Quantum Methodologies in Beam, Fluid and Plasma Physics

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Abstract

The quantum methodologies useful for describing in a unified way several problems of nonlinear and collective dynamics of fluids, plasmas and beams are presented. In particular, the pictures given by the Madelung fluid and the Moyal-Ville-Wigner phase-space quasidistribution, including the related quantum tools such as marginal distributions for the tomographic representations, are described. Some relevant applications to soliton and modulational instability theory are presented.

I. INTRODUCTION

A. General role of the quantum methodologies

The quantum methodologies, such as the tools provided by Schrödinger-like equations, Madelung fluid picture [1], Moyal-Ville-Wigner transform [2]–[4] or von Neumann-Weyl formalism [5], [6], quantum tomography [7] and n-waves parametric processes [8], are widely used in almost all branches of nonlinear physics. In particular, they are frequently encountered in dispersive media such as laboratory, space and astrophysical plasmas, fluids, Kerr media, optical fibers, electrical transmission lines and many other physical systems including cosmological and biological systems; in optical beam physics and charged particle beam physics they play a very relevant role.

Since the recently past years, the quantum methodologies are intensively applied in all the above branches as result of international collaborations belonging to the frontiers of the physics researches and they are one of the main topics of several important interdisciplinary scientific international conferences [9]. In fact, each of the above physical systems exhibit behavior that can be described with a quantum formalism. Typically, their evolution in space and time is governed by suitable linear or nonlinear Schrödinger-like equations for complex functions that are coupled, through an effective potential, with a set of equations describing the interaction of the system with the surroundings. For instance, in plasmas, the nonlinearity arises from the harmonic generation and the ponderomotive force [10], while in nonlinear optics it originates due to a Kerr nonlinear refractive index [11]. The nonlinear collective charged-particle beam dynamics in accelerating machines is due to the interaction between beam and the surroundings by means of both image charges and image currents of the beam created on the walls of the accelerator vacuum chamber. This interaction is conveniently described in terms of the so-called "coupling impedance", whose imaginary part accounts for both the space charge blow up and the magnetic self attraction, and whose real part accounts for the resistive effects occurring on the walls. In the physics of the surface gravity waves the nonlinearity is introduced by the high values of the wave elevation.

The system dynamics governed by a linear or nonlinear Schrödinger equation can be described in an equivalent way by means of the Madelung fluid equations.

Alternatively, one may use the Moyal-Ville-Wigner transform that allows to transit from the configuration space description to the phase space one, providing this way a kinetic approach.

Additionally, there is a tomographic map which provides a description in terms of a marginal distribution (classical probability function), starting from the quasidistribution.

In this scenario, the study of the quantum methodologies have been recognized as very important for a synergetic developments of the above branches of physics with very powerful multidisciplinary as well as interdisciplinary approaches. For instance, the intense study on nonlinear and collective effects in the several physical systems have stimulated a number of interdisciplinary approaches and transfer of know how from one discipline to another, obtaining, in turn, a big growing of importance of the methodologies used to investigate very different physical phenomena governed by formally identical equations. Two advantages of this interdisciplinary strategy are absolutely fruitful. One is that communities of physicists from different areas are stimulated to collaborate more an more exchanging their own experiences and make available their own expertise; the other one is the subsequent very rapid improvement of both methodologies to be used and goals to be reached in each specific discipline. This aspect is connected with the efforts done during the last decades in transferring know how and methodologies from one discipline to another trying to predict new effects as well as to give answers to scientific and technological problems of international expectation. For instance, the applications of the quantum methodologies:

(i) to gravity ocean waves, touche the very important and hot problem of the environmental risk due to natural catastrophes, as the one that recently took place in the South-Est of Asia;

(ii) to beam physics, open up the possibility to develop an emerging area of physics, called Quantum Beam Physics, which in the limit of very low temperature should provide the realization of non-classical (but collective and nonlinear) states of charged particle beams fully similar to the ones obtained for the light (optical beams) and for Bose-Einstein condensation;

(iii) to nonlinear optics (f.i., optical fibers) and electric transmission lines deal with important and modern aspects of telecommunications;

(iv) to discrete systems, is relevant for the very recent development of nanotechnologies.

B. Some important aspects of the phenomenological platform investigated by means of the quantum methodologies

The modulational instability (MI), also known as Benjamin-Feir instability, is a general phenomenon encountered when a quasi-monochromatic wave is propagating in a weak non-linear medium. It has been predicted and experimentally observed in almost all field of physics where these conditions are present. We mention especially the wave propagation in deep waters (ocean gravity waves), in plasma physics (electrostatic and electromagnetic plasma waves), in particle accelerators (high-energy charged-particle beam dynamics), in nonlinear optics (Kerr media, optical fibers), in electrical transmission lines, in matter wave physics (Bose-Einstein condensates), in lattice vibrations physics (molecular crystals) and in the physics of antiferromagnetism (dynamics of the spin waves). For ocean gravity waves, the modulational instability has been discovered independently by Benjamin and Feir and by Zakharov in the Sixties; the instability predicts that in deep water a monochromatic wave is unstable under suitable small perturbations. This phenomenon is well described by the nonlinear Schrödinger equation (NLSE). In this framework, it has been established that the MI can be responsible for the formation of freak waves. In plasmas, finite amplitude Langmuir waves can be created when some free energy sources, such as electron and laser beams, are available in the system as a result of a nonlinear coupling between high-frequency Langmuir and low-frequency ion-acoustic waves. Under suitable physical conditions, the dynamics can be described by a NLSE and the MI can be analyzed directly with this equation. In nonlinear optics, the propagation of large amplitude electromagnetic waves produces a modification of the refractive index which, in turn, affects the propagation itself and makes possible the formation of wave envelopes. In the slowly-varying amplitude approximation, this propagation is governed again by suitable NLSE and the MI plays a very important role. In electrical transmission lines, the propagation of modulated non-linear waves is governed by discrete equations of the LC circuit which, in turn, can be reduced to single or two coupled NLSEs.

The large amplitude wave propagation is common of many environmental and technological processes. In nature, wave interactions exhibit a random character. Therefore, in order to predict the wave behavior, taking into account the statistical properties of the medium, an adequate statistical description is needed. This kind of approach started in physics of

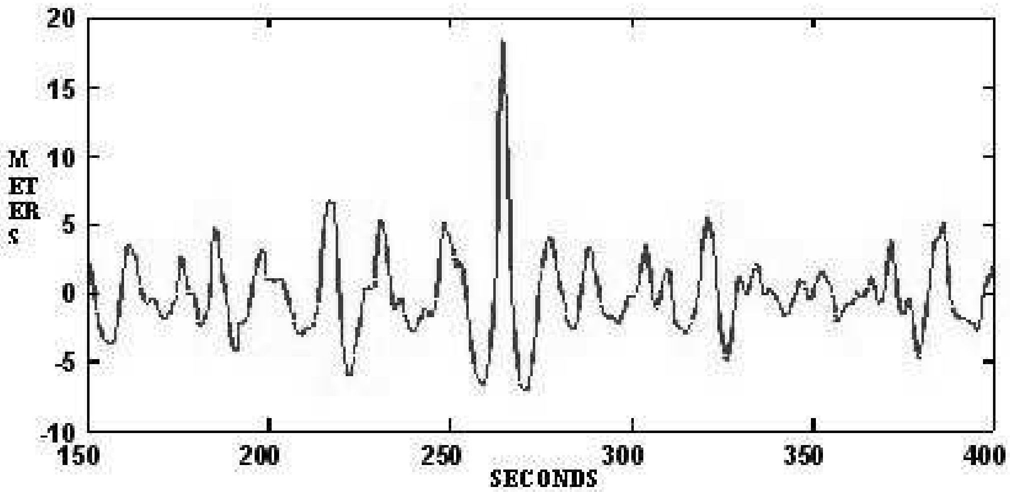

FIG. 1: *Freak wave measured in the North Sea the first of January 1995 from Draupner platform (Statoil operated platform, Norway)* [12].

fluids at least forty years ago to study the random behavior of the large amplitude surface gravity waves. The time evolution of the small amplitude surface gravity waves is characterized by a Gaussian statistics for most of the time. But, nevertheless, in still quite unknown conditions, the behavior of these waves may become highly non-Gaussian and the wave field is characterized by the appearance of very large amplitude waves whose effects can be devastating for fixed off-shore structures, for boats and also for coastal regions. Figure 1 shows one of the most spectacular recordings of a freak wave measured in the North Sea the first of January 1995 from Draupner platform (Statoil operated platform, Norway). The time series of the surface elevation shows a single wave whose height from crest to through is 26 meters (a 9 floor building) in a 10 meters height sea state. In deep water those large amplitude have been recently attributed to modulational instability (MI) [12].

On the other hand, MI is of fundamental importance for the formation of robust nonlinear excitations of the medium. In fact, the asymptotic behaviour of MI may be characterized by the formation of very stable localized solutions, such as envelope solitons, cavitons, holes, etc., which, in turn, are involved in a long timescale dynamics that have been shown to be of great importance in all nonlinear systems. Furthermore, as a peculiarity of any nonlinear system, MI can be understood as a four wave parametric process.

C. Impacts produced by the quantum methodology applications to nonlinear physics

In this paper, we confine our attention on three subjects only, namely, physics of fluids, plasma physics and beam physics (optical and charged particle beams). In order to present in the next section some relevant applications of the quantum methodologies involved in these three subjects, we discuss here the impact that they have produced in nonlinear physics with special emphasis for the methodological transfer from one discipline to another.

- One of the most relevant example of using quantum methodologies in nonlinear physics is surely given by the inverse scattering method [13] which allows to find soliton solutions of the standard Korteweg-de Vries equation (KdVE) by constructing a correspondence between the latter and a linear Schrödinger equation (LSE) whose potential term coincides with the solution of the KdVE. This way, the problem of solving the KdVE is reduced to a quantumlike problem, i.e., to an inverse eigenvalue problem of the LSE. Very important theorems have been found for this method [14] and it has been successfully extended to NLSE [15]. The capability and the richness of similar methods currently applied to nonlinear partial differential equations for solving a number of physical problems have produced an autonomous research activity in mathematical physic called "inverse problems".

- More recently, outside of the inverse scattering method framework, a correspondence between solitonlike and envelope solitonlike solutions, in the form of travelling waves, of wide families of generalized Korteweg-de Vries equation (gKdVE) and generalized nonlinear Schrödinger equation (gNLSE), respectively, has been constructed within the framework of the Madelung fluid [16], [17]. Under suitable constraints, this correspondence can be made invertible. Starting from the gNLSE, the travelling wave solutions of the associated gKdVE are found taking the square modulus of the travelling envelope wave solutions of the gNLSE. Viceversa, starting from the gKdVE, an arbitrary non-negative definite travelling wave solutions can be used to construct a map which, apart from an arbitrary linear phase, gives a travelling envelope wave solutions of the associated gNLSE. Remarkably, this correspondence has been used to find envelope solitonlike solutions of a wide family of gNLSE. In particular: (i) soliton solutions, in the form of bright and/or gray/dark envelope solitons of the cubic NLSE [17] and the modified one with the nonlinearity of the type $|\Psi|^{2\beta}\Psi$ (β being an arbitrary positive real number) have been recovered or found, respectively [16], [18]; (ii) bright and gray/dark envelope solitons of "cubic-quintic" NLSE (i.e., $(c_1|\Psi|^2\Psi + c_2|\Psi|^4\Psi)$ have been found, as well [16]; the method, extended to the cubic-quintic NLSE with and additional "anti-cubic" nonlinearity (i.e., $|\Psi|^{-4}\Psi$), has allowed to find new soliton-like solutions [19].

- Quantum methodologies have been recently employed to describe the charged-particle beam optics and dynamics in terms of a Schrödinger-like equation for a complex function whose square modulus is proportional to the beam density. The related model is called Thermal Wave Model (TWM) [20]– [22]. In general, the potential term of this equation accounts for both external and self-consistent field forces. The self-consistent fields are due to collective effects (reactive and resistive interaction schematized by the coupling impedance). The inclusion of the collective effects leads to a sort of NLSE [23]. For purely reactive coupling impedance [24], the NLSE becomes formally identical to the one governing the propagation of optical beams with a cubic nonlinearity [11]. So that, a transferring of know how from nonlinear optics to accelerator physics has allowed to predict new results, such as soliton density structures associated with the longitudinal dynamics of a charged-particle bunch in a circular high-energy accelerating machine that the conventional approach, based of Vlasov equation, was not yet capable to predict [24]. Later on, by including the resistive part of the coupling impedance, the resulting integro-differential NLSE was capable to describe the nonlocal and distortion effects , non dissipative shock waves and wave breaking on an initially given soliton-like particle beam density profile [25].

- A further methodological transfer from nonlinear optics to accelerator physics was done with the analysis of modulational instability of macroscopic matter waves as described by the TWM. The results of these transferring may be summarized as follows. (i) The well known coherent instability (for instance, positive or negative mass instability), described by the Vlasov theory, is nothing but a sort of MI predicted by TWM for macroscopic matter waves with the above integro-differential NLSE [23]. (ii) The phenomenon of Landau damping [26] and its stabilizing role against the coherent instability was recovered [27], and then extended in a more general framework [28], [29], [30]. It has been arrived to these results by adding a new element to the TWM macroscopic matter wave description that at that time was missing in nonlinear optics: by extending the TWM description to phase space with the use of the Moyal-Ville-Wigner quasidistribution in the context of TWM, whose evolution is governed by a quantumlike kinetic equation (von Neumann equation). Actually this approach was first introduced by Klimontovich and Silin in plasma theory [31] and, later on, in nonlinear fluid theory to develop the modulational instability of the surface gravity waves (such as in the nonlinear dynamics of deep waters waves [32]). However, it should be pointed out that the notion of "density matrix", also referred to as "statistical operator", very useful to give the definition of the quasidistribution, has been introduced for the first time by Landau [33] in 1927 and later, in 1932, it was mathematically treated by von Neumann [5].

