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Homeomorphisms with finite distortion and applications to Γ-convergence

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Introduction

During the last 10 years there has been an intensive study of mappings with finite distortion, since they naturally arise in the theory of the non-uniformly elliptic equations and in the elasticity theory. We refer the reader for instance to [IM2], [FKZ], [IKM] or [IM1] and the references therein, for the basic literature on the subject.

We will be mainly concerned with homeomorphisms with finite distortion. Let Ω be a planar domain, recall that a homeomorphism $f \in W^{1,1}_{\rm loc}(\Omega, \mathbb{R}^2)$ has *finite distortion* if there is a measurable function $K(z) \geq 1$, finite almost everywhere, such that

$$|Df(z)|^2 \leq K(z) J_f(z) \ \text{ for a.e. } z \in \Omega.$$

Such K is called *distortion* of f. The smallest such distortions is denoted by K_f and is called the *distortion function* of f.

Very recently there has been also a growing interest in studying properties of homeomorphisms, which can be proved also for the inverse maps (see [MPS], [HMPS], [HKO2], [HKM], [HK], [GST], [HKO1]). A first result in this direction is contained in the paper by Hencl-Koskela and states that if Ω and Ω' are planar domains and if $f : \Omega \xrightarrow{\text{onto}} \Omega'$ is a homeomorphism belonging to Sobolev space $W^{1,1}_{\rm loc}(\Omega, \mathbb{R}^2)$ and the differential Df vanishes almost everywhere on the zero set of Jacobian J_f of f, then also $f^{-1} \in W^{1,1}_{\rm loc}(\Omega', \mathbb{R}^2)$ and the differential Df^{-1} vanishes almost everywhere on the zero set of Jacobian $J_{f^{-1}}$ of f^{-1} (see [HK]).

Moreover, if f is K-*quasiconformal* i.e. $K_f \in L^\infty(\Omega)$ and $K_f(z) \leq K$ for a.e. $z \in \Omega$, then also f^{-1} is K-quasiconformal i.e. $K_{f^{-1}} \in L^\infty(\Omega')$ and $K_{f^{-1}}(w) \leq K$ for a.e. $w \in \Omega'$ (see [AIM], Theorem 3.1.2).

A part of the present thesis is devoted to the study of the integrability of distortion function $K_{f^{-1}}$ of the inverse mapping f^{-1} under more general assumptions.

Indeed, denoting by $\mathrm{Hom}(\Omega, \Omega')$ the set of all homeomorphisms between Ω and Ω' planar domains, we prove that if $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2) \cap \mathrm{Hom}(\Omega, \Omega')$ has finite distortion with distortion function K_f satisfying the condition

$$\mathrm{dist}_{EXP}(K_f, L^\infty) < 1,$$

then

$$K_{f^{-1}} \in L^1_{\mathrm{loc}}(\Omega').$$

Moreover, we show that this result is optimal in sense that the conclusion fails if

$$\mathrm{dist}_{EXP}(K_f, L^\infty) = 1.$$

In fact, we exhibit an example of homeomorphism $f \in W^{1,1}_{\mathrm{loc}}$ with finite distortion such that

$$\mathrm{dist}_{EXP}(K_f, L^\infty) = 1,$$

while

$$K_{f^{-1}} \notin L^1_{\mathrm{loc}}.$$

Moreover, we prove that if K_f satisfies the condition

$$\mathrm{dist}_{EXP}(K_f, L^\infty) = \lambda \ \text{ for some } \ \lambda > 0,$$

then

$$K_{f^{-1}} \in L^p_{\mathrm{loc}}(\Omega') \ \text{ for every } \ p \in \left(0, \frac{1}{2\lambda}\right).$$

As special case of this result we show that if K_f satisfies the condition

$$\mathrm{dist}_{EXP}(K_f, L^\infty) = 0,$$

then

$$K_{f^{-1}} \in \bigcap_{p \geq 1} L^p_{\mathrm{loc}}(\Omega').$$

The definition of $\mathrm{dist}_{EXP}(\varphi, L^\infty)$ is given in Chapter 1 (see Section 1.4) and we will prove such results in Chapter 2 (see Section 2.2).

The previous results are contained in [C2].

In Chapter 3 we are concerned with weak continuity results for Jacobians. The utility of weak convergence of Jacobians was clearly recognized in quasi-conformal geometry [IM1], calculus of variations [Mo2] and elasticity theory.

Our main result in this setting states that if $f_k, f \in W^{1,2}(\Omega, \mathbb{R}^2)$, where Ω is a bounded open subset of $\mathbb{R}^2$ sufficiently smooth, satisfy the following asymmetric assumption on the components

$$f_k = (u_k, v_k) \rightharpoonup f = (u, v) \text{ weakly in } W^{1,L\log^{1/2}L}(\Omega) \times W^{1,2}(\Omega),$$

then

$$J_{f_k} \overset{*}{\rightharpoonup} J_f \text{ in the sense of measures.}$$

(see [AC]).

This is a generalization of the well know result due to Morrey [Mo1], [Mo2] and Caccioppoli [C] that tell us that if $f_k, f \in W^{1,2}(\Omega, \mathbb{R}^2)$ then

$$f_k \rightharpoonup f \text{ weakly in } W^{1,2}(\Omega, \mathbb{R}^2)$$

implies

$$J_{f_k} \overset{*}{\rightharpoonup} J_f \text{ in the sense of measures.}$$

In Chapter 4 we confine ourselves to dimension one and we extend to the weaker topology $\sigma(L^1, L^\infty)$ a classical result of G-convergence relative to the $\sigma(L^\infty, L^1)$ topology (see [C1]). More precisely, we prove that if $a_j = a_j(x)$ ($j = 1, 2, \ldots$) and $a = a(x)$ are non-negative functions belonging to Lebesgue space $L^1(0,1)$, $p > 1$, $a_j^{-1/(p-1)}$ is a bounded sequence in $L^1(0,1)$ and $a_j^{-1/(p-1)}$ is equi-integrable, then the sequence of non-linear degenerate non-uniformly elliptic operators of the type

$$\mathcal{A}_j = -\frac{d}{dx}\left(a_j(x)\left|\frac{d}{dx}\right|^{p-2}\frac{d}{dx}\right)$$

G-converges to the operator

$$\mathcal{A} = -\frac{d}{dx}\left(a(x)\left|\frac{d}{dx}\right|^{p-2}\frac{d}{dx}\right)$$

if and only if

$$\frac{1}{a_j^{1/(p-1)}} \rightharpoonup \frac{1}{a^{1/(p-1)}} \text{ weakly in } L^1(0,1).$$

The definition of G-convergence is given in Section 4.1 and we will prove such result in Section 4.3.

Finally, in the last chapter (Chapter 5) we are concerned with a suitable continuity property of the map

$$f \to A_f$$

when f varies in the class of homeomorphisms having exponentially integrable distortion and A_f is the coefficient matrix of the Laplace-Beltrami operator associated to f. It is known that A_f satisfies the ellipticity condition

$$\frac{|\xi|^2}{K(z)} \leq \langle A_f(z)\xi, \xi\rangle \leq K(z)|\xi|^2 \tag{1}$$

for a.e. $z \in \Omega$ and for any $\xi \in \mathbb{R}^2$, where K is the distortion of f, and moreover

$$\det A_f(z) = 1 \text{ for a.e. } z \in \Omega.$$

Precisely in collaboration with M. Carozza (see [CC]) we have proved that if Ω and Ω' are bounded planar domains, with Ω sufficiently smooth, if $f_j \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ is a sequence of homeomorphisms with finite distortion K_j such that

$$\int_\Omega e^{\frac{K_j(z)}{\lambda}}\, dz \leq c_0 \text{ for every } j \in \mathbb{N},$$

for some $\lambda \in (0, 1/2)$ and $c_0 > 0$ and if

$$f_j \rightharpoonup f \text{ weakly in } W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$$

where $f \in \mathrm{Hom}(\Omega, \Omega')$, then f has finite distortion, its distortion function K_f satisfies the same condition

$$\int_\Omega e^{\frac{K_f(z)}{\lambda}}\, dz \leq c_0$$

and

$$A_{f_j} \xrightarrow{\Gamma_{L^2 \log L}} A_f.$$

The definition of $\Gamma_{L^2 \log L}$-convergence is given in Section 5.2.

S. Spagnolo in [Sp2] was the first to prove the result above under the stronger assumption

$$1 \leq K_j(z) \leq K \text{ for every } j \in \mathbb{N}$$

for a.e. $z \in \Omega$, in which A_{f_j} are bounded and uniformly elliptic and Γ-convergence and G-convergence, in the sense of L^2-convergence of solutions of Dirichlet problems, are equivalent. Later in [Fo] it was proved an analogous result with higher degree of exponential integrability assumption for K_j. Namely in [Fo] the author uses a method introduced for $n > 2$ by [DD] in the case n=2 under the assumption

$$\int_\Omega e^{\left(\frac{K_j(z)}{\lambda}\right)^\alpha} dz \leq c_0 \text{ for every } j \in \mathbb{N},$$

for some $\alpha > 1$, $\lambda > 0$ and $c_0 > 0$. Here we keep on the same issue by using recent optimal regularity results for mappings having exponentially integrable distortion given in [IKMS] and [AGRS].

Chapter 1

Functional spaces

In this chapter we introduce some functional spaces which occur in recent developments of the regularity theory for PDE's or to study subtle integrability properties of Jacobians.

First of all we give a self-contained presentation of Orlicz spaces. Next we list some special case of Orlicz spaces like Zygmund spaces and the spaces of exponentially integrable functions.

The Zygmund spaces naturally arise in the study of the regularity of Jacobians of orientation preserving mappings. In fact the mapping $f \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R})$ orientation preserving, i.e. $J_f(z) \geq 0$ for a.e. $z \in \Omega$, has Jacobian J_f not only belongs to $L^1_{\mathrm{loc}}(\Omega)$, as it is obvious from Hadamard's inequality $J_f(z) \leq |Df(z)|^2$, but actually J_f belongs to Zygmund space $L \log L_{\mathrm{loc}}(\Omega)$. This is a surprising result due to S. Müller in '89 (see [Mü2]) which arouse new interest in the field of the regularity of Jacobians (see [AGRS], [FKZ], [Mos]).

1.1 Orlicz spaces

An *Orlicz function* is a continuously increasing function

$$\mathcal{P} : [0, \infty) \to [0, \infty)$$

verifying

$$\mathcal{P}(0) = 0 \quad \text{and} \quad \lim_{t \to \infty} \mathcal{P}(t) = \infty.$$

A convex Orlicz function $\mathcal{P}$ is called *Young function.* The Orlicz space, denoted by $L^{\mathcal{P}}(\Omega)$, consists of those Lebesgue measurable functions φ defined in $\Omega \subset \mathbb{R}^2$

and valued in $\mathbb{R}$ such that

$$\int_\Omega \mathcal{P}\left(\frac{|\varphi(z)|}{\lambda}\right) dz < \infty$$

for some $\lambda = \lambda(\varphi) > 0$. $L^{\mathcal{P}}(\Omega)$ is a complete linear metric space with respect to the distance defined by

$$\mathrm{dist}_{\mathcal{P}}(\varphi, \psi) = \inf\left\{\lambda > 0 : \int_\Omega \mathcal{P}\left(\frac{|\varphi(z) - \psi(z)|}{\lambda}\right) dz \leq \lambda\right\}.$$

We shall also make use of the non-linear functional on $L^{\mathcal{P}}(\Omega)$, called the *Luxemburg functional*,

$$||\varphi||_{L^{\mathcal{P}}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \mathcal{P}\left(\frac{|\varphi(z)|}{\lambda}\right) dz \leq 1\right\}.$$

It is homogeneous, but in general fails to satisfy the triangle inequality. If $\mathcal{P}$ is a Young function, then the functional $||\cdot||_{L^{\mathcal{P}}(\Omega)}$ is a norm and $L^{\mathcal{P}}(\Omega)$ endowed with this norm is a Banach space.

One can easily check that

$$\int_\Omega \mathcal{P}\left(\frac{|\varphi(z)|}{||\varphi||_{L^{\mathcal{P}}(\Omega)}}\right) = 1.$$

As a first example, if we put $\mathcal{P}(t) = t^p$, with $p \in (0, \infty)$, then the space $L^{\mathcal{P}}(\Omega)$ coincides with the usual Lebesgue space $L^p(\Omega)$. Note that $L^p(\Omega)$ is a Banach space only when $p \geq 1$.

A pair of Orlicz functions $(\mathcal{P}, \mathcal{Q})$ are called a *Hölder conjugate couple* if we have Hölder's inequality

$$\left|\int_\Omega \langle\varphi, \psi\rangle\right| \leq C||\varphi||_{L^{\mathcal{P}}(\Omega)}||\psi||_{L^{\mathcal{Q}}(\Omega)}$$

for $\varphi \in L^{\mathcal{P}}(\Omega)$ and $\psi \in L^{\mathcal{Q}}(\Omega)$.

To define the dual space, we must assume a *doubling condition* on $\mathcal{P}$:

$$\mathcal{P}(2t) \leq 2^\alpha \mathcal{P}(t)$$

for some constant $\alpha \geq 1$ and all $t > 0$. In this case we have the following

Theorem 1.1. ***(Riesz representation)*** *Let $(\mathcal{P}, \mathcal{Q})$ be a Hölder conjugate couple of Young functions with $\mathcal{P}$ satisfying a doubling condition. Then every bounded linear functional defined on $L^{\mathcal{P}}(\Omega)$ is uniquely represented by a function $\psi \in L^{\mathcal{Q}}(\Omega)$ as*

$$\varphi \to \int_\Omega \langle\varphi, \psi\rangle.$$

For a general Hölder conjugate couple $(\mathcal{P}, \mathcal{Q})$, if both $\mathcal{P}$ and $\mathcal{Q}$ satisfy a doubling condition, then $L^{\mathcal{P}}(\Omega)$ and $L^{\mathcal{Q}}(\Omega)$ are duals of each other and both are reflexive Banach spaces.

The relevance of the doubling condition on Orlicz functions is well understood with the following theorem.

Theorem 1.2. *Let $\mathcal{P}$ be an Orlicz function (not necessarily convex) satisfying a doubling condition. Then the space $C_0^\infty(\Omega)$ is dense in the metric space $L^{\mathcal{P}}(\Omega)$.*

Without the doubling condition $L^\infty(\Omega)$ need not be dense in $L^{\mathcal{P}}(\Omega)$. Of course, if $L^\infty(\Omega)$ is dense in $L^{\mathcal{P}}(\Omega)$, then so is $C_0^\infty(\Omega)$.

Having introduced Orlicz spaces, we now turn to Orlicz-Sobolev spaces. Given an Orlicz function $\mathcal{P}$, the space $W^{1,\mathcal{P}}(\Omega)$ can be defined in much the same way as in the classical case $\mathcal{P}(t) = t^p$. In order to speak of the distributional derivatives it is necessary that functions in $L^{\mathcal{P}}(\Omega)$ are at least locally integrable. This forces upon us the assumption that for all sufficiently large t,

$$\mathcal{P}(t) \geq \alpha t \text{ for some } \alpha > 0.$$

Under this assumption we make the following definition.

Definition 1.1. A distribution $\varphi \in \mathcal{D}'(\Omega)$ belongs to Orlicz-Sobolev space $W^{1,\mathcal{P}}(\Omega)$ if $\varphi \in L^{\mathcal{P}}(\Omega)$ and $\partial\varphi/\partial x$, $\partial\varphi/\partial y$ exist in the weak sense and belong to $L^{\mathcal{P}}(\Omega)$.

It is evident that many of the basic notions and results in the theory of Sobolev spaces carry over to this more general setting without any difficulty.

Finally the corresponding local space $W^{1,\mathcal{P}}_{\text{loc}}(\Omega)$ is defined as the space of functions φ such that

$$\varphi \in W^{1,\mathcal{P}}(S) \text{ for any } S \subset\subset \Omega,$$

where we write

$$S \subset\subset \Omega$$

if S is an open subset of Ω and $S \subset \overline{S} \subset \Omega$ and $\overline{S}$ is compact.

1.2 Zygmund spaces

The *Zygmund space*, denoted by $L^p \log^\beta L(\Omega)$, is the Orlicz space generated by to the Orlicz function

$$\mathcal{P}(t) = t^p \log^\beta (e+t)$$

with $p \in [1,\infty)$ and $\beta \in \mathbb{R}$. Hence the Zygmund space $L^p \log^\beta L(\Omega)$ consists of all measurable functions $\varphi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ such that

$$\int_\Omega \left(\frac{|\varphi(z)|}{\lambda}\right)^p \log^\beta \left(e + \frac{|\varphi(z)|}{\lambda}\right) dz < \infty$$

for some $\lambda = \lambda(\varphi) > 0$ and it is equipped with the Luxemburg functional

$$(1.1)\quad \|\varphi\|_{L^p \log^\beta L(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \left(\frac{|\varphi(z)|}{\lambda}\right)^p \log^\beta \left(e + \frac{|\varphi(z)|}{\lambda}\right) dz \leq 1 \right\}.$$

For $p = 1$ we will write $L \log^\beta L(\Omega)$ instead of $L^1 \log^\beta L(\Omega)$.

Observe that if $\beta \geq 1 - p$, then the defining function $\mathcal{P}(t) = t^p \log^\beta (e+t)$ is a Young function. Therefore the functional (1.1) is a norm and $L^p \log^\beta L(\Omega)$ endowed with this norm becomes a Banach space.

