DOTTORATO DI RICERCA

in

MATEMATICA APPLICATA E INFORMATICA

Ciclo XVI

Consorzio tra Università di Catania, Università di Napoli Federico II,

Seconda Università di Napoli, Università di Palermo, Università di Salerno

SEDE AMMINISTRATIVA: UNIVERSITÀ DI NAPOLI FEDERICO II

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Fractional Integral Equations and Applications to Point Interaction Models in Quantum Mechanics

TESI DI DOTTORATO DI RICERCA

Contents

Introduction

The present work is devoted to the study of dynamical systems described by integral equations of Abel kind.

This equation - introduced by N. H. Abel in order to describe the mechanical problem of *isochrone* curves - have found a wide range of applications in the framework of Continuum Mechanics (e.g. in [1]) as well as in Quantum Physics. Usually Abel, or more general fractional integral equations, are expected to appear in connection with singular perturbations of the Laplace operator, when source or potential terms supported by set of null measure are added to the laplacian. A classical example is given by the the problem of heating an infinite rod by an influx of heat through a pointwise source of strenght $p(t)$ placed in the origin. The temperature field $u(t,x)$ is decribed by the equation:

$$\begin{cases} \frac{d}{dt}u(t,x) \overset{w}{=} \frac{d^2}{dx^2}u(t,x) + p(t)\delta(x) \\ u(0,x) = u_0(x) \end{cases}$$

which has to be intended in the weak sense. Using Dhuamel's formula, we may write the solution $u(t,x)$ of this problem in the form:

$$u(t,x) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}p(s)\delta(x)\,ds$$

where $e^{t\Delta}$ is the propagator associated to the one dimensional Laplace operator, whose action is defined by the relation:

$$e^{t\Delta}f = \frac{1}{(\pi t)^{\frac{1}{2}}}\int_{-\infty}^{+\infty} e^{-\frac{|x-x'|^2}{4t}} f(x')\,dx'$$

By definition of the Dirac delta δ, we get:

$$e^{t\Delta}\delta(x) = \frac{1}{\sqrt{\pi t}}e^{-\frac{x^2}{4t}}$$

Then, assuming a vanishing initial temperature, we obtain:

$$u(t,x) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{e^{-\frac{x^2}{4(t-s)}}}{\sqrt{t-s}} p(s)\, ds$$

If the interior boundary temperature: $\theta(t) = \lim_{|x|\to 0} u(t,x)$ is known (for example a result of an experimental continuous monitoring), we may determine the corrisponding unknown influx strenght as the solution of the following Abel equation:

$$\frac{1}{\sqrt{\pi}} \int_0^t \frac{p(s)}{\sqrt{t-s}}\, ds = \theta(t)$$

In this work we will consider the energy transfer between discrete and continuous spectral components in nonautonomous quantum systems generated by time dependent point interaction operators. The properties of these operators, as we shall see, allow to describe energy exchanges in terms of a system of two coupled integral equations of Abel kind with non constant coefficients, reducing in this way the real degree of freedom of the problem.

In the first Chapter, after a brief introduction to fractional integrals equations of Abel kind, we study the finite time and the large time asymptotic behavior of the solutions, taking into account the case of nonconstant coefficients. Results presented there form the main contribution of this work and will be largely applied in the subsequent chapters.

In the second Chapter we introduce the operators of point interaction in Quantum Mechanics giving the main properties of the related quantum systems. It will be shown that the state ψ of a quantum particle subjected to the action of a 3-D point interaction is determined by the system:

$$\begin{cases} \psi(t,x) = e^{it\Delta}\psi_0(x) + \frac{i}{(\pi i)^{\frac{3}{2}}} \displaystyle\int_0^t \frac{e^{-i\frac{x^2}{4(t-s)}}}{(t-s)^{\frac{3}{2}}} q(s) ds \\ q(t) + 4\sqrt{\pi i} \displaystyle\int_0^t \frac{\alpha(s)q(s)}{\sqrt{t-s}} ds = 4\sqrt{\pi i} \displaystyle\int_0^t \frac{\left(e^{is\Delta}\psi_0\right)(x=\underline{0})}{\sqrt{t-s}} ds \end{cases}$$

where ψ_0 is the initial state and $\alpha(t)$ is the dynamical parameter characterizing the interaction as a function of time. It is immediate to recognize that the auxiliary variable $q(t)$ - which will be defined as the "charge" of the system - satisfies an Abel integral equation with a nonconstant coefficient $\alpha(t)$ in the integral kernel.

Chapters three and four are devoted to the applications of Abel kind equations in the context of quantum system generated by point interactions. In the third Chapter we consider the problem of energy-mass transfer from scattering to bound states for a one body quantum system uder the action of a time dependent point interaction. Under suitable assumptions on the initial state of the particle and using results on the finite time asymptotic behavior for solutions of Abel equations, we prove a theorem of local controllability of this process. In the fourth Chapter we study the time evolution of a three dimensional quantum particle under the action of a time-dependent point interaction fixed at the origin. We assume that the "strength" of the interaction $\alpha(t)$ is a periodic function with an arbitrary mean. Under very weak conditions on the Fourier coefficients of $\alpha(t)$, and making use of large time asymptotic results of Ch.1, we prove that there is complete ionization as $t \to \infty$, starting from a bound state at time $t = 0$.

Chapter 1

Fractional Integral Equations

In this chapter we shall consider some basic integral equations of fractional order; in particular, we will focus our attention on the asymptotic behaviour of the solutions. This analysis will be extended also to the the case of equations with time periodic coefficients.

A rather complete exposition on the subject of integral and differential problems of fractional order may be found in [2], [3]. For the specific case of fractional integral equation with time dependent coefficients, we refer to [4].

1.1 Introduction to Abel integral equations

According to the Riemann-Liouville definition, the fractional integral of order α ($\alpha \in \mathbb{R}^+$), of a function f, is given by:

$$J^\alpha f \equiv \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds \tag{1.1}$$

where $\Gamma(z)$ denotes the Euler Gamma function. This expression may be thought as a natural generalization of the well known Cauchy formula:

$$J^n f = \frac{1}{(n-1)!} \int_0^t \frac{f(s)}{(t-s)^{1-n}} ds$$

representing the n-fold primitive of the function f - $J^n f$ in our notation - in terms of a convolution integral.

The simplest equations involving the operators J^α are the Abel equations of the first and of the second kind:

1.1.1 Abel integral equation of the first kind

The equation is:

$$\frac{1}{\Gamma(\alpha)}\int_0^t \frac{u(s)}{(t-s)^{1-\alpha}}ds = f(t) \qquad \alpha \in (0,1) \tag{1.2}$$

Let us solve (1.2) using Laplace transform '$\mathcal{L}$'; first we observe that the $\mathcal{L}$-transform of the kernel function is given by:

$$\mathcal{L}\left[\frac{1}{\Gamma(\alpha)}\frac{1}{t^{1-\alpha}}\right] = \frac{1}{p^{\alpha}} \qquad \forall\alpha > 0 \tag{1.3}$$

from which we get the following relation for the Laplace transform of (1.2):

$$\tilde{u}(p) = p^{\alpha}\tilde{f}(p) \tag{1.4}$$

where $\tilde{u}$ and $\tilde{f}$ denote the transforms of u and f respectively. Next observe that there are different ways to perform the inverse Laplace transform of (1.4); in fact we may write this relation in the form:

$$\tilde{u}(p) = p\frac{\tilde{f}(p)}{p^{1-\alpha}} \tag{1.5}$$

obtaining, under the regularity condition:

$$\begin{cases} f \in L^1_{loc}(0,+\infty) \\ f \underset{t\to 0}{\sim} t^{-\nu} \qquad \nu \in [0, 1-\alpha) \end{cases} \tag{1.6}$$

the following solution:

$$u(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{f(s)}{(t-s)^{\alpha}}ds \tag{1.7}$$

Otherwise, equation (1.4) could also be set in the form:

$$\tilde{u}(p) = \frac{\left[p\tilde{f}(p) - f(0^+)\right]}{p^{1-\alpha}} + \frac{f(0^+)}{p^{1-\alpha}} \tag{1.8}$$

from which, under the condition:

$$\begin{cases} f \in L^1_{loc}(0,+\infty) \\ |f(0^+)| < \infty \end{cases} \tag{1.9}$$

we get an alternative form of the solution:

$$u(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f'(s)}{(t-s)^{\alpha}}ds + f(0^+)\frac{1}{\Gamma(1-\alpha)}\frac{1}{t^{\alpha}} \tag{1.10}$$

We notice that the expressions (1.7) and (1.10) are not eqiuvalent, and, in order to write the solution in the last form, a stronger regularity condition on the source term f is required.

1.1.2 Abel integral equation of the second kind

The equation is:

$$u(t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t \frac{u(s)}{(t-s)^{1-\alpha}} ds = f(t) \qquad \alpha \in (0,1),\ \lambda \in \mathbb{C} \tag{1.11}$$

Its solution may be represented by the Picard series:

$$\begin{cases} u(t) = \sum_{n=0}^{+\infty} u_n(t) \\ u_0(t) = f(t) \\ u_n = -\lambda J^\alpha u_{n-1} \end{cases} \tag{1.12}$$

The explicit expression of u_n is:

$$u_n(t) = (-\lambda)^n\, J^{\alpha n} f(t) \tag{1.13}$$

and, using Laplace transform, this reads as:

$$u_n = (-\lambda)^n\, \frac{t^{\alpha n - 1}}{\Gamma(\alpha n)} * f \tag{1.14}$$

where '$*$' indicates the convolution product. Then, the solution $u(t)$ of equation (1.11) is, at least at a formal level, given by the limit:

$$u(t) = f(t) + \lim_{n\to\infty} \sum_{k=1}^{n} \left((-\lambda)^k \frac{t^{\alpha k-1}}{\Gamma(\alpha k)} * f \right) = f(t) + \lim_{n\to\infty} \left(\sum_{k=1}^{n} (-\lambda)^k \frac{t^{\alpha k-1}}{\Gamma(\alpha k)} \right) * f \tag{1.15}$$

Next we observe that, due to the rapid growth of the gamma function, the series $\sum_{k=0}^{n} (-\lambda)^k \frac{t^{\alpha k-1}}{\Gamma(\alpha k)}$ is uniformly convergent to a continuous function in every bounded interval $t \in [t_0, T]$ such that $t_0 > 0$. Moreover, this sum diverges as $\frac{1}{t^{1-\alpha}}$ for $t \to 0^+$. This implies that there exists a solution:

$$u(t) = f(t) + \left(\sum_{n=1}^{\infty} (-\lambda)^n \frac{t^{\alpha n-1}}{\Gamma(\alpha n)} \right) * f \tag{1.16}$$

of equation (1.11) for any function f fulfilling the following regularity condition:

$$\begin{cases} f \in L^1_{loc}(0,+\infty) \\ f \underset{t\to 0}{\sim} t^{-\nu} \qquad \nu \in [0,\alpha) \end{cases} \tag{1.17}$$

The uniqueness of this solution is a straightforward consequence of expression (1.16).

A compact form for the solution of equation (1.11) can be given in terms of Mittag-Leffler functions. These functions are defined on the complex plane by the relation:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \qquad \alpha > 0,\ z \in \mathbb{C}$$

The convolution kernel in (1.16) may be expressed as the derivative of a Mittag-Leffler function as follows:

$$\frac{d}{dt} E_\alpha(-\lambda t^\alpha) = \frac{d}{dt} \sum_{n=0}^{\infty} \frac{(-\lambda)^n\, t^{\alpha n}}{\Gamma(\alpha n + 1)} = \sum_{n=1}^{\infty} \frac{(-\lambda)^n\, (\alpha n)\, t^{\alpha n - 1}}{\Gamma(\alpha n + 1)} = \sum_{n=1}^{\infty} \frac{(-\lambda)^n\, t^{\alpha n - 1}}{\Gamma(\alpha n)} \tag{1.18}$$

then, the solution (1.16) reads as:

$$u(t) = f(t) + E'_\alpha(-\lambda t^\alpha) * f\,(t) \tag{1.19}$$

The same solution may still be given into different forms; the $\mathcal{L}$-transform of (1.11) reads:

$$\tilde{u}(p) + \lambda \frac{\tilde{u}(p)}{p^\alpha} = \tilde{f}(p) \tag{1.20}$$

from which we find two alternative forms for $\tilde{u}$:

$$\begin{cases} \tilde{u}(p) = p \left[\frac{p^{\alpha-1}}{p^\alpha + \lambda} \tilde{f}(p) \right] \\ \tilde{u}(p) = \frac{p^{\alpha-1}}{p^\alpha + \lambda} \left[\tilde{f}(p) - f(0^+) \right] + \frac{p^{\alpha-1}}{p^\alpha + \lambda} f(0^+) \end{cases}$$

Recalling that:

$$\mathcal{L}\left[E_\alpha(-\lambda t^\alpha)\right] = \frac{p^{\alpha-1}}{p^\alpha + \lambda} \qquad \mathrm{Re}\, p > |\lambda|^{\frac{1}{\alpha}} \tag{1.21}$$

we obtain:

$$u(t) = \frac{d}{dt} \int_0^t f(t-s) E_\alpha(-\lambda s^\alpha)\, ds \tag{1.22}$$

and

$$u(t) = f(0^+)\, E_\alpha(-\lambda t^\alpha) + \int_0^t f'(t-s) E_\alpha(-\lambda s^\alpha)\, ds \tag{1.23}$$

Again we notice that the expressions (1.19), (1.22) and (1.23) are not equivalent because they require different condition of regularity on the source term f. In particular, the validity of (1.23) implies the condition:

$$\begin{cases} f \in L^1_{loc}(0, +\infty) \\ |f(0^+)| < \infty \end{cases} \tag{1.24}$$

which is stronger than (1.17), needed for the validity of (1.19) and (1.22).

