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Cycle XXII

Variable Annuities and Embedded Options: models and tools for fair valuation and solvency appraisal

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Contents

List of Tables

List of Figures

Introduction

The objective of this thesis is to provide an analysis of the Variable Annuity (VA) products. In particular, we focus on the actuarial and financial valuation of two guarantees embedded in VAs, the Guaranteed Minimum Death Benefit option (GMDB) and the Guaranteed Lifelong Withdrawal Benefit option (GLWB), derive No-arbitrage pricing models and study the impact of mortality risk.

We have decided to deal with this products in the light of significant international success obtained by VAs and we believe that perspectives of their development in Italy market and throughout Europe and Asia are favourable. One of the reasons of this success is the presence of guarantees which offer partial protection against the downside movements of the interest rates or the equity market, an attractive feature for the individual retirement security. The shift from defined benefit to self-directed defined contribution plans and the reform of the Social Retirement System in many countries, so that it includes personal accounts, have encouraged the proliferation of new kind of products. Owing to the long term horizon of their commitments, pension funds are exposed to important financial risk due to the volatility of interest rates and equity markets. At this regard, VAs were first introduced by insurance companies in the 1970s in the United States to compete with mutual funds.

Over the years, many practical and academic contributions have been offered for describing the VAs and the guarantees embedded. Most of the earlier literature (e.g., Rentz Jr. (1972) and Green (1973)) is constituted by

empirical works dealing with product comparisons rather than pricing and hedging issues. It was not until recently that some guarantees were discussed by practitioners (e.g., J.P.Morgan (2004), Lehman Brothers (2005), Milliman (2007)); they highlight the growing opportunities to introduce VAs in new markets. Recently, the academic literature has shown a fervent interest to the topic too (cf. Bauer et al. (2006), Chen et al. (2008), Coleman et al. (2006), Dai (2008), Holz (2006), Milevsky and Panyagometh (2001), Milevsky and Posner (2001), Milevsky M.A and Promislow S.D (2001), Milevsky and Salisbury(2002)., Milevsky and Salisbury (2006), Nielsen and Sandmann (2003)). Bauer et al.(2006) offer the first universal general framework in which any design of options and guarantees currently offered within Variable Annuities can be modeled. Besides the valuation of a contract assuming that the policyholder follows a given strategy with respect to surrender and withdrawals, they are able to price contracts with different embedded options.
Milevsky und Posner (2001) price basic form and enhanced versions of guaranteed minimum death benefits. They present closed form solutions for this option in case of an exponential mortality law and offer numerical results for the pricing under the hypothesis of the Gompertz-Makeham law. They find that in general these guarantees are overpriced in the market.
In Milevsky und Salisbury (2006), the authors price GMWB options. Besides a static approach, where deterministic withdrawal strategies are assumed, they calculate the value of the option in a dynamic approach. Here, the option is valuated under optimal policyholder behavior. They show that under realistic parameter assumptions and according to the optimal strategy at least the annually guaranteed withdrawal amount should be withdrawn. Furthermore, they find that such options are usually underpriced in the market. This result is in contrast with the common belief that the guarantees embedded in variable annuity policies are overpriced (see Clements (2004)).
This thesis aims at following this literature by proposing some theoretical and practical innovative works. Our original contributions lie in:

- describe how the value of Guaranteed Minimum Death Benefit (GMDB) options evolves over time and in the presence of mortality changes and produce an application to Italian data,
- study the insurance surplus over time for a portfolio of Variable Annuities with GMDB Options and offer a model that can be used for an evaluation of the adequacy of solvency,
- develop a sensitivity analysis for the value of Guaranteed Lifelong Withdrawal Benefit (GLWB) options under the hypothesis of a static withdrawal strategy,
- decompose a VA with a GLWB option into a life annuity plus a portfolio of Quanto Asian Put Options, with decreasing strikes and increasing expiration dates, and verify that this product is underpriced on US market.

The outline of the thesis is as follows:

Chapter 1: The Variable Annuities and the GMxB features. This chapter has an introductory role and aims to present the basic structures of VAs. We offer an historical review of the development of the VA contracts and describe the payoff of the embedded guarantees, examine the main life insurance markets in order to highlight the international development of VAs and their strong potential growth, retrace the main contributions of the literature on the topic. In the concluding remarks, we explain the motivations have urged us to write this thesis and to deal with the actuarial valuation of VAs and the related searches of the impact of mortality risk and the surplus analysis.

Chapter 2: The Guaranteed Minimum Death Benefit Option. We describe the payoff of GMDB options embedded in annuity contracts. These put options have stochastic maturity dates due to the involuntary exercise at the moment of death. We value the GMDB as a weighted average price of a set of deterministic put options with different maturity dates, where the weights are the probability of death at every date. We take into account the mortality risk and investigate the sensitivity of the price of the option to changes in mortality probability using both deterministic and stochastic approaches. In the first part of the chapter, we use the methodology of tilting to modify the observed probability of mortality and the projection is realized using assumptions based on historical data. Recently, it has become evident that deterministic mortality projections are inadequate, because unanticipated changes over time in the mortality rates have been observed. For this reason, in the second part of the chapter we use a stochastic mortality approach, that is necessary in order to avoid underestimation or overestimation of the expected present value of insurance and annuity contracts. We propose a simplified version of the stochastic model suggested by Cox and Lin (2005) and developed by Ballotta, Esposito, Haberman (2006) and provide a detailed application to the Italian market, where the first Variable Annuity has been issued in September 2007 with a GMDB option.

Chapter 3: Surplus analysis for the GMDB option. In this chapter, we analyze the insurance surplus for a Variable Annuity contract with a GMDB option. There are 2 theoretical foundations for this work: on the one hand, we take into account the actuarial literature concerning the valuation of the Variable Annuity and GMDB option (Bauer, Kling and Russ (2006); Coleman, Yuying and Patron (2006); Milevsky M. and Posner (2001), Milevsky M. and Salisbury(2002), Milevsky M.A. and Promislow (2001)); on the other hand, we look at the actuarial research literature on insurance surplus and insolvency probability (Coppola et al. (2003), Dahl (2004), Hoedemakers et al. (2005), Lysenko and Parker (2007), Marceau and Gaillardetz (1999) Parker (1996) and Parker (1994)). The abovementioned papers deal with the stochastically discounted value of future cash flows in respect of life insurance and life annuity contracts. The innovative contribution of our work is to apply this methodology to a new product like a Variable Annuity with a GMDB option, extending the models appearing in the literature in order to study a product with a payments linked to a fund account. Initially, we derive the first two moments of the distribution of the surplus; and subsequently, we develop the whole distribution using a stochastic model which involves an integrated analysis of financial and mortality risk. We offer a model according which the premium can be modified as per the forecasts of mortality probabilities, interest rate and fund evolution. Moreover, the study enables us to determine the premium that leads to a required probability of insolvency, and so it can be used for an evaluation of the adequacy of solvency. Numerical examples illustrate the results.

Chapter 4: the Guaranteed Lifelong Withdrawal Benefit Option. We develop a pricing model and define a fair price for a GLWB, using the standard No-arbitrage models of mathematical finance, in line with the tradition of Boyle and Schwartz (1997) that extend the Black-Scholes framework to insurance contracts. The approach follows the recent actuarial literature on the valuation of VA products: Bauer et al. (2006); Chen et al. (2008), Coleman et al. (2006); Holz (2006), Milevsky and Posner (2001), Milevsky M.A and Promislow S.D (2001), Milevsky and Salisbury(2002). First, we adopt a static approach that assumes policyholders take a static strategy, i.e. the withdrawal amount is always equal to the guaranteed amount. One of our main original contributions is to show that in the static case the product can be decomposed into a life annuity plus a portfolio of Quanto Asian Put Options, with decreasing strikes and increasing expiration dates. We believe that this decomposition has not been previously proposed in actuarial literature. In this regard, Milevsky and Salisbury (2006) decompose the GMWB option into a Quanto Asian Put plus a generic term-certain annuity. Our paper differs from that of Milevsky and Salisbury since the lifelong guarantee of GLWB makes necessary the introduction of the survival probabilities in the pricing model; in this regard, we show that the weights of the composition of the portfolio consisting of many put options are the deferred probabilities of death. In the second approach, we describe the GLWB payoff if the policyholder assumes a different strategy, according which he can lapse (i.e. withdraw more or less than the guaranteed amount from the found) and surrender the contract when he prefers. Milevsky and Salisbury (2006) prove that for a GMWB policyholder can be optimal to withdraw either nothing or the guaranteed amount or the total account value. Instead, Holz et al. (2007) show that for a GLWB withdrawing nothing can never be optimal, unless roll-ups or other options are included, and the rational policyholder withdraws the amount guaranteed until he decides to surrender. Therefore, in this dynamic approach we deal with an optimal stopping problem; we solve it with the definition of a probability function of the optimal surrender time and its construction on a practical side with a Monte Carlo simulation. Finally, we develop an application of our model and verify that the GLWB issued on the USA market are underpriced.

At the end of the thesis, we report a general bibliography, which includes all references cited in the Introduction and in the following chapters and other works not explicitly cited in the thesis but consulted during the study for this research. In addition, at the end of each chapter a more specific bibliography is presented in order to facilitate the deepening of the discussed topics.

Chapter 1

The VA products: the GMxB features

1.1 Introduction

Varriable Annuities (VA) were introduced in the 1970s in the United States (cf. [38]). The term Variable Annuity stands for a wide category of products and it is difficult to trace a comprehensive definition: "*As variable annuities are essentially a new product class in the U.K., an industry standard definition does not yet exist. For the reasons set out below, we shall define a variable annuity as any unit-linked or managed fund vehicle which offers optional guarantee benefits as a choice for the customer*"(cf. [22]). In the U.S.A. the National Association of Variable Annuity Writers (cf. [33]) explain that "*with a variable annuity, contract owners are able to chose from a wide range of investment options called enabling them to direct some assets into investment fund*".For this reason, the VA contracts are defined the "*close cousins of mutual fund, but they are formally classified as an insurance policy in addition to being registered as a security*".(cf. [30]). The VA, whose benefits are based on the performance of a underlying fund, are very attractive, because they provide a participation in the stock market and also a partial protection against the downside movements of the interest rates or the equity market.

As it is clear, VAs are life insurance saving products and they have become highly popular as retirement management vehicles. The key of the success of VAs is that they bridge the gap between traditional guarantee based life insurance saving policies and unit-linked investment fund, providing a mix of investment flexibility and risk protection. Since the 1990s, two kinds of embedded guarantees are offered in such policies (cf. [17]): Guaranteed Minimum Death Benefit (GMDB), offering a guaranteed amount in the event of the death of the policyholder, as well as Guaranteed Minimum Living Benefit (GMLB). There are three main products which guarantee some living benefits: the two earliest form, the Guaranteed Minimum Accumulation Benefit (GMAB) and the Guaranteed Minimum Income Benefit (GMIB), offer the policyholder a guaranteed minimum at maturity T of contract; however, with the GMIB, this guarantee only applies if the account value is annuitized at time T. In 2002 Hartford issued a new type of GMLB: the Guaranteed Minimum Withdrawal Benefit (GMWB), which gives the insured the possibility to withdraw a pre-specified amount annually, even if the account value has fallen below this amount. In 2004, each of the 15 largest Variable Annuity providers offered this guarantee and 69% of the Variable Annuities sold included a GMWB option; in 2007 the percentage was 86% (cf. [22]). The latest GMLB option is the Guaranteed Minimum Withdrawal Benefit for Life or Guaranteed Lifelong Withdrawal Benefit option (GLWB). As the name suggests, it offers a lifelong withdrawal guarantee; the first VA with a withdrawal benefit guaranteed for the life was introduced in the U.S.A. market in 2003. Since 2006 nine of ten VA products offered guaranteed living benefit; GLWB options captured some GMIB markets and represented the 35% of the whole market in early 2006 (cf. [2]).

The chapter is organized as follows: in section 1.2 and 1.3 we describe respectively the guarantees and the VA markets; in section 1.4 we produce an overview and retrace the main contributions of the literature; in section 1.5 we offer concluding remarks.

1.2 The Variable Annuity Guarantees

In the previous section, we have highlighted that the key attraction of VAs is the presence of guarantees. These include both death (GMDB) and living (GMLB) benefits. In the following, we describe them briefly and schematically. This section is preparatory for the next chapters, where the guarantees are studied from a technical point of view.

- GMDB:the Guaranteed Minimum Death Benefit Option is an increasing-strike put option with a stochastic maturity date. If the insured dies during the deferment period, the beneficiary obtains a death benefit, that is equal, in the basic form of the product, to the maximum of the invested premium accrued at the guaranteed rate and the account value linked to the fund. There are also variations to this contract. In the case of a *Roll-up* option, the minimum benefit is equal to the single premium compounded with a constant interest rate (the *roll-up rate*);an enhanced version of the option provides *rising-floor guarantee.* When the contract contains an *Annual Ratchet Death Benefit*, the minimum amount guaranteed is compared every years with the account value, and then this that becomes the new amount guaranteed if it is greater; finally, when there is a *look back* guarantee, a guaranteed death benefit is based on a suitably defined highest anniversary account value; some policies offer an annual reset, others require a five year wait and so on.

- *GMAB* :the Guaranteed Minimum Accumulation Benefit Option is the simplest form of guaranteed living benefits. The Guarantee is similar to GMDB but bites if the policyholder is still in force at a specified date. Different versions of this guarantee offer minimum roll-up rates, ratchets or resets, which enable the policyholder to underwrite a new GMAB on the expiry of the first one.

- *GMIB* :the Guaranteed Minimum Income Benefits options, as well as the Guaranteed Minimum Accumulation Benefit options, offer the policyholder a guaranteed minimum at maturity T of contract; however, with the GMIB, this guarantee only applies if the account value is annuitized at time T. The amount of the guaranteed minimum income benefit may be fixed in absolute terms at outset, or could be expressed as a percentage of the premiums invested by the policyholder.

- *GMWB* :the Guaranteed Minimum Withdrawal Benefit option gives the policyholder the possibility to annually withdrawal a certain percentage g of the single premium, that is invested in one or several mutual funds. The guarantee consists in the entitlement to withdrawal until an amount equal to the premium paid even if the account value falls to zero. Instead, if the account value does not vanish, at maturity the policyholder can take out or annuitize any remaining fund. GMWBs differ from GMIBs in that the remaining fund is paid to the estate of the deceased on death. The latest version is the Guaranteed Lifelong Withdrawal Benefit (GLWB) option or GMWB for Life. It offers a lifelong guarantee: the maximum amount to be withdrawn is specified but the total amount is not limited and the insured can annually request a portion of the premium paid until he is still alive, even if the fund value drops to zero. Any remaining account value at the time of death is paid to the beneficiary as death benefit. Many additional features can be added on this base contract: in the case of a *Roll-up* option, the annual guaranteed withdrawal amount is increased by a fixed percentage every year during a certain time period but only if the policyholder has not started withdrawing money. Therefore, *Roll-ups* are commonly used as a disincentive to withdraw during the first years. Finally, in the case of a deferred version of the contract, the product is fund linked during the deferment and the account value at the end of this period, or a guaranteed amount if greater, is treated like a single premium paid for an immediate GLWB.

This is a simplified description of the basic design of the guarantees embedded in VAs; a complete description of all possible variants would be beyond the scope of this thesis, focused on the actuarial and financial valuation of this kind of contracts. Thus, some products offered in the market may have features different from those investigated above or may be combination of two or more guarantees; however, their valuation can be carried out along the lines of the models considered in the following chapters with opportune modifications or extensions.

1.3 The International markets

In this section, we examine U.S., Japanese and European life insurance markets in order to highlight the international development of VAs and their strong potential growth. In the report by Hanif et al ([17]), the authors summarize the reasons of the global popularity of VAs in the following factors:

- *Equity exposure:* VAs provide higher expected returns than fixed annuities during a period of low bond yields and protection again inflation for policyholders approaching retirement.

- *Longevity Protection:* living benefits like GMIB and GLWB allow VA to offer a protection for pensioners against living longer than expected.

- *Transparency and Flexibility:* policyholders value the transparency of explicit charges required for guarantees and the possibility to customize them to suit particular needs, such as income planning and inheritance. VAs often allow a huge choice of underlying funds according the preferences and the risk tolerance of the investors. VAs also provide flexibility in retirement, due to the possibility to chose the withdrawal amount and the option to lapse or surrender the contract.

- *Profitability and Capital Efficiency:* Hanif et al. ([17]) note that, from an insurance company's perspective, variable annuities are relatively profitable and capital efficient under economic capital measures required by Solvency II, due to the fact that the guarantees are hedged and the rider charge for them is sufficient to finance the cost of the hedging scheme.

- *External Factors:* uncertainty relating to future inflation, the regulatory environment, the increasing burden on State pension provision and the balance between state and privately funded pensions, customer preferences for guarantees end so on.

In the following, we briefly consider the development of VAs in the main international markets.

1.3.1 U.S. Market

VAs have existed in the U.S.A. since the 1970s. The first guarantee embedded in a VA was a GMDB option; GMIB options have been issued since 1996 and GMAB and GMWB options were introduced in 2000. Figure 1.1 shows the general growth curve that the U.S. industry experienced: the growth was rapid during the 1990s, related to the growth of the stock markets during the so-called "tech boom". The stock market decline in 2001-2002 produced a temporary fall in VA sales, but in the following years the market expanded. Extensive market research is beyond the scope of this thesis; more precise data are available from Variable Annuity Research Data Services (VARDS) and an analysis of U.S. market has been offered by Abkemeier et al. (cf. [1]).

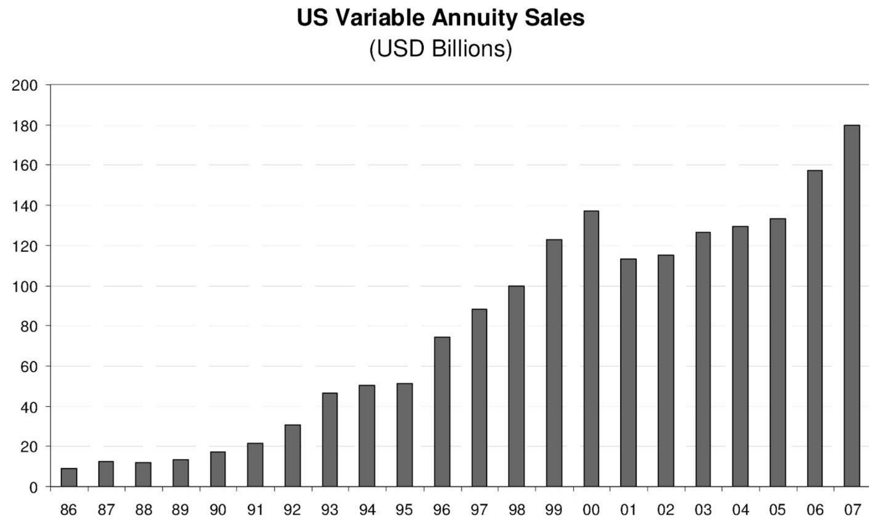

Figure 1.1: US Variable Annuity Sales 1986-2007

1.3.2 Japanese Market

The Japanese VA industry is described by Ino (cf. [21]). Variable annuities have a relatively short history in Japan. After the financial deregulation that permitted the sale of variable annuities, ING Life started selling variable annuities with GMDB's in 1999. Hartford Life, leader in U.S.A., entered in the Japanese market in 2000; in 2002 bank were allowed to sell annuity product. Figure 1.2 shows the growth in VA sales:

Ino explains this growth is due to many factors: the demographic trend (Japan is one of the most rapidly ageing societies), the economic environment, characterized by extremely low interest rates, the saving culture and deregulation.