For the reader's convenience let us give the proof of the following estimates

$$(1.2)\qquad \|\varphi\|_{L \log L(\Omega)} \leq [\varphi]_{L \log L(\Omega)} \leq 2\|\varphi\|_{L \log L(\Omega)}$$

where

$$[\varphi]_{L \log L(\Omega)} = \int_\Omega |\varphi(z)| \log \left(e + \frac{|\varphi(z)|}{\|\varphi\|_{L^1(\Omega)}}\right) dz.$$

Proof of the estimates (1.2)**.** First we observe that the equality

$$\int_\Omega \frac{|\varphi(z)|}{\|\varphi\|_{L \log L(\Omega)}} \log \left(e + \frac{|\varphi(z)|}{\|\varphi\|_{L \log L(\Omega)}}\right) dz = 1$$

implies

$$\begin{aligned} \|\varphi\|_{L \log L(\Omega)} &= \int_\Omega |\varphi(z)| \log \left(e + \frac{|\varphi(z)|}{\|\varphi\|_{L \log L(\Omega)}}\right) dz \\ &\geq \int_\Omega |\varphi(z)|\, dz = \|\varphi\|_{L^1(\Omega)} \end{aligned}$$

and therefore

$$\|\varphi\|_{L \log L(\Omega)} = \int_\Omega |\varphi(z)| \log \left(e + \frac{|\varphi(z)|}{\|\varphi\|_{L \log L(\Omega)}}\right) dz$$

$$\leq \int_\Omega |\varphi(z)| \log\left(e + \frac{|\varphi(z)|}{\|\varphi\|_{L^1(\Omega)}}\right) dz = [\varphi]_{L\log L(\Omega)}.$$

On the other hand, by elementary inequalities

$$\log(e + x\,y) \leq \log(e+x) + \log(1+y) \ \text{ for any } \ x, y > 0$$

and

$$\log(1+y) \leq y \ \text{ for any } \ y > 0,$$

we obtain

$$[\varphi]_{L\log L(\Omega)} = \int_\Omega |\varphi(z)| \log\left(e + \frac{|\varphi(z)|}{\|\varphi\|_{L\log L(\Omega)}} \frac{\|\varphi\|_{L\log L(\Omega)}}{\|\varphi\|_{L^1(\Omega)}}\right) dz$$

$$\leq \int_\Omega |\varphi(z)| \log\left(e + \frac{|\varphi(z)|}{\|\varphi\|_{L\log L(\Omega)}}\right) dz + \int_\Omega |\varphi(z)| \log\left(1 + \frac{\|\varphi\|_{L\log L(\Omega)}}{\|\varphi\|_{L^1(\Omega)}}\right) dz$$

$$\leq \int_\Omega |\varphi(z)| \log\left(e + \frac{|\varphi(z)|}{\|\varphi\|_{L\log L(\Omega)}}\right) dz + \int_\Omega |\varphi(z)| \frac{\|\varphi\|_{L\log L(\Omega)}}{\|\varphi\|_{L^1(\Omega)}} dz = 2\|\varphi\|_{L\log L(\Omega)}.$$

□

More generally, for $p \in [1, \infty)$ and $\beta \geq 0$ the non-linear functional

$$[\varphi]_{L^p \log^\beta L(\Omega)} = \left[\int_\Omega |\varphi(z)|^p \log^\beta\left(e + \frac{|\varphi(z)|}{\|\varphi\|_{L^p(\Omega)}}\right) dz\right]^{\frac{1}{p}}$$

is comparable with the Luxemburg norm given at (1.1) and the following estimates are straightforward

$$\|\varphi\|_{L^p \log^{-1} L(\Omega)} \leq \|\varphi\|_{L^p(\Omega)} \leq \|\varphi\|_{L^p \log^\beta L(\Omega)} \leq [\varphi]_{L^p \log^\beta L(\Omega)} \leq 2\|\varphi\|_{L^p \log^\beta L(\Omega)}.$$

Hölder's inequality for Zygmund spaces takes the form

$$(1.3) \qquad ||\varphi_1 \ldots \varphi_k||_{L^p \log^\beta L(\Omega)} \leq c\, ||\varphi_1||_{L^{p_1} \log^{\beta_1} L(\Omega)} \cdots ||\varphi_k||_{L^{p_k} \log^{\beta_k} L(\Omega)}$$

where $p_1, \ldots, p_k > 1$, $\beta_1, \ldots, \beta_k \in \mathbb{R}$,

$$\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_k}$$

and

$$\frac{\beta}{p} = \frac{\beta_1}{p_1} + \ldots + \frac{\beta_k}{p_k}.$$

The constant here does not depend on the functions $\varphi_i \in L^{p_i} \log^{\beta_i} L(\Omega)$ ($i = 1, \ldots, k$).

Note that by (1.3) in particular we have

$$||\varphi_1\,\varphi_2||_{L^p \log^\beta L(\Omega)} \leq c\,||\varphi_1||_{L^{p_1} \log^{\beta_1} L(\Omega)}||\varphi_2||_{L^{p_2} \log^{\beta_2} L(\Omega)}$$

where $p_1, p_2 > 1$, $\beta_1, \beta_2 \in \mathbb{R}$,

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$$

and

$$\frac{\beta}{p} = \frac{\beta_1}{p_1} + \frac{\beta_2}{p_2},$$

$$||\varphi_1\,\varphi_2||_{L^1(\Omega)} \leq c\,||\varphi_1||_{L^2 \log^{-\beta} L(\Omega)}||\varphi_2||_{L^2 \log^\beta L(\Omega)},$$

$$||\varphi_1\,\varphi_2||_{L \log^\beta L(\Omega)} \leq c\,||\varphi_1||_{L^2 \log^\beta L(\Omega)}||\varphi_2||_{L^2 \log^\beta L(\Omega)}, \tag{1.4}$$

and

$$||\varphi_1^2||_{L \log^\beta L(\Omega)} \leq c\,||\varphi_1||^2_{L^2 \log^\beta L(\Omega)}.$$

Moreover, the Zygmund-Sobolev spaces $W^{1,L \log^{1/2} L}(\Omega)$, $W^{1,L^2 \log^{-1} L}(\Omega)$ and $W^{1,L^2 \log L}(\Omega)$ are defined as the spaces of functions φ such that

$$\varphi \in L \log^{1/2} L(\Omega) \text{ and } |\nabla\varphi| \in L \log^{1/2} L(\Omega),$$

$$\varphi \in L^2 \log^{-1} L(\Omega) \text{ and } |\nabla\varphi| \in L^2 \log^{-1} L(\Omega)$$

and

$$\varphi \in L^2 \log L(\Omega) \text{ and } |\nabla\varphi| \in L^2 \log L(\Omega)$$

respectively. We endow these spaces with the norms

$$\|\varphi\|_{W^{1,L \log^{1/2} L}(\Omega)} = \|\varphi\|_{L \log^{1/2} L(\Omega)} + \||\nabla\varphi|\|_{L \log^{1/2} L(\Omega)},$$

$$\|\varphi\|_{W^{1,L^2 \log^{-1} L}(\Omega)} = \|\varphi\|_{L^2 \log^{-1} L(\Omega)} + \||\nabla\varphi|\|_{L^2 \log^{-1} L(\Omega)}$$

and

$$\|\varphi\|_{W^{1,L^2 \log L}(\Omega)} = \|\varphi\|_{L^2 \log L(\Omega)} + \||\nabla\varphi|\|_{L^2 \log L(\Omega)}$$

respectively.

1.3 The spaces of exponentially integrable functions

The *space of exponentially integrable functions*, denoted by $EXP_\alpha(\Omega)$, is the Orlicz space generated by the Orlicz function

$$\mathcal{Q}(t) = e^{t^\alpha} - 1$$

with $\alpha > 0$. So $EXP_\alpha(\Omega)$ consists of all measurable functions $\varphi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ such that

$$\int_\Omega e^{\left(\frac{|\varphi(z)|}{\lambda}\right)^\alpha} dz < \infty$$

for some $\lambda = \lambda(\varphi) > 0$ and it is equipped with the Luxemburg norm

$$\|\varphi\|_{EXP_\alpha(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega e^{\left(\frac{|\varphi(z)|}{\lambda}\right)^\alpha} dz \leq 2 \right\}. \tag{1.5}$$

For $\alpha = 1$ we will write $EXP(\Omega)$ instead of $EXP_1(\Omega)$.

Furthermore the following continuous embeddings

$$L^\infty(\mathbb{D}) \hookrightarrow EXP(\mathbb{D}) \hookrightarrow L^p(\mathbb{D}) \hookrightarrow L \log L(\mathbb{D}) \hookrightarrow L^1(\mathbb{D})$$

hold for all $p \in (1, \infty)$ and $\mathbb{D}$ disk of $\mathbb{R}^2$ (see [BS]).

Recall that $(\mathcal{P}, \mathcal{Q})$ where

$$\mathcal{P}(t) = t^p \log^{1/\alpha}(e + t)$$

and

$$\mathcal{Q}(t) = e^{t^\alpha} - 1$$

is an Hölder conjugate couple, i.e.

$$\left| \int_\Omega \varphi\, \psi \right| \leq c\, ||\varphi||_{L \log^{1/\alpha} L(\Omega)} ||\psi||_{EXP_\alpha(\Omega)} \tag{1.6}$$

for $\varphi \in L \log^{1/\alpha} L(\Omega)$ and $\psi \in EXP_\alpha(\Omega)$. In particular we have

$$\left| \int_\Omega \varphi\, \psi \right| \leq c\, ||\varphi||_{L \log L(\Omega)} ||\psi||_{EXP(\Omega)}$$

for $\varphi \in L \log L(\Omega)$ and $\psi \in EXP(\Omega)$ and

$$\left| \int_\Omega \varphi\, \psi \right| \leq c\, ||\varphi||_{L \log^{1/2} L(\Omega)} ||\psi||_{EXP_2(\Omega)} \tag{1.7}$$

for $\varphi \in L\log^{1/2} L(\Omega)$ and $\psi \in EXP_2(\Omega)$. Moreover, for $\varphi, \psi \in L^2\log^{1/\alpha} L(\Omega)$, by (1.4) we obtain $\varphi\,\psi \in L\log^{1/\alpha} L(\Omega)$ and

$$||\varphi\,\psi||_{L\log^{1/\alpha} L(\Omega)} \leq c_0\,||\varphi||_{L^2\log^{1/\alpha} L(\Omega)}||\psi||_{L^2\log^{1/\alpha} L(\Omega)}, \tag{1.8}$$

let $\gamma \in EXP_\alpha(\Omega)$, by (1.6) and (1.8) we have

$$\left|\int_\Omega \varphi\,\psi\,\gamma\,\right| \leq c\,||\varphi||_{L^2\log^{1/\alpha} L(\Omega)}||\psi||_{L^2\log^{1\alpha}(\Omega)}||\gamma||_{EXP_\alpha(\Omega)}.$$

In particular

$$\left|\int_\Omega \varphi^2\,\psi\right| \leq c\,||\varphi||^2_{L^2\log^{1/\alpha} L(\Omega)}||\psi||_{EXP_\alpha(\Omega)}$$

for $\varphi \in L^2\log^{1/\alpha} L(\Omega)$ and $\psi \in EXP_\alpha(\Omega)$ and

$$\left|\int_\Omega \varphi^2\,\psi\right| \leq c\,||\varphi||^2_{L^2\log L(\Omega)}||\psi||_{EXP(\Omega)} \tag{1.9}$$

for $\varphi \in L^2\log L(\Omega)$ and $\psi \in EXP(\Omega)$.

Since $\mathcal{P}$ and $\mathcal{Q}$ are both Young functions with $\mathcal{P}$ satisfying a doubling condition, by Theorem 1.1, we have that the dual to the Zygmund space $L\log^{1/\alpha} L(\Omega)$ is the space $EXP_\alpha(\Omega)$, i.e.

$$(L\log^{1/\alpha} L(\Omega))' = EXP_\alpha(\Omega),$$

but not conversely. In particular, for $\alpha = 1$ and $\alpha = 1/2$ we have

$$(L\log L(\Omega))' = EXP(\Omega)$$

and

$$(L\log^{1/2} L(\Omega))' = EXP_2(\Omega).$$

Observe that $\mathcal{Q}$ does not satisfy a doubling condition and that the dual to space $EXP_\alpha(\Omega)$ is not $L\log^{1/\alpha} L(\Omega)$ and that $L^\infty(\Omega)$ is not dense in $EXP_\alpha(\Omega)$ (see [RR], Chapter 3).

1.4 Distance formula to L^∞ in EXP_α

Let Ω be a bounded open subset of $\mathbb{R}^2$, the space of exponentially integrable functions $EXP_\alpha(\Omega)$ ($\alpha > 0$) can also be defined as the set of all measurable functions $\varphi : \Omega \to \mathbb{R}$ such that

$$⨍_\Omega e^{\left(\frac{|\varphi(z)|}{\lambda}\right)^\alpha} dz < \infty$$

for some $\lambda = \lambda(\varphi) > 0$ and be equipped with the norm

$$\|\varphi\|_{EXP_\alpha} = \inf\left\{\lambda > 0 : \fint_\Omega e^{\left(\frac{|\varphi(z)|}{\lambda}\right)^\alpha} dz \le 2\right\}$$

where

$$\fint_\Omega \quad \text{stands for} \quad \frac{1}{|\Omega|}\int_\Omega .$$

It will be useful in the sequel to remember that in [CS] (see also [FLS]) the authors established the following distance formula to $L^\infty(\Omega)$ in $EXP_\alpha(\Omega)$. Let $\varphi \in EXP_\alpha(\Omega)$

$$\begin{aligned}\text{dist}_{EXP_\alpha}(\varphi, L^\infty) &= \inf\{\psi \in L^\infty(\Omega) : ||\varphi - \psi||_{EXP_\alpha}\} \\ &= \inf\left\{\lambda > 0 : \fint_\Omega e^{\left(\frac{|\varphi(z)|}{\lambda}\right)^\alpha} dz < \infty\right\} \\ &= e \limsup_{p\to\infty} \frac{1}{p}\left[\fint_\Omega |\varphi(z)|^{\alpha p} dz\right]^{\frac{1}{p}}.\end{aligned}$$

We observe that for every $\varphi, \psi \in EXP_\alpha(\Omega)$, $\phi \in L^\infty(\Omega)$ and $\lambda \in \mathbb{R}$ we have

$$\text{dist}_{EXP_\alpha}(\varphi, L^\infty) \le \|\varphi\|_{EXP_\alpha}$$

$$\text{dist}_{EXP_\alpha}(\lambda\,\varphi, L^\infty) = |\lambda|\,\text{dist}_{EXP_\alpha}(\varphi, L^\infty)$$

$$\text{dist}_{EXP_\alpha}(\varphi + \psi, L^\infty) \le \text{dist}_{EXP_\alpha}(\varphi, L^\infty) + \text{dist}_{EXP_\alpha}(\psi, L^\infty)$$

$$\text{dist}_{EXP_\alpha}(\varphi - \phi, L^\infty) = \text{dist}_{EXP_\alpha}(\varphi, L^\infty)$$

$$\text{dist}_{EXP_\alpha}(\phi, L^\infty) = 0$$

and

$$\text{dist}_{EXP_\alpha}(\varphi, L^\infty) = \lim_{j\to\infty} \|\varphi_j - \varphi\|_{EXP_\alpha}$$

where

$$\varphi_j(z) = \begin{cases} \varphi(z) & \text{if } |\varphi(z)| \le j \\ 0 & \text{if } |\varphi(z)| > j \end{cases}$$

(see [CS], [FLS]). In particular, for $\alpha = 1$ we have

$$\begin{aligned}\text{dist}_{EXP}(\varphi, L^\infty) &= \inf\left\{\lambda > 0 : \fint_\Omega e^{\frac{|\varphi(z)|}{\lambda}} dz < \infty\right\} \\ &= e \limsup_{p\to\infty} \frac{1}{p}\left[\fint_\Omega |\varphi(z)|^p dz\right]^{\frac{1}{p}}.\end{aligned} \tag{1.10}$$

Denoting with $exp(\Omega)$ the closure of $L^\infty(\Omega)$ in $EXP(\Omega)$, i.e.

$$exp(\Omega) = \text{clos}_{EXP} L^\infty(\Omega),$$

by (1.10) we obtain that

$$\varphi \in exp(\Omega) \Leftrightarrow \text{dist}_{EXP}(\varphi, L^\infty) = 0 \Leftrightarrow e^{\frac{\varphi}{\lambda}} \in L^1(\Omega) \text{ for every } \lambda > 0.$$

Finally, we recall that the dual to space $exp(\Omega)$ is the Zygmund space $L \log L(\Omega)$, i.e.

$$(exp(\Omega))' = L \log L(\Omega).$$

Chapter 2

Mappings with finite distortion

In this chapter we will denote with Ω, Ω' and Ω'' planar domains.

2.1 Differentiability

We recall that a mapping $f = (u, v) : \Omega \to \mathbb{R}^2$ is differentiable at $z = (x, y) \in \Omega$ if there is a linear map $Df(z) : \mathbb{R}^2 \to \mathbb{R}^2$, called the *pointwise differential* of f, such that

$$\lim_{h \to 0} \frac{|f(z + h) - f(z) - Df(z)h|}{|h|} = 0.$$

The pointwise differential is uniquely determined by the formula

$$Df(z)h = \lim_{t \to 0} \frac{f(z + th) - f(z)}{t}. \tag{2.1}$$

Moreover, we recall that a mapping $f : \Omega \to \mathbb{R}^2$ is *open* if $f(U)$ is open for every open $U \subset \Omega$.

Formula (2.1) ensures the existence of the partial derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

of f at point z. The converse is not true. However, every continuous open mapping (a homeomorphism, for example) defined on Ω having finite first partial derivatives almost everywhere in Ω, it is differentiable almost everywhere in Ω in the classical sense (see [GL]).

As every continuous mapping $f \in W^{1,1}(\Omega, \mathbb{R}^2)$ is absolutely continuous on almost every line parallel to the coordinate axes (see [R]) and therefore has finite first partial derivatives almost everywhere in Ω we have the following

Lemma 2.1. *Let $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^2) \cap Hom(\Omega, \Omega')$. Then f is differentiable almost everywhere in Ω in the classical sense.*

We recall the following properties of the pointwise differential and of the Jacobian.

- *Composition.* If $f : \Omega \to \Omega'$ is differentiable at $z \in \Omega$ and if $g : \Omega' \to \Omega''$ is differentiable at $w = f(z)$, then $g \circ f$ is differentiable at z and

$$D(g \circ f)(z) = Dg(f(z)) \circ Df(z)$$

  and

$$J_{g \circ f}(z) = J_g(f(z)) \circ J_f(z).$$

- *Inverses.* If $f : \Omega \to \Omega'$ is a homeomorphism differentiable at $z \in \Omega$ with $J_f(z) \neq 0$, then the inverse mapping $f^{-1} : \Omega' \to \Omega$ is differentiable at $w = f(z)$ and

$$Df^{-1}(w) = (Df(f^{-1}(w)))^{-1}$$

  and

$$J_{f^{-1}}(w) = \frac{1}{J_f(f^{-1}(w))}.$$

We now remember the following result (see [AIM], Theorem 3.3.4).

Theorem 2.2. *Let $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^2) \cap Hom(\Omega, \Omega')$. Then the Jacobian J_f does not change sign, that is, either*

- *$J_f(z) \geq 0$ for a.e. $z \in \Omega$ or*
- *$J_f(z) \leq 0$ for a.e. $z \in \Omega$.*

Given measurable functions $f : \Omega \to \Omega'$ and $g : \Omega' \to \Omega''$, in general their composition is not a measurable function. However, in the geometric study of mappings it is necessary to avoid all unnecessary constraints on such natural operations as the composition. It is for this reason, among many others, that the following Lusin's condition arises.

Definition 2.1. Let $f : \Omega \to \mathbb{R}^2$ be a measurable mapping. We say that f *satisfies Lusin's condition $\mathcal{N}$* if for every measurable set $E \subset \Omega$

$$|E| = 0 \qquad \Rightarrow \qquad |f(E)| = 0.$$

Recall that if a measurable mapping f satisfies Lusin's condition $\mathcal{N}$, then (and only then) f takes measurable sets to measurable sets.

Naturally, one frequently needs to study mappings that preserve measurability under inverse images. This leads us to the following condition.

