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ON A GRAPH OF GROUPS RELATED TO CYCLIC SUBGROUPS

Tesi di Dottorato

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Introduction

"*On a graph of groups related to cyclic subgroups*" is a dissertation on a problem related to the so-called *cyclic graph* associated with a group. The increasing number in literature of papers on graphs associated with groups shows that the use of a graphical representation to study group theoretical properties, became an interesting research topic in last years. Essentially when we assign a graph to a group we provide a method to visualize it and we can study algebraic properties using the graph theoretical concepts. Already in 1878, Cayley associated the so-called *Cayley Digraph* (see, for instance, [17]) with a group given by a set of generators and relations. Another important typical example is given by the *Degree Graph* (see, for instance, [21]) associated with a finite group G, where we consider as set of vertices the set of all primes dividing some character degree of G and we join two primes by an edge if their product divides some degree. An interesting example of graph associated with a non-abelian group is the *non-commuting graph*. If G is a non-abelian group and $Z(G)$ is its center, the non-commuting graph of G, denoted by Δ_G, is the graph whose vertex set is $G \setminus Z(G)$ and such that two distinct vertices are joined if they do not commute. This graph has been studied by many group theorists (see, for instance, [1], [39], [40]).
Paul Erdös, who was the first to consider the non-commuting graph of a group, posed the following problem in 1975 (see [40]): Let G be a group whose non-commuting graph Δ_G has no infinite clique. Is it true that the

clique number of Δ_G is finite?

B.H. Neumann, in [40], answered positively Erdös' question. In fact he proved that the non-commuting graph Δ_G of a group G has no infinite clique if and only if $G/Z(G)$ is finite and the clique number of Δ_G is just $|G/Z(G)|$. It could be natural to consider the graph which is the complement of the non-commuting graph and call this the *commuting graph*. This has been done, for instance, in [26] and [27]. In some other papers, as [13] and [47], the commuting graph of a group G has been defined as the graph whose vertex set is the set of non-identity elements of G and whose edges are pairs of commuting elements.

Furthermore A. Abdollahi and A. Mohammadi Hassanabadi (see [2] and [3]) associated a graph Γ_G to a non-locally cyclic group G (called the *non-cyclic graph of* G) defined as follows: the vertex set is $G \setminus Cyc(G)$, where $Cyc(G) = \{y \in G \mid \langle x, y\rangle \ is\ cyclic\ for\ all\ x \in G\}$ and two vertices are joined if they do not generate a cyclic subgroup. Anyway, in the third chapter of this dissertation, the reader can find a summary of some of the most relevant properties of graphs associated with groups.

In this dissertation we consider the so-called *cyclic graph*. Let G be a non-trivial group. The cyclic graph associated with G, denoted by $\Gamma(G)$, is defined in the following way: the set of vertices of $\Gamma(G)$ is the set of all non-identity elements of G; two distinct vertices x and y are joined if they generate a cyclic subgroup.

Obviously in a locally cyclic group G two vertices are always joined, therefore $\Gamma(G)$ is connected and $diam\,(\Gamma(G)) = 1$. Conversely, it is easy to see that if $\Gamma(G)$ is connected and $diam\,(\Gamma(G)) = 1$, then G is locally cyclic.

In 3.8 we investigate the connectivity of the cyclic graph. The first result in this direction is Theorem 3.8.1, in which we show that the cyclic graph $\Gamma(G)$ associated with a finite nilpotent group G is connected if and only if G is a non-primary group or G is a cyclic primary group or G is a generalized quaternion group.

Furthermore we remark that if a and b are non-identity elements of a group G such that a is periodic and b is not periodic, then a and b are not connected in the cyclic graph $\Gamma(G)$. Therefore a group G with connected cyclic graph is either periodic or torsion-free. In the latter case we prove that $\Gamma(G)$ is connected if and only if, for any finitely generated subgroup H of G, $Z(H)$ is cyclic and $H/Z(H)$ is periodic (see Theorem 3.8.2). Consequently we prove in Theorem 3.8.3 that the cyclic graph associated with a torsion-free solvable group G is connected if and only if G is an infinite locally cyclic group. It is also remarkable the fact that any finite group G can be embedded in a finite group H with $\Gamma(H)$ connected.

The investigation on the cyclic graph suggests the definition of the following group theoretical property. A group G is said to be *cyclic-transitive* if the following condition holds: if x, y, z are elements of $G \setminus \{1\}$ such that $\langle x, y\rangle$ and $\langle y, z\rangle$ are both cyclic, then also $\langle x, z\rangle$ is cyclic. Clearly, in terms of the cyclic graph associated with G, the property of cyclic transitivity means that every its connected component is a complete graph. In [24] and [25] we studied the influence of this condition on the structure of a group G belonging to some well-known classes of groups. The obtained results are essentially collected in Chapter 4 and Chapter 5 of this dissertation. A first interesting property occurs in the nilpotent case; we establish that if G is a nilpotent cyclic-transitive group, then G is either periodic or torsion-free (see Proposition 4.3.1). Then, in Proposition 4.3.2, we prove that every torsion-free nilpotent group is cyclic-transitive. Moreover in Proposition 4.3.3 we prove that if G is a periodic nilpotent group which is cyclic-transitive, then either G is a p-group or G is locally cyclic. In order to study the structure of primary nilpotent groups that are cyclic-transitive, we describe in Theorem 4.3.5 the structure of cyclic-transitive hypercentral p-groups. In the supersolvable case we observe that there exist cyclic-transitive supersolvable groups, that are neither torsion-free nor periodic; moreover we show in Proposition 4.4.3 the structure of these groups. Furthermore we prove in Lemma 4.4.5 that the class of torsion-free cyclic-transitive groups is "partially" closed respect to forming quotients. Then we show that if a group G is torsion free, supersolvable and cyclic-transitive, then G is nilpotent (see Theorem 4.4.7).

A really remarkable property of the class of cyclic-transitive groups is the fact that every cyclic transitive group can be seen as partitioned group; more precisely, in Theorem 5.1.2, we prove that a group G is cyclic-transitive if and only if it has a partition of locally cyclic subgroups. Then, using a well-known result of Suzuki that determines the structure of all non-solvable finite groups with a partition, we obtain a complete classification non-solvable finite groups cyclic-transitive (see Theorem 5.1.5). In order to study the structure of finite cyclic-transitive solvable groups we consider the case of a finite solvable group of order $p^{\alpha}q^{\beta}$, where p, q are two distinct primes and α, β are both positive integers; in Theorem 5.2.6 we obtain a complete characterization of finite solvable groups of this type, that are cyclic-transitive. Using a well-known result of Baer that describes the structure of all finite solvable groups with a partition, we finally prove Theorem 5.3.1, that gives a classification of finite cyclic transitve solvable groups.

Chapter 1

Some basic topics in Group Theory

The purpose of this first chapter is essentially recalling some classical results in Group Theory. Many of the results presented in this first chapter can be studied in a more general context than this considered here. Anyway this first part provides a summary of the most significant group theoretic prerequisites needed in the rest of dissertation.

1.1 Finite solvable groups

The purpose of this section is recalling some classical results in the theory of finite solvable groups. First, we give some basic definitions.
Let G be a finite group. If π is a non-empty set of primes, a π-number is a positive integer whose prime divisors belong to π. A subgroup H of G is called a *Hall* π*-subgroup* of G if $|H|$ is a π-number and $|G : H|$ is a π'-number, where π' is the set consisting of all prime numbers that do not belong to π. The (possibly empty) set of Hall π-subgroups of G is denoted by $Hall_\pi(G)$. When π is a singleton, say $\pi = \{p\}$, we write $Hall_p(G)$ instead of $Hall_\pi(G)$ and p' instead of $\{p\}'$.

A subgroup H of G is called a *Hall subgroup* of G if it is a Hall π-subgroup, for some set π of primes. Clearly H is a Hall subgroup of G if and only if $(|G:H|,|H|)=1$; a Hall π-subgroup is a maximal π-subgroup. Notice that, if p is a prime, then $Hall_p(G) = Syl_p(G)$, in view of Sylow's theorem. Moreover, if p is a prime, P is in $Syl_p(G)$ and H is in $Hall_{p'}(G)$, then $G = HP$ and $H \cap P = 1$. Therefore Hall p'-subgroups are sometimes called *Sylow p-complements*. Although by Sylow's theorem a finite group G has Sylow p-subgroups, for each prime p dividing its order, G need not contain a Hall π-subgroup for some set π of primes. For example, it is easy to see that the alternating group A_5 has not Hall $\{3,5\}$-subgroups.

A fundamental theorem of Philip Hall states that Hall π-subgroups exist in a group G for all set of primes π if and only if the group considered is solvable. In fact, the following theorem is true (for more details see, for instance, 1.3 of [15]).

Theorem 1.1.1 (P.Hall)**.** *If G is a finite solvable group, then every π-subgroup is contained in a Hall π-subgroup of G. Moreover all Hall π-subgroups of G are conjugate. Conversely, suppose that a finite group G has Hall π-subgroups, for every set of primes π. Then G is solvable.*

The previous results suggest that in the study of finite solvable groups, Hall subgroups have an important role. The central concept is that of Sylow systems, which has been introduced by Philip Hall in 1937 in his fundamental paper ([19]).

Let G be a finite group and let $p_1, ..., p_k$ be the distinct prime divisors of $|G|$. Suppose that Q_i is a Hall p_i'-subgroup of G, for $i \in \{1, \cdots, k\}$. Then the set $\{Q_1, ..., Q_k\}$ is called a *Sylow system* of G. As a direct consequence of 1.1.1 we get the following result.

Proposition 1.1.2. *Let G be a finite group. Then G has a Sylow system if and only if it is solvable.*

It is a remarkable fact that a Sylow system determines a set of permutable Sylow subgroups (for more details, see, for instance, 9.2.1 in [44]). In fact, the following result holds.

Proposition 1.1.3. *Let G be a finite solvable group and let $p_1, ..., p_k$ be the distinct prime divisors of the order of G. Assume that $\{Q_1, ..., Q_k\}$ is a Sylow system of G. Then:*

(i) *If π is any set of primes, then $\bigcap_{p_i \notin \pi} Q_i$ is a Hall π-subgroup of G. In particular $P_i = \bigcap_{j=i} Q_j$ is a Sylow p_i-subgroup of G;*

(ii) *The set $\{P_1, ..., P_k\}$ is a set of permutable Sylow subgroups, that is, $P_iP_j = P_jP_i$, for any $i, j \in \{1, \cdots, k\}$.*

A set of mutually permutable Sylow subgroups, one for each prime dividing the order of the group, is called a *Sylow basis*. Then by the previous proposition, if $\mathcal{Q} = \{Q_1, ..., Q_k\}$ is a Sylow system of a finite solvable group G, then there exists a corresponding Sylow basis $\mathcal{Q}^* = \{P_1, ..., P_k\}$, where $P_i = \bigcap_{j=i} Q_j$. The converse is also true: each Sylow basis determines a Sylow system. Consequently, if G is an arbitrary finite solvable groups, then the function $\mathcal{Q} \to \mathcal{Q}^*$ is a bijection between the set of all Sylow systems of G and the set of all Sylow bases of G (see, for instance, [44], pag. 262, 9.2.2). Finally, we recall that two Sylow systems $\{Q_1, ..., Q_k\}$ and $\{\bar{Q}_1, ..., \bar{Q_k}\}$ of a group G are said to be *conjugate* if there exists an element x in G such that $Q_i^x = \bar{Q}_i$, for every $i \in \{1, 2, ..., k\}$. In the same way we can define conjugate Sylow bases. The importance of the two introduced notions of conjugacy is given by the following result (for the proof see, for instance, [44], pag. 262, 9.2.3).

Theorem 1.1.4. *In a finite solvable group G any two Sylow systems are conjugate, as any two Sylow bases.*

1.2 Frobenius groups

The aim of this section is recalling some basic definitions and properties concerning the so-called Frobenius groups. There are many equivalent ways to give the definition of Frobenius groups. To introduce one of these, we recall the following result of Wielandt, which is a famous criterion for non-semplicity.

Theorem 1.2.1 (Wielandt)**.** *Let G be a finite group. Suppose that H and K are subgroups of G such that $K \lhd H$ and $H \cap H^x \leq K$, for every x in $G \setminus H$. Let N be the set of all elements of G which do not belong to any conjugate of $H \setminus K$. Then N is a normal subgroup of G such that $G = HN$ and $H \cap N = K$.*

The elegant proof of the above theorem involves character theory (see, for instance, [44], pag. 248, 8.5.4). Moreover the most important case is when K is the identity subgroup, case already studied by Frobenius.

Theorem 1.2.2 (Frobenius)**.** *If G is a finite group with a subgroup M such that $M \cap M^x = \{1\}$, for all x in $G \setminus M$, then $N = \left(G \setminus \bigcup_{x \in G} M^x\right) \cup \{1\}$ is a normal subgroup of G such that $G = MN$ and $M \cap N = 1$.*

A group G with a non-trivial subgroup M satisfying the above property is called a *Frobenius group.* The subgroup M is called a *Frobenius complement* and N the *Frobenius kernel.*
In the next we describe some major results related to the structure of Frobenius groups. First examples of such groups are some dihedral groups D_{2n}, precisely D_{2n} is a Frobenius group if and only if n is odd and greater than 1 (in this case a complement has order 2). More generally if K is any abelian group of odd order and H is a group of order 2 acting on K by inversion, then the semidirect product of K and H is a Frobenius group. Other useful equivalent definitions of Frobenius group can be given by

using some theoretical properties, in particular some conditions on centralizers. If $G = MN$, where M, N are subgroups of G such that $N \lhd G$ and $M \cap N = \{1\}$ (i.e. G is the semidirect product of the normal subgroup N by M), the following conditions are equivalent to G being a Frobenius group with kernel N and complement M (see [28], pag. 121, problem (7.1)):

(i) $C_G(g) \subseteq N, \forall g \in N \setminus \{1\}$;

(ii) $C_G(h) \subseteq M, \forall h \in M \setminus \{1\}$;

(iii) $C_M(g) = 1, \forall g \in N \setminus \{1\}$.

Furthermore we recall the following fundamental result on Frobenius groups (for more details see also [44], pag. 308, 10.5.6).

Theorem 1.2.3. *Let G be a finite Frobenius group with kernel N and complement M. Then:*

(i) *the kernel N is always nilpotent (Thompson);*

(ii) *the Sylow p-subgroups of M are cyclic if $p > 2$ and cyclic or generalized quaternion if $p = 2$ (Burnside).*

Moreover the Frobenius kernel N is uniquely determined by G as it is the Fitting subgroup of G and Frobenius complements are conjugated by the Schur-Zassenhaus Theorem. Furthermore the center of a Frobenius group is always trivial (see, for instance, [44], pag. 251, Exercises 8.5) and the center of a complement of a Frobenius group is always non-trivial (see, for instance, [22], pag. 506, Satz 8.18 (c)). We can also view a Frobenius group in term of permutation groups. Suppose that G is a permutation group on a set X, where G and X are both finite. Then G is said to be transitive on X if, chosen any pair of elements x, y of X, there exists a permutation π in G such that the image of x in π is equal to y. If Y is a non-empty subset of X, the stabilizer $St_G(Y)$ of Y in G is the set of all permutations in G that leave fixed every element of Y; of course we can write $St_G(\{x\}) = St_G(x)$. The permutation group G is said to be semiregular if $St_G(x) = 1$, for every x in X. A regular permutation group is a permutation group G that is both transitive and semiregular. Then the following result holds (see also, for instance, [44], pag. 250, 8.5.6).

