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Some developments in Pre-equilibrium and Equilibrium analysis of Variational Inequalities

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To Francesca and to our Totem.

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Abstract

In the present work, the notion of equilibrium and pre-equilibrium of variational inequalities (but also some for some quasi-variational inequalities)is developed in Weighted Hilbert spaces, in strictly convex and smooth Banach spaces and in reflexive Banach spaces. The concept of Weighted variational inequality is introduced, some associated questions as regularity,delayed equilibrium and Lagrangian duality are developed and applied to the traffic equilibrium problem. The more recent notion of pre-equilibrium very important in time dependent equilibrium must be understood as the optimal path from an arbitrarily point to reach the equilibrium (critical point of the system). The notion of Non pivot and Implicit Dynamical system is introduced, an existence result is given (in Hilbert spaces with linear duality mapping) as application an existence result is given also for a specific quasi-variational inequality (translated set) without using the classical assumption for the projection (Lipschitz) [This assumption is wrong a very simple case and a counter example is provided]. The notion of projected dynamical systems is extended to strictly convex and smooth Banach spaces and reflexive Banach spaces and the equivalence between critical points of such PDS and equilibrium of Variational inequalities is proved. Some applications will also be given to the traffic equilibrium problem, an elementary design of an industrial application will be also illustrated.

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Chapter 1

Introduction

The aim of this thesis is to provide a detailed study of the traffic equilibrium problem focusing the attention on the management of the major and minor congestions of the traffic flows and on the choose of the more convenient distribution of the traffic demand on the paths.
The congestion on the network can be modeled by introductions some weights acting on the paths and on the path cost functions. This model will let to the introduction of weighted variational inequalities and weighted projected dynamical systems. Moreover, a more realistic model of the weighted traffic equilibrium problem requests the use of the delay because the transmission of the data does not happen with infinity speed. Therefore this characteristic justifies the introduction of retarded weighted variational inequalities and retarded weighted projected dynamical system (this last point point has not been included into this thesis because it is still under investigation). The study of the new models has requested a generalization of existing theories. Moreover, for a more complete analysis of the problem, computational methods has been also generalized and a new visualization method as been set up. Since the critical points to projected dynamical systems are the solutions to evolutionary variational inequalities, our study starts generalizing the known results on the projected dynamical systems. Precisely, our extensions regards the use of non-pivot Hilbert spaces and Reflexive Banach Spaces. In Banach Spaces we have no inner product but only duality pairing and apparently we have no projection concept that can be considered as an extension of the usual projection operator. But looking more deeply some interesting results Alber (1996, 2000) regarding an extension of the projection operator into Strictly Convex and

Smooth Banach Spaces have been found. Strictly Convex and Smooth Banach Spaces seems to be the nearest subclass of Banach Space to Hilbert Spaces (L^p spaces for $1 < p < +\infty$ are strictly convex and smooth Banach spaces). In fact the unit Ball of an Hilbert space is round (and the duality mapping $J = Id$ and the space can be identified with its dual space) the unit ball of a Strictly convex and smooth Banach space is not round (the duality mapping is an Isometry but $J \neq Id$ and J is nonlinear). It has been possible to obtain in this framework interesting results in the sense of an existence result for Projected Dynamical Systems (see section 3.3) but the non linearity of the duality paring coupled with the non linearity of the generalized projection operator seems to be a non banal obstacle at least using constructive methods used in Cojocaru (2002). A new problem appeared indirectly, if the $J \neq Id$ and J is linear, to treat that case, it is necessarily to introduce the framework of Non pivot spaces and therefore with M.G. Cojocaru the problem of the existence of a solution in such spaces, has been tackled. Also implicit Projected dynamical Systems are analyzed and the result is used to obtain an interesting existence result for quasi-variational inequalities (see 3.2). As it has been said Projected Dynamical Systems are strictly related to Variational inequalities theory (see Proposition 3.2.6) in the sense that a critical point of a PDS is an equilibrium point of a Variational Inequality and vice versa.

The new framework, Non Pivot Hilbert Spaces, opens the path to the study of different aspects to variational inequality (such has the regularity and numerical methods) but also to go further in detail on this enhanced traffic equilibrium problem (the study of weighted traffic equilibrium with delay and the study of the duality of the traffic equilibrium problem). We propose during WICOM 08 (see Cojocaru & Pia (2008)) an industrial application of this research to Intelligent GPS systems. Even if we extend the theory of PdS and Variational inequality to Weighted Hilbert Spaces there is still some very interesting work to do as some new aspects of the theory jump up from these "old" problems. Recently the primary goal to enhance the PDS theory to a subclass of Banach spaces found new possibilities to be attained. A very hidden paper (military research) produced by Eduardo Zarantonello (see Zarantonello (1977)) change the paternity of the generalization of projection operator in Banach spaces. In fact, even if Yakov Alber introduced this concept in Alber (1996) and he gave some very interesting properties for the restricted framework of Strictly convex and smooth Banach spaces, actually widely used, Zarantonello introduced a projector in Reflexive Banach spaces in order to use this notion in an unachieved study, basically to develop the spectral synthesis on cones (an extension of his well known paper Zarantonello (1971)) .

This work has three Chapters an two Appendixes. In Chapter 2, the relationship between Projected Dynamical System and Variational Inequalities are discussed and this first discussion has the goal to clarify the concept of equilibrium and the concept of pre-equilibrium. In Chapter 3 the theoretical results are exposed and among others we have, existence results, regularity result for weighted Hilbert spaces and equivalence theorem between critical points and equilibrium points moreover in this Chapter we present also open problems and possible directions for future researches. In Chapter 4 we present the results obtained looking more deeply into one specific application: Traffic Equilibrium problem. Of course the results obtained in this section can be easily transposed to other networks as for example Financial Network but to avoid dispersion we focus only on traffic equilibrium problem and develop different aspects (how weights can be obtained, problem with delay, numerical method, analysis of the dual problem and industrial application). To facilitate the reading and to separate in a clear way the results obtained, published and pertinent with the thesis topics, major results for existing theory of Variational Inequalities and Projected dynamical Systems (and some results obtained but not enough pertinent with the scope of the thesis), have been grouped in two Appendixes. Even if sometimes it is difficult to separate in a clear manner the results, the intention has always been to offer a clear reading and to communicate guiding ideas of the research process.

Chapter 2

Notions of Equilibrium and Pre-Equilibrium

2.1 Variational Inequalities

2.1.1 Variational Inequalities and the concept of Equilibrium

Variational Inequalities (VI) are a powerful generalization of a fundamental basic fact in Analysis: The study of stationary point. This statement is of course an extreme simplification but useful to have an intuitive understanding of the importance of the theory and all possible applications. A very good introduction is given in Stampacchia (1997), but for an easier reading we will remind some basics facts. For an overview of the theory of (VI), the reader can refer to Appendix B.

Example 2.1.1. *Let $f \in \mathcal{C}^1$ with $f : [a, b] \to \mathbb{R}$. We wish to determine those points x_0 for which:*

$$f(x_0) = \min_{a \leq x \leq b} f(x).$$

It's clear that there exists at least one such point x_0.
The following cases can occur.

- *If $a < x_0 < b$, then $f^{'}(x_0) = 0$,*
- *If $x_0 = a$, then $f^{'}(x_0) \geq 0$,*
- *If $x_0 = b$, then $f^{'}(x_0) \leq 0$.*

Therefore it is clear that for any such x_0 *we have, for all* $x \in [a, b]$,

$$f^{'}(x_0)(x - x_0) \geq 0$$

this is the first example of variational inequality.

Example 2.1.2. *Let* $\mathbb{K}$ *be a closed, convex set in* $\mathbb{R}^n$ *and let*

$$f : \mathbb{K} \to \mathbb{R}, \; f \in \mathcal{C}^2(\mathbb{R})$$

Let $x_0 \in \mathbb{K}$ *be such that* $f(x_0) = \min_{x \in \mathbb{K}} f(x)$*. Since* $\mathbb{K}$ *is convex we have for each* $y \in \mathbb{K}$ *that*

$$\lambda x_0 + (1 - \lambda)y \in \mathbb{K}, \; 0 \leq \lambda \leq 1.$$

Define

$$F : [0, 1] \to \mathbb{R},$$

$$F(\lambda) = f(\lambda x_0 + (1 - \lambda)y).$$

Then $F(1) = \min_{\mathbb{K}} f$*. It follows from (2.1.1) that* $F^1(\lambda - 1) \geq 0$ *for all* $0 \leq \lambda \leq 1$*; but this is equivalent to* $F^{'} \leq 0$*. Therefore, for* $x_0 \in \mathcal{K}$,

$$[grad \; f(x_0)](y - x_0) \geq 0, \; \forall y \in \mathbb{K}$$

becomes the variational inequality for this example

Example 2.1.3. *Now let* V *be a real Hilbert space and let* $\mathbb{K}$ *be a closed convex subset of* V*. Let* $f \in V$*. If* $u_0 \in \mathbb{K}$ *such that*

$$||u_0 - f|| = \min_{\mathbb{K}} ||u - f||$$

we will say $u_0 = P_{\mathbb{K}}(f)$ *and call* u_0 *the projection of* f *onto* $\mathbb{K}$*. Now clearly*

$$||u_0 - f|| \leq ||v - f||, \; \forall v \in \mathbb{K}$$

Let us define

$$F(\lambda) = (\lambda u_0 + (1 - \lambda)v - f, \lambda u_0 + (1 - \lambda)v - f) = ||\lambda u_0 + (1 - \lambda)v - f||_V^2$$

where $(.,.)$ *denotes the inner product of* V*.*
Then $F : [0, 1] \to \mathbb{R}$*. From example (2.1.1) we have* $F^{'}(1) \leq 0$*. That mean we have*

$$u_0 \in \mathbb{K}, \; (u_0, v - u_0) \geq (f, v - u_0), \; \forall v \in \mathbb{K}. \tag{2.1.1}$$

We can rewrite the previous inequality as $(u_0 - f, v - u_0) \geq 0$*. In fact (2.1.1)implies* $u_0 = P_{\mathbb{K}}(f)$*. Here also an existence result exist.*

From these classical examples, up to now, a lot of directions has been explored and a lot of applications has been given.

The usage of VI in specific infinite dimensional (functional) spaces permits to prove interesting results as done for example in Daniele & Maugeri (2001) with the introduction of the concept of evolutionary equilibrium (applied to the traffic equilibrium problem). Using these framework, the equilibrium (the solution of the variational inequality) it is not anymore a "point" but a functional (i.e a point in the functional space), which means that the solution of the VI is time dependent. Of course a specific focus on this "evolutionary" set up open a full range of new problems as, regularity or the differentiability of the solution with respect to the time variable, but also a large number of applications (see for example Nagurney & Dong (2002)).

2.2 Projected Dynamical Systems

These systems are non-smooth dynamical systems and are defined as the solutions to a class of ordinary differential equations with a discontinuous right-hand side. We refer to this class of equations as projected differential equations. The word projected indicates the use of a projection operator in defining a projected differential equation. This operator restrains the whole Hilbert space X onto a non-empty, closed and convex subset $K \subset X$. Suppose to be in a non equilibrium situation, i.e based on the definition we give in Chapter 3 (section 3.1.3), we have the variational inequality not satisfied but using an existence theorem we know that under certain conditions, the equilibrium exists. The study of the optimal trajectory to reach the equilibrium starting from an arbitrary point into our constraints set (the convex K) is precisely one of the purposes of PDS. the projected differential equation is given in Hilbert case by

$$\frac{dx}{dt} = P_{T_K(x)}(-F(x(t))),\ a.a.\ t \in I$$

Where K is a closed convex set and F a vector field.

If we are dealing with time dependent functions, that is if for instance if the variational inequality is defined on a functional space, the study of desequilibirum behaviour is described using a different timescale. Solving the PDS means in that case to find and Absolutely Countinuous function for $[0, T[,\ T > 0$ with values in a functional space X.

$$\frac{dx_\gamma}{dt} = P_{T_{K_{x_\gamma}}}(-F(x_\gamma(t))),\ a.a.\ t \in I$$

and our trajectory is in a certain dependent of the time parameter γ depending on the choice of the functional space in which we set up the Variational inequality problem.

The coexistence of two times scales has been called double-layered dynamics in Cojocaru *et al.* (2006) , check also A.3.3.

The concept of the double time scale emerge naturally, from an historical point of view in fact PDS theory has been introduced to study the dynamical part of Variational inequality, when the variational inequality were only treated in $\mathbb{R}^n$.

The motivation for studying a projected dynamical system is that it can be used in the study of dynamics of perturbed steady states of problems arising from Economic Theory, Physics and Engineering A micro-time scale which is used to describe the pre-equilibrium situation using the projected Dynamical system setting and a macro-time scale which is used to describe the evolution of the equilibrium situation. We can even affirm that this is the more interesting problem as in most applied problem condition changes so quickly that it is very difficult to maintain the equilibrium state on evolutionary problems but the possibility to have the optimal path to the equilibrium can represent a valid alternative to follow.

2.2.1 Projected Dynamical Systems and the concept of Pre-Equilibrium

When we deal with time dependent problems, equilibrium states have to be considered as we said before as time dependent. So the question to answer is, how looks the time dependent functional that describes equilibrium? Answer to that question is the main goal when we treat a variational equilibrium problem. Obviously in general it is very difficult to have an exact solution, and the approximation of a solution can be also far from reality. The reason is simple: In general it is very difficult to forecast the variables we have to deal with.

This simple fact justifies by itself the introduction of the Projected Dynamical systems (PDS). We will see later on 3.2.4.2, 3.3.2.3 and 3.4.4 that PDS (associated to a VI) and Variational Inequalities have as a contact point the equivalence theorems. Therefore we can say that the solution of PDS is the “best” trajectory to reach an equilibrium point from a given point in the constraints set. In abstract, in an infinite constraints set there are an infinity of possible states that precede an equilibrium if we start from an arbitrary point in such set. Nevertheless we can prove under certain regularity conditions that among all possible trajectories the is one better that others, in the sense that finding the solutions of a PDS is equivalent in finding the "slow" solution (the solution of minimal norm) to a differential variational inequality (see Section 3.3.2.4) The study of this exceptional trajectory is the main goal of PDS theory, but several difficulties are difficult to by-pass in Banach spaces, nevertheless we provide some elements to built a solution to this problem.

Chapter 3

Results in Weighted Hilbert Spaces and Reflexive Banach Spaces

This chapter is mainly dedicated to abstract results. We illustrate the results obtained in Giuffré *et al.* (2006b), Cojocaru & Pia (2008), Giuffré & Pia (2008), Giuffré *et al.* (2006a), Barbagallo & Pia (2009b) and Barbagallo & Pia (2009a). Basically we focus on existence and regularity results in Weighted Hilbert spaces for Variational inequalities and PDS including some extensions of this concept. Then we introduce some options to set up the Problem for Strictly Convex and Smooth Banach spaces. Intuitively a weighted Hilbert Space is an Hilbert Space in which not all the 'directions' are equivalents, and this basically means that the unit ball is not round. This framework offer the possibility to extend, let's say by compensation, the existence domain of functional on some directions if singularity a appears (to be understood as a point in which a given class of functions is not $L^2(\Omega, \mathbb{R}^n, (.,.))$, where $\Omega \subset \mathbb{R}^p$ is open and $(.,.)$ denotes an inner product). Of course this enhancement works if we can switch from one space to another by multiplying the components by a continuous and strictly positive function defined on Ω. In addition to the weights on the functional space we introduce also weights for the bilinear form, this setting will be used in next chapter for an application to traffic equilibrium problem, but has potentially a large number of applications. It seems that weighted bilinear forms can be used to treat conservative equations (Gao (2000)) but nothing has been explored in such direction in this thesis.

It is divided in the following sections.
In 3.1 we treat Variational Inequalities in Weighted spaces and in order to do that, we recall some basic material (subsection 3.1.1) then we provide some existence (subsection 3.1.2) and regularity results (subsection 3.1.4) all that notions will be used in chapter 4, mainly dedicated to applications.
In section 3.2 we extend the known results on PDS to non-pivot Hilbert spaces and in 3.2.4 to introduce implicit PDS and provide and existence results for a particular quasi-variational inequality with weaker assumptions than the one used in Noor (2003), basically without using Lipchitz conditions on the projection operator . In section 3.3 we set-up PDS in strictly convex and smooth Banach spaces an equivalence theorem 3.3.2.3 between critical points of PDS and solutions of VI, in that way it is possible to justify the notion of pre-equilibrium. We provide also some results that permits to make a bridge between PDS theory and Differential inclusions theory 3.3.2.4. In the last section 3.4, some results given in Zarantonello (1977) are exposed. They offers a large potential for future research. This last section has been included in this thesis because the paper Zarantonello (1977) introduce for the first time the notion of projection in reflexive Banach spaces and give very interesting results (in particular a decomposition theorem) use in Section 3.4.4 to prove an equivalence result. Paper Zarantonello (1977) is almost forgotten and the introduction of the projection operator in Banach Space and the decomposition theorem (only in strictly convex and smooth Banach space) is attributed to Yakov Alber Alber (1996, 2000). Even if Alber obtained very interesting estimates and as developped, several applications of the projection concept in SCS Banach space, we think It is important to point out when such concepts has been introduced for the first time.

3.1 Weighted Variational Inequalities in Weighted Spaces

3.1.1 Dual realization of a Hilbert space

Each time we work with a Hilbert space V, it is necessary to decide whether or not we identify the topological dual space $V^* = \mathcal{L}(V, \mathbb{R})$ with V. Commonly this identification is made, one of the reasons for this being that the vectors of the polar of a set of V are in V. In some cases the identification does not make sense (see Example 3.1.8).

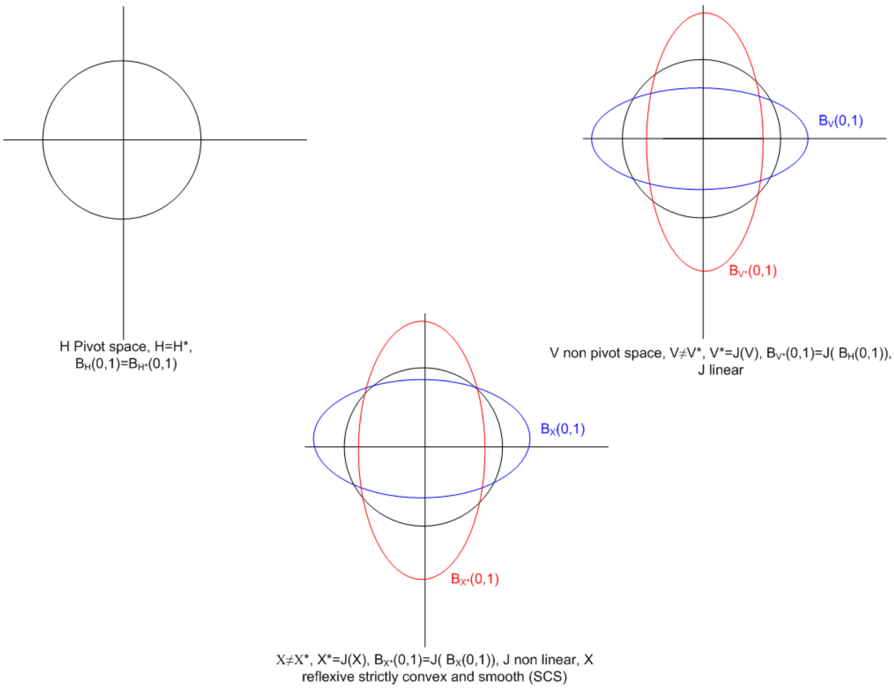

Figure 3.1: A simple view of Balls

For clarity of presentation, we remind below the basic results regarding the dual realization of a Hilbert space. The readers can refer to Aubin (1987) for additional details.

First, consider a pre-Hilbert space V with an inner-product $((x, y))$, and its topological dual $V^* = \mathcal{L}(V, \mathbb{R})$. It is well known that V^* is a Banach space for the classical dual norm ($\|f\|_* = \sup_{x \in V} \frac{|f(x)|}{\|x\|}$). It is also known that there exists an isometry $J : V \to V^*$ such that J is linear and for all $x \in V$, $J(x) = grad(\frac{\|x\|^2}{2})$. This mapping J is called a duality mapping of (V, V^*). The gradient formulation of the duality mapping allows to easily determine the mapping from the norm, as shown by the following examples.

Example 3.1.1. *R^2 endowed with the norm*

$$\|(x_1, x_2)\| = ({x_1}^2 + T{x_2}^2)^{\frac{1}{2}}, T > 0$$

is a reflexive, uniformly convex and uniformly smooth Banach space. And we have:

$$J((x_1, x_2)) = (x_1, Tx_2)$$

R^3 endowed with the norm

$$\|(x_1, x_2, x_3)\| = ({x_1}^2 + {x_2}^2)^{\frac{1}{2}} + ({x_2}^2 + {x_3}^2)^{\frac{1}{2}}$$

is a reflexive, strictly convex and smooth Banach space. And we have:

$$J((x_1, x_2, x_3)) = (x_1(1 + \frac{x_2^2 + x_3^2}{\omega}), x_2(1 + \frac{x_1^2 + 2x_2^2 + x_3^2}{\omega}), x_3(1 + \frac{x_1^2 + x_2^2}{\omega}))$$

where

$$\omega = (({x_1}^2 + {x_2}^2)({x_2}^2 + {x_3}^2))^{\frac{1}{2}}$$

Example 3.1.2. *If $X = L^p(\Omega, \mathbb{R})$ with $1 < p < \infty$ then*

$$J(x) = \|x\|^{2-p}|x|^{p-1}sgn(x)$$

and

$$J^*(x) = \|x\|^{\frac{p-2}{p-1}}|x|^{\frac{1}{1-p}}sgn(x)$$

where $sgn(x) = \chi_{[x>0]} - \chi_{[x<0]}$.

we remind also that the duality mapping J enjoys the following properties:

- J is monotone in arbitrary Banach space.

- J is strictly monotone in strictly convex Banach spaces.
- $J(x) = grad(\|x\|^2/2)$ in smooth Banach spaces.
- J is continuous in Uniformly smooth Banach spaces.
- $J = Id_X \Leftrightarrow X$ is an Hilbert space

Theorem 3.1.3 (Theorem 1 page 68, Aubin (1987))**.** *Let V be a Hilbert space with the inner product $((x, y))$ and $J \in \mathcal{L}(V, V^*)$ the duality mapping above. Then J is a surjective isometry from V to V^*. The dual space V^* is a Hilbert space with the inner product:*

$$((f, g))_* = ((J^{-1}f, J^{-1}g)) = f(J^{-1}g).$$

Theorem 3.1.4 (Theorem 2 page 69, Aubin (1987))**.** *Let V be a pre-Hilbert space. Then there exists a completion $\hat{V}$ of V, that is, an isometry j from V to the Hilbert space $\hat{V}$ such that $j(V)$ is dense in $\hat{V}$.*

Definition 3.1.5. *Let V be a Hilbert space. We call $\{F, j\}$, where*

i) F is a Hilbert space and

ii) j is an isometry from F to $\mathcal{L}(V, \mathbb{R})$

a dual realization of V. We then set

$$\langle f, x \rangle = j \circ f(x), \forall f \in F, \ \forall x \in V,$$

where $\langle f, x \rangle$ is the duality pairing for $F \times V$.

Remark 3.1.6. *The duality pairing is a non degenerate bilinear form on $F \times V$ and $\|f\|_F = \sup_{x \in V} \frac{|\langle f, x \rangle|}{\|x\|}$. These properties permit us to prove that F is isomorphic to V^*.*

We deduce from Theorems 3.1.3 and 3.1.4 that $k = j^{-1} \circ J \in \mathcal{L}(V, F)$ is a surjective isometry such that

$$(x, y) = \langle k(x), y \rangle$$

We use the following convention here: when a dual realization $\{F, j\}$ of a space has been chosen, we set $F = V^*$ and $j \circ f(x) = \langle f, x \rangle$. We say that the isometry $k : V \to V^*$ is the duality operator associated to the inner product on V and to the duality pairing on $V^* \times V$ by the relation

$$(x, y) = \langle k(x), y \rangle$$

A special but most frequent case is to choose as a dual realization of V the couple $\{V, J\}$; in this case the Hilbert space V is called a *pivot space*. To be more precise, we introduce the following definition.

Definition 3.1.7. *A Hilbert space H with an inner product (x, y) is called a pivot space, if we identify H^* with H. In that case*

$$H^* = H, \ j = J, \ \langle x, y \rangle = (x, y)$$

Sometimes it does not make sense to identify the space itself with its topological dual, as the following example shows.

Example 3.1.8. *Let us consider $V = L^2(\mathbb{R}, (1 + |x|)) \subset L^2(\mathbb{R})$ (dense subspace of $L^2(\mathbb{R})$) endowed with the inner product:*

$$(u, v)_V = \int_{\mathbb{R}} (1 + |x|) u(x) v(x) dx$$

an element $\varphi \in L^2(\mathbb{R})^*$ is also an element of V^*. If we identify φ to an element $f \in L^2(\mathbb{R})$, this function does not define a linear form on V and the expression $\varphi(v) = \langle f, v \rangle_V$ has no meaning on V. In this situation it is necessary to work in a non-pivot Hilbert space.

We provide now some useful examples of non-pivot H-spaces.

Let $\Omega \subset \mathbb{R}^n$ be an open subset of, $a : \Omega \to R^+ \backslash \{0\}$ a continuous and strictly positive function called "weight" and $s : \Omega \to R^+ \backslash \{0\}$ a continuous and strictly positive function called "real time density". The bilinear form defined on $\mathcal{C}_0(\Omega)$ (continuous functions with compact support on Ω) by

$$(x, y)_{a,s} = \int_{\Omega} x(\omega) y(\omega) a(\omega) s(\omega) d\omega$$

is an inner product. We remark here that if a is a weight, then $a^{-1} = 1/a$ is also a weight. Let us introduce the following

Definition 3.1.9. *We call $L^2(\Omega, a, s)$ a completion of $\mathcal{C}_0(\Omega)$ for the inner product $\langle x, y \rangle_{a,s}$.*

We now introduce an n-dimensional version of the previous space. If we denote by $V_i = L^2(\Omega, \mathbb{R}, a_i, s_i)$ and $V_i^* = L^2(\Omega, \mathbb{R}, a_i^{-1}, s_i)$, the space

$$V = \prod_{i=1}^{m} V_i \tag{3.1.1}$$

is a non-pivot Hilbert space with the inner product

$$(F, G)_V = (F, G)_{\mathbf{a},\mathbf{s}} = \sum_{i=1}^{m} \int_\Omega F_i(\omega) G_i(\omega) a_i(\omega) s_i(\omega) d\omega.$$

The space

$$V^* = \prod_{i=1}^{m} V_i^* \tag{3.1.2}$$

is clearly a non-pivot Hilbert space for the following inner product

$$(F, G)_{V^*} = (F, G)_{\mathbf{a}^{-1},\mathbf{s}} = \sum_{i=1}^{m} \int_\Omega \frac{F_i(\omega) G_i(\omega) s_i(\omega)}{a_i(\omega)} d\omega$$

and the following bilinear form:

$$V^* \times V \to \mathbb{R}$$

$$\langle f, x \rangle_{V^* \times V} = \langle f, x \rangle_{\mathbf{s}} = \sum_{i=1}^{m} \int_\Omega f_i(\omega) x_i(\omega) s_i(\omega) d\omega \tag{3.1.3}$$

defines a duality between V and V^*. More precisely we have:

Proposition 3.1.10. *The bilinear form (3.1.3) defines a duality mapping between $V^* \times V$, given by*

$$J(F) = (a_1 F_1, \ldots, a_m F_m).$$

Proof. By Definition (3.1.9), for each i, $V_i(\Omega) = \overline{C_0(\Omega)}^{\{a_i, s_i\}}$ and V is complete if and only if for each i, V_i is complete. Then it is enough to take F and G in $C_0^n(\Omega)$. Using Cauchy-Schwartz inequality for fine sums and integrals we get

$$\langle F, G \rangle_{\mathbf{s}} \le \sum_{i=1}^{n} \int_\Omega |F_i(\omega) \sqrt{s_i(\omega)} \sqrt{a_i(\omega)} \frac{G_i(\omega)\sqrt{s_i(\omega)}}{\sqrt{a_i(\omega)}}| d\omega$$

$$\le \sum_{i=1}^{n} (\int_\Omega F_i^2(\omega) s_i(\omega) a_i(\omega) d\omega)^{\frac{1}{2}} (\int_\Omega \frac{G_i^2(\omega) s_i(\omega)}{a_i(\omega)} d\omega)^{\frac{1}{2}}$$

$$\le (\sum_{i=1}^{n} \int_\Omega F_i^2(\omega) s_i(\omega) a_i(\omega) d\omega))^{\frac{1}{2}} (\sum_{i=1}^{n} \int_\Omega \frac{G_i^2(\omega) s_i(\omega)}{a_i(\omega)} d\omega)^{\frac{1}{2}}$$

$$= \|F\|_{\mathbf{a},\mathbf{s}} \|G\|_{\mathbf{a}^{-1},\mathbf{s}}$$

where $\|.\|_{\mathbf{a},\mathbf{s}}$ and $\|.\|_{\mathbf{a}^{-1},\mathbf{s}}$ denote respectively the norm in V and V^*.
But if $F \in V$ then $\mathbf{a}F = (a_1F_1, ..., a_nF_n) \in V^*$ and $\|\mathbf{a}F\|_{\mathbf{a}^{-1},s} = \|F\|_{\mathbf{a},s}$ that means

$$\|G\|_{\mathbf{a}^{-1},\mathbf{s}} = \sup_{F \in V} \frac{|\langle F, G \rangle_{\mathbf{s}}|}{\|F\|_{\mathbf{a},\mathbf{s}}}.$$

So $\langle \cdot, \cdot \rangle_{\mathbf{s}}$ is a duality pairing and

$$\langle F, G \rangle_{\mathbf{a},\mathbf{s}} = \sum_{i=1}^{n} \int_{\Omega} F_i(\omega) G_i(\omega) a_i(\omega) s_i(\omega) d\omega = \langle \mathbf{a}F, G \rangle_{\mathbf{s}}.$$

□

For applications of these spaces, the reader can refer to Giuffré & Pia (2009) or Chapter 4.

3.1.2 Variational analysis in non-pivot H-spaces

Let X be a Hilbert space of arbitrary (finite or infinite) dimension and let $K \subset X$ be a non-empty, closed, convex subset. We assume the reader is familiar with *tangent and normal cones to K at $x \in K$* ($T_K(x)$, respectively $N_K(x)$), and with *the projection operator of X onto K*, $P_K : X \to K$ given by $||P_K(z) - z|| = \inf_{x \in K} ||x - z||$. Moreover we use here the following characterization (called Variational Principle) of $P_K(x)$ (see Alber (1996));

$$\bar{x} = P_K(x) \Leftrightarrow \langle J(x - \bar{x}), y - \bar{x} \rangle \leq 0, \ \forall y \in K \tag{3.1.4}$$

The directional derivative of the operator P_K is defined, for any $x \in K$ and any element $v \in X$, as the limit (for a proof see Zarantonello (1971) or in a more general case Lemma 3.1.16):

$$\pi_K(x, v) := \lim_{\delta \to 0^+} \frac{P_K(x + \delta v) - x}{\delta}; \text{ moreover } \pi_K(x, v) = P_{T_K(x)}(v).$$

Let $\pi_K : K \times X \to X$ be the operator given by $(x, v) \mapsto \pi_K(x, v)$. Note that π_K is nonlinear and discontinuous on the boundary of the set K. In Dupuis & Ishii (1990); Isac & Cojocaru (2004) several characterizations of π_K are given.

The following theorem has been proved in the framework of reflexive strictly convex and smooth Banach spaces. It express the possibility to decompose any element of base space (or its dual) into the sum of elements belonging to mutually polar cones. We will use it to obtain a decomposition theorem in non-pivot Hilbert spaces (for a proof see Alber (2000), Th. 2.4).

Theorem 3.1.11. *Let X be a real reflexive strictly convex and smooth Banach space, and C a non-empty, closed and convex cone of X. Then $\forall x \in X$ and $\forall f \in X^*$ the following decompositions hold:*

$$\begin{aligned} x &= P_C(x) + J^{-1}\Pi_{C^0}J(x) \text{ and } \langle \Pi_{C^0}J(x), P_C(x)\rangle = 0 \\ f &= P_{C^0}(f) + J\Pi_C J^{-1}(f) \text{ and } \langle P_{C^0}(f), \Pi_C J^{-1}(f)\rangle = 0. \end{aligned} \tag{3.1.5}$$

Here P_C is the metric projection operator on K and Π_{C^0} is the generalized projection operator on the polar cone of C that is C^0 (for a definition of Π_{C^0} see Alber (1996)).

Remark 3.1.12. *It is known that that P_C and Π_C coincide whenever the cone C belongs to a Hilbert space. This observation implies the following the result.*

Corollary 3.1.13. *Let C be a nonempty closed convex cone of a non-pivot Hilbert space X. Then for all $x \in X$ and $f \in X^*$ the following decompositions hold:*

$$x = P_C(x) + J^{-1}P_{C^0}J(x) \text{ and } \langle P_{C^0}J(x), P_C(x)\rangle = 0$$

$$f = P_{C^0}(f) + JP_C J^{-1}(f) \text{ and } \langle P_{C^0}(f), P_C J^{-1}(f)\rangle = 0$$

We highlight that Zarantonello has shown in Zarantonello (1977) a similar decomposition result in reflexive Banach spaces, see also Corollary 3.4.29.

Lemma 3.1.14 (Zarantonello (1971), Lemma 4.5)**.** *For any closed convex set K,*

$$P_K(x+h) = x + h + \circ(\|h\|),\ x \in K,\ h \in T_K(x)$$

where $\circ(\|h\|)/\|h\| \to 0$ as $h \to 0$ over any locally compact cone of increments.

Remark 3.1.15. *To prove Lemma 3.1.14 only the properties of the norm in Hilbert spaces are used, therefore the proof is valid in the non-pivot setting.*

The following lemma as been proved in the pivot case in Zarantonello (1971). We give below a similar proof in non-pivot spaces.

Lemma 3.1.16. *For any $x \in K$,*

$$P_K(x+h) = x + P_{T_K(x)}(h) + \circ(\|h\|)$$

where $\circ(\|h\|)/\|h\| \to 0$ as $h \to 0$ over any locally compact cone of increments.

Proof. Clearly,

$$\|x+h-P_K(x+h)\|^2 = \|x+h-P_{x+T_K(x)}(x+h)\|^2 + \|P_{x+T_K(x)}(x+h)-P_K(x+h)\|^2$$
$$+2(x+h-P_{x+T_K(x)}(x+h), x+h-P_{x+T_K(x)}(x+h))$$

but

$$(x+h-P_{x+T_K(x)}(x+h), x+h-P_{x+T_K(x)}(x+h))$$
$$= \langle J(x+h-P_{x+T_K(x)}(x+h)), x+h-P_{x+T_K(x)}(x+h)\rangle \geq 0$$

using the variational principle (3.1.4) applied to $P_{x+T_K(x)}(x+h)$. By definition of the projection operator we have

$$\|x+h-P_K(x+h)\|^2 \leq \|x+h-P_K[(P_{x+T_K(x)}(x+h)]\|^2$$

therefore we have

$$\|x+h-P_{x+T_K(x)}(x+h)\|^2+\|P_{x+T_K(x)}(x+h)-P_K(x+h)\|^2 \leq \|x+h-P_K[(P_{x+T_K(x)}(x+h)]\|^2$$

As $P_{x+T_K(x)}(x+h) = P_{T_K(x)}(h)$ (just apply the definition and the variational principle (3.1.4)), we have

$$\|h-P_{T_K(x)}(h)\|^2 + \|x+P_{T_K(x)}(h)-P_K(x+h)\|^2 \leq \|x+h-P_K(x+P_{T_K(x)}(x))\|^2,$$

but using the Corollary 3.1.13 we have $h = P_{T_C(x)}(h) + J^{-1}P_{N_K(x)}(J(h))$ and therefore,

$$\|P_K(x+h)-x-P_{T_K(x)}(h)\|^2 \leq \|J^{-1}P_{N_K(x)}(J(h))+x+P_{T_K(x)}(h)-P_K(x+P_{T_K(x)}(h))\|^2$$
$$-\|J^{-1}P_{N_K(x)}(J(h))\|^2$$
$$\leq \|x+P_{T_K(x)}(h)-P_K(x+P_{T_K(x)}(h))\|^2$$
$$+2\|J^{-1}P_{N_K(x)}(J(h))\|\|x+P_{T_K(x)}(h)-P_K(x+P_{T_K(x)}(h))\|$$

But by Lemma 3.1.14, $x+P_{T_K(x)}(h)-P_K(x+P_{T_K(x)}(h)) = o(\|P_{T_K(x)}(h)\|)$ so we can write

$$\|P_K(x+h)-x-P_{T_K(x)}(h)\|^2 \leq (2\|J^{-1}P_{N_K(x)}(J(h))\| + o(\|P_{T_K(x)}(h)))o(\|P_{T_K(x)}(h)\|)$$

Therefore we have,

$$\|P_K(x+h)-x-P_{T_K(x)}(h)\|^2 \leq o(\|h\|)^2$$

□

3.1.3 Variation Inequality in Non Pivot Hilbert spaces

Let us consider two vector $a, s \in \mathbb{R}^n$, and let a_i, s_i be the components, for $i = 1, \dots, n$. Denoting by $V_i = L^2(\Omega, \mathbb{R}, a_i, s_i)$ and $V_i^* = L^2(\Omega, \mathbb{R}, a_i^{-1}, s_i)$, the space $V = \prod_{i=1}^n V_i$ is a Hilbert space with respect to the inner product

$$(F, G)_V = (F, G)_{\mathbf{a},\mathbf{s}} = \sum_{i=1}^m \int_\Omega F_i(\omega) G_i(\omega) a_i(\omega) s_i(\omega) d\omega, \quad \forall F, G \in V,$$

and the space $V^* = \prod_{i=1}^n V_i^*$ is a Hilbert space with respect to the inner product

$$(F, G)_{V^*} = (F, G)_{\mathbf{a}^{-1},\mathbf{s}} = \sum_{i=1}^m \int_\Omega \frac{F_i(\omega) G_i(\omega) s_i(\omega)}{a_i(\omega)} d\omega, \quad \forall F, G \in V^*.$$

Moreover, the following bilinear form, defined into $V^* \times V$ by

$$\langle f, x \rangle_{V^* \times V} = \langle f, x \rangle_{\mathbf{s}} = \sum_{i=1}^m \int_\Omega f_i(\omega) x_i(\omega) s_i(\omega) d\omega, \quad \forall f \in V^*, \ \forall x \in V,$$

represents a duality between V and V^* and the duality mapping is given by $J(F) = (a_1 F_1, \dots, a_n F_n)$.