Definition 2.2. Let $f : \Omega \to \Omega'$ be a measurable mapping. We say that f *satisfies Lusin's condition* $\mathcal{N}^{-1}$ if for every measurable set $E \subset \Omega'$

$$|E| = 0 \qquad \Rightarrow \qquad |f^{-1}(E)| = 0.$$

Notice that if $f : \Omega \to \Omega'$ is a measurable mapping satisfying Lusin's condition $\mathcal{N}^{-1}$, then f^{-1} takes measurable sets to measurable sets. In particular, the composition $u \circ f$ of f with any measurable function u on Ω' is measurable.

Moreover, if $f \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^2) \cap \mathrm{Hom}(\Omega, \Omega')$, then f satisfies Lusin's condition $\mathcal{N}$ (see [AIM], Theorem 3.3.7).

It is known that each $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ is approximatively differentiable almost everywhere [F, Theorem 3.1.4] and that the set of approximative differentiability can be exhausted up to a set of measure zero by sets the restriction to which of f is Lipschitz [F, Theorem 3.1.8]. Hence we can decompose Ω into pairwise disjoint sets

$$\Omega = Z \cup \bigcup_{k=1}^{\infty} \Omega_k \tag{2.2}$$

such that $|Z| = 0$ and $f_{|\Omega_i}$ is Lipschitz.

Let $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2) \cap \mathrm{Hom}(\Omega, \Omega')$, $B \subset \Omega$ a Borel set and let η a non-negative Borel-measurable function on $\mathbb{R}^2$, without any additional assumption we have

$$\int_B \eta(f(z))|J_f(z)|\, dz \le \int_{f(B)} \eta(w)\, dw. \tag{2.3}$$

This follows from the area formula for Lipschitz mappings and (2.2). The equality

$$\int_B \eta(f(z))|J_f(z)|\, dz = \int_{f(B)} \eta(w)\, dw$$

is satisfied if $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2) \cap \mathrm{Hom}(\Omega, \Omega')$ satisfies Lusin's condition $\mathcal{N}$.

From (2.3) we deduce that the Jacobian J_f is locally integrable and for every Borel set $B \subset \Omega$

$$\int_B |J_f(z)|\, dz \le |f(B)|.$$

In particular, if $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2) \cap \mathrm{Hom}(\Omega, \Omega')$ is an orientation preserving mapping satisfying Lusin's condition $\mathcal{N}$ we have

$$\int_B \eta(f(z)) J_f(z)\, dz = \int_{f(B)} \eta(w)\, dw, \tag{2.4}$$

so

$$\int_B J_f(z)\, dz = |f(B)|$$

and

$$J_f(z) > 0 \ \text{ for a.e. } z \in \Omega.$$

2.2 Integrability of distortion functions

We see from basic results that the minimal analytic assumptions necessary for a viable theory of mappings with finite distortion appear to be encapsulated in the following definition.

Definition 2.3. We say that a mapping $f : \Omega \to \mathbb{R}^2$ belonging to Sobolev space $W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ is a mapping with *finite distortion* if

i) $J_f \in L^1_{\mathrm{loc}}(\Omega)$;

ii) there is a measurable function $K(z) \geq 1$, finite almost everywhere, such that

$$|Df(z)|^2 \leq K(z) J_f(z) \ \text{ for a.e. } z \in \Omega. \tag{2.5}$$

Such K is called *distortion* of f. Here $|Df(z)|$ stands for the operator norm of the differential matrix $Df(z) \in R^{2\times 2}$ defined by

$$|Df(z)| = \sup_{|h|=1} |Df(z)h|.$$

We observe that the conditions **i)** and **ii)** above are not enough to imply $f \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$, unless of course K is a bounded function.

In two dimensions the mappings of finite distortion are intimately related to elliptic PDE's (see Section 5.1). For equations with non-smooth coefficients the request that f has locally integrable distributional first partial derivatives is the smallest degree of smoothness where one can begin to discuss what it means to be a (weak) solution to such an equation.

The first condition is a regularity property which is automatically satisfied by all homeomorphisms $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2)$ (see Section 2.1).

Inequality (2.5) is called *distortion inequality* for f. Observe that this inequality merely asks that the pointwise Jacobian $J_f(z) \geq 0$ for a.e. $z \in \Omega$ and that the differential $Df(z)$ vanishes at those points z where $J_f(z) = 0$.

In two dimensions the distortion inequality (2.5) is the equivalent to the following

$$\max_{|h|=1} |Df(z)h| \leq K(z) \min_{|h|=1} |Df(z)h| \text{ for a.e. } z \in \Omega.$$

Geometrically, it means that at almost every point $z \in \Omega$ the differential $Df(z) : \mathbb{R}^2 \to \mathbb{R}^2$ deforms the unit disk onto an ellipse whose eccentricity is controlled by $K(z)$. Thus, in particular, the case $K = 1$ results in conformal deformations.

Given a mapping $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2)$ with finite distortion, we define the *distortion function* of f, K_f, as

$$K_f(z) = \begin{cases} \dfrac{|Df(z)|^2}{J_f(z)} & \text{if } Df(z) \text{ exists and } J_f(z) > 0 \\ 1 & \text{otherwise.} \end{cases} \tag{2.6}$$

Notice that K_f is the smallest function $K(z) \geq 1$ for which the distortion inequality (2.5) holds.

We are mainly concerned with *homeomorphisms having finite distortion.*

If $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2) \cap \text{Hom}(\Omega, \Omega')$ and $K_f \in L^\infty(\Omega)$, $K_f(z) \leq K$ for a.e. $z \in \Omega$, we say that f is K*-quasiconformal.* Clearly, in this case $f \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^2)$ and it is well known that also f^{-1} is K-quasiconformal i.e. $K_{f^{-1}} \in L^\infty(\Omega')$ and $K_{f^{-1}}(w) \leq K$ for a.e. $w \in \Omega'$ (see [AIM], Theorem 3.1.2).

Our results deal with the integrability of the distortion function $K_{f^{-1}}$ of f^{-1} under more general assumptions.

Let $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2) \cap \text{Hom}(\Omega, \Omega')$ with finite distortion, the minimal assumption in order to have that the inverse $f^{-1} \in W^{1,2}_{\text{loc}}(\Omega', \mathbb{R}^2)$ is that $K_f \in L^1(\Omega)$ (see [HK]). In fact Hencl-Koskela show that if K_f belongs to $L^{1-\delta}(\Omega)$, with $\delta \in (0, 1)$, then we may have that f^{-1} does not belong to $W^{1,1+\delta}_{\text{loc}}(\Omega', \mathbb{R}^2)$ (see example 1.4 in [HK]). On the other hand in [HMPS] the authors prove that if $f \in W^{1,\alpha}(\Omega, \mathbb{R}^2) \cap \mathrm{Hom}(\Omega, \Omega')$, for some $\alpha \in (1, 2]$, has finite distortion with distortion function K_f satisfying

$$M = \sup_{\delta \in (0,1)} \left(\delta \int_\Omega K_f(z)^{1-\delta} \, dz \right)^{\frac{1}{1-\delta}} < \infty,$$

then $|Df^{-1}|$ belongs to grand Lebesgue space $L^{2)}(\Omega')$, i.e

$$\| |Df^{-1}| \|_{L^{2)}(\Omega')} = \sup_{\varepsilon \in (0,1)} \left(\varepsilon \int_{\Omega'} |Df^{-1}(w)|^{2-\varepsilon} \, dw \right)^{\frac{1}{2-\varepsilon}} < \infty.$$

Combining Theorems 1.3 and 6.1 of [HK], Theorem 2.1 of [HKO1] and a result due to Greco-Sbordone-C.Trombetti (see [GST]) we can state the following result.

Theorem 2.3. *If $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^2) \cap Hom(\Omega, \Omega')$ has finite distortion with*

$$K_f \in L^1(\Omega),$$

then

i) $J_f > 0$ *a.e. in* Ω*;*

ii) $f^{-1} \in W^{1,2}_{loc}(\Omega', \mathbb{R}^2)$ *has finite distortion and*

$$\int_{\Omega'} |Df^{-1}(w)|^2 \, dw = \int_\Omega K_f(z) \, dz;$$

iii) $K_{f^{-1}}$ *has the form*

$$K_{f^{-1}}(w) = K_f(f^{-1}(w)) \text{ for a.e. } w \in \Omega'. \tag{2.7}$$

Observe that, since $f \in \mathrm{Hom}(\Omega, \Omega')$, K_f and $K_{f^{-1}}$ defined at (2.6) and (2.7), are Borel-measurable functions. Moreover, if we assume only that the homeomorphism f belongs to $W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$, we may have that f^{-1} does not belongs to $W^{1,1}_{\mathrm{loc}}(\Omega', \mathbb{R}^2)$. Indeed, consider the mapping

$$f : (0, 2) \times (0, 1) \to (0, 1) \times (0, 1)$$

defined by

$$f(x, y) = (g^{-1}(x), y),$$

where g^{-1} is the inverse map of

$$g : (0,1) \to (0,2)$$

defined by

$$g(t) = t + \varphi(t),$$

where $\varphi : (0,1) \to (0,1)$ is the Cantor ternary function. We have that f is a homeomorphism in $W^{1,\infty}_{\mathrm{loc}}$ whose inverse f^{-1} is of bounded variation, but it does not belong to $W^{1,1}_{\mathrm{loc}}$. On the other hand in [HK] the authors prove that if $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2) \cap \mathrm{Hom}(\Omega, \Omega')$ has Jacobian $J_f(z) > 0$ for a.e. $z \in \Omega$, then $f^{-1} \in W^{1,1}_{\mathrm{loc}}(\Omega', \mathbb{R}^2)$.

Recently, in [AGRS] the authors obtained the following optimal regularity for Jacobian and for differential of a mapping with exponentially integrable distortion function.

Theorem 2.4. *Let $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^2)$ be a mapping with finite distortion. Assume that the distortion function $K_f(z)$ satisfies the condition*

$$e^{\frac{K_f}{\lambda}} \in L^1_{loc}(\Omega) \ \text{ for some } \ \lambda > 0.$$

Then we have

$$J_f \log^{\beta}(e + J_f) \in L^1_{loc}(\Omega) \ \text{ for every } \ \beta \in \left(0, \frac{1}{\lambda}\right)$$

and

$$|Df|^2 \log^{\beta - 1}(e + |Df|) \in L^1_{loc}(\Omega) \ \text{ for every } \ \beta \in \left(0, \frac{1}{\lambda}\right).$$

Moreover this result is sharp in sense that the conclusion fails for $\beta = \dfrac{1}{\lambda}$ for every $\lambda > 0$.

As a special case of Theorem 2.4 we have

Corollary 2.5. *Let $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^2)$ be a mapping with finite distortion. Assume that the distortion function K_f satisfies the condition*

$$e^{\frac{K_f}{\lambda}} \in L^1_{loc}(\Omega) \ \text{ for some } \ \lambda \in (0,1).$$

Then

$$f \in W^{1,2}_{loc}(\Omega, \mathbb{R}^2).$$

Remark 2.1. *If* $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^2)$ *is a mapping with finite distortion such that*

$$\int_\Omega e^{\frac{K_f(z)}{\lambda}}\, dz < \infty \ \text{ for some } \ \lambda \in \left(0, \frac{1}{2}\right)$$

and therefore

$$dist_{EXP}(K_f, L^\infty) < \frac{1}{2},$$

by Theorem 2.4 we obtain

$$|Df| \in L^2 \log L_{loc}(\Omega).$$

We recall that given a square matrix A, the adjugate adj$A$ of A satisfies

$$A \,\mathrm{adj}A = I \,\det A \tag{2.8}$$

where I is the identity matrix and det$A$ denotes the determinant of A.

Let us start by proving the following theorem.

Theorem 2.6. *Let* $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^2) \cap Hom(\Omega, \Omega')$ *with finite distortion. If the distortion function* $K_f \in EXP(\Omega)$ *satisfies the condition*

$$dist_{EXP}(K_f, L^\infty) < 1,$$

then

$$K_{f^{-1}} \in L^1_{loc}(\Omega').$$

This result is optimal in sense that the conclusion fails if $dist_{EXP}(K_f, L^\infty) = 1$.

Proof. By hypothesis in particular K_f belongs to $L^1(\Omega)$, by Theorem 2.3 we have that $f^{-1} \in W^{1,2}_{\mathrm{loc}}(\Omega', \mathbb{R}^2)$ has finite distortion. Since $f^{-1} \in W^{1,2}_{\mathrm{loc}}(\Omega', \mathbb{R}^2)$, then f^{-1} satisfies Lusin's condition $\mathcal{N}$. From (2.4) we then deduce that

$$J_{f^{-1}}(w) > 0 \quad \text{for a.e. } w \in \Omega'.$$

By Lemma 2.1 we know that f^{-1} is differentiable almost everywhere in Ω' in the classical sense. Moreover, we know that at each point of differentiability of f^{-1} such that $J_{f^{-1}}(w) > 0$ we have that f is differentiable at $z = f^{-1}(w)$ and

$$Df(z) = (Df^{-1}(f(z)))^{-1}. \tag{2.9}$$

Let $T \subset\subset \Omega'$, we have

$$\int_T K_{f^{-1}}(w)\,dw = \int_T \frac{|Df^{-1}(w)|^2}{J_{f^{-1}}(w)}\,dw = \int_T \frac{|\mathrm{adj} Df^{-1}(w)|^2}{J_{f^{-1}}(w)}\,dw.$$

Using (2.8) we get

$$\int_T K_{f^{-1}}(w)\,dw = \int_T |(Df^{-1}(w))^{-1}|^2\, J_{f^{-1}}(w)\,dw.$$

Applying (2.4) we obtain

$$\int_T K_{f^{-1}}(w)\,dw = \int_{f^{-1}(T)} |(Df^{-1}(f(z)))^{-1}|^2\, J_{f^{-1}}(f(z))\, J_f(z)\,dz.$$

By (2.9) we conclude

$$\int_T K_{f^{-1}}(w)\,dw = \int_{f^{-1}(T)} |Df(z)|^2\,dz. \tag{2.10}$$

Since $\mathrm{dist}_{EXP}(K_f, L^\infty) < 1$, then there exists $\lambda \in (0,1)$ such that $e^{\frac{K_f}{\lambda}} \in L^1_{\mathrm{loc}}(\Omega)$. By Corollary 2.5, we have that $f \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ and therefore

$$K_{f^{-1}} \in L^1_{\mathrm{loc}}(\Omega').$$

□

To show that the conclusion of this theorem fails if $\mathrm{dist}_{EXP}(K_f, L^\infty) = 1$ we consider the following mapping (see [AGRS]):

$$f(z) = \begin{cases} \dfrac{z}{|z|} \dfrac{1}{\sqrt{\log\left(e + \dfrac{1}{|z|}\right) \log\log\left(e + \dfrac{1}{|z|}\right)}} & \text{for } z \in \mathbb{D}(0,1) \setminus \{0\} \\ 0 & \text{for } z = 0. \end{cases}$$

Note that

$$f : \mathbb{D}(0,1) \to \mathbb{D}(0,R),$$

where $\mathbb{D}(0,1)$ denotes the disk of $\mathbb{R}^2$ centered at 0 with radius 1 and $\mathbb{D}(0,R)$ denotes the disk of $\mathbb{R}^2$ centered at 0 with radius

$$R = \frac{1}{\sqrt{\log(e+1)\log\log(e+1)}}.$$

Moreover f is a homeomorphism belonging to $W^{1,1}_{\mathrm{loc}}(\mathbb{D}(0,1), \mathbb{R}^2)$ with finite distortion and its distortion function K_f satisfies

$$\mathrm{dist}_{EXP}(K_f, L^\infty) = 1.$$

In fact

$$e^{K_f} \in L^1(\mathbb{D}(0,1)) \text{ and } e^{\frac{K_f}{\lambda}} \notin L^1(\mathbb{D}(0,1)) \text{ for every } \lambda \in (0,1).$$

Recall that in the orientation preserving case, given a *radial stretching* h, i.e. a mapping h defined by

$$h(z) = \frac{z}{|z|}\,\rho(|z|),$$

we find that at points where the derivative ρ' exists

$$K_h(z) = \max\left\{\frac{|z|\rho'(|z|)}{\rho(|z|)}, \frac{\rho(|z|)}{|z|\rho'(|z|)}\right\}$$

and

$$J_h(z) = \frac{\rho(|z|)\rho'(|z|)}{|z|}$$

(see Chapter 11 of [IM1]). Since our mapping f is a radial stretching with

$$\rho(|z|) = \frac{1}{\sqrt{\log\left(e+\frac{1}{|z|}\right)\log\log\left(e+\frac{1}{|z|}\right)}}$$

for $|z| = r$ we obtain

$$K_f(r) = \frac{2(1+e\,r)\log\left(e+\frac{1}{r}\right)\log\log\left(e+\frac{1}{r}\right)}{1+\log\log\left(e+\frac{1}{r}\right)}$$

and

$$J_f(r) = \frac{1+\log\log\left(e+\frac{1}{r}\right)}{2r^2(1+e\,r)\left(\log\left(e+\frac{1}{r}\right)\log\log\left(e+\frac{1}{r}\right)\right)^2}.$$

So

$$|Df(r)|^2 = K_f(r)\,J_f(r) = \frac{1}{r^2\log\left(e+\frac{1}{r}\right)\log\log\left(e+\frac{1}{r}\right)}$$

which is not summable at zero under the measure $r\,dr$. By (2.10), we conclude that

$$K_{f^{-1}} \notin L^1_{\mathrm{loc}}(\mathbb{D}(0,R)).$$

Our aim now is to prove the following theorem.

Theorem 2.7. *Let $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^2) \cap Hom(\Omega, \Omega')$ with finite distortion. If the distortion function $K_f \in EXP(\Omega)$ satisfies the condition*

$$dist_{EXP}(K_f, L^\infty) = \lambda \ \text{ for some } \ \lambda > 0,$$

then

$$K_{f^{-1}} \in L^p_{loc}(\Omega') \ \text{ for every } \ p \in \left(0, \frac{1}{2\lambda}\right).$$

Proof. As in Theorem 2.6 the L^1-integrability of K_f implies that $f^{-1} \in W^{1,2}_{\mathrm{loc}}(\Omega', \mathbb{R}^2)$ has finite distortion. Hence f^{-1} satisfies Lusin's condition $\mathcal{N}$, f^{-1} is differentiable almost everywhere in Ω' in the classical sense and

$$J_{f^{-1}}(w) > 0 \quad \text{for a.e. } w \in \Omega'.$$

Moreover, we know that at each point of differentiability of f^{-1} such that $J_{f^{-1}}(w) > 0$ we have that f is differentiable at $z = f^{-1}(w)$,

$$Df(z) = (Df^{-1}(f(z)))^{-1} \tag{2.11}$$

and

$$J_f(z) = \frac{1}{J_{f^{-1}}(f(z))}. \tag{2.12}$$

Let $T \subset\subset \Omega'$ and let $p > 0$, we have

$$\int_T K_{f^{-1}}(w)^p \, dw = \int_T \frac{|Df^{-1}(w)|^{2p}}{J_{f^{-1}}(w)^p} \, dw = \int_T \frac{|\mathrm{adj} Df^{-1}(w)|^{2p}}{J_{f^{-1}}(w)^p} \, dw.$$

Using (2.8) we get

$$\int_T K_{f^{-1}}(w)^p \, dw = \int_T |(Df^{-1}(w))^{-1}|^{2p} \, J_{f^{-1}}(w)^p \, dw.$$

Applying (2.4) we obtain

$$\int_T K_{f^{-1}}(w)^p \, dw = \int_{f^{-1}(T)} |(Df^{-1}(f(z)))^{-1}|^{2p} \, J_{f^{-1}}(f(z))^p \, J_f(z) \, dz.$$

By (2.11) and (2.12) we conclude

$$\int_T K_{f^{-1}}(w)^p \, dw = \int_{f^{-1}(T)} \frac{|Df(z)|^{2p}}{J_f(z)^p} \, J_f(z) \, dz = \int_{f^{-1}(T)} K_f(z)^p J_f(z) \, dz.$$

By inequality

$$K^p J \le J \log^{2p}(e + J) + c(p, \lambda) \, e^{\frac{K}{\lambda}} \qquad (K, J, p, \lambda > 0)$$

(see [HK], Lemma 5.1), we arrive at

$$(2.13)\quad \int_T K_{f^{-1}}(w)^p\, dw \le \int_{f^{-1}(T)} (J_f(z)\log^{2p}(e+J_f(z)) + c(p,\lambda)\, e^{\frac{K_f(z)}{\lambda}})\, dz.$$

Since $\text{dist}_{EXP}(K_f, L^\infty) = \lambda$, then $e^{\frac{K_f}{\lambda}} \in L^1_{\text{loc}}(\Omega)$, by Theorem 2.4 we conclude

$$K_{f^{-1}} \in L^p_{\text{loc}}(\Omega') \ \text{ for every } \ p \in \left(0, \frac{1}{2\lambda}\right).$$

□

Finally we prove the following corollary.