Proposition 1.2.4. *Let G be a transitive but non-regular permutation group in which no non-trivial element has more than one fixed point. Then G is a Frobenius group, whose Frobenius kernel consists of 1 and all elements of G with no fixed points.*

In other words, the above result states that if G is a permutation group such that no non-trivial element fixes more than one point and some non-trivial element fixes a point, then G is a Frobenius group. Conversely, every Frobenius group admits a representation as a transitive non-regular permutation group in which no non-trivial element has more than one fixed point (see, for instance, [44], pag. 250, 8.5.6).

1.3 Some topics on the theory of groups with a partition

The purpose of this section is essentially recalling some fundamental results in the theory of groups admitting a partition. First of all we give the definitions of *covering* and *partition* of a group.

Definition 1.3.1. Let G be a group and let $\mathfrak{A} = \{A_i\}_{i \in I}$ be a collection of subgroups of G (where I is a set of indices, eventually infinite). The set $\mathfrak{A}$ is said to be a covering (or cover) of G if $\bigcup_{i \in I} A_i = G$.

Probably one of the first results on coverings is contained in the paper [38] of G. A. Miller. Then, we can recall the following, more general, definition.

Definition 1.3.2. A cover $\mathfrak{A} = \{A_i\}_{i\in I}$ of a group G is said to be a partition if $A_i \cap A_j = 1$, for all $i, j \in I$, with $i = j$.

Young in [54] used, for the first time, the terminology *partition*.

The subgroups A_i of $\mathfrak{A}$ are said to be the components of the partition; moreover $\mathfrak{A}$ is called non-trivial if every its component is a proper non-trivial subgroup of G (that is $1 < A_i < G$, for every index i). In other words a partition is non-trivial if it contains more than one component.

Examples 1.3.1. Let G be a finite group, p a prime, N a proper subgroup of G such that every element in $G \setminus N$ has order p. Then any partition of N together with the set of all cyclic subgroups of G not contained in N is a non-trivial partition of G.
In particular, if N is a non-trivial normal subgroup such that $|G : N| = p$ and every element of $G \setminus N$ has order p, then N together with the set of all cyclic subgroups of G not contained in N is a partition of G.
Every Frobenius group is partitioned by its complements and its kernel.
If $G = PGL(2, p^n)$, with $p^n > 3$, then the set of all its maximal cyclic subgroups is a partition (see, for instance, [22], pag. 185, II.7 or [46], pag. 145, 3.5.1).
If G is the symmetric group S_4 on four letters, then the set of all its maximal cyclic subgroups of G is a partition (see, for instance, [46], pag.145, 3.5.1).
Also for $G = PSL(2, p^n)$, with $p^n > 3$, the projective special linear group of dimension 2 over the field with p^n elements, there exists a non-trivial partition (see, for instance, [22], pag.191, II.8).
Another class of finite simple groups with a non-trivial partition is given by the *Suzuki groups Sz(q)*, with $q = 2^{2n+1}$ (for more details, see, for instance [23], pag. 190).

It was shown that every finite group with a non-trivial partition is one of the above examples. This was proved in 1961 for solvable groups by Baer and in the general case by Suzuki. This result of Suzuki can be considered one of the most important contribution to the classification of finite simple groups, in particular in his use of character theory to show that a finite non-solvable group with a non-trivial partition has even order.
We recall these fundamental results (for the proofs, the reader can refer, for instance, to [46], pag. 152, 3.5.10 - 3.5.11).

Theorem 1.3.1 (Baer)**.** *Let G be a finite solvable group with a non-trivial partition Σ. Then one of the following occurs:*

(a) *G is a p-group for some prime p and Σ contains a component X such that every element in $G \setminus X$ has order p.*

Furthermore $|\Sigma| \equiv 1 (mod\ p)$;

(b) *G has a nilpotent normal sugroup N such that $N \in \Sigma$, $|G : N|$ is a prime p and every element in $G \setminus N$ has order p;*

(c) *G is a Frobenius group;*

(d) $G \cong S_4$

Theorem 1.3.2 (Suzuki)**.** *Let G be a non-solvable finite group with a non-trivial partition. Then one of the following holds:*

(a) *G is a Frobenius group.*

(b) *$G \cong PGL(2, q)$, q a prime power, $q \geq 4$;*

(c) *$G \cong PSL(2, q)$, q a prime power, $q \geq 4$;*

(d) *$G \cong Sz(q)$, $q = 2^{2n+1}$, $n \in \mathbb{N}$;*

Finally notice the following corollary of the previous result.

Corollary 1.3.3. *The only finite simple groups with a non-trivial partition are the groups* $PSL(2, p^n)$*,* $p^n > 3$*, and* $Sz(2^{2n+1})$*,* $n \in \mathbb{N}$*.*

Chapter 2

Basic concepts in Graph Theory

This chapter gives an introduction to some of the most basic and useful terminology of Graph Theory. For more advanced topics the reader can refer to specific textbooks, for instance [11] or [14].

2.1 First definitions

Before giving the definition of graph we notice that, if V is a set, the notation $P_k(V)$ (or $[V]^k$) stands for the set of all k-elements subsets of V. Then we have the following definition of graph.

Definition 2.1.1. A graph Γ (or an undirected graph) is a pair $\Gamma = (V, E)$, where:

- V is a set, called the set of vertices of Γ;
- E is a subset of $P_2(V)$ (in other words, E is a set of two-elements subsets of V), called the set of the edges of Γ.

Given a graph $\Gamma = (V, E)$, if $x, y \in V$ and $e = \{x, y\} \in E$, we say that the edge e joins the vertices x and y. In this case we say that the vertex x (or y) and the edge e are incident; we say also that e is an edge at x

(or y). The two vertices incident with an edge are called its *endvertices* or its *ends*; therefore we can say that an edge joins its ends. An edge $e = \{x, y\} \in E$ is usually written as xy (or yx). Two vertices are called *adjacent* (or *neighbours*) if there exists an edge between them; likewise two edges are *adjacent* if they have a vertex in common. An equivalent definition of graph is the following.

Definition 2.1.2. A graph is a triple $\Gamma = (V, E, \phi)$, where:

- V is a set, called the set of vertices of Γ;
- E is a set, called the set of the edges of Γ;
- ϕ is an injective function with domain E and codomain $P_2(V)$.

Sometimes the function ϕ is said to be the *incidence function* of the graph. According to this definition, the two elements of $\phi(x) = \{u, v\}$, for any edge x in E, are the *endvertices* of the edge x. The injectivity of the function ϕ allows the identification of E with a subset of $P_2(V)$. Conversely, suppose that $\Gamma = (V, E)$ is graph, defined as in 2.1.1. Define $\phi : E \to P_2(V)$ to be the inclusion map. The graph $\Gamma' = (V, E, \phi)$ is essentially the same as Γ. Notice that in both definitions of graph given above, the order of the vertices is not of remarkable importance (in the next we show that this is not true in other notions of graphs, like digraphs). A graph with vertex set V is called a graph on V. Sometimes, given a graph $\Gamma = (V, E)$, the set of vertices V is written as $V(\Gamma)$, the edge set E as $E(\Gamma)$. Notice that we will not always distinguish between a graph and its vertex set or its edge set; namely, for example, sometimes we speak on a vertex v in Γ (instead of v in $V(\Gamma)$).
The cardinality of the set of the vertices of a graph Γ is its *order*, denoted by $|\Gamma|$. Graphs can be *finite*, *infinite*, *countable* and so on, according to their order. The empty graph $(\emptyset,\emptyset)$ is the graph of order 0; a graph of order 0 or 1 is called *trivial*. Now we give some other basic definitions.

Definition 2.1.3. Let $\Gamma = (V, E)$ a graph. Given any $v \in V$, then

1. the set of the edges incident with v is written as $E(v)$;
2. the set of vertices adjacent to v is written as $N(v)$ (the letter "N" recall the first letter of the word "neighbours").

Definition 2.1.4. A set of vertices (likewise of edges) is said to be *independent* if no two of its element are adjacent.

Definition 2.1.5. A graph Γ is called *complete* if all its vertices are pairwise adjacent.

Usually, a complete graph on n vertices is denoted by K^n.

2.2 Isomorphisms of graphs

Let $\Gamma = (V, E)$ and $\Gamma' = (V', E')$ be graphs. Then Γ and Γ' are said to be *isomorphic* (and we write $\Gamma \cong \Gamma'$) if there exists a *graph-isomorphism* between them, that is a bijection

$$\psi : V \longrightarrow V'$$

such that $xy \in E$ if and only if $\psi(x)\psi(y) \in E'$. Such a map ψ is also said shortly to be an *isomorphism*; if $\Gamma = \Gamma'$, then ψ is called an *automorphism*. For the definition of *graph-isomorphism* according to the definition 2.1.2, the reader can refer any Graph Theory textbook.
A *graph property* is a class of graphs that is closed under isomorphisms. In general, deciding whether or not two graphs are isomorphic can be very difficult. Sometimes it is useful using some *graph invariants* to discover that two graphs are not isomorphic. Recall that a graph invariant is a map having graph as arguments and assigning tha same values to isomorphic graphs.

Two easy examples of graph invariants are the number of vertices and the number of edges of a graph. Other examples of graph invariants are the *minimum degree* and the *maximum degree*, that can be defined as follows.

Definition 2.2.1. Let $\Gamma = (V, E)$ be a graph and $v \in V$ a vertex. The *degree* of v, denoted by $d(v)$, is the number of the edges $e \in E$ incident on v. The number

$$\delta(G) := min\{d(v) \mid v \in V\}$$

is the *minimum degree* of Γ, the number

$$\Delta(G) := max\{d(v) \mid v \in V\}$$

is its *maximum degree.*

It can be proved that the number of vertices of odd degree in a graph is always even.
Furthermore we have the following definition.

Definition 2.2.2. A graph $\Gamma = (V, E)$ is *regular* if every vertex in V has the same degree, i.e. each vertex has the same number of neighbors.
If $\Gamma = (V, E)$ is a regular graph with vertices of degree k, Γ is called a *k-regular* graph or regular graph of degree k.

2.3 Subgraphs

Let $\Gamma = (V, E)$ be a graph. A graph $\Gamma' = (V', E')$ is a *subgraph* of Γ (and we write $\Gamma' \subseteq \Gamma$) if $V' \subseteq V$ and $E' \subseteq E$.
Moreover $\Gamma' = (V', E')$ is called an *induced subgraph* of Γ if $\Gamma' \subseteq \Gamma$ and if E' contains all the edges $xy \in E$, for all $x, y \in V'$. We write also $\Gamma' := \Gamma[V']$ and we say that V' *induces* or *span* Γ' in Γ. Then, if we consider a subset of vertices $U \subseteq V$, then $\Gamma[U]$ is the graph on U such that $xy \in E(\Gamma[U])$, whenever $xy \in E$.

Moreover, if H is a subgraph of Γ (not necessarily induced), we write $\Gamma[H]$, rather than $\Gamma[V(H)]$.
Finally if $\Gamma' = (V', E')$ is a subgraph of Γ, we say that Γ' is a *spanning subgraph* if V' spans Γ, i.e. $V = V'$. Then we have the following definition.

Definition 2.3.1. Let $\Gamma = (V, E)$ be a graph. A subset X of the vertices is said to be a *clique* if the induced subgraph on X is a complete graph. The *clique number* of a graph Γ, denoted by $\omega(\Gamma)$, is defined as the maximum size of a clique (if it exists) in Γ.

If $\Gamma = (V, E)$ is a graph, we can define the *complement* of Γ, denoted by $\bar{\Gamma}$ and defined as the graph on the same vertices such that two vertices of $\bar{\Gamma}$ are adjacent if and only if they are not adjacent in Γ. Clearly every clique in a graph Γ corresponds to an independent set in the complement graph $\bar{\Gamma}$.

2.4 Paths and connectivity

A *path* is a non-empty graph of the form $P = (V, E)$, where

$$V = \{x_0, x_1, ..., x_k\}, E = \{x_0x_1, x_1x_2, ..., x_{k-1}x_k\}$$

(the vertices x_i are pairwise distinct). The vertices x_0 and x_k are called the *ends* of the path P. Usually we say that x_0 and x_k are linked by the path P. The number of edges of a path P is its *length.* Often a path P is denoted by the sequence

$$x_0e_1x_1e_2....e_kx_k$$

whose terms are alternately distinct vertices and distinct edges such that for any i $(1 \leq i \leq k)$ the ends of e_i are x_{i-1} and x_i. We can also refer to a path by the sequence of its vertices

$$P = x_0x_1...x_k.$$

A *cycle* is a path such that the start vertex and end vertex are the same. A path in a graph is a sequence of vertices such that from each of its vertices there is an edge to the next vertex in the sequence. More formally, given a graph $\Gamma = (V, E)$ and two different vertices $x, y \in V$, a path (in Γ) between x and y is a sequence $x = x_0x_1...x_k = y$, such that $x_0x_1, x_1x_2, ..., x_{k-1}x_k$ are edges in the graph Γ. If $x = y$, then the above sequence $x = x_0x_1...x_k = y$ is a cycle (in Γ).
Some authors introduce also the concept of *walk* (of length k) in a graph $\Gamma = (V, E)$ as a non-empty alternating sequence $x_0e_1x_1e_2....e_kx_k$ of vertices and edges in Γ, such that the ends of e_i are x_{i-1} and x_i, for any i ($1 \leq i \leq k$). If $x_0 = x_k$, the walk is *closed*. Clearly if the vertices in a walk are all distinct, it defines a path in Γ. Moreover, in general, every walk between two vertices contains a path between these vertices.
If we consider two different vertices x, y in a graph $\Gamma = (V, E)$ we define the *distance* *d(x,y)* between x and y as the length of the shortest path between them. If no such path exists, we set $d(x, y) := \infty$. The largest distance between all pairs of distinct vertices of Γ is called the *diameter* of Γ and is denoted by *diam(*Γ*)*. The following concepts are of crucial importance.

Definition 2.4.1. Let $\Gamma = (V, E)$ be a non-empty graph. We say that Γ is *connected* if any two distinct vertices are linked by a path in Γ. If $U \subseteq V$ is a subset of vertices, we say that U is connected in Γ, if $\Gamma[U]$ is connected (in Γ). A maximal connected subgraph of Γ is called a *connected component* (or simply *component*) of Γ.