1.3.3 European Markets

VAs have been imported from U.S. market, where they have enjoyed success, to European markets, firstly in U.K. There, the most recent launched guarantees are offered in the pension market, giving a more efficient trade-off between risk and return than the conventional annuity or income drawdown

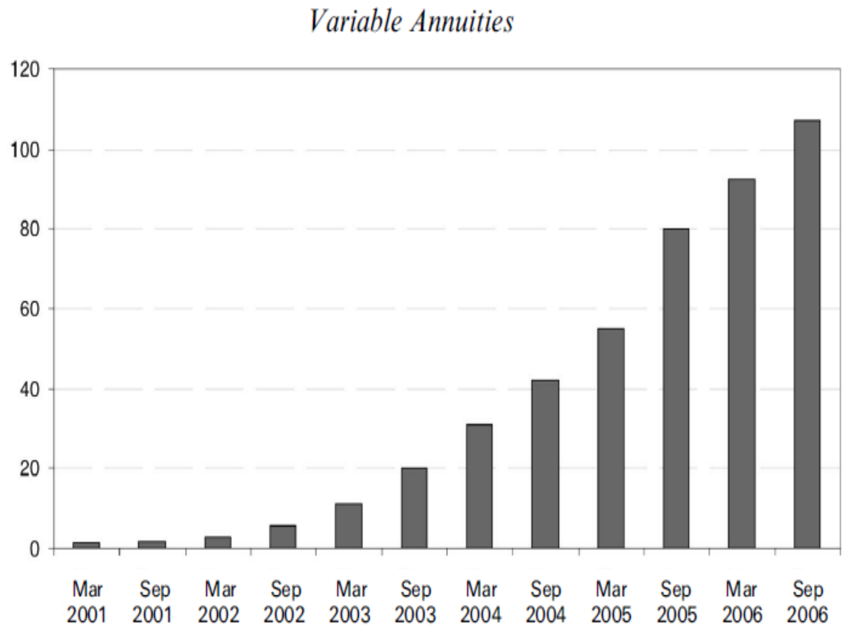

Figure 1.2: Japanese Variable Annuity Sales 1986-2007

products. Since 2006, there have been six different VA propositions launched in UK. Variable annuities are now also appearing across Europe. Some of the more significant launches have been offered by Axa in France, Germany, Spain, Italy and Belgium, as well as by ING in Spain, Hungary and Poland, by Generali (December 2007) in Italy and Ergo (February 2008) in Germany, by Aegon (Scottish Equitable), Hartford, Metlife and Lincoln in the U.K. A number of other multinationals have announced their intention to launch VAs within Europe. Some factors encourage the development of VAs in Europe: first of all, Europe currently has the oldest demographic profile in the world, with around 35% of the population projected to be aged over 60 by 2050. Ageing populations substantially increases the demand for pensions; in addition, public spending on pension is high and this situation is unsustainable, so governments are encouraging private retirement saving. Hanif et al. (cf. [17]) provide some detailed analysis of the market segmentation and growth opportunities in the main European markets. The various life and pensions saving markets in Europe are quite different and there are diverse factors that affect VA potential growth. For example, Hanif et al. note that in Northern Europe the focus is on GMWB and GMIB guarantees for the retirement market. This is in contrast to Southern Europe (Spain and Italy), as well as Eastern Europe (Hungary and Poland), where the focus has been on the more traditional accumulation and death guarantees (GMAB's and GMDB's). In these markets, there is generally an attractive state pension system in place, and investment in the less developed retirement savings market segment is seen as a longer term investment.

1.4 Overview

Over the years, many practical and academic contributions have been offered for describing the VAs and the guarantees embedded. Most of the earlier literature (cf. [14] and [36]) is constituted by empirical works dealing with product comparisons rather than pricing and hedging issues. It was not until recently that some guarantees were discussed by practitioners (cf. [15], [23], [32]); they highlight the growing opportunities to introduce VAs in new markets. Recently, the academic literature has shown a fervent interest to the topic too (cf. [4],[8],[10],[13],[19], [25],[27],[28],[29],[30]). The first universal general framework in which any design of options and guarantees currently offered within Variable Annuities can be modeled has been offered by Bauer et al. ([4]). Besides the valuation of a contract assuming that the policyholder follows a given strategy with respect to surrender and withdrawals, they are able to price contracts with different embedded options. The pricing models proposed in the actuarial literature are based on the standard No-arbitrage formulas of mathematical finance, in line with the tradition of Boyle and Schwartz (see [7]) that extend the Black-Scholes framework (see [6]) to insurance contract. The main difference is that for the option embedded in VA products the fee is deducted ongoing as fraction of asset, instead in the Black and Scholes approach the premium is paid up-front. In order to price options embedded in Variable Annuity contracts many authors use numerical PDE methods (see [13],[8],[26],[30],[34]), others exploit Monte Carlo simulations (see [25],[27]). Pennacchi ([35]), Sherris ([37]) and Cox et al ([11]) use option pricing technique to value options embedded in pension funds or structured insurance products. Within the last 10 years, more than 60 scientific papers on the financial valuation of guarantees embedded in insurance policies have been published. For a selected bibliography on the topic, we refer to [18]. In the last years, the attention of academics and practitioners has been fixed on the guarantees embedded on VA. In the following we cite some contributions on the valuation of GMxB options. Milevsky and Posner ([26]) price various types of guaranteed minimum death benefit option treated as a Titanic Option. They present closed form solutions for this option in case of an exponential mortality law and numerical results for the Gompertz-Makeham law. They find that in general these guarantees are overpriced in the market. Milevsky and Salisbury ([29]) adopt a framework for the valuation of GMDB where the insured has a Real Option to Lapse, i.e. the possibility to surrender the policy. Belanger et al ([5]) value the GMDB option and consider an additional common feature included in many contract, the possibility of partial withdrawals. They determine how this clause affects the insurance fee and produce a pricing model based on the impulse control problem. Haberman and Piscopo ([16]) discuss the valuation of GMDB options using data for the Italian male population as a case study; they take into account the mortality risk and investigate the sensitivity of the price of the option to changes in mortality probability using both deterministic and stochastic approaches. In [29], a model for the valuation of certain GMLB and GMDB options is presented in a framework where the insured has the possibility to partially surrender the policy. The authors call this a "Real Option to Lapse". They present closed form solution in the case of an exponential mortality law, constant surrender fees and no maturity benefits. In [30], the same authors price GMWB options. Besides a static approach, where deterministic withdrawal strategies are assumed, they calculate the value of the option in a dynamic approach. Here, the option is valuated under optimal policyholder behavior.

They show that under realistic parameter assumptions optimally at least the annually guaranteed withdrawal amount should be withdrawn. Furthermore, they find that such options are usually underpriced in the market. This result is in contrast with the common belief that the guarantees embedded in variable annuity policies are overprice (cf. [9]). Chen et al ([8]) and Ho ([20]) note that sub-optimal policyholder behavior considerably reduces the value of the GMWB rider. Cramer et al. ([12]) suggests to describe the sub-optimal policyholder behavior with a function of how much the embedded option is in the money. Wang ([39]) offers a dynamic lapse function that more reduces the lapse when the GLWB is more in-the-money.

1.5 Conclusions and our motivations

As it is clear from the previous section, great attention is currently devoted to the study of VAs, both from a theoretical and a practical point of view. One of the reason of their success is the growing importance of annuity benefits paid by private pension scheme. The shift from defined benefit to self-directed defined contribution plans and the reform of the Social Retirement System in many countries, so that it includes personal accounts, have encouraged the proliferation of this kind of products. In this chapter, we have described the guarantees offered in VAs and the markets where they are issued and we have retraced the main contributions of the literature. We have highlighted the significant international popularity obtained by VAs, believing in the perspectives of their favourable development in Italy market and throughout Europe and Asia. In the light of the fervent interest oriented towards VAs, in this thesis we focus on the actuarial and financial valuation of guarantees embedded in VAs, derive No-arbitrage pricing models and study of the mortality risk. Among the risks which affect insurance products, particular attention has to be directed to mortality risk, whose impact on living and death benefits is considerable due to the long maturity of the life annuity portfolio and pension plan. In this regards, in the following chapters we present the results of the study of the impact of mortality risk on both GMDBs and GLWBs. We choose to use a very general model, a simplified version of the stochastic model suggested by Cox and Lin ([11]) and developed by Ballotta, Esposito, Haberman ([3]), because the focus of this work is not on the study of longevity risk per se, but on the possible effects of improvements or worsening in life expectancy on the VAs. In this way, we offer a broad view of the potential impact of shifts of mortality functions, regardless of the particular mortality projections; the scope is to understand the influence of mortality risk on this products, before to quantify them with opportune projected mortality tables for different countries and times. Of course, the analysis can be further developed with the quantification of the impact of longevity risk using specific models, such as the Lee-Carter and its extensions.

Another aspect we investigate in the following is the surplus analysis. Allowing for randomness in mortality urges us to study the random fluctuations in the portfolio behavior. We offer to insurance companies a model useful to manage a portfolio of VAs to respect the solvency requirements. We choose to follow Lysenko and Parker ([13]), adopting a definition of surplus as the difference between the retrospective gains and the prospective loss. There are two reasons for selecting this setting. Firstly, this is one of the most recent development in the field of surplus analysis and its accuracy can be helpful to practitioners, regardless of its complexity from a computational point of views. Secondly, an advantage of this model is that it allows an ex ante assessment of the insurer's solvency throughout the duration of contract and it enables us to determine the premium that leads to a required probability of insolvency, consistent with recent regulatory changes.

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Chapter 2

The GMDB Option

2.1 Introduction

The typical VA is a unit-linked deferred annuity contract, which is normally purchased by a single premium payment up-front which is invested in one of several funds. The VA also typically contains some embedded guarantees. One of these is the Guaranteed Minimum Death Benefit, which is an increasing-strike put option with a stochastic maturity date. If the insured dies during the deferment period, the beneficiary obtains a death benefit , that is equal, in the basic form of the product, to the maximum of the invested premium and the account value linked to the fund. An enhanced version of the product returns at least the originally investment accrued at a minimally guaranteed interest rate or the account value, if greater. These guarantees are paid for by the policyholder in the form of a perpetual fee that is deducted regularly from the account value linked to the underlying assets.

In this chapter we define a fair price for a GMDB in a market consistent manner and describe how the value of a GMDB evolves over time and in the presence of mortality changes. Our work develops the standard pricing model of mathematical finance and uses the Black and Scholes formula to price this insurance contract. The approach follows the recent actuarial literature on the valuation of VA products (cf. [4], [5], [10], [12]). Thus, Milevsky and Posner ([10]) price various types of guaranteed minimum death benefit treated as a *Titanic Option* and find that in general these products are over-priced in the market. Milevsky and Salisbury ([12]) adopt a framework for the valuation of GMDB where the insured has a *Real Option to Lapse*,i.e. the possibility to surrender the policy.

The contribution of this work is the study of the impact of mortality risk on the value of a GMDB under both deterministic and stochastic approaches. At first, we use the methodology of tilting to modify the observed probability of mortality and the projection is realized using assumptions based on historical data. Recently, it has become evident that deterministic mortality projections are inadequate, because unanticipated changes over time in the mortality rates have been observed. For this reason, a stochastic mortality approach is necessary in order to avoid underestimation or overestimation of the expected present value of insurance and annuity contracts. In this work, we propose a simplified version of the stochastic model suggested by Cox and Lin ([6]) and developed by Ballotta, Esposito, Haberman ([1]). We provide a detailed application to the Italian market, where the first Variable Annuity has been issued in September 2007 with a GMDB option. Finally, we develop a surplus analysis for a portfolio of GMDB options. We offer a model according which the premium can be modified as per the forecasts of mortality probabilities, interest rate and fund evolution. Moreover, the model enables us to determine the premium that leads to a required probability of insolvency, and so it can be used for an evaluation of the adequacy of solvency.

The chapter is organized as follows: in section 2.2 we describe the product. Section 2.3 develops the model for the pricing of a GMDB. In Section 2.4, we study the impact of mortality risk on the value of the contract and show an application to Italian data following a deterministic framework. Mindful of the limits of this approach, we develop, in the section 2.5, a simulation-based stochastic mortality model and consider the effects on the GMDB.

Concluding remarks are offered in the section 2.6.

2.2 Product description

The GMDB provides for the beneficiary a guaranteed benefit at the time of death that may increase as the fund value grows. It is a put option with a stochastic maturity. There are many kinds of option:

- the basic form of a death benefit is the *Return of Premium Death Benefit*,that ensures the maximum of the current account value at time of death and the single premium paid;
- in the case of a *Roll-up* option, then the minimum benefit is equal to the single premium compounded with a constant interest rate (*Roll-up rate*);
- an enhanced version of the option provides a *rising-floor guarantee*: then the returns is at least the premium paid accrued at a minimally certain interest rate and the payoff is

  $$Max\left[Min\left[S_0e^{rT}, MS_0\right], S_T\right]$$

  where r is the continuously compounded fixed guaranteed rate and M is the cap on the guaranteed return;
- when the contract contains an *Annual Rachet Death Benefit*,the minimum amount guaranteed is compared every years with the account value, and then this that becomes the new amount guaranteed if it is greater;
- when there is a *look back guarantee*, a guaranteed death benefit is based on a suitably defined highest anniversary account value; some policies offer an annual reset, others require a five year wait and so on. The payoff is $Max\left[S_{it}, S_T\right]$, where S_{it} is a defined anniversary.

In general, we can classify the GMDB in two groups:

- *interest guarantees*, which refer to a contract in which the amount guaranteed is the premium accumulated at a fixed rate of return;
- *market guarantees*, which ensure the highest market return during a certain period. Most Variable Annuities provide a combination of both categories.

In this chapter we consider a Variable Annuity within a simple *Roll-up* GMDB, and we assume that the policyholder does not have an option to lapse, for the sake of simplicity. The policyholder pays a single premium P, that is invested in a fund; we denote the account value by V_t. As far as the put option is concerned, in contrast to the other derivatives where payments are made on acquisition, the GMDB option is paid by deducting a fixed proportional amount from the account value on a continuous basis. Milevsky and Posner ([10]) calculate the fair charge considering that its expected present value has to be equal to the value of a put option with a stochastic maturity date. We note that American options also have a stochastic maturity, but the methodology used to price these derivatives cannot be used for the GMDB, because there is a difference between the two products: in the first case, the investor decides when he exercises the option, in the second one the put will expire at the moment of death. For this reason, the only way to price a GMDB is based on its decomposition into other simpler instruments, as we illustrate in the next section.

2.3 The model

Let T_x be the future lifetime random variable expressed in continuous time, $F_x(t)$ be its cdf and $f_x(t)$ be its *pdf*; therefore, for an individual aged x the probability of death before time t is

$$F_x(t) = P(T_x \leq t) = 1 -_t p_x = 1 - exp\{-\int_0^t \zeta(x+s)ds\} \qquad (2.1)$$

where ζ denotes the force of mortality .
Let V_t be the account value at time t linked to fund value. Following the standard assumptions in the literature, we model the evolution of the account as:

$$dV_t = (\mu - \eta) V_t dt + \sigma V_t dW_t \tag{2.2}$$

where μ is the drift rate, η is the insurance fee paid for the GMDB option, σ is the fund volatility , W_t is a standard Brownian motion .
The risk neutral process for V_t is:

$$dV_t = (r - \eta) V_t dt + \sigma V_t dZ_t^Q \tag{2.3}$$

where r is the risk free rate and Z_t^Q is a Brownian motion under a new Girsanov transformed measure Q. The solution of the SDE is:

$$V_t = V_0 exp[(r - \eta - \frac{\sigma^2}{2}) + \sigma Z_t^Q] \tag{2.4}$$

Now we describe the GMDB payoff. At the random date of death τ the beneficiary will receive

$$D_\tau = max(e^{g\tau} V_0, V_\tau) = e^{g\tau} max(V_0 - e^{-g\tau} V_\tau, 0) + V_\tau \tag{2.5}$$

where g is the guaranteed rate .
The value of the GMDB option at τ is the sum of the fund value and a put option whose strike price is the initial value V_0, with an underlying asset V_τ discounted by the guaranteed growth rate g. Since the maturity is stochastic and τ and V_τ are independent, the present value of GMDB is given by the expectations under τ and V_τ :

$$D_0 = E_t\{E^Q\{e^{-r\tau} D_\tau | \tau = t\}\} \tag{2.6}$$

If we fixe the date τ, we have at τ an European option , whose value can be calculated with Black and Scholes formula. Therefore, the previous formula can be interpreted as a decomposition of the actual value of GMDB into the actual value of a continuous sequence of European put options. Substituting the expression for D_τ

$$D_0 = E_t\{E^Q\{e^{-(r-g)\tau} max(V_0 - e^{-g\tau} V_\tau, 0) + e^{-r\tau} V_\tau | \tau = t\}\} \tag{2.7}$$

We can observe that

$$E_0^Q\{e^{-r\tau}V_\tau\} = e^{-\eta\tau}V_0 \tag{2.8}$$

since we have supposed that V_t is a geometric Brownian motion with drift equal to $r-\eta$ and so its expected value is:

$$E_0^Q\{V_\tau\} = e^{(r-\eta)\tau}V_0 \tag{2.9}$$

Consequently:

$$\begin{aligned} D_0 &= E_t\{E^Q\{e^{-(r-g)\tau}max(V_0 - e^{-g\tau}V_\tau, 0) + e^{-r\tau}V_\tau|\tau = t\}\} = \\ &= E_t\{E^Q\{e^{-(r-g)\tau}max(V_0 - e^{-g\tau}V_\tau, 0)e^{-\eta\tau}|\tau = t\}\} \end{aligned} \tag{2.10}$$

We observe that for a fixed date T

$$\begin{aligned} & E^Q\{e^{-(r-g)T}max(V_0 - e^{-gT}V_T, 0) + e^{-\eta T}\} \equiv \\ & \equiv V_0[e^{-\tilde{r}T}N(-d_2) - e^{-\eta T}N(-d_1) + e^{-\eta T} \equiv \\ & \equiv V_0[BS(\tilde{r}, \eta, \sigma, T) + e^{-\eta T}] \end{aligned} \tag{2.11}$$

where $d_1 = \frac{(\tilde{r}-\eta+\frac{\sigma^2}{2})\sqrt{T}}{\sigma}$; $d_2 = d_1 - \sigma\sqrt{T}$; $\tilde{r} = r - g$; $N(.)$ is the cumulative probability function for a random variable normally distributed. If we consider both the expectations, we obtain:

$$D_0 = \int_0^{\omega-x} f_x(t)V_0[BS(\tilde{r}, \eta, \sigma, T) + e^{-\eta T}]dt \tag{2.12}$$

where $BS(T)$ is a put option with maturity T. In the discrete case we have:

$$D_0 = \sum_{t=1}^{\omega-x} {}_tp_x q_{x+t} V_0[BS(\tilde{r}, \eta, \sigma, T) + e^{-\eta T}] \tag{2.13}$$

for a policyholder aged x at inception of the contract.
Thus, the value of the GMDB is a weighted average of the values of $\omega - x$ European put options, where the weights are the postponed probability of death in t, i.e. the probability of survival until t and death between t and $t+1$.

