Implicit Partial Differential Equations: different approaching methods

Giovanni Pisante

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Preface

In the last few decades the interest of scientists in nonlinear analysis has been constantly increasing and nonlinear partial differential equations have become one of the main tools of modern mathematical analysis.

In this thesis we deal with a family of nonlinear partial differential equations, the so called *"implicit partial differential equations"*. The problem of solving this type of equations has been approached looking at it from different points of view and consequently different approaching methods were developed. The choice of the method to apply depends on what properties we require on the solutions of the equation.

The principal aim of this work is to present some different ways to establish the existence of solutions of implicit partial differential equations. In particular we focus our attention on two methods: the Baire category method and the viscosity method.

The first one is a functional analytic method based essentially on the Baire category theorem. It was introduced by Cellina in 1980 to prove density properties of solutions of some differential inclusions and it has been extensively studied and extended by many authors. In particular Dacorogna and Marcellini in a series of papers extended the method to the framework of implicit differential equations looking at these as differential inclusions. We present this theory, making a survey on the recent results that we can find in literature and we present new proofs of general existence theorems. We would point out that the Baire category method has the "defect" to be purely existential, that is it can establish only the existence of solutions (in fact it ensures the existence of infinitely many solutions), but it does not give any other information.

The viscosity approach for this type of equations is one of the oldest methods applied in this field and it has received much attention since its introduction by Crandall and Lions in 1982. It deals essentially with scalar problems, i.e. the Hamilton-Jacobi equations, and it is less general than the previous one. Nevertheless it has the advantage that it gives much more information than existence of solutions; for instance, uniqueness, stability, maximality, and, under suitable hypotheses, explicit formulas.

Here we discuss about the existence of viscosity solutions for the Hamilton-Jacobi equations looking at it from a *"geometrical"* point of view. The interest in finding geometrical conditions comes out from the idea to compare the two above methods. Indeed, if we want to use the viscosity method as a criterium to select, among the infinitely many solutions given by the Baire category method, a preferred one, we immediately deal with restrictive geometrical compatibility conditions. In particular, generalizing these results, we present new geometrical conditions sufficient and, in some cases, necessary for the existence of viscosity solution of Hamilton-Jacobi equations with non necessarily convex hamiltonian.

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Notations

$\mathbb{R}^n$	the euclidean n-dimensional space
$\partial\Omega$	the boundary of the set Ω
$\operatorname{int} E$	the interior of the set E
$\overline{E}$	the closure of the set E
Cf	the convex envelope of the function f
Rf	the rank-one convex envelope of the function f
Pf	the polyconvex envelope of the function f
Qf	the closure of the quasiconvex envelope of the function f
$\operatorname{co} E$	the convex hull of the set E
$\operatorname{Rco} E$	the rank-one convex hull of the set E
$\operatorname{Pco} E$	the polyconvex hull of the set E
$\operatorname{Qco} E$	the closure of the quasiconvex hull of the set E
meas E	the Lebesgue measure of the set E
$T_K(x)$	the generalized tangent cone to the compact set K at the point x
$N_K(x)$	the generalized outward normal cone to the compact set K at the point x
$C_K(x)$	the Clarke's tangent cone to K at x
$\operatorname{dist}(x, y)$	for $x, y, \in \mathbb{R}^n$, the euclidean distance between x and y
$\operatorname{dist}(x, E)$	for $x \in \mathbb{R}^n$ and $E \subset \mathbb{R}^n$, $\operatorname{dist}(x, E) = \inf_{y\in E} \operatorname{dist}(x, y)$
$\operatorname{supp} u$	the support of the function u, i.e. the closure of the set $\{x : u(x) \neq 0\}$
$\nabla u(x)$	the gradient of the function u at x, i.e. $\nabla u(x) = \left(\frac{\partial u}{\partial x_1}(x), \ldots, \frac{\partial u}{\partial x_n}(x)\right)$
Du(x)	the distributional gradient of the function u at x.
$D^+u(x), D^-u(x)$	the super and subdifferential of u at x
Δu	the laplacian of the function u , i.e., $\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$
$u'(x, q)$	the one-sided directional derivative of u at x in the direction q
$u^0(x, q); u_0(x, q)$	the generalized directional derivatives of u at x in the direction q

$\partial u(x)$	the generalized gradient (or Clarke's gradient) of u
$Aff_{piec}(\Omega;\mathbb{R}^n)$	the set of piecewise affine function defined on Ω with values in $\mathbb{R}^n$
$C^1_{piec}(\Omega;\mathbb{R}^n)$	the set of piecewise C^1 function defined on Ω with values in $\mathbb{R}^n$
$C(\Omega)$	the set of continuous functions $u : \Omega \to \mathbb{R}$
$C_0(\Omega)$	the set of continuous function with compact support in Ω
$C^k(\Omega)$	for $k \geq 1$ and Ω open subset of $\mathbb{R}^n$, the subspace of $C(\Omega)$ of functions with continuous partial derivatives in Ω up to order k.
$C^k_0(\Omega)$	for $k \geq 1$ and Ω open subset of $\mathbb{R}^n$, the subspace of $C_0(\Omega)$ of functions with continuous partial derivatives in Ω up to order k.
$L^p(\Omega)$	for $p \geq 1$, the spaces of p-summable functions in Ω
$W^{1,p}(\Omega)$	for $1 \leq p \leq \infty$, the Sobolev spaces
$W^{1,p}_0(\Omega)$	for $1 \leq p < \infty$ the closure of $C^\infty_0(\Omega)$ in $W^{1,p}(\Omega)$
$W^{1,\infty}_0(\Omega)$	the intersection $W^{1,1}_0(\Omega) \cap W^{1,\infty}(\Omega)$
H^*	the lagrangian or the dual convex function of the convex function H

Chapter 1

Introduction

1.1 Implicit PDE

The purpose of this thesis is to study the Dirichlet problem for a class of non linear differential equations, the so called *implicit partial differential equations*.

Namely we deal with the following problem:

$$\begin{cases} F_i(x, u(x), Du(x)) = 0 & x \in \Omega \quad i = 1, ..., I \\ u(x) = \varphi(x) & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is an open set, $u : \Omega \to \mathbb{R}^m$ and therefore $Du \in \mathbb{R}^{m\times n}$, $F_i : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ are given and the prescribed boundary condition φ, depending on the context, will be either continuosly differentiable or only Lipschitz-continuous. If $m = 1$ we say that the problem is *scalar* and it reduces to the Dirichlet problem for the classical Hamilton-Jacobi equation; otherwise we say that it is *vectorial*.

In particular we will investigate the existence of $W^{1,\infty}(\Omega; \mathbb{R}^m)$ solutions of problem (1.1). We recall that, as is well known, it is not reasonable to expect the solution to be $C^1(\Omega; \mathbb{R}^m)$ even in the simplest case $m = n = 1$.

We note that the nature of the question, as it is stated, excludes automatically from our investigation quasilinear problems, since solutions of such problems cannot satisfy the Dirichlet boundary condition.

The problem (1.1) has been studied for long time using different methods in different contexts, starting with the theory of Hamilton-Jacobi equations to arrive at dealing with the vectorial case using convex analysis and functional analytic methods. In the next section we will briefly present some of the possible ways to approach the problem (1.1).

1.2 Different Methods

As we have said, the problem (1.1) was approached in many ways; in literature we find also some ad hoc methods developed to deal with some particular examples, as, for instance, the pyramidal construction of Cellina (see [?] or [?]). However, roughly speaking, there are three general methods to deal with it. In particular here we will see in detail two of them: the Baire category method and the viscosity method. The third one is a method due to Gromov (see [?], [?]), and usually we refer at it as "Convex integration". It was introduced to solve some problems of differential geometry and recently Müller and Šverák have applied this theory, in an analytical context, to obtain general existence theorems for partial differential inclusions. We will not enter into the details of this method and we refer to [?], [?] and [?] for a complete description and further references. We only want to point out that the result obtained with this method are essentially the same that we can obtain using the Baire category one.

1.2.1 The Baire category method

The Baire category method was for the first time applied to differential equations by Cellina in [?] to prove the density of solutions for the differential inclusion

$$\begin{cases} x'(t) \in \{-1, 1\}, & \text{a.e. } t \geq 0 \\ x(0) = x_0. \end{cases}$$

This method was further developed by De Blasi and Pianigiani (see [?], [?], [?]), Bressan and Flores (see [?]) and Dacorogna and Marcellini (see [?], [?], [?]).

We roughly want to present the idea of how this method can be applied and which are the main ingredients of the proof of existence of solutions.

We start looking at the scalar problem

$$\begin{cases} F(Du(x)) = 0, & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x) & x \in \partial\Omega, \end{cases}$$

where $F : \mathbb{R}^n \to \mathbb{R}$ is a continuous function and $\varphi \in W^{1,\infty}(\Omega)$. We will see that the optimal formulation for this kind of problem is obtained rewriting it as a differential inclusion as follows:

$$\begin{cases} Du(x) \in E, & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x) & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where we have denoted by E the set of the zeroes of F, i.e.

$$E := \{\xi \in \mathbb{R}^n \ : \ F(\xi) = 0\}.$$

Moreover we suppose that E is a compact subset of $\mathbb{R}^n$.

We start by introducing the functional space

$$V := \left\{ u \in \varphi + W_0^{1,\infty}(\Omega) \ : \ Du(x) \in E \cup \operatorname{int} \operatorname{co} E \ \text{a.e.} \ x \in \Omega \right\}$$

where by $\operatorname{int}\operatorname{co} E$ we denote the interior of the convex hull of E. We then denote by $\overline{V}$ the complete metric space obtained by making the closure of V with respect to the $L^\infty(\Omega)$ metric.

To avoid $\overline{V}$ being the empty set, we impose the compatibility condition on the boundary datum

$$D\varphi(x) \in E \cup \operatorname{int}\operatorname{co} E \quad \text{a.e.} \ x \in \Omega. \tag{1.3}$$

The main idea is now to construct for every integer k a subset V_k of $\overline{V}$ such that

$$V_k \subset \left\{ u \in \overline{V} \ : \ \int_\Omega \operatorname{dist}(Du(x); E) < \frac{1}{k} \right\}$$

and such that V_k is open and dense in the complete metric space $\overline{V}$ for every $k \in \mathbb{N}$. Then by Baire category theorem we have that the set

$$\bigcap_{k=1}^{\infty} V_k \subset \left\{ u \in \overline{V} \ : \ \int_\Omega \operatorname{dist}(Du(x); E) = 0 \right\} \subset \overline{V}$$

is a dense set in $\overline{V}$ and so non empty. Finally, since E is compact, we deduce that any function $u \in \bigcap_{k=1}^{\infty} V_k$ is solution of problem (1.2).

We should point out that in order to prove existence of solutions of the scalar problem (1.2) we don't need to require the compactness of the set E. In fact, it can be proved that the condition (1.3) is sufficient for the existence of $W^{1,\infty}(\Omega)$ solutions of (1.2) for a general set $E \subset \mathbb{R}^n$ and it is very close to the necessary one (cf. Theorem 2.2.5 and Section 2.2.3).

However our aim here was to give a simple outline of the method that we will use to prove the main existence result for vectorial differential inclusions (cf. Theorem 2.3.15). Indeed we will deal with the problem to find a solution $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ of

$$\begin{cases} Du(x) \in E & \text{a.e} \, x \in \Omega \\ u(x) = \varphi(x) & x \in \partial\Omega \end{cases} \tag{1.4}$$

with $E \subset \mathbb{R}^{m \times n}$ a compact set; in this case we will see that, in order to have the existence of solutions, the compatibility condition (1.3) will be replaced by

$$D\varphi(x) \in E \cup K \quad \text{a.e.} \ x \in \Omega$$

where the set $K \subset \operatorname{Qco} E$ (for the definition of the closure of the quasiconvex hull of E, $\operatorname{Qco} E$, see Definition 2.3.8) is related to E by the *relaxation property* (cf. Section 2.3.1).

We should spend some words to underline the big gap of difficulty between the scalar and the vectorial cases.

In the scalar case, the problem to find almost everywhere solutions of problem (1.2), also for general sets $E \subset \mathbb{R}^n$ non necessarily compact, is well understood and the existence of solutions under the only hypothesis (1.3) can be proved applying the Baire category method (see [**?**], [**?**]) or using different approaches, as for example, the constructive method of pyramids introduced by Cellina (see [**?**] and also [**?**], [**?**]).

The generalization to systems of the results available for a single partial differential inclusion turns out to be more delicate. Let us now point out why.

First of all there is the problem to generalize the compatibility condition (1.3) when $E \subset \mathbb{R}^{m\times n}$. The natural generalization should be

$$D\varphi(x) \in E \cup \operatorname{int} \operatorname{Qco} E \quad \text{a.e. } x \in \Omega;$$

unfortunately, we are not able yet to deal with this type of condition, because of the problems in well understanding the notion of quasiconvexity and consequently in finding a good definition for $\operatorname{Qco} E$.

Another delicate issue in proving existence using the Baire category method, is to find an appropriate way to define the sets V_k and to show that they are open and dense in $\overline{V}$. In the general theory for solving implicit partial differential equations with the Baire method, developed systematically by Dacorogna and Marcellini in [**?**] (see also [**?**], [**?**], [**?**]) the main tool used to define the sets V_k and to prove that they are open in $\overline{V}$ was the weak lower semicontinuity of integral functionals. This explains why they always deal with problems where the set E can be described as a level set of a quasiconvex function.

In Chapter 2 we will show how this hypothesis of quasiconvexity can be removed, using fine properties of continuity of the gradient operator, instead of weak lower semicontinuity arguments (cf. Theorem 2.3.15).

We should note also that the Baire category method is purely "existential"; moreover, when it ensures the existence, it in fact, ensures the existence of infinitely many solutions; consequently the problem to find a way to select among them a preferred solution comes out.

Finally we want to underline that the Baire method is a general method that can be applied also to solve equations involving higher order derivatives as we show in Section 2.3.3.

1.2.2 The viscosity method

The viscosity method is one of the most studied to approach the problem (1.1). Also if there are some results on some particular vectorial equations, it deals essentially with scalar problems and it is in this framework that we are going to apply it here. The viscosity method starts with the idea to search the solution of the problem

$$\begin{cases} F(x,u(x),Du(x)) = 0 & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x) & x \in \partial\Omega \end{cases} \tag{1.5}$$

as a limit for $\varepsilon \to 0$ of solutions of the approximate problems

$$\begin{cases} F(x,u_\varepsilon(x),Du_\varepsilon(x)) = \varepsilon\Delta u_\varepsilon & \text{a.e. } x \in \Omega \\ u_\varepsilon(x) = \varphi(x) & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is an open set, $F : \Omega \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ is a continuous function and φ is given. This method was later generalized and a precise definition of Lipschitz viscosity solution was given by Crandall and Lions in [?] (see Section 3.3 for definitions and also [?], [?], [?] for further references).

The notion of viscosity solution comes out naturally in optimal control theory since the value function of certain problems turns out to be a viscosity solution of an Hamilton-Jacobi equation like (1.5) (see [?], [?]).

The study of viscosity solutions is an active field of research and many generalizations are available (the definition has been extended to functions that are even discontinuous); nevertheless here we deal only with locally Lipschitz viscosity solutions.

The advantage of this method over the Baire category one is that it gives much more information than existence of solutions; for instance it ensures uniqueness, stability, maximality and, under suitable hypotheses, explicit formulas. These features suggest us to use the viscosity method as a criterium to select, among the infinitely many solutions that we may find with the abstract Baire category method, a preferred one.

In this direction goes the work of P. Cardaliaguet, B. Dacorogna, W. Gangbo and N. Georgy in [?], where they make a comparison between the two methods in finding solutions of the generalized eikonal equation

$$\begin{cases} F(Du(x)) = 0 & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x) & x \in \partial\Omega. \end{cases} \tag{1.6}$$

They show that if we want to select among the solutions of (1.6), found with the Baire category method, then the viscosity approach is too restrictive and in many cases it cannot be applied. Nevertheless the techniques used to make this comparison turn out to be useful, in the viscosity framework, to establish sufficient conditions for the existence of solutions.

To explain how this can be done, we start recalling briefly the result proved in [?]. If we suppose that Ω is a convex domain and that $\varphi \in C^1(\overline{\Omega})$ is such that

$$D\varphi(x) \in E \cup \operatorname{int} \operatorname{co} E \quad \forall\, x \in \Omega, \tag{1.7}$$

where $E := \{\xi \in \mathbb{R}^n \ : \ F(\xi) = 0\}$, then the Baire category method ensures us that the problem (1.6) has infinitely many almost everywhere solution in $W^{1,\infty}(\Omega)$; but in general there are no viscosity solutions unless strong geometrical restrictions on Ω and φ are assumed (cf. Section **??**, Examples **??**, **??**).

To have an idea of these restrictions one could consider, for example, the problem (1.6) with zero boundary data; in this case, the geometrical restriction can be written as

- (G1) $\forall\, y \in \partial\Omega$ where the inward normal , $\nu(y)$, is uniquely defined, there exists $\lambda(y) > 0$ such that

$$\lambda(y)\nu(y) \in E,$$

i.e. the inward normal vector to the boundary of Ω at the points where it is well defined, have to point in a direction contained in the set of the zeroes of the Hamiltonian F.

This result shows that the existence of viscosity solutions of problem (1.6) strongly depends on geometrical relations between the domain Ω and the boundary datum φ. The investigation of these relations will be the main purpose of Chapter **??**. Indeed we will see that the geometrical hypotheses, that we need to assume to compare the Baire category method and the viscosity one, can be generalized in order to obtain sufficient conditions for the existence of viscosity solutions of (1.6). As before, to have a flavor of how this generalization can be made, we can say that, dealing with the simple case of zero boundary data, the condition (G1) will be replaced by

- (G2) $\forall\, y \in \partial\Omega$ where $N^N_{\mathbb{R}^n \setminus \Omega}(y) \neq \emptyset$, $\forall\, \nu \in N^N_{\mathbb{R}^n \setminus \Omega}(y)$ there exists a unique $\lambda_\nu > 0$ such that

$$\lambda_\nu \nu \in E$$

where $N^N_{\mathbb{R}^n \setminus \Omega}(y)$ is the normal cone to the set $\mathbb{R}^n \setminus \Omega$ (see Definitions **??**).

In particular we will prove the sufficiency of (G2) without assuming any other regularity on φ, but Lipschitz regularity on $\partial\Omega$ and any hypothesis of convexity on the domain Ω (cf. Section **??** and Theorem **??**).

We should say that these geometrical sufficient conditions, in some cases turn out to be also necessary for the existence of viscosity solutions, it is the case, for instance, when φ is an affine function (cf. Section **??**). We will also show how to read in an analytical way these conditions under suitable hypotheses on the domain Ω.

Finally, we want to point out that we will present a constructive proof of the existence of solution, i.e. we are able to find, under suitable hypotheses, an explicit Hopf-Lax type formula for the viscosity solution of (1.6).

Chapter 2

Generalized solutions: the Baire category method

2.1 Introduction

In this chapter we introduce a functional analytic method: the so called Baire category method, to find solutions of the Dirichlet problem

$$\begin{cases} F_i(x, u(x), Du(x)) = 0 & \text{a.e. } x \in \Omega \quad i = 1, ..., I \\ u(x) = \varphi(x) & x \in \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^n$ is an open set, $u : \Omega \to \mathbb{R}^m$ and therefore $Du \in \mathbb{R}^{m \times n}$, $F_i : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \to \mathbb{R}$ and $\varphi : \overline{\Omega} \to \mathbb{R}$ are given.

We divide the discussion in two parts. In the first one we deal with the scalar case, i.e. if $m = 1$. We observe that if $I = 1$ the problem (2.1) reduces to the Dirichlet problem for the classical Hamilton-Jacobi equation. Following the work of Dacorogna and Marcellini [**?**], we will give the idea of the Baire method applying it to a model problem which has the advantage to make the procedure transparent and not burdened by too many technical details. The structure of the model problem presented here allows us to apply the Baire category method using lower semicontinuity techniques, which are the main tools in the general theory developed by Dacorogna and Marcellini to prove existence of a.e. solutions also in the vectorial case, i.e. if $m > 1$. The use of these tools, in reality, restrict the range of problems that can be solved by Baire category method (see Section 1.2.1). In the second part of this chapter we try to solve this problem giving a generalization of the existence results available in literature. In particular we provide an abstract existence result using a new proof, which combines the Baire category method with fine properties of continuity of the gradient operator. We provide also some extensions to problems involving higher order derivatives.

Moreover we state some more practical conditions that ensures the hypotheses of the abstract theorem to obtain existence of $W^{1,\infty}(\Omega)$ solutions of problem (2.1).

2.2 The scalar case

This section is devoted to the study of first order scalar partial differential equations, i.e. we consider the problem

$$\begin{cases} F(x, u(x), Du(x)) = 0 & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x) & x \in \partial\Omega \end{cases} \tag{2.2}$$

where $\Omega \subset \mathbb{R}^n$ is an open set, $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is continuous and $\varphi \in W^{1,\infty}(\Omega)$.

In particular we will study in details only the model case where there is no dependence on (x, u) and the function $F : \mathbb{R}^n \to \mathbb{R}$ is convex. Applying the Baire category method to this simple problem we will see, in a transparent way, what are the principal steps of this method.

In this case it is more convenient to rewrite (2.2) as a differential inclusion, namely

$$\begin{cases} Du(x) \in E & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x) & x \in \partial\Omega \end{cases} \tag{2.3}$$

where E is the set

$$E := \{\xi \in \mathbb{R}^n \;:\; F(\xi) = 0\}.$$

We will find that the Dirichlet problem has solution under the sole compatibility condition:

$$D\varphi(x) \in E \cup \operatorname{int}\operatorname{co} E\,, \text{ a.e. } x \in \Omega \tag{2.4}$$

where $\operatorname{int}\operatorname{co} E$ denotes the interior of the convex hull of the set E. We will also spend some words on the necessity in some sense of the condition (2.4) for the existence of $W^{1,\infty}$ solutions of (2.3).

Finally we will recall some existence results for the problem with explicit dependence on (x, u) and for systems of equations that can be obtained with the same method.

2.2.1 The model problem

Let us start giving the definition of coercivity in a direction and making some remarks.

Definition 2.2.1. *We say that a scalar function $F(x, s, \xi)$ defined on $\Omega \times \mathbb{R} \times \mathbb{R}^n$ is coercive with respect to ξ in the direction $\lambda \in \mathbb{R}^n$ if, for every*

$x \in \Omega$ and every bounded set of $\mathbb{R} \times \mathbb{R}^n$, there exists constants $m, q > 0$, such that

$$F(x, s, \xi + t\lambda) \geq m|t| - q$$

for every $t \in \mathbb{R}$, $x \in \Omega$ and for every (s, ξ) that vary on the bounded set of $\mathbb{R} \times \mathbb{R}^n$.

Remark 2.2.2. *(i) The above definition can be weakened in the following way: for every $x \in \Omega$ and for every bounded set of $\mathbb{R} \times \mathbb{R}^n$ there exists $\omega : \mathbb{R}_+ \to \mathbb{R}$ continuous, strictly increasing and satisfying $\lim_{t \to +\infty} \omega(t) = +\infty$, such that*

$$F(x, s, \xi + t\lambda) \geq \omega(|t|)$$

for every $t \in \mathbb{R}$, $x \in \Omega$ and for every (s, ξ) that vary on the bounded set of $\mathbb{R} \times \mathbb{R}^n$.

(ii) If F does not depends on (x, s), we easily see that the coercivity condition in the direction $\lambda \in \mathbb{R}^n$, $|\lambda| = 1$, implies (with different constants m and q) the following property: for every $\xi_0 \in \mathbb{R}^n$ and $r > 0$ there exists constants $m = m(r)$, $q = q(r) > 0$, such that

$$F(\xi) \geq m|\langle \xi; \lambda \rangle| - q$$

for every $\xi \in \mathbb{R}^n$ such that $|\xi - \xi_0 - \langle \xi - \xi_0; \lambda \rangle| \leq r$.

Here we deal with the following model problem

$$\begin{cases} F(Du(x)) = 0, & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{2.5}$$

This problem has been intensively studied in many relevant articles (see for example [?], [?], [?], [?], [?], [?] [?], [?]) and there are many monographs (see [?], [?], [?], [?], [?]) where one can find a more complete bibliography. So here we don't want to rewrite all the theory on this subject, we only take this problem as a model that allows us to outline in a simple way the main idea of our work. With this aim in mind we prove the following lemma:

Lemma 2.2.3. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $F : \mathbb{R}^n \to \mathbb{R}$ be a convex function, coercive in a direction $\lambda \in \mathbb{R}^n$. Let φ be an affine function in Ω (i.e., $D\varphi(x) = \xi_0$ for some $\xi_0 \in \mathbb{R}^n$ and for every $x \in \Omega$) such that*

$$F(D\varphi) = F(\xi_0) \leq 0. \tag{2.6}$$

Then there exists (a dense set of) $u \in W^{1,\infty}(\Omega)$ solution of problem

$$\begin{cases} F(Du(x)) = 0, & a.e. \ x \in \Omega \\ u = \varphi, & on \ \partial\Omega. \end{cases} \tag{2.7}$$

Proof. We assume without loss of generality that Ω is bounded, since we can cover it with a countable family of bounded sets and prove the lemma on each of these sets. We can also assume that $F(\xi_0) < 0$, otherwise φ is a solution of our problem.

The coercivity of F and the Remark 2.2.2 (ii) ensure us that for every $r > 0$ there exist positive constants m, q, such that

$$F(\xi) \geq m|\langle \xi; \lambda \rangle| - q \tag{2.8}$$

for every $\xi \in \mathbb{R}^n$ such that $|\xi - \xi_0 - \langle \xi - \xi_0; \lambda \rangle| \leq r$.

For $r > 0$ we define a cylinder K by

$$K := \left\{ \xi \in \mathbb{R}^n \ : \ |\xi - \xi_0 - \langle \xi - \xi_0; \lambda \rangle| \leq r \ , \ \ |\langle \xi; \lambda \rangle| \leq \frac{q}{m} \right\},$$

where m, q are the constant that appear in the coercivity condition (2.8). We easily see that the set K is compact and convex, moreover by (2.8) the following inclusion holds:

$$\left\{ \xi \in \mathbb{R}^n \ : \ |\xi - \xi_0 - \langle \xi - \xi_0; \lambda \rangle| \leq r \ , \ \ F(\xi) \leq 0 \right\} \subset K. \tag{2.9}$$

Finally we observe also that $\xi_0 \in \operatorname{int} K$.

Now we define the functional set V as

$$V := \left\{ u \in \varphi + W_0^{1,\infty}(\Omega) \ : \ Du(x) \in K, \ F(Du(x)) \leq 0, \ \text{a.e. } x \in \Omega \right\}.$$

The set V is non empty since $\varphi \in V$. We endow V with the $L^\infty-$norm so that V can be seen as a metric space. We will prove that V is closed in $C^0(\overline{\Omega})$ and thus V is a complete metric space.

To prove that the set V is closed in $C^0(\overline{\Omega})$, we consider a sequence u_k in V that converges in $L^\infty(\Omega)$ to a function u. Since the set K is bounded, by (2.9) also V is bounded in $W^{1,\infty}(\Omega)$; then u_k contains a subsequence, which we will continue to denote by u_k, which converges in the weak$*$ topology of $W^{1,\infty}(\Omega)$ to $u \in \varphi + W_0^{1,\infty}(\Omega)$. Since F and K are convex, we obtain

$$F(Du(x)) \leq 0, \ Du(x) \in K, \ \text{a.e. } x \in \Omega; \tag{2.10}$$

in fact, for example to prove the first condition of (3.7), if $\eta \in C^0(\Omega)$, $\eta \geq 0$, since $F(Du_k) \leq 0$, then by the lower semicontinuity of the integral we have

$$\int_\Omega \eta(x) F(Du(x))\, dx \leq \liminf_{k \to +\infty} \int_\Omega \eta(x) F(Du_k(x))\, dx,$$

and thus

$$\int_\Omega \eta(x) F(Du(x))\, dx \leq 0, \ \ \forall\, \eta \in C^0(\Omega), \ \eta \geq 0,$$

which implies (2.10). Therefore V is closed in $C^0(\overline{\Omega})$ and it is a complete metric space.

Now for every $k \in \mathbb{N}$ we define the subset of V

$$V^k := \left\{ u \in V \ : \ \int_\Omega F(Du(x))\, dx > -\frac{1}{k} \right\}.$$

The set V^k is open in V, indeed by the boundedness in $W^{1,\infty}(\Omega)$ of V and by the lower semicontinuity in the weak-$*$ topology of $W^{1,\infty}$ of the integral of F, we deduce that the complement set

$$V \setminus V^k = \left\{ u \in V \ : \ \int_\Omega F(Du(x))\, dx \leq -\frac{1}{k} \right\}$$

is closed in V.