Notice that a component always exists and it is non-empty if $E = \emptyset$; otherwise, the graph has no components.
In the next we introduce some definitions that will be useful in the next chapter.

Definition 2.4.2. A *Hamiltonian path* is a path that visits each vertex exactly once. A graph is *Hamilton-connected* if for every pair of vertices there is a Hamiltonian path between the two vertices.

Definition 2.4.3. A *Hamiltonian cycle* or *Hamiltonian circuit* is a cycle that visits each vertex exactly once (except the vertex which is both the start and end). A graph that contains a Hamiltonian cycle is called a *Hamiltonian graph*.

Finally we recall the following notion of connectivity for a graph.

Definition 2.4.4. A closed walk in a graph is called an *Euler tour* if it traverses every edge of the graph exactly once. A graph is *Eulerian* it admits an Euler tour.

Theorem 2.4.1 (Euler,1736)**.** *A connected graph is Eulerian if and only if every vertex has even degree.*

Directed graphs

A *directed graph* (or *digraph*) is a pair $D = (V, E)$, where V is a set, called the vertex set, and E is a set of ordered pairs of vertices ($E \subseteq V \times V$), called the (directed) edge set.
If $e = (x, y)$ is an edge, then x is the *initial vertex* of e and y is the *terminal vertex*. Sometimes, in the finite case, we also say that a *directed graph* is a triple $D = (V, E, \phi)$ where V and E are finite sets and ϕ is a function with domain E and codomain $V \times V$. Again, we call E the set of edges of the digraph D and call V the set of vertices of D.
We can also define a directed graph as a pair (V, E) of disjoint sets (whose elements are called respectively the vertices and edges of the directed graph) together with two maps $i : E \to V$ and $t : E \to V$ assigning to

every edge e an *initial vertex* $i(e)$ and a *terminal vertex* $t(e)$. We say that the edge e is directed from $i(e)$ to $t(e)$.
Notice that a directed graph may have more than one edge between the same two vertices. Such edges are said to be *multiple edges*; moreover multiple edges are called *parallel* if they have the same direction. Finally, if $i(e) = t(e)$, the edge e is called a *loop*.

Multigraphs

A *multigraph* is a pair (V, E) of disjoint sets (of vertices and edges) together with a map from E to $V \cup P_2(V)$ assigning to every edge either one or two vertices, called its ends. This definition implies that multigraphs too can have loops and multiple edges: we may view a multigraph as a directed graph whose edge directions have been 'forgotten'. To express that x and y are the ends of an edge e we still write $e = xy$, though this no longer determines e uniquely. Finally some authors define a *simple graph* as an undirected graph without loops and multiple edges.

2.5 Graph Colouring

In this section we introduce the concept of *graph colouring*. Graph colouring can be seen as a *graph labeling*, that is an assignment of labels to the edges or vertices, or both, of a graph. Under this point of view, graph colouring is a special case of graph labelings, such that adjacent vertices and coincident edges must have different labels.
A *vertex colouring* of a graph $\Gamma = (V; E)$ is a map

$$c : V \to S$$

for some set S, such that $c(v) = c(w)$, if v and w are adjacent vertices. The set S is called the *set of available colours*. Typically the size of the

set S is of relevant interest.
In fact, a colouring using at most k colours is called a (proper) *k-colouring* The smallest number of colours needed to colour a graph G is called its chromatic number; it is denoted by $\chi(\Gamma)$. A graph Γ with $\chi(\Gamma) = k$ is called *k-chromatic*; if $\chi(\Gamma) \leq k$, we call Γ *k-colourable.*
A subset of vertices assigned to the same colour is called a *colour class.* Thus, a k-colouring is the same as a partition of the vertex set into k disjoint sets, so that the terms k-partite and k-colourable have the same meaning.
An *edge colouring* of a graph $\Gamma = (V; E)$ is a map

$$c : E \to S$$

for some set S, such that $c(e) = c(f)$, for any adjacent edges e, f. An edge colouring with k colours is called a *k-edge-colouring.*
The smallest number of colours needed for an edge colouring of a graph Γ is the *chromatic index*, or *edge chromatic number*; it is denoted by $\chi'(\Gamma)$. An edge colouring with k colours is equivalent to the problem of partitioning the edge sets into k sets of edges without common vertices (such sets are sometimes called *matchings* or *edge-independent sets*).
Finally we refer the reader to [11] (or [14]) for other graph thoretical concepts.

Chapter 3

Graphs associated with groups

The main purpose of this section is providing an introduction to the problem of associating graphs to groups. The use of a graphical representation to study group theoretical properties is an interesting research topic. When we assign a graph to a group we provide a method to visualize it and we can study algebraic properties using the graph theoretical concepts. There are many papers concerning interesting graphs associated with a group (see, for example, [2]-[4], [8]-[10], [12], [13], [20], [31] and [33]). Already in 1878, Cayley associated the so called *Cayley Digraph* with a group given by a set of generators and a set of relations.

3.1 Cayley Digraph

Let $G = \langle S|R\rangle$ be a presentation of a group G with generators S and relations R. We define a directed graph $Cay(S : G)$, called the *Cayley Digraph with generating set S*, as follows:

1. The vertex set of $Cay(S : G)$ is identified with G.

2. If x and y are elements of G, there is an edge between x and y if and only if $xs = y$ for some $s \in S$.

In other words, the vertices of the Cayley graph are precisely the elements of G and two elements of G are connected by an edge if some generator in S maps the one to the other.

Sometimes it is useful to consider the *Coloured Cayley Digraph*. In this case to each generator s of S is assigned a colour $c(s)$. Then if x is an element of G and s is a generator, the directed edge joining x to xs is coloured with the colour assigned to s. There exist some interesting results about the connectivity of Cayley Digraph. In the following we state some of these (for the proofs the reader can refer, for instance, to [17], Chapter 31).

Theorem 3.1.1. *The Cayley Digraph of* $\mathbb{Z}_m \oplus \mathbb{Z}_n$ *is not Hamiltonian when* m *and* n *are relatively prime and greater than 1.*

Theorem 3.1.2. *The Cayley Digraph of* $\mathbb{Z}_m \oplus \mathbb{Z}_n$ *is Hamiltonian when* n *divides* m.

Theorem 3.1.3. *Let G be a finite non-identity abelian group and let S be any generating set for G. Then Cay(S:G) has an Hamiltonian path.*

3.2 Degree Graph

Another important example of graph associated with a group is given by the *Degree Graph* (see, for instance, [21]).
Let G be a finite group and let $Irr(G)$ be the set of irreducible characters of G. Let $\Gamma(G)$ be the graph defined in the following way:

1. The vertices of $\Gamma(G)$ are all primes in $\rho(G)$, i.e. the primes dividing some $\chi(1)$, where $\chi \in Irr(G)$.

2. We connect two distinct primes $p, q \in \rho(G)$ by an edge, if there exists some $\chi \in Irr(G)$ such that pq divides $\chi(1)$.

The graph constructed as above is called the *Degree Graph* associated to a finite group G. In the next we recall two deep results on Degree Graph (for the proof, the reader can refer, for instance, to [21]).

Theorem 3.2.1 (O. Manz). *If G is a finite solvable group, then $\Gamma(G)$ has at most two components.*

Theorem 3.2.2 (O. Manz). *Suppose that $\chi(1)$ is a power of some prime for every $\chi \in Irr(G)$. Then the following statements hold:*

a) *G is solvable if and only if $|\rho(G)| \leq 2$;*

b) *If $G'' = 1$, then $|\rho(G)| \leq 2$.*

There are many results concerning the Degree Graph summarized in [31].

3.3 Prime Graph

Let G be a finite group and let $\pi(G)$ be the set of all primes dividing the order of G. The *prime graph* of G is the graph denoted by $\Gamma(G)$ and defined as follows: the vertex set is $\pi(G)$ and two primes p, q in $\pi(G)$ are joined by an edge, and we write $p \sim q$, if G contains an element of order pq. The prime graph of G is sometimes called *Kegel-Gruenberg* graph. We denote the connected components of the graph by $\pi_i,\ for\ i = 1, ..., t$ and, if $2 \in \pi(G)$, we denote the component containing 2 by π_1.

The concept of prime graph appeared in 1975 during the investigation of certain cohomological problems related to integral representation of finite groups. In fact, in [18], it turned out that $\Gamma(G)$ is not connected if and only if the augmentation of G is decomposable as module. The first classification of groups whose prime graphs have two or more components is an unpublished result of Gruenberg and Kegel. This deep result was published by J. S. Williams in [51]. Actually, in this paper, the following result was proved.

Theorem 3.3.1. *Let G be a finite simple group whose prime graph $\Gamma(G)$ is disconnected and let Δ be a connected component of $\Gamma(G)$ whose vertex set does not cointain 2. Then Δ is a clique.*

This theorem was obtained as a corollary of the following stronger theorem, proved by J. S. Williams using the classification of finite simple groups.
Recall that a subgroup H of a group G is *isolated* if $H \cap H^g = 1$ or $H \cap H^g = H$, for every element g in G, and $C_G(x) \leq H$, for every non-identity element x in H.

Theorem 3.3.2. *Let G be a finite simple group whose prime graph $\Gamma(G)$ is disconnected and let Δ be a connected component of $\Gamma(G)$ whose vertex set does not cointain 2. Denote the vertex set of Δ by δ. Then G contains an isolated nilpotent Hall δ-subgroup H.*

It is easy to see that the existence of such a subgroup H in any finite group G implies that the corresponding connected component of $\Gamma(G)$ is a clique.
In [49], M. Suzuki proved Theorem 3.3.2 without using the classification. Moreover this Suzuki's paper is among the most influential precursor of the so-called "Odd Order Theorem" of W. Feit and J. G. Thompson ([16]). Other results concernig the prime graph are, for instance, [32]-[35].
In the next we recall a result about the diameter of the prime graph. We denote by $d(p,q)$ the distance between two elements p, q if they are in the same connected component of the prime graph $\Gamma(G)$. Then we define the diameter of $\Gamma(G)$ as follows:

$$diam(\Gamma(G)) = max\{d(p,q) \mid p, q \textit{ in the same connected component of } \Gamma(G)\}$$

In [33] (see Theorem 10) it was proved that $diam(\Gamma(G)) \leq 5$ and $diam(\Gamma(G)) \leq 3$ if G is solvable.

3.4 A theorem of B.H. Neumann on non-commuting graph

Let G be a non-abelian group and let $Z(G)$ be the center of G. The *non-commuting graph of* G, denoted by Δ_G, is defined as follows:

the vertex set is $G \setminus Z(G)$;

two distinct vertices are joined if they do not commute.

One of the most recent papers on this graph is, for instance, [1].

Paul Erdös, who was the first to consider the non-commuting graph of a group, posed the following problem in 1975 (see [40]): Let G be a group whose non-commuting graph Δ_G has no infinite clique. Is it true that the clique number of Δ_G is finite?

In the above question recall that a subset X of the vertices of a graph Γ is said to be a *clique* if the induced subgraph on X is a complete graph. The *clique number* of a graph Γ, denoted by $\omega(\Gamma)$, is defined as the maximum size of a clique (if exists) in Γ.

The following theorem of B.H. Neumann answers positively Erdös' question .

Theorem 3.4.1 (B.H. Neumann, [40])**.** *The non-commuting graph* Δ_G *of a group* G *has no infinite clique if and only if* $G/Z(G)$ *is finite. In this case, the clique number of* Δ_G *is finite.*

3.5 Commuting graph associated with a group

In the previous section we introduced the non-commuting graph associated with a non-abelian group. It could be natural to consider the graph which is the complement of the non-commuting graph and call this the *commuting graph* (or *commutativity graph*). This has been done, for instance, in [26] and [27]. In this section we consider the commuting graph $\Gamma(G)$ associated with a group G defined as in [13] and [47]. In these papers the commuting graph of a group G has been defined as the graph whose vertex set is the set of non-identity elements of G and whose edges are pairs of commuting elements.

Furthermore, in [43], A. S. Rapinchuck, Y. Segev and G. M. Seitz showed that there exists a close connection between finite quotients of the multiplicative group of finite-dimensional division algebras and the commuting graph of certain finite groups. In the case of the diameter of $\Gamma(G)$ being equal to 1, we have the class of *commutative-transitive* groups, that will be discussed in the next section.

In [13], C. Delizia and C. Nicotera looked at the cases of the diameter of $\Gamma(G)$ being 2; they proved that if G is a group whose commutativity graph $\Gamma(G)$ has diameter 2 and more than one connected component, then it is either Frobenius or a simple non-abelian group. The case that $\Gamma(G)$ has only one connected component is more complicated. Notice that the commutativity graph of every group G having non-trivial center has diameter 2 and only one component. On the other hand, in [13], it was showed an example of a finite group G such that $Z(G) = 1$ and $\Gamma(G)$ is connected of diameter 2.

It is also easy to prove that the finitary group (i. e. the group of all permutations with finite support on a countable infinite set) is an example of locally finite infinite group having trivial center and connected commutativity graph of diameter 2.
As we remarked above, the case of the diameter of $\Gamma(G)$ being 1 is linked to the class of *commutative-transitive* groups (briefly *CT-groups*).

3.6 Commutative-transitive groups

A group G is said to be *commutative-transitive* (briefly *CT-group*) if $[x, y] = 1$ and $[y, z] = 1$ imply that $[x, z] = 1$, for all non-trivial elements $x, y, z \in G$. In other words, the relation of commutativity is transitive on the set of all non-identity elements of G.

Clearly in terms of the commuting graph associated with G, the above property means that every its connected component is a complete graph. The structure of commutative-transitive groups has been studied in [52], [50], [48]. Clearly every abelian group is commutative-transitive. Furthermore, if a group G having a non-trivial center is commutative-transitive, then it is abelian. In particular, this implies that if G is a nilpotent group of class $c > 1$, then G is not commutative transitive. The first example of a non-abelian commutative-transitive group is the symmetric group S_3 of degree 3. The class of commutative-transitive groups is clearly closed under taking subgroups, but it is not closed under taking homomorphic images, since every free group is a CT-group. The Tarski groups are also CT- groups (see [42]). This shows that the structure of CT-groups can be really complicated.

Suzuki proved, in 1957 (see [48]), by using character theory, that every non-abelian simple CT-group is isomorphic to some $PSL(2, 2^t)$, for some $t \geq 2$. The classification of locally finite commutative-transitive groups is due to Yu-Fen Wu.