2.4 The impact of mortality on the GMDB value: a deterministic approach

In this section, we illustrate the relationship between the GMDB value and the age of policyholder at the inception of the contract.
We price a simple form of the death benefit; we consider g equal to 0, so that the GMDB option ensures the maximum of the current account value at the beginning of the year of death and the single premium paid is given by:

$$D_0 = \sum_{t=1}^{\omega-x} {}_tp_x q_{x+t} max(V_t, V_0)e^{(-rt)} = \sum_{t=1}^{\omega-x} {}_tp_x q_{x+t}[max(V_0 - V_t, 0) + V_t]e^{(-rt)} \tag{2.14}$$

and where the parameters in the model take the following specific values: the risk free rate r is 7%, the fee η is 1% , the underlying volatility σ is 10%, the strike V_0 is 100 and the fund value follows a geometric Brownian motion. We make reference to the Black and Scholes framework for option pricing.
We consider two different mortality tables based on the experience of the Italian male population for 2001 and 2004. Figure 2.1 shows the probability of survival and mortality rate for a policyholder aged 50 occurred in the 2004. The graphs show the characteristic features. We note a kink in the q_{50+t} curve for value of t equal to 55. The function ${}_tp_x q_{x+t}$ for discrete values of t represents the probability function of the discrete random variable K_x for $t = 0, 1, 2$. Thus

$$ {}_tp_x q_{x+t} = Pr[t < K_x \leq t + 1]$$

Figure 2.2 shows an unusual feature: it has two modes at t equal to 29 and 34. It depends on the fluctuation in the fitted curve of q_{50+t}, which has a rather strange behaviour between the ages of 79 and 84[1].
Let $F_x(t)$ be the cumulative distribution function (cdf) of the random variable time to death for Italian male policyholders aged x based on the 2004 mortality table, as in equation (2.1).

[1]The 2-modal feature can be found also in a recent Belgian males table ([15]).

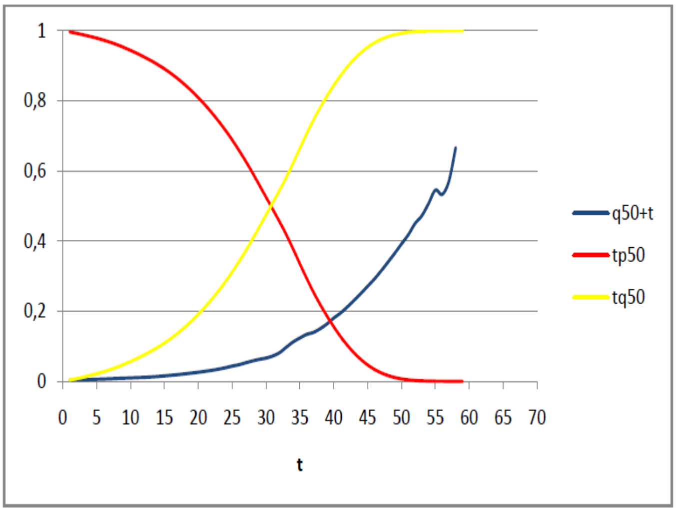

Figure 2.1: The Mortality probability function

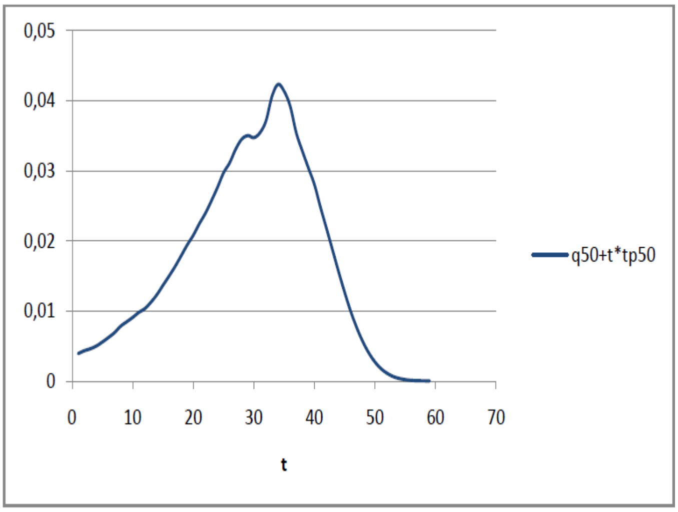

Figure 2.2: The Postponed Mortality probability for a fifty-years old

We operate a tilting of $F_x(t)$ to create a new function $F_x^*(t)$, characterized by a reduction of mortality:

$$F_x^*(t) = h[F_x(t)] \tag{2.15}$$

where h is modeled on a historical basis and projects forward the same reduction of mortality that happened between 2001 and 2004. We can think of $F_x^*(t)$ as an adjusted mortality cumulative distribution function , which takes into account projected improvements in life expectancy. Figure 2.3 provides an example of the tilted *cdf* from age 0 onwards.

The assumption is strong: for the sake of simplicity, we are assuming that

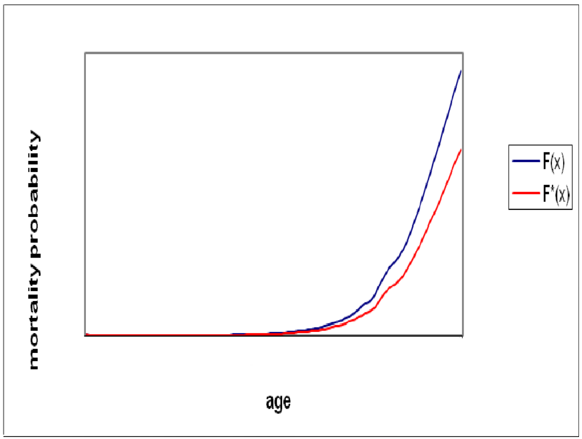

Figure 2.3: Tilting of mortality *Cdf*

there will be in the future the same improvement in life expectancy that occurred in the past 3 years.

In our application, we use the above procedure in order to derive a modified probability function at each age between 50 and 95. In Figure 2.4, we report only the discrete probability function for a policyholder aged 50 at inception. We calculate the GMDB value for different policyholders with ages from 50 to 95 at inception. At first, we consider only the discrete mortality probability density function in order to study the way in which the GMDB value varies when the age of policyholder at the inception of the contract increases. Then, we analyze the impact of mortality improvements on the GMDB value.

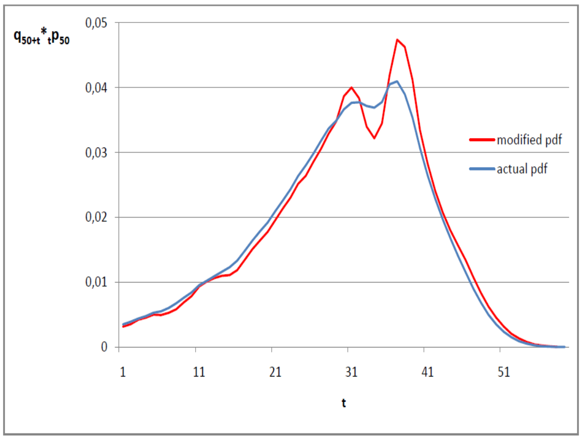

Figure 2.4: Tilting of mortality probability function

The GMDB option is composed of a sequence of put options with different maturities. For example, we report the calculation of the GMDB value for a policyholders aged 50:

$$D_0 = \sum_{t=1}^{\omega-50} {}_tp_50q_{50+t}[max(V_0 - V_t, 0) + V_t]e^{(-rt)} \tag{2.16}$$

In order to study the relation between the GMDB value and age at inception, we need to take into account two different effects: on one hand, the weights change because the probability function changes with age; on the other hand, as age at inception increases, the number of put options that compose the GMDB product decreases. Moreover, we have to consider that the value of the put decreases with time. The combination of these effects generates the relation represented in Figure 2.5: as age at inception increases the value of the GMDB increases.

Next, we compare the value of the GMDB under the real and modified probability functions for different policyholders aged between 50 and 95 at the inception of the contract (see Figure 2.6). In order to explore the consequences of an improvement in life expectancy on the GMDB value, we have to take into account the fact that the probability function changes in response to two different effects: at each time point the survival probability increases and the mortality probability decreases. As we can see from Figure 2.6, the

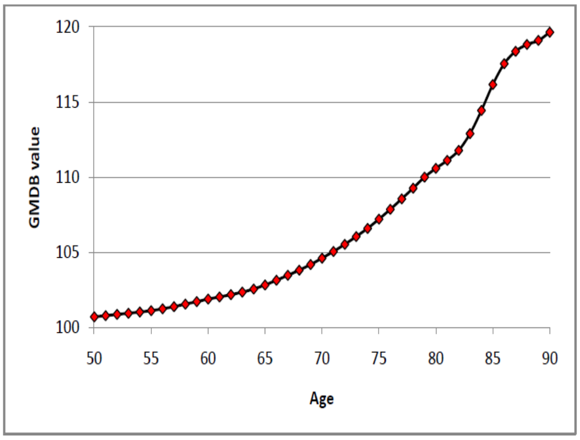

Figure 2.5: The GMDB value and dependency on age

second effect prevails between ages 50 and 80 and between ages 82 and 86. For this reason, the GMDB values under the modified probability function are smaller than that under the real probability function at almost all of the ages considered.

At the end of this section, we reflect upon what happens if g is different

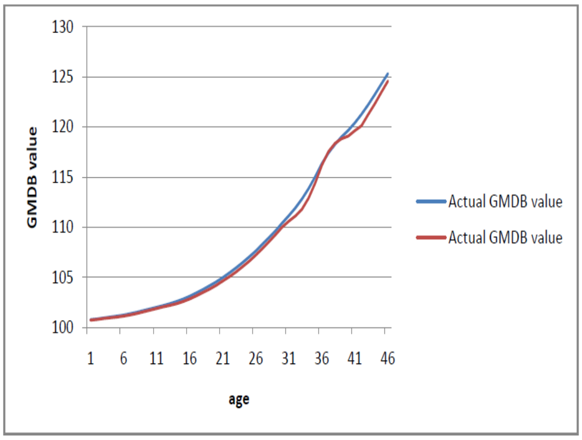

Figure 2.6: The comparison of GMDB value under real and modified mortality probability function

from zero. In this case, the GMDB option provides the maximum of the current account value at the beginning of the year of death and the single premium capitalized at the rate g:

$$D_0 = \sum_{t=1}^{\omega-x} {}_tp_x q_{x+t} max(V_t, V_0 e^{gt}) e^{(-rt)} = \sum_{t=1}^{\omega-x} {}_tp_x q_{x+t} [max(V_0 - e^{-gt} V_t, 0) + V_t] e^{(-rt)} \tag{2.17}$$

As g increases the spot price of the underlying $(e^{-gt}V_t)$ decreases and the value of each put option increases; furthermore, it is capitalized at the rate g, so as g increases the GMDB value increases. Figure 2.7 shows the relationship between the guaranteed rate g and the GMDB value for a policyholder aged 50.

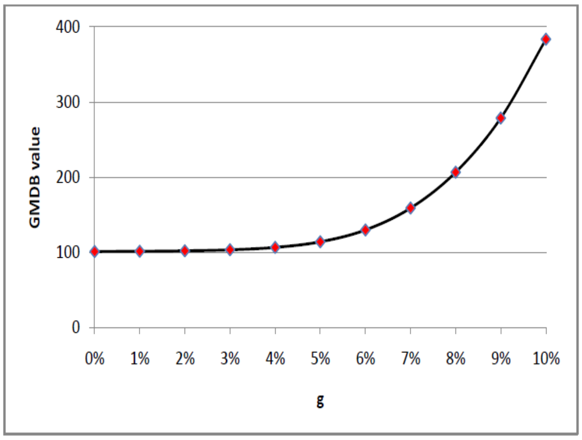

Figure 2.7: The relation between the GMDB value and g for a fifty-years old

2.5 The impact of mortality risk on the GMDB value: a stochastic approach

In the previous section we have modified the mortality distribution using a tilting method based on historical observations. Recently, it has become evident that deterministic mortality projections are an inadequate approach to dealing with risk, i.e. unanticipated changes over time in the mortality rates and other indices. For this reason, a stochastic mortality approach is necessary in order to avoid underestimation or overestimation of expected present value of life insurance contracts with a significant mortality component. In this section, we propose a simplified version of the stochastic mortality model suggested by Cox and Lin ([6]) and developed by Ballotta, Esposito, Haberman ([1]).

Our calculation is based on the survival model used before; our purpose is to develop an adjusted survival model (or mortality table), which takes into account possible mortality shocks. In this regard, we estimate the expected value of the number of survivors at age $x + t$, $E[l(x + t)]$, in a stochastic framework. It is possible to prove that $l(x + t)$ is approximately distributed as a normal random variable with mean equal to $l(x)_t p_x$ and variance equal to $l(x)_t p_x(1 -_t p_x)$. However, the latest actuarial literature highlights that the empirical data show perturbations in the survival probabilities due to random shocks. Accordingly, we simulate the survival probabilities adjusted for shocks as follows:

$$p'_{x+t} = p_{x+t}^{(1-\epsilon_t)} \tag{2.18}$$

where ϵ_t is the shock in the expected probability at time t. Ballotta, Esposito, Haberman (2006) assume that ϵt follows a beta distribution with parameter a and b and the sign of the shocks depends on the random number $k(t)$ simulated from the uniform distribution $U(0, 1)$. In particular, we set:

$$\begin{aligned} \epsilon(t) \text{ if } k(t) < c \\ -\epsilon(t) \text{ if } k(t) \geq c \end{aligned} \tag{2.19}$$

where c is a parameter which depends on the user's expectation of the future mortality trend.

The importance of assigning a random sign to ϵ_t is that, in this way, the model captures not only the long period variations in mortality rates, but also the short period fluctuations due to exceptional circumstances.

In our application, we consider two opposite cases for the value of c: $c = 1$ and $c = 0$. In the first case, there will be improvements in life expectancy at every date; in other words, all shocks are expected to be positive. Conversely, in the second case further improvements of an already high expectancy of life are impossible and all shocks are expected to be negative. So, we simulate the value of $p_x^{'}$ for a policyholder aged $x = 50$ at inception of the contract under the two different hypotheses and then we calculate the expected number of survivors $l^{'}(x + t + 1)$ as follows:

$$l^{'}(x + t + 1) = l(x + t)p^{'}(x + t) \tag{2.20}$$

We are then able to calculate the other mortality functions that we need.
In order to analyze the impact of different variations in mortality probabilities, we consider two different expected value for ϵ_t:

$$E[\epsilon_t] = 0.10$$

$$E[\epsilon_t] = 0.30$$

We carry out two calculation procedures: in the first one, we fix $a = 0.5$ and $b = 4.5$, so that shocks have expected value equal to 0.10 and standard deviation equal to 0.12; in the second one, we fix $a = 1.5$ and $b = 3.5$, so that shocks have expected value equal to 0,30 and standard deviation equal to 0.19. In both cases, we simulate 1000 paths of evolution of mortality using the Monte Carlo method and consider the alternative hypotheses $c = 0$ and $c = 1$. Then, we calculate the price of the GMDB option and compare the results under the different scenarios.
At first, we report the graphics relating to only one path simulated under the hypothesis $E[\epsilon_t] = 0.30$, in order to reflect upon the impact on the GMDB value of an improvement or a worsening in life expectancy; In Appendix A, we show the more general results of our simulations.
In the Figure 2.8, we compare the actual survival function[2] with those simulated under the hypothesis $c = 1$ and $c = 0$. In the first case, we expect that there will be only improvements in life expectancy and, consequently, the simulated function lies above the actual survival function. Instead, in

[2]The actual survival function, which we refer, is based on SIM2002 mortality table .

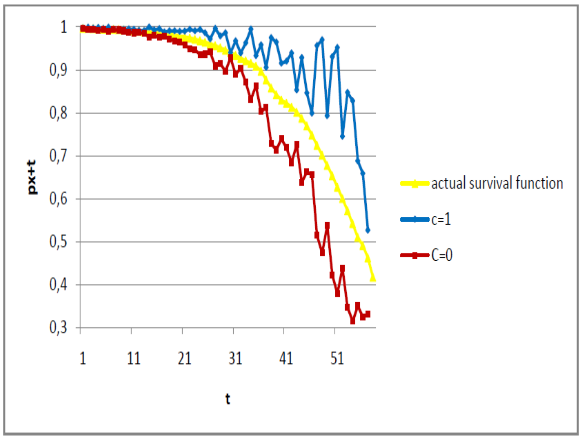

Figure 2.8: Survival function under different mortality hypothesis

the second case we expect there will be only deteriorations and the simulated function lies below the actual survival function. In the same manner, in Figure 2.9 we compare the actual mortality function with those simulated under hypothesis $c = 1$ and $c = 0$ and we see a complementary picture.

The purpose of this simulation is to quantify the impact of mortality risk on the GMDB value; in this regard, we have to consider the projected postponed probabilities of death[3]. In Figure 2.10, we compare the actual mortality probability function for a policyholder aged 50 and the simulated distribution under the hypotheses $c = 1$ and $c = 0$. We have to keep in mind that the probability function changes because of two different effects: if $c = 1$ the survival probability increases and the mortality probability decreases at every time point; on the contrary, if $c = 0$ the mortality probability decreases and the survival probability increases. The consequences are that, under the hypothesis $c = 0$, the probability function is translated so that the left tail bacomes fatter and the right tail less fat than fot the actual probability function[4]. On the contrary, if $c = 1$, the probability function is translated so that

[3] In Figure 2.8 we have reported only the results relating to a policyholder aged 50 at inception of the contract, but we have simulated the postponed probability of death for every age of inception between 50 and 110.

[4] In Figure 2.10 we have constructed a smooth function with a polynomial regression to make this translation more clear.

the left tail becomes less fat and the right tail more fat than for the actual probability function. The consequence is that there will be improvementes in life expectancy, the probability of death during a given year will decrease at younger ages and will increase at older ages.

The effects on the GMDB value[5] are described in Figure 2.11.

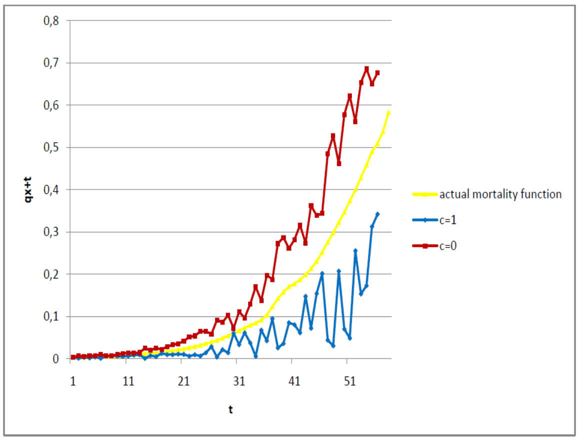

Figure 2.9: Mortality function under different mortality hypothesis

Under the hypothesis $c = 1$, the weights of the valuation formula (i.e. the

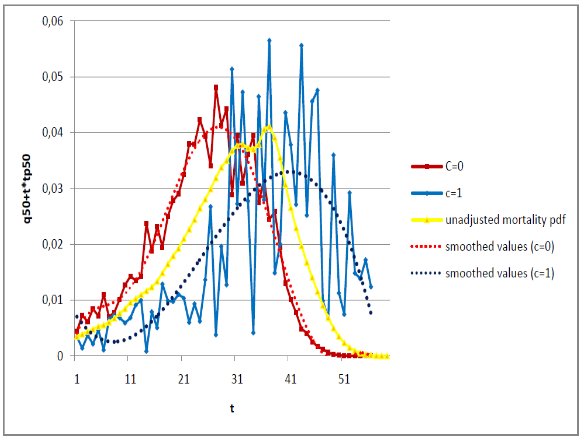

Figure 2.10: Mortality probability function under different mortality hypothesis

[5]We still refer to the basic form of the GMDB option, that ensures the maximum of the current account value at time of death and the single premium paid.