Let us introduce weighted variational inequalities defined into a non-pivot Hilbert space V.

Let K be a nonempty, convex and closed subset of V and let $C : K \to V^*$ be a vector-function. The weighted variational inequality is the problem to find a vector $x \in K$, such that

$$\langle C(x), y - x \rangle_{\mathbf{s}} \geq 0, \quad \forall y \in K. \tag{3.1.6}$$

3.1.4 Regularity of Variational Inequalities in Non pivot Hilbert Spaces

In order to prove a continuity result our methodology needs to introduce the finite-dimensional weighted variational inequality associated to the infinite-dimensional weighted variational inequality (3.1.6). Let us introduce the following norm in $\mathbb{R}^m$

$$\|x\|_{m,\mathbf{a},\mathbf{s}}^2 = \sum_{i=1}^m x_i^2 a_i s_i$$

where $a, s \in \mathbb{R}_+^m$. We introduce the following bilinear form:

$$(\mathbb{R}^m, \|\cdot\|_{m,\mathbf{a}^{-1},\mathbf{s}}) \times (\mathbb{R}^m, \|\cdot\|_{m,\mathbf{a},\mathbf{s}}) \to \mathbb{R}$$

$$\langle y, x\rangle_{m,\mathbf{s}} = \sum_{i=1}^{m} y_i x_i s_i,$$

it is easy to prove (same method that the one used in for Proposition 3.1.10) that it is a duality pairing between $(\mathbb{R}^m, \|\cdot\|_{m,\mathbf{a}^{-1},\mathbf{s}})$ and $(\mathbb{R}^m, \|\cdot\|_{m,\mathbf{a},\mathbf{s}})$. We set

$$K(t) = \{f(t) \in \mathbb{R}^m : \ f \in K\},$$

and we remark that $K(t)$ is closed and convex, then we can introduce the finite-dimensional weighted variational inequality associated to (3.1.6)

$$\text{Find } x(t) \in K(t) : \ \langle C(x(t)), y(t) - x(t)\rangle_{m,\mathbf{s}(t)} \geq 0, \quad \forall y(t) \in K(t), \ \text{a.e. in } \Omega. \quad (3.1.7)$$

Under our hypothesis we can prove the following result

$$x \text{ is solution of (3.1.6)} \Leftrightarrow x(t) \text{ is solution of (3.1.7) for almost every } t \in \Omega.$$

In fact, we suppose that the integral form of the variational inequality problem holds. If the pointed form is false, we have

$$\exists I \subseteq \Omega, \ \mu(I) > 0, \ \exists \bar{y}(t) \in K(t) : \ \langle C(x(t)), \bar{y}(t) - x(t)\rangle_{m,\mathbf{s}(t)} < 0, \quad \forall t \in I.$$

Setting

$$y^*(t) = \begin{cases} \bar{y}(t) & t \in I \\ x(t) & t \in \Omega \setminus I \end{cases},$$

we obtain

$$\begin{aligned} \int_\Omega \langle C(x(t)), y^*(t) - x(t)\rangle_{m,\mathbf{s}(t)} dt &= \int_{\Omega\setminus I} \langle C(x(t)), x(t) - x(t)\rangle_{m,\mathbf{s}(t)} dt \\ &+ \int_I \langle C(x(t)), \bar{y}(t) - x(t)\rangle_{m,\mathbf{s}(t)} dt < 0 \end{aligned}$$

that is a contradiction.

We can now show the following regularity theorem, we can observe that the point to point variational problem is a finite-dimensional problem.

Theorem 3.1.17. *Let V be as in 3.1.1, let $\Omega \subseteq \mathbb{R}^p$, let $t \in \Omega$ and let $K(t)$ be a nonempty, closed, convex and bounded subset of $\mathbb{R}^m$ verifying Kuratowski's convergence assumptions, let $C : \Omega \times K \to V^*$ be a continuous function and $C(t, \cdot)$ strongly pseudo-monotone with degree $\alpha > 1$. Then the solution map $x : \Omega \ni t \to x(t) \in \mathbb{R}^m$ of (3.1.7) is continuous on Ω.*

Proof. Let $x(t_n)$ be the unique solution of the weighted variational inequality

$$\langle C(t_n, x(t_n)), y(t_n) - x(t_n)\rangle_{m,\mathbf{s}(t_n)} \geq 0, \quad \forall y(t_n) \in K(t_n),\ \forall n \in \mathbb{N} \tag{3.1.8}$$

Fixing $t = (t_1, \dots, t_p) \in \Omega$, it suffices to verify that for any $\{t_n\}_{n\in\mathbb{N}} = \{(t_1^n, \dots, t_p^n)\}_{n\in\mathbb{N}} \subseteq \Omega$ such that $t_n \to t$, we have that $x(t_n) \to x(t)$. Under our hypothesis the generalized version of Minty-Browder Lemma (see for instance Maugeri *et al.* (1997)) holds, that is, for any $t \in \Omega$ we have

$$\langle C(t, y(t)), y(t) - x(t)\rangle_{m,\mathbf{s}(t)} \geq 0, \quad \forall y(t) \in K(t).$$

Using the set convergence by Kuratowski, we know that for $x(t) \in K(t)$, there exists a sequence $\{v(t_n)\}_{n\in\mathbb{N}}$ such that $v(t_n) \in K(t_n)$ for n large enough and, moreover, $v(t_n) \to x(t)$. It follows that $C(t_n, v(t_n)) \to C(t, x(t))$ because of the continuity hypothesis on C. Setting, for n large enough, $y(t_n) = v(t_n)$ in (3.1.8), we have

$$\langle C(t_n, x(t_n)), v(t_n) - x(t_n)\rangle_{m,\mathbf{s}(t_n)} \geq 0.$$

From the strongly pseudo-monotone with degree $\alpha > 1$ assumption we obtain

$$\nu \|v(t_n) - x(t_n)\|^{\alpha}_{m,\mathbf{a}(t_n),\mathbf{s}(t_n)} \leq \langle C(t_n, v(t_n)), v(t_n) - x(t_n)\rangle_{m,\mathbf{s}(t_n)}$$

$$\leq \|C(t_n, v(t_n))\|_{m,\mathbf{a}^{-1}(t_n),\mathbf{s}(t_n)} \|v(t_n) - x(t_n)\|_{m,\mathbf{a}(t_n),\mathbf{s}(t_n)}$$

and, consequently,

$$\|v(t_n) - x(t_n)\|_{m,\mathbf{a}(t_n),\mathbf{s}(t_n)} \leq \nu^{\frac{1}{1-\alpha}} \|C(t_n, v(t_n))\|^{\frac{1}{\alpha-1}}_{m,\mathbf{a}^{-1}(t_n),\mathbf{s}(t_n)}.$$

It follows that

$$\begin{aligned} \|x(t_n)\|_{m,\mathbf{a}(t_n),\mathbf{s}(t_n)} &\leq \|x(t_n) - v(t_n)\|_{m,\mathbf{a}(t_n),\mathbf{s}(t_n)} + \|v(t_n)\|_{m,\mathbf{a}(t_n),\mathbf{s}(t_n)} \\ &\leq \nu^{\frac{1}{1-\alpha}} \|C(t_n, v(t_n))\|^{\frac{1}{\alpha-1}}_{m,\mathbf{a}^{-1}(t_n),\mathbf{s}(t_n)} + \|v(t_n)\|_{m,\mathbf{a}(t_n),\mathbf{s}(t_n)}, \end{aligned}$$

so that $\{x(t_n)\}_{n\in\mathbb{N}}$ is bounded. There exists $v \in \mathbb{R}^m$ and there exists a subsequence denoted again by $\{x(t_n)\}_{n\in\mathbb{N}}$, such that $x(t_n) \in K(t_n)$, and, moreover, $x(t_n) \to v$. Using again the sets convergence by Kuratowski we get that $v \in K(t)$. Now we prove that $v = x(t)$. Applying again the generalized version of Minty-Browder lemma to any $x(t_n)$ we obtain

$$\langle C(t_n, y_n), y(t_n) - x(t_n)\rangle_{m,\mathbf{s}(t_n)} \geq 0, \quad \forall y(t_n) \in K(t_n).$$

Using again the proprieties of the Kuratowski's convergence,for any $y(t) \in K(t)$, one can find $\{y(t_n)\}_{n\in\mathbb{N}}$ such that $y(t_n) \in K(t_n)$ for n large enough and, moreover, $y(t_n) \to y(t)$. As

$$\langle C(t_n, y(t_n)), y(t_n) - x(t_n)\rangle_{m,\mathbf{s}(t_n)} = \langle \mathbf{s}(t_n)C(t_n, y_n), y(t_n) - x(t_n)\rangle_m \geq 0, \quad \forall y(t_n) \in K(t_n).$$

where $\langle \cdot, \cdot\rangle_m$ is the standard inner product of $\mathbb{R}^m$, letting $n \to +\infty$ it follows that:

$$\langle C(t, y(t)), y(t) - v\rangle_{m,\mathbf{s}(t)} \geq 0, \quad \forall y(t) \in K(t).$$

Applying the generalized version of Minty-Browder's lemma once more we obtain

$$\langle C(t, v), y(t) - v\rangle_{m,\mathbf{s}(t)} \geq 0, \quad \forall y(t) \in K(t).$$

From the uniqueness of solution to (3.1.7) it follows that $v = x(t)$ and that $x(t_n) \to x(t)$. $\square$

Now, we want to prove that the unique solution to a variational inequality related to a strictly pseudo-monotone operator, in a non-pivot Hilbert space, is a continuous mapping on Ω. In order to obtain this result, it is necessary to make a remark concerning generic variational inequalities with strictly pseudo-monotone operators.

Remark 3.1.18. Let V be as in (3.1.1) and let $K(t) \subseteq \mathbb{R}^m$ be a given nonempty closed convex and bounded set for any fixed $t \in \Omega$. For every $\varepsilon > 0$ and for any fixed $t \in \Omega$, let us consider the following perturbed variational inequality

$$\langle C(t, x(t)) + \varepsilon J_m(x(t)), y(t) - x(t)\rangle_{m,\mathbf{s}(t)} \geq 0, \quad \forall y(t) \in K(t), \tag{3.1.9}$$

where J_m is the duality mapping between $(\mathbb{R}^m, \|\cdot\|_{m,\mathbf{a},\mathbf{s}})$ and $(\mathbb{R}^m, \|\cdot\|_{m,\mathbf{a}^{-1},\mathbf{s}})$. We note that the map J_m is a monotone operator. If this inequality admits a unique solution x_ε, then by virtue of Theorem 3.1.17, this solution is continuous on Ω.

With this in mind, we can now prove the continuity result for variational inequalities with strictly pseudo-monotone operators. For any fixed $t \in \Omega$, let us consider the variational inequality

$$\langle C(t, x(t)), y(t) - x(t)\rangle_{m,\mathbf{s}(t)} \geq 0, \quad \forall y(t) \in K(t). \tag{3.1.10}$$

We suppose that the operator $C(t, \cdot)$ is strictly pseudo-monotone and all the hypotheses that guarantee the existence and uniqueness of a solution to (3.1.10) are satisfied, refer for this purpose to Section B.4. Then, the following result holds.

Theorem 3.1.19. *Let V be as in (3.1.1), let $\Omega \subseteq \mathbb{R}^n$, let $t \in \Omega$ and let $K(t)$ be a nonempty closed convex and uniformly bounded with respect to $t \in \Omega$ subset of $\mathbb{R}^m$, verifying the Kuratowski's convergence. Let $C : \Omega \times K \to V^*$ be a continuous function such that $C(t, \cdot)$ is strictly pseudo-monotone uniformly with respect to $t \in \Omega$. Then the solution map $x : \Omega \ni t \to x(t) \in \mathbb{R}^m$ of (3.1.10) is continuous on Ω.*

Proof. Let us consider the solution $x(t)$ to weighted variational inequality (3.1.10) and the solution $x(t_n)$ to the following variational inequality

$$\langle C(t_n, x(t_n)), y(t_n) - x(t_n)\rangle_{m,\mathbf{s}(t_n)} \geq 0, \quad \forall y(t_n) \in K(t_n), \ \forall n \in \mathbb{N}. \tag{3.1.11}$$

Fixing $t \in \Omega$, it suffices to verify that for any $\{t_n\}_{n \in N} \subseteq \Omega$ such that $t_n \to t$, it results $x(t_n) \to x(t)$, as $n \to +\infty$.

Let $x_\varepsilon(t)$ be the unique solution of perturbed strongly pseudo-monotone variational inequality (3.1.9), namely $x_\varepsilon(t) \in K(t)$ and

$$\langle C(t, x_\varepsilon(t)) + \varepsilon J_m(x_\varepsilon(t)), y(t) - x_\varepsilon(t)\rangle_{m,\mathbf{s}(t_n)} \geq 0, \quad \forall y(t) \in K(t). \tag{3.1.12}$$

Taking into account Theorem 3.1.17, it results that $x_\varepsilon(t)$ is a continuous function on Ω. Then the solutions $x_\varepsilon(t_n)$ to the following weighted variational inequalities

$$\langle C(t_n, x_\varepsilon(t_n)) + \varepsilon J_m(x_\varepsilon(t_n)), y(t_n) - x_\varepsilon(t_n)\rangle_{m,\mathbf{s}(t_n)} \geq 0, \quad \forall y(t_n) \in K(t_n), \tag{3.1.13}$$

$\forall n \in \mathbb{N}$, converge to $x_\varepsilon(t)$, as $n \to +\infty$.

Moreover, we remark that $x_\varepsilon(t) \to x(t)$, as $\varepsilon \to 0$, in Ω. In fact, let $x_\varepsilon(t)$ be the unique solution to (3.1.9), namely $x_\varepsilon \in K(t)$ and

$$\langle C(t, x_\varepsilon(t)) + \varepsilon J_m(x_\varepsilon(t)), y(t) - x_\varepsilon(t)\rangle_{m,\mathbf{s}(t)} \geq 0, \quad \forall y(t) \in K(t). \tag{3.1.14}$$

Setting $y(t) = x(t)$ in (3.1.12) we get

$$\langle C(t, x_\varepsilon(t)), x(t) - x_\varepsilon(t)\rangle_{m,\mathbf{s}(t)} + \varepsilon\langle J_m(x_\varepsilon(t)), x(t) - x_\varepsilon(t)\rangle_{m,\mathbf{s}(t)} \geq 0. \tag{3.1.15}$$

Moreover, setting $y = x_\varepsilon(t)$ in (3.1.10) we have

$$\langle C(t, x(t)), x_\varepsilon(t) - x(t)\rangle_{m,\mathbf{s}(t)} \geq 0. \tag{3.1.16}$$

From the strict pseudo-monotonicity of $C(t, \cdot)$, uniformly with respect to $t \in \Omega$, and relation (3.1.16) it follows that

$$\langle C(t, x_\varepsilon(t)), x_\varepsilon(t) - x(t)\rangle_{m,\mathbf{s}(t)} > 0.$$

Then, by (3.1.15), we obtain

$$\varepsilon\langle J_m(x_\varepsilon(t)), x(t)-x_\varepsilon(t)\rangle_{m,\mathbf{s}(t)} \geq 0,$$

and dividing by $\varepsilon > 0$, we have

$$\langle J_m(x_\varepsilon(t)), x(t)-x_\varepsilon(t)\rangle_{m,\mathbf{s}(t)} \geq 0. \tag{3.1.17}$$

Taking into account (3.1.17), one has

$$\begin{aligned}
\|x_\varepsilon(t)\|^2_{m,\mathbf{a}(t),\mathbf{s}(t)} &= \sum_{i=1}^m a_i(t)s_i(t)\big(x^i_\varepsilon(t)\big)^2 \\
&\leq \Big(\sum_{i=1}^m s_i(t)a_i(t)\Big) \\
&\leq \langle J_m(x_\varepsilon(t)), x(t)\rangle_{m,\mathbf{s}(t)} \\
&\leq \left(\sum_{i=1}^m s_i(t)a_i(t)\big(x^i_\varepsilon(t)\big)^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^m s_i(t)a_i(t)\big(x^i(t)\big)^2\right)^{\frac{1}{2}} \\
&= \|x(t)\|_{m,\mathbf{a}(t),\mathbf{s}(t)}\|x_\varepsilon(t)\|_{m,\mathbf{a}(t),\mathbf{s}(t)},
\end{aligned}$$

which implies

$$\|x_\varepsilon(t)\|_{m,\mathbf{a}(t),\mathbf{s}(t)} \leq \|x(t)\|_{m,\mathbf{a}(t),\mathbf{s}(t)}.$$

Since $x(t) \in K(t)$, and $K(t)$ is a family of uniformly bounded sets of $\mathbb{R}^m$ it results

$$\|x(t)\|_{m,\mathbf{a}(t),\mathbf{s}(t)} \leq C_1,$$

with C_1 a constant independent on ε, so that $\{x_\varepsilon(t)\}_\varepsilon$ is bounded therefore there exists $v \in V$ and there exists a subsequence denoted again by $\{x_\varepsilon(t)\}_\varepsilon$, such that $x_\varepsilon(t) \in K(t)$, and, moreover, $x_\varepsilon(t) \to v$. Taking into account the closeness of $K(t)$ we get that $v \in K(t)$.
Now we prove that $v = x(t)$, therefore we consider the following variational inequality

$$\langle C(t, y(t)) + \varepsilon J_m(y(t)), y(t) - x_\varepsilon(t)\rangle_{m,\mathbf{s}(t)} \geq 0, \quad \forall y(t) \in K(t),$$

and letting $\varepsilon \to 0$, it results

$$\langle C(t, y(t)), y(t) - v\rangle_{m,\mathbf{s}(t)} \geq 0, \quad \forall y(t) \in K(t). \tag{3.1.18}$$

From the generalized version of Minty's Lemma, we have that (3.1.18) is equivalent to the following variational inequality

$$\langle C(t, v), y(t) - v\rangle_{m,\mathbf{s}(t)} \geq 0, \quad \forall y(t) \in K(t). \tag{3.1.19}$$

Hence (3.1.19) implies that v is a solution to (3.1.10). Since the solution to (3.1.10) is unique, then we concluded that the sequence $\{x_\varepsilon(t)\}_\varepsilon$ converges strongly to $x(t)$, as $\varepsilon \to 0$.

Now, we set $y(t_n) = x(t_n)$, $\forall n \in \mathbb{N}$, in (3.1.13),

$$\langle C(t_n, x_\varepsilon(t_n)), x(t_n) - x_\varepsilon(t_n)\rangle_{m,\mathbf{s}(t_n)} + \varepsilon\langle J_m(x_\varepsilon(t_n)), x(t_n) - x_\varepsilon(t_n)\rangle_{m,\mathbf{s}(t_n)} \geq 0, \quad (3.1.20)$$

and $y(t_n) = x_\varepsilon(t_n)$, $\forall n \in \mathbb{N}$, in (3.1.11) it results, $\forall n \in \mathbb{N}$

$$\langle C(t_n, x(t_n)), x_\varepsilon(t_n) - x(t_n)\rangle_{m,\mathbf{s}(t_n)} \geq 0,$$

but, from the strict pseudo-monotonicity assumption on the function $C(t, \cdot)$, uniformly with respect to $t \in [0, T]$, it follows that

$$\langle C(t_n, x_\varepsilon(t_n)), x_\varepsilon(t_n) - x(t_n)\rangle_{m,\mathbf{s}(t_n)} > 0, \quad \forall n \in \mathbb{N}.$$

Then, from (3.1.20) we have

$$\varepsilon\langle J_m(x_\varepsilon(t_n)), x(t_n) - x_\varepsilon(t_n)\rangle_{m,\mathbf{s}(t_n)} \geq 0, \quad \forall n \in \mathbb{N},$$

and proceeding as above, we have

$$\|x_\varepsilon(t_n)\|_{m,\mathbf{a}(t_n),\mathbf{s}(t_n)} \leq C_2, \quad (3.1.21)$$

where C_2 is a constant independent on ε and on $n \in \mathbb{N}$. Therefore we have

$$x_\varepsilon(t_n) \to \widetilde{x}(t_n), \quad \text{as } \varepsilon \to 0, \ \forall n \in \mathbb{N},$$

with $\widetilde{x}(t_n) \in K(t_n)$ and such that

$$\langle C(t_n, \widetilde{x}(t_n)), y(t_n) - \widetilde{x}(t_n)\rangle_{m,\mathbf{s}(t_n)} \geq 0, \quad \forall y(t_n) \in K(t_n), \ \forall n \in \mathbb{N}.$$

Since the solution to (3.1.11) is unique, it results

$$\widetilde{x}(t_n) = x(t_n), \quad \forall n \in \mathbb{N},$$

therefore we have

$$\|x(t_n)\|_{m,\mathbf{a}(t_n),\mathbf{s}(t_n)} \leq C_2, \quad \forall n \in \mathbb{N}.$$

Then, the sequence $\{x(t_n)\}_{n\in\mathbb{N}}$ is bounded, that implies the existence of a subsequence denoted again by $\{x(t_n)\}_{n\in\mathbb{N}}$, such that $x(t_n) \in K(t_n)$, $\forall n \in \mathbb{N}$, converging strongly in Ω to an element $\overline{x}(t)$ of $K(t)$, namely

$$x(t_n) \to \overline{x}(t), \quad \text{as } n \to +\infty.$$

Taking into account the variational inequality

$$\langle C(t_n, y(t_n)), y(t_n) - x(t_n)\rangle_{m,\mathbf{s}(t_n)} \geq 0, \quad \forall y(t_n) \in K(t_n),$$

and passing to the limit as $n \to +\infty$, it follows

$$\langle C(t, y(t)), y(t) - \overline{x}(t)\rangle_{m,\mathbf{s}(t)} \geq 0, \quad \forall y(t) \in K(t).$$

For the generalized version of Minty-Browder Lemma, we have that $\overline{x}(t)$ is a solution to (3.1.10), since this variational inequality has a unique solution, it results

$$\overline{x}(t) = x(t).$$

The same result holds for each subsequence and therefore

$$x(t_n) \to x(t),$$

namely our assert. The proof is now complete. □

3.2 Projected Dynamical Systems in Weighted Hilbert Spaces

In this section we study the existence of solutions for a class of differential equations with discontinuous and non-linear right-hand side on the class of non-pivot Hilbert spaces. This class of equations (called projected differential equations) was first introduced in the form we use in Dupuis & Ishii (1990), however it has been other studies of a similar formulation has been known since Aubin & Cellina (1984); Brezis (1967); Henri (1973). The formulation of the flow of such equations as dynamical systems in $\mathbb{R}^n$ is due to Dupuis & Ishii (1990); Dupuis & Nagurney (1993) and it has been applied to study the dynamics of solutions of finite-dimensional variational inequalities in Dupuis & Nagurney (1993); Nagurney & Zhang (1996).

Finite-dimensional variational inequalities theory provides solutions to a wide class of equilibrium problems in mathematical economics, optimization, management science, operations research, finance, etc. (see for example Aubin & Cellina (1984); Dafermos (1980); Nagurney & Siokos (1997); Nagurney & Zhang (1996) and the references therein). Therefore there has been a steady interest over the years in studying the stability of solutions to finite-dimensional variational inequalities (and consequently the stability of equilibria for various problems). In general, such a study is done by associating a projected dynamical system to a variational inequality problem, however in the past few years the applied problems, as well as the theoretical results, have progressed to a qualitative study of stability of solutions to variational inequality problems on Hilbert spaces and even on Banach spaces. Examples of the kind of variational problems (and their applications) can be found in (see Cojocaru (2007)-Cojocaru (2005),Barbagallo (2007a); Barbagallo & Cojocaru (2009a,b); Cojocaru (2006); Daniele (2006); Isac & Cojocaru (2002b); Johnston & Cojocaru (2008)) and the references therein).

In this paper we present a new step in this study: we show that a projected differential equation has solutions on a non-pivot Hilbert space of any dimension. We prove the existence and uniqueness of integral curves and show they remain in a given constraint set of interest. As in the finite-dimensional case, a dynamics given by solutions to a projected differential equation is interesting because it describes these problems as dynamical systems. Moreover, as shown in this section, new results as been developped for the study of the weighted traffic equilibrium problem (see Cojocaru & Pia (2008); Giuffré & Pia (2009)). Our goal in this section is to present the mathematical techniques involved in proving the existence of solutions to projected differential equations in a non-pivot setting, which is in fact similar to the one in Cojocaru & Jonker (2004), but adapted to a non-pivot space; in addition, there are a number of preliminary results needed prior to obtaining our main result, which are remarkable since they also hold in a larger setting, namely that of a reflexive Banach space (see the results in Giuffré & Pia (2008); Giuffré *et al.* (2006b)). Last but not least, we also present a projected system formulation called *implicit*. These kinds of systems have been introduced in the literature in Noor (2003), but without any existence result being presented in their case. We thus solve this additional problem as well.

3.2.1 PDS in pivot H-spaces

Let X be a pivot Hilbert space of arbitrary (finite or infinite) dimension and let $K \subset X$ be a non-empty, closed, convex subset. The following result has been shown (see Cojocaru & Jonker (2004)).

Theorem 3.2.1. *Let X be a Hilbert space and K be a non-empty, closed, convex subset. Let $F : K \to X$ be a Lipschitz continuous vector field and $x_0 \in K$. Then the initial* *value problem associated to the projected differential equation (PrDE)*

$$\frac{dx(\tau)}{d\tau} = \pi_K(x(\tau), -F(x(\tau)), \; x(0) = x_0 \in K \tag{3.2.22}$$

has a unique absolutely continuous solution on the interval $[0, \infty)$.

This result is a generalization of the one in Nagurney (1993), where $X := \mathbb{R}^n$, K was a convex polyhedron and F had linear growth.

Definition 3.2.2. *A* **projected dynamical system** *then is given by a mapping* $\phi : \mathbb{R}_+ \times K \to K$ *which solves the initial value problem:* $\dot{\phi}(t,x) = \Pi_K(\phi(t,x), -F(\phi(t,x)))$ *a.a.* t, $\phi(0,x) = x_0 \in K$.

3.2.2 Existence of Solutions

In this subsection we show that, with minor modifications, the existence of PDS in non-pivot H-spaces can be obtained. We first introduce non-pivot projected dynamical systems (NpPDS) and then show their existence. In analogy with Cojocaru & Jonker (2004) we first introduce

Definition 3.2.3. *A* ***non-pivot projected differential equation*** *(NpPrDE) is a discontinuous ODE given by:*

$$\frac{dx(t)}{dt} = \pi_K(x(t), -(J^{-1} \circ F)(x(t))) = P_{T_K(x(t))}(-(J^{-1} \circ F)(x(t))). \tag{3.2.23}$$

Consequently the associated Cauchy problem is given by:

$$\frac{dx(t)}{dt} = \pi_K(x(t), -(J^{-1} \circ F)(x(t))), \;\; x(0) = x_0 \in K. \tag{3.2.24}$$

Next we define what we mean by a solution for a Cauchy problem of type (3.2.24).

Definition 3.2.4. *An absolutely continuous function* $x : \mathfrak{I} \subset \mathbb{R} \to X$, *such that*

$$\begin{cases} x(t) \in K, \;\; x(0) = x_0 \in K, \forall t \in \mathfrak{I} \\ \dot{x}(t) = \pi_K(x(t), -(J^{-1} \circ F)(x(t))), \;\; a.e. \;\; on \; \mathfrak{I} \end{cases} \tag{3.2.25}$$

is called a solution for the initial value problem (3.2.24).

Finally, assuming problem (3.2.24) has solutions as described above, then we are ready to introduce:

Definition 3.2.5. *A* **non-pivot projected dynamical system** *(NpPDS) is given by a mapping $\phi : \mathbb{R}_+ \times K \to K$ which solves the initial value problem:*
$\dot{\phi}(t,x) = \Pi_K(\phi(t,x), -(J^{-1} \circ F)(\phi(t,x))),\;$ *a.a.* $t,\; \phi(0,x) = x_0 \in K$.

To end this section we show how problem (3.2.24) can be equivalently (in the sense of solution set coincidence) formulated as a differential inclusion problem. Finally, in subsection 3.2.3 we show that solutions for this new differential inclusion problem exist. We introduce the following differential inclusion:

$$\dot{x}(t) \in J^{-1}(-F(x) - N_K(x)),\, x(0) = x_0 \in K, \tag{3.2.26}$$

and we call $x : \mathfrak{I} \subset \mathbb{R} \to X$ absolutely continuous a solution to (3.2.26) if

$$\begin{cases} x(t) \in K,\;\; x(0) = x_0 \in K,\, \forall t \in \mathfrak{I} \\ \dot{x}(t) \in J^{-1}(-F(x) - N_K(x)),\;\; \text{a.a. } t. \end{cases} \tag{3.2.27}$$

We introduce also the following differential inclusion:

$$\dot{x}(t) \in J^{-1}(-F(x) - \tilde{N}_K(x)),\, x(0) = x_0 \in K, \tag{3.2.28}$$

where

$$\tilde{N}_K(x) = \{n \in N_K(x) \mid ||n|| \leq ||F(x)||\}.$$

Obviously, we call $x : \mathfrak{I} \subset \mathbb{R} \to X$ absolutely continuous a solution to (3.2.28) if

$$\begin{cases} x(t) \in K,\;\; x(0) = x_0 \in K,\, \forall t \in \mathfrak{I} \\ \dot{x}(t) \in J^{-1}(-F(x) - \tilde{N}_K(x)),\;\; \text{a.a. } t. \end{cases} \tag{3.2.29}$$

Proposition 3.2.6. *The solution set of problem (3.2.24) coincides with the solution set of problem (3.2.29).*

Proof. (3.2.24) $\Rightarrow$ (3.2.29). Let $x(.)$ be an absolutely continuous function on K such that $x(.)$ is a solution to (3.2.24). Then $x(t) \in K$, $\forall t \in T$ and $\dot{x}(t) = \pi_K(x(t), -(J^{-1} \circ F)(x(t)))$, a.e. on $\mathfrak{I}$, therefore using (3.1.13) we get $\dot{x}(t) = -J^{-1}(F(x)) - J^{-1}P_{N_K(x)}(-F(x))$, a.e $\in I$. Evidently, $P_{N_K(x)}(-F(x)) \in N_K(x)$. Moreover as $N_K(x)$ is a closed, convex cone, we get that

$$\|P_{N_K(x)}(-F(x))\|_{X^*} \leq \| - F(x)\|_{X^*}$$

($N_K^0(x) = T_K(x)$ and both contains 0). Therefore $\exists \tilde{n}_K(x) \in \tilde{N}_K(x), \tilde{n}_K(x) := P_{N_K(x)}(-F(x))$ such that $\dot{x}(t) = -J^{-1}(F(x(t)) - \tilde{n}_K(x))$ for a.a $t \in I$ so we have $\dot{x}(t) \in -J^{-1}(F(x(t)) - \tilde{N}_K(x))$ for a.a $t \in I$ and $x(.)$ is a solution to (3.2.29).

(3.2.29) $\Rightarrow$ (3.2.24).

As the trajectory remains in K it is clear that $\dot{x}(t) \in T_K(x(t))$. First we show that for almost all $t \in I$ we have

$$\dot{x}(t) \in N_K^{\perp}(x(t)) \tag{3.2.30}$$

Let us consider three different cases, first suppose that $x(t) \in int(K)$, we have then $N_K(x(t)) = \{0_{X^*}\}$ and then $N_K^{\perp}(x(t)) = X^*$ and (3.2.30) is automatically satisfied. Suppose now that $x(t) \in \partial K$ and in $x(t)$, ∂K is smooth. In that case $T_K(x(t))$ is flat and $N_K^{\perp}(x(t)) \subsetneq T_K(x(t))$ with $N_K^{\perp}(x(t))$ not reduced to $\{0_{X^*}\}$, if $\dot{x}(t) \notin N_K^{\perp}(x(t))$ then in a neighbourhood $\mathcal{V}(t)$ the trajectory $x(t')$, $t' \in \mathcal{V}(t)$ goes in $int(K)$ so we are in the first case and we can exclude time t. Suppose now that $x(t) \in \partial K$ and $x(t)$ is in a corner point. In that case $N_K^{\perp}(x(t)) = \{0\}$ therefore if $\dot{x}(t) = 0$ (3.2.30) is satisfied. If $\dot{x}(t) \neq 0$ it means that $x(t') \neq x(t)$ for $t' \in \mathcal{V}(t)$, with $x(t')$ in one of the tow previous cases, as we can "exclude" time t, we have (3.2.30). As we can write $\dot{x}(t) = J^{-1}(-F(x) - \tilde{n}_K(x))$ we have

$$\langle J(\dot{x}(t)) - JJ^{-1}(-F(x)), \dot{x}(t)\rangle = 0$$

Using the polarity between $N_K(x(t))$ and $T_K(x(t))$ and the variational principle (3.1.4) we deduce (3.2.24). $\square$

3.2.3 Existence of NpPDS

In this section we show that problem (3.2.24) has solutions, and consequently that NpPDS exist in the sense of Definition 3.2.4, by showing that problem (3.2.28) has solutions, in the sense of (3.2.27). To obtain the main result of this paper, we need some preliminary ones, according to the following steps:

1) we first prove the existence of a sequence of approximate solutions with "good" properties such that

$$\forall k \geq k_0,\ (x_k(t), \dot{x}_k(t)) \in graph(J^{-1}(-F - \tilde{N}_K)) + \mathcal{M},$$

for any neighbourhood $\mathcal{M}$ of 0 in $X \times X$. This step constitutes Theorem 3.2.9;

2) we prove next that the sequence obtained in the first step converges to a solution of problem (3.2.28), and that it has a weakly convergent subsequence whose derivative converges to $\dot{x}(.)$.

The methodology of the proofs is completely analogous to that used for pivot Hilbert spaces in Cojocaru & Jonker (2004). Therefore we present the results with summary proofs, pointing out where the they need to be updated for the case of a non-pivot H-space. The main difference in all proofs is made by the presence of the linear mapping J.

The main result can be stated as follows:

Theorem 3.2.7. *Let X be a Hilbert space and X^* its topological dual and let $K \subset X$ be a non-empty, closed and convex subset. Let $F : K \to X^*$ be a Lipschitz continuous vector field with Lipschitz constant b. Let $x_0 \in K$. Then the initial value problem (3.2.24) has a unique solution on $\mathbb{R}_+$.*

Proof. **Existence of a solution on an interval $[0, l]$, $l < \infty$.**

For this part of the proof, we need two major results, as follows:

Proposition 3.2.8. *Let X be a nonpivot H-space, X^* its topological dual and $K \subset X$ a non-empty, closed and convex subset. Let $F : K \to X^*$ be a Lipschitz continuous vector field with Lipschitz constant b, so that on $K \cap B_X(x_0, L)$, with $L > 0$ and $x_0 \in K$ arbitrarily fixed, we have $\|F(x)\| \leq M := \|F(x_0)\| + bL$.*

Then the set-valued mapping $\mathcal{N}_p : K \cap B_X(x_0, L) \to \mathbb{R}$ given by

$$x \mapsto \langle F - \tilde{N}_K(x), p\rangle$$

has a closed graph.

Proof. The proof is similar to the one in Cojocaru & Jonker (2004). We show first that the mapping $\mathcal{N}_p : K \cap B_X(x_0, L) \to \mathbb{R}$ given by $x \mapsto \langle -\tilde{N}_K(x), p\rangle$ has a closed graph. It is clear that for each $p \in X$, the set-valued map $\mathcal{N}_p : K \cap B_X(x_0, L) \to \mathbb{R}$ maps $K \cap B_X(x_0, L)$ into $2^{[-M\|p\|, -M\|p\|]}$. Let $\{(x_n, z_n)\}_n \in graph(\mathcal{N}_p)$ such that $(x_n, z_n) \to (x, z) \in X \times 2^{[-M\|p\|, -M\|p\|]}$. We want to show that $(x, y) \in graph(\mathcal{N}_p)$. From $z_n \in graph(\mathcal{N}_p)$, for all n, we deduce that there exists $y_n \in -\tilde{N}_K(x_n)$ such that $z_n = \langle y_n, p\rangle$. Since the set $-\tilde{N}_K(x) \subset \overline{B}_{X^*}(0, M)$ and $\overline{B}_{X^*}(0, M)$ is weakly compact, then there exists a subsequence y_{n_k} and $y \in X^*$ such that

$$y_{n_k} \rightharpoonup y$$

for the weak topology $\sigma(X^*, X^{**}) \overset{\text{by reflexivity}}{=} \sigma(X^*, X)$, which is equivalent to

$$\langle y_{n_k}, \beta\rangle \to \langle y, \beta\rangle, \forall \beta \in X$$

Suppose now that $y \notin -\tilde{N}_K(x)$. This implies that at least one of the following two alternatives should be satisfied:

1) There exists $w \in K$ such that $\langle y, w - x\rangle < \lambda < 0$

2) $\|y\| > \mu > \|F(x)\|$

In the first case as $\langle y_{n_k}, \beta\rangle \to \langle y, \beta\rangle, \forall \beta \in X$ for $k > k_0$ we have $\langle y_{n_k}, w - x\rangle < \frac{\lambda}{2}$. But $\langle y_{n_k}, w - x_{n_k}\rangle = \langle y_{n_k}, w - x\rangle + \langle y_{n_k}, x - x_{n_k}\rangle$ and as $x_{n_k} \to x$, there exists $k_1 > 0$ such that $\forall k \geq k_1$, we have $\langle y_{n_k}, x - x_{n_k}\rangle \leq \|x - x_{n_k}\| \, \|y_{n_k}\| < \frac{|\lambda|}{4M} M = \frac{|\lambda|}{4}$. Thus $\langle y_{n_k}, w - x_{n_k}\rangle < \frac{\lambda}{4} < 0$, for all $k > max(k_0, k_1)$. But this contradicts the fact that $y_{n_k} \in -\tilde{N}_K(x_{n_k})$.
In the second case as $\langle y_{n_k}, \beta\rangle \to \langle y, \beta\rangle, \forall \beta \in V$, we have (Brezis (1993a), Proposition III.12) $\|F(x)\| < \|y\| \leq \liminf_{k\to\infty} \|y_{n_k}\|$ which is a contradiction because $y_n \in -\tilde{N}_K(x_n)$, $\forall n \in \mathbb{N}$. The continuity of F and the first part of the proof implies that

$$x \mapsto \langle F - \tilde{N}_K(x), p\rangle$$

has non-empty, closed and convex values for each $x \in K$ and has a closed graph.