- Until few years ago, the modulational instability description in nonlinear optics was not yet capable to include the stabilizing effects, as in the coherent instability description in accelerator physics. However, the results given by TWM were soon transferred back to nonlinear optics to extend the standard modulational instability theory of optical beams and bunches to the context of ensemble of partially incoherent waves whose dynamics include the statistical properties of the medium. At the present time, as result of the quantum methodology advances described above, we can say that two distinct ways to treat MI are possible. The first and the most used one, corresponding to the standard one, is a deterministic approach, where the linear stability analysis around a carrying wave is considered. This corresponds to consider the stability/instability of monochromatic wave trains (system of coherent waves). The second one, is a statistical approach and its main goal is to introduce the statistical properties of the medium (whether continuum or discrete). In these physical conditions, the stability analysis cannot be carried out as in the monochromatic case. An ensemble of partially incoherent waves must, in fact, to be taken into account. Thus, MI reveals to be strongly dependent on the parameters characterizing the initial conditions (initial wavenumber distribution or initial momentum distribution of waves). This second approach stimulated very recently a new branch of investigation devoted to MI of ensemble of partially incoherent waves with both theoretical and experimental aims [34], [35]. It was rapidly applied to Kerr media [36], [37] and soon extended to plasma physics (ensemble of partially incoherent Langmuir wave envelopes) [38] and physics of lattice vibrations [39], [40] where the discrete self-trapping equation represents an useful model for several properties of one-dimensional nonlinear molecular crystals. New improvements were also registered in the statistical formulation of MI for large amplitude surface gravity waves [41].

- In the very recent years, the deterministic approach to modulational instability has been developed to the matter wave physics of Bose Einstein condensates, as well, thanks to a methodological transfer from nonlinear optics and plasma physics to this discipline. It has been both predicted and experimentally confirmed. For instance, MI conditions for the phonon spectrum takes place for an array of traps containing Bose-Einstein condensates (BEC) with each trap linked to adjacent traps by tunneling [42]; additionally, MI of matter waves of BECs periodically modulated by a laser beam takes place in a number of physical situations, as well [43]. The 3D dynamics of BECs is, in fact, governed by the well known Gross-Pitaevskii equation [44] which is a sort of NLS equation. In the same effort of methodological and know how transfer, several valuable predictions and experimental confirmations concerning the formation of solinton-like structures in Bose Einstein condensates should be mentioned [45], [46]. Remarkably, a very valuable scientific and technological feedback of this transfer was a production of dark-bright BEC solitons within the framework of the nanotechnologies.

II. THE MADELUNG FLUID PICTURE

A. Hydrodynamical description of quantum mechanics

During the period 1924-1925, L. de Broglie elaborated his theory of "pilot waves" [47], introducing the very fruitful idea of wave-particle dualism, funding the theory of matter waves. However, until 1926 a wave equation for particles, thought as waves, was not yet proposed. In that year, Schrödinger proposed a wave equation that today has his name (the Schrödinger equation), funding the wave mechanics [48]. On the pilot waves de Broglie published a series of articles during 1927 [49] but they did not produce great excitation within the scientific community. During October of the same year, in fact, de Broglie presented a simplified version of his recent studies on pilot waves at the Fifth Physical Conference of the Solvay Institute in Brussels. The criticism received pushed him to abandon this theory to start to study the complementary principle. He came back to the pilot waves during the period 1955-1956, proposing a more organic theory [50]. Nevertheless, a very valuable seminal contribution to quantum mechanics was given by de Broglie while developing the pilot wave theory with the concept of "quantum potential", but a systematic presentation of this idea came only several years later [51]. At the beginning of Fifties Bohm also have considered the concept of quantum potential [52]. However, the concept was naturally appearing in a hydrodynamical description meanwhile proposed in 1926 by Madelung [1] (first proposal of a hydrodynamical model of quantum mechanics), followed by the proposal of Korn in 1927 [53]. The Madelung fluid description of quantum mechanics revealed to be very fruitful in a number of applications: from the pilot waves theory to the hidden variables theory, from stochastic mechanics to quantum cosmology.

In the Madelung fluid description, the wave function, say Ψ, being a complex quantity, is represented in terms of modulus and phase which, substituted in the Schrödinger equation, allow to obtain a pair of nonlinear fluid equations for the "density" $\rho = |\Psi|^2$ and the "current velocity" $\mathbf{V} = \nabla \mathrm{Arg}(\Psi)$: one is the continuity equation (taking into account the probability conservation) and the other one is a Navier-Stokes-like motion equation, which contains a force term proportional to the gradient of the quantum potential, i.e., $(\nabla^2|\Psi|)/|\Psi| = (\nabla^2\rho^{1/2})/\rho^{1/2}$. The nonlinear character of these system of fluid equations naturally allows to extend the Madelung description to systems whose dynamics is governed by one ore more NLSEs. Remarkably, during the last four decades, this quantum methodology was imported practically into all the nonlinear sciences, especially in nonlinear optics [34], [54], [55] and plasma physics [56], [57] and it revealed to be very powerful in solving a number of problems. Let us consider the following (1+1)D nonlinear Schrödinger-like equation (NLSE):

$$i\alpha \frac{\partial \Psi}{\partial s} = -\frac{\alpha^2}{2}\frac{\partial^2 \Psi}{\partial x^2} + U\left[|\Psi|^2\right]\Psi \quad , \tag{1}$$

where $U\left[|\Psi|^2\right]$ is, in general, a functional of $|\Psi|^2$, the constant α accounts for the dispersive effects, and s and x are the timelike and the configurational coordinates, respectively. Let us assume

$$\Psi = \sqrt{\rho(x,s)}\exp\left[\frac{i}{\alpha}\Theta(x,s)\right] \quad , \tag{2}$$

then substitute (2) in (1). After separating the real from the imaginary parts, we get the following Madelung fluid representation of (1) in terms of pair of coupled fluid equations:

$$\frac{\partial\rho}{\partial s}+\frac{\partial}{\partial x}(\rho V)=0\,, \tag{3}$$

(continuity)

$$\left(\frac{\partial}{\partial s}+V\frac{\partial}{\partial x}\right)V=-\frac{\partial U}{\partial x}+\frac{\alpha^2}{2}\frac{\partial}{\partial x}\left[\frac{1}{\rho^{1/2}}\frac{\partial^2\rho^{1/2}}{\partial x^2}\right], \tag{4}$$

(motion) where the current velocity V is given by

$$V(x,s)=\frac{\partial\Theta(x,s)}{\partial x}\,. \tag{5}$$

In the next subsections, we present solme relevant applications of Madelung fluid methodology to the soliton and MI theories.

B. Applications to soliton theory

In order to apply Madelung description to soliton theory, let us manipulate the system of equations (3) and (4), in such a way to transform the motion equation into a third-order partial differential equaion for ρ.

By multiplying Eq. (3) by V, the following equation can be obtained:

$$\rho\left(\frac{\partial}{\partial s}+V\frac{\partial}{\partial x}\right)V\;=\;-V\frac{\partial\rho}{\partial s}\;-\;V^2\frac{\partial\rho}{\partial x}\;+\;\rho\frac{\partial V}{\partial s}\quad. \tag{6}$$

Note that:

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho^{1/2}}\frac{\partial^2\rho^{1/2}}{\partial x^2}\right)\;=\;\frac{1}{\rho}\left(\frac{1}{2}\frac{\partial^3\rho}{\partial x^3}\;-\;4\,\frac{\partial\rho^{1/2}}{\partial x}\,\frac{\partial^2\rho^{1/2}}{\partial x^2}\right)\quad. \tag{7}$$

Furthermore, multiplying Eq. (4) by ρ and combining the result with (6) and (7) one obtains

$$\rho\left(\frac{\partial}{\partial s}+V\frac{\partial}{\partial x}\right)V=-\frac{\partial U}{\partial x}\rho+\frac{\alpha^2}{4}\frac{\partial^3\rho}{\partial x^3}\;-\;2\alpha^2\,\frac{\partial\rho^{1/2}}{\partial x}\,\frac{\partial^2\rho^{1/2}}{\partial x^2}\quad, \tag{8}$$

which combined again with Eq. (6) gives:

$$-V\frac{\partial\rho}{\partial s}\;-\;V^2\frac{\partial\rho}{\partial x}\;+\;\rho\frac{\partial V}{\partial s}=-\frac{\partial U}{\partial x}\rho+\frac{\alpha^2}{4}\frac{\partial^3\rho}{\partial x^3}\;-\;2\alpha^2\,\frac{\partial\rho^{1/2}}{\partial x}\,\frac{\partial^2\rho^{1/2}}{\partial x^2}\quad. \tag{9}$$

On the other hand, by integrating Eq. (4) with respect to x and multiplying the resulting equation by $\rho^{1/2}\;\left(\partial\rho^{1/2}/\partial x\right)$, we have:

$$-2\alpha^2\,\frac{\partial\rho^{1/2}}{\partial x}\,\frac{\partial^2\rho^{1/2}}{\partial x^2}\;=\;-\,2\frac{\partial\rho}{\partial x}\int\left(\frac{\partial V}{\partial s}\right)\,dx-\;V^2\frac{\partial\rho}{\partial x}\;-2\,U\frac{\partial\rho}{\partial x}\;+\;2\;c_0(s)\;\frac{\partial\rho}{\partial x}\;, \tag{10}$$

where $c_0(s)$ is an arbitrary function of s. By combining (9) and (10) the following equation is finally obtained:

$$-\left(\frac{\partial V}{\partial s}\right)\rho + V\frac{\partial \rho}{\partial s} + 2\left[c_0(s) - \int\left(\frac{\partial V}{\partial s}\right)\,dx\right]\frac{\partial \rho}{\partial x} - \left(\frac{\partial U}{\partial x}\rho + 2\,U\frac{\partial \rho}{\partial x}\right) \;+\; \frac{\alpha^2}{4}\frac{\partial^3 \rho}{\partial x^3} \;=\; 0 \quad . \tag{11}$$

Note that now Eq. (1), or the pair of equations (3) and (4), is reduced to the pair of equations (3) and (11).

Let us denote with $\mathcal{E} = \{\Psi\}$ the set of all the envelope solutions of the generalized nonlinear Schrödinger equation (gNLSE), Eq. (1), in the form of travelling wave envelope, i.e. $\Psi(x,s) = \sqrt{\rho(\xi)}\exp\left\{\frac{i}{\alpha}\,\Theta(x,s)\right\}$, where $\xi = x - u_0 s$ (u_0 being a real constant).

Let us also denote with $\mathcal{S} = \{u(\xi) \geq 0\}$ the set of all non-negative stationary-profile solutions (travelling waves) of the following generalized Korteweg-de Vries equation (gKdVE)

$$a\,\frac{\partial u}{\partial s} \;-\; G\left[u\right]\,\frac{\partial u}{\partial x} \;+\; \frac{\nu^2}{4}\,\frac{\partial^3 u}{\partial x^3} \;=\; 0 \quad . \tag{12}$$

In order to construct a correspondence between $\mathcal{E}$ and $\mathcal{S}$, we observe that if $\Psi \in \mathcal{E}$, thus ρ and V have the form $\rho = \rho(\xi)$ and $V = V(\xi)$, respectively.
Under the above hypothesis, it is easy to see that: (a). $c_0(s)$ becomes constant (so that, let us put $c_0(s) \equiv c_0$); (b). continuity equation (3) becomes:

$$u_0\frac{d\rho}{d\xi} = \frac{d}{d\xi}\left(\rho V\right) \quad , \tag{13}$$

which integrated gives:

$$V(\xi) = u_0 + \frac{A_0}{\rho(\xi)} \quad , \tag{14}$$

where A_0 is an arbitrary constant. By combining (11) and (13), we easily have:

$$\left(u_0^2 + 2c_0\right)\frac{d\rho}{d\xi} \;-\; \mathcal{I}\left[\rho\right]\frac{d\rho}{d\xi} \;+\; \frac{\alpha^2}{4}\frac{d^3\rho}{d\xi^3} \;=\; 0 \quad , \tag{15}$$

where the functional $\mathcal{I}\left[\rho\right]$ is defined as:

$$\mathcal{I}\left[\rho\right] \;=\; \rho\frac{dU\left[\rho\right]}{d\rho} \;+\; 2U\left[\rho\right] \quad . \tag{16}$$

On the other hand, for stationary-profile solution $u = u(\xi)$, Eq. (12) becomes:

$$-u_0 a\,\frac{du}{d\xi} \;-\; G\left[u\right]\frac{du}{d\xi} \;+\; \frac{\nu^2}{4}\frac{d^3u}{d\xi^3} \;=\; 0 \quad . \tag{17}$$

Consequently, (15) and (17) have the same solutions, if the same boundary conditions are taken for them and provided that their coefficients are respectively proportional. In particular, it follows that $u(\xi)$ is a non-negative travelling wave solution of the following gKdVE ($u_0 \neq 0$):

$$-\frac{u_0^2 + 2c_0}{u_0}\frac{\partial \rho}{\partial s} - \mathcal{I}\left[\rho\right]\frac{\partial \rho}{\partial x} + \frac{\alpha^2}{4}\frac{\partial^3 \rho}{\partial x^3} = 0 \quad , \tag{18}$$

where the following identies have been used:

$$a \equiv -\left(u_0^2 + 2c_0\right)/u_0 \; , \qquad G\left[u\right] \equiv \mathcal{I}\left[u\right] \; , \qquad \nu \equiv \alpha \quad . \tag{19}$$

Thus, it results that, starting from Eq. (1), we have constructed the following correspondence:

$$\begin{aligned} \mathcal{F} \quad &: \quad \Psi \in \mathcal{E} \rightarrow u \in \mathcal{S} \quad , \\ u \quad &= \quad \mathcal{F}\left[\Psi\right] \quad = \quad |\Psi|^2 \quad = \quad \rho \quad . \end{aligned} \tag{20}$$

$\mathcal{F}$ associates a travelling wave envelope solution of (1) to a travelling wave solution of the associated gKdVE (12). In particular, it may associate an envelope solitonlike solution of (1) with a solitonlike solution of (11).

It can be proven that [16]:

$$\Theta(x,s) \; = \; \phi_0 \; - \; \left(c_0 + u_0^2\right) s \; + \; u_0 x \; + \; A_0 \int \frac{d\xi}{\rho(\xi)} \quad , \tag{21}$$

where ϕ_0 is an arbitrary real constant.