Theorem 3.6.1 ([52])**.** *If G is a solvable locally finite CT-group, then $G = H \ltimes F$, where $F = Fit(G)$ is abelian and H is a locally cyclic group of fixed-point-free automorphisms of F. Moreover, any two complements of F are conjugate in G. Conversely, if F is an abelian locally finite group and H is a locally cyclic group of fixed-point-free automorphisms of F, then $G = H \ltimes F$ is a solvable locally finite CT-group.*

Theorem 3.6.2 ([52]). *A non-solvable locally finite group G is commutative-transitive if and only if $G \simeq PSL(2, F)$ for some locally finite field F of characteristic 2 with $|F| \geq 4$.*

Using the above mentioned results, Yu-Fen Wu also proved that the class of locally finite CT-groups is closed respect to forming quotients, a property that does not hold for CTgroups in general as noticed before. Moreover the structure of polycyclic CT-groups and torsion-free solvable CT-groups is explored in [52]. It is interesting to observe that there is a connection between polycyclic CT-groups and algebraic number fields.
Actually if G is an abelian-by-finite polycyclic CT-group, then there are two possibilities: G is a split extension of the Fitting subgroup F by a finite cyclic fixed-point-free group of automorphisms of F or G is an extension of F by a generalized quaternion group Q in which every extension of F by a quaternion subgroup of Q is non-split. Moreover, if G is a non abelian-by-finite polycyclic CT-group, then it is a finite extension of an extension of a free abelian subgroup by another with fixed-point-free action. Essentially, every polycyclic CT-group of this sort is constructed from algebraic number fields.
On the other hand, it is possible to exhibit examples of finitely generated torsion-free solvable CT-groups with arbitrary derived length using standard wreath products; such groups have complicated structure.

3.7 Non-cyclic graph associated with a group

In [2] and [3], A.Abdollahi and A. Mohammadi Hassanabadi defined the so called *non-cyclic graph* associated with a group and studied the properties of this graph. In this section we recall the most relevant results about this graph.

Let G be a group. If x is an element of G we can define the *cyclicizer* of x in G, denoted by $Cyc_G(x)$, as follows

$$Cyc_G(x) = \{y \in G \mid \langle x, y \rangle \ is\ cyclic\}$$

Then we define the *cyclicizer* of G, denoted by $Cyc(G)$, as

$$Cyc(G) = \{y \in G \mid \langle x, y \rangle \ is\ cyclic\ for\ all\ x \in G\}$$

The reader can find some interesting properties of cyclicizers in the second section of [2] or in [41].
Now consider a non-locally cyclic group G. We define the *non-cyclic graph* associated with G, denoted by Γ_G as follows:

(i) the set of vertices is $G \setminus Cyc(G)$;

(ii) two distinct vertices are joined if they do not generate a cyclic subgroup.

In [2] the authors proved for the non-cyclic graph the following result, similar to 3.4.1.

Theorem 3.7.1. *The non-cyclic graph Γ_G of a group G has no infinite clique if and only if $G/Cyc(G)$ is finite. In this case, the clique number of Γ_G is finite.*

The following theorem establish the regularity of the non-cyclic graph of a group.

Theorem 3.7.2 ([2])**.** *Let G be a non-cyclic finite group. Then the non-cyclic graph of G is regular if and only if G is isomorphic to one of the following groups:*

(1) *$Q_8 \times \mathbb{Z}_n$, where n is an odd integer and Q_8 is the quaternion group of order 8 ;*

(2) *$P \times \mathbb{Z}_m$, where P is a finite non-cyclic group of prime exponent p and $m > 0$ is an integer such that $(m, p) = 1$.*

It is interesting the fact that the proof of above theorem needs the following result due to I.M. Isaacs on equally partitioned groups.

Theorem 3.7.3 ([29]). *Let A be a finite non-trivial group and let $n > 1$ be an integer such that $\{A_i | i = 1, ..., n\}$ is a set of subgroups of A with the property that $A = \bigcup_{i=1}^{n} A_i$, $|A_i| = |A_j|$ and $A_i \cap A_j = 1$ for any two distinct indices i, j. Then A is a group of prime exponent.*

3.8 Cyclic graph associated with a group

The main purpose of this section is introducing the *cyclic graph* associated with a group G.
Consider a non-trivial group G. We can define the *cyclic graph* associated with G, denoted by Γ_G and constructed by the following positions:

the set of vertices of Γ_G is the set of all non-identity elements of G;

two distinct vertices x and y are joined by an edge if they generate a cyclic subgroup.

Obviously in a locally cyclic group G two vertices are always joined, therefore Γ_G is connected and $diam\,(\Gamma_G) = 1$. Conversely, it is easy to see that if Γ_G is connected and $diam\,(\Gamma_G) = 1$, then G is locally cyclic.
If $a, b \in G \setminus \{1\}$ are directly connected in the cyclic graph, and $|a| = p^n, |b| = p^m$, p a prime, $n, m > 1$, then obviously $\langle a^{p^{n-1}} \rangle = \langle b^{p^{m-1}} \rangle$. Furthermore if G is a nilpotent group and $|a| = p^n, |b| = q^m$, where p and q are different primes, then obviously a, b commute and $\langle a, b \rangle$ is cyclic, therefore a, b are directly connected in the cyclic graph. Starting from these remarks we have the following result.

Theorem 3.8.1. *Let G be a finite nilpotent group. The graph Γ_G is connected if and only if one of the following holds: G is not a p-group, or G is a cyclic p-group, or G is generalized quaternion.*

Proof. If G is not a p-group, let $a, b \in G \setminus \{1\}$. If $|a|, |b|$ are coprime, then $\langle a, b \rangle$ is cyclic, and a, b are directly connected in Γ_G. If there exists a prime q dividing $|a|$ and $|b|$, let a_1 be an element of $\langle a \rangle$ of order q, b_1 an element of $\langle b \rangle$ of order q, let r be a prime different from q dividing the order of G and c be an element of G of order r. Then we have: a connected to a_1, a_1 connected to c, c connected to b_1, b_1 connected to b, therefore a and b are connected in Γ_G.

If G is a p-group and it is either cyclic or a generalized quaternion group, then G has only one subgroup of order p, say $H = \langle d \rangle$. Then if $a, b \in G \setminus \{1\}$, we have $H \subseteq \langle a \rangle \cap \langle b \rangle$ and a is connected with d, that is connected with b; hence a, b are connected.

Conversely, assume Γ_G connected and G a p-group. If $a, b \in G$ of order p, then there exist elements in G, $x_1 = a, x_2, \cdots, x_n = b$, such that $\langle x_i, x_{i+1} \rangle$ is cyclic, for any $i \in \{1, \cdots, n-1\}$. Then there exists exactly a subgroup H of order p contained in $\langle x_i, x_{i+1} \rangle$, for any i. Then $\langle a \rangle = H = \langle b \rangle$. Therefore G has exactly one subgroup of order p and then it is either cyclic or a generalized quaternion group. □

Obviously a periodic element different from 1 and a non-periodic element are not directly connected in Γ_G, it follows easily that if $a, b \in G \setminus \{1\}$, a is periodic and b is not periodic, then a, b are not connected in Γ_G. Therefore a group G with connected cyclic graph Γ_G is either periodic or torsion-free. In the latter case we have:

Theorem 3.8.2. *Let G be a torsion-free group. Γ_G is connected if and only if, for any finitely generated subgroup H of G, $Z(H)$ is cyclic and $H/Z(H)$ is periodic.*

Proof. Assume that Γ_G is connected. First we show that for any $a, b \in G \setminus \{1\}$ there exist non-zero integers m, n such that $a^n = b^m$. If a, b are directly connected then $\langle a, b\rangle$ is cyclic and the result is true. In the general case, there exist $x_1, \cdots, x_n \in G$ such that $a = x_1, b = x_n, \langle x_i, x_{i+1}\rangle$ is cyclic, for any $i \in \{1, \cdots, n-1\}$. Then $x_i^{\alpha_i} = x_{i+1}^{\beta_i}$, for some non-zero integers α_i, β_i. Hence $a^{\alpha_1 \cdots \alpha_i} = x_i^{\beta_1 \cdots \beta_{i-1}}$, for any i, and $a^{\alpha_1 \cdots \alpha_n} = b^{\beta_1 \cdots \beta_{n-1}}$.

Now let H be a finitely generated subgroup of G, $H = \langle y_1, \cdots, y_t\rangle$. Then, for any $a \in H$, a^{γ_i} centralizes y_i, for some non-zero integer γ_i. Thus $a^{\gamma_1, \cdots \gamma_t} \in Z(H)$. Therefore $H/Z(H)$ is periodic. Furthermore $Z(H)$ is cyclic, since $\langle a, b\rangle$ has rank 1, for any commuting elements $a, b \in G$.

Conversely, assume $H/Z(H)$ periodic and $Z(H)$ cyclic, for every finitely generated subgroup H of G and let $a, b \in G \setminus \{1\}$. Then, for suitable non-zero integer m, n, $a^m, b^n \in Z(\langle a, b\rangle) = \langle d\rangle$. Then we have: a connected to a^m, a^m connected to d, d connected to b^n and b^n connected to b; hence a, b are connected, as required. □

In particular, if G is soluble we have the following theorem.

Theorem 3.8.3. *Let G be a torsion-free soluble group. Γ_G is connected if and only if G is an infinite locally cyclic group.*

Proof. If H is a finitely generated subgroup of G, from $H/Z(H)$ periodic it follows $H/Z(H)$ finite, then H' is finite by a theorem of Schur and H' is trivial, since G is torsion-free. Then $H = Z(H)$ is cyclic. Therefore G is locally cyclic. □

We notice that A. Yu. Ol'shanskii constructed in [42] an infinite torsion-free group G with $Z(G)$ infinite cyclic and $G/Z(G)$ isomorphic to the infinite Burnside group $B(n, p)$ of exponent p. Then Γ_G is connected, by proposition. Hence the hypothesis of solubility in the previous theorem cannot be omitted. Notice also that any finite group G can be embedded in a finite group H with Γ_H connected, in fact, if $|G| = n$ and q is a prime not dividing n, then the group $H = G \times \langle c \rangle$, with $\langle c \rangle$ of order q has connected cyclic graph.

3.9 Cyclic-transitive groups

In order to study the cyclic graph associated with a group G, we introduce the following property.

Definition 3.9.1. Let G be a group. Then G is said to be *cyclic-transitive* if the following condition holds: if x, y, z are elements of $G \setminus \{1\}$ such that $\langle x, y \rangle$ and $\langle y, z \rangle$ are both cyclic, then also $\langle x, z \rangle$ is cyclic.

Clearly, in terms of the cyclic graph associated with G, the property of cyclic-transitivity means that every its connected component is a complete graph. The purpose of the next two chapters of the dissertation is studying the influence of this condition on the structure of groups belonging to some well-known classes of groups.

Chapter 4

A first approach to the class of cyclic-transitive groups

The purpose of this chapter is studying the influence of the condition defined in 3.9.1 on the structure of a group G in the following cases: G abelian, more generally G nilpotent and finally G supersolvable. The reader can find the results contained in this chapter in the paper *On a graph associated with a group* (see [24]). The proofs in this chapter are direct and could be simpler using some results of the next chapter. Anyway we give them to improve understanding the intrinsic stucture of cyclic-transitive groups. First recall that, according to Definition 3.9.1, a group G is said to be *cyclic-transitive* if the following condition holds: if x, y, z are elements of $G \setminus \{1\}$ such that $\langle x, y \rangle$ and $\langle y, z \rangle$ are both cyclic, then also $\langle x, z \rangle$ is cyclic.

4.1 Some examples and properties

In this section we first give some examples of cyclic-transitive groups. Then, we study some properties of these groups. Clearly, every locally cyclic group is cyclic-transitive. The Hamilton's quaternions group Q_8 is not cyclic-transitive. More generally, every generalized quaternion group Q_{2^n} $(n \geq 4)$ is not cyclic-transitive. It is easy to prove that any group of exponent p, where p is a prime, is cyclic-transitive. Moreover in Proposition 4.2.4 it will be proved that every abelian torsion-free group is cyclic-transitive. In the following proposition we give another important class of examples of such groups.

Proposition 4.1.1. *Let F be a non-identity free group, $F = 1$. Then F is cyclic-transitive.*

Proof. Let suppose that x, y, z are non-trivial elements of F such that $\langle x, y\rangle$ and $\langle y, z\rangle$ are both cyclic. Then there exist an element $d \in G \setminus \{1\}$ and non-zero integers α, β such that $x = d^{\alpha}$ and $y = d^{\beta}$. Analogously, $y = g^{\gamma}$, $z = g^{\delta}$, for suitable $g \in G$, $\gamma, \delta = 0$. Hence we get

$$x^{\beta\delta} = d^{\alpha\beta\delta} = d^{\beta\alpha\delta} = y^{\alpha\delta} = g^{\gamma\alpha\delta} = g^{\delta\gamma\alpha} = z^{\gamma\alpha}.$$

As a consequence, if we write $K := \langle x, y, z\rangle$, we can deduce that x^r, y, z^t are all elements of $Z(K)$, for suitable non-zero integers r, t. Moreover K is a free group, by the *Nielsen-Schreier Theorem* (see, for instance, [44], pag. 159, 6.1.1). If *rank*(K)=1, then $K \cong \mathbb{Z}$ and therefore $\langle x, z\rangle$ is cyclic. If *rank*(K)>1, then K has a trivial center and therefore $x^r = 1 = y = z^t$, a contradiction. □

Certainly the class consisting of all cyclic-transitive groups is S-closed, because it is clear that every subgroup of a cyclic-transitive group is also cyclic-transitive. The class of cyclic-transitive groups is not closed with respect to forming factor groups. A counterexample is given by free groups, since any group is a quotient of a suitable free group and, for example, Q_8 is not cyclic-transitive. The direct product of two cyclic-transitive groups needs not to be cyclic-transitive, as shown by the following proposition.

Proposition 4.1.2. *Let consider*

$$G = A \times B$$

with A cyclic,

$$|A| = p^2, |B| = p,$$

p prime. Then G is not cyclic-transitive.

Proof. Let $A = \langle a \rangle$ and $B = \langle b \rangle$. Then $\langle a, a^p \rangle$ and $\langle a^p, ab \rangle$ are cyclic, while $G = \langle a, ab \rangle$ is not cyclic. □

Another important example of a cyclic-transitive group is given by the infinite dihedral group D_∞, as shown by the following proposition.

Proposition 4.1.3. *The infinite dihedral group D_∞ is cyclic-transitive.*

Proof. Recall that the infinite dihedral group D_∞ can be realized as a semidirect product

$$D_\infty = \langle b \rangle \ltimes \langle a \rangle,$$

where $\langle b \rangle \cong \mathbb{Z}_2$, $\langle a \rangle \cong \mathbb{Z}$ and b maps any element of $\langle a \rangle$ into its inverse. Therefore

$$D_\infty = \left\{ b^\epsilon a^i | \epsilon \in \{0, 1\}, i \in \mathbb{Z} \right\}.$$

First of all, we observe that if $g \in D_\infty \setminus \langle a \rangle$, then $|g| = 2$. For, $g = ba^i$ and we can write

$$g^2 = \left(ba^i\right)\left(ba^i\right) = \left(ba^i b\right) a^i = \left(b^{-1} a^i b\right) a^i = \left(a^i\right)^b a^i = a^{-i} a^i = 1,$$

which implies $|g| = 2$. Now let us consider three different non-identity elements x, y, z such that $\langle x, y \rangle$ and $\langle y, z \rangle$ are both cyclic. Then, $x, y, z \in \langle a \rangle$, therefore $\langle x, z \rangle$ is cyclic. □

Arguing similarly we get the following result.

