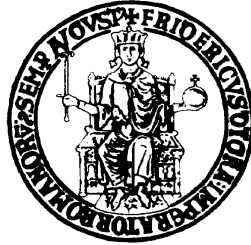


Università degli Studi di Napoli Federico II



Dottorato di Ricerca in Fisica Fondamentale ed Applicata
XXII ciclo

**Non-commutative Field Theory,
Translational Invariant Products and
Ultraviolet/Infrared Mixing**

Dissertation submitted for the degree of Philosophiae Doctor
January 13, 2010

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Introduction

There are several motivations to consider a non-commutative structure of space-time. One of them is that at short distances that is, at distances of the order of the Planck length:

$$\ell_P = \sqrt{\frac{G\hbar}{c^3}} \approx 10^{-33} \text{cm}$$

the geometry of space-time has to be described by a different theory because its points become no longer localizable. Therefore, one is forced to deal with a “pointless” geometry and this leads in a natural way to the introduction of non-commutative geometry [1, 2, 3, 4]. A historical motivation [5, 6] to consider a non-commutative structure of space-time was the hope that a modification of the short distance properties of space-time, by means of a deformation parameter, could resolve the problem of the infinities of quantum field theory, like the introduction of the fundamental quantity \hbar solved the so-called ultraviolet catastrophe of the black-body radiation.

The simplest kind of non-commutativity is the so-called canonical one which is characterized by the following commutation relation between the coordinate functions on the space-time:

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}$$

where $(\theta^{\mu\nu})$ is constant matrix i.e. it does not depend on the x 's. There are several reasons to consider such a kind of non-commutativity, going from the localizability of events in space-time [7, 8] to the string theory [9]. Moreover, field theories on a space-time equipped with the canonical non-commutativity have interesting renormalization properties [10, 11, 12]. What is generally done to construct a non-commutative field theory [13, 14] is to deform the ordinary pointwise commutative product among functions on space-time with the introduction of a star product which is non-commutative and reduces to

the usual one in certain limit. The choice of the star product compatible with the canonical non-commutativity is not unique and throughout this thesis we discuss two different products, the Moyal product [15, 16] and the Wick-Voros one [17, 18, 19, 20] and investigate their ultraviolet behaviour and compare their “physical predictions”.

In the case of a field theory with the Moyal product the hope that the product resolve the problem of the infinities of quantum field theory is not fulfilled. Indeed, in this case instead of the elimination (at least partial) of the ultraviolet divergences, we encounter the phenomenon of ultraviolet/infrared mixing [21], one of the novel features of a non-commutative field theory. Therefore, while the ultraviolet properties of the theory are changed in the sense of a mitigation of the infinities, the price paid is the appearance of new kind of infinity. We show that the ultraviolet/infrared mixing persists in an unchanged way as well as for a field theory with the Wick-Voros product [22] which can be seen as a variant of the Moyal one. This is to be expected because heuristically this is consequence of commutation relation which is, of course, the same for both products. Our analysis is centered mainly on the one-loop correction to the propagator which is the source of all mixing and we discuss only the scalar ϕ^4 theory, but the results are more general. Moreover, we show that the ultraviolet/infrared mixing for the Moyal product is a generic feature of any translation invariant associative product [23].

We have to note that the two field theories with the Moyal and Wick-Voros products are not completely equivalent because their Green’s functions are different and this leads to a contradiction. In fact, one can heuristically reason as follows. What really counts is the non-commutative structure of space-time and the star product is just a way to express such a structure so that one can always choose the most convenient star product. As long as one is describing the same field theory, the results should be the same as already noted in [24]. We see that this contradiction is only apparent. Indeed, Green’s functions are not observable quantities and what is observable is the S -matrix.

Discussions of the properties of the S -matrix naturally go together with the issue of Poincaré invariance. The canonical non-commutativity relation is not Poincaré invariant and this can cast doubts on its being fundamental. However, it is possible to preserve the Poincaré symmetry at a deformed level, as a non-commutative and non-cocommutative Hopf algebra because both the Moyal and Wick-voros products come from a Drinfeld twist [25, 26, 27].

In other words, the theory has a twisted Poincaré symmetry [28, 29, 30].

The presence of a twist forces us to reconsider all of the steps in a field theory which has to be built in a coherent twisted way and we show that there is equivalence between the Moyal and Wick-Voros field theories at the level of S -matrix only if a consistent procedure of twisting all products is applied. There can be some ambiguity in the issue of twisting and in an ideal context one should let experiments resolve these ambiguities. However, the non-commutative theory is not yet mature for a confrontation with experiments. Thus what we do is just to use the field theories built with the Moyal and Wick-Voros products to check each other. This gives us the indication on the procedure to follow for non-commutative theories coming from a twist.

The thesis is organized as follows. In the first chapter we will introduce the Moyal and Wick-Voros products and show that they are both coming from a “Weyl map”. In particular, we will see that the Moyal product comes from the usual Weyl map while the Wick-Voros one comes from a generalization of Weyl map called a weighted Weyl map.

In the second chapter we will discuss that the Moyal and Wick-Voros product can be set in a more general framework. Indeed, we will show that both products can be derived from a general quantization scheme as well. In particular, we will see that the Moyal product derives from the so-called Weyl-Wigner quantization scheme.

In the third chapter we will investigate the ultraviolet behaviour of a non-commutative field theory obtained from a commutative one replacing the ordinary product with the Moyal one. To this end, we will discuss the one-loop corrections to the two- and four-point Green’s functions and see that in the non-planar cases the Moyal product softens the ultraviolet divergences, but it is responsible for the infrared divergence. Therefore, the Moyal product presents the phenomenon of ultraviolet/infrared mixing.

In the last three chapters we will present our original work. In the fourth chapter we will investigate the ultraviolet behaviour of a non-commutative field theory with the Wick-Voros product. We will show that the ultraviolet properties in this case is the same as in the Moyal one and in particular they present the same ultraviolet/infrared mixing as heuristically expected. However, we will find that the two theories are not equivalent since their Green’s functions are different.

In the fifth chapter we will proceed to the discussion of the relationship

between the translation invariance and the ultraviolet/infrared mixing and show that the ultraviolet/infrared mixing found for the Moyal and Wick-Voros products is not specific of the two products, but it is a generic feature of any translation invariant associative product.

In the last chapter we will present a comparison of the non-commutative field theories with the Moyal and Wick-Voros products in the framework of the twisted non-commutativity and see that the two theories are equivalent at level of S -matrix by means of a consistent procedure of twisting all products, in agreement with our physical intuition, although the Green's functions are different.

Finally, there is an appendix in which we will recall the principal notions of Hopf algebras that we will use throughout the thesis.

Chapter 1

The Moyal and Wick-Voros products from a Weyl map

In this chapter we introduce the Moyal and Wick-Voros products and show that the two products can be cast in the same general framework in that they are both coming from a “Weyl map”. More precisely, we show that the Moyal product comes from a map, called the Weyl map, which associates operators to functions with symmetric ordering, while the Wick-Voros one comes from a similar map, a weighted Weyl map, which associates operators to functions with normal ordering. Furthermore, we exhibit the integral form of the two products.

1.1 The Weyl map

For the sake of simplicity, we consider the Weyl map on the plane since its generalization to a several dimension is straightforward. The Weyl map [31] is the map which associates to a function on the plane an operator according to

$$\hat{\Omega}_M(f) = \frac{1}{2\pi\theta} \int d^2\alpha \tilde{f}(\alpha) e^{i\theta_{ij}\hat{x}^i\alpha^j} \quad (1.1.1)$$

where θ is a real constant parameter of dimensions of a square length,

$$\tilde{f}(\alpha) = \frac{1}{2\pi\theta} \int d^2x f(x) e^{-i\theta_{ij}x^i\alpha^j} \quad (1.1.2)$$

is the symplectic Fourier transform of the function f ,

$$\theta_{ij} = \theta^{-1}\varepsilon_{ij} \quad \text{with} \quad (\varepsilon_{ij}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (1.1.3)$$

and the \hat{x} 's are operators which satisfy the commutation relation

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij} \quad (1.1.4)$$

where the matrix (θ^{ij}) is the inverse of the matrix (θ_{ij}) . In general, it is always possible to consider the operators \hat{x} 's in an abstract way and define them as

$$\begin{aligned} \hat{x}^1 &= \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}} \\ \hat{x}^2 &= \frac{\hat{a} - \hat{a}^\dagger}{i\sqrt{2}} \end{aligned} \quad (1.1.5)$$

where \hat{a} and \hat{a}^\dagger are two operators which satisfy the commutation relation

$$[\hat{a}, \hat{a}^\dagger] = \theta. \quad (1.1.6)$$

Therefore, the Weyl map can be explicitly written as

$$\hat{\Omega}_M(f) = \frac{1}{(2\pi\theta)^2} \int d^2x d^2\alpha f(x) e^{-i\theta_{ij}x^i\alpha^j} e^{i\theta_{ij}\hat{x}^i\alpha^j}. \quad (1.1.7)$$

It can be equivalently written as

$$\hat{\Omega}_M(f) = \frac{1}{(2\pi\theta)^2} \int d^2x d^2\alpha f(x) e^{-i\theta_{ij}x^i\alpha^j} W(\alpha) \quad (1.1.8)$$

where

$$W(\alpha) = e^{i\theta_{ij}\hat{x}^i\alpha^j}. \quad (1.1.9)$$

This last formula has the advantage to involve the operators $W(\alpha)$ which form a Weyl system [31, 32]. Indeed, by using the Baker-Campbell-Hausdorff formula¹, it is easy to verify that

$$W(\alpha)W(\beta) = W(\alpha + \beta)e^{\frac{i}{2}\theta_{ij}\alpha^i\beta^j}. \quad (1.1.10)$$

¹If \hat{A} and \hat{B} are two operators such that $[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0$, then

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}}e^{\frac{1}{2}[\hat{A}, \hat{B}]}$$

from which follows that

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{B}}e^{\hat{A}}e^{[\hat{A}, \hat{B}]}$$

The Weyl map is linear and invertible and its inverse is given by the Wigner map

$$\Omega_M^{-1} \left(\hat{\Omega}_M(f) \right) = \frac{1}{2\pi\theta} \int d^2\alpha e^{i\theta_{ij}x^i\alpha^j} \text{Tr} \left(\hat{\Omega}_M(f) W^\dagger(\alpha) \right). \quad (1.1.11)$$

In fact,

$$\begin{aligned} \Omega_M^{-1} \left(\hat{\Omega}_M(f) \right) &= \frac{1}{(2\pi\theta)^2} \int d^2\alpha d^2\beta \tilde{f}(\alpha) e^{i\theta_{ij}x^i\beta^j} \text{Tr} (W(\alpha)W^\dagger(\beta)) \\ &= \int d^2\alpha d^2\beta \tilde{f}(\alpha) e^{i\theta_{ij}x^i\beta^j} \delta^{(2)}(\alpha - \beta) \\ &= \int d^2\alpha \tilde{f}(\alpha) e^{i\theta_{ij}x^i\alpha^j} = f(x) \end{aligned} \quad (1.1.12)$$

since in the last line appears the symplectic Fourier antitransform of $\tilde{f}(\alpha)$. Moreover, it can be show that [33] the Weyl map is an isomorphism between $L^2(\mathbb{R}^2)$ i.e the Hilbert space of square-integrable functions on the plane and $\mathcal{HS}(L^2(\mathbb{R}))$ i.e. the Hilbert space of Hilbert-Schmidt operators on $L^2(\mathbb{R})$.

1.2 The Moyal product from the Weyl map

The Moyal product [15, 16], often called the Grönewold-Moyal product, is defined by the relation

$$\hat{\Omega}_M(f \star_M g) = \hat{\Omega}_M(f) \hat{\Omega}_M(g). \quad (1.2.1)$$

We can very easily obtain the integral form of the Moyal product. Indeed, from (1.1.8) follows that the left-hand side of (1.2.1) can be written as

$$\hat{\Omega}_M(f \star_M g) = \frac{1}{(2\pi\theta)^2} \int d^2x d^2\alpha (f \star_M g)(x) e^{-i\theta_{ij}x^i\alpha^j} W(\alpha) \quad (1.2.2)$$

and the right-hand side of (1.2.1) as

$$\begin{aligned} \hat{\Omega}_M(f) \hat{\Omega}_M(g) &= \frac{1}{(2\pi\theta)^4} \int d^2y d^2\beta d^2z d^2\gamma f(y) g(z) \\ &\quad e^{-i\theta_{ij}y^i\beta^j} e^{-i\theta_{ij}z^i\gamma^j} W(\beta) W(\gamma) \end{aligned} \quad (1.2.3)$$

which can be written because of (1.1.10) as

$$\begin{aligned} \hat{\Omega}_M(f) \hat{\Omega}_M(g) &= \frac{1}{(2\pi\theta)^4} \int d^2y d^2\beta d^2z d^2\gamma f(y) g(z) \\ &\quad e^{-i\theta_{ij}y^i\beta^j} e^{-i\theta_{ij}z^i\gamma^j} e^{\frac{i}{2}\theta_{ij}\beta^i\gamma^j} W(\beta + \gamma). \end{aligned} \quad (1.2.4)$$

By means of the linear transformation

$$\begin{aligned}\beta &= \alpha - 2x + 2y \\ \gamma &= 2x - 2y\end{aligned}\tag{1.2.5}$$

with y constant, it takes the form

$$\hat{\Omega}_M(f)\hat{\Omega}_M(g) = \frac{2^2}{(2\pi\theta)^4} \int d^2x d^2\alpha d^2y d^2z f(y)g(z) e^{-2i\theta_{ij}(x^i-y^i)(x^j-z^j)} e^{-i\theta_{ij}x^i\alpha^j} W(\alpha).\tag{1.2.6}$$

By confronting (1.2.2) with (1.2.6) we obtain the integral form of the Moyal product

$$(f \star_M g)(x) = \frac{1}{(\pi\theta)^2} \int d^2y d^2z f(y)g(z) e^{-2i\theta_{ij}(x^i-y^i)(x^j-z^j)}.\tag{1.2.7}$$

Other integral expressions are possible some of which can be found in the appendix of [34]. Note that the Moyal product can be expressed as well as in a differential form which is an asymptotic expansion of the integral one [35]. However, the integral form has the advantage to be defined on a set wider than the one on which is defined the differential form.

1.3 The Wick-Voros product from a weighted Weyl map

A weighted Weyl map is a generalization of the Weyl map defined as

$$\hat{\Omega}(f) = \frac{1}{(2\pi\theta)^2} \int d^2x d^2\alpha f(x)w(\alpha)e^{-i\theta_{ij}x^i\alpha^j} W(\alpha)\tag{1.3.1}$$

where $w(\alpha)$ is an invertible function, called weighted function. A general weighted Weyl map is linear and invertible and its inverse is

$$\Omega^{-1}\left(\hat{\Omega}(f)\right) = \frac{1}{2\pi\theta} \int d^2\alpha w^{-1}(\alpha)e^{i\theta_{ij}x^i\alpha^j} \text{Tr}\left(\hat{\Omega}(f)W^\dagger(\alpha)\right).\tag{1.3.2}$$

Here we are interesting to the weighted Weyl map given by

$$\hat{\Omega}_V(f) = \frac{1}{(2\pi\theta)^2} \int d^2x d^2\alpha f(x)e^{-\frac{1}{4\theta}\alpha^2} e^{-i\theta_{ij}x^i\alpha^j} W(\alpha)\tag{1.3.3}$$

which leads to the Wick-Voros product. This weighted Weyl map can be written in complex coordinates:

$$x^\pm = \frac{x^1 \pm ix^2}{\sqrt{2}} \quad (1.3.4)$$

as

$$\hat{\Omega}_V(f) = \frac{1}{(2\pi\theta)^2} \int d^2x d^2\alpha f(x) e^{-\frac{1}{2\theta}\alpha^+\alpha^-} e^{-\frac{1}{\theta}(\alpha^+x^- - \alpha^-x^+)} W(\alpha) \quad (1.3.5)$$

where

$$\alpha^\pm = \frac{\alpha^1 \pm i\alpha^2}{\sqrt{2}}. \quad (1.3.6)$$

In these coordinates the Weyl system (1.1.9) takes the form

$$W(\alpha) = e^{\frac{1}{\theta}(\alpha^-\hat{a} - \alpha^+\hat{a}^\dagger)} \quad (1.3.7)$$

and the relation (1.1.10) is given by

$$W(\alpha)W(\beta) = W(\alpha + \beta) e^{\frac{1}{2\theta}(\alpha^+\beta^- - \alpha^-\beta^+)}. \quad (1.3.8)$$

The Wick-Voros product is then defined by the relation

$$\hat{\Omega}_V(f \star_V g) = \hat{\Omega}_V(f)\hat{\Omega}_V(g). \quad (1.3.9)$$

and it is possible to show that it reads

$$(f \star_V g)(x) = \int \frac{d^2y}{\pi\theta} f(x^-, y^+) g(y^-, x^+) e^{-\frac{1}{\theta}(x^- - y^-)(x^+ - y^+)}. \quad (1.3.10)$$

This product, like the Moyal one, can be expressed as well as in a differential form which is an asymptotic expansion of the integral one as we will see in the following.

Chapter 2

The Moyal and Wick-Voros products from a quantization scheme

In this second chapter we show how the Moyal and Wick-Voros products can be derived from a general quantization scheme. To this end, we first review a general quantization scheme for associating operators with functions and vice versa and producing new star products. In particular, we describe the duality symmetry of a quantization scheme and the notion of dual star product. We finally introduce the Weyl-Wigner and Wick-Voros quantization schemes for the Moyal and Wick-Voros products respectively.

2.1 Quantization schemes and star products

We begin with a review of a general scheme to associate operators with functions and vice versa [36] and produce new star products [37] for operator symbols. In this scheme the symbols of the operators are defined in terms of a family of operators, called dequantizers, while the reconstruction of operators in terms of their symbols is determined using another family of operators, called quantizers.

Let us consider a Hilbert space \mathcal{H} and two sets of operators $\hat{U}(x)$ and $\hat{V}(x)$ on \mathcal{H} parameterized by an n -dimensional vector $x = (x_1, x_2, \dots, x_n)$ and suppose they satisfy the consistency condition

$$\mathrm{Tr} \left(\hat{U}(x) \hat{V}(x') \right) = \delta^{(n)}(x - x'). \quad (2.1.1)$$

With these two families of operators we can construct an invertible map which associates to each operator \hat{A} on \mathcal{H} a function $f_{\hat{A}}(x)$, called the symbol of the operator \hat{A} , defined by

$$f_{\hat{A}}(x) = \text{Tr} \left(\hat{A} \hat{V}(x) \right) \quad (2.1.2)$$

and to each function $f_{\hat{A}}(x)$ an operator \hat{A} on \mathcal{H} defined by

$$\hat{A} = \int f_{\hat{A}}(x) \hat{U}(x) d^n x. \quad (2.1.3)$$

Indeed, multiplying both sides of equation (2.1.3) by the operator $\hat{V}(x')$ and taking the trace, we have

$$\text{Tr} \left(\hat{A} \hat{V}(x') \right) = \text{Tr} \int f_{\hat{A}}(x) \hat{U}(x) \hat{V}(x') d^n x \quad (2.1.4)$$

and, assuming it is possible to exchange the trace with the integral, we have

$$\text{Tr} \left(\hat{A} \hat{V}(x') \right) = \int f_{\hat{A}}(x) \text{Tr} \left(\hat{U}(x) \hat{V}(x') \right) d^n x = f_{\hat{A}}(x') \quad (2.1.5)$$

where we have used (2.1.1). Therefore, the operators $\hat{V}(x)$ associate to the operator \hat{A} (quantum observable) a function $f_{\hat{A}}(x)$ (classical observable) i.e. they “dequantize” the quantum observable, while the role of the other operators $\hat{U}(x)$ is opposite; they associate to the function $f_{\hat{A}}(x)$ an operator \hat{A} i.e. they “quantize” the classical observable. For this reason we call the operators $\hat{U}(x)$ and $\hat{V}(x)$ quantizers and dequantizers respectively. However, there is an ambiguity in defining the operators $\hat{U}(x)$ and $\hat{V}(x)$. Indeed, we can make a scaling transformation of the operators $\hat{U}(x)$ and $\hat{V}(x)$ without violating the consistency of the quantization scheme i.e. the condition (2.1.1). Moreover, if we require that the symbol of identity operator $\mathbb{1}$ is equal to the constant function 1, this ambiguity is removed because the operators $\hat{V}(x)$ have to satisfy the condition

$$\text{Tr} \hat{V}(x) = 1 \quad (2.1.6)$$

and the operators $\hat{U}(x)$ the condition

$$\int \hat{U}(x) d^n x = \mathbb{1}. \quad (2.1.7)$$

Now we can introduce the star product of the symbols $f_{\hat{A}}(x)$ and $f_{\hat{B}}(x)$ of two operators \hat{A} and \hat{B} on \mathcal{H} by the relationships

$$f_{\hat{A}}(x) * f_{\hat{B}}(x) = f_{\hat{A}\hat{B}}(x) \quad (2.1.8)$$

that is,

$$f_{\hat{A}}(x) * f_{\hat{B}}(x) = \text{Tr} \left(\hat{A}\hat{B}\hat{V}(x) \right). \quad (2.1.9)$$

In other words, the star product of the symbols $f_{\hat{A}}(x)$ and $f_{\hat{B}}(x)$ of two operators \hat{A} and \hat{B} on \mathcal{H} is the symbol of the their product. The star product is associative due to the associativity of the operator product. Indeed,

$$\begin{aligned} (f_{\hat{A}}(x) * f_{\hat{B}}(x)) * f_{\hat{C}}(x) &= f_{\hat{A}\hat{B}}(x) * f_{\hat{C}}(x) = f_{(\hat{A}\hat{B})\hat{C}}(x) = f_{\hat{A}(\hat{B}\hat{C})}(x) \\ &= f_{\hat{A}}(x) * f_{\hat{B}\hat{C}}(x) = f_{\hat{A}}(x) * (f_{\hat{B}}(x) * f_{\hat{C}}(x)). \end{aligned} \quad (2.1.10)$$

However, it is not commutative. From the definition of the star product follows that

$$f_{\hat{A}}(x) * f_{\hat{B}}(x) = \text{Tr} \int f_{\hat{A}}(x') f_{\hat{B}}(x'') \hat{U}(x') \hat{U}(x'') \hat{V}(x) d^n x' d^n x'' \quad (2.1.11)$$

and, assuming once again it is possible to exchange the trace with the integral, we have

$$f_{\hat{A}}(x) * f_{\hat{B}}(x) = \int f_{\hat{A}}(x') f_{\hat{B}}(x'') \text{Tr} \left(\hat{U}(x') \hat{U}(x'') \hat{V}(x) \right) d^n x' d^n x''. \quad (2.1.12)$$