Now, let $u(\xi)$ be a positive stationary-profile solution of (12). Thus, u satisfies an equation similar to (17) and, provided that (15) and (17) have still proportional coefficients, in correspondence to the same boundary conditions, u is also solution of (15). Thus, by defining the phase $\Theta(x,s)$ given by (21), one can go back defining $V = \partial\Theta(x,s)/\partial x$. It follows that $\rho(\xi) = u(\xi)$ and V are solutions of the following system of coupled equations:

$$\frac{\partial u}{\partial s} + \frac{\partial}{\partial x}\left(uV\right) = 0 \, , \tag{22}$$

$$-\left(\frac{\partial V}{\partial s}\right) u + V\frac{\partial u}{\partial s} + 2\left[c_0(s) - \int \left(\frac{\partial V}{\partial s}\right)\, dx\right]\frac{\partial u}{\partial x} - \left(\frac{\partial \mathcal{U}}{\partial x} u + 2\,\mathcal{U}\frac{\partial u}{\partial x}\right) + \frac{\nu^2}{4}\frac{\partial^3 u}{\partial x^3} = 0 \quad , \tag{23}$$

where the functional $\mathcal{U}$ is solution of the following differential equation:

$$u\,\frac{d\mathcal{U}}{du} \; + \; 2\mathcal{U} \; = \; G\left[u\right] \, , \tag{24}$$

namely

$$\mathcal{U}\left[u\right] \;=\; \frac{1}{u^2}\left[K_0 \;+\; \int G\left[u\right]\; u\; du\right] \;, \tag{25}$$

where K_0 is an arbitrary real integration constant. Consequently, the complex function

$$\Psi = \sqrt{u(\xi)}\; \exp\left[\frac{i}{\alpha}\Theta(x,s)\right] \tag{26}$$

is a travelling envelope wave solution of the following gNLSE:

$$i\nu\frac{\partial\Psi}{\partial s} \;+\; \frac{\nu^2}{2}\frac{\partial^2\Psi}{\partial x^2} \;-\; \left[\frac{K_0 \;+\; \int\; G\left[|\Psi|^2\right]\; |\Psi|^2\; d|\Psi|^2}{|\Psi|^4}\right]\Psi \;=\; 0 \quad . \tag{27}$$

The substitution of (26) in (27) (after separating real and imaginary parts) gives the following equation for u:

$$-\frac{\nu^2}{2}\frac{d^2u^{1/2}}{d\xi^2} \;+\; \frac{K_0}{u^{3/2}} \;+\; \frac{1}{u^{3/2}}\int\; G\left[u\right]\; u\; du \;=\; \left(c_0 \;+\; \frac{u_0^2}{2}\right)u^{1/2} \;-\; \frac{A_0^2}{2u^{3/2}} \quad . \tag{28}$$

It results that for each given $u \in \mathcal{S}$ and for each given set of constants ϕ_0, c_0, and A_0, the modulus and the phase of Ψ are uniquely determined and, consequently, the solution of (27) is uniquely determined.

In conclusion, starting from the gKdVE (12), we have constructed the following correspondence:

$$\begin{aligned} &\mathcal{H}\;:\; u \in \mathcal{S} \rightarrow \Psi \in \mathcal{E} \quad , \\ &\Psi = \mathcal{H}\left[u\right] = \sqrt{u(\xi)}\exp\left\{\frac{i}{\nu}\left[\phi_0 - \left(c_0 + u_0^2\right)s + u_0 x + A_0\int\frac{d\xi}{u(\xi)}\right]\right\} \quad , \end{aligned} \tag{29}$$

which, for each given set of real constants ϕ_0, c_0, and A_0, associates a positive stationary-profile solution $u(\xi)$ of (12) to a stationary-profile envelope solution $\Psi(x,s)$ of (27) which is of the type (1). It is clear that, as the above parameters vary over all their accessible ranges of values, $\mathcal{H}\left[u\right]$ describes the subset of stationary-profile envelope solutions of (27) whose squared modulus equals $u(\xi)$. In particular, if $u(\xi)$ is a localized solution of (12), thus $\mathcal{H}\left[u\right]$ describes the subset of envelope localized solutions of the associated equation (27), where ϕ_0 is still arbitrary and the values allowed for c_0 and A_0 are determined by the specific boundary conditions required for such a kind of localized solution. It has been proven that for bright and dark soliton solutions, satisfying the standard boundary conditions, i.e., $\lim_{\xi\rightarrow\pm\infty} u(\xi) = 0$, $K_0 = A_0 = 0$ and therefore the phase of Ψ is linear (see Eq. (29)) [16]. If $K_0 = 0$, the case $A_0 \neq 0$ is compatible with gray and up-shifted solitons [16], while the case of $K_0 \neq 0$ and $A_0 \neq 0$ exhibits some new insights [19].

In the next subsections we present some examples where the above mapping is applied to find solitonlike solution taking some special forms of the nonlinear potential $U[|\Psi|^2]$ and the ones of the corresponding nonlinear term $G[u]$. We divide the analysis into two cases: the case of $K_0 = 0$ and the one of $K_0 \neq 0$.

1. Case of $K_0 = 0$.

(a). Let us take the following nonlinear term of the gNLSE (1): $U\left[|\Psi|^2\right] = q_0|\Psi|^{2\beta}$, where q_0 and β are real and positive real numbers, respectively. Thus, according to Eq.s (16) and (19), the corresponding nonlinear term of the gKdVE (12) is $G[u] = p_0 u^\beta$, where $p_0 = (\beta+2)q_0$. For $au_0 > 0$ and $p_0 < 0$, Eq. (12) admits the following positive solution in the form of bright travelling solitons (for any positive real β):

$$u(\xi) = \left[\frac{au_0\,(\beta+2)\,(\beta+1)}{2|p_0|}\right]^{1/\beta} \mathrm{sech}^{2/\beta}\left[\frac{\beta\sqrt{au_0}}{|\nu|}\xi\right] \tag{30}$$

(note that $\beta = 1$ recovers the bright KdV soliton [17]).

Consequently, by virtu of (29), we can write the following travelling bright envelope soliton solutions of (1) for the case under discussion (note that here $A_0 = 0$):

$$\Psi(x,s) = \left[\frac{|E_0|\,(\beta+1)}{|q_0|}\right]^{1/2\beta} \mathrm{sech}^{1/\beta}\left[\frac{\beta\sqrt{2|E_0|}}{|\alpha|}\xi\right] \exp\left\{\frac{i}{\alpha}\left[\phi_0 - \left(E_0 + u_0^2/2\right)s + u_0 x\right]\right\}, \tag{31}$$

where $E_0 < 0$ and $q_0 < 0$. In Figure 2 are plotted the modulus of the wave function Ψ and the one of the current velocity V, respectively. They show the solution of a cubic NLSE ($\beta = 1$) in the form of a grey soliton.

(b). As second example, let us find envelope solitonlike solutions of the Eq. (1) with the following nonlinear potential: $U = U\left[|\Psi|^2\right] = q_1|\Psi|^2 + q_2|\Psi|^4$ (cubic-quintic NLSE). Thus the corresponding nonlinear terms in Eq. (12) is: $G[u] = p_1 u + p_2 u^2$, where $p_1 = 3q_1$ and $p_2 = 5q_2$. This term can be also cast in the form: $G[u] = p_0(u - \overline{u})^2 + \delta_0$, where $p_0 = 4q_2$, $\overline{u} = -3q_1/(8q_2)$, and $\delta_0 = -9q_1^2/16q_2$. By using the results of the previous example for $\beta = 2$, travelling solitons of the modified KdVE under discussion are easily found [16]:

$$u(\xi) = \overline{u}\ \left[1\ +\ \epsilon\ \mathrm{sech}\,(\xi/\Delta)\right] \quad , \tag{32}$$

where

$$\epsilon = \pm\sqrt{1 - 32|q_2|\,(u_0 - V_0)^2\,/\,(3q_1^2)} \quad ,$$

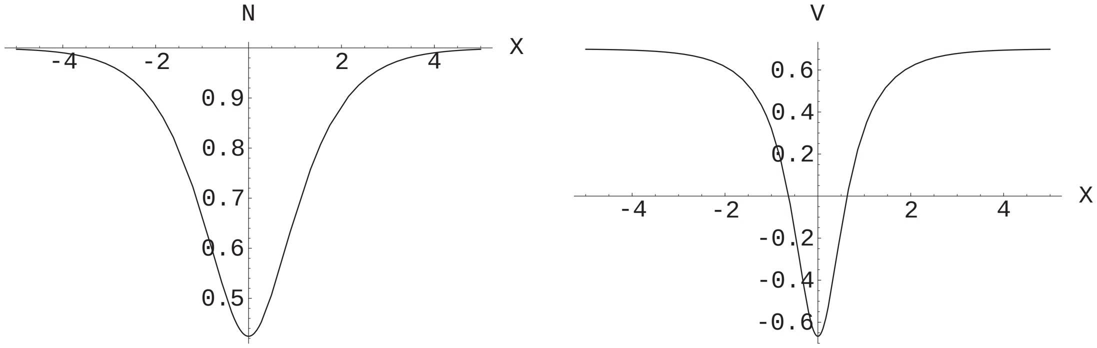

FIG. 2: *Grey soliton for cubic NLSE ($\beta = 1$). $N \equiv |\Psi|/\sqrt{\rho_0} = \sqrt{\rho/\rho_0}$ versus $X \equiv \xi/|\alpha|$ (picture at the left) and V versus $X \equiv \xi/|\alpha|$ (picture at the right), where $\xi = x - u_0 s$.* [17] The parameters are: $\rho_0 q_0 = 0.5$, $u_0 = 1$, and $V_0 = 0.7$.

$$\Delta = |\alpha|/\left(2\sqrt{2|E_0'|}\right) \quad ,$$

and

$$E_0' = -3q_1^2/\left(64|q_2|\right) \ + \ \left(u_0 - V_0\right)^2/2 \quad ,$$

provided that $E_0' < 0$ and $q_2 < 0$ and

$$-\sqrt{\frac{3q_1^2}{32|q_2|}} + V_0 < u_0 < \sqrt{\frac{3q_1^2}{32|q_2|}} + V_0 \quad .$$

Eq. (32) shows that we can distinguish the following four cases.

(i). $0 < \epsilon < 1$ ($u_0 - V_0 \neq 0$):

$$u(\xi = 0) = \overline{u}(1 + \epsilon) \ , \quad \text{and} \quad \lim_{\xi \to \pm\infty} u(\xi) = \overline{u}$$

which corresponds to a bright soliton of maximum amplitude $(1 + \epsilon)\overline{u}$ and up-shifted by the quantity $\overline{u}$. We could call it *up-shifted bright soliton.*

(ii).$- < \epsilon < 0$ ($u_0 - V_0 \neq 0$):

$$u(\xi = 0) = \overline{u}(1 - \epsilon) \ , \quad \text{and} \quad \lim_{\xi \to \pm\infty} u(\xi) = \overline{u}$$

which is a dark soliton with minimum amplitude $(1 - \epsilon)\overline{u}$ and reaching asimptotically the upper limit $\overline{u}$. It corresponds to a standard *gray soliton.*

(iii). $\epsilon = 1$ ($u_0 - V_0 = 0$):

$$u(\xi = 0) = 2\overline{u} \ , \quad \text{and} \quad \lim_{\xi \to \pm\infty} u(\xi) = \overline{u}$$

which corresponds to a bright soliton of maximum amplitude $2\overline{u}$ and up-shifted by the maximum quantity $\overline{u}$. We could call it *upper-shifted bright soliton.*

(iv).$\epsilon = -1$ $(u_0 - V_0 = 0)$:

$$u(\xi = 0) = 0 \ , \quad \text{and} \quad \lim_{\xi \to \pm\infty} u(\xi) = \overline{u}$$

which is a dark soliton (zero minimum amplitude), reaching asimptotically the upper limit $\overline{u}$. It correspond to a standard *dark soliton.*

Consequenlty, by using mapping (29), we can conclude that the cubic-quintic NLSE under discussion has the following travelling envelope solitons:

$$\Psi(x,s) \;=\; \sqrt{\overline{u}\ [1 \ + \ \epsilon \ \mathrm{sech}\,(\xi/\Delta)]}\, \exp\left\{\frac{i}{\alpha}\,[\phi_0 - As + u_0 x]\right\}$$
$$\exp\left\{\frac{iB}{\alpha}\left[\frac{\xi}{\Delta} + \frac{2\epsilon}{\sqrt{1-\epsilon^2}} \arctan\left(\frac{(\epsilon - 1)\tanh{(\xi/2\Delta)}}{\sqrt{1-\epsilon^2}}\right)\right]\right\} \quad , \tag{33}$$

where ϕ_0 still plays the role of arbitrary constant,

$$A \;=\; \frac{15 q_1^2}{64|q_2|} + \frac{(u_0 - V_0)^2}{2} + \frac{u_0^2}{2} \quad , \tag{34}$$

and

$$B \;=\; -\frac{|\alpha|\,(u_0 - V_0)}{2\sqrt{2|E_0'|}} \quad . \tag{35}$$

In Table I the correspondence between modified Korteweg-de Vries equations (mKdVEs) and modified nonlinear Schrödinger equations (mNLSEs) and the corresponding soliton-like solution is shown for several values of β.