Proposition 4.1.4. *The dihedral group D_{2n} (with $n > 1$) is cyclic-transitive.*

4.2 The abelian case

In this section we investigate the structure of abelian groups belonging to the class of cyclic-transitive groups. Our first result is a technical lemma.

Lemma 4.2.1. *Let G be a cyclic-transitive group. If there exist two permutable elements x,y $\in G$ such that $|x| = p^n$ and $|y| = p$, with p a prime and $n > 1$, then $\langle y \rangle \leq \langle x \rangle$.*

Proof. Suppose that $\langle y \rangle \nleq \langle x \rangle$, then $\langle x \rangle \cap \langle y \rangle = \{1\}$ and $\langle x, y \rangle = \langle x \rangle \times \langle y \rangle$. The subgroup $\langle x^{p^{n-2}} \rangle \times \langle y \rangle$ is not cyclic-transitive, by Proposition 4.1.2, a contradiction. □

Notice that if $n = 1$ the result of the previous Lemma does not hold, since V_4 is a cyclic-transitive group. More generally, every elementary abelian p-group is obviously cyclic-transitive. Conversely we have the following result.

Proposition 4.2.2. *Let G be an abelian p-group. Then G is cyclic-transitive if and only if G is either an elementary abelian p-group or a locally cyclic group.*

Proof. Assume G is cyclic-transitive. If G is not elementary abelian, then there exists an element $a \in G$ of order p^2. For every element $b \in G$ of order p, we have $\langle b \rangle \leq \langle a \rangle$, by Lemma 4.2.1 . Then we get that G is locally cyclic since it has only a subgroup of order p and $G \simeq Q_{2^n}$, for any $n \geq 3$. □

In general we have the following proposition.

Proposition 4.2.3. *Let G be a group. If G is abelian and cyclic-transitive, then G is either periodic or aperiodic.*

Proof. By contradiction suppose that there exist in $G \setminus \{1\}$ an element z of finite order, say n, and an element a of infinite order. Then both $\langle a\rangle$ and $\langle az\rangle$ have infinite order. Moreover

$$(az)^n = a^n z^n = a^n,$$

which implies $a^n \in \langle az\rangle$ and $\langle a^n, az\rangle$ is cyclic. Obviously $\langle a, a^n\rangle$ is cyclic. Applying the cyclic-transitivity, from these two statements it follows that $K := \langle a, az\rangle$ is cyclic. But K contains a and z, a contradiction. □

In the torsion-free case we have the following result.

Proposition 4.2.4. *Any torsion-free abelian group is cyclic-transitive.*

Proof. If we consider $a, b, c \in G \setminus \{1\}$ such that $\langle a, b\rangle$ and $\langle b, c\rangle$ are cyclic, then $a = d^\alpha$ and $b = d^\beta$, for suitable $d \in G, \alpha, \beta = 0$. Analogously, $b = g^\gamma$ and $c = g^\delta$, for suitable $g \in G$, $\gamma, \delta = 0$. Hence we get that

$$a^{\beta\delta} = d^{\alpha\beta\delta} = d^{\beta\alpha\delta} = b^{\alpha\delta} = g^{\gamma\alpha\delta} = g^{\delta\gamma\alpha} = c^{\gamma\alpha}$$

So we can deduce that there exists a positive integer s such that the quotient group

$$\frac{\langle a, c\rangle}{\langle a^s\rangle}$$

is finite. Then, if we consider the 0-rank of $\langle a, c\rangle$, we can write

$$r_0\left(\langle a, c\rangle\right) = r_0\left(\langle a^s\rangle\right) + r_0\left(\frac{\langle a, c\rangle}{\langle a^s\rangle}\right)$$

Clearly the 0-rank of $\langle a, c\rangle/\langle a^s\rangle$ is zero, so that

$$r_0\left(\langle a, c\rangle\right) = r_0\left(\langle a^s\rangle\right).$$

Hence $r_0\left(\langle a, c\rangle\right) = 1$. Since $\langle a, c\rangle$ is a finitely generated torsion-free abelian group, then

$$d\left(\langle a, c\rangle\right) = r_0\left(\langle a, c\rangle\right) = 1.$$

Therefore $\langle a, c\rangle$ is cyclic. We deduce that G is a cyclic-transitive group, as required. □

Now we are able to prove the following characterization of all cyclic-transitive abelian groups.

Theorem 4.2.5. *Let G be an abelian group. Then G is cyclic-transitive if and only if one of the following holds:*

(*i*) *G is locally cyclic;*

(*ii*) *G is an elementary abelian p-group;*

(*iii*) *G is torsion-free.*

Proof. Suppose that G is a cyclic-transitive group. If G is a torsion group, we can write

$$G = \mathbf{X}_{i \in I} P_i,$$

where P_i is a Sylow p_i-subgroup of G. If each P_i is locally cyclic, then G is locally cyclic. Assume now that there exists an index $\bar{i}$ such that $P_{\bar{i}}$ is not locally cyclic. Then $P_{\bar{i}}$ is an elementary abelian $p_{\bar{i}}$-group, by Proposition 4.2.2. If there exists in G an element c of order prime to $p_{\bar{i}}$, then, for any $a, b \in P_{\bar{i}}$, we get that $\langle a, c \rangle$ and $\langle c, b \rangle$ are both cyclic. Since G is cyclic-transitive, $\langle a, b \rangle$ is cyclic. So we can deduce that, for any $a, b \in P_{\bar{i}}$, the subgroup $\langle a, c \rangle$ is cyclic; hence $P_{\bar{i}}$ is locally cyclic, a contradiction. Therefore $G = P_{\bar{i}}$ is an elementary abelian $p_{\bar{i}}$-group, as required. □

An immediate consequence of the Theorem 4.2.5 is the following result.

Corollary 4.2.6. *Let G be a periodic abelian group. If G is cyclic-transitive, then either G is locally cyclic or G is an elementary abelian p-group.*

4.3 The nilpotent case

It is natural trying to extend the results obtained in the abelian case to the class of nilpotent groups. In order to study the structure of nilpotent groups satisfying the property of being cyclic-transitive, first we notice that the following generalization of Proposition 4.2.3 holds.

Proposition 4.3.1. *Let G be a nilpotent group. If G is cyclic-transitive, then G is either periodic or aperiodic.*

Proof. By contradiction suppose that there exist in $G \setminus \{1\}$ an element of finite order and another element of infinite order. Since G is nilpotent, the center $Z(G)$ of G is not trivial, so there exixts $z \in Z(G)$ such that $z = 1$. Moreover the subgroup T of all periodic elements of G has non-trivial intersection with $Z(G)$. So, without loss of generality, we can suppose that z is a periodic element of $Z(G) \setminus \{1\}$, say $|z|= n$. Let $a \in G$ be an element of infinite order. Since z is a central element, z and a commute, and we get a contradiction as in the proof of Proposition 4.2.3. □

As in the abelian case (see Proposition 4.2.4), any torsion-free nilpotent group is cyclic-transitive.

Proposition 4.3.2. *Let G be a torsion-free nilpotent group. Then G is cyclic-transitive.*

Proof. Suppose that $\langle x, y\rangle$ and $\langle y, z\rangle$ are both cyclic, where x, y, z are non-trivial elements of G. We have to prove that $\langle x, z\rangle$ is also cyclic. Clearly $x^{\alpha} = y^{\beta}$ and $y^{\gamma} = z^{\delta}$, for suitable non-zero integers $\alpha, \beta, \gamma, \delta$. Moreover

$$x^{\alpha\gamma} = y^{\beta\gamma} = y^{\gamma\beta} = z^{\delta\beta}.$$

Therefore, if we write $K := \langle x, y, z\rangle$, we get that $y^s \in Z(K)$, for a suitable positive integer s. Analogously $x^t, z^r \in Z(K)$, for suitable positive integers t, r. Therefore

$$\frac{K}{Z(K)} = \langle xZ(K), yZ(K), zZ(K)\rangle$$

where x, y, z are all elements of finite order modulo $Z(K)$. Thus $K/Z(K)$ is finite, since it is a finitely generated torsion solvable group. Therefore

K is a central-by-finite group; thus, by a well-known theorem of Schur, K' is finite. Since K is torsion-free, $K' = \{1\}$, which implies that K is an abelian group. Then K is a torsion-free abelian group and therefore K is cyclic-transitive, by Lemma 4.2.4. Thus, if $\langle x, y\rangle$ and $\langle y, z\rangle$ are both cyclic, also $\langle x, z\rangle$ is cyclic, as required. □

Moreover we have the following immediate extension of Corollary 4.2.6.

Proposition 4.3.3. *Let G be a periodic nilpotent group. If G is cyclic-transitive, then either G is a p-group or G is locally cyclic.*

In order to study the structure of nilpotent groups which belong to the class of cyclic-transitive groups, in the next results we consider the class of hypercentral groups.

Lemma 4.3.4. *Let G be a non-abelian hypercentral p-group such that $exp(G) = p$. If G is cyclic-transitive, then the center $Z(G)$ of G has order p, in particular $Z(G) = \langle x^p\rangle$, where x is a non-central element of order p^2, and is contained in any cyclic subgroup of G of order greater than p.*

Proof. Let G be a cyclic-transitive hypercentral p-group, with $exp(G) = p$. Suppose that G is not a group of exponent p. Then there exists an element of G of order p^n, with $n \geq 2$. Let $x \in G$ such that $|x| = p^2$. Suppose that x lies in the center of G and consider two elements $g, g_1 \in G$. We have $\langle x, g\rangle$ and $\langle x, g_1\rangle$ abelian not of exponent p, then $\langle x, g\rangle$ and $\langle x, g_1\rangle$ are cyclic by Proposition 4.2.2 and $\langle g, g_1\rangle$ is cyclic, by the cyclic-transitivity. Therefore g and g_1 commute. Then G is abelian, a contradiction. Hence in $Z(G)$ there are no elements of order greater than p. Then, if we consider an element $y \in Z(G)$, we have $|y| \leq p$ and $\langle y\rangle \leq \langle x\rangle$, by Lemma 4.2.1. Thus we get $Z(G) = \langle x^p\rangle$ and $|Z(G)|$=p, as required. Moreover, by Lemma 4.2.1 $x^p \in \langle a\rangle$, for any $a \in G$ such that $|a| > p$. □

The following result gives a classification of cyclic-transitive nilpotent p-groups.

Theorem 4.3.5. *Let G be a hypercentral p-group. Then G is cyclic-transitive if and only if one of the following holds:*

(i) G is a group of exponent p;

(ii) G is locally cyclic;

(iii) $G{=}A \rtimes \langle y\rangle$, where A is either cyclic or the Prufer 2-group $\mathbb{Z}(2^\infty)$, $|y| = 2$ and y inverts any element of A.

Proof. Let G be a cyclic-transitive hypercentral p-group. Suppose that G is not a group of exponent p. First assume that $p = 2$.
In this case we shall prove that G is abelian, then locally cyclic. Assume not. Then, as a consequence of Lemma 4.3.4, there exists an element $x \in G\backslash Z(G)$ such that $|x|{=}p^2$ and $Z(G) = \langle x^p\rangle$ (in particular $|Z(G)| = p$). Now we prove that if $x \in Z_2(G)$, then $\langle a\rangle{\leq}\langle x\rangle$, for any element $a \in G$ such that $|a| = p$. In fact, certainly we can write

$$(ax)^p = a^p x^p \,[x, a]^{\frac{p(p-1)}{2}} = x^p \,[x, a]^{\frac{p(p-1)}{2}}$$

Since $x^p \in Z(G)$, then $|[x, a]| = p$.
Moreover, since $p = 2$, p divides $p(p - 1)/2$. We can deduce that $(ax)^p = x^p$ and, applying cyclic-transitivity, we get that $\langle ax, x\rangle = \langle x\rangle$ is cyclic, which implies $\langle a\rangle \leq \langle x\rangle$. Now we shall prove that there is an element in $Z_2(G) \setminus Z(G)$ of order p^2. Let us suppose that every element g of $Z_2(G)$ has order p. Then $(xg)^p = x^p$, which implies $\langle xg\rangle{=}\langle x\rangle$ and hence $g \in \langle x\rangle$; since g has order p, $g \in\langle x^p\rangle$; hence $g \in Z(G)$, a contradiction. The last remarks can be summarized in the following properties:

$$\exists x \in Z_2(G) \setminus Z(G) \;\; \textit{such that} \; |x| = p^2; \tag{4.3.1}$$

$$\langle g \rangle < \langle x \rangle, \forall g \in G \textit{ such that } |g| = p. \tag{4.3.2}$$

Therefore $Z(G)$ is the only subgroup of order p and G is locally cyclic, a contradiction.

Finally, suppose $p = 2$. If $exp(G) = 2$ and G is not abelian, then, by Lemma 4.3.4, there exists an element $x \in G \setminus Z(G)$ such that $|x| = 4$ and $\langle x^2 \rangle = Z(G)$ (in particular $|Z(G)| = 2$). Let A be a maximal abelian subgroup of G containing $\langle x \rangle$. Then A is locally cyclic; therefore either A is cyclic or $A \cong \mathbb{Z}(2^\infty)$. Let $y \in G \setminus A$. If $|y| > 2$, then from $\langle y, x^2 \rangle$ abelian we get $\langle y, x^2 \rangle$ cyclic, and $\langle y, a \rangle$ cyclic for every $a \in A$, since $\langle a, x^2 \rangle$ is cyclic. Then $y \in C_G(A) = A$, a contradiction. Therefore $|y| = 2$ and $|ay| = 2$, for any $a \in A$.

We deduce that $\langle A, y \rangle = A \rtimes \langle y \rangle$ and y inverts any element of A. Consider now an element $g \in G \setminus \langle A, y \rangle$. Then $|g| = 2$ and g inverts any element of A. Therefore $g^{-1}y \in A$ and $g \in \langle A, y \rangle$, a contradiction. We obtain that $G = \langle A, y \rangle = A \rtimes \langle y \rangle$, where y inverts any element of A. The converse is clear. □

Since all finite p-groups are hypercentral, as an immediate consequence, we get the following result.

Proposition 4.3.6. *Let G be a finite p-group. Then G is cyclic-transitive if and only if one of the following holds:*

(i) *G is a group of exponent p;*

(ii) *G is cyclic;*

(iii) *G is a dihedral 2-group.*

4.4 The supersolvable case

In the previous section we have proved that if G is a nilpotent cyclic-transitive group, then G is either periodic or torsion-free (see Proposition

4.3.1). But there exist supersolvable groups cyclic-transitive which are neither torsion-free nor periodic.
A first example of such groups is given by the infinite dihedral group D_∞. Notice that $D_\infty=\langle a\rangle\rtimes\langle b\rangle$, where b is an involution, a is torsion-free and b inverts any element of $\langle a\rangle$.