Now we show that V^k is dense in V. So let $v \in V$, we have

$$Dv(x) \in K, \ \ F(Dv(x)) \leq 0, \ \ \text{a.e. } x \in \Omega.$$

For $t \in (0,1)$ we consider the convex combination

$$v_t = tv + (1-t)\varphi;$$

then, by the standard convexity inequalities, since $D\varphi = \xi_0 \in \operatorname{int} K$ and $F(\xi_0) < 0$, we obtain

$$Dv_t(x) \in \operatorname{int} K, \ \ F(Dv_t(x)) \leq 0, \ \ \text{a.e. } x \in \Omega, \ \forall t \in (0,1).$$

Since v_t converges in $L^\infty(\Omega)$ to v as $t \to 1$, we can approximate v by v_t and thus, without denoting explicitly the dependence on t, we can reduce ourselves to the conditions

$$Dv(x) \in \operatorname{int} K, \ \ F(Dv(x)) \leq 0, \ \ \text{a.e. } x \in \Omega.$$

Now we apply the Lemma 2.4.1 with $B = \emptyset$ and with A given by the open set

$$A := \{\xi \in \operatorname{int} K \subset \mathbb{R}^n \ : \ F(\xi) < 0\}.$$

Then, for every $\varepsilon > 0$, there exists $v_\varepsilon \in W^{1,\infty}(\Omega)$ such that

$$\begin{cases} v_\varepsilon \text{ is piecewise affine on } \Omega; \\ v_\varepsilon = u \text{ on } \partial\Omega; \\ \|v_\varepsilon - u\|_{L^\infty(\Omega)} < \varepsilon; \\ Dv(x) \in A, \text{ a.e. } x \in \Omega. \end{cases}$$

Therefore there exist disjoint open sets Ω_j, $j \in \mathbb{N}$, so that $v_\varepsilon|_{\Omega_j}$ is affine. More precisely

$$Dv_\varepsilon = \xi_j \ \in \operatorname{int} K, \ \ F(\xi_j) < 0, \ \ \text{in } \Omega_j, \ \forall j \in \mathbb{N}.$$

Let us consider the function which at every $t \in \mathbb{R}$ associates $\xi_j(t) = \xi_j + t\lambda \in \mathbb{R}^n$. By (2.9) $\xi_j(t) \in K$ for $t \in \mathbb{R}$ as long as $F(\xi_j(t)) \le 0$. Since $F(\xi_j) < 0$, by (2.8) we can find $t_1 < 0 < t_2$ such that $F(\xi_j(t_1)) = F(\xi_j(t_2)) = 0$. By the continuity of F for ε sufficiently small, we can find δ_1, $\delta_2 > 0$ such that

$$F\left(\xi_j(t_1+\delta_1)\right) = F\left(\xi_j(t_2-\delta_2)\right) = -\varepsilon.$$

We then apply the Lemma 2.4.2 with $\xi = \xi_j(t_1+\delta_1)$, $\eta = \xi_j(t_2-\delta_2)$ and $t = (t_2-\delta_2)/(t_2-\delta_2-t_1-\delta_1)$, with φ replaced by v_ε and ε replaced by $\min\{\delta, \varepsilon/2^j\}$ with δ to be chosen below. We find functions $v_{\varepsilon,j} \in W^{1,\infty}(\Omega)$ and sets $\widetilde{\Omega}_j (= \Omega_\xi \cup \Omega_\eta) \subset \Omega_j$ such that

$$\begin{cases} \operatorname{meas}(\Omega_j - \widetilde{\Omega}_j) \le \varepsilon/2^j; \\ v_{\varepsilon,j}(x) = v_\varepsilon(x),\ x \in\ \partial\Omega_j; \\ \|v_{\varepsilon,j} - v_\varepsilon\|_{L^\infty(\Omega_j)} \le \varepsilon/2^j \le \varepsilon/2; \\ F\left(Dv_{\varepsilon,j}(x)\right) = -\varepsilon \text{ a.e. } x \in \Omega_\xi \cup \Omega_\eta = \widetilde{\Omega}_j; \\ Dv_{\varepsilon,j}(x) \in \operatorname{int} K,\ F\left(Dv_{\varepsilon,j}(x)\right) < 0 \,\text{a.e.}\, x \in \Omega_j. \end{cases}$$

The fact that $Dv_{\varepsilon,j}(x) \in \operatorname{int} K$ and $F\left(Dv_{\varepsilon,j}(x)\right) < 0$ are consequences of

$$\begin{cases} \operatorname{co}\{\xi,\eta\} \subset \operatorname{int} K \\ F|_{\operatorname{co}\{\xi,\eta\}} \le -\varepsilon < 0 \\ \operatorname{dist}\left(Dv_{\varepsilon,j}, \operatorname{co}\{\xi,\eta\}\right) < \delta, \end{cases}$$

and the possibility of choosing δ arbitrarily small; note that $F(\xi) = F(\eta) = -\varepsilon < 0$ and, by convexity, $F(\zeta) < 0$ for every $\zeta \in \operatorname{co}\{\xi,\eta\}$. Then we define a function u_ε by

$$u_\varepsilon(x) = v_{\varepsilon,j}(x) \text{ if } x \in \overline{\Omega}_j,\ \forall\, j \in \mathbb{N}.$$

We have $u_\varepsilon \in V$ and $\|u_\varepsilon - v\|_{L^\infty(\Omega)} \le \varepsilon$. It remains to show that $u_\varepsilon \in V^k$. To this end we compute

$$\begin{aligned} \int_\Omega F(Du_\varepsilon(x))\,dx &= \sum_{j=1}^\infty \int_{\Omega_j} F(Dv_{\varepsilon,j}(x))\,dx \\ &= \sum_{j=1}^\infty \int_{\Omega_j - \widetilde{\Omega}_j} F(Dv_{\varepsilon,j}(x))\,dx - \varepsilon \sum_{j=1}^\infty \operatorname{meas}(\widetilde{\Omega}_j). \end{aligned}$$

We use the inequality

$$\sum_{j=1}^\infty \operatorname{meas}(\widetilde{\Omega}_j) < \varepsilon,$$

and the fact that $Dv_{\varepsilon,j}(x)$ belongs to the compact set K, a.e. $x \in \Omega$, to deduce, since F is continuous on K, that

$$\int_\Omega F(Du_\varepsilon(x))\,dx > -\frac{1}{k}$$

for ε sufficiently small. Therefore $u_\varepsilon \in V^k$ and the density of V^k in V has been established.

By the Baire category theorem we have that the functional set

$$\bigcup_{k\in\mathbb{N}} V^k = \left\{ u \in V \ : \ \int_\Omega F(Du(x))\, dx \geq 0 \right\}$$

is dense in V, in particular it is not empty. Since every $u \in V$ satisfies the condition $F(Du(x)) \leq 0$ a.e. $x \in \Omega$, we obtain that every element u of this intersection solves the equation $F(Du(x)) = 0$ a.e. $x \in \Omega$, and we have the claim. □

From the proof of Lemma 2.2.3 we understand what we mean with *dense set of solutions*. Indeed we construct the set V such that

$$\varphi \in V := \left\{ u \in \varphi + W_0^{1,\infty}(\Omega) \ : \ Du(x) \in K, \ F(Du(x)) \leq 0, \ \text{a.e.}\, x \in \Omega \right\},$$

and we have seen that the set of solutions of (2.7) is dense, in the L^∞ norm, in V. So, in particular, for every $\varepsilon > 0$, we can find $u_\varepsilon \in W^{1,\infty}(\Omega)$, a solution of (2.7), so that

$$\|u_\varepsilon - \varphi\|_{L^\infty} \leq \varepsilon.$$

Remark 2.2.4. *We want here to make precise in which sense the boundary condition, $u = \varphi$ on $\partial\Omega$, that appears in the previous lemma, is to be understood.*

- *If Ω is bounded, we mean that*

$$u - \varphi \in W_0^{1,\infty}(\Omega).$$

where, we recall, $W_0^{1,\infty}(\Omega) = W^{1,\infty}(\Omega) \cap W_0^{1,1}(\Omega)$. If Ω is a "good set", for example convex, then $W^{1,\infty}(\Omega)$ is the set of Lipschitz functions u with Lipschitz constant $\|Du\|_{L^\infty}$ (see for example [?], [?]); therefore $u = \varphi$ on $\partial\Omega$ in the classical sense, i.e., $u = \varphi$ pointwise.

- *If Ω is unbounded, then $u = \varphi$ on $\partial\Omega$ means*

$$(u - \varphi)\psi \in W_0^{1,\infty}(\Omega \cap \operatorname{int}\operatorname{supp}\psi)$$

for every $\psi \in C_0^\infty(\mathbb{R}^n)$; where $\operatorname{int}\operatorname{supp}\psi$ is the interior of the support of ψ. Moreover, the method showed in the previous proof can also give existence of a solution u that not only satisfies the boundary condition in the above sense, but also verifies the limit condition

$$\lim_{|x|\to\infty} |u(x) - \varphi(x)| = 0.$$

It is sufficient to apply the existence theorem with the bounded open sets

$$\begin{aligned}\Omega_0 &= \{x \in \Omega \,:\, |x| < 1\}\\ \Omega_k &= \{x \in \Omega \,:\, k < |x| < k+1\}\,,\ k \in \mathbb{N}\end{aligned}$$

and using the density find u_k, a solution of (2.7) in Ω_k, such that $|u_k(x) - \varphi(x)| \leq 1/k$ for every $k \in \mathbb{N}$.

2.2.2 The general case

The Lemma 2.2.3 is in fact a particular case of a more general theorem that can be proved using essentially the same argument. Indeed we first observe that a more convenient way to look at the problem (2.5) is as differential inclusion, that is, setting $E = \{\xi \in \mathbb{R}^n \,:\, F(\xi) = 0\}$, the problem (2.5) is equivalent to

$$\begin{cases} Du(x) \in E, & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{2.11}$$

With this notation the following theorem holds

Theorem 2.2.5. *Let $\Omega \subset \mathbb{R}^n$ be open and $E \subset \mathbb{R}^n$. Let $\varphi \in W^{1,\infty}(\Omega)$ satisfy the compatibility condition*

$$D\varphi(x) \in E \cup \operatorname{int} \operatorname{co} E,\ \ a.e.\ x \in \Omega \tag{2.12}$$

then there exists (a dense set of) $u \in W^{1,\infty}(\Omega)$ solution of problem (2.11).

The theorem 2.2.5 has been proved in [**?**], [**?**] using the Baire category method, but it can be also proved by different arguments as, for example, the constructive method of pyramids introduced by Cellina [**?**] (see also [**?**], [**?**] for a proof).

Remark 2.2.6. *If F is convex we have that $\overline{\operatorname{int} E}$ is convex and $E = \partial(\operatorname{int} \operatorname{co} E)$, and this implyies that the condition (2.12) is equivalent to $F(D\varphi) \leq 0$, that is the compatibility condition of Lemma 2.2.3.*

Finally we want to mention that the Baire method in the scalar case can also be applied to find solutions of more general problems, for example for Hamiltonians with explicit dependence on (x, u). Indeed the following general result holds (for a detailed proof see [**?**] [**?**])

Theorem 2.2.7. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $F_i : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ functions continue and convex with respect to the last variable for every $i = 1, 2, ..., I$. Let*

$$E = \{(x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \mid F_i(x, s, \xi) = 0,\ \forall\, i = 1, ..., I\}.$$

If $\operatorname{co} E$ *(where with* $\operatorname{co} E$ *we denote the convex hull of* E *with respect to the last variable) is uniformly bounded for* (x,s) *in a bounded subset of* $\Omega \times \mathbb{R}$ *and if* $\varphi \in W^{1,\infty}(\Omega)$ *is such that*

$$(x, \varphi(x), D\varphi(x)) \in E \cup \operatorname{int} \operatorname{co} E, \ \textit{a.e.}\ x \in \Omega$$

then there exists a (dense set of) $u \in W^{1,\infty}(\Omega)$*, solutions of*

$$\begin{cases} F_i(x,u,Du) = 0 & \textit{a.e. } x \in \Omega \ \ \forall\, i = 1, ..., I, \\ u = \varphi & \textit{on } \partial\Omega. \end{cases}$$

Example 2.2.8. *As an application of the previous theorems we can show, under suitable hypotheses on the initial condition, the existence of generalized solutions for the classical Hamilton-Jacobi-Bellman equation. Consider the problem*

$$\begin{cases} u_t + f(D_x u) = 0 & \textit{a.e. } x \in \mathbb{R}^n, \ t \in (0,T) \\ u(0,x) = \psi(x) & x \in \mathbb{R}^n. \end{cases} \tag{2.13}$$

where $T > 0$*,* $u = u(t,x)$*,* $Du = (u_t, D_x u) \in \mathbb{R}^{n+1}$*,* $f : \mathbb{R}^n \to \mathbb{R}$ *a continuous function and* $\psi \in W^{1,\infty}(\mathbb{R}^n)$*.*

Applying Theorem 2.2.5 with $F : \mathbb{R}^{n+1} \to \mathbb{R}$ *defined as*

$$F(\alpha, \beta) = \alpha + f(\beta), \ (\alpha, \beta) \in \mathbb{R} \times \mathbb{R}^n,$$

and

$$E := \{(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}^n \,:\, F(\alpha,\beta) = 0\} = \{(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}^n \,:\, \alpha + f(\beta) = 0\}$$

we deduce that if

$$(0, D\psi(x)) \in E \cup \operatorname{int} \operatorname{co} E \quad \textit{a.e.}\ x \ \in \mathbb{R}^n,$$

then there exists a solution $u \in W^{1,\infty}((0,T) \times \mathbb{R}^n)$ *of 2.13.*

Moreover if f *is convex and coercive in a direction* $\lambda \in \mathbb{R}^n$ *then we can prove existence of solutions without any hypotheses on* ψ*. Indeed first we note that* F *is convex and coercive in the direction* $(0, \lambda)$*. Then, since* $\psi \in W^{1,\infty}(\mathbb{R}^n)$*, we can find* $R > 0$ *such that*

$$f(D\psi(x)) \le R\,, \quad \textit{a.e. } x \in \mathbb{R}^n,$$

and so, if we define $\varphi(t,x) = -Rt + \psi(x)$*, we have* $\varphi \in W^{1,\infty}((0,T) \times \mathbb{R}^n)$ *and*

$$F(D\varphi(t,x)) = -R + f(D\psi(x)) \le 0\,, \quad \textit{a.e. } (t,x) \in (0,T) \times \mathbb{R}^n.$$

Therefore, applying Theorem 2.2.7 we get the existence of solutions.

2.2.3 The compatibility condition

Before passing to the vectorial case, we want to make some remarks on the compatibility condition (2.12). We have, in some sense that we will made precise, that (2.12) is a condition necessary for the existence of solutions of (2.11).

We motivate our claim studying the case of affine boundary data. Let

$$\varphi(x) = \langle \xi_0; x \rangle + q$$

for certain $\xi_0 \in \mathbb{R}^n$ and $q \in \mathbb{R}$. We assume that there exists a solution $u \in W^{1,\infty}(\Omega)$ of problem (2.11). Then since $Du(x) \in E$, we have by the Jensen inequality that, for any convex function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f|_E = 0$, the following inequality holds:

$$\begin{aligned} f(\xi_0) &= f\left(\frac{1}{\operatorname{meas}\Omega}\int_\Omega D\varphi(x)\,dx\right) = f\left(\frac{1}{\operatorname{meas}\Omega}\int_\Omega Du(x)\,dx\right) \\ &\le \frac{1}{\operatorname{meas}\Omega}\int_\Omega f(Du(x))\,dx = 0. \end{aligned}$$

Since (see Definition 2.3.8)

$$\overline{\operatorname{co} E} = \left\{ \begin{array}{c} \xi \in \mathbb{R}^n \ : \ f(\xi) \le 0, \ \forall f : \mathbb{R}^n \to \mathbb{R}, \\ f|_E = 0, \ f \text{ convex} \end{array} \right\}$$

we deduce that

$$D\varphi(x) = \xi_0 \in \overline{\operatorname{co} E}. \tag{2.14}$$

In the particular case of a coercive convex function F this is exactly (2.6). However, in the general nonconvex case, (2.14) cannot replace (2.12) for the existence of solutions as we can see in the following example.

Example 2.2.9. *Let $n = 2$,*

$$E := \left\{\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \ : \ |\xi_1| = |\xi_2| = 1\right\},$$
$$\operatorname{co} E := \left\{\xi \in \mathbb{R}^2 \ : \ |\xi|_\infty = \max\{|\xi_1|, |\xi_2|\} \le 1\right\},$$

$\Omega = (0,1)^2$ and $\varphi(x_1, x_2) = x_1 + \beta x_2$ with $|\beta| < 1$. Note that (2.14) is satisfied while (2.12) is not. We claim that the problem

$$\left\{ \begin{array}{ll} Du(x) \in E, & a.e.\ x \in \Omega \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{array} \right. \tag{2.15}$$

has no $W^{1,\infty}(\Omega)$ solution. Indeed observe that any solution should satisfy

$$\begin{aligned} \int_0^1\int_0^1 \left[\left|\frac{\partial u}{\partial x_1}\right| - \frac{\partial u}{\partial x_1}\right] dx_1 dx_2 &= \int_0^1\int_0^1 \left[1 - \frac{\partial u}{\partial x_1}\right] dx_1 dx_2 \\ &= \int_0^1 [1 - u(1, x_2) + u(0, x_2)] dx_2 \\ &= \int_0^1 [1 - (1 + \beta x_2) + \beta x_2] dx_2 = 0. \end{aligned}$$

This leads immediately to

$$\left|\frac{\partial u}{\partial x_1}\right| = \frac{\partial u}{\partial x_1} = 1, \quad a.e \ in \ \Omega.$$

Therefore we deduce that (2.15) is equivalent to finding $u(x_1, x_2) = x_1 + \psi(x_2)$*, where* ψ *satisfies*

$$\begin{cases} |\psi'(x_2)| = 1, & a.e. \ x_2 \in (0,1) \\ \psi(x_2) = \beta x_2, & (x_1, x_2) \in \partial\Omega. \end{cases}$$

Then we get an absurd, since the second equation $\psi(x_2) = \beta x_2$ *contradicts the first one, since* $|\beta| < 1$*.*

The general case of non linear boundary data can also be dealt. For example, if we assume that

$$\begin{cases} \Omega \text{ is bounded and convex} \\ \operatorname{co} E \text{ is compact} \\ 0 \in \operatorname{int} \operatorname{co} E \end{cases} \tag{2.16}$$

and we denote by ρ and ρ^0 the gauge associated to $\operatorname{co} E$ (cf. Appendix **??**) and its polar respectively, i.e.

$$\rho(\xi) = \inf\{\lambda \geq 0 \ : \ \xi \in \lambda \operatorname{co} E\},$$

$$\rho^0(\xi^*) = \inf\{\lambda^* \geq 0 \ : \ \langle \xi^*; \xi \rangle \leq \lambda^* \rho(\xi) \ , \ \ \forall \xi \in \mathbb{R}^n\},$$

then the following result holds (cf. [**?**] Theorem 2.17)

Theorem 2.2.10. *Let (2.16) be satisfied, then*

(i) Necessary condition. Let $u \in W^{1,\infty}(\Omega)$ *be a solution of*

$$\begin{cases} Du(x) \in E, & a.e. \ x \in \Omega \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{2.17}$$

then necessarily

$$\varphi(x) - \varphi(y) \leq \rho^0(x - y) \ , \ \ \forall x, y \in \partial\Omega. \tag{2.18}$$

Conversely, if φ *satisfies (2.18), then there exists* $\widetilde{\varphi} \in \varphi + W^{1,\infty}(\Omega)$ *such that* $\rho(D\widetilde{\varphi}) \leq 1$*, a.e. in* Ω*, i.e.,*

$$\begin{cases} D\widetilde{\varphi}(x) \in \operatorname{co} E, & a.e. \ x \in \Omega \\ \widetilde{\varphi}(x) = \varphi(x), & x \in \partial\Omega. \end{cases}$$

(ii) Sufficient condition. Let φ *satisfy*

$$\varphi(x) - \varphi(y) \leq \gamma \rho^0(x - y) \ , \ \ \forall x, y \in \partial\Omega.$$

for some $0 < \gamma < 1$*; then there exists a solution* $u \in W^{1,\infty}(\Omega)$ *of (2.17).*

We note that necessary or sufficient conditions on φ should be stated in terms of functions defined only on $\partial\Omega$. This can be achieved in the scalar case while in the vectorial setting it is still an open problem (see Chapter **??** or [**?**] for more details).

We will see in the last chapter how we can weakened this compatibility condition for existence of viscosity solutions of problem (2.11) (see also [**?**]).

2.3 The vectorial case

Here we deal with systems of partial differential equations and we will discuss the generalization of the existence results, showed in the previous section for the scalar case, to the vectorial one. In particular, we will investigate the existence of $W^{1,\infty}$ solutions for the Dirichlet problem involving systems written in an implicit way as differential inclusions, i.e. we search for solution $u \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ of the problem

$$\begin{cases} (x,u,Du) \in E & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x) & x \in \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is open, $\varphi : \partial\Omega \to \mathbb{R}$ is a given function and $E \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m$ is a given set.

We have seen that if $E \subset \mathbb{R}^n$ and $\varphi \in W^{1,\infty}(\Omega;\mathbb{R})$ satisfies

$$D\varphi(x) \in E \cup \operatorname{int}\operatorname{co} E, \quad \text{a.e. } x \in \Omega$$

then there exists $u \in W^{1,\infty}(\Omega;\mathbb{R})$ such that

$$\begin{cases} Du(x) \in E & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x) & x \in \partial\Omega. \end{cases}$$

This shows that, in the scalar case, the notion of convexity plays a crucial role in establishing sufficient conditions for the existence of solutions; moreover this notion comes out naturally also in the necessary condition, that turns out to be, when properly formulated,

$$D\varphi(x) \in \overline{\operatorname{co} E}, \quad \text{a.e. in } \Omega.$$

Our aim is to extend this result to the vectorial case, that is the case where $u : \Omega \to \mathbb{R}^m$. This turns out to be much more delicate and no result with such a degree of elegancy as in the scalar case, is available. The natural generalization to the vectorial case of the above result should be the following. Let $E \in \mathbb{R}^{m\times n}$ and $\varphi \in W^{1,\infty}(\Omega,\mathbb{R}^m)$ satisfy

$$D\varphi(x) \in E \cup \operatorname{int}\overline{\operatorname{Qco}} E, \quad \text{a.e. } x \in \Omega \tag{2.19}$$

(where $\overline{\operatorname{Qco}} E$ stands for the closure of the quasiconvex hull of E, see Section 2.3.1 for definitions), then there exists $u \in W^{1,\infty}(\Omega, \mathbb{R}^m)$ such that

$$\begin{cases} Du(x) \in E & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x) & x \in \partial\Omega. \end{cases} \tag{2.20}$$

Unfortunately we are not able yet to prove such general sufficiency, also because of the problems in well understanding the notion of quasiconvexity and consequently in finding a good definition for $\overline{\operatorname{Qco}} E$. The first general results in this direction were obtained by B. Dacorogna and P. Marcellini (see [?]) using the Baire category method. They proved an existence theorem with the set $\overline{\operatorname{Qco}} E$ replaced, in the compatibility condition (2.19), by a set K which verify the *relaxation property* with respect to E (see Section 2.3.1 for definitions) and under the further hypothesis that E is the set of zeros of quasiconvex functions, i.e.

$$E := \left\{ \xi \in \mathbb{R}^{m \times n} \ : \ F_i(\xi) = 0 \ ; \ i = 1, 2, ..., I \right\}$$

where $F_i : \mathbb{R}^{m \times n}$, for any $i = 1, 2, ..., I$, are quasiconvex. The hypothesis of quasiconvexity on E was later removed by Sichev (see [?] and also [?]) using a different approach based on the Gromov convex integration.

In this section we want to show how this hypothesis can be dropped also using the Baire category approach and we will provide a sharp theorem generalizing the results of Dacorogna and Marcellini (see also [?]).

The main result will be

Theorem 2.3.1. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $E, K \subset \mathbb{R}^{m \times n}$ be such that E is compact and K is bounded. Assume that K has the relaxation property with respect to E. Let $\varphi \in Aff_{piec}\left(\overline{\Omega}; \mathbb{R}^m\right)$ (i.e φ is piecewise affine in $\overline{\Omega}$) such that*

$$D\varphi\left(x\right) \in E \cup K, \ \text{a.e. in } \Omega.$$

Then there exists (a dense set of) $u \in \varphi + W_0^{1,\infty}\left(\Omega; \mathbb{R}^m\right)$ such that

$$Du\left(x\right) \in E, \ \text{a.e. in } \Omega.$$

We will provide a proof of this result in Section 2.3.2 and we will present also some generalizations to the non-homogeneous case and involving higher order derivatives. We will use in the proof some classical theorem on Baire one functions (see Appendix for more details) following an idea of Kirchheim (cf. [?]).

2.3.1 Preliminaries

In this section we recall some definitions that we will use in the sequel. We start recalling some different notions generalizing the convexity (i.e. rank-one convexity, quasiconvexity and polyconvexity) and we will see how these notions allow us to define different notions of hull of a set. Then we introduce the so called *relaxation property*, that will be one of the main hypotheses for the abstract existence theorem.

Different notions of convexity

We start giving the different notions of convexity, introduced by Morrey [?] (see also [?] and [?] for our terminology), that are used in the calculus of variations.

Definition 2.3.2. *(i) A function $f : \mathbb{R}^N \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is said to be convex if*

$$f(tA + (1-t)B) \leq tf(A) + (1-t)f(B)$$

for every $t \in [0,1]$ and every $A, B \in \mathbb{R}^N$.

(ii) A function $f : \mathbb{R}^{m\times n} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is said to be polyconvex if there exists a function $g : \mathbb{R}^{\tau(m,n)} \to \overline{\mathbb{R}}$ convex and such that

$$f(A) = g(T(A))$$

where $T : \mathbb{R}^{m\times n} \to \mathbb{R}^{\tau(m,n)}$ is defined as

$$T(A) = (A, adj_2 A, \ldots, adj_{m\wedge n} A);$$

$adj_s A$ stands for the matrix of all $s \times s$ subdeterminants of the matrix A, for $1 \leq s \leq m \wedge n = \min\{m, n\}$, and

$$\tau(m,n) = \sum_{s=1}^{m\wedge n} \binom{m}{s}\binom{n}{s}.$$

(iii) A Borel measurable function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is said to be quasiconvex if

$$f(A) \leq \frac{1}{\text{meas }\Omega} \int_\Omega f(A + D\varphi(x))\,dx$$

for every bounded domain $\Omega \subset \mathbb{R}^n$, every $A \in \mathbb{R}^{m\times n}$, and every $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^m)$.

(iv) A function $f : \mathbb{R}^{m\times n} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is said to be rank one convex if

$$f(tA + (1-t)B) \leq tf(A) + (1-t)f(B)$$

for every $t \in [0,1]$ and every $A, B \in \mathbb{R}^{m\times n}$ with $\operatorname{rank}(A - B) = 1$.

Remark 2.3.3. *Note that in the definition of quasiconvexity, contrary to the other ones, we only consider functions that take finite values. In fact there is an intrinsic difficulty in defining the notion of quasiconvexity for functions that take the value $+\infty$. In literature we can find this notion given in the* *case where f is allowed to be defined with values in $\overline{\mathbb{R}}$, but no examples of this type of functions are known (see [**?**] and [**?**]). However, although such definition have been shown to be necessary for weak lower semicontinuity, it has not been proved that it is sufficient and this seems to be a difficult problem.*

The notion collected in the previous definitions are related by the following theorem (c.f. [**?**]):

Theorem 2.3.4. *(i) Let $f : \mathbb{R}^{m\times n} \to \mathbb{R}$; then*

$$f \, convex \Rightarrow f \, polyconvex \Rightarrow f \, quasiconvex \Rightarrow f \, rank \ one \ convex.$$

(ii) Let $f : \mathbb{R}^{m\times n} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$; then

$$f \, convex \Rightarrow f \, polyconvex \Rightarrow f \, rank \ one \ convex.$$

(iii) If $m = 1$ or $n = 1$, then all these notions are equivalent.

(iv) If $f \in C^2(\mathbb{R}^{m\times n})$ then the rank one convexity is equivalent to Legendre-Hadamard (or ellipticity) condition

$$\sum_{i,j=1}^{m} \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 f(A)}{\partial A^i_\alpha \partial A^j_\beta} \lambda^i \lambda^j \mu_\alpha \mu_\beta \geq 0$$

for every $\lambda \in \mathbb{R}^m$, $\mu \in \mathbb{R}^n$ and $A = \{A^i_j\} \in \mathbb{R}^{m\times n}$.

(v) If $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is convex, polyconvex, quasiconvex or rank one convex, then f is locally Lipschitz.

