0$ and

$\mathcal{L}_1$ are defined on $\mathbb{D}$ and on $F(\mathbb{D})$, respectively. On the other hand, even after composition with most natural map F, the coefficient matrix $A_1 \circ F$ is not close to A_0 in the sense of natural distance between the coefficients as it is shown in the following example

EXAMPLE 5.2.1. Let $Q = [0,1] \times [0,1]$ be the unit cube of $\mathbb{R}^2$ (analogous result can be obtained by replacing Q with the unit disc $\mathbb{D} \subset \mathbb{R}^2$ with just technical adjustments). Let $F : Q \to Q$ be defined by the rule

$$F(x,y) = \left(\int_0^x a(\chi)d\chi, \int_0^y b(\eta)d\eta \right), \tag{5.15}$$

where a, b are non negative measurable functions defined for $(x,y) \in Q$. We assume that

i) $\frac{1}{K} \leqslant \frac{a(x)}{b(y)} \leqslant K$

so that F is K-quasiconformal. Now, let A_0 be the pull-back of the Laplacian defined by

$$A_0(x,y) = \left[\frac{{}^tDF\, DF}{J_F(x,y)} \right]^{-1} = \begin{pmatrix} \frac{b(y)}{a(x)} & 0 \\ 0 & \frac{a(x)}{b(y)} \end{pmatrix}.$$

The reader may wish to observe that writing $F(x,y) = (h(x), k(y)) = (s,t)$, we have the following formula for the inverse map

$$F^{-1}(s,t) = \left(\int_0^s \frac{1}{a(h^{-1}(\tau))} d\tau, \int_0^t \frac{1}{b(k^{-1}(\sigma))} d\sigma \right).$$

We then compute,

$$A_1(s,t) = \left[\frac{{}^tDF^{-1}\, DF^{-1}}{J_{F^{-1}}} \right] = \begin{pmatrix} \frac{a(h^{-1}(s)}{b(k^{-1}(t))} & 0 \\ 0 & \frac{b(k^{-1}(t))}{a(h^{-1}(s))} \end{pmatrix}.$$

Composing $A_1(s,t)$ with $F(x,y)$ we are able to compute the *gap function* $\varepsilon(x,y)$ (see [FKP]) between matrices A_0 and $A_1 \circ F$. (In this case $A_1 \circ F$ is also equal to the inverse matrix A_0^{-1}, because DF is a symmetric matrix). More precisely, we have

$$\varepsilon(x,y) = A_0(x,y) - A_1 \circ F(x,y) = \begin{pmatrix} \frac{b(y)}{a(x)} - \frac{a(x)}{b(y)} & 0 \\ 0 & \frac{a(x)}{b(y)} - \frac{b(y)}{a(x)} \end{pmatrix}.$$

Assuming"closeness" between coefficients, as in [FKP], Theorems 2.3 - 2.5, we find that $\varepsilon(\cdot) \equiv 0$ on the boundary. So let us consider points $(x,0) \in \partial Q$,

$$\varepsilon(x,0) = 0 \Leftrightarrow \left(\frac{a(x)}{b(0)} - \frac{b(0)}{a(x)}\right) = 0 \Leftrightarrow a(x) = b(0), \quad \text{a.e.} \quad x \in [0,1].$$

This occurs iff $a(x)$ is a constant function. In particular, it must be that $a(x) \equiv 1$. Analogously, $b(y) \equiv a(0),$; that is, $b(y) \equiv 1$, for almost every $y \in [0,1]$. Then, unless $F(x,y)$ is the identity map, the closeness hypotheses fails.

Let us proceed to the proof of the Theorem 5.0.5:

Proof. of Theorem 5.0.5 Let $\mathcal{L}_0 = \Delta_F = \operatorname{div}(\mathcal{A}_0 \nabla)$, where $F : \mathbb{D} \to \mathbb{D}$ is a K-quasiconformal mapping and $\mathcal{A}_0$ is defined by

$$\mathcal{A}_0(x) = \left[\frac{{}^t DF(x)\, DF(x)}{J_F(x)}\right]^{-1}. \tag{5.16}$$

Moreover, let $h : \partial\mathbb{D} \to \partial\mathbb{D}$ be the orientation preserving homeomorphism on $\partial\mathbb{D}$ induced by F, $h(\sigma) = F|_{\partial\mathbb{D}}(\sigma)$.

To see the harmonic measure $\omega_{\mathcal{L}_0}$ of the operator $\mathcal{L}_0$ (following an idea contained in [CFK]), let us choose and fix an arbitrary continuous function f defined on $\partial\mathbb{D}$. We solve the Dirichlet problem

$$\begin{cases} \operatorname{div}(\mathcal{A}_0 \nabla u) = 0 \quad \text{in } \mathbb{D} \\ u_{|\partial\mathbb{D}} = f \end{cases} \tag{5.17}$$

Let us observe that u is the solution to problem (5.17) if and only if the function $v =$

$u \circ F^{-1}$ is a solution of the Dirichlet problem

$$\begin{cases} \Delta v = 0 \quad \text{in } \mathbb{D} \\ v_{|\partial\mathbb{D}} = g \end{cases} \tag{5.18}$$

where $g = f \circ F^{-1}$. Indeed, assuming $F = (F_1, F_2)$, $x = (x_1, x_2)$, we have,

$$DF = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{pmatrix}, \qquad (DF)^{-1} = \frac{1}{J_F} \begin{pmatrix} \frac{\partial F_2}{\partial x_2} & -\frac{\partial F_1}{\partial x_2} \\ -\frac{\partial F_2}{\partial x_1} & \frac{\partial F_1}{\partial x_1} \end{pmatrix}$$

and then

$$({}^t DF)^{-1} = \frac{1}{J_F} \begin{pmatrix} \frac{\partial F_2}{\partial x_2} & -\frac{\partial F_2}{\partial x_1} \\ -\frac{\partial F_1}{\partial x_2} & \frac{\partial F_1}{\partial x_1} \end{pmatrix}. \tag{5.19}$$

So we can compute

$$J_F (DF)^{-1} ({}^t DF)^{-1} = \frac{1}{J_F} \begin{pmatrix} |\nabla F_2|^2 & -(\nabla F_1, \nabla F_2) \\ -(\nabla F_1, \nabla F_2) & |\nabla F_1|^2 \end{pmatrix}.$$

Assuming $y = F(x)$ and $u = v \circ F$ we have

$$\frac{\partial u}{\partial x_i} = \frac{\partial v}{\partial y_1} \frac{\partial F_1}{\partial x_i} + \frac{\partial v}{\partial y_2} \frac{\partial F_2}{\partial x_i}$$

and

$$\nabla_x u = ({}^t DF) \nabla_y v \tag{5.20}$$

Combining (5.16), (5.19) and (5.20) we have

$$\mathcal{A}_0 \nabla_x u = J_F (DF)^{-1} \nabla_y v = \begin{pmatrix} \frac{\partial F_2}{\partial x_2}\frac{\partial v}{\partial y_1} - \frac{\partial F_1}{\partial x_2}\frac{\partial v}{\partial y_2} \\ \\ -\frac{\partial F_2}{\partial x_1}\frac{\partial v}{\partial y_1} + \frac{\partial F_1}{\partial x_1}\frac{\partial v}{\partial y_2} \end{pmatrix}$$

and then it holds

$$\mathcal{L}u = \operatorname{div}_x(\mathcal{A}_0 \nabla_x u) = J_F \cdot \Delta_y v.$$

Hence,

$$\operatorname{div}_x(\mathcal{A}_0 \nabla_x u) = 0 \qquad \Longleftrightarrow \qquad \Delta_y v = 0.$$

So, by the Poisson integral formula for the unit disc, we have (see [K])

$$v(x) = u \circ F^{-1}(x) = \int_{\partial \mathbb{D}} \frac{1 - |x|^2}{2\pi |x - \sigma|^2} g(\sigma) d\sigma$$

Then, for all $z \in \overset{\circ}{\mathbb{D}}$ it holds

$$u(z) = v \circ F(z) = \int_{\partial \mathbb{D}} \frac{1 - |F(z)|^2}{2\pi |F(z) - \sigma|^2} g(\sigma) d\sigma.$$

We now change variables, $\sigma = F(\tau)$, and recall that $h(\sigma) = F(\sigma)$ on $\partial\mathbb{D}$ to obtain,

$$u(z) = \int_{\partial \mathbb{D}} \frac{1 - |F(z)|^2}{2\pi |F(z) - h(\tau)|^2} f(\tau) dh(\tau).$$

Since $F(0,0) = (0,0)$ and $|h(\sigma)| = 1$ on $\partial\mathbb{D}$, we have

$$\omega(z) = \frac{1}{2\pi} dh(z). \tag{5.21}$$

Combining (5.21), Theorem 1.4.1 and (1.21) we have that $\mathcal{L}_0$ resolutive implies

$$dh \in A_\infty. \tag{5.22}$$

Moreover, the measure dh is absolutely continuous with respect to the arc length $d\sigma$, so we have the Radon-Nikodym derivative $k_0 = dh/d\sigma$.

Now, let us consider the operator $\mathcal{L}_1 = \Delta_{F^{-1}} = \operatorname{div}(\mathcal{A}_1 \nabla\ )$, where

$$\mathcal{A}_1(y) = \left[\frac{{}^t DF^{-1}(y) DF^{-1}(y)}{J_{F^{-1}}(y)}\right]^{-1}.$$

The orientation preserving homeomorphism on $\partial\mathbb{D}$, induced by F^{-1}, coincides with h^{-1} and, by similar considerations as before, the harmonic measure of the divergence form of the uniformly elliptic operator $\mathcal{L}_1$ is equivalent to $d(h^{-1})$. Now, we are ready to prove that (5.22) implies

$$d(h^{-1}) \in A_\infty \tag{5.23}$$

This is a consequence of equations between the various A_∞- constants defined in (1.22), (1.24). Namely, we prove now (5.7), i.e.

$$\tilde{A}_\infty(\omega_{\mathcal{L}_0}) = \tilde{B}_1(\omega_{\mathcal{L}_1}), \tag{5.24}$$

$$\tilde{A}_\infty(\omega_{\mathcal{L}_1}) = \tilde{B}_1(\omega_{\mathcal{L}_0}). \tag{5.25}$$

In fact, by Definition 1.3.2, (5.22) implies that there exist $0 < \alpha \leqslant 1 \leqslant M$ such that

$$\frac{\sigma(F)}{\sigma(\Lambda)} \leqslant M \left(\frac{\int_F k_0 d\sigma}{\int_\Lambda k_0 d\sigma}\right)^\alpha, \tag{5.26}$$

for any rectifiable set $F \subset \Lambda$ arc on $\partial\mathbb{D}$.