The first result of this section is the following technical lemma.

Lemma 4.4.1. *Let G be a cyclic-transitive group and let c, x be elements in G such that c is torsion-free, $|x| = 2$ and $\langle c\rangle\trianglelefteq\langle c, x\rangle$. Then $c^x=c$.*

Proof. Since the only automorphisms of an infinite cyclic group are the identity and the inversion, we have either $c^x = c$ or $c^x = c^{-1}$. If $c^x = c^{-1}$, then $(c^{-1})^x = c$ and

$$(cx)^2 = cxcx = cc^{-1}x^2 = x^2$$

which implies that $\langle cx, x^2\rangle$ is cyclic. Obviously $\langle x^2, x\rangle$ is cyclic; since G is cyclic-transitive, we get that $\langle cx, x\rangle=\langle c, x\rangle$ is cyclic; thus $\langle c, x\rangle$ is abelian and $c^x = c$. We can deduce that the only possibility is $c^x = c$, as required. □

We can generalize the construction of D_∞ to obtain other examples of supersolvable groups which are cyclic-transitive and neither torsion-free nor periodic. In fact we can prove the following result.

Proposition 4.4.2. *Let A be a finitely generated torsion-free abelian group and let G be the following group*

$$G = A \rtimes \langle x\rangle$$

where x is an involution and $a^x = a^{-1}$, for any a in A. Then G is cyclic-transitive.

Proof. As in the case of D_∞ we can prove that for any element g in $G \setminus A$, the order of g is equal to 2. In fact, g can be written as $g = ax$, for a suitable $a \in A$, so that

$$g^2 = axax = ax^{-1}ax = aa^x = aa^{-1} = 1,$$

which implies that $|g| = 2$. Now let us consider three elements in $G \setminus \{1\}$, say s, t, v, such that $\langle s, t\rangle$, $\langle t, v\rangle$ are both cyclic. If t is torsion-free, then $t \in A$, moreover s, v are torsion-free too, hence s and v are in A too. Since any torsion-free abelian group is cyclic-transitive (see 4.2.4), we get that $\langle s, v\rangle$ is cyclic. If t is periodic, then $t \notin A$; by the above remark, we get $|t| = 2$. We deduce that also $|s| = |v| = 2$; thus $\langle s\rangle = \langle t\rangle = \langle v\rangle$ and therefore $\langle s, v\rangle$ is cyclic. □

The following proposition inverts the previous result.

Proposition 4.4.3. *Let G be an infinite supersolvable group neither torsion-free nor periodic. Then G is cyclic-transitive if and only if $G = A \rtimes \langle x\rangle$, where A is a finitely generated torsion-free group, x is an involution and $a^x = a^{-1}$, for any a in A.*

Proof. Let suppose that G is a cyclic-transitive supersolvable group neither periodic nor torsion-free. Then, there exists in G an infinite cyclic normal subgroup, say C. Thus we have $C = \langle c\rangle \trianglelefteq G$, with $|c| = \infty$, for a suitable c in G. Write $A = C_G(\langle c\rangle)$ and consider an element $a \in A$. If a has finite order, then $\langle a, c\rangle = \langle a\rangle \times \langle c\rangle$, which implies that G is not cyclic-transitive. Then A is a torsion-free group.

Therefore $A < G$ and $|G : A| = 2$. Moreover if x is an element of G of finite order, then $G = A\langle x\rangle$, with $\langle x\rangle \cap A = \{1\}$ and $|x| = 2$. Furthermore, for every $a \in A$, we have, by Lemma 4.4.1, that ax has order 2. Therefore $(ax)^2 = 1$, which implies that $axax = 1$, so that $a^x = x^{-1}ax = a^{-1}$. We deduce that A is abelian and G has the required structure. The converse is Proposition 4.4.2. □

The next step will be considering the case of a torsion-free supersolvable group, that is cyclic-transitive. In this case the following lemmas will be crucial.

Lemma 4.4.4. *Let G be a torsion-free cyclic-transitive group. If $a \in G$ and $a^n \in Z(G)$, where n is a positive integer, then $a \in Z(G)$.*

Proof. First we show that if $a, b \in G$ are such that $|aZ(G)| = |bZ(G)| = p$, where p is a prime, then $[a, b] = 1$. In fact, if p is odd, then $\langle aZ(G), bZ(G)\rangle$ is finite, since in a supersoluble group the elements of odd order form a finite subgroup (see, for instance, [44], pag 151, 5.4.9). Thus $\langle a, b\rangle Z(G)$ is finite and $\langle a, b\rangle'$ is finite by a well-known theorem of Schur, thus $\langle a, b\rangle' = 1$, since G is torsion free.

Now assume $p = 2$, thus $\langle aZ(G), bZ(G)\rangle$ is a dihedral group. If it is not finite, then the element $abZ(G)$ is aperiodic and inverted by $aZ(G)$. Write $c = ab$, then we have $c^a = c^{-1}z$, where $z \in Z(G)$, thus $(c^2)^a = c^{-2}z^2$ and

$$(c^2a)^2 = c^2ac^2a = ac^{-2}z^2c^2a = a^2z^2 = (az)^2 = 1$$

since G is torsion free and $c^2Z(G) = aZ(G)$. Then $\langle c^2a, (az)^2\rangle$, $\langle (az)^2, az\rangle$ are both cyclic and therefore $\langle c^2a, az\rangle$ is cyclic, thus az centralizes c^2a and a centralizes c^2, a contradiction, since $c^2Z(G)$ is aperiodic and inverted by $aZ(G)$. Therefore $\langle aZ(G), bZ(G)\rangle$ is finite, and, arguing as before, we obtain that $[a, b] = 1$.

Now assume $a^n \in Z(G)$; we show that $a \in Z(G)$. Obviously we can suppose that $n = p$, where p is a prime. Then, for any $x \in G$, we have $a^p, (a^p)^x \in Z(G)$, then $[a, a^x] = 1$, hence $[a, x]$ commutes with a; then we have $1 = [a^p, x] = [a, x]^p$ and $[a, x] = 1$, as required. □

Lemma 4.4.5. *Let G be a torsion-free cyclic-transitive group and let A be a central infinite cyclic subgroup of G. Then $\frac{G}{A}$ is cyclic-transitive too.*

Proof. Let consider $xA, yA, zA \in G/A \setminus \{A\}$ such that $\langle xA, yA\rangle$ and $\langle yA, zA\rangle$ are both cyclic. We want to prove that $\langle xA, zA\rangle$ is cyclic too. If yA has finite order, then xA and zA are of finite order too. Suppose that $\frac{\langle y\rangle A}{A}$ has finite order. If we consider the torsion free rank of $\frac{\langle x,y\rangle A}{A}$, we get

$$r_0\left(\frac{\langle x, y\rangle A}{A}\right) = 0,$$

which implies that

$$r_0\left(\frac{\langle x, y\rangle}{\langle x, y\rangle \cap A}\right) = 0.$$

Furthermore the group $\langle x, y\rangle$ is abelian, since $\langle xA, yA\rangle$ is cyclic and $A \cap \langle x, y\rangle \subseteq Z(\langle x, y\rangle)$, and we have

$$r_0(\langle x, y\rangle) = r_0\left(\frac{\langle x, y\rangle}{\langle x, y\rangle \cap A}\right) + r_0(\langle x, y\rangle \cap A).$$

Since $r_0\left(\frac{\langle x,y\rangle}{\langle x,y\rangle\cap A}\right) = 0$, we deduce that $r_0(\langle x, y\rangle) \leq 1$ and therefore $\langle x, y\rangle$ is cyclic. The same argument shows that $\langle y, z\rangle$ is cyclic; applying the cyclic transitivity of G, we get that $\langle x, z\rangle$ is cyclic and , as a consequence, $\langle xA, zA\rangle$ is cyclic.

Suppose now that $\frac{\langle y\rangle A}{A}$ has infinite order. Then $r_0\left(\frac{\langle y\rangle A}{A}\right) = 1$ and $\langle y, A\rangle = \langle y\rangle \times A$. Notice that $\langle x, y\rangle$ and $\langle y, z\rangle$ are both abelian, since $\langle xA, yA\rangle$ and $\langle yA, zA\rangle$ are both cyclic. Then, there exist elements b and c such that

$$\frac{\langle x\rangle A}{A} = \frac{\langle b^i A\rangle}{A}, \ \frac{\langle y\rangle A}{A} = \frac{\langle b^j A\rangle}{A}, \ \frac{\langle y\rangle A}{A} = \frac{\langle c^h A\rangle}{A}, \ \frac{\langle z\rangle A}{A} = \frac{\langle c^k A\rangle}{A},$$

for suitable positive integers *i,j,h,k*.

Then $x^j, z^h \in \langle y\rangle \times A \subseteq Z(\langle x, y, z, A\rangle)$. Applying Lemma 4.4.4, we get $x, z \in Z(\langle x, y, z, A\rangle)$, thus $\langle x, y, z, A\rangle$ is abelian.

Moreover $r_0(\langle x, z\rangle A/A) = r_0(\langle y\rangle A/A) = 1$; consequently $\langle x, z\rangle A/A$ is cyclic and $\langle xA, zA\rangle$ is cyclic, as required.

□

The previous Lemma is of independent interest; in the first section we have pointed out that the class of cyclic-transitive groups is not closed with respect to forming factor groups; by Lemma 4.4.5, we can observe that this class is *partially* closed with respect to forming quotients.

Lemma 4.4.6. *Let G be a torsion-free supersolvable group and let $H=\langle h\rangle$ be an infinite cyclic normal subgroup of G. If G is cyclic-transitive, then the factor group G/H is either torsion-free or periodic.*

Proof. Evidently G/H is supersolvable. Moreover $H \subseteq Z(G)$ by Lemma 4.4.1. Suppose, by contradiction, that G/H is neither torsion-free nor periodic. Then, by Proposition 4.4.3, G/H would be of the form

$$\frac{G}{H} = \langle y_1 H, ..., y_t H\rangle \rtimes \langle xH\rangle,$$

where $\langle y_1 H, ..., y_t H\rangle$ is a finitely generated torsion free group, xH is an involution and inverts any element of $\langle y_1 H, ..., y_t H\rangle$.

Clearly, there exists in G/H an infinite cyclic normal subgroup, say $\langle aH\rangle$. We deduce that the semidirect product $\langle aH\rangle \rtimes \langle xH\rangle$, where xH inverts any element of $\langle aH\rangle$, is an isomorphic copy of D_∞. Clearly $|(ax)H| = 2$, which implies $(ax)^2 \in H = \langle h\rangle$ where $(ax)^2 = h^\alpha$, for a suitable positive integer α. Therefore $\langle (ax)^2, h\rangle$ is cyclic. Since $\langle ax, (ax)^2\rangle$ is clearly cyclic and G is cyclic-transitive, we conclude that $\langle ax, h\rangle$ is cyclic.

Moreover $|xH| = 2$, which implies $x^2 \in H$ and $x^2 = h^\beta$, for a suitable non-zero integer β; thus $\langle x^2, h\rangle$ is cyclic. By a similar argument as above, we obtain that $\langle x, h\rangle$ is cyclic. Summarizing, we have proved that $\langle x, h\rangle$ and $\langle ax, h\rangle$ are both cyclic and therefore $\langle ax, x\rangle$ is cyclic, which implies that ax and x commute, a contradiction. □

The previous two lemmas allow us to prove the following theorem on the structure of a torsion-free supersolvable group which is cyclic-transitive.

Theorem 4.4.7. *Let G be a torsion-free supersolvable group. If G is cyclic-transitive, then G is nilpotent.*

Proof. Since G is supersolvable and torsion-free, by a result of Zappa, (see, for instance, 5.4.8 in [44]), G has a normal series

$$1 = H_0 < H_1 < ... < H_h < H_{h+1} < ...H_n = G \qquad (4.4.1)$$

in which the first h factors are cyclic infinite and the others are cyclic of order 2. To show that G is nilpotent it suffices to prove that the infinite factors of the series above are central. We argue by induction on the Hirsch number h of G, i.e. the number of the infinite cyclic factors in (4.4.1).

If $h = 1$, then H_1 is infinite cyclic and G/H_1 is finite. Moreover $H_1 \leq Z(G)$ by Lemma 4.4.1. Then, from $G/Z(G)$ finite, we get G' finite by a well-known result of Schur and $G' = \{1\}$, since G is torsion-free. Now assume $h > 1$. Consider the group G/H_1. Then $H_1 \leq Z(G)$ by Lemma 4.4.1, and G/H_1 is cyclic-transitive, by Lemma 4.4.5. Moreover G/H_1 is not periodic, therefore G/H_1 is torsion-free, by Lemma 4.4.6. By induction we get the required conclusion. □

In the last part of the chapter we investigate the structure of finite cyclic-transitive supersolvable groups. Before proving a result on the structure of such groups, in the following proposition we give an example of cyclic-transitive Frobenius group.

Proposition 4.4.8. *Let G be a Frobenius group with a cyclic complement H and kernel K which is either cyclic or of exponent p, where p is a prime. Then G is cyclic-transitive.*

Proof. Assume that G is a Frobenius with the structure in our hypotheses. Every element of G is either in K or in H^g for some $g \in G$. Now, let $x, y, z \in G$, with $\langle x, y \rangle$ and $\langle y, z \rangle$ cyclic. If $y \in H^g$, then $x, z \in H^g$ since every element of H^g acts fixed point freely on K, thus $x, z \in H^g$ and $\langle x, z\rangle$ is cyclic. If $y \in K$, then $x, z \in K$ since every element of $G \setminus K$ acts fixed point freely on K, then again $\langle x, z\rangle$ is cyclic since K is cyclic-transitive. □

In the next result we give a futher example of a cyclic-transitive group related to the class of Frobenius groups.

Proposition 4.4.9. *Let consider the following group*

$$G = Z(G) \times A,$$

where $|Z(G)| = p$, p a prime, A is a Frobenius group with cyclic kernel K and a cyclic complement H, $|H| = p$. Then G is cyclic-transitive.

Proof. Suppose that G has the structure in the above hypotheses. First notice that every element of G either is in $Z(G)K$ or has order p. Now we show that if $x, y \in G \setminus \{1\}$ with $\langle x, y\rangle$ cyclic, then either $x, y \in Z(G)K$ or $|x| = |y| = p$. In fact, if, for example $|x| = p$ and $|y| = p$, then $(xy)^p = x^p = 1$, thus $x, xy \in Z(G)K$ and $y \in Z(G)K$, as required. Now, let $x, y, z \in G$ with $\langle x, y\rangle, \langle y, z\rangle$ cyclic, then $|x| = |y| = |z| = p$ or $x, y \in Z(G)K$ or $y, z \in Z(G)K$. In the first case $\langle x\rangle = \langle y\rangle = \langle z\rangle$ and $\langle x, z\rangle$ is cyclic, in the second case either $z \in Z(G)K$ and $\langle x, z\rangle$ is cyclic as $Z(G)K$ is cyclic-transitive or $|y| = |z| = p$ and $\langle y\rangle = \langle z\rangle$ is cyclic, and again $\langle x, z\rangle$ is cyclic. Similarly we can argue if $y, z \in Z(G)K$. □

Finally we prove the following characterization of finite cyclic-transitive supersolvable groups.