1.1.3 Fractional integral equation with time dependent coefficients

Here we shall consider a fractional integral equation of order $\frac{1}{2}$ where the solution appears multiplied by a time dependent coefficient:

$$u(t) + \int_0^t \frac{\alpha(s)\; u(s)}{\sqrt{t-s}} ds = f(t) \tag{1.25}$$

We will study the properties of equation (1.25) on a finite time interval: $t \in [0, T]$ under the assumptions:

$$\begin{cases} \alpha \in L^\infty(0, T;\, \mathbb{C}) \\ f \in L^\infty(0, T;\, \mathbb{C}) \end{cases} \tag{1.26}$$

Firts observe that, by a simple iteration procedure, the solution $u(t)$ may be formally expressed using the Picard series:

$$\begin{cases} u(t) = \sum_{n=0}^{+\infty} u_n(t) \\ u_0(t) = f(t) \\ u_n(t) = \int_0^t \frac{\alpha(s)\, u_{n-1}(s)}{\sqrt{t-s}} ds \end{cases} \tag{1.27}$$

which admit the following estimate:

$$\sum_{n=0}^{+\infty} |u_n(t)| \leq \|f(t)\|_{L^\infty(0,T)} \left[1 + \sum_{n=1}^{+\infty} \|\alpha\|^n_{L^\infty(0,T)}\; A_n\, \pi^{\frac{n}{2}}\, t^{\frac{n}{2}} \right] \tag{1.28}$$

with:

$$A_n = \begin{cases} \frac{1}{\left(\frac{n}{2}\right)!} & n \text{ even} \\ \frac{2^{\left(\frac{n}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1}{2}\right)} \frac{1}{n!!} & n \text{ odd} \end{cases} \tag{1.29}$$

Thus (1.27) defines a sum of continuous functions which is, by virtue of our assumptions (1.26) and of the estimates (1.28), uniformely convergent into $[0, T]$. It remains proved the following result:

Theorem 1 *Let α and f satisfy the assumptions (1.26). Then equation (1.25) admits an unique continuous solution on the interval $t \in [0, T]$, given by the Picard series (1.27), (1.29), for which holds the estimate:*

$$\|u\|_\infty \leq \|f\|_{L^\infty(0,T)} \left[1 + \sum_{n=1}^{+\infty} \|\alpha\|^n_{L^\infty(0,T)}\; A_n\, \pi^{\frac{n}{2}}\, T^{\frac{n}{2}} \right] \tag{1.30}$$

Proposition 2 *The homogeneous equations associated to (1.25) with* $\alpha \in L^\infty(\mathbb{R})$ *has no non-zero solution in* $L^p_{\text{loc}}(\mathbb{R}^+)$*,* $1 \le p \le \infty$*.*

Proof The proof (see e.g. [26]) exploits the fact that, due to the estimate (1.28), the homogeneous equation associated to (1.25) with $\alpha \in L^\infty(\mathbb{R})$ has a null solution in any $L^p(0, T_n)$ with T_n increasing to infinity for increasing n.

□

1.2 Small Time Asymptotics

Our aim is to study the small time behaviour of the solution $u(t)$ of the equation (1.11) with $\alpha = \frac{1}{2}$:

$$u(t) + \frac{\lambda}{\Gamma(\frac{1}{2})} \int_0^t \frac{u(s)}{\sqrt{t-s}} ds = f(t) \qquad \lambda \in \mathbb{C} \tag{1.31}$$

under the following hypothesis on the source term f:

$$f(t) = u_0^{in} E_{\frac{1}{2}}(\lambda t^{\frac{1}{2}}) + g(t) \tag{1.32}$$

$$g(t) = a_m\, t^{m+\frac{1}{2}} + o(t^{m+\frac{3}{2}});\ a_m \neq 0 \tag{1.33}$$

with:

$$\left| o(t^{m+\frac{3}{2}}) \right| \le c\, t^{m+\frac{3}{2}}, \quad c > 0 \tag{1.34}$$

for $t \in [0, \delta)$, $\delta \in \mathbb{R}^+$, $m \in \mathbb{N}$.

First we notice that the solution of (1.31) may be expressed as the sum of the solutions u_0 and u_g of the equations:

$$u_0(t) + \frac{\lambda}{\Gamma(\frac{1}{2})} \int_0^t \frac{u_0(s)}{\sqrt{t-s}} ds = u_0^{in} E_{\frac{1}{2}}(\lambda t^{\frac{1}{2}}) \tag{1.35}$$

$$u_g(t) + \frac{\lambda}{\Gamma(\frac{1}{2})} \int_0^t \frac{u_g(s)}{\sqrt{t-s}} ds = g(t) \tag{1.36}$$

Using the Laplace transform (1.35) become:

$$\tilde{u}_0(p) \left(1 + \frac{\lambda}{p^{\frac{1}{2}}} \right) = u_0^{in} \frac{p^{-\frac{1}{2}}}{p^{\frac{1}{2}} + \lambda} \Rightarrow \tilde{u}_0(p) = \frac{u_0^{in}}{p - \lambda^2}$$

then we have for $u_0(t)$ an exponential solution:

$$u_0(t) = u_0^{in}\, e^{\lambda^2 t} \tag{1.37}$$

For the second equation we have:

$$\tilde{u}_g(p)\left(1+\frac{\lambda}{p^{\frac{1}{2}}}\right) = \tilde{g}(p) \Rightarrow \tilde{u}_g(p) = \frac{p^{\frac{1}{2}}}{p^{\frac{1}{2}}+\lambda}\tilde{g}(p) = p\frac{p^{-\frac{1}{2}}}{p^{\frac{1}{2}}+\lambda}\tilde{g}(p)$$

whose solution of reads as:

$$u_g(t) = \int_0^t E'_{\frac{1}{2}}(-\lambda t^{\frac{1}{2}})g(t-s)\,ds \tag{1.38}$$

Thus, the solution of equation (1.31) admits the following representation:

$$u(t) = u_0^{in}\, e^{\lambda^2 t} + \int_0^t E'_{\frac{1}{2}}(-\lambda t^{\frac{1}{2}})g(t-s)\,ds \tag{1.39}$$

Recalling that, from definition (1.18), the function $E'_{\frac{1}{2}}(-\lambda t^{\frac{1}{2}})$ may be expressed as:

$$E'_{\frac{1}{2}}(-\lambda t^{\frac{1}{2}}) = -\frac{\lambda}{\Gamma(\frac{1}{2})}\frac{1}{\sqrt{t}} + \lambda^2 + o(t^{\frac{1}{2}}) \tag{1.40}$$

with:

$$\left|o(t^{\frac{1}{2}})\right| \le c_1\, t^{\frac{1}{2}}, \quad c_1 > 0 \tag{1.41}$$

in a suitable right neighbourhood of the origin, by substitution in (1.39), we have:

$$u(t) - u_0^{in}\, e^{\lambda^2 t} = \int_0^t E'_{\frac{1}{2}}(-\lambda t^{\frac{1}{2}})g(t-s)\,ds =$$

$$= -\frac{\lambda\, a_m}{\Gamma(\frac{1}{2})}\int_0^t \frac{s^m}{\sqrt{t-s}}ds + \lambda^2\, a_m \int_0^t s^m ds + a_m \int_0^t s^m o((t-s)^{\frac{1}{2}})ds - \frac{\lambda}{\Gamma(\frac{1}{2})}\int_0^t \frac{o(s^{m+1})}{\sqrt{t-s}}ds +$$

$$+\lambda^2 \int_0^t o(s^{m+1})ds + \int_0^t o((t-s)^{\frac{1}{2}})o(s^{m+1})ds$$

which, after explicit calculations, becomes:

$$u(t) - u_0^{in}\, e^{\lambda^2 t} = -\frac{\lambda\, a_m}{\Gamma(\frac{1}{2})}\frac{2^{m+1}m!}{(2m+1)!!}t^{m+\frac{1}{2}} + \lambda^2\, a_m \frac{t^{m+1}}{m+1} + o(t^{m+\frac{3}{2}}) \tag{1.42}$$

with:

$$\left|o(t^{m+\frac{3}{2}})\right| \le c_2 t^{m+\frac{3}{2}}, \quad c_2 > 0 \tag{1.43}$$

for any t in a neighbourhood $[0, \delta_1)$.

1.3 Large Time Asymptotics

In this section we will perform the analysis of the long time behaviour of the solution of an (1.25)-type equation of order $\frac{1}{2}$:

$$q(t) + \frac{\lambda}{\Gamma(\frac{1}{2})} \int_0^t \frac{\alpha(s)\ q(s)}{\sqrt{t-s}} ds = f(t) \tag{1.44}$$

under the following assumptions for the source term and the coefficients:

$$\lambda = 4\pi\sqrt{i} \tag{1.45}$$

$$f(t) = 4\pi\sqrt{2\,|\alpha(0)|}\, E_{\frac{1}{2}}(\lambda\, \alpha(0)\, t^{\frac{1}{2}}) \tag{1.46}$$

In the physical applications - as we shall see - the meaningful parameter of the system is the negative lower bound of $\alpha(t)$. Hence we require that:

$$\alpha(0) < 0 \tag{1.47}$$

Moreover we shall assume that $\alpha(t)$ is a real periodic continuous function of period T. The continuity of $\alpha(t)$ guarantees that it can be decomposed in a Fourier series, for each $t \in \mathbb{R}^+$, and the series converges uniformly on every compact subset of the real line. In terms of the Fourier coefficients $\alpha(t)$ is given by:

$$\begin{gathered} \alpha(t) = \sum_{n\in\mathbb{Z}} \alpha_n e^{-i\omega n t},\ \{\alpha_n\} \in \ell_1(\mathbb{Z}),\ \omega = \tfrac{2\pi}{T} \\ \alpha_n = \alpha^*_{-n} \end{gathered} \tag{1.48}$$

In order to justyfie our use of Laplace transform in the analysis of equation (1.44), our next task is to prove the following:

Lemma 3 *Let $q(t)$ be the solution of equation (1.25) with $\alpha, f \in L^\infty(\mathbb{R})$. Then, the laplace transform $\mathcal{L}q(p)$ exists and is analytic at least in the open half plane of $\mathbb{C}$ defined by the condition:*

$$p \in \mathbb{C} : \mathrm{Re}\, p > |\lambda|^2 \, \|\alpha\|_\infty^2 \tag{1.49}$$

Proof Consider the function: $q'(t) = e^{-p\,t} q(t)$ with $p \in \mathbb{C}$ and $\mathrm{Re}\, p > 0$, it satisfies the equation:

$$q'(t) + \frac{\lambda}{\Gamma(\frac{1}{2})} \int_0^t \frac{\alpha(s)\, e^{-p(t-s)}\, q(s)}{\sqrt{t-s}} ds = e^{-pt} f(t) = \varphi(t) \tag{1.50}$$

where, from our hypothesis, we have $\varphi(t) \in L^1(0, +\infty)$. Then, applying the Young's inequality:

$$\|f * g\|_1 \leq \|f\|_1 \, \|g\|_1$$

to the convolution in equation (1.50), we obtain the following estimate:

$$\|q'\|_1 \left(1 - \frac{|\lambda|}{\Gamma(\frac{1}{2})} \left\|\frac{e^{-p\,t}}{\sqrt{t}}\right\|_1 \|\alpha\|_\infty\right) \leq \|\varphi\|_1$$

which provide an effective bound for the norm $\|q'\|_1$ if the coefficient $\left(1 - \frac{|\lambda|}{\Gamma(\frac{1}{2})} \left\|\frac{e^{-p\,t}}{\sqrt{t}}\right\|_1 \|\alpha\|_\infty\right)$ is positive. Recalling that for $\mathrm{Re}\, p > 0$:

$$\left\|\frac{e^{-\,p\,t}}{\sqrt{t}}\right\|_1 = \sqrt{\frac{\pi}{\mathrm{Re}\, p}}$$

we get the condition:

$$1 > \frac{|\lambda|}{\Gamma(\frac{1}{2})} \sqrt{\frac{\pi}{\mathrm{Re}\, p}} \|\alpha\|_\infty \Rightarrow \mathrm{Re}\, p > |\lambda|^2 \|\alpha\|_\infty^2 \qquad (1.51)$$

Following the same line, it's easy to prove that the partial derivatives of the function q'w.r.t $\mathrm{Re}\, p$ and $\mathrm{Im}\, p$:

$$\partial_{\mathrm{Re}\, p} q' = \partial_{\mathrm{Im}\, p} q' = -te^{-p\,t} q(t)$$

are bounded by measurable functions of t if the condition (1.51) is fulfilled:

$$\left\|te^{-p\,t} q(t)\right\|_1 \leq \left\|te^{-\left(16\pi^2 \|\alpha\|_\infty^2 + iy\right)\,t} q(t)\right\|_1 < \infty$$

Then $e^{-p\,t} q(t)$ is C^1 measurable w.r.t. $t \in [0, +\infty)$ for any p in the domain (1.49) and, in the same hypothesis, its partial derivatives w.r.t. p are bounded by measurable functions of t. This allows us to conclude that the Laplace integral:

$$\mathcal{L}q(p) = \int_0^{+\infty} q(t) e^{-p\,t}\, dt$$

defines a C^1 class function for p in the domain (1.49)

□

Lemma 3 allow us to say that, under the hypothesis (1.45) and (1.46), the Laplace transform of the solution $q(t)$ exist analytic at least for:

$$p \in \mathbb{C} : \Re\,(p) > 16\pi^2 \|\alpha\|_\infty^2$$

In the notation:

$$\mathcal{L}f = \tilde{f}$$

the Laplace transform to equation (1.44), reads as:

$$\tilde{q}(p) = -4\pi\sqrt{\frac{i}{p}} \sum_{k\in\mathbb{Z}} \alpha_k \, \tilde{q}(p + i\omega k) + \tilde{f}(p) \tag{1.52}$$

where

$$\tilde{f}(p) \equiv 4\pi\sqrt{2|\alpha(0)|}\, \mathcal{L}\left[E_{\frac{1}{2}}(\lambda\, \alpha(0)\, t^{\frac{1}{2}})\right](p) = 4\pi\sqrt{2|\alpha(0)|}\, \frac{p^{-\frac{1}{2}}}{p^{\frac{1}{2}} - 4\pi\alpha(0)\sqrt{i}}$$

with the choice of the branch cut for the square root along the negative real line: if $p = \varrho\, e^{i\vartheta}$,

$$\sqrt{p} = \sqrt{\varrho}\; e^{i\vartheta/2} \tag{1.53}$$

with $-\pi < \vartheta \leq \pi$.