2. Case of $K_0 \neq 0$

It is clear from (28) that, for $K_0 \neq 0$, a family of solitary wave solutions of (27) can be obtained by imposing the following condition

$$A_0 \;=\; \pm\sqrt{-2K_0}\,, \tag{36}$$

which implies that such a kind of family of solutions exists for negative values of K_0. In fact, condition (36) select the suitable values of the arbitrary constant A_0 for finding solitonlike solutions, by providing the cancellation of the term $K_0/u^{3/2}$, located at left hand side of

β	**mKdVE**	u	**mNLSE**	$\lvert\Psi\rvert$
1/10	$a\frac{\partial u}{\partial s} - \frac{21}{10} q_0 u^{1/10}\frac{\partial u}{\partial x} + \frac{\alpha^2}{4}\frac{\partial^3 u}{\partial x^3} = 0$	$\left(\frac{11\lvert E_0\rvert}{10\lvert q_0\rvert}\right)^{10}\mathrm{sech}^{20}\left(\frac{\sqrt{2\lvert E_0\rvert}}{10\lvert\alpha\rvert}\xi\right)$	$i\alpha\frac{\partial\Psi}{\partial s} = -\frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2} - q_0\lvert\Psi\rvert^{1/5}\Psi$	$\left(\frac{11\lvert E_0\rvert}{10\lvert q_0\rvert}\right)^{5}\mathrm{sech}^{10}\left(\frac{\sqrt{2\lvert E_0\rvert}}{10\lvert\alpha\rvert}\xi\right)$
1/8	$a\frac{\partial u}{\partial s} - \frac{17}{8} q_0 u^{1/8}\frac{\partial u}{\partial x} + \frac{\alpha^2}{4}\frac{\partial^3 u}{\partial x^3} = 0$	$\left(\frac{9\lvert E_0\rvert}{8\lvert q_0\rvert}\right)^{8}\mathrm{sech}^{16}\left(\frac{\sqrt{2\lvert E_0\rvert}}{8\lvert\alpha\rvert}\xi\right)$	$i\alpha\frac{\partial\Psi}{\partial s} = -\frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2} - q_0\lvert\Psi\rvert^{1/4}\Psi$	$\left(\frac{9\lvert E_0\rvert}{8\lvert q_0\rvert}\right)^{4}\mathrm{sech}^{8}\left(\frac{\sqrt{2\lvert E_0\rvert}}{8\lvert\alpha\rvert}\xi\right)$
1/3	$a\frac{\partial u}{\partial s} - \frac{7}{3} q_0 u^{1/3}\frac{\partial u}{\partial x} + \frac{\alpha^2}{4}\frac{\partial^3 u}{\partial x^3} = 0$	$\left(\frac{4\lvert E_0\rvert}{3\lvert q_0\rvert}\right)^{3}\mathrm{sech}^{6}\left(\frac{\sqrt{2\lvert E_0\rvert}}{3\lvert\alpha\rvert}\xi\right)$	$i\alpha\frac{\partial\Psi}{\partial s} = -\frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2} - q_0\lvert\Psi\rvert^{2/3}\Psi$	$\left(\frac{4\lvert E_0\rvert}{3\lvert q_0\rvert}\right)^{3/2}\mathrm{sech}^{3}\left(\frac{\sqrt{2\lvert E_0\rvert}}{3\lvert\alpha\rvert}\xi\right)$
1/2	$a\frac{\partial u}{\partial s} - \frac{5}{2} q_0 u^{1/2}\frac{\partial u}{\partial x} + \frac{\alpha^2}{4}\frac{\partial^3 u}{\partial x^3} = 0$	$\left(\frac{3\lvert E_0\rvert}{2\lvert q_0\rvert}\right)^{2}\mathrm{sech}^{4}\left(\frac{\sqrt{2\lvert E_0\rvert}}{2\lvert\alpha\rvert}\xi\right)$	$i\alpha\frac{\partial\Psi}{\partial s} = -\frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2} - q_0\lvert\Psi\rvert\Psi$	$\left(\frac{3\lvert E_0\rvert}{2\lvert q_0\rvert}\right)\mathrm{sech}^{2}\left(\frac{\sqrt{2\lvert E_0\rvert}}{2\lvert\alpha\rvert}\xi\right)$
1	$a\frac{\partial u}{\partial s} - 3q_0 u\frac{\partial u}{\partial x} + \frac{\alpha^2}{4}\frac{\partial^3 u}{\partial x^3} = 0$	$\left(\frac{2\lvert E_0\rvert}{\lvert q_0\rvert}\right)\mathrm{sech}^{2}\left(\frac{\sqrt{2\lvert E_0\rvert}}{\lvert\alpha\rvert}\xi\right)$	$i\alpha\frac{\partial\Psi}{\partial s} = -\frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2} - q_0\lvert\Psi\rvert^{2}\Psi$	$\left(\frac{2\lvert E_0\rvert}{\lvert q_0\rvert}\right)^{1/2}\mathrm{sech}\left(\frac{\sqrt{2\lvert E_0\rvert}}{\lvert\alpha\rvert}\xi\right)$
3/2	$a\frac{\partial u}{\partial s} - \frac{7}{2} q_0 u^{3/2}\frac{\partial u}{\partial x} + \frac{\alpha^2}{4}\frac{\partial^3 u}{\partial x^3} = 0$	$\left(\frac{5\lvert E_0\rvert}{2\lvert q_0\rvert}\right)^{2/3}\mathrm{sech}^{4/3}\left(\frac{3\sqrt{2\lvert E_0\rvert}}{2\lvert\alpha\rvert}\xi\right)$	$i\alpha\frac{\partial\Psi}{\partial s} = -\frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2} - q_0\lvert\Psi\rvert^{3}\Psi$	$\left(\frac{5\lvert E_0\rvert}{2\lvert q_0\rvert}\right)^{1/3}\mathrm{sech}^{2/3}\left(\frac{3\sqrt{2\lvert E_0\rvert}}{2\lvert\alpha\rvert}\xi\right)$
2	$a\frac{\partial u}{\partial s} - 4q_0 u^{2}\frac{\partial u}{\partial x} + \frac{\alpha^2}{4}\frac{\partial^3 u}{\partial x^3} = 0$	$\left(\frac{3\lvert E_0\rvert}{\lvert q_0\rvert}\right)^{1/2}\mathrm{sech}\left(\frac{2\sqrt{2\lvert E_0\rvert}}{\lvert\alpha\rvert}\xi\right)$	$i\alpha\frac{\partial\Psi}{\partial s} = -\frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2} - q_0\lvert\Psi\rvert^{4}\Psi$	$\left(\frac{3\lvert E_0\rvert}{\lvert q_0\rvert}\right)^{1/4}\mathrm{sech}^{1/2}\left(\frac{2\sqrt{2\lvert E_0\rvert}}{\lvert\alpha\rvert}\xi\right)$
5/2	$a\frac{\partial u}{\partial s} - \frac{9}{2} q_0 u^{5/2}\frac{\partial u}{\partial x} + \frac{\alpha^2}{4}\frac{\partial^3 u}{\partial x^3} = 0$	$\left(\frac{7\lvert E_0\rvert}{2\lvert q_0\rvert}\right)^{2/5}\mathrm{sech}^{4/5}\left(\frac{5\sqrt{2\lvert E_0\rvert}}{2\lvert\alpha\rvert}\xi\right)$	$i\alpha\frac{\partial\Psi}{\partial s} = -\frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2} - q_0\lvert\Psi\rvert^{5}\Psi$	$\left(\frac{7\lvert E_0\rvert}{2\lvert q_0\rvert}\right)^{1/5}\mathrm{sech}^{2/5}\left(\frac{5\sqrt{2\lvert E_0\rvert}}{2\lvert\alpha\rvert}\xi\right)$
3	$a\frac{\partial u}{\partial s} - 5q_0 u^{3}\frac{\partial u}{\partial x} + \frac{\alpha^2}{4}\frac{\partial^3 u}{\partial x^3} = 0$	$\left(\frac{4\lvert E_0\rvert}{\lvert q_0\rvert}\right)^{1/3}\mathrm{sech}^{2/3}\left(\frac{3\sqrt{2\lvert E_0\rvert}}{\lvert\alpha\rvert}\xi\right)$	$i\alpha\frac{\partial\Psi}{\partial s} = -\frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2} - q_0\lvert\Psi\rvert^{6}\Psi$	$\left(\frac{4\lvert E_0\rvert}{\lvert q_0\rvert}\right)^{1/6}\mathrm{sech}^{1/3}\left(\frac{3\sqrt{2\lvert E_0\rvert}}{\lvert\alpha\rvert}\xi\right)$
10	$a\frac{\partial u}{\partial s} - 12q_0 u^{10}\frac{\partial u}{\partial x} + \frac{\alpha^2}{4}\frac{\partial^3 u}{\partial x^3} = 0$	$\left(\frac{11\lvert E_0\rvert}{\lvert q_0\rvert}\right)^{1/10}\mathrm{sech}^{1/5}\left(\frac{\sqrt{2\lvert E_0\rvert}}{10\lvert\alpha\rvert}\xi\right)$	$i\alpha\frac{\partial\Psi}{\partial s} = -\frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2} - q_0\lvert\Psi\rvert^{20}\Psi$	$\left(\frac{11\lvert E_0\rvert}{\lvert q_0\rvert}\right)^{1/20}\mathrm{sech}^{1/10}\left(\frac{10\sqrt{2\lvert E_0\rvert}}{\lvert\alpha\rvert}\xi\right)$

TABLE I: *Correspondence between the mNLSE and the mKdVE and their bright soliton-like solutions for several values of* β [18]. *Note that here* $\xi = x - u_0 s$, $E_0 = -au_0 < 0$ *and* $q_0 < 0$.

(28), with $-A_0^2/u^{3/2}$, located a the right hand side. Consequently, equation (28) becomes the following NLSE for stationary states

$$-\frac{\alpha^2}{2}\frac{d^2u^{1/2}}{d\xi^2} + \mathcal{U}[u]\,u^{1/2} = E_0u^{1/2} \quad , \tag{37}$$

where $E_0 = c_0 + u_0^2/2$. Note that, by virtue Eq. (24) the functional G[u] remains unchanged switching from the case of $K_0 = 0$ to the one of $K_0 \neq 0$. Consequently, the corresponding gKdVE equations are undistinguishable. This implies that the travelling envelope soliton solutions of (27) with $K_0 \neq 0$ can be constructed by using the ones known in the case of $K_0 = 0$. Recently, this implication has allowed to find travelling envelope solitons of (1) with a nonlinear potential of the form: $q_1|\Psi|^2 + q_2|\Psi|^4 + Q_0|\Psi|^{-4}$ (cubic-quintic-anticubic

NLSE) [19]. It has been shown that for any $Q_0 \ < \ 0$, under condition (36) ($Q_0 \equiv K_0$), the cubic-quintic NLSE under discussion, with the additional anti-cubic nonlinear term, has the following envelope soliton solutions:

$$\Psi_\pm(x,s) = \sqrt{\overline{u}\ \left[1 \ + \ \epsilon \ \mathrm{sech}\left(\frac{\xi}{\Delta}\right)\right]}$$
$$\exp\left\{\frac{i}{\alpha}\left[\phi_0 - \left(E_0 - \frac{u_0^2}{2}\right)s + u_0\xi \ \pm \sqrt{2|Q_0|}\ \int\frac{d\xi}{\sqrt{\overline{u}\ [1 \ + \ \epsilon \ \mathrm{sech}\,(\xi/\Delta)]}}\right]\right\}, \quad (38)$$

where here, in principle, according to Ref. [19], ϵ should be taken in the following range

$$-1 \ < \epsilon \ \leq 1 \quad , \quad (39)$$

which excludes the standard "dark" solitary waves ($\epsilon = -1$), namely the condition for which the modulus of Ψ vanishes at $\xi = 0$.

Actually, the direct substitution of $u = |\Psi|^2$ given by (38) into the eigenvalue equation (37) allows us to find

$$\epsilon \ = \ 1 \quad , \quad (40)$$

$$\overline{u} \ = \ -\frac{3q_1}{8q_2} \quad , \quad (41)$$

$$q_1 \ > \ 0 \quad , \qquad q_2 \ < \ 0 \quad , \quad (42)$$

$$E_0 \ = \ -\frac{15q_1^2}{64q_2} \quad , \quad (43)$$

$$\Delta = \frac{2|\alpha|}{q_1}\sqrt{\frac{2|q_2|}{3}} \quad . \quad (44)$$

Consequently, solution (38) can be cast as

$$\Psi_\pm(x,s) = \sqrt{\frac{3q_1}{8|q_2|}\ \left[1 \ + \ \mathrm{sech}\left(\frac{\xi}{\Delta}\right)\right]}$$
$$\exp\left\{\frac{i}{\alpha}\left[\phi_0 - \left(\frac{15q_1^2}{64|q_2|} - \frac{u_0^2}{2}\right)s + u_0\xi \ \pm \sqrt{2|Q_0|}\ \frac{16|\alpha||q_2|}{3q_1^2}\sqrt{\frac{2|q_2|}{3}}\left(\frac{\xi}{\Delta} - \tanh\left(\frac{\xi}{2\Delta}\right)\right)\right]\right\} \quad , \quad (45)$$

where u_0 is a fully arbitrary soliton velocity. According to the classification of the solitary waves used in Ref. [16], equation (45) represents an upper-shifted bright envelope solitonlike solution, provided that the coefficients Q_0, q_1 and q_2 satisfy the conditions $Q_0 \ < \ 0$ and (42), respectively. It is clear that equation (40), which does not contradict condition (39), implies that also gray solitary solutions do not exist in the solution form (38).

C. Application to the modulational instability theory in the presence of nonlocal effects

In this subsection, we present an example of using the Madelung fluid description to analyze the MI of NLSE in the presence of nonlocal effects [58]. We consider the case of longitudinal charged-particle beam dynamics in high-energy accelerating machines.

1. The NLSE governing the longitudinal dynamics of a charged particle beam including nonlocal effects in the framework of TWM.

Within the TWM framework, the longitudinal dynamics of particle bunches is described in terms of a complex wave function $\Psi(x, s)$, where s is the distance of propagation (time-like coodinate) and x is the longitudinal instantaneous location of a generic particle, measured in the moving frame of reference. The particle density, $\lambda(x, s)$, is related to the wave function according to $\lambda(x, s) = |\Psi(x, s)|^2$, [20]. The collective longitudinal evolution of the beam in a circular high-energy accelerating machine is governed by the Schrödinger-like equation

$$i\epsilon \frac{\partial \Psi}{\partial s} + \frac{\epsilon^2 \eta}{2} \frac{\partial^2 \Psi}{\partial x^2} + U(x, s)\Psi = 0 \ , \tag{46}$$

where ϵ is the longitudinal beam emittance and η is the slip factor, [59], defined as $\eta = \gamma_T^{-2} - \gamma^{-2}$ (γ_T being the transition energy, defined as the inverse of the momentum compaction, [59], and γ being the relativistic factor); $U(x, s)$ is the effective dimensionless (with respect to the nominal particle energy, $E_0 = m\gamma c^2$) potential energy given by the interaction between the bunch and the surroundings. Note that η can be positive (above transition energy) or negative (below transition energy). Above transition energy, in analogy with quantum mechanics, $1/\eta$ plays the role of an effective mass associated with the beam as a whole. Below transition energy, $1/\eta$ plays the role of a "negative mass".