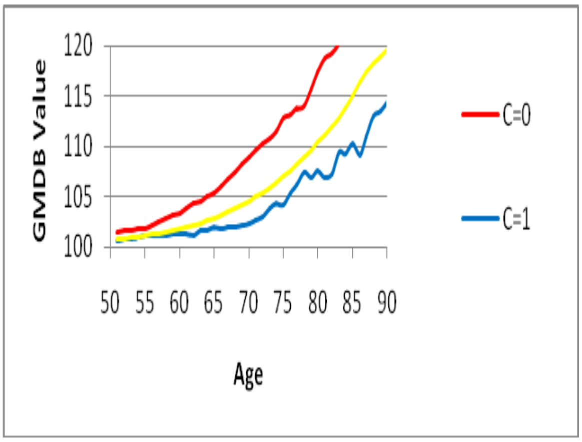

Figure 2.11: The comparison between actual and simulated GMDB value

mortality probability function) are lower at the beginning and higher at the end of the time period than the actual weights; consequently, the earlier put options, that have a large value, are weighted less than under the actual distribution and the final put options, that have a small value, are weighted more. Furthermore, in the valuation formula there is also a term linked to the fund value, $V_t e^{(-rt)}$, which decreases as t increases [6]. It is weighted less than under the actual distribution during the first years, when it is higher, and it is weighted more at later time, when it is smaller. For these reasons, if there will be improvements in life expectancy the GMDB value will decrease and the liabilities of the insurer will shrink. On the contrary, under the hypothesis $c = 0$, the weights of the valuation formula are higher at the beginning and lower at the end than the actual weights; consequently, the earlier put options, that have a large value, are weighted more than under the actual distribution and the final put options, that have a small value, are weighted less. Furthermore, the term linked to the fund value is weighted more than under the actual distribution during the first years, when it is higher, and it is weighted less at later time, when it is smaller. For these reasons, if there were a worsening in life expectancy the GMDB value will

[6] In this application, we have considered a risk neutral process for V_t, whit a drift rate $r - \eta = 0.06$, so the term in the valuation formula $V_t e^{(-rt)}$ decreases as t increases.

increase and the liabilities of the insurer will rise[7].

Up to this time we have shown the results for a particular single simulated path of mortality. Now, we report the more general results from our simulations. We have simulated 10000 values of ϵ_t for each t from a beta distribution, and then we have calculated the mean of the shocks at every time and, on this base, have calculate the expected postponed probabilities of death. Subsequently, we have considered the extreme shocks that can occur by choosing upper and lower percentiles. In particular, we have cut the beta distribution at the 95^{th} and 5^{th} percentile and have projected the postponed probabilities of death under both scenarios. We note from Figure 2.12 and

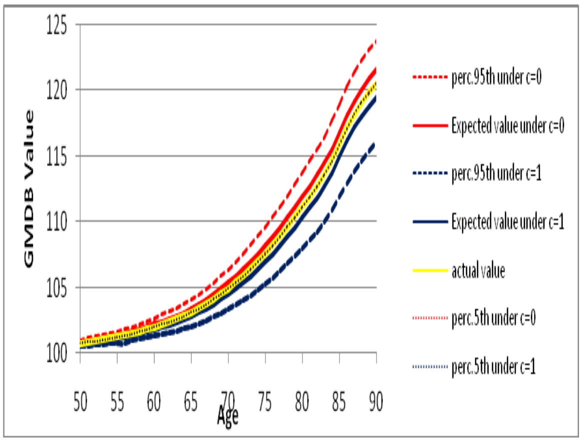

Figure 2.12: GMDB Value under the hypothesis $E(\epsilon_t) = 0.10$ and $\sigma(\epsilon_t) = 0.12$

2.13 that, with a probability of 0.95, the GMDB value fluctuates between the dashed bands; therefore, we can easily derive a measure of Value at Risk for the product. We point out that greater is the expected value of the shocks larger is the impact on the GMDB value.

[7]We point out that the effects of an improvement or a worsening in life expectancy can be different as the assumptions change; for example, if the drift of the process of the fund value is higher than r, the value of $V_t e^{(-rt)}$ increases as t increases. In order to study what happens under the hypothesis $c = 0$ and $c = 1$ it is necessary to observe the interaction between the variations of the value of put option, of $V_t e^{(-rt)}$ and of the weights in the valuation formula. However, a complete description of this interaction is outside of the scope of this work.

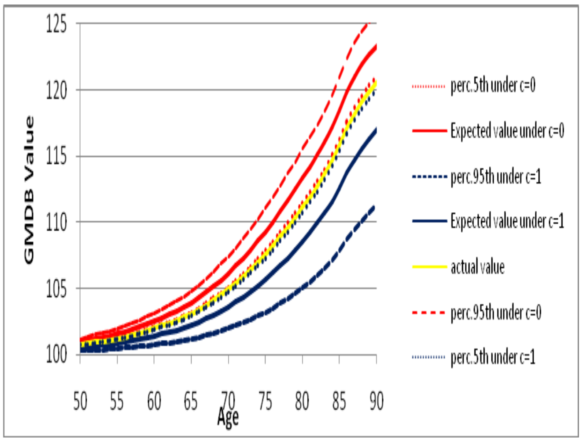

Figure 2.13: GMDB Value under the hypothesis $E(\epsilon_t) = 0.30$ and $\sigma(\epsilon_t) = 0.19$

Up to this time we have considered the expected impact of mortality on the GMDB value; now we carry out a sensitivity analysis, in which we analyze the effect of changes of variance in the distribution of mortality shocks. In particular, we fix a and b such that the shocks have a beta distribution with expected value equal to and standard deviation twice as much those of the previous example; therefore we set $a = 0.056$ and $b = 0.5$, so that $E(\epsilon_t) = 0.10$ and $\sigma(\epsilon_t) = 0.24$.
As in the prior procedure of calculation, we simulate 10000 values of $epsilon_t$ for every t from the new beta distribution, and then we calculate the largest shocks that can occur with a probability of 95%.
If we consider a new beta distribution with the same expected value as before but with double the standard deviation, the simulated pdf under the considered prudential scenario moves to the right under the hypothesis $c = 1$ and to the left under the hypothesis $c = 0$. The consequences for the GMDB value are illustrated in Figure 2.14. We point out that greater is the variance of shocks the larger is the possible oscillation of GMDB value around the expected value and the higher is the risk.

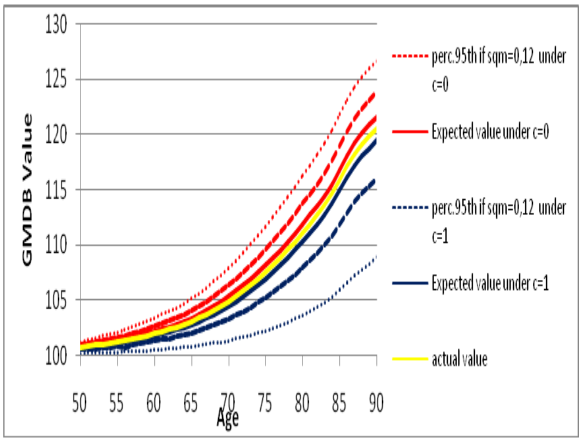

Figure 2.14: GMDB Value under the hypothesis $E(\epsilon_t) = 0.10$, $\sigma(\epsilon_t) = 0.12$ and $E(\epsilon_t) = 0.10$, $\sigma(\epsilon_t) = 0.24$

2.6 Conclusion

In this chapter we have described Guaranteed Minimum Death Benefit options embedded in Variable Annuities. We have dealt with the problem of valuation of these put options, which have stochastic maturity due to the involuntary exercise at the moment of death. We have introduced a theoretical model for the valuation of GMDB as a weighted average price of a set of deterministic put options with different maturity dates, where the weights are the deferred probabilities of death at each date. The contribution of this work has been to analyze the impact of mortality on the value of the GMDB with an application based on Italian data. We have shown that this product is sensitive to mortality risk, which impacts on the GMDB value through the weights in the valuation formula. We also need to keep in mind that the value of puts decreases with maturity. Since the fluctuation in the GMDB value depends on the interaction of all of the abovementioned factors, it is necessary to implement a simulation to measure and manage mortality risk. The results obtained in this work are not general, but depend on the hypothesis about the parameters of the financial and mortality models. Moreover, our valuation formula, Eq. (2.14), relates to an expected present value obtained by the methodology of risk-neutral valuation. It would be interesting to study the full distribution of the random present value of the GMDB option and the impact of mortality risk on it.

In the light of the analysis presented here, we identify areas where there is scope for further work. A limitation of the model developed is the assumption of a flat yield curve; we have made this hypothesis for the sake of simplicity and a complete description of the financial market was outside of the scope of this work, being focused on mortality risk. Certainly, in a further work the model can be improved by introducing an additional hypothesis of a stochastic interest rate term structure.

One problem left open is the definition of an efficient risk management strategy for the GMDB option. The valuation formula expressed in Eq.(2.14) shows that this product is affected by financial risk, due to the changes in the fund value and in the level of interest rates over time, and by mortality risk. The hedging of financial risk is troublesome because of the long maturity of these contracts; this feature increases in the presence of the longevity risk. Also, our study highlights that the mispricing due to neglecting mortality improvements or worsening is noticeable over the long-term horizon. For this reason, a stochastic mortality approach is necessary in order to avoid underestimation or overestimation of the expected present value of this insurance contract which has a significant mortality component.

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Chapter 3

Surplus analysis for GMDB option

3.1 Introduction

In this chapter, we analyze the insurance surplus for a Variable Annuity contract with a Guaranteed Minimum Death Benefit (GMDB) option. Initially, we derive the first two moments of the distribution of the surplus ; and subsequently, we develop the whole distribution using a stochastic model which involves an integrated analysis of financial and mortality risk for a portfolio of annuities with GMDB embedded options. We offer a model according which the premium can be modified as per the forecasts of mortality probabilities, interest rate and fund evolution. Moreover, the study enables us to determine the premium that leads to a required probability of insolvency, and so it can be used for an evaluation of the adequacy of solvency .

There are 2 theoretical foundations for this work: on the one hand, we take into account the actuarial literature concerning the valuation of the Variable Annuity and GMDB option (cf. [3], [6], [16], [17], [18]); on the other hand, we look at the actuarial research literature on insurance surplus and insolvency probability (cf. [7], [9], [11], [13], [15], [19],[20]). The abovementioned papers deal with the stochastically discounted value of future cash flows in respect of life insurance and life annuity contracts. The innovative contribution of our work is to apply this methodology to a new product like a Variable Annuity with a GMDB option, extending the models appearing in the literature in order to study a product with a payments linked to a fund account. In the manner of Lysenko and Parker ([13]), we adopt a definition of surplus as the difference between the Retrospective Gain and Prospective Loss : if we fix a valuation date r, the accumulated value to time r of the insurance cash flows that occurred between times 0 and r represents the retrospective gain and the present value at time r of the cash flows that occur after r is the prospective loss. We modify the model proposed by Lysenko and Parker ([13]) in order to capture the uncertainty of a death benefit linked to a fund account. Further, we do not approximate the true probability function of surplus by its limiting distribution as in Lysenko and Parker, which takes into account the investment risk but treats the cash flows as given and equal to their expected value. Instead, in order to explore the longevity risk, we simulate the impact of both the financial and mortality factors on the retrospective gains and prospective losses. We adopt the same financial assumptions as in the Black and Scholes framework. The mortality hypothesis is based on the stochastic mortality model suggested by Cox and Lin ([8]) and developed by Ballotta, Esposito and Haberman ([1]).

The chapter is organized as follows: in section 3.2 we describe the model; in section 3.3 we develop the financial model. Numerical results are shown in section 3.4 under a deterministic approach. In section 3.5, we develop the simulations and construct the surplus distribution following a stochastic approach and, in particular, we identify three components, relating respectively to interest, fund and mortality risks. Concluding remarks are offered in section 3.6.

3.2 The model

we consider a portfolio of identical Variable Annuities with a GMDB option, which are issued to a group of m policyholders who are aged x with the same risk characteristics, and whose survival probability distribution are independent and identical; the final age is n. The product is composed of an annuity, with annual payment R, and a GMDB option; there is a single premium, paid at time 0 and invested in a fund. Let V_t be the value of the account at time t, which is linked to a unit fund. The payoff of the GMDB option at the time $t = \tau$ is:

$$G_\tau = Max[e^{g\tau}V_0, V_\tau], 0 \leq \tau \leq n \tag{3.1}$$

where τ is the stochastic time of death and g is the guaranteed rate . The premium is calculated according to the equivalence principle:

$$P = Ra_{n,i} + D_0 \tag{3.2}$$

where $a_{n,i}$ is the actuarial value of an annuity, i is the technical rate used to price the annuity, n is the final age and D_0 is the value of the GMDB option at $t = 0$ calculated according the model developed in the previous chapter.
VAs, like unit linked contracts, can be structured in different ways: both of the constituent living and death benefits or just one of them can be linked to a fund account. In our case, only the death benefit is invested in a fund and so the premium can be ideally decomposed into a sterling part and a unit part:

$$P = P' + P'' \tag{3.3}$$

where P' is the sterling part, relating to the annuity, and P'' is the unit part, relating to the GMDB option and which is invested in a fund.
Let r be a valuation date at which we estimate the surplus linked to this contract.
Let $RC_j(r)$ be the net cash flow at time j for $0 \leq j \leq r$; it is called *retro-*

spective cash inflow at time r. It is given by:

$$
\begin{aligned}
RC_j^{(r)} &= \sum_{i=1}^{m}[P\mathbf{1}_{\{j=0\}} - R\alpha_{i,j}\mathbf{1}_{\{j>0\}} - G_j\delta_{i,j}\mathbf{1}_{\{j>0\}}] = \\
&= mP\mathbf{1}_{\{j=0\}} - R(\sum_{i=1}^{m}\alpha_{i,j})\mathbf{1}_{\{j>0\}} - G_j(\sum_{i=1}^{m}\delta_{i,j})\mathbf{1}_{\{j>0\}} = \\
&= mP\mathbf{1}_{\{j=0\}} - R\alpha_j\mathbf{1}_{\{j>0\}} - G_j\delta_j\mathbf{1}_{\{j>0\}}
\end{aligned}
\tag{3.4}
$$

where

$$
\alpha_{i,j} = \begin{cases} 1 & \text{if policyholder i is alive at time j} \\ 0 & \text{otherwise} \end{cases};
$$

$$
\delta_{i,j} = \begin{cases} 1 & \text{if policyholder i is died at time j} \\ 0 & \text{otherwise} \end{cases};
$$

α_j is the number of people from the initial group of m policyholder who survive to time j and δ_j is the number of deaths in year j. Let m_r be the size of the portfolio at time r; for $0 < j \leq n - r$ we have:

$$
\{\alpha_j|\alpha_r = m_r\} \approx BIN(m_r,_j\, p_{x+r}\}
$$

$$
\{\delta_j|\alpha_r = m_r\} \approx BIN(m_r,_{j-1}\, q_{x+r}\}
$$

We consider $r = 0$, since we study all cash flows as viewed from time 0. We have for $k < j$:

$$
E_0[\alpha_{i,j}] = m_j p_{x_i}
$$

$$
E_0[\delta_{i,j}] = m_{j-1|}q_{x_i}
$$

$$
Var_0[\alpha_{i,j}] = m_j p_{x_i}(1 -_j p_{x_i})
$$

$$
Var_0[\delta_{i,j}] = m_{j-1|}q_{x_i}(1 -_{j-1|} q_{x_i})
$$

$$
Cov_0[\alpha_{i,k}, \alpha_{i,j}] = m_j p_{x_i}(1 -_k p_{x_i})
$$

$$
Cov_0[\delta_{i,k}, \delta_{i,k}] = -m_{j-1|}q_{x_i k-1|}q_{x_i}
$$

$$Cov_0[\delta_{i,j}, \alpha_{i,j}] = -m_{j-1|}q_{x_i j}p_{x_i}$$

$$Cov_0[\delta_{i,k}, \alpha_{i,j}] = -m_{k-1|}q_{x_i j}p_{x_i}$$

$$Cov_0[\alpha_{i,k}, \delta_{i,j}] = -m(1 -_k p_{x_i})_{j-1|}q_{x_i}$$

Calculation of the cash flow moments is straightforward. Under the reasonable assumption of independence between G_j and δ_j or α_j we have:

$$E[RC_j^{(r)}] = mP\mathbf{1}_{\{j=0\}} - RE[\alpha_j]\mathbf{1}_{\{j>0\}} - E[G_j]E[\delta_j]\mathbf{1}_{\{j>0\}} \tag{3.5}$$

where $E[G_j] = E[Max(e^{gj}V_0, V_j)]$.
Moreover, we can calculate the variance of the retrospective cash flow:

$$Var[RC_j^{(r)}] = R^2 Var[\alpha_j]\mathbf{1}_{\{j>0\}} + Var[G_j\delta_j]\mathbf{1}_{\{j>0\}} + 2RCov[\alpha_j, G_j\delta_j]\mathbf{1}_{\{j>0\}} \tag{3.6}$$

and the covariance of the retrospective cash flows:

$$\begin{aligned} Cov[RC_k^{(r)}; RC_j^{(r)}] &= R^2 Cov[\alpha_k, \alpha_j] + Cov[G_k, \delta_k, G_j, \delta_j] \\ &+ RCov[\alpha_k, G_j\delta_j] + RCov[\alpha_j, G_k\delta_k] \end{aligned} \tag{3.7}$$

Now we fix our attention on the time period after r. Let $PC_j(r)$ be the net cash flow plus the value of the shares invested in the fund that occurs j time units after r for $0 \leq j \leq n - r$, where n is the final age underlying the life table; this is called the *prospective cash outflow* at time r. It is given by:

$$\begin{aligned} PC_j^{(r)} &= \sum_{i=1}^{m} [R\alpha_{i,(r+j)}\mathbf{1}_{\{j>0\}} + G_{r+j}\delta_{i,(r+j)}\mathbf{1}_{\{j>0\}}] = \\ &= R(\sum_{i=1}^{m} \alpha_{i,(r+j)})\mathbf{1}_{\{j>0\}} + G_{r+j}(\sum_{i=1}^{m} \delta_{i,(r+j)})\mathbf{1}_{\{j>0\}} = \\ &= R\alpha_{r+j}\mathbf{1}_{\{j>0\}} + G_{r+j}\delta_{r+j}\mathbf{1}_{\{j>0\}} \end{aligned} \tag{3.8}$$

We can derive formulae for the moments of the cash flow in the same manner as before:

$$E[PC_j^{(r)}] = RE[\alpha_{r+j}]\mathbf{1}_{\{j>0\}} - +E[G_{r+j}]E[\delta_{r+j}]\mathbf{1}_{\{j>0\}} \tag{3.9}$$

$$\begin{aligned} Var[PC_j^{(r)}] = R^2 Var[\alpha_{r+j}]\mathbf{1}_{\{j>0\}} + Var[G_{r+j}\delta_{r+j}]\mathbf{1}_{\{j>0\}} + \\ + 2RCov[\alpha_{r+j}, G_{r+j}\delta_{r+j}]\mathbf{1}_{\{j>0\}} \end{aligned} \tag{3.10}$$

$$\begin{aligned} Cov[PC_j^{(r)}; PC_k^{(r)}] = R^2 Cov[\alpha_{r+k}, \alpha_{r+j}] + Cov[G_{r+k}, \delta_{r+k}, G_{r+j}, \delta_{r+j}] + \\ + RCov[\alpha_{r+k}, G_{r+j}\delta_{r+j}] + RCov[\alpha_{r+j}, G_{r+k}\delta_{r+k}] \end{aligned} \tag{3.11}$$

Next, we introduce two random variables, the retrospective gain and the prospective loss, which will be used to define the surplus.