Corollary 2.8. *Let $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^2) \cap Hom(\Omega, \Omega')$ with finite distortion. If the distortion function $K_f \in EXP(\Omega)$ satisfies the condition*

$$dist_{EXP}(K_f, L^\infty) = 0,$$

then

$$K_{f^{-1}} \in \bigcap_{p\ge 1} L^p_{loc}(\Omega').$$

Proof. Since $\text{dist}_{EXP}(K_f, L^\infty) = 0$, we have

$$e^{\frac{K_f}{\lambda}} \in L^1_{\text{loc}}(\Omega) \ \text{ for every } \ \lambda > 0.$$

By (2.13) and by Theorem 2.4 we conclude

$$K_{f^{-1}} \in \bigcap_{p\ge 1} L^p_{\text{loc}}(\Omega').$$

□

2.3 Compactness for families of mappings with exponentially integrable distortion function

In this section we start by recalling the following theorem concerning the compactness of the family of mappings with exponentially integrable distortion function (see [IM1], Theorem 8.14.1).

Let us state a special case concerning the planar situation.

Theorem 2.9. *Denote by $\mathcal{F}$ the family of all mappings $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^2)$ having finite distortion with distortion function K_f such that*

$$\int_\Omega e^{\frac{K_f(z)}{\lambda}}\, dz \le c_0$$

for some $\lambda > 0$ and $c_0 > 0$. Then

i) *$\mathcal{F}$ is bounded in $W^{1,L^2 \log^{-1} L}_{loc}(\Omega, \mathbb{R}^2)$;*

ii) *$\mathcal{F}$ is closed with respect to the weak convergence in $W^{1,L^2 \log^{-1} L}_{loc}(\Omega, \mathbb{R}^2)$;*

iii) *$\mathcal{F}$ is locally equicontinuous in Ω', for any $\Omega' \subset\subset \Omega$;*

iv) *the limit of a locally uniformly convergent sequence of mappings in $\mathcal{F}$ belongs to $\mathcal{F}$.*

On the other hand in [IKO] the authors prove the following result.

Theorem 2.10. *Denote by $\mathcal{G}$ the family of all mappings $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^2)$ with finite distortion K such that*

$$\int_\Omega e^{\mathcal{A}(K(z))}\, dz \le c_0$$

for some $c_0 > 0$, where the Orlicz function $\mathcal{A}$ satisfies the divergence condition

$$\int_1^\infty \frac{\mathcal{A}(t)}{t^2}\, dt = \infty$$

and the technical conditions

$$\lim_{t\to\infty} t\mathcal{A}'(t) = \infty$$

the function $t \to e^{\mathcal{A}(t)}$ is convex for $t \ge 1$.

Moreover we assume that

$$\int_\Omega J_f(z)\, dz \le c_1$$

for some $c_1 > 0$. Then for each $\alpha \in [1, 2)$ we have that

$\mathcal{G}$ is closed with respect to the weak convergence in $W^{1,\alpha}_{loc}(\Omega, \mathbb{R}^2)$.

As practical examples, Theorem 2.10 allows for

$$\mathcal{A}(t) = \frac{t}{\lambda},$$

$$\mathcal{A}(t) = \frac{t}{\lambda \log(e+t)},$$

$$\mathcal{A}(t) = \frac{t}{\lambda \log(e+t) \log\log(e^e+t)},$$

$$\cdots$$

for any string of iterated logarithms and every $\lambda > 0$. Regarding the sharpness, Iwaniec-Koskela-Onninen prove, in particular, that

$$\mathcal{A}(t) = \frac{t^{1-\varepsilon}}{\lambda},$$

$$\mathcal{A}(t) = \frac{t}{\lambda \log^{1+\varepsilon}(e+t)},$$

$$\mathcal{A}(t) = \frac{t}{\lambda \log(e+t) \log^{1+\varepsilon} \log(e^e+t)},$$

$$\cdots$$

are not sufficient, for any $\varepsilon > 0$ and for every $\lambda > 0$. This is contained in the following

Theorem 2.11. *Let $\mathcal{B}$ be a strictly increasing non-negative function such that*

$$\int_1^\infty \frac{\mathcal{B}(t)}{t^2}\, dt < \infty.$$

Then there exists a sequence of mappings $f_j \in W^{1,1}((-1,1)^2, \mathbb{R}^2)$ with finite distortion K_j and a continuous mapping $f \in W^{1,1}((-1,1)^2, \mathbb{R}^2)$ such that for each $j \in \mathbb{N}$

$$\int_{(-1,1)^2} e^{\mathcal{B}(K_j(z))} + J_{f_j}(z) \leq c$$

for some $c > 0$ and for each $\alpha \in [1,2)$

$$f_j \rightharpoonup f \text{ weakly in } W^{1,\alpha}((-1,1)^2, \mathbb{R}^2)$$

but f is not a mapping with finite distortion.

Finally, we recall the following theorem concerning the sequential compactness of the family of homeomorphisms with exponentially integrable distortion function (see [IM1], Theorem 11.14.1).

Theorem 2.12. *Let f_j be a sequence of homeomorphisms belonging to $W^{1,1}_{loc}(\Omega, \mathbb{R}^2)$ with finite distortion such that*

$$\sup_j \|K_{f_j}\|_{EXP(\Omega)} \leq M$$

and that

$$f_j(a) = a, \; f_j(b) = b \;\; and \;\; f_j(c) = c$$

for 3 distinct points $a, b, c \in \Omega$. Then there exists a subsequence f_{j_r} converging locally uniformly to a homeomorphism f with

$$\|K_f\|_{EXP(\Omega)} \leq M.$$

Chapter 3

Weak continuity results for Jacobians

In a recent paper (see [FLM]) a general weak continuity result for determinants of $W^{1,N}(\Omega, \mathbb{R}^N)$-Sobolev maps has been established (Ω an open bounded subset of $\mathbb{R}^N$). We will state it here in the particular case $N = 2$.

Theorem 3.1. *If*

$$f_k = (u_k, v_k) \in W^{1,2}(\Omega) \times W^{1,2}(\Omega), f = (u, v) \in W^{1,1}(\Omega) \times W^{1,1}(\Omega)$$

satisfy the following conditions:

$$f_k \rightharpoonup f \text{ weakly in } W^{1,1}(\Omega) \times W^{1,1}(\Omega) \tag{3.1}$$

and

$$J_{f_k} \overset{*}{\rightharpoonup} \mu \text{ in the sense of measures}$$

then

$$d\mu = J_f \, dz + d\mu^s \tag{3.2}$$

where μ^s is a singular measure with respect to the Lebesgue measure on Ω.

This is a generalization of the classical results (Morrey [Mo1], [Mo2], Caccioppoli [C]) that tell us that if

$$f_k, f \in W^{1,2}(\Omega) \times W^{1,2}(\Omega),$$

then the stronger assumption than (3.1)

$$f_k \rightharpoonup f \text{ weakly in } W^{1,2}(\Omega) \times W^{1,2}(\Omega)$$

implies the stronger conclusion

$$J_{f_k} \overset{*}{\rightharpoonup} J_f \text{ in the sense of measures}$$

i.e. the measure μ^s defined by (3.2) satisfies

$$d\mu^s = 0.$$

In other words, in Theorem 3.1 the authors relax both the weak convergence in $W^{1,2} \times W^{1,2}$ into weak convergence in $W^{1,1} \times W^{1,1}$ and the regularity of the limit $f \in W^{1,2} \times W^{1,2}$ into $f \in W^{1,1} \times W^{1,1}$.

The fact that the singular part $d\mu^s$ may be non zero (also under stronger convergence assumptions than (3.1)) is clarified by an example due to Dacorogna-Murat (see [DM]). In fact the authors show that there exist

$$f_k, f \in W^{1,2}(\Omega) \times W^{1,2}(\Omega)$$

such that

$$f_k \rightharpoonup f \text{ weakly in } W^{1,\frac{4}{3}}(\Omega) \times W^{1,\frac{4}{3}}(\Omega)$$

and

$$J_{f_k} \overset{*}{\rightharpoonup} \mu \text{ in the sense of measures}$$

where

$$d\mu = J_f\, dz + d\mu^s \text{ with } d\mu^s \neq 0.$$

Observe that the example by Dacorogna-Murat has further feature: the limit function f belongs to $W^{1,2}(\Omega) \times W^{1,2}(\Omega)$ as well. Moreover, they prove that if

$$f_k, f \in W^{1,2}(\Omega) \times W^{1,2}(\Omega)$$

satisfy the following conditions:

$$f_k \rightharpoonup f \text{ weakly in } W^{1,\alpha}(\Omega) \times W^{1,\alpha}(\Omega) \text{ for some } \alpha > \frac{4}{3}$$

and

$$J_{f_k} \overset{*}{\rightharpoonup} \mu \text{ in the sense of measures}$$

then

$$d\mu = J_f \, dz.$$

On the other hand, the authors show that there exist

$$f_k, f \in W^{1,2}(\Omega) \times W^{1,2}(\Omega)$$

such that

$$f_k \rightharpoonup f \;\text{ weakly in }\; W^{1,\alpha}(\Omega) \times W^{1,\alpha}(\Omega) \;\text{ for some }\; 1 \le \alpha < \frac{4}{3}$$

and there exists $\varphi \in C_0^0(\Omega)$ such that

$$\lim_k \int_\Omega J_{f_k}(z)\, \varphi(z)\, dz = \infty.$$

In fact, let $z = (x, y) \in \mathbb{R}^2$, $r = |z| = \sqrt{x^2 + y^2}$, $\mathbb{D} = \{z \in \mathbb{R}^2 : |z| < 1\}$, ε a sequence which tends to 0 and

$$f_\varepsilon(z) = (u_\varepsilon(z), v_\varepsilon(z)) = \left(\rho_\varepsilon(r), \frac{\rho_\varepsilon(r)\, y}{r}\right).$$

We find at points where the derivative ρ'_ε exists

$$Df_\varepsilon(z) = \begin{pmatrix} \dfrac{\rho'_\varepsilon(r)\, x}{r} & \dfrac{\rho'_\varepsilon(r)\, y}{r} \\ \left[\rho'_\varepsilon(r) - \dfrac{\rho_\varepsilon(r)}{r}\right] \dfrac{xy}{r^2} & \left[\rho'_\varepsilon(r) - \dfrac{\rho_\varepsilon(r)}{r}\right] \dfrac{y^2}{r^2} + \dfrac{\rho_\varepsilon(r)}{r} \end{pmatrix}$$

and

$$J_{f_\varepsilon}(z) = \frac{\rho'_\varepsilon(r)\, \rho_\varepsilon(r)\, x}{r^2}.$$

Since

$$\left|\frac{\partial u_\varepsilon}{\partial x}\right|, \left|\frac{\partial u_\varepsilon}{\partial y}\right|, \left|\frac{\partial v_\varepsilon}{\partial x}\right|, \left|\frac{\partial v_\varepsilon}{\partial y}\right| \le |\rho'_\varepsilon(r)| + \left|\frac{\rho_\varepsilon(r)}{r}\right|,$$

choosing for $\varepsilon > 0$ small enough and $\alpha \ge 1$

$$\rho_\varepsilon(r) = \begin{cases} \varepsilon^{-2/\alpha}\, r & \text{if } r \in [0, \varepsilon] \\ \varepsilon^{-2/\alpha}\, (2\varepsilon - r) & \text{if } r \in [\varepsilon, 2\varepsilon] \\ 0 & \text{if } r \in [2\varepsilon, 1] \end{cases}$$

we have

$$f_\varepsilon \in W^{1,2}(\mathbb{D}) \times W^{1,2}(\mathbb{D})$$

and

$$f_\varepsilon \rightharpoonup 0 \;\text{ weakly in }\; W^{1,\alpha}(\mathbb{D}) \times W^{1,\alpha}(\mathbb{D}).$$

Now, we choose

$$\varphi(z) = -x\,\psi(z)$$

where

$$\psi \in C_0^0(\mathbb{D}) \ \text{ and } \ \psi(z) = 1 \ \text{ if } |z| < 1/2.$$

Hence $\varphi \in C_0^0(\mathbb{D})$ and using that $J_{f_\varepsilon}(z) = 0$ if $|z| > 1/2$, we obtain

$$\int_{\mathbb{D}} J_{f_\varepsilon}(z)\,\varphi(z)\,dz = -\int_{|z|<1/2} x\,J_{f_\varepsilon}(z)\,dz$$

$$= -\int_{|z|<1/2} \frac{\rho_\varepsilon'(|z|)\rho_\varepsilon(|z|)\,x^2}{|z|^2}\,dz = \frac{\pi}{3}\,\varepsilon^{3-4/\alpha}$$

and therefore

$$\int_{\mathbb{D}} J_{f_k}(z)\,\varphi(z)\,dz \to \begin{cases} \dfrac{\pi}{3} & \text{if } \alpha = \dfrac{4}{3} \\ \infty & \text{if } 1 \leq \alpha < \dfrac{4}{3}. \end{cases}$$

3.1 Distributional determinant DetDf under asymmetric assumptions

From now on we will assume that Ω is an open bounded subset of $\mathbb{R}^2$ sufficiently smooth.

Remark 3.1. *As far as we know, up to now, the distributional determinant $DetDf$ of a planar mapping $f = (u, v)$, has been defined under the same assumptions on the two components u and v. Actually for $u, v \in W^{1,\frac{4}{3}}(\Omega)$ the two expression*

$$(3.3) \qquad T_1 = \frac{\partial}{\partial x}\left(u\,\frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial y}\left(u\,\frac{\partial v}{\partial x}\right) = div\begin{bmatrix} u\,\dfrac{\partial v}{\partial y} \\ -u\,\dfrac{\partial v}{\partial x} \end{bmatrix}$$

and

$$(3.4) \qquad T_2 = \frac{\partial}{\partial x}\left(-v\,\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(v\,\frac{\partial u}{\partial x}\right) = div\begin{bmatrix} -v\,\dfrac{\partial u}{\partial y} \\ v\,\dfrac{\partial u}{\partial x} \end{bmatrix}$$

are well defined in the sense of distributions and they agree. This follows by Sobolev embedding which imply $u, v \in L^4(\Omega)$ and thus $|f||Df|$ is integrable.

Our aim is to allow different assumptions on the two components u and v of f. We have the following

Proposition 3.2. *If*

$$f = (u, v) \in W^{1,\alpha}(\Omega) \times W^{1,2}(\Omega)$$

for some $\alpha \in (1, 2)$, *then the two expression* (3.3) *and* (3.4) *are well defined in the sense of distributions and they agree. Hence we define*

$$DetDf = T_1 = T_2.$$

Proof. By Sobolev Embedding Theorem $u \in L^2(\Omega)$ and thus (3.3) has a meaning as a distribution because it is the divergence of $L^1(\Omega) \times L^1(\Omega)$ vector function.

Also (3.4) is well defined in the sense of distribution, because by Trudinger Embedding Theorem (see [T]) $v \in EXP_2(\Omega)$, moreover $\partial u/\partial y$ and $\partial u/\partial x$ belong to Orlicz space $L \log^{1/2} L(\Omega)$, hence using (1.7) we deduce

$$-v \frac{\partial u}{\partial y} \in L^1(\Omega) \text{ and } v \frac{\partial u}{\partial x} \in L^1(\Omega).$$

Let us check that the two distributions T_1 defined in (3.3) and T_2 defined in (3.4) agree.

It is sufficient to check that for any $\varphi \in C_0^\infty(\Omega)$

$$\langle T_1, \varphi\rangle = \langle T_2, \varphi\rangle. \tag{3.5}$$

We will prove (3.5) in case $u, v \in C^\infty(\Omega)$; the general case follows by a standard approximation argument. We have

$$\langle T_1, \varphi\rangle = \langle \frac{\partial}{\partial x}\left(u \frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial y}\left(u \frac{\partial v}{\partial x}\right), \varphi\rangle$$

$$= \langle \frac{\partial}{\partial x}\left(u \frac{\partial v}{\partial y}\right), \varphi\rangle - \langle \frac{\partial}{\partial y}\left(u \frac{\partial v}{\partial x}\right), \varphi\rangle$$

$$= \langle \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + u \frac{\partial^2 v}{\partial x \partial y}, \varphi\rangle - \langle \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + u \frac{\partial^2 v}{\partial y \partial x}, \varphi\rangle$$

$$= \langle \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}, \varphi\rangle,$$

where we use the equality

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$$

that holds for smooth functions.

Hence

$$\langle T_1,\varphi\rangle = \langle \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x},\varphi\rangle. \tag{3.6}$$

Similarly

$$\begin{aligned}\langle T_2,\varphi\rangle &= \langle \frac{\partial}{\partial x}\left(-v\,\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(v\,\frac{\partial u}{\partial x}\right),\varphi\rangle \\ &= \langle \frac{\partial}{\partial x}\left(-v\,\frac{\partial u}{\partial y}\right),\varphi\rangle + \langle \frac{\partial}{\partial y}\left(v\,\frac{\partial u}{\partial x}\right),\varphi\rangle \\ &= \langle -\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} - v\,\frac{\partial^2 u}{\partial x\partial y},\varphi\rangle + \langle \frac{\partial v}{\partial y}\frac{\partial u}{\partial x} + v\,\frac{\partial^2 u}{\partial y\partial x},\varphi\rangle \\ &= \langle -\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial x},\varphi\rangle\end{aligned}$$

so

$$\langle T_2,\varphi\rangle = \langle -\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial x},\varphi\rangle. \tag{3.7}$$

By (3.6) and (3.7) we conclude the proof. □

Remark 3.2. *If we assume $\alpha = 1$ in Proposition 3.2, then only the expression* (3.3) *is well defined as a distribution. On the other hand if we assume*

$$f = (u,v) \in W^{1,L\log^{1/2}L}(\Omega) \times W^{1,2}(\Omega),$$

then the two expression (3.3) *and* (3.4) *are well defined in the sense of distributions and they agree. Notice that* (3.3) *is well defined as a distribution, because $u \in L^2(\Omega)$ by an Orlicz-Sobolev embedding theorem (see [Ci2]).*

3.2 Weak convergence of Jacobians under asymmetric assumptions

Now, we compare the Jacobian $J_f = detDf$ with the weak Jacobian of f, which is distributional determinant $DetDf$.