Equation (46) has to be coupled with an equation for U. If no external sources of electromagnetic fields are present and the effects of charged-particle radiation damping is negligible, the self-interaction of the beam with the surroundings, due to the image charges and the image currents originated on the walls of the vacuum chamber, makes U a functional of the beam density. It can proven that, in a torus-shaped accelerating machine, characterized by a toroidal radius R_0 and a poloidal radius a, for a coasting beam of radius $b << a$ travelling at velocity βc ($\beta \leq 1$ and c being the speed of light), the self-interaction potential energy is given by [25] (a more general expression is given in Ref. [60]):

$$U[\lambda_1(x,s)] = \frac{q^2\beta c}{E_0}\left(R_0 Z_I' \lambda_1(x,s) + Z_R' \int_0^x \lambda_1(x',s)\,dx'\right), \tag{47}$$

where $\lambda_1(x,s)$ is an (arbitrarily large) line beam density perturbation, q is the charge of the particles, ϵ_0 is the vacuum dielectric constant, Z_R' and Z_I' are the resistive and the total reactive parts, respectively, of the longitudinal coupling impedance per unit length of the machine. Thus, the coupling impedance per unit length can be defined as the complex quantity $Z' = Z_R' + iZ_I'$. In our simple model of a circular machine, it is easy to see that [59], [60]:

$$Z_I' = \frac{1}{2\pi R_0}\left(\frac{g_0 Z_0}{2\beta\gamma^2} - \omega_0 \mathcal{L}\right) \equiv \frac{Z_I}{2\pi R_0}, \tag{48}$$

where Z_0 is the vacuum impedance, $\omega_0 = \beta c/R_0$ is the nominal orbital angular frequency of the particles and $\mathcal{L}$ is the total inductance. This way, Z_I represents the total reactance as the difference between the total space charge capacitive reactance, $g_0 Z_0/(2\beta\gamma^2)$, and the total inductive reactance, $\omega_0\mathcal{L}$. Consequently, in the limit of negligible resistance, Eq. (47) reduces to

$$U[\lambda_1] = \frac{q^2\beta c}{2\pi E_0}\left(\frac{g_0 Z_0}{2\beta\gamma^2} - \omega_0 \mathcal{L}\right)\lambda_1. \tag{49}$$

By definition, an unperturbed coasting beam has the particles uniformly distributed along the longitudinal coordinate x. Denoting by $\rho(x,s)$ the line density and by $\rho(x,0)$ the unperturbed one, in the TWM framework we have the following identifications: $\rho(x,s) = |\Psi(x,s)|^2$, $\rho_0 = |\Psi(x,0)|^2 \equiv |\Psi_0|^2$, where Ψ_0 is a complex function and, consequently, $\lambda_1(x,s) = |\Psi(x,s)|^2 - |\Psi_0|^2$. Thus, the combination of Eq. (46) and Eq. (47) gives the following evolution equation for the beam

$$i\frac{\partial\Psi}{\partial s} + \frac{\alpha}{2}\frac{\partial^2\Psi}{\partial x^2} + \mathcal{X}\left[|\Psi|^2 - |\Psi_0|^2\right]\Psi + \mathcal{R}\,\Psi\int_0^x \left[|\Psi(x',s)|^2 - |\Psi_0|^2\right]dx' = 0, \tag{50}$$

where

$$\alpha = \epsilon\eta = \epsilon\left(\gamma^{-2} - \gamma_T^{-2}\right), \tag{51}$$

$$\mathcal{X} = \frac{q^2\beta c R_0}{\epsilon E_0} Z_I', \tag{52}$$

$$\mathcal{R} = \frac{q^2\beta c}{\epsilon E_0} Z_R'. \tag{53}$$

Equation (50) belongs to the family of NLSEs governing the propagation and dynamics of wave packets in the presence of nonlocal effects. The modulational instability of such an integro-differential equation has been investigated for the first time in literature in Ref. [23]. Some nonlocal effects associated with the collective particle beam dynamics have been recently described with this equation. Note that Eq. (50) can be cast in the form of Eq. (1), provided that (51)-(53) are taken and the following expression for the nonlinear potential is assumed, i.e.,

$$U[|\Psi|^2] = -\alpha \left\{ \mathcal{X} \left[|\Psi|^2 \; - \; |\Psi_0|^2 \right] \; + \; \mathcal{R} \int_0^x \; \left[|\Psi(x', s)|^2 \; - \; |\Psi_0|^2 \right] \; dx' \right\} \; . \tag{54}$$

2. MI analysis of a monochromatic coasting beam

Under the conditions assumed above, let us consider a monochromatic coasting beam travelling in a circular high-energy machine with the unperturbed velocity V_0 and the unperturbed density $\rho_0 = |\Psi_0|^2$ (equilibrium state). In these conditions, all the particles of the beam have the same velocity and their collective interaction with the surroundings is absent. In the Madelung fluid representation, the beam can be thought as a fluid with both current velocity and density uniform and constant. In this state, the Madelung fluid equations (3) and (4) vanish identically. Let us now introduce small perturbations in $V(x, s)$ and $\rho(x, s)$, i.e.,

$$V \; = \; V_0 + V_1 \, , \quad |V_1| \; << \; |V_0| \, , \tag{55}$$

$$\rho \; = \; \rho_0 + \rho_1 \, , \quad |\rho_1| \; << \; \rho_0 \, . \tag{56}$$

By introducing (55) and (56) in the pair of equations (3) and (4), after linearizing, we get the following system of equations:

$$\frac{\partial \rho_1}{\partial s} + V_0 \frac{\partial \rho_1}{\partial x} + \rho_0 \frac{\partial V_1}{\partial x} = 0 \, , \tag{57}$$

$$\frac{\partial V_1}{\partial s} + V_0 \frac{\partial V_1}{\partial x} = \alpha \mathcal{R} \rho_1 + \alpha \mathcal{X} \frac{\partial \rho_1}{\partial x} + \frac{\alpha^2}{4\rho_0} \frac{\partial^3 \rho_1}{\partial x^3} \, . \tag{58}$$

In order to find the linear dispersion relation, we take the Fourier transform of the system of equations (57) and (58), i.e. we express the quantities $\rho_1(x, s)$ and $V_1(x, s)$ in terms of their Fourier transforms $\tilde{\rho}_1(k,\omega)$, respectively,

$$\rho_1(x,s) = \int dk\, d\omega\, \tilde{\rho}_1(k,\omega) e^{ikx - i\omega s}\,, \tag{59}$$

$$V_1(x,s) = \int dk\, d\omega\, \tilde{V}_1(k,\omega) e^{ikx - i\omega s}\,, \tag{60}$$

and, after substituting in (57) and (58), we get the following system of algebraic equations:

$$-\rho_0 k \tilde{V}_1 \;=\; (kV_0 - \omega)\,\tilde{\rho}_1\,, \tag{61}$$

$$i\,(kV_0 - \omega)\,\tilde{V}_1 \;=\; \left(\alpha \mathcal{R} + i\alpha k \mathcal{X} - i\frac{\alpha^2}{4\rho_0}k^3\right)\tilde{\rho}_1\,. \tag{62}$$

By combining (61) and (62) we finally get the dispersion relation

$$\left(\frac{\omega}{k} - V_0\right)^2 = i\alpha\rho_0\left(\frac{\mathcal{Z}}{k}\right) + \frac{\alpha^2 k^2}{4}\,, \tag{63}$$

where we have introduced the complex quantity $\mathcal{Z} = \mathcal{R} + ik\mathcal{X} \equiv \mathcal{Z}_R + i\mathcal{Z}_I$, proportional to the longitudinal coupling impedance per unity length of the beam. In general, in Eq. (63), ω is a complex quantity, i.e., $\omega \equiv \omega_R + i\omega_I$. If $\omega_I \neq 0$, the modulational instability takes plase in the system. Thus, by substituting the complex form of ω in Eq. (63), separating the real from the imaginary parts and using (51), we finally get:

$$\mathcal{Z}_I = -\eta\frac{\epsilon k\rho_0}{4\omega_I^2}\mathcal{Z}_R^2 + \frac{1}{\eta}\frac{\omega_I^2}{\epsilon k\rho_0} + \eta\frac{\epsilon k^3}{4\rho_0}\,. \tag{64}$$

This equation fixes, for any values of the wavenumber k and any values of the growth rate ω_I a relationship between real and imaginary parts of the longitudinal coupling impedance. For each $\omega_I \neq 0$, running the values of the slip factor η, it describes two families of parabolas in the complex plane ($\mathcal{Z}_R - \mathcal{Z}_I$). Each pair ($\mathcal{Z}_R, \mathcal{Z}_I$) in this plane represents a working point of the accelerating machine. Consequently, each parabola is the locus of the working points associated with a fixed growth rate of the MI. According to Figure 3, below the transition energy ($\gamma < \gamma_T$), η is positive and therefore the instability parabolas have a negative concavity, whilst above the transition energy ($\gamma > \gamma_T$), since η is negative the instability parabolas have a positive concavity (negative mass instability). It is clear from Eq. (64) that, approaching $\omega_I = 0$ from positive (negative) values, the two families of parabolas reduce asymptotically to a straight line upper (lower) unlimited located on the imaginary axis. The straight line represent the only possible region below (above) the transition) energy where the system is modulationally stable against small perturbations

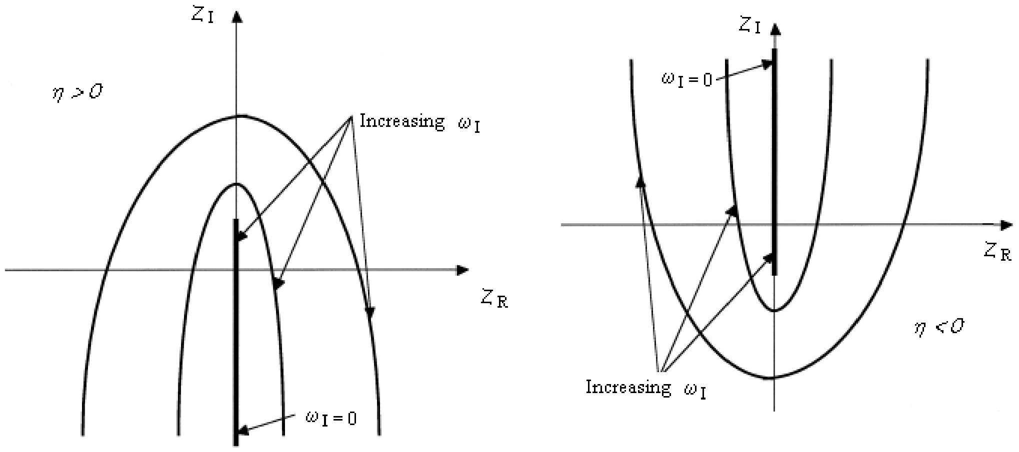

FIG. 3: *Qualitative plots of the modulational instability curves in the plane* $(\mathcal{Z}_R - \mathcal{Z}_I)$ *of a coasting beam below the transition energy,* $(\eta > 0)$ *and above the transition energy,* $(\eta < 0)$*, respectively.* [23] *The bold face vertical straight lines represent the stability region* $(\omega_I = 0)$*.*

in both density and velocity of the beam, with respect to their unperturbed values ρ_0 and V_0, respectively (note that density and velocity are directly connected with amplitude and phase, respectively, of the wave function Ψ). Any other point of the complex plane belongs to a instability parabola ($\omega_I \neq 0$).

In the limit of weak dispersion, i.e., $\epsilon k << 1$, the second term of the right hand side of Eq. (63) can be neglected and Eq. (64) reduces to

$$\mathcal{Z}_I \approx -\eta \frac{\epsilon k \rho_0}{4\omega_I^2} \mathcal{Z}_R^2 + \frac{1}{\eta} \frac{\omega_I^2}{\epsilon k \rho_0} \, . \tag{65}$$

Furthermore, for purely reactive impedances ($\mathcal{Z}_R \equiv 0$), Eq. (50) reduces to the cubic NLSE and the corresponding dispersion relation gives (note that in this case $\omega_R = V_0 k$)

$$\frac{\omega_I^2}{k^2} = -\epsilon \eta \rho_0 \left(\frac{\mathcal{Z}_I}{k} \right) + \frac{\alpha^2 k^2}{4} \, , \tag{66}$$

from which it is easily seen that the system is modulationally unstable ($\omega_I^2 > 0$) under the following conditions

$$\eta \, \mathcal{Z}_I \; > \; 0 \tag{67}$$

$$\rho_o \; > \; \frac{\epsilon \eta k^2}{4 \mathcal{X}_I} \, . \tag{68}$$

Condition (67) is a well known coherent instability condition for purely reactive impedances which coincides with the well known "Lighthill criterion" [61] associated with the cubic

NLSE. This aspect has been pointed out for the first time in Ref.s [62], [24]. According to Table II, this condition implies that the system is modulationally unstable below (above) transition energy and for capacitive (inductive) impedances and stable in the other different possible circumstances. Condition (68) implies that the instability threshold is given by

	$\mathcal{Z}_I > 0$ (capacitive)	$\mathcal{Z}_I < 0$ (inductive)
$\eta > 0$ (below transition energy)	**stable**	**unstable**
$\eta < 0$ (above transition energy)	**unstable**	**stable**

TABLE II: *Coherent instability scheme of a monochromatic coasting beam in the case of a purely reactive impedance* ($\mathcal{Z}_R = 0$). *The instability corresponding to* $\eta < 0$ *is usually referred to as "negative mass instability".*

the nonzero minimum intensity $\rho_{0m} = \epsilon\eta k^2/4\mathcal{X}_I$.

3. MI analysis of a non-monochromatic coasting beam

The dispersion relation (63) allows to write an expression for the admittance of the coasting beam $\mathcal{Y} \equiv 1/\mathcal{Z}$:

$$k\mathcal{Y} = \frac{i\alpha\rho_0}{(\omega/k - V_0)^2 - \alpha^2 k^2/4} . \qquad (69)$$

Let us now consider a non-monochromatic coasting beam. Such a system may be thought as an ensemble of coasting beams with different unperturbed velocities. Let us call $f_0(V)$ the distribution function of the velocity at the equilibrium. The subsystem corresponding to a coasting beam collecting the particles having velocities between V and $V + dV$ has an elementary admittance $d\mathcal{Y}$. Provided, in Eq. (69), to replace ρ_0 with $f_0(V)dV$, the expression for the elementary admittance is easily given:

$$kd\mathcal{Y} = \frac{i\alpha\, f_0(V)\, dV}{(V - \omega/k)^2 - \alpha^2 k^2/4} . \qquad (70)$$

All the elementary coasting beams in which we have divided the system suffer the same electric voltage per unity length along the longitudinal direction. This means that the total admittance of the system is the sum of the all elementary admittances, as it happens for a system of electric wires connected all in parallel. Therefore,

$$k\mathcal{Y} = i\alpha \int \frac{f_0(V)\, dV}{(V-\omega/k)^2 - \alpha^2 k^2/4} \,. \tag{71}$$

Of course, this dispersion relation can be cast also in the following way:

$$1 = i\alpha \left(\frac{\mathcal{Z}}{k}\right) \int \frac{f_0(V)\, dV}{(V-\omega/k)^2 - \alpha^2 k^2/4} \,, \tag{72}$$

where we have introduced the total impedance of the system which is the inverse of the total admittance, i.e., $\mathcal{Z} = 1/\mathcal{Y}$.