□

The next result is constructing the sequence of approximate solutions for Problem (3.2.28).

Theorem 3.2.9. *Let X be a Hilbert space and X^* its topological dual, let $K \subset X$ be a non-empty, closed and convex subset. Let $F : K \to X^*$ be a Lipschitz continuous vector field so that on $K \cap B_X(x_0, L)$, with $L > 0$ and $x_0 \in K$, we have $\|F(x)\| \leq M := \|F(x_0)\| + bL$. Let $l := \frac{L}{M}$ and $\mathcal{I} := [0, l]$. Then there exists a sequence $\{x_k(.)\}$ of absolutely continuous functions defined on $\mathcal{I}$, with values in K, such that for all $k \geq 0$, $x_k(0) = x_0$ and for almost all $t \in \mathcal{I}$, $\{x_k(t)\}$ and $\{\dot{x}_k(t)\}$ (the sequence of its derivatives) have the following property: for every neighbourhood $\mathcal{M}$ of 0 in $X \times X$ there exists $k_0 = k_0(t, \mathcal{M})$ such that*

$$\forall k \geq k_0, \ (x_k(t), \dot{x}_k(t)) \in graph(-F - \tilde{N}_K) + \mathcal{M}$$

Proof. The proof, based on topological properties of the space X, can be found in Cojocaru & Jonker (2004). However, given we are now working in non-pivot H-spaces, then instead of
$z_p := P_K(x - h_p F(x))$ we now construct $z_p := P_K(x - h_p J^{-1} \circ F(x))$. □

Next we show that the sequence $\{x_k(.)\}$ built in Theorem 3.2.9 is uniformly convergent to some $x(.)$. Again, following closely Cojocaru & Jonker (2004), by Theorem 3.2.9 there exists a pair
$(u_k, -F(u_k) - n_k) \in \; graph(-F - \tilde{N}_K)$ such that

$$x_k(t) - u_k(t) = \epsilon_{1,k}(t) \text{ and } \dot{x}_k(t) + J^{-1}(F(u_k(t) + n_k) = \epsilon_{2,k}(t)$$

where $\epsilon_{1,k}(t)$ and $\epsilon_{2,k}(t)$ are vector functions, not necessarily continuous, satisfying $\|\epsilon_{1,k}(t)\| < \epsilon_k$ and $\|\epsilon_{2,k}(t)\| < \epsilon_k$ where $\epsilon_k \to 0$ as $k \to \infty$ and $n_k \in \tilde{N}_K(u_k)$ and $n_m \in \tilde{N}_K(u_m)$.
Let k, m be two indexes. Then we evaluate

$$\frac{1}{2}\frac{d}{dt}\|x_k(t) - x_m(t)\|^2 = \langle J(\dot{x}_k(t) - \dot{x}_m(t)), x_k(t) - x_m(t)\rangle$$
$$= \langle -F(u_k(t)) + F(x_k(t)) + F(u_m(t)) - F(x_m(t)), x_k(t) - x_m(t)\rangle$$
$$+\langle -F(x_k(t)) + F(x_m(t)), x_k(t) - x_m(t)\rangle$$
$$+\langle -n_k + n_m, u_k(t) - u_m(t)\rangle + \langle -n_k + n_m, -u_k(t) + x_k(t) + u_m(t) - x_m(t)\rangle$$
$$+\langle J(\epsilon_{1,k}(t) - \epsilon_{2,m}(t)), x_k(t) - x_m(t)\rangle$$

But using the monotonicity of $x \mapsto N_K(x)$, the isometry property of J and the b-Lipschitz continuity of F we get that

$$\frac{1}{2}\frac{d}{dt}\|x_k(t) - x_m(t)\|^2 \le b||x_k(t) - x_m(t)||^2 + (\epsilon_k + \epsilon_m)\|u_k(t) - u_m(t)\| + (1+b)(\epsilon_k + \epsilon_m)\|x_k(t) - x_m(t)\|$$

We now let $\phi(t) := \|x_k(t) - x_m(t)\|$ so from the previous inequalities we get

$$\dot{\phi}(t)\phi(t) \le b\phi(t)^2 + (\epsilon_k + \epsilon_m)[(1+b)\phi(t) + 2M]$$

Using the same technique as in Cojocaru & Jonker (2004) we get

$$\phi(t)^2 \le \frac{a}{b}(\epsilon_k + \epsilon_m)(e^{-2bt} - 1) \le \frac{a}{b}(\epsilon_k + \epsilon_m)(e^{-2bl} - 1)$$

where l is the length of $\mathcal{I}$. So the Cauchy criteria is satisfied uniformly and we get the conclusion.

From the previous step we know that $\{x_k(.)\}$ is uniformly convergent to $x(.)$ and as $(x_k(t), \dot{x}(t)) \in \; graph(-F - \tilde{N}_K + \mathcal{M})$, we now deduce that there exists a θ such that $\|\dot{x}(t)\| \le \theta$. Using the arguments in Cojocaru & Jonker (2004) and the result of S.Heikkila (1994), we deduce the existence of a subsequence of $\{x_k\}$ weakly*-convergent to $\dot{x}(.) \in L^{\infty}(I, X)$.

Finally, we finish this part of the proof by showing that $x(.)$ is indeed a solution of the differential inclusion (3.2.28). From Theorem 3.9, for each $k \geq k_0$ and almost every $t \in \mathfrak{I}$ there exists a pair

$$(u_k(t), v_k(t)) \in \; graph(-F - \tilde{N}_K)$$

such that $\|x_k(t) - u_k(t)\| < \epsilon_k$ and $\|\dot{x}_k(t) - v_k(t)\| < \epsilon_k$ where $\epsilon_k \to 0$ when $k \to \infty$. Let $p \in X$ arbitrarily fixed. Then for almost all $t \in \mathfrak{I}$

$$(u_k(t), \langle v_k(t), p\rangle) \in \; graph(\langle -F - \tilde{N}_K, p\rangle)$$

and

$$\|\langle \dot{x}_k(t), p\rangle - \langle v_k(t), p\rangle\| \leq \|p\|\epsilon_k.$$

So $u_k(t) \to x(t)$ for every $t \in \mathfrak{I}$ and $\langle v_k(t), p\rangle \to \langle \dot{x}_k(t), p\rangle$ for almost all $t \in \mathfrak{I}$. By Proposition 3.8, we know that $graph(\langle -F - \tilde{N}_K, p\rangle)$ is closed, so it follows that for almost all $t \in \mathfrak{I}$,

$$(x(t), \langle \dot{x}_k(t), p\rangle) \in graph(\langle -F - \tilde{N}_K, p\rangle).$$

Since the set $F(x(t)) - \tilde{N}_K(x(t))$ is convex and closed it follows that

$$\dot{x}(t) \in J^{-1}(-F(x(t) - \tilde{N}_K)(x(t)).$$

By Proposition 3.2.8, $x(t)$ is a solution of Problem (3.2.25).

Uniqueness of solutions on $[0, l]$

Step 4: $x(.)$ is the unique solution. Suppose that we have two solutions $x_1(.)$ and $x_2(.)$ starting at the same initial point. For any fixed $t \in \mathfrak{I}$ we get

$$\frac{1}{2}\frac{d}{dt}\|x_1(t) - x_1(t)\|^2 = \langle J(\dot{x}_1(t) - \dot{x}_2(t)), x_1(t) - x_2(t)\rangle$$

$$= \langle J(\dot{x}_1(t)) - J(\dot{x}_2(t)), x_1(t) - x_2(t)\rangle$$

$$\leq \|J^{-1}(-F(x_1(t)) + F(x_2(t))), x_1(t) - x_2(t) >\leq b\|x_1(t) - x_2(t)\|^2$$

because the metric projection is a nonexpansive operator in X, J is a linear isometry and F is b-Lipschitz. By Gronwall's inequality we obtain $\|x_1(t) - x_2(t)\|^2 \leq 0$, so we have $x_1(t) = x_2(t)$ for any $t \in \mathfrak{I}$.

Existence of solutions on $\mathbb{R}_+$.

From above we can assert the existence of a solution to Problem (3.2.24) on an interval

$[0; l]$, with $b > 0$ fixed and $L > 0$ arbitrary. We note that we can choose L such that $l \geq \frac{1}{1+b}$ in the following way: if $||F(x_0)|| = 0$ we let $L = 1$ and if $||F(x_0)|| \neq 0$, then we let $L \geq ||F(x_0)||$. In both cases we obtain $l \geq \frac{1}{1+b}$. Therefore beginning at each initial point $x_0 \in K$ problem (3.2.24) has a solution on an interval of length at least $[0; \frac{1}{1+b}]$. Now if we consider problem (3.2.24) with $x_0 = x(\frac{1}{1+b})$, applying again all the above, we obtain an extension of the solution on an interval of length at least $\frac{1}{1+b}$. By continuing this solution we obtain a solution on $[0, \infty)$. □

3.2.4 Implicit Projected Dynamical System

3.2.4.1 Introduction and Existence

In this section we consider a generic Hilbert space X, where *generic* is taken to mean that the dimensionality could be either finite or infinite, and the space could be either a pivot or a non-pivot space. Let us introduce the following definition:

Definition 3.2.10. *Let X be a generic H-space and $K' \subset X$ be a non-empty, closed subset. Consider a pair (g, K) such that K is convex and $g : K' \to K = r(K') \subset X$, is continuous, injective, and g^{-1} is Lipschitz continuous. Consider $F : X \to X^*$ satisfying $(F \circ g)(y) = F(y),\ \forall y \in K'$. Then the pair (g, K) is called a **convexification pair of** (F, K').*

Example. Here is an example of such a convexification pair in $\mathbb{R}^2$. Let $K' = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1,\ 0 \leq y \leq x\}$ and g be the map of K' into $K = [0, 1] \times [0, 1]$, namely

$$g(x, y) = (x, \frac{2}{1+x}y + \frac{1-x}{1+x})$$

We can easily check that g is continuous and monotone. Now take F to be $F(x, y) = (x, a)$, where a is an arbitrary constant in $\mathbb{R}$. Then we have $F \circ g(x, y) = (x, a) = F(x, y)$.

We now introduce another type of a projected equation as follows:

Definition 3.2.11. *Let X be a generic H-space and $K' \subset X$ be a non-empty, closed subset. An **implicit projected differential equation** (ImPrDE) is a (PrDE) given by (3.2.23) where $x(t) := g(y(t))$, $g : K' \to K \subset X$, i.e.*

$$\frac{dg(y(t))}{dt} = P_{T_K(g(y(t)))}(-J^{-1} \circ F \circ g(y(t))). \tag{3.2.31}$$

The motivation for the introduction of such an equation comes from the desire to study the dynamics on a set $K' \subset X$, where K' could be non-convex, and to study as well some dynamic problems on a so-called translated set (see Application 3.2.21).

Considering now an equation (3.2.31) and a convexification pair (g, K) of a nonempty, closed $K' \subset X$, then the Cauchy problem associated to (3.2.31) and the pair (g, K) is given by:

$$\frac{dg(y(t))}{dt} = \pi_K(g(y(t)), -(J^{-1} \circ F \mid_{K'})(y(t)), \;\; g(y(0)) = x_0 \in K. \tag{3.2.32}$$

Next we define what we mean by a solution for a Cauchy problem of type (3.2.32).

Definition 3.2.12. *An absolutely continuous function $y : \mathfrak{I} \subset \mathbb{R} \to X$, such that*

$$\begin{cases} y(t) \in K', \;\; g(y(0)) = x_0 \in K, \; \forall t \in \mathfrak{I} \\ \frac{dg(y(t))}{dt} = \pi_K(g(y(t)), -(J^{-1} \circ F \mid_{K'})(y(t))), \;\; a.e. \;\; on \; \mathfrak{I} \end{cases} \tag{3.2.33}$$

is called a solution for the initial value problem 3.2.32.

We claim that problem (3.2.32) has solutions by Theorem 3.2.9. It is obvious that by a change of variable $x(.) := g(y(.))$, problem (3.2.32) has solutions on K, in the sense of Definition 3.2.4. But since g is assumed continuous and strictly monotone, then g is invertible and so $y(.) = g^{-1}(x(.))$; moreover, we see that such a y is a solution to problem (3.2.32) in the above sense.

Now we are ready to introduce:

Definition 3.2.13. *An* **implicit projected dynamical system** *(ImPDS) is given by a mapping $\phi : \mathbb{R}_+ \times K' \to K$ which solves the initial value problem:*

$$\dot{\phi}(t, g(y(t))) = \Pi_K(\phi(t, g(y(t))), -(J^{-1} \circ F)(\phi(t, y(t))), \; a.a. \; t, \;\; \phi(0, g(y(0))) = x_0 \in K \tag{3.2.34}$$

where (g, K) is a convexification pair.

Theorem 3.2.14. *Let X be a generic Hilbert space, and let $K^{'}$ be a non-empty closed subset of X. Let K be non-empty, closed and convex, $g : K^{'} \to K$ be continuous and strictly monotone, and $F : K^{'} \to X$ be Lipschitz continuous such that $(F \circ g)_{|K'} = F$. Let also $x_0 \in K$ and $L > 0$ such that $\|x_0\| \leq L$. Then the initial value problem (3.2.32), has a unique solution on the interval $[0, l]$, where $l = \frac{L}{\|F(x_0)\| + bl}$*

Proof. The proof consists in the modification of a few easy steps of the proof given in Cojocaru & Jonker (2004) combined with the results exposed in section 3.2.2 of the present Chapter. □

3.2.4.2 Applications

The relation between an ImPDS and a VI problem is more interesting, as has been considered before in the literature, but with superfluous conditions on the projection operator P_K.
We describe this relation next.

Definition 3.2.15. *Let X be a generic H-space and $K' \subset X$ be a non-empty, closed subset. Let $F : X \to X^*$ be a mapping. Then we call g-variational inequality on the set K' the problem of*

$$finding\ y \in K',\ < F \circ g(y), z - g(y) > \geq 0, \forall z \in K \tag{3.2.35}$$

where (g, K) is a convexification pair of $(F, K^{'})$.

We highlight the importance of the relation $F \circ g(y) = F(y)$ from Definition 3.2.10 in order for the inequality (3.2.35) to make sense. Under (3.2.10) we can rewrite (3.2.35) as

$$find\ y \in K',\ < F(y), z - g(y) > \geq 0, \forall z \in K \tag{3.2.36}$$

Remark 3.2.16. *In Noor (2003), inequality (3.2.36) is considered in an usual Hilbert space (pivot) and is called a "general variational inequality". We prefer to use the term "g-variational inequality" in relation to (3.2.36), in order to avoid confusion with the commonly accepted "generalized variational inequality" which involves multi-mappings.*

Theorem 3.2.17. *If the problems (3.2.36) and (3.2.32) admit a solution, then the equilibrium points of (3.2.36) coincide with the critical points of (3.2.32).*

Proof. Suppose $x^* \in K^{'}$ is a solution of (3.2.36); then by definition we have

$$< F(y^*), z - g(y^*) > \geq 0,\ \forall z \in K$$

so by multiplying by a strictly positive constant λ and using the bilinearity of the inner product we get

$$< -F(y^*), z > \leq 0,\ \forall y \in T_K(g(y^*))$$

so we deduce that $-F(y^*) \in N_K(g(y^*))$; using the decomposition theorem (3.1.11) we get $P_{T_K(g(y^*))}(-J^{-1}F(y^*)) = 0$ and so y^* is a critical point of (3.2.32).

Now suppose that y^* is a critical point of (3.2.32); then by definition we have

$$P_{T_K(g(y^*))}(-J^{-1}F(y^*)) = 0$$

and by the decomposition theorem we get $-F(y^*) \in N_K(g(y^*))$. By the definition of the normal cone to K in $g(y^*)$, the following inequality is satisfied

$$< -F(y^*), z - g(y^*) > \leq 0, \forall z \in K$$

which is exactly (3.2.36). □

Let X be a generic H-space, D closed, convex, nonempty in X. Let $\mathcal{K} : D \to 2^X$ with $\mathcal{K}(x)$ convex for all $x \in D$ and $F :\to 2^{X^*}$ a mapping. Let us introduce the following variational inequality:

$$find\ x \in \mathcal{K}(x),\ \langle F(x), y - x\rangle \geq 0,\ \forall y \in \mathcal{K}(x). \tag{3.2.37}$$

Note that in this case the set in which we are looking for the solution depends on x. For problem (3.2.37) we can refer to Tian & Zhou (1991) or to Section B.6.2 for an existence result. In order to study the disequilibrium behavior of (3.2.37), we introduce now the following projected differential equation.

Definition 3.2.18. *We call projected dynamical system associated to the quasi-variational inequality (3.2.37) the solution set of the projected differential equation*

$$\frac{dx(t)}{dt} = \lim_{\delta \to 0^+} \frac{P_{\mathcal{K}(x)}(x - \delta J^{-1}F(x)) - x}{\delta} = P_{T_{\mathcal{K}(x)}(x)}(-J^{-1}F(x)),\ x(0) = x_0 \in \mathcal{K}$$

Remark 3.2.19. *In general there are no existence results for problem (3.2.18). An existence result for a particular case of (3.2.18) has been given in Noor (2003), assuming the following fact:*

Assumption 3.2.20. *Let X be a pivot H-space. For all $u, v, w \in X$, $P_{\mathcal{K}(u)}$ satisfies the condition*

$$\|P_{\mathcal{K}(u)}(w) - P_{\mathcal{K}(v)}(w)\| \leq \lambda \|u - v\| \tag{3.2.38}$$

where $\lambda > 0$ is a constant.

However, this assumption fails to be true. One counterexample is as follows. We denote by C a closed convex set and we take $u, v \in C$; we denote by $\mathcal{K}(u) = T_C(u)$ and by $\mathcal{K}(v) = T_C(v)$ the tangent cones of C at u and v.

In fact, $w \in X$ can only be chosen in one of the following four situations:

1. $w \in \mathcal{K}(u) \bigcap \mathcal{K}(v)$

2. $w \in \mathcal{K}(u) \setminus \mathcal{K}(v)$

3. $w \in \mathcal{K}(v) \setminus \mathcal{K}(u)$

4. $w \in X \setminus (\mathcal{K}(u) \bigcup \mathcal{K}(v))$

Suppose now that we have $w \in \mathcal{K}(u)\setminus\mathcal{K}(v)$; then by Moreau's decomposition theorem we get

$$\|P_{\mathcal{K}(u)}(w) - P_{\mathcal{K}(v)}(w)\| = \|w - P_{\mathcal{K}(v)}(w)\| = \|P_{N_C(v)}(w)\| \leq \lambda\|u - v\| \tag{3.2.39}$$

where $N_C(v)$ is the normal cone of C at v. Consider now $X = \mathbb{R}^2$, $C = [0,\epsilon]^2$, $u = (0,0)$ and $v = (\epsilon,\epsilon)$. It is clear that we have the following:

$$T_C(u) = \mathbb{R}^2_+ \tag{3.2.40}$$

$$T_C(v) = \mathbb{R}^2_- \tag{3.2.41}$$

$$N_C(v) = \mathbb{R}^2_+ = T_C(u) \tag{3.2.42}$$

So for any $w \in N_C(v)$ we get

$$\|w\| \leq \lambda\|u - v\| = \sqrt{2}\epsilon\lambda.$$

Since w is arbitrary, let now $w := \mu w$, for any $\mu > 0$. Then

$$\|\mu w\| \leq \lambda\|u - v\| = \sqrt{2}\epsilon\lambda$$

should be true for any $\mu > 0$. However this does not hold.

Application 3.2.21. *Consider now the special case of a set-valued mapping $\mathcal{K}$ which is the translation of a closed, convex subset K:*

$$\mathcal{K}: \; x \to K + v(x)$$

where $v(x)$ is a vector linearly dependant on x, then problems (3.2.37) and (3.2.18) can be studied, under certain conditions, respectively as a g-VI and an implicit PDS as shown bellow.
If $\mathcal{K}(x) = K + p(x)$ as done by Noor in Noor (2003) we have the following equivalent formulations:

$$\begin{aligned}\frac{dx(t)}{dt} &= P_{T_{K+p(x)}(x)}(-J^{-1}F(x)) \\ &= P_{T_K}(g(x))(-J^{-1}F(x)), \; x(0) = x_0 \in K\end{aligned} \tag{3.2.43}$$

where $g(x) = x - p(x)$, *assuming* $F(g(x)) = F(x-p(x)) = F(x)$. *We can observe that if* $\frac{dp(x)}{dt} = 0$, *then* (3.2.43) *is equal to the implicit projected differential equation* (3.2.31), *and therefore Theorem* 3.2.14 *provide an existence result without assuming any kind of Lipschitz condition of the projection operator.*

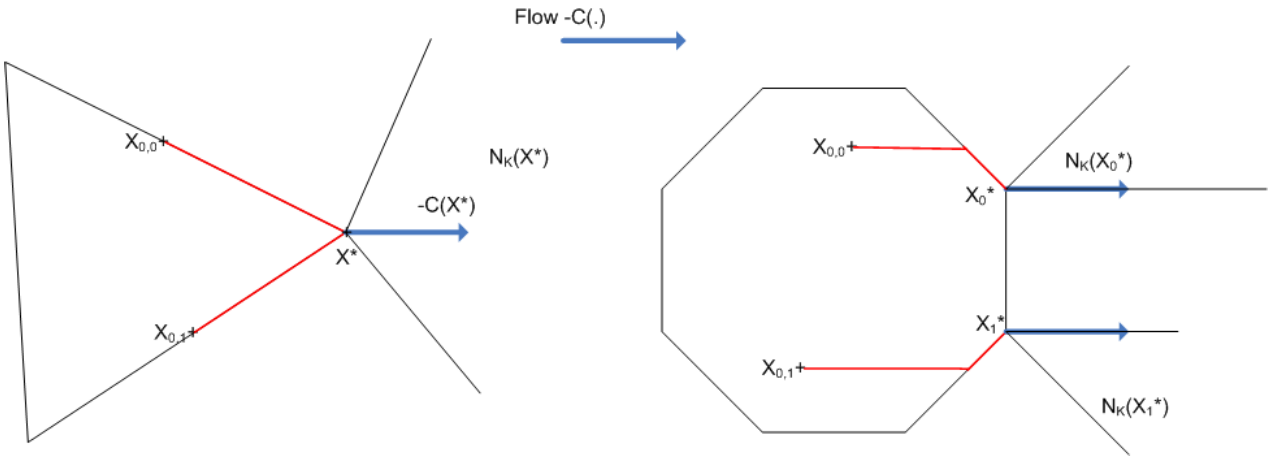

Figure 3.2: Simple Representation of Existence results

3.3 Extensions to Strictly Convex and Smooth Banach Spaces

3.3.1 Introduction to Generalized Projection

We denote by X a Banach space with dual space X^* and by $\|.\|$ and $\|.\|_*$ the respective norms. We denote also the duality pairing between X^* and X by $< f, x >$ for $f \in X^*$ and $x \in X$, $< x, f >$ the duality pairing between X^* and X for $f \in X^*$ and $x \in X$.

We define the duality mapping $J : X \to X^*$ by

$$J(x) = \{f \in X^* :< f, x >= \|f\|_*^2 = \|x\|^2\}, \ \forall x \in X$$

In the same manner we have the duality mapping $J^* : X^* \to X$ defined by:

$$J^*(f) = \{x \in X :< x, f >= \|x\|^2 = \|f\|_*^2\}, \ \forall f \in X^*$$

The existence of J and J^* is a corollary of the Hahn-Banach analytic form (see for instance Brezis (1993b)).

Remark 3.3.1. *If X is an Hilbert space, we have $J = Id_X = J^*$.*

Example 3.3.2. *If $X = L^p(\Omega, \mathbb{R})$ with $1 < p < \infty$ then*

$$J(x) = \|x\|^{2-p}|x|^{p-1}sgn(x)$$

and

$$J^*(x) = \|x\|^{\frac{p-2}{p-1}}|x|^{\frac{1}{1-p}}sgn(x)$$

where $sgn(x) = \chi_{[x>0]} - \chi_{[x<0]}$.
This result could be usefully applied to Time Dependent Traffic Equilibria problems (see Daniele et al. (1999a)).

Now we recall two definitions we need in the sequel.

Definition 3.3.3 (see Diestel (1975))**.** *A space $(X, \|.\|)$ is strictly convex if*

$$\forall x \in X,\ \forall y \in X : \|x\| = \|y\| = 1,\ x \neq y \Rightarrow \|tx + (1-t)y\| < 1, \forall t \in]0,1[$$

Let us denote by $S(X) = \{x \in X : \|x\| = 1\}$.

Definition 3.3.4 (see Diestel (1975))**.** *A Banach space X is said to be smooth at $x_0 \in S(X)$ whenever there exists a unique $f \in S(X^*)$ such that $f(x_0) = 1$. If X is smooth at each point of $S(X)$ then we say that X is smooth.*

From Diestel (1975) we have also the following characterization criteria: A Banach space $(X, \|.\|)$ is smooth if and only if the norm $\|.\|$ admits a Gâteaux derivative in each direction.

Remark 3.3.5. *Hilbert spaces and L^p spaces ($1 < p < \infty$) are reflexive, strictly convex and smooth.*

From Barbu & Precapanu (1978) we know that if we have X reflexive, strictly convex and smooth then J, J^* are one-to-one single-valued operators and $J^{-1} = J^*$. More precisely we have:

- X is reflexive if and only if J is surjective;
- X is smooth if and only if J is single-valued;

- X is strictly convex if and only if J is injective.

Besides the notion of projection operator in Hilbert space, it is possible to give an effective projection operator definition in a more general framework. Let us recall the following definition of metric projection operator (for more details see for instance Song & Cao (2004)).

Definition 3.3.6 (see Song & Cao (2004))**.** *Let X be a Banach space and K a closed convex subset of X. We call the metric projection operator from X on K the set valued mapping $\pi(K|.) : X \to C$ defined by*

$$x \to \pi(K|x) = \{y \in K : \|x - y\| = d_K(x)\}$$

where $d_K(x) = \inf_{z \in K} \|x - z\|$.

Note that for $x \in K$, $\pi(K|x)$ is the set of optimal solution of the following minimization problem:

$$\inf_{y \in C} \|x - y\|^2 \tag{3.3.44}$$

From now on and unless otherwise stated, we make the following assumptions: X reflexive, strictly convex and smooth Banach space.

Then these additional assumptions ensure that $\pi(K|.) = P_K(.)$ is single valued and P_K is called the best approximate operator. Moreover we have the following characterization of $P_K(x)$:

$$\bar{x} = P_K(x) \Leftrightarrow < J(x - \bar{x}), y - \bar{x} > \leq 0, \;\forall y \in K \tag{3.3.45}$$

As an extension of what we have on Hilbert spaces, (3.3.45) is called the basic variational principle for P_K in X. This characterization plays a fundamental role for our application.

Another possibility to generalize the notion of projection is to use, as done by Alber in Alber (1996), the *Lyapunov function.*

The Lyapunov function is the strictly convex function in y, $V(x, y)$ given by:

$$V(x, y) := \|x\|^2 - 2 < J(x), y > + \|y\|^2$$

We remark that if K is a closed convex subset of X and if $x \in K$ then the problem

$$\min_{y \in K} V(x, y)$$

is uniquely solvable (apply for instance Brezis (1993b),Corollary III.20), then we can give the following definition:

Definition 3.3.7 (see Alber (1996) or Song & Cao (2004)). *We call generalized projection of x on C the following value:*

$$\Pi_K(x) := arg \min_{y \in K} V(x, y)$$

Remark 3.3.8 (see Alber (1996)).

- *The operator $\Pi_K : X \to K \subset X$ is the identity on K, i.e. for every $x \in K, \Pi_K(x) = x$.*
- *In a Hilbert space, $V(x, y) = \|x - y\|^2$, Π_K coincides with the projection operator P_K.*

As stated in Alber (2000) we have the following characterization of $\Pi_K(x)$.

Lemma 3.3.9. *Assume that K is a closed convex subset of X, then:*

$$\hat{x} = \Pi_K(x) \Leftrightarrow < J(x) - J(\hat{x}), y - \hat{x} > \leq 0, \ \forall y \in K \tag{3.3.46}$$

Here again the variational characterization plays a fundamental role for our application.

From Corollary 1, page 22, Diestel (1975) we know that if X is reflexive then:

$$\text{X strictly convex} \Leftrightarrow X^* \text{ smooth,}$$
$$\text{X smooth} \Leftrightarrow X^* \text{ strictly convex.}$$

3.3.2 Set-up the problem on Strictly convex and smooth Banach Spaces

3.3.2.1 Set-up

Our aim is to introduce in the framework of Reflexive, smooth, and strictly convex Banach space an operator with a lot of properties of $\pi_K(x, -F(x))$ and apply this new framework. We propose the following new definitions:

Definition 3.3.10. *We call the Metric Projected Dynamical System the operator*

$$\Lambda_K^m : K \times X^* \to X$$

defined by setting:

$$\Lambda_K^m(x, h) = P_{T_K(x)}(J^*(h))$$

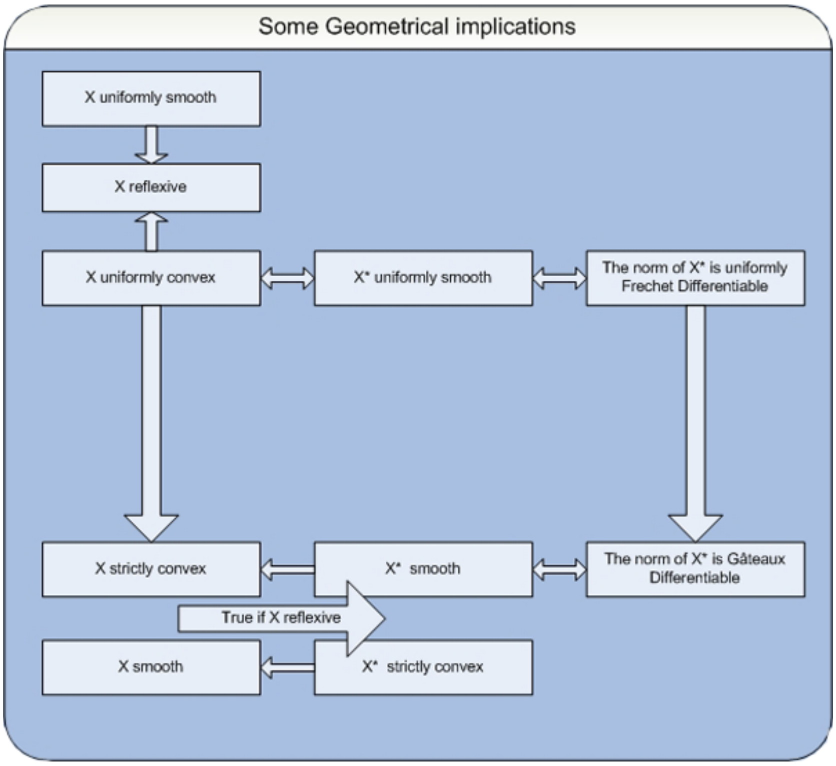

Figure 3.3: Geometrical Relationships

So we can define as done in Nagurney (1993) and in Cojocaru *et al.* (2005) the differential equation with a discontinuous right hand side.

Definition 3.3.11. *We call M-Projected Dynamical System (m-PDS), the discontinuous right hand side differential equation given by:*

$$\frac{dx}{dt} = \Lambda_K^m(x, -F(x)) = P_{T_K(x)}(J^*(-F(x))) \tag{3.3.47}$$

where F is a mapping from $K \to X^*$.

Consequently the associated Cauchy problem is given by:

$$\frac{dx}{dt} = \Lambda_K^m(x, -F(x)) = P_{T_K(x)}(J^*(-F(x))),\ x(0) = x_0 \in K \tag{3.3.48}$$

Definition 3.3.12. *We call the Generalized Projected-Dynamical System the operator*

$$\Lambda_K^g : K \times X^* \to X$$

defined by setting:

$$\Lambda_K^g(x, h) = \Pi_{T_K(x)}(J^*(h))$$

Definition 3.3.13. *We call Generalized Projected Dynamical System (g-PDS), the discontinuous right hand side differential equation given by:*

$$\frac{dx}{dt} = \Lambda_K^g(x, -F(x)) = \Pi_{T_K(x)}(J^*(-F(x))) \tag{3.3.49}$$

where F is a mapping from $K \to X^*$.

The associated Cauchy problem is given by:

$$\frac{dx}{dt} = \Lambda_K^g(x, -F(x)) = \Pi_{T_K(x)}(J^*(-F(x))),\ x(0) = x_0 \in K \tag{3.3.50}$$

3.3.2.2 Decomposition Theorem

In this section we provide a result proved in (Alber (2000)) which generalize Moreau's Theorem (see Moreau (1962)).

Theorem 3.3.14. *[Alber (2000), Theorem 2.4] Assume that X is a real reflexive strictly convex and smooth Banach space, and C a non-empty, closed and convex cone of X then: $\forall x \in X$ and $\forall f \in X^*$ the decompositions*

$$\begin{aligned} x &= P_C(x) + J^*\Pi_{C^0}J(x) \ \textit{and} \ <\Pi_{C^0}J(x), P_C(x)>= 0 \\ f &= P_{C^0}(f) + J\Pi_C J^*(f) \ \textit{and} \ <P_{C^0}(f), \Pi_C J^*(f)>= 0 \end{aligned} \tag{3.3.51}$$

hold.

Remark 3.3.15. *If X is an Hilbert space the decomposition $x = P_C(x) + J^*\Pi_{C^0}J(x)$ reduces to $x = P_C(x) + P_{C^0}(x)$.*

Corollary 3.3.16. *For each $v \in X^*$ we have:*

$$\Lambda_K^m(x, v) = J^*(v) - J^*\Pi_{N_K(x)}(v) \tag{3.3.52}$$

Proof: From Theorem 3.3.14 with $K = T_K(x)$ and $K^0 = N_K(x)$, we get:

$$J^*(v) = P_{T_K(x)}(J^*(v)) + J^*\Pi_{N_K(x)}J(J^*(v))$$

as $JJ^* = Id_{X^*}$ and $P_{T_K(x)}(J^*(v)) = \Lambda_K^m(x, v)$ we deduce immediately the result.

Corollary 3.3.17. *For each $v \in X^*$ we have:*

$$\Lambda_K^g(x, v) = J^*(v - P_{N_K(x)}(v)) \tag{3.3.53}$$

Proof: From Theorem 3.3.14 with $C = T_K(x)$ and $C^0 = N_K(x)$, we get:

$$v = P_{N_K(x)}(v) + J\Pi_{T_K(x)}(J^*(v)).$$

As $\Pi_{T_K(x)}(J^*(v)) = \Lambda_K^g(x, v)$ we deduce immediately the result.

3.3.2.3 Equivalence Theorems

We present the main results of our work, namely we show that the critical points of $m - PrDS$ (3.3.47) and $g - PrDS$ (3.3.49) are the equilibrium points of following variational inequality:

$$x \in K : < F(x), v - x > \geq \; 0, \quad \forall v \in K \tag{3.3.54}$$

where $F : K \to X^*$.
Let us recall some results regarding the existence of equilibria for (3.3.54). There are two standard approaches to the existence of equilibria, namely, with and without a monotonicity requirement.
We shall employ the following definitions.

Definition 3.3.18. *(see Daniele et al. (1999a)) Let E be a real topological vector space, $K \subset E$ convex. Then $F : K \to E^*$ is said to be:*

(i) *pseudomonotone iff, for all* $x,\ y \in K$, $< F(x), y - x > \geq 0 \Rightarrow < F(y), x - y > \leq 0$;

(ii) *hemicontinous iff, for all* $y \in K$, *the function* $\xi \rightarrow < F(\xi), y - \xi >$ *is upper semicontinous on* K;

(iii) *hemicontinous along line segments iff, for all* x,$y \in K$, *the function* $\xi \rightarrow < F(\xi), y - x >$ *is upper semicontinous on the line segment* $[x, y]$.

Then we have the following result, se also Section B.4

Theorem 3.3.19. *(see Daniele et al. (1999a)) Let* E *be a real topological vector space, and let* $K \subseteq E$ *be convex and nonempty. Let* $F : K \rightarrow E^*$ *be given such that:*

(i) *there exist* $A \subseteq K$, *compact, and* $B \subseteq K$ *compact, convex such that, for every* $x \in K \setminus A$, *there exists* $y \in B$ *with* $< F(x), y - x >< 0$;
either ***(ii)*** *or* ***(iii)*** *below holds:*

(ii) F *is hemicontinous;*

(iii) F *is pseudomonotone and hemicontinous along line segments.*

Then, there exists $\bar{x} \in A$ *such that* $< F(\bar{x}), y - \bar{x} > \geq 0$, *for all* $y \in K$.

Theorem 3.3.20. *Assume that the hypotheses of Theorems (3.3.14) and (3.3.19) hold. Then each equilibrium point of (3.3.54) is a critical point of (3.3.47) and, if (3.3.47) admits critical points then they are equilibrium points of (3.3.54).*

Proof. Let x^* be a solution of (3.3.54), since J is bijective there exists an unique $u_{x^*} \in X$ such that $-F(x^*) = J(u_{x^*})$.
So we have

$$< -J(u_{x^*}), x - x^* > \geq 0,\ \forall x \in K$$

and then

$$< -J(u_{x^*}), \lambda(x - x^*) > \geq 0,\ \forall x \in K\ \forall \lambda > 0$$

which is equivalent to write:

$$< J(u_{x^*} - 0_X), y - 0_X > \leq 0,\ \forall y \in T_K(x^*)$$

So using the variational principle (3.3.45) for $P_{T_K(x^*)}$ we get

$$P_{T_K(x^*)}(u_{x^*}) = 0_X = P_{T_K(x^*)}(J^*(-F(x^*)))$$

and we deduce that x^* is a critical point of (3.3.47).□

Now suppose that x^* is a critical point of (3.3.47).
We have $P_{T_K(x^*)}(J^*(-F(x^*))) = 0_X$ and by Corollary 3.3.16 we get

$$J^*(-F(x^*)) = J^*\Pi_{N_K(x^*)}(-F(x^*))$$

as $(J^*)^{-1} = J$ we get

$$-F(x^*) = \Pi_{N_K(x^*)}(-F(x^*))$$

If $x^* \in ri(K)$: then $N_K(x^*) = 0_{X^*}$ so we get:

$$\Pi_{N_K(x^*)}(w) = \Pi_{0_{X^*}}(w) = 0_{X^*} = -F(x^*), \ \forall w \in X^*$$

so we deduce that x^* is solution of (3.3.54).

