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Ovoids and spreads of $\mathcal{Q}^+(7, q)$

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Contents

Preface

This thesis concerns with slices of the unitary spread and of the unitary ovoid. The unitary spread and the unitary ovoid are geometric objects contained in the hyperbolic quadric $\mathcal{Q}^+(7,q)$, if $q \equiv 2\,(mod\,3)$ and in the parabolic quadric $\mathcal{Q}(6,q)$, if $q \equiv 0\,(mod\,3)$; these were introduced by W.M. Kantor in [14] and J.A. Thas in [22]. A slice of a spread (of an ovoid) of an orthogonal polar space is the intersection of the spread (of the ovoid) with a hyperplane of the relevant projective space. In this work, it is proved that the slices of the unitary spread of $\mathcal{Q}^+(7,q)$ $q \equiv 2\,(mod\,3)$ can be divided into five classes. Slices belonging to different classes are inequivalent with respect to the action of the subgroup of $P\Gamma O^+(8,q)$ fixing the unitary spread. When q is even, there is a connection between spreads of $\mathcal{Q}^+(7,q)$ and symplectic spreads of $PG(5,q)$ originally pointed out by Dillon [7] and Dye [8].

In Chapter 1 we recall all the necessary introductory material about finite polar spaces. A section is dedicated to review quadrics and hermitian curves and their projective classification. In the last section is explained some introductorial material about spreads and ovoids of PG$(2n+1)$, that will be deeply considered later for the case $n = 3$. In Chapter 2 using the construction of the unitary ovoid of $\mathcal{Q}^+(7,q)$ due to Kantor ([13]) and some results due to Lunardon ([16]) on a connection between the ovoids of $\mathcal{Q}^+(7,q)$, $q \equiv 2\,(mod\,3)$ and intersection sets of hermitian curves, we determine all possible inequivalent symplectic spreads arising from the unitary spread of $\mathcal{Q}^+(7,q)$, $q = 2^{2h+1}$. Some of these were already discovered in [14]. When $q = 3^h$, the slices of the unitary ovoid of $\mathcal{Q}(6,q)$ with respect to singular hyperplanes and hyperplanes intersecting $\mathcal{Q}(6,q)$ in a hyperbolic quadric were studied in [13]. Here we complete this study by classifying, up to the action of the subgroup of $P\Gamma O(7,q)$ fixing the unitary ovoid, all slices of the unitary ovoid of $\mathcal{Q}(6,q)$ with respect to non–singular hyperplanes.

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Chapter 1

Introduction

1.1 Polar spaces

Let K be a field and V be a vector space over K. We will denote by $\mathrm{PG}(V, K)$ the projective space defined by the lattice of the vector subspaces of V. We will say that a projective subspace W has rank t and dimension $t-1$ if W has dimension t as vector space over K. If K has finite order $q = p^h$ with p prime and $h > 0$, and V is a vector space of finite dimension $n+1$, $n \geq 1$, over K, we can write $\mathrm{PG}(n, q)$ instead of $\mathrm{PG}(V, K)$.

Let g be a bijective function of $\mathrm{PG}(n, q)$ on itself; if g preserves the inclusion then it is a *collineation*; or else if g reverses the inclusions then it is a *correlation*. A correlation of order 2 is called a *polarity*.

Let $K = \mathbb{F}_q$ with $q = p^h$. A function $f : V \times V \to K$ is a *sesquilinear form* if it is linear in the first variable and semilinear in the second and f is called:

- *symmetric* if $f(u,v) = f(v,u)$, $\forall u, v \in V$:
- *alternating* if $f(v,v) = 0$, $\forall v \in V$;
- *hermitian* if $f(u,v) = \sigma(f(v,u))$, $\forall u, v \in V$, where σ is an automorphism of the field different from the identity.

Observe that if f is a hermitian form, then the order of K must be a square.

A sesquilinear form f is said to be *non-degenerate* if $f(u,v) = 0$ for all $u \in V$ implies $v = 0$ or, equivalently, $f(u,v) = 0$ for all $v \in V$ implies $u = 0$; and it is *reflexive* if $f(u,v) = 0$ implies $f(v,u) = 0$ for all u,$v \in V$. A non-degenerate sesquilinear form f of V induces a correlation π in $\mathrm{PG}(V,K)$, in particular if f is reflexive then π is a polarity. We have the following important result:

Theorem 1.1.1. *(Birkhoff-Von Neumann)*
If dim$V \geq 3$ and if π is a polarity of $\mathrm{PG}(V,K)$*, then π arises from a non-degenerate reflexive sesquilinear form f which must be alternating, symmetric or hermitian.*

A pair $(\mathrm{PG}(V,K), \pi)$, where π is a polarity of $\mathrm{PG}(V,K)$, is a *polar geometry*, known as a *symplectic, orthogonal* or *unitary* geometry according as π is alternating, symmetric or hermitian.

A *quadratic form* $Q : V \longrightarrow K$ is a map which satisfies the following:

- $Q(\alpha v) = \alpha^2 Q(v)$, for $\alpha \in K, v \in V$;
- the map $f_Q : V \times V \longrightarrow K$ given by $f_Q(u,v) = Q(u+v) - Q(u) - Q(v)$ is a symmetric bilinear form, called the *polar form* associated to Q.

When $p \neq 2$ symmetric sesquilinear forms and polar forms correspond, otherwise we are not interested in symmetric sesquilinear forms which don't arise from a quadratic form. So we can say that a a *polar space* is a vector space equipped with an alternating, hermitian or quadratic form, and it is said to be symplectic, unitary or orthogonal respectively.

If f is a sesquilinear form on V, we say that a non-zero vector u of V is *isotropic* if $f(u,u) = 0$ and that a vector subspace of V is *totally isotropic* if all of its vectors are isotropic with respect to f. Now, consider a quadratic form Q on V with associated polar form f_Q. We say that two points u and v of V are *orthogonal* if $f_Q(u,v) = 0$. The polar space is non-degenerate if the polar form has the property that $f_Q(u,v) = 0$ for all $v \in V$ implies $u = 0$. A vector $u \neq 0$ is *singular* if $Q(u) = 0$, and a subspace U is *totally singular* if all of its vectors are singular. Furthermore, a totally isotropic/singular subspace is said to be a *maximal* if it is not properly contained in any totally isotropic/singular subspace. Note that all maximals of a polar space have the same rank,said the *rank* of $\mathcal{P}$. The set of isotropic points of a unitary space is called a hermitian variety, while the set of singular points of an orthogonal space is called a quadric. Because of the importance of quadrics and hermitian varieties in this work it is useful to introduce them also from another point of view.

1.1.1 Quadrics and hermitian curves

Let $(x_0, \ldots, x_n)$ be the homogeneus projective coordinates of a point X of $\mathbb{P} = \mathrm{PG}(n, q)$. Consider the sesquilinear form f, the variety $V(f)$ is the set of zeros respect to f.

A *quadric* $\mathcal{Q}$ is a variety $V(Q)$, where Q is a quadratic form, that is

$$Q = \sum_{i,j=0}^{n} c_{ij} x_i x_j$$

not all c_{ij} equal to zero.

If q is a square, a *Hermitian variety* $\mathcal{H}$ is a variety $V(H)$, where H is a hermitian form, that is

$$H = \sum_{i,j=0}^{n} a_{ij} x_i^{\sqrt{q}} x_j$$

with $a_{ij} \in F_{\sqrt{q}}$ not all zero and such that $a_{ij}^q = a_{ji}$.

The form and the variety are *singular* if there is a change of coordinate system which reduces the form to one in fewer variables; otherwise they are called *non singular*.

Two variety V_1 and V_2 are projectively equivalent if they can be obtained each other by a change of coordinate system, in this case we will write $V_1 \sim V_2$

Theorem 1.1.2. *In $\mathbb{P}$ the number of projectively distinct non-singular quadrics* *is one or two according as n is even or odd. They have the following canonical forms:*

- $n = 2s,\ s \geq 0,$

  parabolic*:* $\mathcal{Q}(2s, q) = V(x_0^2 + x_1x_2 + \cdots + x_{2s-1}x_{2s})$*;*

- $n = 2s + 1,\ s \geq 1$

  hyperbolic*:* $\mathcal{Q}^+(2s + 1, q) = V(x_0x_1 + x_2x_3 + \cdots + x_{2s}x_{2s+1})$*;*

  elliptic*:* $\mathcal{Q}^-(2s + 1, q) = V(f(x_0, x_1) + x_2x_3 + \cdots + +x_{2s}x_{2s+1})$ *where* f *is an irreducible quadratic form.*

Supposed q is a square. For every value of n there exist $n + 1$ projectively distinct hermitian varieties in the projective space $\mathrm{PG}(n, q)$; but only one of these is non-singular.

Theorem 1.1.3. *A non singular hermitian variety in* $\mathrm{PG}(n, q)$*, with q square, has the canonical form*

$$\mathcal{H}_n = V(x_0^{\sqrt{q}+1} + \cdots + x_n^{\sqrt{q}+1}).$$

Now focus our attention on the hermitian varieties of the projective plane, the hermitian curves. Let consider the projective plane $\pi = \mathrm{PG}(2, q^2)$. A *unital* in π is a set $\mathcal{U}$ of $q^3 + 1$ points of π such that every line of π meets $\mathcal{U}$ in either 1 or $q + 1$ points, and it is said to be *classical* if it consists of all the self-conjugate points of a unitary polarity of π. A *Hermitian curve* $\mathcal{H}$ is the set of zeros in $PG(2, q^2)$ of a Hermitian form. A *chord* of $\mathcal{H}$ is a line of π meeting it in $q+1$ points. Let now (x_0, x_1, x_2)be the homogeneous coordinates of a point of π. $\mathcal{H}$ is projectively equivalent to one of the following forms:

i) a *non singular Hermitian curve* $\mathcal{H}_1 : X_0^{q+1} + X_1^{q+1} + X_2^{q+1} = 0$;

ii) a *Hermitian cone* $\mathcal{H}_2 : X_0^{q+1} - X_2^{q+1} = 0$

iii) a *line repeated* $q+1$ *times* $\mathcal{H}_3 : X_0^{q+1} = 0$

There exists a straight connection between intersections of hermitian curves (that has been classified by Kestenbandt in [15]) and unitary ovoids, that will be investigated in the next chapter.

1.1.2 Finite classical polar space

In what follows we consider the so called *classical finite polar spaces*, that are the only polar spaces we are interested in.

Let $\perp$ be a symplectic polarity of $PG(n, q)$, with n odd and $n \geq 3$, then the pointset of $PG(n, q)$ together with its totally isotropic subspaces is a symplectic polar space of rank $r = (n+1)/2$ denoted by $W_n(q)$.

Let $\mathcal{H}$ be a non-singular hermitian variety of $PG(n, q^2)$, $n \geq 3$, then the points of $\mathcal{H}$ together with the projective subspaces lying on it form a unitary polar space of rank $r = (n+1)/2$ for n odd otherwise the rank is $r = n/2$. We will denote it by $\mathcal{U}(n, q^2)$.

Finally let $\mathcal{Q}$ be a non-singular quadric of $PG(n,q)$, then the points of $\mathcal{Q}$ together with the projective subspaces lying on it form an orthogonal polar space. We have three different type of orthogonal polar space, according as $\mathcal{Q}$ is a parabolic, hyperbolic or elliptic quadric and we will use the same notation for the quadric and the space. In details if $n = 2s$, $s \geq 0$, then $\mathcal{Q}$ is a parabolic quadric so the polar space is denoted by $\mathcal{Q}(2s,q)$ and has rank $r = s$; if $n = 2s+1$ and $\mathcal{Q}$ is hyperbolic the polar space is denoted by $\mathcal{Q}^+(2s+1,q)$ and has rank $r = s+1$, otherwise if $\mathcal{Q}$ is elliptic we will denote the space by $\mathcal{Q}^-(2s+1,q)$ and the rank is $r = s$.

It could be useful to remaind the two following theorems about numbers of points and maximals of classical finite polar spaces.