For any rectifiable set $F \subset \partial\mathbb{D}$ we have

$$\int_F k_0 d\sigma = \int_F \frac{dh}{d\sigma} d\sigma = \sigma(h(F))$$

where $h(F)$ stands for the image of the set $F \subset \partial\mathbb{D}$ under the mapping $h : \partial\mathbb{D} \to \partial\mathbb{D}$. Then (5.26) can be rewritten as

$$\frac{\sigma(F)}{\sigma(\Lambda)} \leqslant M \left(\frac{\sigma(h(F))}{\sigma(h(\Lambda))}\right)^\alpha. \tag{5.27}$$

for arbitrary rectifiable set contained in the arbitrary arc $\Lambda \subset \partial\mathbb{D}$.

Since $h : \partial\mathbb{D} \to \partial\mathbb{D}$ is a homeomorphism, inequality (5.27) can be reformulated as

$$\frac{\sigma(h^{-1}(E))}{\sigma(h^{-1}(\Gamma))} \leqslant M \left(\frac{\sigma(E)}{\sigma(\Gamma)}\right)^{\alpha}, \tag{5.28}$$

(where $h^{-1}(E)$ denotes the inverse image of E via h) for any rectifiable set E contained in an arc $\Gamma \subset \partial\mathbb{D}$. Hence, by (5.28) the measure μ_1, defined by $\mu_1(E) = \sigma \circ h^{-1}(E)$, is absolutely continuous with respect to σ. If we introduce its Radon-Nikodym derivative $k_1 = d\mu_1/d\sigma$, then (5.28) can be restated as

$$\frac{\int_E k_1 d\sigma}{\int_\Gamma k_1 d\sigma} \leqslant M \left(\frac{\sigma(E)}{\sigma(\Gamma)}\right)^{\alpha}, \tag{5.29}$$

Hence (5.24) and (5.25) follow directly from (1.24) and (1.22). In fact (5.26) holds for arbitrary rectifiable subset F of the arbitrary arc Λ, if and only if (5.29) holds for arbitrary rectifiable subset E of the arbitrary arc $\Gamma \subset \partial\mathbb{D}$. Combining (5.24), (5.25), Theorem 1.4.1 and (1.21), the simultaneous solvability of Dirichlet problems for $\mathcal{L}_0$ and $\mathcal{L}_1$ follows.

In order to obtain more precise information about the L^p- solvability for $\mathcal{L}_0$ and L^r-solvability for $\mathcal{L}_1$, let us preliminary observe that

$$\omega_{\mathcal{L}_0} \in B_q \quad \Leftrightarrow \quad \omega_{\mathcal{L}_1} \in A_p \tag{5.30}$$

where $q = p/(p-1)$, and that

$$B_q(\omega_{\mathcal{L}_0}) = A_p(\omega_{\mathcal{L}_1}). \tag{5.31}$$

In fact, by (5.21) $\omega_{\mathcal{L}_0}$ is equivalent to dh, where $h = F_{|\partial\mathbb{D}}$ is the trace on $\partial\mathbb{D}$ of the K-quasiconformal mapping $F : \mathbb{D} \to \mathbb{D}$ and, as already mentioned, $\omega_{\mathcal{L}_1}$ is equivalent to $d(h^{-1})$. Then, according to Lemma 2.3 [JN], we deduce that

$$dh \in B_q \quad \Leftrightarrow \quad d(h^{-1}) \in A_p$$

and

$$B_q(dh) = A_p(d(h^{-1}))$$

establishing (5.30) and (5.31).

Now, assume L_0 to be L^p- resolutive for an exponent $p > 1$, and $B = B_q(\omega_{L_0})$. By (5.31) we obtain $A_p(\omega_{L_1}) = B$. Combining Theorem 1 in [V] about the optimal connection between the A_p and B_q-classes, and Theorem 1.3.3 we are able to determine the sharp B_θ in the Gehring class pertaining to ω_{L_1} as a function of B and p. More precisely, let us denote by $x = x(B,p) \in (0,1)$, the (unique) solution to the algebraic equation

$$\left(1 - \frac{x}{p}\right)^p = B(1-x) \tag{5.32}$$

and use Theorem 1 in [V] and Theorem 1.3.3 to conclude that for $1 \leqslant \theta < \frac{p-x}{x(p-1)}$ it holds

$$B_\theta(\omega_{L_1}) \leqslant B^{\frac{\theta-1}{p}} \frac{(1-x)^\gamma}{1-\gamma x}$$

where $\gamma = (p\theta - \theta + 1)/p$. By Theorem 1.4.1 we deduce that L_1 is L^r- resolutive, whenever r satisfies

$$r > \frac{p-x}{p(1-x)}. \tag{5.33}$$

To see that the result is sharp, we bound ourselves to the case $p = q = 2$. Let us consider the mappings $(1 < K < 2)$

$$F(z) = \frac{z}{|z|^{1-\frac{1}{K}}}, \qquad z = (x,y) \in \mathbb{R}^2_+,$$

$$G(w) = \frac{w}{|w|^{1-K}}, \qquad w = (s,t) \in \mathbb{R}^2_+.$$

These are the standard radial stretchings and they arise as extremals in many problems for K- quasiconformal mappings and planar PDE's. We notice that

$$G = F^{-1}.$$

Let $\mathcal{A}_0$ be the coefficient matrix (5.4) of the pull-back under F of the Laplacian:

$$\mathcal{A}_0(z) = \frac{I}{K} + \left(K - \frac{1}{K}\right)\frac{z \otimes z}{|z|^2}.$$

$z = (x,t)$, where we have used the shorthand notation

$$z \otimes z = \begin{pmatrix} x^2 & xt \\ xt & t^2 \end{pmatrix}$$

and let $\mathcal{A}_1$ be the coefficient matrix (5.4) of the pull-back under $G = F^{-1}$ of the Laplacian

$$\mathcal{A}_1(w) = KI + \left(\frac{1}{K} - K\right)\frac{w \otimes w}{|w|^2}.$$

The harmonic measures $\omega_{\mathcal{A}_0}$ and $\omega_{\mathcal{A}_1}$ are given by

$$\omega_{\mathcal{A}_0}(x) \sim \frac{1}{|x|^{1-\frac{1}{K}}}$$

$$\omega_{\mathcal{A}_1}(s) \sim \frac{1}{|s|^{1-K}}$$

and we assume

$$B_2(\omega_{\mathcal{A}_0}) = B < \infty.$$

An elementary calculation shows that

$$B = \frac{1}{K(2-K)}$$

and that for $1 < q < \sqrt{\frac{B}{B-1}} = \frac{1}{K-1}$

$$B_q(\omega_{\mathcal{A}_1}) < \infty,$$

while

$$B_{\frac{1}{K-1}}(\omega_{\mathcal{A}_1}) = \infty,$$

hence L^r- solvability fails for $\mathcal{L}_1$ if r verifies equality in (5.33). $\square$

Corollary 5.2.1. *Let $\mathcal{A}_0$ be a matrix in $\mathcal{E}_1(K)$ and let $\mathcal{L}_0 = \mathrm{div}(\mathcal{A}_0\nabla\,)$. Then there exists a K- quasiconformal mapping $F : \mathbb{D} \to \mathbb{D}$ such that $\mathcal{L}_0$ is resolutive iff $\mathcal{L}_1 = \mathrm{div}(\mathcal{A}_1\nabla)$ is resolutive, where*

$$\mathcal{A}_1(y) = \left[\frac{DF\,{}^tDF}{J_F}\right] \circ F^{-1}(y)$$

Proof. Let $\mathcal{A} \in \mathcal{E}_1(K)$. By the *measurable Riemann mappings theorem* (see Theorem 5.1.1) we can find a K- quasiconformal mapping $F : \mathbb{D} \to \mathbb{D}$ such that

$$\mathcal{A}_0 = \left[\frac{{}^tDFDF}{J_F}\right]^{-1}$$

so that $\mathcal{L}_0 = \Delta_F$. The statement follows by observing that $\mathcal{L}_1 = \Delta_{F^{-1}}$.

$\square$

Let $\mathcal{A}_0 \in \mathcal{E}_1(K)$; our goal is to find a connection between the solvability of the Dirichlet problem for the operator $\mathcal{L}_0 = \mathrm{div}(\mathcal{A}_0\nabla\,)$ and the operator $\mathcal{L}_1 = \mathrm{div}(\mathcal{A}_1\nabla\,)$, where $\mathcal{A}_1 = \mathcal{A}_0^{-1}(ix)$. We have the following

Theorem 5.2.2. *Let $\mathcal{A}_0 \in \mathcal{E}_1(K)$. Then the problem*

$$\begin{cases} \mathrm{div}(\mathcal{A}_0\nabla u) = 0 \quad \mathit{in}\ \mathbb{D} \\ u_{|\partial\mathbb{D}} = f \end{cases} \tag{5.34}$$

is L^p- solvable, $p > 1$, if and only if the problem

$$\begin{cases} \mathrm{div}(\mathcal{A}_1\nabla u) = 0 \quad \mathit{in}\ \mathbb{D} \\ u_{|\partial\mathbb{D}} = f \end{cases} \tag{5.35}$$

is L^p- solvable, where

$$\mathcal{A}_1(y) = \mathcal{A}_0^{-1}(iy) \quad a.e. \quad y \in \mathbb{D}.$$

Proof. Let us observe that the hypothesis $\mathcal{A}_0 \in \mathcal{E}_1(K)$ implies that (see Theorem 5.1.3) there exists a K-quasiconformal mapping $F : \mathbb{D} \to \mathbb{D}$ such that

$$\mathcal{A}_0 = \left[\frac{{}^tDF\,DF}{J_F}\right]^{-1} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}.$$