If $x^* \in rb(K)$ and $J^*(-F(x^*)) \notin T_K(x^*)$ we get $N_K(x^*) \neq 0_{X^*}$ and taking into account that $-F(x^*) = \Pi_{N_K(x^*)}(-F(x^*))$, we deduce that $-F(x^*) \in N_K(x^*)$ and so, using the definition of $N_K(x^*)$ we obtain

$$< F(x^*), x - x^* > \geq 0, \ \forall x \in K$$

which implies that x^* is solution of (3.3.54).

If $x^* \in rb(K)$ and $J^*(-F(x^*)) \in T_K(x^*)$ we derive immediately

$$P_{T_K(x^*)}(J^*(-F(x^*))) = 0_X = J^*(-F(x^*))$$

but J^* is an isometry and so $-F(x^*) = 0_{X^*}$. Then again x^* is solution of (3.3.54). □

Remark 3.3.21. *In the previous proof, it is possible to avoid the use of $ri(K)$, but this notion permits to have an easier approach to geometrical aspects of the theorem.*

Theorem 3.3.22. *Assume that the hypotheses of Theorems (3.3.14) and (3.3.19) hold. Then each equilibrium point of (3.3.54) is a critical point of (3.3.49) and, if (3.3.49) admits critical points then they are equilibrium points of (3.3.54).*

Proof. Let x^* be a solution of (3.3.54), since J is bijective there exists an unique $u_{x^*} \in X$ such that $-F(x^*) = J(u_{x^*})$.
So we have

$$< -J(u_{x^*}), x - x^* > \geq 0, \ \forall x \in K$$

and then
$$< -J(u_{x^*}), \lambda(x - x^*) > \geq 0, \ \forall x \in K \ \forall \lambda > 0,$$
which is equivalent to write:
$$< J(u_{x^*}) - J(0_X), y - 0_X > \leq 0, \ \forall y \in T_K(x^*).$$
So using the variational principle (3.3.46) for $\Pi_{T_K(x^*)}$ we get
$$\Pi_{T_K(x^*)}(u_{x^*}) = 0_X = \Pi_{T_K(x^*)}(J^*(-F(x^*)))$$
from which we deduce that x^* is a critical point of (3.3.49).

Now suppose that x^* is a critical point of (3.3.49).
$\Pi_{T_K(x^*)}(J^*(-F(x^*))) = 0_X$ and by Corollary 3.3.17 we get
$$J^*(-F(x^*) - P_{N_K(x^*)}(-F(x^*))) = 0_X \Leftrightarrow -F(x^*) = P_{N_K(x^*)}(-F(x^*))$$

If $F(x^*) = 0_{X^*}$ then (3.3.54) is trivially verified. Now we suppose that $F(x^*) \neq 0_{X^*}$. Then as $-F(x^*) = P_{N_K(x^*)}(-F(x^*))$ we get $-F(x^*) \in N_K(x^*)$ which means
$$< -F(x^*), y - x^* > \leq 0, \ \forall y \in K$$
and this is exactly (3.3.54). □

3.3.2.4 Projected Dynamical Systems, Unilateral Differential Inclusions

We consider also the two following differential inclusions:
$$-\dot{x} \in J^*(F(x) + N_{T_K(x)}(\dot{x})) \tag{3.3.55}$$

$$-\dot{x} \in J^*(F(x) + N_K(x)) \tag{3.3.56}$$

Proposition 3.3.23. *Let C be a non empty closed convex cone of X. For any s and v in X the following relations are equivalent:*
$$s = \Pi_C(v) \tag{3.3.57}$$

$$J(v) - J(s) \in N_C(s) \tag{3.3.58}$$

$$s \in C,\ \ J(v) - J(s) \in C^o,\ \ < J(v) - J(s), s >= 0 \tag{3.3.59}$$

$$J(v) - J(s) \in C^o,\ \ and\ \forall \nu \in C^o,\ \ \|s\|^2 \leq < J(v) - \nu, s > \tag{3.3.60}$$

Proof. : Using the variational characterization of the generalized projection operator (3.3.46) we get that (3.3.57) is equivalent to:

$$s \in C,\ \ < J(v) - J(s), y - s > \leq 0, \forall y \in C$$

and by definition of a normal cone we get (3.3.58). Before the next step, first let us prove that $N_C(s) = C^o \cap \{s\}^\perp$.
By definition of $N_C(s)$, C^o and $\{s\}^\perp$ we get immediately that $C^o \cap \{s\}^\perp \subset N_C(s)$. Now suppose that $y \in N_C(s)$ then we have

$$< y, \eta - s > \leq 0,\ \ \forall \eta \in C$$

If $< y, \eta >> 0$, as C is a cone, we get $\forall \lambda > 0$, $< y, \lambda\eta > \leq < y, s >$ which implies a contradiction. Then $< y, \eta > \leq 0$ and $y \in C^o$. As $s \in C$ we get $< y, s > \leq 0$ and as $0 \in C$ we conclude that $< y, s >= 0$ and $y \in \{s\}^\perp$. From the previous result we can conclude that

$$J(v) - J(s) \in N_C(s) \Leftrightarrow s \in C,\ \ J(v) - J(s) \in C^o,\ \ < J(v) - J(s), s >= 0$$

Now suppose that (3.3.59) holds, take $\nu \in C^o$, as $< \nu, s > \leq 0 =< J(v) - J(s), s >$ we get $< \nu, s > \leq < J(v), s > - < J(s), s >$ and by definition of J we get:

$$\|s\|^2 \leq < J(v) - \nu, s >,\ \ \forall \nu \in C^o$$

Now suppose that (3.3.60) holds, in particular we get

$$< \nu, s > \leq < J(v), s > - \|s\|^2, \forall \nu \in C^o$$

If $< \nu, s >> 0$ we have a contradiction. In fact $< \nu, s >$ is bounded by $< J(v), s > - \|s\|^2$ and C^o is a cone, so we get that $< \nu, s > \leq 0,\ \forall \nu \in C^o$
But $J(v) - J(s) \in C^o$ then $< J(v) - J(s), s > \leq 0$ if we take $\nu = 0$ in (3.3.60) we get exactly (3.3.59). □

Remark 3.3.24. *A proof of the previous result in $\mathbb{R}^n$ space can be found in Acary* et al. *(2004).*

Corollary 3.3.25. *The following statements are equivalent:*

$$\dot{x} = \Pi_{T_K(x)}(J^*(-F(x))) \tag{3.3.61}$$

$$-\dot{x} \in J^*(F(x) + N_{T_K(x)}(\dot{x})) \tag{3.3.62}$$

$$\begin{cases} -\dot{x} \in J^*(F(x) + N_K(x)) \\ -\dot{x} = J^*(F(x) + P_{N_K(x)}(-F(x)) \\ -\dot{x} = J^*(P_{N_K(x)+F(x)}(0)) \end{cases} \tag{3.3.63}$$

Proof. We apply Proposition 3.3.23 with $C = T_K(x)$, $v = J^*(-F(x))$ and $s = \dot{x}$, so we get immediately (3.3.61) from (3.2.23). From (3.3.58) we get

$$JJ^*(-F(x)) - J(\dot{x}) \in N_{T_K(x)}(\dot{x})$$

As $JJ^* = Id_{X^*}$ we have the equivalence with (3.3.62).

From Albert's theorem we deduce that (3.3.61) is equivalent to

$$\dot{x} = J^*(-F(x) - P_{N_K(x)}(-F(x)))$$

so using the variational principle for metric projection we get:

$$< J^*(-F(x) + J(\dot{x}) + F(x)), y + J(\dot{x}) + F(x) >\leq 0, \ \forall y \in N_K(x)$$

and this is equivalent to

$$-\dot{x} = J^*(P_{N_K(x)+F(x)}(0))$$

And this means that the vector $J(-\dot{x})$ is of minimum norm in $(F(x) + N_K(x))$. □

3.4 Extensions to Reflexive Banach Spaces

In July 1977, in Zarantonello (1977), the author introduces the concept of projectors on convex sets in reflexive Banach spaces, in the report an extension to Reflexive Banach spaces of the results obtained in Zarantonello (1971) is explored but unfortunately the theory obtained is it not satisfactory as in Hilbert spaces. In fact in the report Zarantonello explores the possibility to extend the process of compounding canonical projectors through integration (i.e. Spectral synthesis) to bigger spaces than Hilbert spaces, but the path is not easy at all, nevertheless the paper contains very interesting results such as the definition of the projectors on convex sets in Reflexive Banach spaces and also interpretations of such projectors, decompositions theorems and other interesting results which has been obtained independently (but in a subclass of Banach spaces as remind in the previous section 3.3.2 in Alber (1996). The main goal of this section is to remind (and provide some proofs) and see how we can set up a pre-equilibrium analysis in reflexive Banach Spaces. The section contains only a part of the work done by Zarantonello which primarily goal was to extend his theory of spectral synthesis onver cones to reflexive Banach spaces . The criteria for the selection is based on the actual knowledge of PDS theory, that means we have included the results that we think are useful to prove some existence theorem. Sometimes we compare the results obtained by Zarantonello and we rewrite them in the context of strictly convex and smooth Banach spaces. As the goal of Zarantonello and Alber were different it is difficult to compare their work. Roughly speaking we can say that the work done by Alber is more analytic (he obtained very interesting estimates) and the work done by Zarantonello is more geometric. Anyway what we can say, is that Alber's contributions have more impact on existing literature. At last but not least, the decision to dedicate a section to the work of Zarantonello is not only guided by a functional need, but we sincerely hope this will contribute in a certain way to rediscovered part of Zarantonello's work.

3.4.1 Introduction to Projectors in Reflexive Banach Spaces

Let X be a reflexive Banach space, J denotes as previously the duality mapping of X onto X^*, therefore we have:

$$J : X \to 2^{X^*} \tag{3.4.64}$$

$$J(x) = \{x^* \in X^* | \langle x^*, x\rangle = ||x||^2 = ||X^*||^2_{X^*}\} \tag{3.4.65}$$

we introduce also J^{-1} the duality mapping of X^* onto X,

$$J^{-1} : X^* \to 2^X \tag{3.4.66}$$

$$J^{-1}(x) = \{x \in X | \langle x^*, x\rangle = ||x||^2 = ||X^*||^2_{X^*}\} \tag{3.4.67}$$

and we have in that case

$$J(x) = \partial \frac{1}{2}||x||^2$$

and

$$J^{-1}(x^*) = \partial\frac{1}{2}||x^*||^2$$

where ∂ denotes the set of all subdifferential.
We remind the definition of a subdifferential

Definition 3.4.1. *A subdifferential of a function $f : \Omega \subset X \to \mathbb{R}$ in $x_0 \in \Omega$ is an element $l \in X^*$ such that*

$$f(x) - f(x_0) \geq l(x - x_0)$$

The subdifferential play a fundamental role in nonsmooth analysis.

Remark 3.4.2. *Mappings even when single valued (always if the Banach space is strictly convex ad smooth), are considered here in the context of multivalued mappings, therefore the inverses always exists.*

The conjugate of a proper lower semicontinuous function $f : X \to]-\infty, +\infty]$ is denoted f^* and it is given by:

Definition 3.4.3. *Let X be a real normed space, and let X^* be the dual space of X. For a function*

$$f : \ X \to \mathbb{R} \cup \{+\infty\}$$

the convex conjugate is given by

$$f^* : \ X^* \to \mathbb{R} \cup \{+\infty\}$$

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) | x \in X\}$$

or equivalently

$$f^*(x^*) = -\inf\{f(x) - \langle x^*, x\rangle | x \in X\}$$

We denote with Q the function $x \to \frac{1}{2}||x||^2$ and with Q^* its conjugate given by $x^* \to \frac{1}{2}||x^*||^2$. If K is a convex set, Ψ_K denotes its indicator function defined by:

$$\Psi_K(x) = \begin{cases} 0 \ if \ x \in K, \\ +\infty \ if \ x \notin K. \end{cases}$$

Definition 3.4.4. *The projector on a closed convex set K in X is the mapping*

$$\Pi_K^B : X^* \to 2^X$$

assigning to each x^* the set of points minimizing the function

$$\frac{1}{2}||x^*||^2 + \frac{1}{2}||x||^2 - \langle x^*, x\rangle$$

over K, i.e.

$$\Pi_K^B(x^*) = \{x \in K | \frac{1}{2}||x||^2 - \langle x^*, x\rangle \leq \frac{1}{2}||y||^2 - \langle x^*, y\rangle, \forall y \in K\} \tag{3.4.68}$$

Remark 3.4.5. *The difference between the definition given in 3.3.7 is that as in strictly convex and smooth Banach spaces the duality mapping J is single valued we can define the generalized projection operator directly as a mapping from X into 2^X. In SCS Banach spaces, the relationship between (3.3.7) and (3.4.4) is the following:*
As for each x^* there exist an unique x such that $x^* = J(x)$,

$$\Pi_K^B(x^*) = \Pi_K^B(J(x)) = \Pi_K(x)$$

This is true for all $x \in X$.
therefore we have

$$\Pi_K^B J = \Pi_K$$

Remark 3.4.6. *Definition 3.4.4 is given in Zarantonello (1977) using instead of Π_K^B the standard notation for projection P_K but we prefer to use P_K for the operator of minimum norm. As this operator still exists. In fact we can use Example 3.4.7 to see that "generalized" and "metric" projection can coexist and be different.*

Example 3.4.7. *As shown by the following example, Metric Projection and the Generalized Projection normally do not coincide in a non Hilbert space.*

In R^3 endowed with the norm

$$\|(x_1, x_2, x_3)\| = ({x_1}^2 + {x_2}^2)^{\frac{1}{2}} + ({x_2}^2 + {x_3}^2)^{\frac{1}{2}}$$

taking

$$K = \{x \in \mathbb{R}^3 | x_2 = x_3 = 0\}$$

We get $P_K((1,1,1)) = (1,0,0)$ but $\Pi_K((1,1,1)) = (2,0,0)$

Remark 3.4.8. *Since $||x||^2 - \langle x^*, x\rangle$ is a lower semi continuous convex function of x tending to $+\infty$ with $||x||$, the infimum is always attained and $P_K(x^*)$ is never empty.*

Theorem 3.4.9. $\Pi_K^B(x^*) = \{x \in K | (Q + \Psi_K)x + (Q + \Psi_K)^* x^* = \langle x^*, x\rangle\} = (J + \partial\Psi_K)^{-1} x^*$

Proof. From (3.4.68) we obtain

$$x \in \Pi_K^B(x^*) \Leftrightarrow$$
$$\langle x^*, x\rangle - (\frac{\|x\|^2}{2} + \Psi_K(x)) = \sup_{y\in K}\{\langle x^*, x\rangle - (\frac{\|y\|^2}{2} + \Psi_K(x))\}$$
$$\Leftrightarrow (Q + \Psi_K)x + (Q + \Psi_K)^* x^* = \langle x^*, x\rangle$$

Using Theorem 23.5 in Rockafellar & Wets (1998) we get that $x \in \partial(Q + \Psi_K)^*(x^*)$ and $x^* \in \partial(Q + \Psi_K)(x)$ but the domains of Q and Ψ_K have of course a common point, therefore using Moreau-Rockafellar theorem, we have $x^* \in \partial Q(x) + \partial\Psi_K(x)$ which is equivalent to $x^* \in J(x) + \partial\Psi_K(x)$ or $x \in (J + \partial\Psi_K)^{-1}(x^*)$ □

Corollary 3.4.10. Π_K^B *is a subdifferential*

Remark 3.4.11. *This result is important. If we look at Lemma 3.1.16, we can notice that in Hilbert spaces the projection operator is a differential. This property plays a crucial role to prove an existence result. In further research we foresee to study the relationship between (3.4.10) and (3.1.16).*

Corollary 3.4.12. *The function* $\frac{1}{2}||x||^2 - \langle x^*, x\rangle$ *remains constant over* $P_K(x^*)$

Remark 3.4.13. *This corollary justifies the notation* $\langle x^*, P_K(x^*)\rangle - \frac{1}{2}||P_K(x^*)||^2$ *for the common value of* $\langle x^*, x\rangle - \frac{1}{2}||x||^2$ *on* $P_K(x^*)$

Corollary 3.4.14. *For each* $x^* \in X^*$*, we have*

$$\langle x^*, \Pi_K^B(x^*)\rangle - \frac{1}{2}||\Pi_K^B(x^*)||^2 = (Q + \Psi_K)^*(x^*) \tag{3.4.69}$$

Proof. The left hand side coincides with the supremum of $\langle x^*, y\rangle - (\frac{1}{2}||y||^2 + \Psi_K(y))$ which is $(Q + \Psi_K)^*(x^*)$. □

Remark 3.4.15. *In SCS Banach Spaces, we can rewrite (3.4.69) in the following way:*

$$\langle J(x), \Pi_K(x)\rangle - \frac{1}{2}||\Pi_K(x)||^2 = (Q + \Psi_K)^*(J^{-1}(x)) \tag{3.4.70}$$

Corollary 3.4.16. Π_K^B *satisfies the subdifferential equation*

$$\Pi_K^B(x^*) = \partial[\langle x^*, \Pi_K^B(x^*)\rangle - \frac{||\Pi_K^B(x^*)||^2}{2}] \tag{3.4.71}$$

Remark 3.4.17. *In SCS Banach Spaces we can rewrite (3.4.71) in the following way*

$$\Pi_K(x) = \partial[\langle J(x), \Pi_K(x)\rangle - \frac{1}{2}||\Pi_K(x)||^2] \tag{3.4.72}$$

Corollary 3.4.18. *For each $x^*, y^* \in X^*$ we have*

$$\Pi_K^B(x^*) \cap \Pi_K^B(y^*) \subset \Pi_K^B(tx^* + (1-t)y^*) \tag{3.4.73}$$

Where $t \in [0,1]$

We present now the variational principle proved by Zarantonello

Corollary 3.4.19. *We have*

$$x \in \Pi_K^B(x^*) \Leftrightarrow \exists \bar{x}^* \in J(x) \mid \langle x^* - \bar{x}^*, x - y\rangle \geq 0, \forall y \in K \tag{3.4.74}$$

Proof. As we have,
$x \in \Pi_K^B(x^*) \Leftrightarrow x^* \in J(x) + \partial\Psi_K(x)$,
we can write $x^* \in J(x) + \partial\Psi_K(x) \Leftrightarrow \exists \bar{x}^* \in J(x) \mid x^* - \bar{x}^* \in \Psi_K(x)$
$\Leftrightarrow x \in K, \bar{x}^* \in J(x) \mid \langle x^* - \bar{x}^*, x - y\rangle \geq 0, y \in K$ □

Remark 3.4.20. *In SCS Banach spaces, setting $x^* = J(x)$, (3.4.74) can be rewrote:*

$$\hat{x} = \Pi_K(x) \Leftrightarrow \langle J(x) - J(\hat{x}), \hat{x} - y\rangle \geq 0, \forall y \in K \tag{3.4.75}$$

therefore we have exactly (3.3.46).

3.4.1.1 Conical Projectors

Projectors on closed convex cones with vertex at the origin (Tangent and Normal Cone are examples of such cones) are called conical projectors. It is clear that a projector on a convex set is positive homogeneous when the set is a cone with vertex at 0, and only then so the class of conical projectors coincides with the class of positive homogeneous projectors.

Definition 3.4.4 for projectors on cones can be expressed in the following way

Theorem 3.4.21. $\Pi_C^B(x^*) = \{x \in C | \langle x^*, x\rangle = \|x\|^2 = [\sup_{u \in C,\ \|u\| \leq 1} \langle x^*, u\rangle]^2$

Proof. If x minimizes $\frac{1}{2}\|y\|^2 - \langle x^*, y\rangle$ over C, then

$$t \to \frac{1}{2}t^2\|y\|^2 - t\langle x^*, y\rangle$$

attains its minimum on the positive real axis at $t = 1$, and hence $\|x\|^2 = \langle x^*, x\rangle$.
Therefore $x \in \Pi_C^B(x^*)$ if and only if $\|x\|^2 = \langle x^*, x\rangle$ and

$$-\frac{\|x\|^2}{2} = \frac{\|x\|^2}{2} - \langle x^*, x\rangle = \inf_{y \in C}(\frac{1}{2}\|y\|^2 - \langle x^*, y\rangle)$$

$$= \inf_{y\in C} \inf_{t\geq 0} (\frac{1}{2}t^2\|y\|^2 - t\langle x^*, y\rangle)$$

$$= \inf_{y\in C} \inf_{t\geq 0} (\frac{1}{2}t^2\|y\|^2 - t\langle x^*, y\rangle)$$

$$= \inf_{y\in C} \begin{cases} 0, & \text{if } \langle x^*, y\rangle \leq 0 \\ -\frac{1}{2}\langle x^*, \frac{y}{\|y\|}\rangle^2, & \text{if } \langle x^*, y\rangle > 0 \end{cases}$$

$$= -\frac{1}{2} \sup_{u\in C, \|u\|\leq 1} \langle x^*, u\rangle^2$$

□

Remark 3.4.22. *Any element $x \neq 0$ in $\Pi_C^B(x^*)$ is of the form $\langle x^*, u\rangle^+ u$, where u is a vector in C maximizing $\langle x^*, u\rangle^+$, so $\Pi_C^B(x^*)$ is obtained by looking for the directions in C making the smallest angle with x^* and projecting them in the ordinary sense. In Hilbert spaces there is coincidence between the least angle mapping and the minimum norm mapping.*

Remark 3.4.23. *Remark 3.4.22 is very important because this geometric approach (least angle mapping), offers, from an intuitive point of view the possibility to affirm that there exists conditions to guarantee, at least in Strictly convex and smooth Banach Spaces, existence of solution to PDS.*

Definition 3.4.24. *Let $f, h : X \to \mathbb{R} \cup \{+\infty\}$ be proper functions, i.e. there exist points in X such that f and h are finite: We call epi-addition or inf-convolution of f and h at $x \in X$ the following operation:*

$$(f\Box h)(x) := \inf_{y+z=x} (f(y) + h(z))$$

The inf-convolution has many important properties for non-linear problems it seems to be also a very useful tool for integration of subdifferentials.

Properties 3.4.25. *We have*

$$(f\Box h)^* = f^* + h^*$$

Theorem 3.4.26. *If we denote by $\delta_{C^*}(x^*)$ the distance from x^* to C^*, we have*

$$\|\Pi_C^B(x^*)\|^2 = \langle x^*, \Pi_C^B(x^*)\rangle = [\sup_{u\in C,\ \|u\|\leq 1} \langle x^*, u\rangle]^2 = \delta_{C^*}^2(x^*)$$

Proof. Using previous result we only have to prove the last equality. Using Theorem 3.4.9 and 3.4.14.

$$\langle x^*, \Pi_C^B(x^*)\rangle - \frac{\|\Pi_C^B(x^*)\|^2}{2} = (Q + \Psi_C)^*(x^*) = (Q^* \Box \Psi_C^*)^*(x^*)$$
$$= (Q^* \Box \Psi_{C^*})(x^*)$$
$$= \inf_{z^*+y^*=x^*} (\frac{\|z^*\|^2}{2} + \Psi_{C^*}(y^*)) = \inf_{y^* \in C^*} \inf_{z^*+y^*=x^*} (\frac{\|z^*\|^2}{2})$$
$$= \inf_{y^* \in C^*} (\frac{\|x^* - y^*\|^2}{2}) = \frac{\delta_{C^*}^2(x^*)}{2}$$

Since $\langle x^*, \Pi_C^B(x^*)\rangle - \frac{\|\Pi_C^B(x^*)\|^2}{2}$ is equal to both $\frac{\|\Pi_C^B(x^*)\|^2}{2}$ and $\langle x^*, \Pi_C^B(x^*)\rangle$ the theorem is proved. □

Corollary 3.4.27. *We have*

$$\Pi_C^B(x^*) = \partial \frac{\|\Pi_C^B(x^*)\|^2}{2} = \partial \frac{\delta_{C^*}^2(x^*)}{2}$$

Proof. Using (3.4.16) and previous result we get the result. □

3.4.2 Decomposition Theorems and Applications

Establishing the relationship between the conical projector and the nearest point mapping we will optain as Corollary 3.4.29 the decomposition theorem. Which is both a generalization of Moreau's decomposition theorem and Albert's decomposition theorem. Corollary 3.4.29 is fundamental to establish the equivalence between critical point of PDS (see 3.4.3) and equilibrium point of Variational inequalities. If an existence result for PDS is extended to Reflexive Banach spaces we can describe the dynamic of pre-equilibrium (seen as the more efficient path to the equilibrium).

Theorem 3.4.28. $(Id_{X^*} - J\Pi_C^B)(x^*) \cap C^*$ *is the set of point in* C^* *closest to* x^*

Proof. If $z^* \in (Id_{X^*} - J\Pi_C^B)(x^*) \cap C^*$ then $x^* - z^* \in J\Pi_C^B(x^*)$ and

$$\|x^* - z^*\| = \|J\Pi_C^B(x^*)\| = \|Pi_C^*(x^*)\| = \delta_{C^*}(x^*)$$

this shows that z^* minimizes the distance from x^* to points in C^*.
Conversely, if $z^*1 \in C^*$ realizes the minimum distance from x^* to C^*, then

$$\delta_{C^*}^2(x^*) = \|x^* - z^*\|^2$$

As we have

$$\delta^2_{C^*}(y^*) = \|y^* - z^*\|^2, \ \forall y^* \in X^*$$

and using (3.4.27),

$$\partial \frac{\delta^2_{C^*}(x^*)}{2} = \Pi^B_C(x^*)$$

we get

$$\begin{aligned}\frac{\|y^* - z^*\|^2}{2} - \frac{\|x^* - z^*\|^2}{2} &\geq \frac{\delta^2_{C^*}(y^*)}{2} - \frac{\delta^2_{C^*}(x^*)}{2} \\ &\geq \langle y^* - x^*, \Pi^B_C(x^*)\rangle, \ \forall y \in X^*\end{aligned}$$

Using the definition of subgradient

$$\Pi^B_C(x^*) \subset \partial \frac{\|x^* - z^*\|^2}{2} = J^{-1}(x^* - z^*)$$

therefore we have

$$z^* \in x^* - J\Pi^B_C(x^*)$$

□

Let us denote by $P_{C^*} : X^* \to 2^{X^*}$ the nearest point mapping on C^* we can express the previous theorem to get a generalization of the theorems proved by Moreau and Alber.

Corollary 3.4.29. *For any $x^* \in X^*$ there are vectors u and v^* such that*

$$x^* = Ju + v^*, \ u \in C, \ v^* \in C^*, \langle v^*, u\rangle = 0 \tag{3.4.76}$$

Moreover if (3.4.76) holds then $u \in \Pi^B_C(x^*)$ and $v^* \in P_{C^*}(x^*)$

3.4.3 Projected Dynamical Systems in Reflexive Banach spaces

Our aim in this section is to propose a Projected Dynamical System which can be a generalization of the concepts exposed earlier in the chapter (see sections 3.2 and 3.3). We will proceed in an analogous way, therefore we set-up:

Definition 3.4.30. *Let K be a close convex set of a Reflexive Banach space X. A* ***Projected differential equation in Reflexive Banach spaces*** *(PDS-RB) is a discontinuous ODE given by:*

$$\frac{dx(t)}{dt} \in \Pi^B_{T_K(x(t))}(-F)(x(t)). \tag{3.4.77}$$

Consequently the associated Cauchy problem is given by:

$$\frac{dx(t)}{dt} \in \Pi^B_{T_K(x(t))}(-F)(x(t)), \;\; x(0) = x_0 \in K. \tag{3.4.78}$$

Next we define what we mean by a solution for a Cauchy problem of type (3.4.78).

Definition 3.4.31. *An absolutely continuous function $x : \mathfrak{I} \subset \mathbb{R} \to X$, such that*

$$\begin{cases} x(t) \in K, \;\; x(0) = x_0 \in K, \forall t \in \mathfrak{I} \\ \dot{x}(t) \in \Pi^B_{T_K(x(t))}(-F)(x(t)), \;\; a.e. \;\; on \; \mathfrak{I} \end{cases} \tag{3.4.79}$$

is called a solution for the initial value problem (3.4.78) if there exist $v \in L^1(\mathfrak{I}, X)$ such that $v \in \Pi^B_{T_K(x(t))}(-F)(x(t))$.

Remark 3.4.32. *No existence results as been proved for problem (3.4.78), therefore we don't know if Definition 3.4.31 is set up correctly.*

Finally, assuming problem (3.4.78) has solutions as described above, then we are ready to introduce:

Definition 3.4.33. *A* **projected dynamical system in Reflexive Banach Space** *(PDS-RB) is given by a mapping $\phi : \mathbb{R}_+ \times K \to K$ which solves the initial value problem:*
$\dot{\phi}(t, x) \in \Pi^B_K(\phi(t, x), -F)(\phi(t, x))), \;\; a.a. \; t, \; \phi(0, x) = x_0 \in K.$

3.4.4 Equivalence Results

Even if we don't provide in this work an answer regarding the existence of solution for PDS-RB, we investigate how far the analogy to the situation present in Hilbert spaces can be pushed on. Another advantage to illustrate the basis of the theory in Reflexive Banach Space is that the results presented by Zarantonello are very promising. The goal of this section is to establish a contact point between the theory of Variational Inequalities and PDS in reflexive Banach spaces.
We remind the following:

Definition 3.4.34. *Let K be a nonempty, convex and closed subset of X, reflexive Banach space and let $F : K \to X^*$ be a vector-function. A variational inequality is the problem to find a vector $x \in K$, such that*

$$\langle F(x), y - x \rangle \geq 0, \quad \forall y \in K. \tag{3.4.80}$$

Definition 3.4.35. *Let call* $M_x = \Pi^B_{T_K(x))}(-F(x))$.
A critical point of (3.4.78) is a point x *such that*

$$\inf_{y \in M_x} \|y\| = 0 \tag{3.4.81}$$

In an equivalent way we can say that x_0 *is a critical point of (3.4.78),*

$$0_X \in \Pi^B_{T_K(x))}(-F(x)) \tag{3.4.82}$$

Remark 3.4.36. *It is important to note that the notion of critical point is weeker that the one used in strictly convex and smooth Banach spaces. This definition is of course a generalization of the previous one.*

Theorem 3.4.37. *Assume that the hypotheses of Theorem 3.3.19 hold. Then each equilibrium point of (3.4.80) is a critical point of (3.4.79) and, if (3.4.79) admits critical points then they are equilibrium points of (3.4.80).*

Proof. Let x_0 be a solution of (3.4.80), by hypothesis we have

$$\langle F(x_0), y - x_0 \rangle \geq 0, \quad \forall y \in K. \tag{3.4.83}$$

since X is reflexive, J is surjective, therefore there exists an element $u \in X$ such that $-F(x_0) = J(u)$.
So we have

$$< -J(u), x - x_0 >\geq 0, \ \forall x \in K$$

and then

$$< -J(u), \lambda(x - x_0) >\geq 0, \ \forall x \in K \ \forall \lambda > 0,$$

which is equivalent to write (J is an Isometry):

$$< J(u) - J(0_X), y - 0_X >\leq 0, \ \forall y \in T_K(x_0).$$

So using the variational principle (3.4.74) it is equivalent to,

$$0_X \in \Pi^B_{T_K(x_0)}(J(u)) = \Pi^B_{T_K(x_0)}(-F(x_0)))$$

from which we deduce that x_0 is a critical point of (3.4.78).

Now suppose that x_0 is a critical point of (3.4.78).
By absurd, if (3.4.80) is not satisfied there exists $y_0 \in K$ such that

$$\langle F(x_0), y_0 - x_0 \rangle < 0 \tag{3.4.84}$$

As there exists $u \in X$ such that $J(u) = -F(x_0)$, we can write (3.4.84) in the following way

$$< J(u) - J(0_X), y_0 - 0_X >> 0$$

But as $x_0 \in K$, we have $y_0 \in K \subset T_K(x_0)$. as by hypothesis

$$0_X \in \Pi^B_{T_K(x_0)}(J(u)) = \Pi^B_{T_K(x_0)}(-F(x_0)))$$

using (3.4.74) we get a contradiction. □

The previous result confirm that Reflexive Banach spaces are a good to study Projected Dynamical system, as they have in this context the propriety to have an equivalence between their critical points and equilibrium point as previously proved in easier frameworkos.
Nevertheless still no existence results has been obtained in such spaces.
The difficulties can be shortly listed as follow:

- J is not linear
- There are no results about the differentiability of the generalized projection operator (projector). The only result we have is given by (3.4.71). A result analogous to (3.1.16) should be very usefull.

Chapter 4

Applications to Weighted traffic equilibrium problem in Weighted Hilbert Spaces

4.1 Introduction

The weighted traffic equilibrium model has been presented in Giuffré & Pia (2009), moreover, its retarded formulation in Barbagallo & Pia (2009a). This problem extends the dynamic traffic model (see Daniele *et al.* (1998, 1999b)) as regards the operator involved in the description of the equilibrium and the spaces used. In particular, we remark that a very important difficulty in the dynamic traffic equilibrium problem is related to the real time cost determination of the flow over the links in the transportation network. More precisely, it is fundamental to know how the distribution of the traffic flow is over routes connecting the same origin-destination pair in order to obtain the optimal distribution of the flow in the transportation network. For this reason, we have to be able to determinate the traffic density over each route. The collection of this information could be very costly and moreover it is very difficult and almost impossible to aggregate data on a real time basis. The smart idea developed by the SENSEable City laboratory at MIT is that this information can be roughly collected using mobile devices connection data. As explained in Ratti *et al.* (2006), it is possible to compute these data in order to estimate the traffic repartition over a monitored area. The authors introduce this information in the duality pairing (cost/flow) involved in the model described in Daniele *et al.* (1999b), so that the operator could act on a more appropriate way. Moreover, it is possible to study the traffic equilibrium problem in more complex situations, for example in the presence of a congestion. In fact, we consider Hilbert spaces not identified with their topological dual, that allow us to examine the problem for a wider class of flows. More precisely, let us consider the non-pivot Hilbert space $L^2(\Omega, \mathbb{R}^m, \mathbf{a}, \mathbf{s})$ and observe that a system of wireless communications allows to obtain information in real time about traffic congestion in the paths. Then, it is necessary to indicate to the user what are the more preferable paths. The novelty is that can happen by means of a system of weights on the paths and on the path cost function. In fact, considering a term of the bilinear form which underlines the problem:

$$\sum_{i=1}^{m} \int_{\Omega} F_i(\omega)\sqrt{a_i(\omega)}\sqrt{s_i(\omega)}G_i(\omega)\frac{1}{\sqrt{a_i(\omega)}}\sqrt{s_i(\omega)}d\omega$$

we can see that if $(\sqrt{a_i})^{-1}$ is the weight acting on the path $G_i(\omega)$, then $\sqrt{a_i(\omega)}$ is the weight acting on the path cost function. In such a way, it is guaranteed the following behaviour if $(a_i(\omega))^{-1}$ is very large then $G_i(\omega)$ must be very small. We can obtain this we observe that $a_i(\omega)$ is very small and then the path cost very large. Then the objective to reduce the flow in this path on the weight acting in a reciprocal way. Moreover, the vector-weight $\mathbf{s}$ is connected with the traffic density on paths of the network. This means that if we fix an Origin-Destination pair w_j on the network and we consider two paths p and q that connect w_j and have the same cost trajectory and two different weights $s_p < s_q$, then the user discards the path q.

In this section we consider a variant case of the model described in Daniele *et al.* (1998, 1999b). The framework of non-pivot Hilbert spaces allows us to solve some "congested" traffic problems, namely problems that have no solution in $L^2(\Omega, \mathbb{R}^m)$. The introduction of a new bilinear form permits to apply the recent research done by the SENSEable City laboratory directed by Carlo Ratti to improve the optimal solution of a traffic problem taking into account a real time observation (for more details see Ratti *et al.* (2006) and Giuffré & Pia (2009)).

Let us introduce a network $\mathcal{N}$, which is represented by a graph $G = [N, L]$, where N is the set of nodes (i.e. cross-roads, airports, railway stations) and L is the set of directed links between the nodes. Let a denote a link of the network connecting a pair of nodes and let r be a path consisting of a finite sequence of links which connect an

Origin-Destination (O/D) pair of nodes. In the network there are n links and m paths. Let W denote the set of O/D pairs with typical O/D pair w_j, $|W| = l$ and $m > l$. The set of paths connecting the O/D pair w_j is represented by $\mathcal{R}_j$ and the entire set of paths in the network by $\mathcal{R}$. Let Ω be an open subset of $\mathbb{R}$, let $\mathbf{a} = \{a_1, \dots, a_m\}$ and $\mathbf{a}^{-1} = \{a_1^{-1}, \dots, a_m^{-1}\}$ be two families of weights such that for each $1 \leq r \leq m$, $a_r \in \mathcal{C}(\Omega, \mathbb{R}^+ \setminus \{0\})$. We introduce also the family called real time traffic density $\mathbf{s} = \{s_1, \dots, s_m\}$ such that for each $1 \leq r \leq m$, $s_i \in \mathcal{C}(\Omega, \mathbb{R}^+ \setminus \{0\})$. We associate to each path r, $r = 1, 2, \dots, m$ the components a_r and s_r of the weights $\mathbf{a}$ and $\mathbf{s}$, respectively. By means of these components, we define the spaces V and V^*, as introduced in Section 3.1.3. Let $F \in L^2(\Omega, \mathbb{R}^m, \mathbf{a}, \mathbf{s})$ denote the path flow vector-function. Let $\lambda, \mu \in L^2(\Omega, \mathbb{R}^m, \mathbf{a}, \mathbf{s})$ be the capacity constraints functions, such that $\lambda < \mu$ and for all $r \in \mathcal{R}$ and for almost all $t \in \Omega$,

$$\lambda_r(t) \leq F_r(t) \leq \mu_r(t).$$

Let Φ be the O/D pairs-paths incidence matrix, whose typical entry ϕ_{jr} is 1 if path r connects the pair w_j and 0 otherwise. We denote by α_j the family of indices r such that $\phi_{jr} = 1$, for $j = 1, \dots, l$, let $d^j = |\alpha_j|$, for $j = 1, \dots, l$, then we set $a_j^* = \max(a_{(\alpha_j)_1}, \dots, a_{(\alpha_j)_{d^j}})$[1], for $j = 1, \dots, l$, and $s_j^* = \max(s_{(\alpha_j)_1}, \dots, s_{(\alpha_j)_{d^j}})$, for $j = 1, \dots, l$ and we group the weights into the vectors $\mathbf{a}^* = (a_1^*, \dots, a_l^*)$ and $\mathbf{s}^* = (s_1^*, \dots, s_l^*)$. Let $\rho_j \in L^2(\Omega, \mathbb{R}, a_j^*, s_j^*)$, for $j = 1, \dots, l$, represent the travel demand associated with the users travelling between O/D pair w_j and let $\rho = (\rho_1, \dots, \rho_j, \dots, \rho_l)^T \in L^2(\Omega, \mathbb{R}^l, \mathbf{a}^*, \mathbf{s}^*) = \prod_{j=1}^{l} L^2(\Omega, \mathbb{R}, a_j^*, s_j^*)$ be the total demand vector-function. The traffic conservation law is

$$\sum_{r=1}^{m} \phi_{jr} F_r(t) = \rho_j(t), \quad \text{a.e. in } \Omega,$$

which can be written in matrix-vector notation as

$$\Phi F(t) = \rho(t), \quad \text{a.e. in } \Omega.$$

Furthermore, we give the cost trajectory C which is a function belonging to $L^2(\Omega, \mathbb{R}^m, \mathbf{a}^{-1}, \mathbf{s})$.