Theorem 1.1.4. *The numbers of points of the finite classical polar spaces are as follows:*

$|W_n(q)| = (q^{n+1}-1)/(q-1)$;

$|\mathcal{U}(n,q^2)| = (q^{n+1}+(-1)^n)(q^n-(-1)^n)/(q^2-1)$;

$|\mathcal{Q}(2s,q)| = (q^{2s}-1)/(q-1)$;

$|\mathcal{Q}^+(2s+1,q)| = (q^s+1)(q^{s+1}-1)/(q-1)$;

$|\mathcal{Q}^-(2s+1,q)| = (q^s-1)(q^{s+1}+1)/(q-1)$.

If a polar space $\mathcal{P}$ has rank r, its $(r-1)$-subspaces are called *generators*. Let $\mathcal{G}(\mathcal{P})$ be the set of the generators of $\mathcal{P}$, we have the following:

Theorem 1.1.5. *The numbers of generators of the finite classical polar spaces are as follows :*

$|\mathcal{G}(W_n(q))| = (q+1)(q^2+1)\cdots(q^{(n+1)/2}-1)$;

$|\mathcal{G}(\mathcal{U}(2s,q^2))| = (q^3+1)(q^5+1)\cdots(q^{2s+1}+1)$;

$|\mathcal{G}(\mathcal{U}(2s+1,q^2))| = (q+1)(q^3+1)\cdots(q^{2s+1}+1)$;

$|\mathcal{G}(\mathcal{Q}(2s,q))| = (q+1)(q^2+1)\cdots(q^s+1)$;

$|\mathcal{G}(\mathcal{Q}^+(2s+1,q))| = 2(q+1)(q^2+1)\cdots(q^s+1)$;

$|\mathcal{G}(\mathcal{Q}^-(2s+1,q))| = (q^2+1)(q^3+1)\cdots(q^{s+1}+1)$.

The importance of the classical polar spaces is totally explicated in this theorem due to Tits :

Theorem 1.1.6. *All finite polar spaces of rank $r \geq 3$ are classical.*

1.2 Spreads and ovoids

A *partial spread* of $\mathbb{P} = PG(rt-1, q)$ is a family $\mathcal{S}$ of isomorphic subspaces of $\mathbb{P}$ mutually skew; if $\mathcal{S}$ is a partition of the pointset of $\mathbb{P}$ we call it a *spread.* If all the subspaces of $\mathcal{S}$ have the same dimension $t-1$ we can say that $\mathcal{S}$ is a $(t-1)$-spread. Embedded $\mathbb{P}$ as a hyperplane in $PG(rt, q)$ we can define a point-line geometry $A(\mathcal{S})$ where the points of $A(\mathcal{S})$ are whose of $PG(rt, q)$ not incident with $\mathbb{P}$, the lines of $A(\mathcal{S})$ are the t-dimensional subspaces of $PG(rt, q)$ which intersect $\mathbb{P}$ in an element of $\mathcal{S}$, and the incidence relation is inherited from $PG(rt, q)$. If $r = 2$, $A(\mathcal{S})$ is a translation plane of order q^t and a spread $\mathcal{S}$ is desarguesian when the corresponding plane is desarguesian. Let $r > 2$, a spread $\mathcal{S}$ is said to be *normal* if $\mathcal{S}$ induces a spread in any subspace generated by two of its elements. A spread $\mathcal{S}$ of $\mathbb{P}$ is said to be *symplectic* with respect to a symplectic polarity of $\mathbb{P}$, if all elements of $\mathcal{S}$ are totally isotropic with respect to this symplectic polarity.

An *ovoid* of a finite polar space $\mathcal{P}$ of rank $r \geq 2$ is a set of points of $\mathcal{P}$ which has exactly one point in common with every maximal totally isotropic subspace or maximal singular subspace of $\mathcal{P}$. A *spread* of $\mathcal{P}$ is a partition of the pointset of $\mathcal{P}$ by maximal totally isotropic subspaces or maximal totally singular subspaces. Two ovoids $\mathcal{O}$ and $\mathcal{O}'$ of $\mathcal{P}$ are said isomorphic if there is a collineation τ of $Aut\mathcal{P}$ such that $\mathcal{O}^\tau = \mathcal{O}$. Two spreads $\mathcal{S}$ and $\mathcal{S}'$ of $\mathcal{P}$ are said isomorphic if there is a collineation τ of $Aut\mathcal{P}$ such that $\mathcal{S}' = \{X^\tau : X \in \mathcal{S}\}$.

If $\mathcal{P}$ is a polar space, the number of points of an ovoid equals the number of subspaces constituiting the spreads, as we can observe in the next tabel, where these number are explained for every type of polar space:

$\mathcal{P}$	$\|\mathcal{O}\| = \|\mathcal{S}\|$
$W_n(q)$	$q^{\frac{n+1}{2}} + 1$
$\mathcal{U}(n, q^2)$	$q^{2n+1} + 1$
$\mathcal{Q}(2s, q)$	$q^s + 1$
$\mathcal{Q}^+(2s+1, q)$	$q^s + 1$
$\mathcal{Q}^-(2s+1, q)$	$q^{s+1} + 1$

Existence or non-existence of ovoids and spreads are strictly connected to the rank of the space and the order of the field.

Let $\mathcal{P}$ be the polar space $W_{2n-1}(q)$ associated with a symplectic polarity $\perp$, then a spread for it is just a spread of $\mathrm{PG}(2n-1, q)$ symplectic with respect to $\perp$.

Spreads of $Q(2n, q)$ and $Q^+(2n+1, q)$ are also called *orthogonal.* If $Q(2n, q)$ is obtained intersecting $Q^+(2n+1, q)$ with a hyperplane of $\mathrm{PG}(2n+1, q)$, then ovoids of $Q(2n, q)$ are also ovoids of $Q^+(2n+1, q)$. If $\mathcal{P} = Q^+(2n+1, q)$ we have two classes, denoted by $\mathbb{M}_1$ and $\mathbb{M}_2$, of totally singular subspaces of $\mathcal{Q}$ of rank n, two subspaces belonging to the same class if and only if the rank of their intersection has the same parity as n. Observe that no spread exists when n is even. n characteristic 2 there exists a wonderful connection between symplectic spreads of $\mathrm{PG}(2n-1,q)$ and spreads of $\mathcal{Q}^+(2n+1,q)$, n odd, that will be investigated in the next chapter in the special case $n = 3$.

Chapter 2

Slicing the unitary ovoids and the unitary spread of $Q(+)(7, q)$

2.1 Ovoids and spreads of $Q^+(7, q)$

Let $\mathcal{Q} = \mathcal{Q}^+(7, q)$ be the polar space associated with the hyperbolic quadric $\mathcal{Q}^+(7, q)$ of $PG(7, q)$. First of all we want to recall some definitions just seen in the previous chapter now for the special case $n = 3$.

An *ovoid* of $\mathcal{Q}$ is a set of points of $\mathcal{Q}^+(7, q)$ which has exactly one point in common with every 3–dimensional totally singular subspace of $\mathcal{Q}$, consequently an ovoid of $\mathcal{Q}$ has $q^3 + 1$ points. Two ovoids $\mathcal{O}$ and $\mathcal{O}'$ of $\mathcal{Q}$ are said to be *isomorphic* if there is a collineation of $P\Gamma O^+(8, q)$ mapping $\mathcal{O}$ into $\mathcal{O}'$. The known examples of ovoids of $\mathcal{Q}^+(7, q)$ with $q = 2^e$ are the desarguesian ovoids, the unitary ovoids for e odd and the Dye ovoid of $\mathcal{Q}^+(7,8)$.

A *spread* $\mathcal{S}$ of $\mathcal{Q}$ is a partition of the pointset of $\mathcal{Q}$ into disjoint 3–dimensional totally singular subspaces. There are two classes of totally singular 3-spaces of $\mathcal{Q}$, denoted by $\mathbb{M}_1$ and $\mathbb{M}_2$, two subspaces belonging to the same class if and only if either they are disjoint or they intersect in a line. Then, the spread $\mathcal{S}$ consists of q^3+1 subspaces of $\mathcal{Q}$ belonging either to $\mathbb{M}_1$ or to $\mathbb{M}_2$. Two spreads $\mathcal{S}$ and $\mathcal{S}'$ of $\mathcal{Q}$ are said to be *isomorphic* if there is a collineation of $P\Gamma O^+(8,q)$ mapping any element of $\mathcal{S}$ into one of $\mathcal{S}'$.

Let $\mathcal{P}$ be the pointset of $\mathcal{Q}$ and let $\mathcal{L}$ be the set of all lines contained in $\mathcal{Q}$. A *triality map* of $\mathcal{Q}$ is a map τ that fixes $\mathcal{L}$ and $\tau\colon \mathcal{P} \to \mathbb{M}_1 \to \mathbb{M}_2 \to \mathcal{P}$, such that τ is of order 3 and preserves incidence on $\mathcal{P} \cup \mathbb{M}_1 \cup \mathbb{M}_2$ (see [23, 24]).

Theorem 2.1.1. *Let τ be a triality map.*

- *If $\mathcal{O}$ is an ovoid of $\mathcal{Q}^+(7,q)$ then $\mathcal{O}^\tau$ is a spread of $\mathcal{Q}^+(7,q)$;*
- *If $\mathcal{S}$ is a spread of $\mathcal{Q}^+(7,q)$ with $\mathcal{S} \subset \mathbb{M}_1$ then $\mathcal{S}^{\tau^2}$ is an ovoid of $\mathcal{S}$, as is $\mathcal{S}^\tau$ if $\mathcal{S}$ is a spread of $\mathcal{Q}^+(7,q)$ with $\mathcal{S} \subset \mathbb{M}_2$.*

If τ is a triality map, $\bar{\tau}$ is the element of $PO^+(8,q)$ induced by conjugating by τ, and $\mathcal{O}$ and $\mathcal{S}$ are respectively an ovoid and a spread of $\mathcal{Q}^+(7,q)$ that correspond via τ, then we observe that the stabilizer of $\mathcal{O}$ is conjugate to the stabilizer of $\mathcal{S}$ under $\bar{\tau}$.

Let, now, Π be any non–singular hyperplane of $PG(7,q)$ with respect to the polarity defined by $\mathcal{Q}$, then the set $\mathcal{S}' = \{\Pi \cap S \colon S \in \mathcal{S}\}$ defines a spread of the parabolic quadric $\mathcal{Q}' = \mathcal{Q}(6,q) = \Pi \cap \mathcal{Q}$; i.e. a set of q^3+1 planes partitioning the points of $\mathcal{Q}'$. We refer to these spreads as the *slices* of the spread $\mathcal{S}$. Conversely, start from a spread $\mathcal{S}'$ of a parabolic quadric $\mathcal{Q}' = \mathcal{Q}(6,q)$ of $PG(6,q)$ and embed $\mathcal{Q}'$ as a hyperplane section of the hyperbolic quadric $\mathcal{Q} = \mathcal{Q}^+(7,q)$ with a non singular hyperplane of $PG(7,q)$. For any spread element consider the 3–dimensional space of $\mathcal{Q}$, of fixed type, passing through it. This set of 3–dimensional subspaces is a spread of $\mathcal{Q}$ and $\mathcal{S}'$ is one of its slices. Let $\mathcal{Q}' = \mathcal{Q}(6,q)$ be the polar space associated with the parabolic quadric of $\mathrm{PG}(6,q)$. If N is the nucleus of $\mathcal{Q}'$, then the projection of $\mathcal{Q}'$ from N onto a hyperplane H not incident with N is a symplectic polar space $W_5(q)$. A plane of H is totally isotropic with respect to the symplectic polarity associated with $W_5(q)$ if and only if it is the projection from N of a singular plane of $\mathcal{Q}'$. Hence any spread $\overline{\mathcal{S}}$ of $W_5(q)$ defines a spread $\mathcal{S}'$ of $\mathcal{Q}(6,q)$ and conversely.Hence when q is even, there is a connection between spreads of the hyperbolic space $\mathcal{Q}^+(7,q)$ and spreads of $W_5(q)$ which was originally pointed out by Dillon [7] and Dye [8]. Moreover, if two spreads of $W_5(q)$ are isomorphic (i.e. equivalent under the action of $P\Gamma Sp(6,q)$), then the associated spreads of $Q^+(7,q)$ also are. The converse is not generally true (see [14]). This fact leads to the following definition in [14]: two spreads of $W_5(q)$, q even, are said to be *cousins* if the associated spreads in the hyperbolic quadric $Q^+(7,q)$, obtained as described above, are equivalent. In the light of this fact one can construct all cousins of a given spread $\overline{\mathcal{S}}$ of $W_5(q)$ by slicing a spread of a hyperbolic quadric; i.e. in the following way: construct the spread $\mathcal{S}$ of $\mathcal{Q}^+(7,q)$ associated with $\overline{\mathcal{S}}$ as described above, then consider the various slices of $\mathcal{S}$. We are interested only in those cousins that are inequivalent under the action of the automorphisms group $P\Gamma O^+(8,q)$ of $\mathcal{Q}^+(7,q)$. This leads to the investigation of the orbits of non-singular points under the action of the stabilizer of $\mathcal{S}$ in $P\Gamma O^+(8,q)$.