So, let us search for T such that,

$$\mathcal{A}_0^{-1} \circ T = \left[\frac{{}^tDG\,DG}{J_G}\right]^{-1} = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{12} & a_{11} \end{pmatrix} \circ T, \quad \text{where } G = F \circ T.$$

Notice that the Beltrami coefficient of F is given by (see Section 5.1)

$$\mu_F = \frac{a_{22} - a_{11} - 2ia_{12}}{a_{22} + a_{11} + 2}.$$

Then, analogously,

$$\mu_G = \left(\frac{a_{11} - a_{22} + 2ia_{12}}{a_{11} + a_{22} + 2}\right) \circ T.$$

So, for all $z \in \mathbb{D}$, it must be that

$$\mu_G(z) = -\mu_F(T(z)). \tag{5.36}$$

Now, let $G(z) = F(iz)$. Taking into account that $\overline{\partial}(iz) = 0$, $\overline{\partial}(-i\overline{z}) = -i$, $\partial(iz) = i$ and $\partial(-i\overline{z}) = 0$, we obtain

$$\overline{\partial}G(z) = (\partial F)(iz)\overline{\partial}(iz) + (\overline{\partial}F)(iz)\overline{\partial}(-iz) = -i(\overline{\partial}F)(iz)$$

and

$$\partial G(z) = (\partial F)(iz)\partial(iz) + (\overline{\partial}F)(iz)\partial(-iz) = i(\partial F)(iz).$$

This means that the Beltrami coefficient of G is given by

$$\mu_G(z) = \frac{\overline{\partial} G}{\partial G}(z) = -\frac{\overline{\partial} F(iz)}{\partial F(iz)} = -\mu_F(iz))$$

so that (5.36) is true.

Now following the proof of Theorem 5.0.5, one can see that the harmonic measures of the operators $\mathcal{L}_0 = \mathrm{div}(\mathcal{A}_0 \nabla\,) = \Delta_F$ and $\mathcal{L}_1 = \mathrm{div}(\mathcal{A}_1 \nabla\,) = \Delta_G$ are $dh(z)$ and $d(h(iz))$, respectively. Then by Theorem 1.4.1 we complete the proof. $\square$

Notice that Theorem 5.2.2 states that, up to a rotation of the unit disc $\mathbb{D}$ by $\frac{\pi}{2}$, the solvability of the Dirichlet problem for a matrix $\mathcal{A} \in \mathcal{E}_1(K)$ is equivalent to the solvability of the Dirichlet problem for the inverse matrix $\mathcal{A}^{-1}$.

We point out that, in this case, the exponent of L^p- solvability for problems (5.34) and (5.35) is the same.

As an application of Theorem 5.0.5 we have a sufficient condition for both operators $\mathcal{L}_0$ and $\mathcal{L}_1$ to be L^2- resolutive. In other words we see that it is possible to deduce from Theorem 5.0.5 that B_q condition is preserved for a fixed $q > 1$. Namely, we give now the

Proof. (of Theorem 5.0.6). By Theorem 1.4.1, we have that $B_2(\omega_{\mathcal{L}_0}) < \infty$. So, let $B = B_2(\omega_{\mathcal{L}_0})$. We find the unique solution $x = x(B, 2) \in (0, 1)$ to the equation (5.32). We have $x = 2\sqrt{B-1}\left(\sqrt{B} - \sqrt{B-1}\right)$ and then, by Theorem 5.0.5, $\mathcal{L}_1$ is L^2- resolutive if the right hand side in (5.33) is less than 2; that is

$$\frac{2-x}{x} = \sqrt{\frac{B}{B-1}} > 2. \tag{5.37}$$

Elementary calculation reveals that $B < \frac{4}{3}$ yields (5.37), thus Theorem 5.0.6 follows. $\square$

5.3 Sharp estimates for harmonic measures on $\mathbb{R}^2_+$

A classical theorem of Beurling and Ahlfors (see [FKP]) states that, given a homeomorphism h from the real line $\mathbb{R}$ onto itself (or from the unit circle $\partial\mathbb{D}$ onto itself), a necessary and sufficient condition for the existence of a quasiconformal mapping from the upper half plane in itself $F : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ such that $F|_{\mathbb{R}} = h$ (or $F : \mathbb{D} \to \mathbb{D}$ such that $F|_{\partial\mathbb{D}} = h$) is that the distributional derivative dh is a *doubling measure* i.e. there is a constant $D \geqslant 1$ such that

$$\frac{1}{D} \leqslant \frac{\int_x^{x+t} dh}{\int_{x-t}^{x} dh} \leqslant D \qquad \forall x \in \mathbb{R}, \forall t > 0. \tag{5.38}$$

We will refer to the infimum of constants D such that (5.38) holds as *doubling constant* $D_2(\omega)$ of the measure $\omega = dh$. Consider the set

$$\mathcal{F} = \left\{ F : \mathbb{R}^2 \to \mathbb{R}^2 \ K\text{-quasiconformal, } F|_{\mathbb{R}} : \mathbb{R} \to \mathbb{R} \text{ fixes the points } -1, 0, \infty \right\}$$

see [L]. We introduce the distortion function

$$\lambda(K) = \max_{F \in \mathcal{F}} F(1).$$

Then, it is possible to evaluate the sharp doubling constant of dh for all such traces $h = F_{|\mathbb{R}}$, as F runs through the class $\Psi(K)$ of all K- quasiconformal mappings. Namely,

$$D_2(dh) \leqslant \lambda(K). \tag{5.39}$$

Conversely, we are able to obtain an explicit estimate for the doubling constant $D_2(\omega)$ of the harmonic measure ω of any elliptic operator whose coefficients matrix belongs to $\mathcal{E}_1(K)$.

Theorem 5.3.1. *Let $\mathcal{A} \in \mathcal{E}_1(K)$ be defined on $\mathbb{R}^2_+$ and let $\mathcal{L} = \operatorname{div}(\mathcal{A}\nabla)$. Then the harmonic measure ω of $\mathcal{L}$ has the form*

$$\omega = \frac{dh}{1+h^2},$$

where the homeomorphism $h : \mathbb{R} \to \mathbb{R}$ verifies:

$$D_2(dh) \leqslant e^{5(K-1)}.$$

Proof. Let us observe that by analogous computations as in the proof of Theorem 5.0.5 (see also [K]), the form of the harmonic measure for an operator $\mathcal{L} = \operatorname{div}(\mathcal{A}\nabla)$, where $\mathcal{A} \in \mathcal{E}_1(K)$ in $\mathbb{R}^2_+$, can be recognized. Namely, let us denote by $F = (\alpha, \beta)$ the K- quasiconformal mapping such that $\mathcal{L} = \Delta_F$, and let $h : \mathbb{R} \to \mathbb{R}$, $h = F_{|\mathbb{R}}$ be the trace of F on the real axis $\mathbb{R}$. We have that if $\Delta v = 0$ in $\mathbb{R}^2_+$, then $u = v \circ F$ satisfies the equation $\mathcal{L}u = 0$ in $\mathbb{R}^2_+$. Hence, if $g \in C_0(\mathbb{R})$ (the set of all continuous functions with compact support) and $f = g \circ F^{-1}$, then, by the Poisson integral formula,

$$\begin{aligned}
&v(x,t) = C\int_{\mathbb{R}} \frac{t}{|x-y|^2 + t^2} f(y)dy, \qquad \text{so that}\\
&u(z,s) = C\int_{\mathbb{R}} \frac{\beta(z,s)}{|\alpha(z,s) - y|^2 + \beta(z,s)^2} f(y)dy =\\
&= C\int_{\mathbb{R}} \frac{\beta(z,s)}{|\alpha(z,s) - h(\xi)|^2 + \beta(z,s)^2} g(\xi)d\xi.
\end{aligned}$$

Hence, the harmonic measure $\omega_{\mathcal{L}}$ of the operator $\mathcal{L}$ evaluated at the point $F^{-1}(0,1)$ is given by

$$\omega_{\mathcal{L}} = \frac{dh}{1+h^2}. \tag{5.40}$$

Moreover, by the Beurling Ahlfors Theorem, the distributional derivative dh is a doubling measure. It satisfies (5.39). The result follows by the estimate on the distortion function $\lambda(K)$ (see formula (2.6) in [L]). □

The last estimate shows that when K tends to 1, the doubling constant of $\mathcal{L}$ tends to 1, i.e. to the doubling constant of the Laplacian. So, we re-obtain a well known result contained in [CFMS], where (5.38) is proved in the general case but with slight more complicated proof.

5.4 The self-improving property of the L^p- solvability for a single operator, a sharp result

As we already mentioned, the so called "openess" property of the reverse Hölder inequality (see after Theorem 1.4.1) imply that if Problem (1.32) is L^p- solvable, then automatically it is also L^r- solvable, for all $r \in (p-\eta, p]$, with sufficiently small positive η.

In this section we want to determine the infimum of exponents $r < p$ such that the L^p- solvability $\Rightarrow$ L^r- solvability as a function of p, and find the constant $B_p(\omega)$ (see (1.25)) of the measure ω, in the case of the unit disc $\mathbb{D} \subset \mathbb{R}^2$.

Let us emphasize that here the hypotheses $\det A = 1$ and $A = {}^tA$ are not necessary, because we will not use quasiconformal mappings. In the following proposition we will adapt a result of L. D'Apuzzo and C. Sbordone [DAS], A. A. Korenovskii [Ko], C. Sbordone ([S], Theorem 2.1) to our needs (Theorem 1.3.3).

Proposition 5.4.1. *Let $A \in \mathcal{E}(K)$ and the operator $\mathcal{L} = \operatorname{div}(A\nabla\,)$ be L^p- resolutive, $1 < p < \infty$. Moreover, let $B = B_q(\omega_{\mathcal{L}})$ be the B_q- constant of the harmonic measure $\omega_{\mathcal{L}}$ ($q = \frac{p}{p-1}$) and $q_1 > q$ the unique solution y of:*

$$\varphi(y) = 1 - B^{q-1}\,\frac{y-q}{y}\left(\frac{y}{y-1}\right)^q = 0. \tag{5.41}$$

Then the operator $\mathcal{L}$ is also L^r- resolutive, for all $r \in (p_1, p]$, where $p_1 q_1 = p_1 + q_1$. The result is sharp.