An interesting equivalent form of Eq. (72) can be obtained. To this end, we first observe that the following identity holds:

$$\frac{1}{(V-\omega/k)^2 - \alpha^2 k^2/4} = \frac{1}{\alpha k} \left[\frac{1}{(V-\alpha k/2) - \omega/k} - \frac{1}{(V+\alpha k/2) - \omega/k}\right] .$$

Then, using this identity in Eq. (72) it can be easily shown that:

$$1 = i \left(\frac{\mathcal{Z}}{k}\right) \frac{1}{k} \left[\int \frac{f_0(V)\, dV}{(V-\alpha k/2) - \omega/k} - \int \frac{f_0(V)\, dV}{(V+\alpha k/2) - \omega/k}\right] , \tag{73}$$

which, after defining the variables $p_1 = V - \alpha k/2$ and $p_2 = V + \alpha k/2$, can be cast in the form:

$$1 = i \left(\frac{\mathcal{Z}}{k}\right) \frac{1}{k} \left[\int \frac{f_0(p_1 + \alpha k/2)\, dp_1}{p_1 - \omega/k} - \int \frac{f_0(p_2 - \alpha k/2)\, dp_2}{p_2 - \omega/k}\right] , \tag{74}$$

and finally in the following form:

$$1 = i\alpha \left(\frac{\mathcal{Z}}{k}\right) \int \frac{f_0(p+\alpha k/2) - f_0(p-\alpha k/2)}{\alpha k} \frac{dp}{p-\omega/k} \,. \tag{75}$$

We soon observe that, assuming that $f_0(V)$ is proportional to $\delta(V - V_0)$, from Eq. (75) we easily recover the dispersion relation for the case of a monochromatic coasting beam (see Eq. (63)). In general, Eq. (75) takes into account the velocity (or energy) spread of the beam particles, but it has not obtained with a kinetic treatment. We have only assumed the existence of an equilibrium state associated with an equilibrium velocity distribution, without taking into account any phase-space evolution of a kinetic distribution function. Our result has been basically obtained within the framework of Madelung fluid description, extending the standard MI analysis for monochromatic wave train to a non-monochromatic wave packets.

Nevertheless, Eq. (75) can be also obtained within the kinetical description provided by the Moyal-Ville-Wigner description, as it has been done for the first time in the context of the TWM [27]. In the next section, we present the quantum methodologies provided by the quantum kinetic formalism where Eq. (75) is recovered. In this perspective, the main features of the Eq. (75) will be also presented through specific examples. In particular, we will show that for the dynamics governed by the NLSE:

- The wavepacket propagation can be also suitably described in phase space with a kinetic-like equation
- MI of a monochromatic wavetrain concides with coherent instability of a coasting beam
- A Landau–type damping for a non-monochromatic wavepacket is predicted

III. WAVE KINETICS AND MOYAL-VILLE-WIGNER PICTURE

As we have pointed out in the Introduction, two distinct ways to treat MI are possible. The first, and the most used one, is a deterministic approach, whilst the second one is a statistical approach [24]. In the latter, the basic idea is to transit from the configuration space description, where the NLS equation governs the particular wave-envelope propagation, to the phase space, where an appropriate kinetic equation is able to show a random version of the MI. This has been accomplished by using the mathematical tool provided by the "quasidistribution" (Fourier transform of the density matrix) that is widely used for quantum systems. For any nonlinear system, whose dynamics is governed by a Nonlinear Schrödinger equation, one can introduce a two-point correlation function which plays the role similar to the one played by the density matrix of a quantum system. Consequently, the governing kinetic equation is nothing but a sort of nonlinear von Neumann-Weyl equation. In the statistical approach to modulational instability, a linear stability analysis of the von Neumann-Weyl equation leads to a phenomenon fully similar to the well known Landau damping, predicted by L.D. Landau in 1946 for plasma waves [25]. This approach has been carried out in several contexts, such as surface gravity waves [26], charged-particle beam dynamics in high-energy accelerators [27], optical beam propagation in Kerr media [28] and

Langmuir envelope wave propagation in plasmas [29]. All these investigations have predicted a stabilizing effect provided by a Landau-type damping against the occurrence of the MI.

A. Wigner picture associated with NLSE

Let us assume a system whose dynamics is governed by the gNLSE (1). This equation provides a description in configuration space. In order to transit to phase space and give a kinetic description, let us introduce the conjugate momentum associated with x, i.e., $p = dx/ds$. Thus, according to the formalism of quantum mechanics, we introduce the Moyal-Ville-Wigner function for a pure state $\Psi(x,s)$, i.e.,

$$w(x,p,s) = \frac{1}{2\pi\alpha}\int_{-\infty}^{\infty} \Psi^*\left(x+\frac{y}{2},s\right)\ \Psi\left(x-\frac{y}{2},s\right)\ \exp\left(i\frac{py}{\alpha}\right)\ dy \quad . \tag{76}$$

In phase space $w(x,p,s)$ plays the role of a distribution function. In fact, the integrals $\int w(x,p,s)\,dp$ and $\int w(x,p,s)\,dx$ are proportional to the probability density in configuration space and momentum space, respectively. However, due to the uncertainty principle, w is not positive definite and for this reason it is usually referred to as "quasidistribution". In particular, (76) implies that

$$\rho(x,s) = |\Psi(x,s)|^2 \ = \ \int_{-\infty}^{\infty} w(x,p,s)\ dp \quad . \tag{77}$$

Consequently, $U[|\Psi|^2]$ appearing in Eq. (1) can be expressed here as a functional of w:

$$U \ = \ U\left[\int_{-\infty}^{\infty} w\ dp\right] . \tag{78}$$

We observe that, if Ψ satisfies Eq. (1), then w satisfies the following nonlinear von Neumann-Weyl equation :

$$\frac{\partial w}{\partial s} + p\frac{\partial w}{\partial x} - \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\alpha}{2}\right)^{2n}\frac{\partial^{2n+1}}{\partial x^{2n+1}}\left(U\left[\int_{-\infty}^{\infty} w\ dp\right]\right)\frac{\partial^{2n+1} w}{\partial p^{2n+1}} \ = \ 0 \quad . \tag{79}$$

B. Landau-type damping in charged-particle beam dynamics

In order to specialize our kinetic description to the longitudinal dynamics of charged-particle beams in a circular high-energy accelerating machine, let us take the governing equation (46), where the functional form for $U[|\Psi|^2]$ is given by Eq. (54). Within the present kinetic picture, the latter reads as:

$$U\left[\int_{-\infty}^{\infty} w\,dp\right] = -\alpha\mathcal{X}\left(\int_{-\infty}^{\infty} w\,dp \;-\; |\Psi_0|^2\right) - \alpha\mathcal{R}\int_0^x \left(\int_{-\infty}^{\infty} w(x',p,s)\,dp \;-\; |\Psi_0|^2\right) dx'\,. \tag{80}$$

Thus, (79) and (80) constitute a set of coupled equations governing the nonlinear longitudinal dynamics of a charged-particle coasting beam in the TWM context. We want to find the linear dispersion relation of this system. The procedure is similar to the one used in plasma physics for Vlasov-Poisson system [63], [26]. We perturb the system around an initial state. To this end, we start from the equilibrium state: $w = w_0(p)$, where $\rho_0 = |\Psi_0|^2 = \int_{-\infty}^{\infty} w_0(p)\,dp$ and $U = U_0 \equiv U\left[\int_{-\infty}^{\infty} w_0\,dp\right] = 0$. Then, we introduce the following small perturbations in w and U, respectively:

$$w(x,p,s) = w_0(p) + w_1(x,p,s) \quad , \tag{81}$$

$$U(x,s) \;=\; U_0 \;+\; U_1(x,s) \;=\; U\left[\int_{-\infty}^{\infty} w_1(x,p,s)\,dp\right] \quad , \tag{82}$$

where $w_1(x,p,s)$ and $U_1(x,s)$ are first-order quantities. Consequently, the resulting linearized set of equation is:

$$\frac{\partial w_1}{\partial s} + p\frac{\partial w_1}{\partial x} - \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\alpha}{2}\right)^{2n}\frac{\partial^{2n+1}U_1}{\partial x^{2n+1}}\,w_0^{(2n+1)} \;=\; 0 \tag{83}$$

$$U_1 = -\alpha\mathcal{X}\int_{-\infty}^{\infty} w_1\,dp \;-\alpha\mathcal{R}\int_0^x \int_{-\infty}^{\infty} w_1(x',p,s)\,dp\;dx'\,. \tag{84}$$

where $w_0^{(2n+1)} \equiv d^{2n+1}w_0/dp^{2n+1}$.

In order to perform the Fourier analysis on the set of equations (83) and (84), let us introduce the Fourier transform of $U_1(x,s)$ and $w_1(x,p,s)$, i.e.:

$$U_1(x,s) = \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\omega\; \widetilde{U_1}(k,\omega)\;\exp\left(ikx - i\omega s\right) \quad , \tag{85}$$

$$w_1(x,p,s) = \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\omega\; \widetilde{w_1}(k,p,\omega)\;\exp\left(ikx - i\omega s\right) \quad . \tag{86}$$

After substituting (85) and (86) in the pair of equation (83) and (84), the following dispersion relation is easily obtained:

$$1 \;=\; i\,\alpha\left(\frac{\mathcal{Z}}{k}\right)\int_{-\infty}^{\infty}\frac{w_0\left(p+\alpha k/2\right) \;-\; w_0\left(p-\alpha k/2\right)}{\alpha k}\;\frac{dp}{p-\omega/k} \quad , \tag{87}$$

which is formally identical to Eq. (75) obtained in the Madelung fluid description. Since the monochromatic case, i.e., $w_0(p) \propto \delta(p-p_0)$, has been already discussed in the Madelung fluid context, we confine here our attention to the non-monochromatic case which corresponds to the assumption that $w_0(p)$ has a finite spread in the p-space.

1. *Case of $\alpha k << 1$*

Since $\alpha k << 1$, we have:

$$\frac{w_0\,(p+\alpha k/2) \;\; - \;\; \rho_0\,(p-\alpha k/2)}{\alpha k} \;\approx dw_0/dp \;\equiv w_0^{'} \quad . \tag{88}$$

Consequently, Eq. (87) becomes:

$$1 \; = \; i\alpha \left(\frac{\mathcal{Z}}{k}\right) \; \int_{-\infty}^{\infty} \frac{w_0^{'}}{p-\omega/k} \; dp \quad . \tag{89}$$

We note that the limit of small wavenumbers here considered allows us to predict a sort of weak Landau damping, as described in the Vlasov theory of charged-particle beam physics [59].

A. If we assume that $\mathcal{Z}_R = 0$, thus $\mathcal{Z} = i\mathcal{Z}_I$, (89) becomes:

$$1 \; = \; -\,\alpha \left(\frac{\mathcal{Z}_I}{k}\right) \; \int_{-\infty}^{\infty} \frac{w_0^{'}}{p-\omega/k} \; dp \quad . \tag{90}$$

Provided that $\alpha \; Z_I < 0$, by replacing $-\alpha \; Z_I$ with the ratio ω_p^2/k^2 (ω_p being the electron plasma frequency), Eq. (90) becomes formally identical to the linear dispersion relation for a warm unmagnetized electron plasma, which predicts the existence of Landau damping. Consequently, following the well known Landau method [26], [63], and using the small wavenumber approximation, we easily get the following dispersion relation:

$$1 \; = \; -\,\alpha \; k \; \mathcal{X} \left[D(\omega,k) \; + \; i\frac{\pi}{k} \; w_0^{'}(\omega/k)\right] \quad , \tag{91}$$

where

$$D(\omega,k) \; \equiv \; \int_{PV} \frac{w_0^{'}}{kp-\omega} \; dp \tag{92}$$

is the principal value of the integral in (90).

B. In the case $\mathcal{Z}_R \neq 0$, the analysis can be carried out as follows. We note that Eq. (90) can be cast in the following way:

$$V_R \; + \; i \; V_I \; \equiv \; \alpha \left(\frac{\mathcal{Z}_R}{k} \; + \; i \; \frac{\mathcal{Z}_I}{k}\right) \; = \; -\left[i \; \int_{PV} \frac{w_0^{'}}{p-\beta_{ph}} \; dp \; + \; \pi \; \rho_0^{'}(\beta_{ph})\right]^{-1} \quad , \tag{93}$$

where $V_R = \alpha\mathcal{Z}_R/k$, $V_I = \alpha\mathcal{Z}_I/k$, and $\beta_{ph} = \omega/k$. This equation determines a relationship between V_R, V_I, and β_{ph}. In principle, β_{ph} is a complex quantity. Thus, we put:

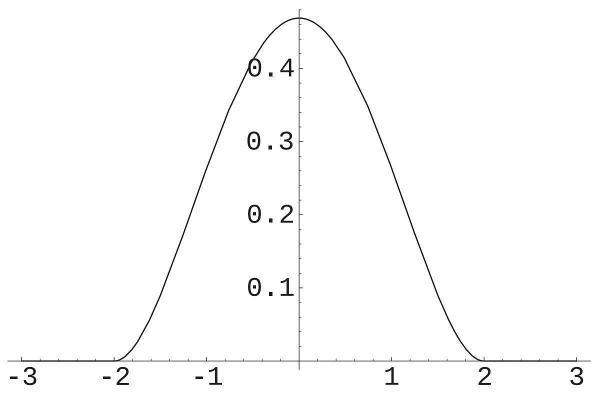

FIG. 4: *Plot of the equilibrium distribution* $\rho_0(p) = \left(1 - p^2/4\right)^2/2.133$*, defined in the intervall of* p $(-2, 2)$*.*

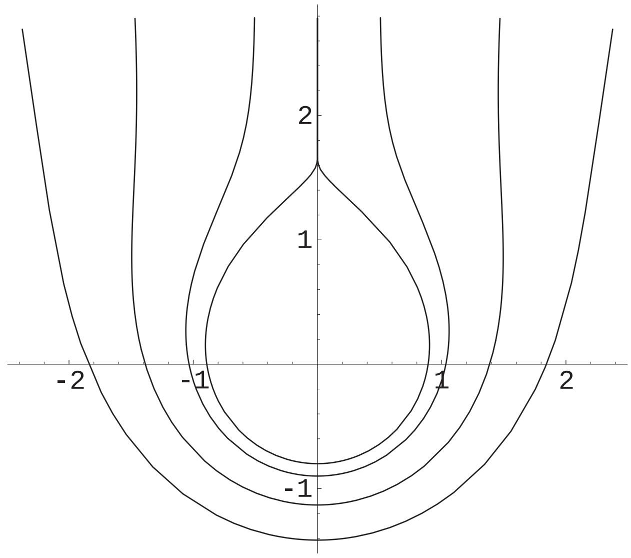

FIG. 5: *Instability chart for* $\alpha k = .01$ *and for* ρ_0 *plotted in Figure 4* [28]*. The area inside the curve with* $\gamma_I = 0$ *represents the stability region. For increasing values of* γ_I $(\gamma_I = 0, .1, .3, .5)$*, the curves plotted cover in the instability region. They are a sort of "deformed parabolas". However, as* γ_I *increases more and more, their shapes become more and more similar to the parabolas described by Eq. (64) as given in the monochromatic case. We note that in the present case the stability region is larger. This effect, together with the deformation of the above parabolas, is due to the stabilizing effect of the quantum-like Landau damping of the wave packet which exists for a p-distribution with non–negligible spread. In fact, in this case, the stabilizing effect is in competition with the modulational instability.*

$\beta_{ph} \equiv \gamma_R + i\,\gamma_I$. Consequently, we can plot curves in the V_R-V_I plane for a given equilibrium distribution funtion $\rho_0(p)$ and for different growth rates γ_I. For instance, we assume $\rho_0(p) = \left(1 - p^2/4\right)^2 /2.133$ which is plotted in Figure 4.