Therefore, we can rewritten the star product as

$$f_{\hat{A}}(x) * f_{\hat{B}}(x) = \int K(x', x'', x) f_{\hat{A}}(x') f_{\hat{B}}(x'') d^n x' d^n x'' \quad (2.1.13)$$

where the kernel is given by

$$K(x', x'', x) = \text{Tr} \left(\hat{U}(x') \hat{U}(x'') \hat{V}(x) \right). \quad (2.1.14)$$

Note that the usual product is also of this kind for

$$K(x', x'', x) = \delta^{(n)}(x' - x) \delta^{(n)}(x'' - x). \quad (2.1.15)$$

Moreover, the expression (2.1.14) is quadratic with respect to $\hat{U}(x)$ and linear with respect to $\hat{V}(x)$ and then there is an asymmetry in the kernel with

respect to quantizers and dequantizers. Furthermore, thanks to the cyclical property of the trace, the kernel and then the star product is invariant under the transformation

$$\hat{U}'(x) = \hat{S}\hat{U}(x)\hat{S}^{-1} \quad (2.1.16)$$

$$\hat{V}'(x) = \hat{S}\hat{V}(x)\hat{S}^{-1} \quad (2.1.17)$$

where \hat{S} is an invertible operator. Finally, the associativity condition for the operator symbols imposes a strong constrain on the kernel $K(x', x'', x)$. Indeed, for associativity

$$\begin{aligned} (f_{\hat{A}}(x) \star f_{\hat{B}}(x)) \star f_{\hat{C}}(x) &= \int K(x_1, x_2, x)(f_{\hat{A}}(x_1) \star f_{\hat{B}}(x_1))f_{\hat{C}}(x_2)d^n x_1 d^n x_2 \\ &= \int K(x_1, x_2, x)K(x_3, x_4, x_1)f_{\hat{A}}(x_3)f_{\hat{B}}(x_4)f_{\hat{C}}(x_2)d^n x_1 d^n x_2 d^n x_3 d^n x_4 \end{aligned} \quad (2.1.18)$$

must be equal to

$$\begin{aligned} f_{\hat{A}}(x) \star (f_{\hat{B}}(x) \star f_{\hat{C}}(x)) &= \int K(x_1, x_2, x)f_{\hat{A}}(x_1)(f_{\hat{B}}(x_2) \star f_{\hat{C}}(x_2))d^n x_1 d^n x_2 \\ &= \int K(x_1, x_2, x)K(x_3, x_4, x_2)f_{\hat{A}}(x_1)f_{\hat{B}}(x_3)f_{\hat{C}}(x_4)d^n x_1 d^n x_2 d^n x_3 d^n x_4 \\ &= \int K(x_3, x_1, x)K(x_4, x_2, x_1)f_{\hat{A}}(x_3)f_{\hat{B}}(x_4)f_{\hat{C}}(x_2)d^n x_1 d^n x_2 d^n x_3 d^n x_4. \end{aligned} \quad (2.1.19)$$

Therefore, the kernel $K(x', x'', x)$ must satisfy the equation

$$\int K(x_1, x_2, x)K(x_3, x_4, x_1)d^n x_1 = \int K(x_3, x_1, x)K(x_4, x_2, x_1)d^n x_1 \quad (2.1.20)$$

which is, of course, satisfied by (2.1.14). Observe that this equation has like symmetry a scaling transform. That is, given a solution $K(x', x'', x)$ of the equation (2.1.20),

$$K'(x', x'', x) = \lambda K(x', x'', x) \quad (2.1.21)$$

is still a solution of (2.1.20), where λ is a non-vanishing complex number. Note that the scaling transform of the kernel can be induced transforming the quantizers and dequantizers as

$$\hat{U}'(x) = \lambda \hat{U}(x) \quad (2.1.22)$$

$$\hat{V}'(x) = \lambda^{-1} \hat{V}(x). \quad (2.1.23)$$

In the next section we will introduce the duality symmetry namely we will show that the role of $\hat{U}(x)$ and $\hat{V}(x)$ can be exchanged without violating the consistency of the quantization scheme. Furthermore, we will define dual star products.

2.2 Dual quantization schemes

As we have already said, the duality symmetry is due to the fact that the role of $\hat{U}(x)$ and $\hat{V}(x)$ can be exchanged without violating the consistency of the quantization scheme. The dual quantization scheme [38, 39] is defined by

$$\hat{U}^{(d)}(x) = \hat{V}(x) \quad (2.2.1)$$

$$\hat{V}^{(d)}(x) = \hat{U}(x) \quad (2.2.2)$$

and condition (2.1.1) is obviously satisfied. In the dual scheme, the symbol $f_A^{(d)}(x)$ of an operator \hat{A} on \mathcal{H} , called the dual symbol of the operator \hat{A} , is given by

$$f_{\hat{A}}^{(d)}(x) = \text{Tr} \left(\hat{A} \hat{U}(x) \right) \quad (2.2.3)$$

and the reconstruction formula for the operator \hat{A} is given by

$$\hat{A} = \int f_{\hat{A}}^{(d)}(x) \hat{V}(x) d^n x. \quad (2.2.4)$$

Therefore, in the dual scheme the operators $\hat{U}(x)$ are used to dequantize, while the operators $\hat{V}(x)$ to quantize. In other words, the dual dequantizers correspond to old quantizers, while the dual quantizers correspond to old dequantizers.

In the dual scheme, the star product of the dual symbols $f_{\hat{A}}^{(d)}(x)$ and $f_{\hat{B}}^{(d)}(x)$ of two operators \hat{A} and \hat{B} on \mathcal{H} , called dual star product, is given by

$$f_{\hat{A}}^{(d)}(x) * f_{\hat{B}}^{(d)}(x) = f_{\hat{A}\hat{B}}^{(d)}(x), \quad (2.2.5)$$

that is,

$$f_{\hat{A}}^{(d)}(x) * f_{\hat{B}}^{(d)}(x) = \text{Tr} \left(\hat{A} \hat{B} \hat{U}(x) \right). \quad (2.2.6)$$

Equivalently, we can rewritten the dual star product as

$$f_{\hat{A}}^{(d)}(x) * f_{\hat{B}}^{(d)}(x) = \int K^{(d)}(x', x'', x) f_{\hat{A}}^{(d)}(x') f_{\hat{B}}^{(d)}(x'') d^n x' d^n x'' \quad (2.2.7)$$

where the kernel, called the dual kernel, is given by

$$K^{(d)}(x', x'', x) = \text{Tr} \left(\hat{V}(x') \hat{V}(x'') \hat{U}(x) \right). \quad (2.2.8)$$

The dual kernel $K^{(d)}(x', x'', x)$, unlike the kernel $K(x', x'', x)$, is linear with respect to $\hat{U}(x)$ and quadratic with respect to $\hat{V}(x)$. So, in general, the star product and its dual are different to each other. We will call the quantization scheme self-dual if

$$K^{(d)}(x', x'', x) = K(x', x'', x). \quad (2.2.9)$$

In other words, a quantization scheme is self-dual if it and its dual scheme produce the same star product.

2.3 The Moyal and Wick-Voros products from a quantization scheme

The main quantization scheme known is the Weyl-Wigner quantization scheme. This scheme is self-dual and the star product associated with it is the Moyal product.

In order to introduce this scheme, let us consider a system with a single degree of freedom. In this case, the Hilbert space \mathcal{H} is identified with $L^2(\mathbb{R})$, the Hilbert space of square-integrable functions on \mathbb{R} , and the quantizers and dequantizers [39] are given respectively by

$$\hat{U}(\lambda) = \frac{1}{2\pi} \hat{V}(\lambda) \quad (2.3.1)$$

$$\hat{V}(\lambda) = 2\hat{D}(\lambda)(-1)^{a^\dagger a} \hat{D}(-\lambda) \quad (2.3.2)$$

where $\lambda = x + ip$, with x and p representing respectively the position and momentum, $(-1)^{a^\dagger a}$ is the parity operator, with the annihilation and creation operators a and a^\dagger given respectively by

$$a = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p}) \quad (2.3.3)$$

$$a^\dagger = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p}) \quad (2.3.4)$$

which satisfy the commutation relation (1.1.6) and the displacement operator $\hat{D}(\lambda)$ given by

$$\hat{D}(\lambda) = e^{\lambda a^\dagger - \lambda^* a} \quad (2.3.5)$$

which is unitary and obeys the relation

$$D^\dagger(\lambda) = D^{-1}(\lambda) = D(-\lambda). \quad (2.3.6)$$

As well-known, the displacement operator creates the coherent states namely, the eigenstates of the annihilation operator a , from the vacuum state:

$$D(\lambda)|0\rangle = |\lambda\rangle \quad (2.3.7)$$

for every complex number λ where the vacuum state is as usual defined by

$$a|0\rangle = 0. \quad (2.3.8)$$

It is not difficult to show that [37, 39] the star product associated with the Weyl-Wigner quantization scheme is the Moyal product (1.2.7). Moreover, this scheme is self-dual and from (2.1.2) follows that:

$$f_1(x, p) = 1 \quad (2.3.9)$$

$$f_q(x, p) = x \quad (2.3.10)$$

$$f_p(x, p) = p. \quad (2.3.11)$$

Eventually, consider the following quantization scheme [36, 39] described by the two families of operators

$$\hat{U}_s(\lambda) = \frac{1}{2\pi} \hat{V}_{-s}(\lambda) \quad (2.3.12)$$

$$\hat{V}_s(\lambda) = \frac{2}{1-s} \hat{D}(\lambda) \left(\frac{s+1}{s-1} \right)^{a^\dagger a} \hat{D}(\lambda) \quad (2.3.13)$$

where s is a real parameter. It is possible to show that the Wick-Voros product (1.3.10) is obtained in the limit $s = 1$. Moreover, it is easy to see that the case $s = 0$ corresponds to the Weyl-Wigner scheme described above.

Chapter 3

Non-commutative Moyal field theory

In this chapter we briefly review a non-commutative field theory obtained from a commutative one replacing the ordinary product with the Moyal one. We show that the free case is the same as the ordinary one, but the interacting case does not. In particular, we show that in the interacting case the Moyal product softens the ultraviolet divergence, but it is responsible for the so-called ultraviolet/infrared mixing.

3.1 The Moyal product

For the sake of simplicity, we consider a $(1 + 2)$ -dimensional space-time and consider exclusively spatial non-commutativity. Therefore, the canonical non-commutative relation between the coordinate functions on the space-time takes the form

$$[x^i, x^j] = i\theta^{ij} \quad (3.1.1)$$

where

$$\theta^{ij} = \theta \varepsilon^{ij} \quad \text{with} \quad (\varepsilon^{ij}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (3.1.2)$$

and θ is a real constant parameter of dimensions of a square length which can be seen as a deformation parameter. Consider now the differential form of the Moyal product [35] which is given by

$$f \star_M g = f e^{\frac{i}{2}\theta^{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j} g \quad (3.1.3)$$

where $\overleftarrow{\partial}_i$ and $\overrightarrow{\partial}_j$ act respectively on the left and on the right. More explicitly, it can be written as

$$f \star_M g = f e^{\frac{i}{2}\theta(\overleftarrow{\partial}_1 \overrightarrow{\partial}_2 - \overleftarrow{\partial}_2 \overrightarrow{\partial}_1)} g \quad (3.1.4)$$

which reduces to the commutative product in the limit when θ goes to 0, namely it is a deformation of the commutative one. The Moyal product is associative, but not commutative. In particular, we have

$$x^1 \star_M x^2 = x^1 x^2 + \frac{i}{2}\theta \quad (3.1.5)$$

$$x^2 \star_M x^1 = x^1 x^2 - \frac{i}{2}\theta. \quad (3.1.6)$$

Hence the Moyal bracket of x^1 and x^2 reads

$$[x^1, x^2]_{\star_M} = x^1 \star_M x^2 - x^2 \star_M x^1 = i\theta \quad (3.1.7)$$

in agreement with the canonical non-commutative relation (3.1.1). More in general, the Moyal bracket of two functions¹

$$[f, g]_{\star_M} = f \star_M g - g \star_M f \quad (3.1.8)$$

to first order in θ reads

$$[f, g]_{\star_M} = i\theta\{f, g\} + \dots \quad (3.1.9)$$

where as usual

$$\{f, g\} = (\partial_1 f)\partial_2 g - (\partial_2 f)\partial_1 g. \quad (3.1.10)$$

Thus the Moyal bracket of two functions to first order in the deformation parameter θ is proportional to the Poisson bracket of the two functions.

¹Like every commutator, the Moyal bracket is bilinear and antisymmetric. Moreover, it satisfies the Jacobi identity

$$[f, [g, h]_{\star_M}]_{\star_M} + [g, [h, f]_{\star_M}]_{\star_M} + [h, [f, g]_{\star_M}]_{\star_M} = 0$$

and the Leibniz rule

$$[f, g \star_M h]_{\star_M} = [f, g]_{\star_M} \star_M h + g \star_M [f, h]_{\star_M}.$$

It is very useful to write the Moyal product in momentum space. Since in Fourier transform

$$f(x) = \int \frac{d^3p}{(2\pi)^3} \tilde{f}(p) e^{-ip \cdot x} \quad (3.1.11)$$

from (3.1.3) we have

$$\begin{aligned} (f \star_M g)(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \tilde{f}(p) \tilde{g}(q) e^{-\frac{i}{2}\theta^{ij} p_i q_j} e^{i(p+q) \cdot x} \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \tilde{f}(p) \tilde{g}(q) e^{-\frac{i}{2}\theta \mathbf{p} \wedge \mathbf{q}} e^{i(p+q) \cdot x} \end{aligned} \quad (3.1.12)$$

where we have set

$$\mathbf{p} \wedge \mathbf{q} = \varepsilon^{ij} p_i q_j \quad (3.1.13)$$

which is, of course, antisymmetric for the exchange of \mathbf{p} and \mathbf{q} . Therefore, the Moyal product in momentum space is the standard convolution of Fourier transforms twisted by a phase. For example, we can calculate the Moyal product of two exponentials. Indeed, by using (3.1.12) we easily get

$$\begin{aligned} e^{-ip \cdot x} \star_M e^{-iq \cdot x} &= \int \frac{d^3r}{(2\pi)^3} \frac{d^3s}{(2\pi)^3} \delta^{(3)}(r-p) \delta^{(3)}(s-q) e^{-\frac{i}{2}\theta \mathbf{r} \wedge \mathbf{s}} e^{i(r+s) \cdot x} \\ &= e^{-\frac{i}{2}\theta \mathbf{p} \wedge \mathbf{q}} e^{i(p+q) \cdot x}. \end{aligned} \quad (3.1.14)$$

It is now easy to see that from (3.1.12) follows that the integral of the Moyal product of two functions is equal to the integral of the ordinary product of the two functions. Indeed,

$$\begin{aligned} \int d^3x f \star_M g &= \int d^3x \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \tilde{f}(p) \tilde{g}(q) e^{-\frac{i}{2}\theta \mathbf{p} \wedge \mathbf{q}} e^{i(p+q) \cdot x} \\ &= \int \frac{d^3p}{(2\pi)^3} \tilde{f}(p) \tilde{g}(-p) = \int d^3x d^3y \frac{d^3p}{(2\pi)^3} f(x) g(y) e^{ip \cdot (x-y)} \\ &= \int d^3x f(x) g(x). \end{aligned} \quad (3.1.15)$$

This property is very important in non-commutative field theory since, as we will show in the next section, it allows to state that the free non-commutative field theory with the Moyal product is the same as the commutative one. Finally, note that from (3.1.15) follows that the Moyal product has the trace property

$$\int d^3x f \star_M g = \int d^3x g \star_M f. \quad (3.1.16)$$

3.2 The Moyal field theory

At this point we proceed to the discussion of a non-commutative field theory obtained from a commutative one replacing the ordinary product with the Moyal one. To this end, consider the commutative field theory described by the action

$$S^{(0)} = S_0^{(0)} + S_{\text{int}}^{(0)}, \quad (3.2.1)$$

where $S_0^{(0)}$ is the free Klein-Gordon action given by

$$S_0^{(0)} = \int d^3x \frac{1}{2} (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) \quad (3.2.2)$$

and $S_{\text{int}}^{(0)}$ is the interacting action given by

$$S_{\text{int}}^{(0)} = \frac{g}{4!} \int d^3x \phi^4. \quad (3.2.3)$$

In order to construct a non-commutative field theory, we replace the ordinary product with the Moyal one. This procedure is part of a general framework called deformation quantization which consists in a modification of a theory in such a way that it reduces to the undeformed one in a certain limit. So the free non-commutative action is given by

$$S_{0_M} = \int d^3x \frac{1}{2} (\partial_\mu \phi \star_M \partial^\mu \phi - m^2 \phi \star_M \phi) \quad (3.2.4)$$

which, of course, is equal to the ordinary one (3.2.2) because of (3.1.15). Then the free non-commutative field theory with the Moyal product is the same as the commutative one. Instead, the interacting non-commutative action is given by

$$S_{\text{int}_M} = \frac{g}{4!} \int d^3x \phi \star_M \phi \star_M \phi \star_M \phi \quad (3.2.5)$$

which is different from the commutative one.

We now move on to the quantum case, we calculate the Green's functions up to one-loop for the two- and four-point cases and discuss the ultraviolet behaviour of the theory. Since the free non-commutative field theory with the Moyal product is the same as the commutative one, the propagator that is, the two-point Green's function is as the usual one

$$\tilde{G}_M^{(2)}(p) = \frac{1}{p^2 - m^2}. \quad (3.2.6)$$

To calculate the four-point Green's function to the tree level in the Moyal case, we first have to determine the vertex. In this case, the vertex is different with respect to the usual one since it acquires a phase [40]. To determine the vertex, let us write down the interacting action (3.2.5) in momentum space. By using the relations (3.1.12) and (3.1.15) we have

$$\begin{aligned}
S_{\text{int}_M} &= \frac{g}{4!} \int dx \frac{dk_1}{(2\pi)^3} \frac{dk_2}{(2\pi)^3} \frac{dk_3}{(2\pi)^3} \frac{dk_4}{(2\pi)^3} \tilde{\phi}(k_1) \tilde{\phi}(k_2) \tilde{\phi}(k_3) \tilde{\phi}(k_4) \\
&\quad e^{-\frac{i}{2}\theta(\mathbf{k}_1 \wedge \mathbf{k}_2 + \mathbf{k}_3 \wedge \mathbf{k}_4)} e^{i(k_1 + k_2 + k_3 + k_4) \cdot x} \\
&= \frac{g}{4!} (2\pi)^3 \int \frac{dk_1}{(2\pi)^3} \frac{dk_2}{(2\pi)^3} \frac{dk_3}{(2\pi)^3} \frac{dk_4}{(2\pi)^3} \tilde{\phi}(k_1) \tilde{\phi}(k_2) \tilde{\phi}(k_3) \tilde{\phi}(k_4) \\
&\quad e^{-\frac{i}{2}\theta[\mathbf{k}_1 \wedge \mathbf{k}_2 + \mathbf{k}_3 \wedge \mathbf{k}_4 + (\mathbf{k}_1 + \mathbf{k}_2) \wedge (\mathbf{k}_3 + \mathbf{k}_4)]} \delta^{(3)}(k_1 + k_2 + k_3 + k_4) \quad (3.2.7)
\end{aligned}$$

which can be rewritten as

$$S_{\text{int}_M} = i \int \prod_{a=1}^4 \frac{d^3 k_a}{(2\pi)^3} \tilde{\phi}(k_a) V_{\star_M} \quad (3.2.8)$$

where

$$V_{\star_M} = V e^{\sum_{a<b} -\frac{i}{2}\theta \mathbf{k}_a \wedge \mathbf{k}_b} = V e^{\sum_{a<b} -\frac{i}{2}\theta^{ij} k_{ai} k_{bj}} \quad (3.2.9)$$

is the Moyal vertex and

$$V = -i \frac{g}{4!} (2\pi)^3 \delta^{(3)} \left(\sum_{a=1}^4 k_a \right) \quad (3.2.10)$$

is the usual vertex which is proportional to the coupling constant multiplying the δ of momentum conservation. Note that the presence of the phase in the vertex (3.2.9) makes it non-invariant for a generic exchange of the momenta. This is a consequence of non-commutativity and of the fact that the integral of Moyal product of more than two functions is not invariant for an exchange of the functions. However, it is invariant for a cyclic exchange of the factors. Eventually, to determine the four-point Green's function to the tree level, we must attach to the vertex (3.2.9) four propagators (3.2.6). We have

$$\tilde{G}_M^{(4)} = -ig(2\pi)^3 \frac{e^{\sum_{a<b} -\frac{i}{2}\theta \mathbf{k}_a \wedge \mathbf{k}_b}}{\prod_{a=1}^4 (k_a^2 - m^2)} \delta^{(3)} \left(\sum_{a=1}^4 k_a \right) \quad (3.2.11)$$

which is different with respect to the usual one.

3.3 UV/IR mixing for the Moyal product

In order to discuss the ultraviolet behaviour of the theory in the Moyal case, we calculate the one-loop corrections to the two- and four-point Green's functions. Because the vertex (3.2.9) is not invariant for a generic exchange of the momenta, we must consider both the planar and non-planar diagrams in the calculation of the corrections. To obtain the one-loop corrections to the propagator, consider first the planar case in figure 3.1(a). The amplitude is obtained using three propagators (3.2.6), two with momentum p , one with momentum q , and the vertex (3.2.9) with assignments

$$k_1 = -k_4 = p \quad \text{and} \quad k_2 = -k_3 = q \quad (3.3.1)$$

and the proper symmetry factor [41]. We have

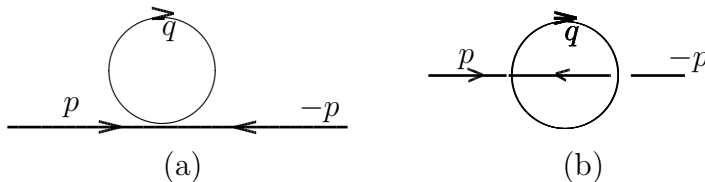


Figure 3.1: The planar (a) and non-planar (b) one-loop two-point diagrams.

$$\tilde{G}_{\text{P}}^{(2)} = -i\frac{g}{3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{(p^2 - m^2)^2(q^2 - m^2)} \quad (3.3.2)$$

which is the same as the ordinary one. Consider now the non-planar case in figure 3.1(b). The structure is the same as in the planar case, but this time the assignments are

$$k_1 = -k_3 = p \quad \text{and} \quad k_2 = -k_4 = q. \quad (3.3.3)$$

We have

$$\tilde{G}_{\text{NP}}^{(2)} = -i\frac{g}{6} \int \frac{d^3q}{(2\pi)^3} \frac{e^{-i\theta\mathbf{p}\wedge\mathbf{q}}}{(p^2 - m^2)^2(q^2 - m^2)}. \quad (3.3.4)$$

Therefore, the q -contribution does not cancel completely and the oscillating factor in the integral softens the ultraviolet divergence because it dampens the functions for high q . However, it is responsible for the infrared divergence. Notice that the persistence of some divergences is more general than the

present calculation and was noted in [42] in the general framework of Connes' non-commutative geometry, while in [43] it is shown that not all divergences can be eliminated in the presence of the commutation relation (3.1.1).