[1]Where we denote by $a_{(\alpha_j)_k}$, for $k = 1, \dots, d_j$ and $j = 1, \dots, l$, the k-th element of the family α_j, for $j = 1, \dots, l$.

The set of feasible flows is the set $\mathbb{K}$ of all the path flows in the network which satisfy the capacity constraints and the conservation law:

$$\mathbb{K} = \Big\{ F \in L^2(\Omega, \mathbb{R}^m, \mathbf{a}, \mathbf{s}) : \; \lambda(t) \leq F(t) \leq \mu(t), \quad \text{a.e. in } \Omega, \\ \Phi F(t) = \rho(t), \quad \text{a.e in } \Omega \Big\}$$

It is to prove that $\mathbb{K}$ is a nonempty, convex, closed and bounded subset of $L^2(\Omega, \mathbb{R}^m, \mathbf{a}, \mathbf{s})$.

The following result holds (see Barbagallo & Pia (2009b)).

Proposition 4.1.1. *Let $\lambda, \mu \in L^2(\Omega, \mathbb{R}^m, \boldsymbol{a}, \boldsymbol{s}) \cap C(\Omega, \mathbb{R}^m_+)$, let $\rho \in L^2(\Omega, \mathbb{R}^l, \boldsymbol{a}^*, \boldsymbol{s}^*) \cap C(\Omega, \mathbb{R}^l_+)$ and let $\{t_n\}_{n\in\mathbb{N}} \subseteq \Omega$ be a sequence such that $t_n \to t \in \Omega$, as $n \to +\infty$. Then, the sequence of sets*

$$\mathbb{K}(t_n) = \Big\{ F(t_n) \in \mathbb{R}^m : \; \lambda(t_n) \leq F(t_n) \leq \mu(t_n), \; \Phi F(t_n) = \rho(t_n) \Big\},$$

$\forall n \in \mathbb{N}$, converges to

$$\mathbb{K}(t) = \Big\{ F(t) \in \mathbb{R}^m : \; \lambda(t) \leq F(t) \leq \mu(t), \; \Phi F(t) = \rho(t) \Big\},$$

as $n \to +\infty$, in the Kuratowski's sense.

In the following, we continue to make use of the weighted norm on $\mathbb{R}^m$

$$\|x(t)\|^2_{m,\mathbf{a},\mathbf{s}} = \sum_{i=1}^{m} x_i^2(t) a_i(t) s_i(t).$$

The next uniformly boundedness result holds (see Barbagallo & Pia (2009b)).

Proposition 4.1.2. *Let $\lambda, \mu \in L^2(\Omega, \mathbb{R}^m, \boldsymbol{a}, \boldsymbol{s}) \cap C(\Omega, \mathbb{R}^m_+)$, let $\rho \in L^2(\Omega, \mathbb{R}^l, \boldsymbol{a}^*, \boldsymbol{s}^*) \cap C(\Omega, \mathbb{R}^l_+)$ and*

$$\|\mu(t)\|_{m,\boldsymbol{a}(t),\boldsymbol{s}(t)} \leq M, \quad \forall t \in \Omega.$$

Then the set

$$\mathbb{K}(t) = \Big\{ F(t) \in \mathbb{R}^m : \; \lambda(t) \leq F(t) \leq \mu(t), \; \Phi F(t) = \rho(t) \Big\},$$

is uniformly bounded for all $t \in \Omega$.

Now, we define the equilibrium condition by means of a weighted variational inequality (see Giuffré & Pia (2009)).

Definition 4.1.3. *$H \in V$ is an equilibrium flow if and only if*

$$H \in \mathbb{K} : \; \langle C(H), F - H \rangle_{\boldsymbol{s}} \geq 0, \; \forall F \in \mathbb{K}. \tag{4.1.1}$$

It is possible to prove the equivalence between condition (4.1.1) and a condition that we call the **weighted Wardrop condition** (see Giuffré & Pia (2009)).

Theorem 4.1.4. *$H \in \mathbb{K}$ is an equilibrium flow in the sense of (4.1.1) if and only if*

$$\forall w \in \mathcal{W}, \ \forall q, m \in \mathcal{R}(w), \ a.e. \ in \ \Omega,$$

$$\begin{aligned} & s_q(t)C_q(t, H(t)) < s_m(t)C_m(t, H(t)) \\ & \Rightarrow H_q(t) = \mu_q(t) \ or \ H_m(t) = \lambda_m(t). \end{aligned} \tag{4.1.2}$$

Proof. Assume that (4.1.2) holds. Let $w \in \mathcal{W}$ and

$$A = \{q \in \mathcal{R}(w) : \ H_q(t) < \mu_q(t) \, a.e. \, in \, \Omega\}$$

$$B = \{m \in \mathcal{R}(w) : \ H_m(t) > \lambda_m(t) \, a.e. \, in \, \Omega\}$$

From (4.1.2) it follows

$$s_q(t)C_q(H(t)) \geq s_m(t)C_m(H(t)) \, \forall q \in A, \, \forall m \in B, \, a.e. \, in \, \Omega.$$

Then there exists a function $\gamma_w(t) : [0, T] \to \mathbb{R}$ such that a.e. in Ω

$$\inf_{q \in A} s_q(t)C_q(H(t)) \geq \gamma_w(t) \geq \sup_{m \in B} s_m(t)C_m(H(t)).$$

Let $F \in K$ be arbitrary. For every $r \in \mathcal{R}(w)$ such that $s_r(t)C_r(H(t)) < \gamma_w(t) \, a.e. \, in \, \Omega$, it results $r \notin A$, that is $H_r(t) = \mu_r(t) \, a.e. \, in \, \Omega$. This implies $F_r(t) - H_r(t) \leq 0 \, a.e. \, in \, \Omega$ and then

$$(s_r(t)C_r(H(t)) - \gamma_w(t))(F_r(t) - H_r(t)) \geq 0 \ a.e. \, in \, \Omega.$$

Likewise for every $r \in \mathcal{R}(w)$ such that $s_r(t)C_r(H(t)) > \gamma_w(t) \, a.e. \, in \, \Omega$, it results $r \notin B$ and

$$(s_r(t)C_r(H(t)) - \gamma_w(t))(F_r(t) - H_r(t)) \geq 0 \, a.e. \, in \, \Omega.$$

It follows

$$\sum_{r=1}^{n} s_r(t)C_r(H(t))(F_r(t) - H_r(t)) \geq \gamma_w(t) \sum_{r=1}^{n} (F_r(t) - H_r(t)) = \gamma_w(t)(\rho_w(t) - \rho_w(t)) = 0$$

and finally we may conclude

$$\langle C(H), F - H \rangle_{\mathbf{s}} = \int_{\Omega} \sum_{i=1}^{n} s_i(\omega)C_i(H(\omega))(F_i(\omega) - H_i(\omega))d\omega \geq 0$$

that is (4.1.1) holds.

Now assume that (4.1.2) does not hold. Then there exists $w \in \mathcal{W}$ and $q, m \in \mathcal{R}(w)$ together with a set $E \subseteq \Omega$ having positive measure such that

$$s_q(t)C_q(H(t)) < s_m(t)C_m(H(t)),\ H_q(t) < \mu_q(t),\ H_m(t) > \lambda_m(t),\ a.e.\, in\, E.$$

For $t \in E$ let $\delta(t) := \min\{\mu_q(t) - H_q(t), H_m(t) - \lambda_m(t)\}$. It results $\delta(t) > 0$ a.e. on E. We construct $F : \Omega \to \mathbb{R}$ in the following way:

$$F_q(t) := H_q(t) + \delta(t),\ F_m(t) := H_m(t) - \delta(t)\, a.e.\, in\, E,$$

$$F_r(t) := H_r(t)\ for\, r \neq q, m,\ a.e.\, in\, E,\ \ F_r(t) := H_r(t)\, a.e.\, in\, \Omega \setminus E.$$

It results that $F \in K$ and

$$\langle C(H), F - H\rangle_{\mathbf{s}} = \int_\Omega \sum_{i=1}^n C_i(H(\omega))(F_i(\omega) - H_i(\omega))s_i(\omega)d\omega =$$

$$\int_E \delta(\omega)[s_q(\omega)C_q(H(\omega)) - s_m(\omega)C_m(H(\omega))]d\omega < 0.$$

Thus H is not an equilibrium. □

4.2 Retarded Weighted traffic equilibrium problem

We suppose now for an easier reading that $\Omega =]0, T[$ and for $h > 0$ we define $\Omega_h =]0, T + h[$. We consider a variant case of the model described in Maugeri (1998) and Raciti (2001). Let us introduce a network $\mathcal{N}$, that means a set $\mathcal{W}$ of origin-destination pair (origin/destination node) and a set $\mathcal{R}$ of routes. Each route $r \in \mathcal{R}$ links exactly one origin-destination pair $w \in \mathcal{W}$. The set of all $r \in \mathcal{R}$ which link a given $w \in \mathcal{W}$ is denoted by $\mathcal{R}(w)$, we consider vector flow $F(t) \in \mathbb{R}^n$.

Let us denote by $n = card(\mathcal{R})$, $\mathbf{a} = \{a_1, \dots, a_n\}$ and by $\mathbf{a}^{-1} = \{a_1^{-1}, \dots, a_n^{-1}\}$ two families of weights such that for each $1 \le i \le n$, $a_i \in \mathcal{C}(\Omega, \mathbb{R}^+ \setminus \{0\})$. We introduce also the family of real time traffic densities $\mathbf{s} = \{s_1, \dots, s_n\}$ such that for each $1 \le i \le n$, $s_i \in \mathcal{C}(\Omega, \mathbb{R}^+ \setminus \{0\})$. We use the framework of a non-pivot Hilbert space which is a multidimensional version of the weighted space $L^2(\Omega, \mathbb{R}, a, s)$. To each element of a and s, let us say a_i and s_i, corresponds a route that we denote by r_i.

As done in Section 3.1.3, we denote by $V_i = L^2(\Omega, \mathbb{R}, a_i, s_i)$ and $V_i^* = L^2(\Omega, \mathbb{R}, a_i^{-1}, s_i)$, the space

$$V = \prod_{i=1}^n V_i$$

is a non pivot Hilbert space for the inner product

$$\langle F, G\rangle_{\mathbf{a},\mathbf{s}} = \sum_{i=1}^{n} \int_{\Omega} F_i(\omega) G_i(\omega) a_i(\omega) s_i(\omega) d\omega.$$

The space

$$V^* = \prod_{i=1}^{n} V_i^*$$

is a non pivot Hilbert space for the following inner product

$$\langle F, G\rangle_{\mathbf{a}^{-1},\mathbf{s}} = \sum_{i=1}^{n} \int_{\Omega} \frac{F_i(\omega) G_i(\omega) s_i(\omega)}{a_i(\omega)} d\omega$$

and the following bilinear form defines a duality between V^* and V:

$$\langle f, x\rangle_{\mathbf{s}} = \sum_{i=1}^{n} \int_{\Omega} f_i(\omega) x_i(\omega) s_i(\omega) d\omega \tag{4.2.3}$$

We suppose that the traffic demand at time t is satisfied after a delay $h > 0$. So if the set of all delayed feasible flows is given by

$$\begin{aligned} \mathbb{K}_h := \{F \in V_h |\ \lambda(t) \le F(t) \le \mu(t), \quad \text{a.e. in } \Omega_h, \\ \Phi F(t+h) = \rho(t), \quad \text{a.e in } \Omega\} \end{aligned} \tag{4.2.4}$$

where $V_h = \prod_{i=1}^{n} L^2(\Omega_h, \mathbb{R}, a_i, s_i)$.

Definition 4.2.1. *$H \in V_h$ is an retarded equilibrium flow if and only if*

$$H \in \mathbb{K}_h : \int_{\Omega} \langle C(H(t)), F(t+h) - H(t+h)\rangle_{s(t)} dt \ge 0, \ \forall F \in \mathbb{K}_h. \tag{4.2.5}$$

We remark that weighted variational inequality (4.2.5) is equivalent to the pointed weighted variational inequality

$$H \in \mathbb{K}_h : \ \langle C(H(t)), F(t+h) - H(t+h)\rangle_{s(t)} \ge 0, \ \forall F(t) \in \mathbb{K}_h(t), \ \text{a.e. in } \Omega,$$

where

$$\mathbb{K}_h(t) := \{F(t) \in \mathbb{R}^n |\ \lambda(t) \le F(t) \le \mu(t), \ \Phi F(t+h) = \rho(t)\}$$

It is possible to prove the equivalence between condition (4.2.5) and what we will call a **weighted retarded Wardrop condition** (4.2.6). More precisely we have:

Theorem 4.2.2. *$H \in \mathbb{K}_h$ is an equilibrium flow in the sense of (4.2.5) if and only if*

$$\forall w \in \mathcal{W},\ \forall r_q, r_m \in \mathcal{R}(w),\ a.e.\ in \Omega,$$

$$\begin{aligned} & s_q(t)C_q(H(t)) < s_m(t)C_m(H(t)) \\ \Rightarrow\ & H_q(t+h) = \mu_q(t+h)\ or\ H_m(t+h) = \lambda_m(t+h). \end{aligned} \tag{4.2.6}$$

Proof. Assume that (4.2.6) holds. Let $w \in \mathcal{W}$ and

$$A = \{q \in \mathcal{R}(w) :\ H_q(t+h) < \mu_q(t+h),\ \text{a.e. in } \Omega\}$$

$$B = \{m \in \mathcal{R}(w) :\ H_m(t+h) > \lambda_m(t+h),\ \text{a.e. in } \Omega\}$$

From (4.2.6) it follows

$$s_q(t)C_q(H(t)) \geq s_m(t)C_m(H(t)),\ \forall q \in A,\ \forall m \in B,\ \text{a.e. in } \Omega.$$

Then there exists a function $\gamma_w(t) : (0,T) \to \mathbb{R}$ such that a.e. in Ω

$$\inf_{q\in A} s_q(t)C_q(H(t)) \geq \gamma_w(t) \geq \sup_{m\in B} s_m(t)C_m(H(t)).$$

Let $F \in K_h$ be arbitrary. For every $r \in \mathcal{R}(w)$ such that $s_r(t)C_r(H(t)) < \gamma_w(t)$, a.e. in Ω_h, it results $r \notin A$, that is $H_r(t+h) = \mu_r(t+h)$, a.e. in Ω_h. This implies $F_r(t+h) - H_r(t+h) \leq 0$, a.e. in Ω_h and then

$$(s_r(t)C_r(H(t)) - \gamma_w(t))(F_r(t+h) - H_r(t+h)) \geq 0,\ \text{a.e. in } \Omega.$$

Likewise for every $r \in \mathcal{R}(w)$ such that $s_r(t)C_r(H(t)) > \gamma_w(t)$ a.e. in Ω, it results $r \notin B$ and

$$(s_r(t)C_r(H(t)) - \gamma_w(t))(F_r(t+h) - H_r(t+h)) \geq 0,\ \text{a.e. in } \Omega.$$

It follows

$$\sum_{r=1}^{n} s_r(t)C_r(H(t))(F_r(t+h) - H_r(t+h)) \geq$$

$$\gamma_w(t)\sum_{r=1}^{n}(F_r(t+h) - H_r(t+h)) = \gamma_w(t)(\rho_w(t) - \rho_w(t)) = 0$$

and finally summing up $\forall w \in \mathcal{W}$ we get the result by integration on Ω.

Now assume that (4.2.6) does not hold. Then there exists $w \in \mathcal{W}$ and $q, m \in \mathcal{R}(w)$ together with a set $E \subseteq \Omega$ having positive measure such that

$$\begin{aligned} & s_q(t)C_q(H(t)) < s_m(t)C_m(H(t)), \\ & H_q(t+h) < \mu_q(t+h),\ H_m(t+h) > \lambda_m(t+h),\ a.e.\ in\ E. \end{aligned}$$

For $t \in E$ let $\delta(t) := \min\{\mu_q(t+h) - H_q(t+h), H_m(t+h) - \lambda_m(t+h)\}$. It results $\delta(t+h) > 0$ a.e. on E. We construct $F : \Omega \to \mathbb{R}$ in the following way:

$$\begin{array}{rcl} F_q(t+h) & := & H_q(t+h) + \delta(t+h), \\ F_m(t+h) & := & H_m(t+h) - \delta(t+h), \text{ a.e. in } E, \\ F_r(t+h) & := & H_r(t+h), \text{ for } r \neq q, m, \text{ a.e. in } E, \\ F_r(t+h) & := & H_r(t+h), \text{ a.e. in } \Omega \setminus E. \end{array}$$

It results that $F \in \mathbb{K}_h$ and

$$\int_\Omega \sum_{i=1}^n C_i(H(\omega))(F_i(\omega+h) - H_i(\omega+h))s_i(\omega)d\omega$$

$$= \int_E \delta(\omega)[s_q(\omega)C_q(H(\omega)) - s_m(\omega)C_m(H(\omega))]d\omega < 0.$$

Thus H is not an equilibrium. □

4.2.1 Existence of Equilibria

In this Section, we obtain an existence result for the retarded weighted model, we can state the following theorem:

Theorem 4.2.3. *Each one of the following conditions is a sufficient condition for the existence of solutions for problem (4.2.5).*

i) $\forall H, F \in \mathbb{K}_h$ *we have*

$$\int_\Omega \langle C(H(t)), F(t+h) - H(t+h)\rangle_{s(t)} dt \geq 0 \Rightarrow \int_\Omega \langle C(F(t)), H(t+h) - F(t+h)\rangle_{s(t)} dt \leq 0$$

ii) $\forall F \in \mathbb{K}_h$ *the function:*

$$H \to \int_\Omega \langle C(H(t)), F(t+h) - H(t+h)\rangle_{s(t)} dt$$

is weakly upper semicontinuous.

iii) $\forall F, G \in \mathbb{K}_h$ *the function:*

$$H \to \int_\Omega \langle C(H(t)), F(t+h) - G(t+h)\rangle_{s(t)} dt$$

is weakly upper semicontinuous on the segment $[F, G]$.

Proof. We remark that $\mathbb{K}_h$ is closed , convex and bounded, hence weakly compact. setting $t+h=y$ from (4.2.5) we get the following problem: Find $H \in K^h$ such that

$$\int_{\Omega_h} \langle C(H(y-h)), F(y)-H(y)\rangle_{\mathbf{s}(t)} dy, \ \forall F \in \mathbb{K}^h \tag{4.2.7}$$

where

$$\begin{aligned} \mathbb{K}^h := \{F \in V^h | \ \lambda(y) \leq F(y) \leq \mu(y), \quad \text{a.e. in } \Omega_{2h}, \\ \Phi F(y) = \rho(y-h), \quad \text{a.e in } \Omega_h\} \end{aligned}$$

where $V^h = \prod_{i=1}^n L^2(\Omega_{2h}, \mathbb{R}, a_i, s_i)$ and $\Omega_{2h} =]0, t+2h[$ We denote by C_h the mapping such that:

$$C(H(y-h)) = C_h(H(s)), \forall y \in \Omega_h$$

So (4.2.7) can be written

$$H \in \mathbb{K}^h, \ \int_{\Omega_h} \langle C_h(H(y)), F(y)-H(y)\rangle_{\mathbf{s(t)}} dy \geq 0, \forall F \in \mathbb{K}^h \tag{4.2.8}$$

we can now apply Corollary 5.1 of Maugeri (1998) and give sufficient condition for the existence of a solution to (4.2.8). But if C satisfies condition (i) on $\mathbb{K}_h$, $\forall H, \ F \in \mathbb{K}_h$ we have

$$\int_{\Omega} \langle C(H(t)), F(t+h)-H(t+h)\rangle_{\mathbf{s}(t)} dt \geq 0 \Rightarrow \int_{\Omega} \langle C(F(t)), H(t+h)-F(t+h)\rangle_{\mathbf{s}(t)} dt \leq 0$$

is pseudomonotone which implies the pseudomonotony of C_h on $\mathbb{K}^h$. If C satisfies condition (ii) on $\mathbb{K}_h$, $\forall F \in \mathbb{K}_h$

$$H \rightarrow \int_{\Omega} \langle C(H(t)), F(t+h)-H(t+h)\rangle_{\mathbf{s}(t)} dt$$

is hemicontinuous which implies the hemicontinuity of C_h on $\mathbb{K}^h$. And if C satisfies condition (iii) on $\mathbb{K}_h$, $\forall F \in \mathbb{K}_h$

$$H \rightarrow \int_{\Omega} \langle C(H(t)), F(t+h)-G(t+h)\rangle_{\mathbf{s}(t)} dt$$

is upper semi-continuous on the segment $[F, G]$ which implies the semi-continuity of C_h on $[F, G]$. Therefore we get the theorem. $\square$

4.3 Study of Equilibrium

4.3.1 Existence and Regularity

The feasible flows have to satisfy the time-dependent capacity constraints and demand requirements, namely for all $r \in \mathcal{R}$, $w \in \mathcal{W}$ and for almost all $t \in \Omega$,

$$\lambda_r(t) \leq F_r(t) \leq \mu_r(t)$$

and

$$\sum_{r \in \mathcal{R}(w)} F_r(t) = \rho_w(t)$$

where $\lambda(t) \leq \mu(t)$ are given, $\rho(t) \in \mathbb{R}^l$, $F_r,\ r \in \mathcal{R}$, denotes the flow in the route r. If $\Phi = (\Phi_{w,r})$ is the pair route incidence matrix, with $w \in \mathcal{W}$ and $r \in \mathcal{R}$, that is

$$\Phi_{w,r} := \chi_{\mathcal{R}(w)}(r),$$

the demand requirements can be written in matrix-vector notation as

$$\Phi F(t) = \rho(t).$$

The set of all feasible flows is given by

$$\begin{aligned} K := \{F \in V |\ \lambda(t) \leq F(t) \leq \mu(t), &\quad \text{a.e. in } \Omega, \\ \Phi F(t) = \rho(t), &\quad \text{a.e in } \Omega\} \end{aligned}$$

We will use again the following norm on $\mathbb{R}^m$

$$\|x(t)\|^2_{m,\mathbf{a},\mathbf{s}} = \sum_{i=1}^{m} x_i^2(t) a_i(t) s_i(t)$$

Proposition 4.3.1. *Let $\lambda, \mu \in C(\Omega, \mathbb{R}^m_+)$, let $\rho \in C(\Omega, \mathbb{R}^l_+)$ and*

$$\|\mu(t)\|_{m,\boldsymbol{a}(t),\boldsymbol{s}(t)} \leq M \ \forall t \in \Omega.$$

Then the set

$$\mathbf{K}(t) = \Big\{F(t) \in \mathbb{R}^m :\ \lambda(t) \leq F(t) \leq \mu(t),\ \Phi F(t) = \rho(t)\Big\},$$

is uniformly bounded in Ω.

Proof Let us take an arbitrary $H(t) \in \mathbb{K}(t)$ therefore we have for $i = 1, ..m$

$$\lambda_i(t) \leq H_i(t) \leq \mu_i(t), \quad \forall t \in \Omega.$$

We have

$$\begin{aligned} \|H(t)\|^2_{m,\mathbf{a},\mathbf{s}} &= \sum_{i=1}^{m} H_i^2(t) a_i(t) s_i(t) \leq \sum_{i=1}^{m} \mu_i(t)^2 a_i(t) s_i(t) \\ &= \|\mu(t)\|^2_{m,\mathbf{a}(t),\mathbf{s}(t)} \leq M^2, \quad \forall t \in \Omega. \end{aligned}$$

that implies the claim. □

It is simply to prove that $\mathbf{K}$ is a nonempty, closed and bounded subset of V, so we can apply Theorem B.4.3 and Theorem B.4.4 to obtain necessary conditions for the existence of the equilibrium solution to the weighted traffic equilibrium problem. Moreover, from Proposition 4.1.1 and Theorem 3.1.19 we deduce that under continuity assumptions of the data, the equilibrium solution to (4.1.1) is continuous.

4.4 Lagrangian theory

This section is devoted to show duality results for the weighted traffic equilibrium problem. In particular, the infinite-dimensional duality theory will be applied in order to obtain the characterization of the weighted traffic equilibrium conditions by means of the Lagrange multipliers. The duality theory has been introduced to solve the unsolved problem of finding, in the infinite dimensional case, the Lagrange multipliers associated to an optimization problem or to a variational inequality subject to possibly nonlinear constraints.

In the papers Daniele *et al.* (2007), Daniele & Giuffré (2007) and Maugeri & Raciti (2009) the authors present an infinite dimensional duality theory which, with the aid of a generalized constraint qualification assumption related to the notion of quasi-relative interior, guarantees the existence of strong duality between a convex optimization problem and its Lagrange dual. The use of quasi relative interior, introduced by Borwein and Lewis Borwein & Lewis (1991), and the notions of tangent and normal cone, allows to overcome the difficulty that in many cases the interior of the set involved in the regularity condition is empty. This is the case of all optimization problems or variational inequalities connected with network equilibrium problems, the obstacle problem, the elastic-plastic torsion problem Barbagallo & Maugeri (appear); Daniele & Giuffré (2007); Daniele *et al.* (2007, to appear); Donato *et al.* (2008); Maugeri & Raciti (2008) which use positive cones of Lebesgue spaces or Sobolev spaces. Then it is not possible to apply the usual duality theory and separation theory which require the Slater assumption. The obstacle was overcome by introducing a new qualification condition called *Assumption S* which allows us to solve the problem of finding the Lagrange multipliers.

First of all, we introduce the concept of cones generated by sets and of tangent cone.

Definition 4.4.1. *Let C be a nonempty subset of a real linear space. Then, the set*

$$cone\ (C) = \{\lambda x : x \in C, \lambda \in \mathbb{R}_+\}$$

is called the cone generated by C.

Let X denote a real normed space, let X^* be the topological dual of all continuous linear functionals on X and let C be a nonempty subset of X.

Definition 4.4.2. *Given an element $x \in X$, the set:*

$$\begin{aligned} T_C(x) \quad = \quad & \Big\{ h \in X : h = \lim_{n\to\infty} \lambda_n(x_n - x),\ \lambda_n \in \mathbb{R} \text{ and } \lambda_n > 0\, \forall n \in \mathbb{N}, \\ & x_n \in C\ \forall n \in \mathbb{N} \text{ and } \lim_{n\to\infty} x_n = x \Big\} \end{aligned}$$

is called tangent cone to C at x.

It results $T_C(x) \subseteq$ cl cone $(C - \{x\})$ and, if C is convex, we get $T_C(x) =$ cl cone $(C - \{x\})$ (see Jahn (1996)).

Following Zarantonello Zarantonello (1971) and Borwein and Lewis Borwein & Lewis (1991), we give the following definition of quasi-relative interior for a convex set.

Definition 4.4.3. *Let C be a convex subset of X. We define quasi-relative interior of C, the set*

$$qri\ C = \{x \in C : \text{cl cone } (C - x) \ \textit{is a linear subspace of}\ X\}$$

or, equivalently,

$$qri\ C = \{x \in C : T_C(x) \ \textit{is a linear subspace of}\ X\}.$$

We define normal cone to C at $x \in X$ the set

$$N_C(x) = \{\xi \in X^* : \langle \xi, y - x \rangle \leq 0, \; \forall y \in C\},$$

then, the following result holds:

Proposition 4.4.4. *Let C be a convex subset of X. Then $x \in C$ belongs to the quasi-relative interior of C, in short, $x \in \operatorname{qri} C$, if and only if $N_C(x)$ is a linear subspace of X^*.*

Using the notion of qri C, in Daniele *et al.* (2007), the following separation theorem is proved.

Theorem 4.4.5. *Let C be a convex subset of X and $x_0 \in C \setminus \operatorname{qri} C$. Then, there exists $\xi \neq \theta_{X^*}$ such that*

$$\langle \xi, x \rangle \leq \langle \xi, x_0 \rangle, \quad \forall x \in C.$$

Viceversa, let us suppose that there exists $\xi \neq \theta_{X^*}$ and a point $x_0 \in X$ such that $\langle \xi, x \rangle \leq \langle \xi, x_0 \rangle$, $\forall x \in C$, and that $Cl(T_C(x_0) - T_C(x_0)) = X$. Then $x_0 \notin \operatorname{qri} C$.

Finally, we remind the definition of convex-like function.

Definition 4.4.6. *Let S be a nonempty subset of a real linear space X and let Y be a real linear space partially ordered by the cone C. A function $f : S \to Y$ is called convex–like if the set $f(S) + C$ is convex.*

For the reader's convenience we present the statement of the duality theorem. Let X be a real linear topological space and let S be a nonempty convex subset of X; let $(Y, \|\cdot\|_Y)$ be a real normed space partially ordered by a convex cone C and let $(Z, \|\cdot\|_Z)$ be a real normed space. Let $f : S \to \mathbb{R}$ and $g : S \to Y$ be two functions such that the function (f, g) is convex-like with respect to the cone $\mathbb{R}_+ \times C$ of $\mathbb{R} \times Y$ and let $h : S \to Z$ be an affine function.

Let us consider the optimization problem

$$\min_{x \in \mathbb{K}} f(x) \tag{4.4.9}$$

where

$$\mathbb{K} = \{x \in S : \; g(x) \in -C, \; h(x) = \theta_Z\}$$

and the dual problem

$$\max_{\substack{u \in C^* \\ v \in Z^*}} \inf_{x \in \mathbb{K}} \{f(x) + \langle u, g(x) \rangle + \langle v, h(x) \rangle\}, \tag{4.4.10}$$

where

$$C^* = \{u \in Y^* : \ \langle u, y \rangle \geq 0 \ \forall y \in C\}$$

is the dual cone of C.

We will say that *Assumption S* is fulfilled at a point $x_0 \in \mathbb{K}$ if it results

$$T_{\widetilde{M}}(f(x_0), \theta_Y, \theta_Z) \cap \Big( ]-\infty, 0[\times \{\theta_Y, \theta_Z\} \Big) = \emptyset, \tag{4.4.11}$$

where

$$\widetilde{M} = \{(f(x) - f(x_0) + \alpha, g(x) + y, h(x)) : \ x \in S \setminus \mathbb{K}, \ \alpha \geq 0, \ y \in C\},$$

The following theorem holds (see Daniele & Giuffré (2007)):

Theorem 4.4.7. *Under the above assumptions, if problem (4.4.9) is solvable and* Assumption S *is fulfilled at the extremal solution $x_0 \in \mathbb{K}$, then also problem (4.4.10) is solvable, the extreme values of both problems are equal and if $(x_0, \overline{u}, \overline{v}) \in \mathbb{K} \times C^* \times Z^*$ is the extremal point of problem (4.4.10), it results:*

$$\langle \overline{u}, g(x_0) \rangle = 0.$$

Using Theorem 4.4.7, we are able to show the usual relationship between a saddle point of the so-called Lagrange functional

$$L(x, u, v) = f(x) + \langle u, g(x) \rangle + \langle v, h(x) \rangle, \quad \forall x \in S, \ \forall u \in C^*, \ \forall v \in Z^*, \tag{4.4.12}$$

and the solution of constraint optimization problem (4.4.9) (see Daniele & Giuffré (2007)).

Theorem 4.4.8. *Let us assume that the assumptions of Theorem 4.4.7 are satisfied. Then $x_0 \in \mathbb{K}$ is a minimal solution to problem (4.4.9) if and only if there exist $\overline{u} \in C^*$ and $\overline{v} \in Z^*$ such that $(x_0, \overline{u}, \overline{v})$ is a saddle point of Lagrange functional (4.4.12), namely*

$$\mathcal{L}(x_0, u, v) \leq \mathcal{L}(x_0, \overline{u}, \overline{v}) \leq \mathcal{L}(x, \overline{u}, \overline{v}), \quad \forall x \in S, \ u \in C^*, \ v \in Z^*$$

and, moreover, it results that

$$\langle \overline{u}, g(x_0) \rangle = 0.$$

4.5 Duality results for weighted traffic equilibrium problem

Let us apply the infinite dimensional duality theorems presented in the previous section in order to characterize the weighted traffic equilibrium conditions in terms of the Lagrange multipliers. At this end, let us consider $H \in \mathbb{K}$ a solution to weighted variational inequality (4.1.1) and let us set

$$\Psi(F) = \langle C(H), F - H \rangle_{\mathbf{s}}, \quad \forall F \in \mathbb{K}.$$

Let us remark that

$$\Psi(F) \geq 0 \quad \forall F \in \mathbb{K}$$

and

$$\min_{F \in \mathbb{K}} \Psi(F) = \Psi(H) = 0. \tag{4.5.13}$$

Before showing the main theorem, we prove some results making use of additional assumptions on the constraint functions of the weighted traffic equilibrium model, more precisely, we suppose that $\lambda = 0$ and $\mu = +\infty$. Let us show the following preliminary lemma having interest in itself.

Lemma 4.5.1. *$H \in \mathbb{K}$ is a weighted traffic equilibrium flow if and only if there exist $\tilde{C} \in L^2(\Omega, \mathbb{R}^m, \boldsymbol{a}^{-1})$ and $\chi \in L^2(\Omega, \mathbb{R}^m, \boldsymbol{a}^{-1})$ such that*

$$C(H) - \Phi^T \tilde{C} = \chi, \quad \langle \chi, H \rangle = 0, \quad \chi \geq 0.$$

Proof. Let us assume that $\mu = +\infty$ and $\lambda = 0$, then from Theorem 4.1.4 we have that $H \in \mathbb{K}$ verifies variational inequality (4.1.1) if and only if for all $i = 1, \ldots, l$, all q, s such that $\phi_{iq} = \phi_{is} = 1$ and a.e. in Ω

$$s_q(t) C_q(t, H(t)) > s_s(t) C_s(t, H(t)) \Longrightarrow H_q(t) = 0. \tag{4.5.14}$$

Setting $\tilde{C}_j(t) = \min\{s_r(t) C_r(t, H(t)) : \phi_{jr} = 1\} \in L^2(\Omega, \mathbb{R}, a_j^{-1})$, $j = 1, \ldots, l$, we can rewrite (4.5.14) in an equivalent form a.e. in Ω as:

$$\left(s_q(t) C_q(t, H(t)) - \tilde{C}_j(t)\right) H_q(t) = 0 \quad \forall q \text{ such that } \phi_{jq} = 1,\ j = 1, \ldots, l. \tag{4.5.15}$$

In fact, if (4.5.14) holds true and $s_q(t) C_q(t, H(t)) - \tilde{C}_j(t) > 0$, then $H_q(t) = 0$, since $\tilde{C}_j(t)$ is equal to some $s_s(t) C_s(t, H(t))$. Vice versa, we suppose that (4.5.15) holds. We assume by contradiction that (4.5.14) does not hold, namely there exists q and s such that $\phi_{jq} = \phi_{js} = 1$ and

$$s_q(t)C_q(t, H(t)) > s_s(t)C_s(t, H(t)), \quad H_q(t) > 0, \quad \text{a.e. in } \Omega.$$

From (4.5.15), it follows

$$s_q(t)C_q(t, H(t)) = \tilde{C}_j(t),$$

that is a contradiction, because $\tilde{C}_j(t) = \min\{s_r(t)C_r(t, H(t)) : \phi_{jr} = 1\}$.

Let us set

$$\begin{cases} s_q(t)C_q(t, H(t)) - \tilde{C}_j(t) = \chi_q(t) \\ \chi_q(t)H_q(t) = 0 \end{cases}. \tag{4.5.16}$$

Denoting by $\tilde{C}(t)$ the vector $\left[\tilde{C}_1(t), \ldots, \tilde{C}_l(t)\right]^T$, χ the vector $[\chi_1(t), \ldots, \chi_m(t)]^T$ and taking into account that in each column of the incidence matrix Φ there is only one entry different from zero, we can rewrite condition (4.5.16) in the form

$$\begin{cases} \mathbf{s}C(H) - \Phi^T \tilde{C} = \chi, \\ \langle \chi, H \rangle = 0, \end{cases}$$

with $\chi \geq 0$, $\chi \in L^2(\Omega, \mathbb{R}^m, \mathbf{a}^{-1})$. □

Now, we are able to prove the following result.

Theorem 4.5.2. *Problem* (4.5.13) *verifies* Assumption S *at the minimal point* $H \in \mathbb{K}$.