In the following sections we shall perform the asymptotic analysis of system (1.44): we shall prove that, under generic conditions on $\alpha(t)$, the solution of equation (1.44) goes asymptoticaly to zero with a polynomial power law. Although the result does not depend on the sign of the mean α_0 of $\alpha(t)$, we have to discuss separately the case $\alpha_0 < 0$ and $\alpha_0 \geq 0$, because of the slightly different form of equation (1.52).

1.4 Case $\alpha_0 < 0$: the $\mathcal{L}$-transform analysis of the problem

In what follows we introduce the analysis of equation (1.44) using as main tool the Laplace transform.

Since $\alpha(0) < 0$, we will assume that $\alpha(t)$ satisfies the normalization condition:

$$\sum_{n\in\mathbb{Z}} \alpha_n = -\frac{1}{4\pi} \tag{1.54}$$

In the framework of applications, this condition represents a change in the energy scale of a physical system; on the other hand, it provide a simplification of the Laplace transform calculus, but does not effects the asymptotic behaviour of the solution.

Moreover we introduce another condition we shall use later on: let $\mathcal{T}$ the right shift operator on $\ell_1(\mathbb{N})$, i.e.

$$\left(\mathcal{T}a\right)_n \equiv a_{n+1} \tag{1.55}$$

we say that $\alpha = \{\alpha_n\} \in \ell_1(\mathbb{Z})$ is *generic* with respect to $\mathcal{T}$, if $\tilde{\alpha} \equiv \{\alpha_n\}_{n>0} \in \ell_1(\mathbb{N})$ satisfies the following condition

$$e_1 = (1, 0, 0, \dots) \in \overline{\bigvee_{n=0}^{\infty} \mathcal{T}^n \tilde{\alpha}} \tag{1.56}$$

For a detailed discussion of genericity condition see [14].
If (1.54) holds, equation (1.52) becomes (at least for $\Re(p) > 0$)

$$\tilde{q}(p) = -\frac{4\pi}{4\pi\alpha_0 + \sqrt{-ip}} \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \alpha_k\, \tilde{q}(p + i\omega k) - \frac{2i\sqrt{2\pi}}{4\pi\alpha_0 + \sqrt{-ip}} \frac{1 - \sqrt{-ip}}{1 + ip} \tag{1.57}$$

Setting $q_n(p) \equiv \tilde{q}(p + i\omega n)$, we obtain a sequence of functions on the strip $\mathcal{I} = \{p \in \mathbb{C},\ 0 \le \Im(p) < \omega\}$. Setting

$$q(p) \equiv \{q_n(p)\}_{n\in\mathbb{Z}}$$

equation (1.57) can be rewritten in the form:

$$q(p) = \mathcal{B}(p)q(p) + g(p) \tag{1.58}$$

where

$$\left(\mathcal{B}q\right)_n(p) \equiv -\frac{4\pi}{4\pi\alpha_0 + \sqrt{\omega n - ip}} \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \alpha_k\, q_{n+k}(p) \tag{1.59}$$

and $g(p) = \{g_n(p)\}_{n\in\mathbb{Z}}$ with

$$g_n(p) \equiv -\frac{2i\sqrt{2\pi}}{4\pi\alpha_0 + \sqrt{\omega n - ip}} \frac{1 - \sqrt{\omega n - ip}}{1 + ip - \omega n} \tag{1.60}$$

We observe that (1.58) defines an equation in the Hilbert space $H = \ell_2(\mathbb{Z})$ for any $p \in \mathcal{I}$. From the explicit expression of the operator (1.59) and (1.60), it is clear that the coefficients of the equation fails to be analytic on the imaginary axis at $\bar{p} = ((4\pi\alpha_0)^2 - \omega\bar{n})i$, for some $\bar{n} \in \mathbb{Z}$ and then the solution may be singular there.
Since $\Im(p) \in [0, \omega)$, one has

$$\frac{(4\pi\alpha_0)^2}{\omega} - 1 < \bar{n} \le \frac{(4\pi\alpha_0)^2}{\omega} \tag{1.61}$$

and then the singularity appears at most in the equation for $q_{\bar{n}}$ (there is only one integer[1] which satisfies the previous inequality) at $\bar{p} = ((4\pi\alpha_0)^2 - \omega\bar{n})i$. For instance, if $\omega > (4\pi\alpha_0)^2$, the pole may be at $\bar{p} = (4\pi\alpha_0)^2 i$ in the equation for q_0.
The next Proposition shows the properties of operator $\mathcal{B}$:

[1] In fact $\bar{n}$ must be non negative.

Proposition 4 *For $p \in \mathcal{I}$, $\Re(p) = 0$, $p \neq 0, \bar{p}$, $\mathcal{B}(p)$ is an analytic operator-valued function and $\mathcal{L}(p)$ is a compact operator on $\ell_2(\mathbb{Z})$.*

Proof: Analyticity on the imaginary axis for $p \neq 0, \bar{p}$ easily follows from the explicit expression of the operator.
Moreover $\mathcal{B}(p)$ can be written

$$\mathcal{B}(p) = b(p) \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \alpha_k \mathcal{T}^{n+k}$$

where $b(p)$ is the operator

$$(b\, q)_n(p) \equiv b_n(p)\, q_n(p) = -\frac{4\pi q_n(p)}{4\pi\alpha_0 + \sqrt{\omega n - ip}}$$

and $\mathcal{T}$ is the right shift operator on $\ell_2(\mathbb{Z})$.
Since $\|\mathcal{T}\| = 1$, the series converges strongly to a bounded operator. Moreover $b(p)$ is a compact operator on the imaginary axis for $p \neq 0, \bar{p}$: $b(p)$ is the norm limit of a sequence of finite rank operators, because $\lim_{n\to\infty} b_n(p) = 0$. Hence the result follows for example from Theorem VI.12 and VI.13 of [27].

□

Let us first consider the behavior of the solution for $\mathrm{Re}\, p > 0$:

Proposition 5 *There exists a unique solution $q_n(p) \in \ell_2(\mathbb{Z})$ of (1.58) and it is analytic for $\mathrm{Re}\, p > 0$.*

Proof: The key point will be the application of the analytic Fredholm theorem to the operator $\mathcal{L}(p)$ (Theorem VI.14 of [27]), in order to prove that $(I - \mathcal{B}(p))^{-1}$ exists for $p : \mathrm{Re}\, p > 0$.
Since there is no non-zero solution in $L^2_{\mathrm{loc}}(\mathbb{R}^+)$ of the homogeneous equation associated to (1.44) (see the Proposition 2), then the homogeneous equation associated to (1.58) has only the trivial solution in $\ell_2(\mathbb{Z})$. Moreover the operator $\mathcal{B}$ is compact and thus analytic Fredholm theorem applies. The result easily follows, because $g(p) \in \ell_2(\mathbb{Z})$ and each $g_n(p)$ is analytic for $p : \mathrm{Re}\, p > 0$.

□

In the following subsections we shall extend the equation (1.57) above to the imaginary axis and study the behavior of the solution there.

1.4.1 Behavior on the imaginary axis at $p \neq 0$

Actually we have to distinguish the so called (see [14]) resonant case, i.e. when

$$(4\pi\alpha_0)^2 = N\omega$$

for some $N \in \mathbb{N}$, because in that case we can have a pole only at $p = 0$ and then the solution is immediately seen to be analytic on the whole imaginary axis except at most for $p = 0$.
At first we consider the behavior of the solution on the imaginary axis for $p \neq 0,\ p \neq \bar{p}$. We are going to prove that the solution is in fact analytic there.

Proposition 6 *There exists a unique solution $q_n(p) \in \ell_2(\mathbb{Z})$ of (1.58) and it is analytic on the imaginary axis for $p \neq 0, \bar{p}$.*

Proof: The Proof of Proposition 5 still applies because each $g_n(p)$ is analytic for $p \neq 0, \bar{p}$ on the imaginary axis.

□

We can now study the equation (1.58) in a neighborhood of $\bar{p}$ (if $\bar{p} \neq 0$). An important preliminary result is the following

Lemma 7 *Let (1.48) and the genericity condition (1.56) be satisfied by $\{\alpha_n\}$. The system of equations*

$$r_n = -\frac{4\pi}{4\pi\alpha_0 + \sqrt{\omega n - ip}} \left\{ \sum_{\substack{k\in\mathbb{Z} \\ k\neq n,\bar{n}}} \alpha_{k-n} r_k + h_n(p) \right\} \tag{1.62}$$

has a unique solution $\{r_n\} \in \ell_2(\mathbb{Z} \setminus \{\bar{n}\})$ in a pure imaginary neighborhood of $\bar{p}$, where $\bar{n} \in \mathbb{Z}$ and $\bar{p} \in \mathcal{I}$, $\Re(\bar{p}) = 0$, are defined by (1.61), for every $h_n(p)$ such that

$$h'_n(p) \equiv \frac{h_n(p)}{4\pi\alpha_0 + \sqrt{\omega n - ip}}$$

belongs to $\ell_2(\mathbb{Z} \setminus \{\bar{n}\})$.
Moreover, if $h_n(p)$ is analytic in a neighborhood of $\bar{p}$, the solution is analytic in the same neighborhood.

Proof: Equation (1.62) is of the form

$$r = \mathcal{B}' r + h'$$

where $h' \equiv \{h'_n\}$ belongs to $\ell_2(\mathbb{Z} \setminus \{\bar{n}\})$ and $\mathcal{B}'$ is a compact operator (see Proposition 4).
In order to apply analytic Fredholm theorem to the operator $\mathcal{B}'$, we need to prove that there is no non-zero solution in a neighborhood of $\bar{p}$ of the homogeneous equation. Suppose that the contrary is true, so that $\{R_n\} \in \ell_2(\mathbb{Z} \setminus \{\bar{n}\})$ is a non-zero solution of

$$R_n = -\frac{4\pi}{4\pi\alpha_0 + \sqrt{\omega n - ip}} \sum_{\substack{k\in\mathbb{Z} \\ k\neq n,\bar{n}}} \alpha_{k-n} R_n$$

Multiplying both sides of equation above by R_n^* and summing over $n \in \mathbb{Z} \setminus \{\bar{n}\}$, one has

$$\sum_{\substack{n\in\mathbb{Z} \\ n\neq\bar{n}}} \sqrt{\omega n - ip}\, |R_n|^2 = -4\pi \sum_{\substack{n,k\in\mathbb{Z} \\ n,k\neq\bar{n}}} {R_n}^* \alpha_{k-n} R_k$$

and, since the right hand side is real,

$$\Im[\sum_{\substack{n\in\mathbb{Z} \\ n\neq\bar{n}}} \sqrt{\omega n - ip}\, |R_n|^2] = 0$$

for $p = i\lambda$, $0 < \lambda < \omega$, and then $R_n = 0$ for $n < 0$. Now suppose that $R \neq 0$ and let $n_0 \in \mathbb{N}$ be such that $R_n = 0$, $n < n_0$, and $R_{n_0} \neq 0$ (hence $n_0 \geq 0$). Fixing $R_{\bar{n}} = 0$, for each $n < n_0$ the homogeneous equation gives

$$\sum_{k=n_0}^{\infty} \alpha_{k-n} R_k = 0$$

or, setting $k = n_0 - 1 + k'$, for $n \geq 0$,

$$\sum_{k'=1}^{\infty} \alpha_{k'+n} R_{n_0-1+k'} = 0$$

which implies (see (1.48)), for each $n \geq 0$,

$$(R', \mathcal{T}^n \alpha)_{\ell_2(\mathbb{N})} = 0$$

where $R'_n = R^*_{n_0-1+n}$ and $(\cdot,\cdot)$ stands for the standard scalar product on $\ell_2(\mathbb{N})$.

If $\{\alpha_n\}$ satisfies the genericity condition (1.56), R' has to be orthogonal also to e_1 and then $R_{n_0} = 0$, which is a contradiction. Therefore $R = 0$.
The first part of the Lemma then follows from analyticity of $\mathcal{B}'(p)$ and analytic Fredholm theorem. Moreover if $\{h_n(p)\}$ is analytic in a neighborhood of $\bar{p}$, analyticity of the solution is a straightforward consequence.