3.2.1 The Restrospective Gain and the Prospective Loss

The *Retrospective Gain* at time r is the difference between the accumulated value to time r of past premiums collected and benefits paid. It can be expressed in terms of $RC_j(r)$ as follows:

$$RG_r = \sum_{j=0}^{r} RC_j^r e^{I(j,r)} \tag{3.12}$$

where $I(s,r)$ denotes the force of interest accumulation function between times s and r if $0 \leq s \leq r$ and the force of interest actualization function if $r \leq s \leq n-r$; it is given by:

$$\begin{cases} \sum_{j=s+1}^{r} \lambda(j) & \text{if } s < r \\ 0 & \text{if } s = r \\ -\sum_{j=s}^{r+1} \lambda(j) & \text{if } s > r \end{cases}$$

and $\lambda(j)$ is the force of interest in period $(j-1, j]$.
It is reasonable to assume independence between the fund value and interest rate. Thus, we obtain:

$$E[RG_r] = \sum_{j=0}^{r} E[RC_j^r]E[e^{I(j,r)}] \tag{3.13}$$

$$
\begin{aligned}
Var[RG_r] = E[RG_r^2][E[RG_r]]^2 = \\
&= \sum_{k=0}^{r}\sum_{j=0}^{r} E[RC_k^r RC_j^r]E[e^{I(k,r)+I(j,r)}] - \{\sum_{j=0}^{r} E[RC_j^r]E[e^{I(j,r)}]\}^2 = \\
&= \sum_{k=0}^{r}\sum_{j=0}^{r}\{Cov[RC_k^r RC_j^r] + E[RC_k^r]E[RC_j^r]\}E[e^{I(k,r)+I(j,r)}] + \\
&- \{\sum_{j=0}^{r} E[RC_j^r]E[e^{I(j,r)}]\}^2
\end{aligned}
\tag{3.14}
$$

The *Prospective Loss* at time r is the difference between the discounted values to time r of future benefits to be paid and premiums to be collected (although, in this case, there are no future premiums since the contract has a single premium at time 0). The prospective loss can be expressed in terms of $PC_j(r)$ as follows:

$$
PL_r = \sum_{j=0}^{n-r} PC_j^r e^{I(r,r+j)} \tag{3.15}
$$

The moments of PL_r can be calculated in a similar way to the moments of RG_r.

3.2.2 The Surplus

Following Lysenko and Parker (Cf. [19]), we define the net stochastic *Surplus* as the difference between the Retrospective Gain and the Prospective Loss:

$$
S_r = RG_r - PL_r = \sum_{j=0}^{n} FC_j^r e^{I(j,r)} \tag{3.16}
$$

where FC_j^r is the generic cash flow (outflow or inflow) at time j.
Thanks to our previous results, we can calculate the expected value and variance of surplus per policy:

$$
E[S_r/m] = E[RG_r/m] - E[PL_r/m] = \frac{1}{m}\sum_{j=0}^{n} E[FC_j^r]E[e^{I(j,r)}] \tag{3.17}
$$

$$Var[S_r/m] = Var[\sum_{j=0}^{n} \frac{FC_j^r e^{I(j,r)}}{m}] =$$
$$\frac{1}{m^2}\{\sum_{j=0}^{n} Var[FC_j^r e^{I(j,r)}] + \sum_{j=0}^{n}\sum_{\substack{k=0\\k\neq j}}^{n} Cov(FC_j^r e^{I(j,r)}, FC_k^r e^{I(k,r)})\} \tag{3.18}$$

In Appendix B, we develop the above formulae.

3.3 The financial hypothesis

Following the standard assumptions in the literature, we model the evolution of the account value as in the previous chapter in equation (2.2):

$$dV_t = (\mu - \eta)\, V_t dt + \sigma V_t dW_t$$

Since W_t is a standard Brownian motion, it follows that:

$$E_0[V_j] = V_0 exp\{(\mu - \eta)j\}$$

$$E_0[V_j^2] = V_0^2 exp\{2(\mu - \eta)j + \sigma^2 j\}$$

$$E_0[G_j] = E[Max(e^{gj}V_0, V_j)] = Max(e^{gj}V_0, E_0[V_j])$$

$$Var[G_j] = Var[V_j]$$

$$Cov[G_j, G_k] = Cov[V_j, V_k] = 0$$

Moreover, we model the force of interest by a conditional autoregressive process $AR(1)$, given the force of interest at time zero. This model is considered by Bellhouse and Panjer (cf. [4]) and Marceau and Gaillardetz (cf. [15]). Let $\lambda(t)$ the force of interest in the period $(t-1, t]$:

$$\lambda(k) - \lambda = \phi[\lambda(k-1) - \lambda] + \gamma\epsilon(k) \tag{3.19}$$

where $\{\epsilon(k)\}$ is a sequence of independent and identically distributed standard normal variables and λ is the long term mean of the process. We assume

$|\phi| < 1$ to ensure the process is stationary in covariance. The moments of the accumulation function are derived in Cairns and Parker (cf. [5]):

$$E[I(s,r)|\lambda(0)=\lambda_0] = (r-s)\lambda + \frac{\phi}{1-\phi}(\phi^s - \phi^r)(\lambda_0 - \lambda) \tag{3.20}$$

$$\begin{aligned} Var[I(s,r)|\lambda(0)=\lambda_0] = \frac{\gamma^2}{1-\phi^2}[r-s+\frac{2\phi}{1-\phi}(r-s-1-\frac{\phi}{1-\phi}(1-\phi^{(r-s-1)}))+ \\ -(\frac{\phi}{1-\phi})^2(\phi^s-\phi^r)^2] \end{aligned} \tag{3.21}$$

In order to derive the covariance between two cash flows at time s and t, we consider three cases:

1. $s < t < r$: both cash flows occur before r. In this case the covariance between the accumulation functions is:

$$\begin{aligned} Cov[I(s,r), I(t,r)] &= \sum_{i=s+1}^{r} \sum_{j=t+1}^{r} Cov[\lambda(i), \lambda(j)] = \\ &= Var[I(t,r)] + \frac{\gamma^2}{1-\phi^2}\frac{\phi}{(1-\phi)^2}(\phi^t - \phi^r)(\phi^{-t} - \phi^{-s}) \end{aligned} \tag{3.22}$$

2. $r < s < t$: both cash flows occur after r. In this case the covariance between the actualization functions is

$$Cov[I(r,s), I(r,t)] = Var[I(r,s)] + \frac{\gamma^2}{1-\phi^2}\frac{\phi}{(1-\phi)^2}(\phi^s - \phi^t)(\phi^{-s} - \phi^{-r}) \tag{3.23}$$

3. $s < r < t$: one cash flow occurs before r and the other one after r. In this case

$$Cov[I(r,s), I(r,t)] = \frac{\gamma^2}{1-\phi^2}\frac{\phi}{(1-\phi)^2}(\phi^r - \phi^t)(\phi^{-r} - \phi^{-s}) \tag{3.24}$$

The conditional covariance terms can be obteined applying the multivariate normal theory (Cf. [14]); the results are shown in Lysenko and Parker

(see[13]).
In the following, we assume the independence between the fund and the force of interest.

3.4 Numerical Results: the first two moments of the Surplus

In this section, we apply the model and show numerical results for a portfolio of identical Variable Annuities with a GMDB option. We consider a group of 1000 policyholders aged 50 with the same risk characteristics, whose survival probability distributions are independent and identically. The mortality table used in our calculation is the SIM2002 based on the Italian male population, with the maximum age $n = 110$. We set $R = 1$, the technical rate used by the insurer in order to price the product $i = 0.04$, the guaranteed rate $g = 0.04$, the drift of the fund process minus the fee $\mu - \eta = 0.06$, the fund volatility $\sigma^2 = 0.03$. Under this hypothesis, the premium calculated according to the equivalence principle is equal to 17, where the sterling part P' is equal to 16 and the unit part $P'' = D_0 = V_0$ is equal to 1 (see eq. (2.14)).
In order to study the first two moments of the suplus, we need to determine the hypothesis concerning the force of interest. In this regard, we set $\lambda = 0.06$, $\lambda_0 = 0.05$, $\phi = 0.8$, $\gamma = 0.01$.
We have carried out 100000 simulations. Figure 3.1 shows the expected value and the variance of the surplus per policy at different dates r: We note that, as the valuation date increases, the standard deviation of the surplus increases. In order to understand this, we have to consider that the standard deviation of the surplus is affected by the uncertainty about the cash flows following the premium and by the variance of the interest rate. When r increases, we have to accumulate a greater number of retrospective cash flows for a longer time and discount a smaller number of prospective cash flows for a shorter period. Consequently, the variance of the capitalized cash gains increases and that of the discounted losses decreases. Numerical

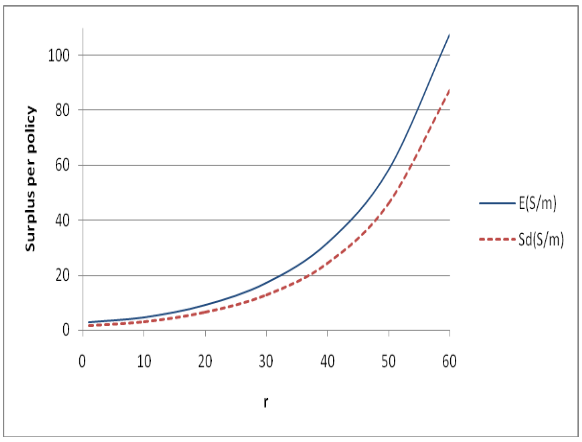

Figure 3.1: Expected Value and Variance of the Surplus per policy

investigation shows that the first effect prevails over the second one.

3.5 Distribution Function of Surplus: a stochastic approach

In the previous section, we have studied the first two moments of the stochastic surplus for a homogeneous portfolio of Variable Annuity contracts with GMDB options. Although the analysis of moments is useful, it is only the first step towards exploring the random behaviour of the surplus. We note that the standard deviation as a risk measure is inappropriate when dealing with asymmetric distributions and it is necessary to study the whole probability function of surplus. Lysenko and Parker ([13]) suggest a recursive method to construct this distribution; the complexity of the product we are considering makes necessary a simulation approach.

One of the objectives of this study is to assess the probability of insolvency, i.e. the probability that it will fall below zero. In order to achieve this purpose, we simulate the evolution of surplus under a mortality and financial stochastic model. Unlike the approach of Lysenko and Parker ([13]), we do not approximate the true probability function of surplus by its limiting dis-

tribution, which takes into account the investment risk but treats cash flows as given and equal to their expected value. Instead, in order to take account also the longevity risk, we simulate the impact of both financial and mortality factors on retrospective gains and prospective losses.
The financial assumptions are the same as described previously. Also, we need a mortality assumption in order to avoid underestimation or overestimation of the surplus. In this respect, we consider the stochastic model suggested by Cox and Lin ([8]) and developed by Ballotta, Esposito and Haberman (cf. [1]); it has been described in Section 2.5.
We carry out 100000 simulations under different financial and mortality hypotheses. The results are shown in the following three subsections concerning the interest rate risk, the fund risk and the mortality risk.

3.5.1 Interest Rate Risk: Numerical Results

We construct the distribution of the surplus per policy at different valuation dates ($\{r = 1, 10, 20, 30, 40, 50, 60\}$) under the hypothesis of the previous section: $g = 0.04$, $\mu - \eta = 0.06$, $\sigma^2 = 0.03$, $\lambda = 0.06$, $\lambda_0 = 0.05$, $\phi = 0.8$, $\gamma = 0.01$. Results are shown in Figure 3.2 and summarized in Table 3.1 We want to verify what happens under a different scenario for the force of

	S1	**S10**	**S20**	**S30**	**S40**	**S50**	**S60**
Min.	-7.116	-8.306	-14.194	-19.904	-36.40	-83.98	-188.80
1st Qu.	1.722	2.615	4.637	8.167	14.65	26.26	47.16
Median	2.833	4.698	8.586	15.590	28.19	50.89	92.25
Mean	2.712	4.767	9.181	17.209	31.80	58.50	107.72
3rd Qu.	3.832	6.828	13.063	24.427	44.90	82.29	150.80
Max.	7.807	20.882	50.022	137.021	289.38	523.09	1057.44

Table 3.1: Summary of the surplus distribution per policy

interest. In particular, we investigate the effect of a reduction in the long mean of the force of interest. We compare the distribution of the Surplus per policy at the valuation date $r = 1$ under the scenarios summarized in Table 3.2. The distributions of surplus are shown and compared in the next

	Scenario I	**Scenario II**
λ	0.06	0.04
λ_0	0.05	0.05
ϕ	0.8	0.8
γ	0.01	0.01

Table 3.2: Scenarios for the force of interest

tables. As the long rate of return of the assets in which the insurer invests

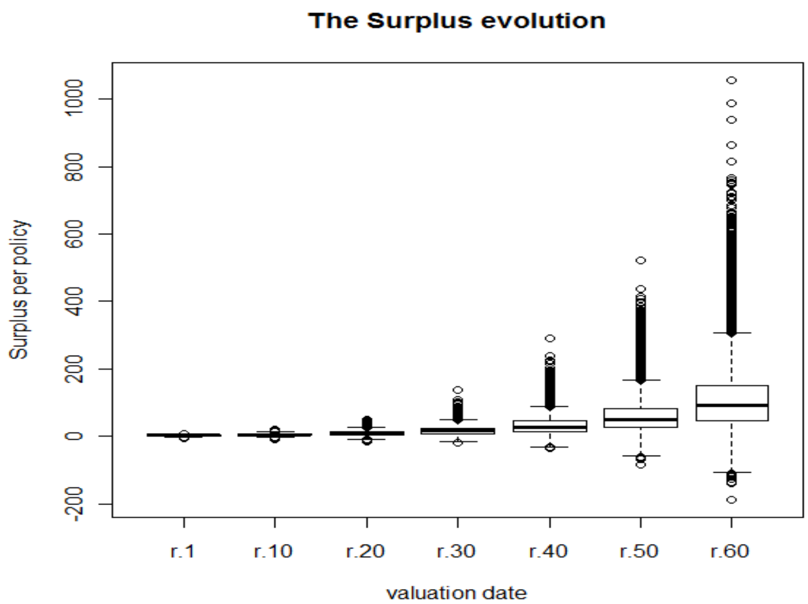

Figure 3.2: Boxplot of surplus per policy at different valuation dates

premium decrases, the cumulative distribution of the surplus moves to the left and, consequently, the probability of insolvency increases.
This comparison highlights the importance of a correct investment strategy

	Scenario I	**Scenario II**
$Prob(S(1)/1000) \leq 0$	**5.48%**	**64.35%**
Quantile 1%	-1.5826	-7.1447
Quantile 5%	-0.0938	-4.9480
Quantile 10%	0.6194	-3.9147
Quantile 90%	4.6504	1.7327
Quantile 95%	5.1123	2.3662
Quantile 99%	5.9288	3.4636

Table 3.3: Quantiles of the Surplus distribution: Interest Risk

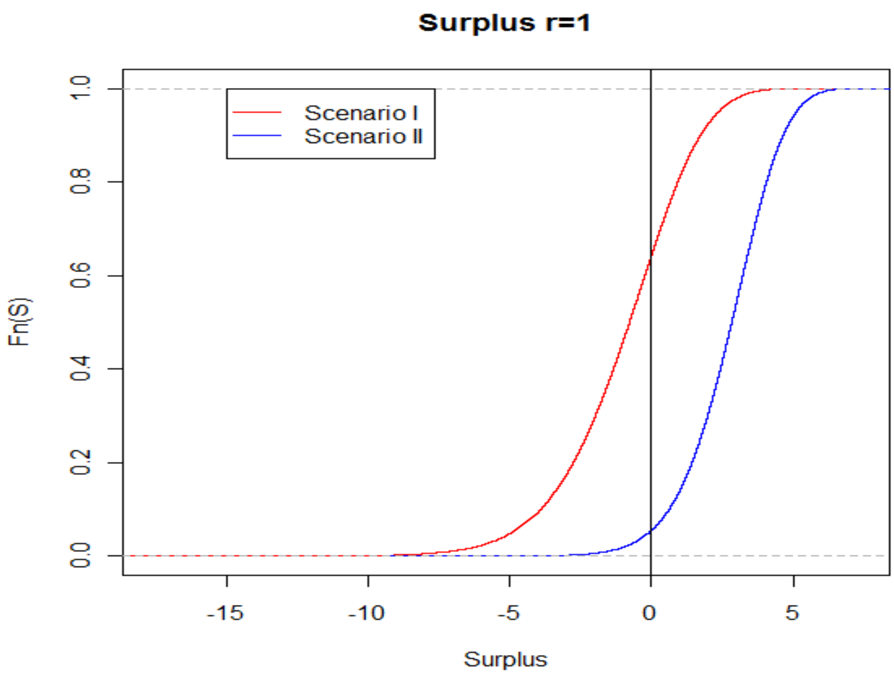

Figure 3.3: The Surplus per policy at r=1 under different scenarios for the forces of interes

in order to avoid the insolvency. In this case, the insurer has to invest the collected premiums into assets with a long mean of the rate of return equal to 0.06 in order to have a positive surplus since the first year and not ask other money to shareholders. We note that if the insurer invests the premiums into assets that in mean yield a return equal to the guaranteed rate on GMDB the probability of insolvency at $r = 1$ is 64.35%.

3.5.2 Fund Risk: Numerical Results

In this section, we study the effect of shifts in the distributions of fund value. As we wish to produce a sensitivity analysis, we fix the hypothesis concerning the interest rate distribution according the parameters used in section 3.5 and change those concerning the fund. In particular, we compare the surplus distributions under the four scenarios summarized in Table 3.4. The results are summarized in Table 3.5 As expected, as the volatility of the

Scenario	**I**	**II**	**III**	**IV**
$\mu - \eta$	0.06	0.06	0.06	0.05
σ^2	0.04	0.04	0.03	0.03
g	0.04	0.04	0.05	0.04

Table 3.4: Scenarios for the Fund Process

fund increases the variance of the surplus increases and as the guaranteed rate increases the mean of the surplus distribution decreases. Moreover, as the drift of the fund process decreases, the distribution of the surplus moves on the right, as shown in Figure 3.4, because the amounts of death benefits paid decrease.