Remark 2.3.5. *All the counter implications of Theorem 2.3.4 are false; For some counterexamples see [**?**], [**?**] and [**?**].*

Remark 2.3.6. *The definitions introduced above carry out to the higher order case and we refer to [**?**] Section 5 for notations and definitions.*

Associated to any notion of convexity we have the related notion of convex envelope of a given function f. Then we define the convex, polyconvex, quasiconvex, rank one convex envelope of f respectively as

$$\begin{aligned} Cf &:= \sup\{g \leq f \ : \ g \text{ is convex}\}, \\ Pf &:= \sup\{g \leq f \ : \ g \text{ is polyconvex}\}, \\ Qf &:= \sup\{g \leq f \ : \ g \text{ is quasiconvex}\}, \\ Rf &:= \sup\{g \leq f \ : \ g \text{ is rank one convex}\}. \end{aligned}$$

By Theorem 2.3.4 we immediately get that

$$Cf \leq Pf \leq Qf \leq Rf \leq f.$$

Remark 2.3.7. *As for the convex envelope we have a representation formula following from the classical Carathéodory's theorem, representation formulas for Pf, Qf and Rf are also available, see for instance [?], [?].*

Now we want to recall the notations for various convex hulls of a given set E.

Definition 2.3.8. *Let $E \subset \mathbb{R}^{m\times n}$ (more generally $E \subset \mathbb{R}^N$ for the convex hull). We define the convex hull of E as*

$$\operatorname{co} E := \left\{ \begin{array}{c} \xi \in \mathbb{R}^N \ : \ f(\xi) \leq 0, \ \ \forall\, f : \mathbb{R}^N \to \overline{\mathbb{R}}, \\ \textit{with} \ \ f|_E \leq 0 \ \ \textit{and} \ f \ \textit{convex} \end{array} \right\},$$

the polyconvex hull of E as

$$\operatorname{Pco} E := \left\{ \begin{array}{c} \xi \in \mathbb{R}^{m\times n} \ : \ f(\xi) \leq 0, \ \ \forall\, f : \mathbb{R}^{m\times n} \to \overline{\mathbb{R}}, \\ \textit{with} \ \ f|_E \leq 0 \ \ \textit{and} \ f \ \textit{polyconvex} \end{array} \right\},$$

the rank one convex hull of E as

$$\operatorname{Rco} E := \left\{ \begin{array}{c} \xi \in \mathbb{R}^{m\times n} \ : \ f(\xi) \leq 0, \ \ \forall\, f : \mathbb{R}^{m\times n} \to \overline{\mathbb{R}}, \\ \textit{with} \ \ f|_E \leq 0 \ \ \textit{and} \ f \ \textit{rank one convex} \end{array} \right\},$$

and the closure of the quasiconvex hull of E as

$$\operatorname{Qco} E := \left\{ \begin{array}{c} \xi \in \mathbb{R}^{m\times n} \ : \ f(\xi) \leq 0, \ \ \forall\, f : \mathbb{R}^{m\times n} \to \mathbb{R}, \\ \textit{with} \ \ f|_E \leq 0 \ \ \textit{and} \ f \ \textit{quasiconvex} \end{array} \right\}.$$

Remark 2.3.9. *(i) The definition of convex hull is equivalent to the classical one, i.e.* $\operatorname{co} E$ *is the smallest convex set that contains E.*

(ii) One of the interesting feature of these definitions is that they establish a natural connection between convex hulls and convex envelopes. Namely, if χ_E is the indicator function of E, i.e.

$$\chi_E(\xi) = \begin{cases} 0 & \textit{if}\, \xi \in E \\ +\infty & \textit{if}\, \xi \notin E, \end{cases}$$

then

$$\begin{aligned} C\chi_E &= \chi_{\operatorname{co} E} \\ P\chi_E &= \chi_{\operatorname{Pco} E} \\ R\chi_E &= \chi_{\operatorname{Rco} E}. \end{aligned}$$

(iii) There are other definitions of these different hulls (see for example [?] and [?]); some of them differ from the ones given here by the fact that they do not allow the functions to take the value $+\infty$. This is not so different for the convex or polyconvex hull, since it amounts to close the set. However, for the rank one convex hull this is drastically different (for counterexamples see [?], [?], [?], [?]). Some authors call the rank one convex hull defined above the laminate convex hull of E.

Relaxation property

We want to present here tha relaxation property, introduced by Dacorogna and Marcellini in [?] (see also [?], [?]), which is the key condition to get the existence of solutions using the Baire category method.

Definition 2.3.10. *[Relaxation property] Let $E, K \subset \mathbb{R}^{m\times n}$. We say that K has the relaxation property with respect to E if for every bounded open set $\Omega \subset \mathbb{R}^n$, for every affine function u_ξ satisfying*

$$Du_\xi\left(x\right) = \xi \in K,$$

there exist a sequence $u_\nu \in Aff_{piec}\left(\Omega;\mathbb{R}^m\right)$ (i.e. the set of piecewise affine functions) such that

$$u_\nu \in u_\xi + W_0^{1,\infty}\left(\Omega;\mathbb{R}^m\right),\ u_\nu \overset{*}{\rightharpoonup} u_\xi \text{ in } W^{1,\infty}$$

$$Du_\nu\left(x\right) \in E \cup K, \text{ a.e. in } \Omega$$

$$\int_\Omega \operatorname{dist}\left(Du_\nu\left(x\right);E\right)dx \to 0 \text{ as } \nu \to \infty.$$

Remark 2.3.11. *(i) It is interesting to note that in the scalar case ($n = 1$ or $m = 1$) then $K = \operatorname{int}\operatorname{co}E$ has the relaxation property with respect to E.*

(ii) In the vectorial case we have that, if K has the relaxation property with respect to E, then necessarily

$$K \subset \overline{\operatorname{Qco}}E.$$

Indeed, if we take $\xi \in K$ and $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ be a quasiconvex function such that $f|_E = 0$ and $u_\nu \in u_\xi + W_0^{1,\infty}\left(\Omega;\mathbb{R}^m\right)$ be as in the Definition 2.3.10, we have

$$f\left(\xi\right)\operatorname{meas}\Omega \le \int_\Omega f\left(Du_\nu\left(x\right)\right)dx \quad \forall\, \nu \in \mathbb{N}. \tag{2.21}$$

Moreover, using the fact that $\{Du_\nu\}$ is uniformly bounded and that, by Theorem 2.3.4 (v), f is locally Lipschitz, we have that there exists a constant C such that for every $\eta \in E$

$$|f(Du_\nu(x))| \le C|Du_\nu(x) - \eta|$$

and so

$$|f(Du_\nu(x))| \le C\operatorname{dist}(Du_\nu(x);E).$$

Then we have, using (2.21) and the last property in the definition of the relaxation property, that

$$f(\xi)\operatorname{meas}\Omega \le C\int_\Omega \operatorname{dist}(Du_\nu(x);E)\,dx.$$

Letting ν go to infinity we get the claim.

Remark 2.3.12. *We want to point out that there are some similarities between the relaxation property and the definition of approximate solutions of the differential inclusion* $Du \in E$ *a.e. in* Ω *(see [?] and the references therein). Indeed we say that a sequence* $u_\nu \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ *is a sequence of approximate solutions if* u_ν *converges weakly* in* $W^{1,\infty}(\Omega;\mathbb{R}^m)$ *and* $dist(Du_\nu;E)$ *goes to 0 in* L^1 *for* ν *goes to infinity.*

Using this definition some authors define the quasiconvexification of E *as the set of all* $A \in \mathbb{R}^{m\times n}$ *for which there exists a sequence of approximate solutions* u_ν *with the additional property* $u_\nu|_{\partial\Omega} = l_A$*, where* l_A *is a linear function wuth the gradient equal to* A *(see [?], [?], [?]).*

In some cases it might be "easier" (cf. Dacorogna-Tanteri [?] Section 3) to construct a sequence, which is not piecewise affine everywhere but only on a large set contained in Ω. The relaxation property can then be replaced by the slightly weaker following condition:

Definition 2.3.13. *[Weak Relaxation property] Let* $E, K \subset \mathbb{R}^{m\times n}$*. We say that* K *has the relaxation property with respect to* E *if for every bounded open set* $\Omega \subset \mathbb{R}^n$*, for every affine function* u_ξ *satisfying*

$$Du_\xi(x) = \xi \in K,$$

there exist a sequence $u_\nu \in C^1_{piec}(\Omega;\mathbb{R}^m)$ *with corresponding open sets* $\Omega_\nu \subset \Omega$ *such that*

$$u_\nu \in u_\xi + W_0^{1,\infty}(\Omega;\mathbb{R}^m),\ u_\nu \overset{*}{\rightharpoonup} u_\xi \text{ in } W^{1,\infty}$$

$$u_\nu \text{ is piecewise affine in } \Omega_\nu,\ Du_\nu(x) \in E \cup K,\ \text{a.e. in } \Omega$$

$$\operatorname{meas}\Omega_\nu \to \operatorname{meas}\Omega,\ \int_\Omega \operatorname{dist}(Du_\nu(x);E)\,dx \to 0 \text{ as } \nu \to \infty.$$

We will see that the relaxation property, that involves only the first order derivatives in its definition, is one of the main hypotheses to get almost everywhere solutions of first order differential inclusions. Now we want to give a generalized version of this notion that will be crucial in getting existence of solutions for problems involving higher orders derivatives. With this in mind we start recalling some notations.

Notations (1) Let $N, n, m \geq 1$ be integers. For $u : \mathbb{R}^n \to \mathbb{R}^m$ we write

$$D^N u = \left(\frac{\partial^N u^i}{\partial x_{j_1}...\partial x_{j_N}}\right)_{1\leq j_1,\dots,j_N\leq n}^{1\leq i\leq m} \in \mathbb{R}_s^{m\times n^N}.$$

(The index s stands here for all the natural symmetries implied by the interchange of the order of differentiation). When $N = 1$ we have

$$\mathbb{R}_s^{m\times n} = \mathbb{R}^{m\times n}$$

while if $m = 1$ and $N = 2$ we obtain

$$\mathbb{R}_s^{n^2} = \mathbb{R}_s^{n \times n}$$

i.e., the usual set of symmetric matrices.

(2) For $u : \Omega \to \mathbb{R}^m$ we let

$$D^{[N]}u = \left(u, Du, ..., D^N u\right)$$

stand for the matrix of all partial derivatives of u up to the order N. Note that

$$D^{[N]}u \in \mathbb{R}_s^{m \times M_N} = \mathbb{R}^m \times \mathbb{R}^{m \times n} \times \mathbb{R}_s^{m \times n^2} \times ... \times \mathbb{R}_s^{m \times n^{(N-1)}},$$

where

$$M_N = 1 + n + ... + n^{(N-1)} = \frac{n^N - 1}{n - 1}\ .$$

Hence

$$D^{[N]}u = \left(D^{[N-1]}u, D^N u\right) \in \mathbb{R}_s^{m \times M_N} \times \mathbb{R}_s^{m \times n^N}.$$

Now we can define:

Definition 2.3.14 (Relaxation property). *Let $E, K \subset \mathbb{R}^n \times \mathbb{R}_s^{m \times M_N} \times \mathbb{R}_s^{m \times n^N}$. We say that K has the relaxation property with respect to E if for every bounded open set $\Omega \subset \mathbb{R}^n$, for every u_ξ, a polynomial of degree N with $D^N u_\xi(x) = \xi$, satisfying*

$$\left(x, D^{[N-1]}u_\xi(x), D^N u_\xi(x)\right) \in K,$$

there exists a sequence $u_\nu \in Aff_{piec}^N\left(\overline{\Omega}; \mathbb{R}^m\right)$ (the set of functions that are piecewise polynomials of degree N) such that

$$u_\nu \in u_\xi + W_0^{N,\infty}(\Omega; \mathbb{R}^m)\ ,\ \ u_\nu \overset{*}{\rightharpoonup} u_\xi \text{ in } W^{N,\infty}(\Omega; \mathbb{R}^m)$$

$$\left(x, D^{[N-1]}u_\nu(x), D^N u_\nu(x)\right) \in E \cup K, \text{ a.e. in } \Omega$$

$$\int_\Omega \operatorname{dist}\left(\left(x, D^{[N-1]}u_\nu(x), D^N u_\nu(x)\right); E\right) dx \to 0 \text{ as } \nu \to \infty.$$

2.3.2 The main theorem

We are now in position to state the main abstract existence theorem.

Theorem 2.3.15. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $E, K \subset \mathbb{R}^{m \times n}$ be such that E is compact and K is bounded. Assume that K has the relaxation property* *with respect to E. Let $\varphi \in Aff_{piec}\left(\overline{\Omega};\mathbb{R}^m\right)$ (i.e. φ is piecewise affine) such that*

$$D\varphi(x) \in E \cup K, \text{ a.e. in } \Omega.$$

Then there exists (a dense set of) $u \in \varphi + W_0^{1,\infty}(\Omega;\mathbb{R}^m)$ such that

$$Du(x) \in E, \text{ a.e. in } \Omega.$$

Proof. We let $\overline{V}$ be the closure in $L^\infty(\Omega;\mathbb{R}^m)$ of

$$V = \left\{u \in Aff_{piec}\left(\overline{\Omega};\mathbb{R}^m\right) : u = \varphi \text{ on } \partial\Omega \text{ and } Du(x) \in E \cup K\right\}.$$

V is non empty since $\varphi \in V$. Let, for $k \in \mathbb{N}$,

$$V^k = \operatorname{int}\left\{u \in \overline{V} : \int_\Omega \operatorname{dist}(Du(x);E)\,dx \le \frac{1}{k}\right\}.$$

We claim that V^k is open, which is obvious by definition, and dense in the complete metric space $\overline{V}$. Postponing the proof of the last fact for the end of the proof, we conclude by Baire category theorem that

$$\bigcap_{k=1}^{\infty} V^k \subset \left\{u \in \overline{V} : \operatorname{dist}(Du(x), E) = 0, \text{ a.e. in } \Omega\right\} \subset \overline{V}$$

is dense, and hence non empty, in $\overline{V}$. The result then follows, since E is compact.

We now show that V^k is dense in $\overline{V}$. So let $u \in \overline{V}$ and $\epsilon > 0$ be arbitrary. We wish to find $v \in V^k$ so that

$$\|u - v\|_{L^\infty} \le \epsilon.$$

- We start by finding $\alpha \in \overline{V}$ a point of continuity of the operator D so that

$$\|u - \alpha\|_{L^\infty} \le \frac{\epsilon}{3}.$$

This is always possible by virtue of Corollary 2.4.17 or 2.4.20. In particular we have that the oscillation $\omega_D(\alpha)$ of the gradient operator at α is zero. We recall (cf. Definition 2.4.12) that

$$\omega_D(\alpha) = \lim_{\delta\to 0} \sup_{v,w \in B_\infty(\alpha,\delta)} \|Dv - Dw\|_{L^1(\Omega)}.$$

- We next approximate $\alpha \in \overline{V}$ by $\beta \in V$ so that

$$\|\beta - \alpha\|_{L^\infty} \le \frac{\epsilon}{3} \text{ and } \omega_D(\beta) < \frac{1}{4k}.$$

This is possible since Proposition 2.4.13 and Theorem 2.4.16 ensure us that for every $\varepsilon > 0$ the set

$$W_D^\varepsilon := \{u \in \overline{V} \;:\; \omega_D(u) < \varepsilon\}$$

is open and dense in $\overline{V}$.

- Finally we use the relaxation property on every piece where $D\beta$ is constant and we then construct $v \in V$, by patching all the pieces together, so that

$$\|\beta - v\|_{L^\infty} \leq \frac{\epsilon}{3} \text{ and } \int_\Omega \operatorname{dist}(Dv(x); E)\, dx < \frac{1}{k}.$$

Moreover again by Proposition 2.4.13 and Theorem 2.4.16 we can choose v such that

$$\omega_D(v) < \frac{1}{2k}$$

and consequently we can find $\delta = \delta(k, v) > 0$ so that

$$\|v - \phi\|_{L^\infty} \leq \delta \Rightarrow \|Dv - D\phi\|_{L^1} \leq \frac{1}{2k}$$

and hence

$$\int_\Omega \operatorname{dist}(D\phi(x); E)\, dx \leq \int_\Omega \operatorname{dist}(Dv(x); E)\, dx + \|Dv - D\phi\|_{L^1} < \frac{1}{k}$$

for every $\phi \in B_\infty(v, \delta)$; which implies that $v \in V^k$.

Combining these three facts we have indeed obtained the desired density result. □

In some cases it can be more useful the following slightly version of Theorem 2.3.15, where the relaxation property is replaced by the weak relaxation property (for example in problems involving the incompressibility constraint in nonlinear elasticity as in [**?**] Section 3).

Theorem 2.3.16. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $E, K \subset \mathbb{R}^{m\times n}$ be such that E is compact and K is bounded and open. Assume that K has the weak relaxation property with respect to E. Let $\varphi \in Aff_{piec}\left(\overline{\Omega}; \mathbb{R}^m\right)$ (i.e. φ is piecewise affine) such that*

$$D\varphi(x) \in E \cup K, \text{ a.e. in } \Omega.$$

Then there exists (a dense set of) $u \in \varphi + W_0^{1,\infty}(\Omega; \mathbb{R}^m)$ such that

$$Du(x) \in E, \text{ a.e. in } \Omega.$$

Proof. We start defining for every $\delta > 0$ the set

$$W_\delta := \left\{ \begin{array}{c} u \in C^1_{piec}(\overline{\Omega}; \mathbb{R}^m) \ : \ \exists\, \Omega_u \subset \Omega \text{ with} \\ \operatorname{meas}(\Omega \setminus \Omega_u) < \delta\,, \ \ u \in Aff_{piec}(\Omega_u), \\ u|_{\partial\Omega} = \varphi \text{ and } Du \in E \cup K \text{ a.e. in } \Omega \end{array} \right\}.$$

We let V be the set of all functions u so that there exists a sequence $\delta_n \to 0$ and a sequence $u_n \in W_{\delta_n}$ such that $u_n \to u$ in $L^\infty(\Omega; \mathbb{R}^{m\times n})$ as $n \to 0$.

We have that V is a complete metric space when endowed with the L^∞ metric and it is non empty since $\varphi \in V$.

Now we can proceed as in the proof of previous theorem replacing $\overline{V}$ by V.

To prove that V_k is dense in V, we first observe that if $Du \in E \cup K$ almost everywhere in Ω then

$$\|\operatorname{dist}(Du(x); E)\|_{L^\infty(\Omega;\mathbb{R}^{m\times n})} \leq C = \sup_{\xi\in K}\{\operatorname{dist}(\xi; E)\}. \tag{2.22}$$

Now take $u \in V$ and fix $\varepsilon > 0$ arbitrarily. We start by finding $v \in V$ a point of continuity of the operator D so that

$$\|u - v\|_{L^\infty} \leq \frac{\varepsilon}{3},$$

as before this is always possible in virtue of Corollary 2.4.17 or 2.4.20. In particular we have that the oscillation $\omega_D(v)$ of the gradient operator at u is zero.

By definition of V we may find $v' \in W_\delta$ with $\delta < \frac{1}{4Ck}$ so that $Dv' \in E \cup K$ a.e. in Ω and

$$\|v - v'\|_{L^\infty} \leq \frac{\varepsilon}{3}.$$

Moreover, by Proposition 2.4.13 and Theorem 2.4.16 we can choose v' such that

$$\omega_D(v') < \frac{1}{4k}$$

Then, using the weak relaxation property on every piece where Dv' is constant, we can construct $w \in W_\delta$ with $w = v'$ in $\Omega \setminus \Omega_{v'}$ so that

$$\|v' - w\|_{L^\infty} \leq \frac{\varepsilon}{3},$$

$$\int_{\Omega_{v'}} \operatorname{dist}\left(Dw(x); E\right)\, dx < \frac{1}{2k}$$

and

$$\omega_D(v') < \frac{1}{2k}.$$

Applying (2.22) we therefore have

$$\begin{aligned}\int_\Omega \operatorname{dist}(Dw(x); E)\, dx &\leq \int_{\Omega\setminus\Omega_{v'}} \operatorname{dist}(Dw(x); E)\, dx + \int_{\Omega_{v'}} \operatorname{dist}(Dw(x); E)\, dx \\ &< C\delta + \frac{1}{2k} < \frac{1}{2k}.\end{aligned}$$

Moreover again by Proposition 2.4.13 and Theorem 2.4.16 we can choose w such that

$$\omega_D(v) < \frac{1}{2k}$$

and consequently we can find $\delta = \delta(k, v) > 0$ so that

$$\|w - \phi\|_{L^\infty} \leq \delta \Rightarrow \|Dw - D\phi\|_{L^1} \leq \frac{1}{2k}$$

and hence

$$\int_\Omega \operatorname{dist}(D\phi(x); E)\, dx \leq \int_\Omega \operatorname{dist}(Dw(x); E)\, dx + \|Dw - D\phi\|_{L^1} < \frac{1}{k}$$

for every $\phi \in B_\infty(w, \delta)$; which implies that $w \in V^k$.

Finally, since $w \in C^1_{piec}(\Omega \setminus \Omega_{v'}; \mathbb{R}^m)$, applying the Lemma 2.4.3 in each part where w is C^1 with $A = \operatorname{int}(E \cup K)$ we can prove that $w \in V$ and therefore the claim. □

Remark 2.3.17. *According to the approximation lemmata in the appendix, the boundary datum φ can be more general, for instance, φ can be taken in $C^1_{piec}\left(\overline{\Omega}; \mathbb{R}^m\right)$, with $D\varphi(x) \in E \cup K$ (see Lemma 2.4.1). Or, if K is open and convex, φ can be taken in $W^{1,\infty}(\Omega; \mathbb{R}^m)$ and*

$$D\varphi(x) \in C, \text{ a.e. in } \Omega$$

where $C \subset K$ is compact (see Lemma 2.4.6).

To conclude this section we give a sufficient condition that ensures the relaxation property. In concrete examples this condition is usually much easier to check than the relaxation property. We start with a definition.

Definition 2.3.18 (Approximation property). *Let $E \subset K(E) \subset \mathbb{R}^{m\times n}$. The sets E and $K(E)$ are said to have the approximation property if there exists a family of closed sets E_δ and $K(E_\delta)$, $\delta > 0$, such that*

(1) $E_\delta \subset K(E_\delta) \subset \operatorname{int} K(E)$ for every $\delta > 0$;

(2) for every $\epsilon > 0$ there exists $\delta_0 = \delta_0(\epsilon) > 0$ such that $\operatorname{dist}(\eta; E) \leq \epsilon$ for every $\eta \in E_\delta$ and $\delta \in [0, \delta_0]$;

(3) if $\eta \in \operatorname{int} K(E)$ then $\eta \in K(E_\delta)$ for every $\delta > 0$ sufficiently small.

We therefore have the following theorem (cf. Theorem 6.14 in [**?**] and for a slightly more flexible one see Theorem 6.15).

Theorem 2.3.19. *Let $E \subset \mathbb{R}^{m\times n}$ be compact and $\operatorname{Rco} E$ has the approximation property with $K(E_\delta) = \operatorname{Rco} E_\delta$, then $\operatorname{int} \operatorname{Rco} E$ has the relaxation property with respect to E.*

Example 2.3.20. *A recent application of Theorem 2.3.15 can be found in [**?**], where is studied the problem to find generalized solutions of the differential inclusion*

$$\begin{cases} Du \in E & \text{a.e. in } \Omega \subset \mathbb{R}^2 \\ u(x) = \varphi(x) & x \in \partial\Omega \end{cases} \tag{2.23}$$

where E is a compact isotropic set of $\mathbb{R}^{2\times 2}$ (that is $AEB \subseteq E$ for every $A, B \in \mathcal{O}(2)$).

The necessity to have a general existence theorem comes out from the fact that in general we don't know if an isotropic set can be written as a level set of some quasiconvex functions.

Using an equivalent way to define an isotropic set E and applying the Theorem 2.3.19 it is possible to find some sufficient conditions for the existence of solutions. Indeed E can be written as

$$E = \{\xi \in \mathbb{R}^{2\times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in K\}$$

for some compact set $K \subset T = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq y\}$, where we have denoted by $(\lambda_1(\xi), \lambda_2(\xi))$ the singular values of the matrix ξ. Then if we assume that

$$\min_{(x,y)\in K} \{x\} > 0$$

and $\varphi \in C^1_{piec}(\overline{\Omega}, \mathbb{R}^2)$ such that

$$D\varphi(x) \in E \cup \operatorname{int} \operatorname{Rco} E \text{ in } \Omega,$$

then there exists a function $u \in \varphi + W_0^{1,\infty}(\Omega; \mathbb{R}^2)$ solution of (2.23).

2.3.3 Some extensions

In the present section we will prove some extensions of the results of the preceding section. The proof will be done essentially following the same argument of the proof of Theorem 2.3.15 and using the standard procedure of freezing the lower order terms as in [**?**] Theorem 6.3.

In the sequel we will denote points of E by (x, s, ξ) with $x \in \mathbb{R}^n$, $s \in \mathbb{R}_s^{m\times M_N}$ and $\xi \in \mathbb{R}_s^{m\times n^N}$.

The following theorem is the main abstract existence theorem.

Theorem 2.3.21. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $E, K \subset \mathbb{R}^n \times \mathbb{R}_s^{m\times M_N} \times \mathbb{R}_s^{m\times n^N}$ be such that E is closed, and both E and K are bounded uniformly for $x \in \Omega$ and whenever s vary on a bounded set of $\mathbb{R}_s^{m\times M_N}$. Assume that K has the relaxation property with respect to E. Let $\varphi \in Aff^N_{piec}\left(\overline{\Omega}; \mathbb{R}^m\right)$ be such that*

$$\left(x, D^{[N-1]}\varphi(x), D^N\varphi(x)\right) \in E \cup K, \text{ a.e. in } \Omega;$$

then there exists (a dense set of) $u \in \varphi + W_0^{N,\infty}(\Omega; \mathbb{R}^m)$ such that

$$\left(x, D^{[N-1]}u(x), D^N u(x)\right) \in E, \text{ a.e. in } \Omega.$$

Remark 2.3.22. *(i) The boundedness of E (or of K) stated in the theorem should be understood as follows. For every $R > 0$, there exists $\gamma = \gamma(R)$ so that*

$$(x, s, \xi) \in E,\ x \in \Omega \text{ and } |x| + |s| \leq R \ \Rightarrow\ |\xi| \leq \gamma.$$

*(ii) As in the previous section, a theorem such as Theorem 10 is also available in the present context, but we do not discuss the details and we refer to Theorem 6.14 and Theorem 6.15 in [**?**].*

Proof. Since $\varphi \in W^{N,\infty}(\Omega; \mathbb{R}^n)$ we can find $R > 0$ so that

$$|D^{[N-1]}\varphi(x)| < R.$$

We let $\overline{V}$ be the closure in $C^{N-1}(\Omega; \mathbb{R}^m)$ of

$$V = \left\{ \begin{array}{c} u \in Aff^N_{piec}\left(\overline{\Omega}; \mathbb{R}^m\right) : u \in \varphi + W_0^{N,\infty}(\Omega; \mathbb{R}^m), |D^{[N-1]}u(x)| < R \\ \text{and } \left(x, D^{[N-1]}u(x), D^N u(x)\right) \in E \cup K \end{array} \right\}.$$

V is non empty since $\varphi \in V$ and it is a complete metric space when endowed with the C^{N-1} norm.

Let, for $k \in \mathbb{N}$,

$$V^k = \operatorname{int}\left\{ u \in \overline{V} : \int_\Omega \operatorname{dist}\left(\left(x, D^{[N-1]}u(x), D^N u(x)\right); E\right) dx \leq \frac{1}{k} \right\}.$$

We claim that V^k, in addition to be open, is dense in the complete metric space $\overline{V}$. Postponing the proof of this fact for the end of the proof, we conclude by Baire category theorem that

$$\bigcap_{k=1}^{\infty} V^k \subset \left\{ u \in \overline{V} : \operatorname{dist}\left(\left(x, D^{[N-1]}u(x), D^N u(x)\right), E\right) = 0, \text{ a.e. in } \Omega \right\} \subset \overline{V}$$

is dense, and hence non empty, in $\overline{V}$. The result then follows, since E is compact.

We finally show that V^k is dense in $\overline{V}$. So let $u \in \overline{V}$ and $\epsilon > 0$ be arbitrary. We wish to find $v \in V^k$ so that

$$\|u - v\|_{N-1,\infty} \leq \epsilon.$$

We recall (cf. the Appendix) that

$$\omega_{D^N}(u) = \lim_{\delta \to 0} \sup_{\varphi, \psi \in B_{C^{N-1}}(u,\delta)} \left\| D^N\varphi - D^N\psi \right\|_{L^1},$$

where $B_{C^{N-1}}(u, \delta) = \{v \in \overline{V} : \|u - v\|_{N-1,\infty} < \delta\}$.

- We start by finding $\alpha \in \overline{V}$ a point of continuity of the operator D^N (in particular $\omega_{D^N}(\alpha) = 0$) so that

$$\|u - \alpha\|_{N-1,\infty} \leq \frac{\epsilon}{3}.$$

This is always possible by virtue of Corollary 2.4.18.