□

Proposition 8 *If $\{\alpha_n\}$ satisfies (1.48) and the genericity condition with respect to $\mathcal{T}$ (1.56), the unique solution $\{q_n\} \in \ell_2(\mathbb{Z})$ of (1.58) in analytic on the imaginary axis except at most for $p = 0$.*

Proof: If $(4\pi\alpha_0)^2 = N\omega$ for some $N \in \mathbb{N}$ (resonant case) there is nothing to prove, since the coefficients of (1.58) fails to be analytic only at $p = 0$. On the other hand, in the non resonant case, Proposition 6 guarantees analyticity on imaginary axis for $p \neq 0, \bar{p}$. Therefore it is sufficient to study the behavior of the solution in a neighborhood of $\bar{p}$, where the coefficients of (1.58) have a singularity. We are going to prove that in fact the solution is analytic at $\bar{p}$. The strategy of the proof is to analyze separately the terms q_n, $n \neq \bar{n}$, $\bar{n}$ being defined in (1.61), and then prove that also $q_{\bar{n}}$ is analytic in a neighborhood of $\bar{p}$.
By Lemma 7 there is a unique solution of the system

$$t_n = -\frac{4\pi}{4\pi\alpha_0 + \sqrt{\omega n - ip}} \sum_{\substack{k\in\mathbb{Z} \\ k\neq n,\bar{n}}} \alpha_{k-n} t_k - \frac{4\pi\alpha_{\bar{n}-n}}{4\pi\alpha_0 + \sqrt{\omega n - ip}} \tag{1.63}$$

Setting $q_n = r_n + t_n q_{\bar{n}}$, $n \neq \bar{n}$, on (1.58), one has

$$r_n + t_n q_{\bar{n}} = -\frac{4\pi}{4\pi\alpha_0 + \sqrt{\omega n - ip}} \left\{ \alpha_{\bar{n}-n} q_{\bar{n}} + \sum_{\substack{k\in\mathbb{Z} \\ k\neq n,\bar{n}}} \alpha_{k-n} \left(r_k + t_k q_{\bar{n}} \right) \right\} +$$

$$-\frac{2i\sqrt{2\pi}}{4\pi\alpha_0 + \sqrt{\omega n - ip}} \frac{1 - \sqrt{\omega n - ip}}{1 + ip - \omega n}$$

and therefore the equation for $\{r_n\}$, $n \neq \bar{n}$, becomes

$$r_n = -\frac{4\pi}{4\pi\alpha_0 + \sqrt{\omega n - ip}} \left\{ \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0,-n}} \alpha_k \, r_{n+k} + \frac{i}{\sqrt{2\pi}} \frac{1 - \sqrt{\omega n - ip}}{1 + ip - \omega n} \right\} \tag{1.64}$$

while $q_{\bar{n}}$ satisfies the equation

$$q_{\bar{n}} = -\frac{4\pi}{4\pi\alpha_0 + \sqrt{\omega\bar{n} - ip}} \left\{ \sum_{\substack{k\in\mathbb{Z}\\ k\neq\bar{n}}} \alpha_{k-\bar{n}} \left(r_k + t_k q_{\bar{n}}\right) + \frac{i}{\sqrt{2\pi}} \frac{1-\sqrt{\omega\bar{n} - ip}}{1+ip-\omega\bar{n}} \right\}$$

or

$$\left[4\pi\alpha_0 + \sqrt{\omega\bar{n} - ip} + 4\pi \sum_{\substack{k\in\mathbb{Z}\\ k\neq\bar{n}}} \alpha_{k-\bar{n}} t_k \right] q_{\bar{n}} = -4\pi \sum_{\substack{k\in\mathbb{Z}\\ k\neq\bar{n}}} \alpha_{k-\bar{n}} r_k - \frac{2i\sqrt{2\pi}}{1+\sqrt{\omega\bar{n} - ip}}$$

Since the last term is analytic in a neighborhood of $\bar{p}$ and $\{t_n\}$, $\{r_n\} \in \ell_2(\mathbb{Z}\setminus\{\bar{n}\})$ are both analytic, as it follows applying Lemma 7 above to (1.63) and (1.64), it is sufficient to prove that

$$\sum_{\substack{k\in\mathbb{Z}\\ k\neq\bar{n}}} \alpha_{k-\bar{n}} \tilde{t}_k \neq 0$$

where

$$\tilde{t}_n \equiv t_n(p)\big|_{p=\bar{p}}$$

Assume that the contrary is true: from equation (1.63) we obtain

$$\sum_{\substack{n\in\mathbb{Z}\\ n\neq\bar{n}}} \left(4\pi\alpha_0 + \sqrt{\omega n - i\bar{p}}\right) \left|\tilde{t}_n\right|^2 = -4\pi \sum_{\substack{n,k\in\mathbb{Z}\\ n,k\neq\bar{n}, n\neq k}} \tilde{t}_n^* \alpha_{k-n} \tilde{t}_k - 4\pi \sum_{\substack{n\in\mathbb{Z}\\ n\neq\bar{n}}} \alpha_{n-\bar{n}}^* \tilde{t}_n^* =$$

$$= -4\pi \sum_{\substack{n,k\in\mathbb{Z}\\ n,k\neq\bar{n}, n\neq k}} \tilde{t}_n^* \alpha_{k-n} \tilde{t}_k$$

where we have used the second condition in (1.48). The previous equation implies (the right hand side is real) $\tilde{t}_n = 0$, $\forall n < \bar{N} = \frac{i\bar{p}}{\omega}$ and then, since $-1 < \bar{N} < 0$, $\tilde{t}_n = 0$, $\forall n < 0$. Hence from (1.63) we have, $\forall n < 0$,

$$\sum_{\substack{k\geq 0\\ k\neq\bar{n}}} \alpha_{k-n} \tilde{t}_k + \alpha_{\bar{n}-n} = 0$$

Now supposing without loss of generality that $\tilde{t}_0 \neq 0$ and setting $T_n = \tilde{t}_{n-1}$, $n \neq \bar{n}+1$, and $T_{\bar{n}+1} = 1$, we obtain, $\forall n \geq 0$,

$$\sum_{k=1}^{\infty} \alpha_{k+n} T_k = 0$$

and using the genericity condition (1.56) (as in the proof of Lemma 7) we get $T_1 = t_0 = 0$, which is a contradiction.
In conclusion $q_{\bar{n}}$ is analytic in a neighborhood of $\bar{p}$: analyticity of q_n, $n \neq \bar{n}$ is then a straightforward consequence of analyticity of $\{r_n\}$, $\{t_n\}$ and decomposition $q_n = r_n + t_n q_{\bar{n}}$. The proof is then completed, since r_n and t_n belong to $\ell_2(\mathbb{Z} \setminus \{\bar{n}\})$ in a neighborhood of $p = \bar{p}$.

□

1.4.2 Behavior at $p = 0$

We shall now study the behavior of the solution of (1.58) on the imaginary axis at the origin. With the choice (1.53) for the branch cut of the square root, it is clear that we must expect branch points of $\tilde{q}(p)$, solution of (1.57), at $p = i\omega n$, $n \in \mathbb{Z}$, which should imply a branch point at $p = 0$ for each q_n in (1.58).
We are going to show that q_n, $n \in \mathbb{Z}$ has a branch point at $p = 0$. The non-resonant case and the resonant one will be treated separately.

Proposition 9 (non resonant case)
If $(4\pi\alpha_0)^2 \neq N\omega$, $\forall N \in \mathbb{N}$ and $\{\alpha_n\}$ satisfies (1.48) and (1.56) (genericity condition), the solution of equation (1.58) has the form $q_n(p) = c_n(p) + d_n(p)\sqrt{p}$, $n \in \mathbb{Z}$, in an imaginary neighborhood of $p = 0$, where the functions $c_n(p)$ and $d_n(p)$ are analytic at $p = 0$.

Proof: Setting $q_n = r_n + t_n q_0$, $n \neq 0$ and choosing a solution $\{t_n\} \in \ell_2(\mathbb{Z} \setminus \{0\})$ of the system of equations (1.63) with $\bar{n} = 0$, we obtain that $\{r_n\}$ must satisfy (1.64). It is easy to see that the result of Lemma 7 holds also in a neighborhood of $\bar{p} = 0$ with $\bar{n} = 0$, so that $\{r_n\}$, $\{t_n\} \in \ell_2(\mathbb{Z} \setminus \{0\})$ are unique and analytic at $p = 0$.
Thus it is sufficient to prove that q_0, which is solution of

$$\left[4\pi\alpha_0 + \sqrt{-ip} + 4\pi \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0}} \alpha_k t_k \right] q_0 = -4\pi \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0}} \alpha_k r_k - \frac{2i\sqrt{2\pi}(1 - \sqrt{-ip})}{1 + ip}$$

has the required behavior near $p = 0$.
First, setting $t_n^0 = t_n(p = 0)$, we have to prove that

$$\sum_{\substack{k\in\mathbb{Z} \\ k\neq 0}} \alpha_k t_k^0 \neq -\alpha_0$$

but, assuming that the contrary is true and multiplying both sides of equation (1.63) by ${t_n^0}^*$ and summing over $n \in \mathbb{Z}$, $n \neq 0$, one has

$$\sum_{n\in\mathbb{Z}} \sqrt{\omega n}\, |t_n^0|^2 = -4\pi \sum_{\substack{n,k\in\mathbb{Z}\\ n,k\neq 0}} {t_n^0}^* \alpha_{k-n} t_k^0 + 4\pi\alpha_0$$

and then, because of genericity condition (1.56), $\{t_n^0\} = 0$, $\forall n \in \mathbb{Z} \setminus \{0\}$, which is impossible, since $\{t_n\}$ solves (1.63).
Now, calling

$$F \equiv 4\pi \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \alpha_k t_k$$

and

$$G \equiv -4\pi \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \alpha_k r_k$$

we have

$$\left[4\pi\alpha_0 + \sqrt{-ip} + F\right] q_0 = G + \frac{2i\sqrt{2\pi}(1-\sqrt{-ip})}{1+ip}$$

and

$$q_0 = F' + \sqrt{p}\, G'$$

where F' is analytic in a neighborhood of $p = 0$, because of analyticity of F and G, and

$$G' \equiv -\frac{2i\sqrt{-2\pi i}\,(4\pi\alpha_0 + F + 1) + \sqrt{-i}\,(1+ip)\,G}{(1+ip)[(4\pi\alpha_0 + F)^2 + ip]} \tag{1.65}$$

□

The resonant case, i.e. $4\pi\alpha_0 = -\sqrt{\omega N}$ for some $N \in \mathbb{N}$, is not so different from the non-resonant one and we shall prove that the solution has the same behavior at the origin. The proof is slightly different because we need to show the absence of a pole at $p = 0$: from (1.58) one has

$$q_N(p) = \frac{4\pi}{\sqrt{\omega N} - \sqrt{\omega N - ip}} \left\{ \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \alpha_k\, q_{n+k}(p) + \frac{i}{\sqrt{2\pi}} \frac{1 - \sqrt{\omega N - ip}}{1 + ip - \omega N} \right\}$$

and the coefficients have a singularity at $p = 0$.
We are going to prove that in fact the solution has no pole at the origin: proceeding as in the proof of Proposition 8, let us begin with a preliminary result, which take the place of Lemma 7:

Lemma 10 *Let (1.48) and the genericity condition (1.56) be satisfied by* $\{\alpha_n\}$*. The system of equations*

$$r_n = \frac{4\pi}{\sqrt{\omega N} - \sqrt{\omega n - ip}} \left\{ \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0,-n}} \alpha_k r_{n+k} + h_n(p) \right\} \tag{1.66}$$

has a unique solution $\{r_n\} \in \ell_2(\mathbb{Z}\setminus\{N\})$ *in a pure imaginary neighborhood of* $p = 0$*, for every* $h_n(p)$ *such that*

$$h'_n(p) \equiv \frac{h_n(p)}{\sqrt{\omega N} - \sqrt{\omega n - ip}}$$

belongs to $\ell_2(\mathbb{Z}\setminus\{N\})$*.*
Moreover, if $h_n(p)$ *is analytic in a neighborhood of* $p = 0$*, the solution is analytic in the same neighborhood.*

Proof: We shall proceed as in the proof of Proposition 8, separating the contribution of r_N, which may be singular: setting $r_n = u_n + v_n r_N$, $n \neq 0, N$, on (1.66), one has

$$u_n + v_n r_N = \frac{4\pi}{\sqrt{\omega N} - \sqrt{\omega n - ip}} \left\{ \alpha_{N-n} r_N + \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0,-n,N-n}} \alpha_k \left(u_{n+k} + v_{n+k} r_N\right) \right\} +$$

$$+ \frac{2i\sqrt{2\pi}}{\sqrt{\omega N} - \sqrt{\omega n - ip}} \frac{1 - \sqrt{\omega n - ip}}{1 + ip - \omega n}$$

and requiring that $\{v_n\}$, $n \neq 0, N$, solves

$$v_n = \frac{4\pi}{\sqrt{\omega N} - \sqrt{\omega n - ip}} \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0,-n,N-n}} \alpha_k v_{n+k} + \frac{4\pi \alpha_{N-n}}{\sqrt{\omega N} - \sqrt{\omega n - ip}} \tag{1.67}$$

the equation for $\{u_n\}$, $n \neq 0, N$, becomes

$$u_n = \frac{4\pi}{\sqrt{\omega N} - \sqrt{\omega n - ip}} \left\{ \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0,-n,N-n}} \alpha_k u_{n+k} + \frac{i}{\sqrt{2\pi}} : \frac{1 - \sqrt{\omega n - ip}}{1 + ip - \omega n} \right\} \tag{1.68}$$

Moreover r_N satisfies

$$r_N = \frac{4\pi}{\sqrt{\omega N} - \sqrt{\omega N - ip}} \left\{ \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0,-N}} \alpha_k \, (u_k + v_k r_N) + \frac{i}{\sqrt{2\pi}} \frac{1-\sqrt{\omega n - ip}}{1+ip-\omega n} \right\}$$

or

$$\left[\sqrt{\omega N} - \sqrt{\omega N - ip} - 4\pi \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0,N}} \alpha_{k-N} v_k \right] r_N =$$

$$= 4\pi \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0,N}} \alpha_{k-N} u_k + \frac{i}{\sqrt{2\pi}} \frac{1-\sqrt{\omega n - ip}}{1+ip-\omega n}$$

Applying the discussion contained in the proof of Lemma 7, it is not difficult to see that the solutions of equations (1.68) and (1.67) are analytic in a neighborhood of the origin and belong to $\ell_2(\mathbb{Z}\setminus\{0,N\})$. Therefore it remains to prove that (setting $v_n^0 = v_n(p=0)$)

$$\sum_{\substack{k\in\mathbb{Z} \\ k\neq 0,N}} \alpha_{k-N} v_k^0 \neq 0$$

but the argument in the proof of Proposition 8 excludes this possibility, if $\{\alpha_n\}$ satisfies the genericity condition. The proof is then completed, because analyticity of r_N implies analyticity of all r_n, $n \neq 0, N$.