Proof. Now, assuming that $H \in \mathbb{K}$ is a solution to (4.1.1), we can rewrite the problem in the form (4.5.13), then we want to prove that *Assumption S* at the minimal point $H \in \mathbb{K}$ is fulfilled. In fact, we have to prove that if $(l, \theta_Y, \theta_Z) \in T_{\widetilde{M}}(\Psi(H), \theta_Y, \theta_Z)$, where $Y = L^2(\Omega, \mathbb{R}^m, \mathbf{a}, \mathbf{s})$ and $Z = L^2(\Omega, \mathbb{R}^l, \mathbf{a}^*, \mathbf{s}^*)$, namely if

$$\begin{aligned} l &= \lim_n \lambda_n(\Psi(F_n) + \alpha_n - \Psi(H)), \\ \theta_Y &= \lim_n \lambda_n(-F_n + G_n), \\ \theta_Z &= \lim_n \lambda_n(\Phi F_n(t) - \rho(t)), \end{aligned} \tag{4.5.17}$$

with $\lambda_n > 0$, $\lim_n(\Psi(F_n) + \alpha_n - \Psi(H)) = 0$, $\lim_n(-F_n + G_n) = \theta_Y$, $\lim_n(\Phi F_n(t) - \rho(t)) = \theta_Z$, l must be nonnegative. In virtue of Lemma 4.5.1 we have

$$\begin{aligned} \Psi(F_n) - \Psi(H) &= \int_\Omega \langle C(t, H(t)), F_n(t) - H(t) \rangle_{\mathbf{s}(t)}\, dt \\ &= \int_\Omega \langle s(t)C(t, H(t)), F_n(t) - H(t) \rangle\, dt \\ &= \int_\Omega \langle \Phi^T \tilde{C}(t) + \chi(t), F_n(t) - H(t) \rangle\, dt \end{aligned}$$

and, taking into account that $\Phi H(t) = \rho(t)$ and $\langle \chi, H \rangle = 0$, we get:

$$\begin{aligned}
&\lambda_n(\Psi(F_n) + \alpha_n - \Phi(H)) \\
&\quad = \lambda_n \int_\Omega \langle \Phi^T \tilde{C}(t), F_n(t) - H(t) \rangle \, dt + \lambda_n \int_\Omega \langle \chi(t), F_n(t) - H(t) \rangle \, dt + \lambda_n \, \alpha_n \\
&\quad = \int_\Omega \langle \tilde{C}(t), \lambda_n(\Phi F_n(t) - \rho(t)) \rangle \, dt + \int_\Omega \langle \chi(t), \lambda_n(F_n(t) - G_n(t)) \rangle \, dt \\
&\qquad + \int_\Omega \langle \chi(t), \lambda_n G_n(t) \rangle \, dt + \lambda_n \alpha_n.
\end{aligned}$$

By means of conditions (4.5.17), we obtain:

$$\lim_n \int_\Omega \langle \tilde{C}(t), \lambda_n(\Phi F_n(t) - \rho(t)) \rangle \, dt = 0, \quad \lim_n \int_\Omega \langle \chi(t), \lambda_n(F_n(t) - G_n(t)) \rangle \, dt = 0,$$

and, being $\chi \geq 0$, $\lambda_n > 0$, $G_n(t) \geq 0$, $\alpha_n \geq 0$, we get:

$$\lim_n \lambda_n(\Psi(F_n) + \alpha_n - \Psi(H)) \geq 0,$$

namely our claim. □

In the following, we obtain the main theorem under the assumptions that the constraint functions are two generic functions belonging to $L^2(\Omega, \mathbb{R}^m, \mathbf{a}, \mathbf{s})$ such that $\lambda < \mu$.

Remark 4.5.3. *We note that if H is a solution to weighted variational inequality* (4.1.1) *Theorem 4.2 in Giuffré & Pia (2009) holds, and, moreover, the following condition is fulfilled: for every $w \in W$, there exists a real-valued function $\gamma_w(\cdot)$ on Ω such that, for all $r \in \mathcal{R}(w)$ and a.e. on Ω,*

$$\begin{aligned}
s_r(t) C_r(t, H(t)) < \gamma_w(t) &\Longrightarrow H_r(t) = \mu_r(t), \\
s_r(t) C_r(t, H(t)) > \gamma_w(t) &\Longrightarrow H_r(t) = \lambda_r(t).
\end{aligned} \tag{4.5.18}$$

Theorem 4.5.4. *Problem* (4.5.13) *verifies* Assumption S *at the minimal point $x^* \in \mathbb{K}$.*

Proof. Let us set

$$\begin{aligned}
A &= \{t \in \Omega : \ s_r(t) C_r(t, H(t)) < \gamma_w(t)\}, \\
B &= \{t \in \Omega : \ s_r(t) C_r(t, H(t)) > \gamma_w(t)\}, \\
C &= \{t \in \Omega : \ s_r(t) C_r(t, H(t)) = \gamma_w(t)\}.
\end{aligned}$$

Let us consider the weighted variational inequality

$$\begin{aligned}
\langle C(H), F-H\rangle_{\mathbf{s}} &= \int_\Omega \langle C(t,H(t)), F(t)-H(t)\rangle_{\mathbf{s}(t)} dt \\
&= \int_\Omega \sum_{w\in W}\sum_{r\in\mathcal{R}(w)} s_r(t)C_r(t,H(t))\Big[F_r(t)-H_r(t)\Big]dt \\
&= \sum_{w\in W}\sum_{r\in\mathcal{R}(w)} \Bigg\{ \int_A s_r(t)C_r(t,x_0(t))\Big[x_r(t)-\mu_r(t)\Big]dt \\
&\quad + \int_B s_r(t)C_r(t,x_0(t))\Big[x_r(t)-\lambda_r(t)\Big]dt \\
&\quad + \int_C \gamma_w(t)\Big[x_r(t)-x_r^0(t)\Big]dt\Bigg\}. \qquad (4.5.19)
\end{aligned}$$

Let us consider, for every $w \in W$, for all $r \in \mathcal{R}(w)$ and a.e. on Ω,

$$s_r(t)C_r(t,H(t)) = \begin{cases} \gamma_w(t) - \tilde{C}_r(t,H(t)), & \text{on } A, \\ \gamma_w(t) + C_r^*(t,H(t)), & \text{on } B, \end{cases}$$

where $\tilde{C}_r(t,H(t))$ and $C_r^*(t,H(t))$ are positive functions.

Making use of the previous statement, (4.5.19) and the traffic conservation law, it follows

$$\begin{aligned}
\langle C(H), F-H\rangle_{\mathbf{s}} &= \sum_{w\in W}\sum_{r\in\mathcal{R}(w)} \Bigg\{ \int_\Omega \gamma_w(t)\Big[F_r(t)-H_r(t)\Big]dt \\
&\quad + \int_A -\tilde{C}_r(t,H(t))\Big[F_r(t)-\mu_r(t)\Big]dt \\
&\quad + \int_B C_r^*(t,H(t))\Big[F_r(t)-\lambda_r(t)\Big]dt\Bigg\} \\
&= \sum_{w\in W}\int_\Omega \gamma_w(t)\sum_{r\in\mathcal{R}(w)}\Big[F_r(t)-H_r(t)\Big]dt \\
&\quad + \sum_{w\in W}\sum_{r\in\mathcal{R}(w)} \Bigg\{ \int_A -\tilde{C}_r(t,H(t))\Big[F_r(t)-\mu_r(t)\Big]dt \qquad (4.5.20) \\
&\quad + \int_B C_r^*(t,H(t))\Big[F_r(t)-\lambda_r(t)\Big]dt\Bigg\} \\
&= \sum_{w\in W}\sum_{r\in\mathcal{R}(w)} \Bigg\{ \int_A -\tilde{C}_r(t,H(t))\Big[F_r(t)-\mu_r(t)\Big]dt \\
&\quad + \int_B C_r^*(t,H(t))\Big[F_r(t)-\lambda_r(t)\Big]dt\Bigg\}.
\end{aligned}$$

Now, we suppose that $H \in \mathbb{K}$ is a solution to (4.1.1), we can rewrite the problem as the optimization problem (4.5.13), we show that *Assumption S* at the minimal point $H \in \mathbb{K}$ is fulfilled. In fact, we have to prove that if $(l, \theta_X, \theta_Y, \theta_Z) \in T_{\widetilde{M}}(\Psi(H), \theta_X, \theta_Y, \theta_Z)$, where $X = Y = L^2(\Omega, \mathbb{R}^m, \mathbf{a}, \mathbf{s})$ and $Z = L^2(\Omega, \mathbb{R}^l, \mathbf{a}^*, \mathbf{s}^*)$, namely if

$$\begin{aligned} l &= \lim_n \lambda_n(\Psi(F_n) + \alpha_n - \Psi(H)), \\ \theta_X &= \lim_n \lambda_n(-F_n + \lambda + G_n), \\ \theta_Y &= \lim_n \lambda_n(F_n - \mu + G_n), \\ \theta_Z &= \lim_n \lambda_n(\Phi F_n(t) - \rho(t)), \end{aligned} \tag{4.5.21}$$

with $\lambda_n > 0$, $\lim_n(\Phi(F_n)+\alpha_n-\Psi(H)) = 0$, $\lim_n(-F_n+G_n) = \theta_Y$, $\lim_n(\Phi F_n(t)-\rho(t)) = \theta_Z$, l must be nonnegative. Taking into account (4.5.20) we have

$$\begin{aligned} \Psi(F_n) - \Psi(H) &= \int_\Omega \langle C(t, H(t)), F_n(t) - H(t)\rangle_{\mathbf{s}(t)} dt \\ &= \sum_{w\in W}\sum_{r\in\mathcal{R}(w)} \left\{ \int_A -\tilde{C}_r(t, H(t))\Big[F_r^n(t) - \mu_r(t)\Big]dt \right. \\ &\quad \left. + \int_B C_r^*(t, H(t))\Big[F_r^n(t) - \lambda_r(t)\Big]dt \right\}. \end{aligned}$$

Hence, it results

$$\begin{aligned} \lambda_n(\Psi(F_n) + \alpha_n - \Psi(H)) &= \sum_{w\in W}\sum_{r\in\mathcal{R}(w)} \lambda_r^n \left\{ \int_A -\tilde{C}_r(t, H(t))\Big[F_r^n(t) - \mu_r(t)\Big]dt \right. \\ &\quad \left. + \int_B C_r^*(t, H(t))\Big[F_r^n(t) - \lambda_r(t)\Big]dt \right\} + \lambda_n\alpha_n \\ &= \sum_{w\in W}\sum_{r\in\mathcal{R}(w)} \lambda_r^n \left\{ \int_A -\tilde{C}_r(t, H(t))\Big[F_r^n(t) - \mu_r(t) + G_r^n(t)\Big]dt \right. \\ &\quad + \int_A \tilde{C}_r(t, H(t))G_r^n(t)dt \\ &\quad + \int_B C_r^*(t, H(t))\Big[F_r^n(t) - \lambda_r(t) - G_r^n(t)\Big]dt \\ &\quad \left. + \int_B C_r^*(t, H(t))G_r^n(t)dt \right\} + \lambda_n\alpha_n \end{aligned}$$

From (4.5.21), we get for every $w \in W$, for all $r \in \mathcal{R}(w)$ and a.e. on Ω

$$\begin{aligned} &\lim_n \int_A -\tilde{C}_r(t, H(t))\lambda_n\Big[F_r^n(t) - \mu_r(t) + G_r^n(t)\Big]dt = 0, \\ &\lim_n \int_B C_r^*(t, H(t))\lambda_n\Big[F_n^r(t) - \lambda_r(t) - G_r^n(t)\Big]dt = 0, \end{aligned}$$

and, since $\lambda_n > 0$, $G_n(t) \geq 0$, $\tilde{C}_r(t, H(t)) \geq 0$, $C_r^*(t, H(t)) \geq 0$, $\alpha_n \geq 0$, we have

$$\lim_n \lambda_n(\Psi(F_n) + \alpha_n - \Psi(H)) \geq 0,$$

this completes the proof. □

Now, we can prove the next result.

Theorem 4.5.5. *$H \in \mathbb{K}$ is a solution to variational problem (4.1.1) if and only if there exist $\alpha^*, \beta^* \in L^2(\Omega, \mathbb{R}^m, \boldsymbol{a}^{-1}, \boldsymbol{s})$ $\delta^* \in L^2(\Omega, \mathbb{R}^m, (\boldsymbol{a}^*)^{-1}, \boldsymbol{s}^*)$ such that:*

(i) $\alpha^*(t), \beta^*(t) \geq 0$ *a.e. in* Ω*;*

(ii) $\alpha^*(t)(\lambda(t) - H(t)) = 0$ *a.e. in* Ω*,*
$\beta^*(t)(H(t) - \mu(t)) = 0$ *a.e. in* Ω*;*

(iii) $\boldsymbol{s}(t)C(t, H(t)) - \boldsymbol{s}(t)\alpha^*(t) + \boldsymbol{s}(t)\beta^*(t) + \Phi^T \boldsymbol{s}^*(t)\rho(t) = 0$ *a.e. in* Ω*.*

Proof. From Theorem 4.4.8 there exists $(\alpha^*, \beta^*, \delta^*) \in C^*$ such that $(H, \alpha^*, \beta^*, \delta^*)$ is a saddle point of the Lagrange functional $\mathfrak{L}$:

$$\mathfrak{L}(H, \alpha, \beta, \delta) \leq \mathfrak{L}(H, \alpha^*, \beta^*, \delta^*) \leq \mathfrak{L}(F, \alpha^*, \beta^*, \delta^*), \tag{4.5.22}$$

$\forall(\alpha, \beta, \delta) \in C^*$ and $\forall F \in L^2(\Omega, \mathbb{R}^m, \mathbf{a}, \mathbf{s})$, and furthermore

$$\begin{aligned} \langle \alpha^*, \lambda - H \rangle_{\mathbf{s}} &= 0, \\ \langle \beta^*, H - \mu \rangle_{\mathbf{s}} &= 0. \end{aligned} \tag{4.5.23}$$

Since $\alpha, \beta, \delta \geq 0$, $\lambda - H, H - \mu \leq 0$, $\Phi H - \rho = 0$, by means of (4.5.23) we obtain

$$\begin{aligned} \alpha^*(t)(\lambda(t) - H(t)) &= 0, \quad \text{a.e. in } \Omega, \\ \beta^*(t)(H(t) - \mu(t)) &= 0, \quad \text{a.e. in } \Omega. \end{aligned}$$

From (4.5.22) it follows, $\forall F \in L^2(\Omega, \mathbb{R}^m, \mathbf{a}, \mathbf{s})$,

$$\begin{aligned} \mathfrak{L}(F, \alpha^*, \beta^*, \delta^*) &= \langle C(H), F - H \rangle_{\mathbf{s}} + \langle \alpha^*, \lambda - F \rangle_{\mathbf{s}} + \langle \beta^*, F - \mu \rangle_{\mathbf{s}} + \langle \delta^*, \Phi F - \rho \rangle_{\mathbf{s}^*} \\ &\geq 0 = \mathfrak{L}(H, \alpha^*, \beta^*, \delta^*), \end{aligned} \tag{4.5.24}$$

Taking into account conditions (4.5.23) and that it results

$$\langle \delta^*, \Phi F - \rho \rangle_{\mathbf{s}^*} = \langle \delta^*, \Phi F - \rho \rangle_{\mathbf{s}^*} - \langle \delta^*, \Phi H - \rho \rangle_{\mathbf{s}^*} = \langle \Phi^T \mathbf{s}^* \delta^*, F - H \rangle,$$

from the right-hand side of (4.5.24), we get

$$\langle \mathbf{s}C(H) - \mathbf{s}\alpha^* + \mathbf{s}\beta^* + \Phi^T \delta^*, F - H \rangle \geq 0, \quad \forall F \in L^2(\Omega, \mathbb{R}^m, \mathbf{a}, \mathbf{s}),$$

Now, we assume

$$F^1 = H + \varepsilon, \quad F^2 = H - \varepsilon, \quad \forall \varepsilon \in L^2(\Omega, \mathbb{R}^m, \mathbf{a}, \mathbf{s}),$$

then, it results, for all $\varepsilon \in L^2(\Omega, \mathbb{R}^m, \mathbf{a}, \mathbf{s})$,

$$\begin{aligned} \mathcal{L}(F^1, \alpha^*, \beta^*, \delta^*) &= -\langle \mathbf{s}C(H) - \mathbf{s}\alpha^* + \mathbf{s}\beta^* + \Phi^T \delta^*, \varepsilon \rangle \geq 0, && (4.5.25) \\ \mathcal{L}(F^2, \alpha^*, \beta^*, \delta^*) &= \langle \mathbf{s}C(H) - \alpha^* + \beta^* + \Phi^T \mathbf{s}^* \delta^*, \varepsilon \rangle \geq 0, && (4.5.26) \end{aligned}$$

Moreover, taking into account (4.5.25), we get, for all $\varepsilon \in C_0^\infty(\Omega)$:

$$\langle \mathbf{s}C(H) - \mathbf{s}\alpha^* + \mathbf{s}\beta^* + \Phi^T \delta^*, \varepsilon \rangle = 0,$$

namely, we obtain

$$\mathbf{s}(t)C(t, H(t)) - \mathbf{s}(t)\alpha^*(t) + \mathbf{s}\beta^*(t) + \Phi^T \mathbf{s}^* \delta^*(t) = 0, \quad \text{a.e. in } \Omega. \tag{4.5.27}$$

Conversely, if there exists

$$H \in \mathbb{K}, \ \alpha^*, \beta^* \in L^2(\Omega, \mathbb{R}^m, \mathbf{a}^{-1}, \mathbf{s})$$

and

$$\delta^* \in L^2(\Omega, \mathbb{R}^m, (\mathbf{a}^*)^{-1}, \mathbf{s}^*)$$

that satisfy the condition $i)$, $ii)$, $iii)$, one has that $(H, \alpha^*, \beta^*, \delta^*)$ is a saddle point of the Lagrange functional $\mathcal{L}$. Then, taking into account Theorem 4.4.8, it results that H is a solution to weighted variational inequality (4.1.1). □

Remark 4.5.6. *The importance of such Lagrange variables is their capacity to describe the behavior of the weighted traffic equilibrium problem. In fact, let us remark that from ii) and if $\alpha^*(t) > 0$, we have that $H(t)$ is given by the flow vector $\lambda(t)$, and if $\beta^*(t) > 0$ then $H(t)$ is given by $\mu(t)$; vice versa if $\alpha^*(t), \beta^*(t) = 0$, it results*

$$\Phi^T \boldsymbol{s}^*(t)\delta^*(t) = -\boldsymbol{s}(t)C(t, H(t)), \quad \text{a.e. in } \Omega.$$

Moreover, assuming that $\beta^*(t) = \delta^*(t) = 0$, we get

$$C(t, H(t)) = \alpha^*(t),$$

namely, α^* represents the equilibrium cost.

Analogously, if $\alpha^*(t) = \delta^*(t) = 0$, we obtain

$$C(t, H(t)) = -\beta^*(t),$$

namely, $-\beta^*_{ij}$ is the equilibrium cost.

4.6 Some considerations about Weights

In this section we propose a way to define the Real Time Traffic Density (RTTD) for a route. This data will be the "weight" of the route considered and it will be used to define the corresponding element of the duality mapping. To define the RTTD we use the smart idea of the Senseable Labo at MIT directed by Carlo Ratti (see Ratti *et al.* (2006) and Ratti *et al.* (2005)). In various contests, using mobile phone connections data, they were able to interpolate and represent graphically, in a continuous way, the density of mobile phone connected over a monitored area. The principle can be generalized to other wireless devices, for instance instead of using mobile phone data it is possible to use also RFID or WiFi devices. It is clear that to weight properly a link is really difficult and it is at least necessary to take into account network's geometry, which means for us the position of network's elements.

We can suppose to have $I \subset \mathbb{R}^2$ closed and large enough to include the monitored area and a parametric continuous function γ_t with $t \in \Omega$ such that:

$$\gamma_t : I \to \mathbb{R}^+$$

$$\gamma_t : (x, y) \to \gamma_t(x, y)$$

This function represent a normalized interpolation obtained using the communication data. We suppose now to have a network that means a set $\mathcal{W}$ of origin-destination pair (origin/destination node) and a set $\mathcal{R}$ of routes. Each route $r \in \mathcal{R}$ links exactly one origin-destination pair $w \in \mathcal{W}$.

For each route we construct a weight in the following way: let us fix $\vartheta \in \mathbb{R}^+ \setminus \{0\}$, a strict positive number called "**resolution**". We introduce the set $r^\vartheta = r \times \vartheta$, $r^\vartheta \subset I$.

We propose now a definition of weight which does not pretend to be exhaustive, all the contrary. We think that the weights should be calibrated case by case. For example one can decide to take into account very exceptional events that are not visible by mobile connection data adding to the definition given below terms that will increase or decrease the weight.

Definition 4.6.1. *Given ϑ a resolution and $\mathcal{N}$ a finite network, we call weight of the route r, the real positive number $\tilde{s}_r(t)$ such that*

$$\tilde{s}_r(t) = \int_{r^\vartheta} \gamma_t(x, y)[\chi_{r^\vartheta \setminus (\bigcup_{p \neq r} p^\vartheta)}(x, y) + m_\vartheta(x, y, t) \sum_{p \neq r} \chi_{r^\vartheta \bigcap p^\vartheta}(x, y)]dxdy \qquad (4.6.28)$$

where $m_\vartheta : I \times \Omega \to \mathbb{R}^+$ *is continuous and called proximity contribution weight and* χ *is the standard characteristic function.*

Remark 4.6.2. *The function* m_ϑ *should be calculated case by case. It has been introduced to balance the action due to the proximity of intersections, roads, etc. In a first approximation we can suppose that* $m_\vartheta(x, y, t) = 1, \ \forall(x, y, t) \in I \times \Omega$

Remark 4.6.3. *The time derivative of* γ_t *in a fixed point* (x, y) *and/or the variation of* γ_t *with respect of to a standard situation* γ_{t_0} *in a same point* (x, y) *could be a very useful information to take into account to set up the real time traffic density.*

Assumption 4.6.4. *We assume that for each* $r \in \mathcal{R}$, $\tilde{s}_r(t) \neq 0$ *for all* $t \in \Omega$.

Definition 4.6.5. *A given family of weights* $\{\tilde{s}_r(t)\}_{r\in\mathcal{R}}$, *is called Normalized Family of Weights if*

$$\sum_{r\in\mathcal{R}} \tilde{s}_r(t) = 1, \forall \ t \ \in \Omega$$

It is clear that each family of weights can be normalized. To define the inner product $\langle\cdot,\cdot\rangle_{\mathbf{a},\mathbf{s}}$ we use a normalized family of weights $\mathbf{s}$.

4.7 Computational Procedure and convergence Analysis

In the present section, we consider the work of Solodov-Svaiter (see Solodov & Svaiter (1999)) developed for Euclidean spaces endowed with the standard inner product in order to extend it in our context. Even if the modifications strictly related to the extension are few but we provide some proofs for reader's convenience. For the detailed description of the method see Solodov & Svaiter (1999).

In particular, to solve a weighted variational inequality we first discretize the time interval, then solve a set of variational inequalities and at least we get the solution by interpolation (the procedure is explained later on). It is clear that the solving method for the variational inequality should be as computationally inexpensive as possible therefore we found the method described by Solodov-Svaiter in Solodov & Svaiter (1999) reduces the computational time because only two projection by iteration are needed. Moreover, this method converges under common assumptions, instead other methods, as extragradient method, request the Lipschitz continuity (see for example Konnov (2001)).

In the following, we will show three steps:

Step 0: We present the Solodov-Svaiter method for the non-pivot setting and we give a convergence result, in $\mathbb{R}^m$ endowed with a weighted norm.

Step 1: We discretize the time interval and we obtain $N+1$ variational inequalities.

Step 2: Then, we apply the Solodov-Svaiter method to $N+1$ variational inequalities.

Step 3: We interpolate the solutions and prove a convergence result of the approximate solution to the exact solution.

Step 0: The purpose is to solve the following weighted variational inequality

$$\langle C(x), y - x\rangle_{\mathbf{s}} \geq 0, \quad \forall y \in K. \tag{4.7.29}$$

where K is a closed convex subset of $V = \mathbb{R}^m$ endowed with a weighted norm, as for example the set that we introduce in Section 5, and we suppose that the norm is given by

$$\|x\|^2_{m,\mathbf{a},\mathbf{s}} = \sum_{i=1}^{m} x_i^2 a_i s_i.$$

It was shown in Auslender & Teboulle (2005) that the projection methods admit different distance functions, although the usual norm distance is the simplest, in this case it is necessary to use the previous one because the problem is modeled by means of weights.

We suppose that the function $C : V \to V^*$ is strictly monotone and hemicontinuous with respect to $\langle \cdot, \cdot\rangle_{\mathbf{s}}$ (as in Definition B.2.3). If we denote by

$$r(x) = x - P_K(x - J_m^{-1}C(x))$$

we can note that

$$r(x) = 0 \Leftrightarrow x \in SVI(C, K),$$

where $SVI(C, K)$ is the set of solutions to weighted variational inequality (4.7.29). For an easier reading we denote by x^i the iteration of order i to find an element of $SVI(C, K)$.

Algorithm 4.7.1. *Choose $x^0 \in K$ and two parameters $\gamma \in]0, 1[$ and $\sigma \in]0, 1[$. Having x^i, compute $r(x^i)$. If $r(x^i) = 0$ stop. Otherwise, compute $z^i = x^i - \eta_i r(x^i)$, where $\eta_i = \gamma^{k_i}$, with k_i the smallest nonnegative integer k satisfying*

$$\langle C(x^i - \gamma^k r(x^i)), r(x^i)\rangle_s \geq \sigma \|r(x^i)\|^2_V \tag{4.7.30}$$

Compute

$$x^{i+1} = P_{K \cap H_i}(x^i)$$

where

$$H_i = \Big\{ x \in V | \ \langle C(z^i), x - z^i \rangle_s \leq 0 \Big\}$$

As done in Solodov & Svaiter (1999) for the finite-dimensional variational inequalities, we need to remind some proprieties of the metric projection operator. We state them for a not necessarily pivot Hilbert space (instead of an Euclidean space), but the proof remains the same. For further details see Solodov & Svaiter (1999) and Zarantonello (1971).

Lemma 4.7.2. *Let V be a non necessarily pivot Hilbert space. Let B be any nonempty closed convex subset of V. For any $x, y \in V$ and any $z \in V$ the following properties hold.*

- $(x - P_B(x), z - P_B(x))_V \leq 0.$
- $\|P_B(x) - P_B(y)\|_V^2 \leq \|x - y\|_V^2 - \|P_B(x) - x + y - P_B(y)\|_V^2$

where $(\cdot, \cdot)_V$ and $\|.\|_V$ are respectively the inner product and the norm of V.

Lemma 4.7.3. *Suppose that the linesearch procedure (4.7.30) of Algorithm (4.7.1) is well-defined. Then it holds that*

$$x^{i+1} = P_{K \cap H_i}(\bar{x}^i)$$

where

$$\bar{x}^i = P_{H_i}(x^i).$$

We also use the following lemma state in an even more general context.

Lemma 4.7.4. *Let X be strictly convex and smooth Banach space, if we denote by f an element of $X^* \setminus \{0\}$, by α a real number and by*

$$K_\alpha = \{x \in V | \ \langle f, x \rangle_s \leq \alpha\},$$

we have

$$P_{K_\alpha}(x) = x^i - max\Big\{0, \frac{\langle f, x \rangle_{X^*, X} - \alpha}{\|f\|_{X^*}^2}\Big\} J^{-1}(f). \tag{4.7.31}$$

Proof. See Theorem 4.2 in Song & Cao (2004). □

Now, we are able to prove the following result.

Corollary 4.7.5. *For x^i construct as specified in Algorithm (4.7.1) and V a not necessarily pivot Hilbert space, if*

$$H_i = \Big\{x \in V|\ \langle C(z^i), x - z^i\rangle_{V^*,V} \leq 0\Big\},$$

then

$$\overline{x}^i = P_{H_i}(x^i) = x^i - \frac{\langle C(z^i), x^i - z^i\rangle_{V^*\times V}}{\|C(z^i)\|^2_{V^*}} J^{-1}(C(z^i)). \tag{4.7.32}$$

Proof. A not necessarily pivot Hilbert space is a strictly convex and smooth Banach space, and the metric and the generalized projection coincide in V because J is linear, by Lemma 4.7.4 we obtain immediately the result taking $\alpha = \langle C(z^i), z^i\rangle_{V^*,V}$ and observing that $x^i \notin H_i$ which implies that $\langle f, x^i\rangle_{V^*,V} - \alpha > 0$. ☐

We can present now the modified convergence theorem, where $V = \mathbb{R}^m$ endowed with a weighted norm. This implies that we have to deal with J the duality mapping between V and V^*.

Theorem 4.7.6. *Let $C(\cdot)$ be a continuous and monotone (with respect to $\langle\cdot,\cdot\rangle_s$ as in Definition B.2.3) function. Suppose $SVI(C,K)$ is nonempty. Then any sequence $\{x^i\}$ generated by Algorithm (4.7.1) converges to a solution of $VI(C,K)$*

Proof. First we show that the linesearch (4.7.30) is well-defined. If $r(x^i) = 0$, then we have that x^i is a solution to the problem. Now, we suppose that $\|r(x^i)\|_V > 0$ and that, for some i, (4.7.30) is not satisfied for any k, this implies

$$\langle C(x^i - \gamma^k r(x^i)), r(x^i)\rangle_{\mathbf{s}} < \sigma\|r(x^i)\|^2_V,\ \forall k \tag{4.7.33}$$

Applying Lemma 4.7.2, we get

$$\begin{aligned} 0 &\geq (x^i - J^{-1}(C(x^i)) - P_K(x^i - J^{-1}(C(x^i))), x^i - P_K(x^i - J^{-1}(C(x^i))))_V \\ &= (r(x^i) - J^{-1}(C(x^i)), r(x^i))_V \\ &= \|r(x^i)\|^2_V - < C(x^i), r(x^i) >_{V^*,V} \end{aligned}$$

Hence

$$\langle C(x^i), r(x^i)\rangle_{\mathbf{s}} \geq \|r(x^i)\|^2_V \tag{4.7.34}$$

Since $x^i - \gamma^k r(x^i) \to x^i$ as $k \to +\infty$, and $C(.)$ is continuous, passing to the limit as $k \to +\infty$ in (4.7.33), we get

$$\langle C(x^i), r(x^i)\rangle_{\mathbf{s}} < \sigma\|r(x^i)\|^2_V$$

So we have a contradiction because $\sigma < 1$ and $\|r(x^i)\|_V > 0$, that means there exists an integer k_i such that (4.7.30) is satisfied. As $x^{i+1} = P_{K\cap H_i}(\bar{x}^i)$, where $\bar{x}^i = P_{H_i}(x^i)$. Using Lemma (4.7.2) for $B = K \cap H_i$, $x = \bar{x}^i$ and $y = x^* \in SVI(C,K) \subset K \cap H_i$ we have by definition of the projection on $K \cap H_i$, $(\bar{x}^i - x^{i+1}, x^* - x^{i+1})_V \leq 0$, but $(\bar{x}^i - x^{i+1}, x^* - x^{i+1})_V = \|x^{i+1} - \bar{x}^i\|_V^2 - (x^{i+1} - \bar{x}^i, x^* - \bar{x}^i)_V$, it follows that

$$(x^* - \bar{x}^i, x^{i+1} - \bar{x}^i)_V \geq \|x^{i+1} - \bar{x}^i\|_V^2.$$

Moreover,

$$\|x^{i+1} - x^*\|_V^2 = \|\bar{x}^i - x^*\|_V^2 + \|x^{i+1} - \bar{x}^i\|_V^2 + 2(x^{i+1} - \bar{x}^i, \bar{x}^i - x^*)_V$$

therefore, we get

$$\|x^{i+1} - x^*\|_V^2 \leq \|\bar{x}^i - x^*\|_V^2 - \|x^{i+1} - \bar{x}^i\|_V^2. \tag{4.7.35}$$

Using Corollary 4.7.5, we get

$$\begin{aligned}
\|\overline{x}^i - x^*\|_V^2 &= \|\overline{x}^i - x^i\|_V^2 + \|x^i - x^*\|_V^2 - 2(\overline{x}^i - x^i, x^i - x^*) \\
&= \Big(\frac{\eta_i\langle C(z^i), r(x^i)\rangle_{\mathbf{s}}}{\|C(z^i)\|_{V^*}^2}\Big)^2 \|J^{-1}(C(z^i))\|_V^2 + \|x^i - x^*\|_V^2 \\
&\quad -2\frac{\eta_i\langle C(z^i), r(x^i)\rangle_{\mathbf{s}}}{\|C(z^i)\|_{V^*}^2}(J^{-1}(C(z^i)), x^i - x^*) \\
&= \frac{\eta_i^2\big(\langle C(z^i), r(x^i)\rangle_{\mathbf{s}}\big)^2}{\|C(z^i)\|_{V^*}^4}\|C(z^i)\|_{V^*}^2 + \|x^i - x^*\|_V^2 \\
&\quad -2\frac{\eta_i\langle C(z^i), r(x^i)\rangle_{\mathbf{s}}}{\|C(z^i)\|_{V^*}^2}\langle C(z^i), x^i - x^*\rangle_{\mathbf{s}}.
\end{aligned} \tag{4.7.36}$$

Moreover, it results

$$\langle C(z^i), x^* - z^i\rangle_{\mathbf{s}} \leq 0 \tag{4.7.37}$$

then, from (4.7.37) we obtain

$$\begin{aligned}
\langle C(z^i), x^i - x^*\rangle_{\mathbf{s}} &= \langle C(z^i), x^i - z^i\rangle_{\mathbf{s}} - \langle C(z^i), x^* - z^i\rangle_{\mathbf{s}} \\
&\geq \eta_i\langle C(z^i), r(x^i)\rangle_{\mathbf{s}}.
\end{aligned} \tag{4.7.38}$$

Now, using (4.7.36), (4.7.38) and (4.7.30), we are able to establish the following inequality

$$\begin{aligned}
\|\overline{x}^i - x^*\|_V^2 &\leq \|x^i - x^*\|_V^2 - \frac{\eta_i^2\big(\langle C(z^i), r(x^i)\rangle_{\mathbf{s}}\big)^2}{\|C(z^i)\|_{V^*}^2} \\
&\leq \|x^i - x^*\|_V^2 - \frac{\eta_i^2\big(\sigma\|r(x^i)\|_V^2\big)^2}{\|C(z^i)\|_{V^*}^2}.
\end{aligned} \tag{4.7.39}$$

Finally, from (4.7.35) and (4.7.39) we get

$$\|\overline{x}^i - x^*\|_V^2 \leq \|x^i - x^*\|_V^2 - \|x^{i+1} - \bar{x}^i\|_V^2 - \frac{\eta_i^2\left(\sigma\|r(x^i)\|_V^2\right)^2}{\|C(z^i)\|_{V^*}^2}. \tag{4.7.40}$$

From the last inequality we can deduce that the sequence $\{\|x^i - x^*\|_V\}_{i\in\mathbb{N}}$ is non increasing, so we deduce that the sequence $\{x^i\}_{i\in\mathbb{N}}$ is bounded, the same holds for $\{z^i\}_{i\in\mathbb{N}}$. So there exists a constant $M > 0$ such that $\|C(z^i)\|_{V^*} \leq M$ for all i. We deduce that

$$\|x^{i+1} - x^*\|_V^2 \leq \|x^i - x^*\|_V^2 - \|x^{i+1} - \bar{x}^i\|_V^2 - \left(\frac{\eta_i\sigma}{M}\right)^2 \|r(x^i)\|_V^4. \tag{4.7.41}$$

Since $\{\|x^i - x^*\|_V\}_{i\in\mathbb{N}}$ converges, we deduce

$$\lim_{i\to\infty} \eta_i\|r(x^i)\|_V = 0$$

Now supposing that $\limsup_{i\to\infty} \eta_i > 0$, we must have in that case $\liminf_{i\to\infty} \|r(x^i)\|_V = 0$. Since $\{x^i\}_{i\in\mathbb{N}}$ is bounded there exists $\hat{x}$ an accumulation point of $\{x^i\}_{i\in\mathbb{N}}$, moreover, being $r(\cdot)$ continuous, we deduce that $r(\hat{x}) = 0$. Then, it follows that $\hat{x} \in SVI(C, K)$ and applying the previous step we deduce that $\{\|x^i - \hat{x}\|_V\}$ converges necessarily to 0, which means that $x^i \to \hat{x} \in SVI(C, K)$.

Suppose now that $\lim_{i\to\infty} \eta_i = 0$, by definition of $\eta_i = \gamma^{k_i}$, we have $\forall k \leq k_i - 1$

$$\langle C(x^i - \gamma^k r(x^i)), r(x^i)\rangle_{\mathbf{s}} < \sigma\|r(x^i)\|_V^2$$

again as $\{x^i\}_{i\in\mathbb{N}}$ is bounded there exists a subsequence again denoted by $\{x^i\}_{i\in\mathbb{N}}$ which converges to $\hat{x}$. So passing to the limit in the previous inequality we get

$$\langle C(\hat{x}), r(\hat{x})\rangle_{\mathbf{s}} < \sigma\|r(\hat{x})\|_V^2,$$

taking into account (4.7.34) we have

$$\langle C(\hat{x}), r(\hat{x})\rangle_{\mathbf{s}} \geq \|r(\hat{x})\|_V^2$$

as $\sigma < 1$ this is possible only if $r(\hat{x}) = 0$ which mean $\hat{x} \in SVI(C, K)$. Using the same method than before we obtain that $\{x^i\}_{i\in\mathbb{N}}$ converges to $\hat{x} \in SVI(C, K)$. □

Step 1 and Step 2: To solve the dynamical case we discretize the open set $\Omega =]0, T[$, in particular we fix $\epsilon > 0$ and we consider the following partition of Ω:

$$0 < t_0^\epsilon < t_1^\epsilon < \ldots < t_r^\epsilon < \ldots < t_N^\epsilon < T$$

where $t_0^\epsilon < \epsilon$ and $T - t_N^\epsilon < \epsilon$. For each value of t_r^ϵ for $r = 0, 1, \dots, N$ we apply the Solodov-Svaiter method, to solve the finite-dimensional weighted variational inequality given by:

$$\langle C(x(t_r^\epsilon), y(t_r^\epsilon) - x(t_r^\epsilon)\rangle_{m,a(t_r),s(t_r)} \geq 0, \quad \forall y(t_r^\epsilon) \in \mathbb{K}(t_r^\epsilon). \tag{4.7.42}$$

Let us denote by $VI(C, K(t_r^\epsilon))$ the variational inequality defined by (4.7.42) and $SVI(C, K(t_r^\epsilon))$ the corresponding set of solutions. It is clear that $SVI(C, K(t_r^\epsilon))$ coincide with the points that satisfy:

$$P_{K(t_r^\epsilon)}(x - J^{-1}C(x)) = x \tag{4.7.43}$$

where $P_{K(t_r^\epsilon)}$ is the metric projection operator associated to the norm induced above, and it is characterized by the following variational principle:

$$\bar{x} = P_K(x) \Leftrightarrow \langle J(x - \bar{x}), y - \bar{x}\rangle_{\mathbf{s}} \leq 0, \ \forall y \in K, \tag{4.7.44}$$

where J is the duality mapping (linear) given in (3.1.10). It results that a point $x \in SVI(F, K(t_r^\epsilon))$ if and only if $r(x) := P_{K(t_r^\epsilon)}(x - J^{-1}(F(x)) - x = 0$.
Generally it is well-known that $x(t)$ solves the variational inequality (4.7.29) if and only if we have $x(t) = P_K(x(t) - \lambda J^{-1}(C(x(t))))$ for all $\lambda > 0$. Where P_K is the metric projection operator on K related to the norm $\|\cdot\|_{\mathbf{a},\mathbf{s}}$. But from the definition,

$$\begin{aligned} P_K(x(t) - \lambda J^{-1}(C(x(t)))) &= \arg\min_{v \in K} \|x - \lambda J^{-1}(C(x)) - v\|^2_{\mathbf{a},\mathbf{s}} \\ &= \arg\min_{v \in K} \left(\frac{1}{2}\langle v, v\rangle_{\mathbf{s}} - \langle x - \lambda J^{-1}(C(x)), v\rangle_{\mathbf{s}} \right) \end{aligned} \tag{4.7.45}$$

In order to solve infinite-dimensional weighted variational inequality (4.7.29) defined into $]0, T[$, we consider a partition of the time interval and the finite-dimensional weighted variational equalities (4.7.42) associated to every point of the partition and we apply the generalized Solodov-Svaiter method to compute the solutions, then, by means of an interpolation procedure, we obtain the solution to infinite-dimensional weighted variational inequality, as it has been done in Barbagallo (2006, 2007b, 2009a,b).