Finally, to get the one-loop correction to the four-point Green's function, consider first the planar case of figure 3.2. The one-loop correction to the four-point Green's function can easily be calculated by properly joining two vertices (3.2.9). We have

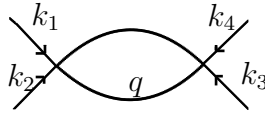


Figure 3.2: The planar one-loop four-point diagram.

$$\tilde{G}_P^{(4)} = \frac{(-ig)^2}{8} (2\pi)^3 \int \frac{d^3q}{(2\pi)^3} \frac{e^{\sum_{a<b} -\frac{i}{2}\theta \mathbf{k}_a \wedge \mathbf{k}_b} \delta^{(3)}(\sum_{a=1}^4 k_a)}{(q^2 - m^2) [(k_1 + k_2 - q)^2 - m^2] \prod_{a=1}^4 (k_a^2 - m^2)}. \quad (3.3.5)$$

Therefore, the internal momentum q appears only in the denominator so that also in this case the planar diagram has the same ultraviolet behaviour as the ordinary one. Consider now the non-planar diagrams shown in figure 3.3. We have

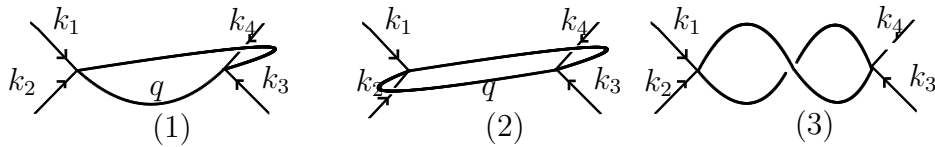


Figure 3.3: The non-planar one-loop four-point diagrams.

$$\tilde{G}_{NP_a}^{(4)} = \frac{(-ig)^2}{8} (2\pi)^3 \int \frac{d^3q}{(2\pi)^3} \frac{e^{\sum_{a<b} -\frac{i}{2}\theta \mathbf{k}_a \wedge \mathbf{k}_b + E_a} \delta^{(3)}(\sum_{a=1}^4 k_a)}{(q^2 - m^2) [(k_1 + k_2 - q)^2 - m^2] \prod_{a=1}^4 (k_a^2 - m^2)} \quad (3.3.6)$$

with

$$\begin{aligned} E_1 &= i\theta \mathbf{k}_1 \wedge \mathbf{q} \\ E_2 &= -i\theta (\mathbf{k}_2 \wedge \mathbf{q} + \mathbf{k}_3 \wedge \mathbf{q}) \\ E_3 &= -i\theta (\mathbf{k}_1 \wedge \mathbf{q} + \mathbf{k}_2 \wedge \mathbf{q}). \end{aligned} \tag{3.3.7}$$

So in non-planar cases the one-loop correction to the two- and four-point Green's function have the same oscillating factor and then in both cases hold similar considerations. In the next chapter we will introduce the Wick-Voros product that is, a variant of the more studied Moyal product and compare the Wick-Voros field theory with the Moyal one.

Chapter 4

Non-commutative Wick-Voros field theory

In this chapter we describe another non-commutative field theory obtained from the same commutative one of the previous chapter replacing the ordinary product with the Wick-Voros one. We show that the free case is the same as the Moyal one (and the ordinary one) while the interacting case is different and, in fact, we find different Green's functions. However, the interacting case present the same kind of ultraviolet/infrared mixing as the Moyal one.

4.1 The Wick-Voros product

Let us consider the differential form of Wick-Voros product which is given by

$$f \star_V g = f e^{\frac{i}{2}\theta \left[\overleftarrow{\partial}_1 \overleftarrow{\partial}_2 - \overleftarrow{\partial}_2 \overleftarrow{\partial}_1 - i \left(\overleftarrow{\partial}_1 \overleftarrow{\partial}_1 + \overleftarrow{\partial}_2 \overleftarrow{\partial}_2 \right) \right]} g \quad (4.1.1)$$

where θ is still a real constant parameter of dimensions of a square length. In this case is more natural working with complex coordinates than real ones and in complex coordinates (1.3.4) the Wick-Voros product takes the form

$$f \star_V g = f e^{\theta \overleftarrow{\partial}_+ \overleftarrow{\partial}_-} g \quad (4.1.2)$$

where

$$\partial_{\pm} = \frac{\partial}{\partial x^{\pm}} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x^1} \mp i \frac{\partial}{\partial x^2} \right) = \frac{1}{\sqrt{2}} (\partial_1 \mp i \partial_2). \quad (4.1.3)$$

Note that the Moyal product (3.1.3) can be rewritten in these coordinates as

$$f \star_M g = f e^{\frac{\theta}{2} \left(\overleftarrow{\partial}_+ \overrightarrow{\partial}_- - \overleftarrow{\partial}_- \overrightarrow{\partial}_+ \right)} g. \quad (4.1.4)$$

Moreover, the Laplacian is

$$\nabla^2 = 2\partial_+ \partial_- \quad (4.1.5)$$

and the d'Alembertian is as usual

$$\square = \partial_0^2 - \nabla^2. \quad (4.1.6)$$

The Wick-Voros product is associative and non-commutative. In particular, we have

$$x^+ \star_V x^- = x^+ x^- + \theta \quad (4.1.7)$$

$$x^- \star_V x^+ = x^+ x^-. \quad (4.1.8)$$

Hence the Wick-Voros bracket of x^+ and x^- reads

$$[x^+, x^-]_{\star_V} = x^+ \star_V x^- - x^- \star_V x^+ = \theta \quad (4.1.9)$$

which gives the canonical non-commutative relation (3.1.1) going back to the x 's. It is very easy to see that the Wick-Voros bracket, like the Moyal one, of two functions:

$$[f, g]_{\star_V} = f \star_V g - g \star_V f \quad (4.1.10)$$

is proportional to the Poisson bracket of the two functions to first order in θ :

$$[f, g]_{\star_V} = i\theta \{f, g\} + \dots \quad (4.1.11)$$

Therefore, both products are a deformation of the ordinary one which reduce to the ordinary one in the limit $\theta \rightarrow 0$ and reproduce to first order in the deformation parameter θ the Poisson structure.

Even in the Wick-Voros case, it is very useful to write the product in momentum space. Since in complex coordinates the Fourier transform can be expressed as

$$f(x) = \int \frac{d^3 p}{(2\pi)^3} \tilde{f}(p) e^{-i(p_+ x^- + p_- x^+)} \quad (4.1.12)$$

with

$$p_{\pm} = \frac{p_1 \pm ip_2}{\sqrt{2}} \quad (4.1.13)$$

from (4.1.2) we have

$$\begin{aligned} (f \star_V g)(x) &= \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \tilde{f}(p) \tilde{g}(q) e^{-\theta p - q_+} e^{i(p+q) \cdot x} \\ &= \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \tilde{f}(p) \tilde{g}(q) e^{-\frac{\theta}{2}(\mathbf{p} \cdot \mathbf{q} + i\mathbf{p} \wedge \mathbf{q})} e^{i(p+q) \cdot x} \end{aligned} \quad (4.1.14)$$

since¹

$$p - q_+ = \frac{1}{2}(\mathbf{p} \cdot \mathbf{q} + i\mathbf{p} \wedge \mathbf{q}). \quad (4.1.15)$$

So the Wick-Voros product in momentum space is the standard convolution of Fourier transforms twisted by a factor which is not just a phase like in the Moyal case, but a phase multiplied by a real exponential. Furthermore, from (4.1.14) follows that

$$\begin{aligned} \int d^3 x f \star_V g &= \int d^3 x \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \tilde{f}(p) \tilde{g}(q) e^{-\theta p - q_+} e^{i(p+q) \cdot x} \\ &= \int \frac{d^3 p}{(2\pi)^3} \tilde{f}(p) \tilde{g}(-p) e^{\theta p - p_+} = \int \frac{d^3 p}{(2\pi)^3} \tilde{f}(p) \tilde{g}(-p) e^{\theta \mathbf{p}^2}. \end{aligned} \quad (4.1.16)$$

Therefore, unlike for the Moyal case, the integral of the Wick-Voros product of two functions is not equal to the integral of the ordinary product of the two functions

$$\int d^3 x f \star_V g \neq \int d^3 x fg. \quad (4.1.17)$$

This has a precise interpretation in non-commutative field theory. It means that also the free non-commutative field theory with the Wick-Voros product could be different from the ordinary one. However, it has the trace property

$$\int d^3 x f \star_V g = \int d^3 x g \star_V f. \quad (4.1.18)$$

Notice that the integral of the Wick-Voros product of two functions can be written as well as

$$\begin{aligned} \int d^3 x f \star_V g &= \int d^3 x \sum_{n=0}^{\infty} \frac{\theta^n}{n!} (\partial_+^n f)(\partial_-^n g) = \int d^3 x \sum_{n=0}^{\infty} \frac{(-\theta)^n}{n!} f \partial_+^n \partial_-^n g = \\ &= \int d^3 x f e^{-\theta \partial_+ \partial_-} g = \int d^3 x f e^{-\frac{\theta}{2} \nabla^2} g \end{aligned} \quad (4.1.19)$$

¹In complex coordinates

$$\mathbf{p} \cdot \mathbf{q} = p_- q_+ + q_- p_+.$$

where we have integrated by parts and neglected all the boundary terms.

We conclude this section by noting that at the algebraic level the Moyal and Wick-Voros products are equivalent in sense that they define the same deformed algebra, namely there exists an invertible map [44, 45] T such that

$$T(f \star_M g) = T(f) \star_V T(g) \quad (4.1.20)$$

where

$$T = e^{\frac{\theta}{4}\nabla^2}. \quad (4.1.21)$$

Note that from invertibility of T follows that

$$T^{-1}(f \star_V g) = T^{-1}(f) \star_M T^{-1}(g). \quad (4.1.22)$$

We can easily show that the Moyal and Wick-Voros products are algebraically equivalent. In fact, by using (4.1.14) the left-hand side of (4.1.20) can be written as

$$\begin{aligned} T(f \star_M g) &= e^{\frac{\theta}{4}\nabla^2} \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \tilde{f}(p) \tilde{g}(q) e^{-\frac{i}{2}\theta\mathbf{p}\wedge\mathbf{q}} e^{i(p+q)\cdot x} \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \tilde{f}(p) \tilde{g}(q) e^{-\frac{\theta}{4}(\mathbf{p}+\mathbf{q})^2} e^{-\frac{i}{2}\theta\mathbf{p}\wedge\mathbf{q}} e^{i(p+q)\cdot x} \end{aligned} \quad (4.1.23)$$

and the right-hand side of (4.1.20) can be written in the same way

$$\begin{aligned} T(f) \star_V T(g) &= e^{\frac{\theta}{4}\nabla^2} \int \frac{d^3p}{(2\pi)^3} \tilde{f}(p) e^{-ip\cdot x} \star_V e^{\frac{\theta}{4}\nabla^2} \int \frac{d^3q}{(2\pi)^3} \tilde{f}(q) e^{-iq\cdot x} \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \tilde{f}(p) \tilde{g}(q) e^{-\frac{\theta}{4}(\mathbf{p}^2+\mathbf{q}^2)} e^{-ip\cdot x} \star_V e^{-iq\cdot x} \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \tilde{f}(p) \tilde{g}(q) e^{-\frac{\theta}{4}(\mathbf{p}^2+\mathbf{q}^2)} e^{-\frac{\theta}{2}(\mathbf{p}\cdot\mathbf{q}+i\mathbf{p}\wedge\mathbf{q})} e^{i(p+q)\cdot x} \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \tilde{f}(p) \tilde{g}(q) e^{-\frac{\theta}{4}(\mathbf{p}+\mathbf{q})^2} e^{-\frac{i}{2}\theta\mathbf{p}\wedge\mathbf{q}} e^{i(p+q)\cdot x} \end{aligned} \quad (4.1.24)$$

since from (4.1.14) follows immediately that the Wick-Voros product of the two exponentials $e^{-ip\cdot x}$ and $e^{-iq\cdot x}$ is given by

$$\begin{aligned} e^{-ip\cdot x} \star_V e^{-iq\cdot x} &= \int d^3r d^3s \delta^{(3)}(r-p) \delta^{(3)}(s-q) e^{-\frac{\theta}{2}(\mathbf{r}\cdot\mathbf{s}+i\mathbf{r}\wedge\mathbf{s})} e^{i(r+s)\cdot x} \\ &= e^{-\frac{\theta}{2}(\mathbf{p}\cdot\mathbf{q}+i\mathbf{p}\wedge\mathbf{q})} e^{i(p+q)\cdot x}. \end{aligned} \quad (4.1.25)$$

4.2 The Wick-Voros classical field theory

We now study a classical field theory with the Wick-Voros product and we describe both the Lagrangian and the Hamiltonian formalisms.

4.2.1 Lagrangian formalism

At this point we proceed to the discussion of another non-commutative field theory obtained from the commutative one described by the action (3.2.1) substituting the ordinary product with the Wick-Voros one. So the free non-commutative action (as well as the Lagrangian and the Lagrangian density) is given by

$$S_{0_V} = \int dt L_{0_V} = \int d^3x \mathcal{L}_{0_V} = \int d^3x \frac{1}{2} (\partial_\mu \phi \star_V \partial^\mu \phi - m^2 \phi \star_V \phi) \quad (4.2.1)$$

which, unlike the Moyal one, does not reduce to the commutative one and the interacting non-commutative action is given by

$$S_{\text{int}_V} = \frac{g}{4!} \int d^3x \phi \star_V \phi \star_V \phi \star_V \phi. \quad (4.2.2)$$

As we have already seen, the Moyal product and the Wick-Voros one are algebraically equivalent. However, this does not mean that any deformation of an action with the two products are the same. Indeed, mapping the free action with the Wick-Voros product (4.2.1) to the corresponding action with the Moyal one by means of (4.1.21)

$$S_{0_V} \rightarrow \int d^3x T^{-1} \mathcal{L}_{0_V} \quad (4.2.3)$$

we have

$$\begin{aligned} S_{0_V} &\rightarrow \int d^3x \frac{1}{2} \left[\left(e^{-\frac{\theta}{4} \nabla^2} \partial_\mu \phi \right) \star_M \left(e^{-\frac{\theta}{4} \nabla^2} \partial^\mu \phi \right) - m^2 \left(e^{-\frac{\theta}{4} \nabla^2} \phi \right) \star_M \left(e^{-\frac{\theta}{4} \nabla^2} \phi \right) \right] \\ &= \int d^3x \frac{1}{2} \left[\left(e^{-\frac{\theta}{4} \nabla^2} \partial_\mu \phi \right) \left(e^{-\frac{\theta}{4} \nabla^2} \partial^\mu \phi \right) - m^2 \left(e^{-\frac{\theta}{4} \nabla^2} \phi \right) \left(e^{-\frac{\theta}{4} \nabla^2} \phi \right) \right] \\ &= \int d^3x \frac{1}{2} \left(\partial_\mu \phi e^{-\frac{\theta}{2} \nabla^2} \partial^\mu \phi - m^2 \phi e^{-\frac{\theta}{2} \nabla^2} \phi \right) \end{aligned} \quad (4.2.4)$$

which is not the free action with the Moyal product namely the ordinary one.

We begin with the discussion of the free case since the free action (4.2.1) is different from the ordinary one. As well-known, the dynamical behaviour of

a dynamical system is determined by the action principle which affirms that the equation of motion, namely the field equation, is obtained demanding that the variation of the action is vanishing under any infinitesimal variation of the field. So given an infinitesimal variation of ϕ :

$$\phi \rightarrow \phi + \delta\phi \quad (4.2.5)$$

the corresponding infinitesimal variation of the action S_{0_V} is given by

$$\delta S_{0_V} = \int d^3x (\partial_\mu \phi \star_V \partial^\mu \delta\phi - m^2 \phi \star_V \delta\phi). \quad (4.2.6)$$

By integrating by parts we obtain, up to boundary terms,

$$\delta S_{0_V} = - \int d^3x \delta\phi \star_V (\square + m^2) \phi \quad (4.2.7)$$

and using the relation (4.1.19) we have

$$\delta S_{0_V} = - \int d^3x \delta\phi e^{-\frac{\theta}{2}\nabla^2} (\square + m^2) \phi. \quad (4.2.8)$$

Since the variation of the action δS_0 has to vanish for any variation of the field $\delta\phi$, we get the equation of motion

$$e^{-\frac{\theta}{2}\nabla^2} (\square + m^2) \phi = 0 \quad (4.2.9)$$

which is different from the ordinary Klein-Gordon equation given by

$$(\square + m^2) \phi = 0. \quad (4.2.10)$$

However, the two equations of motion have exactly the same solutions due to the invertibility of the operator $e^{-\frac{\theta}{2}\nabla^2}$. Moreover, the on shell condition is not deformed. That is, the dispersion relation is the same as the ordinary one. In fact, in Fourier transform the equation of motion (4.2.9) becomes

$$\begin{aligned} e^{-\frac{\theta}{2}\nabla^2} (\square + m^2) \phi(x) &= e^{-\frac{\theta}{2}\nabla^2} (\square + m^2) \int \frac{d^3k}{(2\pi)^3} \tilde{\phi}(k) e^{-ik \cdot x} \\ &= \int \frac{d^3k}{(2\pi)^3} e^{\frac{\theta}{2}\mathbf{k}^2} (-k^2 + m^2) \tilde{\phi}(k) e^{-ik \cdot x} = 0. \end{aligned} \quad (4.2.11)$$

Hence on shell condition is given by

$$e^{\frac{\theta}{2}\mathbf{k}^2} (k^2 - m^2) \tilde{\phi}(k) = 0 \quad (4.2.12)$$

which reduces to the ordinary one

$$(k^2 - m^2) \tilde{\phi}(k) = 0 \quad (4.2.13)$$

because the exponential never vanishes. Therefore, at classical level the free non-commutative field theory with the Wick-Voros product, like that with the Moyal product, is the same as the commutative one.

4.2.2 Hamiltonian formalism

Before proceeding further in our analysis, we investigate the Hamiltonian formalism in the free case. To begin with, let us find the field conjugate to ϕ . As well-known, it is defined by

$$\pi = \frac{\delta L_{0_V}}{\delta \dot{\phi}} \quad (4.2.14)$$

where we have set $\dot{\phi} = \partial_0 \phi$. To determine the functional derivative of the Lagrangian L_{0_V} with respect $\dot{\phi}$, consider an infinitesimal variation of $\dot{\phi}$:

$$\dot{\phi} \rightarrow \dot{\phi} + \delta \dot{\phi}. \quad (4.2.15)$$

The corresponding infinitesimal variation of the Lagrangian L_{0_V} is given by

$$\delta L_{0_V} = \int d^2x \delta \dot{\phi} \star_V \dot{\phi} \quad (4.2.16)$$

and using the relation (4.1.19) we have

$$\delta L_{0_V} = \int d^2x \delta \dot{\phi} e^{-\frac{\theta}{2} \nabla^2} \dot{\phi}. \quad (4.2.17)$$

So the field conjugate to ϕ is given by

$$\pi = e^{-\frac{\theta}{2} \nabla^2} \dot{\phi}. \quad (4.2.18)$$

We now assume that the free non-commutative Hamiltonian in the Wick-Voros case is given by

$$H_{0_V} = \int d^2x \mathcal{H}_{0_V} = \int d^2x (\pi \dot{\phi} - \mathcal{L}_{0_V}) \quad (4.2.19)$$

expressing $\dot{\phi}$ as a function of the conjugate field π . The choice of Hamiltonian can be justified thinking of our theory as just as a theory with infinitely many

numbers of derivatives without any consideration about non-commutative geometry. Therefore, by using the relation (4.1.19) the free Hamiltonian can be written as

$$\begin{aligned} H_{0_V} &= \int d^2x \left[\pi e^{\frac{\theta}{2}\nabla^2} \pi - \frac{1}{2} \left(e^{\frac{\theta}{2}\nabla^2} \pi \star_V e^{\frac{\theta}{2}\nabla^2} \pi + \partial_i \phi \star_V \partial^i \phi - m^2 \phi \star_V \phi \right) \right] \\ &= \int d^2x \frac{1}{2} \left(\pi e^{\frac{\theta}{2}\nabla^2} \pi - \partial_i \phi \star_V \partial^i \phi + m^2 \phi \star_V \phi \right) \end{aligned} \quad (4.2.20)$$

which is different from the one that we would write if we started directly from Hamiltonian formalism rather Lagrangian one². We now are ready to derive the Hamilton equations:

$$\dot{\phi} = \frac{\delta H}{\delta \pi} \quad (4.2.21)$$

$$\dot{\pi} = -\frac{\delta H}{\delta \phi}. \quad (4.2.22)$$

To evaluate the time evolution of ϕ , consider an infinitesimal variation of π :

$$\pi \rightarrow \pi + \delta\pi. \quad (4.2.23)$$

The corresponding infinitesimal variation of the Hamiltonian is given by

$$\delta H_{0_V} = \int d^2x \frac{1}{2} \left(\pi e^{\frac{\theta}{2}\nabla^2} \delta\pi + \delta\pi e^{\frac{\theta}{2}\nabla^2} \pi \right). \quad (4.2.24)$$

That is,

$$\delta H_{0_V} = \int d^2x \delta\pi e^{\frac{\theta}{2}\nabla^2} \pi \quad (4.2.25)$$

where we have integrated by parts and neglected all the boundary terms. So the time evolution of ϕ is given by

$$\dot{\phi} = e^{\frac{\theta}{2}\nabla^2} \pi \quad (4.2.26)$$

which is, of course, consistent with (4.2.18). Instead to calculate the time evolution of π , consider an infinitesimal variation of ϕ :

$$\phi \rightarrow \phi + \delta\phi. \quad (4.2.27)$$

²Indeed, in such a case we would write the free non-commutative Hamiltonian as

$$H_{0_V} = \int d^2x \frac{1}{2} \left(\pi \star_V \pi - \partial_i \phi \star_V \partial^i \phi + m^2 \phi \star_V \phi \right)$$

where the field conjugate to ϕ is easily seen to be $\pi = e^{\frac{\theta}{2}\nabla^2} \dot{\phi}$.