□

Proposition 11 (resonant case)
If $(4\pi\alpha_0)^2 = N\omega$, for some $N \in \mathbb{N}$ and $\{\alpha_n\}$ satisfies (1.48) and (1.56) (genericity condition), the solution of equation (1.58) has the form $q_n(p) = c_n(p) + d_n(p)\sqrt{p}$, $n \in \mathbb{Z}$, in an imaginary neighborhood of $p = 0$, where the functions $c_n(p)$ and $d_n(p)$ are analytic at $p = 0$.

Proof: See the proof of Proposition 9 and Lemma 10 above.

□

1.4.3 Asymptotic behaviour of the solution in the generic case

Summing up the results about the behavior of the Laplace transform $\tilde{q}(p)$ of $q(t)$ we can state the following

Theorem 12 *If $\{\alpha_n\}$ satisfies (1.48) and the genericity condition (1.56) with respect to $\mathcal{T}$, as $t \to \infty$,*

$$|q(t)| \le A\,t^{-\frac{3}{2}} + R(t) \tag{1.69}$$

where $A \in \mathbb{R}$ and $R(t)$ has an exponential decay, $R(t) \sim Ce^{-Bt}$ for some $B > 0$.

Proof: Propositions 6, 8 and 9 guarantee that $\tilde{q}(p)$ is analytic on the closed right half plane, except branch point singularities on the imaginary axis at $p = i\omega n$, $n \in \mathbb{Z}$.
Therefore we can chose a integration path for the inverse of Laplace transform of $\tilde{q}(q)$ along the imaginary axis like in [14].
Proposition 9 implies that the contribution of the branch point at $p = 0$ is given by the integral

$$2i \int_0^\infty dp\,\sqrt{p}\,G'(-p)\,e^{-pt}$$

where G', defined in (1.65), is a bounded analytic function on the negative real line: from explicit expression of F and G and equations (1.64) and (1.63), it is clear that G' is analytic and $\lim_{p\to\infty} G'(-p) = 0$ on the real line. So that the corresponding asymptotic behavior as $t \to \infty$ is

$$\left| \int_0^\infty dp\,\sqrt{p}\,G'(-p)\,e^{-pt} \right| \le C \int_0^\infty dp\,\sqrt{p}\,e^{-pt} = A\,t^{-\frac{3}{2}}$$

Let us consider now the contribution of branch points at $p = i\omega n$, $n \neq 0$: from Propositions 9 and 11 it follows that, in a neighborhood of $p = 0$,

$$q_n(p) = c_n(p) + d_n(p)\,\sqrt{p}$$

where $c_n(p)$ and $d_n(p)$ are analytic at $p = 0$. Moreover using the decomposition $q_n = r_n + t_n q_0$, $n \neq 0$, as in the proof of Proposition 9 and 11, and studying the equation (1.63) for t_n, we immediately obtain $\{d_n\} \in \ell_1(\mathbb{Z}\backslash\{0\})$, because of condition 2 in (1.48). Since $q_n(p) = \tilde{q}(p + i\omega n)$, the contribution of singularities at $p = i\omega n$, $n \neq 0$, is then given by

$$2\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \int_{i\omega n-\infty}^{i\omega n} dp\, d_n(p - i\omega n)\sqrt{p - i\omega n}\, e^{pt} =$$

$$= 2i \int_0^\infty dp \{ \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} d_n(-p)\, e^{i\omega n t} \} \sqrt{p}\, e^{-pt} =$$

and the series

$$\sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} d_n(-p)\, e^{i\omega n t}$$

converges uniformly to a bounded function of t, because $\{d_n\} \in \ell_1(\mathbb{Z} \setminus \{0\})$. Adding up the contributions of every branch cut, one obtain the required leading term in the asymptotic behavior. Indeed the rest function $R(t)$ is given by the contribution of poles outside the imaginary axis and then shows an exponential decay as $t \to \infty$.

□

1.5 Case II: $\alpha_0 = 0$

If $\alpha(t) = \alpha_0 = 0$ does not depend on time, the problem has a simple solution: the spectrum of $H_{\alpha(t)}$ is absolutely continuous and equal to the positive real line, with a resonance at the origin; hence there is no bound state and the system shows complete ionization independently on the initial datum.
On the other hand if $\alpha(t)$ is a zero mean function, we shall see that the genericity condition (1.56) is still needed to have complete ionization.
So let us assume that $\alpha_0 = 0$, the normalization (1.54) holds and the initial datum is given by (4.4): equation (1.52) then becomes

$$\tilde{q}(p) = -4\pi \sqrt{\frac{i}{p}} \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \alpha_k\, \tilde{q}(p + i\omega k) - 2i \sqrt{\frac{2\pi i}{p}} \frac{1 - \sqrt{-ip}}{1 + ip} \tag{1.70}$$

with the choice (1.53) for the branch cut of $\sqrt{p}$. By Proposition 5 the solution is analytic on the open right half plane. In the following section we shall study the singularities on the imaginary axis.

1.5.1 Singularities on the imaginary axis

Setting $q_n(p) \equiv \tilde{q}(p + i\omega n)$, $p \in \mathcal{I} = [0, \omega)$, as in Section 3.1, equation (1.70) assumes the form (1.58),

$$q(p) = \mathcal{M}(p) q(p) + o(p) \tag{1.71}$$

with

$$
(\mathcal{M}q)_n(p) \equiv -\frac{4\pi}{\sqrt{\omega n - ip}} \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \alpha_k \, q_{n+k}(p) \tag{1.72}
$$

and $o(p) = \{o_n(p)\}_{n\in\mathbb{Z}}$,

$$
o_n(p) \equiv -\frac{2i\sqrt{2\pi}}{\sqrt{\omega n - ip}\,(1+\sqrt{\omega n - ip})} \tag{1.73}
$$

Proposition 13 *For $p \in \mathcal{I}$, $\Re(p) = 0$, $p \neq 0$, $\mathcal{M}(p)$ is an analytic operator-valued function and $\mathcal{M}(p)$ is a compact operator on $\ell_2(\mathbb{Z})$.*

Proof: See the proof of Proposition 4.

□

Proposition 14 *There exists a unique solution $q_n(p) \in \ell_2(\mathbb{Z})$ of (1.71) and it is analytic on the imaginary axis for $p \neq 0$.*

Proof: See the proof of Proposition 6.

□

Proposition 15 *If $\{\alpha_n\}$ satisfies (1.48) and the genericity condition (1.56), the solution of equation (1.71) has the form $q_n(p) = c_n(p) + d_n(p)\sqrt{p}$, $n \in \mathbb{Z}$, in a neighborhood of $p = 0$, where the functions $c_n(p)$ and $d_n(p)$ are analytic at $p = 0$.*

Proof: Let us proceed as in the proof of Proposition 9: setting $q_n = r_n + t_n q_0$, $n \in \mathbb{Z} \setminus \{0\}$, where $\{t_n\}$ is the solution of

$$
t_n = -\frac{4\pi}{\sqrt{\omega n - ip}} \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0,-n}} \alpha_k t_{n+k} - \frac{4\pi\alpha_{-n}}{\sqrt{\omega n - ip}} \tag{1.74}
$$

A slightly different version of Lemma 7 guarantees that the solution $\{t_n\} \in \ell_2(\mathbb{Z} \setminus \{0\})$ is unique and analytic at $p = 0$.
By means of this substitution we obtain

$$
r_n = -\frac{4\pi}{\sqrt{\omega n - ip}} \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0,-n}} \alpha_k r_{n+k} - \frac{2i\sqrt{2\pi}}{\sqrt{\omega n - ip}\,(1+\sqrt{\omega n - ip})} \tag{1.75}
$$

and

$$q_0 = -\frac{4\pi}{\sqrt{-ip}} \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \alpha_k \left(r_k + t_k q_0 \right) - \frac{2i\sqrt{2\pi}}{\sqrt{-ip}\,(1+\sqrt{-ip})}$$

or

$$\left(\sqrt{-ip} + F\right) q_0 = G - \frac{2\sqrt{2\pi}}{1+\sqrt{-ip}}$$

where (like in the proof of Proposition 9)

$$F \equiv 4\pi \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \alpha_k t_k$$

and

$$G \equiv -4\pi \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \alpha_k r_k$$

Moreover $F(0) \neq 0$, because of genericity condition (1.56) (see the proof of Proposition 9), F and G are analytic in a neighborhood of $p = 0$ (see Lemma 7), so that

$$q_0 = F' + \sqrt{p}\, G'$$

where F' and G' are analytic and

$$G' \equiv \frac{2\sqrt{-2\pi i}(F+1) - \sqrt{-i}(1+ip)G}{(1+ip)(F^2+ip)}$$

□

As in section 4 of this Chapter, we can now state the main result:

Theorem 16 *If $\{\alpha_n\}$ satisfies (1.48) and the genericity condition (1.56) with respect to $\mathcal{T}$, as $t \to \infty$,*

$$|q(t)| \leq A\, t^{-\frac{3}{2}} + R(t) \tag{1.76}$$

where $A \in \mathbb{R}$ and $R(t)$ has an exponential decay, $R(t) \sim Ce^{-Bt}$ for some $B > 0$.

Proof: See the proof of Theorem 12.

□

1.6 Case III: $\alpha_0 > 0$

To complete the analysis of the problem, we shall consider the case of mean greater than 0: taking the normalization (1.54) and the initial condition (4.4), (1.52) assumes the form (1.57):

$$\tilde{q}(p) = -\frac{4\pi}{4\pi\alpha_0 + \sqrt{-ip}} \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0}} \alpha_k\, \tilde{q}(p + i\omega k) - \frac{2i\sqrt{2\pi}}{4\pi\alpha_0 + \sqrt{-ip}} \frac{1-\sqrt{-ip}}{1+ip} \quad (1.77)$$

Analyticity of the solution on the open right half plane is a consequence of Proposition 5.
Moreover, following the discussion contained in section 4 and setting $q_n(p) \equiv \tilde{q}(p + i\omega n)$, $\Im(p) \in [0, \omega)$, the equation assumes the form (1.58).
Let us now consider the behavior on the imaginary axis: singularities for $\Re(p) = 0$ are associated to zeros of $4\pi\alpha_0 + \sqrt{\omega n + s}$, $s \in [0, \omega)$, but, since $\alpha_0 > 0$, it is clear that the expression can not have zeros on the imaginary axis. Hence the proof of Proposition 6 can be extended to the closed right half plane except the origin:

Proposition 17 *If $\{\alpha_n\}$ satisfies (1.48), the solution $\tilde{q}(p)$ of (1.77) is unique and analytic for $\Re(p) \geq 0$, $p \neq i\omega n$, $n \in \mathbb{Z}$.*

Proof: See the proof of Proposition 6, Propositions 4 and 2 and the previous discussion.

□

Moreover the behavior at the origin is described by the following

Proposition 18 *If $\{\alpha_n\}$ satisfies (1.48) and the genericity condition with respect to $\mathcal{T}$ (1.56), then, in an imaginary neighborhood of $p = i\omega n$, $n \in \mathbb{Z}$, the solution of equation (1.77) has the form $\tilde{q}(p) = c_n(p) + d_n(p)\sqrt{p - i\omega n}$, where the functions $c_n(p)$ and $d_n(p)$ are analytic at $p = i\omega n$.*

Proof: The proof of Proposition 9 still applies with only one difference: since, independently on ω, the solution can not have a pole on the imaginary axis, we need not to distinguish between the resonant case and the non-resonant one.

□

We can now prove the asymptotic result:

Theorem 19 *If $\{\alpha_n\}$ satisfies (1.48) and the genericity condition (1.56) with respect to $\mathcal{T}$, as $t \to \infty$,*

$$|q(t)| \leq A\, t^{-\frac{3}{2}} + R(t) \tag{1.78}$$

where $A \in \mathbb{R}$ and $R(t)$ has an exponential decay, $R(t) \sim Ce^{-Bt}$ for some $B > 0$.

Proof: See the proof of Theorem 12.