Proof. Let $\omega_{\mathcal{L}}$ be the harmonic measure of the operator $\mathcal{L}$. By the hypothesis and Theorem 1.4.1, we know that $\omega_{\mathcal{L}} \in B_q$. So, let us start by proving that for all $q \leqslant \theta < q_1$, we have $\omega_{\mathcal{L}} \in B_\theta$ with:

$$[B_\theta(\omega_{\mathcal{L}})]^{\frac{1}{\theta'}} \leqslant B^{\frac{1}{q'}}\left[\frac{q}{\theta\varphi(\theta)}\right]^{\frac{1}{q}}, \quad \frac{1}{\theta} + \frac{1}{\theta'} = 1,\ \frac{1}{q} + \frac{1}{q'} = 1. \tag{5.42}$$

To this aim, let $k = \frac{d\omega_{\mathcal{L}}}{d\sigma}$, and let

$$v : \mathbb{R} \to [0, +\infty)$$

be defined by

$$v(\sigma) = k(e^{i\sigma}).$$

Thus v is defined on the whole $\mathbb{R}$, periodic with period 2π. For any $\Gamma \subset \partial\mathbb{D}$ let $\alpha, \beta \in \mathbb{R}$, $\beta - \alpha \leqslant 2\pi$ be such that $\Gamma = (e^{i\alpha}, e^{i\beta})$. Then, by (1.25) we have

$$\begin{aligned}
&\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta v^q(\sigma)d\sigma\right)^{\frac{1}{q}} = \left(\frac{1}{\sigma(\Gamma)}\int_\Gamma k^q(z)d\sigma(z)\right)^{\frac{1}{q}} \\
&\leqslant B^{\frac{1}{q'}}\left(\frac{1}{\sigma(\Gamma)}\int_\Gamma k(z)d\sigma(z)\right) \\
&= B^{\frac{1}{q'}}\frac{1}{\beta-\alpha}\int_\alpha^\beta v^q(\sigma)d\sigma
\end{aligned}$$

By Theorem 2.1, [S], applied to the weight function v restricted to any interval $[a, b] \subset \mathbb{R}$ with $(b - a) = 2\pi$, we obtain that (5.42) holds. Moreover $\omega \in B_q$ implies $\omega \in B_\theta$, for all $q \leqslant \theta < q_1$. The result is sharp. Notice that, by the periodicity of v, the result does not depend by the particular choice of the interval $[a, b]$.

Combining (5.42) and Theorem 1.4.1 completes the proof of Proposition 5.4.1 . □

Let us observe that in case $p = 2$, we can give an explicit value of exponents $r < 2$ for which the L^2- solvability $\Rightarrow$ L^r- solvability as a function of the constant $B_2(\omega_{\mathcal{L}})$.

Corollary 5.4.2. *Let $A \in \mathcal{E}(K)$ and let problem (1.32) be L^2- solvable, and $B = B_2(\omega_{\mathcal{L}})$ be the B_2- constant of the harmonic measure $\omega_{\mathcal{L}}$. Then problem (1.32) is also L^r- solvable for all $r > 1$ such that*

$$1 + \sqrt{\frac{B-1}{B}} < r. \tag{5.43}$$

The result is sharp.

Proof. For $q = 2$ equation (5.41) admits the solution $q_1 = 1 + \sqrt{\frac{B}{B-1}}$; hence $B_\theta(\omega_{\mathcal{L}}) < \infty$ for $2 \leqslant \theta < 1 + \sqrt{\frac{B}{B-1}}$. By Theorem 1.4.1 we deduce L^r- solvability for $r > \frac{q_1}{q_1-1} = 1 + \sqrt{\frac{B-1}{B}}$.

In order to verify that the result is sharp, fix $K > 1$ and consider the K-quasiconformal mapping on $\mathbb{R}^2_+$ (see [FKP])

$$F(x,t) = (x^2 + t^2)^{\frac{1-K}{2K}}(x,t) = (\alpha(x,t), \beta(x,t)). \tag{5.44}$$

Let $\mathcal{A} = \mathcal{A}(F)$ be the coefficient matrix (5.4) of the *pull-back* under F of the Laplacian. We know that $\mathcal{A}$ belongs to $\mathcal{E}_1(K)$, and has the following expression ([IS]), $z = (x,t)$:

$$\mathcal{A}(z) = \frac{I}{K} + \left(K - \frac{1}{K}\right)\frac{z \otimes z}{|z|^2}. \tag{5.45}$$

The harmonic measure $\omega_{\mathcal{A}}$ is locally equivalent to $h'(x)dx$, where $h : \mathbb{R} \to \mathbb{R}$ is the increasing homeomorphism defined by

$$F(x,0) = (h(x),0), \tag{5.46}$$

hence $h(x) = |x|^\gamma x$ and $\omega_{\mathcal{A}} \sim |x|^\gamma$, with $\gamma = \frac{1}{K} - 1 > -1$. An elementary calculation shows that

$$B_2(|x|^{\frac{1-K}{K}}) = \frac{1}{K(2-K)} = B.$$

The self improving property of Gehring's classes implies that for $2 < q < 1 + \sqrt{\frac{B}{B-1}}$,

$$B_q(|x|^{\frac{1-K}{K}}) \leqslant \frac{\sqrt{2B}(q-1)}{\sqrt{q[(q-1)^2 - Bq(q-2)]}}$$

(see [S]) and the value $q_0 = 1 + \sqrt{\frac{B}{B-1}}$ cannot be attained. $\square$

Proposition 5.4.3. *Let $K > 1$ and let $\mathcal{A}$ be defined by (5.45). Then the operator*

$$\mathcal{L} = \operatorname{div}(\mathcal{A}\nabla u)$$

is L^p- resolutive if and only if $p > K$. Moreover

$$\tilde{B}_1(\omega_{\mathcal{A}}) = K. \tag{5.47}$$

Proof. We begin by observing that (see [IS], pag. 532) $z = (x, t)$

$$\mathcal{A}(F)(z) = \frac{I}{K} + \left(K - \frac{1}{K}\right) \frac{z}{|z|} \otimes \frac{z}{|z|}$$

and that locally $\omega_{\mathcal{A}} \sim dh \sim d\left(\frac{1}{K}|x|^{\frac{1}{K}-1}\right)$. On the other hand we have

$$B_q(\omega_{\mathcal{A}}) = \frac{1}{K[K - q(K-1)]^{\frac{1}{q-1}}}. \tag{5.48}$$

In fact, for $\omega(E) = \int_E t^{-\gamma}$, an elementary calculation gives,

$$B_q(\omega) = \frac{(1-\gamma)^{q'}}{(1-\gamma q)^{\frac{1}{q-1}}}, \quad \text{for } q < \frac{1}{\gamma}.$$

In our case we have $\gamma = 1 - \frac{1}{K}$, hence (5.48) holds. Then, $B_q(\omega_{\mathcal{A}}) < +\infty \Leftrightarrow q < \frac{K}{K-1}$. But $q = p' < \frac{K}{K-1} \Leftrightarrow p > K$ and then Proposition 5.4.3 follows. The equality (5.47) follows by Remark 1.3.1.

□

We end this section by observing that all the results contained in this chapter can be easily extended in the context of the Orlicz functional spaces by similar arguments as in Section 4.1.

Chapter 6

Sequences of Dirichlet problems

In this chapter, we examine a sequence of operators $\mathcal{L}_j = \operatorname{div} \mathcal{A}_j \nabla$ where $\mathcal{A}_j \in \mathcal{E}_1(K)$. We formulate a necessary and sufficient condition in order to ensure that the harmonic measures $\omega_{\mathcal{A}_j}$ converge weakly to $\omega_{\mathcal{A}}$ with some $\mathcal{A} \in \mathcal{E}_1(K)$. These conditions are formulated in terms of G- convergence, if $\mathcal{A}_j$ in a certain subclass of $\mathcal{E}_1(K)$. Here G- convergence is understood in the sense of De Giorgi and Spagnolo [DGS], [MT].

6.1 Harmonic measures and the G- convergence

Let us consider a sequence $A_j \in \mathcal{E}^s(K)$ and denote by ω_{A_j} (or $\omega_{\mathcal{L}_j}$) the harmonic measures associated with the operators

$$\mathcal{L}_j = \operatorname{div}(A_j \nabla\,). \tag{6.1}$$

In [K], [KP1] it is proved that if

$$A_j(x) \to A(x) \tag{6.2}$$

a.e. for $A \in \mathcal{E}^s(K)$, then

$$\omega_{A_j} \rightharpoonup \omega \qquad \text{weakly} \tag{6.3}$$

in the sense of measures.

The converse statement (6.3) $\Rightarrow$ (6.2) is not true. Moreover, if we replace the a.e.

convergence in (6.2) by the weak convergence:

$$A_j \rightharpoonup A \qquad \text{weakly in } \sigma(L^\infty, L^1) \tag{6.4}$$

then (6.3) may fail, as the following example shows. It is convenient to construct examples in $\mathbb{R}^2_+$ rather than in the unit disc $\mathbb{D}$.