This process can be performed also starting from an ovoid of $\mathcal{Q}$ or an ovoid of $\mathcal{Q}'$, i.e. a set of points of $\mathcal{Q}' = \mathcal{Q}(6,q)$ which has exactly one point in common with every totally singular plane of $\mathcal{Q}'$ (also in this case an ovoid of $\mathcal{Q}'$ necessarily has q^3+1 points). Let $\mathcal{O}$ be an ovoid of $\mathcal{Q}$ and let P be a point of $\mathcal{Q}$ not belonging to $\mathcal{O}$; the polar hyperplane T_P tangent to $\mathcal{Q}$ at P is a cone with vertex P whose projection from P is a hyperbolic quadric $\mathcal{Q}^+(5,q)$. The hyperplane T_P contains exactly q^2+1 points of $\mathcal{O}$ whose projection from P defines an ovoid $\mathcal{O}_P$ of $\mathcal{Q}^+(5,q)$, called a *slice* of $\mathcal{O}$ with respect to P. The ovoid $\mathcal{O}_P$ of $\mathcal{Q}^+(5,q)$ corresponds, under a triality τ, to the spread $\{P^\tau \cap M : M \in \mathcal{O}^\tau\, P^\tau \cap M \neq \emptyset\}$ of the 3–dimensional space P^τ. Indeed, in [14] and [13], the author considers the so called *unitary spread* and *unitary ovoid* of $\mathcal{Q}^+(7,q)$ and $\mathcal{Q}(6,q)$, respectively when $q \equiv 2(mod\, 3)$ and $q \equiv 0(mod\, 3)$. The stabilizers of these geometric objects both contain, up to isomorphism, the group $PGU(3,q)$. In the case $q \equiv 2(mod\, 3)$, in [13] the intersection of the unitary ovoid of $\mathcal{Q}$ with some singular hyperplanes of $PG(7,q)$ which are polar hyperplanes, with respect to the polarity defined by $\mathcal{Q}$, of points of $\mathcal{Q}$ not belonging to the ovoid, is studied. These intersections, project into ovoids of the parabolic quadric $Q(4,q)$. On the other hand, in the same paper when $q \equiv 0(mod\,3)$, is considered the intersection of the unitary ovoid of $\mathcal{Q}'$ with hyperpalnes intersecting $\mathcal{Q}'$ in a hyperbolic quadric $\mathcal{Q}^+(5,q)$. This gives a ovoid of $\mathcal{Q}^+(5,q)$ and hence through the Klein correspondence a translation plane of order q^2.

Regarding spreads, the case $q = 2^{2h+1}$ is particularly interesting; indeed, in [14] the author exhibits three slices of the unitary spread of $\mathcal{Q}$, inequivalent under the action of $PGU(3,q)$, and hence three symplectic spreads of $PG(5,q)$.

For our pourpose, in the next section we will see in details the construction of the unitary ovoid and of the unitary spread exhibited by W.M. Kantor in [13].

2.2 Kantor's construction of the unitary ovoids

Let $\mathbb{M}$ be the 9–dimensional vector space of all the 3×3–matrices $M = (\mu_{ij})$, $\mu_{ij} \in \mathbb{F}_{q^2}$, and for any $M \in \mathbb{M}$, set $\overline{M} = (\mu_{ij}^q)$ and $Tr(M) = \Sigma\mu_{ii}$; then denote by M^t the transpose of an element of $\mathbb{M}$. Set $J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and consider $\mathbb{V}$ the vector subspace of $\mathbb{M}$, of those matrices M such that $Tr(M) = 0$ and $J^{-1}MJ = \overline{M}^t$, i.e.

$$\mathbb{V} = \left\{ \begin{pmatrix} x & y & c \\ z & a & y^q \\ b & z^q & x^q \end{pmatrix} : x, y, z \in \mathbb{F}_{q^2},\ a, b, c \in \mathbb{F}_q \text{ and } a + x + x^q = 0 \right\}.$$

Thus, $\mathbb{V}$ is an 8–dimensional $\mathbb{F}_q$–vector subspace of $\mathbb{M}$. Let $\mathbb{P} = PG(7, q)$ be the projective space underling $\mathbb{V}$, i.e. the lattice of all vector subspaces of $\mathbb{V}$, and let

$$Q(M) = -\sum_{i<j} \mu_{ii}\mu_{jj} + \sum_{i<j} \mu_{ij}\mu_{ji}$$

So if $Tr\colon x \in \mathbb{F}_{q^2} \mapsto x + x^q \in \mathbb{F}_q$ and $N\colon x \in \mathbb{F}_{q^2} \mapsto x^{q+1} \in \mathbb{F}_q$ are the trace and the norm of the field, we can explain

$$Q(M) = Tr(x)^2 - N(x) + Tr(yz) + bc,$$

. Then, $Q(M) = 0$ is a quadric of $\mathbb{P}$ with associated bilinear form $Q(M + N) - Q(M) - Q(N) = tr(MN)$. Now, $Q(M) = 0$ is a hyperbolic quadric

$\mathcal{Q} = \mathcal{Q}^+(7,q)$ of $PG(7,q)$ if and only if $q \equiv 2(mod\,3)$. Moreover, if $q = 3^h$, the quadric $Q(M) = 0$ is a cone, say $\mathcal{C}$, of $\mathbb{P}$ with vertex $\langle I \rangle$, where I is the identity matrix, having as a base the parabolic quadric $\mathcal{Q}(6,q)$.

If $q \equiv 2\,(mod\,3)$ the set $\Omega = \{\langle X \rangle \in \mathbb{V} | X^2 = 0\}$ consists of q^3+1 points of $\mathcal{Q}$ pairwise non–perpendicular, that is Ω is an ovoid of $\mathcal{Q}$, while if $q = 3^h$ it projects onto an ovoid, say Ω', of the nonsingular parabolic quadric $\mathcal{Q}(6,q)$. The ovoids Ω and Ω' are called the unitary ovoids of $\mathcal{Q}$ and $\mathcal{Q}(6,q)$, respectively.

Precisely, Ω consists of the points

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} \alpha & \alpha\beta^q & \alpha^{q+1} \\ \beta & \beta^{q+1} & \alpha^q\beta \\ 1 & \beta^q & \alpha^q \end{pmatrix}$$

with $\alpha, \beta \in \mathbb{F}_{q^2}$ such that $Tr(\alpha) + N(\beta) = 0$.

While Ω' consists of the points

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} \alpha + Tr(\alpha) & \alpha\beta^q & \alpha^{q+1} \\ \beta & 0 & \alpha^q\beta \\ 1 & \beta^q & \alpha^q + Tr(\alpha) \end{pmatrix}$$

with $\alpha, \beta \in \mathbb{F}_{q^2}$ such that $Tr(\alpha) + N(\beta) = 0$.

Here the parabolic quadric $\mathcal{Q}(6,q)$ containing Ω' has equation

$$x^2 + Tr(yz) + bc = 0;$$

i.e. $\mathcal{Q}(6,q) = \Pi \cap \mathcal{C}$ where Π is the hyperplane of $\mathbb{P}$ with equation $Tr(x) = 0$.

Let $GU(3,q)$ be the *unitary group* of all the non–singular 3×3 matrices A over $\mathbb{F}_{q^2}$ such that $JAJ = (\overline{A}^t)^{-1}$. The group $GU(3,q)$ acts on $\mathbb{V}$ by conjugation inducing $PGU(3,q)$ on $PG(7,q)$; nevertheless, $GU(3,q)$ preserves the quadric $Q(M) = 0$ and acts 2–transitively on the set Ω ([14]).

Let $T(X) = \{M \in \mathbb{V} \colon XM = MX = 0\}$, where X is a point of Ω. Then, $T(X)$ is a totally singular plane, and $T(X)$ is disjoint from $T(Y)$ if X and Y are distinct points of Ω. If $q \equiv 2\,(mod\,3)$, we can fix one type of maximal totally singular subspaces of $\mathcal{Q}$, denote by $F(X)$ the subspace of fixed type containing $T(X)$ for any matrix $X \in \Omega$, and set $\mathcal{S}_U = \{F(X) \colon X \in \Omega\}$. The spread $\mathcal{S}_U$ is called the unitary spread of $\mathcal{Q}$.

On the other hand, if $q = 3^h$, the projection of any $T(X)$ from the vertex $\langle I \rangle$ of $\mathcal{C}$ defines a totally singular plane, say $T(X)'$, of $\mathcal{Q}(6,q)$. The set $\mathcal{S} = \{T(X)' \colon X \in \Omega\}$ is defined in [13] as the unitary spread of $\mathcal{Q}(6,q)$.

Moreover, we can embed $\mathcal{Q}(6,q)$ in a hyperbolic quadric $\mathcal{Q}^+(7,q)$ of a $PG(7,q)$ as intersection of $\mathcal{Q}^+(7,q)$ with a non–singular hyperplane of $PG(7,q)$; the set Ω' is an ovoid of $\mathcal{Q}^+(7,q)$ as well and $\tilde{\mathcal{S}} = \{M(X), X \in \Omega\}$, where $M(X)$ is the totally singular 3–dimensional subspace of a fixed type of $\mathcal{Q}^+(7,q)$ containing $T(X)'$, is a spread of $\mathcal{Q}^+(7,q)$. It will be useful for our purposes to consider also this spread and we will refer to it as the *unitary spread* of $\mathcal{Q}^+(7,q)$ when $q = 3^h$.

Remark For $q \equiv 1(mod\ ,3)$,$Q(M) = 0$ is an elliptic quadric $\mathcal{Q}^-(7,q)$ of

$PG(7,q)$ with $Q(M) = Tr(x)^2 - N(x) + Tr(yz) + bc$; hence the set Ω is a partial ovoid of $\mathcal{Q}^-(7,q)$ and the set $\mathcal{S} = \{T(x)\, X \in \Omega\}$ is a partial spread of $\mathcal{Q}^-(7,q)$.

2.3 Hermitian curves and unitary ovoids

Let $\Sigma^* = PG(5,q^2)$ and let $(x_0, x_1, x_2, x_3, x_4, x_5)$ be the projective homogeneous coordinates of a point of Σ^*. Denote by σ the involutory collineation of Σ^* defined by $(x_0, x_1, x_2, x_3, x_4, x_5)^\sigma = (x_3^q, x_4^q, x_5^q, x_0^q, x_1^q, x_2^q)$. The set of points fixed by σ is a canonical subgeometry of Σ^*, i.e.

$$\Sigma = \{(x_0, x_1, x_2, x_0^q, x_1^q, x_2^q) \colon x_0, x_1, x_2 \in \mathbb{F}_{q^2}\}.$$

Let $\pi \subset \Sigma^*$ be a plane with equations $x_3 = x_4 = x_5 = 0$. Then π is disjoint from Σ and the plane π^σ has equations $x_0 = x_1 = x_2 = 0$. For each point x of π, let $L(x) =< x, x^\sigma >$, be the line joining the points x and x^σ and put $\mathcal{S}^* = \{L(x) \colon x \in \pi\}$. Then, $\mathcal{S} = \{L(x) \cap \Sigma \colon x \in \pi\}$ is a line spread of Σ which turns out to be a normal spread. It is easy to show that the Grassmannian map g from the lines of Σ^* into the points of $\Lambda^* = PG(14, q^2)$ maps the set $\mathcal{S}^* = \{L(x) \colon x \in \pi\}$ into an 8–dimensional projective subspace Δ^* of Λ^*. Precisely Δ^* has equations $p_{01} = p_{02} = p_{12} = p_{34} = p_{35} = p_{45} = 0$, and any of its point has homogenous coordinates $(p_{03}, p_{04}, p_{05}, p_{13}, p_{14}, p_{15}, p_{23}, p_{24}, p_{25})$.