Figure 5 shows these curves, for $\alpha k = .01$ and for the smooth distribution plotted in Figure 4. This picture describes the coherent instability (for instance, the negative-mass instability) in circular accelerating machines in which coherent instabilities competes with the stabilizing effect of the Landau damping. We would like to stress that Figure 5 represents a sort of universal stability chart of the MI predicted by NLSE (50). Any impedance $\mathcal{Z}$ leading to a (V_R, V_I) pair belonging the area surrounded by the curve with $\gamma_I = 0$ corresponds to a stable operation.

2. *Case of arbitrary αk*

When the αk assumes arbitrary values, approximation (88) is no longer valid. In this case, the instability analysis must be carried out directly with Eq. (87). To perform the integration, the residui theory can be applied as in the previous case. Figure 6 shows the instability curves for $\alpha k = 0.4$ for the equilibrium distribution $w_0(p)$ shown in Figure 4, as well. It is evident that the stabilizing effect of the Landau damping is enhanced in comparison to the case shown in Figure 5. In fact, the stability region is sensitively enlarged.

C. Landau-type damping of partially-incoherent Langmuir wave envelopes

We now describe the MI of partially incoherent Langmuir wave envelopes in a unmagnetized, collisionless, warm plasma with unperturbed number density n_0. To this end, we start from the pair of equations governing the nonlinear propagation of a Langmuir wavepackets (the so-called "Zacharov system of equations"), i.e.,

$$i\frac{\partial E}{\partial t} + \frac{3v_{te}^2}{2\omega_{pe}}\frac{\partial^2 E}{\partial x^2} - \omega_{pe}\frac{\delta n}{2n_0}E = 0 \quad , \tag{94}$$

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2\frac{\partial^2}{\partial x^2}\right)\frac{\delta n}{2n_0} = -\frac{c_s^2}{2}\frac{\partial^2}{\partial x^2}\left(\frac{|E|^2 - |E_0|^2}{4\pi n_0 T_e}\right) \quad , \tag{95}$$

where E is the slowly varying complex electric field amplitude, δn is the density fluctuation, v_{te} is the electron thermal velocity, ω_{pe} is the electron plasma frequency, T_e is the electron temperature, c_s is the sound speed, and E_0 is the unperturbed electric field amplitude.

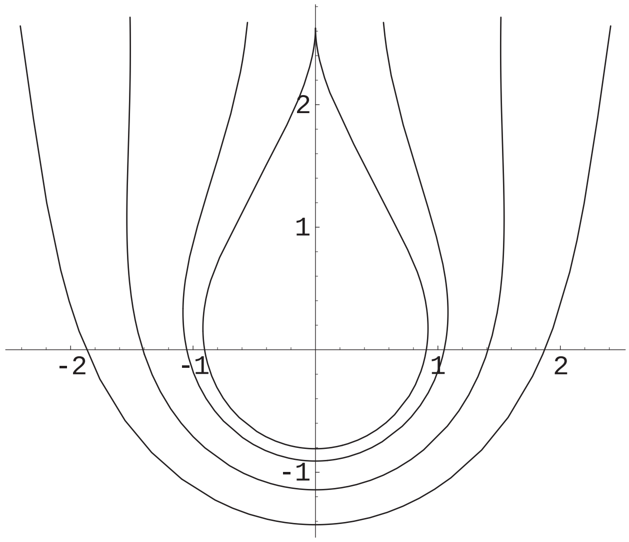

FIG. 6: *Instability chart for $\alpha k = .4$, for ρ_0 plotted in Figure 4 and for increasing values of γ_I ($\gamma_I = 0, .1, .3, .5$)* [28]. *In this case, the area inside of the stability region ($\gamma_I = 0$) is larger than the one of $\alpha k << 1$ (compare with Figure 5). The "deformed parabolas" do not coincide with the case of Figure 4. However, as γ_I is increasing, also in this case the deformation becomes more and more negligible and the parabolas shown is Figure 3 are recovered for very large values of this parameter.*

Moreover, x and t play the role of configurational space coordinate (longitudinal coordinate with respect to the wave packet centre) and time, respectively. The above equations have been obtained by assuming that the electric field of the wave has the envelope form

$$\mathrm{E} = E(x,t)\exp\left(-i\Omega t\right) \quad ,$$

where $\Omega \leq \omega_{pe}$ and $\mid \Omega^{-1}\partial E(x,t)/\partial t \mid \ll 1$. The above system of equations (94) and (95) can be cast in the form

$$i\alpha\frac{\partial \Psi}{\partial s} + \frac{\alpha^2}{2}\frac{\partial^2 \Psi}{\partial x^2} - U\,\Psi = 0 \quad , \tag{96}$$

$$\left(\frac{\partial^2}{\partial s^2} - \mu^2\frac{\partial^2}{\partial x^2}\right)U = -\frac{\partial^2}{\partial x^2}\left(\langle|\Psi|^2 - |\Psi_0|^2\rangle\right) \quad , \tag{97}$$

where the angle brackets account for the statistical ensemble average due to the partial incoherence of the waves, $s = \sqrt{3}\lambda_{De}\omega_{pe}t$ ($\lambda_{De} \equiv v_{te}/\omega_{pe}$ is the electron Debye length), $\alpha = \sqrt{3}\lambda_{De}$, $U = \delta n/2n_0$, $\mu = \sqrt{m_e/3m_i}$ (m_e and m_i are the electron and the ion masses, respectively), and

$$\Psi = \frac{\mu E}{\sqrt{8\pi n_0 T_e}} \quad , \tag{98}$$

$$\Psi_0 = \frac{\mu E_0}{\sqrt{8\pi n_0 T_e}} \quad . \tag{99}$$

Note that (99) can be written as

$$\rho_0 \equiv |\Psi_0|^2 = \mu^2 \frac{\epsilon}{\epsilon_T} \quad , \tag{100}$$

where $\epsilon \equiv E_0^2/8\pi$ and $\epsilon_T \equiv n_0 T_e$ are the electric energy density and the electron thermal energy density, respectively. According to the theory modelled by the Zakharov system (96) and (97), $\epsilon/\epsilon_T \gg 1$ (i.e, the ponderomotive effect is larger than the thermal pressure effect). Note also that Eq. (97) implies that U is a functional of $|\Psi|^2$, namely $U = U\left[|\Psi|^2\right]$. Consequently, once (97) is combined with (96), the latter becomes a nonlinear Schrödinger equation (NLSE) with the nonlinear term $U\left[|\Psi|^2\right]$. One can introduce the correlation function (whose corresponding meaning in Quantum Mechanics is nothing but the density matrix for mixed states)

$$\rho(x, x', s) = \langle \Psi^*(x, s) \Psi(x', s) \rangle \quad , \tag{101}$$

where the statistical ensemble average takes into account the incoherency of the waves. The technique was successfully introduced in the field of statistical quantum mechanics to describe the dynamics of a system in the classical space language [32], [64], [65]. Thus, one can define the following Moyal-Ville-Wigner transform

$$w(x, p, s) = \frac{1}{2\pi\alpha} \int_{-\infty}^{\infty} \rho\left(x + \frac{y}{2},\ x - \frac{y}{2},\ s\right)\ \exp\left(i\frac{py}{\alpha}\right)\ dy \quad , \tag{102}$$

which extends to the "mixed states" the definition of w given in the previous subsection for "pure states". Consequently, it is easy to show that the Zacharov system is transformed into the following pair of equatons:

$$\frac{\partial w}{\partial s} + p\frac{\partial w}{\partial x} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\alpha}{2}\right)^{2n} \frac{\partial^{2n+1} U}{\partial x^{2n+1}} \frac{\partial^{2n+1} w}{\partial p^{2n+1}} = 0 \quad . \tag{103}$$

and

$$\left(\frac{\partial^2}{\partial s^2} - \mu^2 \frac{\partial^2}{\partial x^2}\right) U = -\frac{\partial^2}{\partial x^2} \left(\int_{-\infty}^{\infty} w\ dp - \int_{-\infty}^{\infty} w_0\ dp\right) \quad , \tag{104}$$

where w_0 is the Moyal-Ville-Wigner transform of Ψ_0 ($\rho_0 = |\Psi_0|^2 = \int_{-\infty}^{\infty} w_0\ dp$).

Note that Eq. (102) implies that

$$\langle |\Psi|^2 \rangle = \int_{-\infty}^{\infty} w(x, p, s)\ dp \quad . \tag{105}$$

The above scheme allows us to carry out a stability analysis of the stationary Langmuir wave by considering the linear dispersion relation of small amplitude perturbations in a way similar to the one presented in the previous subsection. Then we find the following dispersion relation:

$$\omega^2 \ - \ \mu^2 k^2 \ = \ k^2 \int_{-\infty}^{\infty} \frac{w_0\,(p+\alpha k/2) \ - \ w_0\,(p-\alpha k/2)}{\alpha k} \, \frac{dp}{p-\omega/k} \quad . \tag{106}$$

Since the assumption $w_0(p) \propto \delta(p-p_0)$ leads to the standard results of MI of monochromatic Langmuir wave packets, we focus here our attention to non-monochromatic wave packets to predict the Landau-type damping.

We consider a Lorenzian spectrum of the form

$$w_0(p) = \frac{\rho_0}{\pi} \frac{\Delta}{\Delta^2 + p^2}, \tag{107}$$

where Δ is the width of the spectrum. The dispersion relation (106) then becomes

$$\left(\omega^2 - \mu^2 k^2\right) \left(\omega^2 - \alpha^2 k^4/4 + 2ik\Delta\omega - k^2\Delta^2\right) \ = \ \rho_0 k^4 \quad , \tag{108}$$

(Note that the limit of $\Delta \rightarrow 0$ corresponds to a coherent monochromatic wave-train and the dispersion relation reduces to $(\omega^2 - \mu^2 k^2)\,(\omega^2 - \alpha^2 k^4/4) \ = \ \rho_0 k^4$.

In the following, we concentrate on the limits $\alpha k \ll 1$ and $\omega/k \gg 1$. We will show that during its nonlinear evolution the Langmuir wave packet can exhibit a phenomenon, similar to the Landau damping [66], between parametrically-driven non-resonant density ripples and Langmuir quasi-particles. It is easy to verify that the condition $\alpha k \ll 1$ (i.e., $k\lambda_{De} \ll 1$, λ_{De} being the electron Debye length) implies that

$$\omega^2 \ = \ k^2 \int \frac{w_0^{'}}{p-\omega/k} \, dp \quad , \tag{109}$$

where, using the assumption $\omega/k \gg 1$, we can neglect the term involving μ^2 because $\mu \ll 1$. Consequently, the dispersion relation (109) allows us to predict a sort of Landau damping [26], as described in plasma physics for linear plasma waves. We can investigate this phenomenon also for large-amplitude Langmuir waves, just using the standard procedure of Landau damping description. The result is

$$\omega^2 \ = \ k^2 \int_{PV} \frac{w_0^{'}}{p-\omega/k} \, dp \ + \ i\pi k^2 w_0^{'}(\beta) \quad , \tag{110}$$

For a symmetric sufficiently smooth stationary background distribution, $w_0(p)$, we have

$$\omega^2 \;=\; \left(1+\frac{3\Delta^2}{\rho_0}\right)k^2 \;+\; i\pi k^2 w_0^{'}(\beta) \quad , \tag{111}$$

where $\Delta \equiv (\langle p^2\rangle)^{1/2}$ denotes the r.m.s. width of $w_0(p)$, and $\beta \equiv \rho_0^{1/4}\,(1+3\Delta^2/4\rho_0^2)$. Note that, according to the above hypothesis, we have $\beta \gg 1$ and $\Delta^2/\rho_0 \ll 1$. This implies that $\rho_0 = |\Psi_0|^2 \gg 1$, and from (100) it follows that $\epsilon/\epsilon_T \gg 1/\mu^2$. Eq. (111) clearly shows that the damping rate is proportional to the derivative of the Wigner distribution w_0. This is formally similar to the expression for the Landau damping of a linear plasma wave in a warm unmagnetized plasma. In fact, writing ω in the complex form $\omega = \omega_R + i\omega_I$, substituting it in (110) and then separating the real and imaginary parts, we obtain

$$\omega_R^2 \;-\; \omega_I^2 \;=\; k^2 \int_{PV} \frac{w_0^{'}}{p-\omega/k}\, dp \quad , \tag{112}$$

and

$$\omega_I \;=\; \frac{\pi}{2}\frac{k^2}{\omega_R} w_0^{'}(\beta) \quad . \tag{113}$$

The Landau damping rate of the Langmuir wave in the appropriate units is $\gamma = \sqrt{3}\omega_{pe}\lambda_{De}\omega_I$. For example, for a Gaussian wave packet spectrum, i.e. $w_0(p) = \left(\rho_0/\sqrt{2\pi\Delta^2}\right)\exp\left(-p^2/2\Delta^2\right)$, one obtains

$$\gamma = -\left(\frac{3\pi}{8}\right)^{1/2} \frac{\mu^{5/2}}{\Delta^3}\left(\frac{\epsilon}{\epsilon_T}\right)^{5/4} \omega_{pe}\lambda_{De}k\ \exp\left[-\frac{\mu}{2\Delta^2}\left(\frac{\epsilon}{\epsilon_T}\right)^{1/2}\right] \quad , \tag{114}$$

where higher-order terms have been neglected.