- We next approximate $\alpha \in \overline{V}$ by $\beta \in V$ so that,

$$\|\beta - \alpha\|_{N-1,\infty} \leq \frac{\epsilon}{3} \text{ and } \omega_{D^{N-1}}(\beta) < 1/3k\,.$$

Since $|D^{[N-1]}\beta(x)| < R$, from now on all the approximations can be supposed, without loss of generality, sufficiently small in order to work always under the hypothesis

$$|D^{[N-1]}u(x)| < R.$$

- By working on each piece where $D^N\beta$ is constant, without loss of generality, we can assume that $\beta \in C^N\left(\overline{\Omega}; \mathbb{R}^m\right)$ with $D^N\beta(x) = \text{constant}$ in $\overline{\Omega}$ and $\left(x, D^{[N-1]}\beta(x), D^N\beta(x)\right) \in E \cup K$. Therefore let

$$\begin{aligned} &\Omega_0 = \left\{x \in \Omega : \left(x, D^{[N-1]}\beta(x), D^N\beta(x)\right) \in E\right\} \\ &\Omega_1 = \Omega \setminus \Omega_0. \end{aligned}$$

It is clear that Ω_0 is closed, since E is compact, hence Ω_1 is open.

- We can now use the relaxation property on Ω_1 to find $v_1 \in Aff_{piec}^N\left(\overline{\Omega_1}; \mathbb{R}^m\right)$ such that

$$\begin{cases} v_1 \in \beta + W_0^{N,\infty}(\Omega_1; \mathbb{R}^m) \\ \|v_1 - \beta\|_{N-1,\infty} \leq \dfrac{\epsilon}{3}; \\ \left(x, D^{[N-1]}v_1(x), D^N v_1(x)\right) \in E \cup K \quad \text{a.e. } x \in \Omega_1 \\ \displaystyle\int_{\Omega_1} \operatorname{dist}\left(\left(x, D^{[N-1]}v_1(x), D^N v_1(x)\right); E\right) dx \leq \frac{1}{3k}. \end{cases}$$

We can now define

$$v(x) = \begin{cases} \beta(x) & \text{if} \quad x \in \Omega_0 \\ v_1(x) & \text{if} \quad x \in \Omega_1. \end{cases}$$

Observe that v is $Aff_{\text{piec}}^N\left(\overline{\Omega}; \mathbb{R}^m\right)$ and

$$\begin{cases} v \in \varphi + W_0^{N,\infty}(\Omega; \mathbb{R}^m) \\ \|v - \beta\|_{N-1,\infty} \leq \dfrac{\epsilon}{3}; \\ \left(x, D^{[N-1]}v(x), D^N v(x)\right) \in E \cup K \quad \text{a.e. } x \in \Omega \\ \displaystyle\int_{\Omega} \operatorname{dist}\left(\left(x, D^{[N-1]}v(x), D^N v(x)\right); E\right) dx \leq \frac{1}{3k}. \end{cases}$$

Moreover by taking a smaller ε if needed we can ensure also that

$$\omega_{D^{N-1}}(v) < \frac{1}{3k},$$

then we can find $h = h(k, v)$ so that

$$\|v - \psi\|_{N-1,\infty} \leq h \Longrightarrow \left\|D^N\psi - D^N v\right\|_{L^1} \leq \frac{1}{3k}.$$

Hence choosing $h < \frac{1}{3k}$ and writing for simplicity of notations

$$\eta_v(x) = \left(x, D^{[N-1]}v(x), D^N v\left(x\right)\right)$$

we have

$$\begin{aligned}\int_\Omega \operatorname{dist}\left(\left(x, D^{[N-1]}\psi, D^N\psi\right); E\right) dx \quad &\leq \quad \int_\Omega \operatorname{dist}\left(\eta_v(x); E\right) dx \\ &+ \quad \|D^{[N-1]}\psi(x) - D^{[N-1]}v(x)\|_{N-1,\infty} \\ &+ \quad \|D^N\psi(x) - D^N v(x)\|_{L^1} \\ &< \quad \frac{1}{3k} + h + \frac{1}{3k} \leq \frac{1}{k},\end{aligned}$$

for every $\psi \in B_{N-1,\infty}(v, h)$; which implies that $v \in V^k$.

Combining these three facts we have indeed obtained the desired density result. □

In the following theorem we show that if K is open (in the relative topology of $\mathbb{R}^n \times \mathbb{R}_s^{m\times M_N} \times \mathbb{R}_s^{m\times n^N}$) then we can aslo take $\varphi \in C_{piec}^N\left(\overline{\Omega}; \mathbb{R}^m\right)$.

Theorem 2.3.23. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $E, K \subset \mathbb{R}^n \times \mathbb{R}_s^{m\times M_N} \times \mathbb{R}_s^{m\times n^N}$ be such that E is closed, K open and both E and K are bounded uniformly for $x \in \Omega$ and whenever s vary on a bounded set of $\mathbb{R}_s^{m\times M_N}$. Assume that K has the relaxation property with respect to E. Let $\varphi \in C_{piec}^N\left(\overline{\Omega}; \mathbb{R}^m\right)$ be such that*

$$\left(x, D^{[N-1]}\varphi\left(x\right), D^N\varphi\left(x\right)\right) \in E \cup K, \text{ a.e. in } \Omega;$$

then there exists (a dense set of) $u \in \varphi + W_0^{N,\infty}\left(\Omega; \mathbb{R}^m\right)$ such that

$$\left(x, D^{[N-1]}u\left(x\right), D^N u\left(x\right)\right) \in E, \text{ a.e. in } \Omega.$$

Proof. Since $\varphi \in W^{N,\infty}(\overline{\Omega}; \mathbb{R}^n)$ we can find $R > 0$ so that

$$|D^{[N-1]}\varphi(x)| < R.$$

We let $\overline{V}$ be the closure in $C^{N-1}\left(\Omega; \mathbb{R}^m\right)$ of

$$V = \left\{\begin{array}{c} u \in C_{piec}^N\left(\overline{\Omega}; \mathbb{R}^m\right) : u \in \varphi + W_0^{N,\infty}\left(\Omega; \mathbb{R}^m\right), |D^{[N-1]}u(x)| < R \\ \text{and } \left(x, D^{[N-1]}u(x), D^N u\left(x\right)\right) \in E \cup K \end{array}\right\}.$$

V is non empty since $\varphi \in V$ and it is a complete metric space when endowed with the C^{N-1} norm.

Let, for $k \in \mathbb{N}$,

$$V^k = \operatorname{int}\left\{u \in \overline{V} : \int_\Omega \operatorname{dist}\left(\left(x, D^{[N-1]}u(x), D^N u\left(x\right)\right); E\right) dx \leq \frac{1}{k}\right\}.$$

We claim that V^k, in addition to be open, is dense in the complete metric space $\overline{V}$. Postponing the proof of this fact for the end of the proof, we conclude by Baire category theorem that

$$\bigcap_{k=1}^{\infty} V^k \subset \left\{u \in \overline{V} : \operatorname{dist}\left(\left(x, D^{[N-1]}u(x), D^N u\left(x\right)\right), E\right) = 0, \text{ a.e. in } \Omega\right\} \subset \overline{V}$$

is dense, and hence non empty, in $\overline{V}$. The result then follows, since E is compact.

We finally show that V^k is dense in $\overline{V}$. So let $u \in \overline{V}$ and $\epsilon > 0$ be arbitrary. We wish to find $v \in V^k$ so that

$$\|u - v\|_{N-1,\infty} \leq \epsilon.$$

We recall (cf. the Appendix) that

$$\omega_{D^N}(u) = \lim_{\delta \to 0} \sup_{\varphi, \psi \in B_{C^{N-1}}(u,\delta)} \left\|D^N \varphi - D^N \psi\right\|_{L^1} .$$

- We start by finding $\alpha \in \overline{V}$ a point of continuity of the operator D^N (in particular $\omega_{D^N}\left(\alpha\right) = 0$) so that

$$\|u - \alpha\|_{N-1,\infty} \leq \frac{\epsilon}{4} \text{ and } |D^{[N-1]}\alpha(x)| < R.$$

This is always possible by virtue of Corollary 2.4.18.

From now on all the approximations can be supposed, without loss of generality, sufficiently small in order to work always under the hypothesis

$$|D^{[N-1]}u(x)| < R.$$

- We next approximate $\alpha \in \overline{V}$ by $\beta \in V$ so that,

$$\|\beta - \alpha\|_{N-1,\infty} \leq \frac{\epsilon}{4} \text{ and } \omega_{D^{N-1}}\left(\beta\right) < 1/3k\,.$$

- By working on each piece where β is C^N, without loss of generality, we can assume that $\beta \in C^N\left(\overline{\Omega}; \mathbb{R}^m\right)$ and $\left(x, D^{[N-1]}\beta(x), D^N\beta\left(x\right)\right) \in E \cup K$. Therefore let

$$\Omega_0 = \left\{x \in \Omega : \left(x, D^{[N-1]}\beta(x), D^N\beta\left(x\right)\right) \in E\right\}$$
$$\Omega_1 = \Omega \setminus \Omega_0.$$

It is clear that Ω_0 is closed, since E is compact, hence Ω_1 is open.

Then by theorem 10.16 in [?] for any fixed $\delta_1 > 0$ we can find $\gamma_1 \in C^N\left(\overline{\Omega_1};\mathbb{R}^m\right)$, an integer J and $\Omega_j \subset \Omega_1$, $j \in \{1,\dots,J\}$ disjoint open sets such that

$$\begin{cases} \gamma_1 = \beta \text{ near } \partial\Omega_1 \\ \|\gamma_1 - \beta\|_{N,\infty} \leq \dfrac{\epsilon}{8} \ ; \ D^N\gamma_1 = \xi_j = \text{constant in } \Omega_j \\ \left(x, D^{[N-1]}\gamma_1(x), D^N\gamma_1(x)\right) \in K \ \text{ a.e. } x \in \Omega_1 \\ \operatorname{meas}\left(\Omega_1 \setminus \cup_{j=1}^J \Omega_j\right) \leq \delta_1. \end{cases}$$

- We can use now the relaxation property on every piece where $D^N\gamma_1$ is constant to find $v_{1,j} \in Aff_{piec}\left(\overline{\Omega_j};\mathbb{R}^m\right)$ such that

$$\begin{cases} v_{1,j} \in \gamma_1 + W_0^{N,\infty}(\Omega_j;\mathbb{R}^m) \\ \|v_{1,j} - \gamma_1\|_{N-1,\infty} \leq \dfrac{\epsilon}{8}; \\ \left(x, D^{[N-1]}v_{1,j}(x), D^N v_{1,j}(x)\right) \in E \cup K \ \text{ a.e. } x \in \Omega_j \\ \displaystyle\int_{\Omega_j} \operatorname{dist}\left(\left(x, D^{[N-1]}v_{1,j}(x), D^N v_{1,j}(x)\right); E\right) dx \leq \frac{\delta}{2}\frac{\operatorname{meas}(\Omega_j)}{\operatorname{meas}(\Omega_1)}. \end{cases}$$

We can now define

$$v(x) = \begin{cases} \beta(x) & \text{if} \quad x \in \Omega_0 \\ \gamma_1(x) & \text{if} \quad x \in \Omega_1 \setminus \cup_{j=1}^J \Omega_j \\ v_{1,j}(x) & \text{if} \quad x \in \Omega_j. \end{cases}$$

Observe that v is $C^N_{\text{piec}}\left(\overline{\Omega};\mathbb{R}^m\right)$ and

$$\begin{cases} v \in \varphi + W_0^{N,\infty}(\Omega;\mathbb{R}^m) \\ \|v - \beta\|_{N-1,\infty} \leq \dfrac{\epsilon}{4}; \\ \left(x, D^{[N-1]}v(x), D^N v(x)\right) \in E \cup K \ \text{ a.e. } x \in \Omega. \end{cases}$$

Writing for simplicity of notations $\eta_v(x) = \left(x, D^{[N-1]}v(x), D^N v(x)\right)$, we have

$$\begin{aligned} \int_\Omega \operatorname{dist}(\eta_v(x), E)\,dx &\leq \int_{\Omega_0} \operatorname{dist}(\eta_v(x), E)\,dx + \int_{\Omega_1} \operatorname{dist}(\eta_v(x), E)\,dx \\ &= \int_{\Omega_1 \setminus \cup_{j=1}^J \Omega_j} \operatorname{dist}(\eta_v(x), E)\,dx \\ &+ \sum_{j=1}^J \int_{\Omega_j} \operatorname{dist}(\eta_v(x), E)\,dx \\ &\leq \operatorname{meas}\left(\Omega_1 \setminus \cup_{j=1}^J\right) \sup\left(|\operatorname{dist}(\eta_v(x), E)|\right) + \frac{\delta}{2} \end{aligned}$$

then we choose $\delta = \frac{1}{3k}$ and $\delta_1 = \frac{\delta}{2B}$ where B is a constant depending on R such that

$$(x, s, \xi) \in E \cup K,\ x \in \Omega \text{ and } |s| \leq R \ \Rightarrow \ |\xi| \leq B,$$

this is possible since E and K are bounded in the sense of Remark 2.3.22. Finally we obtain

$$\int_\Omega \operatorname{dist}\left(\left(x, D^{[N-1]}v(x), D^N v(x)\right), E\right) dx \leq \frac{1}{3k}.$$

Moreover by taking a smaller ε if needed we can ensure also that

$$\omega_{D^{N-1}}(v) < \frac{1}{3k},$$

then we can find $h = h(k, v)$ so that

$$\|v - \psi\|_{N,\infty} \leq h \Longrightarrow \left\|D^N\psi - D^N v\right\|_{L^1} \leq \frac{1}{3k}.$$

Hence choosing $h < \frac{1}{3k}$ we have

$$\begin{aligned}\int_\Omega \operatorname{dist}\left(\left(x, D^{[N-1]}\psi(x), D^N\psi(x)\right); E\right) dx \ &\leq \ \int_\Omega \operatorname{dist}\left(\eta_v(x); E\right) dx \\ &+ \ \|D^{N-1}\psi(x) - D^{N-1}v(x)\|_{N-1,\infty} \\ &+ \ \|D^N\psi(x) - D^N v(x)\|_{L^1} \\ &< \ \frac{1}{3k} + h + \frac{1}{3k} \leq \frac{1}{k},\end{aligned}$$

for every $\psi \in B_{N-1,\infty}(v, h)$; which implies that $v \in V^k$.

Combining these three facts we have indeed obtained the desired density result. □

2.4 Appendix

2.4.1 Approximation lemmata

In this section we recall some approximation lemmata that are well known in the theory of Calculus of Variations (see [?] Section 10 for detailed proofs)

We start stating some notations. Let Ω be an open set of $\mathbb{R}^n$. We say that a function $v \in W^{1,\infty}(\Omega)$ is *piecewise affine* in Ω if there exists an (at most) countable partition of Ω into open sets Ω_k, $k \in \mathbb{N}$ and a set of measure zero, i.e.,

$$\Omega_h \cap \Omega_k = \emptyset, \ \ \forall h,\ k \in \mathbb{N},\ h \neq k,$$

$$\operatorname{meas}\left(\Omega - \bigcup_{k\in\mathbb{N}} \Omega_k\right) = 0,$$

such that v is affine on each Ω_k, i.e., there exists $\xi_k \in \mathbb{R}^n$ and $q_k \in \mathbb{R}$ such that

$$v(x) = \langle \xi_k; x \rangle + q_k, \quad \forall\, x \in \Omega_k,\ k \in \mathbb{N}.$$

We say that a function $u \in W^{N,\infty}(\Omega; \mathbb{R}^m)$ is *piecewise polynomial of degree* $N \geq 1$ in Ω if the derivative $D^N u$ is piecewise constant in Ω; i.e. if there exists an (at most) countable partition of Ω into open sets Ω_k, $k \in \mathbb{N}$ and a set of measure zero, i.e.,

$$\Omega_h \cap \Omega_k = \emptyset, \quad \forall h,\ k \in \mathbb{N},\ h \neq k,$$

$$\text{meas}\left(\Omega - \bigcup_{k \in \mathbb{N}} \Omega_k \right) = 0,$$

such that v is affine on each Ω_k, i.e., there exists $\xi_k \in \mathbb{R}^n$ and $q_k \in \mathbb{R}$ such that

$$D^N u(x) = \xi_k, \quad \forall\, x \in \Omega_k,\ k \in \mathbb{N}.$$

The first result that we recall in this section is a consequence of the so called Vitali covering theorems[1]

Lemma 2.4.1. *Let Ω be an open set of $\mathbb{R}^n$. Let A, B be disjoint sets of $\mathbb{R}^n$, with A open and B possibly empty. Let $u \in W^{1,\infty}(\Omega)$ such that*

$$Du(x) \in A \cup B < \ a.e.\ x \in \Omega.$$

Then, for every $\varepsilon > 0$, there exists a function $v \in W^{1,\infty}(\Omega)$ and an open set $\Omega' \subset \Omega$ ($\Omega' = \Omega$ if $B = \emptyset$) such that

$$\begin{cases} v \text{ is piecewise affine on } \Omega'; \\ v = u \text{ on } \partial\Omega; \\ \|v - u\|_{L^\infty(\Omega)} < \varepsilon; \\ Dv(x) \in A,\ a.e.\, x \in \Omega' \\ Dv(x) = Du(x) \in B,\ a.e.\, x \in \Omega - \Omega'. \end{cases}$$

The following result is a classical tool in the calculus of variations to obtain necessary conditions (c.f. for example [?], [?] and in this form [?], [?], [?], [?])

Lemma 2.4.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set with finite measure. Let $t \in [0,1]$ and ξ, $\eta \in \mathbb{R}^n$. Let φ be an affine function in Ω (i.e. with constant gradient in Ω) such that*

$$D\varphi(x) = t\xi + (1-t)\eta, \ \forall\, x \in \Omega.$$

[1]The first covering result of this type was proved by Vitali [?], and, also if the most common covering theorem is due essentially to Lebesgue [?] in the litterature this type of results are commonly called Vitali covering theorems. A presentation of these result with modern notations and with recent results can be founded in [?], [?], [?].

Then, for every $\varepsilon > 0$, there exists $u \in W^{1,\infty}(\Omega)$ and there exist disjoint open sets Ω_ξ, $\Omega_\eta \subset \Omega$ such that

$$\begin{cases} |meas\Omega_\xi - t meas\Omega|, |meas\Omega_\eta - (1-t) meas\Omega| \le \varepsilon \\ u(x) = \varphi(x), \ \forall\, x \in \partial\Omega \\ |u(x) - \varphi(x)| \le \varepsilon, \ \forall\, x \in \Omega \\ Du(x) = \begin{cases} \xi \ a.e. \ in\, \Omega_\xi \\ \eta \ a.e. \ in\, \Omega_\eta \eta \end{cases} \\ dist(Du(x), co\{\xi, \eta\}) \le \varepsilon \ a.e. \ in\, \Omega, \end{cases}$$

where $co\{\xi,\eta\} = [\xi,\eta]$ is the convex hull of $\{\xi,\eta\}$, that is the closed segment joining ξ and η.

Now we want to recall what are the last available results of approximation with piecewise affine functions for vector valued functions. As we will see in the following lemmata, to have results similar to Lemma 2.4.1 we have to require some extra hypotheses on the constraint set A.

The following results holds.

Lemma 2.4.3. *Let $\Omega \subset \mathbb{R}^n$ and $A \subset \mathbb{R}^{m\times n}$ be open sets. Let $u \in C^1(\Omega;\mathbb{R}^m) \cap W^{1,\infty}(\Omega;\mathbb{R}^m)$ with*

$$Du(x) \in A \ \ \forall\, x \in \Omega.$$

Then there exists a sequence of functions $\{v_k\}_{k\in\mathbb{N}} \subset W^{1,\infty}(\Omega;\mathbb{R}^m)$ such that

$$\begin{cases} each\ v_k \ is\ piecewise\ affine\ on\ \Omega; \\ v_k = u, \ on\ \partial\Omega \ \ \forall\, k \in \mathbb{N}; \\ Dv_k(x) \in A, \ a.e. \ \ x \in \Omega; \\ \|v_k - u\|_{W^{1,\infty}_{loc}(\Omega;\mathbb{R}^m)} \to 0 \ as\ k \to +\infty; \\ \|v_k - u\|_{L^{1,\infty}(\Omega;\mathbb{R}^m)} \to 0 \ as\ k \to +\infty. \end{cases}$$

Moreover, if $u \in C^1(\overline{\Omega};\mathbb{R}^m)$ and $Du(x) \in A$ for all $x \in \overline{\Omega}$, then

$$\|v_k - u\|_{W^{1,\infty}(\Omega;\mathbb{R}^m)} \to 0 \ as\ k \to +\infty.$$

Lemma 2.4.4. *Let $\Omega \subset \mathbb{R}^n$ be bounded and open,$K \in \mathbb{R}^{m\times n^N}_s$ be compact and $u \in C^N(\overline{\Omega};\mathbb{R}^m)$ such that*

$$D^N u(x) \in \operatorname{int} K, \quad \forall\, x \in \Omega.$$

Let $\varepsilon > 0$; then there exists $u_\varepsilon \in C^N(\overline{\Omega};\mathbb{R}^m)$, an integer $I = I(\varepsilon)$, $\xi_{\varepsilon,i} \in \mathbb{R}^{m\times n^N}_s$ and $\Omega_{\varepsilon,i} \subset \Omega$, $1 \le i \le I$, disjoint open sets such that

$$\begin{cases} u_\varepsilon \equiv u \ near\ \partial\Omega; \\ \|u_\varepsilon - u\|_{W^{N,\infty}} \le \varepsilon, \ in\ \Omega; \\ D^N u_\varepsilon(x) \in \operatorname{int} K, \ a.e. \ \ x \in \Omega; \\ \operatorname{meas}\left(\Omega - \bigcup_{i=1}^{I} \Omega_{\varepsilon,i}\right) \le \varepsilon; \\ D^N u_\varepsilon(x) = \xi_{\varepsilon,i} = constant, \forall\, x \in \Omega_{\varepsilon,i}. \end{cases}$$

Remark 2.4.5. *The same result holds if* $K \subset \mathbb{R}^n \times \mathbb{R}_s^{m\times M_N} \times \mathbb{R}_s^{m\times n^N}$ *is compact and* $u \in C^N(\overline{\Omega};\mathbb{R}^m)$ *satisfies*

$$\left(x, D^{[N-1]}u(x), D^N u(x)\right) \in \operatorname{int} K, \quad \forall\, x \in \Omega.$$

The conclusion is then that

$$\left(x, D^{[N-1]}u_\varepsilon(x), D^N u_\varepsilon(x)\right) \in \operatorname{int} K, \ \text{a.e.} \ \ x \in \Omega.$$

Lemma 2.4.6. *Let* Ω *be a bounded open set of* $\mathbb{R}^n$*, let* $u \in W^{N,\infty}(\Omega;\mathbb{R}^m)$ *for some* $N \geq 1$ *and let* K *be a convex bounded set in* $\mathbb{R}_s^{m\times n^N}$ *such that* $D^N u(x)$ *is compactly contained in* $\operatorname{int} K$*, for almost every* $x \in \Omega$*, i.e., there exists a compact set* $K' \subset K$ *such that*

$$\operatorname{meas}\left\{x \in \Omega \;:\; D^N u(x) \notin K'\right\} = 0.$$

Then there exists a sequence of functions $\{v_k\}_{k\in\mathbb{N}} \subset W^{N,\infty}(\Omega;\mathbb{R}^m)$ *such that*

$$\begin{cases} \text{each } v_k \text{ is piecewise polynomial of degree } N \text{ in } \Omega; \\ v_k \in u + W_0^{N,\infty}(\Omega;\mathbb{R}^m), \ \ \forall\, k \in \mathbb{N}; \\ D^N v_k(x) \in \operatorname{int} K, \ \text{a.e.} \ \ x \in \Omega; \\ \|v_k - u\|_{W^{N,p}(\Omega;\mathbb{R}^m)} \to 0 \text{ as } k \to +\infty \ \ \forall\, p \in [1,+\infty); \\ \|v_k - u\|_{W^{N-1,\infty}(\Omega;\mathbb{R}^m)} \to 0 \text{ as } k \to +\infty. \end{cases}$$

2.4.2 Function of first class

In this section we investigate some properties of the set where a given function $f : X \to Y$, where X and Y are complete metric spaces, is continuous and we show some applications to the gradient operator . More precisely we give here a proof of the Baire one function theorem (see also [?], [?]).

We start recalling some abstract topologically definitions. Let X a topological space.

Definition 2.4.7. *A set* $A \subset X$ *is nowhere dense if the interior of its closure is empty, that is, if for every non-empty open set* G *there is a non-empty open set* H *contained in* $G - A$*. In other words* A *is nowhere dense if and only if its complement* A^c *contains a dense open set.*

The class of nowhere dense sets is closed under certain operation, namely any subset of a nowhere dense set, any finite union of nowhere dense sets and the closure of a nowhere dense set are nowhere dense. Nevertheless a countable union of nowhere dense sets is not in general nowhere dense, indeed it may even be dense. For instance, the set of rational numbers is a countable union of singletons that are nowhere dense.

Definition 2.4.8. *A set* $A \subset X$ *is of first category in the sense of Baire if it can be represented as a countable union of nowhere dense sets; otherwise, it is said of second category.*

Definition 2.4.9. *A topologically space X is called a Baire space if every non empty open set in X is of second category, or equivalently, if the complement of every set of first category is dense in X. In a Baire space, the complement of any set of first category is called a residual set.*

Definition 2.4.10. *A metric space (X, ρ) is topologically complete if it is homeomorphic to some complete space, i.e. if it can be remetrized with a topologically equivalent metric so as to be complete*

An important property of topologically complete space is that the Baire category theorem holds. We give here a proof for sake of completeness.

Theorem 2.4.11. *Any topologically complete metric space X is a Baire space. That is if $A \subset X$ is of first category, then $X - A$ is dense in X.*

Proof. Let $A = \bigcup_{n\in\mathbb{N}} A_n$, where A_n is nowhere dense, let ρ be a metric with respect to which X is complete, and let $S_0 \subset X$ be a non-empty open set. Choose a nested sequence of balls S_n of radius $r_n < 1/n$ such that $\overline{S}_n \subset S_{n-1} - A_n$ for $n \geq 1$ [2]. This can be done step by step, taking for S_n a ball with center x_n in $S_{n-1} - A_n$ (which is non-empty because $\overline{A}_n$ is nowhere dense) and with sufficiently small radius. Then $\{x_n\}$ is a Cauchy sequence, since

$$\rho(x_i, x_j) \leq \rho(x_i, x_n) + \rho(x_n, x_j) < 2r_n \ \text{ for } i, j \geq n.$$

Hence $x_n \to x$ for some $x \in X$. Since $x_i \in \overline{S}_n$ for $i \geq n$, if follows that $x \in \bigcap_{n\in\mathbb{N}} \overline{S}_n \subset S_0 - A$. This show that $X - A$ is dense in X and the claim. □

To handle with the continuity properties of functions we should introduce the following notion of oscillation that allows us to "measure" in some sense the continuity of a given function in a point.

Definition 2.4.12. *Let X, Y complete metric spaces and $f : X \to Y$. We define the oscillation of f in $x_0 \in X$ as*

$$\omega_f(x_0) = \lim_{\delta\to 0} \sup_{x,y \in B(x_0,\delta)} d_Y\left(f(y), f(x)\right)$$

where $B(x_0, \delta) := \{x \in : \ d_X(x, x_0) < \delta\}$ is the open ball centered in x_0 and d_X, d_Y are the metric on the spaces X and Y respectively.

In the next proposition we state some useful propertied of ω_f.

[2] The proof of Theorem 2.4.11 implicitly uses the axiom of choice in the weak form, know as the principle of dependent choices, however it should be emphasized that in a complete *separable* metric space the Baire category theorem does not require any form of the axiom of choice (see [?], [?]) for more details

Proposition 2.4.13. *Let X, Y metric spaces, and $f : X \to Y$.*
(i) f is continuous in $x_0 \in X$ if and only if $\omega_f(x_0) = 0$
(ii) The set $\Omega_f^\varepsilon := \{x \in X \ : \ \omega_f(x) < \varepsilon\}$ is an open set in X.

Proof. We start by proving (i). If $\omega_f(x_0) = 0$ then for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that , for every $\delta \leq \delta_\varepsilon$ we have

$$0 \leq \sup_{x,y \in B(x_0,\delta)} d_Y\left(f(y), f(x)\right) < \varepsilon. \tag{2.24}$$

Let $x_n \to x_0$, then there exists $n_{\delta_\varepsilon} \in \mathbb{N}$ such that for every $n \geq n_{\delta_\varepsilon}$, $x_n \in B\left(x_0, \dfrac{\delta_\varepsilon}{2}\right)$. Then by (2.24) we have

$$d_Y\left(f(x_n), f(x_0)\right) < \varepsilon \ \ \forall\, n \geq n_{\delta_\varepsilon}.$$

On the other side, if f is continuous in $x_0 \in X$ then

$$\forall\, \varepsilon > 0\ \exists \delta_\varepsilon \ : \ \forall\, y \in B(x_0, \delta_\varepsilon) \ \ d_Y\left(f(y), f(x_0)\right) < \frac{\varepsilon}{2}$$

then

$$d_Y\left(f(y), f(x)\right) < d_Y\left(f(y), f(x_0)\right) + d_Y\left(f(x), f(x_0)\right) < \varepsilon$$

for every $x, y \in B(x_0, \delta)$ with $\delta \leq \delta_\varepsilon$ and taking the supremum in $B(x_0, \delta_\varepsilon)$ we have the claim.