3.5.3 Mortality Risk: Numerical Results

In this section, we study the effect of shifts in the parameters of stochastic mortality model. In the same manner of the previous sections, we aim to

Scenario	**I**	**II**	**III**	**IV**
Min.	19.904	-40.903	-20.354	-14.547
1st Qu.	8.167	8.013	7.611	9.485
Median	15.590	15.434	15.019	16.901
Mean	17.209	17.022	16.642	18.532
3rd Qu.	24.427	24.251	23.849	25.726
Max.	137.021	136.487	136.568	138.754

Table 3.5: Quantiles of the Surplus distribution: Fund Risk

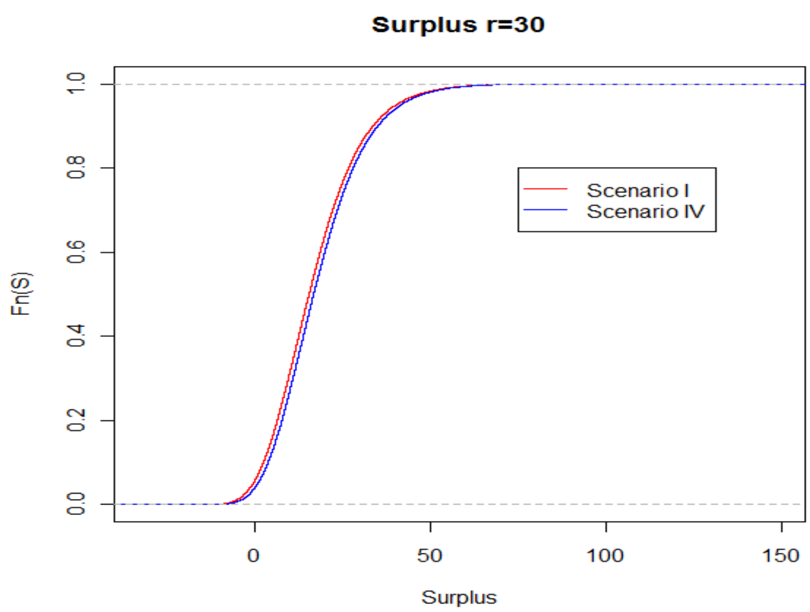

Figure 3.4: The distribution of surplus per policy under two different hypothesis for the fund process

produce a sensitivity analysis, and so we fix the hypothesis concerning the interest rate and the fund evolution as in Section 3.5 and change the mortality table. In particular, we use the mortality model described and set $a = 0.5$ and $b = 4.5$. We evaluate the surplus per policy at $r = 30$. We consider three cases for the value of c: $c = \{0, 0.5, 1\}$. In the first case, there will be improvements in life expectancy at every date; in other words, all of the shocks are expected to be positive. Conversely, in the second case further improvements of an already high expectancy of life are impossible and all shocks are expected to be negative.
As c increases, the outflows linked to the annuity increase; moreover, the payments related to the GMDB option increase too, because they are rolled over and they are linked to a fund value that increases with time. Consequently, the cumulative distribution of the surplus moves on the left and the probability of insolvency increases. These results are illustrated in Figure 3.5 and Table 3.6.

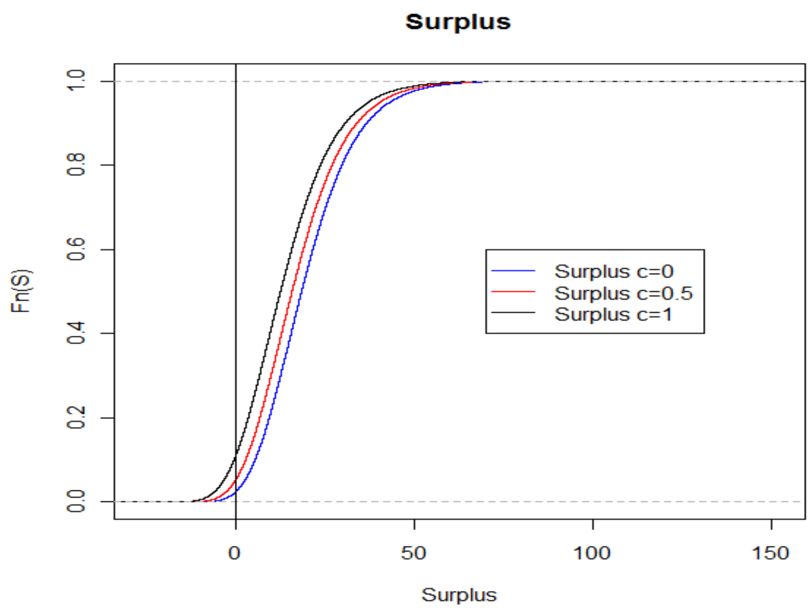

Figure 3.5: The distribution of the Surplus per policy under different mortality hypothesis

	c=0	**c=0.5**	**c=1**
$Prob(S(r)/1000) \leq 0$	**2.36%**	**5.17%**	**10.85%**
Quantile 1%	-2.4539	-5.1057	-8.2820
Quantile 5%	2.5213	-0.1077	-3.3526
Quantile 10%	5.4728	2.8347	-0.3719
Quantile 90%	36.7409	34.009	30.7354
Quantile 95%	43.3471	40.6048	37.3256
Quantile 99%	57.6183	54.8743	51.6409

Table 3.6: Quantiles of the Surplus distribution: Mortality risk.

3.6 Conclusions

The surplus is an important indicator of an insurance company's financial position and there exists a considerable actuarial literature on the topic (cf. [7], [11], [13], [15], [20]). The contribution of this paper has been to analyze the behaviour of the insurance surplus for a portfolio of Variable Annuities with GMDB options. In order to achieve this purpose, we have simulated the evolution of the surplus under a mortality and financial stochastic model. We believe that the paper is useful in enhancing an insurer's understanding of the stochastic behaviour underlying a Variable Annuity product with a GMDB option and that it provides the first study of the surplus in respect of this recently developed insurance product. Indeed, up to this time, the literature has offered only pricing models for GMDB, but has not studied the evolution of cash flows. We deem this consideration is important in the perspective of the liquidity and insolvency risk management. We have considered both financial and mortality risk and have outlined a comprehensive description of the interaction of different risk factors on the GMDB value.As general rule, if a death benefit is added to an annuity, there is a sort of "mortality natural hedging effect", i.e. the impact of longevity risk is reduced because the annuity is paid for a longer period but the actual value of the

death benefit decreases. The GMDB options can represent an exception to this rule; in Section 3.6 we have shown that as the estimated life extends the outflows linked to the annuity increase and at the same time the payments related to the GMDB option increase too, because they are rolled over and they are linked to a fund value that increases with time. Therefore, under the hypothesis of a growing fund, the effect of "natural hedging" is nullified. Hence, it is not sufficient to study the impact of each risk factor on the GMDB value, but it is necessary to examine their interaction.
In the paper, numerical examples show a significant impact of the interest, fund and mortality risks on the surplus distribution, insomuch as the insolvency probability increases considerably in many cases. With regard to this point, an advantage of the model used is that it allows an ex ante assessment of the insurer's solvency throughout the duration of contract. Consequently, a change to the design of the product can be made, and, in particular, the premium can be modified according to the forecasts of mortality probabilities, interest rate and fund evolution. Moreover, the model enables us to determine the premium that leads to a required probability of insolvency, and so it can be used for an evaluation of the adequacy of solvency, which is consistent with recent regulatory changes.

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Chapter 4

The GLWB Option

4.1 Introduction

In 2002 Hartford issued a new type of Guaranteed Minimum Living Benefit Option: the Guaranteed Minimum Withdrawal Benefit (GMWB), which gives the insured the possibility to withdraw a pre-specified amount annually, even if the account value has fallen below this amount. In 2004, each of the 15 largest Variable Annuity providers offered this guarantee and 69% of the Variable Annuities sold included a GMWB option; in 2007 the percentage was 86% (cf. [18]). The latest GMLB option is the Guaranteed Minimum Withdrawal Benefit for Life or Guaranteed Lifelong Withdrawal Benefit option (GLWB). As the name suggests, it offers a lifelong withdrawal guarantee; therefore, there is no limit for the total amount that is withdrawn over the term of the policy, because if the account value becomes zero while the insured is still alive ha can continue to withdraw the guaranteed amount annually until death. The first VA with a withdrawal benefit guaranteed for the life was introduced in the U.S.A. market in 2003. Since 2006 nine of ten VA products offered guaranteed living benefit; GLWB options captured some GMIB markets and represented the 35% of the whole market in early 2006 (cf. [1]).

In the light of the growing diffusion of the GLWBs, the aim of this chapter is twofold: on one hand, we intend to develop a pricing model and define a fair price for a GLWB in a market consistent manner; on the other hand we want to verify if the current GLWB price on the USA market is fair. In order to achieve the first objective, our work use the standard No-arbitrage models of mathematical finance, in line with the tradition of Boyle and Schwartz (cf. [5]) that extend the Black-Scholes framework (cf. [4]) to insurance contract. The main difference is that for the option embedded in VA products the fee is deducted ongoing as fraction of asset, instead in the Black and Scholes approach the premium is paid up-front. The approach follows the recent actuarial literature on the valuation of VA products (cf. [2]; [7]; [9]; [16]; [21]; [19]; [22]). In order to price options embedded in Variable Annuity contracts many authors use numerical PDE methods (cf. [11]; [7]; [21]; [22]; [24]), others exploit Monte Carlo simulations (cf. [19]; [20]). We choose to follow the latter approach. Therefore, we develop two extreme valuation formula in order to price the GLWB option, both within the framework of No Arbitrage pricing (see [3]). First, we adopt a static approach that assumes policyholders take a static strategy, i.e. the withdrawal amount is always equal to the guaranteed amount. One of our main contributions is to show that in the static case the product can be decomposed into a life annuity plus a portfolio of Quanto Asian Put Options, with decreasing strikes and increasing expiration dates (cf. [25]). We believe that this decomposition has not been previously proposed in actuarial literature. In this regard, Milevsky and Salisbury ([22]) decompose the GMWB option into a Quanto Asian Put plus a generic term-certain annuity. Our paper differs from that of Milevsky and Salisbury since the lifelong guarantee of GLWB makes necessary the introduction of the survival probabilities in the pricing model. We show that the weights of the composition of the portfolio consisting of many put options are the deferred probabilities of death. In the second approach, we describe the GLWB payoff if the policyholder assumes a dynamic strategy, according which he can lapse (i.e. withdraw more or less than the guaranteed amount from the found) and surrender the contract when he prefers.

Milevsky and Salisbury ([22]) prove that for a GMWB policyholder can be optimal to withdraw either nothing or the guaranteed amount or the total account value. Instead, Holz et al ([16]) show that for a GLWB withdrawing nothing can never be optimal, unless roll-ups or other options are included, and the rational policyholder withdraws the amount guaranteed until he decides to surrender. Therefore, in this dynamic approach we deal with an optimal stopping problem; we solve it with the definition of a probability function of the optimal surrender time and its construction on a practical side with a Monte Carlo simulation (see [13]). As far as it is within our knowledge, we think this procedure has not been formerly used in the pricing of the option embedded in the VA contracts.

In order to achieve the second objective, we develop an application of our model to the US market and derive the fair insurance for an illustrative policyholder in both the static and dynamic cases. In line with the actuarial literature, we think that the real fee has to lie between the static and dynamic embedded option cost; in fact, the policyholder can behave with a high level of irrational lapsation. Chen et al. ([7]) and Ho ([15]) note that sub-optimal policyholder behavior considerably reduces the value of the GMWB rider. Cramer et al. ([10]) suggests to describe the sub-optimal policyholder behavior with a function of how much the embedded option is in the money. Wang (cf. [28]) offers a dynamic lapse function that more reduces the lapse when the GLWB is more in-the-money. However, our conclusion is that the GLWB issued on the USA market are underpriced and this appears regardless of whether we take a static or dynamic approach. For example, our numerical results show that the No Arbitrage cost of a GLWB issued to a policyholder aged 60 would range between 79 and 145 basis points assuming a sub-account volatility in line with the average of the sub-account volatility for the universe of variable annuity products, while most products in the US market only charge 50-70 basis points. Our results are in contrast to the common belief that the guarantees embedded in VA contracts are all overpriced (cf. [8]); similar conclusions have been proposed for other options:

Milevsky and Salisbury ([22]) show that GMWB are underpriced on the US market; also Chen et al. ([7]) verify that the market fee are inadequate if the underlying risky asset follows a jump diffusion process.
The remainder of the chapter is organized as follows. In Section 4.2 we describe the product. Section 4.3 develops the stochastic model for the pricing of a GLWB option under the No Arbitrage approach. In Section 4.4 we trace the static framework and decompose the product into a life annuity plus a portfolio of Quanto Asian Put Options; some numerical results are offered in Section 4.5. In the same way, Section 4.6 develops the dynamic framework and Section 4.7 provides some numerical results. Concluding remarks are offered in Section 4.8.

4.2 Product description

The GLWB is the latest variant of the GMWB option recently introduced in US, Asia and Europe. Products with a GMWB option give the policyholder the possibility to annually withdrawal a certain percentage g of the single premium, that is invested in one or several mutual funds. The guarantee consists in the entitlement to withdrawal until an amount equal to the premium paid even if the account value falls to zero. Instead, if the account value does not vanish, at maturity the policyholder can take out or annuitize any remaining fund. Products with a GLWB option offers a lifelong guarantee: the maximum amount to be withdrawn is specified but the total amount is not limited and the insured can annually request a portion of the premium paid until he is still alive, even if the fund value drops to zero. Any remaining account value at the time of death is paid to the beneficiary as death benefit. The insurer charges a fee for this guarantee, which is usually a pre-specified annual percentage of the account value.

- in the case of a *Roll-up* option, then the annual guaranteed withdrawal amount is increased by a fixed percentage every year during a certain time period but only if the policyholder has not started withdrawing money. Therefore, (*Roll-ups*) are commonly used as a disincentive to withdraw during the first years;

- when the contract contains a *Step-up*,at pre-specified points in time *step-up dates* the guaranteed withdrawal amount is increased if the percentage g of the account value exceeds the previous guaranteed amount. Therefore, *Step-ups* occur if the fund has a high performance and the account value has not been overmuch decreased with previous withdrawals;

- in the case of a deferred version of the contract, the product is fund linked during the deferment and the account value at the end of this period, or a guaranteed amount if greater, is treated like a single premium paid for an immediate GLWB.

In order to explain the operation of a GMWB and GLWB options, we provide a numerical example. Let V_t the market value of the underling fund at time t and let 100 the initial investment, so $V_0 = 100$. In a typical GMWB contract without Roll-up or Step-up the amount guaranteed is gV_0, let g be equal to 7%. The guarantee continues until the 100 has been withdrawn, so the minimum period is $100/7 = 14.28$ years. If during this period Wt collapses to zero the investor will withdraw 7 per year until $T = 14.28$. Instead, if the account value does not vanish, at the expiration date the policyholder can take out or annuitize any remaining fund. Anyway, in any given year the policyholder is entitled to withdraw an amount less than 7 and extend the life of guarantee or an amount greater than 7 and reduce it. In a typical GLWB contract there is no limit for the total amount that is withdrawn over the term of the policy, because if the account value becomes zero while the insured is still alive he will continue to withdraw 7 until death. Any remaining account value at the time of death is paid to the beneficiary as death benefit. If the policyholder withdraws an amount less or greater than 7, this has effect not on the life of guarantee but on the death benefit linked to the account value.

4.3 The model

In the same manner of the previous chapters, let T be the future lifetime random variable expressed in continuous time and let its *cdf* be described by eq. 2.2.
Let $(\Omega, \mathcal{F}, P)$ the real real probability space. Let V_t be the account value at time t linked to fund value. Following the standard assumptions in the literature, we model the evolution of the fund as:

$$\begin{aligned} dV_t &= (\mu - \eta)V_t dt - \gamma_t dt + \sigma V_t dW_t \\ V_0 &= \omega_0 \end{aligned} \tag{4.1}$$

where W_t is a standard Brownian motion, μ is the drift rate, η is the insurance fee paid for the GLWB option, γ_t is the discretionary withdrawal from the account at time t, which can range from a low of zero to as high as the account value V_t. This dynamic model for the underlying investment is consistent with the actuarial literature on pricing insurance guarantees (cf. [7]; [12]; [22]; [29]).
We assume that exists a risk neutral probability space $(\Omega, \mathcal{F}, \{F_t\}_{t\geq 0}, Q)$with a filtration $\{F_t\}_{t\geq 0}$ and a risk neutral measure Q, under which payment streams can be valued as expected discounted value using the risk-neutral valuation formula (cf. [3]); existence of this measure implies the existence of an arbitrage free market. If we consider this new space, the evolution of the fund is:

$$dV_t = (r - \eta)V_t dt - \gamma_t dt + \sigma V_t dZ_t^Q$$

where r is the risk free rate and

$$Z_t = V_t + \frac{\mu - r}{\sigma} t$$

is a Brownian motion under Q.
The product offers a lifelong guarantee: the maximum amount to be withdrawn is specified but the total amount is not limited and the insured can annually request a portion of the premium paid until he is still alive, even if

the fund value drops to zero.
Let F_t be the value of the guarantee of minimum withdrawal at time t:

$$F_t = \begin{cases} 0 & \text{if } V_t \geq G \\ G - V_t & \text{otherwise} \end{cases}$$

where $G = gV_0$ be the minimum guaranteed withdrawal ang g the guaranteed rate.
Any remaining account value at the time of death is paid to the beneficiary as death benefit. Let DB_t be the death benefit account, where we credit the death benefit payment occurred between 0 and t and compounded with the risk-free rate up to t. Since the policyholder is alive at time zero we have $DB_t = 0$ for every $t \neq \tau$ where τ is the random time of death.

$$DB_t = max\{V_t, 0\}$$

4.4 Static modeling framework

In this section we describe the GLWB payoff if the policyholder assumes a static strategy, i.e. he withdraws the same amount $G = gV_0$ each year, so the evolution of the fund in the real probability space is described by the following SDE:

$$\begin{aligned} dV_t &= (\mu - \eta)V_t dt - G_t dt + \sigma V_t dW_t \\ V_0 &= \omega_0 \end{aligned} \tag{4.2}$$

Equation (4.2) holds while $V_t > 0$. We consider a simple form of product, without Roll-up option, Step-up or deferred period. The policyholder receives the amount guaranteed until he is still alive; moreover, at the date of death the beneficiary will receive any remaining account value. The discounted value at $t = 0$ of the GLWB $GLWB_0$ is the sum of the discounted values of the living and death benefits:

$$GLWB_0 = LB_0 + DB_0 \tag{4.3}$$

LB_0 is the well-know discounted value of a life annuity; DB_0 can be calculated considering the payoff that the beneficiary will receive at the random time of death τ. Since the maturity is stochastic and τ and V_τ are independent, the discounted value at $t = 0$ of the death benefit is given by the expectation under τ and V_τ:

$$DB_0 = E_t\{E\{e^{-rt}DB_\tau|\tau = t\}\} \tag{4.4}$$

If we fixe the date T, the death benefit can be calculated by Ito's lemma; the solution to equation (4.2) is:

$$DB_T = e^{(\mu-\eta-\frac{\sigma^2}{2})T+\sigma W_T} max[(\omega_0 - G\int_0^T e^{-(\mu-\eta-\frac{\sigma^2}{2})T-\sigma W_T} dt); 0] \tag{4.5}$$

The integral in equation (4.5) is monotonically increasing in T; thus, once it grows into the amount $\frac{\omega_0}{G}$ the fund value resets to zero and can never become positive, unlike the geometric Brownian motion. In like manner of Milevsky and Salisbury (cf. [22]), we describe the death benefit at time T with the payoff of a Quanto Asian Put (QAP) Option; in fact equation (4.5) can be rewritten as:

$$DB_T = e^{(\mu-\eta-\frac{\sigma^2}{2})T+\sigma W_T} GT max[(\frac{\omega_0}{GT} - \frac{1}{T}\int_0^T e^{-(\mu-\eta-\frac{\sigma^2}{2})T-\sigma W_T} dt); 0] \tag{4.6}$$

The option differs from that used by Milevsky and Salisbury for the strike. The QAP has a strike price equal to $\frac{\omega_0}{GT}$ and is defined on an underlying security Y_t, whose process is:

$$Y_t = e^{-(\mu-\eta-\frac{\sigma^2}{2})T-\sigma W_T} \tag{4.7}$$

Using a standard technique in literature, the No-arbitrage time-zero value of death benefit at time t is:

$$DB_0(\tau = T) = e^{-rT} E^Q[QAP_T] \tag{4.8}$$

where QAP_T is the value of a Quanto Asian Put in T:

$$E^Q[QAP_T] = E^Q[e^{(r-\eta-\frac{\sigma^2}{2})T+\sigma Z_T} GT max[(\frac{\omega_0}{GT} - \frac{1}{T}\int_0^T e^{-(r-\eta-\frac{\sigma^2}{2})T-\sigma Z_T} dt); 0]]$$

Q is the risk neutral measure (cf. [3]), r is the risk free rate and Z_t is a Brownian motion under Q

If we consider both the expectations in equation (4.4), we obtain:

$$DB_0 = \int_0^{\omega-x} f_x(t) E_0^Q(QAP_t) dt \tag{4.9}$$

where $f_x(t)$ has been defined in chapter 3.