EXAMPLE 6.1.1. Consider, for $x = (x_1, x_2) \in \mathbb{R}^2_+$, the sequence of matrix fields

$$A_j(x) = \begin{pmatrix} \frac{1}{a_j(x_1)} & 0 \\ 0 & a_j(x_1) \end{pmatrix}$$

with

$$\frac{1}{\sqrt{K}} \leqslant a_j(t) \leqslant \sqrt{K} \qquad \text{a.e. } t \in \mathbb{R}$$

such that

$$a_j \rightharpoonup a, \qquad \frac{1}{a_j} \rightharpoonup \frac{1}{a_\infty} \qquad \sigma(L^\infty, L^1) \tag{6.5}$$

with $a_\infty < a$. Hence the matrix fields A_j converge weakly to the matrix field A_∞ defined by

$$A_\infty(x) = \begin{pmatrix} \frac{1}{a_\infty(x_1)} & 0 \\ 0 & a_\infty(x_1) \end{pmatrix}.$$

Now define:

$$h_j(x_1) = \int_0^{x_1} a_j(\tau)d\tau, \qquad h(x_1) = \int_0^{x_1} a(\tau)d\tau. \tag{6.6}$$

Then we know that the harmonic measures for $\mathbb{R}^2_+$ associated with the operators

$$\mathcal{L}_j = \operatorname{div}(A_j \nabla\ )$$

and

$$\mathcal{L} = \operatorname{div}(A \nabla\ )$$

where $A(x)$ is the matrix

$$A(x) = \begin{pmatrix} \frac{1}{a(x_1)} & 0 \\ 0 & a(x_1) \end{pmatrix}$$

are, respectively, given by

$$d\omega_{A_j}(t) = \frac{h_j'(t)}{1 + h_j^2(t)} dt$$

and

$$d\omega_A(t) = \frac{h'(t)}{1 + h^2(t)} dt.$$

Hence, by (6.5), (6.6) we deduce

$$d\omega_{A_j} \rightharpoonup d\omega_A \qquad \text{weakly in the sense of measures} \tag{6.7}$$

while the sequence $A_j(x)$ does not converge a.e. to $A(x)$. It nevertheless weakly converges to $A_\infty(x) \neq A(x)$. Notice that $\det A_\infty > 1$.

The above example suggests to consider another type of convergence of operators to compare with the weak convergence (6.7) of harmonic measures. To this effect, let Ω be an open set in $\mathbb{R}^2$. Here and below we will denote by $\mathcal{E}^s(K;\Omega)$ the class of symmetric matrices $A = A(x)$, $x \in \Omega$, satisfying (1.1) a.e. $x \in \Omega$, and by $\mathcal{E}_1(K;\Omega)$ the subset of $\mathcal{E}^s(K;\Omega)$ whose elements satisfy the condition $\det A(x) = 1$ a.e. $x \in \Omega$.

Let us recall now the definition and properties of G- convergence of elliptic operators in $\mathcal{E}^s(K;\mathbb{R}^2)$. According to De Giorgi- Spagnolo ([DGS], [MT]) we write

Definition 6.1.1. *Given A_j, $A \in \mathcal{E}^s(K,\mathbb{R}^2)$, we say that A_j G- converges to A and denote*

$$A_j \xrightarrow{G} A$$

if for every bounded open subset Ω of $\mathbb{R}^2$ and for every $f \in L^2(\Omega)$ one has

$$u_j \rightharpoonup u \quad \textit{weakly in} \;\; W_0^{1,2}(\Omega),$$

where u_j and u are defined by

$$\begin{cases} -\operatorname{div}(A_j(x)\nabla u_j) = f & \text{in } \mathcal{D} \\ u_j \in W_0^{1,2}(\Omega) \end{cases}$$

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) = f & \text{in } \mathcal{D} \\ u \in W_0^{1,2}(\Omega) \end{cases}$$

The fundamental compactness theorem of S. Spagnolo asserts that

Theorem 6.1.1. *The class $\mathcal{E}^s(K;\mathbb{R}^2)$ is sequentially compact with respect to G- convergence.*

It is interesting to note that $\mathcal{E}_1(K;\mathbb{R}^2)$ is a G- closed (and G- compact) subset of $\mathcal{E}^s(K;\mathbb{R}^2)$ [FM].

The following result provides another sufficient condition for the weak convergence of harmonic measures in (6.7).

Theorem 6.1.2. *Let A_j and A be matrices from the class $\mathcal{E}^s(K;\mathbb{D})$. Assume that*

$$A_j \xrightarrow{G} A.$$

Then

$$\omega_{A_j} \rightharpoonup \omega_A \qquad \text{weakly in the sense of measures.}$$

Proof. Assume $A_j \xrightarrow{G} A$. Since

$$\int_{\partial\mathbb{D}} d\omega_{A_j} = 1, \qquad \forall j \in \mathbb{N}, \qquad\qquad \int_{\partial\mathbb{D}} d\omega_A = 1,$$

in order to obtain the condition $\omega_{A_j} \rightharpoonup \omega_A$ weakly in the sense of measures, that is

$$\lim_j \int_{\partial\mathbb{D}} f d\omega_{A_j} = \int_{\partial\mathbb{D}} f d\omega_A, \qquad \forall\ f \in C(\partial\mathbb{D}) \tag{6.8}$$

it will be sufficient to prove (6.8) for every $f \in C^\infty(\bar{\mathbb{D}})$.

Let $f \in C^\infty(\bar{\mathbb{D}})$ and u_j be the unique solution in $W^{1,2}_{loc}(\mathbb{D}) \cap C(\bar{\mathbb{D}})$ to the Dirichlet problem

$$\begin{cases} \operatorname{div}(A_j \nabla u_j) = 0 & \text{in } \mathbb{D} \\ {u_j}_{|\partial\mathbb{D}} = f \end{cases}$$

Similarly, let u be the unique solution in $W^{1,2}_{loc}(\mathbb{D}) \cap C(\bar{\mathbb{D}})$ of the Dirichlet problem

$$\begin{cases} \operatorname{div}(A \nabla u) = 0 & \text{in } \mathbb{D} \\ u_{|\partial\mathbb{D}} = f \end{cases}$$

In view of G- convergence $A_j \xrightarrow{G} A$ we have

$$u_j(0) \to u(0). \tag{6.9}$$

Therefore, by the definition of harmonic measures ω_{A_j}, ω_A we conclude that

$$u_j(0) = \int_{\partial\mathbb{D}} f d\omega_{A_j},$$

$$u(0) = \int_{\partial\mathbb{D}} f d\omega_A.$$

Thus (6.8) follows (see [DGS]). $\square$

The converse implication $(\omega_{A_j} \rightharpoonup \omega_A) \Rightarrow (A_j \xrightarrow{G} A)$ also holds under some restrictions. Let us denote by $\mathcal{S}(K)$ the subset of $\mathcal{E}_1(K, \mathbb{R}^2_+)$ consisting of matrices of the form

$$\mathcal{A} = \left[\frac{({}^t DF)(DF)}{J_F}\right]^{-1}$$

where $F = \mathcal{BA}(h)$ is the *Beurling-Ahlfors* extension to $\mathbb{R}^2_+$ of a normalized homeomorphism $h : \mathbb{R} \to \mathbb{R}$ (i.e. such that $h(0) = 0$, $h(1) = 1$, $h(\infty) = \infty$) with dh doubling. Namely

$F = F(x)$ is defined for $x = (x_1, x_2) \in \mathbb{R}^2_+$ by

$$F(x) = \frac{1}{2}\left(\alpha(x) + \beta(x), \alpha(x) - \beta(x)\right), \tag{6.10}$$

where

$$\alpha(x) = \alpha(x_1, x_2) = \fint_{x_1}^{x_1+x_2} h(t)dt, \tag{6.11}$$

$$\beta(x) = \beta(x_1, x_2) = \fint_{x_1-x_2}^{x_1} h(t)dt. \tag{6.12}$$

Then

$$F = \mathcal{BA}(h) \in \mathcal{Q}_0(K) = \{F : F \text{ is } K\text{-quasiconformal; } F(0,0) = (0,0),$$

$$F(1,0) = (1,0),\ F(\infty) = \infty\}.$$

Theorem 6.1.3. *Let $\mathcal{A}_j$ and $\mathcal{A}$ belong to $\mathcal{S}(K)$. Then*

$$\omega_{\mathcal{A}_j} \rightharpoonup \omega_{\mathcal{A}} \quad \Rightarrow \quad \mathcal{A}_j \xrightarrow{G} \mathcal{A}.$$

Proof. Assume $\omega_{\mathcal{A}_j} \rightharpoonup \omega_{\mathcal{A}}$. We know that

$$\mathcal{A}_j = \left[\frac{({}^tDF_j)(DF_j)}{J_{F_j}}\right]^{-1}, \tag{6.13}$$

$$\mathcal{A} = \left[\frac{({}^tDF)(DF)}{J_F}\right]^{-1},$$

and that (see (7.5)) the corresponding expressions of the harmonic measures are

$$\omega_{\mathcal{A}_j} = \frac{dh_j}{1+h_j^2}, \qquad \omega_{\mathcal{A}} = \frac{dh}{1+h^2},$$

where $F_j = \mathcal{BA}(h_j)$, $F = \mathcal{BA}(h)$, $F_j(x,0) = (h_j(x), 0)$ and $F(x,0) = (h(x), 0)$.

Since $\omega_{\mathcal{A}_j} \rightharpoonup \omega_{\mathcal{A}}$ in the sense of the measures, then $dh_j \rightharpoonup dh$ in the sense of measures as well. Moreover, $h_j(0) = 0$ and $h_j(1) = 1$. Then, we have

$$h_j \to h \quad \text{locally uniformly.} \tag{6.14}$$

Since the K- quasiconformal mappings F_j are normalized with the condition $F_j(0,0) = (0,0)$, $F_j(1,0) = (1,0)$, then by a Montel's theorem $\{F_j\}$ is a *normal family*; that is, it contains a subsequence $\{F_{j_r}\}$ converging locally uniformly to a mapping $F_0 \in \Psi_0(K)$. According to (6.10)-(6.14) we deduce for the whole sequence that

$$F_j \to F_0 = F$$

locally uniformly, and weakly in $W^{1,2}_{loc}(\mathbb{R}^2_+, \mathbb{R}^2_+)$.