Theorem 4.4.10. *Let G be a finite supersolvable group. Then G is cyclic-transitive if and only if one of the following holds:*

(1) *G is a nilpotent cyclic-transitive group;*

(2) *G is a Frobenius group with cyclic complements and kernel which is either cyclic or of exponent p, where p is a prime;*

(3) *$G = Z(G) \times A$, where $|Z(G)| = p$, p a prime, A is a Frobenius group with cyclic kernel and cyclic complements of order p.*

Proof. Clearly any group G with the structure either (2) or (3) is cyclic-transitive, by Proposition 4.4.8 and Proposition 4.4.9. Conversely, let G be a finite supersolvable cyclic-transitive group. Write F the Fitting subgroup of G. Then F is nilpotent, hence, by Proposition 4.3.3, either F is cyclic or F is a p-group (with p prime). First assume that F is a p-group. Then p is odd and F is a Sylow p-subgroup of G, since G is supersolvable. Moreover either F is cyclic or F has exponent p, by Theorem 4.3.5. Furthermore, by Schur-Zassenhaus Theorem, $G = H \ltimes F$ and p does not divide $|H|$. We show that every non-identity element of H acts fixed-point-freely on F. Assume in fact that there exist an element $a \in F \setminus \{1\}$ of order p and an element $h \in H$ of order q (where p and p are prime and $p = q$) such that $a^h = a$. Then $\langle a, h \rangle$ is cyclic.

If $F = \langle c \rangle$ is cyclic, then, from $\langle c, a \rangle$ cyclic, it follows $\langle c, h \rangle$ cyclic and $h \in C_G(F) \leq F$, a contradiction. If F has exponent p and it is not cyclic, then we can choose $b \in F \setminus \langle a \rangle$ such that $[b, a]$= 1 (for , if $a \notin Z(F)$ we can take any $b \in Z(F) \setminus \{1\}$). If $b^h = b$, then $\langle b, h \rangle$ is cyclic; also $\langle a, h \rangle$ is cyclic, thus $\langle a, b \rangle$ is cyclic and $\langle a \rangle$=$\langle b \rangle$, a contradiction. Hence $b^h = b$. Then $\langle a, bh \rangle$ is abelian and it is not a p-group; since $|a| = p$, $(bh)^q \in F$, thus $\langle a, bh \rangle$ is cyclic. But $\langle a, h \rangle$ is cyclic, thus $\langle bh, h \rangle$=$\langle b, h \rangle$ is cyclic, a contradiction. Notice that G' is nilpotent, therefore $G' \leq F$ and H is abelian. Moreover, since H is a Frobenius complement, H has all Sylow subgroups cyclic. We get that H is cyclic and G is a Frobenius group with the required structure.

Now assume that F is not a p-group. Then F is cyclic, say $F = \langle c \rangle$. First we show that if $a \in F \setminus \{1\}$ is an element of prime order and $y \in G \setminus F$

is an element of prime-power order, say $|y| = q^\beta$ (q prime), and $[a, y] = 1$, then $|a| = q$. In fact, if $|a| = p$, with $p = q$, then $\langle a, y\rangle$ is cyclic; but $\langle a, c\rangle$ is also cyclic, thus $\langle c, y\rangle$ is cyclic and $y \in C_G(F) \leq F$, a contradiction. Then $|a| = q$. Assume that there exists $y \in G \setminus F$ of prime-power order and $a \in F \setminus \{1\}$ such that $[a, y] = 1$. Then $\langle a, y\rangle$ is either cyclic or a group of exponent prime. If we suppose that $\langle a, y\rangle$ is cyclic, arguing as in the first part of the proof, we reach a contradiction. Then $\langle a, y\rangle$ is a group of exponent prime, which implies $|a| = |y| = q$, a Sylow q-subgroup of G has exponent q, a Sylow q-subgroup of F has order q and $F = \langle a\rangle \times S$, where q does not divide $|S|$, $\langle a\rangle \trianglelefteq G$ and $S \trianglelefteq G$.

Every element $t \in G \setminus F$ acts fixed-point-freely on S, otherwise there exist $b \in S$, $t \in G \setminus F$ such that $[b, t] = 1$ and $|b| = p = |t|$, with p prime and $p = q$. If we assume, without loss of generality, $p < q$, then we have that $|G/C_G(\langle b\rangle)|$ divides $p - 1$ and $y \in C_G(\langle b\rangle)$, thus $p = q$, a contradiction. Then, for any $s \in S$ of prime order, we have $C_G(\langle s\rangle) \leq F$, $|S|$ odd and $G/C_G(\langle s\rangle)$ cyclic. Therefore G/F is cyclic, say $G/F = \langle gF\rangle$.

Write $G = F\langle g\rangle$. Clearly we can assume $\langle y\rangle \leq \langle g\rangle$ and $\langle g\rangle = \langle y\rangle \times V$, where $(|V|, q) = 1$. If $V = \{1\}$, let v be a non-identity element of $V$and write $|v| = n$. Then $[a, v] = 1$, since q does not divide n, and $\langle a, v\rangle = \langle v\rangle \ltimes \langle a\rangle$ is not abelian, in particular $|av| = |v| = n$. Moreover we have $\langle y, v\rangle$ cyclic, $\langle y, av\rangle$ abelian and then cyclic, since $|y| = q$, $|av| = n$ and q does not divide n; hence $\langle v, av\rangle = \langle a, v\rangle$ is cyclic, a contradiction. Consequently $V = \{1\}$, $G = F\langle y\rangle$, $\langle a\rangle \leq Z(G)$ and $G = \langle a\rangle \times S\langle y\rangle$ has the required structure. Therefore we can assume that every $y \in G \setminus F$ of prime power order acts fixed-point-freely on F. In particular $(|F|, |G/F|) = 1$ and $G = F \rtimes H$, for some H, by Schur-Zassenhaus Theorem. Every element of H acts fixed-point-freely on F, so G is a Frobenius group with the required structure. □

Chapter 5

Finite cyclic-transitive solvable groups

In this chapter we investigate the structure of finite solvable groups that are cyclic-transitive. The reader can find the results contained in this chapter in the paper *A condition in finite solvable groups related to cyclic subgroups* (see [25]). We first remark some useful properties of an arbitrary cyclic-transitive group.

5.1 Cyclic-transitive groups as partitioned groups

If G is a cyclic-transitive group, then it is easy to see that we get an equivalence relation on $G \setminus \{1\}$ by saying two elements are equivalent if they generate a cyclic subgroup. Our first result is a lemma on partitioned groups.

Lemma 5.1.1. *Let G be a group. Suppose that there exists a partition $\mathfrak{F}$ of G. Then G is cyclic-transitive if and only if every subgroup of $\mathfrak{F}$ is cyclic-transitive.*

Proof. Suppose that every subgroup of $\mathfrak{F}$ is cyclic-transitive. Consider three non-identity elements x, y, z of G such that $\langle x, y\rangle$ and $\langle y, z\rangle$ are both cyclic; we must show that $\langle x, z\rangle$ is cyclic. Say, for instance, $\langle x, y\rangle=\langle a\rangle$ and $\langle y, z\rangle=\langle b\rangle$. By hypothesis $\mathfrak{F}$ is a partition of G, hence there exist two subgroups H and K in $\mathfrak{F}$ such that $\langle a\rangle\leq H$ and $\langle b\rangle\leq K$. Then $y \in \langle a\rangle\cap\langle b\rangle \subseteq H \cap K$. Since $\mathfrak{F}$ is a partition, the only possibility is $H = K$ and therefore $\langle x, z\rangle$ is cyclic, as required. The converse is clear since the class of cyclic-transitive groups is S-closed. □

In our next result we prove that any cyclic-transitive group has always a partition.

Theorem 5.1.2. *Let G be a group. Then G is cyclic-transitive if and only if it has a partition of locally cyclic subgroups.*

Proof. Assume G is cyclic-transitive. Then we get an equivalence relation on $G\setminus\{1\}$ by saying a and b equivalent if $\langle a, b\rangle$ is cyclic. For every element $a \in G \setminus \{1\}$, let $[a]$ be the equivalence class of a.

We claim that $[a] \cup \{1\}$ forms a subgroup of G. Obviously, $a^{-1} \in [a]$ since $\langle a, a^{-1}\rangle = \langle a\rangle$. If a, b lie in $[a]$, then $\langle a, ab\rangle = \langle a, b\rangle$ is cyclic so either $ab = 1$ or $ab \in [a]$. In any case, $ab \in [a] \cup \{1\}$. Therefore $[a] \cup \{1\}$ is a subgroup of G. Now, either $[a] = [b]$ or $[a] \cap [b]$ is empty. Thus, G is partitioned by the subgroups given by these equivalence classes together with $\{1\}$.

Moreover $[a]\cup\{1\}$ is locally cyclic, for any $a \in G$. In fact, for every x, y in $[a]$, we get that $\langle x, a\rangle$ and $\langle y, a\rangle$ are both cyclic and therefore, since G is cyclic-transitive, $\langle x, y\rangle$ is cyclic. Conversely, if G has a partition consisting of locally cyclic subgroups, then is cyclic-transitive by Lemma 5.1.1. □

As an immediate application of Lemma 5.1.1, we have that every projective special linear group of type $PSL(2, q)$ is cyclic-transitive because, as we recalled in 1.3, it has a partition consisting of cyclic-transitive subgroups. If $G = PGL(2, p^n)$, with $p^n > 3$, then the set of maximal cyclic subgroups of G is a partition (see also 1.3). Consequently G is cyclic-transitive, by Lemma 5.1.1. Analogously the symmetric group on four letters S_4 is cyclic-transitive, since it is partitioned by all its maximal cyclic subgroups, as we recalled in 1.3.

In 1.2 we recalled some useful properties of Frobenius groups. In particular, we know that every Frobenius group is partitioned by an its complement and its kernel; so that, as immediate corollary of Lemma 5.1.1, we get the following result.

Proposition 5.1.3. *Let G be a Frobenius group with Fobenius complement M and Frobenius kernel N. If M and N are both cyclic-transitive, then G is cyclic-transitive.*

Moreover we have the following result on Frobenius groups that are cyclic-transitive.

Corollary 5.1.4. *Let G be a Frobenius group. If G is cyclic-transitive, then G is solvable.*

Proof. If G is a non-solvable Frobenius group, then a Frobenius complement of G has a subgroup isomorphic to $SL_2(5)$ (see, for instance, [21], pag. 605, 46.7). Now, $SL_2(5)$ has a unique involution. Since it has more than one Sylow 2-subgroup, it cannot be partitioned. Thus, $SL_2(5)$ is not cyclic transitive and G is not cyclic-transitive. □

In the first chapter (see Theorem 1.3.2) we recalled a result of Suzuki that determines the structure of non-solvable groups with a partition. If we notice that Suzuki groups are not cyclic-transitive, the above theorem implies immediately the following result.

Theorem 5.1.5. *Let G be a non-solvable finite group. Then G is cyclic transitive if and only if one of the following occurs:*

(1) *$G \cong PGL(2,q)$, q a prime power, $q \geq 4$;*

(2) *$G \cong PSL(2,q)$, q a prime power, $q \geq 4$.*

5.2 Some properties of finite cyclic-transitive $\{p,q\}$-groups

In this section we show some properties of finite $\{p,q\}$-groups, which belong to the class of cyclic-transitive groups. We recall that a finite group G is a $\{p,q\}$-group if its order is a $\{p,q\}$-number, i.e. $|G| = p^{\alpha}q^{\beta}$, α, β integers. Since in 4.3.6 we have given a complete characterization of finite cyclic-transitive p-groups, we assume that $|G| = p^{\alpha}q^{\beta}$, with α and β both greater than zero. To study the structure of such groups under the condition of cyclic-transitivity, we will argue on the center of G and we will distinguish the following two cases:

$$Z(G) = 1 \tag{5.2.1}$$

$$Z(G) = 1 \tag{5.2.2}$$

First we study the case $Z(G) = 1$. Without loss of generality, in this case we can assume that p divides $|Z(G)|$. The first result in this case is the following lemma.

Lemma 5.2.1. *Let G be a finite $\{p,q\}$-group such that p divides the order of the center $Z(G)$ of G. If G is cyclic-transitive and not cyclic, then the following statements hold:*

1. *G has a unique normal cyclic Sylow q-subgroup Q;*

2. *$C_Q(a)$=1, for every element a of p-power order such that $\langle x, a\rangle$ is not cyclic, where x is a central element of order p;*

3. *A Sylow p-subgroup P of G is not cyclic.*

Proof. By the hypothesis, we can choose an element x in the center $Z(G)$ of G such that $|x|$=p. Let y and z be two elements of G of q-power orders. Then $\langle x, y\rangle$ and $\langle x, z\rangle$ are cyclic because $[x, y] = 1 = [y, z]$ and $(|x|,|y|)$=1=$(|x|,|z|)$. Since G is cyclic-transitive, we get that $\langle y, z\rangle$ is cyclic. Furthermore, since y and z are arbitrary elements of G of q-power order, we obtain that G has a unique normal cyclic Sylow q-subgroup Q. Write $Q = \langle y\rangle$.

Suppose now that there exists in G an element a with p-power order such that $\langle x, a\rangle$ is not cyclic. We shall prove that $C_Q(a)$=1. In fact, if $C_Q(a)$= 1, since $Q = \langle y\rangle$, $y^\alpha \in C_Q(a)$, for a suitable integer α. Then $\langle y^\alpha, a\rangle$ and $\langle y^\alpha, x\rangle$ are both cyclic and therefore $\langle a, x\rangle$ is cyclic, that is a contradiction. Thus $C_Q(a)$=1. Now let P a Sylow p-subgroup of G. If P is cyclic, then $P = \langle a\rangle$, for suitable $a \in$G. Clearly $G = \langle a, y\rangle$. Now $x \in P$; therefore $\langle a, x\rangle(=\langle a\rangle)$ and $\langle x, y\rangle$ are both cyclic, which implies $\langle a, y\rangle$ is cyclic and thus G is cyclic, that is a contradiction.

□

Proposition 5.2.2. *Let G be a finite $\{p, q\}$-group such that p divides the order of the center $Z(G)$ of G. If G is cyclic-transitive, then one of the following occurs:*

1. *G is cyclic;*
2. *G can be written in the following form $G = \langle x\rangle \times (Q \rtimes \langle a\rangle)$, where Q is a cyclic Sylow q-subgroup of G, $|x| = |a| = p > 2$, $\langle x\rangle = Z(G)$ and a acts fixed-point-freely on Q;*
3. *G is dihedral.*

Proof. Suppose that G is not cyclic. By the hypothesis we can choose an element x in the center $Z(G)$ of G such that $|x|$=p. Applying Lemma

5.2.1, we get that G has a unique normal cyclic Sylow q-subgroup Q, say $Q = \langle y \rangle$, and $C_Q(a) = 1$, for every element a in G of p-power order such that $\langle x, a \rangle$ is not cyclic.