Now, let $\mathcal{V} = g(\mathcal{S})$, i.e. let $\mathcal{V}$ be the representation of $\mathcal{S}$ on the Grassmannian $\mathcal{G}$ of the lines of Σ; this is an algebraic variety of a canonical subgeometry $\Lambda \simeq PG(14, q)$ of Λ^*. It is easy to show that Δ^* is a subspace of Λ as well, i.e. $\Delta = \Delta^* \cap \Lambda$ has rank 9; precisely,

$$\Delta := \{(x_0, x_1, x_2, x_1^q, x_4, x_5, x_2^q, x_5^q, x_8), x_0, x_4, x_8 \in \mathbb{F}_q, x_1, x_2, x_5 \in \mathbb{F}_{q^2}\}.$$

Also, in [16], it has been proven that $\mathcal{V}$ is the complete intersection of the Grasmannian $\mathcal{G}$ with Δ.

Note that the vector space Δ underlies an 8–dimensional projective space containing the projective space $\mathbb{P}$ associated with $\mathbb{V}$ as a hyperplane. Moreover, a point p of Δ belongs to $\mathcal{V}$ if and only if

$$p = (a_0^{1+q}, a_0a_1^q, a_0a_2^q, a_1a_0^q, a_1^{1+q}, a_1a_2^q, a_2a_0^q, a_2a_1^q, a_2^{1+q}),$$

where a_0, a_1 and $a_2 \in \mathbb{F}_{q^2}$.

Now, let $m = \langle x, y\rangle$ be a line of π, $S^* = \langle L(x), L(y)\rangle$, $S = S^* \cap \Sigma$, and let $\mathcal{N}$ be the spread of the 3–dimensional projective space S induced by $\mathcal{S}$, then the image of $\mathcal{N}$ under g is an elliptic quadric $\mathcal{Q}_m = \mathcal{Q}^-(3, q)$ complete intersection of $\mathcal{V}$ with a 3–dimensional projective subspace contained in Δ [16, Theorem 1]. Hence, the incidence structure having as points the points of $\mathcal{V}$, as lines the quadrics $\mathcal{Q}_m$ contained in $\mathcal{V}$ and whose incidence is the natural one, is isomorphic to $PG(2, q^2)$ via the isomorphism β defined by the following rules $x \mapsto g(L(x))$ and $m \mapsto \mathcal{Q}_m$, where x and m belong to the pointset and to the lineset of $PG(2, q^2)$, respectively. If $\mathcal{H}(2, q^2)$ is a non-singular hermitian curve of $PG(2, q^2)$ with equation $x_0x_2^q + x_1^{q+1} + x_0^qx_2$, than the image of $\mathcal{H}(2, q^2)$ under β is $\Omega = \mathcal{V} \cap \mathbb{P}$ [16, Theorem 6]. This result was also independently obtained by B. Cooperstein [5, Lemma 2.3].

Denote by H, both the stabilizer of Ω in the orthogonal group $P\Gamma O^+(8, q)$,

$q \equiv 2(mod\,3)$ and the stabilizer of Ω' in $P\Gamma O(7,q)$, $q = 3^h$. The stabilizer of the classical unital $\mathcal{H}(2,q^2)$ is the group $PGU(3,q) \rtimes Aut(\mathbb{F}_{q^2})$, induced by $GU(3,q)$ and, because of the above arguments, it is isomorphic to H. Precisely, by using the isomorphism β, one can see that the linear part $\overline{H}$ of H is isomorphic to $PGU(3,q) \rtimes C_2$, where C_2 is the subgroup of $Aut(\mathbb{F}_{q^2})$ of order two.

Now, denote by G both the stabilizer of $\mathcal{S}_U$, $q \equiv 2\,(mod\,3)$, and the stabilizer of $\tilde{\mathcal{S}}$, $q = 3^h$, in the orthogonal group associated with the relevant hyperbolic quadrics. Note that when $q \equiv 0(mod\,3)$ the stabilizer G_Π of the hyperplane Π in G coincides with the stabilizer of the spread $\mathcal{S}$ of $Q(6,q) = \Pi \cap \mathcal{Q}^+(7,q)$ in $P\Gamma O(7,q)$. Denote by $\overline{G}$ and $\overline{G_\Pi}$ the linear part of G and G_Π, respectively. We have the following

Proposition 2.3.1. *The group $\overline{G}$ is isomorphic to $PGU(3,q)$ and the group $\overline{G_\Pi}$ is isomorphic to $PGU(3,q) \rtimes C_2$ where C_2 is the subgroup of $Aut(\mathbb{F}_{q^2})$ of order two.*

Proof. As seen in 2.1 the unitary ovoid and the unitary spread of $\mathcal{Q}^+(7,q)$ are related each other by a triality map, denoted by τ, of $\mathcal{Q}^+(7,q)$. Suppose $\Omega = \mathcal{S}_U^{\tau^2}$ (or $\Omega' = \tilde{\mathcal{S}}^{\tau^2}$). This means that $\tau\overline{G}\tau^{-1}$ is a subgroup of $\overline{H}$. Also, by [14, Proposition 6.15 (iii)], the groups $\overline{G}$ and $\overline{G_\Pi}$, both contain a subgroup isomorphic to $PGU(3,q)$. This means that $\overline{G}$ and $\overline{G_\Pi}$ are either isomorphic to $PGU(3,q)$ or to $PGU(3,q) \rtimes C_2$. Now, when $q \equiv 0(mod\,3)$, consider $X =$

$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then

$$T(X)' = \left\{ \begin{pmatrix} 0 & y & c \\ 0 & 0 & y^q \\ 0 & 0 & 0 \end{pmatrix} \mid y \in \mathbb{F}_{q^2}, c \in \mathbb{F}_q \right\}.$$

The group C_2, fixes $T(X)'$. Nevertheless, suppose $q \equiv 2 (mod\, 3)$, then $q = p^{2h+1}$ with p a prime number such that $p \equiv 2\,(mod\, 3)$. Hence, since $\mathbb{F}_q$ does not contain primitive cube roots of unity, the polynomial $t^2 + t + 1$ is irreducible over $\mathbb{F}_q$. Let ω be a root of $t^2 + t + 1$ in $\mathbb{F}_{q^2}$, then any element $x \in \mathbb{F}_{q^2}$ can be uniquely written as $x = x_0\omega + x_1\omega^q$, where $x_0, x_1 \in \mathbb{F}_q$. So $T(x) = -(x_0 + x_1)$ and $N(x) = x_0^2 - x_0x_1 + x_1^2$ and it is easy to show that the two maximal totally singular subspaces containing $T(X) = T(X)'$ are

$$F_1(X) = \left\{ \begin{pmatrix} x_0\omega & y & c \\ 0 & x_0 & y^q \\ 0 & 0 & x_0\omega^q \end{pmatrix} \mid y \in \mathbb{F}_{q^2}, c, x_0 \in \mathbb{F}_q \right\}$$

$$F_2(X) = \left\{ \begin{pmatrix} x_1\omega^q & y & c \\ 0 & x_1 & y^q \\ 0 & 0 & x_1\omega \end{pmatrix} \mid y \in \mathbb{F}_{q^2}, c, x_1 \in \mathbb{F}_q \right\}.$$

These are mapped one into the other by C_2. Since, up to isomorphisms,

$PGU(3,q)$ acts transitively on the elements $\mathcal{S}_U$, on the elements of the spread $\mathcal{S}$ of $Q(6,q)$ and it is normal in $PGU(3,q) \rtimes C_2$, the assert follows.

□

The next proposition can be exstracted from [16] in fact, it slightly generalizes Theorem 4 of that paper:

Proposition 2.3.2. *Any hermitian curve (possibly singular) of $PG(2,q^2)$ is isomorphic, via β, to the intersection $W \cap \mathcal{V}$, where W is a hyperplane of Δ.*

Hence, we have the following

Proposition 2.3.3. *Let K be any hyperplane of $\mathbb{P}$. Then the intersection $K \cap \Omega$ is isomorphic to the intersection set of a pencil of Hermitian curves of $PG(2,q^2)$, one of them being $\mathcal{H}(2,q^2)$.*

Proof. Remaind that, by [16], $\Omega = \mathbb{P} \cap \mathcal{V}$. Since K is a 6–dimensional subspace of Δ, $K = W_1 \cap W_2 \cap \ldots \cap W_{q+1}$, where W_i $i = 1, \ldots, q+1$, is a hyperplane of Δ and we can put $W_1 = \mathbb{P}$. Hence, we have $K \cap \Omega = W_1 \cap W_2 \cap \ldots \cap W_{q+1} \cap \Omega = W_1 \cap W_2 \cap \ldots \cap W_{q+1} \cap \mathcal{V}$. By Proposition 2.3.2, $K \cap \Omega$ is then isomorphic to the intersection set of a pencil of $q+1$ Hermitian curves of $PG(2,q^2)$ and $\mathbb{P} \cap \mathcal{V}$ corresponds to $\mathcal{H}(2,q^2)$. □

Lemma 2.3.4. *A collineation $h \in H$ fixes a hyperplane K of $\mathbb{P}$ (a hyperplane U of Π) if and only if h fixes the intersection $K \cap \Omega$ (the intersection $U \cap \Omega'$).*

Proof. We only need to prove the sufficient condition. To this purpose let h be a collineation of H fixing $K \cap \Omega$ and suppose $K \neq K^h$. Then,

$$K = W_1 \cap W_2 \cap \cdots \cap W_{q+1} \quad \text{and} \quad K^h = W_1' \cap W_2' \cap \cdots \cap W_{q+1}',$$

where W_i and W_i' $\quad i = 1, \ldots, q+1$ are the hyperplanes of $\Delta = PG(8, q)$ containing K and K^h, respectively. We can suppose $W_1 = W_1' = \mathbb{P}$. Now, by Proposition 2.3.2 each W_i and W_i' $i = 1, \ldots, q+1$ corresponds, via the isomorphism β, to a hermitian curve (possibly degenerate) of $PG(2, q^2)$; moreover this set of $q+1$ hermitian curves both define a pencil in $PG(2, q^2)$, whose base is $(K \cap \Omega)^{\beta^{-1}}$. Since $K \cap \Omega = K^h \cap \Omega$, there exist two pencils of hermitian curves both containing the curve $\mathcal{H}(2, q^2)$ with the same base; a contradiction. Hence, if h fixes $K \cap \Omega$, then h fixes K. The same holds if we start by considering h fixing $U \cap \Omega'$ were U is a hyperplane of $\Pi = PG(6, q)$; indeed, it is enough to observe that any such 5–dimensional projective space can be uniquely extended to a hyperplane, say K of $\mathbb{P}$ passing through the vertex $\langle I \rangle$ of the cone $\mathcal{C}$ and that the group H fixes the vertex $\langle I \rangle$ of the cone. $\square$

2.3.1 Intersection of Hermitian curves

Let $\mathcal{H}$ denote a non singular hermitian curve, i.e. an hermitian curve with equation $\mathcal{H}_1$as described in section 1.1.1 and denote by $\mathcal{H}'$ any other hermitian curve of $\mathrm{PG}(2, q^2)$, possibly singular. Also, denote by $\mathcal{E} = \mathcal{H} \cap \mathcal{H}'$, the intersection of $\mathcal{H}$ and $\mathcal{H}'$, and by $|\mathcal{E}|$ the size of $\mathcal{E}$. The set $\mathcal{E}$ defines a pencil of $q+1$ hermitian curves of $\mathrm{PG}(2, q^2)$ which is independent of the choice of $\mathcal{H}$ and $\mathcal{H}'$ in the pencil.