The corresponding infinitesimal variation of the Hamiltonian is given by

$$\delta H_{0_V} = - \int d^2x (\partial_i \phi \star_V \partial^i \delta \phi - m^2 \phi \star_V \delta \phi) \quad (4.2.28)$$

By integrating by parts we get, up to boundary terms,

$$\delta H_{0_V} = - \int d^2x \delta \phi \star_V (\nabla^2 - m^2) \phi \quad (4.2.29)$$

and using the relation (4.1.19) we have

$$\delta H_{0_V} = - \int d^2x \delta \phi e^{-\frac{\theta}{2} \nabla^2} (\nabla^2 - m^2) \phi. \quad (4.2.30)$$

So the time evolution of π is given by

$$\dot{\pi} = e^{-\frac{\theta}{2} \nabla^2} (\nabla^2 - m^2) \phi. \quad (4.2.31)$$

It is now easy to show that combining (4.2.26) with (4.2.31) we obtain the ordinary equations of motion for ϕ and π fields. This result confirms that at classical level the free non-commutative field theory with the Wick-Voros product is the same as the commutative one.

4.3 The Wick-Voros quantum field theory

We now move on to the quantum case and proceed to the determination of Green's functions in the Wick-Voros case. To calculate the propagator, we can start from its general definition:

$$e^{-\frac{\theta}{2} \nabla^2} (\square + m^2) G_V^{(2)}(x - x') = -\delta^{(3)}(x - x'). \quad (4.3.1)$$

In Fourier transform it becomes

$$\begin{aligned} e^{-\frac{\theta}{2} \nabla^2} (\square + m^2) G_V^{(2)}(x - x') &= e^{-\frac{\theta}{2} \nabla^2} (\square + m^2) \int \frac{d^3p}{(2\pi)^3} \tilde{G}_V^{(2)}(p) e^{-ip \cdot (x - x')} \\ &= \int \frac{d^3p}{(2\pi)^3} e^{\frac{\theta}{2} \mathbf{p}^2} (-p^2 + m^2) \tilde{G}_V^{(2)}(p) e^{-ip \cdot (x - x')} \\ &= - \int \frac{d^3p}{(2\pi)^3} e^{-ip \cdot (x - x')} \end{aligned} \quad (4.3.2)$$

from which follows that

$$e^{\frac{\theta}{2} \mathbf{p}^2} (-p^2 + m^2) \tilde{G}_V^{(2)}(p) = -1 \quad (4.3.3)$$

and then

$$\tilde{G}_V^{(2)}(p) = \frac{e^{-\frac{\theta}{2}p^2}}{p^2 - m^2}. \quad (4.3.4)$$

Note that to get the propagator, we can proceed as well as follows. The free action (4.2.1) can be written as

$$\begin{aligned} S_{0_V} &= \int d^3x \frac{1}{2} \left(\partial_\mu \phi e^{-\frac{\theta}{2}\nabla^2} \partial_\mu \phi - m^2 \phi e^{-\frac{\theta}{2}\nabla^2} \phi \right) \\ &= \int d^3x \frac{1}{2} \phi e^{-\frac{\theta}{2}\nabla^2} (-\partial_\mu^2 - m^2) \phi \end{aligned} \quad (4.3.5)$$

where we have used (4.1.19), integrated by parts and neglected the boundary terms. So it can be rewritten as

$$S_{0_V} = \int d^3x d^3x' \phi(x) K(x, x') \phi(x') \quad (4.3.6)$$

with

$$K(x, x') = e^{-\frac{\theta}{2}\nabla^2} (-\partial_\mu^2 - m^2) \delta^{(3)}(x - x') \quad (4.3.7)$$

or equivalently

$$\begin{aligned} K(x, x') &= e^{-\frac{\theta}{2}\nabla^2} (-\partial_\mu^2 - m^2) \int \frac{d^3p}{(2\pi)^3} e^{-ip \cdot (x-x')} \\ &= \int \frac{d^3p}{(2\pi)^3} e^{\frac{\theta}{2}p^2} (p^2 - m^2) e^{-ip \cdot (x-x')}. \end{aligned} \quad (4.3.8)$$

Therefore, the propagator, the inverse of the operator $K(x, x')$, is

$$G_V^{(2)}(x, x') = \int \frac{d^3p}{(2\pi)^3} \frac{e^{-\frac{\theta}{2}p^2}}{p^2 - m^2} e^{-ip \cdot (x-x')}. \quad (4.3.9)$$

from which we can read off the propagator in momentum space (4.3.4). Since the poles in the propagator (4.3.4) are the same as in the Moyal and ordinary cases (3.2.6), the free field theory in the Wick-Voros case is the same as in the two cases at the quantum level as well. Nevertheless, the two propagators are not identical. Moreover, for infinite momentum there is an essential singularity or a zero of the propagator according to the sign of θ . The meaning of the essential singularity is not clear, but the oddity is that the sign of θ has no physical meaning since it can be changed by the exchange of the two coordinates.

To calculate the four-point Green's function to the tree level in the Wick-Voros case, we first must determine the vertex. To this end, let us write down the interacting action (4.2.2) in momentum space. By using the relations (4.1.14) and (4.1.16) we have

$$\begin{aligned}
S_{\text{int}_V} &= \frac{g}{4!} \int dx \frac{dk_1}{(2\pi)^3} \frac{dk_2}{(2\pi)^3} \frac{dk_3}{(2\pi)^3} \frac{dk_4}{(2\pi)^3} \tilde{\phi}(k_1) \tilde{\phi}(k_2) \tilde{\phi}(k_3) \tilde{\phi}(k_4) \\
&\quad e^{-\theta(k_1 - k_2 + k_3 - k_4 +)} e^{i(k_1 + k_2) \cdot x} \star_V e^{i(k_3 + k_4) \cdot x} \\
&= \frac{g}{4!} (2\pi)^3 \int \frac{dk_1}{(2\pi)^3} \frac{dk_2}{(2\pi)^3} \frac{dk_3}{(2\pi)^3} \frac{dk_4}{(2\pi)^3} \tilde{\phi}(k_1) \tilde{\phi}(k_2) \tilde{\phi}(k_3) \tilde{\phi}(k_4) \\
&\quad e^{-\theta(k_1 - k_2 + k_3 - k_4 - k - k_+)} \delta^{(3)}(k_1 + k_2 + k) \delta^{(3)}(k_3 + k_4 - k) \\
&= \frac{g}{4!} (2\pi)^3 \int \frac{dk_1}{(2\pi)^3} \frac{dk_2}{(2\pi)^3} \frac{dk_3}{(2\pi)^3} \frac{dk_4}{(2\pi)^3} \tilde{\phi}(k_1) \tilde{\phi}(k_2) \tilde{\phi}(k_3) \tilde{\phi}(k_4) \\
&\quad e^{-\theta[k_1 - k_2 + k_3 - k_4 + (k_1 - + k_2 -)(k_3 + + k_4 +)]} \delta^{(3)}(k_1 + k_2 + k_3 + k_4) \quad (4.3.10)
\end{aligned}$$

where

$$V_{\star_V} = V e^{\sum_{a < b} -\theta k_a - k_{b+}} = V e^{\sum_{a < b} -\frac{\theta}{2} (\mathbf{k}_a \cdot \mathbf{k}_b + i \mathbf{k}_a \wedge \mathbf{k}_b)} \quad (4.3.11)$$

is the Wick-Voros vertex and V is the ordinary vertex (3.2.10). To calculate the four-point Green's function at the tree level, we have to attach to the vertex (4.3.11) four propagators (4.3.4). We have

$$\begin{aligned}
\tilde{G}_V^{(4)} &= -ig(2\pi)^3 \frac{e^{-\theta(\sum_{a=1}^4 k_a - k_{a+} + \sum_{a < b} k_a - k_{b+})}}{\prod_{a=1}^4 (k_a^2 - m^2)} \delta^{(3)} \left(\sum_{a=1}^4 k_a \right) \\
&= -ig(2\pi)^3 \frac{e^{-\frac{\theta}{2} [\sum_{a=1}^4 \mathbf{k}_a^2 + \sum_{a < b} (\mathbf{k}_a \cdot \mathbf{k}_b + i \mathbf{k}_a \wedge \mathbf{k}_b)]}}{\prod_{a=1}^4 (k_a^2 - m^2)} \delta^{(3)} \left(\sum_{a=1}^4 k_a \right) \\
&= -ig(2\pi)^3 \frac{e^{-\frac{\theta}{4} [\sum_{a=1}^4 \mathbf{k}_a^2 + (\sum_{a=1}^4 \mathbf{k}_a)^2 + 2i \sum_{a < b} \mathbf{k}_a \wedge \mathbf{k}_b]}}{\prod_{a=1}^4 (k_a^2 - m^2)} \delta^{(3)} \left(\sum_{a=1}^4 k_a \right) \\
&= -ig(2\pi)^3 \frac{e^{-\frac{\theta}{4} (\sum_{a=1}^4 \mathbf{k}_a^2 + 2i \sum_{a < b} \mathbf{k}_a \wedge \mathbf{k}_b)}}{\prod_{a=1}^4 (k_a^2 - m^2)} \delta^{(3)} \left(\sum_{a=1}^4 k_a \right) \quad (4.3.12)
\end{aligned}$$

since the δ of conservation of momentum kills the mid term in the exponential. The presence of a real exponent in the Green's functions signifies that the ultraviolet behaviour of the theory could be different from the Moyal and ordinary ones. In order to investigate ultraviolet behaviour of the theory in the Wick-Voros case, we will calculate in the next section the one-loop correction to the two- and four-point Green's functions.

4.4 UV/IR mixing for the Wick-Voros product

We now proceed to the calculation of the one-loop corrections to the two- and four-point Green's functions in the Wick-Voros case. In order to get the one-loop corrections to the propagator, consider first the planar case in figure 3.1(a). The amplitude is then obtained using three propagators (4.3.4), two with momentum p , one with momentum q , the vertex (4.3.11) with assignments (3.3.1) and the proper symmetry factor. We have

$$\begin{aligned}\tilde{G}_P^{(2)} &= -i\frac{g}{3} \int \frac{d^3q}{(2\pi)^3} \frac{e^{-\theta(2p-p_++q-q_+)} e^{-\theta(p-q_+-p-q_+-p-p_+-q-q_+-q-p_++q-p_+)}}{(p^2 - m^2)^2(q^2 - m^2)} \\ &= -i\frac{g}{3} \int \frac{d^3q}{(2\pi)^3} \frac{e^{-\theta p-p_+}}{(p^2 - m^2)^2(q^2 - m^2)}\end{aligned}\quad (4.4.1)$$

where the first exponential is due to the propagators and the second one to the vertex. Therefore, the q -contribution cancels completely that is, there is no change in the convergence of the integral with respect the ordinary case. Consider now the non-planar case in figure 3.1(b). The structure is the same as in the planar case, but with assignments given by (3.3.3). We have

$$\begin{aligned}\tilde{G}_{NP}^{(2)} &= -i\frac{g}{6} \int \frac{d^3q}{(2\pi)^3} \frac{e^{-\theta(2p-p_++q-q_+)} e^{-\theta(p-q_+-p-p_+-p-q_+-q-p_+-q-q_++p-q_+)}}{(p^2 - m^2)^2(q^2 - m^2)} \\ &= -i\frac{g}{6} \int \frac{d^3q}{(2\pi)^3} \frac{e^{-\theta(p-p_++i\mathbf{p}\wedge\mathbf{q})}}{(p^2 - m^2)^2(q^2 - m^2)}\end{aligned}$$

since

$$p-q_+ - q-p_+ = i\mathbf{p} \wedge \mathbf{q}.\quad (4.4.2)$$

Therefore, the q -contribution does not cancel completely. We can conclude that the ultraviolet divergence of the planar diagram in both the Moyal and Wick-Voros cases is unchanged with respect to the ordinary one. Instead, in the non-planar diagram the presence of the oscillating factor in the integral softens the ultraviolet divergence, because it dampens the functions for high values of q . However, it is responsible for infrared divergences. Therefore, the ultraviolet behaviour in both cases is exactly the same and this can be seen like a consequence of the canonical non-commutative relation (3.1.1) which is unchanged between the Moyal and Wick-Voros cases. Nevertheless, the Green's functions for the two cases are not the same.

Finally, to get the one-loop correction to the four-point Green's function, consider first the planar case of figure 3.2. We have

$$\tilde{G}_P^{(4)} = \frac{(-ig)^2}{8} (2\pi)^3 \int \frac{d^3q}{(2\pi)^3} \frac{e^{-\theta(\sum_{a=1}^4 k_a - k_{a+} + \sum_{a < b} k_a - k_{b+})} \delta^{(3)}(\sum_{a=1}^4 k_a)}{(q^2 - m^2) [(k_1 + k_2 - q)^2 - m^2] \prod_{a=1}^4 (k_a^2 - m^2)} \quad (4.4.3)$$

Therefore, the internal momentum q appears only in the denominator, like in the Moyal case, that is, the planar diagram has the same ultraviolet behaviour as the ordinary one. In the determination of the one-loop correction to the four-point Green's function we must consider the non-planar diagrams shown in figure 3.3 as well. We have

$$\tilde{G}_{\text{NP}_a}^{(4)} = \frac{(-ig)^2}{8} (2\pi)^3 \int \frac{d^3q}{(2\pi)^3} \frac{e^{-\theta(\sum_{a=1}^4 k_a - k_{a+} + \sum_{a < b} k_a - k_{b+} + E_a)} \delta^{(3)}(\sum_{a=1}^4 k_a)}{(q^2 - m^2) [(k_1 + k_2 - q)^2 - m^2] \prod_{a=1}^4 (k_a^2 - m^2)} \quad (4.4.4)$$

where E_a 's are still given by (3.3.7). We can conclude that the ultraviolet properties in the Wick-Voros case remain unchanged with respect to the Moyal case and, in particular, the ultraviolet/infrared mixing is unchanged. This is to be expected since heuristically this is consequence of commutation relation which is, of course, the same in both theories. However, the two theories are not equivalent, despite the physical intuition, since the Green's functions differ for a factor can be considered as a momentum dependent coupling constant. In the last chapter we will discuss this problem and show that the Moyal and Wick-Voros field theories describe the same physics at the level of S -matrix and in the framework of twisted non-commutativity. Instead, in the next chapter we will introduce a general translation invariant associative product and show that the ultraviolet/infrared mixing is still the same as for the Moyal and Wick-Voros products.

Chapter 5

UV/IR mixing for a general translation invariant product

In this chapter we introduce a general translation invariant associative product and describe a non-commutative field theory with such a product in order to investigate the relationship between the translation invariance and the ultraviolet/infrared mixing. We show that the ultraviolet/infrared mixing for the Moyal and Wick-Voros products is a generic feature of any translation invariant associative product.

5.1 A general translation invariant product

In this section we introduce a general associative star product¹ between functions on \mathbb{R}^d and then we discuss the condition of translational invariance in order to investigate the relationship between the translation invariance and the ultraviolet/infrared mixing. Notice that we contemplate the possibility that the star product be commutative although in general it is not so.

Consider a generalization of the Moyal product (3.1.12) given by

$$(f \star g)(x) = \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{d^d r}{(2\pi)^d} \tilde{f}(q) \tilde{g}(r) K(p, q, r) e^{ip \cdot x} \quad (5.1.1)$$

where K is in general a distribution. Note that the ordinary product is also of this kind for

$$K(p, q, r) = \delta^{(d)}(r - p + q). \quad (5.1.2)$$

¹General star products were first introduced in [46, 47] in the framework of deformation quantization of Poisson manifolds.

In order to have an associative product, we have to impose

$$\begin{aligned}
((f \star g) \star h)(x) &= \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{d^d r}{(2\pi)^d} \widetilde{(f \star g)}(q) \tilde{h}(r) K(p, q, r) e^{ip \cdot x} \\
&= \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{d^d r}{(2\pi)^d} \frac{d^d s}{(2\pi)^d} \frac{d^d t}{(2\pi)^d} \tilde{f}(s) \tilde{g}(t) \tilde{h}(r) K(p, q, r) K(q, s, t) e^{ip \cdot x} \\
&= \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{d^d r}{(2\pi)^d} \frac{d^d s}{(2\pi)^d} \frac{d^d t}{(2\pi)^d} \tilde{f}(r) \tilde{g}(s) \tilde{h}(t) K(p, q, t) K(q, r, s) e^{ip \cdot x}
\end{aligned} \tag{5.1.3}$$

is equal to

$$\begin{aligned}
(f \star (g \star h))(x) &= \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{d^d r}{(2\pi)^d} \tilde{f}(q) \widetilde{(g \star h)}(r) K(p, q, r) e^{ip \cdot x} \\
&= \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{d^d r}{(2\pi)^d} \frac{d^d s}{(2\pi)^d} \frac{d^d t}{(2\pi)^d} \tilde{f}(q) \tilde{g}(s) \tilde{h}(t) K(p, q, r) K(r, s, t) e^{ip \cdot x} \\
&= \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{d^d r}{(2\pi)^d} \frac{d^d s}{(2\pi)^d} \frac{d^d t}{(2\pi)^d} \tilde{f}(r) \tilde{g}(s) \tilde{h}(t) K(p, r, q) K(q, s, t) e^{ip \cdot x}.
\end{aligned} \tag{5.1.4}$$

In other words, we have to impose

$$\int d^d q K(p, q, t) K(q, r, s) = \int d^d q K(p, r, q) K(q, s, t). \tag{5.1.5}$$

We can show that this condition is nothing but the usual cocycle condition in the Hochschild cohomology. To this end, we recall that the 2-cochain $c \in C^2(\mathcal{A})$ is the map

$$c : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \tag{5.1.6}$$

defined by

$$c(f, g) = f \star g \tag{5.1.7}$$

where \mathcal{A} is the non-commutative algebra of functions with the product (5.1.1) and the coboundary operator is the map

$$\partial : C^k(\mathcal{A}) \rightarrow C^{k+1}(\mathcal{A}) \tag{5.1.8}$$

which transforms the k -cochain $c(f_1, \dots, f_k)$ in the $(k+1)$ -cochain defined by

$$\begin{aligned}
\partial c(f_0, \dots, f_k) &= f_0 \star c(f_1, \dots, f_k) + \sum_{i=0}^{k-1} (-1)^{i+1} c(f_0, \dots, f_i \star f_{i+1}, \dots, f_k) \\
&\quad + (-1)^{k+1} c(f_0, \dots, f_{k-1}) \star f_k.
\end{aligned} \tag{5.1.9}$$

In order for the 2-cochain (5.1.7) to be a 2-cocycle, it has to be

$$\begin{aligned} 0 = \partial c(f, g, h) &= f \star c(g, h) - c(f \star g, h) + c(f, g \star h) - c(f, g) \star h \\ &= 2(f \star (g \star h)) - (f \star g) \star h \end{aligned} \quad (5.1.10)$$

which gives (5.1.5). In general, the product (5.1.21) is non-commutative. However, it becomes commutative if we impose a constraint on K . Indeed, imposing

$$\begin{aligned} (f \star g)(x) &= \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{d^d r}{(2\pi)^d} \tilde{f}(q) \tilde{g}(r) K(p, q, r) e^{ip \cdot x} \\ &= \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{d^d r}{(2\pi)^d} \tilde{g}(q) \tilde{f}(r) K(p, r, q) e^{ip \cdot x} = (g \star f)(x) \end{aligned} \quad (5.1.11)$$

we get the commutativity condition

$$K(p, q, r) = K(p, r, q). \quad (5.1.12)$$

This condition means that the 2-cochain c given by (5.1.7) is a 2-coboundary. That is,

$$c(f, g) = \partial b(f, g) = f \star b(g) + g \star b(f) - b(f \star g) \quad (5.1.13)$$

where the 1-cochain b is simply given by the identity map. In other words, the commutativity condition (5.1.12) is a coboundary condition in the Hochschild cohomology.