The present damping mechanism differs from the standard Landau damping in that here $w_0(p)$ does not represent the equilibrium velocity distribution of the plasma electrons, but it can be considered as the “kinetic” distribution of all Fourier components of the partially incoherent large-amplitude Langmuir wave in the warm plasma. In terms of the plasmons, we realize that $w_0(p)$ represents the distribution of the partially incoherent plasmons that are distributed in p-space (i.e. in k-space) with a finite “temperature”. Consequently, the Landau damping described here is due to the partial incoherence of the wave which corresponds to a finite-width Wigner distribution of the plasmons (ensemble of partially incoherent plasmons) which acts in competition with the modulational instability.

IV. RELATED TOOLS: MARGINAL DISTRIBUTIONS FOR TOMOGRAPHIC REPRESENTATIONS

The picture presented in the previous section provides a phase space description in terms of the quasidistribution w. However, as we have already pointed out, it can be negative and it does not match with usual classical picture that is usually given by the Boltzmann-Vlasov description. Actually, there is a possibility to transit from the quasidistribution to a positive definite function, called "marginal distribution", that has the features of a classical probability distribution [7]. It is widely employed for a number of tomographic applications. In particular, in quantum optics and quantum mechanics it is involved in both optical [67], [7] and symplectic tomographic [68] methods and it has been suggested for measuring quantum states. The marginal distribution application reveals to be important in spin tomography, as well [69], [70]. Its definitions establishes an invertible map with the quasidistribution. For instance, in the symplectic tomographic methods, the marginal distribution is defined as the Fourier transform, say $F_w(X,\mu,\nu,s)$, of the quasidistribution $w(x,p,s)$:

$$F_w(X,\mu,\nu,s) = \int \exp\left[-ik(X-\mu x-\nu p)\right] w(x,p,s) \frac{dk\,dx\,dp}{(2\pi)^2}. \tag{115}$$

The function $F_w(X,\mu,\nu,s,)$ is a real function of the random variable X with the properties: $F_w(X,\mu,\nu,s) \geq 0$, and $\int F_w(X,\mu,\nu,s)\, dX = 1$. If the marginal distribution is known, the quasidistribution can be obtained by the inverse Fourier transform of (115). It has been shown in [68] that, for a suitable choice of the parameters μ and ν, the expression for the quasidistribution in terms of the marginal distribution can be reduced to the Radon transform [71] used in optical tomography.

Furthermore, the marginal distribution $F_w(X,\mu,\nu,s)$ obeys to a sort of Fokker–Planck-like equation [72] that can be also nonlinear [73], [74]. In fact, the use of marginal distributions seems to be important in providing a tomographic representation of several nonlinear processes. In particular, it has been very recently applied to the envelope soliton propagation governed by some kinds of NLSEs.

In the next subsections, we briefly introduce the concept of tomogram and discuss its properties and give some relevant examples of tomogram of envelope solitons.

1. Tomogram map

One of the main reasons to use the tomogram technique is justified by the natural possibility of measuring the states usually described by the complex wave function Ψ, in principle solution of Eq. (1). In fact, it is possible to prove that, by using the mapping between F_w and w and the one between w and Ψ, the following direct connection between F_w and Ψ holds (expression of the tomogram in terms of the wave function) [73], [74]:

$$F_w(X,\mu,\nu) = \frac{1}{2\pi|\nu|}\left|\int \psi(y)\exp\left(\frac{i\mu}{2\nu}y^2 - \frac{iX}{\nu}y\right)dy\right|^2 . \tag{116}$$

One can prove that the tomogram has the following homogeneity property, very useful for the optical tomography [73]:

$$F_w(\lambda X,\lambda\mu,\lambda\nu,s) = \frac{1}{|\lambda|}F_w(X,\mu,\nu,s). \tag{117}$$

Actually, a relation among the parameters can be in principle assumed. In particular, one can take $\mu = \cos\theta$ and $\nu = \sin\theta$ and the optical tomogram becomes:

$$F_w(X,\theta) = \frac{1}{2\pi|\sin\theta|}\left|\int \psi(y)\exp\left(\frac{i\,\cot\theta}{2}y^2 - \frac{iX}{\sin\theta}y\right)dy\right|^2 . \tag{118}$$

In the next subsection, we use this formula to provide tomograms of envelope solitons.

2. Soliton envelopes in tomographic representation

For the case of a modified NLSE, with $U[|\psi|^2] = q_0|\psi|^{2\beta}$, for $q_0 < 0$ (β being an arbitrary real positive value) one has the following bright envelope soliton solutions (see section II):

$$\Psi(x,s) = \left[\frac{|E|\,(1+\beta)}{|q_0|}\right]^{1/2\beta}\mathrm{sech}^{1/\beta}\left[\beta\sqrt{2\,|E|}\;\xi\right]\exp\left\{i\left[V_0x - \left(E + \frac{V_0^2}{2}\right)s\right]\right\}. \tag{119}$$

Thus, by virtue of Eq. (118), the corresponding tomograms can be computed, respectively. 3D plots and density plots of both tomograms and Wigner functions for bright solitons for $\beta = 0.5$, 1.0, 2.0, and 2.5 are given in Figures 7 and 8. The free parameters have been fixed as follows: $V_0 = 0$, $E = -1$ and $q_0 = -1$. The left picture in Figure 7 represents 3D plots of the tomogram of solitons for different values of β. The corresponding density plots are displayed in the right picture of Figure 7. Then left picture in Figure 8 displays 3D plots of the quasidistribution of solitons for different values of β. For such solitons, the phase-space

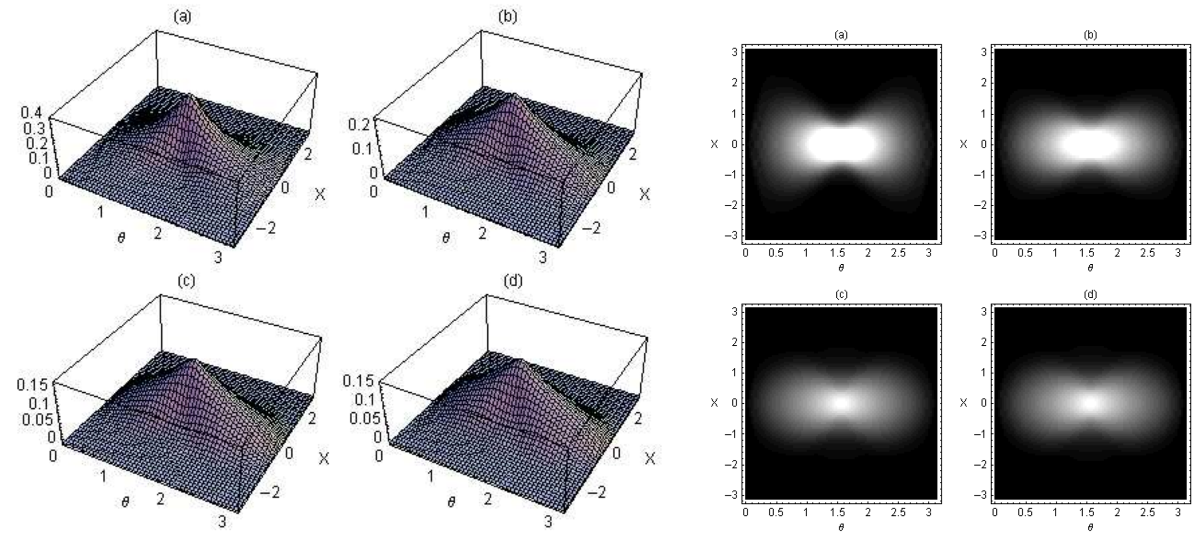

FIG. 7: *Tomogram of the soliton for various β: 3D plots (at the left hand side) and density plots (at the right hand side)* [73]. *For both sides: (a) $\beta = 0.5$, (b) $\beta = 1.0$, (c) $\beta = 2$, (d) $\beta = 2.5$.*

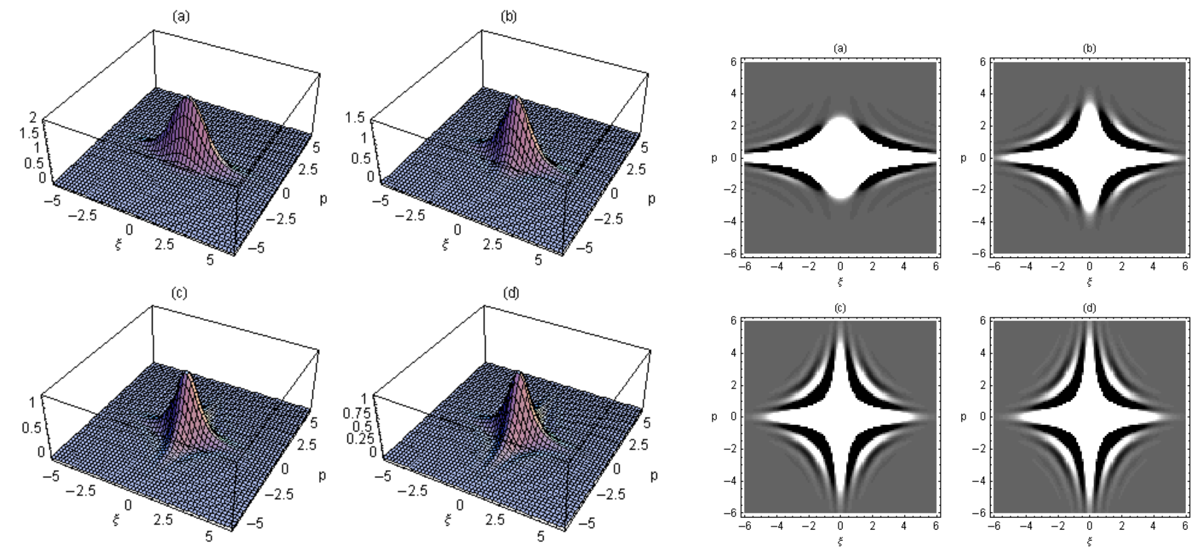

FIG. 8: *Quasidistribution of the soliton for various β: Left hand side: 3D plots of the quasidistribution of the soliton for various β: 3D plots (at the left hand side) and density plots (at the right hand side)* [73]. *For both sides: (a) $\beta = 0.5$, (b) $\beta = 1.0$, (c) $\beta = 2$, (d) $\beta = 2.5$.*

domains where the Moyal-Ville-Wigner function is negative are not easily visible in this Figure, whilst the corresponding density plots displayed by the right picture in Figure 8 show clearly this behaviour (black and grey regions). The "deepness" of the negativity is

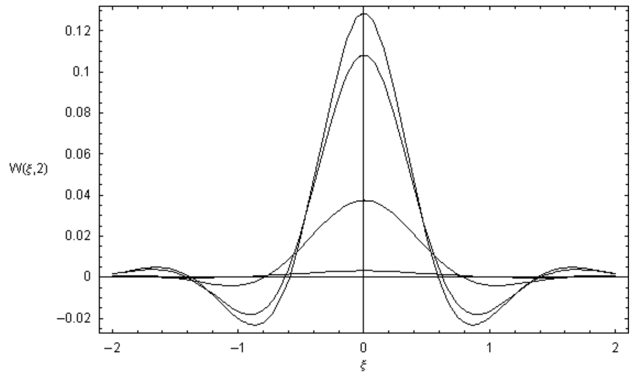

FIG. 9: *Cross section of the quasidistribution of the soliton for* $p = 2$ *and for (a)* $\beta = 0.5$*, (b)* $\beta = 1.0$*, (c)* $\beta = 2$*, (d)* $\beta = 2.5$ *. The four cross section plots are drawn for increasing values of the parameter* β *; the greater* β *the greater the amplitude* [73].

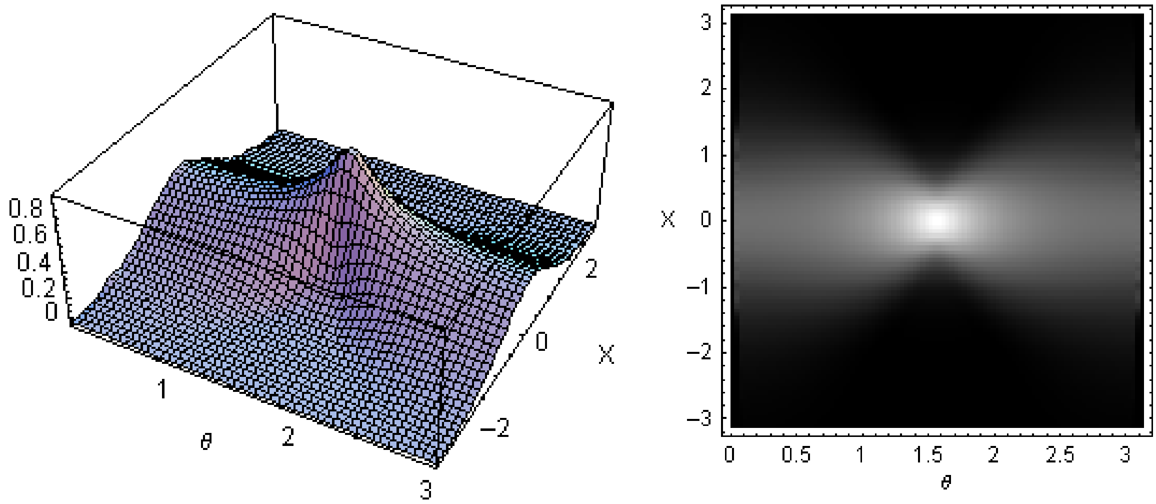

FIG. 10: *(a) Tomogram of the bright soliton-like solution as function of* X *and* θ*; (b) Density plot in the* (X,θ) *plane* [74]. *For both plots* $\gamma = L/\ell_z \approx 0.82$ *(*$L = 1.4$ μm*,* $\ell_z = 1.7$ μm*), according to the BEC experimental conditions reported in* [75].

represented in scale of grey. Black regions correspond to the deepest negative parts. The size of these negative parts of the quasidistribution are, for instance, clearly represented by the cross section shown in Figure 9. Additionally, the probability description of collective states associated with Bose-Einstein condensates has been recently given in terms of tomographic map [73]. A tomogram of a quasi-1D bright soliton, which is solution of the Gross-Pitaevskii equation, is displayed in Figure 10 [74].

V. CONCLUSIONS

In this paper, we have presented the main quantum methodologies as tools useful for describing a number of problems in nonlinear physics. We have first discussed the role played by the quantum methodologies in the development of the physical theories. In particular, we have considered the very-recently progress registered in the modulational instability and soliton theories involving quantum tools given by the Madelung fluid description, the Moyal-Ville-Wigner kinetic approaches and the tomographic techniques. Valuable methodological transfer among physics of fluids, plasma physics, nonlinear optics and particle accelerator physics have been discussed in terms of recently done applications.

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