The following are all the possible geometric configurations for $\mathcal{E}$ as has been studied by Kestenbandt in [15]:

(I) $\mathcal{H}'$ is a Hermitian cone with vertex $V \notin \mathcal{H}$ and each of its generator is a chord of $\mathcal{H}$, so $|\mathcal{E}| = (q+1)^2$;

(II) $\mathcal{H}'$ is a Hermitian cone with vertex $V \in \mathcal{H}$ and each of its generator is a chord of $\mathcal{H}$, so $|\mathcal{E}| = q^2 + q + 1$;

(III) $\mathcal{H}'$ is a Hermitian cone with vertex $V \notin \mathcal{H}$ and two of its generator are tangent to $\mathcal{H}$ while all the others are chords, so $|\mathcal{E}| = q^2 + 1$;

(IV) $\mathcal{H}'$ is a Hermitian cone with vertex $V \in \mathcal{H}$ and one of its generator are tangent to $\mathcal{H}$ while all the others are chords, so $|\mathcal{E}| = q^2 + 1$;

(V) $\mathcal{H}'$ is a line repeated $q+1$ times and is a chord of $\mathcal{H}$, so $|\mathcal{E}| = q + 1$;

(VI) $\mathcal{H}'$ is a line repeated $q+1$ times and is tangent to $\mathcal{H}$, so $|\mathcal{E}| = 1$;

(VII) $\mathcal{H}'$ is a non singular Hermitian curve and so $|\mathcal{E}| = q^2 - q + 1$.

In what follows we will denote by $\mathcal{E}_i$, $i \in \{I, II, \ldots, VII\}$, the intersection set whose geometric structure ensue from $(I), (II), \ldots, (VII)$, respectively.

Lemma 2.3.5. *Consider the non–singular hermitian curve $\mathcal{H}$ of π. We have that:*

1. *there are $\frac{q^3(q^2-q+1)(q-1)(q-2)}{6}$ sets of type $\mathcal{E}_I$;*
2. *there are $q^2(q^3+1)(q-1)$ sets of type $\mathcal{E}_{II}$;*
3. *there are $\frac{q^4(q^3+1)}{2}$ sets of type $\mathcal{E}_{III}$;*
4. *there are $q(q^3+1)(q+1)$ sets of type $\mathcal{E}_{IV}$;*
5. *there are $q^2(q^2-q+1)$ sets of type $\mathcal{E}_V$;*
6. *there are q^3+1 sets of type $\mathcal{E}_{VI}$;*
7. *there are $\frac{q^3(q+1)^3(q-1)}{3}$ sets of type $\mathcal{E}_{VII}$.*

Proof. The number of distinct intersection sets $\mathcal{E}_V$ equals the number of chords of $\mathcal{H}$, while the number of distinct $\mathcal{E}_{VI}$ equals the number of points of $\mathcal{H}$. These can be easily computed, proving points 5. and 6., respectively. Since the pencils with intersection sets $\mathcal{E}_{II}$, $\mathcal{E}_{III}$ and $\mathcal{E}_{IV}$ contain exactly one cone $\mathcal{H}_1$ of the type described , counting the number of these intersection sets is equivalent to counting the number of these cones. On the other hand, since there are three cones of the same type in a pencil having as intersection set one of type $\mathcal{E}_I$, the number of such intersection sets is the number of the cones described divided by three. Let V be the vertex of the cone, if $V \in \mathcal{H}$ and ℓ is a line of $PG(2, q^2)$ not through V, then there exists exactly one point $P \in \ell$ such that the line $\langle V, P \rangle$ is a tangent line to $\mathcal{H}$. Hence, the number of cones defining a sets $\mathcal{E}_{II}$ is $(q^3 + 1)N_1$ and the number of cones defining intersection sets $\mathcal{E}_{IV}$ is $(q^3 + 1)N_2$, where N_1 and N_2 are the number of the Baer sublines of ℓ not through P and the number of the Baer sublines of ℓ through P, respectively. On the other hand, if $V \notin \mathcal{H}$, then there exists a Baer subline of ℓ, say ℓ', such that the lines joining V with any of the points of ℓ' are tangent, the others being chords. Hence, the number of cones defining intersection sets $\mathcal{E}_I$ is $q^2(q^2 - q + 1)N_3/3$ and the number of cones defining intersection sets $\mathcal{E}_{III}$ is $q^2(q^2 - q + 1)N_4$, where N_3 is the number of the Baer sublines of ℓ skew to ℓ' and N_4 is the number of the Baer sublines of ℓ having two points in common with ℓ'. Finally, the numbers $N_i, i = 1, \ldots, 4$, can be easily computed using the isomorphism between the projective line $PG(1, q^2)$ and the elliptic quadric $\mathcal{Q}^-(3, q)$ (see [10], ch. 15). Finally, as a consequence of Proposition 2.3.3, we get that the number of remaining intersection sets, i.e. intersection sets $\mathcal{E}_{VII}$, is $\frac{q^3(q+1)^3(q-1)}{3}$. This concludes the proof. □

Now we will determine the subgroup of the unitary group $PGU(3, q)$ associated with the non-singular hermitian curve $\mathcal{H}(2,q^2)$, fixing each $\mathcal{E}_i$, $i \in \{I, II, \ldots, VII\}$. In what follows we will denote by $\mathbb{Z}_h$ a cyclic group of order h. The linear automorphism group $Aut(\mathcal{E}_i)$ (i.e. the subgroup of $PGL(3,q^2)$ fixing $\mathcal{E}_i$) has been computed in [9], for all $i \in \{I, II, \ldots, VII\}$. It is easy to see that, up to isomorphism, $Aut(\mathcal{E}_i) \leq PGU(3,q)$ whenever $i \in \{III, IV, VII\}$ and we have that $Aut(\mathcal{E}_{III}) \simeq \mathbb{Z}_2 \rtimes \mathbb{Z}_{q^2-1}$, $Aut(\mathcal{E}_{IV}) \simeq E_q \times AGL(1,q)$, where E_q is an elementary abelian group of order q, and finally $Aut(\mathcal{E}_{VII}) \simeq \mathbb{Z}_3 \rtimes \mathbb{Z}_{q^2-q+1}$. By [9, Lemma 2.6], $Aut(\mathcal{E}_{II}) \cap PGU(3,q) \simeq E_q \rtimes \mathbb{Z}_{q+1}$. Also, $Aut(\mathcal{E}_V)$ is the subgroup of $PGU(3,q)$ fixing a chord of $\mathcal{H}_1$, and $Aut(\mathcal{E}_{VI})$ is the subgroup of $PGU(3,q)$ fixing a point of $\mathcal{H}_1$. Finally, regarding the stabilizer of $\mathcal{E}_I$ in $PGU(3,q)$, we have the following result.

Proposition 2.3.6. *Let $\mathcal{E}$ be an intersection set of type (I) in $\mathcal{H}$ and denote by E the group $Aut(\mathcal{E}) \cap PGU(3,q)$. Then, we have the following possibilities:*

1. *if $q = 2^{2h}$, then either $E \simeq (\mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1}) \rtimes \mathbb{Z}_3$ or $E \simeq \mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1}$;*
2. *if $q = 2^{2h+1}$, then $E \simeq \mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1}$;*
3. *if $q = 3^h$, then either $E \simeq (\mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1}) \rtimes Sym_3$ or $E \simeq \mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1}$.*
4. *if $q = p^h$ and $p \neq 3, 2$, then either $E \simeq \mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1}$ or $E \simeq (\mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1}) \rtimes \mathbb{Z}_3$, or $E \simeq (\mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1}) \rtimes \mathbb{Z}_2$;*

Proof. In [9], the author reconstruct the intersection set $\mathcal{E}$ with geometric structure of type (I) using as fixed non–singular hermitian curve $\mathcal{H}$ of $PG(2,q^2)$

containing $\mathcal{E}$, that with equation $X_0^{q+1}+X_1^{q+1}+X_2^{q+1}=0$. He proves that the group $Aut(\mathcal{E})$ is isomorphic to $(\mathbb{Z}_{q+1}\times\mathbb{Z}_{q+1})\rtimes Sym_3$, where Sym_3 is the symmetric group acting on three elements. It is easy to see that the unitary group associated with $\mathcal{H}$ contains the subgroup of $Aut(\mathcal{E})$ isomorphic to $\mathbb{Z}_{q+1}\times\mathbb{Z}_{q+1}$. Now, the subgroup Sym_3 is generated by the following collineations of $PG(2,q^2)$

$$\sigma_1\colon (X_0,X_1,X_2)\mapsto (X_2,aX_0,bX_1),$$

$$\sigma_2\colon (X_0,X_1,X_2)\mapsto (cX_1,c^{-1}X_0,X_2),$$

where $a^{q+1}=\lambda(1-\lambda)$, $b^{q+1}=-\frac{(1-\lambda)^2}{\lambda}$, $c^{q+1}=-\frac{1}{\lambda}$. Here λ is an element of $\mathbb{F}_q\setminus\{0,1\}$ such that the hermitian cones $\mathcal{K}_i, i=1,2,3$ with equations

$$\mathcal{K}_1:\ \lambda X_0^{q+1}+X_1^{q+1}=0,$$

$$\mathcal{K}_2:(\lambda-1)X_1^{q+1}+\lambda X_2^{q+1}=0,$$

$$\mathcal{K}_3:\ (1-\lambda)X_1^{q+1}+X_2^{q+1}=0$$

belong to the pencil with base $\mathcal{E}_I$. Hence $Sym_3\simeq\{1,\sigma_1,\sigma_1^2,\sigma_2,\sigma_3,\sigma_4\}$, where

$$\sigma_3\colon (X_0,X_1,X_2)\mapsto (caX_0,c^{-1}X_2,bX_1)\ \text{ and}$$

$$\sigma_4\colon (X_0,X_1,X_2)\mapsto (X_2,acX_1,bc^{-1}X_0).$$

Now, the collineation σ_1 fixes $\mathcal{H}$ if and only if $\lambda(1-\lambda)=-\frac{(1-\lambda)^2}{\lambda}=1$, that is if and only if $\lambda^2-\lambda+1=0$. Moreover, σ_2 fixes $\mathcal{H}$ if and only if $\lambda=-1$. Nevertheless, σ_3 fixes $\mathcal{H}$ if and only if $2\lambda=1$ and finally, σ_4 fixes $\mathcal{H}$ if and only if $\lambda = 2$. If q is even, then σ_2, σ_3 and $\sigma_4 \notin E$. Moreover, if $q = 2^{2h+1}$, then $\sigma_1 \notin E$ as well; while if $q = 2^{2h}$, then $\sigma_1 \in E$ if and only if $\lambda^2 - \lambda + 1 = 0$. This proves points 1. and 2. If $q = p^h$ and $p \neq 3, 2$, then there are three possibilities according to $\lambda^2 - \lambda + 1 = 0$, $\lambda \in \{-1, 2, \frac{1}{2}\}$ or $\lambda^2 - \lambda + 1 \neq 0$ and $\lambda \notin \{-1, 2, \frac{1}{2}\}$ and, as a consequence, we have three stated forms for the group E. Finally, if $q = 3^h$ then E has the described form according with $\lambda = -1$ or $\lambda \neq -1$. $\square$

2.4 Slices of the unitary spread

2.4.1 The case $q \equiv 2(mod\, 3)$

Let $q \equiv 2\,(mod\, 3)$ and let $\mathcal{S}_U$ and Ω be the unitary spread and the unitary ovoid of the hyperbolic quadric $\mathcal{Q} = \mathcal{Q}^+(7, q)$ of $\mathbb{P}$ defined by the Quadratic form $\mathcal{Q}$, respectively. Let K be a non–singular hyperplane of $\mathbb{P}$; the slice of $\mathcal{S}_U$ with respect to K is the 2–spread induced by $\mathcal{S}_U$ in the parabolic quadric obtained intersecting $\mathcal{Q}$ with K. Note that the stabilizer G_K of K in G coincides with the stabilizer, in the orthogonal group associated with the parabolic quadric, of the slice determined by K. As observed in the previous section, any hyperplane of $\mathbb{P}$ intersects Ω in a set of points isomorphic, via the map β, to a set $\mathcal{E}_i$, where i varies in the set $\{I, II, III, IV, V, VI, VII\}$. We say that a hyperplane K of $\mathbb{P}$, is of *type* i for $i \in \{I, II, III, IV, V, VI, VII\}$, if $(\Omega \cap K)^{\beta^{-1}} = \mathcal{E}_i$. We prove the following