□

Remark: If $\alpha(t) \geq 0$, $\forall t \in \mathbb{R}^+$, Proposition 18 holds without the genericity condition on the Fourier coefficients of $\alpha(t)$: for instance the genericity condition enters (see the proof of Proposition 9) in the proof of absence of non-zero solutions of the homogeneous equation

$$t_n = -\frac{4\pi}{4\pi\alpha_0 + \sqrt{\omega n + s}} \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0,-n}} \alpha_k\, t_{n+k}$$

where $s \in [0, \omega)$. Let us suppose that there exists a non-zero solution $\{T_n\} \in \ell_2(\mathbb{Z})$. Multiplying both sides of the equation by T_n^*, one has

$$\sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \sqrt{\omega n + s}\, |T_n|^2 = -4\pi \sum_{\substack{n,k\in\mathbb{Z} \\ n,k\neq 0}} T_n^* \alpha_{k-n} T_k$$

Since the right hand side is real, $T_n = 0$, $\forall n < 0$. Moreover, fixing $T_0 = 0$ and setting

$$T(t) \equiv \sum_{n\in\mathbb{Z}} T_n\, e^{-i\omega n t}$$

it follows that

$$-4\pi \sum_{n,k\in\mathbb{Z}} T_n^* \alpha_{k-n} T_k = -4\pi \left(T(t), \alpha(t)T(t)\right)_{L^2([0,T])} \leq 0$$

because $\alpha(t) \geq 0$, $\forall t \in [0, T]$, but the left hand side is positive and then $Q_n = 0$, $\forall n \in \mathbb{Z}$.

Chapter 2

Point Interactions in Quantum Mechanics

2.1 Introduction

A perturbation of the laplacian supported by a finite set of points $\{y_i\}_{i=1}^n$ in $\mathbb{R}^d$ - with $d \leq 3$ - defines a special case of singular perturbation referred to as point interaction. At a formal level, the associated Schrödinger operator can be written as[1]:

$$H = -\Delta + \sum_{i=1}^{n} \alpha_i \delta(x - y_i) \tag{2.1}$$

A point interaction hamiltonian, then, is intended as the selfadjoint realization in $L^2(\mathbb{R}^d)$ of the formal expression (2.1) [5].

These operators appeared first in Theoretical Physics during the 30's. They were introduced in order to realize a model for the interaction of particles in nuclei [6]. After then, they became a natural tool to describe short range forces or "small" obstacles for scattering of waves and particles.

In the applications perspective, the main reason of interest of this subject rests upon the fact that point ineractions often lead to models which are explicitely solvables. It turns out that the spectral characteristics (eigenvalues and eigenfunctions) of operators (2.1), and then all the physical relevant quantities related to, can be explicitely computed [7]. This circumstance motivates an increasing attention on this subject in the application of mathematics in various sciences, e.g. in physics, chemistry, biology, and in technology.

[1] Note that this expression is always to be intended in a weak sense when working in dimension greater than one.

The rigorous definition of a one point interaction - due to F.A. Berezin and L.D. Fadeev [5] - is based on the theory of selfadjoint extensions of symmetric operators. Let us consider the set of all selfadjoint extentions of the operator:

$$\begin{cases} H = -\triangle \\ D(H) = C^{\infty}(\mathbf{R}^3/\{\underline{0}\}) \end{cases}$$

it may be expressed in the form:

$$\begin{cases} H = H_\alpha \\ D(H_\alpha) = \left\{ \psi \in L^2(\mathbf{R}^3) \,\middle|\, \psi = \phi + q_\lambda G^\lambda,\ \phi \in H^2(\mathbf{R}^3),\ \lim_{\underline{x}\to\underline{0}} \phi = q_\lambda(\alpha + \frac{\sqrt{\lambda}}{4\pi}) \right\} \\ (H_\alpha + \lambda)\,\psi = (-\Delta + \lambda)\,\phi \end{cases} \tag{2.2}$$

where G^λ is the Green function of the laplacian with respect to $\lambda > 0$:

$$(-\Delta - \lambda)G^\lambda = \delta(\underline{x})$$

whose explicit expression is:

$$G^\lambda(\underline{x}) = \frac{e^{-\sqrt{\lambda}\,|\underline{x}|}}{4\pi\ |\underline{x}|} \tag{2.3}$$

while α is a real parameter. From definition (2.2) results that every function in the domain of H_α is then composed of a regular part ϕ in $H^2$and a singular part given by G^λ. Moreover it can be shown that the domain $D(H_\alpha)$ doesn't depend on the choice of λ. In particular for $\alpha < 0$, we may choose the parameter λ such that: $\left(\bar{\alpha} + \frac{\sqrt{\lambda}}{4\pi}\right) = 0$, giving rise to the following domain's representation:

$$D(H_{\bar{\alpha}}) = \left\{ \psi \in L^2(\mathbb{R}^3) \,\middle|\, \psi = \varphi + q\psi_{\bar{\alpha}},\ \varphi \in H^2(\mathbb{R}^3),\ \varphi(\underline{0}) = 0,\ q \in \mathbb{C} \right\} \tag{2.4}$$

which will be extensively used in this work.

By projecting the action of H_α on a space of test functions we get[2]:

$$\begin{aligned} ((H_\alpha + \lambda)\,\psi, \varphi) &= ((-\Delta + \lambda)\,\phi, \varphi) = \\ &= \left(-\Delta\psi + \lambda\psi + \Delta q_\lambda G^\lambda - q_\lambda \lambda G^\lambda, \varphi\right) = \\ &= \left(-\Delta\psi + \lambda\psi + q_\lambda \lambda G^\lambda - q_\lambda\delta - q_\lambda\lambda G^\lambda, \varphi\right) = (-\Delta\psi + \lambda\psi - q_\lambda\delta, \varphi) \end{aligned}$$

then we have:

$$(H_\alpha\psi, \varphi) = (-\Delta\psi - q_\lambda\delta, \varphi) \qquad \forall\varphi \in \text{space of test functions} \tag{2.5}$$

[2]Here $(\cdot,\cdot)$ denotes the usual L^2 scalar product

This relation shows that, in a weak sense, we may consider H_α as the operator associated to a **delta shaped potential** placed in $\underline{x} = 0$. We will refer to H_α as a **point interaction** Schrödinger operator.

In the following some basic properties of point interactions operators are resumed.

2.2 Spectral Properties

Assume that the couple $\lambda \in \mathbf{R}$ and $\psi \in D(H_\alpha)$ satisfies the eigenvalues equation for the operator H_α.

$$(H_\alpha + \lambda)\,\psi = 0$$

By definition (2.2), we have:

$$\begin{cases} (-\Delta + \lambda)\,\phi = 0 \\ \phi \in H^2(\mathbf{R}^3) \end{cases}$$

whose unique solution in $H^2(\mathbf{R}^3)$ is: $\phi = 0$. Then, the eigenfunction related to λ should be proportional to the Green function G^λ:

$$\psi = q_\lambda G^\lambda$$

and the boundary condition in (2.2) implies that:

$$q_\lambda(\alpha + \frac{\sqrt{\lambda}}{4\pi}) = \phi(\underline{0}) \Rightarrow \begin{cases} \sqrt{\lambda} = -4\pi\alpha \\ or \\ q_\lambda = 0 \end{cases} \tag{2.6}$$

Relation (2.6) implies that the discrete spectrum of H_α is empty if $\alpha \geq 0$, otherwise, if $\alpha < 0$, it consists of one eigenvalue $\lambda_\alpha = -16\,\pi^2\,\alpha^2$. The corrispondent normalized eigenfunction is:

$$\psi_\alpha = \frac{\sqrt{2\,|\alpha|}e^{4\pi\,\alpha\,|\underline{x}|}}{|\underline{x}|} \tag{2.7}$$

The risolvent operator associated to H_α, may be expressed as follows:

$$\frac{1}{H_\alpha + z}\varphi(x) = \int_{\mathbb{R}^3} R_z^\alpha(x, y)\varphi(y)\,dy \tag{2.8}$$

where the integral kernel $R_z^\alpha(x, y)$ - obtained by an application of Krein's formula to H_α (e.g. [7])- is given explicitely by the relation:

$$R_z^\alpha(x, y) = G^z(x - y) + \frac{4\pi}{\sqrt{z} + 4\pi\alpha} G^z(x)G^z(y) \tag{2.9}$$

where the Green function $G^z(x)$ -with the usual definition- provide the integral kernel for the laplacian resolvent $\frac{1}{-\Delta+z}$. From (2.9) easily follows that the continuous spectrum of H_α is purely continuous and coincides with the interval equal to $[0,+\infty)$.

The generalized eigenfunctions of H_α are defined by the relation:

$$\varphi_{\underline{k}}(\underline{x}) = e^{i\underline{k}\,\underline{x}} - \frac{1}{\bar{\alpha} - \frac{i|\underline{k}|}{4\pi}} \frac{e^{i|\underline{k}|\,|\underline{x}|}}{|\underline{x}|}$$

with $\underline{k} \in \mathbb{R}^3$. Then all scattering states may be expressed in the integral form:

$$\forall \phi \in D(H_{\bar{\alpha}}) : (\phi, \psi_{\bar{\alpha}}) = 0 \Rightarrow \exists \tilde{\phi} \in L^2(\mathbb{R}^3) \Big|$$

$$\Rightarrow \phi(\underline{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \tilde{\phi}(\underline{k}) \left(e^{i\underline{k}\,\underline{x}} - \frac{1}{\alpha - \frac{i\,|\underline{k}|}{4\pi}} \frac{e^{i\,|\underline{k}|\,|\underline{x}|}}{|\underline{x}|} \right) d\underline{k} \qquad (2.10)$$

2.3 Quantum Dynamics of Point Interactions

The dynamics of a quantum particle subjected to the action of a point interaction H_α is described by the evolution of a state function ψ which satisfies the equation:

$$\begin{cases} i\frac{d}{dt}\psi = H_\alpha \psi \\ \psi(0) = \psi_0 \in D(H_\alpha) \end{cases} \qquad (2.11)$$

The existence of the dynamics for Schrödinger operator of type H_α is a well established fact (e.g. [20]); this implies that H_α is the generator of a strongly continuous unitary group of operators $e^{-i\,t\,H_\alpha}$ acting on the space $D(H_\alpha)$. The time evolution of system (2.11) is determined by the relation:

$$\psi(t) = e^{-i\,t\,H_\alpha}\psi_0$$

An explicit expression of the time propagator may be achieved setting equation (2.11) in the weak form:

$$(i\partial_t\psi, \varphi) = (H_\alpha\psi, \varphi) \quad \forall\,\varphi \in C_c^\infty(\mathbb{R}^3) \qquad (2.12)$$

From (2.5) and (2.12) we get:

$$(i\partial_t\psi, \varphi) = (-\Delta\psi - q_\lambda\delta, \varphi) \quad \forall\,\varphi \in C_c^\infty(\mathbb{R}^3) \qquad (2.13)$$

Let us denote with $e^{-it\Delta}$ the unitary group generated by the laplacian operator. Its action is well defined on the space $H^2(\mathbb{R}^3)$ and may be represented in the following integral form:

$$e^{-it\Delta}\varphi(\underline{x}) = \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{-i\,\frac{|\underline{x}'-\underline{x}|}{4t}} \varphi(\underline{x}')\,d\underline{x}' \qquad (2.14)$$

In what follows we will refer to the integral kernel of (2.14) also as $U(t, \underline{x})$. The same letter will be used also to indicate the operator itselfs:

$$e^{-it\Delta}\varphi = U_t\varphi$$

Setting: $e^{it\Delta}\tilde{\psi} = \psi$, and taking into account the relation:

$$i\partial_t e^{it\Delta}\tilde{\psi} = -\Delta e^{it\Delta}\tilde{\psi} + e^{it\Delta} i\partial_t\tilde{\psi}$$

we get:

$$\left(i\partial_t\tilde{\psi}, \varphi\right) = \left(-q_\lambda\, e^{-i\,t\,\Delta}\,\delta, \varphi\right) \quad \forall\, \varphi \in C_c^\infty(\mathbb{R}^3) \tag{2.15}$$

Observing that $\tilde{\psi}$ and ψ have the same initial conditions, a direct integration of (2.15) gives the following solution:

$$\tilde{\psi}(\underline{x}) \overset{w}{=} \psi_0(\underline{x}) + i\int_0^t q_\lambda(s)\, e^{-i\,s\,\Delta}\,\delta(\underline{x})\, ds \tag{2.16}$$

where the time dependece of the parameter 'q_λ' is connected with the time variation of the regular part of the state via the boundary condition:

$$\lim_{\underline{x}\to\underline{0}} \phi = q_\lambda(\alpha + \frac{\sqrt{\lambda}}{4\pi}) \tag{2.17}$$

From (2.16) we obtain the equation:

$$(\psi, \varphi) = \left(e^{it\Delta}\psi_0 + i\int_0^t q_\lambda(s)\, e^{i\,(t-s)\,\Delta}\,\delta\, ds, \varphi\right) \quad \forall\, \varphi \in C_c^\infty(\mathbb{R}^3)$$

from which we get the weak solution of (2.11)[3]:

$$\begin{aligned}
\psi(t, \underline{x}) &= e^{it\Delta}\psi_0(\underline{x}) + i\int_0^t e^{i(t-s)\Delta} q_\lambda(s)\,\delta(\underline{x})\, ds = \\
&= e^{it\Delta}\psi_0(\underline{x}) + i\int_0^t \int_{R^n} U(t-s, \underline{x}-\underline{x}')\, q_\lambda(s)\,\delta(\underline{x}')\, ds = \\
&= e^{it\Delta}\psi_0(\underline{x}) + i\int_0^t U(t-s, \underline{x})\, q_\lambda(s)\; ds \qquad (2.18)
\end{aligned}$$

which, due to the existence of the dynamics for the system (2.11), is a strong solution as well.