Let us now give a direct proof of the following G- convergence result (for a proof which uses the G- compactness theorem see [Sp])

$$\mathcal{A}_j \xrightarrow{G} \mathcal{A}. \tag{6.15}$$

Let $u_j \in W^{1,2}_{loc}(\Omega)$ be a weak solution in a bounded open set $\Omega \subset \mathbb{R}^2_+$ to the equation

$$\operatorname{div} \mathcal{A}_j(z) \nabla u_j = 0 \quad \text{in} \quad \Omega \tag{6.16}$$

and assume that

$$u_j \rightharpoonup u \qquad W^{1,2}_{loc}(\Omega). \tag{6.17}$$

It is plain, in view of the G- convergence, that (6.15) will follows once we obtain the equation

$$\operatorname{div} \mathcal{A}(x) \nabla u = 0 \quad \text{in} \;\; \Omega. \tag{6.18}$$

Denote by $v_j \in W^{1,2}_{loc}(\Omega)$ the stream function defined via the following relation between the gradients

$$\nabla v_j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathcal{A}_j(x) \nabla u_j \tag{6.19}$$

and set

$$G_j = u_j + i v_j. \tag{6.20}$$

We may assume, passing to a subsequence if necessary, that

$$v_j \rightharpoonup v \qquad W^{1,2}_{loc}(\Omega) \tag{6.21}$$

The mapping G_j is K-quasiregular, that is

$$|DG_j|^2 \leqslant \left(K + \frac{1}{K} \right) J_{G_j} \quad \text{a.e. } x \in \Omega \tag{6.22}$$

as one can easily check. Hence by Stoilow Factorization Theorem [IM] there exists H_j, holomorphic on $F_j(\Omega)$, such that

$$G_j(x) = H_j \circ F_j(x). \tag{6.23}$$

We need the equicontinuity properties of both factors in (6.23). For the factor H_j note that

$$\int_{F_j(\Omega)} |DH_j(w)| dw = \int_\Omega |DH_j(F_j(x))| J_{F_j} dx \leqslant$$

$$\leqslant \left(\int_\Omega |DH_j(F_j(x))|^2 |DF_j(x)|^2 dx \right)^{\frac{1}{2}} \left(\int_\Omega |DF_j(x)|^2 dx \right)^{\frac{1}{2}} =$$

$$= \left(\int_\Omega |DG_j(w)|^2 dw \right)^{\frac{1}{2}} \left(\int_\Omega |DF_j(x)|^2 dx \right)^{\frac{1}{2}}$$

Let $B = B(x, r)$ be a fixed disk containing Ω, then

$$\left(\int_B |DF_j(z)|^2 dz \right)^{\frac{1}{2}} \leqslant c |F_j(B)|^{\frac{1}{2}} \leqslant c_o < \infty$$

Consequently

$$\sup_{j\in N}\int_{F_j(\Omega)}|DH_j(w)|dw<\infty$$

Quasiconformality is equivalent to the uniform quasisymmetry [LV], meaning that

$$\frac{|F_j(y)-F_j(x)|}{|F_j(z)-F_j(x)|}\leqslant\gamma\left(\frac{|y-x|}{|z-x|}\right) \tag{6.24}$$

for distinct points x, y, $z\in\mathbb{R}^2$. Here γ is an increasing homeomorphism of $[0,\infty)$ onto itself, we deduce that domains $F_j(\Omega)$ converge in the Hausdorff metric to the domain $F(\Omega)$. Since $H_j(0)=0$ by (6.24) we obtain that $\{H_j : F_j(\Omega)\longrightarrow\mathbb{R}^2\}$ is a normal family. Hence by choosing a further subsequence we can assume $H_j\longrightarrow H$ and $H'_j\longrightarrow H'$, uniformly on compact sets of $F(\Omega)$, where H is analytic on $F(\Omega)$. Then $H'_j(F(z))\longrightarrow H'(F(z))$ uniformly in compact subsets of Ω. It follows that

$$DG_j(x)\rightharpoonup D(H\circ F(x))\ \text{ in } L^2(\Omega) \tag{6.25}$$

Moreover,

$$G_j=H_j\circ F_j\longrightarrow G=H\circ F \tag{6.26}$$

and so, by (6.17), (6.20), (6.21) and (6.26), we infer

$$G_j\longrightarrow G=H\circ F=u+iv \tag{6.27}$$

and

$$DG=D(H\circ F)$$

This last equality implies that

$$\Delta_G=\Delta_F=\mathcal{A} \tag{6.28}$$

because the holomorphic function H does not affect the pullback of the Laplacian Δ_F.

A simple computation reveals that (6.28) is equivalent to

$$\operatorname{div} \mathcal{A}(z)\nabla u = 0$$

$$\nabla v = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathcal{A}(z)\nabla u$$

and the proof is complete.

□

In the following, we give also a sufficient condition in order that the G- limit of a sequence of resolutive operators be resolutive as well.

Definition 6.1.2. *Let $A_j \in \mathcal{E}_1(K)$. We say that A_j are L^2- equiresolutive if there exists $C_0 > 0$ such that*

$$\|Nu_j\|_{L^2(\partial\mathbb{D})} \leqslant C_0 \|f\|_{L^2(\partial\mathbb{D})} \tag{6.29}$$

for any $f \in C(\partial\mathbb{D})$.

We conclude this chapter with the following

Theorem 6.1.4. *Let $\mathcal{A}_j$, $\mathcal{A} \in \mathcal{E}_1(K)$, $\mathcal{A}_j \xrightarrow{G} \mathcal{A}$ and $\mathcal{A}_j$ be L^2- equiresolutive. Then, $\mathcal{A}$ is L^2- resolutive.*

Proof. The uniform bound (6.29) induces a similar bound for the weighted maximal operator M_{ω_j}, where ω_j is the harmonic measure of $\mathcal{A}_j$ relative to $\mathbb{D}$: for any $j \in \mathbb{N}$

$$\left\|M_{\omega_j} f\right\|_{L^2(\partial\mathbb{D}, d\sigma)} \leqslant C_1 \|f\|_{L^2(\partial\mathbb{D}, d\sigma)} \tag{6.30}$$

on the unweighted L^2 space.

From (6.30) one can deduce a uniform bound for the B_2- constants of all harmonic measures ω_j:

$$B_2(\omega_j) \leqslant C_2, \qquad \forall j \in \mathbb{N}. \tag{6.31}$$

By Theorem 6.1.2 we know that

$$\omega_j \rightharpoonup \omega = \omega_{\mathcal{A}}, \quad \text{in the sense of measures.} \tag{6.32}$$

Now, fix an open arc $\Gamma \subset \partial\mathbb{D}$ and use (6.31) to write

$$⨍_\Gamma \omega_j^2 \leqslant C_2 \left(⨍_\Gamma \omega_j \right)^2 .$$

Passing to the weak limit in the right-hand side, yields

$$\liminf_{j \to +\infty} ⨍_\Gamma \omega_j^2 \leqslant C_2 \left(⨍_\Gamma \omega \right)^2 .$$

By lower semicontinuity of L^2- norms with respect to weak convergence we find

$$⨍_\Gamma \omega^2 \leqslant C_2 \left(⨍_\Gamma \omega \right)^2$$

and this inequality holds for any open arc Γ. Hence

$$B_2(\omega) \leqslant C_2$$

and $\mathcal{A}$ is L^2- resolutive. □

Chapter 7

Neumann and Dirichlet problems with Orlicz data

In this chapter we prove in the Orlicz context, a relation between the solvability of Dirichlet and Neumann problems in the half-plane for special class of operators $\mathcal{L} = \text{div}\,(A\nabla)$ where A is a real, symmetric, 2×2 uniformly elliptic matrix and $\det A = 1$.

7.1 Neumann problem: definitions and preliminary results

Let $\mathbb{D}$ denote the unit disc in $\mathbb{R}^2$. As usual, we will denote

$$W^{\frac{1}{2},2}(\partial\mathbb{D}) = \left\{ u \in L^2(\partial\mathbb{D}) : \int_{\partial\mathbb{D}}\int_{\partial\mathbb{D}} \frac{|u(P)-u(Q)|^2}{|P-Q|^2} d\sigma(P) d\sigma(Q) < \infty \right\}.$$

Throughout this section we will assume $A \in \mathcal{E}^s(K,\mathbb{D})$. Now our purpose is to introduce the Neumann problem for an operator $\mathcal{L} = \text{div}(A\nabla)$. To this aim, let $g \in W^{-\frac{1}{2},2}(\partial\mathbb{D}) = (W^{\frac{1}{2},2}(\partial\mathbb{D}))^*$ with $\langle 1, g\rangle = 0$. A Sobolev function $u \in W^{1,2}(\mathbb{D})$ is said the variational solution to the Neumann problem

$$\begin{cases} \mathcal{L}u = \text{div } A\nabla u = 0, \text{ in } \mathbb{D} \\ A\nabla u \cdot \vec{N}|_{\partial\mathbb{D}} = g \end{cases} \tag{7.1}$$

if, given any $\varphi \in W^{1,2}(\mathbb{D})$, $\int_{\mathbb{D}} \varphi = 0$, it holds

$$\int_{\mathbb{D}} A\nabla u \cdot \nabla\varphi = \langle Tr(\varphi), \mu\rangle$$

(Here for any $Q \in \partial\mathbb{D}$, $\vec{N}(Q)$ denotes the unit normal at Q on $\partial\mathbb{D}$).

Clearly, the Lax Milgram lemma shows that given $g \in W^{-\frac{1}{2},2}(\partial\mathbb{D})$, $\langle 1, g\rangle = 0$, there exists a unique (modulo constants) $u \in W^{1,2}(\mathbb{D})$ which solves (7.1). For more details we refer the reader to [K].

Let $u \in L^2_{loc}(\mathbb{D})$ and let us introduce a modified non non-tangential maximal function

$$\widetilde{N}(u)(Q) = \sup_{X\in\Gamma(Q)} \left(\fint_{B\left(X, \frac{\delta(X)}{2}\right)} |u(z)|^2 \, dz \right)^{1/2}.$$

where $Q \in \partial\mathbb{D}$ and $\Gamma(Q) \subseteq \mathbb{D}$ is the non-tangential approach region (see formula (1.9)) with vertex at Q and $\delta(X) = \text{dist}(X, \partial\mathbb{D})$.

In analogy with (4.1) we have the following (see [KP1])

Definition 7.1.1. *Let* $1 < p < \infty$. *We say that the Neumann problem* (7.1) *is* L^p*-solvable if, whenever* $g \in L^2(\partial\mathbb{D}, d\sigma) \cap L^p(\partial\mathbb{D}, d\sigma)$, *and* $\int_{\partial B} g d\sigma = 0$, *the solution* u *to* (7.1), *verifies*

$$\|\widetilde{N}(\nabla u)\|_{L^p(\partial\mathbb{D},d\sigma)} \leqslant C\|f\|_{L^p(\partial\mathbb{D},d\sigma)} \tag{7.2}$$

Roughly speaking, the L^p-solvability of the Neumann problem (7.1) says that, for solutions, the whole gradient is controlled by the "conormal derivative" $a_{i,j}(Q)\frac{\partial u}{\partial X_i} n_j(Q)$, $Q \in \partial\mathbb{D}$, where $\overrightarrow{N}(Q) = n_j(Q)$ denotes the inward unit normal to $\partial\mathbb{D}$.