We recall that, if

$$f \in W^{1,2}(\Omega) \times W^{1,2}(\Omega)$$

or if

$$f \in W^{1,\alpha}(\Omega) \times W^{1,\alpha}(\Omega) \text{ for some } \alpha \geq \frac{4}{3} \text{ and } \mathrm{Det}Df \in L^1(\Omega),$$

then

$$\text{Det}Df = \text{det}Df, \tag{3.8}$$

(see [Mü1]). Moreover, considering the grand Lebesgue space $L^{2)}(\Omega)$, into-duced by Iwaniec-Sbordone in [IS1], defined as

$$L^{2)}(\Omega) = \left\{ \varphi : \Omega \subset \mathbb{R}^2 \to \mathbb{R} \mid \sup_{\varepsilon \in (0,1)} \left(\varepsilon \int_\Omega |\varphi(z)|^{2-\varepsilon} dz \right)^{\frac{1}{2-\varepsilon}} < \infty \right\}$$

and denoting by $\Sigma^2(\Omega)$ the subclass of $L^{2)}(\Omega)$ defined as

$$\Sigma^2(\Omega) = \left\{ \varphi \in L^{2)}(\Omega) \mid \lim_{\varepsilon \to 0^+} \varepsilon \int_\Omega |\varphi(z)|^{2-\varepsilon} dz = 0 \right\},$$

it is well known that (3.8) holds if

$$\text{det}Df \geq 0 \ \text{ a.e. in } \Omega \ \text{ and } \ |Df| \in \Sigma^2(\Omega)$$

(see [G1]).

We observe that the identity (3.8) fails under the weaker assumption

$$f \in W^{1,\alpha}(\Omega) \times W^{1,\alpha}(\Omega) \ \text{ for any } \ \alpha < 2.$$

To this aim it suffices to consider the mapping

$$f(z) = \frac{z}{|z|} \qquad \text{for } z \in \mathbb{D}$$

(see [Mü1]). In fact, we have

$$\text{det}Df = 0 \quad \text{a.e.},$$

while

$$\text{Det}Df = \pi \delta_0,$$

where δ_0 is the Dirac mass at 0.

Now, we are able to prove that if

$$f_k, f \in W^{1,2}(\Omega) \times W^{1,2}(\Omega) \tag{3.9}$$

and

$$J_{f_k} \overset{*}{\rightharpoonup} \mu \ \text{ in the sense of measures,} \tag{3.10}$$

together with an asymmetric assumption on the components, rules out [DM]-example of the previous section and guarantees that

$$d\mu = J_f \, dz. \tag{3.11}$$

Theorem 3.3. *Under the assumptions* (3.9) *and* (3.10)*, if*

$$f_k = (u_k, v_k) \rightharpoonup f = (u, v) \ \textit{ weakly in } \ W^{1,\alpha}(\Omega) \times W^{1,2}(\Omega)$$

for some $\alpha \in (1, 2)$*, then* (3.11) *holds true.*

Proof. Since $f_k, f \in W^{1,2}(\Omega) \times W^{1,2}(\Omega)$, then

$$J_{f_k} = \det Df_k = \mathrm{Det} Df_k$$

$$J_f = \det Df = \mathrm{Det} Df.$$

For any $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega J_{f_k} \varphi \, dz = \langle \mathrm{Det} Df_k, \varphi \rangle = \int_\Omega \left[\frac{\partial}{\partial x} \left(u_k \frac{\partial v_k}{\partial y} \right) - \frac{\partial}{\partial y} \left(u_k \frac{\partial v_k}{\partial x} \right) \right] \varphi \, dz$$

$$= - \int_\Omega u_k \left(\frac{\partial v_k}{\partial y} \frac{\partial \varphi}{\partial x} - \frac{\partial v_k}{\partial x} \frac{\partial \varphi}{\partial y} \right) dz.$$

We can easily pass to the limit on the right-hand side, because $u_k \to u$ strongly in $L^2(\Omega)$ by Sobolev Compact Embedding Theorem and $v_k \rightharpoonup v$ weakly in $W^{1,2}(\Omega)$. As result we obtain

$$- \int_\Omega u \left(\frac{\partial v}{\partial y} \frac{\partial \varphi}{\partial x} - \frac{\partial v}{\partial x} \frac{\partial \varphi}{\partial y} \right) dz$$

$$= \int_\Omega \left[\frac{\partial}{\partial x} \left(u \frac{\partial v}{\partial y} \right) - \frac{\partial}{\partial y} \left(u \frac{\partial v}{\partial x} \right) \right] \varphi \, dz = \langle \mathrm{Det} Df, \varphi \rangle = \int_\Omega J_f \varphi \, dz.$$

We conclude that

$$d\mu = J_f \, dz.$$

□

Theorem 3.4. *Under the assumptions* (3.9) *and* (3.10)*, if*

$$f_k = (u_k, v_k) \rightharpoonup f = (u, v) \ \textit{ weakly in } \ W^{1, L \log^{1/2} L}(\Omega) \times W^{1,2}(\Omega),$$

then (3.11) *holds true.*

Proof. The proof is almost the same as that of Theorem 3.3 with the difference that it requires the application of an Orlicz-Sobolev compact embedding theorem (see [Ci2]). □

The following is a particular case of Theorem 3.1.

Theorem 3.5. *If*

$$f_k = (u_k, v_k) \in W^{1,2}(\Omega) \times W^{1,2}(\Omega), f = (u, v) \in W^{1,\alpha}(\Omega) \times W^{1,2}(\Omega),$$

for some $\alpha \in (1, 2)$, satisfy the following conditions:

$$f_k \rightharpoonup f \text{ weakly in } W^{1,\alpha}(\Omega) \times W^{1,2}(\Omega)$$

and

$$J_{f_k} \overset{*}{\rightharpoonup} \mu \text{ in the sense of measures}$$

then

$$d\mu = J_f\, dz + d\mu^s$$

where μ^s is a singular measure with respect to the Lebesgue measure on Ω.

In order to provide a proof simpler than that in [FLM], we use a method intoduced by Zhikov in [Z] to prove a generalization of the Tartar-Murat compensated compactness lemma.

First we need to recall the notion of a Lebesgue point. Let

$$Q_r(z_0) = z_0 + \left(\frac{r}{2}, \frac{r}{2}\right)^2$$

be the square with edge length $r > 0$ centered at a point $z_0 \in \Omega$. If $f \in L^\gamma(\Omega)$, with $\gamma \geq 1$, then

$$\lim_{r\to 0} \fint_{Q_r(z_0)} |f(z) - f(z_0)|^\gamma\, dz = \lim_{r\to 0} \int_{Q_1(0)} |f(z_0 + r\zeta) - f(z_0)|^\gamma\, d\zeta = 0$$

for a.e. $z_0 \in \Omega$. In particular

$$\lim_{r\to 0} \int_{Q_1(0)} f(z_0 + r\zeta)\, \varphi(\zeta)\, d\zeta = \lim_{r\to 0} \fint_{Q_r(z_0)} f(z)\, \varphi_r(z)\, dz = f(z_0) \int_{Q_1(0)} \varphi(\zeta)\, d\zeta$$

for $\varphi \in C_0^\infty(Q_1(0))$, where $\varphi_r(z) = \varphi\left(\dfrac{z - z_0}{r}\right)$.

Moreover, we recall the classical theorem on the differentiation of a measure μ with respect to the Lebesgue measure (see [DS, Chap.III]). We will state it here in the particular case of measure μ_{r,z_0} defined on the unit square $Q_1(0)$ by the relation

$$\int_{Q_1(0)} \varphi\, d\mu_{r,z_0} = \fint_{Q_r(z_0)} \varphi_r\, d\mu$$

for $\varphi \in C_0^\infty(Q_1(0))$, where $\varphi_r(z) = \varphi\left(\dfrac{z - z_0}{r}\right)$.

Theorem 3.6. *For a.e. $z_0 \in \Omega$ (with respect to the Lebesgue measure), the relation*

$$d\mu_{r,z_0} \overset{*}{\rightharpoonup} a(z_0)\, dz \qquad \text{as } r \to 0$$

holds, where $d\mu^a = a(z)\, dz$ is the absolutely continuous component of the measure μ. In other words,

$$\lim_{r\to 0} \fint_{Q_r(z_0)} \varphi_r \, d\mu = a(z_0) \int_{Q_1(0)} \varphi \, d\zeta.$$

Proof of Theorem 3.5. Since $f_k \in W^{1,2}(\Omega) \times W^{1,2}(\Omega)$, then

$$J_{f_k} = \det Df_k = \operatorname{Det} Df_k.$$

As in Theorem 3.3, for any $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega \varphi \, d\mu_k = \int_\Omega J_{f_k} \varphi \, dz$$

$$\downarrow$$

$$\int_\Omega \varphi \, d\mu = -\int_\Omega u \left(\frac{\partial v}{\partial y} \frac{\partial \varphi}{\partial x} - \frac{\partial v}{\partial x} \frac{\partial \varphi}{\partial y} \right) dz = -\int_\Omega u w \cdot \nabla \varphi \, dz \tag{3.12}$$

where

$$w = \left(\frac{\partial v}{\partial y}, -\frac{\partial v}{\partial x} \right).$$

We consider

$$I = \fint_{Q_r(z_0)} \varphi_r \, d\mu$$

where $\varphi_r(z) = \varphi\left(\dfrac{z - z_0}{r} \right)$ with $\varphi \in C_0^\infty(Q_1(0))$. Let $t \in \mathbb{R}$ and $C \in \mathbb{R}^2$, by (3.12) e by the fact that $\operatorname{div} w = 0$ we have

$$I = -\fint_{Q_r(z_0)} u w \cdot \nabla \varphi_r \, dz = -\fint_{Q_r(z_0)} (u - t) w \cdot \nabla \varphi_r \, dz$$

$$= -\fint_{Q_r(z_0)} (u - t)(w - C) \cdot \nabla \varphi_r \, dz - \fint_{Q_r(z_0)} (u - t)\, C \cdot \nabla \varphi_r \, dz$$

$$= -\fint_{Q_r(z_0)} (u - t)(w - C) \cdot \nabla \varphi_r \, dz + \fint_{Q_r(z_0)} C \cdot \nabla u \, \varphi_r \, dz = I_1 + I_2$$

Let z_0 be a Lebesgue point of the functions w and ∇u, $C = w(z_0)$, $t = \fint_{Q_r(z_0)} u \, dz$ and $k_0 = \max |\nabla \varphi|$. By Hölder's inequality, Poincaré-Sobolev inequality and by the properties of Lebesgue points, we obtain

$$|I_1| \le k_0 \fint_{Q_r(z_0)} |w - w(z_0)| \left| \frac{u - t}{r} \right| dz$$

$$\leq k_0 \left(\fint_{Q_r(z_0)} |w - w(z_0)|^2 \, dz \right)^{1/2} \left(\fint_{Q_r(z_0)} \left| \frac{u - t}{r} \right|^2 \, dz \right)^{1/2}$$

$$\leq k_1 \left(\fint_{Q_r(z_0)} |w - w(z_0)|^2 \, dz \right)^{1/2} \left(\fint_{Q_r(z_0)} |\nabla u|^\alpha \, dz \right)^{1/\alpha} \to 0.$$

Hence

$$\lim_{r \to 0} I = \lim_{r \to 0} I_2 = \lim_{r \to 0} \fint_{Q_r(z_0)} w(z_0) \cdot \nabla u \, \varphi_r \, dz.$$

By the properties of Lebesgue points, we have

$$\lim_{r \to 0} I = w(z_0) \cdot \nabla u(z_0) \int_{Q_1(0)} \varphi \, d\zeta.$$

By Theorem 3.6, we conclude

$$d\mu^a = w(z) \nabla u(z) \, dz = J_f(z) \, dz$$

is the absolutely continuous component of the measure μ, so

$$d\mu = J_f \, dz + d\mu^s$$

where μ^s is a singular measure with respect to the Lebesgue measure on Ω.

□

Chapter 4

G-convergence and Γ-convergence in dimension one

The aim of this chapter is to present some extension to degenerate functionals of the one dimensional Calculus of Variations some Γ-convergence and G-convergence results, which are well know under more restrictive assumptions.

One of the result presented states that if $a_j = a_j(x)$ $(j = 1, 2, \ldots)$ and $a = a(x)$ are non-negative functions belonging to Lebesgue space $L^1(0,1)$, $p > 1$, $a_j^{-1/(p-1)}$ is a bounded sequence in $L^1(0,1)$ and $a_j^{-1/(p-1)}$ is equi-integrable, then the sequence of functionals defined on $W^{1,p}(0,1)$

$$F_j(u) = \int_0^1 a_j(x)\,|u'|^p\,dx \tag{4.1}$$

Γ-converges in $W^{1,1}(0,1)$ with respect to weak topology to the functional

$$F(u) = \int_0^1 a(x)\,|u'|^p\,dx \tag{4.2}$$

if and only if

$$\frac{1}{a_j^{1/(p-1)}} \rightharpoonup \frac{1}{a^{1/(p-1)}} \quad \text{weakly in } \; L^1(0,1).$$

This result is an extension to the weaker topology $\sigma(L^1, L^\infty)$ of the result of [S1] relative to the $\sigma(L^\infty, L^1)$ topology.

In this general setting it is convenient to compare the G-convergence, i.e. the weak-$W^{1,1}$ convergence of the solutions of boundary value problems, with the Γ-convergence of functionals (4.1) in the space $W^{1,1}(0,1)$ equipped with its weak convergence.

Despite the fact that, enlarging the space where the functional (4.2) is defined, the Γ-limit may degenerate into a non integral functional (see [MS1], [B], see also Section 4.3), we confirm the equivalence which is well know when the natural coerciveness space is $W^{1,1+\varepsilon}$ ($\varepsilon > 0$).

Moreover, we prove that the G-convergence implies the convergence of minima values of functionals (4.1).

We point out another interesting feature of the functional (4.2) when we assume only

$$a = a(x) \geq 0 \text{ and } a \in L^1(0,1), \frac{1}{a^{1/(p-1)}} \in L^1(0,1). \tag{4.3}$$

Thanks to the formula (see [M])

$$\inf_{v \in V} \int_0^1 a(x)\,|v'|^p\,dx = \frac{1}{\left(\int_0^1 \frac{1}{a(x)^{1/(p-1)}}\,dx\right)^{p-1}}, \tag{4.4}$$

where

$V = \{v \in C^\infty(0,1) : v$ is non-decreasing, $v(0) = 0$, $v(1) = 1$, $\operatorname{supp} v' \subset (0,1)\}$,

we have that if $a = a(x)$ satisfies the assumptions (4.3), then the infimum in (4.4) equals the minimum value of the same functional on the natural domain $id + W_0^{1,1}(0,1)$, where id is the identity function.

4.1 Definitions of Γ-convergence and G-convergence in dimension one

In dimension one by Γ-convergence we mean the following (see [B]).

Let (X, d) be a metric space, $F_j = F_j(u)$ ($j = 1, 2, \ldots$) and $F = F(u)$ functions from X into $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$

Definition 4.1. We say that the sequence F_j Γ-converges in X to F, and we write $F_j \xrightarrow{\Gamma_X} F$, if the following two conditions are verified

i) for every $u_j, u \in X$ such that $u_j \xrightarrow{d} u$,

$$F(u) \leq \liminf_j F_j(u_j); \tag{4.5}$$

ii) for every $u \in X$ there exists a sequence $\{u_j\} \subset X$ such that $u_j \xrightarrow{d} u$ and

$$F(u) = \lim_j F_j(u_j). \tag{4.6}$$

The function F is called Γ-*limit* of the sequence F_j.

Definition 4.2. Let $u \in X$. We say that the sequence F_j Γ-converges at u to the value $F(u)$, and we write $F(u) = \Gamma\text{-}\lim_j F_j(u)$, if the following two conditions are verified

i) for every $u_j \in X$ such that $u_j \xrightarrow{d} u$, the inequality (4.5) holds;

ii) there exists a sequence $\{u_j\} \subset X$ such that $u_j \xrightarrow{d} u$ and (4.6) is satisfied.

With this notation, F_j Γ-converges in X to F if and only if $F(u) = \Gamma\text{-}\lim_j F_j(u)$ at all $u \in X$.

Definition 4.3. Let $u \in X$. The quantity

$$\Gamma\text{-}\liminf_j F_j(u) = \inf\{\liminf_j F_j(u_j) : u_j \xrightarrow{d} u\}$$

is called the Γ-*lower limit* of the sequence F_j at u. The quantity

$$\Gamma\text{-}\limsup_j F_j(u) = \inf\{\limsup_j F_j(u_j) : u_j \xrightarrow{\tau} u\}$$

is called the Γ-*upper limit* of the sequence F_j at u. If we have the equality

$$\Gamma\text{-}\liminf_j F_j(u) = \lambda = \Gamma\text{-}\limsup_j F_j(u)$$

for some $\lambda \in \overline{\mathbb{R}}$, then we write

$$\lambda = \Gamma\text{-}\lim_j F_j(u) \tag{4.7}$$

and we say that λ is the Γ-limit of sequence F_j at u.

Remark 4.1. *Clearly, the Γ-lower limit and the Γ-upper limit exist at every point $u \in X$. Definition 4.3 is in agreement with Definition 4.2, and we can say that a sequence F_j Γ-converges in X to F if and only if for fixed $u \in X$ the Γ-limit exists and we have $\lambda = F(u)$ in (4.7).*

For our purposes it will be convenient to introduce the following notion of G-convergence.