The proof of (ii) follows directly from definition of ω_f. □

Using the notion of oscillation and the Proposition 2.4.13 we can write the set $\mathcal{D}_f$ of all point at which a given function f is discontinuous as an F_σ, i.e. as a countable union of closed sets, as follows

$$\mathcal{D}_f := \bigcup_{n=1}^{\infty} \left\{ x \in X \ : \ \omega_f(x) \geq \frac{1}{n} \right\}. \tag{2.25}$$

Definition 2.4.14. *A function f is said to be of first class (of Baire) if it can be represented as the limit of an everywhere convergent sequence of continuous functions.*

Remark 2.4.15. *The functions of first class need not to be continuous, for instance, the functions*

$$f_n(x) = \max\{0, 1 - n|x|\}$$

are continuous on $\mathbb{R}$ and the sequence f_n converges pointwise to the discontinuous first class function

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0. \end{cases}$$

Now we are in the position to prove the Baire theorem for function of first class.

Theorem 2.4.16. *Let X, Y complete metric spaces and $f : X \to Y$. If f is a function of first class, then $\mathcal{D}_f$ is a set of first category, i.e. the set where f is continuous is residual in X.*

Proof. Using the representation (2.25) of $\mathcal{D}_f$ it suffices to show that, for each $\varepsilon > 0$ the set $F := \{x \in X \; : \; \omega_f(x) \geq 5\varepsilon\}$ is nowhere dense.

Let $f(x) = \lim_{n\to\infty} f_n(x)$, with f_n continuous and define the sets

$$E_n := \bigcap_{i,j \geq n} \{x \; : \; d_Y\left(f_i(x), f_j(x)\right) \leq \varepsilon\} \quad \forall\, n \in \mathbb{N}.$$

Then E_n is closed in X and $E_n \subset E_{n+1}$ by continuity of f_n, moreover $\bigcup_{n\in\mathbb{N}} E_n = X$ since for every $x \in X$ the sequence $\{f_n(x)\}$ is convergent and then a Cauchy sequence in Y.

Consider any closed set with non-empty interior $I \subset X$. Since $I = \bigcup(E_n \cap I)$, the sets $E_n \cap I$ cannot all be nowhere dense, since in this case the complement of I in X ,I^c, should be a dense set as a complement of a set of first category by Theorem 2.4.11 and this is a contradiction. Hence for some positive integer n, $E_n \cap I$ contains an open subset J.

We have $d_Y(f_j(x), f_i(x)) \leq \varepsilon$ for all $x \in J$ and for all $i, j \geq n$. Putting $j = n$ and letting i goes to ∞, we find that $d_Y(f_n(x), f(x)) \leq \varepsilon$ for all $x \in J$. By continuity of f_n for any $x_0 \in J$ there exists a neighborhood $I(x_0) \subset J$ such that $d_Y(f_n(x), f_n(x_0)) \leq \varepsilon$ for all $x \in I(x_0)$. Hence

$$d_Y(f(x), f_n(x_0)) \leq 2\varepsilon \;\; \forall\, x \in I(x_0).$$

Therefore

$$d_Y(f(x), f(y)) \leq d_Y f(x), f_n(x_0) + d_Y f(y), f_n(x_0) \leq 4\varepsilon \;\; \forall\, x, y \in I(x_0),$$

then $\omega_f(x_0) \leq 4\varepsilon$, and so no point of J belongs to F. Thus for every closed set with non-empty interior there is an open interval $J \subset I - F$. This shows that F is nowhere dense and therefore $\mathcal{D}_f$ is of first category. □

Now we want to apply this theory in order to investigate some properties of continuity of the gradient operator $D : V \to L^1(\Omega)$ where V is a given subspace of $W^{1,\infty}(\Omega)$, complete with respect to the $L^\infty(\Omega)$ metric. We will prove here two corollaries that ensures the same density result for the set where the operator D is continuous under different hypotheses on V. In the first one we assume some uniform boundary data and in the second one we assume that there is a uniform bound for the gradient of functions in V. It is interesting to note that also if to establish the two results we use the same type of tools, the first one can be obtained really as a corollary of Theorem 2.4.16, while for the second one we need to adapt directly the proof of the Baire Theorem.

Corollary 2.4.17. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $u_0 \in W^{1,\infty}(\mathbb{R}^n)$. Let $V \subset u_0 + W_0^{1,\infty}(\Omega)$ be a non empty complete space with respect to the L^∞ metric. Then the set of points of continuity of the gradient operator $D : V \to L^p(\Omega;\mathbb{R}^n)$, where $1 \le p < \infty$, is dense in V.*

Proof. We will establish the result only when $p = 1$, the general case being handled similarly.

Step 1. We start with some notations.

1) For every $u \in V$, we let $\widetilde{u} \in W^{1,\infty}(\mathbb{R}^n)$ be defined by

$$\widetilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega \\ u_0(x) & \text{if } x \notin \Omega. \end{cases}$$

2) For $h \neq 0$, we let

$$D^h = \left(D_1^h, ..., D_n^h\right) : V \to L^p(\Omega;\mathbb{R}^n)$$

be defined, for every $u \in V$ and $x \in \Omega$, by

$$D_i^h u(x) = \frac{\widetilde{u}(x + he_i) - u(x)}{h},\ i = 1, ..., n$$

where $e_1, ..., e_n$ stand for the vectors from the Euclidean basis.

Step 2. The claim will follow from Theorem 2.4.16, once we will have proved the two following facts.

1) The operator D^h is continuous. Indeed let us prove that, for every $i = 1, ..., n$, $\epsilon > 0$ and $u, v \in V$, then

$$\left\|D_i^h u - D_i^h v\right\|_{L^1(\Omega)} \le \frac{2\,\text{meas}\,\Omega}{|h|} \|u - v\|_{L^\infty(\Omega)} .$$

Letting $w = u - v$ and noting that $w \equiv 0$ outside Ω, we get the claim, since

$$\begin{aligned} \left\|D_i^h u - D_i^h v\right\|_{L^1(\Omega)} &\le \frac{1}{|h|}\int_\Omega [|\widetilde{u}(x + he_i) - \widetilde{v}(x + he_i)| + |w(x)|]\,dx \\ &\le \frac{2\,\text{meas}\,\Omega}{|h|}\|w\|_{L^\infty(\Omega)} . \end{aligned}$$

2) Let $i = 1, ..., n$ and $u \in V$. We wish to show that for any sequence $h \to 0$, we have

$$\lim_{h\to 0}\left\|D_i^h u - D_i u\right\|_{L^1(\Omega)} = 0. \tag{2.26}$$

Let $h \neq 0$ and $x \in \Omega$. Using the facts that in Ω, $u = \widetilde{u}$ and $D_i u = D_i\widetilde{u}$, we find

$$\begin{aligned} \epsilon^h(x) = \left|D_i^h u(x) - D_i u(x)\right| &= \left|\frac{\widetilde{u}(x + he_i) - u(x)}{h} - D_i u(x)\right| \\ &= \left|\frac{\widetilde{u}(x + he_i) - \widetilde{u}(x)}{h} - D_i\widetilde{u}(x)\right|. \end{aligned}$$

Observe that since $\widetilde{u} \in W^{1,\infty}(\mathbb{R}^n)$, we have that, for almost every $x \in \Omega$,

$$\epsilon^h(x) \to 0, \text{ as } h \to 0.$$

Moreover, for almost every $x \in \Omega$ and every $h \neq 0$, we have

$$0 \le \epsilon^h(x) \le \|D_i \widetilde{u}\|_{L^\infty(\mathbb{R}^n)} + |D_i \widetilde{u}(x)| .$$

The function $\widetilde{u}$ being in $L^1(\Omega) \cap L^\infty(\mathbb{R}^n)$ we can apply Lebesgue dominated convergence theorem to get (2.26). □

Using the same type of argument it is easy to prove the following generalization

Corollary 2.4.18. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $u_0 \in W^{N,\infty}(\mathbb{R}^n)$. Let $V \subset u_0 + W_0^{N,\infty}(\Omega)$ be a non empty complete space with respect to the C^{N-1} metric. Then the set of points of continuity of the gradient operator $D^N : V \to L^p(\Omega; \mathbb{R}_s^{n^N})$, where $1 \le p < \infty$, is dense in V.*

In particular the set

$$\Omega^\varepsilon_{D^N} := \{u \in V : \omega_{D^N}(u) < \varepsilon\}$$

is open and dense in V for every ε.

Now we want to use the representation (2.25) with f replaced by the gradient operator D. With this in mind, let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $V \subset W^{1,\infty}(\Omega)$ be complete when endowed with the $L^\infty(\Omega)$ metric. We consider the operator $D : V \to L^1(\Omega)$ which associates to every function $u \in V$ its weak gradient $Du \in L^1(\Omega)$. Since V and $L^1(\Omega)$ are complete metric space when endowed with the $L^\infty(\Omega)$ and the $L^1(\Omega)$ metric respectively, we have, by definition, that the oscillation of the gradient at $u \in V$ is

$$\omega_D(u) = \lim_{\delta \to 0} \sup_{v,w\, \in B_\infty(u,\delta)} \|Dv - Dw\|_{L^1(\Omega)}$$

where $B_\infty(u,\delta) := \{v \in V \ : \ \|v-u\|_{L^\infty(\Omega)} < \delta\}$, moreover by (2.25), the set of all points of V at which the operator D is discontinuous can be written as

$$\mathcal{D}_D := \bigcup_{n=1}^{\infty} \left\{ u \in V \ : \ \omega_D(u) \ge \frac{1}{n} \right\}. \tag{2.27}$$

Before to state the second corollary we need to recall some properties of difference quotients of weak differentiable functions. Let $u \in W^{1,\infty}(\Omega)$ we define

$$D_h u(x) := \frac{u(x+h) - u(x)}{|h|}.$$

The following lemma holds

Lemma 2.4.19. *Let $V \subset W^{1,\infty}(\Omega)$ be complete with respect to the $L^\infty(\Omega)$ distance and such that there exists a constant $\gamma > 0$ with $\|Du\|_{L^\infty(\Omega)} \le \gamma$ for every $u \in V$. Then for every open set $\Omega' \subset\subset \Omega$ and for every $u \in V$ we have $D_h u \to Du$ in $L^1(\Omega')$ for $h \to 0$.*

Proof. Let $u \in V$ and fix $\Omega' \subset\subset \Omega$. Since $u \in W^{1,\infty}(\Omega)$, then u is differentiable a.e. in Ω and its gradient equals its weak gradient a.e. in Ω, that is for a.e. $x \in \Omega$ the weak gradient of u, $Du(x)$, is the limit of the difference quotients $D_h u(x)$ (see Evans [**?**] Theorem 5 pag. 280). We observe that

$$\|D_h u\|_{L^\infty(\Omega')} \le \|Du\|_{L^\infty(\Omega)} \le \gamma$$

for every h such that $|h| \le d(\Omega', \partial\Omega)$ (see. Brezis [**?**] Theorem IX.3). Then we can apply the dominated convergence theorem to obtain the desired convergence. □

Now we are in the position to prove the following

Corollary 2.4.20. *Let $V \subset W^{1,\infty}(\Omega)$ be complete when endowed with the $L^\infty(\Omega)$ metric. If there exists a constant $\gamma > 0$ such that $\|Du\|_{L^\infty(\Omega)} \le \gamma$ for every $u \in V$, then the set of all points of continuity of the gradient operator $D : V \to L^1(\Omega)$ is dense in V.*

Proof. We will prove that the set $\mathcal{D}_D$ of all points of V at which D is discontinuous is of first category in the sense of Baire and so, by Baire theorem, its complement is dense in V. For this, using the representation (2.27), it is sufficient to show that for every $\varepsilon > 0$ the set

$$F := \{u \in V \ : \ \omega_D(u) \ge 6\varepsilon\}$$

is nowhere dense in V.

Let us start fixing $\varepsilon > 0$ and let $\Omega' \subset\subset \Omega$ be such that $meas(\Omega - \Omega') < \frac{\varepsilon}{2\gamma}$. We know by Lemma 2.4.19 that $D_h u \to Du$ in $L^1(\Omega')$ for $h \to 0$, then if we choose a sequence $\{h_n\}$ such that $h_n \to 0$ and $|h_n| < d(\Omega', \partial\Omega)$ we have, for every $u \in V$, $D_{h_n} u \to Du$ in $L^1(\Omega')$ for $n \to \infty$.

Now we define the countable family of subsets of V setting for every $n \in \mathbb{N}$

$$E_n := \bigcap_{i,j \ge n} \Big\{u \in V \ : \ \|D_{h_i} u - D_{h_j} u\|_{L^1(\Omega')} \le \varepsilon\Big\}.$$

We observe that, since $D_{h_n} : V \to L^1(\Omega')$ is a continuous linear operator for every $n \in \mathbb{N}$ the set E_n is closed in V for every $n \in \mathbb{N}$. Moreover $\bigcup_{n\in\mathbb{N}} E_n = V$, since for every $u \in V$ the sequence $D_{h_n} u$ is convergent in $L^1(\Omega')$ and so it is a Cauchy sequence.

Consider now a closed set $I \subset V$ with non empty interior. As $I = \bigcup_{n\in\mathbb{N}}(E_n \cap I)$, the sets $E_n \cap I$ cannot all be nowhere dense, since in this case the complement of I in V ,I^c, should be a dense set as a complement of a set of first category by the Baire theorem and this is a contradiction. Hence for some positive integer n, $E_n \cap I$ contains an open subset J.

We have $\|D_{h_i}u - D_{h_j}u\|_{L^1(\Omega')} \le \varepsilon$ for all $u \in J$ and for all $i, j \ge n$. Putting $j = n$ and letting i goes to ∞, we find that

$$\|D_{h_n}u - Du\|_{L^1(\Omega')} \le \varepsilon \quad \forall\, u \in J. \tag{2.28}$$

Now, by continuity of $D_{h_n} : V \to L^1(\Omega')$, for any $u_0 \in J$ there exists a neighborhood $I(u_0) \subset J$ such that

$$\|D_{h_n}v - D_{h_n}u_0\|_{L^1(\Omega')} \le \varepsilon \quad \forall\, v \in I(u_0). \tag{2.29}$$

Hence combining (2.28) and (2.29) we have for every $v, w \in I(u_0)$ that

$$\|Dv - Dw\|_{L^1(\Omega')} \le \|Dv - D_{h_n}u_0\|_{L^1(\Omega')} + \|D_{h_n}u_0 - Dw\|_{L^1(\Omega')} \le 4\varepsilon$$

and so for all $v, w \in I(u_0)$

$$\|Dv - Dw\|_{L^1(\Omega)} \le 4\varepsilon + 2\gamma\, meas(\Omega - \Omega') \le 5\varepsilon.$$

then $\omega_D(u_0) \le 5\varepsilon$ for every $u_0 \in J$, and so no point of J belongs to F. Thus for every closed set with non-empty interior there is an open set J contained in $I - F$ and this shows that F is nowhere dense. □

We should point out that the same proof holds with the space $L^1(\Omega)$ replaced by $L^p(\Omega)$ with $1 < p < \infty$.

Chapter 3

Continuous viscosity solutions of Hamilton-Jacobi Equation

3.1 Introduction

In this chapter we expose the basic theory of continuous viscosity solutions of the Hamilton-Jacobi equation

$$F(x, u(x), Du(x)) = 0 \quad \text{in } \Omega, \tag{3.1}$$

where Ω is an open domain of $\mathbb{R}^n$ and the Hamiltonian $F = F(x, r, p)$ is a continuous real valued function on $\Omega \times \mathbb{R} \times \mathbb{R}^n$.

The central role played by the Hamilton-Jacobi equations in the classical setting of the Calculus of Variations was recognized in 1935 by C. Carathéodory (see [?] or [?] for a recent survey).

In the '60s, after the introduction of the Dynamical Programming method by Bellman, the study of these equations became a standard topic in deterministic optimal control theory (see [?], [?]). However, the difficulty to give an appropriate sense to the Hamilton-Jacobi equation in the Optimal Control framework (i.e. to find an appropriate notion of solution of the Hamilton-Jacobi equation related to an optimal control problem, which characterize the unique value function), caused a considerable restriction to the range of applicability of the Hamilton-Jacobi theory.

Then several non classical notions of solutions of Hamilton-Jacobi type equations have been proposed in literature, as for instance the theory developed by Kružkov (see [?], [?], [?], [?]) or Subbotin (see [?], [?], [?]).

A new impulse to the rigorous mathematical justification of Dynamical Programming method was originated by the introduction of the notion of viscosity solution for general Hamiltonians.

The notion of viscosity solution has aroused much interest since its introduction by M.G. Crandall and P.L. Lions in [?] and the viscosity method is one of the most applied in optimal control theory and differential game theory. This method is also one of the most used criterion to select, among all the generalized solution of (3.1), a preferred one. In fact, as we will see, when we can prove the existence of a viscosity solution, we can, at the same time, deduce for instance, under suitable hypotheses, properties of uniqueness, stability and maximality for the solution.

Our first aim here is to make a survey of the different equivalents definitions of viscosity solution and to underline, with some examples, how, in different contexts, one definition can be more convenient than the others.

Another important feature of the viscosity method is that it allows us to write an explicit formula for the viscosity solution of the Dirichlet problem for the equation (3.1), under suitable compatibility hypotheses on the Hamiltonian and on the boundary data. We will discuss in particular some compatibility conditions that will ensure the existence of a so-called Hopf-Lax formula for the viscosity solution of the Dirichlet problem associated to a convex autonomous Hamiltonian. The case of a convex Hamiltonian is the classical one, nevertheless the hypothesis of convexity play a crucial role in establishing explicit formulas for the solutions. In fact no explicit formula, as simple as the one that we have in the convex case, is available for the viscosity solution of the Dirichlet problem related to a general non-convex Hamiltonian. In the next chapter we will discuss some additional compatibility conditions that will allow us to write an Hopf-Lax type formula also in the case when the Hamiltonian is not convex.

We should point out that here we deal only with continuous viscosity solutions but a parallel theory can be developed also for discontinuous solutions (see for instance [?] and the bibliography contained).

3.2 Motivations: the vanishing viscosity method

In this section we want to motivate the introduction of the notion of viscosity solution looking how it comes out naturally from some properties of the solution founded with the classical vanishing viscosity method.

Before the introduction of the viscosity solutions, almost all existence results concerning the Hamilton-Jacobi equations (3.1) were obtained with the help of the so-called vanishing viscosity method; this method consists in solving first the approximate problem[1]

$$-\varepsilon\Delta u_\varepsilon + F(x, u_\varepsilon, Du_\varepsilon) = 0 \quad \text{in } \Omega \tag{3.2}$$

[1] The approximate problem (3.2) also if it is an equation of second order, a priori, it looks simpler then (3.1) since it is no more as strongly non linear; the equation (3.2) is said to be a quasilinear elliptic equation (see [?],[?],[?],[?],[?]).

for all $\varepsilon > 0$ and then to pass to the limit as ε goes to 0 and its name comes from a classic method in fluid dynamics where a term like $-\varepsilon\Delta$ represents physically a viscosity.

Now suppose that u_ε is a solution of the approximate problem and that u_ε converges to a function u in a suitable way, what we want to do is to investigate the properties of the limit function u. More precisely, we assume that $u_\varepsilon \in C^2(\Omega)$ solves (3.2) and that

$$u_\varepsilon \to u \quad \text{in } C(\Omega) \ \text{ as } \varepsilon \to 0 \tag{3.3}$$

where the convergence in $C(\Omega)$ means uniform convergence on every compact subset of Ω.

Let φ be a non negative test function, i.e. $\varphi \in C_0^\infty(\Omega)$ and $\varphi \geq 0$, and fix $k \in \mathbb{R}$, we start by localizing the equation (3.2), that is we consider the equation satisfied by the product $\varphi(u_\varepsilon - k)$ on the support of φ. For this we first observe that, since $D(u_\varepsilon - k) = Du_\varepsilon$, (3.2) implies

$$\Delta(u_\varepsilon - k) = \frac{1}{\varepsilon} F\left(x, u_\varepsilon, D(u_\varepsilon - k)\right) \tag{3.4}$$

Moreover simple computations shows that

$$D(u_\varepsilon - k) = \frac{D(\varphi\,(u_\varepsilon - k)) - D\varphi\,(u_\varepsilon - k)}{\varphi} \tag{3.5}$$

and

$$\Delta(\varphi\,(u_\varepsilon - k)) = \Delta\varphi\,(u_\varepsilon - k) + 2D\varphi \cdot D(u_\varepsilon - k) + \varphi\,\Delta(u_\varepsilon - k). \tag{3.6}$$

Then from (3.4), (3.5), (3.6) we deduce the localized equation

$$\begin{aligned}
-\frac{\varepsilon}{\varphi}\,\Delta\left(\varphi(u_\varepsilon - k)\right) \quad &+ \quad F\left(x, u_\varepsilon, \frac{1}{\varphi}\left[D\left(\varphi(u_\varepsilon - k)\right) - D\varphi(u_\varepsilon - k)\right]\right) = \\
&= \quad -\varepsilon\frac{\Delta\varphi}{\varphi}(u_\varepsilon - k) - \frac{2\varepsilon}{\varphi} D\varphi \cdot D(u_\varepsilon - k) = \\
&= \quad -\varepsilon\frac{\Delta\varphi}{\varphi}(u_\varepsilon - k) - \frac{2\varepsilon}{\varphi^2} D\varphi \cdot D[\varphi \cdot (u_\varepsilon - k)] \\
&+ \quad 2\varepsilon\frac{|D\varphi|^2}{\varphi^2}(u_\varepsilon - k).
\end{aligned}$$

Next we suppose that $\max_\Omega \varphi(u - k) > 0$, then, by (3.3), for ε small enough we have $\max_\Omega \varphi(u_\varepsilon - k) \geq \alpha$, for some positive α. Thus there exists $x_\varepsilon \in \Omega$ such that $\varphi(x_\varepsilon)(u_\varepsilon(x_\varepsilon) - k) = \max_\Omega \varphi(u_\varepsilon - k)$, since φ has compact support in Ω, and obviously we have that $\varphi(x_\varepsilon) \geq \alpha > 0$, i.e. $x_\varepsilon \in supp(\varphi)$.

Then we may assume that there exists $x_0 \in \Omega$ such that $x_\varepsilon \to x_0$ as ε goes to 0 and such that

$$\varphi(x_0)(u(x_0) - k) = \max_\Omega \varphi(u - k) > 0.$$

Now, if we look at the localized equation at the point x_ε, we find, since $D(\varphi(u_\varepsilon - k))(x_\varepsilon) = 0$, that

$$-\frac{\varepsilon}{\varphi(x_\varepsilon)}\,\Delta\left(\varphi(u_\varepsilon - k)\right)(x_\varepsilon) + F\left((x_\varepsilon), u_\varepsilon(x_\varepsilon), -\frac{D\varphi}{\varphi}(u_\varepsilon - k)(x_\varepsilon)\right) \leq C\varepsilon$$

for a constant

$$\infty > C > \sup_\varepsilon \left(-\frac{\Delta\varphi}{\varphi}(u_\varepsilon - k)(x_\varepsilon) + 2\frac{|D\varphi|^2}{\varphi^2}(u_\varepsilon - k)(x_\varepsilon)\right).$$

Moreover, by the maximum principle, we have also that

$$-\Delta\left(\varphi(u_\varepsilon - k)\right)(x_\varepsilon) \geq 0,$$

thus we obtain:

$$F\left(x_\varepsilon, u_\varepsilon(x_\varepsilon), -\frac{D\varphi(x_\varepsilon)}{\varphi(x_\varepsilon)}(u_\varepsilon(x_\varepsilon) - k)\right) \leq C\varepsilon.$$

Using (3.3) we may pass to the limit for $\varepsilon \to 0$ to obtain that u satisfies the following subsolution property:

Property 3.2.1. (subsolution) *For all $\varphi \in C_0^\infty(\Omega)$, with $\varphi \geq 0$ and for any $k \in \mathbb{R}$ such that $\max_\Omega \varphi(u - k) > 0$, there exists $x_0 \in \Omega$ satisfying $\varphi(u - k)(x_0) = \max_\Omega \varphi(u - k)$, such that :*

$$F\left(x_0, u(x_0), -\frac{D\varphi(x_0)}{\varphi(x_0)}(u(x_0) - k)\right) \leq 0.$$

By similar argument we can prove that the limit function u also satisfy the following subsolution property:

Property 3.2.2. (supersolution) *For all $\varphi \in C_0^\infty(\Omega)$, with $\varphi \geq 0$ and for any $k \in \mathbb{R}$ such that $\min_\Omega \varphi(u - k) < 0$, there exists $x_0 \in \Omega$ satisfying $\varphi(u - k)(x_0) = \min_\Omega \varphi(u - k)$, such that :*

$$F\left(x_0, u(x_0), -\frac{D\varphi(x_0)}{\varphi(x_0)}(u(x_0) - k)\right) \geq 0.$$

In the literature we find the properties (3.2.1) and (3.2.2) as definition of the viscosity subsolution and supersolution of the equation (3.1) (see [**?**],[**?**]). In the next section we will introduce some different equivalent definitions of viscosity solutions of the equation (3.1).

3.3 Definitions and properties

Here we recall two equivalent definitions of viscosity solution for Hamilton-Jacobi equation and we state some properties of this type of solutions underlying as in different contests is more convenient use the first one rather than the second one.

Definition 3.3.1. *A function $u \in C(\Omega)$ is:*
(i) a viscosity subsolution of (3.1) if, for any $\varphi \in C^1(\Omega)$,

$$F(x_0, u(x_0), D\varphi(x_0)) \leq 0 \tag{3.7}$$

at any local maximum point $x_0 \in \Omega$ of $u - \varphi$;
(ii) a viscosity supersolution of (3.1) if, for any $\varphi \in C^1(\Omega)$,

$$F(x_1, u(x_1), D\varphi(x_1)) \geq 0 \tag{3.8}$$

at any local minimum point $x_1 \in \Omega$ of $u - \varphi$;
(iii) a viscosity solution of (3.1) if it is simultaneously a viscosity sub and supersolution.

We want to point out that (i) and (ii) of Definition 3.3.1 are equivalent to the (3.2.2) and (3.2.1) of the previous section (see [**?**], [**?**], [**?**] for further details).

Remark 3.3.2. *In the definition of subsolution we can always assume that x_0 is a local strict maximum point for $u - \varphi$ (by replacing $\varphi(x)$ by $\varphi(x) + |x - x_0|^2$). Moreover, since (3.7) depends only on the value of $D\varphi$ at x_0, it is not restrictive to assume that $u(x_0) = \varphi(x_0)$. In a similar way we can adapt these remarks to the definition of supersolution. We note also that by density argument the space $C^1(\Omega)$ of test functions in Definition 3.3.1 can be replaced by $C^\infty(\Omega)$. Geometrically, this means that the validity of the subsolution condition (3.7) for u can be tested on smooth functions "touching from above" the graph of u at x_0 (cf. figure 3 in Appendix* **??***).*

The notion of viscosity solution has a local character and it is consistent with the classical pointwise definition, as we can deduce from the following proposition (cf. [**?**] Proposition 1.3).

Proposition 3.3.3. *(a) If $u \in C(\Omega)$ is a viscosity solution of (3.1) in Ω, then u is a viscosity solution of (3.1) in Ω', for any open set $\Omega' \subset \Omega$;*
(b) if $u \in C^1(\Omega)$ is a classical solution of (3.1), that is, u is differentiable at any $x \in \Omega$ and

$$F(x, u(x), Du(x)) = 0 \;\; \forall\, x \in \Omega, \tag{3.9}$$

then u is a viscosity solution of (3.1);
(c) if $u \in C^1(\Omega)$ is a viscosity solution of (3.1), then u is a classical solution.

An important feature that has to be stressed is that viscosity solutions are not preserved by change of sign in the equation, as we can see from the Example **??**. Indeed, since any local maximum of $u-\varphi$ is a local minimum of $-u-(-\varphi)$, u is a viscosity subsolution of (3.1) if and only if $v=-u$ is a viscosity supersolution of $-F(x,-v,-Dv)=0$ in Ω; similary, u is a viscosity supersolution of (3.1) if and only if $v=-u$ is a viscosity subsolution of $-F(x,-v,-Dv)=0$ in Ω.