Proposition 2.4.1. *Let* $\mathcal{Q} = \mathcal{Q}^+(7, q)$*,* $q \equiv 2\,(mod\, 3)$*; there are five disjoint classes of slices of the unitary spread* $\mathcal{S}_U \subset \mathcal{Q}$*. These are obtained intersecting* $\mathcal{S}_U$ *with hyperplanes of* $\mathbb{P}$ *of types* i*, where* $i \in \{I, II, III, V, VII\}$*. Slices obtained intersecting* $\mathcal{Q}$ *with hyperplanes of different types are not equivalent under the action of the group* G*.*

Proof. By Proposition 2.3.1, we have that the linear part $\overline{G}$ of G is isomorphic to the unitary group $PGU(3, q)$. Let K be any hyperplane of $\mathbb{P}$. By Lemma

2.3.4, the stabilizer in the group $\overline{G}$ of K, i.e. the linear stabilizer of the slice determined by K, coincides with the stabilizer in $PGU(3,q)$ of $K \cap \Omega$ and by Proposition 2.3.2, it is isomorphic to the stabilizer in the relevant projective unitary group of one of the intersection sets $\mathcal{E}_i$ $i \in \{I, II, \ldots, VI, VII\}$. These groups and their orders have been described and discussed in Section 2. In what follows we will determine which intersection sets $\mathcal{E}_i$, $i \in \{I, II, III, IV, V, VI, VII\}$, correspond, through the map β, to the intersection of Ω with non–singular hyperplanes of $\mathbb{P}$. To this aim, we first observe that if K is a singular hyperplane polar of a point $P \in \Omega$, then $K \cap \Omega = P$. Hence, the hyperplane K corresponds, via the isomorphism β, to an intersection set $\mathcal{E}_{VI}$. This provides an orbit of such hyperplanes of length $q^3 + 1$ under the action of G. On the other hand, if K is a singular hyperplane polar of a point $P \notin \Omega$, then $K \cap \Omega$ projects into an ovoid of a $\mathcal{Q}^+(5,q)$ [13]. So, $|K \cap \Omega| = q^2 + 1$. There are two types of intersection sets of this size namely, the $\mathcal{E}_{III}$'s and the $\mathcal{E}_{IV}$'s, (see Table 1). Now, let $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$; then $P \in \mathcal{Q}^+(7,q) \setminus \Omega$ and $K = P^\perp$ has equation $Tr(z) = 0$. So, we have

$$K \cap \Omega =$$

$$\left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \alpha & \alpha\beta^q & \alpha^{q+1} \\ \beta & \beta^{q+1} & \alpha^q\beta \\ 1 & \beta^q & \alpha^q \end{pmatrix} \mid Tr(\beta) = 0 \text{ and } Tr(\alpha) + N(\beta) = 0 \right\}.$$

It is easy to show that, $K \cap \Omega$ is isomorphic to the intersection, in $PG(2, q^2)$, between the hermitian curve $\mathcal{H}_1$ and the hermitian cone $\mathcal{K}$ with equation $X_1X_2^q + X_1^qX_2 = 0$. The cone $\mathcal{K}$ has vertex $V = \langle(1, 0, 0)\rangle_{\mathbb{F}_q}$, hence $V \in \mathcal{H}_1$ and so the hyperplane $K = P^\perp$ corresponds to a subset $\mathcal{E}_{IV}$ of $\mathcal{H}_1$. The subgroup of $PGU(3, q)$ fixing such intersection has order $q^2(q-1)$ (see [9]), hence the orbit of K under the action of this group has length $\frac{q^3(q^3+1)(q^2-1)}{q^2(q-1)} = q(q^3+1)(q+1)$. There are $q^3(q^3+1)$ remaining singular points; since the subgroup of $PGU(3, q)$ fixing a intersection set $\mathcal{E}_{III}$ has order $2(q^2 - 1)$, the orbit of any of this point, under the action of the mentioned group, has length $\frac{q^3(q^3+1)}{2}$. So, by Lemma 2.3.5, we conclude that there are two orbits of singular hyperplanes and $q-2$ orbits of non–singular hyperplanes intersecting Ω in a set corresponding through β to an intersection set of type $\mathcal{E}_{III}$ in $\mathcal{H}_1$. The above arguments show that non–singular hyperplanes correspond to intersection sets of type $\mathcal{E}_i$, $i \in \{I, II, III, V, VII\}$ and, naturally, hyperplanes corresponding to different intersection sets are not equivalent under the action of G. □

Note that a slightly different version of the above theorem is stated in [5] (Theorem 3.9, page 194) where the author says that under the same hypothesis of Proposition 2.4.1, the possibility $|\Omega \cap K| = 1$ does occur for some non–

singular hyperplane K of $\mathbb{P}$ while $|\Omega \cap K| = q^2 + 1$ does not. This would mean that there exist slices of the unitary spread $\mathcal{S}_U$ with respect to non–singular hyperplanes of $\mathbb{P}$ of type VI and none with respect to non–singular hyperplanes of type III. The argument used in the prof of Proposition 2.4.1 shows that this can not be the case.

As mentioned in 2.1.1, when q is even, i.e. when $q = 2^{2h+1}$, there is a connection between spreads of $\mathcal{Q}^+(7, q)$ and spreads of $W(5, q)$.

In [13] the author exhibits three slices of $\mathcal{S}_U \subset \mathcal{Q}^+(7, 2^{2h+1})$ non isomorphic with respect to $\overline{G} \simeq PGU(3, q)$. Precisely they are defined by the following non singular point of $\mathbb{P}$

i. $N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, in this case the stabilizer in $PGU(3, q)$ of the corresponding slice is $\mathbb{Z}_{q+1} \times PGU(2, q)$;

ii. $N' = \begin{pmatrix} a & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & a^q \end{pmatrix}$ with $a \in \mathbb{F}_{q^2}$ such that $Tr(a) = 1$, in this case the stabilizer in $PGU(3, q)$ of the corresponding slice is $\mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1}$;

iii. all points of an anisotropic line ℓ, in this case the slices corresponding to any point of ℓ has as stabilizer a cyclic group of order $q^2 - q + 1$.

The translation planes arising from these spreads are also investigated.

According to the terminology used in [14], referred to the Desarguesian spread of $\mathcal{Q}^+(2n+1,q)$, we say that spreads of $W(5,q)$ obtained from the unitary spread S_U of $\mathcal{Q}^+(7,q)$ are *cousins*. We are here mainly interested in those cousins that are non–equivalent under the action of the stabilizer in $P\Gamma O^+(8,q)$ of S_U. In what follows we will use the same symbol $\mathcal{S}$ to denote both the slices and the symplectic spreads of $PG(5,q)$ they produce. Moreover, we denote by $Sp(6,q)_{\mathcal{S}}$ the stabilizer of $\mathcal{S}$ in the group $Sp(6,q)$ associated with the symplectic polarity of $PG(5,q)$. As a consequence of Proposition 2.4.1, we have the following

Theorem 2.4.2. *There are five classes of non–isomorphic symplectic spreads of $PG(5,q)$ which can be obtained from the unitary spread of $Q^+(7,q)$ $q = 2^{2h+1}$; precisely*

1. $Sp(6,q)_{\mathcal{S}} \cong \mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1}$*; there are at least* $\frac{q-2}{6}\frac{d}{2h+1}$ *cousins in this class, where d is a divisor of $2h+1$;*

2. $Sp(6,q)_{\mathcal{S}} \cong E_q \rtimes \mathbb{Z}_{q+1}$ *where E_q is an elementary abelian group of order q; there is a unique cousin in this class;*

3. $Sp(6,q)_{\mathcal{S}} \cong \mathbb{Z}_2 \rtimes \mathbb{Z}_{q^2-1}$*; there are at least* $(q-2)\frac{d'}{2h+1}$ *cousins in this class, where d' is a divisor of $2h+1$;*

4. $Sp(6,q)_{\mathcal{S}} \cong SL(2,q) \times \mathbb{Z}_{q+1}$*; there is a unique cousin in this class;*

5. *$Sp(6,q)_{\mathcal{S}} \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_{q^2-q+1}$; there are $(q+1)\frac{d''}{2h+1}$ cousins in this class, where d'' is a divisor of $2h+1$.*

Proof. In [13] the slices of $\mathcal{S}_U$ defined by the non–singular points of $\mathbb{P}$

$$N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } N' = \begin{pmatrix} a & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & a^q \end{pmatrix} \text{ with } Tr(a) = 1,$$

have been studied. Regarding point N it is proven that the stabilizer in $\overline{G} \simeq PGU(3,q)$ of the corresponding slice $\mathcal{S}$ is isomorphic to $\mathbb{Z}_{q+1} \times SL(2,q)$. Indeed, it is easy to show that $N^{\perp} \cap \Omega$ is isomorphic to an intersection set $\mathcal{E}_V$. Moreover, these slices form a unique orbit under the action of the full stabilizer G of S_U. Nevertheless, regarding point N', in [13] it is proven that the stabilizer in $\overline{G}$ of the corresponding slice $\mathcal{S}$ is isomorphic to $\mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1}$, indeed straightforward calculation show that $N'^{\perp} \cap \Omega$ is isomorphic to an intersection set $\mathcal{E}_I$. These slices are partitioned into $\frac{q-2}{6}$ orbits under the action of $\overline{G}$ (see point 1. of Lemma 2.3.5).

Now, let $N'' = \begin{pmatrix} 0 & a & 1 \\ 1 & 0 & a^q \\ 0 & 1 & 0 \end{pmatrix}$ with $a \in \mathbb{F}_{q^2}$ such that the polynomial $x^3 + Tr(a)x + 1$ is irreducible over $\mathbb{F}_q$; we observe that it is always possible to chose an element in $\mathbb{F}_{q^2}$ with this property, in fact this is equivalent to the existence of an element $u \in \mathbb{F}_{q^3} \setminus \mathbb{F}_q$ whose trace and norm over $\mathbb{F}_q$ are 0 and

1, respectively and, indeed, such an element exists for any prime power q (for instance, see[17]).

The hyperplane $(N'')^\perp$ has equation $Tr(az) + Tr(y) + b = 0$. Hence,

$$(N'')^\perp \cap \Omega = \{X\} \cup$$

$$\left\{ \begin{pmatrix} \alpha & \alpha\beta^q & \alpha^{q+1} \\ \beta & \beta^{q+1} & \alpha^q\beta \\ 1 & \beta^q & \alpha^q \end{pmatrix} : Tr(\alpha\beta^q) + 1 + Tr(a\beta) = 0 \text{ and } Tr(\alpha) + N(\beta) = 0 \right\}$$

Since $q = 2^{2h+1}$, the polynomial $t^2 + t + 1 = 0$ is irreducible over $\mathbb{F}_q$.

Let $i \in \mathbb{F}_{q^2}$ such that $i^2 + i + 1 = 0$ and let $\{i, i^q\}$ be a normal basis of $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$. Any element $\alpha \in \mathbb{F}_{q^2}$ can be uniquely written as follows $\alpha = \alpha_1 i + \alpha_2 i^q$, where $\alpha_1, \alpha_2 \in \mathbb{F}_q$; hence we have $Tr(\alpha) = \alpha_1 + \alpha_2$ and $N(\alpha) = \alpha_1^2 + \alpha_2^2 + \alpha_1\alpha_2$. So, the system

$$\begin{cases} Tr(\alpha\beta^q) + 1 + Tr(a\beta) & = & 0 \\ Tr(\alpha) + N(\beta) & = & 0 \end{cases}$$

can be written as follows

$$\begin{cases} \alpha_1\beta_2 + \alpha_2\beta_1 + 1 + a\beta + a^q\beta^q & = & 0 \\ \alpha_1 + \alpha_2 & = & N(\beta). \end{cases}$$

This system has solutions only when $\beta \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$; this implies that $|(N'')^\perp \cap \Omega| = q^2 - q + 1$. Hence, $(N'')^\perp \cap \Omega$ is isomorphic to an intersection set $\mathcal{E}_{VII}$ of $\mathcal{H}_1$. The stabilizer of the corresponding slice is then isomorphic to $\mathbb{Z}_3 \rtimes \mathbb{Z}_{q^2-q+1}$. We note that the slice corresponding to the non-singular point N'' is one of the examples stabilized by a cyclic group of order $q^2 - q + 1$ discussed by Kantor in [14, Example 7.6]. These slices are partitioned into $q + 1$ orbits under the action of $\overline{G}$ (see point 7. of Lemma 2.3.5).