We now proceed to the discussion of translation invariance of the product (5.1.1). We recall that the translation by a vector a is defined by

$$\mathcal{T}_a(f)(x) = f(x + a). \quad (5.1.14)$$

Since in Fourier transform

$$\int \frac{d^d p}{(2\pi)^d} \widetilde{\mathcal{T}_a(f)}(p) e^{ip \cdot x} = \int \frac{d^d p}{(2\pi)^d} \tilde{f}(p) e^{ip \cdot (x+a)}, \quad (5.1.15)$$

we have

$$\widetilde{\mathcal{T}_a(f)}(p) = e^{ia \cdot p} \tilde{f}(p). \quad (5.1.16)$$

By translation invariant product we mean as usual a product which satisfies the property

$$\mathcal{T}_a(f) \star \mathcal{T}_a(g) = \mathcal{T}_a(f \star g). \quad (5.1.17)$$

For the translational invariance of the product (5.1.1) we have to impose

$$\mathcal{T}_a(f \star g) = \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{d^d r}{(2\pi)^d} e^{ip \cdot (x+a)} \tilde{f}(q) \tilde{g}(r) K(p, q, r) \quad (5.1.18)$$

is equal to

$$\begin{aligned} \mathcal{T}_a(f) \star \mathcal{T}_a(g) &= \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{d^d r}{(2\pi)^d} e^{ip \cdot x} \widetilde{\mathcal{T}_a(f)}(q) \widetilde{\mathcal{T}_a(g)}(r) K(p, q, r) \\ &= \int dp dq dr e^{ip \cdot x} e^{ia \cdot q} \tilde{f}(q) e^{ia \cdot r} \tilde{g}(r) K(p, q, r). \end{aligned} \quad (5.1.19)$$

This is achieved by setting

$$K(p, q, r) = e^{\alpha(p,q)} \delta^{(d)}(r - p + q) \quad (5.1.20)$$

where α is a generic function. Therefore, because of translation invariance, the product (5.1.1) takes the form

$$(f \star g)(x) = \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \tilde{f}(q) \tilde{g}(p - q) e^{\alpha(p,q)} e^{ip \cdot x}. \quad (5.1.21)$$

The ordinary product is given by $\alpha = 0$, the Moyal product (3.1.12) by

$$\alpha_M(p, q) = -\frac{i}{2} \theta^{ij} q_i (p_j - q_j) = \frac{i}{2} \theta p \wedge q \quad (5.1.22)$$

and the Wick-Voros product (4.1.14) by

$$\alpha_V(p, q) = -\theta q_- \cdot (p_+ - q_+) = \alpha_M(p, q) - \frac{\theta}{2} (p - q) \cdot q. \quad (5.1.23)$$

We can express the associativity condition (5.1.5) in terms of α . Indeed, from (5.1.5) and (5.1.20) follows that

$$\begin{aligned} \int d^d q e^{\alpha(p,q)} \delta^{(d)}(t - p + q) e^{\alpha(q,r)} \delta^{(d)}(s - q + r) = \\ \int d^d q e^{\alpha(p,r)} \delta^{(d)}(q - p + r) e^{\alpha(q,s)} \delta^{(d)}(t - q + s) \end{aligned} \quad (5.1.24)$$

That is,

$$\int d^d p e^{\alpha(p,r+s) + \alpha(r+s,r)} \delta^{(d)}(r+s+t-p) = \int d^d p e^{\alpha(p,r) + \alpha(p-r,s)} \delta^{(d)}(r+s+t-p). \quad (5.1.25)$$

Therefore, α has to satisfy the condition

$$\alpha(p, r + s) + \alpha(r + s, r) = \alpha(p, r) + \alpha(p - r, s) \quad (5.1.26)$$

which can be rewritten as

$$\alpha(p, q) = \alpha(p, r) - \alpha(q, r) + \alpha(p - r, q - r). \quad (5.1.27)$$

Note that we can get the associativity condition by starting directly from (5.1.21). Indeed, imposing

$$\begin{aligned} ((f \star g) \star h)(x) &= \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \widetilde{(f \star g)}(q) \tilde{h}(p - q) e^{\alpha(p, q)} e^{ip \cdot x} \\ &= \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{d^d r}{(2\pi)^d} \tilde{f}(r) \tilde{g}(q - r) \tilde{h}(p - q) e^{\alpha(p, q) + \alpha(q, r)} e^{ip \cdot x} \end{aligned} \quad (5.1.28)$$

is equal to

$$\begin{aligned} (f \star (g \star h))(x) &= \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \tilde{f}(q) \widetilde{(g \star h)}(p - q) e^{\alpha(p, q)} e^{ip \cdot x} \\ &= \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{d^d r}{(2\pi)^d} \tilde{f}(q) \tilde{g}(r) \tilde{h}(p - q - r) e^{\alpha(p, q) + \alpha(p - q, r)} e^{ip \cdot x} \\ &= \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{d^d r}{(2\pi)^d} \tilde{f}(r) \tilde{g}(q - r) \tilde{h}(p - q) e^{\alpha(p, r) + \alpha(p - r, q - r)} e^{ip \cdot x} \end{aligned} \quad (5.1.29)$$

we reobtain the condition (5.1.27). We also require

$$f \star 1 = \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \tilde{f}(q) \delta^{(d)}(p - q) e^{\alpha(p, q)} e^{ip \cdot x} = \int \frac{d^d p}{(2\pi)^d} \tilde{f}(p) e^{\alpha(p, p)} e^{ip \cdot x} = f \quad (5.1.30)$$

and

$$1 \star f = \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \delta^{(d)}(q) \tilde{f}(p - q) e^{\alpha(p, q)} e^{ip \cdot x} = \int \frac{d^d p}{(2\pi)^d} \tilde{f}(p) e^{\alpha(p, 0)} e^{ip \cdot x} = f. \quad (5.1.31)$$

That is, the identity of the algebra of functions with the product (5.1.21) is the constant function with value 1. These conditions impose

$$\alpha(p, p) = 0 \quad (5.1.32)$$

$$\alpha(p, 0) = 0. \quad (5.1.33)$$

In particular,

$$\alpha(0, 0) = 0 \quad (5.1.34)$$

Moreover, we require the algebra to be a $*$ -algebra. That is, there must be a map which satisfies the following conditions [48]:

$$(f^*)^* = f \quad (5.1.35)$$

$$(\lambda f + \mu g)^* = \bar{\lambda} f^* + \bar{\mu} g^* \quad (5.1.36)$$

$$(f \star g)^* = g^* \star f^* \quad (5.1.37)$$

for any complex numbers λ and μ where the bar denotes complex conjugation. In our case the involution $*$ is given by complex conjugation and in particular the last relation imposes a constrain on α . Indeed, since

$$(f \star g)^* = \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \tilde{f}(q)^* \tilde{g}(p - q)^* e^{\alpha(p, q)^*} e^{-ip \cdot x} \quad (5.1.38)$$

has to be equal to

$$\begin{aligned} g^* \star f^* &= \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \tilde{g}^*(q) \tilde{f}^*(p - q) e^{\alpha(p, q)} e^{ip \cdot x} \\ &= \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \tilde{g}(-q)^* \tilde{f}(q - p)^* e^{\alpha(p, q)} e^{ip \cdot x} \\ &= \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \tilde{f}(q)^* \tilde{g}(p - q)^* e^{\alpha(-p, q - p)} e^{-ip \cdot x} \end{aligned} \quad (5.1.39)$$

then α has to satisfy the condition

$$\alpha(p, q)^* = \alpha(-p, q - p). \quad (5.1.40)$$

We can express the commutativity condition (5.1.12) in terms of α as well. Indeed, from (5.1.12) and (5.1.20) follows that

$$\begin{aligned} e^{\alpha(p, q)} \delta^{(d)}(r - p + q) &= e^{\alpha(p, r)} \delta^{(d)}(q - p + r) \\ &= e^{\alpha(p, r)} \delta^{(d)}(r - p + q) \\ &= e^{\alpha(p, p - q)} \delta^{(d)}(r - p + q). \end{aligned} \quad (5.1.41)$$

That is,

$$\alpha(p, q) = \alpha(p, p - q). \quad (5.1.42)$$

Note that we can get the commutativity condition by starting directly from (5.1.21). Indeed, imposing

$$\begin{aligned} (f \star g)(x) &= \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \tilde{f}(q) \tilde{g}(p-q) e^{\alpha(p,q)} e^{ip \cdot x} \\ &= \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \tilde{g}(q) \tilde{f}(p-q) e^{\alpha(p,p-q)} e^{ip \cdot x} = (g \star f)(x) \end{aligned} \quad (5.1.43)$$

we reobtain the condition (5.1.42). From the associativity condition (5.1.27), we can derive some very useful relations. For $q = r = p$ we have

$$\alpha(p, p) = \alpha(0, 0) \quad (5.1.44)$$

and for $q = r = 0$ we have

$$\alpha(p, 0) = \alpha(0, 0) \quad (5.1.45)$$

in agreement with (5.1.32) and (5.1.33) respectively. For $q = 0$ and $r = p$ we have

$$\alpha(0, -p) = \alpha(0, p). \quad (5.1.46)$$

For $r = p$ we have

$$\alpha(p, q) = -\alpha(q, p) + \alpha(0, q - p). \quad (5.1.47)$$

Moreover, from (5.1.27) we have

$$\alpha(0, q) = \alpha(0, p) - \alpha(q, p) + \alpha(-p, q - p) \quad (5.1.48)$$

and by using (5.1.47) we obtain a very important relation

$$\alpha(p, q) = -\alpha(0, p) + \alpha(0, q) + \alpha(0, p - q) - \alpha(-p, q - p). \quad (5.1.49)$$

With a symbolic manipulation programme and a little work is not difficult to construct polynomial expression for α . For example, the following expression in two-dimensions

$$\begin{aligned} \alpha &= Ap_2q_1 + Bp_1q_2 - (A + B)q_1q_2 + C(p_2q_2^2 - p_2^2q_2) \\ &\quad + D \left(\frac{p_2^2q_1 - p_1q_2^2}{2} + p_1p_2q_2 - p_2q_1q_2 \right) \end{aligned} \quad (5.1.50)$$

gives rise to an associative product for any complex numbers A, B, C and D . However, the condition (5.1.40) imposes some conditions on these coefficients. Indeed, it is easy to see that

$$B^* = A, \quad C^* = -C \quad \text{and} \quad D^* = -D. \quad (5.1.51)$$

Note that the integral of the product of two functions can be written as

$$\begin{aligned} \int d^d x f \star g &= \int d^d x \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \tilde{f}(q) \tilde{g}(p-q) e^{\alpha(p,q)} e^{ip \cdot x} \\ &= \int \frac{dq}{(2\pi)^d} \tilde{f}(q) \tilde{g}(-q) e^{\alpha(0,q)}. \end{aligned} \quad (5.1.52)$$

Therefore, the integral of the product of two functions is not equal to the integral of the ordinary product of the two functions. However, from (5.1.46) follows the trace property

$$\int d^d x f \star g = \int d^d x g \star f. \quad (5.1.53)$$

5.2 Cohomology

We now proceed to show that it is possible to define an “ α -cohomology” with respect to which α is a 2-cocycle in the associative case while it is a 2-coboundary in the commutative one. Let $\alpha \in A^2(\tilde{\mathcal{A}})$ be the map

$$\alpha : (p, q) \in \tilde{\mathcal{A}} \otimes \tilde{\mathcal{A}} \longrightarrow \tilde{\mathcal{A}} \quad (5.2.1)$$

with $\tilde{\mathcal{A}}$ the algebra of Fourier transforms (to be more precise α is defined on translations, realised as linear functions in $\tilde{\mathcal{A}}$) and the coboundary operator

$$\partial : A^k(\tilde{\mathcal{A}}) \rightarrow A^{k+1}(\tilde{\mathcal{A}}) \quad (5.2.2)$$

defined by

$$\begin{aligned} \partial \gamma(p_0, \dots, p_k) &= \sum_{i=0}^k (-1)^i \gamma(p_0, \dots, p_{i-1}, p_i, p_{i+1}, \dots, p_k) \\ &\quad - (-1)^k \gamma(p_0 - p_k, p_i - p_k, \dots, p_{k-1} - p_k). \end{aligned} \quad (5.2.3)$$

Note that a straightforward calculation verifies that

$$\partial^2 = 0. \quad (5.2.4)$$

In order for α in (5.2.1) to be a 2-cocycle in the α -cohomology, it has to be

$$\begin{aligned} 0 = \partial\alpha(p, q, r) &= \alpha(q, r) - \alpha(p, r) + \alpha(p, q) - \alpha(p - r, q - r) \\ &= 2(f \star (g \star h) - (f \star g) \star h) \end{aligned} \quad (5.2.5)$$

that is (5.1.27). Therefore, the associativity condition (5.1.27) is a cocycle condition in the α -cohomology. Analogously the commutativity condition (5.1.42) is a coboundary condition. Indeed, for α to be a 2-coboundary in the α -cohomology, it has to be

$$\alpha(p, q) = \partial\beta(p, q) = \beta(q) - \beta(p) + \beta(p - q) \quad (5.2.6)$$

which implies the commutativity condition (5.1.42). If α is a 2-coboundary in the α -cohomology, then it is a 2-cocycle because of (5.2.4) so that the product (5.1.21) is associative and commutative. However, the coboundary condition in the α -cohomology (5.2.6) is not equivalent to the commutative condition (5.1.42). Indeed, the function

$$\alpha(p, q) = A\beta(p) + \beta(q) + \beta(p - q) \quad (5.2.7)$$

with A a complex number and $A \neq -1$ is not a 2-coboundary, but it makes the product (5.1.21) commutative and, of course, non-associative. As we have already seen, the Moyal and Wick-Voros products, both non-commutative, are respectively given by (5.1.22) and (5.1.23) which are both 2-cocycles in the α -cohomology and more interestingly they differ by a term which is a α -coboundary according to (5.2.6) with β so defined

$$\beta(q) = q^2 \quad (5.2.8)$$

Indeed, it is easy to verify that

$$\alpha_V(p, q) = \alpha_M(p, q) + \frac{\theta}{4}\partial\beta(p, q). \quad (5.2.9)$$

5.3 Differential form of a general product

Another way to get a general star product is generalizing the differential form of the Moyal product (3.1.3). To this end, consider a product which is a series in a deformation parameter which we call again θ :

$$f \star g = \sum_{r=0}^{\infty} C_r(f, g)\theta^r \quad (5.3.1)$$

where C 's are in general bidifferential operators i.e. bilinear maps which are differential operators with respect to each argument of globally bounded order. To recover the ordinary product in the limit $\theta \rightarrow 0$ we need to impose

$$C_0(f, g) = fg. \quad (5.3.2)$$

Therefore, we can rewrite the product which we are considering as

$$f \star g = fg + \sum_{r=1}^{\infty} C_r(f, g)\theta^r. \quad (5.3.3)$$

Instead, in order to ensure associativity the remaining C_r 's have to satisfy the following conditions

$$\begin{aligned} fC_r(g, h) - C_r(fg, h) + C_r(f, gh) - C_r(f, g)h = \\ = \sum_{j+k=r} (C_j(C_k(f, g), h) - C_j(f, C_k(g, h))) \end{aligned} \quad (5.3.4)$$

for all $r > 0$. Note that a problem with the product (5.3.1) is that it is defined on the space of formal series in the coordinates and then there is in general no control on the convergence of the series after the product has been taken. This kind of product is considered in the very general framework of deformation quantization [49, 50]. This approach consists in finding a deformation of the algebra of functions on a Poisson manifold² with the additional property that to first order in the deformation parameter θ the star commutator of two functions:

$$[f, g]_{\star} = f \star g - g \star f \quad (5.3.5)$$

reduces, or better is proportional, to the Poisson bracket of the two functions. This requires that

$$\{f, g\} = C_1(f, g) - C_1(g, f). \quad (5.3.6)$$

²In general, a Poisson manifold is a manifold endowed with a Poisson bracket that is, a bilinear and antisymmetric map $\{\cdot, \cdot\}$ which satisfies the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

and the Leibniz rule

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

Since in general the Poisson bracket of two function can be written as

$$\{f, g\} = \Lambda^{ij} \partial_i f \partial_j g \quad (5.3.7)$$

we have

$$\Lambda^{ij} = C_1(x^i, x^j) - C_1(x^j, x^i) \quad (5.3.8)$$

which can be easily proved by setting $f = x^i$ and $g = x^j$. Note that if

$$C_1(f, g) = C_1(g, f) \quad (5.3.9)$$

then the product (5.3.1) is commutative. The proof is the following. First consider (5.3.4) for $r = 2$, $f = h = x^m$ and $g = x^n$. Then relation (5.3.4) becomes

$$\begin{aligned} f C_2(g, f) - C_2(fg, f) + C_2(f, gf) - C_2(f, g)f &= \\ = x^m (C_2(x^n, x^m) - C_2(x^m, x^n)) - C_2(x^{m+n}, x^m) + C_2(x^m, x^{m+n}) &= \\ = (C_1(C_1(f, g), f) - C_1(f, C_1(g, f))) &= 0 \end{aligned} \quad (5.3.10)$$

because of the symmetry of C_1 . The second line of the above equation has to hold for all x 's and therefore it must be

$$C_2(x^{m+n}, x^m) = C_2(x^m, x^{m+n}) \quad (5.3.11)$$

for generic m and n . This implies that

$$C_2(f, g) = C_2(g, f). \quad (5.3.12)$$

It is then possible to prove exactly in the same way that if

$$C_l(f, g) = C_l(g, f) \quad (5.3.13)$$

for $l < r$ then all the terms in the right hand side of (5.3.4) pairwise cancel and we are left to the equivalent of (5.3.10) with a generic r proving that

$$C_r(f, g) = C_r(g, f). \quad (5.3.14)$$

In general, given a Poisson manifold it is not easy to prove that it is always possible to find a star product whose commutator reduces, to first order in the deformation parameter θ , to the Poisson bracket because the associativity conditions (5.3.4) are difficult to satisfy. A general result of Kontsevich [51] in

the context of formal series solves this problem for a generic Poisson manifold. Moreover, it proves that two products with the same Poisson structure are equivalent in sense that there exists a map T such that

$$T(f \star g) = T(f) \star' T(g). \quad (5.3.15)$$

We have seen an instance of such an equivalence for the Moyal and Wick-Voros products for which the equivalence map is given by (4.1.21).

To conclude this section, we briefly discuss the translation invariance of the product (5.3.1). Since the C_r 's are bidifferential operators, the product (5.3.1) becomes translation invariant if and only if the C_r 's are combinations of derivatives only. Therefore, it can be written as

$$f \star g = fg + \sum_{r=1}^{\infty} \sum_{i_1, \dots, i_r, j_1, \dots, j_r} \theta^r c^{i_1 \dots i_r j_1 \dots j_r} (\partial_{i_1} \dots \partial_{i_r} f) (\partial_{j_1} \dots \partial_{j_r} g) \quad (5.3.16)$$

where c 's are complex constant coefficients. By using (5.3.1) and the fact that the C_r 's are combinations of derivatives only, we easily get

$$[x^i, x^j]_{\star} = x^i \star x^j - x^j \star x^i = C_1(x^i, x^j) - C_1(x^j, x^i). \quad (5.3.17)$$

That is, the commutator of coordinates reads

$$[x^i, x^j]_{\star} = \Lambda^{ij}. \quad (5.3.18)$$

Therefore, it reproduces the Poisson structure of the underlying space.

5.4 Field theory with a general translation invariant product

We now proceed to the discussion of a non-commutative field theory with the general translation invariant product (5.1.21) in order to investigate its ultraviolet behaviour. So let us consider a field theory described by the action

$$S = S_0 + S_{\text{int}} \quad (5.4.1)$$

where S_0 is the free action given by

$$S_0 = \int d^d x \frac{1}{2} (\partial_{\mu} \phi \star \partial^{\mu} \phi - m^2 \phi \star \phi) \quad (5.4.2)$$

and S_{int} is the interacting one given by

$$S_{\text{int}} = \frac{g}{4!} \int d^d x \phi \star \phi \star \phi \star \phi. \quad (5.4.3)$$

To begin with, we calculate the equation of motion. To this end, we consider an infinitesimal variation of ϕ :

$$\phi \rightarrow \phi + \delta\phi. \quad (5.4.4)$$

The corresponding infinitesimal variation of the action S_0 is given by

$$\delta S_0 = \int d^d x (\partial_\mu \phi \star \partial^\mu \delta\phi - m^2 \phi \star \delta\phi). \quad (5.4.5)$$

By integrating by parts we obtain, up to boundary terms,

$$\delta S_0 = - \int d^d x (\square + m^2) \phi \star \delta\phi \quad (5.4.6)$$

and using the relation (5.1.52) we have

$$\delta S_0 = - \int d^d p e^{\alpha(0,p)} (-p^2 + m^2) \tilde{\phi}(p) \widetilde{\delta\phi}(-p). \quad (5.4.7)$$

Since the variation of the action δS_0 have to be vanishing for every variation of the field $\delta\phi$, we get the equation of motion in Fourier transform

$$e^{\alpha(0,p)} (p^2 - m^2) \tilde{\phi}(p) = 0 \quad (5.4.8)$$

which reduces to the same ordinary equation of motion in Fourier transform due to the invertibility of the exponential factor. Therefore, at classical level the free non-commutative field theory with the general translation invariant product (5.1.21) is the same as the commutative one.

We now move on to the quantum case and proceed to the calculation of Green's functions. The propagator can be easily obtained by solving the equation

$$e^{\alpha(0,p)} (p^2 - m^2) \tilde{G}^2(p) = 1 \quad (5.4.9)$$

which gives

$$\tilde{G}^2(p) = \frac{e^{-\alpha(0,p)}}{p^2 - m^2}. \quad (5.4.10)$$

Note that in general the presence of the exponential in the propagator (5.4.10) modifies its properties. In order to calculate the vertex, let us write down

the interacting term of the action in momentum space. Using the relations (5.1.21) and (5.1.52) we have

$$\begin{aligned}
S_{\text{int}} &= \frac{g}{4!} \int d^d x \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \frac{d^d k_3}{(2\pi)^d} \frac{d^d k_4}{(2\pi)^d} \tilde{\phi}(k_2) \tilde{\phi}(k_1 - k_2) \tilde{\phi}(k_4) \tilde{\phi}(k_3 - k_4) \\
&\quad e^{\alpha(k_1, k_2)} e^{\alpha(k_3, k_4)} e^{ik_1 \cdot x} \star e^{ik_3 \cdot x} \\
&= \frac{g}{4!} (2\pi)^d \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \frac{d^d k_3}{(2\pi)^d} \frac{d^d k_4}{(2\pi)^d} \tilde{\phi}(k_2) \tilde{\phi}(k_1 - k_2) \tilde{\phi}(k_4) \tilde{\phi}(k_3 - k_4) \\
&\quad e^{\alpha(k_1, k_2)} e^{\alpha(k_3, k_4)} \int \frac{d^d k}{(2\pi)^d} e^{\alpha(0, k)} \delta^{(d)}(k_1 - k) \delta^{(d)}(k_3 + k) \\
&= \frac{g}{4!} (2\pi)^d \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \frac{d^d k_3}{(2\pi)^d} \frac{d^d k_4}{(2\pi)^d} \tilde{\phi}(k_2) \tilde{\phi}(k_1 - k_2) \tilde{\phi}(k_4) \tilde{\phi}(k_3 - k_4) \\
&\quad e^{\alpha(k_1, k_2) + \alpha(k_3, k_4) + \alpha(0, k_1)} \delta^{(d)}(k_1 + k_3) \quad (5.4.11)
\end{aligned}$$

which can be rewritten as well as

$$\begin{aligned}
S_{\text{int}} &= \frac{g}{4!} (2\pi)^d \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \frac{d^d k_3}{(2\pi)^d} \frac{d^d k_4}{(2\pi)^d} \tilde{\phi}(k_1) \tilde{\phi}(k_2) \tilde{\phi}(k_3) \tilde{\phi}(k_4) \\
&\quad e^{\alpha(k_1 + k_2, k_1) + \alpha(k_3 + k_4, k_3) + \alpha(0, k_1 + k_2)} \delta^{(d)}(k_1 + k_2 + k_3 + k_4). \quad (5.4.12)
\end{aligned}$$