In the discrete case we have:

$$DB_0 = \sum_{t=0}^{\omega-x} {}_t p_x q_{x+t} E_0^Q(QAP_t) \tag{4.10}$$

for a policyholder aged x at inception of the contract. Thus, the discounted value at $t = 0$ of the death benefit is a weighted average of the values of $\omega - x$ QAP options with decreasing strikes and increasing expiration dates, where the weights are the deferred probability of death in t, i.e. the probability of survival until t and death between t and $t + 1$.

The policyholder is also entitled to withdraw the guaranteed amount each year until he is still alive; therefore, the actual value of the GLWB option if the policyholder assumes a static strategy is:

$$GLWB_0 = \sum_{t=0}^{\omega-x} [{}_t p_x G e^{-rt} +_{t/1} q_x E_0^Q(QAP_t)] \tag{4.11}$$

Giving the value of other parameters, the fair insurance fee can be obtained making equal the initial price to the zero value of the future cash flows

$$\omega_0 = \sum_{t=0}^{\omega-x} [{}_t p_x G e^{-rt} +_{t/1} q_x E_0^Q(QAP_t)] \tag{4.12}$$

Our main contribution lies in bifurcating the GLWB option into a life annuity plus a portfolio of QAP option with decreasing strikes, where the weights of composition are the deferred probabilities of death (see eq.(4.11)).

4.5 Numerical Results: the static case

We apply our pricing model to GLWB options issued in the USA market, where the top 25 VA contracts with GLWB option sold in 2007 are cited in table 5.1.

The main product futures in this market are summarized in Table 4.2. The

Variable Annuity contracts with GLWB
American Skandia Highest Daily Lifetime Five Income Benefit
AXA Guaranteed Withdrawal Benefit for Life
Genworth RetireReady IRA Annuity
The Hartford's Lifetime Income Foundation/ Lifetime Income Builder II
ING LifePay Plus Minimum Guaranteed Withdrawal Benefit
Jackson National Life LifeGuard Freedom
John Hancock Income Plus For Life
Lincoln National SmartSecurity Advantage 1 Year
Lincoln Lifetime Income(SM) Advantage
MassMutual Lifetime Payment Plus
Merrill Lynch Life Variable Annuity Separate Account D GMWB for life
MetLife LWG II
Pacific Life Flexible Lifetime Income Rider/Foundation 10
Phoenix Lifetime GMWB 2007
Prudential Highest Daily Lifetime Seven
Protective SecurePay With Roll-up
RiverSource SecureSource
Sun Life Secured Returns for Life Plus Living Benefit Rider/Retirement Income Escalator
VALIC IncomeLock

Figure 4.1: USA GLWB products in 2007

guaranteed rate offered increases with the age of policyholder at the inception of the contract.

The insurance fee in the market ranges from 50 to 70 b.p. According to

Guaranteed Rate	Polycyholder's Age
5%	60-69
6%	70-79
7%	80-85

Table 4.1: The guaranteed rate of GLWB issued in USA

Morningstar Principia Pro, the average of the sub-account volatility for the universe of variable annuity products is 18%, the 25^{th} percentile is 16% and the 90^{th} percentile is 25We consider a policyholder aged 60 at the inception of the contract, the final age is $\omega = 110$; in order to price the GLWB option we use the latest USA mortality table downloaded from the Human Mortality Database. We set $\omega_0 = 100$ and $r = 0.05$. We carry out many Monte Carlo simulations generating for each of them 10000 paths of evolution of the fund. Let $P(\xi_t)$ the probability that V_t hits zero at some point $t < \omega - x$. The first step of our numerical application is to calculate the probability $P(\xi_t)$. We highlight that the guarantee to withdraw G per annum has a positive value if and only if the process V_t hits zero prior to time $\omega - x$; for those paths for which V_t hits zero after $\omega - x$ the withdrawal is satisfied "endogenously" from the fund, without an amount paid by the insurer. We compute the probability that the insurer has to pay the amount guaranteed, if the guaranteed rate is 5% and the insurance fee is 60 b.p., which are hypothesis consistent with the current market. The following table shows this probability simulated with the Monte Carlo method under a variety of real word drift and volatility assumption.

If the expected investment return is $\mu = 8\%$ and the volatility is $\sigma = 18\%$

$P(\xi)$	$\mu = 4\%$	$\mu = 6\%$	$\mu = 8\%$	$\mu = 10\%$	$\mu = 12\%$
$\sigma = 10\%$	52.1%	45.1%	39.2%	30.9%	24.6%
$\sigma = 15\%$	58.5%	52.9%	48.0%	43.7%	39.5%
$\sigma = 18\%$	61.9%	57.0%	52.1%	48.1%	44.5%
$\sigma = 20\%$	64.3%	60.0%	54.9%	51.0%	48.0%

Table 4.2: The probability of the insurer's payment

we find $P(\xi_t) = 52.1\%$; in other words, there is a probability of 47.9% that the insurer does not have to pay the amount guaranteed. If we, ceteris paribus, increase the volatility to $\sigma = 20\%$ the $P(\xi_t)$ increases to 54.9% and the probability that the insurer does not have to pay decreases to 45.1%. Instead if we increase the investment return and set $\mu = 12\%$ and $\sigma = 18\%$ we find $P(\xi_t) = 44.5\%$ and the probability that the insurer does not have to

pay increases to 55.5%. It is clear that a fair insurance fee has to increase when the volatility of sub-account increases. In order to reduce the fee and make competitive the product insurance companies could impose restriction on the asset allocation within the variable annuity contracts, investing part of the fund into fixed income bonds. Moreover, it is easy to verify that the probability $P(\xi_t)$ is increasing in the withdrawal rate if the age of policyholder at the inception of the contract does not change; in fact, in this case the life of contract does not change but V_t decreases more quickly and, consequently, it resets zero in a greater number of paths.
The second step of our numerical analysis is to calculate the fair insurance fee according the pricing model developing in the previous section. Our main result is the decomposition of the GLWB option into the sum of a life annuity paying G per annum until the policyholder is still alive and a portfolio of Quanto Asian Put with decreasing strikes on the reciprocal variable annuity account. Giving the value of the other parameters of the pricing model the insurance fee can be calculated by solving equation (4.12). In the static case, the time-zero value of the living benefit is not affected by the insurance fee; the costs of life annuities with different annual payments are summarized in Table 4.4.

For an initial deposit of $\omega_0 = 100$, an interest rate of $r = 0.05$ and a

g	**Cost of Annuity component**
0.04	46.17
0.05	57.71
0.06	69.26
0.07	80.79

Table 4.3: The cost of annuity component for a sixty-years old

guaranteed rate of $g = 0.04$ the cost of the annuity component is 46.17. The remaining 53.83 has to be put to purchase the portfolio of the Quanto Asian Put and the fee has to be calculated so that the fair price of the portfolio is exactly 53.83. Given the guaranteed rate, if the interest rate increases the cost of the annuity decreases and a greater amount has to invested in the portfolio. Instead, given the interest rate, if the guaranteed rate increases the weight of the annuity component increases and decreases that of the portfolio of Options. Table 4.5 displays the fair insurance fee under different guaranteed rate and sub-account volatility.

The results are obtained with Monte Carlo simulation yet again. Once the

g	$\sigma = 18\%$	$\sigma = 20\%$	$\sigma = 25\%$
0.04	43 b.p.	54.5 b.p.	83 b.p.
0.05	79 b.p.	96.5 b.p.	138 b.p.
0.06	143 b.p.	167 b.p.	226 b.p.
0.07	270 b.p.	308 b.p.	389 b.p.

Table 4.4: The impact of g and σ on the fair fee

interest rate, the volatility and the guaranteed rate have been fixed, we have searched the fair value of the fee with an iterative procedure: if the time-zero cost of the whole product turned out to be higher than ω_0 we increased the fee up to decrease the cost to ω_0; vice-versa, if the time-zero cost of the whole product turned out to be smaller than ω_0 we decreased the fee.

As expected, the fair guarantee is increasing in the volatility, because options are more expensive when volatility increases. In the same way, we can verify that if, ceteris paribus, we increase the interest rate the fair fee decrease because the risk neutral value of the guarantee decreases.

We pay attention to the fair insurance fee under the hypothesis $g = 5\%$ and $\sigma = 0.18$, which are consistent with the market (see Table 4.2 and remember the valuation of the Morningstar Principia Pro). In this case the fair insurance is equal to 79 b.p., whereas the current market fee ranges between 50 b.p. and 70 b.p. Although there is a common belief that the guarantees embedded in variable annuity policies are overpriced (see [8]), our analysis shows that the USA market of GLWB is underpriced, in line with

the results obtained by Milevsky and Salisbury (cf. [22]) for the GMWB market.
The last remark that we suggest is a comparison with the fair insurance fee for GMWB product calculated by Milevsky and Salisbury (2005). Figures 4.5. and 4.6 show the fair fee for GMWB and GLWB assuming $r = 5\%$ and $\sigma = 20\%$. As expected, the GLWB option is more expensive than the

	GMWB Maturity $\frac{1}{T}$	**GMWB Fair Fee**
g=4%	25	23 b.p.
g=5%	20	37 b.p.
g=6%	16.67	54 b.p.
g=7%	14.29	73 b.p.

Table 4.5: The fair fee for GMWB option

	GLWB Maturity $\omega - x$	**GLWB Fair Fee**
g=4%	50	54.5 b.p.
g=5%	50	95.5 b.p.
g=6%	50	167 b.p.
g=7%	50	308 b.p.

Table 4.6: The fair fee for GLWB option

GMWB because it has a longer maturity. The difference between the fee for the two products is increasing in the interest rate, because the maturity of the GMWB is decreasing in g, instead that of GLWB does not vary if g changes but only if the age of policyholder at the inception of the contract varies. Moreover, in the GMWB the total amount that can be withdrawn is restricted, whereas is not restricted in the GLWB. As a consequence, the influence of the guaranteed rate on the fee for a GLWB is considerably higher than in a GMWB contract.

4.6 Dynamic Model

In this section we describe the GLWB payoff if the policyholder assumes a dynamic strategy: he can lapse (i.e. withdraw more or less than the guaranteed amount from the found) and surrender the contract when he prefers. Recall that the most variable annuities impose a penalty if the product is lapsed or surrendered prior to maturity. Supposing a proportional penalty charge k is applied on the portion of γ above G, the net amount received by the policyholder is $\gamma - k(\gamma - G)$ if $\gamma > G$. Let $\ell(\gamma)$ the function of the cash flows received by the policyholder:

$$\ell(\gamma_t) = \begin{cases} \gamma_t & \text{if } 0 \leq \gamma_t \leq G \\ \\ G + (1-k)(\gamma_t - G) & \text{otherwise} \end{cases}$$

Following the notation in Holz at al. (cf. [16]), any withdrawal strategy can be described by using a withdrawal vector $\overline{\gamma} = (\gamma_1, ..., \gamma_T)$, where γ_t denotes the discretionary withdrawal amount at the year t, if the insured is still alive. A full surrender strategy at time t is represented by allowing. Every deterministic strategy is such that it is specified at time $t = 0$, so it is $F_0 - measurable$. Instead, the policyholder assumes a stochastic strategy if the decision whether and how much withdraw at time t depends on the account value and other information available at time t. Each stochastic strategy can be represented by a $F_t - measurable$ process (X). Therefore, the value of the contract following the stochastic strategy (X) is given by:

$$V_0((X)) = \sum_{t=1}^{\omega - x} \{ {}_{t-1}p_x q_{x+t-1} E_Q[e^{-\int_0^t r_s ds}(\gamma(t, X) + DB(t, X))] \} \qquad (4.13)$$

Let Ξ be the set of all admissible stochastic strategy. The value V_0 of the contract assuming a rational policyholder is

$$V_0 = \sup_{(X) \in \Xi} V_0((X)) \qquad (4.14)$$

In this optimization problem the control variable for the policyholder is γ_t ; he can choose to withdraw the guaranteed amount or less or more than it, considering that there is a positive relation between the amount withdrawn and the living benefit and a negative one between γ_t and the death benefit.
In contrast to a GMWB (see [22]), for a GLWB withdrawing nothing or less than G can never be optimal. In fact, for a GMWB this strategy extends the life of guarantee; instead, in a GLWB there is a lifelong guarantee and no adjustments are made for future guaranteed withdrawals. Hence, when the policyholder withdraws less than G, the future guarantees are the same, but their values are lower because V_t is greater. In addition, we have to consider that withdraw less than G involves a smaller γ_t and a greater D_t. However, due to the martingale property of the fund process and the fee deducted from the account value, the expected value of the additional death benefit is never greater than the withdrawal amount. So, the rational policyholder withdraws at least G.
In this thesis, we consider the basic form of the GLWB, without possibility of partial surrender. In this case, the the rational policyholder would withdraw exactly the annual guaranteed amount until the value of the fund less the penalty exceeds the value of future benefits; then, he would surrender the contract.
It has to be highlighted that this strategy is not necessarily optimal from the point of view of maximization of policyholder's utility function; it is the worst case for the insurer who has to hedge the issued policy, because it is the strategy that maximizes the zero-value of the contract for every possible scenarios.
In this dynamic approach we deal with an optimal stopping problem. We define the optimal time τ at which the rational policyholder surrenders the contract as:

$$\tau : V_\tau - k(V_\tau - G) \geq E_\tau^Q\{\sum_{t=\tau+1}^{\omega-x} [{}_t p_x G e^{-\int_\tau^t r_s ds} +_{t/1} q_x max(V_t; o) e^{-\int_\tau^t r_s ds}] | \mathcal{F}_\tau\} \tag{4.15}$$

The strategy adopted is stochastic and $F_\tau - measurable$: at each date the policyholder observes the fund value and consequently modifies the expectation of future benefits. The hypothesis at the basis of this model is a perfect information with respect to both financial and demographic risk factors. However, this hypothesis is not so strong if we analyze the scope of the model: it is not a true decision rule for the policyholder, but rather it is a way for the insurer to define the worst case in order to hedge the policy.
In order to price the GLWB option in a dynamic framework, we need to define a probability function of the optimal surrender time ; on a practical side, it can be constructed with a Monte Carlo simulation.
Let S_t be the event that the policyholder surrenders the contract at time t. In this dynamic contest the pricing formula turns into the following:

$$V_0 = \sum_{t=1}^{\omega-x}\{E_0^Q[\sum_{\tau=1}^{t-1}[LB_\tau + DB_\tau]|\overline{S}_\tau] + E_0^Q[{}_tp_x e^{-rt} max(V_t - k(V_t - G); 0)|S_t]\} \tag{4.16}$$

4.7 Numerical Results: the dynamic case

As in the previous numerical application, we consider a policyholder aged 60 at the inception of the contract and in order to price the GLWB option use the same USA mortality table. We set $\omega_0 = 100$ and $r = 5\%$; in addition, we set $k = 10\%$. The first step of this application is to construct the simulated probability function of the optimal surrender time. We carry out many Monte Carlo simulations under different values of the volatility and guaranteed rate, generating for each of them 1000 paths of evolution of the fund. For each path, at each date we control if the inequality (4.15) is verified; as soon as the value of the fund less the penalty exceeds the value of future benefits we stop the simulation of the evolution of the fund. Figures 4.2 and 4.3 show the simulated probability function of the surrender time:
As expected, the surrender time is increasing in the guaranteed rate: when the amount withdrawn is greater the policyholder aims to keep the guaran-

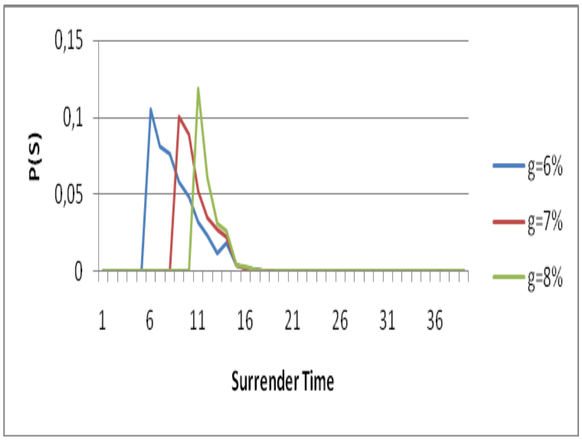

Figure 4.2: Distribution function of the surrender time ($r = 5\%$, $\sigma = 18\%$, $\delta = 60b.p.$)

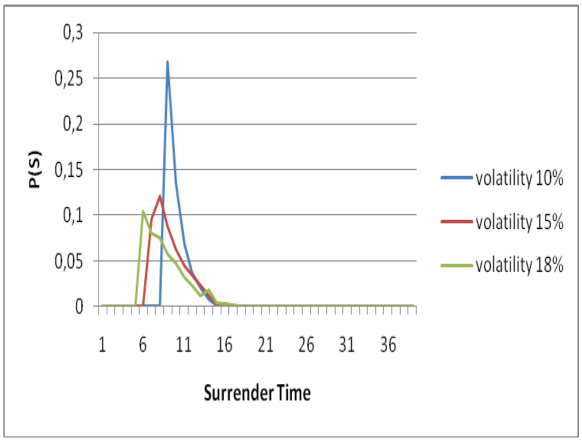

Figure 4.3: Distribution function of the surrender time ($r = 5\%$, $g = 6\%$, $\delta = 60b.p.$)

tee for a longer time. Moreover, when the volatility increases the fund is affected by larger variations and W_t hits zero in a short time; consequently, the probability of surrender decreases and the *pdf* moves to the left of the graph because if the policyholder surrenders he has to do it in a shorter time in order to avoid the zeroing of the fund.