Now let P be a Sylow p-subgroup of G. By Lemma 5.2.1, P is not cyclic, thus P is either a group of exponent p or a dihedral 2-group. Then there exists an element $a \in P \setminus \langle x \rangle$ such that $\langle a, x \rangle$ is not cyclic, and, by above remark, $C_Q(a) = 1$, which implies that $Z(G) \leq P$. We deduce that $Z(G)=\langle x \rangle$ and $|Z(G)|= p$. By previous remark, $C_P(Q)=\langle x \rangle=Z(G)$ and therefore $C_G(Q)=Z(G)$Q. Moreover, since Q is a normal subgroup of G, $N_G(Q) = G$ and therefore

$$\frac{G}{C_G(Q)} \leq Aut(Q)$$

If q is odd, since Q is cyclic, then Aut(Q) is cyclic. If q is even, then p is odd and a Sylow p-subgroup of Aut(Q) is cyclic. Applying well-known results on isomorphisms of groups, we obtain that

$$\frac{G}{C_G(Q)} \cong \frac{P}{Z(G)}$$

is a cyclic group of order p. Therefore $G = Q \rtimes (\langle x \rangle \times \langle a \rangle)$. Since $Z(G)=\langle x \rangle$, we can also write

$$G = \langle x \rangle \times (Q \rtimes \langle a \rangle)$$

where $|a| = p$. Moreover $Q \rtimes \langle a \rangle$ is a Frobenius group, since $C_Q(a) = 1$, as we proved in the first part of the proof. Finally, if p is even we can easily deduce that G is dihedral. □

Our next purpose will be investigating the structure of a group of order $p^\alpha q^\beta$, which is cyclic-transitive and such that $|Z(G)| = 1$.
First recall that if G is a group of order $p^\alpha q^\beta$,then, by the so-called "Burnside's $p^\alpha q^\beta$-theorem", G is a solvable group, so that we can consider the Fitting subgroup $F = Fit(G)$ of G. Obviously we suppose that G is not

trivial, thus also F is not trivial. In the following two lemmas we prove that under our hypotheses the structure of Fitting subgroup is deeply conditioned.

Lemma 5.2.3. *Let G be a cyclic-transitive group such that $|G|=p^\alpha q^\beta$ and $Z(G) = \{1\}$. Write F=Fit(G). Assume that p divides |F|. Then F is a p-group either cyclic or of exponent p.*

Proof. Assume that F is not a p-group. Since F is nilpotent and cyclic transitive, F is cyclic, by Proposition 4.3.1. Now write F in the following way

$$F = A \times B$$

where $|A| = p^\gamma > 1$ and $|B| = q^\sigma > 1$. Write $r = min\{p, q\}$. Clearly there exists a subgroup S of F such that $|S| = r$. Since the Fitting subgroup of G is characteristic in G, we have that $S \trianglelefteq G$ and therefore $S \leq Z(G)$, a contradiction since $Z(G) = \{1\}$. Furthermore F is not a dihedral group, otherwise $Z(F) \trianglelefteq G$ and $|Z(F)| = 2$, which implies $Z(F) \leq Z(G)$, a contradiction. Thus by Proposition 4.3.6 F is a cyclic p-group or a group of exponent p. □

The following theorem determines the structure of the group in the case that the Fitting subgroup is a cyclic p-group.

Theorem 5.2.4. *Let G be a cyclic-transitive group such that $|G|=p^\alpha q^\beta$ and $Z(G) = \{1\}$. Let Fit(G) the Fitting subgroup of G and write F=Fit(G). Suppose that F is cyclic and p divides $|F|$. Then one of the following occurs:*

(1) *G is cyclic;*

(2) *F is a Sylow p-subgroup of G and G is a Frobenius group with Frobenius kernel F;*

(3) *G is dihedral.*

Proof. We can assume that F is a cyclic p-group and $p > 2$. If $|F| = p$, then G is either cyclic or a Frobenius group with F as its Frobenius kernel. In fact, if G is not cyclic, we can write $G = H \ltimes F$, where H is a q-subgroup of G. Suppose that there exists a non-identity element x in F and an element y in H such that $x^y = x$; then $\langle x, y\rangle$ is cyclic. Since F is cyclic, say $F = \langle a\rangle$, applying cyclic-transitivity, we obtain that $\langle a, y\rangle$ is cyclic, which implies $y \in C_G(F)$ and therefore $y \in F$, a contradiction. We deduce that G is a group of type (2).

Now assume that p^2 divides $|F|$. Let P be a Sylow p-subgroup of G. Since F is cyclic and p^2 divides $|F|$, P has a cyclic subgroup of order divisible by p^2. This implies that P does not have exponent p, so that P is cyclic since p is odd. Take x such that $P = \langle x\rangle$. Notice

$$P \leq C_G(F) \leq F$$

which implies $F = P = \langle x\rangle$. Since F is cyclic, we know that $F \leq C_G(x)$. Remark that, under our hypotheses, $C_G(x)$ is a p-group, thus $C_G(x) = P$. Therefore $C_G(x) = C_G(F) = F$. Now we know that F is a p-group and $F = \langle x\rangle$, with $|x| \geq p^2$. Consider a non-identity element y in F such that q divides $|C_G(y)|$. Then there exists an element c in $C_G(y)$ such that $|c| = q$. We obtain that $\langle x, y\rangle$ (that is equal to $\langle x\rangle$) and $\langle y, c\rangle$ are both cyclic and therefore, by cyclic-transitivity, $\langle x, c\rangle$ is cyclic, which implies that $c \in C_G(x)$, a contradiction ($C_G(x)$ is a p-group). We deduce that $C_G(y)$ is a p-group, for every non-identity element y in F. As a consequence, we obtain that G is a Frobenius group, whose Frobenius kernel is $P = F$ and a Frobenius complement is $Q \in Syl_q(G)$. This completes the proof when F is cyclic. □

Now we assume that F is not cyclic, and so it must have exponent p. In this case we have the following result.

Proposition 5.2.5. *Let G be a cyclic-transitive group and assume $|G| = p^\alpha q^\beta$, where p and q are primes, $\alpha > 0, \beta > 0$. Assume that $F = Fit(G)$ is a non-cyclic group of exponent p. Then one of the following holds:*

(i) $G \simeq S_4$;

(ii) *F is a Sylow p-subgroup of G, and G is a Frobenius group with the kernel F and a cyclic complement Q of order q^β.*

Proof. Let P be a Sylow p-subgroup of G and Q a Sylow q subgroup of G. Then $F \subseteq P$, therefore P cannot be cyclic, thus either P has exponent p, or P is dihedral and F is elementary abelian of order 4. In this latter case G/F is a subgroup of S_3 and G is either S_4 or A_4, then either (i) or (ii) holds. Then we can assume that P has exponent p. First we show that $C_G(x)$ has a non-cyclic Fitting subgroup, for any $x \in F \setminus \{1\}$. In fact, if $x \in Z(F)$, then $F \subseteq C_G(x)$, then F is contained in the Fitting subgroup of $C_G(x)$ and we have the result. Now suppose $x \notin Z(F)$. Then $\langle x\rangle \leq Fit(C_G(x))$. Since $x \in F$, we have $Z(F) \leq C_G(x)$. Moreover, since $Z(F)$ is normal and nilpotent in G, we get $Z(F) \leq Fit(C_G(x))$; thus $Z(F)\langle x\rangle \subseteq Fit(C_G(x))$, which implies that $Fit(C_G(x))$ is not cyclic. This completes the proof of the claim.

By Proposition 5.2.2, $C_G(x)$ must be a p-group for all $x \in F\setminus\{1\}$. Since F is the normal Sylow p-subgroup of FQ, this implies that $C_{FQ}(x) \subseteq F$ for all $x \in F \setminus \{1\}$. Therefore FQ is a Frobenius group with kernel F. In particular Q is a cyclic group. Now we show that FQ is normal in G. In fact, if $F_1/F = Fit(G/F)$, then we have $F_1/F = \{1\}$, and obviously p does not divide $|F_1/F|$, otherwise there exists a non-trivial p-subgroup S/F normal in G/F, and S would be a normal p-subgroup of G, which is impossible since $F \subset S$.

Thus F_1/F is a q-group, and $QF/F \subseteq C_{G/F}(F_1/F) \subseteq F_1/F$. Therefore $F_1 = FQ$ and FQ is normal in G. Obviously G/FQ is a p-group. We show that $|G/FQ|$ divides p. We have G/FQ cyclic, since $G/FQ \simeq (G/F)/C_{G/F}(FQ/F)$ is a subgroup of $Aut(FQ/F)$ which is cyclic. Then $|G/FQ|$ divides p since P has exponent p.

Now we show that $G = FQ$. Assume not. Then there exists $a \in G \backslash FQ$ such that $|a| = p$, since P has exponent p. If there is an element $y \in F \backslash FQ$ of order pq, then, if Q_1 is a Sylow q-subgroup of G containing y^p, we have $\langle Q_1, y^p \rangle$ and $\langle y^p, y \rangle$ cyclic, hence $\langle Q_1, y \rangle$ is cyclic and $y \in C_G(Q_1) \subseteq FQ$.

Therefore every element of $G \setminus FQ$ has order p. Then $1 = (ca^{-1})^p = cc^a c^{a^2} \cdots c^{a^{p-1}}$, for any $c \in FQ$, thus a acts regularly on FQ and FQ is nilpotent by a famous theorem of Hughes, Kegel, Thompson (see [22], pag 502, Hauptsatz 8.13). Then $FQ = F$, a contradiction. Therefore $G = FQ$, $F = P$ and (ii) holds. □

The previous results can be summarized in the following theorem.

Theorem 5.2.6. *Let G be a finite group and assume $|G| = p^\alpha q^\beta$, where p and q are primes, $\alpha > 0, \beta > 0$. Then G is a cyclic-transitive group if and only if one of the following holds:*

(i) *G is a cyclic group;*

(ii) *G is a dihedral group;*

(iii) *$G \simeq S_4$;*

(iv) *G is a Frobenius group with cyclic complements and kernel which is either cyclic or of prime exponent;*

(v) *$G = \langle a \rangle \times H$, where $\langle a \rangle$ has prime order $r > 2$ and H is a Frobenius group with complements of order r and cyclic kernel.*

Proof. If G is cyclic-transitive and $Z(G) = \{1\}$, then, by Proposition 5.2.2, G satisfies (i) or (ii) or (iv). If G is cyclic-transitive and $Z(G) = \{1\}$, then $F = Fit(G)$ is either cyclic or is a primary group of prime exponent. In the first case, G is either dihedral or has the structure (*iii*) by Proposition 5.2.4, in the latter case G has the structure in (*iii*) by Proposition 5.2.5. Conversely, assume that (*i*) or (*ii*) or (*iii*) or (*iv*) or (*v*) holds. If G is either cyclic or dihedral, then we know that G is cyclic-transitive. If $G \simeq S_4$, then G has a partition consisting of cyclic subgroups and G is cyclic-transitive, by Theorem 5.1.2.

If (*iv*) holds, then the kernel has a partition consisting of cyclic subgroups which, together with all the complements, form a partition of G of cyclic subgroups. Therefore G is cyclic-transitive, by Theorem 5.1.2. If (*v*) holds, then G has a non-trivial normal cyclic subgroup N of index r such that every element of $G \setminus N$ has order r, then (see Examples 1.3.1) again G has a partition of cyclic subgroups and G is cyclic-transitive, by Theorem 5.1.2. □

5.3 Finite solvable cyclic-transitive groups

In this section we use the results of the previous section to investigate the structure of the finite solvable groups, which are cyclic-transitive. We prove the following result:

Theorem 5.3.1. *Let G be a finite soluble non-primary group. Then G is a cyclic-transitive group if and only if one of the following holds:*

(i) *G is a cyclic group;*

(ii) *G is a dihedral group;*

(iii) *$G \simeq S_4$;*

(iv) *G is a Frobenius group, whose complements are cyclic and kernel is either cyclic or of prime exponent;*

(v) *$G = \langle a \rangle \times H$ where $\langle a \rangle$ has prime order $p = 2$ and H is a Frobenius group with cyclic kernel and complements of order p.*

In order to prove theorem 5.3.1 we show the following proposition.

Proposition 5.3.2. *Let G be a finite solvable group, let $p_1, ..., p_n$ denote the distinct prime divisors of $|G|$ and let $\{P_1, ..., P_n\}$ be a corresponding Sylow basis. If G is cyclic-transitive, then every Sylow subgroup of $\{P_1, ..., P_n\}$ is cyclic except at most one.*

Proof. Suppose $|G| = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$. Suppose that P_1 is not cyclic. We want to prove that P_i is cyclic, for every $i \in \{2, ..., n\}$. Consider an arbitrary i in $\{2, ..., n\}$ and the corresponding subgroup P_i in the Sylow basis. Since P_1 and P_i commute, then $P_1 P_i$ is a cyclic-transitive $\{p_1, p_i\}$-group, with $p_i = p_1$. Then the result follows immediately from 5.2.6 $\square$

Now we can prove theorem 5.3.1

Proof. (*of Theorem 5.3.1)* Assume G cyclic-transitive. Then G has a partition consisting of cyclic groups by Theorem 5.1.2. We can apply a famous result of Baer, recalled in 1.3.1, and therefore $G \simeq S_4$, or G is a Frobenius group, or there exists a normal nilpotent subgroup N of G with G/N of prime order. In this latter case, G is supersoluble and G has the required structure by Theorem 4.4.10. Thus we can assume that G is a Frobenius group. Then $F = Fit(G)$ is the kernel of G and G has a complement H, whose all Sylow subgroups are cyclic. If every Sylow subgroup of G is cyclic, then G is supersoluble (see [44] , pag. 290, 10.1.10), and either G is cyclic or it is a Frobenius group with cyclic complements and cyclic kernel, by Theorem 4.4.10.

Hence we can assume that there exists a Sylow subgroup P_1 which is not cyclic. Then $P_1 \subseteq F$ and $F = P_1$, since a finite nilpotent cyclic-transitive group is either a primary group or is cyclic. Notice that F is not a dihedral 2-group, otherwise $Z(F) = 2$, and $Z(F) \subseteq Z(G)$, a contradicton, since G is a Frobenius group. Therefore F is of prime exponent. Moreover H is either cyclic or a Frobenius group. Since H is a Frobenius complement of G, $Z(H) = 1$ (see [22], pag. 506, Satz 8.18 (c)) and therefore H cannot be a Frobenius group. Consequently H is cyclic, as required.

$\square$

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