Therefore, the vertex is given by

$$V_\star = V e^{\alpha(k_1 + k_2, k_1) + \alpha(k_3 + k_4, k_3) + \alpha(0, k_1 + k_2)} \quad (5.4.13)$$

where

$$V = -i \frac{g}{4!} (2\pi)^d \delta^{(d)} \left(\sum_{a=1}^4 k_a \right) \quad (5.4.14)$$

is the ordinary d -dimensional vertex. It is possible to show that the vertex loses the invariance for arbitrary exchanges of the external momenta, but it maintains invariance for cyclic permutations. Finally, to calculate the four-point Green's function to the tree level, we must attach to the vertex (5.4.13) four propagators (5.4.10). We have

$$\begin{aligned}
\tilde{G}^{(4)} &= -ig (2\pi)^d e^{\alpha(k_1 + k_2, k_1) + \alpha(k_3 + k_4, k_3) + \alpha(0, k_1 + k_2)} \prod_{a=1}^4 \frac{e^{-\alpha(0, k_a)}}{k_a^2 - m^2} \delta^{(d)} \left(\sum_{a=1}^4 k_a \right) \\
&= -ig (2\pi)^d \frac{e^{\alpha(k_1 + k_2, k_1) + \alpha(k_3 + k_4, k_3) + \alpha(0, k_1 + k_2) - \sum_{a=1}^4 \alpha(0, k_a)}}{\prod_{a=1}^4 (k_a^2 - m^2)} \delta^{(d)} \left(\sum_{a=1}^4 k_a \right). \quad (5.4.15)
\end{aligned}$$

5.5 UV/IR mixing for a general translation invariant product

We now proceed to the calculation of the one-loop corrections to the propagator in order to investigate the ultraviolet behaviour of the theory. Consider then both diagrams of figure 3.1. The correction for the planar case (a) is obtained using three propagators (5.4.10), one with momentum p , one with momentum $-p$, one with momentum q and the vertex (5.4.13) with assignments given by (3.3.1) and the integration in q . We have up to a constant

$$\begin{aligned}
G_P^{(2)} &= \int d^d q \frac{e^{-\alpha(0,p)-\alpha(0,-p)-\alpha(0,q)}}{(p^2 - m^2)^2(q^2 - m^2)} e^{\alpha(p+q,p)+\alpha(-p-q,-q)+\alpha(0,p+q)} \\
&= \int d^d q \frac{e^{-2\alpha(0,p)-\alpha(0,q)+\alpha(p+q,p)+\alpha(-p-q,-q)+\alpha(0,p+q)}}{(p^2 - m^2)^2(q^2 - m^2)} \\
&= \int d^d q \frac{e^{-\alpha(0,p)}}{(p^2 - m^2)^2(q^2 - m^2)} \tag{5.5.1}
\end{aligned}$$

since by using (5.1.49) we have

$$\alpha(p+q, p) = -\alpha(0, p+q) + \alpha(0, p) + \alpha(0, q) - \alpha(-p-q, -q). \tag{5.5.2}$$

We see that with respect to the commutative case the only correction is in the factor $e^{-\alpha(0,p)}$ which is the correction of the free propagator. Therefore, the ultraviolet divergence of the planar diagram is the same as the ordinary one. Consider now the non-planar case in figure 3.1(b). The structure is the same as in the planar case, but this time the assignments are given by (3.3.3). We have up to a constant

$$\begin{aligned}
G_{NP}^{(2)} &= \int d^d q \frac{e^{-\alpha(0,p)-\alpha(0,-p)-\alpha(0,q)}}{(p^2 - m^2)^2(q^2 - m^2)} e^{\alpha(p+q,p)+\alpha(-p-q,-p)+\alpha(0,p+q)} \\
&= \int d^d q \frac{e^{-2\alpha(0,p)-\alpha(0,q)+\alpha(p+q,p)+\alpha(-p-q,-p)+\alpha(0,p+q)}}{(p^2 - m^2)^2(q^2 - m^2)} \\
&= \int d^d q \frac{e^{-\alpha(0,p)+\alpha(p+q,p)-\alpha(p+q,q)}}{(p^2 - m^2)^2(q^2 - m^2)} \tag{5.5.3}
\end{aligned}$$

since by using again (5.1.49) we have

$$\begin{aligned}
\alpha(-p-q, -p) &= -\alpha(0, -p-q) + \alpha(0, -p) + \alpha(0, -q) - \alpha(p+q, q) \\
&= -\alpha(0, p+q) + \alpha(0, p) + \alpha(0, q) - \alpha(p+q, q). \tag{5.5.4}
\end{aligned}$$

The one-loop correction to the propagator in the non-planar case can be rewritten as

$$G_{NP}^{(2)} = \int d^d q \frac{e^{-\alpha(0,p)+\omega(p,q)}}{(p^2 - m^2)^2(q^2 - m^2)} \quad (5.5.5)$$

where we have set

$$\omega(p, q) = \alpha(p + q, p) - \alpha(p + q, q). \quad (5.5.6)$$

Note that for both the Moyal and Wick-Voros products this term is given by

$$\omega_M(p, q) = \omega_V(p, q) = -i\theta^{ij} p_i p_j. \quad (5.5.7)$$

The function ω has some important properties which will be useful in the discussion of the ultraviolet behaviour of the non-planar diagram. Using (5.1.49) it is easy to get the following relations:

$$\omega(p, p) = 0 \quad (5.5.8)$$

$$\omega(p, 0) = 0 \quad (5.5.9)$$

$$\omega(0, q) = 0 \quad (5.5.10)$$

$$\omega(p, q) = -\omega(q, p) \quad \text{antisymmetry} \quad (5.5.11)$$

$$\omega(-p, -q) = \omega(p, q) \quad \text{parity} \quad (5.5.12)$$

$$\omega(-p, q) = \omega(p, -q). \quad (5.5.13)$$

Moreover, ω satisfies the associativity condition (5.1.27) and then it satisfies the condition (5.1.49) from which follows

$$\omega(p, q) = \omega(p - q, p), \quad (5.5.14)$$

where we have used antisymmetry (5.5.11) and parity (5.5.12). It can be written as well as

$$\omega(p, q) = \omega(q, q - p), \quad (5.5.15)$$

where we have exchanged p and q and used once again antisymmetry (5.5.11). Finally, from (5.1.27) we have

$$\alpha(p + q, p) = \alpha(p + q, r) - \alpha(p, r) + \alpha(p + q - r, p - r) \quad (5.5.16)$$

and by setting $r = q$ we get

$$\omega(p, q) = \alpha(p, p - q) - \alpha(p, q). \quad (5.5.17)$$

This quantity vanishes if the product is commutative because of (5.1.42). This means that the non-planar diagram captures the non-commutativity of the product. In other words, no change in the ultraviolet can come from a commutative product.

We now prove that the contribution to the correction to the non-planar one-loop two-point diagram must necessarily be of the form

$$\omega(p, q) = -i\theta^{ij}p_i p_j \quad (5.5.18)$$

exactly like in the Moyal and Wick-Voros cases. We only need the extra assumption that α and so ω can be expanded in a power series of p and q . The parity relation (5.5.12) requires the series to be composed only of even monomials. Let us express the function ω with a multi-index notation as

$$\omega(p, q) = \sum_{\mathbf{i}\mathbf{j}} a_{\mathbf{i}\mathbf{j}} p^{\mathbf{i}} q^{\mathbf{j}} \quad (5.5.19)$$

with $\mathbf{i} = (i_1, i_2, \dots, i_d)$ and

$$p^{\mathbf{i}} = p_1^{i_1} p_2^{i_2} \dots p_d^{i_d}. \quad (5.5.20)$$

Note that from (5.5.12) follows that

$$\sum_{\mathbf{i}\mathbf{j}} a_{\mathbf{i}\mathbf{j}} p^{\mathbf{i}} q^{\mathbf{j}} = - \sum_{\mathbf{i}\mathbf{j}} a_{\mathbf{i}\mathbf{j}} q^{\mathbf{i}} p^{\mathbf{j}} = - \sum_{\mathbf{i}\mathbf{j}} a_{\mathbf{j}\mathbf{i}} p^{\mathbf{i}} q^{\mathbf{j}}. \quad (5.5.21)$$

That is,

$$\sum_{\mathbf{i}\mathbf{j}} (a_{\mathbf{i}\mathbf{j}} + a_{\mathbf{j}\mathbf{i}}) p^{\mathbf{i}} q^{\mathbf{j}} = 0 \quad (5.5.22)$$

Because of the independence of p and q , we have

$$a_{\mathbf{i}\mathbf{j}} = -a_{\mathbf{j}\mathbf{i}}. \quad (5.5.23)$$

Therefore, the coefficients a 's are antisymmetric for the exchange of the multi-indices \mathbf{i} and \mathbf{j} . Now from (5.5.15) follows that

$$\sum_{\mathbf{i}\mathbf{j}} a_{\mathbf{i}\mathbf{j}} p^{\mathbf{i}} q^{\mathbf{j}} = \sum_{\mathbf{i}\mathbf{j}} a_{\mathbf{i}\mathbf{j}} q^{\mathbf{i}} (q-p)^{\mathbf{j}} = \sum_{\mathbf{i}\mathbf{j}} a_{\mathbf{j}\mathbf{i}} (q-p)^{\mathbf{i}} q^{\mathbf{j}} = - \sum_{\mathbf{i}\mathbf{j}} a_{\mathbf{i}\mathbf{j}} (q-p)^{\mathbf{i}} q^{\mathbf{j}} \quad (5.5.24)$$

That is,

$$\sum_{\mathbf{i}\mathbf{j}} a_{\mathbf{i}\mathbf{j}} [p^{\mathbf{i}} + (q-p)^{\mathbf{i}}] q^{\mathbf{j}} = 0. \quad (5.5.25)$$

This condition implies, because of the independence of p and q , that the coefficients a 's must vanish except in the case in which all of the j_a 's but one vanish. In this case the antisymmetry of the a 's ensures that (5.5.25) vanishes without putting further constraints on the coefficients. Using antisymmetry the same reasoning can be done for the first multi-index and this shows that ω is of the kind

$$\omega(p, q) = A\varepsilon^{ij}p_i p_j \quad (5.5.26)$$

where A is in general a complex number. Moreover, since from (5.1.40) follows that $\bar{A} = -A$, we can set $A = -i\theta$ with θ real number and this concludes the proof. Therefore, like in the Moyal and Wick-Voros cases, the non-planar diagram presents the phenomenon of ultraviolet/infrared mixing. In other words, for high internal momentum the ultraviolet divergences are damped by a phase, but these divergences reappear in the infrared namely for low incoming momenta.

We now want to highlight that ω is related to the commutator of the coordinates. To this end, we must first derive the commutation relations and for this it is more useful to rewrite (5.1.21) as

$$(f \star g)(x) = \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \tilde{f}(p) \tilde{g}(q) e^{\alpha(p+q,p)} e^{i(p+q)\cdot x} \quad (5.5.27)$$

with a change of variables. We have

$$\begin{aligned} x^i \star x^j &= - \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \left(\frac{\partial}{\partial p_i} \delta^{(d)}(p) \right) \left(\frac{\partial}{\partial q_j} \delta^{(d)}(q) \right) e^{\alpha(p+q,p)} e^{i(p+q)\cdot x} \\ &= \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \delta^{(d)}(p) \left(\frac{\partial}{\partial q_j} \delta^{(d)}(q) \right) \frac{\partial}{\partial p_i} [e^{\alpha(p+q,p)} e^{i(p+q)\cdot x}] \\ &= \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \delta^{(d)}(p) \left(\frac{\partial}{\partial q_j} \delta^{(d)}(q) \right) \left(\frac{\partial \alpha}{\partial p_i}(p+q, p) + ix^i \right) e^{\alpha(p+q,p)} e^{i(p+q)\cdot x} \\ &= \int \frac{d^d q}{(2\pi)^d} \left(\frac{\partial}{\partial q_j} \delta^{(d)}(q) \right) \left(\frac{\partial \alpha}{\partial p_i}(q, 0) + ix^i \right) e^{iq\cdot x} \\ &= - \int \frac{d^d q}{(2\pi)^d} \delta^{(d)}(q) \frac{\partial}{\partial q_j} \left[\left(\frac{\partial \alpha}{\partial p_i}(q, 0) + ix^i \right) e^{iq\cdot x} \right] \\ &= - \int \frac{d^d q}{(2\pi)^d} \delta^{(d)}(q) \left[\frac{\partial^2 \alpha}{\partial p_i \partial q_j}(q, 0) + \left(\frac{\partial \alpha}{\partial p_i}(q, 0) + ix^i \right) ix^j \right] e^{iq\cdot x} \\ &= x^i x^j - i \frac{\partial \alpha}{\partial p_i}(0, 0) x^j - \frac{\partial^2 \alpha}{\partial p_i \partial q_j}(0, 0). \end{aligned} \quad (5.5.28)$$

In a similar way we get

$$x^j \star x^i = x^i x^j - i \frac{\partial \alpha}{\partial p_j}(0, 0) x^i - \frac{\partial^2 \alpha}{\partial p_j \partial q_i}(0, 0). \quad (5.5.29)$$

Therefore, the commutator of the coordinates is given by

$$[x^i, x^j]_\star = -i \frac{\partial \alpha}{\partial p_i}(0, 0) x^j + i \frac{\partial \alpha}{\partial p_j}(0, 0) x^i - \frac{\partial^2 \alpha}{\partial p_i \partial q_j}(0, 0) + \frac{\partial^2 \alpha}{\partial p_j \partial q_i}(0, 0). \quad (5.5.30)$$

The first two terms of this expression vanish because α has no linear term because of (5.1.33) and (5.1.46). So the commutator of the coordinates is simply given by

$$[x^i, x^j]_\star = -\frac{\partial^2 \alpha}{\partial p_i \partial q_j}(0, 0) + \frac{\partial^2 \alpha}{\partial p_j \partial q_i}(0, 0). \quad (5.5.31)$$

Expanded α in a power series of p and q as³

$$\alpha(p, q) = \alpha^{ij} p_i q_j + \dots \quad (5.5.32)$$

we have

$$[x^i, x^j]_\star = -\alpha^{ij} + \alpha^{ji}. \quad (5.5.33)$$

Moreover, from the definition of ω we have

$$\omega(p, q) = \alpha^{ij} (p_i + q_i) p_j - \alpha^{ij} (p_i + q_i) q_j = \alpha^{ij} (p_i + q_i) (p_j - q_j) \quad (5.5.34)$$

in which survive only the mixed terms because of (5.5.9) and (5.5.10). That is,

$$\omega(p, q) = -\alpha^{ij} p_i q_j + \alpha^{ji} q_i p_j = (-\alpha^{ij} + \alpha^{ji}) p_i q_j. \quad (5.5.35)$$

Therefore, the term appearing in the exponent of the one-loop correction to the propagator in the non-planar case is just the commutator of the x 's multiplied by the external and internal momenta like in the Moyal and Wick-Voros cases. In other words, the Moyal and Wick-Voros cases are generic and their behaviour is replicated by any translation invariant associative product. More in general we can conclude that star products with the same commutator and hence the same Poisson structure, which are equivalent in the sense of Kontsevich, have the same structure of ultraviolet/infrared mixing.

³Note that α has no constant term because of (5.1.34).

Chapter 6

The Moyal and Wick-Voros products as twisted products

In this chapter we present a comparison of the non-commutative field theories built using the Moyal and Wick-Voros products in the context of the twisted non-commutativity. The comparison is made at the level of Green's functions and S -matrix and we show that while the Green's functions are different for the two theories, the S -matrix is the same in both cases and is different from the commutative case.

6.1 Towards twisted non-commutativity

We have seen that the vertex and Green's functions of the Wick-Voros field theory are different from the Moyal ones. This leads to a contradiction. Indeed, with the introduction of another star product which gives the same commutation relation, one can heuristically reason as follows. The presence of the non-commutativity described by (3.1.1) gives the non-commutative structure of space-time regardless of the realization of the product one uses. Therefore, the star product is just a way to express the structure of space-time and so the results should be the same.

The element that we need consider to solve this puzzle is symmetries. The commutation relation (3.1.1) breaks the Poincaré symmetry and this is not a desirable feature for a fundamental theory. In particular, it breaks the Lorentz symmetry, but retain the translational one. However, the symmetry can be reinstated at a deformed level since both products can be seen as

coming from a Drinfeld twist [25, 26]. The non-commutativity described by the two star products is therefore a twisted non-commutativity.

The presence of a twist forces us to reconsider all of the steps in a field theory which has to be built in a coherent twisted way. We will see that there is equivalence between the Moyal and Wick-Voros field theories at the level of S -matrix. This is in agreement with our physical intuition since Green's functions are not observable quantities while S -matrix is. Furthermore, the equivalence is only obtained if a consistent procedure of twisting all products is applied. In this way the Poincaré symmetry, which appears to be broken in (3.1.1), is preserved, though in a deformed way, as a non-commutative and non-cocommutative Hopf algebra. However, there is some ambiguity in the issue of twisting [52, 53, 54, 55, 56, 57, 58, 59, 60, 61] and what we do in this chapter is just to use the field theories built with the Moyal and Wick-Voros products to check each other. This gives us an indication on the procedure to follow for non-commutative theories coming from a twist [28, 29, 30, 62].

6.2 The Moyal and Wick-Voros products as twisted products

Given the Poincaré algebra \mathcal{P} and its universal enveloping algebra $U(\mathcal{P})$, a twist \mathcal{F} is an invertible element of $U(\mathcal{P}) \otimes U(\mathcal{P})$ such that

$$\mathcal{F}_{12}(\Delta \otimes \text{id})\mathcal{F} = \mathcal{F}_{23}(\text{id} \otimes \Delta)\mathcal{F} \quad (6.2.1)$$

$$(\epsilon \otimes \text{id})\mathcal{F} = (\text{id} \otimes \epsilon)\mathcal{F} = \mathbb{1} \quad (6.2.2)$$

where

$$\mathcal{F}_{12} = \mathcal{F} \otimes \mathbb{1} \quad \text{and} \quad \mathcal{F}_{23} = \mathbb{1} \otimes \mathcal{F}. \quad (6.2.3)$$

Observe that the condition (6.2.1) nothing other than a cocycle condition¹. Before to proceed further, we recall that the Poincaré algebra \mathcal{P} describes the symmetries of ordinary Minkowski space-time \mathcal{M} and it is characterized by the following commutation relations:

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\sigma}M_{\mu\rho} + \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma}) \quad (6.2.4)$$

$$[M_{\mu\nu}, P_\rho] = i(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu) \quad (6.2.5)$$

$$[P_\mu, P_\nu] = 0 \quad (6.2.6)$$

¹See appendix A.

where $M_{\mu\nu}$ and P_μ are respectively the Lorentz generators and translation generators which are represented on the algebra of functions on Minkowski space-time \mathcal{M} by

$$P_\mu = -i\partial_\mu \quad (6.2.7)$$

$$M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu). \quad (6.2.8)$$

Moreover, its universal enveloping algebra $U(\mathcal{P})$ has a non-commutative, but cocommutative Hopf algebra structure with the coproduct, the counit and the antipode given respectively by

$$\Delta_0(X) = X \otimes \mathbb{1} + \mathbb{1} \otimes X \quad (6.2.9)$$

$$\varepsilon_0(X) = 0 \quad (6.2.10)$$

$$S_0(X) = -X \quad (6.2.11)$$

where X stands for $M_{\mu\nu}$ and P_μ . For the Moyal and Wick-Voros cases the twist is given respectively by

$$\mathcal{F}_M = e^{-\frac{i}{2}\theta^{ij}\partial_i\otimes\partial_j} \quad (6.2.12)$$

$$\mathcal{F}_V = e^{-\theta\partial_+\otimes\partial_-}. \quad (6.2.13)$$

We now assume the following point of view in agreement with [29, 63, 64, 65]. The non-commutativity can be seen a consequence of twisting of all products. Thus, for example, the ordinary commutative product between functions² on space-time \mathcal{M}

$$m_0 : \text{Fun}(\mathcal{M}) \otimes \text{Fun}(\mathcal{M}) \rightarrow \text{Fun}(\mathcal{M}) \quad (6.2.14)$$

defined by

$$m_0(f \otimes g) = fg \quad (6.2.15)$$

is consistently deformed by composing it with the twist \mathcal{F} which can be viewed as well as a map

$$\mathcal{F} : \text{Fun}(\mathcal{M}) \otimes \text{Fun}(\mathcal{M}) \rightarrow \text{Fun}(\mathcal{M}) \otimes \text{Fun}(\mathcal{M}). \quad (6.2.16)$$

obtaining a deformed version m_\star of the ordinary product m_0 . In other words, the star product can be seen as the composition of the ordinary product m_0 with the twist \mathcal{F} :

$$m_\star = m_0 \circ \mathcal{F}^{-1}. \quad (6.2.17)$$

²At this level we need not specify which kind of algebra of functions we are considering. The algebra of formal series of the generators is adequate, but more restricted algebras such as algebra of Schwarzian functions can also be considered.

In particular, the Moyal and Wick-Voros products are given respectively by

$$f \star_M g = m \circ \mathcal{F}_{\star_M}^{-1}(f \otimes g) \quad (6.2.18)$$

$$f \star_V g = m \circ \mathcal{F}_{\star_V}^{-1}(f \otimes g). \quad (6.2.19)$$

Note the the associativity of the product is ensured by the cocycle condition (6.2.1) [63, 64]. Moreover, notice that the twist \mathcal{F} deforms the structure of the universal enveloping algebra of Poincaré algebra $U(\mathcal{P})$ which becomes a non-cocommutative Hopf algebra. Indeed, the twist \mathcal{F} changes the coproduct of $U(\mathcal{P})$ according to [26]

$$\Delta_\star(X) = \mathcal{F}\Delta_0(X)\mathcal{F}^{-1}. \quad (6.2.20)$$

Since the translation generators P_μ are commutative, their coproduct is not deformed:

$$\Delta_{\mathcal{F}}(P_\mu) = \Delta_0(P_\mu). \quad (6.2.21)$$

However, the coproduct of the Lorentz generators is changed [28]:

$$\Delta_{\mathcal{F}}(M_{\mu\nu}) = \Delta_0(M_{\mu\nu}) - \frac{1}{2}\theta^{\rho\sigma} [(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \otimes P_\sigma + P_\rho \otimes (\eta_{\mu\sigma}P_\nu - \eta_{\nu\sigma}P_\mu)]. \quad (6.2.22)$$

In the untwisted case the action of the Poincaré generators X on the ordinary product of two functions f and g is given by

$$X(fg) = (Xf)g + fXg \quad (6.2.23)$$

which can be rewritten as

$$X(fg) = m_0(\Delta(X)(f \otimes g)). \quad (6.2.24)$$

In other words, their action is applied through the original coproduct (6.2.9). Instead, in the twisted case their action has to be applied through the twisted coproduct (6.2.20). That is,

$$X(f \star g) = m_\star(\Delta_\star(X)(f \otimes g)). \quad (6.2.25)$$

It is now possible to show that the Poincaré symmetry is retained as a twisted symmetry (see [66] for a discussion of twisted conformal symmetry). Indeed, it is not difficult to see that the canonical non-commutative relation (3.1.1) is a twisted Poincaré invariant [28]:

$$X([x^\mu, x^\nu]_\star) = 0. \quad (6.2.26)$$

Eventually, we also introduce the universal \mathcal{R} -matrix which represents the permutation group in non-commutative case

$$\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1} \quad (6.2.27)$$

with

$$\mathcal{F}_{21}(a \otimes b) = \tau \circ \mathcal{F} \circ \tau(a \otimes b) \quad (6.2.28)$$

and τ the usual exchange operator

$$\tau(a \otimes b) = b \otimes a. \quad (6.2.29)$$

For the cases at hand, it is easy to see that

$$\mathcal{R}_{\star_V} = \mathcal{R}_{\star_M} = \mathcal{F}_{\star_M}^{-2}. \quad (6.2.30)$$

Therefore, the exchange operator and so the statistics are the same in both the Moyal and Wick-Voros cases.