We consider the non-linear degenerate non-uniformly elliptic operators

$$\mathcal{A}_j = -\frac{d}{dx}\left(a_j(x)\left|\frac{d}{dx}\right|^{p-2}\frac{d}{dx}\right) : W^{1,p}(0,1) \to W^{-1,p'}(0,1)$$

and

$$\mathcal{A} = -\frac{d}{dx}\left(a(x)\left|\frac{d}{dx}\right|^{p-2}\frac{d}{dx}\right) : W^{1,p}(0,1) \to W^{-1,p'}(0,1),$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$, with $a_j = a_j(x)$ and $a = a(x)$ under the assumptions

$$a_j \geq 0 \ \text{ and } \ a_j \in L^1(0,1), \frac{1}{a_j^{1/(p-1)}} \in L^1(0,1) \tag{4.8}$$

$$a \geq 0 \ \text{ and } \ a \in L^1(0,1), \frac{1}{a^{1/(p-1)}} \in L^1(0,1). \tag{4.9}$$

Definition 4.4. We say that the sequence $\mathcal{A}_j$ G-converges to $\mathcal{A}$, and we write $\mathcal{A}_j \xrightarrow{G} \mathcal{A}$, if

$$u_j \rightharpoonup u \ \text{ weakly in } \ W^{1,1}(0,1)$$

where u_j and u are the solutions of the Dirichlet problems

$$\begin{cases} \mathcal{A}_j[u_j] = 0 & \text{in } (0,1) \\ u_j \in id + W_0^{1,1}(0,1) \end{cases}$$

$$\begin{cases} \mathcal{A}[u] = 0 & \text{in } (0,1) \\ u \in id + W_0^{1,1}(0,1) \end{cases}$$

respectively.

4.2 Existence of minima for degenerate functionals

We will follow an idea of [DV] to prove the existence of the minimum point for degenerate functional (4.2).

Theorem 4.1. *Under the assumptions* (4.9) *there exists an unique solution of the following variational problem*

$$\min_{v\in id+W_0^{1,1}(0,1)} \int_0^1 a(x)\,|v'|^p\,dx.$$

Proof. Let v_j be a minimizing sequence in $id+W_0^{1,1}(0,1)$ for the functional

$$\int_0^1 a(x)\,|v'|^p\,dx, \tag{4.10}$$

that is

$$\{v_j\}\subset id+W_0^{1,1}(0,1)$$

and

$$\int_0^1 a(x)\,|v_j'|^p\,dx \to \inf_{v\in id+W_0^{1,1}(0,1)} \int_0^1 a(x)\,|v'|^p\,dx = I. \tag{4.11}$$

For any measurable subset E of $(0,1)$, using Hölder's inequality and arguing as in [DV] we have

$$\int_E |v_j'|\,dx = \int_E \frac{1}{a(x)^{1/p}}\,a(x)^{1/p}\,|v_j'|\,dx$$

$$\leq \left(\int_E \left(\frac{1}{a(x)^{1/p}}\right)^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \left(\int_E (a(x)^{1/p}\,|v_j'|)^p\,dx\right)^{\frac{1}{p}}$$

then

$$\int_E |v_j'|\,dx \leq \left(\int_E \frac{1}{a(x)^{1/(p-1)}}\,dx\right)^{\frac{p-1}{p}} \left(\int_0^1 a(x)\,|v_j'|^p\,dx\right)^{\frac{1}{p}}$$

so

$$\left(\int_E |v_j'|\,dx\right)^{\frac{p}{p-1}} \leq \int_E \frac{1}{a(x)^{1/(p-1)}}\,dx \left(\int_0^1 a(x)\,|v_j'|^p\,dx\right)^{\frac{1}{p-1}}.$$

By (4.11) we obtain

$$\left(\int_E |v_j'|\,dx\right)^{\frac{p}{p-1}} \leq c_\sigma \int_E \frac{1}{a(x)^{1/(p-1)}}\,dx \qquad \text{for } j>j_0(\sigma) \tag{4.12}$$

where $c_\sigma=(\sigma+I)^{\frac{1}{p-1}}$.

Since $\dfrac{1}{a^{1/(p-1)}}\in L^1(0,1)$, by the absolutely continuity of the integral we have that for any $\varepsilon>0$ there exists $\delta>0$ such that

$$E\subset(0,1),\ |E|<\delta \quad\Rightarrow\quad \int_E \frac{1}{a(x)^{1/(p-1)}}\,dx<\varepsilon. \tag{4.13}$$

Therefore (4.12) and (4.13) imply

$$\left(\int_E |v_j'|\,dx\right)^{\frac{p}{p-1}} \le c_\sigma\,\varepsilon \qquad \text{for } j > j_0(\sigma),$$

consequently

$$v_j' \text{ is equi-integrable.}$$

Moreover, we have

$$\left(\int_0^1 |v_j'|\,dx\right)^{\frac{p}{p-1}} \le c_\sigma \int_0^1 \frac{1}{a(x)^{1/(p-1)}}\,dx \qquad \text{for } j > j_0(\sigma)$$

so

$$\sup_j \|v_j'\|_{L^1(0,1)} \le\, c.$$

Hence, by Dunford-Pettis Theorem, upon extracting a subsequence we may suppose that

$$v_j' \rightharpoonup w \;\text{ weakly in }\; L^1(0,1). \tag{4.14}$$

Since $v_j \in id + W_0^{1,1}(0,1)$, by Poincaré inequality and by (4.14) we obtain

$$\sup_j \|v_j\|_{W^{1,1}(0,1)} \le\, c' \sup_j \|v_j'\|_{L^1(0,1)} \le\, c''$$

and therefore by the classic Sobolev Imbedding Theorem in dimension one, upon extracting a subsequence we may suppose that

$$v_j \rightharpoonup v_0 \;\text{ strongly in }\; L^q(0,1) \qquad \text{for any } q \ge 1.$$

We then have for all $\varphi \in C_0^1(0,1)$

$$\int_0^1 v_0\,\varphi' dx = \lim_j \int_0^1 v_j\,\varphi' dx = -\lim_j \int_0^1 v_j'\,\varphi\,dx = -\int_0^1 w\,\varphi\,dx,$$

which shows that v_0 is weakly differentiable and $v_0' = w$, so that

$$v_j \rightharpoonup v_0 \;\text{ weakly in }\; W^{1,1}(0,1). \tag{4.15}$$

Since $\{v_j\} \subset id + W_0^{1,1}(0,1)$, by (4.15) we infer

$$v_0 \in id + W_0^{1,1}(0,1).$$

Thanks to lower semicontinuity of the integral functional

$$\int_0^1 a(x)\,|v'|^p\,dx$$

with respect to the weak convergence in $W^{1,1}(0,1)$ we deduce that v_0 is a minimum point for the functional (4.10).

The uniqueness follows from the strict convexity of our functional. □

Theorem 4.2. *Under the assumptions* (4.9) *we have*

$$\min_{v\in id+W_0^{1,1}(0,1)} \int_0^1 a(x)\,|v'|^p\,dx = \frac{1}{\left(\displaystyle\int_0^1 \frac{1}{a(x)^{1/(p-1)}}\,dx\right)^{p-1}}.$$

Proof. Let $v \in id + W_0^{1,1}(0,1)$ we have

$$\begin{aligned} 1 = \int_0^1 v'\,dx \le \int_0^1 |v|'\,dx &= \int_0^1 \frac{1}{a(x)^{1/p}}\,a(x)^{1/p}\,|v'|\,dx \\ &\le \left(\int_0^1 \frac{1}{a(x)^{1/(p-1)}}\,dx\right)^{\frac{p-1}{p}} \left(\int_0^1 a(x)\,|v'|^p\,dx\right)^{\frac{1}{p}} \end{aligned}$$

then

$$\int_0^1 a(x)\,|v'|^p\,dx \ge \frac{1}{\left(\displaystyle\int_0^1 \frac{1}{a(x)^{1/(p-1)}}\,dx\right)^{p-1}}$$

so

$$\min_{v\in id+W_0^{1,1}(0,1)} \int_0^1 a(x)\,|v'|^p\,dx \ge \frac{1}{\left(\displaystyle\int_0^1 \frac{1}{a(x)^{1/(p-1)}}\,dx\right)^{p-1}}.$$

On the other hand

$$\inf_{v\in V} \int_0^1 a(x)\,|v'|^p\,dx \ge \min_{v\in id+W_0^{1,1}(0,1)} \int_0^1 a(x)\,|v'|^p\,dx,$$

where

$$V = \{v \in C^\infty(0,1) : v \text{ is non-decreasing, } v(0)=0,\ v(1)=1,\ \operatorname{supp} v' \subset (0,1)\},$$

since

$$\inf_{v\in V} \int_0^1 a(x)\,|v'|^p\,dx = \frac{1}{\left(\displaystyle\int_0^1 \frac{1}{a(x)^{1/(p-1)}}\,dx\right)^{p-1}}$$

(see [M]) we conclude

$$\min_{v\in id+W_0^{1,1}(0,1)} \int_0^1 a(x)\,|v'|^p\,dx = \frac{1}{\left(\displaystyle\int_0^1 \frac{1}{a(x)^{1/(p-1)}}\,dx\right)^{p-1}}.$$

□

4.3 The G-convergence results

Let us start by proving the following theorem.

Theorem 4.3. *If* $a_j = a_j(x)$ $(j = 1, 2, \ldots)$ *and* $a = a(x)$ *satisfy the assumptions* (4.8) *and* (4.9), $p > 1$, $a_j^{-1/(p-1)}$ *is a bounded sequence in* $L^1(0,1)$ *and* $a_j^{-1/(p-1)}$ *is equi-integrable, then*

$$\mathcal{A}_j = -\frac{d}{dx}\left(a_j(x)\left|\frac{d}{dx}\right|^{p-2}\frac{d}{dx}\right) \xrightarrow{G} \mathcal{A} = -\frac{d}{dx}\left(a(x)\left|\frac{d}{dx}\right|^{p-2}\frac{d}{dx}\right)$$

if and only if

$$\frac{1}{a_j^{1/(p-1)}} \rightharpoonup \frac{1}{a^{1/(p-1)}} \quad \textit{weakly in} \quad L^1(0,1).$$

Proof. By compactness it will be enough to prove that

$$\frac{1}{a_j^{1/(p-1)}} \rightharpoonup \frac{1}{a^{1/(p-1)}} \quad \text{weakly in} \quad L^1(0,1)$$

implies

$$\mathcal{A}_j \xrightarrow{G} \mathcal{A}.$$

Let u_j be the solution of the Dirichlet problem

$$\begin{cases} -\dfrac{d}{dx}\left(a_j(x)\,|u_j'(x)|^{p-2}\,u_j'(x)\right) = 0 \quad \text{in} \quad (0,1) \\ u_j \in id + W_0^{1,1}(0,1). \end{cases}$$

By

$$-\frac{d}{dx}\left(a_j(x)\,|u_j'(x)|^{p-2}\,u_j'(x)\right) = 0$$

we obtain

$$a_j(x)\,|u_j'(x)|^{p-2}\,u_j'(x) = c_j$$

then

$$u_j'(x)^{p-1} = \frac{c_j}{a_j(x)}$$

so

$$u_j'(x) = \frac{c_j^{1/(p-1)}}{a_j(x)^{1/(p-1)}}.$$

Since

$$1 = \int_0^1 u_j'(x)\,dx = c_j^{1/(p-1)} \int_0^1 \frac{1}{a_j(x)^{1/(p-1)}}\,dx$$

we have

$$c_j^{1/(p-1)} = \frac{1}{\displaystyle\int_0^1 \frac{1}{a_j(x)^{1/(p-1)}}\,dx}$$

so

$$u_j'(x) = \frac{\dfrac{1}{a_j(x)^{1/(p-1)}}}{\displaystyle\int_0^1 \frac{1}{a_j(x)^{1/(p-1)}}\,dx}$$

and

$$u_j(x) = \int_0^x u_j'(t)\,dt = \frac{\displaystyle\int_0^x \frac{1}{a_j(t)^{1/(p-1)}}\,dt}{\displaystyle\int_0^1 \frac{1}{a_j(t)^{1/(p-1)}}\,dt}.$$

By hypothesis we obtain

$$u_j(x) = \frac{\displaystyle\int_0^x \frac{1}{a_j(t)^{1/(p-1)}}\,dt}{\displaystyle\int_0^1 \frac{1}{a_j(t)^{1/(p-1)}}\,dt} \to \frac{\displaystyle\int_0^x \frac{1}{a(t)^{1/(p-1)}}\,dt}{\displaystyle\int_0^1 \frac{1}{a(t)^{1/(p-1)}}\,dt} = u(x) \quad \text{q.o.}$$

$$u_j' = \frac{\dfrac{1}{a_j^{1/(p-1)}}}{\displaystyle\int_0^1 \frac{1}{a_j^{1/(p-1)}}\,dx} \rightharpoonup \frac{\dfrac{1}{a^{1/(p-1)}}}{\displaystyle\int_0^1 \frac{1}{a^{1/(p-1)}}\,dx} = u' \ \text{ weakly in } \ L^1(0,1)$$

so

$$u_j \rightharpoonup u \ \text{ weakly in } \ W^{1,1}(0,1),$$

where $u(x)$ is the solution of the Dirichlet problem

$$\begin{cases} -\dfrac{d}{dx}\left(a(x)\,|u'(x)|^{p-2}\,u'(x)\right) = 0 \quad \text{in } (0,1) \\ u \in id + W_0^{1,1}(0,1). \end{cases}$$

□

As consequence of Theorem 4.3 we have

Corollary 4.4. *Under the assumptions of Theorem 4.3 we have*

$$\mathcal{A}_j = -\frac{d}{dx}\left(a_j(x)\left|\frac{d}{dx}\right|^{p-2}\frac{d}{dx}\right) \xrightarrow{G} \mathcal{A} = -\frac{d}{dx}\left(a(x)\left|\frac{d}{dx}\right|^{p-2}\frac{d}{dx}\right)$$

if and only if

$$F_j(u) = \int_0^1 a_j(x)\,|u'|^p\,dx \xrightarrow{\Gamma_{W^{1,1}}} F(u) = \int_0^1 a(x)\,|u'|^p\,dx.$$

Finally we prove the following theorem.

Theorem 4.5. *Under the assumptions of Theorem 4.3 we have that if*

$$\mathcal{A}_j = -\frac{d}{dx}\left(a_j(x)\left|\frac{d}{dx}\right|^{p-2}\frac{d}{dx}\right) \xrightarrow{G} \mathcal{A} = -\frac{d}{dx}\left(a(x)\left|\frac{d}{dx}\right|^{p-2}\frac{d}{dx}\right),$$

then

$$\min_{v\in id+W_0^{1,1}(0,1)}\int_0^1 a_j(x)\,|v'|^p\,dx \to \min_{v\in id+W_0^{1,1}(0,1)}\int_0^1 a(x)\,|v'|^p\,dx.$$

Proof. The assert follows by Theorem 4.3 and Theorem 4.2. □

For the sake of clarity let us develop a well know example of a sequence of functionals of type (4.1) under assumptions of type (4.8) where not only the Γ-limit's domain is not the same as the one of the starting functionals, but the form of the Γ-limit is not an integral if one extends it to larger space where the functionals are defined.

Consider the functionals

$$F_j(u) = \begin{cases} \displaystyle\int_{-1}^1 a_j(x)\,|u'|^2\,dx & \text{if } u \in W^{1,2}(-1,1) \\ +\infty & \text{otherwise} \end{cases}$$

where

$$a_j(x) = \begin{cases} \dfrac{1}{j} & \text{if } |x| \le 1/2j \\ 1 & \text{if } |x| > 1/2j \end{cases}$$

Then

$$a_j > 0,\ a_j \in L^1(-1,1),\ \frac{1}{a_j} \in L^1(-1,1)$$

and

$$\int_{-1}^1 \frac{1}{a_j(x)}\,dx = \int_{-1}^{-\frac{1}{2j}} dx + \int_{-\frac{1}{2j}}^{\frac{1}{2j}} j\,dx + \int_{\frac{1}{2j}}^1 dx = 2 - \frac{1}{j} \le 2$$

for every $j \in \mathbb{N}$.

We want to compute the Γ-limit with respect to the L^2-convergence.

We observe that if

$$u_j \to u \text{ strongly in } L^2(-1,1) \text{ and } \sup_j F_j(u_j) < \infty,$$

then u_j is weakly compact in $W^{1,2}((-1,-1/k)\cup(1/k,1))$ for every $k>1$, and

$$\sup_k \|u'\|_{L^2((-1,-1/k)\cup(1/k,1))} \leq \sup_j F_j(u_j) \leq c$$

indipendently of k, so that $u \in W^{1,2}((-1,1)\setminus\{0\})$. In particular the values $u(0\pm)$ are well defined and we have

$$\lim_j u_j\left(\pm\frac{1}{2j}\right) = \lim_j u\left(\pm\frac{1}{2j}\right) = u(0\pm). \tag{4.16}$$

For each fixed k we have

$$\liminf_j F_j(u_j) \geq \liminf_j \int_{-1}^{-\frac{1}{k}} |u_j'|^2\,dx + \liminf_j \frac{1}{j}\int_{-\frac{1}{2j}}^{\frac{1}{2j}} |u_j'|^2\,dx + \liminf_j \int_{\frac{1}{k}}^{1} |u_j'|^2\,dx. \tag{4.17}$$

By Jensen's inequality

$$\fint_{-\frac{1}{2j}}^{\frac{1}{2j}} |u_j'|^2\,dx \geq \left(\fint_{-\frac{1}{2j}}^{\frac{1}{2j}} u_j'\,dx\right)^2$$

so

$$\int_{-\frac{1}{2j}}^{\frac{1}{2j}} |u_j'|^2\,dx \geq \frac{1}{j}\left|j\int_{-\frac{1}{2j}}^{\frac{1}{2j}} u_j'\,dx\right|^2 = j\left|u_j\left(\frac{1}{2j}\right) - u_j\left(-\frac{1}{2j}\right)\right|^2. \tag{4.18}$$

Therefore (4.17) and (4.18) imply

$$\liminf_j F_j(u_j) \geq \int_{-1}^{-\frac{1}{k}} |u'|^2\,dx + \int_{\frac{1}{k}}^{1} |u'|^2\,dx + \lim_j \left|u_j\left(\frac{1}{2j}\right) - u_j\left(-\frac{1}{2j}\right)\right|^2.$$

By (4.16) we obtain

$$\liminf_j F_j(u_j) \geq \int_{-1}^{-\frac{1}{k}} |u'|^2\,dx + \int_{\frac{1}{k}}^{1} |u'|^2\,dx + |u(0+)-u(0-)|^2.$$

By taking the supremum over all k we get that

$$\Gamma\text{-}\liminf_j F_j(u) \geq \int_{(-1,1)\setminus\{0\}} |u'|^2\,dx + |u(0+)-u(0-)|^2$$

if $u \in W^{1,2}((-1,1)\setminus\{0\})$.

Conversely, if $u \in W^{1,2}((-1,1)\setminus\{0\})$ a recovery sequence is constructed by taking

$$u_j(x) = \begin{cases} j(u(0+)-u(0-))x + \dfrac{u(0+)+u(0-)}{2} & \text{if } |x| \leq 1/2j \\ u\left(x - \dfrac{x}{2j|x|}\right) & \text{if } |x| > 1/2j \end{cases}$$

to show that

$$\Gamma\text{-}\limsup_j F_j(u) \leq \int_{(-1,1)\setminus\{0\}} |u'|^2\, dx + |u(0+) - u(0-)|^2.$$

Therefore

$$\Gamma\text{-}\lim_j F_j(u) = \int_{(-1,1)\setminus\{0\}} |u'|^2\, dx + |u(0+) - u(0-)|^2.$$

Chapter 5

Γ-convergence of quadratic functionals in the plane

In this last chapter we dealt with *Laplace-Beltrami operator* in the plane associated to homeomorphisms with finite distortion.