Example 3.3.4. *Consider the one dimensional equation*

$$1-|u'(x)|=0, \quad x\in(-1,1). \tag{3.10}$$

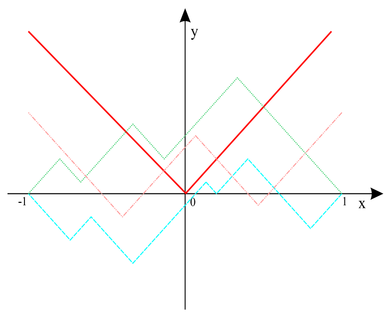

fig.1

*The function $u(x)=|x|$ is a viscosity solution of the equation (**??**)(cf. figure 1). Indeed, if $x\neq 0$ is a local extremum for $u-\varphi$, then $u'(x)=\varphi'(x)$, therefore at those points both the supersolution and the subsolution conditions are trivially satisfied. Also, if 0 is a local minimum for $u-\varphi$, we can easily show that $|\varphi'(0)|\leq 1$ and the supersolution condition holds. For the subsolution condition it is sufficint to observe that 0 cannot be a local maximum for $u-\varphi$ with $\varphi\in C^1((-1,1))$ since this would imply $-1\geq\varphi(0)\geq 1$.*

On the other hand, $u(x)=|x|$ is not a viscosity solution of

$$|u'(x)|-1=0, \quad x\in(-1,1)$$

since the supersolution condition is not fulfilled at $x_0=0$ which is a local minimum for $|x|-(-x^2)$. Moreover we observe that there are infinitely many almost everywhere solutions (cf. figure 1).

Using the definition we can easily deduce the following theorem that shows the consistency of the notion of viscosity solutions.

Theorem 3.3.5. *(i) Suppose that $u_\varepsilon\in C^2(\Omega)$ solves*

$$-\varepsilon\Delta u_\varepsilon+F_\varepsilon(x,u_\varepsilon,Du_\varepsilon)=0 \ \ in\,\Omega$$

and that, as $\varepsilon \to 0$*,* u_ε *converges in* $C(\Omega)$ *to some function* u *and* $F_\varepsilon(x,t,p)$ *converges to* $F(x,t,p)$ *uniformly on compact subsets of* $\Omega \times \mathbb{R} \times \mathbb{R}^n$*; then* u *is a viscosity solution of*

$$F(x, u(x), Du(x)) = 0 \quad in\, \Omega. \tag{3.11}$$

(ii) If $u_\varepsilon \in C(\Omega)$ *is a viscosity solution of*

$$F_\varepsilon(x, u_\varepsilon, Du_\varepsilon) = 0 \quad in\, \Omega,$$

u_ε *converges in* $C(\Omega)$ *to some function* u*, as* $\varepsilon \to 0$*, and* $F_\varepsilon(x,t,p)$ *converges to* $F(x,t,p)$ *uniformly on compact subsets of* $\Omega \times \mathbb{R} \times \mathbb{R}^n$*; then* u *is a viscosity solution of (**??**).*

We want to give now a new definition of viscosity solution that is more in the spirit of nonsmooth analysis. With this in mind we introduce the following notations.

Let $u : \Omega \to \mathbb{R}$ be a continuous function. For every $x \in \Omega$ we define the sets

$$D^+u(x) := \left\{ p \in \mathbb{R}^n \ : \ \limsup_{y \to x, y \in \Omega} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|x - y|} \leq 0 \right\},$$

$$D^-u(x) := \left\{ p \in \mathbb{R}^n \ : \ \liminf_{y \to x, y \in \Omega} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|x - y|} \geq 0 \right\}.$$

$D^+u(x)$ and $D^-u(x)$ are called respectively the *superdifferential* of u at the point x and the *subdifferential* of u at the point x (see appendix for more details and properties of this sets).

Now we can give the following definition

Definition 3.3.6. *A function* $u \in C(\Omega)$ *is:*

(i) a viscosity subsolution of (3.1) in Ω *if*

$$F(x, u(x), p) \leq 0\,, \quad \forall\, x \in \Omega, \forall\, p \in D^+u(x); \tag{3.12}$$

(ii) a viscosity supersolution of (3.1) in Ω *if*

$$F(x, u(x), p) \geq 0\,, \quad \forall\, x \in \Omega, \forall\, p \in D^-u(x); \tag{3.13}$$

(iii) a viscosity solution of (3.1) if it is simultaneously a viscosity sub and supersolution.

The above definition turn out to be equivalent to the previous one as a direct consequence of Lemma **??** in the appendix and it is sometimes easier to handle. We will see that this definition is more convenient for our purposes in the following chapters, but let show now a first example of how it can be more flexible than the first one. The following consistency result improves Proposition 3.3.3.

Proposition 3.3.7. *(i) If $u \in C(\Omega)$ is a viscosity solution of (3.1), then*

$$F(x, u(x), Du(x)) = 0$$

at any point $x \in \Omega$ where u is differentiable;

(ii) If u is locally Lipschitz continuous and it is a viscosity solution of (3.1), then

$$F(x, u(x), Du(x)) = 0 \ \ a.e. \ in\, \Omega.$$

Proof. If x is a point of differentiability for u then by Proposition **??** (ii) we have $\{Du(x)\} = D^+u(x) = D^-u(x)$. Hence, by Definition **??**

$$0 \geq F(x, u(x), Du(x)) \geq 0,$$

which proves (i). The statement (ii) follows immediately from (i) and the Rademacher's theorem on the almost everywhere differentiability of Lipschitz continuous functions. □

Note that (ii) of Proposition **??** says that any viscosity solution of (3.1) is also a generalized solution, i.e. a locally Lipschitz continuous function u such that

$$F(x, u, Du) = 0 \ \text{ a.e. in } \Omega.$$

The converse is false in general, indeed there are many generalized solutions which are not viscosity solutions (see for instance Chapter **??** and Example **??**).

3.4 Existence results for convex Hamiltonian

In this section we recall some classical existence results for viscosity solutions of Dirichlet problem in the case where the Hamiltonian F is convex in the last variable. Since a complete presentation of these topics can be founded in many books (for instance the Lions's monography [**?**]), far from giving here all the detailed proofs, we want to focus our attention on some particular problems where an explicit formula for the solution can be obtained since this will be the main tool to prove the existence result in non convex case, that we will present in the next chapter.

We start recalling the main existence theorem. To simplify the presentation we consider the Hamiltonian of the form

$$F(x, t, p) = H(p) + \lambda t - n(x) \ ; \quad (x, t, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n.$$

We assume that H is a convex continuous function from $\mathbb{R}^n$ into $\mathbb{R}$, n is continuous and bounded on $\mathbb{R}$ and $\lambda \geq 0$. We want to find $u \in W^{1,\infty}_{loc}(\Omega) \cap C(\overline{\Omega})$ solution of

$$\begin{cases} H(Du) + \lambda u = n & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases} \tag{3.14}$$

where $\varphi \in C(\partial\Omega)$.

Remark 3.4.1. *(i) The assumption $\lambda \geq 0$ is not really a restriction, to avoid it we can consider $-u$ instead of u as unknown.*

(ii) Since H is convex and continuous, it is locally Lipschitz and its differential is monotone, i.e.

$$\langle \nabla H(p) - \nabla H(q); p - q \rangle \geq 0 \; ; \quad a.e.\; p, q \; in \mathbb{R}^n.$$

Under these hypotheses the following holds

Theorem 3.4.2. *(i) Let $\Omega \neq \mathbb{R}^n$ and suppose that $H(p) \to \infty$ as $|p| \to \infty$. If there exists $v \in W^{1,\infty}_{loc}(\Omega) \cap C(\overline{\Omega})$ subsolution of (**??**), that is such that*

$$\begin{cases} H(Dv(x)) + \lambda v(x) \leq n(x) & in\; \Omega \\ v(x) = \varphi(x) & on\; \partial\Omega \end{cases}$$

*then there exists a viscosity solution $u \in W^{1,\infty}_{loc}(\Omega) \cap C(\overline{\Omega})$ of problem (**??**).*

*(ii) Let $\Omega = \mathbb{R}^n$ and $\lambda > 0$. If $H(p) \to \infty$ as $|p| \to \infty$, or $n \in W^{1,\infty}(\mathbb{R}^n)$, then there exists a viscosity solution $u \in W^{1,\infty}(\mathbb{R}^n)$ of problem (**??**).*

Let us emphasize that for the existence it is not necessary to assume anything on the regularity of the boundary of Ω.

This type of existence results started with the work of Kružkov [**?**] where it is assumed much more regularity on H and n. Here we present a generalized version due to Lions (see for example [**?**] for a proof and for some extensions in the case of more general Hamiltonians)

Remark 3.4.3. *We should say something about the boundary conditions in the case where Ω is unbounded. In this case we should prescribe what is the behavior of $u(x)$ at $x \in \Omega$ and $|x| \to \infty$. Indeed this has to be precised for uniqueness purposes only when $\lambda = 0$; if $\lambda > 0$ we can simply require the solution to be bounded on Ω.*

Now we want to focus our attention on the problem

$$\begin{cases} H(Du) = n(x) & \text{in} \quad \Omega \\ u = \varphi & \text{on} \quad \partial\Omega \end{cases} \tag{3.15}$$

where $H : \mathbb{R}^n \to \mathbb{R}$ is convex, continuous and satisfies

$$H(p) \to \infty \, , \text{ as } |p| \to \infty, \tag{3.16}$$

and $n \in C(\overline{\Omega})$ is such that

$$n \geq \inf_{\mathbb{R}^n} H(p) \quad \text{in } \overline{\Omega}. \tag{3.17}$$

Our purposes is to show how, under suitable compatibility conditions on the boundary data φ, we can ensure the existence and write an explicit formula for the viscosity solution of problem (**??**).

Before giving the main result concerning the problem (**??**), we want to show some motivations for our conditions.

We denote by H^* the lagrangian of H, that is the dual convex function defined by

$$H^*(p) := \sup_{p\in\mathbb{R}^n} \{(p,q) - H(p)\}.$$

We note that the set $K := \{q \in \mathbb{R}^n \,:\, H(q) < +\infty\}$ is a closed convex set in $\mathbb{R}^n$ and $0 \in \operatorname{int} K$, since (**??**) implies

$$H(p) \geq \alpha|p| - C \quad \text{for some } \alpha, C > 0.$$

By definition of H^* we see that

$$H^*(q) \geq (p,q) - H(p) \quad \forall\, p \in \mathbb{R}^n. \tag{3.18}$$

Assume that exists $u \in W^{1,\infty}(\Omega)$ generalized solution of (**??**). Let $x, y \in \overline{\Omega}$ and let $\xi : [0, T_0] \to \overline{\Omega}$ be any Lipschitz path joining x and y, i.e. $\xi(0) = x$ and $\xi(T_0) = y$. Formally, we have:

$$u(y) - u(x) = \int_0^{T_0} Du(\xi(s)) \cdot \frac{d\xi}{dt}(s)\, ds.$$

and then by (**??**)

$$\begin{aligned} u(x) - u(y) &= \int_0^{T_0} Du(\xi(s)) \cdot \left(-\frac{d\xi}{dt}(s)\right) ds \\ &\leq \int_0^{T_0} H(Du(\xi(s))) + H^*\left(-\frac{d\xi}{dt}(s)\right) ds. \end{aligned}$$

Finally, if we define

$$L(x,y) := \inf_{\widetilde{S}_{x,y}} \left\{ \int_0^{T_0} n(\xi(s)) + H^*\left(-\frac{d\xi}{dt}, p\right) ds \right\} \tag{3.19}$$

where

$$\widetilde{S}_{x,y} := \left\{ \xi : [0,T_0] \to \overline{\Omega} \mid \xi(0) = x\,,\, \xi(T_0) = y\,,\, \frac{d\xi}{dt} \in K \right\},$$

we have

$$u(x) - u(y) \leq L(x,y) \quad \forall\, x, y \in \overline{\Omega} \times \overline{\Omega},$$

and in particular on the boundary $\partial\Omega$

$$\varphi(x) - \varphi(y) \leq L(x,y) \quad \forall\, x, y \in \partial\Omega. \tag{3.20}$$

The condition (**??**) which is "necessary"[2] for the existence of viscosity solution of (**??**), turns out to be also sufficient and it allows us to write an explicit Hopf-Lax type formula for the solution. The function L plays in fact a crucial role in the construction of the explicit solution of problem **??**.

Remark 3.4.4. *In the definition of L the infimum*

$$\inf_{\widetilde{S}_{x,y}} \int_0^{T_0} H^*\left(-\frac{d\xi}{dt}, p\right) ds$$

*has a meaning in $\mathbb{R} \cup +\infty$ since H^* is bounded from below and it is not difficult to verify that $L(x,y)$ is finite as soon as $n \geq \inf_{\mathbb{R}^n} H$ in $\overline{\Omega}$, and this motivates the hypothesis (**??**)*

In the following proposition we recall some properties of the function L that will be useful in the sequel. We will also understand, looking at these properties, why the role of L is crucial in the construction of the viscosity solution of (**??**).

Proposition 3.4.5. *Let $\Omega \subset \mathbb{R}^n$ be bounded connected domain and let L be defined by (**??**), then we have:*

(i) $L(x,x) = 0$ for all $x \in \overline{\Omega}$ and $L(x,z) \leq L(x,y) + L(y,z)$ for all $x,y,z \in \overline{\Omega}$

(ii) $L \in W^{1,\infty}(\Omega \times \Omega)$ and for fixed $y_0 \in \Omega$, if $L(x,y_0)$ is differentiable at $x = x_0 \in \Omega$ then

$$H(D_x L(x_0,y_0)) = n(x_0),$$

respectively for fixed $x_0 \in \Omega$, if $L(x_0,y)$ is differentiable at $y = y_0 \in \Omega$ then

$$H(-D_y L(x_0,y_0)) = n(y_0).$$

In particular $H(D_x L(x,y)) = n(x)$ and $H(-D_y L(x,y)) = n(y)$ almost everywhere in $\Omega \times \Omega$.

(iii) We have $\forall\, x,y, \in \overline{\Omega}$ the following characterization of L:

$$L(x,y) := \inf_{\xi \,\in\, S_{x,y}} \left\{ \int_0^1 \max_{p \,\in\, P_{\xi,t}} \left\langle -\frac{d\xi}{dt}, p \right\rangle dt \right\} \tag{3.21}$$

where

$$P_{\xi,t} := \Big\{ p \in \mathbb{R}^n \mid H(p) = n\big(\xi(t)\big) \Big\},$$

$$S_{x,y} := \left\{ \xi : [0,1] \to \bar{\Omega} \mid \xi(0) = x\,,\, \xi(1) = y\,,\, \frac{d\xi}{dt} \in L^\infty(0,1) \right\}.$$

(iv) $L(\cdot,y)$ is a viscosity solution of

[2] The condition (**??**) is in fact necessary for the existence of viscosity solution of (**??**) under suitable hypotheses on the regularity of the domain Ω (see for instance [**?**], [**?**]).

$$\begin{cases} H(Du) & = & n & in & \Omega_y \\ u(y) & = & 0 \end{cases}$$

and respectively $L(y,\cdot)$ *is a viscosity solution of*

$$\begin{cases} H(-Du) & = & n & in & \Omega_y \\ u(y) & = & 0 \end{cases}$$

where $\Omega_y = \Omega - \{y\}$.

Remark 3.4.6. *Many authors refer to the function* L *as optical length; let us point out why. For an admissible path* ξ *(i.e a function* $\xi : [0,1] \to \overline{\Omega}$ *such that* $\xi(0) = x$ *and* $\xi(1) = y$*) we define the* optical length *of* ξ *as*

$$L(\xi) = \int_0^1 \max_{p \in P_{\xi,t}} \big\langle -\frac{d\xi}{dt}, p \big\rangle \, dt$$

*and this denomination introduced by Kruzkov in [**?**] is motivated by the fact that in the very special case* $H(p) = |p|^2$, $n(x) = const.$, *this coincide with the optical length introduced by Born and Wolf in [**?**].*

Now we can state the classical (cf. [**?**] Theorem 5.2)

Theorem 3.4.7. *(Hopf-Lax formula) Let* Ω *be a bounded, connected domain of* $\mathbb{R}^n$ *with Lipschitz boundary,* $\partial\Omega$. *Let* $\varphi \in Lip(\partial\Omega)$. *If* φ *verifies the compatibility condition*

$$\varphi(x) - \varphi(y) \leq L(x,y)\,, \ \ \forall\, x,y \in \partial\Omega, \tag{3.22}$$

then the function

$$u(x) = \inf_{y \in \partial\Omega} \{\varphi(y) + L(x,y)\}$$

is the unique $W^{1,\infty}(\Omega)$ *viscosity solution of the problem (**??**).*

Before seeing how we can apply this theorem with an example, lets make some bibliographical notes. Representation formulas are classical issues since the works of Hopf [**?**] and Kružkov [**?**], while the connection between Hopf's formulas and viscosity solution was intensively studied in [**?**], [**?**] and [**?**]. Moreover some extensions of Hopf like formulas can be founded also in [**?**], [**?**], [**?**], [**?**] and [**?**].

Example 3.4.8. *In the next chapter we will apply the Theorem* **??** *in the particular case where the Hamiltonian* H *is the gauge function of a convex set. For this reason now we want to investigate how can be rewritten* L *in the special case where* $n(x) = 1$ *and* H *is a gauge function, that is* H *is convex and*

$$\begin{cases} H(\xi) > 0 & \forall\, \xi \neq 0 \\ H(t\xi) = tH(\xi) & \forall\, \xi \in \mathbb{R}^n \quad \forall\, t > 0. \end{cases}$$

Under these assumption the function L can be rewritten as

$$L(x,y) := \inf_{S_{x,y}} \left\{ \int_0^1 \max_{H(p)=1} \left\langle -\frac{d\xi}{dt}, p \right\rangle dt \right\} \tag{3.23}$$

and by definition of polar[3] *function of a gauge, (??) is equivalent to*

$$L(x,y) := \inf_{S_{x,y}} \left\{ \int_0^1 H^0 \left(-\frac{d\xi}{dt} \right) dt \right\} \tag{3.24}$$

where H^0 is the polar function of H.

3.5 Appendix

3.5.1 One-sided differentials

In this section we introduce the notion of sub- and superdifferential for a continuous function and we recall some properties of these sets.

Definition 3.5.1. *Let $u : \Omega \to \mathbb{R}$ be a continuous function. For every $x \in \Omega$ we define the sets*

$$D^+u(x) := \left\{ p \in \mathbb{R}^n \ : \ \limsup_{y \to x, y \in \Omega} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|x - y|} \leq 0 \right\},$$

$$D^-u(x) := \left\{ p \in \mathbb{R}^n \ : \ \liminf_{y \to x, y \in \Omega} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|x - y|} \geq 0 \right\}.$$

The sets $D^+u(x)$ and $D^-u(x)$ are called respectively superdifferential and subdifferential (or semidifferentials) of u at x.

In other words, a vector $p \in \mathbb{R}^n$ is in the superdifferential of u at x if and only if the plane $y \mapsto u(x) + p \cdot (y - x)$ is tangent from above to the graph of u at point x (cf. figure 2.a) and p is in the subdifferential of u at x if and only if the plane $y \mapsto u(x) + p \cdot (y - x)$ is tangent from below to the graph of u at the point x (cf. figure 2.b).

[3] The polar of a gauge H is defined as

$$H^0(\xi^*) = \inf \{ \lambda \geq 0 \ : \ \langle \xi, \xi^* \rangle \leq \lambda H(\xi) \ , \ \forall \, \xi \in \mathbb{R}^n \}$$

and it is characterized by

$$H^0(\xi^*) = \sup_{\xi \neq 0} \left\{ \frac{\langle \xi, \xi^* \rangle}{H(\xi)} \right\}.$$

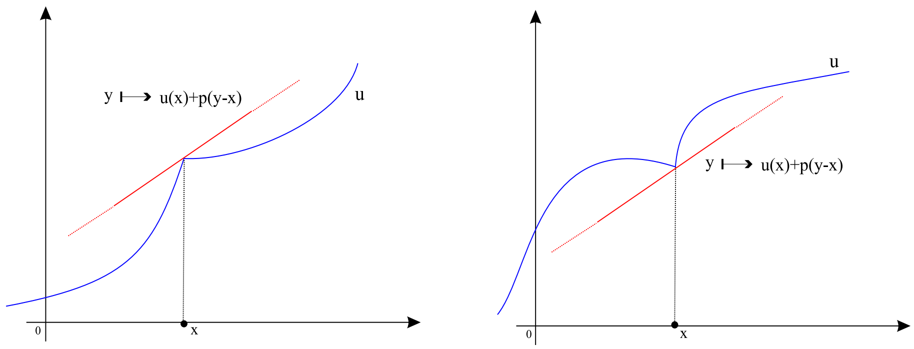

fig. 2

Example 3.5.2. *Consider the function*

$$u(x) := \begin{cases} 0 & \textit{if} \quad x < 0, \\ \sqrt{x} & \textit{if} \quad x \in [0,1], \\ 1 & \textit{if} \quad x > 1. \end{cases}$$

In this case we have

$$D^+u(0) = \emptyset \textit{ and } D^-u(0) = [0, \infty[,$$

$$D^+u(1) = \left[0, \frac{1}{2}\right] \textit{ and } D^-u(1) = \emptyset,$$

$$D^+u(x) = D^-u(x) = \left\{\frac{1}{2\sqrt{x}}\right\}.$$

The following lemma provides a useful characterization of $D^+u(x)$ and $D^-u(x)$ in terms of test functions (cf. [**?**] Lemma 1.7).

Lemma 3.5.3. *Let $u \in C(\Omega)$. Then,*

(i) $p \in D^+u(x)$ if and only if there exists $\varphi \in C^1(\Omega)$ such that $\nabla\varphi(x) = p$ and $u - \varphi$ has a local maximum at x;

(ii) $p \in D^-u(x)$ if and only if there exists $\varphi \in C^1(\Omega)$ such that $\nabla\varphi(x) = p$ and $u - \varphi$ has a local minimum at x.

We want to point out that by adding a constant, it is not restrictive to assume that $\varphi(x) = u(x)$. In this case, we are saying that $p \in D^+u(x)$ if and only if there exists a smooth function $\varphi \geq u$ with $\nabla\varphi(x) = p$, $\varphi(x) = u(x)$. Geometrically this means that the graph of φ touches the graph of u from above at the point x. Clearly a similar property holds for subdifferentials, that is $p \in D^-u(x)$ if and only if there exists a smooth function $\varphi \leq u$, with $\nabla\varphi(x) = p$, whose graph touches from below the graph of u at the point x (cf. figure 3).

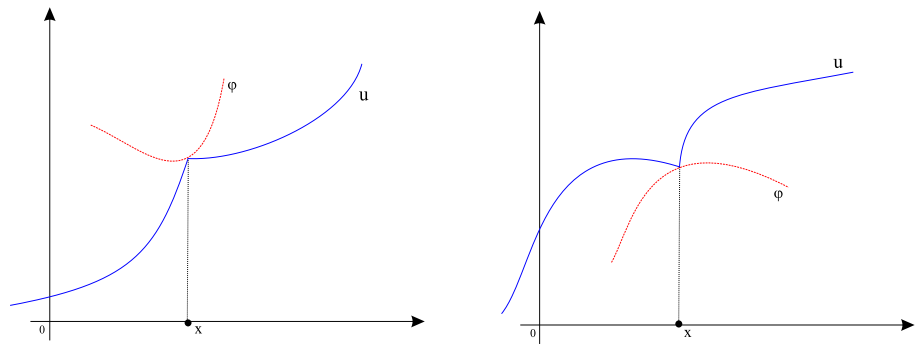

fig. 3

We note also that, by possibly replacing the function $\varphi(y)$ with $\widetilde{\varphi}(y) = \varphi(y) \pm |y-x|^2$, it is clear that in the above lemma we can require that $u - \varphi$ attains a *strict* local maximum or local minimum at the point x. This is of particular importance since it ensure some stability result as we see from the following lemma.

Lemma 3.5.4. *Let $u \in C(\Omega)$, assume that, for some $\varphi \in C^1(\Omega)$, the function $u - \varphi$ has a strict local minimum (a strict local maximum) at the point $x \in \Omega$. If $u_m \to u$ uniformly, then there exists a sequence of points $x_m \to x$ with $u_m(x_m) \to u(x)$ and such that $u_m - \varphi$ has a local minimum (a local maximum) at x_m.*

In the following proposition we recall some useful properties of $D^+u(x)$ and $D^-u(x)$ that we will need in the sequel:

Proposition 3.5.5. *Let $u \in C(\Omega)$ and $x \in \Omega$. Then*

(i) $D^+u(x)$ and $D^-u(x)$ are closed, convex (possibly empty) subsets of $\mathbb{R}^n$.

(ii) If u is differentiable at x, then

$$D^+u(x) = D^-u(x) = \{Du(x)\}. \tag{3.25}$$

*(iii) If for some x both $D^+u(x)$ and $D^-u(x)$ are nonempty then (**??**) holds.*

(iv) the sets of points where a one-sided differential exists:

$$A^+ := \{x \in \Omega \,:\, D^+u(x) \neq \emptyset\},$$

$$A^- := \{x \in \Omega \,:\, D^-u(x) \neq \emptyset\}$$

are both non empty. Indeed, they are dense in Ω.

(v) If $u \in W^{1,\infty}(\Omega)$, then

$$D^+u(x) \cup D^-u(x) \subseteq \mathrm{co}\left\{ p \in \mathbb{R}^n \;\mid\; p = \lim_{n\to\infty} Du(x_n)\,,\, x_n \to x \right\}, \tag{3.26}$$

where the limit is taken over all the sequence $x_n \to x$, such that $Du(x_n)$ exists and the sequence $\{Du(x_n)\}$ converges.

Chapter 4

Geometric Conditions for the existence of viscosity solutions

4.1 Introduction

In the previous chapter we have discussed some existence results for viscosity solutions of the Dirichlet problem related to the Hamilton-Jacobi equation $F(Du) = 0$, where $F : \mathbb{R}^n \to \mathbb{R}$ is a continuous convex function. In particular we have shown that it is also possible to find an explicit Hopf-Lax type formula for the solution (cf. Theorem **??**).

When the Hamiltonian F is non convex, the problem to establish the existence of a viscosity solution became more difficult and there are, in general, no explicit formulas available. In fact there are some geometrical relations between the domain Ω, the Hamiltonian F and the boundary data φ that should be satisfied in order to get the existence of viscosity solutions.

This will be the main topic of this chapter; indeed we will establish some geometrical compatibility conditions sufficient and, in some cases, necessary (cf. Theorem **??**) for the existence of viscosity solutions of the Dirichlet problem

$$\begin{cases} F(Du) & = & 0 & \text{in} & \Omega \\ u & = & \varphi & \text{on} & \partial\Omega \end{cases} \tag{4.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, $F : \mathbb{R}^n \to \mathbb{R}$ is continuous and $\varphi \in Lip(\partial\Omega)$ (with the notation $\varphi \in Lip(\partial\Omega)$ we mean that there exists a constant C such that $|\varphi(x) - \varphi(y)| \leq C|x - y|$ for all $x, y \in \partial\Omega$).

The interest in finding geometrical conditions comes out from the idea of compare the Baire category method (cf. Chapter 2) and the viscosity approach (cf. Chapter 3) to find solutions of (**??**).

Since the Baire category approach is purely "existential" and moreover it establishes the existence of infinitely many generalized solutions, the problem was to find a criterium to select one preferred solution among them. One possible way to follow is to use the viscosity method, which, as we have seen in the previous chapter, when it ensures existence, it gives us also uniqueness an many other nice properties for the solution. This approach was used by P. Cardaliaguet, B. Dacorogna, W. Gangbo and N. Georgy in [?] (see also [?] Section 4) where they showed, under restrictive hypotheses on Ω and φ, that the existence of the viscosity solution of problem (**??**) is equivalent to a geometrical condition. In particular they showed that if Ω is convex, $\varphi \in C^1(\bar{\Omega})$ and verifies the compatibility condition

$$D\varphi(x) \in E \cup \operatorname{int} \operatorname{co} E\ , \quad \forall\, x \in \Omega \tag{4.2}$$

where $E = \{\xi \in \mathbb{R}^n \mid F(\xi) = 0\}$ and $\operatorname{int}\operatorname{co} E$ is the interior of the convex hull of E, then the following geometrical condition

- (G1) $\forall\, y \in \partial\Omega$ where the inward normal , $\nu(y)$, is uniquely defined, there exists $\lambda(y) > 0$ such that

$$D\varphi(y) + \lambda(y)\nu(y) \in E$$

is necessary and sufficient for the existence of $W^{1,\infty}(\Omega)$ viscosity solution of (**??**).

We should remark that the compatibility condition (**??**) is sufficient for the existence of infinitely many $W^{1,\infty}(\Omega)$ a.e. solutions of problem (**??**); and under this hypothesis the geometrical condition (G1) characterizes, among all the convex domains Ω, the ones for which the problem (**??**) admits a viscosity solution.

Our aim here is to show that the same type of techniques used in [?] can be refined to obtain a more general result in a more general framework. That is we want to follow the geometrical approach to find sufficient conditions for the existence of viscosity solutions of problem (**??**) trying to weak the hypotheses on the domain Ω and on the boundary data.