The second step of our numerical analysis is to calculate the fair insurance fee of the GLWB option according the pricing model developing in the dynamic framework. In the same way of the previous section, the results are obtained with Monte Carlo simulation. Once the interest rate, the volatility and the guaranteed rate have been fixed, we have searched the fair value of the fee with an iterative procedure: if the time-zero cost of the whole product turned out to be higher than ω_0 we increased the fee up to decrease the cost to ω_0; vice-versa, if the time-zero cost of the whole product turned out to be smaller than ω_0 we decreased the fee. Table 4.7 compares the fair insurance fee under the static and dynamic pricing model for a policyholder aged 60 if $r = 5\%$, $g = 5\%$ and $k = 10\%$.

The purpose of this comparison is to highlight the higher required fee if

Fair Fee	**Static**	**Dynamic**
$\sigma = 0.15\%$	54 b.p.	105 b.p.
$\sigma = 0.18\%$	79 b.p.	145 b.p.
$\sigma = 0.20\%$	96 b.p.	178 b.p.
$\sigma = 0.25\%$	138 b.p.	258 b.p.
$\sigma = 0.30\%$	177 b.p.	342 b.p.

Table 4.7: The fair fee under static and dynamic framework

we assume the individual will surrender the option at an optimal time. Our results is in line with the analysis conducted by Milevsky and Salisbury ([22]) for the GMWB market. As Milevsky and Salisbury note, the real fee has to lie between the static and dynamic embedded option cost; in fact, the policyholder can behave with a high level of irrational lapsation. Chen et al. ([7])

and Ho et al. (cf. [15]) note that sub-optimal policyholder behavior considerably reduces the value of the GMWB rider. Cramer et al. ([10]) suggests to describe the sub-optimal policyholder behavior with a function of how much the embedded option is in the money. Wang ([28]) offers a dynamic lapse function that more reduces the lapse when the GLWB is more in-the-money. These approaches involve a fair valuation of the insurance fee that lies between our static and dynamic estimated option price. In this regard, we have to remember that the dynamic strategy defined in the previous section is not necessarily optimal from the point of view of maximization of policyholder's utility function; it is the worst case for the insurer who has to hedge the issued policy, because it is the strategy that maximizes the zero-value of the contract for every possible scenarios. However, on a practical side our numerical results show that the USA market of GLWB is underpriced (like that of GMWB) and this is regardless of we use a static or a dynamic pricing model.

4.8 Conclusions

In this chapter we have developed two formulas in order to price the GLWB option, both within the framework of No Arbitrage pricing. First, we have taken a static approach that assumes policyholders take a static strategy, i.e. the withdrawal amount is always equal to the guaranteed amount. In this case we have shown the product can be decomposed in a life annuity plus a portfolio of Quanto Asian Put Options, with decreasing strikes and increasing expiration dates. The opposite assumption we have considered is that all investors are rational and maximize the embedded option value by surrendering the product at an optimal time, when the surrender value exceeds the value of future benefits. In this dynamic approach we have dealt with an optimal stopping problem and we have resolved it with Monte Carlo simulation.

Up to this time the literature has not offered a specific model for GLWB pricing, but only a general pricing-framework for the universe of VA or papers on the pricing of other particular embedded options, like GMWB and GMDB. Our work fits in the actuarial literature on VA and investigate two aspects, which have not been previously discussed: the definition of a pricing model for the latest GLWB option, which takes in account both financial and actuarial aspects, and the verification of the fairness of the current GLWB price on the USA market. Our main contribution lies in bifurcating the product into the life annuity component and the derivatives components and calculate the fair insurance fee. Our conclusion is that the GLWB issued on the USA market are underpriced and this appears regardless of whether we take a static or dynamic approach. On a practical side, our numerical results show that the No Arbitrage cost of a GLWB issued to a policyholder aged 60 would range between 79 and 145 basis points assuming a sub-account volatility in line with the average of the sub-account volatility for the universe of variable annuity products, while most products in the USA market only charge 50-70 basis points. We compute the probability of ruin, i.e. the probability that the fund hits zero and the insurer has to pay the amount guaranteed, if the insurance fee is that normally charged by the market. This probability oscillates between 48% and 57% under different expected investment returns and increases with the sub-account volatility; given the long-dated nature of the embedded option it is likely that the volatility would increase and thus the risk for the insurer to pay the guaranteed amount is very high. This results indicate that the market fees are not sufficient to cover the market hedging cost of the guarantee. Of course, our pricing model does not allow for more sophisticated financial hypothesis, such as stochastic volatility or jumps in the fund process and term-structure effects, but as Milevsky and Salisbury ([22]) we are confident that these considerations will only increases the price of the embedded option. The same effect would be obtained with the introduction of an actuarial model allowing for the longevity risk. So, we conclude by arguing that the current price of GLWB is not sustainable for insurers and the fees have to increase in order to avoid arbitrage opportunities. Future researches will examine realistic hedging strategy for GLWB, taking in account the possibility to create portfolios of VA with different embedded options and to exploit the effect of partial natural hedging between them.

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Appendix A

Graphical Results on mortality simulations

In Chapter 2 we have studied the impact of mortality risk on the GMDB value. In this regard, we have considered the projected postponed probabilities of death calculated according a stochastic model suggested by Cox and Lin (2005).

$$p'_{x+t} = p_{x+t}^{(1-\epsilon_t)}$$

where ϵ_t is the shock in the expected probability at time t. Ballotta, Esposito, Haberman (2006) assume that ϵt follows a beta distribution with parameter a and b and the sign of the shocks depends on the random number $k(t)$ simulated from the uniform distribution $U(0,1)$:

$$\epsilon(t) \text{ if } k(t) < c$$

$$-\epsilon(t) \text{ if } k(t) \geq c$$

where c is a parameter which depends on the user's expectation of the future mortality trend.

We have carried out two calculation procedures: in the first one, we have fixed $a = 0.5$ and $b = 4.5$, so that shocks have expected value equal to 0.10 and standard deviation equal to 0.12; in the second one, we have fixed $a = 1.5$ and $b = 3.5$, so that shocks have expected value equal to 0.30 and standard

deviation equal to 0.19. In both cases, we have simulated 1000 paths of evolution of mortality using the Monte Carlo method and have considered the alternative hypotheses $c = 0$ and $c = 1$.
The following figures show the results.

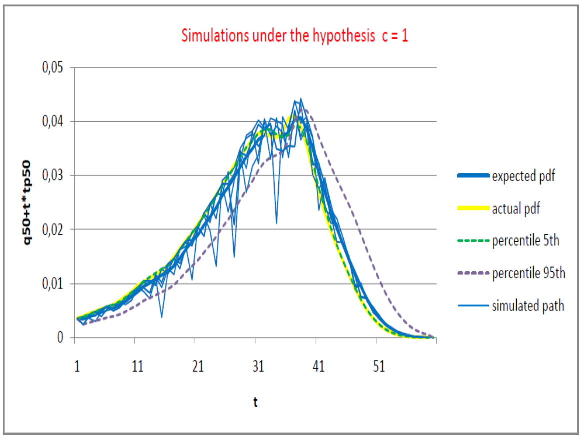

Figure A.1: Actual, expected and prudential projected mortality probability function under the hypothesis $E(\epsilon_t) = 0.10$ and $\sigma(\epsilon_t) = 0.12$; $c = 1$

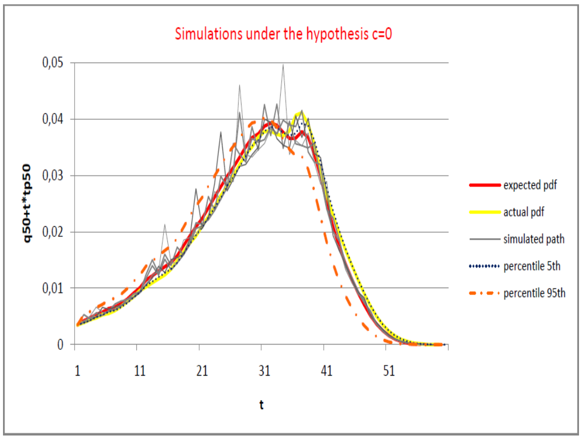

Figure A.2: Simulated mortality probability function under the hypothesis $E(\epsilon_t) = 0.10$ and $\sigma(\epsilon_t) = 0.12$; $c = 0$

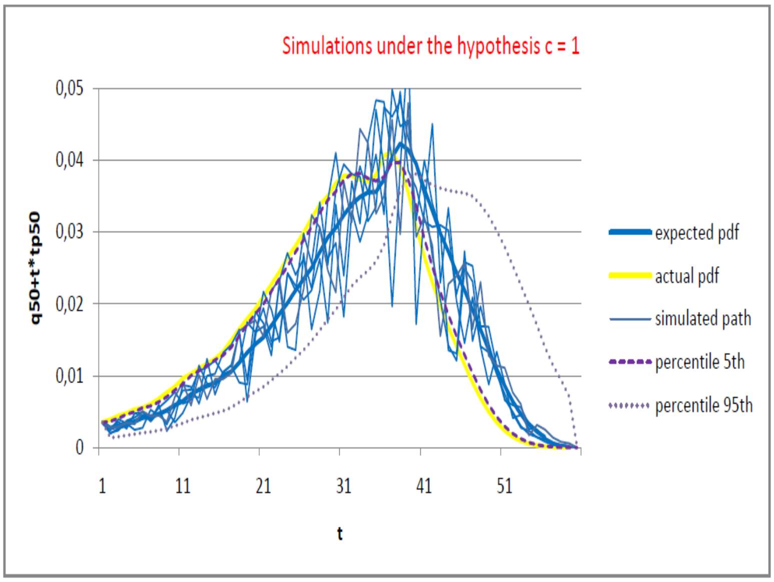

Figure A.3: Simulated mortality probability function under the hypothesis $E(\epsilon_t) = 0.30$ and $\sigma(\epsilon_t) = 0.19$; $c = 1$

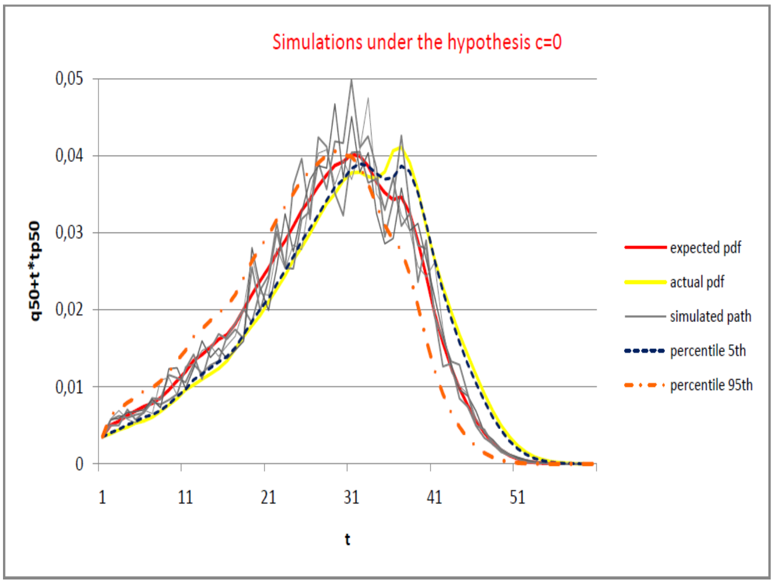

Figure A.4: Actual, expected and prudential projected mortality probability function under the hypothesis $E(\epsilon_t) = 0.30$ and $\sigma(\epsilon_t) = 0.19$; $c = 0$

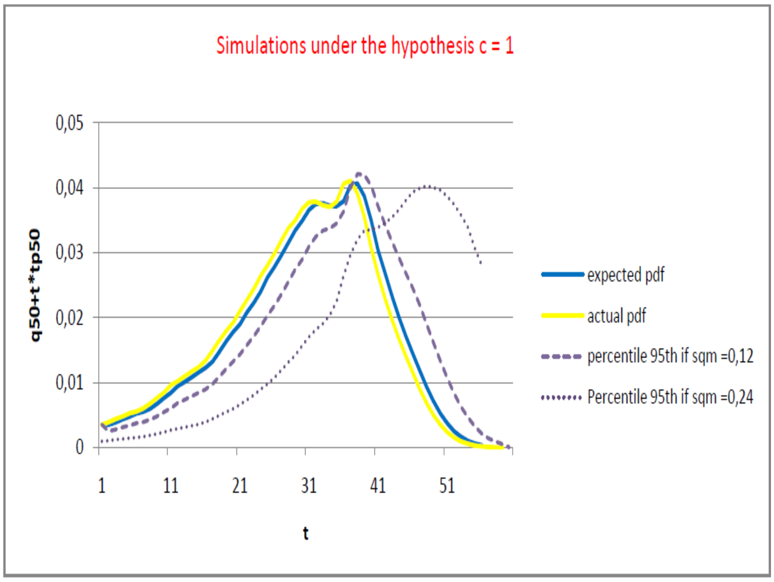

Figure A.5: Actual, expected and prudential projected mortality probability function under the hypothesis $E(\epsilon_t) = 0.30$ and $\sigma(\epsilon_t) = 0.24$; $c = 1$

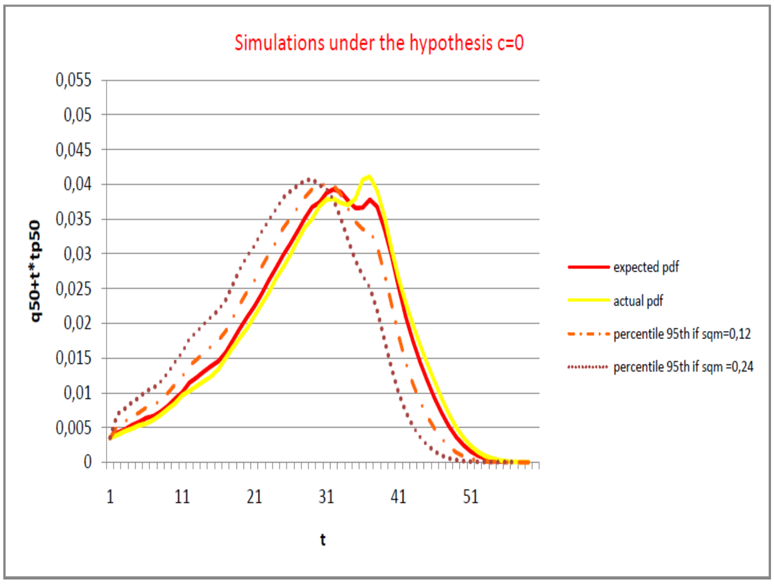

Figure A.6: Actual, expected and prudential projected mortality probability function under the hypothesis $E(\epsilon_t) = 0.30$ and $\sigma(\epsilon_t) = 0.24$; $c = 0$

Appendix B

The derivation of the Surplus Variance

In Chapter 3 we have adopted a definition of surplus as the difference between the Retrospective Gain and Prospective Loss and have derived the first two moments of the surplus distribution. In the following, we deepen the calculations suggested in Chapter 3. We have assumed the independence between the fund process and the force of interest process

The variance of the cash flows (both retrospective or prospective) is given by the following formula:

$$Var[FC_j^r] = R^2 Var[\alpha_j] + Var[G_j \delta_j] + 2RCov[\alpha_j, G_j \delta_j] \tag{B.1}$$

where

$$Var[G_j \delta_j] = E[G_j^2] E[\delta_j^2] - (E[G_j])^2 (E[\delta_j])^2 \tag{B.2}$$

$$\begin{aligned} Cov[\alpha_j, G_j \delta_j] &= E[\alpha_j G_j \delta_j] - E[\alpha_j] E[G_j \delta_j] = E[G_j] E[\alpha_j \delta_j] - E[\alpha_j] E[G_j] E[\delta_j] \\ &= E[G_j] \{E[\alpha_j \delta_j] - E[\alpha_j] E[\delta_j]\} = E[G_j] Cov[\alpha_j, \delta_j] \end{aligned} \tag{B.3}$$

The covariance of the cash flows is:

$$\begin{aligned} Cov[FC_k^{(r)}, FC_j^{(r)}] = {} & R^2 Cov[\alpha_k, \alpha_j] + Cov[G_k \delta_k, G_j \delta_j] + RCov[\alpha_k, G_j \delta_j] + \\ & RCov[\alpha_j, G_k \delta_k] \end{aligned} \tag{B.4}$$

where

$$
\begin{aligned}
&Cov[G_k\delta_k, G_j\delta_j] = E[G_k\delta_k G_j\delta_j] - E[G_k\delta_k]E[G_j\delta_j] = \\
&= E[G_kG_j]E[\delta_k\delta_j] - E[G_k]E[\delta_k]E[G_j]E[\delta_j] = \\
&= E[G_k]E[G_j]E[\delta_k\delta_j] - E[G_k]E[\delta_k]E[G_j]E[\delta_j] = \\
&= E[G_k]E[G_j]\{E[\delta_k\delta_j] - E[\delta_k]E[\delta_j]\} = \\
&= E[G_k]E[G_j]Cov[\delta_k, delta_j]
\end{aligned} \tag{B.5}
$$

$$
Cov[\alpha_k, G_j\delta_j] = E[G_j]Cov[\alpha_k, \delta_j]
$$

$$
Cov[\alpha_j, G_k\delta_k] = E[G_k]Cov[\alpha_j, \delta_k]
$$

Finally, the variance of the surplus can be calculated:

$$
\begin{aligned}
&Var[\frac{S_r}{m}] = Var[\sum_{j=0}^{n} \frac{FC_j^r e^{I(j,r)}}{m}] = \\
&\frac{1}{m^2}\{\sum_{j=0}^{n} Var[FC_j^r e^{I(j,r)}] + \sum_{j=0}^{n}\sum_{\substack{k=0\\k\neq j}}^{n} Cov(FC_j^r e^{I(j,r}), FC_k^r e^{I(k,r)})\}
\end{aligned} \tag{B.6}
$$

where

$$
\begin{aligned}
&Var[FC_j^r e^{I(j,r)}] = E[FC_j^{(r)2} e^{I(j,r)2}] - \{E[FC_j^r e^{I(j,r)}]\} \\
&= E[FC_j^{(r)2}]E[e^{I(j,r)2}] - E[FC_j^{(r)}]^2 E[e^{I(j,r)}]^2
\end{aligned} \tag{B.7}
$$

and

$$
\begin{aligned}
&Cov[FC_j^r e^{I(j,r)}, FC_k^r e^{I(k,r)}] = \\
&= E[FC_j^r FC_k^r e^{I(j,r)+I(k,r)}] - E[FC_j^r]E[e^{I(j,r)}]E[FC_k^r]E[e^{I(k,r)}] = \\
&= E[FC_j^r FC_k^r]E[e^{I(j,r)+I(k,r)}] - E[FC_j^r]E[e^{I(j,r)}]E[FC_k^r]E[e^{I(k,r)}] = \\
&= E[FC_j^r FC_k^r]\{Cov[e^{I(j,r)}, e^{I(k,r)}] - E[e^{I(j,r)}]E[e^{I(k,r)}]\}+ \\
&- E[FC_j^r]E[e^{I(j,r)}]E[FC_k^r]E[e^{I(k,r)}]
\end{aligned} \tag{B.8}
$$

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