We conclude this section by noting that it is possible to introduce an unified notation for the Moyal and Wick-Voros products which will be very useful in the following. Indeed, we can write the vertex in both cases as

$$V_{\star} = V \prod_{a < b} e^{k_a \bullet k_b} \quad (6.2.31)$$

and the four-point Green's functions as

$$\tilde{G}_0^{(4)} = -ig(2\pi)^3 \frac{e^{\sum_{a < b} k_a \bullet k_b}}{\prod_{a=1}^4 (k_a^2 - m_a^2)} \delta^{(3)}\left(\sum_{a=1}^4 k_a\right) \quad (6.2.32)$$

where

$$k_a \bullet k_b = \begin{cases} -\frac{i}{2}\theta^{ij}k_{ai}k_{bj} & \text{Moyal} \\ -\theta k_{a-}k_{b+} & \text{Wick-Voros.} \end{cases} \quad (6.2.33)$$

The one-loop corrections to the propagator can be written for the planar case as

$$\tilde{G}_P^{(2)} = -i\frac{g}{3} \int \frac{d^3q}{(2\pi)^3} \frac{e^{p \bullet p}}{(p^2 - m^2)^2(q^2 - m^2)} \quad (6.2.34)$$

and for the non-planar one as

$$\tilde{G}_{NP}^{(2)} = -i\frac{g}{6} \int \frac{d^3q}{(2\pi)^3} \frac{e^{p \bullet p + p \bullet q - q \bullet p}}{(p^2 - m^2)^2(q^2 - m^2)}. \quad (6.2.35)$$

Furthermore, one-loop correction to the four-point Green's function can be written for the planar case as

$$\tilde{G}_{\text{P}}^{(4)} = \frac{(-ig)^2}{8} (2\pi)^3 \int \frac{d^3q}{(2\pi)^3} \frac{e^{\sum_{a \leq b} k_a \bullet k_b} \delta^{(3)}(\sum_{a=1}^4 k_a)}{(q^2 - m^2) [(k_1 + k_2 - q)^2 - m^2] \prod_{a=1}^4 (k_a^2 - m^2)} \quad (6.2.36)$$

and for the non-planar one as

$$\tilde{G}_{\text{NP}_a}^{(4)} = \frac{(-ig)^2}{8} (2\pi)^3 \int \frac{d^3q}{(2\pi)^3} \frac{e^{\sum_{a \leq b} k_a \bullet k_b + E_a} \delta^{(3)}(\sum_{a=1}^4 k_a)}{(q^2 - m^2) [(k_1 + k_2 - q)^2 - m^2] \prod_{a=1}^4 (k_a^2 - m^2)} \quad (6.2.37)$$

where in unified notation

$$\begin{aligned} E_1 &= q \bullet k_1 - k_1 \bullet q \\ E_2 &= k_2 \bullet q - q \bullet k_2 + k_3 \bullet q - q \bullet k_3 \\ E_3 &= k_1 \bullet q - q \bullet k_1 + k_2 \bullet q - q \bullet k_2. \end{aligned} \quad (6.2.38)$$

6.3 Twist-deformed products

We have now the necessary ingredients to calculate a physical process, like the S -matrix for the elastic scattering of two particles. We recall that one of the crucial ingredients in the study of the S -matrix is the issue of Poincaré invariance. If we naïvely insert the Green's functions found for the Moyal and Wick-Voros cases into the calculation of the S -matrix, we find a dependence of it from the external momenta, something like a momentum dependence of the coupling constant. Furthermore, we find that the result is different for the two cases in contradiction with the heuristic reasoning we made in the introduction. We also find a breaking of Poincaré invariance. The reason for the breaking of Poincaré invariance is that the commutator (3.1.1) breaks this invariance³.

The invariance can be reinstated considering it as a twisted symmetry i.e. as a symmetry described by a non-commutative and non-cocommutative Hopf algebra [28, 29, 30]. Our purpose is therefore to show, with an explicit calculation of scattering amplitudes, that the naïve procedure which leads to a difference among the two cases can be corrected by a coherent twisting

³We are considering θ^{ij} to be constant. Another possibility which preserves Poincaré invariance is to have it a tensor [7, 67] or to have it transform together with the product [68].

procedure. We see that if the twisted symmetry is properly implemented, then the final “physical” result will be the same in the Wick-Voros and Moyal cases, despite the presence of different propagators and vertices.

Consider the elastic scattering of two particles as described in figure 6.1. The first consequence of non-commutativity is that, since the vertex is not

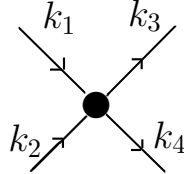


Figure 6.1: The two-particles elastic scattering.

invariant for noncyclic exchange of the particles, we have to twist-symmetrize the incoming and outgoing states using the \mathcal{R} -matrix. Several aspects of this twist-symmetrization and the consequences for spin and statistics have been discussed in [53, 69, 70]. In the commutative case the order of the propagators into the vertex is irrelevant, but in our case there are several twists at work and we must be careful in considering all of them. Therefore, let us become with the definition of multiparticle states as twisted tensor products. Consider first the one-particle state. It is defined as usual as

$$|k\rangle = a_k^\dagger |0\rangle \quad (6.3.1)$$

where the operators a_k and a_k^\dagger can be expressed in terms of the free field of Klein-Gordon equation

$$\phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \left(a_k e^{-ik \cdot x} + a_k^\dagger e^{ik \cdot x} \right) \quad (6.3.2)$$

respectively as

$$\begin{aligned} a_k &= \frac{i}{\sqrt{(2\pi)^3 2\omega_k}} \int d^3x e^{ik \cdot x} \overleftrightarrow{\partial}_0 \phi(x) \\ a_k^\dagger &= -\frac{i}{\sqrt{(2\pi)^3 2\omega_k}} \int d^3x e^{-ik \cdot x} \overleftrightarrow{\partial}_0 \phi(x) \end{aligned} \quad (6.3.3)$$

with

$$f \overleftrightarrow{\partial}_0 g = f \partial_0 g - (\partial_0 f) g. \quad (6.3.4)$$

Since a_k and a_k^\dagger may be regarded, for any fixed k , as functionals of the fields, their star product can be obtained as in [65] obtaining in the Moyal case

$$a(k) \star_M a(k') = e^{-\frac{i}{2}\theta^{ij}k_i k'_j} a(k)a(k') \quad (6.3.5)$$

$$a(k) \star_M a^\dagger(k') = e^{\frac{i}{2}\theta^{ij}k_i k'_j} a(k)a^\dagger(k') \quad (6.3.6)$$

$$a^\dagger(k) \star_M a(k') = e^{\frac{i}{2}\theta^{ij}k_i k'_j} a^\dagger(k)a(k') \quad (6.3.7)$$

$$a^\dagger(k) \star_M a^\dagger(k') = e^{-\frac{i}{2}\theta^{ij}k_i k'_j} a^\dagger(k)a^\dagger(k') \quad (6.3.8)$$

and in the Wick-Voros one

$$a(k) \star_V a(k') = e^{-\theta k_- k'_+} a(k)a(k') \quad (6.3.9)$$

$$a(k) \star_V a^\dagger(k') = e^{\theta k_- k'_+} a(k)a^\dagger(k') \quad (6.3.10)$$

$$a^\dagger(k) \star_V a(k') = e^{\theta k_- k'_+} a^\dagger(k)a(k') \quad (6.3.11)$$

$$a^\dagger(k) \star_V a^\dagger(k') = e^{-\theta k_- k'_+} a^\dagger(k)a^\dagger(k'). \quad (6.3.12)$$

Consider now the two-particle state. In the ordinary case it is defined by

$$|k_a, k_b\rangle = |k_a\rangle \otimes |k_b\rangle \quad (6.3.13)$$

and the symmetrized state, which is an eigenstate of the exchange operator

$$\tau |k_a\rangle \otimes |k_b\rangle = |k_b\rangle \otimes |k_a\rangle \quad (6.3.14)$$

with eigenvalue 1, is

$$|k_a, k_b\rangle_{\text{sim}} = \frac{|k_a\rangle \otimes |k_b\rangle + |k_b\rangle \otimes |k_a\rangle}{2}. \quad (6.3.15)$$

Note however that for the comparison we are going to make later we will not actually use the fact that the state has to be symmetrized. In fact, inserting the two expressions (6.3.13) or (6.3.15) in the calculation of the S -matrix does not make a difference (for the connected diagrams) because of the invariance for exchange on the incoming momenta and we are discussing the issue of symmetrization of states just for completeness. The symmetries for identical particles change for the non-commutative case [53, 69, 70] and we must take into account the fact that the tensor product is twisted as well as the exchange of particles. Therefore, we define

$$|k_a, k_b\rangle_\star = \tilde{\mathcal{F}}^{-1} |k_a\rangle \otimes |k_b\rangle = \tilde{\mathcal{F}}^{-1} |k_a, k_b\rangle \quad (6.3.16)$$

where by $\tilde{\mathcal{F}}$ we indicate the twist acting in momentum space:

$$\tilde{\mathcal{F}}_{\star M}^{-1} |k_a\rangle \otimes |k_b\rangle = e^{-\frac{i}{2}\theta^{ij}k_{ai}\otimes k_{bj}} |k_a\rangle \otimes |k_b\rangle \quad (6.3.17)$$

$$\tilde{\mathcal{F}}_{\star V}^{-1} |k_a\rangle \otimes |k_b\rangle = e^{\theta k_{a-}\otimes k_{b+}} |k_a\rangle \otimes |k_b\rangle. \quad (6.3.18)$$

This is not the only change we have to make to the state (6.3.13). Indeed, it has to be eigenstate of the twist-exchange, given by the \mathcal{R} -matrix acting in momentum space. The properly symmetrized state is

$$\begin{aligned} |k_a, k_b\rangle_{\text{simm}\star} &= \frac{1}{2} \left(\tilde{\mathcal{F}}^{-1} |k_a\rangle \otimes |k_b\rangle + \tilde{\mathcal{F}}^{-1} \tilde{\mathcal{R}}^{-1} |k_a\rangle \otimes |k_b\rangle \right) \\ &= \frac{1}{2} \left(\tilde{\mathcal{F}}^{-1} |k_a\rangle \otimes |k_b\rangle + \tilde{\mathcal{F}}^{-1} \tilde{\mathcal{F}} \tilde{\mathcal{F}}_{21}^{-1} |k_a\rangle \otimes |k_b\rangle \right). \end{aligned} \quad (6.3.19)$$

We can reexpress equation (6.3.16) as

$$|k_a, k_b\rangle_{\star} = a_{k_a}^{\dagger} \star a_{k_b}^{\dagger} |0\rangle \quad (6.3.20)$$

and equation (6.3.19) as

$$|k_a, k_b\rangle_{\text{simm}\star} = \frac{a_{k_a}^{\dagger} \star a_{k_b}^{\dagger} + a_{k_b}^{\dagger} \star a_{k_a}^{\dagger}}{2} |0\rangle. \quad (6.3.21)$$

The next step is to twist the inner product among one-particle states. In the commutative case we have

$$\langle k|k'\rangle = \langle 0|a_k a_{k'}^{\dagger}|0\rangle = \delta^{(3)}(k - k'). \quad (6.3.22)$$

We twist this product in the usual way composing it with the twist

$$\left\langle k_1 \left| \star \right. k_2 \right\rangle = \langle \cdot | \cdot \rangle \circ \mathcal{F}^{-1}(|k\rangle \otimes |k'\rangle) = \tilde{\mathcal{F}}^{-1}(k, k') \langle k|k'\rangle = \langle 0|a_k \star a_{k'}^{\dagger}|0\rangle \quad (6.3.23)$$

where we have set

$$\tilde{\mathcal{F}}_{\star M}^{-1}(k, k') = e^{-\frac{i}{2}\theta^{ij}k_i k'_j} \quad (6.3.24)$$

$$\tilde{\mathcal{F}}_{\star V}^{-1}(k, k') = e^{-\theta k_- k'_+}. \quad (6.3.25)$$

for the Moyal and Wick-Voros products respectively. We finally have to twist the inner product among two-particle states. In the commutative case:

$$\langle k_1, k_2|k_3, k_4\rangle = \delta^{(3)}(k_1 - k_3)\delta^{(3)}(k_2 - k_4). \quad (6.3.26)$$

Instead, in the non-commutative case we have to twist the two-particle state according to (6.3.16) and then we have to twist the inner product according to the two-particle generalization of (6.3.23). To this end, we must consider the action of the twist on two-particle states which is given by the coproduct of the Hopf algebra. Given a representation of an element of the Hopf algebra on a space, the representation of the element on the product of states is given in the undeformed case by

$$\Delta_0(u)(f \otimes g) = (\mathbb{1} \otimes u + u \otimes \mathbb{1})(f \otimes g) \quad (6.3.27)$$

For the twisted Hopf algebra the coproduct is deformed according to the fact that it is the \mathcal{R} -matrix which realizes the permutations:

$$\Delta_\star(u)(f \otimes g) = (\mathbb{1} \otimes u + \mathcal{R}^{-1}(u \otimes \mathbb{1}))(f \otimes g) \quad (6.3.28)$$

However, the twists we are considering are built out of translations whose coproduct is undeformed

$$\Delta_{\star_M}(\partial_i) = \Delta_{\star_V}(\partial_i) = \Delta_0(\partial_i) = \mathbb{1} \otimes \partial_i + \partial_i \otimes \mathbb{1}. \quad (6.3.29)$$

Since we are acting on two-particle states we need to define also

$$\Delta_\star(\partial_i \otimes \partial_j) = \Delta_0(\partial_i \otimes \partial_j) = \mathbb{1} \otimes \mathbb{1} \otimes \partial_i \otimes \partial_j + \partial_i \otimes \partial_j \otimes \mathbb{1} \otimes \mathbb{1}. \quad (6.3.30)$$

Therefore, the twisted inner product among two-particle states

$$\left\langle k_1, k_2 \left| \begin{smallmatrix} \star \\ k_3, k_4 \end{smallmatrix} \right. \right\rangle = \langle \cdot | \cdot \rangle \circ \Delta_\star(\mathcal{F}^{-1})(|k_1, k_2\rangle \otimes |k_3, k_4\rangle) \quad (6.3.31)$$

may be easily computed to be

$$\begin{aligned} \left\langle k_1, k_2 \left| \begin{smallmatrix} \star_M \\ k_3, k_4 \end{smallmatrix} \right. \right\rangle &= e^{\frac{i}{2}\theta^{ij}(k_{1i}+k_{2i})(k_{3j}+k_{4j})} \langle k_1, k_2 | k_3, k_4 \rangle \\ \left\langle k_1, k_2 \left| \begin{smallmatrix} \star_V \\ k_3, k_4 \end{smallmatrix} \right. \right\rangle &= e^{\theta(k_{1-}+k_{2-})(k_{3+}+k_{4+})} \langle k_1, k_2 | k_3, k_4 \rangle. \end{aligned} \quad (6.3.32)$$

We can now calculate the twisted inner product of twisted states. Combining (6.3.32) with (6.3.16) we obtain the simple expressions

$$\left\langle k_1, k_2 \left| \begin{smallmatrix} \star_M \\ k_3, k_4 \end{smallmatrix} \right. \right\rangle_{\star_M} = e^{\frac{i}{2}\theta^{ij} \sum_{a<b} k_{ai}k_{bj}} \langle k_1, k_2 | k_3, k_4 \rangle \quad (6.3.33)$$

$$\left\langle k_1, k_2 \left| \begin{smallmatrix} \star_V \\ k_3, k_4 \end{smallmatrix} \right. \right\rangle_{\star_V} = e^{\theta \sum_{a<b} k_{a-}k_{b+}} \langle k_1, k_2 | k_3, k_4 \rangle. \quad (6.3.34)$$

That is, in unified notation

$$\left\langle k_1, k_2 \left| \begin{array}{c} \star \\ k_3, k_4 \end{array} \right. \right\rangle = e^{-\sum_{a < b} k_a \bullet k_b} \langle k_1, k_2 | k_3, k_4 \rangle \quad (6.3.35)$$

which can be cast in the form

$$\left\langle k_1, k_2 \left| \begin{array}{c} \star \\ k_3, k_4 \end{array} \right. \right\rangle = \langle 0 | a_{k_1} \star a_{k_2} \star a_{k_3}^\dagger \star a_{k_4}^\dagger | 0 \rangle. \quad (6.3.36)$$

This is in some sense also a consistency check. We could have started with the commutative expression

$$\langle k_1, k_2 | k_3, k_4 \rangle = \langle 0 | a_{k_1} a_{k_2} a_{k_3}^\dagger a_{k_4}^\dagger | 0 \rangle \quad (6.3.37)$$

and twisted the product among the creation and annihilation operators a_k and a_k^\dagger obtaining the above result. We decided to follow a longer procedure to highlight the appearance of the various twists.

6.4 The twisted S -matrix

Let $|f\rangle$ and $|i\rangle$ denote a collection of free asymptotic states at $t = \pm\infty$ respectively. We also assume that we can define in some way the one-particle incoming and outgoing states. This is a very nontrivial assumption since in a theory in which localization is impossible, the concept of asymptotic state may not be well defined. Nevertheless, is it reasonable to expect that also in this theory for small θ and for large distances and times, it is possible to talk on incoming and outgoing states expandable in terms of momentum eigenstates $|k\rangle$.

As in standard books in quantum field theory, we define the S -matrix as the matrix which describes the scattering of the initial $|i\rangle$ states into the final $|f\rangle$ states

$$S_{fi} = \left\langle f \left| \begin{array}{c} \star \\ i \end{array} \right. \right\rangle_{\text{in}\star} = \left\langle f \left| S \left| \begin{array}{c} \star \\ i \end{array} \right. \right. \right\rangle_{\text{out}\star} = \left\langle f \left| S \left| \begin{array}{c} \star \\ i \end{array} \right. \right. \right\rangle_{\text{in}\star}^{\star\text{in}}. \quad (6.4.1)$$

The one-particle asymptotic state is defined as in (6.3.1) to be

$$|k\rangle_{\text{in}} = N_\star(k) a_k^\dagger |0\rangle_{\text{in}} = -N_\star(k) \frac{i}{\sqrt{(2\pi)^3 2\omega_k}} \int d^3x e^{-ik \cdot x} \overleftrightarrow{\partial}_0 \phi_{\text{in}}(x) |0\rangle_{\text{in}} \quad (6.4.2)$$

for the in states and an analogous formula for the out states where $N_\star(k)$ is a normalization factor to be determined for the Moyal and Wick-Voros cases separately. Moreover, we assume, as in the commutative case, that the matrix elements of the interacting field $\phi(x)$ approaches those of the free asymptotic field as time goes to $\pm\infty$. That is,

$$\lim_{x^0 \rightarrow \pm\infty} \langle f | \phi(x) | i \rangle = Z^{1/2} \langle f | \phi_{\text{out}}(x) | i \rangle \quad (6.4.3)$$

with Z a renormalization factor. To be definite, let us consider an elastic process of two particles in two particles. According to the previous section we have

$$S_{fi_\star}(k_1, \dots, k_4) = \left\langle k_1, k_2 \mid k_3, k_4 \right\rangle_{\text{in}\star} = e^{\sum_{a<b} k_a \bullet k_b} \left\langle k_1, k_2 \mid k_3, k_4 \right\rangle_{\text{out}} \quad (6.4.4)$$

which can be expressed in terms of Green's functions following the same procedure as in the commutative case [41]. On repeatedly using (6.4.2) and (6.4.3) we arrive at

$$\begin{aligned} S_{fi} &= \left\langle k_1, k_2 \mid k_3, k_4 \right\rangle_{\text{out}\star} = \text{disconnected graphs} \\ &+ \bar{N}_\star(k_1) \bar{N}_\star(k_2) N_\star(k_3) N_\star(k_4) (iZ^{-1/2})^2 e^{-\sum_{a<b} k_a \bullet k_b} \\ &\int \prod_{a=1}^4 \frac{d^3 x^a}{\sqrt{(2\pi)^3 2\omega_{k_a}}} e^{-ik_a \cdot x^a} (\partial_\mu^2 + m^2)_a G^{(4)}(x_1, x_2, x_3, x_4) \end{aligned} \quad (6.4.5)$$

where $G^{(4)}(x_1, x_2, x_3, x_4)$ is the four-point Green's function. In order to fix the normalization of the asymptotic states, let us compute the scattering amplitude for one particle going into one particle at zeroth order. Up to the undeformed normalization factors $N(p_a)$, this has to give a delta function

$$\begin{aligned} \bar{N}(k) N(p) \delta^{(3)}(k - p) &= N_\star^*(k) N_\star(p) \left\langle k \mid p \right\rangle_{\text{out}\star} \\ &= N_\star^*(k) N_\star(p) e^{-k \bullet p} \left\langle k \mid p \right\rangle_{\text{out}} \\ &= N_\star^*(k) N_\star(p) e^{-k \bullet p} \delta^{(3)}(k - p) \end{aligned} \quad (6.4.6)$$

from which follows

$$N_{\star_M}(p) = N(p) \quad (6.4.7)$$

$$N_{\star_V}(p) = e^{-\frac{\theta}{4} p^2} N(p). \quad (6.4.8)$$

Let us now compute the scattering amplitude for the process above that is, the scattering of two-particles in two particles at one loop. We have two kinds of contribution to (6.4.5), one coming from the planar term (6.2.36) which in spatial coordinates reads

$$G_{\mathbb{P}}^{(4)}(x_1, x_2, x_3, x_4) = \int \prod_{a=1}^4 \frac{d^3 k_a}{\sqrt{(2\pi)^3 2\omega_{k_a}}} e^{ik_a \cdot x^a} \tilde{G}_{\mathbb{P}}^{(4)}(k_1, k_2, k_3, k_4) \quad (6.4.9)$$

and the other coming from non-planar terms (6.2.37)

$$G_{\mathbb{NP}}^{(4)}(x_1, x_2, x_3, x_4) = \int \prod_{a=1}^4 \frac{d^3 k_a}{\sqrt{(2\pi)^3 2\omega_{k_a}}} e^{ik_a \cdot x^a} \tilde{G}_{\mathbb{NP}}^{(4)}(k_1, k_2, k_3, k_4). \quad (6.4.10)$$