Moreover we will see that it is also possible to weak the compatibility condition (**??**) considering a weak version of it, localized on the boundary $\partial\Omega$.

We will prove that if Ω is bounded and connected, not necessarily convex, $\varphi \in Lip(\partial\Omega)$ and verifies a compatibility condition like (**??**) only on the boundary $\partial\Omega$ (the precise meaning of this condition will be also clarified in the sequel), then the geometrical condition (G1) can be replaced by

- (G2) $\forall\, y \in \partial\Omega$ where $N^N_{\mathbb{R}^n\setminus\Omega}(y) \neq \emptyset$ there exists $h \in D^+\varphi(y)$ such that $\forall \nu \in N^N_{\mathbb{R}^n\setminus\Omega}(y)$ there exists a unique $\lambda_{\nu,h} > 0$ such that

$$h + \lambda_{\nu,h}\nu \in E$$

  where $N^N_{\mathbb{R}^n\setminus\Omega}(y)$ is the normal cone to the set $\mathbb{R}^n \setminus \Omega$ and $D^+\varphi(y)$ is the superdifferential of φ in y (see Definitions **??**).

In particular we will see that (G2) is a sufficient condition for the existence of $W^{1,\infty}(\Omega)$ viscosity solutions of (**??**) and if φ is an affine function, then (G2) is also necessary.

We should remark that (G2) strictly extends (G1): indeed, if Ω is convex, $\forall\, y \in \partial\Omega$ where the inward normal, $\nu(y)$, is uniquely defined we have $N^N_{\mathbb{R}^n\setminus\Omega}(y) = \{\nu(y)\}$ and if $\varphi \in C^1$ then $D^+\varphi(y) = \{D\varphi(y)\}$ (see Remark **??** and Proposition **??**).

To understand better the conditions (G1) and (G2) one should keep in mind the following examples.

Example 4.1.1. *Let*

$$F_1(\xi_1, \xi_2) = -(\xi_1^2 - 1)^2 - (\xi_2^2 - 1)^2 \; ; \; \varphi = 0.$$

Clearly

$$\begin{cases} E_1 &= \{\xi \in \mathbb{R}^2 \,:\, \xi_1^2 = \xi_2^2 = 1\} = \{\xi \in \mathbb{R}^2 \,:\, F_1(\xi) = 0\} \\ \operatorname{co} E_1 &= \{\xi \in \mathbb{R}^2 \,:\, |\xi_1| \le 1\,,\, |\xi_2| \le 1\} \\ E_1 &\subset \partial \operatorname{co} E_1 \text{ and } E_1 \ne \partial \operatorname{co} E_1. \end{cases}$$

For this classical example the condition (G1) allows us to say that the only convex Ω for which exists a $W^{1,\infty}(\Omega)$ viscosity solution of

$$\begin{cases} F_1(Du) &= 0 \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{4.3}$$

*are rectangles whose normals are in E_1. The condition (G2) instead allows us to make this selection among all the sets Ω, convex and not; in particular there are no $W^{1,\infty}(\Omega)$ viscosity solution of problem (**??**) if Ω is a non convex domain (see figure 1).*

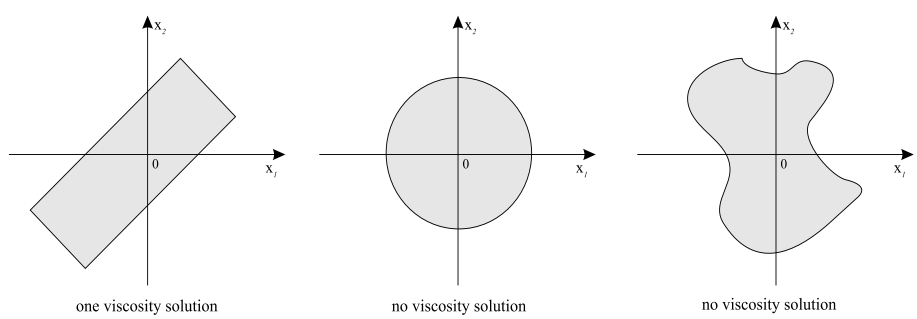

fig. 1

*We should remark that since $0 \in \operatorname{int} \operatorname{co} E_1$, the existence of $W^{1,\infty}(\Omega)$ almost everywhere solution is, in general, guaranteed. Moreover this example shows us that the existence of viscosity solutions do not depend on the smoothness of the data, since when Ω is the unit disk, F, φ and $\partial\Omega$ are analytic, but we do not have any viscosity solution of problem (**??**).*

In the previous example we have seen a non existence result, we now want to give an example where the condition (G2) can ensure us the existence of a viscosity solution for the Dirichlet problem involving a non convex Hamiltonian in a non convex domain.

Example 4.1.2. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a positive continuous function which is zero only on the vertical segment $S = \{(\xi_1, \xi_2) \ : \ \xi_1 = 1, \xi_2 \in [-1, 1]\}$; for instance we can consider*

$$f(\xi_1,\xi_2) = \begin{cases} \xi_2 - 1 & \text{if} \quad (\xi_1,\xi_2) \in Q_1 \\ \xi_1 - 1 & \text{if} \quad (\xi_1,\xi_2) \in Q_2 \\ -\xi_1 + 1 & \text{if} \quad (\xi_1,\xi_2) \in Q_3 \\ -\xi_2 - 1 & \text{if} \quad (\xi_1,\xi_2) \in Q_4 \end{cases}$$

where Q_i, $i = 1...4$ is a partition of the plane as in figure 2.

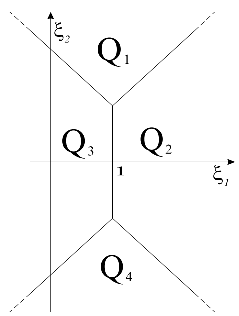

fig. 2

Let

$$F_2(\xi_1,\xi_2) = f(\xi_1,\xi_2)F_1(\xi_1,\xi_2) = f(\xi_1,\xi_2)[-(\xi_1^2-1)^2 - (\xi_2^2-1)^2]$$

where $F_1(\xi_1, \xi_2)$ is the function defined in the previous example and $\varphi = 0$.

Clearly we have

$$\begin{cases} E_2 &= E_1 \cup S = \{\xi \in \mathbb{R}^2 \ : \ F_2(\xi) = 0\} \\ coE_2 &= \{\xi \in \mathbb{R}^2 \ : \ |\xi_1| \le 1 \, , \, |\xi_2| \le 1\} \\ E_2 &\subset \partial coE_2 \text{ and } E_2 \neq \partial coE_2. \end{cases}$$

If we consider the problem

$$\begin{cases} F_2(Du) &= 0 \quad in \quad \Omega \\ u &= 0 \quad on \quad \partial\Omega, \end{cases} \tag{4.4}$$

where Ω is the non convex domain as in figure 3

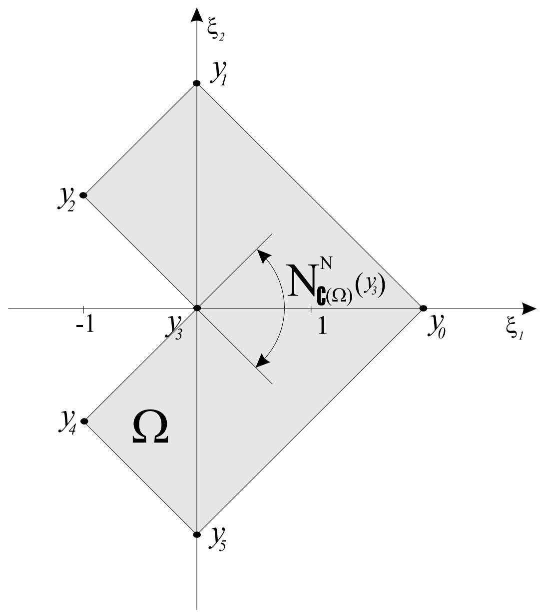

fig. 3

we can easily verify the condition (G2) that ensures the existence of viscosity solutions. Indeed, since $\varphi = 0$*, to verify (G2) it is sufficient to show that the sets of directions of the internal normal cone to* $\partial\Omega$ *at* y *,* $N^N_{\mathbb{R}^n\setminus\Omega}(y)$*, is contained in* E_2 *for every* $y \in \partial\Omega$*. In order to see this, we start by observing that in the points of regularity of* $\partial\Omega$ *the inward unit normal is in* E_1*. Then we have to consider* $N^N_{\mathbb{R}^n\setminus\Omega}(y_i)$ *for* $i = 0...5$*. The only point at which* $N^N_{\mathbb{R}^n\setminus\Omega}(y_i) \neq \emptyset$ *is* y_3*, since at the other points* Ω *is convex and* $N^N_{\mathbb{R}^n\setminus\Omega}(y_i)$ *is empty; moreover we can easily see (cf. Section* **??***) that*

$$N^N_{\mathbb{R}^n\setminus\Omega}(y_3) = S,$$

and this proves (G2).

Finally we want to point out that our method to establish the existence of viscosity solution is in fact based on comparing the non convex problem (**??**) with the convex one

$$\begin{cases} \rho(Du) &= 1 \quad \text{in} \quad \Omega \\ u &= \varphi \quad \text{on} \quad \partial\Omega, \end{cases} \tag{4.5}$$

where ρ is the gauge function of the convex hull of $E = \{\xi \in \mathbb{R}^n \ : \ F(\xi) = 0\}$ (cf. Section **??**). We will show that the geometrical condition (G2), under suitable hypotheses, ensures us that the viscosity solution of problem (**??**) is in fact also a viscosity solution of the initial problem (**??**). Then we are able to exhibit an Hopf-Lax type explicit formula for the solution of (**??**), using Theorem **??** applied to the problem (**??**).

4.2 Preliminaries: normal and tangent cones

In this section we recall several definitions of generalized normal and tangent cones to a given set K and we investigate some properties that will be useful to understand the statement of the main result and its proof.

We start giving some definitions that generalize the concepts of tangent and normal vectors.

Definition 4.2.1. *Let K be a locally compact subset of $\mathbb{R}^n$ and $x \in K$. A vector $v \in \mathbb{R}^n$ is a* generalized tangent *to K at x if there are $h_n \to 0^+$, $v_n \to v$ such that $x + h_n v_n \in K$, $\forall\, n \in \mathbb{N}$. The set of all generalized tangent vectors to K at x is denoted by $T_K(x)$, that is*

$$T_K(x) := \left\{ v \in \mathbb{R}^n \,|\, \exists\, h_n \to 0^+ \,,\, v_n \to v \;:\; x + h_n v_n \in K \right\}.$$

A vector $\nu \in \mathbb{R}^n$ is a generalized outward normal *to K at x if for every generalized tangent v to K at x, $\langle v, \nu\rangle \leq 0$. We denote by $N_K(x)$ the set of generalized normals to K at x. That is*

$$N_K(x) := \{\nu \in \mathbb{R}^n \,|\, \langle v, \nu\rangle \leq 0 \;\; \forall\, v \in T_K(x)\}.$$

The set $T_K(x)$ is a closed cone containing the origin and we will refer at it as *tangent cone*[1] to K at x; by duality we will call $N_K(x)$ the *outward normal cone* to K at x. Moreover we denote by $N_K^N(x)$ the set of directions of the normal cone to K at x, that is

$$N_K^N(x) := \left\{ \frac{\nu}{|\nu|}, v \in N_K(x) \setminus \{0\} \right\}.$$

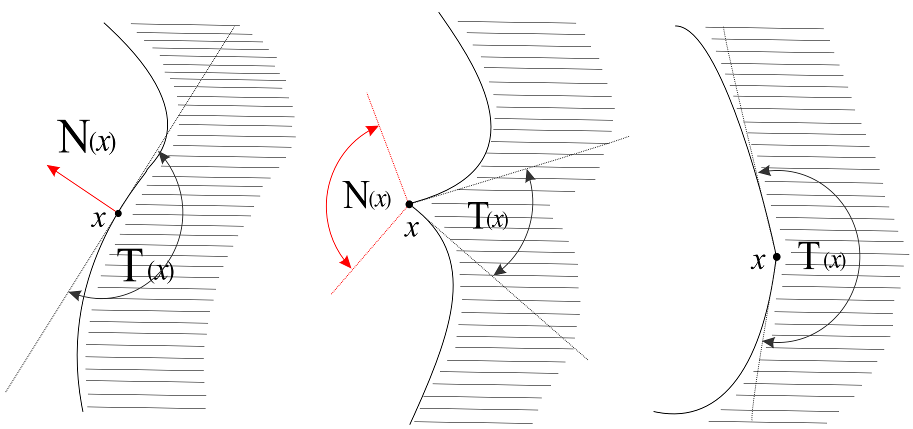

fig. 4

[1] The set $T_K(x)$ was introduced in the 30th by Bouligand with the name of *contingent cone* and it was studied for the theory of derivations of functions on $\mathbb{R}^2$(see [?]). Later in the theory of optimal control it was called simply *tangent cone* (see for example [?]).

Remark 4.2.2. *i) If the boundary of K is piecewise C^1, then $N_K(x)$ is reduced to a single vector ν_x, where ν_x is the usual outward normal, at any $x \in \partial K$ where the normal exists (see figure 4).*

ii) If Ω is an open subset of $\mathbb{R}^n$ and $x \in \partial\Omega$, then a generalized normal $\nu \in N_{\mathbb{R}^n \setminus \Omega}(x)$ can be regarded as an interior normal to Ω at x (see Figure 4).

Another useful set that can be defined is the *Clarke's tangent cone* to K at x (see [**?**],[**?**]). It is defined by[2].

$$C_K(x) := \left\{ v \in \mathbb{R}^n : \forall x_n \to x, \forall t_n \to 0^+, \exists\, v_n \to v : x_n + t_n v_n \in K, \forall n \in \mathbb{N} \right\}.$$

Definition 4.2.3. *A set K is said to be* regular *in the sense of Clarke at x provided $T_K(x) = C_K(x)$.*

To have an idea of the relations between the two definitions of tangent cones $T_K(x)$ and $C_K(x)$ we can take a look at figure 5.

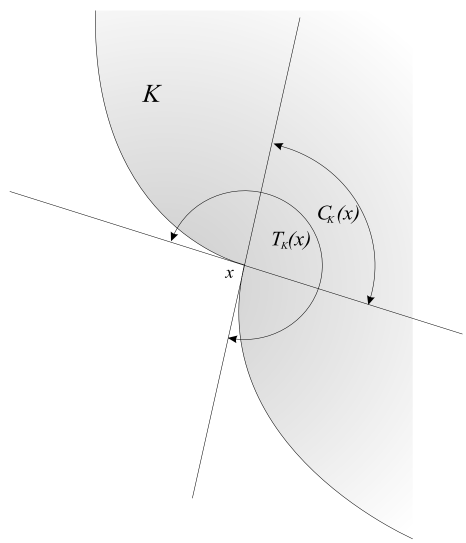

fig. 5

Remark 4.2.4. *(i) $C_K(x)$ is always a closed convex cone contained in $T_K(x)$ (see for example [**?**]). For this reason many authors prefer $C_K(x)$ instead of $T_K(x)$ as definition of tangent cone in many applications of non convex analysis, optimal control theory and calculus of variations.*

ii) If $T_K(x)$ is convex, then $N_K(x)$ is in fact the polar cone of $T_K(x)$ in the sense of convex analysis. It is the case for example of a set K regular in the Clarke's sense at x for which we have.

$$N_K(x) = T_K^0(x) = C_K^0(x), \tag{4.6}$$

[2]The original definition of $C_K(x)$ was given by Clarke in a slightly different way, more indirectly, but the two definitions are equivalent (see [**?**])

where $C^0_K(x)$ *and* $T^0_K(x)$ *denote the polar cones of* $C_K(x)$ *and* $T_K(x)$ *in the sense of convex analysis.*

(iii) Any convex set is regular in the sense of Clarke.

4.3 Main results

4.3.1 Sufficient condition

In this section we establish a sufficient condition for the existence of a $W^{1,\infty}(\Omega)$ viscosity solution of the problem (**??**) under the following hypotheses:

- (H1) Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ be continuous and such that

$$E = \{\xi \in \mathbb{R}^n \,:\, F(\xi) = 0\} \subset \partial(\operatorname{co} E),$$

with E bounded, $0 \in \operatorname{int}\operatorname{co} E$ and $F(\xi) < 0$ for every $\xi \in$ *int coE*.

Remark 4.3.1. *If* F *is convex and coercive, as in the classical literature, then*

$$\operatorname{co} E := \{\xi \in \mathbb{R}^n \,:\, F(\xi) \le 0\}$$

and (H1) is satisfied with $E = \partial \operatorname{co} E$*.*

Following an idea used in [**?**], we want to compare the solution of the following problem

$$\begin{cases} F(Du) &= 0 \quad \text{in} \quad \Omega \\ u &= \varphi \quad \text{on} \quad \partial\Omega \end{cases} \tag{4.7}$$

with the viscosity solution of the equation

$$\begin{cases} \rho(Du) &= 1 \quad \text{in} \quad \Omega \\ u &= \varphi \quad \text{on} \quad \partial\Omega, \end{cases} \tag{4.8}$$

where ρ is the gauge associated to $\operatorname{co} E$ defined as

$$\rho(\xi) = \inf\{\lambda \ge 0 \mid \xi \in \lambda \operatorname{co} E\}.$$

We start by observing that ρ is well defined since by (H1) $0 \in \operatorname{int}\operatorname{co} E$ and $\operatorname{co} E$ is compact; moreover ρ is, by definition, convex and positively homogeneous of degree 1. Therefore we are in the hypotheses of the Example **??** of the previous chapter, then in particular we can write the 'optical length' $L(x, y)$ related to the problem (**??**) as follows

$$L(x, y) := \inf_{S_{x,y}} \left\{ \int_0^1 \rho^0 \left(-\frac{d\xi}{dt} \right) \, dt \right\} \tag{4.9}$$

where ρ^0 is the polar function of ρ in the sense of convex analysis and therefore ρ^0 is in its turn convex and positively homogeneous.

Before stating the main result we need to set our hypotheses on the boundary datum φ,

- (H2) Let $\varphi \in Lip(\partial\Omega)$, with

$$\emptyset \neq D^+\varphi(x) \subseteq E \cup \operatorname{int}\operatorname{co} E \;\; \forall\, x \in \partial\Omega \tag{4.10}$$

and satisfying the compatibility condition

$$\varphi(x) - \varphi(y) \leq \rho^0(x, y) \;\; \forall\, x, y \in \partial\Omega \tag{4.11}$$

Remark 4.3.2. *We should note that the notion of $D^+\varphi(x)$ in (H2) must be considered thinking at the Lipschitz extension of φ in the sense of lemma* **??***. Moreover we can prove that $D^+\varphi(x) \subseteq \overline{\operatorname{co} E}$ for all $x \in \Omega$ (see the proof of Theorem* **??***).*

Remark 4.3.3. *The condition (**??**) is more restrictive then (**??**),which ensure the existence of a viscosity solution for the problem (**??**), since using Jensen's inequality we can easily prove that*

$$L(x, y) \geq \rho^0(x - y).$$

Moreover we should note that if the segment $[x, y]$ is an admissible path for the definition of L (that is it is completely contained in $\overline{\Omega}$) then $L(x, y) = \rho^0(x - y)$, it is the case, for example, when Ω is convex.

Finally the Theorem **??**, the Remark **??** and (H2) allow us to write the $W^{1,\infty}(\Omega)$ viscosity solution of equation (**??**) as follows

$$u(x) = \inf_{y \in \partial\Omega} \{\varphi(y) + L(x, y)\}, \quad x \in \overline{\Omega}. \tag{4.12}$$

Now we are in the position to state the main theorem of this section.

Theorem 4.3.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set. Let F and φ satisfy (H1), and (H2). If $\forall\, y \in \partial\Omega$ where $N^N_{\mathbb{R}^n \setminus \Omega}(y) \neq \emptyset$, there exists $h \in D^+\varphi(y)$ such that $\forall \nu \in N^N_{\mathbb{R}^n \setminus \Omega}(y)$ there exists a unique $\lambda_{\nu,h} > 0$ that verifies*

$$h + \lambda_{\nu,h}\nu \in E,$$

*then there exists $u \in W^{1,\infty}(\Omega)$ viscosity solution of (**??**).*

Remark 4.3.5. *Let $h \in D^+\varphi(y)$ as in the hypotheses of the Theorem* **??** *and $\nu \in N^N_{\mathbb{R}^n \setminus \Omega}(y)$, then, since $E \subset \partial \operatorname{co} E$, the unique $\lambda_{\nu,h} > 0$ such that $h + \lambda_{\nu,h}\nu \in E$ is determined by the equality*

$$\rho(h + \lambda_{\nu,h}\nu) = 1.$$

We will prove that, under the hypotheses of the Theorem **??**, the function $u : \overline{\Omega} \to \mathbb{R}$ defined by (**??**) is actually the viscosity solution of (**??**). Before starting the proof we need to investigate the properties of u. Let's start proving the following key lemma and making some remarks.

Lemma 4.3.6. *Let Ω be a bounded connected open set of $\mathbb{R}^n$ with Lipschitz boundary, $\partial\Omega$, and $\varphi \in Lip(\partial\Omega)$ verify (H2). Let u be defined by (**??**) and $y(x) \in \partial\Omega$ be such that $u(x) = \varphi(y(x)) + L(x, y(x))$. Then $\forall\, p \in D^- u(x)$ and $\forall\, h \in D^+\varphi(y(x))$,*

$$\langle p - h, q\rangle \leq 0 \quad \forall\, q \in T_{\mathbb{R}^n \setminus \Omega}(y(x)),$$

that is $p - h \in N_{\mathbb{R}^n \setminus \Omega}(y(x))$.

Proof. Let $x_0 \in \Omega$, $y_0 \in \partial\Omega$ such that $u(x_0) = \varphi(y_0) + L(x_0, y_0)$ and $q \in T_{\mathbb{R}^n \setminus \Omega}(y_0)$. Let $q_k \to q$, as in the Definition **??**, such that $y_0 + \varepsilon_k q_k \notin \Omega$ and $x_0 + \varepsilon_k q_k \in \Omega$. By definition of $L(x_0, y_0)$ for every $\varepsilon > 0$ we can find $\xi_0 \in S_{x_0, y_0}$ (that is $\xi_0 : [0,1] \to \overline{\Omega} \mid \xi(0) = x_0\,,\ \xi(1) = y_0\,,\ \frac{d\xi_0}{dt} \in L^\infty(0,1)$) such that

$$L(x_0, y_0) + \varepsilon \geq \int_0^1 \rho^0 \left(-\frac{d\xi_0}{dt}(t) \right) dt. \tag{4.13}$$

Next we define for every $k \in \mathbb{N}$, $\xi_k(t) = \xi_0(t) + \varepsilon_k q_k$; clearly we have $\xi_k(0) = x_0 + \varepsilon_k q_k$, $\xi_k(1) = y_0 + \varepsilon_k q_k$ and $\frac{d\xi_k}{dt} = \frac{d\xi_0}{dt}$.

Since ξ_k and $\partial\Omega$ are continuous, there exists $t_k \in (0,1)$ and $y_k \in \partial\Omega$ such that $\xi_k(t_k) = y_k$ and $\xi_k(t) \in \overline{\Omega}\ \forall\, t < t_k$ (see figure 6).

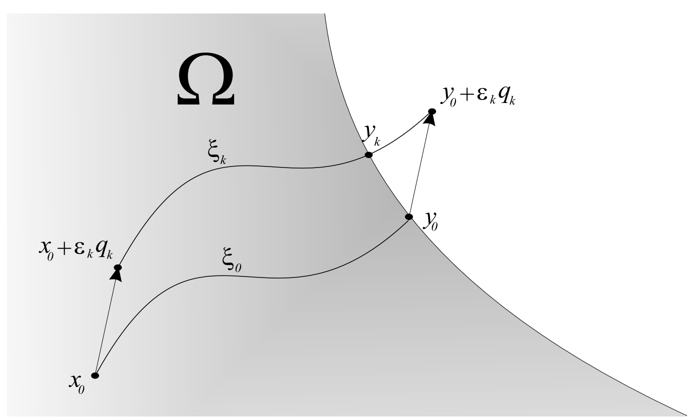

fig.6

Using (**??**), the properties of ξ_k and the definition (**??**) of u, we have

$$\begin{aligned}
u(x_0) &= \varphi(y_0) + L(x_0, y_0) \\
&\geq \varphi(y_0) + \int_0^1 \rho^0 \left(-\frac{d\xi_0}{dt}(t) \right) dt - \varepsilon \\
&= \varphi(y_0) - \varphi(y_0 + \varepsilon_k q_k) \\
&+ \varphi(y_0 + \varepsilon_k q_k) - \varphi(y_k) + \int_{t_k}^1 \rho^0 \left(-\frac{d\xi_0}{dt}(t) \right) dt \\
&+ \varphi(y_k) + \int_0^{t_k} \rho^0 \left(-\frac{d\xi_0}{dt}(t) \right) dt - \varepsilon \\
&\geq \varphi(y_0) - \varphi(y_0 + \varepsilon_k q_k) + u(x_0 + \varepsilon_k q_k) \\
&+ \varphi(y_0 + \varepsilon_k q_k) - \varphi(y_k) + \int_{t_k}^1 \rho^0 \left(-\frac{d\xi_0}{dt}(t) \right) dt - \varepsilon
\end{aligned} \tag{4.14}$$

where we have used the homogeneity of ρ^0 to establish

$$\int_0^{t_k} \rho^0 \left(-\frac{d\xi_0}{dt}(t) \right) dt = \int_0^1 \rho^0 \left(-\frac{d(\xi_k(t_k s))}{ds} \right) ds \geq L(x_0 + \varepsilon_k q_k, y_k).$$

We claim that

$$\varphi(y_0 + \varepsilon_k q_k) - \varphi(y_k) + \int_{t_k}^1 \rho^0 \left(-\frac{d\xi_0}{dt}(t) \right) dt \geq 0. \tag{4.15}$$

Indeed Lemma (**??**) ensures us that

$$\varphi(y_0 + \varepsilon_k q_k) - \varphi(y_k) \geq -\rho^0(y_k - y_0 - \varepsilon_k q_k); \tag{4.16}$$

moreover by Jensen's inequality we have

$$\begin{aligned}
\int_{t_k}^1 \rho^0 \left(-\frac{d\xi_0}{dt}(t) \right) dt &= \int_0^1 \rho^0 \left(-\frac{d\xi_k((1-t_k)s + t_k)}{ds} \right) ds \\
&\geq \rho^0(y_k - y_0 - \varepsilon_k q_k).
\end{aligned} \tag{4.17}$$

Combining (**??**) and (**??**) we obtain the claim.

Now using (**??**) and (**??**) we can write, letting $\varepsilon \to 0$

$$u(x_0) \geq u(x_0 + \varepsilon_k q_k) - \big(\varphi(y_0 + \varepsilon_k q_k) - \varphi(y_0)\big). \tag{4.18}$$

Therefore, taking $h \in D^+\varphi(y_0)$ and $p \in D^-u(x_0)$, we have by definition that

$$\begin{aligned}
\varphi(y_0 + \varepsilon_k q_k) - \varphi(y_0) &\leq \langle h, \varepsilon_k q_k \rangle + o(\varepsilon_k) \\
u(x_0 + \varepsilon_k q_k) - u(x_0) &\geq \langle p, \varepsilon_k q_k \rangle + o(\varepsilon_k),
\end{aligned}$$

and in light of (**??**), we can say that

$$\begin{aligned}\langle p, \varepsilon_k q_k\rangle &\leq u(x_0+\varepsilon_k q_k) - u(x_0) + o(\varepsilon_k)\\ &\leq \varphi(y_0+\varepsilon_k q_k) - \varphi(y_0) + o(\varepsilon_k)\\ &\leq \langle p, \varepsilon_k q_k\rangle + o(\varepsilon_k).\end{aligned}$$

Finally, dividing both side of last inequality by ε_k and taking the limit for $k \to \infty$ we obtain

$$\langle p - h, q\rangle \leq 0;$$

the arbitrariness of q, h, p gives the desired result. □

Remark 4.3.7. *If we fix $p \in D^-u(x)$ and $h \in D^+\varphi(y(x))$ with $h \neq p$, then there exist $\nu_{p,h} \in N^N_{\mathbb{R}^n\setminus\Omega}(y(x))$ and a unique $\lambda_{p,h} > 0$ such that $p = h + \lambda_{p,h}\nu_{p,h}$.*

We now give the proof of main theorem.

Proof of Theorem **??**. Let u defined as in (**??**); by definition u is a viscosity solution of (**??**). We claim that u is also a viscosity solution of (**??**). We divide the proof into two steps: first we show that u in in fact a supersolution of (**??**) and then that u is also a subsolution.