Step 3: We interpolate the stationary equilibrium solution, in order to do that, we assume that all hypothesis to have the continuity of the solution to (4.7.29) and the convergence of the method to compute solutions to finite-dimensional variational inequalities hold. Let us introduce a sequence of $\{\pi_n\}_{n\in\mathbb{N}}$ of partitions of time interval $]0,T[$ such that

$$\pi_n = (t_n^0, \dots, t_n^r, \dots, t_n^{N_n})$$

where

$$0 < \epsilon_n = t_n^0 < \dots < t_n^r < \dots < t_n^{N_n} = T - \epsilon_n < T$$

where $\{\epsilon_n\}_{n\in\mathbb{N}}$ is a strictly positive and decreasing sequence. We consider a sequence of equidistant partitions, such that

$$k_n := max\{t_n^r - t_n^{r-1} | r = 1, 2, \dots, N_n\}$$

approaches zero for $n \to +\infty$. We consider an approximation of the solution by mean of piecewise constant functions.

We denote by $\|\cdot\|_{m,\mathbf{a},\mathbf{s}}$ the norm associated to the inner product before introduced. Under some additional conditions on the weights, we can show the following result:

Theorem 4.7.7. *Assume that the conditions of Theorem 3.1.19 and Theorem 4.7.6 are satisfied, then the approximate solution, given by*

$$u_k(t) = \begin{cases} 0 & if\ t \in]0, \epsilon_n[\\ \sum_{r=1}^{N_k} u(t_k^r)\chi_{[t_k^{r-1}, t_k^r[}(t) & if\ t \in [t_k^0, t_k^{N_k}[\\ \\ 0 & if\ t \in]T - \epsilon_k, T[\end{cases},$$

converges to $u(t)$ in $L^2(]0,T[, \mathbb{R}^m, \boldsymbol{a}, \boldsymbol{s})$ sense

Proof. Let us estimate the following integral

$$\|u - u_k\|_{\mathbf{a},\mathbf{s}}^2 = \int_0^T \|u(t) - u_k(t)\|_{m,a,s}^2 dt$$

$$= \int_0^{\epsilon_k} \|u(t)\|_{m,a,s}^2 dt + \int_{\epsilon_k}^{T-\epsilon_k} \|u(t) - u_k(t)\|_{m,a,s}^2 dt + \int_{T-\epsilon_k}^{T} \|u(t)\|_{m,a,s}^2 dt$$

$$\leq \int_0^{\epsilon_k} \|u(t)\|_{m,a,s}^2 dt + \sum_{r=1}^{N_k} \int_{t_k^{r-1}}^{t_k^r} \|u(t) - u(t_k^r)\|_{m,a,s}^2 dt + \int_{T-\epsilon_k}^{T} \|u(t)\|_{m,a,s}^2 dt$$

$$\leq 2\epsilon_k \|u\|_{\mathbf{a},\mathbf{s}}^2 + \epsilon$$

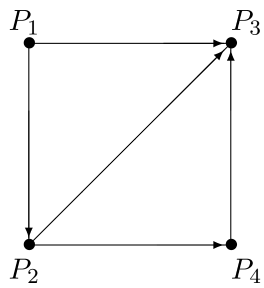

Figure 4.1: A network model.

because u is uniformly continuous on $[\epsilon_k, T - \epsilon_k]$, we have that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $t \in [t_n^{r-1}; t_n^r]$ satisfies the condition $|t - t_n^r| < \delta$, it results

$$\|u(t) - u(t_n^r)\|^2_{m,a,s} < \frac{\epsilon}{T}, \; for \; r = 1, 2, \ldots, N_n, \forall n \in \mathbb{N}$$

Choosing n large enough in such way that $k_n < \delta$, we get

$$\sum_{r=1}^{N_k} \int_{t_k^{r-1}}^{t_k^r} \|u(t) - u(t_k^r)\|^2_{m,a,s} dt \leq \epsilon$$

therefore we get the result. □

4.7.1 Computational methods to solve Projected Dynamical Systems

Actually there are no published work regarding a computational analysis of a scheme in order to determine the trajectory of the pre-equilibrium, it will be object of a future publication by the author. There exists indeed computational methods in order to calculate critical points of a PDS which is equivalent to calculate the solutions of a variational inequality.
The first difficulty that we have to overcome is that calculation of the tangent cone related to a convex K in a generic point x. In a large quantity of problems the convex K is quite simple and even if multidimensional it is possible to calculate it .

4.8 Numerical Example

Let us consider a network as Figure 4.1. The network consists of four nodes and five links. The origin-destination pair is $w = (P_1, P_3)$, which is connected by the paths $R_1 = (P_1, P_3)$, $R_2 = (P_1, P_2) \cup (P_2, P_3)$ and $R_3 = (P_1, P_2) \cup (P_2, P_4) \cup (P_4, P_3)$. Let us consider the cost operator on the path C defined by

$$\begin{aligned} C_1(t,H(t)) &= \frac{t+2}{t}H_1(t)+2t, \\ C_2(t,H(t)) &= \frac{t+3}{2-t}H_2(t)+1, \\ C_3(t,H(t)) &= tH_2(t)+(2t+3)H_3(t)+3t+1. \end{aligned} \tag{4.8.46}$$

The set of feasible flows is given by

$$\mathbb{K} = \Big\{ F \in L^2(]0,2[,\mathbb{R}^3_+) : \ (0,0,0) \le (F_1(t),F_2(t),F_3(t)) \le (20t, 10t+3, 20t+5), \\ F_1(t)+F_2(t)+F_3(t) = 4t+3, \quad \text{a.e. in }]0,2[\Big\}$$

We compute the solution for two different real time densities (using 20 nodes in the interval $]0,2[$) using two different real time densities. The first real time density is given by:

$$\begin{aligned} s_1(t) &= \frac{6}{5}t, \\ s_2(t) &= \frac{6}{5}(2-t), \\ s_3(t) &= 1. \end{aligned} \tag{4.8.47}$$

The cost function is strictly monotone with the previous weight, in fact for all $F(t) \neq H(t)$, a.e. in $]0,2[$ it results

$$\begin{aligned} \langle C(t,F(t))-C(t,H(t)), F(t)-H(t)\rangle_{\mathbf{s}(t)} &= \sum_{i=1}^{3} s_i(t)(C_i(t,F(t))-C_i(t,H(t)))(F_i(t)-H_i(t)) \\ &= \frac{6}{5}(t+2)(F_1(t)-H_2(t))^2+\frac{6}{5}(t+3)(F_2(t)-H_2(t))^2 \\ &\quad +t(F_2(t)-H_2(t))(F_3(t)-H_3(t))+(2t+3) \\ &\quad (F_3(t)-H_3(t))^2 \\ &\ge \frac{6}{5}(t+2)(F_1(t)-H_1(t))^2+\Big(\frac{7}{10}t+\frac{18}{5}\Big) \\ &\quad (F_2(t)-H_2(t))^2+\Big(\frac{3}{2}t+3\Big)(F_3(t)-H_3(t))^2 > 0. \end{aligned}$$

We get the graphical distribution of the traffic flows in Figure 4.2.

The second real time density used is given by increasing the previous one by the real time density (RTD) on the first path by 25% and the RTD on the third path by

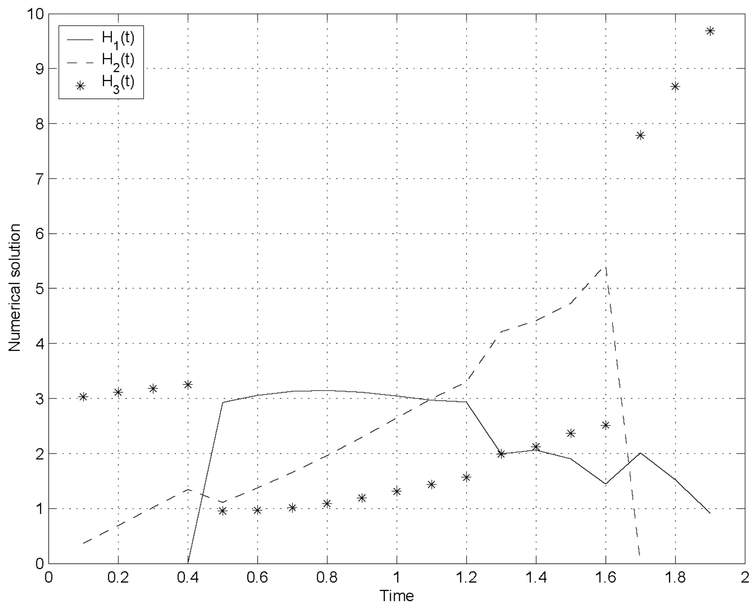

Figure 4.2: Curves of equilibria.

50%. Therefore the RTD that we consider now is

$$\begin{aligned} s_1(t) &= \frac{3}{2}t, \\ s_2(t) &= \frac{6}{5}(2-t), \\ s_3(t) &= \frac{3}{2}. \end{aligned} \tag{4.8.48}$$

In the following, we prove that the cost function is also strictly monotone with the weight above, in fact for all $F(t) \neq H(t)$, a.e. in $]0,2[$ it results

$$\begin{aligned} \langle C(t,F(t)) - C(t,H(t)), F(t) - H(t)\rangle_{\mathbf{s}(t)} &= \sum_{i=1}^{3} s_i(t)(C_i(t,F(t)) - C_i(t,H(t)))(F_i(t) - H_i(t)) \\ &= \frac{3}{2}(t+2)(F_1(t) - H_2(t))^2 + \frac{6}{5}(t+3)(F_2(t) - H_2(t))^2 \\ &\quad + \frac{3}{2}t(F_2(t) - H_2(t))(F_3(t) - H_3(t)) + \frac{3}{2}(2t+3) \\ &\quad (F_3(t) - H_3(t))^2 \\ &\geq \frac{6}{5}(t+2)(F_1(t) - H_1(t))^2 + \Big(\frac{9}{20}t + \frac{18}{5}\Big) \\ &\quad (F_2(t) - H_2(t))^2 + \Big(\frac{9}{4}t + \frac{9}{2}\Big)(F_3(t) - H_3(t))^2 > 0. \end{aligned}$$

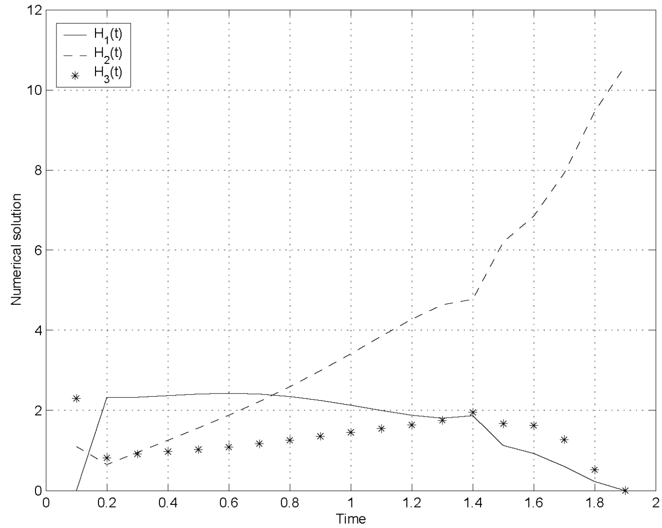

Figure 4.3: Curves of equilibria.

We obtain the equilibrium distribution of the traffic flows in Figure 4.3.

We can visualize the densities and the flows in the following way: We present

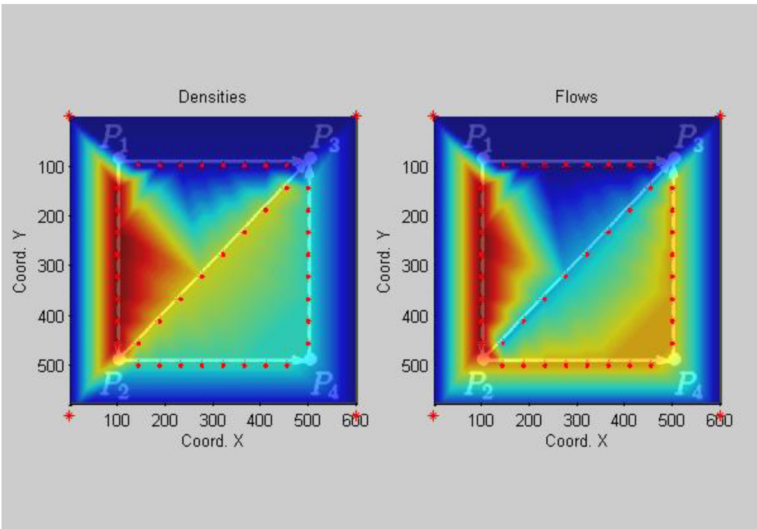

Figure 4.4: Densities and Flows ♯ 1

also this more suggestive visual representation. The graphics has been generated by Mathlab. It is possible to observe that when the density is high on a route, then the flows are redistributed in an equivalent route with lower density.

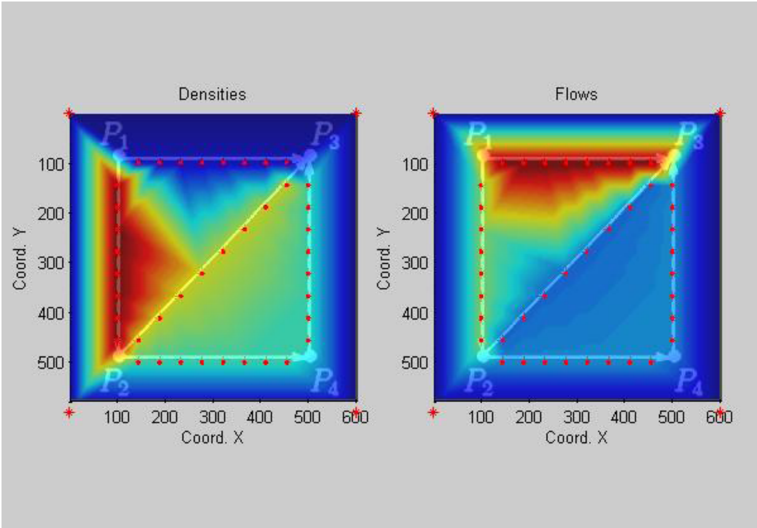

Figure 4.5: Densities and Flows ♯ 2

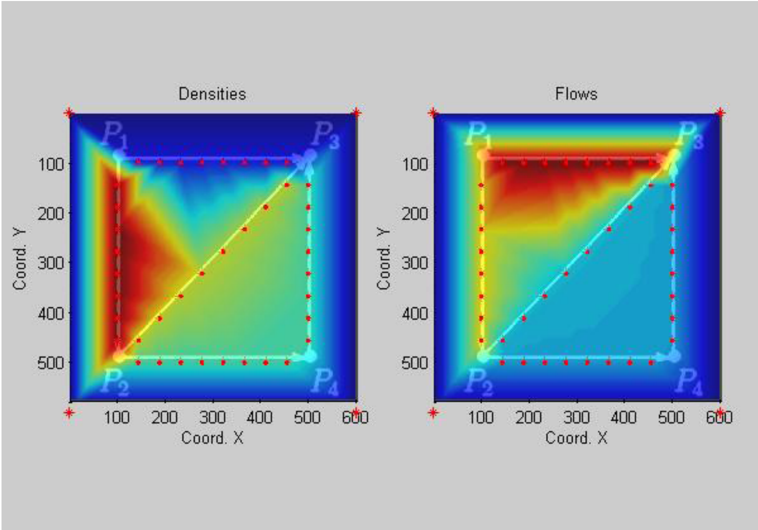

Figure 4.6: Densities and Flows ♯ 3

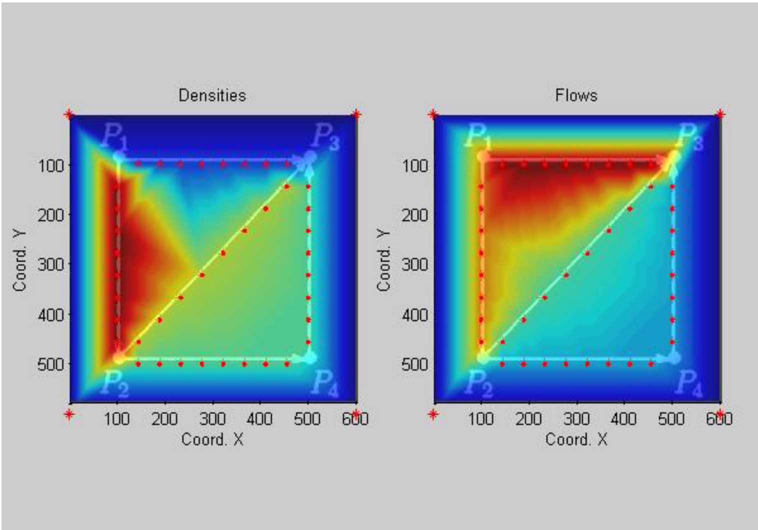

Figure 4.7: Densities and Flows ♯ 4

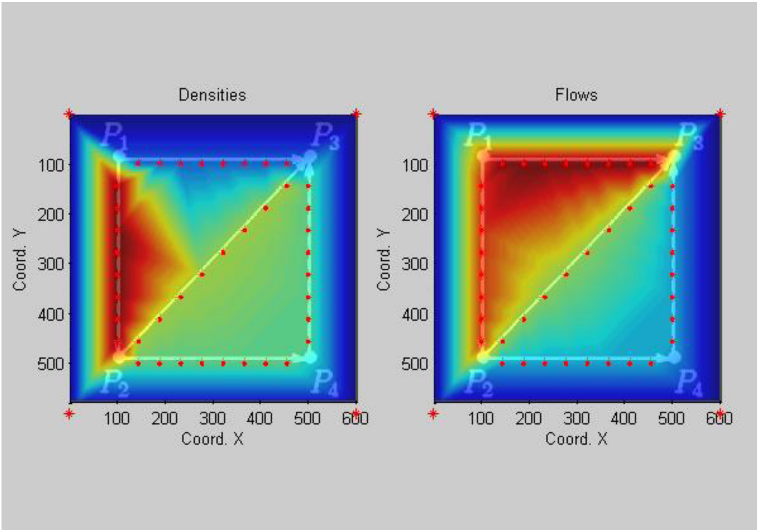

Figure 4.8: Densities and Flows ♯ 5

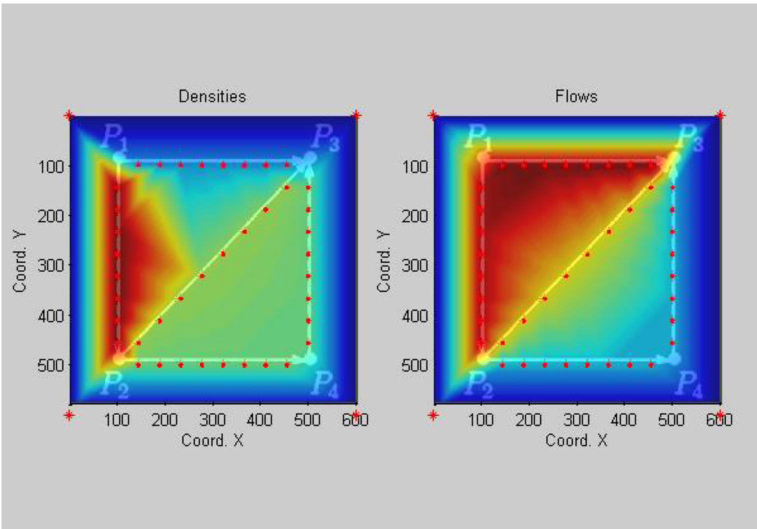

Figure 4.9: Densities and Flows ♯ 6

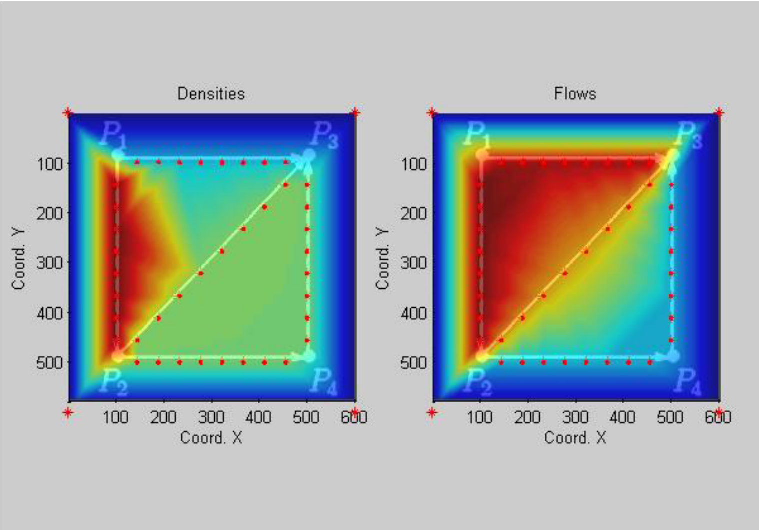

Figure 4.10: Densities and Flows ♯ 7

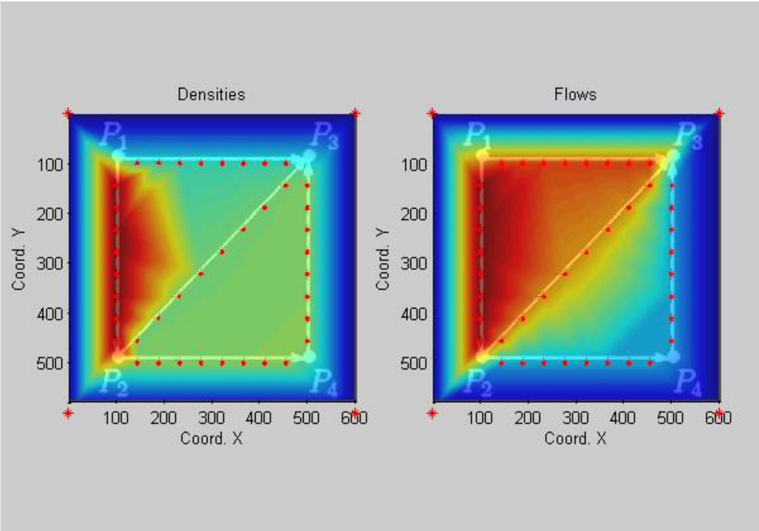

Figure 4.11: Densities and Flows ♯ 8

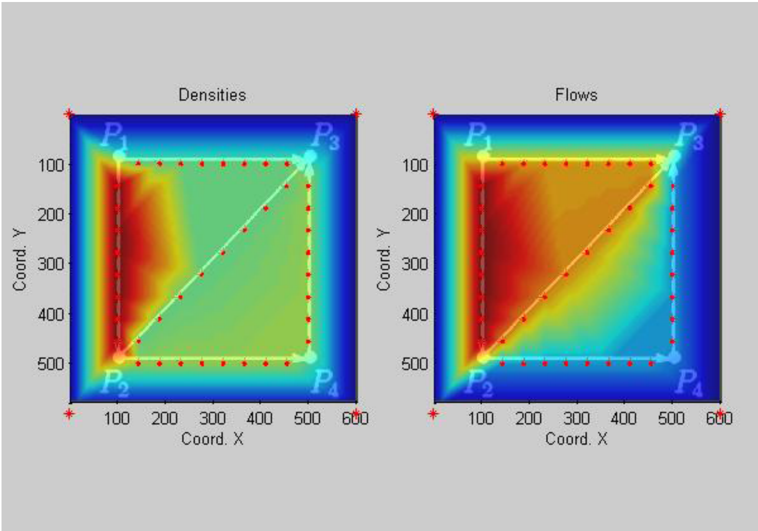

Figure 4.12: Densities and Flows ♯ 9

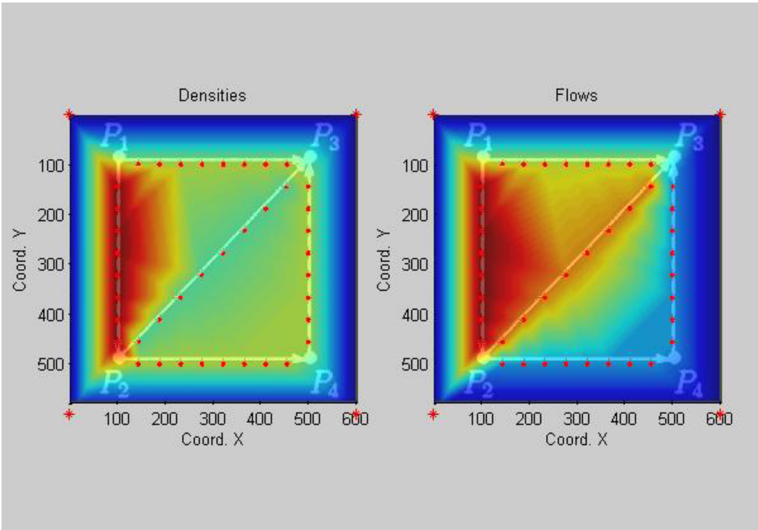

Figure 4.13: Densities and Flows ♯ 10

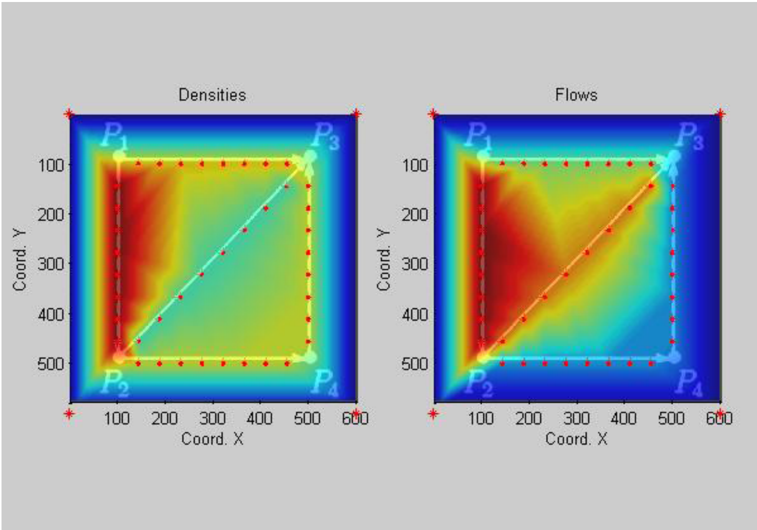

Figure 4.14: Densities and Flows ♯ 11

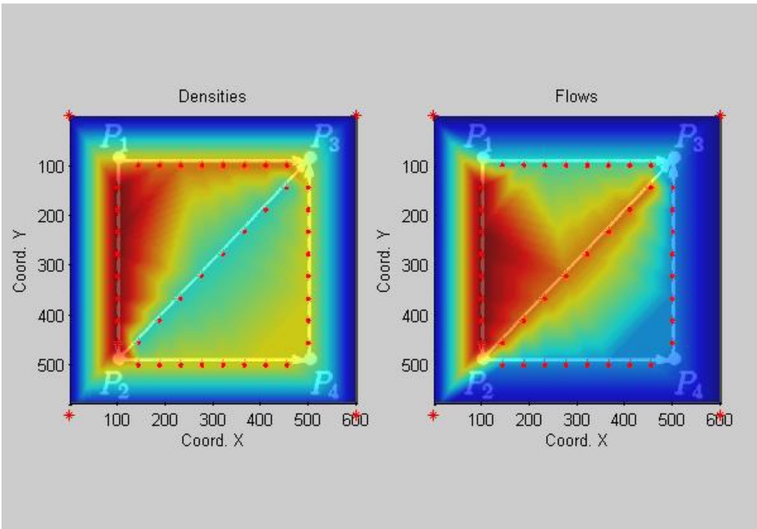

Figure 4.15: Densities and Flows ♯ 12

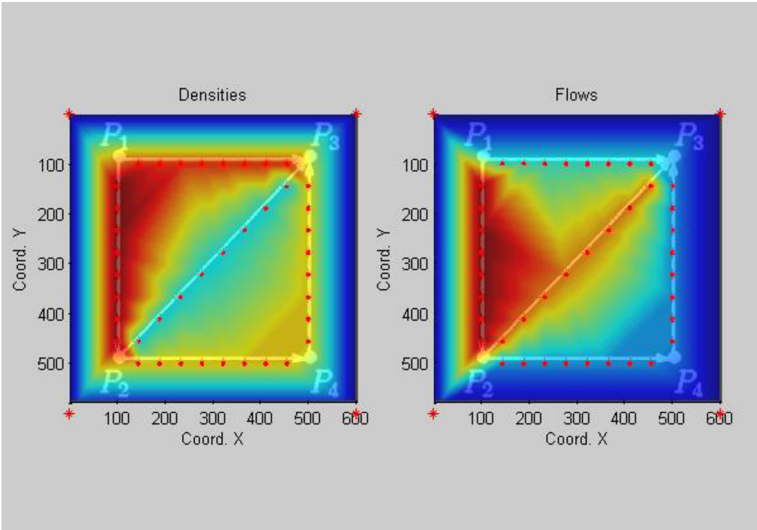

Figure 4.16: Densities and Flows ♯ 13

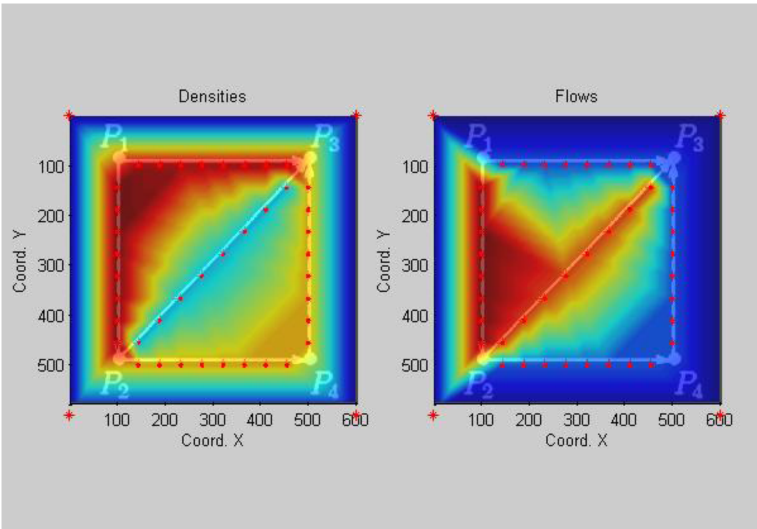

Figure 4.17: Densities and Flows ♯ 14

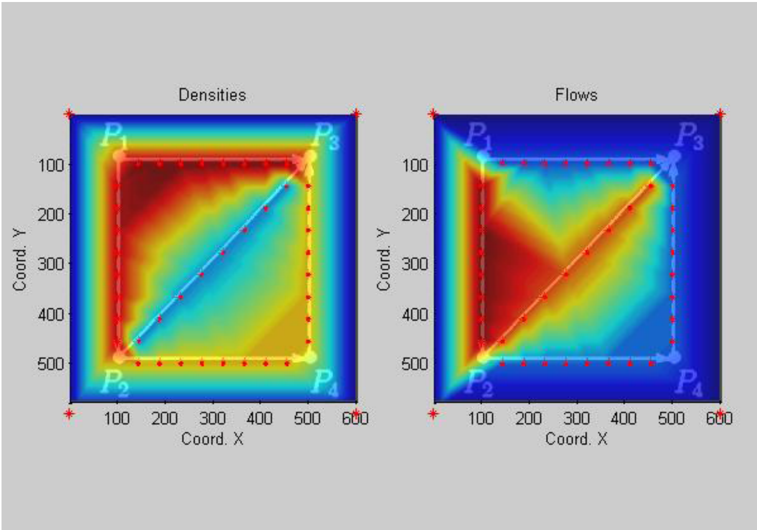

Figure 4.18: Densities and Flows ♯ 15

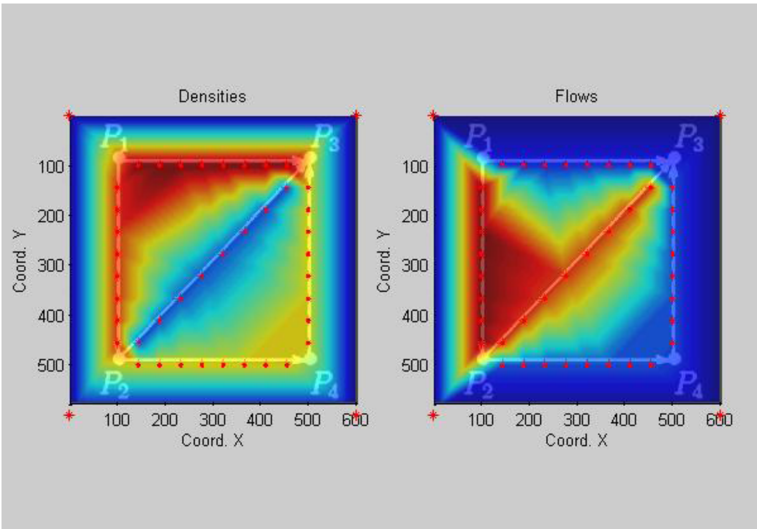

Figure 4.19: Densities and Flows ♯ 16

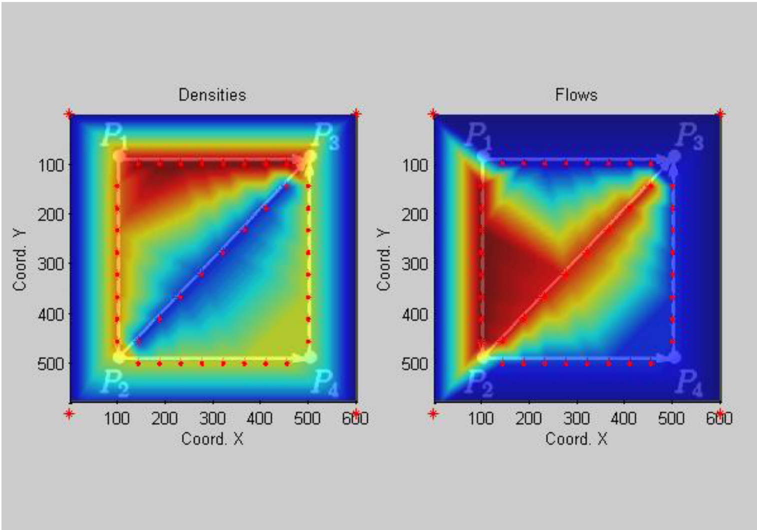

Figure 4.20: Densities and Flows ♯ 17

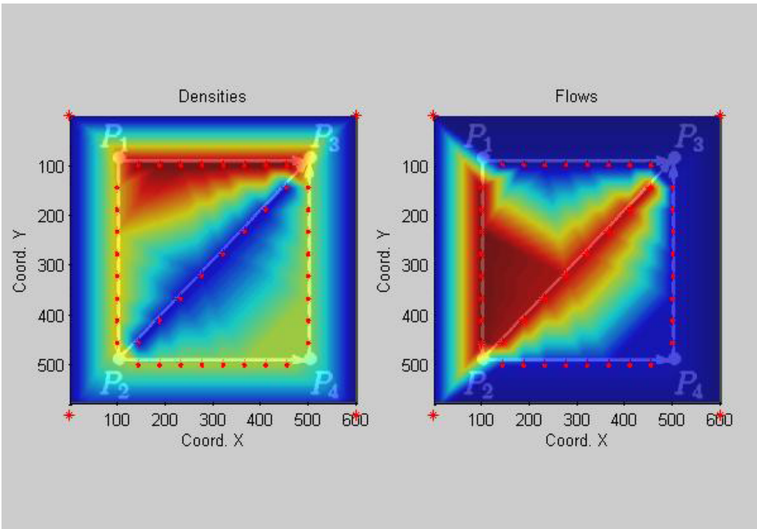

Figure 4.21: Densities and Flows ♯ 18

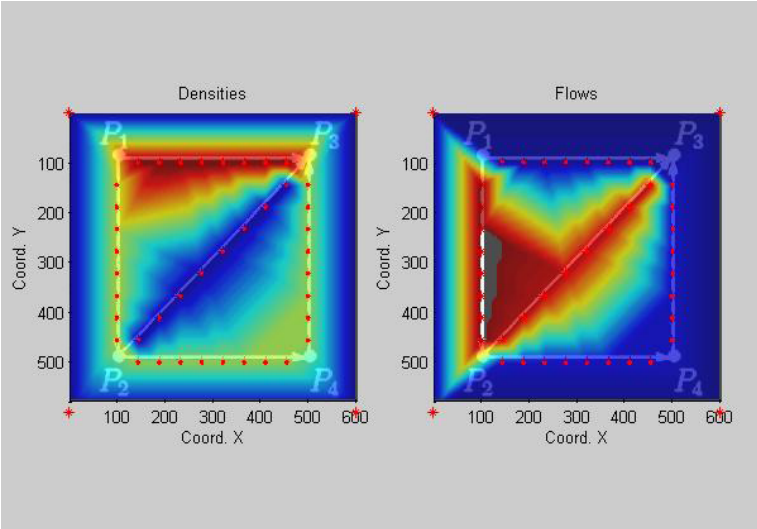

Figure 4.22: Densities and Flows ♯ 19

We can observe, as foreseen, a redistribution of the traffic flows with a clear increase on the flow on the path R_2. We can highlight also a new and interesting problem to study, the sensitivity of the equilibrium with respect to the real time density, and this point will be part of our future publications.

4.9 Industrial Application - Intelligent GPS

It is quite difficult to foresee all the necessarily steps to make productive an industrial project. There are many unknown factors that can convert an idea into a successful application or into something unusable. The decision to include a possible industrial application is in our work as been induced by the willing to prove that even well studied domain as for instance the traffic equiibrium problem can be renewed from a theoretical and practical point of view, integrating an interdisciplinary knowledge. We know that portable GPS systems have a capillary diffusion in modern societies, as well as portable wired devices. The idea is therefore to integrate previous work into portable GPS devices, producing in a certain sense an intelligent GPS device, which is able to propose "the best" routing according preferred criteria.