[3]Here the function U is the kernel of the free propagator in n dimensions:

$$U(t, x) = \frac{1}{(4\pi i t)^{\frac{n}{2}}} e^{i\frac{|x|^2}{4t}}$$

2.4 The Charge Equation

As shown in equation (2.18), the dynamics of a quantum particle, subjected the action of the Schrödinger operator H_α, is determined by the time evolution of the boundary condition (2.17). From the decomposition properties of the domain $D(H_\alpha)$ we see that relation (2.17) may also be written in the form:

$$\lim_{\underline{x}\to\underline{0}}\left(\psi - q_\lambda G^\lambda\right) = q_\lambda(\alpha + \frac{\sqrt{\lambda}}{4\pi}) \tag{2.19}$$

Acting on (2.19) with the Laplace transform operator '$\mathcal{L}$' and making use of (2.18), we get the equation for the coefficient q_λ:

$$\lim_{\underline{x}\to\underline{0}}\left(\mathcal{L}e^{it\Delta}\psi_0 + i\mathcal{L}\int_0^t U(t-s,\underline{x})\, q_\lambda(s)\; ds - \mathcal{L}q_\lambda G^\lambda\right) = \mathcal{L}q_\lambda\alpha + \frac{\sqrt{\lambda}}{4\pi}Lq_\lambda$$

from which follows:

$$\mathcal{L}\left[e^{it\Delta}\psi_0\right](\underline{0}) + \lim_{\underline{x}\to\underline{0}}\left(i\mathcal{L}U(t,\underline{x})\,\tilde{q}_\lambda(p) - q_\lambda(p)G^\lambda(\underline{x})\right) = \mathcal{L}\left[q_\lambda\alpha\right] + \frac{\sqrt{\lambda}}{4\pi}\tilde{q}_\lambda(p) \tag{2.20}$$

where $\tilde{q}_\lambda(p)$ defines the Laplace trensform of $q_\lambda(t)$. Observing that the transform of the free propagator kernel U is:

$$\mathcal{L}U(t,\underline{x})(p) = -i\frac{e^{-\sqrt{\frac{p}{i}}|\underline{x}|}}{4\pi\,|\underline{x}|}$$

the limit at first member of (2.20) is explicitely given by:

$$\lim_{\underline{x}\to\underline{0}}\left(i\mathcal{L}U(t,\underline{x})\,\tilde{q}_\lambda(p) - \tilde{q}_\lambda(p)G^\lambda(\underline{x})\right) = \tilde{q}_\lambda(p)\lim_{\underline{x}\to\underline{0}}\left(\frac{e^{-\sqrt{\frac{p}{i}}|\underline{x}|}}{4\pi\,|\underline{x}|} - \frac{e^{-\sqrt{\lambda}|\underline{x}|}}{4\pi\,|\underline{x}|}\right) = \frac{\tilde{q}_\lambda(p)}{4\pi}(\sqrt{\lambda}+\sqrt{\frac{p}{i}})$$

then, by substitution into (2.20) we get:

$$\frac{1}{\sqrt{p}}\mathcal{L}\left[e^{it\Delta}\psi_0\right](\underline{0}) + \frac{\tilde{q}_\lambda(p)}{4\pi\sqrt{i}} = \frac{1}{\sqrt{p}}\mathcal{L}\left[q_\lambda\alpha\right] \tag{2.21}$$

Applying the inverse transform and observing that $\mathcal{L}^{-1}\frac{\sqrt{\pi}}{\sqrt{p}} = \frac{1}{\sqrt{t}}$, an equation for the parameter $q(t)$ is obtained:

$$4\sqrt{\pi i}\int_0^t \frac{\left[e^{is\Delta}\psi_0\right](\underline{0})}{\sqrt{t-s}}\, ds - q(t) = 4\,\alpha\sqrt{\pi i}\int_0^t \frac{q(s)}{\sqrt{t-s}}\, ds \tag{2.22}$$

In what follows, the complex scalar field $q(t)$ will be referred to as the **charge** associated to the point interaction H_α and to the initial state ψ_0. It is worthwile notice that dynamics of q does not on the particular choice of λ in the domain representation.

The charge evolution in time is defined by an Abel integral equation of II kind; the existence and uniqueness of solutions for this kind of problem - depending from the regularity of the nonhomogeneous term - have been analyzed in Chapter 1.

As an aside we notice that by a simple iteration it is possible to regularize the integral kernel of the charge equation:

$$q(t) + 4\alpha\sqrt{\pi i}\int_0^t \frac{q(s)}{\sqrt{t-s}}\,ds = 4\sqrt{\pi i}\int_0^t \frac{\left[e^{is\Delta}\psi_0\right](\underline{0})}{\sqrt{t-s}}\,ds \Rightarrow$$

$$\Rightarrow q(t)+4\alpha^2\sqrt{\pi i}\int_0^t \frac{1}{\sqrt{t-s}}\left[4\sqrt{\pi i}\int_0^s \frac{q(s')}{\sqrt{s-s'}}\,ds' + 4\sqrt{\pi i}\int_0^s \frac{\left[e^{is'\Delta}\psi_0\right](\underline{0})}{\sqrt{t-s'}}\,ds'\right] ds =$$

$$= 4\sqrt{\pi i}\int_0^t \frac{\left[e^{is\Delta}\psi_0\right](\underline{0})}{\sqrt{t-s}}\,ds$$

Applying Dirichlet formula to the double integral we obtain the equation:

$$q(t)+16\,\alpha^2\,\pi^2\,i\int_0^t q(s)\,ds = -16\,\alpha^2\,\pi^2\,i\int_0^t \left[e^{is\Delta}\psi_0\right](\underline{0})\,ds + 4\sqrt{\pi i}\int_0^t \frac{\left[e^{is\Delta}\psi_0\right](\underline{0})}{\sqrt{t-s}}\,ds \tag{2.23}$$

which describes the motion of a forced armonic obscillator in the complex plane.

2.4.1 Solving charge equation

The charge equation (2.22) may be considered as a particular case of the following problem:

$$q(t) + 4\sqrt{\pi i}\,\alpha\int_0^t \frac{q(s)}{\sqrt{t-s}}ds = 4\sqrt{\pi i}\int_0^t \frac{f(s)}{\sqrt{t-s}}ds \tag{2.24}$$

Oparating a Laplace transform of (2.24), we have:

$$\tilde{q}(p)\left(1+\frac{4\pi\bar{\alpha}\sqrt{i}}{\sqrt{p}}\right) = \frac{4\pi\sqrt{i}}{\sqrt{p}}\tilde{f}(p) \Rightarrow \tilde{q}(p) = \frac{4\pi\sqrt{i}}{\sqrt{p}+4\pi\alpha\sqrt{i}}\tilde{f}(p)$$

and taking into account the relation:

$$\mathcal{L}^{-1}\frac{1}{\sqrt{p}+4\pi\alpha\sqrt{i}} = \frac{1}{\sqrt{\pi t}} - (4\pi\alpha\sqrt{i})e^{i(4\pi\alpha)^2 t}\, erfc(4\pi\alpha\sqrt{it}) \equiv K_\alpha(t) \quad (2.25)$$

an explicit expression for the solution $q(t)$ is obtained:

$$q(t) = 4\pi\sqrt{i}\int_0^t K_\alpha(t-s)\, f(s)\, ds \quad (2.26)$$

From (2.26) we get an explicit expressions of the charge q in terms of the initial state function ψ_0:

$$q(t) = 4\pi\sqrt{i}\int_0^t K_\alpha(t-s)\, U_s\psi_0(\underline{0})ds \quad (2.27)$$

2.4.2 The charge associated to the bound state

If the particle is in the bound state (2.7) at $t = 0$, the nonhomogeneous term in (2.23) may be explicitely computed. The Fourier integral of the function $e^{i\,t\,\Delta}\psi_\alpha$ evaluated in the point $\underline{x} = 0$ is:

$$e^{i\,t\,\Delta}\psi_\alpha(\underline{0}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}\mathcal{F}\psi_\alpha(k)\, e^{-i\,k^2\,t}\, d\underline{k} \quad (2.28)$$

where we denoted with $\mathcal{F}\psi$ the Fourier transform of ψ. As already mentioned, the bound state ψ_α is proportional to the Green function (2.3) with $\lambda = 16\pi^2\alpha^2$, whose Fourier transform is given by:

$$G^{16\pi^2\alpha^2}(\underline{k}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{k^2+16\pi^2\alpha^2}$$

By substitution into (2.28) and taking into account (2.7) we get:

$$e^{i\,t\,\Delta}\psi_\alpha(\underline{0}) = \frac{4\pi\sqrt{2\,|\alpha|}}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{1}{k^2+16\pi^2\alpha^2}\, e^{-i\,k^2\,t}\, d\underline{k} \quad (2.29)$$

where C is a suitable constant. Passing to the Laplace Transform $\mathcal{L}$ w.r.t. time and using Fubini's theorem to change the order of space and time integrations, equation (2.29) reads as:

$$\mathcal{L}\left[e^{i\,t\,\Delta}\psi_\alpha(\underline{0})\right](p) = \frac{4\pi\sqrt{2\,|\alpha|}}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{1}{k^2+16\pi^2\alpha^2}\,\frac{1}{ik^2+p}\, d\underline{k}$$

and, after some calculation, the following Laplace transform is deduced:

$$\mathcal{L}\left[e^{i\,t\,\Delta}\psi_\alpha(\underline{0})\right](p) = \frac{4\pi\sqrt{2\,|\alpha|}}{(2\pi)^3}\frac{\pi i}{\left(\sqrt{ip} - 4\pi\alpha\sqrt{i}\right)} \tag{2.30}$$

Expression (2.30) may be used to evaluate the nonhomogeneous term in equation (2.22) when $\psi_0 = \psi_\alpha$; recalling that:

$$\mathcal{L}\frac{1}{\sqrt{t}} = \sqrt{\frac{\pi}{p}}$$

the Laplace transform of the source nonhomogeneous term is:

$$\mathcal{L}\left[4\sqrt{\pi i}\int\limits_0^t \frac{\left[e^{is\Delta}\psi_\alpha\right](\underline{0})}{\sqrt{t-s}}\,ds\right](p) = 4\pi\sqrt{2\,|\alpha|}\frac{p^{-\frac{1}{2}}}{p^{\frac{1}{2}} - 4\pi\alpha\sqrt{i}} \tag{2.31}$$

from which we have:

$$4\sqrt{\pi i}\int\limits_0^t \frac{\left[e^{is\Delta}\psi_\alpha\right](\underline{0})}{\sqrt{t-s}}\,ds = 4\pi\sqrt{2\,|\alpha|}\,E_{\frac{1}{2}}(4\pi\alpha\sqrt{i}\,t^{\frac{1}{2}}) \tag{2.32}$$

Then, the charge equation associated to the bound state reads as:

$$q(t) + \frac{4\pi\sqrt{i}\,\alpha}{\Gamma(\frac{1}{2})}\int_0^t \frac{q(s)}{\sqrt{t-s}}ds = q_\alpha(0)\,E_{\frac{1}{2}}(4\pi\alpha\sqrt{i}\,t^{\frac{1}{2}}) \tag{2.33}$$

whose solution is given by (1.37) with $\lambda = 4\pi\sqrt{i}\,\alpha$:

$$q(t) = q_\alpha(0)\,e^{i16\pi^2\alpha^2 t} \tag{2.34}$$

We stress out that relation (2.34) could also be recovered directely from the Schrodinger equation (2.11). In fact, once assumed the initial condition $\psi_0 = \psi_\alpha$, the state function at time t is:

$$e^{-itH_\alpha}\psi_\alpha = e^{-i\,t\,\lambda_\alpha}\psi_\alpha$$

with $\lambda_\alpha = -16\,\alpha^2\,\pi^2$. Then from the operator domain representation, (2.4), it is immediate to conclude once more that the charge associated to the bound state is given by (2.34).