One of the first natural questions is whether the condition $\widetilde{N}(\nabla u) \in L^p(\partial\mathbb{D}, d\sigma)$ has any bearing on the existence of "boundary values" of ∇u. An answer is provided in the following

Theorem 7.1.1. **[KP1]** *Assume that* $\mathcal{L}u = 0$ *in* $\mathbb{D}$ *and let* $u \in L^p(\mathbb{D})$, $\widetilde{N}(\nabla u) \in L^p(\partial\mathbb{D}, d\sigma)$, $1 < p < \infty$. *Then,*

i) u *converges non-tangentially to* $f \in W^{1,p}(\partial\mathbb{D}, d\sigma)$, *and* $(\nabla_T u)_r(Q) = \fint_{B(rQ,(1-r)/2} \nabla u(X) \cdot \overrightarrow{T}(Q) dX$ *converges weakly in* L^p *to* $\nabla_T f$ *(here* $\overrightarrow{T}(Q)$ *denotes a basis of tangential vectors on* $\partial\mathbb{D}$, *and* $\nabla_T f = \nabla F(Q) \cdot \overrightarrow{T}(Q)$, *where* $F_{|\partial\ D} = f$*).*

ii) There exists a unique $g \in L^p(\partial\mathbb{D}, d\sigma)$, $\int_{\partial\mathbb{D}} g = 0$ *such that*

$$\int_{\mathbb{D}} A\nabla u \nabla\varphi = \int_{\partial\mathbb{D}} g\varphi d\sigma, \qquad \forall\varphi \in \ Lip(\bar{\mathbb{D}})$$

and

$$(A\nabla u \cdot \overrightarrow{N})_r = \fint_{B(rQ,(1-r)/2} A(X)\nabla u(X) \cdot \overrightarrow{N}(Q) dX,$$

converges weakly in L^p *to* g.

iii) If $f \equiv 0$, $u \equiv 0$.

iv) If there exists $\{u_j\} \subset W^{1,2}(\mathbb{D})$, $\mathcal{L}u_j = 0$ *in* $\mathbb{D}$ *for any* $j \in \mathbb{N}$, *with* $u_j \to u$ *uniformly on compact sets, with* $\|\widetilde{N}(\nabla u)\|_p \leqslant C$, *then* $u(X) = \int_{\partial\mathbb{D}} N(X,Q)g(Q)d\sigma(Q) + C$, *and hence, if* $g \equiv 0$, $u \equiv C$.

Let now $F : \mathbb{D} \to \mathbb{D}$ be a quasiconformal map and let $\mathcal{L}$ be the *pull-back* of the Laplacian under F, i.e. $\mathcal{L} = \Delta_F$. For this special class of operators the following result holds true (see [KP1], Section 4)

Theorem 7.1.2. *Let* $\mathcal{L} = \Delta_F$. *If Neumann problem* (7.1) *is* L^p*-solvable then Dirichlet problem* (1.32) *is* L^q*-solvable,* $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 7.1.1. As observed in [KP1], it can be shown that, in Theorem 7.1.2, the converse is also true. That is, by the special geometric properties of quasiconformal mappings one can see that the Neumann problem is L^p-solvable if and only if the Dirichlet problem is L^q-solvable ($\frac{1}{p} + \frac{1}{q} = 1$) for operators which arise as the pull-back of the Laplacian under a quasiconformal change of variables (see Remark 4.3, [KP1]).

In this chapter, we extend Theorem 7.1.2 in the context of Orlicz spaces (see Theorem 7.2.1). Moreover, as we will see, we partially give an answer to a question contained in [K] about planar operators of the type $\mathcal{L} = \mathrm{div}(\mathcal{A}\nabla)$ with $\mathcal{A} \in \mathcal{E}_1(K)$ (see Problem 3.2.6, [K]).

In analogy with L^p-case, we give the following

Definition 7.1.2. *The Neumann problem (7.1) is L^Ψ-solvable if, whenever $g \in L^2(\partial\mathbb{D}, d\sigma) \cap L^\Psi(\partial\mathbb{D}, d\sigma)$, and $\int_{\partial\mathbb{D}} g d\sigma = 0$, there exists a unique solution $u \in W^{1,2}_{loc}(\mathbb{D})$ to (7.1), verifying the uniform estimate*

$$\int_{\partial\mathbb{D}} \Psi[\widetilde{N}(\nabla u)]d\sigma \leqslant C \int_{\partial\mathbb{D}} \Psi[|g|]d\sigma, \tag{7.3}$$

where $\widetilde{N}$ is the non-tangential maximal function.

Previous definitions require obvious modifications in case the underlying space $(\partial\mathbb{D}, \sigma)$ is replaced by $\mathbb{R}^2_+ = \{(x,t) : t > 0\}$, the arc length σ by Lebesgue area and arcs Γ by intervals I contained in $\mathbb{R}$.

7.2 A relation between Dirichlet and Neumann problem

Let $\mathcal{A} \in \mathcal{E}_1(K)$ and let us consider the following Neumann problem with L^Ψ data

$$\begin{cases} \mathrm{div}\ A\nabla u = 0, \ \text{in } \mathbb{R}^2_+ \\ \mathcal{A}\nabla u \cdot \vec{N}|_{\mathbb{R}} = g. \end{cases} \tag{7.4}$$

We have the following

Theorem 7.2.1. **[CZ]** *Let $\mathcal{L} = \mathrm{div}\ (\mathcal{A}\nabla)$, $\mathcal{A} \in \mathcal{E}_1(K)$ and let $1 < p < \infty$. Moreover, let $\Psi \in \nabla_2$ be a Young function verifying*

i)$\underline{\alpha}(\Psi) = p$

ii)$\exists c_1$, c_2 *such that* $c_1 t^{p-1} \leqslant \Psi'(t) \leqslant c_2 t^{p-1}$, *for any* $t > 0$.

If Neumann problem (7.4) *is* L^{Ψ}*-solvable then Dirichlet problem* (1.32) *is* L^{Θ}*-solvable, for any Young function* $\Theta \in \nabla_2$ *such that* $\underline{\alpha}(\Theta) = q$, $\frac{1}{p} + \frac{1}{q} = 1$.

REMARK 7.2.1. It is worth to point out that in last Theorem if we assume $\Psi(t) = t^q$, $1 < q < \infty$, then we get Theorem 7.1.2.

Proof. Let $\mathcal{A} \in \mathcal{E}_1(K)$. By the *measurable Riemann mappings theorem* (see Theorem 5.1.1) we can find a K- quasiconformal mapping $F : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ such that

$$\mathcal{A} = \left[\frac{{}^t DFDF}{J_F}\right]^{-1}.$$

By Theorem 5.3.1, we have that the form of the harmonic measure of the operator $\mathcal{L} = \text{div } (\mathcal{A}\nabla)$ can be recognized. Namely, let $h : \mathbb{R} \to \mathbb{R}$, $h = F_{|\mathbb{R}}$ be the trace of F on the real axis $\mathbb{R}$. Hence, the harmonic measure $\omega_{\mathcal{L}}$ of the operator $\mathcal{L}$ evaluated at the point $F^{-1}(0,1)$ verifies

$$\omega_{\mathcal{L}} \sim dh. \tag{7.5}$$

Now, let us consider the Neumann problem (7.4) and let v be the solution to the problem

$$\begin{cases} \Delta v = 0, \text{ in } \mathbb{R}^2_+ \\ \frac{\partial v}{\partial t}|_{\mathbb{R}} = f. \end{cases} \tag{7.6}$$

We get a solution to (7.4) by composing v and F. Now we want compute the Neumann data for u. To this aim, let $(y,t) = F(x,s)$. We have, by the chain rule formula

$$\nabla u = \nabla(v \circ F) = ({}^t DF)[(\nabla v) \circ F]$$

so that

$$\mathcal{A}\nabla u = J_F (DF)^{-1}({}^t DF)^{-1}({}^t DF)[(\nabla v) \circ F] = J_F (DF)^{-1}[(\nabla v) \circ F].$$

Moreover,

$$DF|_{\mathbb{R}} = \begin{pmatrix} h'(x) & a(x) \\ 0 & b(x) \end{pmatrix}$$

where $a(x)$ and $b(x)$ are functions depending only on x. Denoting by e_2 the vertical unit vector, $e_2 = (0,1)$ it holds

$$(\mathcal{A}\nabla u)\cdot e_2|_{\mathbb{R}} = [(J_F(DF)^{-1})|_{\mathbb{R}}(\nabla v)\circ F|_{\mathbb{R}}]\cdot e_2 = \left[(J_F(DF)^{-1})|_{\mathbb{R}} \begin{pmatrix} \frac{\partial v}{\partial y}|_{\mathbb{R}}\circ h \\ \frac{\partial v}{\partial t}|_{\mathbb{R}}\circ h \end{pmatrix}\right]\cdot e_2$$

and then, by (7.6) we have

$$(\mathcal{A}\nabla u)\cdot e_2|_{\mathbb{R}} = \begin{pmatrix} b(x)\frac{\partial v}{\partial y}|_{\mathbb{R}}\circ h - a(x) f\circ h(x) \\ h'(x) f\circ h(x) \end{pmatrix}\cdot e_2 = h'(x)(f\circ h(x)).$$

By definition of L^{Ψ}-solvability, we know that $g = (f\circ h)h' \in L^{\Psi}(\mathbb{R})$, i.e.

$$\int_{\mathbb{R}} \Psi[(f\circ h)h']dx < \infty.$$

Changing variables $y = h(x)$, we get

$$\int_{\mathbb{R}} \Psi[(f\circ h)h']dx = \int_{\mathbb{R}} \Psi\left[f(y)\cdot\frac{1}{(h^{-1})'(y)}\right](h^{-1})'(y)dy < \infty. \tag{7.7}$$