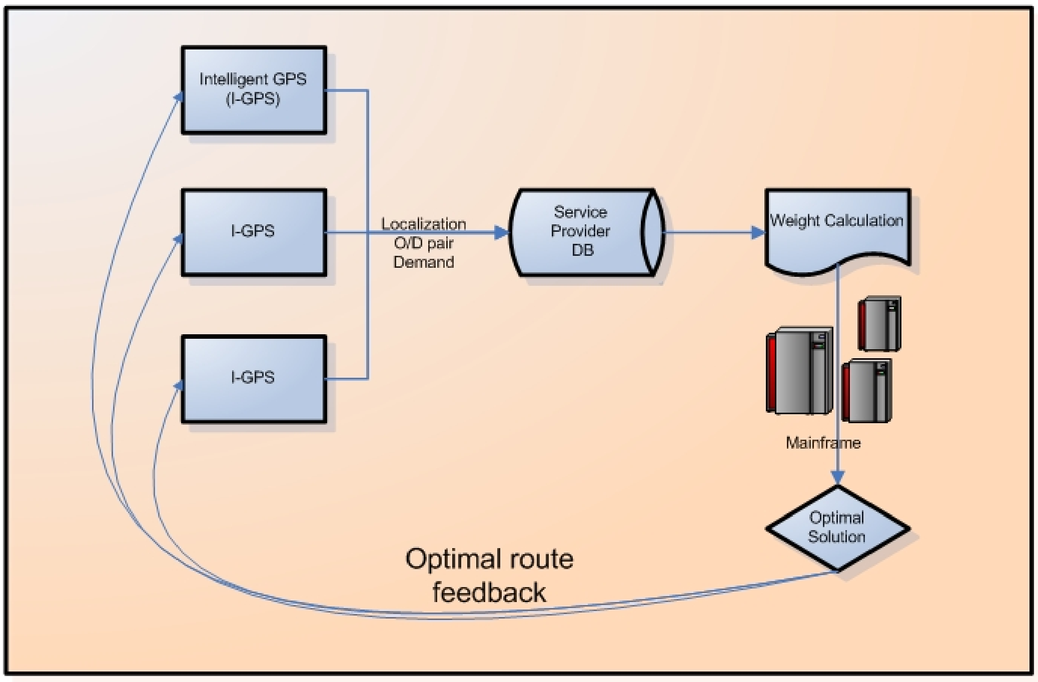

Figure 4.23: Basic design of Intelligent GPS systems.

Chapter 5

Conclusion

In the following work, we present some achievements in the two main directions. First we extend the notion of Projected Dynamical Systems in Weighted Hilbert Spaces, and, as the projection operator is strictly related to the inner product (or the duality paring and the duality mapping), the work done generalizes existing results. We introduce then a framework to extend PDS theory to Banach spaces proving an equivalence theorem in Reflexive Banach spaces. Nevertheless we still don't have an existence result in such spaces, even if the think there are good perspective of results in that direction. Implicit Non pivot PDS has been introduced, and using them we prove and existence result for a quasi-variational inequality with any assumption on the projection operator. A generalization of the traffic equilibrium model has been introduced to manage flows according to the real time urban density (obtained from mobile device connexion data). Moreover some problems have been studied related to this weighted traffic equilibrium model, among others, regularity of solution, dual problem, retarded traffic equilibrium model. There are numbers of paths still to be studied. In particular we can highlight the needs to develop a model for very large Networks (as urban Network) in order to design in a concrete way a prototype for industry. There are several ideas to make that possible, using exactly this model but mapping only critical routes, trying to describe the complex network using a topological approach or using a stochastic approach. On the other hand, it is necessarily to get an existence result for PDS in Banach space: this point is still a big challenge, but recently some interesting perspectives can be investigated . There are also some on going activities related to a deeper analysis of the double layered phenomena. On the VI front line, some generalizations are under study, to include for example a relationship between densities and flows. Finally we hope that our contribution is useful to show how a classical problem can be renewed both theoretically and from the point of view of applications following advances (technological for instance) of real life.

Appendix A

Variational Geometry and PDS

A.1 Tangent Cones, Normal Cones

Definition A.1.1. *Let be $C \subset X$ convex, we call **General Tangent Cone** to C at $\bar{x}$ the set given by:*

$$T_C(\bar{x}) = \limsup_{\lambda \to 0} \frac{1}{\lambda}(C - \bar{x})$$

Remark A.1.2. *The definition A.1.1 is valid also if C is non convex. If C is a convex subset of X, the definition A.1.1 is equivalent to:*

$$T_C(\bar{x}) = \overline{\bigcup_{\lambda > 0} \lambda(C - \bar{x})}$$

Definition A.1.3. *We call **Regular Tangent Cone** to C at $\bar{x}$ the set given by:*

$$\hat{T}_C(\bar{x}) = \liminf_{\lambda \to 0,\ x \to \bar{x},\ x \in C} \frac{1}{\lambda}(C - \bar{x}) \tag{A.1.1}$$

*This cone is also called **Clarke Tangent Cone**.*

Remark A.1.4. *We always have $\hat{T}_C(\bar{x}) \subset T_C(\bar{x})$. If C is convex then $\hat{T}_C(\bar{x}) = T_C(\bar{x})$.*

Definition A.1.5. *We call **Regular Normal Cone** to C at $\bar{x}$ the set given by:*

$$\hat{N}_C(\bar{x}) = \{v | < v, x - \bar{x} > \leq \circ(\|x - \bar{x}\|) \text{ per } x \in C\} \tag{A.1.2}$$

Where $\|.\|$ *is the norm on* X *and "*$\circ$*" means*

$$\limsup_{x \to \bar{x}, x \in C, x \neq \hat{x}} \frac{< v, x - \bar{x} >}{\|x - \bar{x}\|} \leq 0 \tag{A.1.3}$$

Definition A.1.6. *We call* ***General Normal Cone*** *to* C *at* $\bar{x}$ *the set given by:*

$$N_C(\bar{x}) = \{v| \; \exists x^\nu \in C, \; v^\nu \in \bar{N}_C(x^\nu), \; con \; (x^\nu, v^\nu) \to (\hat{x}, v)\} \tag{A.1.4}$$

Note: As done in Rockafellar & Wets (1998) we use ν indexes to indicate the elements of a suite.

Definition A.1.7. *We call* ***Clarke normal Cone*** *the set given by:*

$$\bar{N}_C(\bar{x}) = \; Closed \; convex \; hull \; of \; N_C(\bar{x}) \tag{A.1.5}$$

Remark A.1.8. $\bar{N}_C(\bar{x})$ *and* $N_C(\bar{x})$ *are closed and convex.*

$\hat{N}_C(\bar{x})$ *is convex if* C *is convex.*

The following inclusions are always true:

$$\hat{N}_C(\bar{x}) \subset N_C(\bar{x}) \subset \bar{N}_C(\bar{x}) \tag{A.1.6}$$

Proposition A.1.9. *We have:*

$$\bar{N}_C(\bar{x}) = \{v| < v, w > \leq 0, \; \forall w \in \hat{T}_C(\hat{x})\}, \tag{A.1.7}$$

$$\hat{T}_C(\hat{x}) = \{w| < v, w > \leq 0, \; \forall v \in \bar{N}_C(\bar{x})\} \tag{A.1.8}$$

We recall for readers utility the following basic definitions and properties.

Definition A.1.10. *Let be* $C \subset X$ *convex, we call Normal cone to* C *in* x *the set given by:*

$$N_C(x) = \{\xi \in X^*, < \xi, y - x > \leq 0, \forall y \in C\}$$

Definition A.1.11. *Let* M *be a cone of* X*, the polar set of* M*, noted* M^0 *is defined by:*

$$M^0 = \{\xi \in X^*, < \xi, x > \leq 0, \forall x \in M\}$$

If **X is reflexive**, then the following relationships hold:

$$(T_C(x))^0 = N_C(x), \forall x \in C$$
$$(N_C(x))^0 = T_C(x), \forall x \in C \tag{A.1.9}$$

T_C and N_C are always closed and if C is non empty and convex they are non empty and convex. These cones are used to introduce the relative interior (see Daniele *et al.* (2007)).

Definition A.1.12. *Let $C \subset X$ be convex. We call the relative interior of C the following set:*

$$ri(C) = \{x \in C : T_C(x) = X\}$$

Definition A.1.13. *Let $C \subset X$ be convex. We call the relative boundary of C the following set:*

$$rb(C) = C \setminus ri(C)$$

Proposition A.1.14 (Proposition 2.2 in Maugeri (1998))**.** *Let us assume that X is a reflexive Banach space and $C \subset X$ convex. If $x \in C$ we have:*

$$x \in ri(C) \Leftrightarrow N_C(x) = \{0_{X^*}\}$$

Proof: Let it be $T_C(x) = X$ then we have:

$$N_C(x) = \{\xi \in X^* :< \xi, x >\leq 0, \forall x \in X\}$$

so we get $\forall x \in X, < \xi, x >\leq 0$ and $< \xi, x >\geq 0$ so we can deduce that $\xi = 0_{X^*}$.
On the other side if $N_C(x) = \{0_{X^*}\}$ then using the polarity we get

$$T_C(x) = \{\xi \in X :< \xi, 0_{X^*} >\leq 0\} = X$$

and by definition $x \in ri(C)$. ■
These notions reveal to be very useful in infinite dimensions because many convex sets used in Variational analysis have a topological interior void and a relative interior non void (see Maugeri (1998)).

A.2 Projected Dynamical systems in $\mathbb{R}^n$

PDS theory in $\mathbb{R}^n$ has been developed in Dupuis & Nagurney (1993). One of the notable features of this tool is its relationship to variational inequality problem. In $\mathbb{R}^n$ it is clear how the the static study of VI is extended by PDS which introduced an additional time dimension in order to analyze desequilibrium behavior that precedes the equilibrium. Supposte to have $K \subset \mathbb{R}^n$ a closed convex set. Given $x \in K$, and $v \in \mathbb{R}^n$ define the the directional derivative of the operator P_K is defined, for any $x \in K$ and any element $v \in X$, as the limit (for a proof see Zarantonello (1971)):

$$\pi_K(x, v) := \lim_{\delta \to 0^+} \frac{P_K(x + \delta v) - x}{\delta}; \text{ moreover } \pi_K(x, v) = P_{T_K(x)}(v).$$

Let $\pi_K : K \times X \to X$ be the operator given by $(x, v) \mapsto \pi_K(x, v)$. Note that π_K is nonlinear and discontinuous on the boundary of the set K.
The class class of ordinary differential equations of interest takes the form:

$$\frac{dx(\tau)}{d\tau} = \pi_K(x(\tau), -F(x(\tau)), \; x(0) = x_0 \in K \tag{A.2.10}$$

K corresponds to the constraint set in a particular application, and $F(x)$ is a vector field defined on K. the right hand side of the ordinary differential equation A.2.10 is associated to an operator and hence it is discontinuous on the boundary of K therefore we need to explicitly state what one means by solution to an ODE with discontinuous right hand side.

Definition A.2.1. *The projected dynamical system (PDS), $X_0(t) : K \times \mathbb{R}^n \to K$ is the family of solutions to the initial value problem A.2.10 for all $x_0 \in K$*

Definition A.2.2. *A Critical point of the PDS is point x^* such that:*

$$0 = \pi_K(x^*, -F(x^*) \tag{A.2.11}$$

Let's give now conditions for existence, we introduce for that purpose (Nagurney & Dong (2002)) the linear growth condition assumption.

Assumption A.2.3. *There exists a $B < \infty$ such that the vector field $-F : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the linear growth condition: $\|F(x)\| \leq B(1 + \|x\|)$ for $x \in K$ and also*

$$\langle -F(x) + F(y), x - y\rangle \leq B\|x - y\|^2, \; \forall x, y \in K \tag{A.2.12}$$

Theorem A.2.4. *Assume A.2.3, then for any $x_0 \in K$, there exists a unique solution $X_0(t)$ to the initial value problem A.2.10 and if $x_k \to x_0$ as $k \to \infty$, then $X^k(t) \to X_0(t)$ uniformly on every compact set of* $[0, \infty]$.

Then second statement of this theorem is sometimes called the continuous dependence of the solution path to A.2.10 on initial values The projected dynamical system (PDS), $x_0(t) : K \times \mathbb{R}^n \to K$ is the family of solutions to the initial value problem A.2.10 for all $x_0 \in K$

A.3 Projected Dynamical System in Hilbert Spaces

In Cojocaru & Jonker (2004) the authors extends the theory of PDS to Hilbert spaces, this extension is very important because instead of dealing with statics problems it is possible to treat dynamic problems, the vectors are time dependent functions. This new light on the problem is described in an excellent way in the paper Cojocaru *et al.* (2006) in which the authors introduce the concept of double layered dynamic. The formulation is exactly the same as before. But we need to define what is a solution for a PDS in Hilbert Spaces.

Definition A.3.1. *A solution for a PDS is an absolutely continuous function $x : I \subset \mathbb{R} \to X$ (X Hilbert space), such that $x(t) \in K$, $\forall t \in I$ and*

$$\frac{dx}{dt} = \pi_K(x(y)), -F(x(y)), \ a.a. \ t \in I$$

A.3.1 Existence result

To obtain the following existence result it is necessary to activate an important machinery (Cojocaru & Jonker (2004)) which is used also in the Chapter 3 to prove the existence result in Non pivot Hilbert spaces.

Theorem A.3.2. *Let X be a Hilbert space of arbitrary dimension and let $K \subset X$ be a non-empty, closed and convex subset. Let $F : K \to X$ be a Lipschitz continuous vector field with Lipschitz constant b. Let $x_0 \in K$ and $L > 0$ such that $\|x\| \leq L$.Then the* *initial value problem* $\frac{dx}{dt} = \pi_K(x(t), -F(x(t)))$, $x(0) = x_0$ *has a unique solution on the interval* $[0, l]$, *where* $I = L\|F(x_0)\| + bL$.

A.3.2 Equivalence Results

In Cojocaru & Jonker (2004) an equivalence result is proven using Moreau's decomposition theorem. The result state that critical points of PDS and equilibrium point of VI are equivalent.

A.3.3 Double layered time

The notion of Double Layered Time has been first time introduced in Cojocaru *et al.* (2006). It is a quite surprising notion. In fact it states that there is a micro time scale, that is the time scale used for PDS system and a macro time scale used in Evolutionary variational Inequalities. The authors of Cojocaru *et al.* (2006) try to answer to the following questions:

1. Is it accurate to expect that for almost all $t \in [0, T]$ given, the trajectories of the PDS at t (which we denote by PDS_t) evolve towards the curve of equilibria?

2. What is the relation between an arbitrarily chosen $t \in [0, T]$ and the time it takes for solutions to PDS_t to actually reach the curve of equilibria?

3. What is the interpretation of the double-layered dynamics for applications?

But we refer directly to the paper for the answers.

A.4 Projected Dynamical Systems for non Convex subsets in Hilbert spaces

Remark A.4.1. *This research has been included in this section even if it has been developed in Giuffré* et al. *(2006a) because my consideration is that the following results are not really connected with the core subject of the thesis. Nevertheless there are* *interesting in the sense that they show that it is possible to set up projected dynamical systems problems over cone that are more general that usual tangent and normal cone...in particular it is possible to set up the problem in non convex sets.*

Let us start introducing the following concepts of projected dynamical system for non convex subsets of an Hilbert space.

Definition A.4.2. *We call the Clarke Generalized Projected-Dynamical System the operator*

$$\Lambda_C^g : C \times X^* \to X$$

defined by setting:

$$\Lambda_C^g(x, h) = \Pi_{\hat{T}_C(x)}(J^*(h))$$

Definition A.4.3. *We call Generalized Projected Dynamical System (g-PDS), the discontinuous right hand side differential equation given by:*

$$\frac{dx}{dt} = \Lambda_C^g(x, -F(x)) = \Pi_{\hat{T}_C(x)}(J^*(-F(x))) \tag{A.4.13}$$

The associated Cauchy problem is given by:

$$\frac{dx}{dt} = \Lambda_C^g(x, -F(x)) = \Pi_{\hat{T}_C(x)}(J^*(-F(x))), \ x(0) = x_0 \in C \tag{A.4.14}$$

Remark A.4.4. *If C is convex then $\hat{T}_C(x) = T_C(x)$ and we obtain the Projected Dynamical system defined in Giuffré & Pia (2009) and if in addition X is an Hilbert Space then (A.4.13) is the Projected dynamical system used in (see Isac & Cojocaru (2002c), Isac & Cojocaru (2002a), Cojocaru (2002), Cojocaru & Jonker (2004), Cojocaru* et al. *(2005)).*

We also introduce a quasi-variational inequality or using a common used denomination (see Rapcsák (2003)) a quasi-complementarity system.

Definition A.4.5. *We call Quasi-Complementarity System based on Clarke tangent cone, the problem given by a subset of a real Hilbert space $\mathbb{H}$, a closed subset C and the* *set value mapping* $D : C \to 2^{\mathbb{H}}$ *such that :*

$$D(x) = x + \hat{T}_C(x)$$

and the following quasi-Variational inequality:

$$x \in C :< F(x), y - x >\geq 0,\ \forall y \in D(x) \tag{A.4.15}$$

Where F *is a mapping from* $C \to \mathbb{H}$.

Then we may obtain the following equivalence results.

Theorem A.4.6. *Assume that* X *is an Hilbert Space. If (A.4.15) and (A.4.14) admits a solution then each equilibrium point of (A.4.15) is a critical point of (A.4.14) and, if (A.4.14) admits critical points then they are equilibrium points of (A.4.15).*

Proof: If x^* is an equilibrium point of (A.4.15), then we get:

$$x^* \in C :< x^* - \lambda F(x*) - x^*, x - x^* >\leq 0,\ \forall x \in x^* + \hat{T}_C(x^*),\ \forall \lambda > 0$$

which can be written in the following way

$$x^* = P_{x^* + \hat{T}_C(x*)}(x^* - \lambda F(x^*)),\ \forall \lambda > 0$$

but as $x^* \in x^* + \hat{T}_C(x^*)$ we deduce that $P_{\hat{T}_C(x^*)}(-F(x^*)) = 0$.□
Now suppose that x^* is a critical point of (A.4.14), using Moreau's theorem we can write that

$$-F(x^*) = P_{\hat{T}_C(x^*)}(-F(x^*) + P_{\bar{N}_C(x^*)}(-F(x^*) = P_{\bar{N}_C(x^*)}(-F(x^*)$$

If $F(x^*) = 0$ then (A.4.15) is trivially verified. Now we suppose that $F(x^*) \neq 0$. Then as $-F(x^*) = P_{\bar{N}_C(x^*)}(-F(x^*))$ we get $-F(x^*) \in \bar{N}_C(x^*)$ which means by polarity

$$< -F(x^*), \omega >\leq 0,\ \forall \omega \in \hat{T}_C(x^*)$$

and this is (A.4.15).■

Appendix B

Variational Inequalities

B.1 Historical development

Variational inequalities proved to be a very useful and powerful tool for investigation and solution of many equilibrium type problems in Economics, Engineering, Operations Research and Mathematical Physics. In fact, variational inequalities for example provide a unifying framework for the study of such diverse problems as boundary value problems, price equilibrium problems and traffic network equilibrium problems. Besides, they are closely related with many general problems of Nonlinear Analysis, such as fixed point, optimization and complementarity problems. As a result, the theory and solution methods for variational inequalities have been studied extensively, and considerable advances have been made in these areas.

The theory of variational inequalities, born in Italy in the sixties, was introduced to study elliptic problems with unilateral conditions at the boundary (the celebrated Signorini problem Signorini (1959)), the obstacle problem, the elastic plastic problem, and other similar problems of mathematical physics. The pioneer works in this field are due to G. Fichera (see Fichera (1964)) and G. Stampacchia (see Stampacchia (1964)) were motivated by concrete problems, the first in mechanics (a problem in elasticity with a unilateral boundary condition) and the second in potential theory (in connection with capacity, a basic concept from electrostatics). A further study of a special case of variational inequalities was done by J.L. Lions and G. Stampacchia in the joint papers, Lions & Stampacchia (1965) and Lions & Stampacchia (1967), with applications to elliptic and parabolic unilateral boundary value problems. In the same period, H. Brezis (see Brezis (1967)) introduced evolutionary variational inequalities.

The existence theorem in the general form stated above (and its extension to semi-monotone operators) was obtained by F.E. Browder (see Browder (1965a)) and P.H. Hartman and G. Stampacchia (see Hartmann & G. Stampacchia (1966)) by using the "monotonicity" approach to nonlinear problems previously developed for operator equations in Hilbert space by E.H. Zarantonello (see Zarantonello (1960)), G. Minty (see Minty (1962)) and F.E. Browder (see Browder (1963c) and Browder (1963b)) and for equations involving operators from a Banach space X to its dual X^* by F.E. Browder (see Browder (1963a) and Browder (1965b)), G. Minty (see Minty (1963)) and J. Leray and J.L. Lions (see Leray & Lions (1965)).

In the following, many other authors worked on the theory of variational inequalities, as D. Kinderleher and G. Satmpacchia (see Kinderleher & G. Stampacchia (1980)).

In the same years, A. Bensoussan and J.L. Lions in a series of papers (see, e.g., Bensoussan & Lions (1973)) introduced a more general mathematical tool, quasi-variational inequalities, in connection with impulse optimal control problems. Then they have been extensively studied in numerous publications, mainly from the viewpoints of existence of solutions and numerical methods; see Baiocchi & Capelo (1984), Chan & Pang (1982), Tan (1985) among others.

In the next sections we present various basic concepts in optimization and variational analysis and recall their properties.

B.2 Preliminary concepts

Let X be a real topological vector space and let S be a subset of X. Moreover let X' be the topological dual space of X.

Definition B.2.1. *A functional $f : S \to \mathbb{R} \cup \{\pm\infty\}$ is said to be upper semi-continuous (briefly u.s.c.) if for each x', we have*

$$\limsup_{x \to x'} f(x) \leq f(x').$$

Definition B.2.2. *A functional $f : S \to \mathbb{R} \cup \{\pm\infty\}$ is said to be lower semi-continuous (briefly l.s.c.) if $-f(x)$ is upper semi-continuous.*

Definition B.2.3. *An operator $f : S \to X'$ is monotone on S if*

$$\langle f(x_1) - f(x_2), x_1 - x_2 \rangle \geq 0, \quad \forall x_1, x_2 \in S.$$

Definition B.2.4. *An operator $f : S \to X'$ is strictly monotone on S if*

$$\langle f(x_1) - f(x_2), x_1 - x_2 \rangle > 0, \quad \forall x_1 \neq x_2.$$

Definition B.2.5. *An operator $f : S \to X'$ is strongly monotone on S if for some $\nu > 0$*

$$\langle f(x_1) - f(x_2), x_1 - x_2 \rangle \geq \nu \|x_1 - x_2\|^2, \quad \forall x_1, x_2 \in S.$$

Definition B.2.6. *An operator $f : S \to X'$ is pseudomonotone on S if for all $x_1, x_2 \in S$*

$$\langle f(x_1), x_1 - x_2 \rangle \geq 0 \Longrightarrow \langle f(x_2), x_1 - x_2 \rangle \leq 0.$$

Definition B.2.7. *An operator $f : S \to X'$ is strongly pseudomonotone with degree $\alpha > 0$ on S if and only if there exists $\nu > 0$ such that for all $x_1, x_2 \in S$*

$$\langle f(x_2), x_1 - x_2 \rangle \geq 0 \Longrightarrow \langle f(x_1), x_1 - x_2 \rangle \leq \nu \|x_1 - x_2\|^\alpha.$$

Let X be a real topological vector space and let $\mathbf{K}$ be a convex subset of X.

Definition B.2.8. *An operator $f : \mathbf{K} \to X'$ is hemicontinuous if for any $x \in \mathbf{K}$, the function*

$$\mathbf{K} \ni \xi \to \langle f(\xi), x - \xi \rangle$$

is upper semi-continuous on $\mathbf{K}$.

Definition B.2.9. *An operator $f : \mathbf{K} \to X'$ is hemicontinuous along line segments if and only if for any $x, y \in \mathbf{K}$, the function*

$$\mathbf{K} \ni \xi \to \langle f(\xi), y - x \rangle$$

is upper semi-continuous on the line segment $[x, y]$.

Let X, Y be two Hausdorff topological vector spaces and let S be a subset of X. Moreover, let X' denote the dual space of X.

Definition B.2.10. *A set-valued map $F : S \to 2^Y$ is upper semi-continuous (briefly u.s.c.) in $x' \in S$ if for any open subset Ω of Y such that $F(x') \subseteq \Omega$, there exists a neighborhood V of x' such that for all $x \in V$*

$$F(x) \subseteq \Omega.$$

Definition B.2.11. *A set-valued map $F : S \to 2^Y$ is lower semi-continuous (briefly l.s.c.) in $x' \in S$ if for any open subset Ω of Y such that $F(x') \cap \Omega \neq \emptyset$, there exists a neighborhood V of x' such that for all $x \in V$*

$$F(x) \cap \Omega \neq \emptyset.$$

Definition B.2.12. *A set-valued map $F : S \to 2^Y$ is continuous if it is both u.s.c. and l.s.c.*

Definition B.2.13. *A set-valued map $F : S \to 2^Y$ is called closed if its graph*

$$G = \{(x, y) : \ x \in S, \ y \in F(x)\}$$

is a closed subset of $X \times Y$.

Remark B.2.14. *It is easy to show that if X and Y are real topological linear locally convex Hausdorff spaces the following statements hold:*

1. *F is closed if and only if for any sequence $\{x_n\}_{n\in\mathbb{N}}$, $x_n \to x$, and any $\{y_n\}_{n\in\mathbb{N}}$, $y_n \in F(x_n)$, $y_n \to y$, then it results that $y \in F(x)$;*

2. *F is l.c.s. in $x \in \mathbf{K}$ if and only if for any $y \in F(x)$ and any $\{x_n\}_{n\in\mathbb{N}}$, $x_n \to x$, there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ such that $y_n \in F(x_n)$ and $y_n \to y$.*

B.3 Finite dimensional variational inequalities

Now, we introduce finite dimensional variational inequalities and we recall some existence results.

Definition B.3.1. *Let $\mathbf{K}$ be a nonempty, convex and closed set of the m-dimensional Euclidean space $\mathbb{R}^m$ and let $C : \mathbf{K} \to \mathbb{R}^m$ be a vector-function. The finite dimensional variational inequality is the problem to find a vector $x \in \mathbf{K}$, such that*

$$\langle C(x), y - x\rangle \geq 0, \quad \forall y \in \mathbf{K}. \tag{B.3.1}$$

Geometrical meaning (B.3.1) states that $C(x)^T$is orthogonal to the set $\mathbf{K}$ at the point x.

Now, we recall some classic conditions showed by Stampacchia for existence of solutions to variational inequality (B.3.1).

Theorem B.3.2. (Hartmann & G. Stampacchia (1966)) *If $\mathbf{K}$ is a nonempty, convex and compact subset of $\mathbb{R}^m$ and $C : \mathbf{K} \to \mathbb{R}^m$ is a continuous operator, then variational inequality (B.3.1) admits at least one solution.*

Theorem B.3.3. (Lions & Stampacchia (1967)) *If $\mathbf{K}$ is a nonempty, convex and compact subset of $\mathbb{R}^m$ and C is continuous on $\mathbf{K}$, then the set of solutions to the variational inequality (B.3.1) is convex and compact.*

Theorem B.3.4. (Mancino & Stampacchia (1972)) *If C is strictly monotone on $\mathbf{K}$, then the solution to variational inequality (B.3.1), if it exists, is unique.*

Whenever the set $\mathbf{K}$ is unbounded, the existence of solutions may also be established under the coercivity condition, as shows the following result.

Theorem B.3.5. (Kinderleher & G. Stampacchia (1980)) *If C satisfies the coercivity condition*

$$\lim_{\|x\|_m \to +\infty} \frac{\langle C(x) - C(x'), x - x' \rangle}{\|x - x'\|_m} = +\infty \tag{B.3.2}$$

for $x \in \mathbf{K}$ and some $x' \in \mathbf{K}$[1]*. Then variational inequality (B.3.1) admits a solution.*

B.4 Infinite dimensional variational inequalities

In this section we give some results for the existence of solutions to variational inequalities in infinite dimensional spaces.

Let X be a reflexive Banach space and let $\mathbf{K} \subseteq X$ be a convex and closed set. Let us denote by $\|\cdot\|$ the norm in X. Let B_R be the closed ball with center in O and radius R and let us consider the closed and convex set $\mathbf{K}_R = \mathbf{K} \cap B_R$. If R is large enough, then $\mathbf{K}_R$ is nonempty. We have the following result.

Theorem B.4.1. (Stampacchia (1969)) *Let $C : \mathbf{K} \to X'$ be a monotone and hemicontinuous along line segments function, the the variational inequality*

$$x \in \mathbf{K} : \ \langle C(x), y - x \rangle \geq 0, \quad \forall y \in \mathbf{K}, \tag{B.4.3}$$

admits a solution if and only if there exists a constant R such that at least one solution of the variational inequality

$$x_R \in \mathbf{K}_R : \ \langle C(x_R), y - x_R \rangle \geq 0, \quad \forall y \in \mathbf{K}_R, \tag{B.4.4}$$

satisfies the condition

$$\|x_R\| < R. \tag{B.4.5}$$

Remark B.4.2. *If the set $\mathbf{K}$ is unbounded, then the following conditions for the existence of solutions are provided:*

[1] From here onward we always denote by $\|\cdot\|_m$ the norm in $\mathbb{R}^m$, for all $m \geq 1$.

1. *let us suppose that* $\exists x_0 \in \mathbf{K}$ *and* $R > \|x_0\|$ *such that*

$$\langle C(y), x_0 - y \rangle < 0,$$

   $\forall y \in \mathbf{K}$, $\|y\| = R$, *then (B.4.5) is verified.*

2. *let us suppose that* $\exists x_0$ *such that* C *satisfies the coercivity condition (B.3.2), then (B.4.4) holds.*

3. *let us suppose that* C *satisfies the weak coercivity requirement:*

$$\lim_{\|y\| \to +\infty} \frac{\langle C(y), y \rangle}{\|y\|} = +\infty$$

   $\forall y \in \mathbf{K}$, *then (B.4.5) is fulfilled.*

We recall Theorems 2 and 3 in Oettli & Schläger (1998).

Theorem B.4.3. *Let* X *be a real topological vector space and let* $\mathbf{K} \subseteq X$ *be a nonempty and convex set. Let* $C : \mathbf{K} \to X'$ *be a given function such that:*

(i) there exist $A \subseteq \mathbf{K}$ *nonempty, compact and* $B \subseteq \mathbf{K}$ *compact, convex such that, for every* $y \in \mathbf{K} \setminus A$, *there exists* $\widehat{x} \in B$ *with* $\langle C(y), \widehat{x} - y \rangle < 0 \rangle$;

(ii) C *is pseudomonotone and hemicontinuous along line segments.*

Then, there exists $x \in A$ *such that* $\langle C(x), y - x \geq 0 \rangle$, *for all* $y \in \mathbf{K}$.

Theorem B.4.4. *Let* X *be a real topological vector space and let* $\mathbf{K} \subseteq X$ *be a nonempty and convex set. Let* $C : \mathbf{K} \to X'$ *be a given function such that:*

(i) there exist $A \subseteq \mathbf{K}$ *nonempty, compact and* $B \subseteq K$ *compact, convex such that, for every* $y \in \mathbf{K} \setminus A$, *there exists* $\widehat{x} \in B$ *with* $\langle C(y), \widehat{x} - y \rangle < 0 \rangle$;

(ii) C *is hemicontinuous.*

Then, there exists $x \in A$ *such that* $\langle C(x), y - x \geq 0 \rangle$, *for all* $y \in \mathbf{K}$.

With a weakened coercivity assumption, we get the following theorem.

Theorem B.4.5. (Ricceri (1995)) *Let X be a Hausdorff real topological vector space and $\mathbf{K} \subseteq X$ be a closed and convex subset with nonempty relative interior (that is the interior of $\mathbf{K}$ in its affine hull) and $C : \mathbf{K} \to X'$ a weakly* continuous function. Moreover, let $\mathbf{K}_1$ and $\mathbf{K}_2$ be two nonempty and compact subset of X with $\mathbf{K}_2 \subseteq \mathbf{K}_1$ and $\mathbf{K}_2$ having finite dimension, such that $\forall x \in X \setminus \mathbf{K}_1$, we have*

$$\sup_{y \in \mathbf{K}_2} \langle C(x), x - y \rangle > 0.$$

Then the variational inequality

$$\langle C(x), y - x \rangle \geq 0, \quad \forall y \in \mathbf{K}$$

admits solutions in $\mathbf{K}$.

In particular, if X is a real Hilbert space and the operator C is affine, the next result, due to Lions and Stampacchia (see Lions & Stampacchia (1967)), holds.

Theorem B.4.6. *Let X be a real Hilbert space, let $\mathbf{K}$ be a nonempty, convex and closed, subset of X and let $A : \mathbf{K} \to X'$ a Lipschitz and coercive operator (not necessarily linear), that is,*

$$\begin{aligned} \|Ax - Ay\|_* &\leq M\|x - y\|, \quad \forall x, y \in \mathbf{K}, \\ \langle Ax - Ay, x - y \rangle &\geq \nu\|x - y\|^2, \quad \forall x, y \in \mathbf{K}, \end{aligned}$$

for some constant $M, \nu > 0$. Then for each $B \in X'$, there exists a unique solution to the variational inequality

$$x \in \mathbf{K} : \ \langle Au + B, y - x \rangle \geq 0, \quad \forall y \in \mathbf{K}.$$

Moreover, the (nonlinear) solution mapping is Lipsichitz continuous, that is, if $x_1, x_2 \in \mathbf{K}$ are the solutions to the variational inequalities related to two different free terms $B_1, B_2 \in X'$, it results

$$\|x_1 - x_2\| \leq \frac{1}{\nu}\|B_1 - B_2\|_*. \tag{B.4.6}$$

B.5 Finite dimensional quasi-variational inequalities

Let us introduce finite dimensional quasi-variational inequalities.

Definition B.5.1. *Let D be a nonempty subset of $\mathbb{R}^m$, let $C : D \to \mathbb{R}^m$ and $\mathbf{K} : D \to 2^D$ be a function and a multifunction, respectively. The quasi-variational inequality is the problem to find a vector $x \in \mathbf{K}(x)$ such that*

$$\langle C(x), y - x \rangle \geq 0, \quad \forall y \in \mathbf{K}(x). \tag{B.5.7}$$

Let us give some theorems concerning the existence of solutions to finite dimensional quasi-variational inequalities.

Theorem B.5.2. (Harker & Pang (1990)) *Let D be a compact and convex set. Let C and $\mathbf{K}$ be a function and a multifunction, respectively, and, for all $x \in D$, let $\mathbf{K}(x)$ be a nonempty, closed and convex subset of $\mathbb{R}^m_+$. Then quasi-variational inequality (B.5.7) admits a solution.*

Theorem B.5.3. (De Luca & Maugeri (1992)) *Let D be a compact and convex set. Let $\mathbf{K}$ be a continuous multifunction such that, for all $x \in D$, $\mathbf{K}(x)$ is a nonempty, closed and convex subset of $\mathbb{R}^m_+$ and let C satisfy the condition*

$$\{x \in X : \ C(x)y \leq 0\} \text{ is closed } \forall y \in D - D.$$

Then quasi-variational inequality (B.5.7) admits a solution.

Theorem B.5.4. (De Luca (1995)) *Let D be a compact and convex set. Let $\mathbf{K}$ be a continuous multifunction such that, for all $x \in D$, $\mathbf{K}(x)$ is a nonempty, closed and convex subset of $\mathbb{R}^m_+$. Let $C : D \to 2^{\mathbb{R}^m_+}$ be a set-valued map (possibly discontinuous) such that:*

$$\forall y \in D - D \text{ the set } G_y = \left\{ x \in D : \inf_{z \in C(x)} zy \leq 0 \right\} \text{ is closed.}$$

Then, there exist $x \in \mathbf{K}(x) \cap D$ and $z \in C(y)$ such that $z(y-x) \geq 0$, for all $y \in \mathbf{K}(x) \cap D$.

B.6 Infinite dimensional quasi-variational inequalities

We may present problem (B.5.7) in an infinite dimensional setting by replacing $\mathbb{R}^m$ with a real topological vector space X and assuming that C is a operator from D to X', where X' is the topological dual of X.

In the following, we recall some results for the existence of solutions to the quasi-variational inequality in infinite dimensional spaces.

Theorem B.6.1. (Tan (1985)) *Let X be a topological linear locally convex Hausdorff space and let $D \subset X$ be a convex, compact and nonempty subset. Let $C : D \to 2^{X'}$ be an u.s.c. multifunction with $C(y)$, $y \in C$, convex, compact and nonempty and let $\mathbf{K} : D \to 2^D$ be a closed l.s.c. set-valued mapping with $\mathbf{K}(y)$, $y \in D$, convex, compact and nonempty and let $\varphi : D \to \mathbb{R}$ a convex l.s.c. function. Then, there exists $x \in C(x)$ such that:*

1. *$x \in \mathbf{K}(x)$,*
2. *there exists $y^* \in C(x)$ for which*

$$\langle y - x, y^* \rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in \mathbf{K}(x).$$

The following theorem relaxes the hypothesis of compactness of the set D requiring the coercivity of the operator.

Theorem B.6.2. (Tian & Zhou (1991)) *Let D be a convex subset in a locally convex Hausdorff topological vector space X. Let us suppose that*

(i) $\mathbf{K} : D \to 2^D$ is a closed l.s.c. correspondence with closed, convex and nonempty values,

(ii) $C : D \to 2^{X'}$ is a monotone, finite continuous and bounded single-valued map,

(iii) there exist a compact, convex and nonempty set $Z \subset D$ and a nonempty subset $B \subset Z$ such that

(iii.a) $\mathbf{K}(B) \subset Z$;

(iii.b) $\mathbf{K}(z) \cap Z \neq \emptyset$, *for all* $z \in Z$;

(iii.c) for every $z \in Z \setminus B$ *there exists* $\widehat{z} \in \mathbf{K}(z) \cap Z$ *with* $\langle C(z), \widehat{z} - z \rangle < 0$.

Then there exists x *such that*

$$x \in \mathbf{K}(x) : \ \langle C(x), y - x \rangle \geq 0, \quad \forall y \in \mathbf{K}(x).$$

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