Let $N''' = \begin{pmatrix} a & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & a^q \end{pmatrix}$ with $Tr(a) = 1$. The polar hyperplane $(N''')^\perp$ has equation $Tr(a^q x) + b = 0$. Hence,

$$(N''')^\perp \cap \Omega = \{X\} \cup$$

$$\left\{ \begin{pmatrix} \alpha & \alpha\beta^q & \alpha^{q+1} \\ \beta & \beta^{q+1} & \alpha^q\beta \\ 1 & \beta^q & \alpha^q \end{pmatrix} : Tr(a^q\alpha) + 1 = 0 \text{ and } Tr(\alpha) + N(\beta) = 0 \right\}.$$

It is easy to show that $(N''')^\perp \cap \Omega$ is isomorphic to the intersection between the hermitian curve $\mathcal{H}_1$ and the hermitian cone with equation $Tr(a^q X_0 X_2^q) + X_2^{q+1}$. Hence it corresponds to an intersection set of type $\mathcal{E}_{III}$ in $\mathcal{H}_1$ and the stabilizer of the corresponding slice is isomorphic to $\mathbb{Z}_2 \rtimes \mathbb{Z}_{q^2-1}$. We have already showed that there are $q - 2$ orbits of such non–singular hyperplanes under the action of $\overline{G}$.

Finally, let $N^{iv} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The polar hyperplane $(N^{iv})^\perp$ has equation $Tr(x) + b = 0$. Hence,

$$(N^{iv})^\perp \cap \Omega = \{X\} \cup$$

$$\left\{ \begin{pmatrix} \alpha & \alpha\beta^q & \alpha^{q+1} \\ \beta & \beta^{q+1} & \alpha^q\beta \\ 1 & \beta^q & \alpha^q \end{pmatrix} : Tr(\alpha) = 1 \text{ and } Tr(\alpha) + N(\beta) = 0 \right\}.$$

It is easy to show that $(N^{iv})^{\perp} \cap \Omega$ is isomorphic to a intersection set $\mathcal{E}_{II}$ and the stabilizer of the corresponding slice is isomorphic to $E_q \rtimes \mathbb{Z}_{q+1}$. By Lemma 2.3.5 we have that there is a unique orbits of such non–singular hyperplanes. This concludes the proof. □

We end the section with the following remark

Remark 2.4.3. *In [13], it is proven that the intersection of the unitary ovoid Ω with a singular hyperplane gives arise to spreads and hence to translation planes. Indeed, if P is a singular point not in Ω, then $P^{\perp} \cap \Omega$ projects into an ovoid of $\mathcal{Q}^+(5, q)$; via the Klein map, an ovoid of $\mathcal{Q}^+(5, q)$ corresponds to a spread of $PG(3, q)$ and hence to a translation plane of order q^2. In [13] some subgroups of the automorphism group of such a spread are studied. By the arguments used in the proof of Theorem 2.4.1, we can see that indeed these subgroups are isomorphic to subgroups of $PGU(3, q)$ fixing intersection sets of types $\mathcal{E}_{III}$ and $\mathcal{E}_{IV}$ of $\mathcal{H}$.*

2.4.2 The case $q = 3^h$

Let $\mathbb{V}$ be the 8–dimensional vector space described in 2.2. If $q = 3^h$, then the quadratic form $\mathcal{Q}$ on $\mathbb{V}$, defines a cone of the associated projective space $\mathbb{P}$ with vertex the point $\langle I \rangle_{\mathbb{F}_q}$ where I is the identity matrix, and with base a parabolic quadric $\mathcal{Q}' = \mathcal{Q}(6, q)$. We can choose as base of the cone the quadric contained in the hyperplane Π with equation $Tr(x) = 0$; i.e. $\mathcal{Q}'$ has equation

$$-N(x) + Tr(yz) + bc = 0.$$

The set Ω then projects into an ovoid say Ω' of $\mathcal{Q}'$, indeed

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} \alpha + Tr(\alpha) & \alpha\beta^q & \alpha^{q+1} \\ \beta & 0 & \alpha^q\beta \\ 1 & \beta^q & \alpha^q + Tr(\alpha) \end{pmatrix}$$

with $\alpha, \beta \in \mathbb{F}_{q^2}$ such that $Tr(\alpha) + N(\beta) = 0$.

In this section we classify the slices of the unitary spread $\mathcal{S} = \{T(X)' \colon X \in \Omega\}$ of $\mathcal{Q}'$ with respect to hyperplanes of Π intersecting $\mathcal{Q}'$ in elliptic quadrics, up to the action of $\overline{G_\Pi} \simeq PGU(3, q) \rtimes C_2$. First, we prove the following

Theorem 2.4.4. *Let U be a hyperplane of $\Pi \simeq PG(6, q)$, then the following possibilities can occur:*

1. *U is the polar hyperplane of a point of Ω'. There is a unique orbit of*

such hyperplanes; also, $U \cap \Omega'$ is isomorphic to an intersection set $\mathcal{E}_{VI}$.

2. *U is the polar hyperplane of a singular point not belonging to Ω'. There is a unique orbit of such hyperplanes; also, $U \cap \Omega'$ is isomorphic to an intersection set $\mathcal{E}_{IV}$.*

3. *U is a non–singular hyperplane intersecting the $\mathcal{Q}(6, q)$ in a $\mathcal{Q}^+(5, q)$; such hyperplanes form a unique orbit; also, $U \cap \Omega'$ is isomorphic to an intersection set $\mathcal{E}_{III}$.*

4. *U is a non–singular hyperplane intersecting the $\mathcal{Q}(6, q)$ in a $\mathcal{Q}^-(5, q)$; there are two orbits of such hyperplanes, say O_1 and O_2, the first of length $\frac{q^3(q^2-q+1)(q-1)}{6}$ and the second one of length $\frac{q^3(q+1)^2(q-1)}{3}$, such that for any $U \in O_1$, $U \cap \Omega'$ is isomorphic to an intersection set $\mathcal{E}_I$, while for any $U \in O_2$, $U \cap \Omega'$ is isomorphic to a intersection set $\mathcal{E}_{VII}$.*

Proof. By Proposition 2.3.3, we know that the intersection $K \cap \Omega$ (K a hyperplane of $\mathbb{P}$) is isomorphic to the intersection of two hermitian curves. Also, any hyperplane U of Π can be uniquely extended as a hyperplane K of Π passing through the vertex of $\mathcal{C}$ and $K \cap \Omega$ is isomorphic to $U \cap \Omega'$.

If U is a hyperplane polar of the point $P \in \Omega'$, then we have that $U \cap \Omega'$ is isomorphic to a subset $\mathcal{E}_{VI}$ of $\mathcal{H}_1$. Consider, on the other hand, $P =$

$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and denote by $\perp$ the polarity defined by the parabolic quadric $\mathcal{Q}(6,q)$, then $P \in \mathcal{Q}(6,q) \setminus \Omega'$ and $P^{\perp}$ has equation $Tr(z) = 0$. Arguing as in the proof of Theorem 2.4.1 we can prove that the intersection of this hyperplane with Ω' is isomorphic to an intersection set $\mathcal{E}_{IV}$ in $\mathcal{H}_1$. The subgroup of $PGU(3,q)$ fixing such an intersection has order $q^2(q-1)$ (see [9]), hence the orbit of U under the action of G_Π has length $\frac{q^3(q^3+1)(q^2-1)}{q^2(q-1)} = q(q^3+1)(q+1)$.

So, we can state that any singular hyperplane which is the polar hyperplane of a point not in Ω' intersects Ω' in a set isomorphic to a set of type $\mathcal{E}_{IV}$ in $\mathcal{H}_1$. Slicing the unitary ovoid Ω' with one of these singular hyperplanes we obtain a set of points which projects into a Kantor ovoid of $\mathcal{Q}(4,q)$ as already proven in [13]. Consider, now, the hyperplane U of $PG(6,q)$ defined by the following points

$$\left\{ \begin{pmatrix} 0 & y & c \\ z & 0 & y^q \\ b & z^q & 0 \end{pmatrix} : y, z \in \mathbb{F}_{q^2},\, b, c \in \mathbb{F}_q \right\}.$$

The intersection of $\mathcal{Q}$ with such a hyperplane is the hyperbolic quadric

$\mathcal{Q}^+(5,q)$ of equation $Tr(yz)+bc=0$. The points of $\Omega'\cap U$ are:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & \beta^{2q+1} & \beta^{2(q+1)} \\ \beta & 0 & \beta^{2+q} \\ 1 & \beta^q & 0 \end{pmatrix},$$

with $\beta\in\mathbb{F}_{q^2}$. Hence $|\Omega'\cap U|=q^2+1$; since the unique orbit of hyperplanes, with respect to the action of G_Π, intersecting Ω' in a set isomorphic to a $\mathcal{E}_{IV}$ consists of singular hyperplanes, the only possibility is that $U\cap\Omega'$ is isomorphic to a set of type $\mathcal{E}_{III}$ and such hyperplanes form a unique orbit as well [13]. Let, now,

$$U=\left\{\begin{pmatrix} x & y & -b \\ z & 0 & y^q \\ b & z^q & x^q \end{pmatrix} : x,y,z\in\mathbb{F}_{q^2}, x+x^q=0,\ b\in\mathbb{F}_q\right\}.$$

In this case, the intersection $U\cap\mathcal{Q}$ is the elliptic quadric $Q^-(5,q)$ of equation $x^2+Tr(yz)-b^2$. The points of $\Omega'\cap U$ are

$$\begin{pmatrix} \alpha+Tr(\alpha) & \alpha\beta^q & -1 \\ \beta & 0 & \alpha^q\beta \\ 1 & \beta^q & \alpha^q+Tr(\alpha) \end{pmatrix},$$

with $\alpha,\beta\in\mathbb{F}_{q^2}$ such that $Tr(\alpha)+N(\beta)=0$ and $\alpha^{q+1}=-1$. So, $|U\cap\Omega'|=(q+1)^2$ and by Proposition 2.3.6 the stabilizer of $U\cap\Omega'$ is isomorphic to $((\mathbb{Z}_{q+1}\times\mathbb{Z}_{q+1})\rtimes Sym_3)\rtimes C_2$; indeed if this was not the case then we would have an orbit of such hyperplanes of length grater then the number of elliptic quadric in $Q(6,q)$. Hence, we have one orbit of length $\frac{q^3(q^2-q+1)(q-1)}{6}$, with respect to the action of $\overline{G_\Pi}$, of hyperplanes containing a $\mathcal{Q}^-(5,q)$ and intersecting Ω' in a set isomorphic to a set $\mathcal{E}_I$. There are, then, $\frac{q^3(q+1)^2(q-1)}{3}$ hyperplanes containing a $\mathcal{Q}^-(5,q)$ left and, by Lemma 2.3.5, the only possibility is that they form one orbit and they intersect Ω' in a set isomorphic to a $\mathcal{E}_{VII}$. □

Let U be a hyperplane of Π intersecting $\mathcal{Q}'$ in a $\mathcal{Q}^+(5,q)$, then the set $\Omega' \cap \mathcal{Q}^+(5,q)$ is an ovoid of $\mathcal{Q}^+(5,q)$ and, by Theorem 2.4.4 and [16], it consists of $q-1$ pairwise disjoint conics and two special points. This set corresponds, via the Klein map, to a spread of the 3–dimensional projective space $PG(3,q)$ containing $q-1$ disjoint reguli and two special lines. In what follows, we explicitly describe such a spread. To this aim, consider the elliptic quadric $\mathcal{Q}^+(5,q)$ with equation $Tr(yz)+bc=0$ and note that the set $\Omega' \cap \mathcal{Q}^+(5,q)$ consists of the points:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & \beta^{2q+1} & \beta^{2(q+1)} \\ \beta & 0 & \beta^{2+q} \\ 1 & \beta^q & 0 \end{pmatrix}$$

with $\beta \in \mathbb{F}_{q^2}$. Let ξ be a fixed non–square element in $\mathbb{F}_q$. Then any element x of $\mathbb{F}_q^2$ can be uniquely written as $x_0 + x_1\sigma$, where $\sigma^2 = \xi$. Consider the following isomorphism $(y,z,b,c) \in \mathbb{F}_q^2 \times \mathbb{F}_q^2 \times \mathbb{F}_q \times \mathbb{F}_q \mapsto (y_0, y_1, b, c, -z_1\xi, -z_0) \in$

$\mathbb{F}_q^6$. Then, the equation $Tr(yz) + bc = 0$ can be written in the following way $-y_0z_0 - y_1z_1\xi + bc = 0$, and applying the above isomorphism it is isomorphic to the klein quadric of equation $x_1x_6 + x_2x_5 + x_3x_4 = 0$.