Now, let us observe that by hypothesis *ii*) we have that there exists a constant $C > 0$ such that

$$\Psi(st) \geqslant C\Psi(s)\Psi(t), \qquad \forall s,t > 0. \tag{7.8}$$

Then, by (7.7) and (7.8) we get

$$\int_{\mathbb{R}} \Psi[f(y)]\Psi\left[\frac{1}{(h^{-1})'(y)}\right](h^{-1})'(y)dy < \infty$$

and, by the second inequality in (2.17), we get

$$\begin{aligned}&\int_{\mathbb{R}} \Psi[f(y)]\frac{1}{(h^{-1})'(y)}\Psi'\left[\frac{1}{(h^{-1})'(y)}\right](h^{-1})'(y)dy\\ &\leqslant C\int_{\mathbb{R}} \Psi[f(y)]\Psi\left[\frac{1}{(h^{-1})'(y)}\right](h^{-1})'(y)dy < \infty\end{aligned}$$

which yields

$$\int_{\mathbb{R}} \Psi[f(y)]\Psi'\left[\frac{1}{(h^{-1})'(y)}\right]dy < \infty.$$

On the other hand $\int_{\mathbb{R}} \Psi[f(y)]\Psi'\left[\frac{1}{(h^{-1})'(y)}\right]dy < \infty$ if and only if $f \in L^{\Psi}(wdy)$, where $w(y) = \Psi'\left[\frac{1}{(h^{-1})'(y)}\right]$.

Since Neumann problem (7.4) is L^{Ψ}-solvable, then we have that all derivatives of u, restricted to $\mathbb{R}$, have to be in L^{Ψ} (see Theorem 7.1.1).

Now, let us observe that $v = v(y,t)$ is harmonic in $\mathbb{R}^2_+$. Hence there exists the conjugate harmonic function v_c on $\mathbb{R}^2_+$ such that

$$\frac{\partial v}{\partial y} = \frac{\partial v_c}{\partial t}, \qquad \frac{\partial v}{\partial t} = -\frac{\partial v_c}{\partial y}$$

i.e. the function $V = v + iv_c$ is holomorphic on $\mathbb{R}^2_+$ respect to $z = y + it$. Then, also the function

$$iV' = \frac{\partial v}{\partial t} + i\frac{\partial v}{\partial y}$$

is holomorphic. Hence, $\frac{\partial v}{\partial t}|_{\mathbb{R}} = f$ implies $\frac{\partial v}{\partial t}|_{\mathbb{R}} = Hf$ where Hf denotes the classical Hilbert transform of f (see (2.21)).

$$Hf(y) = \lim_{\varepsilon\to 0^+}\frac{1}{\pi}\int_{|x-y|>\varepsilon}\frac{f(x)}{y-x}dx.$$

This implies, in particular, that if $f \in L^{\Psi}(wdy)$, then

$$\frac{\partial v}{\partial y} = Hf \in L^{\Psi}(wdy),$$

and

$$\int_{\mathbb{R}} \Psi(Hf)w(y)dy \leqslant C \int_{\mathbb{R}} \Psi(|f|)w(y)dy. \tag{7.9}$$

To prove (7.9) let us start by observing that $u(x,s) = v(F(x,s)) = v(y,t)$. By considering the restriction of u to $\mathbb{R}$, we have $u(x,0) = v(F(x,0)) = v(h(x),0)$. Hence, the derivative of u with respect to x on $\mathbb{R}$ is given by,

$$\frac{\partial u}{\partial x}(x,0) = \frac{\partial v}{\partial y}(h(x),0)h'(x),$$

so that

$$\int_{\mathbb{R}} \Psi(\frac{\partial u}{\partial x}(x,0))dx = \int_{\mathbb{R}} \Psi(\frac{\partial v}{\partial y}(h(x),0)h'(x))dx.$$

Now, changing variables $x = h(y)$, $dy = h'(x)dx$, by (2.17) and (7.8) we have

$$\begin{aligned}
&\int_{\mathbb{R}} \Psi(\frac{\partial u}{\partial x}(x,0))dx = \\
&= \int_{\mathbb{R}} \Psi(\frac{\partial v}{\partial y}(y,0)h'(h^{-1}(y)))\frac{1}{h'(h^{-1}(y))}dy \\
&\geqslant C\int_{\mathbb{R}} \Psi(\frac{\partial v}{\partial y}(y,0))\Psi'(h'(h^{-1}(y)))dy.
\end{aligned}$$

This means

$$\int_{\mathbb{R}} \Psi(\frac{\partial v}{\partial y}(y,0))w(y)dy \leqslant C \int_{\mathbb{R}} \Psi(\frac{\partial u}{\partial x}(x,0))dx. \tag{7.10}$$

Now, by the L^{Ψ} solvability of the Neumann problem it holds in particular

$$\int_{\mathbb{R}} \Psi(\frac{\partial u}{\partial x}(x,0))dx \leqslant C \int_{\mathbb{R}} \Psi(|f|)w(y)dy,$$

so that, by (7.10) inequality (7.9) follows.

By a classical result (see Theorem 2.1.4), (7.9) holds if and only if $w \in A_{\Psi}$. Then, by

Theorem 2.3.6, $w \in A_p$-class, where $p = \underline{\alpha}(\Psi)$. Hence, there exists $A \geqslant 1$ such that

$$\fint_J \Psi'\left(\frac{1}{(h^{-1})'(y)}\right) dy \left(\fint_J \Psi'\left(\frac{1}{(h^{-1})'(y)}\right)^{-\frac{1}{p-1}} dy\right)^{p-1} \leqslant A.$$

Changing variables $y = h(x)$ we have

$$\left(\frac{1}{|h(I)|}\int_I \Psi'(h'(x))h'(x)dx\right)\left(\frac{1}{|h(I)|}\int_I (\Psi'(h'(x)))^{-\frac{1}{p-1}} h'(x)dx\right)^{p-1} \leqslant A,$$

for all bounded interval $I \subset \mathbb{R}$.

By the assumption *ii*) we get

$$\left(\frac{1}{|h(I)|}\int_I h'(x)^p dx\right)\left(\frac{1}{|h(I)|}\int_I h'(x)^{-1} h'(x)dx\right)^{p-1} \leqslant A,$$

so that,

$$\left(\frac{1}{|h(I)|}\int_I (h'(x))^p dx\right)\left(\frac{|I|}{|h(I)|}\right)^{p-1} \leqslant A.$$

Hence

$$\left(\fint_I (h'(x))^p dx\right)^{\frac{1}{p}} \leqslant A \fint_I h'(x)dx \tag{7.11}$$

and then, by (7.5) and (7.11) we obtain that $\omega_{\mathcal{L}} \in B_p$.

In conclusion by Theorem 4.1.1 the L^{Θ}-solvability of the Dirichlet problem follows, for any $\Theta \in \nabla_2$ with $\underline{\Theta}(\Phi) = q$.

□

7.3 Examples

In this section we present some examples of Young functions verifying the hypotheses of Theorem 7.2.1.

Example 7.3.1. Let $a, b \in \mathbb{R}_+$ and let $1 < p, q < \infty$. Let us consider the following Young function:

$$\Psi_1(t) = \begin{cases} t^p, & 0 \leqslant t \leqslant a \\ a^{p-q} t^q, & a \leqslant t \leqslant b \\ (\frac{a}{b})^{p-q} t^p, & t \geqslant b \end{cases}$$

By Theorem 2.3.4 [FK2] we can easily compute the fundamental indices of Ψ_1. More precisely we have

$$\underline{\alpha}(\Psi_1) = \overline{\alpha}(\Psi_1) = p$$

so that by (2.31) we have $\Psi_1 \in \nabla_2$.

In order to see that Ψ_1 satisfies condition ii) of Theorem 7.2.1 let us observe that the derivative Ψ_1' of Ψ_1 is given by

$$\Psi_1'(t) = \begin{cases} pt^{p-1}, & 0 \leqslant t < a \\ qa^{p-q} t^{q-1}, & a \leqslant t < b \\ p(\frac{a}{b})^{p-q} t^{p-1}, & t \geqslant b. \end{cases}$$

Then, when $0 \leqslant t < a$ or $t \geqslant b$, condition ii) is obvious. On the other hand, whenever $a \leqslant t < b$ we have

$$\Psi_1'(t) = qa^{p-q} t^{q-1} \leqslant c_2 t^{p-1} \Longleftrightarrow c_2 \geqslant q \max\left\{1, \left(\frac{b}{a}\right)^{q-p}\right\},$$

and

$$\Psi_1'(t) = qa^{p-q} t^{q-1} \geqslant c_1 t^{p-1} \Longleftrightarrow c_1 \leqslant q \min\left\{1, \left(\frac{b}{a}\right)^{q-p}\right\}.$$

Hence, Ψ_1 verifies the hypotheses of Theorem 7.2.1.

Example 7.3.2. Let us consider (see [FS])

$$\Psi_2(t) = \begin{cases} et^3, & 0 \leqslant t \leqslant e \\ t^{4+\sin\log\log t}, & e \leqslant t \leqslant e^e \\ (e^{e+e\sin 1})t^3, & t \geqslant e^e \end{cases}$$

By Theorem 2.3.4 we have

$$\underline{\alpha}(\Psi_2) = \overline{\alpha}(\Psi_2) = 3,$$

and by (2.31) we have $\Psi_2 \in \nabla_2$.

The derivative of Ψ_2 is given by

$$\Psi_2'(t) = \begin{cases} 3et^2, & 0 \leqslant t < e \\ t^{3+\sin\log\log t}(\cos\log\log t + 4 + \sin\log\log t), & e \leqslant t < e^e \\ 3(e^{e+e\sin 1})t^2, & t \geqslant e^e \end{cases}$$

so that when $0 \leqslant t < e$ or $t \geqslant e^e$ condition ii) is simply verified. On the other hand, if $e \leqslant t < e^e$ we can choose $c_1 \leqslant 2e^e$ and $c_2 \geqslant 6e^{e+e\sin 1}$ and condition ii) of Theorem 7.2.1 is verified.

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