In Section 5.3, assuming that Ω and Ω' are bounded planar domains, with Ω sufficiently smooth, we prove that if a sequence of homeomorphisms $f_j : \Omega \xrightarrow{\text{onto}} \Omega'$ of Sobolev space $W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2)$ with finite distortion K_j satisfying

$$\int_\Omega e^{\frac{K_j(z)}{\lambda}}\, dz \le c_0 \;\text{ for every }\; j \in \mathbb{N} \tag{5.1}$$

for some $\lambda \in (0, 1/2)$ and $c_0 > 0$ and if f_j weakly converges in $W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2)$ to a homeomorphism f, then the matrices A_{f_j} of the Laplace-Beltrami operators associated to the sequence f_j Γ-converge in the Zygmund-Sobolev space $W^{1,L^2 \log L}$ to the matrix A_f of the Laplace-Beltrami operator associated to f .

Moreover, we show that the limit homeomorphism $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2)$ of a weakly convergent sequence of homeomorphisms $f_j \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2)$ with finite distortion K_j satisfying (5.1) also has finite distortion with distortion function K_f satisfying the same condition

$$\int_\Omega e^{\frac{K_f(z)}{\lambda}}\, dz \le c_0.$$

5.1 Laplace-Beltrami operator in the plane

An quantity associated to a mapping $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ with finite distortion is the mapping

$$G_f : \Omega \to \mathbb{R}^{2\times 2},$$

called *distortion tensor*, defined by

$$G_f(z) = \begin{cases} \dfrac{Df(z)^t Df(z)}{J_f(z)} & \text{if } Df(z) \text{ exists and } J_f(z) > 0 \\ I & \text{otherwise} \end{cases}$$

where $Df(z)^t$ denotes the transpose of the differential matrix of f and I denotes the identity matrix.

It is easy to check that G_f is a symmetric matrix with

$$\det G_f(z) = 1 \ \text{ for a.e. } z \in \Omega$$

and that the distortion inequality for f

$$|Df(z)|^2 \le K(z) J_f(z) \ \text{ for a.e. } z \in \Omega$$

is equivalent to the condition

$$\frac{|\xi|^2}{K(z)} \le \langle G_f(z)\xi, \xi\rangle \le K(z)|\xi|^2$$

for a.e. $z \in \Omega$ and for any $\xi \in \mathbb{R}^2$. In fact, for any matrix $F \in \mathbb{R}^{2\times 2}$ with $\det F > 0$, we can consider

$$G = \frac{F^t F}{\det F}.$$

Then, obviously G is a symmetric matrix and

$$\det G = 1.$$

Moreover, denoting by $\|\cdot\|$ the Hilbert-Schmidt norm of F, i.e.

$$\|F\|^2 = \mathrm{trace}(F^t F),$$

the inequality

$$|F|^2 \le K \det F$$

is equivalent to

$$\|F\|^2 \le \left(K + \frac{1}{K}\right) \det F. \tag{5.2}$$

One can easily check that (5.2) is equivalent to

$$\operatorname{trace}(G) \le K + \frac{1}{K}. \tag{5.3}$$

Let λ and $\frac{1}{\lambda}$ be the eigenvalues of G. Then the inequality (5.3) means

$$\lambda + \frac{1}{\lambda} \le K + \frac{1}{K}$$

which implies

$$\frac{1}{K} \le \lambda \le K.$$

We are interested into the inverse matrix of G_f

$$A_f(z) = G_f(z)^{-1} = J_f(z) \left[Df(z)^t Df(z)\right]^{-1}$$

which obviously is a symmetric matrix with

$$\det A_f(z) = 1 \text{ for a.e. } z \in \Omega$$

and satisfies the same condition

$$\frac{|\xi|^2}{K(z)} \le \langle A_f(z)\xi, \xi\rangle \le K(z)|\xi|^2 \tag{5.4}$$

for a.e. $z \in \Omega$ and for any $\xi \in \mathbb{R}^2$.

Connections between mappings with finite distortion and PDE's are established via the Laplace-Beltrami operator

$$\mathcal{L}_f = \operatorname{div}(A_f(z)\nabla).$$

Notice that the components f^i $(i = 1, 2)$ of f solve the equations

$$\begin{cases} \mathcal{L}_f[f^i] = 0 & \text{in } \Omega \\ \langle A_f(z)\nabla f^i, \nabla f^j\rangle = \delta_{ij} J_f(z) & \text{for a.e. } z \in \Omega \end{cases}$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}$$

(see [BI], [Sp2]).

Now observe that, given a quadratic integral functional

$$F(u) = \int_\Omega \langle A(z)\nabla u, \nabla u\rangle \, dz$$

under the assumption

$$\frac{|\xi|^2}{K(z)} \leq \langle A(z)\xi, \xi\rangle \leq K(z)|\xi|^2$$

for a.e. $z \in \Omega$ and for any $\xi \in \mathbb{R}^2$, with $1 \leq K \in EXP(\Omega)$ and by duality between $EXP(\Omega)$ and $L\log L(\Omega)$, there exist constants $c, C > 0$ such that

$$c\int_\Omega |\nabla u|^2 \log^{-1}\left(e + \frac{|\nabla u|}{\|\nabla u\|_{L^2(\Omega)}}\right) dz \leq F(u) \leq C\int_\Omega |\nabla u|^2 \log\left(e + \frac{|\nabla u|}{\|\nabla u\|_{L^2(\Omega)}}\right) dz.$$

5.2 Definitions of Γ-convergence in dimension two

In this section we will use the De Giorgi's notion of Γ-convergence (see [DF]).

Let $\Omega \subset \mathbb{R}^2$, $A_j = A_j(z)$ (j=1,2,...) and $A = A(z)$ be symmetric 2×2 matrix functions satisfying the conditions

$$0 \leq \langle A_j(z)\xi, \xi\rangle \leq K_j(z)|\xi|^2 \tag{5.5}$$

$$0 \leq \langle A(z)\xi, \xi\rangle \leq K(z)|\xi|^2 \tag{5.6}$$

for a.e. $z \in \Omega$ and for any $\xi \in \mathbb{R}^2$, with $K_j, K \geq 1$ defined in Ω belonging to the space $EXP(\Omega)$.

Definition 5.1. We say that the sequence A_j $\Gamma_{L^2\log L}$-converges to A, and we write $A_j \xrightarrow{\Gamma_{L^2\log L}} A$, if the following two conditions are verified:

i) for every $u_j, u \in W^{1,L^2\log L}(\Omega)$ such that $u_j \to u$ in $L^2\log L(\Omega)$,

$$\int_\Omega \langle A(z)\nabla u, \nabla u\rangle \, dz \leq \liminf_j \int_\Omega \langle A_j(z)\nabla u_j, \nabla u_j\rangle \, dz; \tag{5.7}$$

ii) for every $u \in W^{1,L^2\log L}(\Omega)$ there exists a sequence $\{u_j\} \subset W^{1,L^2\log L}(\Omega)$ such that $u_j \to u$ in $L^2\log L(\Omega)$ and

$$\int_\Omega \langle A(z)\nabla u, \nabla u\rangle \, dz = \lim_j \int_\Omega \langle A_j(z)\nabla u_j, \nabla u_j\rangle \, dz. \tag{5.8}$$

Remark 5.1. *The assumptions K_j and K belonging to $EXP(\Omega)$ guarantee that the integrals above are finite. In fact by* (5.6) *and* (1.9) *we have*

$$\int_\Omega \langle A(z)\nabla u, \nabla u\rangle \, dz \leq \int_\Omega K(z)|\nabla u|^2 \, dz \leq c \, \|K\|_{EXP(\Omega)} \||\nabla u|\|^2_{L^2 \log L(\Omega)}.$$

If one assumes only that K and K_j belong to $L^1(\Omega)$, then one must confine to Lipschitz functions. In this case we speak of Γ-convergence. Precisely, let $A_j = A_j(z)$ (j=1,2,...) and $A = A(z)$ be symmetric 2×2 matrix functions satisfying (5.5) and (5.6) respectively, with $K_j, K \geq 1$ defined in Ω belonging to $L^1(\Omega)$.

Definition 5.2. We say that the sequence A_j Γ-converges to A, and we write $A_j \xrightarrow{\Gamma} A$, if the following two conditions are verified:

j) for every $u_j, u \in Lip(\Omega)$ such that $u_j \to u$ in $L^1(\Omega)$, the inequality (5.7) holds;

jj) for every $u \in Lip(\Omega)$ there exists a sequence $\{u_j\} \subset Lip(\Omega)$ such that $u_j \to u$ in $L^1(\Omega)$ and the condition (5.8) is satisfied.

Remark 5.2. *Well know general properties of* Γ*-convergence ensure that the conditions* ***j)*** *and* ***jj)*** *remain valid if we replace* Ω *by any open subset of* Ω.

A compactness result concerning Γ-convergence due to Marcellini-Sbordone (see [MS1] and [CS]), will be useful in the following

Theorem 5.1. *Let A_j be a sequence of symmetric 2×2 matrix functions satisfying* (5.5)*. If*

$$K_j \rightharpoonup K \ \text{ weakly in } \ L^1(\Omega),$$

then there exists a subsequence A_{j_r} Γ-converging to a symmetric matrix function A. Moreover, this matrix A also satisfies (5.6)*.*

We emphasize that, in the special case where f_j and f are K-quasiconformal, then the coefficient matrices $A_{f_j}(z)$ and $A_f(z)$ of the Laplace-Beltrami operators associated to f_j and f satisfy

$$\frac{|\xi|^2}{K} \leq \langle A_{f_j}(z)\xi, \xi\rangle \leqslant K|\xi|^2$$

$$\frac{|\xi|^2}{K} \leqslant \langle A_f(z)\xi, \xi\rangle \leqslant K|\xi|^2$$

for a.e. $z \in \Omega$ and for any $\xi \in \mathbb{R}^2$, with K constant greater or equal to 1. Therefore A_{f_j} and A_f are bounded and uniformly elliptic and Γ-convergence and G-convergence, in the sense of L^2-convergence of solutions of the Dirichlet problems (see [Sp1] and [Sp2]), are equivalent. More precisely, let Ω be a bounded domain of $\mathbb{R}^2$, $A_j = A_j(z)$ (j=1,2,...) and $A = A(z)$ be symmetric 2×2 matrix functions satisfying the conditions

$$\frac{|\xi|^2}{K} \leqslant \langle A_j(z)\xi, \xi \rangle \leqslant K|\xi|^2$$

$$\frac{|\xi|^2}{K} \leqslant \langle A(z)\xi, \xi \rangle \leqslant K|\xi|^2$$

for a.e. $z \in \Omega$ and for any $\xi \in \mathbb{R}^2$, with $K \geq 1$. We can consider the functionals

$$F_j(u) = \int_\Omega \langle A_j(z)\nabla u, \nabla u \rangle \, dz$$

$$F(u) = \int_\Omega \langle A(z)\nabla u, \nabla u \rangle \, dz$$

and the elliptic operators

$$\mathcal{L}_j = -\mathrm{div}(A_j(z)\nabla) : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$$

$$\mathcal{L} = -\mathrm{div}(A(z)\nabla) : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega).$$

We have that F_j Γ_{L^2}-converges to F if and only if $\mathcal{L}_j$ G-converges to $\mathcal{L}$ on Ω, that is if and only if $u_j(\varphi) \to u(\varphi)$ in $L^2(\Omega)$ for any $\varphi \in W^{-1,2}(\Omega)$, where $u_j = u_j(\varphi)$ and $u = u(\varphi)$ are the unique solutions of the Dirichlet problems

$$\begin{cases} \mathcal{L}_j[u_j] = \varphi & \text{in } \Omega \\ u_j \in W_0^{1,2}(\Omega) \end{cases}$$

$$\begin{cases} \mathcal{L}[u] = \varphi & \text{in } \Omega \\ u \in W_0^{1,2}(\Omega) \end{cases}$$

respectively.

5.3 The Convergence Theorem

In this section we assume that Ω and Ω' are bounded planar domains, with Ω sufficiently smooth and consider a sequence of homeomorphisms $f_j = (f_j^1, f_j^2)$:

$\Omega \xrightarrow{\text{onto}} \Omega'$ of Sobolev space $W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2)$ with finite distortion K_j that is satisfying the distortion inequality

$$|Df_j(z)|^2 \leq K_j(z) J_{f_j}(z) \text{ for a.e. } z \in \Omega. \tag{5.9}$$

We will make the following assumptions:

- there exists $\lambda > 0$ and $c_0 > 0$ such that

$$\int_\Omega e^{\frac{K_j(z)}{\lambda}}\, dz \leq c_0 \text{ for every } j \in \mathbb{N} \tag{5.10}$$

-

$$f_j \rightharpoonup f = (f^1, f^2) \text{ weakly in } W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2) \tag{5.11}$$

  where $f \in \text{Hom}(\Omega, \Omega')$.

Notice that we are not assuming $f_j \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^2)$. Actually $f_j \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2)$ with finite distortion $K_j \in EXP(\Omega)$ implies

$$|Df_j| \in L^2 \log^{-1} L_{\text{loc}}(\Omega)$$

according to the following result which clarifies why it is convenient to develop the theory of mappings with exponentially integrable distortion using the space $W^{1, L^2 \log^{-1} L}_{\text{loc}}(\Omega)$. For the sake of completeness let us give the proof of the following

Proposition 5.2. *If $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^2)$ has finite distortion K such that*

$$\int_\Omega e^{\frac{K(z)}{\lambda}}\, dz < \infty$$

for some $\lambda > 0$, then $|Df| \in L^2 \log^{-1} L_{loc}(\Omega)$ and

$$\int_S \frac{|Df(z)|^2}{\log(e + |Df(z)|)}\, dz \leq 2\lambda \left(\int_S J_f(z)\, dz + \int_S (e^{\frac{K(z)}{\lambda}} - 1)\, dz \right)$$

for any $S \subset\subset \Omega$.

Proof. Thanks to the distortion inequality

$$|Df(z)|^2 \leq K(z) J_f(z) \text{ for a.e. } z \in \Omega,$$

to the fact that $t \mapsto \dfrac{t}{\log(e+t)}$ is an increasing function and to the elementary inequality

$$a\,b \leq a \log(1+a) + e^b - 1 \qquad (a, b \geq 0)$$

we have that for every $\lambda > 0$

$$\frac{|Df(z)|^2}{\log(e+|Df(z)|^2)} \leq \frac{K(z)J_f(z)}{\log(e+K(z)J_f(z))} \leq \lambda \frac{J_f(z)}{\log(e+J_f(z))} \frac{K(z)}{\lambda}$$
$$\leq \lambda \left(\frac{J_f(z)}{\log(e+J_f(z))} \log\left(1 + \frac{J_f(z)}{\log(e+J_f(z))}\right) + e^{\frac{K(z)}{\lambda}} - 1 \right)$$
$$\leq \lambda \left(J_f(z) + e^{\frac{K(z)}{\lambda}} - 1\right).$$

We now integrate over $S \subset\subset \Omega$ the previous estimate to obtain

$$\int_S \frac{|Df(z)|^2}{\log(e+|Df(z)|)} \, dz \leq 2\lambda \left(\int_S J_f(z)\, dz + \int_S (e^{\frac{K(z)}{\lambda}} - 1)\, dz \right),$$

by hypothesis we conclude

$$|Df| \in L^2 \log^{-1} L_{loc}(\Omega)$$

. □

We now remember the following result (see [Mos] and [IM1] Theorem 8.4.1)

Theorem 5.3. *Let $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation preserving mapping belonging to the Orlicz-Sobolev space $W^{1,\mathcal{P}}(\Omega, \mathbb{R}^2)$ with $\mathcal{P}$ satisfying*

$$\int_1^\infty \frac{\mathcal{P}(t)}{t^3} \, dt = \infty \tag{5.12}$$

and

$$\text{the function } t \to \mathcal{P}(t^{5/8}) \text{ is convex.} \tag{5.13}$$

Then the Jacobian of f belongs to the space $L^\psi_{loc}(\Omega)$, where ψ is defined by

$$\psi(t) = \mathcal{P}(t^{1/2}) + 2t \int_0^{t^{1/2}} \frac{\mathcal{P}(s)}{s^3} \, ds.$$

Moreover, we have the uniform bound

$$\|J_f\|_{L^\psi(Q)} \leq c\, \||Df|\|^2_{L^{\mathcal{P}}(2Q)} \tag{5.14}$$

for any square $Q \subset 2Q \subset \Omega$.

Remarks 5.1. *By Theorem 5.3 we can obtain Müller's result [Mü2]. In fact if*

$$\mathcal{P}(t) = t^2, \text{ for any } t \geq 1$$

we have

$$\psi(t) \sim t \log(e + t)$$

and therefore if

$$|Df| \in L^2(\Omega)$$

then

$$J_f \in L \log L_{loc}(\Omega).$$

We also remark that by Theorem 5.3 we can deduce the result obtained in [BFS]. In fact if

$$\mathcal{P}(t) = t^2 \log^{-\alpha}(e + t), \text{ for any } t \geq 1$$

for some $\alpha \in (0, 1)$*, we get*

$$\psi(t) \sim t \log^{1-\alpha}(e + t)$$

and therefore if

$$|Df| \in L^2 \log^{-\alpha} L(\Omega)$$

then

$$J_f \in L \log^{1-\alpha} L_{loc}(\Omega).$$

Finally we observe that if

$$\mathcal{P}(t) = t^2 \log^{-1}(e + t), \text{ for any } t \geq 1$$

then, by an easy calculation, we deduce that

$$\psi(t) \sim t \log \log(e + t)$$

and therefore if

$$|Df| \in L^2 \log^{-1} L(\Omega)$$

then

$$J_f \in L \log \log L_{loc}(\Omega)$$

and by (5.14) *we have*

$$\|J_f\|_{L\log\log L(Q)} \le c\,\||Df|\|^2_{L^2\log^{-1}L(2Q)} \tag{5.15}$$

for any square $Q \subset 2Q \subset \Omega$*. This last result improve the result of [IS1].*

An important result related to weak convergence of Jacobians is the following (see [IM1] Theorem 8.4.2)

Theorem 5.4. *Let* f_j *be a sequence of orientation preserving mappings weakly converging in* $W^{1,\mathcal{P}}(\Omega,\mathbb{R}^2)$ *to* f*, where* $\mathcal{P}$ *satisfies* (5.12) *and* (5.13)*. Then* f *is an orientation preserving mapping and the Jacobians* J_{f_j} *weakly converge in* $L^1_{loc}(\Omega)$ *to* J_f*.*

Very recently in [FMS] the following result concerning sequences of homeomorphisms with finite distortion has been proved.