In the planar case we find the same result in the Moyal and Wick-Voros cases which coincide with the ordinary result:

$$\begin{aligned} S_{fi_{\star P}}(k_1, \dots, k_4) &= \frac{(-ig)^2}{8} (2\pi)^3 \bar{N}(k_1) \bar{N}(k_2) N(k_3) N(k_4) \prod_{a=1}^4 e^{\frac{\theta}{4} k_a^2} \\ &e^{-\sum_{a<b} k_a \bullet k_b} \int \prod_{a=1}^4 \frac{d^3 x^a}{\sqrt{(2\pi)^3 2\omega_{k_a}}} e^{-ik_a \cdot x^a} \int \prod_{a=1}^4 \frac{d^3 p_a}{\sqrt{(2\pi)^3 2\omega_{p_a}}} e^{ip_a \cdot x^a} (-p_a^2 + m^2) \\ &\int \frac{d^3 q}{(2\pi)^3} \frac{e^{\sum_{a \leq b} p_a \bullet p_b} \delta^{(3)}(\sum_{a=1}^4 p_a)}{(q^2 - m^2) [(p_1 + p_2 - q)^2 - m^2] \prod_{a=1}^4 (p_a^2 - m^2)}. \end{aligned} \quad (6.4.11)$$

The integration over the x^a variables yields factors of $(2\pi)^3 \delta^{(3)}(k_a - p_a)$ and so the propagators of the external legs cancel as in the standard case; as well as the factor

$$\prod_{a=1}^4 e^{\frac{\theta}{4} k_a^2} e^{-\sum_{a<b} k_a \bullet k_b} e^{\sum_{a \leq b} p_a \bullet p_b} \delta^{(3)}(k_a - p_a) \rightarrow 1 \quad (6.4.12)$$

we are left with the usual commutative expression so that

$$S_{fi_{\star P}}(k_1, \dots, k_4) = S_{fi}(k_1, \dots, k_4). \quad (6.4.13)$$

In the non-planar case instead we find

$$\begin{aligned}
S_{fi_{\star NP}}(k_1, \dots, k_4) &= \frac{(-ig)^2}{8} (2\pi)^3 \bar{N}(k_1) \bar{N}(k_2) N(k_3) N(k_4) \prod_{a=1}^4 e^{\frac{\theta}{4} k_a^2} \\
e^{-\sum_{a<b} k_a \bullet k_b} &\int \prod_{a=1}^4 \frac{d^3 x^a}{\sqrt{(2\pi)^3 2\omega_{k_a}}} e^{-ik_a \cdot x^a} \int \prod_{a=1}^4 \frac{d^3 p_a}{\sqrt{(2\pi)^3 2\omega_{p_a}}} e^{ip_a \cdot x^a} (-p_a^2 + m^2) \\
&\int \frac{d^3 q}{(2\pi)^3} \frac{e^{\sum_{a \leq b} p_a \bullet p_b + E_a} \delta^{(3)}(\sum_{a=1}^4 p_a)}{(q^2 - m^2) [(p_1 + p_2 - q)^2 - m^2] \prod_{a=1}^4 (p_a^2 - m^2)}. \quad (6.4.14)
\end{aligned}$$

After integrating over x^a the propagators of the external legs cancel and the simplification (6.4.12) continues to hold, but we are left with the exponential of E_a which does not simplify. This factor is an imaginary phase and it has the same expression in the Moyal and Wick-Voros cases. It depends on the q so that it gets integrated and modifies the ultraviolet behaviour of the loop. Furthermore, it is responsible for the UV/IR mixing [21]. Therefore, we can conclude that

$$S_{fi_{\star M} NP}(k_1, \dots, k_4) = S_{fi_{\star V} NP}(k_1, \dots, k_4) \neq S_{fi}(k_1, \dots, k_4). \quad (6.4.15)$$

Conclusions

Throughout this thesis we have investigated the ultraviolet behaviour of a non-commutative field theory obtained from an ordinary one substituting the commutative product with a non-commutative one that is, a star product. In particular, we have considered the scalar ϕ^4 field theory deformed with the Wick-Voros product, a variant of the well-known Moyal product. We have discussed both the classical and the quantum field theory and calculated the vertex, the two- and four-point Green's functions and their corrections up to one-loop. We have found that the vertex, like in the Moyal case, is not anymore invariant for the exchange of the external momenta, but it maintains invariance for cyclic permutations. Thus the planar and non-planar diagrams for the calculation of the one-loop corrections to the Green's functions behave differently. Indeed, the planar diagrams have the same behaviour as the commutative ones, while the non-planar diagrams present the phenomenon of ultraviolet/infrared mixing, like in the Moyal case [21]. That is, for high internal momentum the ultraviolet divergences are damped by a phase, but these divergences reappear in the infrared (for low incoming momenta). This is to be expected because heuristically this is consequence of commutation relation which is, of course, the same in both theories.

More in general, we have shown that the ultraviolet/infrared mixing found for the Moyal and Wick-Voros products is a generic feature of any translation invariant associative product. To this end, we have introduced a general associative product and then discussed its translational invariance properties. We have found that the vertex is changed by an exponential which maintains invariance for cyclic permutation of the external momenta, but not for any arbitrary exchange. So, like in the Moyal and Wick-Voros cases, the planar and non-planar diagrams behave differently. In particular, the non-planar diagram present the same kind of ultraviolet/infrared mixing. Moreover, we have showed that the phase appearing in the exponent in the non-planar

diagram is related to the commutator of the coordinates so that we can state that the mixing is given by the Poisson structure of the underlying space.

Going back to the discussion about the Moyal product versus the Wick-Voros one, the two products are not equivalent at first sight. Indeed, we have found different Green's functions despite both the physical intuition and the fact that the two star products are algebraically equivalent, in the sense that they define exactly the same deformed algebra [44, 45] and as such describe the same non-commutative geometry.

The element we have used in order to solve this puzzle is symmetries. Indeed, the commutation relation (3.1.1) breaks the Poincaré symmetry, but it can be easily reinstated at a deformed level, as a non-commutative and non-cocommutative Hopf algebra as described in [28, 29, 30], since both products can be seen as coming from a Drinfeld twist [25, 26]. We have showed how the presence of a twist forces us to reconsider all of the steps in a field theory which has to be built in a coherent twisted way. We have found that there is equivalence between the Moyal and Wick-Voros field theories at the level of S -matrix in agreement with our physical intuition, since Green's functions are not observable quantities while S -matrix is. Moreover, this equivalence is obtained only if a consistent procedure of twisting all products is applied. Therefore, we have used the field theories built with the Moyal and Wick-Voros products to check each other and to obtain an indication on the procedure to follow for non-commutative theories coming from a twist.

Appendix A

An elementary introduction to the Hopf algebras

In the following appendix we present a very elementary introduction to the theory of Hopf algebras. We just collect some very essential definitions of Hopf algebras [71] that we have used throughout the thesis, mainly in the last chapter.

A.1 Algebras and coalgebras

We begin with the notion of algebra for completeness. A complex algebra is a complex vector space \mathcal{A} equipped with a linear map called multiplication

$$m : a \otimes b \in \mathcal{A} \otimes \mathcal{A} \rightarrow m(a \otimes b) = ab \in \mathcal{A}$$

associative namely which satisfies the condition

$$m \circ (m \otimes \text{id}) = m \circ (\text{id} \otimes m) \tag{A.1.1}$$

or equivalently the condition

$$(ab)c = a(bc) \tag{A.1.2}$$

for any $a, b, c \in \mathcal{A}$, where id denotes the identity map of \mathcal{A} . An algebra \mathcal{A} is commutative if

$$m \circ \tau = m \tag{A.1.3}$$

or equivalently if

$$ab = ba \tag{A.1.4}$$

for any $a, b \in \mathcal{A}$, where

$$\tau : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$$

defined by

$$\tau(a \otimes b) = b \otimes a \tag{A.1.5}$$

for any $a, b \in \mathcal{A}$ is the exchange map. An algebra \mathcal{A} is unitary if there exists a linear map called unit

$$\eta : \mathbb{C} \rightarrow \mathcal{A}$$

which satisfies the condition

$$m \circ (\eta \otimes \text{id}) = m \circ (\text{id} \otimes \eta) = \text{id} \tag{A.1.6}$$

or equivalently the condition

$$\eta(1)a = a\eta(1) = a \tag{A.1.7}$$

for any $a \in \mathcal{A}$. In what follows we set

$$\eta(1) = \mathbb{1}. \tag{A.1.8}$$

Finally, a homomorphism between two algebras \mathcal{A} and \mathcal{A}' is a linear map

$$f : \mathcal{A} \rightarrow \mathcal{A}'$$

such that

$$f \circ m = m' \circ (f \otimes f) \tag{A.1.9}$$

or equivalently such that

$$f(ab) = f(a)f(b) \tag{A.1.10}$$

for any $a, b \in \mathcal{A}$. Moreover, if \mathcal{A} and \mathcal{A}' are unitary algebras, we assume that

$$f \circ \eta = \eta' \tag{A.1.11}$$

or equivalently that

$$f(\mathbb{1}) = \mathbb{1}'. \tag{A.1.12}$$

A complex coalgebra is a complex vector space \mathcal{A} equipped with a linear map called comultiplication

$$\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$$

coassociative namely which satisfies the condition

$$(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta \quad (\text{A.1.13})$$

or equivalently the condition

$$a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)} = a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)} \quad (\text{A.1.14})$$

for any $a \in \mathcal{A}$, where we have used the generalized Sweedler's notation¹:

$$\Delta(a) = a_{(1)} \otimes a_{(2)} \quad (\text{A.1.15})$$

for any $a \in \mathcal{A}$. A coalgebra \mathcal{A} is cocommutative if

$$\tau \circ \Delta = \Delta \quad (\text{A.1.16})$$

or equivalently if

$$a_{(1)} \otimes a_{(2)} = a_{(2)} \otimes a_{(1)} \quad (\text{A.1.17})$$

for any $a \in \mathcal{A}$. A coalgebra \mathcal{A} is counitary if there exists a linear map called counit

$$\varepsilon : \mathcal{A} \rightarrow \mathbb{C}$$

which satisfies the condition

$$(\varepsilon \otimes \text{id}) \circ \Delta = (\text{id} \otimes \varepsilon) \circ \Delta = \text{id} \quad (\text{A.1.18})$$

or equivalently the condition

$$\varepsilon(a_{(1)}) a_{(2)} = a_{(1)} \varepsilon(a_{(2)}) = a \quad (\text{A.1.19})$$

for any $a \in \mathcal{A}$. Finally, a homomorphism between two coalgebras \mathcal{A} and \mathcal{A}' is a linear map

$$f : \mathcal{A} \rightarrow \mathcal{A}'$$

¹For any $a \in \mathcal{A}$, the Sweedler's notation consists of writing

$$\Delta(a) = \sum_i a_{(1)}^i \otimes a_{(2)}^i$$

with $a_{(1)}^i, a_{(2)}^i \in \mathcal{A}$ or more simply

$$\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}.$$

such that

$$\Delta' \circ f = (f \otimes f) \circ \Delta \quad (\text{A.1.20})$$

or equivalently such that

$$\Delta'(f(a)) = f(a_{(1)}) \otimes f(a_{(2)}) \quad (\text{A.1.21})$$

for any $a \in \mathcal{A}$. Moreover, if \mathcal{A} and \mathcal{A}' are counitary coalgebras, we assume as well

$$\varepsilon' \circ f = \varepsilon. \quad (\text{A.1.22})$$

A.2 Bialgebras and Hopf algebras

A complex bialgebra is a complex vector space \mathcal{A} that is at the same time a complex unitary algebra and a complex counitary coalgebra in a compatible way namely the multiplication, the comultiplication, the unit and the counit satisfy the conditions

$$\begin{aligned} \Delta \circ m &= (m \otimes m) \circ (\text{id} \otimes \tau \otimes \text{id}) \circ (\Delta \otimes \Delta) \\ \varepsilon \circ m &= \varepsilon \otimes \varepsilon \\ \Delta \circ \eta &= \eta \otimes \eta \\ \varepsilon \circ \eta &= \text{id}_{\mathbb{C}} \end{aligned} \quad (\text{A.2.1})$$

where $\text{id}_{\mathbb{C}}$ denotes the identity map of \mathbb{C} . Equivalently the compatibility relations between the two structures can be written as²

$$\begin{aligned} \Delta(ab) &= \Delta(a)\Delta(b) \\ \varepsilon(ab) &= \varepsilon(a)\varepsilon(b) \\ \Delta(\mathbb{1}) &= \mathbb{1} \otimes \mathbb{1} \\ \varepsilon(\mathbb{1}) &= 1 \end{aligned} \quad (\text{A.2.2})$$

for any $a, b \in \mathcal{A}$. A bialgebra \mathcal{A} is commutative if it is commutative like an algebra and it is cocommutative if it is cocommutative like a coalgebra. Finally, a homomorphism between two bialgebras \mathcal{A} and \mathcal{A}' is linear map

$$f : \mathcal{A} \rightarrow \mathcal{A}'$$

which is both an unitary algebra and counitary coalgebra homomorphism.

² $\Delta(a)\Delta(b) = a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}$ for any $a, b \in \mathcal{A}$.

A complex Hopf algebra is a complex bialgebra \mathcal{A} equipped with a linear map called antipode

$$S : \mathcal{A} \rightarrow \mathcal{A}$$

such that

$$m \circ (S \otimes \text{id}) \circ \Delta = m \circ (\text{id} \otimes S) \circ \Delta = \eta \circ \varepsilon \quad (\text{A.2.3})$$

or equivalently such that

$$S(a_{(1)}) a_{(2)} = a_{(1)} S(a_{(2)}) = \varepsilon(a) \mathbb{1} \quad (\text{A.2.4})$$

for any $a \in \mathcal{A}$. The role of the antipode is like that of an inverse. However, we do not demand that $S^2 = \text{id}$. The antipode is unique and satisfies the conditions

$$\begin{aligned} S \circ m &= m \circ \tau \circ (S \otimes S) \\ \Delta \circ S &= (S \otimes S) \circ \tau \circ \Delta \\ S \circ \eta &= \eta \\ \varepsilon \circ S &= \varepsilon \end{aligned} \quad (\text{A.2.5})$$

or equivalently the conditions

$$\begin{aligned} S(ab) &= S(b)S(a) \\ \Delta(S(a)) &= S(a_{(2)}) \otimes S(a_{(1)}) \\ S(\mathbb{1}) &= \mathbb{1} \\ \varepsilon(S(a)) &= \varepsilon(a) \end{aligned} \quad (\text{A.2.6})$$

for any $a, b \in \mathcal{A}$. Therefore, the antipode is an unitary algebra and counitary coalgebra antihomomorphism. Like for bialgebras, a Hopf algebra is commutative if it is commutative like an algebra and it is cocommutative if it is cocommutative like a coalgebra. Notice that if \mathcal{A} is a commutative or cocommutative Hopf algebra, then

$$S^2 = \text{id}. \quad (\text{A.2.7})$$

Moreover, a bialgebra homomorphism between two Hopf algebras \mathcal{A} e \mathcal{A}'

$$f : \mathcal{A} \rightarrow \mathcal{A}'$$

is automatically a Hopf algebra homomorphism i.e it satisfies the condition

$$f \circ S = S' \circ f. \quad (\text{A.2.8})$$

An example of Hopf algebra is given by universal enveloping algebra $U(g)$ of a Lie algebra g , where the comultiplication is defined by

$$\Delta(\xi) = \xi \otimes \mathbb{1} + \mathbb{1} \otimes \xi \quad (\text{A.2.9})$$

the counit is defined by

$$\varepsilon(\xi) = 0 \quad (\text{A.2.10})$$

and the antipode is defined by

$$S(\xi) = -\xi \quad (\text{A.2.11})$$

for any $\xi \in U(g)$. Furthermore, the comultiplication and the counit are extended as unitary algebra homomorphisms while the antipode is extended as a counitary coalgebra antihomomorphism.

A.3 Cocycles and twists

Let \mathcal{A} be a Hopf algebra. Consider the maps

$$\Delta_i : \underbrace{\mathcal{A} \otimes \mathcal{A} \dots \otimes \mathcal{A}}_{n\text{-times}} \rightarrow \underbrace{\mathcal{A} \otimes \mathcal{A} \dots \otimes \mathcal{A}}_{(n+1)\text{-times}} \quad (\text{A.3.1})$$

defined by

$$\Delta_i = \text{id} \otimes \text{id} \dots \otimes \text{id} \otimes \Delta \otimes \text{id} \otimes \text{id} \dots \otimes \text{id} \quad (\text{A.3.2})$$

with Δ is in the i th position and $i = 1, 2, \dots, n$. Moreover, we define

$$\Delta_0 = \mathbb{1} \otimes () \quad \text{and} \quad \Delta_{n+1} = () \otimes \mathbb{1}. \quad (\text{A.3.3})$$

An n -cochain is an invertible element

$$\chi \in \underbrace{\mathcal{A} \otimes \mathcal{A} \dots \otimes \mathcal{A}}_{n\text{-times}} \quad (\text{A.3.4})$$

and its coboundary as the $(n+1)$ -cochain

$$\partial\chi = \left(\prod_{i \text{ even}} \Delta_i \chi \right) \left(\prod_{i \text{ odd}} \Delta_i \chi \right) \quad (\text{A.3.5})$$

and the products are each taken in increasing order. An n -cochain χ is an n -cocycle if

$$\partial\chi = \mathbb{1} \quad (\text{A.3.6})$$

and it is counitary if

$$\varepsilon_i \chi = \mathbb{1} \quad (\text{A.3.7})$$

for all

$$\varepsilon_i = \text{id} \otimes \text{id} \dots \otimes \text{id} \otimes \varepsilon \otimes \text{id} \otimes \text{id} \dots \otimes \text{id} \quad (\text{A.3.8})$$

with ε is in the i th position. For example, a 1-cocycle is an invertible element $\chi \in \mathcal{A}$ such that

$$\chi \otimes \chi = \Delta \chi \quad (\text{A.3.9})$$

and it is automatically counitary. Instead, a 2-cocycle is an invertible element $\chi \in \mathcal{A} \otimes \mathcal{A}$ such that

$$(\mathbb{1} \otimes \chi)(\text{id} \otimes \Delta)\chi = (\chi \otimes \mathbb{1})(\Delta \otimes \text{id})\chi \quad (\text{A.3.10})$$

and it is counitary if

$$(\varepsilon \otimes \text{id})\chi = (\text{id} \otimes \varepsilon)\chi = \mathbb{1}. \quad (\text{A.3.11})$$

A twist is a counitary 2-cocycle which is usually denoted by \mathcal{F} and with the generalized Sweedler's notation it can be written as

$$\mathcal{F} = \mathcal{F}_{(1)} \otimes \mathcal{F}_{(2)}. \quad (\text{A.3.12})$$

A.4 Quasi-triangular Hopf algebras

A Hopf algebra \mathcal{A} is quasi-triangular if there exists an invertible element $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$ called a quasi-triangular structure or universal \mathcal{R} -matrix which satisfies the following conditions:

$$(\Delta \otimes \text{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23} \quad (\text{A.4.1})$$

$$(\text{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \quad (\text{A.4.2})$$

$$\tau \circ \Delta(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1} \quad \forall a \in \mathcal{A} \quad (\text{A.4.3})$$

where with the generalized Sweedler's notation

$$\mathcal{R} = \mathcal{R}_{(1)} \otimes \mathcal{R}_{(2)} \quad (\text{A.4.4})$$

and

$$\mathcal{R}_{12} = \mathcal{R}_{(1)} \otimes \mathcal{R}_{(2)} \otimes \mathbb{1} \quad (\text{A.4.5})$$

$$\mathcal{R}_{13} = \mathcal{R}_{(1)} \otimes \mathbb{1} \otimes \mathcal{R}_{(2)} \quad (\text{A.4.6})$$

$$\mathcal{R}_{23} = \mathbb{1} \otimes \mathcal{R}_{(1)} \otimes \mathcal{R}_{(2)}. \quad (\text{A.4.7})$$

Note that it is possible to show that

$$(\varepsilon \otimes \text{id})\mathcal{R} = (\text{id} \otimes \varepsilon)\mathcal{R} = \mathbb{1}. \quad (\text{A.4.8})$$

Moreover,

$$(S \otimes \text{id})\mathcal{R} = \mathcal{R}^{-1} \quad (\text{A.4.9})$$

$$(\text{id} \otimes S)\mathcal{R}^{-1} = \mathcal{R} \quad (\text{A.4.10})$$

and hence

$$(S \otimes S)\mathcal{R} = \mathcal{R}. \quad (\text{A.4.11})$$

Finally, it is easy to see that \mathcal{R} obeys the abstract Yang-Baxter equation:

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \quad (\text{A.4.12})$$

To conclude this section, we recall that given a Hopf algebra \mathcal{A} , we can get a new Hopf algebra by means of a twist \mathcal{F} that is, by twisting the initial Hopf algebra \mathcal{A} . Indeed, it is not difficult to see that there is a new Hopf algebra $\mathcal{A}_{\mathcal{F}}$ with the same algebra structure and counit of \mathcal{A} and the comultiplication, the antipode and the universal \mathcal{R} -matrix given respectively by

$$\Delta_{\mathcal{F}}(a) = \mathcal{F}\Delta(a)\mathcal{F}^{-1} \quad (\text{A.4.13})$$

$$S_{\mathcal{F}}(a) = US(a)U^{-1} \quad (\text{A.4.14})$$

$$\mathcal{R}_{\mathcal{F}} = \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1} \quad (\text{A.4.15})$$

for any $a \in \mathcal{A}_{\mathcal{F}}$ where U is invertible and given by

$$U = \sum \mathcal{F}^{(1)}S(\mathcal{F}^{(2)}) \quad (\text{A.4.16})$$

and

$$\mathcal{F}_{21} = \mathcal{F}_{(2)} \otimes \mathcal{F}_{(1)}. \quad (\text{A.4.17})$$

Notice that if \mathcal{A} is just a Hopf algebra, then so is $\mathcal{A}_{\mathcal{F}}$.

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