- Since u is a supersolution of (**??**), we have that for all $x \in \Omega$ and for all $p \in D^-u(x)$ $\rho(p) \geq 1$. Moreover, since u is also a viscosity subsolution of (**??**), in particular we have $\rho(Du(x)) \leq 1$ (i.e. $Du(x) \in \overline{coE}$) $\forall\, x \in \Omega$ where $Du(x)$ exists, since in such points $D^+u(x) = \{Du(x)\}$. The continuity of ρ ensures us that

  $$\rho\left(\lim_{n\to\infty} Du(x_n)\right) \leq 1$$

  for all $x_n \to x$ such that $Du(x_n)$ is well defined and $Du(x_n)$ converges, that is the following inclusion holds

  $$\left\{p \in \mathbb{R}^n \mid p = \lim_{n\to\infty} Du(x_n) \;:\; x_n \to x\right\} \subseteq \overline{coE}. \tag{4.19}$$

  Therefore, by (iv) of Proposition **??** and (**??**) we can say that

  $$D^-u(x) \subseteq co\left\{p \in \mathbb{R}^n \mid p = \lim_{n\to\infty} Du(x_n) \;:\; x_n \to x\right\} \subseteq \overline{coE},$$

  that is $\rho(p) \leq 1\ \forall\, p \in D^-u(x)$. We finally have $\rho(p) = 1$.

  Let $y(x) \in \partial\Omega$ such that $u(x) = \varphi(y(x)) + L(x, y(x))$ and $h \in D^+\varphi(y(x))$ as in the hypotheses. We distingue two cases.

  If $h = p$ then $\rho(h) = 1$; since $h \in E \cup int\, coE$, we have $h \in E$ and so $p \in E$ that is $F(p) = 0$.

If $h \neq p$, by Remark **??**, there exist $\nu_{p,h} \in N^N_{\mathbb{R}^n \setminus \Omega}(y(x))$ and a unique $\lambda_{p,h} > 0$ such that

$$p = h + \lambda_{p,h}\nu_{p,h}; \tag{4.20}$$

moreover $\lambda_{p,h}$ is uniquely determined by $\rho(h + \lambda_{p,h}\nu_{p,h}) = 1$. The hypothesis made on h and (**??**) imply $p \in E$, that is, as before, $F(p) = 0$.

In particular u is a viscosity supersolution of (**??**).

- Since u is also a viscosity subsolution of (**??**), then for every $x \in \Omega$ and $p \in D^+u(x)$ we have $p \in \overline{coE}$ (i.e. $\rho(p) \leq 1$). As (H1) is satisfied and as F is continuous, it follows that $F(p) \leq 0$. So u is a viscosity subsolution of (**??**).

The two above observations complete the proof. □

4.3.2 Necessary condition

We have seen in the previous section that the geometrical condition (G2), under suitable hypotheses, is sufficient for the existence of a unique viscosity solution of problem (**??**). In fact, in some cases, the condition (G2) turn out to be also necessary, at least when the boudary datum is an affine function. This result could be deduced reading the section 3 in [**?**] (cf Theorem 3.4 in [**?**]), nevertheless we give here a direct proof.

We start recalling a useful consequence of the viability theorem (cf. Lemma 3.6 in [**?**], Theorem 3.3.2 and Theorem 3.2.4 in [**?**]).

Lemma 4.3.8. *Let $\Omega \subset \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}$ be continuous and such that the set $E := \{\xi \in \mathbb{R}^n \ : \ F(\xi) = 0\}$ is compact. If there is some $\nu \in \mathbb{R}^n \setminus \{0\}$ such that*

(i) $\forall \lambda \geq 0,\ F(\lambda\nu) < 0$,

(ii) $\exists\, y \in \partial\Omega$ such that $\nu \in N_{\mathbb{R}^n \setminus \Omega}(y)$,

then there is no $W^{1,\infty}(\Omega)$ viscosity supersolution to

$$\begin{cases} F(Du) = 0 & in \quad \Omega \\ u = 0 & on \quad \partial\Omega. \end{cases}$$

Now we can prove the following

Theorem 4.3.9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set. Let F satisfy (H1) and φ an affine function with $D\varphi = b \in \operatorname{int} \operatorname{co} E$. If there exists $u \in W^{1\infty}(\Omega)$, viscosity solution of problem (**??**), then $\forall\, y \in \partial\Omega$ where $N^N_{\mathbb{R}^n \setminus \Omega}(y) \neq \emptyset$, there exists $h \in D^+\varphi(y)$ such that $\forall \nu \in N^N_{\mathbb{R}^n \setminus \Omega}(y)$ there exists a unique $\lambda_{\nu,h} > 0$ that verifies*

$$h + \lambda_{\nu,h}\nu \,\in\, E,$$

Proof. Let us use the notation $\varphi(y) = \langle b; y\rangle + a$. The condition (H1) ensure us that $F(\xi) > 0$ for $|\xi|$ sufficiently large. Next we observe that u is in particular a viscosity supersolution of

$$\begin{cases} F(Du) &= 0 \quad \text{in} \quad \Omega \\ u &= \varphi \quad \text{on} \quad \partial\Omega. \end{cases}$$

Then, if we define $\widetilde{F}(\xi) = F(\xi + b)$ and $\widetilde{u}(x) = u(x) - \langle b; x\rangle - a$, we easily see that $\widetilde{u}$ is a supersolution of

$$\begin{cases} \widetilde{F}(D\widetilde{u}) &= 0 \quad \text{in} \quad \Omega \\ \widetilde{u} &= 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Lets take now $\nu \in N^N_{\mathbb{R}^n\setminus\Omega}(y)$ for some $y \in \partial\Omega$, the lemma **??** ensure us that there exists a $\lambda_0 \geq 0$ such that $\widetilde{F}(\lambda_0\nu) \geq 0$, i.e. $F(D\varphi + \lambda_0\nu) \geq 0$.

Since $F(D\varphi) < 0$, there exists $\lambda > 0$ such that $F(D\varphi + \lambda\nu) = 0$, i.e.

$$D\varphi + \lambda\nu \in E.$$

Moreover by (H1) λ is uniquely determined by $\rho(D\varphi + \lambda\nu) = 1$ and this conclude the proof. □

4.4 Corollaries

This section is divided into two parts. In the first one we focus our attention on the differentiability properties of Lipschitz and semiconcave functions with the aim to relate the notions of normal and tangent cones to the sets described by such type of functions (as epigraphs or level-sets) to their generalized gradients. In the second one we state two corollaries of Theorem **??** in which the hypotheses on the geometry of the domain Ω can be written in a nicer way in terms of the differential property of the functions that represent the boundary $\partial\Omega$.

4.4.1 Lipschitz continuity and semiconcavity

Let us recall briefly some definitions and some relevant differential properties of locally Lipschitz continuous functions that we will use in the sequel. By the Rademacher theorem such functions are almost everywhere differentiable with locally bounded gradient (see [**?**]). Hence, if $u \in Lip_{\text{loc}}(\Omega)$, we can consider the set

$$D^*u(x) := \{p \in \mathbb{R}^n \; : \; p = \lim_{n\to\infty} Du(x_n), x_n \to x\}$$

where x_n is a sequence of points of differentiability for u. We note that $D^*u(x)$ is non empty and closed for any $x \in \Omega$.

Let $u : \Omega \to \mathbb{R}$ be Lipschitz in a neighborhood of a given point x, and let $q \in S^{n-1}$ be a direction in $\mathbb{R}^n$. We define:

- The *one-sided directional derivative* of u at x in the direction q as

$$u'(x,q) = \lim_{t \to 0^+} \frac{u(x+tq) - u(x)}{t}.$$

- The *generalized directional derivatives* of u at x in the direction q as

$$u^0(x,q) = \limsup_{y \to x, t \to 0^+} \frac{u(y+tq) - u(y)}{t},$$

$$u_0(x,q) = \liminf_{y \to x, t \to 0^+} \frac{u(y+tq) - u(y)}{t}.$$

- The generalized gradient (or Clarke's gradient) of u at x as

$$\begin{aligned} \partial u(x) &= \{p \in \mathbb{R}^n \,:\, u^0(x,q) \geq p \cdot q\,,\, \forall q \in \mathbb{R}^n\} \\ &= \{p \in \mathbb{R}^n \,:\, u_0(x,q) \leq p \cdot q\,,\, \forall q \in \mathbb{R}^n\}. \end{aligned}$$

In the following proposition we collect some well-known properties of Lipschitz functions (see [**?**], [**?**]).

Proposition 4.4.1. *Let $u : \Omega \to \mathbb{R}$ be locally Lipschitz continuous in the open set Ω, then*
(i) $u_0(x,q) = -u^0(x,-q)$ for all $x \in \Omega$, $q \in \mathbb{R}^n$;
(ii) For all $x \in \Omega$ the function $q \mapsto u^0(x,q)$ is finite, positively homogeneous, subadditive, convex (and locally Lipschitz continuous);
(iii) The map $(x,q) \mapsto u^0(x,q)$ is upper semicontinuous;
(iv) For all $x \in \Omega$ we have $\mathrm{co}\, D^*u(x) = \partial u(x)$;
(v) $D^+u(x)$ and $D^-u(x)$ are bounded for all $x \in \Omega$ and

$$D^+u(x) \cup D^-u(x) \subseteq \partial u(x);$$

(vi) For all $q \in S^{n-1}$ there exists the classical one-sided directional derivative $u'(x,q)$ at any $x \in \Omega$ where $D^+u(x) = \partial u(x)$ and the following equality holds

$$u'(x,q) = \min_{p \in D^+u(x)} p \cdot q = u_0(x,q). \tag{4.21}$$

Remark 4.4.2. *Looking at the definition of $D^*u(x)$ and at Proposition* **??** *(iv), one can observe that Proposition* **??** *(v) is just a reformulation of Proposition* **??** *(iv).*

Now we introduce a definition of regularity of functions that is in some way related to regularity of sets in the Clarke's sense (from which the name derives). It will be useful for stating some hypotheses that allow us to write the normal cone of a set in a nicer way.

Definition 4.4.3. *A function $u : \Omega \to \mathbb{R}$ is said to be* regular *at x (in the sense of Clarke) provided*

(i) $\forall\, q \in \mathbb{R}^n$ the one-sided directional derivative $u'(x, q)$ exists;

(ii) $\forall\, q \in \mathbb{R}^n$ the equality holds $u'(x, q) = u^0(x, q)$.

The following theorem (a proof of which can be found in [**?**]) and its corollaries, give us a useful characterization of normal cone to the level sets of regular functions.

Theorem 4.4.4. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz near a given point x and suppose that $0 \notin \partial f(x)$. If K is defined as*

$$K := \{y \in \mathbb{R}^n \,:\, f(y) \le f(x)\},$$

then

$$C_K^0(x) \subset \bigcup_{\lambda \ge 0} \lambda \partial f(x).$$

If in addition f is regular in the sense of Clarke at x, then equality holds and K is Clarke's regular at x, that is

$$N_K(x) = C_K^0(x) = \bigcup_{\lambda \ge 0} \lambda \partial f(x). \tag{4.22}$$

Remark 4.4.5. *The first equality in (**??**) follows by (**??**) of Remark **??** (iii), since K is regular.*

Remark 4.4.6. *The above Proposition holds also in a more general framework, that is for functions defined in a general Banach space, as is stated in [**?**].*

Corollary 4.4.7. *Let $\Omega := \{y \in \mathbb{R}^n \,:\, f(y) > 0\}$, where f is a Lipschitz continuous function. Let $y_0 \in \partial\Omega$ and suppose that f verifies the following properties in y_0*

(i) f is regular in the Clarke's sense;

(ii) $0 \notin \partial f(y_0) = D^- f(y_0) \cup D^+ f(y_0)$;

Then

$$N^N_{\mathbb{R}^n \setminus \Omega}(y_0) = \left(D^- f(y_0) \cup D^+ f(y_0)\right)^N. \tag{4.23}$$

Proof. We note first that $\partial\Omega \subseteq \{y \in \mathbb{R}^n \,:\, f(y) = 0\}$, then $y_0 \in \partial\Omega$ imply $f(y_0) = 0$. So we can write

$$\mathbb{R}^n \setminus \Omega := \{y \in \mathbb{R}^n \,:\, f(y) \le f(y_0)\}.$$

Hence we can apply the Theorem **??**, and in particular, since f is Clarke's regular, by (**??**) we have

$$N_{\mathbb{R}^n \setminus \Omega}(y_0) = \bigcup_{\lambda \ge 0} \lambda \partial f(y_0).$$

Finally we can conclude, using hypothesis (ii) that

$$N^N_{\mathbb{R}^n\setminus\Omega}(y_0) = \left(\bigcup_{\lambda\geq 0} \lambda \partial f(y_0) \right)^N = (\partial f(y_0))^N = \left(D^- f(y_0) \cup D^+ f(y_0) \right)^N .$$

□

In order to prove a second Corollary of Theorem **??** equally useful, we need to recall the definition and some relevant properties of semiconcave and semiconvex functions (see [**?**] for further details).

Definition 4.4.8. *We say that $u : \Omega \to \mathbb{R}$ is* semiconcave *on an open convex set Ω if there exists a constant $C > 0$ such that*

$$\lambda u(x) + (1-\lambda)u(y) \leq u\left(\lambda x + (1-\lambda)y\right) + \frac{1}{2}C\lambda(1-\lambda)|x-y|^2, \qquad (4.24)$$

or equivalently if the application $x \mapsto u(x) - \frac{1}{2}C|x|^2$ is concave.

We say that $u : \Omega \to \mathbb{R}$ is semiconvex if $-u$ is semiconcave.

If u is continuous an equivalent way to express the condition (**??**) is to require that

$$u(x+h) - 2u(x) + u(x-h) \leq C|h|^2$$

for any $x \in \Omega$ and $h \in \mathbb{R}^n$ with sufficiently small $|h|$.

Remark 4.4.9. *It can be proved (see for example [**?**]) that a semiconcave function u in Ω is in fact locally Lipschitz continuous and for all $x \in \Omega$ we have*

$$D^+u(x) = \partial u(x) = co\, D^* u(x),$$

while

$$D^- u(x) \neq 0 \;\Rightarrow\; u \textit{ is differentiable in } x.$$

Now we can prove the following

Corollary 4.4.10. *Let $\Omega := \{y \in \mathbb{R}^n \,:\, f(y) \leq 0\}$ where f is a semiconcave function, if $y_0 \in \partial\Omega$ and $0 \notin D^+ f(y_0)$ then*

$$N^N_{\mathbb{R}^n\setminus\Omega}(y_0) = -(D^+ f(y_0))^N.$$

Proof. We first note that, from Remark **??** f is locally Lipschitz continuous and $D^+ f(y_0) = \partial f(y_0)$. So we can say, by (vi) of Proposition **??**, that

$$f'(y_0, q) = f_0(y_0, q) \;\; \forall\, q \in S^{n-1}.$$

Moreover using the definition of generalized derivatives we have

$$-\left(-f'(y_0,q)\right) = f'(y_0,q) = f_0(y_0,q) = -\left(-f^0(y_0,q)\right) \;\; \forall\, q \in S^{n-1},$$

that is $-f$ is regular at y_0 in the sense of Clarke. We now observe that, since

$$-D^-(-f)(y_0) = D^+f(y_0) = \partial f(y_0) = -\partial(-f)(y_0),$$

f verifies the hypothesis of the Corollary **??** with $\Omega := \{y \in \mathbb{R}^n : -f(y) > 0\}$ and so we have

$$N^N_{\mathbb{R}^n \setminus \Omega}(y_0) = \left(D^-(-f)(y_0)\right)^N = -(D^+f(y_0))^N.$$

□

Remark 4.4.11. *The two above corollaries hold also if the hypothesis are verified only locally, that is if for $y_0 \in \partial\Omega$ there exists a ball $B(y_0, r)$ centered in y_0 such that $\Omega \cap B(y_0, r)$ can be represented as sublevel or superlevel set of a function defined on $B(y_0, r)$ satisfying the hypothesis required.*

The last result that we want to recall can be found in [**?**] and it gives us a useful relation between the generalized gradient of a locally Lipschitz function f and the Clarke's normal cone $C^0_{epi\,f}$ to its epigraph.

Proposition 4.4.12. *Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be Lipschitz continuous near a given point x, then $\xi \in \mathbb{R}^n$ belongs to $\partial f(x)$ if and only if $(\xi, -1)$ belongs to $C^0_{epi\,f}(x, f(x))$.*

4.4.2 Corollaries

In the two following corollaries we consider some hypotheses on the geometry of the domain Ω that allow us to write the Theorem **??** in a nicer way.

Let Ω be a Lipschitz domain, we have that Ω can be locally represented as the epigraph of a Lipschitz function, that is $\forall y \in \partial\Omega$ there exists a direction ν_y and a function ω_y defined on the hyperplane orthogonal to ν_y such that in a neighborhood of y, Ω is the epigraph of ω_y.

Definition 4.4.13. *We will say that Ω is convex [concave] at $y \in \partial\Omega$ if there exists a $\nu_y \in S^{n-1}$ such that the function ω_y, that represent Ω in the direction ν_y, is convex [concave].*

Corollary 4.4.14. *Let Ω be a locally Lipschitz domain and denote by J the set of the points of non differentiability of $\partial\Omega$. Suppose that Ω is convex or concave at y for all $y \in J$. Let F and φ satisfy (H1) and (H2).*

If $\forall y \in \partial\Omega$ where $D^+\omega_y(y) \neq \emptyset$, there exists $h \in D^+\varphi(y)$ such that $\forall \xi \in D^+(\omega_y)(y)$ there exists a unique $\lambda_{h,\xi}$ that verify

$$h - \lambda_{h,\xi}(\xi + \nu_y) \in E, \tag{4.25}$$

*then there exists $u \in W^{1,\infty}(\Omega)$ viscosity solution of (**??**).*

Remark 4.4.15. *We have to note that in (**??**) ξ has to be considered as a point of $\mathbb{R}^n$ using the classical immersion in $\mathbb{R}^n$ of the hyperplane orthogonal to ν_y, to witch ξ belongs by definitions.*

Remark 4.4.16. *In the statement of Corollary **??** we have used the functions ω_y; with this notations it seems that we have to change ω_y for all $y \in \partial\Omega$, but we can simply observe that the compactness of $\partial\Omega$ ensures us that we need only a finite number of ω_y. In fact, we can consider for every $y \in \partial\Omega$ a neighborhood Ω_y of y in which Ω is represented by the function ω_y. From this cover we can extract a finite one $\cup_{i=1}^{k}\Omega_{y_i}$ where $\omega_y = \omega_{y_i}$ for every $y \in \Omega_{y_i} \cap \partial\Omega$.*

Remark 4.4.17. *If we consider an orthogonal basis $\{e_1, ..., e_n\}$ for $\mathbb{R}^n$, with $e_n = \nu_y$, we note that ξ lives in the space spanned by $\{e_1, ..., e_{n-1}\}$ and (**??**) can be rewritten as*

$$h - \lambda_{h,\xi}(\xi, 1) \in E \ \ \forall \xi \in D^+(f_y)(y).$$

Proof of Corollary **??** Looking at the proof of Theorem **??** we only need to work with the points on $\partial\Omega$ that realize the minimum in the definition (**??**). Now let $x \in \Omega$ and $y \in \partial\Omega$ be such that $u(x) = \varphi(y) + L(x, y)$. If $D^+\omega_y(y) \neq \emptyset$ then Ω is convex in y and we can prove, using the same argument of Lemma 2.9 in [**?**], that y must be a point of differentiability for $\partial\Omega$ and this is a contradiction. Hence we have that all the points that realize the minimum in (**??**) have $D^+\omega_y \neq \emptyset$.

Now we want to identify the set $N_{\mathbb{R}^n\setminus\Omega}(y)$ and write it in terms of superdifferential of ω_y in order to apply the Theorem **??**.

We first observe that if ω_y is differentiable in y then $N_{\mathbb{R}^n\setminus\Omega}(y)$ reduces to the classic interior normal to $\partial\Omega$ given by $(D\omega(y) + \nu_y)$ and there is nothing to prove.

The last case that we have to consider is if Ω is concave at y and $D^+\omega_y(y)$ does not reduce to a single vector. In this case we have that $-\omega_y$ is convex near y and it represent $\mathbb{R}^n \setminus \Omega$ in the direction $-\nu_y$. Hence $\mathbb{R}^n \setminus \Omega$ is convex near y and so, by Remark **??** (iii), is Clarke regular and we have

$$N_{\mathbb{R}^n\setminus\Omega}(y) = C^0_{\mathbb{R}^n\setminus\Omega}(y) = C^0_{epi\,(-\omega_y)}(y). \tag{4.26}$$

Moreover, by the Proposition **??** we can write

$$C^0_{epi\,(-\omega_y)}(y) = \{(\xi, -1) \,:\, \xi \in \partial(-\omega_{y(y)})(y)\}. \tag{4.27}$$

We now observe that, since $-\omega_y$ is convex, we have

$$\partial(-\omega_{y(y)})(y) = D^-(-\omega_y)(y) = -D^+\omega_y(y), \tag{4.28}$$

and finally, by (**??**), (**??**) and (**??**) we have

$$N^N_{\mathbb{R}^n\setminus\Omega}(y) = \{(\xi, -1) \,:\, \xi \in -D^+\omega_y(y)\}^N.$$

The conclusion follows by Theorem **??**. □

Another way to represent a domain is like a sublevel or superlevel set of a given function. Also in such case, we have an appropriate version of the Theorem **??**. It is clear that if $\partial\Omega$ is regular in a neighborhood of a point $y \in \partial\Omega$ we can locally (near y) write Ω as sublevel-set of a regular function f_y^Ω, suppose moreover that Ω verifies the hypothesis

- (H3) Let Ω be a locally Lipschitz domain and denote by J the set of the points of non differentiability of $\partial\Omega$. Suppose that if for $y \in J$ there exists an $x \in \Omega$ such that $u(x) = \varphi(y) + L(x, y)$ (that is y realizes the minimum in the definition (**??**)) then Ω can be represented near y as the sublevel-set of a semiconcave function f_y^Ω (see Remark **??**).

The following corollary is an easily consequence of Theorem **??** and Corollary **??**.

Corollary 4.4.18. *Let Ω, F and φ satisfy (H1),(H2) and (H3).*

If $\forall y \in \partial\Omega$ where $D^+ f_y^\Omega(y) \neq \emptyset$, there exists $h \in D^+\varphi(y)$ such that $\forall \xi \in D^+ f_y^\Omega(y)$ there exists a unique $\lambda_{h,\xi}$ that verify

$$h - \lambda_{h,\xi}\xi \in E, \tag{4.29}$$

*then there exists $u \in W^{1,\infty}(\Omega)$ viscosity solution of (**??**).*

4.5 Appendix

4.5.1 Lipschitz extensions

The aim of this section is to recall a Mac-Shane type extension lemma for Lipschitz functions, which is in fact a consequence of the Hopf-Lax formula. With this in mind we start recalling some facts about gauge functions and their polars (see [**?**] for further details).

Definition 4.5.1. *(i) Let $K \subset \mathbb{R}^n$ be a convex set; then the gauge associated to K is defined as*

$$\rho(\xi) = \inf\{\lambda \geq 0 \ : \ \xi \in \lambda K\}.$$

(ii) The polar of a gauge ρ is defined as

$$\rho^0(\xi^*) = \inf\{\lambda^* \geq 0 \ : \ \langle \xi^*; \xi\rangle \leq \lambda^* \rho(\xi) \ , \ \ \forall\, \xi \in \mathbb{R}^n\}.$$

In the following proposition we recall some useful properties of gauge functions that can be easily deduced from the definition.

Proposition 4.5.2. *Let $K \subset \mathbb{R}^n$ be a compact and convex set with $0 \in \operatorname{int} K$. The following properties then hold.*

(i) The gauge ρ associated to K is finite everywhere, convex and satisfyies

$$\begin{array}{ll} (a) & \rho(\xi) > 0, \quad \forall\, \xi \neq 0 \\ (b) & \rho(t\xi) = t\rho(\xi), \quad \forall\, \xi \in \mathbb{R}^n \ \ \forall\, t > 0. \end{array}$$

(ii) We have $K = \{\xi \in \mathbb{R}^n \ : \ \rho(\xi) \leq 1\}$.

(iii) The polar function of ρ, ρ^0, is characterized by the following relation

$$\rho^0(\xi^*) = \sup_{\xi \neq 0} \left\{ \frac{\langle \xi^*; \xi \rangle}{\rho(\xi)} \right\}.$$

(iv) The following identity holds: $\rho^{00} = \rho$.

Remark 4.5.3. *(i) Note that if $0 \notin \operatorname{int} K$ then, in general, ρ is not finite everywhere. Similarly, if K is unbounded, then we may have $\rho(\xi) = 0$ for some $\xi \neq 0$.*

(ii) The notion of gauge and its polar are aimed at generalizing the Cauchy-Schwarz inequality, indeed we have

$$\langle \xi^*; \xi \rangle \leq \rho(\xi)\rho^0(\xi^*).$$

However we should note that in general we do not have $\rho(\xi) = \rho(-\xi)$.

Example 4.5.4. *The classical examples are those involving Hölder norms. Namely, if $1 \leq p \leq \infty$ and if $\frac{1}{p} + \frac{1}{p'} = 1$ and*

$$\rho(\xi) = |\xi|_p = \begin{cases} \left(\displaystyle\sum_{i=1}^{n} |\xi_i|^p \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \\ \displaystyle\max_{i \leq i \leq n} \{|\xi_i|\} & \text{if } p = \infty, \end{cases}$$

then $\rho^0(\xi^*) = |\xi^*|_{p'}$.

Now we can state the main extension lemma

Lemma 4.5.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded closed set. Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a gauge function, that is a positively homogeneous convex function, let ρ^0 its polar. If $\varphi \ : \ \partial\Omega \to \mathbb{R}$ satisfy*

$$\varphi(x) - \varphi(y) \leq \rho^0(x - y) \ \ \forall\, x, y \, \in \, \partial\Omega$$

then the function

$$\tilde{\varphi}(x) = \inf_{y \in \partial\Omega} \left\{ \varphi(y) + \rho^0(x - y) \right\}$$

is a Lipschitz extension of φ to the whole $\mathbb{R}^n$ and moreover it satisfy

$$\tilde{\varphi}(z) - \tilde{\varphi}(x) \leq \rho^0(z - x) \ \ \forall\, x, z \, \in \, \mathbb{R}^n \tag{4.30}$$

and

$$\rho(D\tilde{\varphi}(x)) \leq 1 \ \ \text{a.e. in } \mathbb{R}^n. \tag{4.31}$$

Proof. We start by proving (**??**). Let $\varepsilon > 0$ and $y_\varepsilon \in \partial\Omega$ such that

$$\varphi(y_\varepsilon) + \rho^0(x - y_\varepsilon) - \varepsilon \leq \tilde{\varphi}(x).$$

We thus have

$$\begin{aligned}\tilde{\varphi}(z) - \tilde{\varphi}(x) &\leq \varphi(y_\varepsilon) + \rho^0(z - y_\varepsilon) - \left[\varphi(y_\varepsilon) + \rho^0(x - y_\varepsilon) - \varepsilon\right] \\ &\leq \varepsilon + \rho^0(z - y_\varepsilon) - \rho^0(x - y_\varepsilon) \leq \varepsilon + \rho^0(z - x),\end{aligned}$$

where we have used the properties of convexity and homogeneity of ρ^0. The arbitrariness of ε implies (**??**).

Now we show that (**??**) implies (**??**). As $\tilde{\varphi}$ is a lipschitz function, we can use the Rademacher theorem and obtain that for almost every $x \in \mathbb{R}^n$

$$\lim_{h \to 0} \frac{\tilde{\varphi}(x+h) - \tilde{\varphi}(x) - \langle D\tilde{\varphi}(x); h\rangle}{|h|} = 0.$$

This means that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\frac{\tilde{\varphi}(x+h) - \tilde{\varphi}(x) - \langle D\tilde{\varphi}(x); h\rangle}{|h|} \leq \varepsilon$$

for every h with $|h| \leq \delta$, and so

$$\frac{\tilde{\varphi}(x+h) - \tilde{\varphi}(x) - \langle D\tilde{\varphi}(x); h\rangle}{\rho^0(-h)} \leq \varepsilon \frac{|h|}{\rho^0(-h)}.$$

From (**??**), we get that

$$-1 - \frac{\langle D\tilde{\varphi}(x); h\rangle}{\rho^0(-h)} \leq \varepsilon \frac{|h|}{\rho^0(-h)}. \tag{4.32}$$

As ρ is convex and homogeneous of degree one, we have

$$\rho(D\tilde{\varphi}(x)) = \rho^{00}(D\tilde{\varphi}(x)) = \sup_{|\lambda| \leq \delta} \frac{\langle D\tilde{\varphi}(x); \lambda\rangle}{\rho^0(\lambda)}. \tag{4.33}$$

Taking the supremum over every h with $|h| \leq \delta$ in (**??**), we obtain

$$-1 + \sup_{|h| \leq \delta} \frac{\langle D\tilde{\varphi}(x); h\rangle}{\rho^0(-h)} \leq \varepsilon \sup_{|h| \leq \delta} \frac{|h|}{\rho^0(-h)}.$$

Letting now ε tend to 0 and using (**??**), we obtain $\rho(D\tilde{\varphi}(x)) \leq 1$ and so the claim. ☐

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