In this setting, $\Omega' \cap \mathcal{Q}^+(5,q) = \{P_\infty, P_\beta\}$ where $P_\infty = (0,0,0,1,0,0)$ and $P_\beta = (\beta^{q+1}\beta_0, -\beta^{q+1}\beta_1, 1, \beta^{2(q+1)}, -\beta_1\xi, -\beta_0)$, $\beta \in \mathbb{F}_{q^2}$. Applying the inverse of the Klein map, we get: $P_\infty \mapsto \ell_\infty =< (0,1,0,0),(0,0,1,0) >_{\mathbb{F}_q}$ and $P_\beta \mapsto \ell_\beta = \langle(1,\beta_1\xi,-\beta_0,0),(0,\beta_0\beta^{q+1},-\beta_1\beta^{q+1},1)\rangle_{\mathbb{F}_q}$. The set $\mathcal{L} = \{\ell_\infty, \ell_\beta\}$, $\beta \in \mathbb{F}_{q^2}$, is a spread of $PG(3,q)$. Consider the hyperbolic quadrics $\mathcal{Q}_d := \mathcal{Q}^+(3,q)$ of equation $x_1^2\xi d + x_2^2 - x_3^2\xi - x_4^2 d^3 = 0$, where d is an element of $\mathbb{F}_q^*$. The line ℓ_β is contained in $\mathcal{Q}_d$ if and only if $\beta^{q+1} = d$, and hence the spread $\mathcal{L}$ contains $q-1$ disjoint reguli. The lines ℓ_0 and ℓ_∞ are not contained in any of the $q-1$ quadrics but they are pairwise polar with respect to the polarity defined by $\mathcal{Q}_d \, \forall \, d \in \mathbb{F}_q^*$. This spread is spawned by a *regular hyperbolic fibration* of $PG(3,q)$. Hyperbolic fibrations were introduced in [1] and in fact they consist of $q-1$ hyperbolic quadrics and two lines such that they form a partition of the point–set of $PG(3,q)$; if the two lines are pairwise polar with respect to the polarity induced by any of the hyperbolic quadric, then the hyperbolic fibration is said to be regular. Choosing one regulus in each quadric, we get a line–spread of $PG(3,q)$.

In [1, Theorem 2.2], the authors exhibit three families of regular hyperbolic fibrations. One of these is the following

$$\mathcal{J}_0 = \{V[t, 0, -\omega t^{p^i}, 1, 0, -\omega] : t \in \mathbb{F}_q\} \cup \{l_0, l_\infty\} \; i \in \{0, 1, 2, \ldots, h\}$$

where ω is a fixed non–square element in $\mathbb{F}_q$ and for any $t \in \mathbb{F}_q^*$

$$V[t, 0, -\omega t^p, 1, 0, -\omega] = tx_1^2 - \omega t^{p^i} x_2^2 + x_3^2 - \omega x_4^2.$$

Straightforward computation show that the hyperbolic fibration spowned by the spread $\mathcal{L}$ is isomorphic to the hyperbolic fibration $\mathcal{J}_0$ when $p = 3$ and $i = 1$.

In [25] the authors also find a linear automorphism group $\mathcal{G}$ in the stabilizer of $\mathcal{J}_0$; the group $\mathcal{G}$ has order $4(q^2 - 1)$ and is proven to be the semidirect product of a cyclic group of order $q^2 - 1$ and a Klein 4–group. The subgroup $\mathcal{G}'$ of $\mathcal{G}$ fixing $\mathcal{L}$ has order $2(q^2 - 1)$, since $\mathcal{G}'$ does not contain the collineation of order two interchanging the two reguli of each hyperbolic quadric belonging to $\mathcal{J}_0$. Also, the authors state that MAGMA computations for $q = 9$ show that the full linear stabilizer of $\mathcal{J}_0$ has order $8(q^2 - 1)$. As a consequence of Theorem 2.4.4 we have that the full linear stabilizer of $\mathcal{J}_0$ has always order at least $8(q^2 - 1)$. Indeed, by Theorem 2.4.4, the automorphism group of $\mathcal{L}$ is isomorphic to $(Aut(\mathcal{E}_{III}) \cap PGU(3, q)) \rtimes C_2$, where $Aut(\mathcal{E}_{III}) \cap PGU(3, q)$ is the semidirect product of a group of order two permuting the two special lines l_0 and l_∞ and leaving the remaining invariant and a cyclic group of order $q^2 - 1$, acting regularly on the lines of the spread different from l_0 and l_∞.

The group C_2 fixes the lines l_0 and l_∞ and fixes each regulus of the fibration. Moreover, C_2 fixes ℓ_β if and only if $\beta \in \mathbb{F}_q$ and this is possible if and only if d is a square in $\mathbb{F}_q$; so, in $\frac{q-1}{2}$ reguli there are no fixed lines while in the remaining ones two fixed lines. Hence, the full linear stabilizer of $\mathcal{L}$ has size $4(q^2-1)$ and we can conclude that for any $q = 3^h$, the full linear stabilizer of a regular fibration which belongs to the family $\mathcal{J}_0$ has order at least $8(q^2-1)$.

Now, let U be a non-singular hyperplane intersecting $\mathcal{Q}$ in an elliptic quadric $\mathcal{Q}^-(5,q)$: the intersection $U \cap \Omega'$ is a partial ovoid of $\mathcal{Q}^-(5,q)$; on the other hand the intersection $U \cap \mathcal{S}$, where $\mathcal{S} = \{T(X)' \colon X \in \Omega\}$, induces a spread say $\mathcal{S}'$ of $\mathcal{Q}^-(5,q)$. Lemma 2.3.4 can be applied to see that the partial ovoid and the spread $\mathcal{S}'$ have the same stabilizer. Also, in the previous Theorem, we have pointed out that there are two type of non–singular hyperplane intersecting $\mathcal{Q}(6,q)$ in a $\mathcal{Q}^-(5,q)$. Let $U \cap \Omega'$ be isomorphic to a set $\mathcal{E}_I$ in $\mathcal{H}_1$; the set $U \cap \Omega'$ consists of $q+1$ pairwise disjoint conics. More precisely, taking into account the structure of the pencil of hermitian curve of $PG(2,q^2)$ with intersection set $\mathcal{E}_I$, one can see that there are three different possible partitions of $U \cap \Omega'$ into a set of $q+1$ disjoint conics, say $\{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3\}$. The subgroup of $\overline{H}$ fixing this set is isomorphic to $((\mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1}) \rtimes Sym_3) \rtimes C_2$, where Sym_3 is the symmetric group over three objects and $\mathbb{Z}_{q+1}$ is a cyclic group of order $q+1$; for any $i \in \{1,2,3\}$ one of the two copies of $\mathbb{Z}_{q+1}$ acts regularly on the conics of $\mathcal{P}_i$, the other one acts regularly on the points of the conics of $\mathcal{P}_i$. Moreover, the group Sym_3 acts on the set $\{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3\}$. Now, looking at the action of this group on the spread $\mathcal{S}'$ we have that it has one orbit of length $(q+1)^2$, which is in fact formed by the lines of $\mathcal{S}'$ containing the points of $\mathcal{Q}^-(5,q) \cap \Omega'$.

Finally, let U be a hyperplane of Π containing $\mathcal{Q}^-(5,q)$ such that $U \cap \Omega'$ is isomorphic to an intersection set of type $\mathcal{E}_{VII}$. In this case $|U \cap \Omega'| = q^2 - q + 1$ and this set of points has the property that never three of them are contained in a conic. The automorphism group is isomorphic to $Aut(\mathcal{E}_{VII}) \rtimes C_2$ and acts transitively on the points of this partial ovoid and hence on the lines of the induced spread containing these points.

It is worth mentioning that since the generalized quadrangle $\mathcal{Q}^-(5,q)$ is isomorphic to the dual of the generalized quadrangle $\mathcal{H}(3,q^2)$ (for more details we remind to [18]), these two classes of spreads of $\mathcal{Q}^-(5,q)$ produce two non–isomorphic classes of ovoids of the hermitian surface $\mathcal{H}(3,q^2)$ admitting $((\mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1}) \rtimes Sym_3) \rtimes C_2$ and $(\mathbb{Z}_3 \rtimes \mathbb{Z}_{q^2-q+1}) \rtimes C_2$.

Bibliography

[1] R.D.Baker, G.L.Ebert and K.L.Wantz, Regular hyperbolic fibrations, *Adv. Geom.* 1 (2001), pp. 119–144.

[2] A.Barlotti and J.Cofman, Finite Sperner spaces constructed from projective and affine spaces, *Abh. Math. Sem. Univ. Hamburg*, 40 (1974), pp. 230–241

[3] E.Boros and T.Szönyi, On the sharpness of a theorem of B. Segre, *Combinatorica*, 6 (1986), pp. 261–268

[4] R. H. Bruck and R.C. Bose, The construction of translation planes from projective space, *J. Algebra*, 1 (1964), pp. 85–102.

[5] B.N.Cooperstein, Hyperplane sections of Kantor's unitary ovoid, *Des. Codes Cryptogr.*, 23 (2001), no. 2, pp. 185–195.

[6] A.Cossidente, On Kestenband–Ebert partitions. *J. Comb. Designs*, 5 (1997), no. 5, pp. 367–375.

[7] J.F. Dillon, Elementary Hadamard difference sets, Ph.D thesis, Univ. of Meryland, College Park, (1974).

[8] R. Dye, Partitions and their stabilizers for line complexes and quadrics, Ann. Mat. (4), (114) (1977).

[9] L.Giuzzi, Collineation groups of the intersection of two classical unitals, *J. Combin. Des.*, 9 (2001), no. 6, pp. 445–459.

[10] J.W.P. Hirschfeld, *Finite Projective Spaces of Three Dimension*, Oxford University Press, USA (1986).

[11] J.W.P. Hirschfeld and J.A. Thas, *General Galois Geometries*, Oxford University Press, Oxford(1991).

[12] W.V.Hodge and D.Pedoe, *Methods of algebraic geometry*, Cambridge University Press, 1947.

[13] W.M. Kantor, Ovoids and transaltion planes, *Can. J. Math.*, 36 (5) (1982), pp. 1195–1207.

[14] W.M. Kantor, Spreads, transaltion planes and Kerdock sets I, *SIAM J. Alg. Disc. Meth.*, 3 (2) (1982), pp. 151–165.

[15] B.C. Kestenband, Unital intersections in finite projective planes, *Geom. Dedicata* 11 (1981), no. 1, 107–117.

[16] G. Lunardon, Normal Spreads, *Geometriae Dedicata*, 75 (1999), pp. 245–261.

[17] M. Moisio, Kloosterman sums, elliptic curves, and irreducible polynomials with prescribed trace and norm, to appear in *Acta Arith.*

[18] S.E. Payne and J.A. Thas, *Finite Generalized Quadrangles*, Pitman Advanced Publishing Program (1984)

[19] T.Penttila and B.Williams, Ovoids of Parabolic Spaces, *Geometriae Dedicate*, 82 (2000), pp. 1–19

[20] B. Segre, Teoria di Galois, fibrazioni proiettive e geometrie non desarguesiane, *Ann. Mat. Pura Appl.*, 64 (1964), pp. 1–76.

[21] D.E. Taylor, *The Geometry of the Classical Groups*, Heldermann Verlag, Berlin, 1992.

[22] J.A. Thas, Polar spaces, generalized hexagons and perfect codes, *J. Comb. Theory (A)*, 29 (1980), pp. 87–93.

[23] J. Tits, Sur la trialité et certains groupes qui s'en déduisent, *Publ. Math. IHES*, (1959), pp. 14–60.

[24] H. Van Maldeghem, *Generalized Polygons*, Birkhäuser, Basel, 1998.

[25] G. Lunardon, L. Parlato, R. Trombetti, V.Pepe, Slices of the unitary spread, to appear.