Theorem 5.5. *Let* $f_j, f \in W^{1,1}_{loc}(\Omega,\mathbb{R}^2) \cap Hom(\Omega,\Omega')$*. Assume that* (5.9) *and* (5.11) *hold true and that*

$$K_j \rightharpoonup K \ \text{ weakly in } \ L^1(\Omega).$$

Then f *has finite distortion and its distortion function* K_f *satisfies*

$$K_f(z) \le K(z) \ \text{ for a.e. } z \in \Omega.$$

From the previous Theorem we derive

Corollary 5.6. *Let* $f_j, f \in W^{1,1}_{loc}(\Omega,\mathbb{R}^2) \cap Hom(\Omega,\Omega')$*. Assuming that* (5.9)*,* (5.10) *and* (5.11) *hold true, then* f *has finite distortion and its distortion function* K_f *satisfies*

$$\int_\Omega e^{\frac{K_f(z)}{\lambda}}\,dz \le c_0.$$

Proof. Thanks to uniform equiboundedness of K_j we can consider a subsequence of K_j, not relabelled, weakly converging in $L^1(\Omega)$ to K, so by lower semicontinuity we obtain

$$\int_\Omega e^{\frac{K(z)}{\lambda}}\,dz \le \liminf_j \int_\Omega e^{\frac{K_j(z)}{\lambda}}\,dz.$$

Thanks to Theorem 5.5, f has finite distortion and and its distortion function K_f satisfies $K_f(z) \leq K(z)$ for a.e. $z \in \Omega$, then

$$\int_\Omega e^{\frac{K_f(z)}{\lambda}}\, dz \leq \int_\Omega e^{\frac{K(z)}{\lambda}}\, dz$$

holds and therefore we can conclude that

$$\int_\Omega e^{\frac{K_f(z)}{\lambda}}\, dz \leq c_0.$$

□

Our main result of this last section is the following

Theorem 5.7. *Let* $f_j, f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^2) \cap Hom(\Omega, \Omega')$. *Assuming that* (5.9), (5.10) *with* $\lambda \in (0, 1/2)$ *and* (5.11) *hold true, then* f *has finite distortion and*

$$A_{f_j} \xrightarrow{\Gamma_{L^2 \log L}} A_f.$$

Observe that such result gives an extension of a previous Γ-stability result (see [FM] e.g.) to the more general class of matrices with determinant equal one.

Due to the loss of uniform pointwise ellipticity conditions of the matrices A_{f_j}, one of the main difficulties in proving our result is the extension of the Γ-convergence from space *Lip* to the Zygmund-Sobolev space $W^{1,L^2 \log L}$. This is overcome by requiring that K_j is a bounded sequence in EXP and the distances of the distortions $K_j \in EXP$ from L^∞ are less than a sufficiently small number. This assumptions guarantees that the solutions of the minimum problems which a-priori would lie only in the coerciveness space $W^{1,L^2 \log^{-1} L}$, actually belonging to the continuity space $W^{1,L^2 \log L}$.

A reason for our exponential integrability assumption on the distortions relies on the following. Denoting by $\mathcal{F}$ the family of all $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2) \cap \mathrm{Hom}(\Omega, \Omega')$ with finite distortion K for which

$$\int_\Omega e^{\frac{K(z)}{\lambda}}\, dz \leq c_0$$

for some $\lambda > 0$ and $c_0 > 0$, then

$\mathcal{F}$ is sequentially compact with respect to the locally uniform convergence,

i.e. every sequence $\{f_j\} \subset \mathcal{F}$ has a subsequence converging locally uniformly to some $f \in \mathcal{F}$ (see Theorem 2.12).

On the other hand, denoting by $\mathcal{G}$ the family of all $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2) \cap \mathrm{Hom}(\Omega, \Omega')$ with finite distortion K for which

$$\int_\Omega K(z)^p \, dz \le c_0$$

for some $p > 0$ and $c_0 > 0$, then

$\mathcal{G}$ may not be sequentially compact with respect to the locally uniform convergence,

(see [PR], [Da]). Moreover, $\mathcal{G}$ may not be equicontinuous and there exists discontinuous mappings in its closure (see [P]).

We recall that there are other results on Γ-convergence of degenerate quadratic functionals under isotropic assumption of the type

$$w(z)|\xi|^2 \leqslant \langle A(z)\xi, \xi\rangle \leqslant \Lambda w(z)|\xi|^2 \quad (\Lambda \ge 1) \tag{5.16}$$

for a.e. $z \in \Omega$ and for any $\xi \in \mathbb{R}^2$. We refer to a result in [De] when in (5.16) $w, w^{-1} \in L^1_{\mathrm{loc}}$, to [SC], [G2] when w belongs to the Muckhenhoupt class A_2. Here we consider a different situation dealing with anisotropic case as in (5.4).

In order to prove Theorem 5.7 we also need the following optimal regularity result for differential of a mapping with exponentially integrable distortion obtained combining a special case of Theorem due to Astala-Gill-Rohde-Saksman (see [AGRS] and Remark 2.1) and Theorem 1 of [IKMS].

Theorem 5.8. *If $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^2)$ has finite distortion K satisfying*

$$\int_\Omega e^{\frac{K(z)}{\lambda}} \, dz < \infty$$

for some $\lambda \in (0, 1/2)$, then $|Df| \in L^2 \log L_{loc}(\Omega)$ and for any concentric disk $\mathbb{D} \subset 2\mathbb{D} \subset \Omega$

$$\int_{\mathbb{D}} |Df(z)|^2 \log\left(e + \frac{|Df(z)|}{|Df(z)|_{\mathbb{D}}}\right) dz \le c \int_{2\mathbb{D}} J_f(z) \, dz$$

where c is an absolute constant.

Notice that this result is sharp in sense that the conclusion fails for $\lambda = 1/2$. To show this we consider the following mapping (see [AGRS]):

$$f(z) = \begin{cases} \dfrac{z}{|z|} \dfrac{1}{\log\left(e + \dfrac{1}{|z|}\right) \log\log\left(e + \dfrac{1}{|z|}\right)} & \text{for } z \in \mathbb{D}(0,1) \setminus \{0\} \\ 0 & \text{for } z = 0. \end{cases}$$

We have that f is a mapping of Sobolev space $W^{1,1}_{\mathrm{loc}}(\mathbb{D}(0,1), \mathbb{R}^2)$ with finite distortion such that

$$\int_{\mathbb{D}(0,1)} e^{2K_f(z)}\, dz < \infty$$

while

$$|Df| \notin L^2 \log L_{\mathrm{loc}}(\mathbb{D}(0,1)).$$

Proof of Theorem 5.7. The proof develops into three steps.

Step 1. By virtue of Theorem 5.1 there exists a subsequence $A_{f_{jr}}$ of A_{f_j} such that

$$A_{f_{jr}} \xrightarrow{\Gamma} A \tag{5.17}$$

where A is a symmetric matrix function satisfying the condition

$$0 \le \langle A(z)\xi, \xi\rangle \le K(z)|\xi|^2 \tag{5.18}$$

for a.e. $z \in \Omega$ and for any $\xi \in \mathbb{R}^2$.

Our aim is to prove that

$$A(z) = A_f(z) = J_f(z)\,[Df(z)^t Df(z)]^{-1}.$$

This will imply that the lower bound in (5.18) can be improved as

$$\frac{|\xi|^2}{K(z)} \le \langle A(z)\xi, \xi\rangle$$

and that the whole sequence

$$A_{f_j} \xrightarrow{\Gamma} A.$$

The assumption (5.9) is equivalent to

$$\frac{|\xi|^2}{K_j(z)} \le \langle A_{f_j}(z)\xi, \xi\rangle \le K_j(z)|\xi|^2 \tag{5.19}$$

for a.e. $z \in \Omega$ and for any $\xi \in \mathbb{R}^2$. By (5.19) and by generalized Hölder's inequality in Orlicz spaces (1.9) we have

$$\int_\Omega \langle A_{f_j}(z)\nabla u, \nabla u\rangle \, dz \le \int_\Omega K_j(z)|\nabla u|^2 \, dz \le c\,\|K_j\|_{EXP(\Omega)} \||\nabla u|\|^2_{L^2\log L(\Omega)}$$

$$\le c' \, \|u\|_{W^{1,L^2\log L(\Omega)}}.$$

where the constant c' only depends on the uniform bound c_0 of the assumption (5.10). Hence the functionals

$$\left(\int_\Omega \langle A_{f_j}(z)\nabla u, \nabla u\rangle \, dz\right)^{\frac{1}{2}}$$

are equilipschitzian in $W^{1,L^2\log L}(\Omega)$. Therefore arguing as in [MS1] (Proposition 3.2) we can pass from Γ-convergence (5.17) to the stronger one

$$A_{f_{j_r}} \xrightarrow{\Gamma_{L^2\log L}} A. \tag{5.20}$$

Step 2. Let us show that for any $\Omega_1 \subset\subset \Omega$ and for $i = 1, 2$ we have

$$\int_{\Omega_1} \langle A(z)\nabla f^i, \nabla f^i\rangle \, dz = \lim_r \int_{\Omega_1} \langle A_{f_{j_r}}(z)\nabla f^i_{j_r}, \nabla f^i_{j_r}\rangle \, dz. \tag{5.21}$$

For $i = 1, 2$ fixed set for simplicity $u_r = f^i_{j_r}$, $u = f^i$ and $A_r = A_{f_{j_r}}$. By Theorem 5.8 we deduce that $u_r, u \in W^{1,L^2\log L}(\Omega_1)$ and $u_r \to u$ in $L^2\log L(\Omega_1)$.

Let now v_r be a sequence in $W^{1,L^2\log L}(\Omega_1)$ such that $v_r \to u$ in $L^2\log L(\Omega_1)$ and

$$\int_{\Omega_1} \langle A(z)\nabla u, \nabla u\rangle \, dz = \lim_r \int_{\Omega_1} \langle A_r(z)\nabla v_r, \nabla v_r\rangle \, dz.$$

Let $S \subset\subset \Omega_1$ and $\varphi \in C_0^\infty(\Omega_1)$ such that $\varphi(z) \in [0, 1]$ and $\varphi \equiv 1$ in S; then for every $t \in (0, 1)$ we obtain as in [DD], [Fo]

$$\int_{\Omega_1} \langle A_r(z)\nabla u_r, \nabla u_r\rangle \, dz \le \int_{\Omega_1} \langle A_r(z)\nabla(\varphi v_r + (1-\varphi)u_r), \nabla(\varphi v_r + (1-\varphi)u_r)\rangle \, dz$$

$$= \int_{\Omega_1} \langle A_r(z) \left\{\frac{t}{t}\nabla\varphi(v_r - u_r) + \frac{1-t}{1-t}(\varphi\nabla v_r + (1-\varphi)\nabla u_r)\right\},$$

$$\left\{\frac{t}{t}\nabla\varphi(v_r - u_r) + \frac{1-t}{1-t}(\varphi\nabla v_r + (1-\varphi)\nabla u_r)\right\}\rangle \, dz$$

$$\le t\int_{\Omega_1} \langle A_r(z) \left\{\frac{1}{t}\nabla\varphi(v_r - u_r)\right\}, \left\{\frac{1}{t}\nabla\varphi(v_r - u_r)\right\}\rangle \, dz +$$

$$(1-t)\int_{\Omega_1}\langle A_r(z)\left\{\frac{1}{1-t}(\varphi\nabla v_r+(1-\varphi)\nabla u_r)\right\},\left\{\frac{1}{1-t}(\varphi\nabla v_r+(1-\varphi)\nabla u_r)\right\}\rangle\,dz$$
$$\le\frac{1}{t}\int_{\Omega_1}K_{j_r}(z)\,|\nabla\varphi|^2\,|v_r-u_r|^2\,dz+\frac{1}{1-t}\int_{\Omega_1}\langle A_r(z)\nabla v_r,\nabla v_r\rangle\,\varphi\,dz$$
$$+\frac{1}{1-t}\int_{\Omega_1}\langle A_r(z)\nabla u_r,\nabla u_r\rangle(1-\varphi)\,dz.$$

This yields

$$(1-t)\int_{\Omega_1}\langle A_r(z)\nabla u_r,\nabla u_r\rangle\,dz\le\frac{1-t}{t}\||\nabla\varphi|\|^2_{L^\infty(\Omega_1)}\|K_{j_r}\|_{EXP(\Omega_1)}\|v_r-u_r\|^2_{L^2\log L(\Omega_1)}$$
$$+\int_{\Omega_1}\langle A_r(z)\nabla v_r,\nabla v_r\rangle\,\varphi\,dz+\int_{\Omega_1}\langle A_r(z)\nabla u_r,\nabla u_r\rangle(1-\varphi)\,dz,$$

that is

$$\int_{\Omega_1}\langle A_r(z)\nabla v_r,\nabla v_r\rangle\,\varphi\,dz\ge\int_{\Omega_1}\langle A_r(z)\nabla u_r,\nabla u_r\rangle(1-t-1+\varphi)\,dz$$
$$-\frac{1-t}{t}\,c\,\||\nabla\varphi|\|^2_{L^\infty(\Omega_1)}\|v_r-u_r\|^2_{L^2\log L(\Omega_1)}.$$

Now, passing to the limit as $r\to\infty$, we obtain

$$\int_{\Omega_1}\langle A(z)\nabla u,\nabla u\rangle\,dz\ge\limsup_r\int_{\Omega_1}\langle A_r(z)\nabla u_r,\nabla u_r\rangle(\varphi-t)\,dz$$

and then passing to the limit as $t\to 0$

$$\int_{\Omega_1}\langle A(z)\nabla u,\nabla u\rangle\,dz\ge\limsup_r\int_{\Omega_1}\langle A_r(z)\nabla u_r,\nabla u_r\rangle\,\varphi\,dz$$
$$\ge\liminf_r\int_S\langle A_r(z)\nabla u_r,\nabla u_r\rangle\,dz\ge\int_S\langle A(z)\nabla u,\nabla u\rangle\,dz.$$

From these inequalities, since S is an arbitrary subdomain of Ω_1, (5.21) follows.

Step 3. That f has finite distortion was already established in Corollary 5.6. Since we wish to identify the $\Gamma_{L^2\log L}$-limit of A_{f_j}, we can assume that in (5.17), (5.20) and (5.21) the convergence of the whole sequence holds.

For $i=1,2$ fixed set $u_j=f_j^i$, $u=f^i$ and $A_j=A_{f_j}$. As in step 2 let $\Omega_1\subset\subset\Omega$. We consider step function

$$\varphi=\sum_{i=1}^n\mu_i\chi_{B_i},\quad \mu_i\ge 0$$

where B_i are pairwise disjoint open subsets of Ω_1 such that

$$|\Omega_1\setminus\bigcup_{i=1}^n B_i|=0.$$

From (5.20), it follows that

$$\liminf_j \int_{\Omega_1} \langle A_j(z)\nabla u_j, \nabla u_j\rangle \,\varphi\, dz \geq \int_{\Omega_1} \langle A(z)\nabla u, \nabla u\rangle \,\varphi\, dz. \tag{5.22}$$

Moreover, the estimate (5.22) still holds if $\varphi \in C^0(\overline{\Omega}_1)$ and $\varphi \geq 0$, since such functions can be approximated in $C^0(\overline{\Omega}_1)$ by functions of the type $\sum_{i=1}^n \mu_i \chi_{B_i}$.

Let us now prove that (5.22) holds as equality for any $\varphi \in C^0(\overline{\Omega}_1)$ not necessarily non-negative. In fact, by (5.15) it follows that there is a subsequence $J_{f_{j_r}} \equiv \langle A_{j_r}\nabla u_{j_r}, \nabla u_{j_r}\rangle$ of $J_{f_j} \equiv \langle A_j\nabla u_j, \nabla u_j\rangle$ weakly converging in $L^1(\Omega_1)$ to a function F, in particular

$$\lim_r \int_{\Omega_1} \langle A_{j_r}(z)\nabla u_{j_r}, \nabla u_{j_r}\rangle \,\varphi(z)\, dz = \int_{\Omega_1} F(z)\,\varphi(z)\, dz \tag{5.23}$$

for any $\varphi \in C^0(\overline{\Omega}_1)$. Thanks to (5.22) we get

$$\int_{\Omega_1} \langle A(z)\nabla u, \nabla u\rangle \,\varphi(z)\, dz \leq \int_{\Omega_1} F(z)\,\varphi(z)\, dz \tag{5.24}$$

Now, let $\{\varphi_k\} \subset C^0(\overline{\Omega}_1)$ a sequence such that $\varphi_k(z) \to \chi_S(z)$ for a.e. $z \in \Omega_1$, where S is a measurable subset of Ω_1. Hence we obtain by (5.24) and Lebesgue Theorem

$$\int_S \langle A(z)\nabla u, \nabla u\rangle \, dz \leq \int_S F(z)\, dz.$$

From (5.21) and (5.23) it follows

$$\int_{\Omega_1} \langle A(z)\nabla u, \nabla u\rangle \, dz = \int_{\Omega_1} F(z)\, dz$$

and then by latter two estimates we get

$$\int_S \langle A(z)\nabla u, \nabla u\rangle \, dz = \int_S F(z)\, dz$$

for any S. Hence, in virtue of Radon-Nikodym Theorem

$$F(z) = \langle A(z)\nabla u, \nabla u\rangle \text{ a.e. in } \Omega_1.$$

Therefore for the whole sequence we have that

$$\lim_j \int_{\Omega_1} \langle A_j(z)\nabla u_j, \nabla u_j\rangle \,\varphi\, dz = \int_{\Omega_1} \langle A(z)\nabla u, \nabla u\rangle \,\varphi\, dz \tag{5.25}$$

for every $\varphi \in C^0(\overline{\Omega}_1)$.

Since the components f_j^i $(i = 1, 2)$ solve the equation

$$\langle A_j(z)\nabla f_j^i(z), \nabla f_j^k(z)\rangle = J_{f_j}(z)\delta_{ik} \tag{5.26}$$

for a.e. $z \in \Omega$ and for $i, k = 1, 2$, by the symmetry of the matrix A_j, (5.25), (5.26) and by Theorem 5.4, we have

$$\int_{\Omega_1} \langle A(z)\nabla f^i, \nabla f^k\rangle\, \varphi\, dz = \lim_j \int_{\Omega_1} \langle A_j(z)\nabla f_j^i, \nabla f_j^k\rangle\, \varphi\, dz$$

$$= \lim_j \int_{\Omega_1} J_{f_j}(z)\, \delta_{ik}\, \varphi\, dz = \int_{\Omega_1} J_f(z)\, \delta_{ik}\, \varphi\, dz,$$

where $\varphi \in C_0^\infty(\Omega_1)$ and $i, k = 1, 2$. Since φ is arbitrary, it follows that

$$\langle A(z)\nabla f^i(z), \nabla f^k(z)\rangle = J_f(z)\, \delta_{ik}$$

for a.e. $z \in \Omega_1$ and for $i, k = 1, 2$. Using the fact that $J_f(z) > 0$ a.e. (see [KM])

$$A(z) = J_f(z)[Df(z)^t Df(z)]^{-1}$$

for a.e. $z \in \Omega_1$. Since Ω_1 is arbitrary, the above equality holds for a.e. $z \in \Omega$ and therefore we conclude that

$$A_{f_j} \xrightarrow{\Gamma_{L^2 \log L}} A_f.$$

□

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