Another relation, which will be useful for further calculations, is obtained applying the general formula (2.27) to the case we're taking into account:

$$q(t) = 4\pi\sqrt{i}\int_0^t K_\alpha(t-s)\,U_s\psi_\alpha(\underline{0})ds$$

from which, using (2.34), we deduce:

$$4\pi\sqrt{i}\int_0^t K_\alpha(t-s)\,U_s\psi_\alpha(\underline{0})ds = e^{i\,16\,\alpha^2\,\pi^2\,t} \tag{2.35}$$

2.4.3 The charge associated to a scattering state

From general properties of generator set of $L^2(\mathbb{R}^3)$, we know that any function $\phi \in L^2(\mathbb{R}^3)$ may be represented as the sum of a radial part $\phi_s(r) \in L^2(\mathbb{R}_+, r^2)$, $r = |\underline{x}|$, plus a term given by a linear combination of spherical harmonics:

$$\phi_p(\underline{x}) = \sum_{\substack{l=1 \\ l\neq 0}}^{+\infty} \sum_{m=-l}^{l} f_{lm}(r) Y_l^m(\vartheta, \varphi)$$

which are both in $L^2(\mathbb{R}^3)$:

$$\phi(\underline{x}) = \phi_S(r) + \phi_P(\underline{x}) \tag{2.36}$$

This implies also that the charge q associated to the hamiltonian H_α and to the initial state ϕ, can be splitted in two contributions:

$$q = q_S + q_P$$

which satisfie the equations[4]:

$$\begin{cases} q_S(t) + 4\alpha\sqrt{\pi i} \int\limits_0^t \frac{q_S(s)}{\sqrt{t-s}}\, ds = 4\sqrt{\pi i} \int\limits_0^t \frac{U_s \phi_S(\underline{0})}{\sqrt{t-s}}\, ds \\ q_P(t) + 4\alpha\sqrt{\pi i} \int\limits_0^t \frac{q_P(s)}{\sqrt{t-s}}\, ds = 4\sqrt{\pi i} \int\limits_0^t \frac{U_s \phi_P(\underline{0})}{\sqrt{t-s}}\, ds \end{cases} \tag{2.37}$$

Next we observe that, due to the ortogonality relation:

$$\int d\Omega\, Y_l^m(\vartheta, \varphi)\, Y_{m'}^{l'}(\vartheta, \varphi) = \delta_{ll'}\delta_{mm'}$$

the free propagator of ϕ_P is always null when evaluated in the point $\underline{x} = 0$. From this follows that the source term in the second of equations (2.37) is zero. By the uniqueness of solution of equation (2.22), we conclude that the charge associated to any state of 'p' kind is zero:

$$q_p = 0 \tag{2.38}$$

As an aside we notice that from (2.38) and from the definition (3.4) follows that a particle, initially placed in a p state, ϕ_p, and subjected to the action

[4] Here we follows the already introduced notation:

$$U_t = e^{it\Delta}$$

for the free propagator at time t

of any hamiltonian of type H_α, simply evolves under the action of the free propagator:

$$\psi(t) = e^{-it\Delta}\phi_p$$

In these conditions, the particle does not 'feel' the interaction and behave like a free particle.

Now consider the case in which the initial state of system (2.11) is a scattering state of radial type:

$$\psi_0 = \phi_r$$

Using Fourier transform, the action of the free propagator on ϕ_r at time t is:

$$U_t\phi_r(\underline{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}\mathcal{F}\phi_r(k')e^{i(\underline{k}'\,\underline{x}-k'^2t)}d\underline{k}' \tag{2.39}$$

Moreover, by definition (2.10), the transform $\mathcal{F}\phi_r$ may be represented as follows:

$$\mathcal{F}\phi_r(k') = \tilde{\phi}(k') - \frac{4\pi}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}\tilde{\phi}(k)\frac{1}{\alpha-\frac{ik}{4\pi}}\frac{1}{k'^2-k^2}d\underline{k} \tag{2.40}$$

where $k = |\underline{k}|$. From (2.39) and (2.40), we obtain:

$$U_t\phi_r(\underline{0}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}\tilde{\phi}(k')e^{-ik'^2t}d\underline{k}' - \frac{4\pi}{(2\pi)^3}\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\tilde{\phi}(k)}{\alpha-\frac{ik}{4\pi}}\frac{e^{-ik'^2t}}{k'^2-k^2}d\underline{k}\,d\underline{k}' \tag{2.41}$$

The integral representation of the charge q_r, then follows from (2.41) and (2.26):

$$q_r(t) = \frac{4\pi\sqrt{i}}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}\tilde{\phi}(k')\int_0^t K_\alpha(t-s)e^{-ik'^2s}ds\,d\underline{k}'+$$
$$-\frac{16\pi^2\sqrt{i}}{(2\pi)^3}\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\tilde{\phi}(k)}{\bar{\alpha}-\frac{ik}{4\pi}}\frac{\int_0^t K_\alpha(t-s)e^{-ik'^2s}ds}{k'^2-k^2}d\underline{k}\,d\underline{k}'$$

2.5 A Time Reversed Point Interaction

Consider a particle which moves back in time, under the action of the Schrödinger operator $H_{\bar{\alpha}}$, starting at $t = T$ from the initial state $\psi_{\bar{\alpha}}$; the state of this system is described by the equation:

$$\begin{cases} -i\partial_t\psi = H_{\bar{\alpha}}\psi \\ \psi(T) = \psi_{\bar{\alpha}} \end{cases} \tag{2.42}$$

which admit the following integral form:

$$\psi(t) = e^{i(T-t)\Delta}\psi_{\bar{\alpha}} + i\int_T^t U(s-t,\underline{x})q(s)\,ds \tag{2.43}$$

where $U(t,\underline{x})$ is the integral kernel of the free propagator associated to the Schrödinger equation. If we set:

$$\begin{cases} \tau = T - t \\ g(\tau) = \psi(t) \end{cases}$$

equation (2.43) become:

$$g(\tau) = e^{i\tau\Delta}g_0 + i\int_0^\tau U(\tau - z)q(T-z)\,dz \tag{2.44}$$

The related charge equation comes from a Laplace transform analysis of the boundary condition on the operator domain. Adopting the representation[5]:

$$D(H_{\bar{\alpha}}) = \left\{ \psi \in L^2(\mathbb{R}^3) \,\middle|\, \psi = \varphi + qG_\lambda,\ \varphi \in H^2(\mathbb{R}^3),\ \varphi(\underline{0}) = q\left(\bar{\alpha} + \frac{\sqrt{\lambda}}{4\pi}\right) \right\}$$

the modified state function, $g(\tau)$, is splitted into a regular and a singular part:

$$\varphi(T-\tau) = g(\tau) - q(T-\tau)G_\lambda$$

and the domain boundary conditions at time τ become:

$$\lim_{\underline{x}\to\underline{0}}\left[e^{i\tau\Delta}g_0 + i\int_0^\tau U(\tau-z,\underline{x})q(T-\tau)\,dz - q(T-\tau)G_\lambda(\underline{x})\right] = q(T-\tau)\left(\bar{\alpha} + \frac{\sqrt{\lambda}}{4\pi}\right)$$

Setting $q'(\tau) \equiv q(T-\tau)$, the Laplace transform analysis of this relation, w.r.t. the variable τ, gives:

$$L\left[e^{i\tau\Delta}g_0\right](\underline{0}) + \lim_{\underline{x}\to 0}(iLU(\tau,\underline{x})\tilde{q}'(p) - \tilde{q}'(p)G_\lambda(\underline{x})) = \tilde{q}'(p)\left(\bar{\alpha} + \frac{\sqrt{\lambda}}{4\pi}\right) \tag{2.45}$$

[5] For the general form of $D(H_{\bar{\alpha}})$ see for instance [7]. Here λ is an arbitrary positive real parameter.

where $\tilde{q}'(p)$ defines the Laplace tensform of the charge. The transform of the free propagator kernel U is given by,

$$LU(t,\underline{x})(p) = -i\frac{e^{-\sqrt{\frac{p}{i}}|\underline{x}|}}{4\pi\,|\underline{x}|}$$

then we have:

$$\begin{aligned}\lim_{\underline{x}\to 0}(iLU(\tau,\underline{x})\tilde{q}'(p) - \tilde{q}'(p)G_\lambda(\underline{x})) &= \tilde{q}'(p)\lim_{\underline{x}\to 0}\left(\frac{e^{-\sqrt{\frac{p}{i}}|\underline{x}|}}{4\pi\,|\underline{x}|} - \frac{e^{-\sqrt{\lambda}|\underline{x}|}}{4\pi\,|\underline{x}|}\right) = \\ &= \frac{\tilde{q}'(p)}{4\pi}(\sqrt{\lambda} + \sqrt{\frac{p}{i}})\end{aligned}$$

By substitution in (2.45) we get:

$$L\left[e^{i\tau\Delta}g_0\right](\underline{0}) + \frac{\tilde{q}'(p)}{4\pi}(\sqrt{\lambda} + \sqrt{\frac{p}{i}}) = \tilde{q}'(p)\left(\bar{\alpha} + \frac{\sqrt{\lambda}}{4\pi}\right) \Rightarrow$$

$$\Rightarrow \frac{1}{\sqrt{p}}L\left[e^{i\tau\Delta}g_0\right](\underline{0}) + \frac{\tilde{q}'(p)}{4\pi\sqrt{i}} = \frac{\tilde{q}'(p)}{\sqrt{p}} \tag{2.46}$$

Applying the inverse transform and observing that $L^{-1}\frac{1}{\sqrt{p}} = \frac{1}{\sqrt{\pi t}}$, we finally obtain the equation for the charge:

$$q'(\tau) + 4\sqrt{\pi i}\bar{\alpha}\int_0^\tau \frac{q'(s)}{\sqrt{\tau - s}}ds = 4\sqrt{\pi i}\int_0^\tau \frac{U_s g_0(\underline{0})}{\sqrt{\tau - s}}ds \tag{2.47}$$

From (2.27) we know that the solution $q'(\tau)$ my be expressed as

$$q'(\tau) = 4\pi\sqrt{i}\int_0^\tau G(\tau - s)U_s g_0(\underline{0})\,ds$$

moreover, being by definition: $q'(\tau) = q(t)$, $g_0 = \psi_{\bar{\alpha}}$, we have:

$$q(t) = 4\pi\sqrt{i}\int_0^{T-t} G(T-t-s)U_s\psi_{\bar{\alpha}}(\underline{0})\,ds \underset{s'=T-s}{=} 4\pi\sqrt{i}\int_t^T G(s'-t)U_{T-s'}\psi_{\bar{\alpha}}(\underline{0})\,ds \tag{2.48}$$

Now, observing that the solution $\psi(t)$ of (2.42) is explicitely given in terms of the time evolution operator associated to $H_{\bar{\alpha}}$:

$$\psi(t) = e^{-i(T-t)H_{\bar{\alpha}}}\psi_{\bar{\alpha}} = e^{-i(T-t)\lambda_{\bar{\alpha}}}\psi_{\bar{\alpha}}$$

and comparing this expression with (2.4), it is easily deduced that the charge associated to equation (2.43) is:

$$q(t) = \frac{4\pi}{\sqrt{2\,|\bar{\alpha}|}}e^{-i(T-t)\lambda_{\bar{\alpha}}} \tag{2.49}$$

From (2.48) and (2.49) we finally obtain:

$$\int_t^T G(s'-t)U_{T-s'}\psi_{\bar{\alpha}}(\underline{0})\,ds = \frac{1}{\sqrt{2i\,|\bar{\alpha}|}}e^{-i(T-t)\lambda_{\bar{\alpha}}} \tag{2.50}$$

2.6 Time Dependent Point Interactions

A time dependent point interaction is defined, at a formal level, by assigning the interaction parameter α as a function of time: $\alpha = \alpha(t)$. The Schrodinger equation associated to the family of hamiltonians $H_{\alpha(t)}$ is:

$$\begin{cases} i\partial_t\psi(t) = H_{\alpha(t)}\psi(t) \\ \psi(0) = \psi_0 \in D(H_{\alpha(0)}) \end{cases} \tag{2.51}$$

In the weak formulation, the solution of (2.51) can be derived, following the same lines of previous sections, from (2.13) and (2.21):

$$\begin{cases} \psi(t) = e^{it\Delta}\psi_0 + i\int_0^t U(t-s,\underline{x})q(s)ds \\ q(t) + 4\sqrt{\pi i}\int_0^t \frac{\alpha(s)q(s)}{\sqrt{t-s}}ds = 4\sqrt{\pi i}\int_0^t \frac{U_s\psi_0(\underline{0})}{\sqrt{t-s}}ds \end{cases} \tag{2.52}$$

The system (2.51) defines properly a quantum dynamics if the condition $\alpha \in C^2_{loc}(0,+\infty)$ is satisfied. It can be proved, indeed, that, under this condition, $H_{\alpha(t)}$ is the generator of continuous flux of unitary operators on $D(H_{\alpha(t)})$ [8]. As a consequence (2.51) has the unique solution:

$$\psi(t) = e^{-i\,t\,H_{\alpha(t)}}\psi_0 \in D(H_{\alpha(t)}) \quad \forall t \tag{2.53}$$

which is described explicitely by system (2.52).

In what follows we recall some basic properties of the charge equation associated to time dependent point interaction.

2.6.1 The charge equation in the time dependent case

The charge associated to a time dependent point interaction $H_{\alpha(t)}$ and to the initial state ψ_0 (ref. (2.52)) is described by the following equation:

$$q(t) + 4\sqrt{\pi i}\int_0^t \frac{\alpha(s)q(s)}{\sqrt{t-s}}ds = 4\sqrt{\pi i}\int_0^t \frac{U_s\psi_0(\underline{0})}{\sqrt{t-s}}ds \tag{2.54}$$

This is a equation of type (1.25) and with a nonhomogeneous term given, once again, by a fractional integral of order $\frac{1}{2}$. From Theorem 1, we have the following result:

Corollary 20 *Let $\alpha \in L^\infty(\mathbb{R})$, $\psi_0 \in D(H_{\alpha(0)})$. Then, equation (2.54) has an unique continuous solution for any finite interval of time $t \in [0,T]$. Moreover, the following estimate holds:*

$$\|q\|_\infty \leq \left\| 4\sqrt{\pi i} \int_0^t \frac{U_s\psi_0(\underline{0})}{\sqrt{t-s}} ds \right\|_{L^\infty(0,T)} \left[1 + \sum_{n=1}^{+\infty} \left|4\sqrt{\pi i}\right|^n \|\alpha\|^n_{L^\infty(0,T)} A_n \, \pi^{\frac{n}{2}} \, T^{\frac{n}{2}} \right] \tag{2.55}$$

Proof From definition (2.4), any function $\psi_0 \in D(H_{\alpha(0)})$ is the sum of a regular part plus a bound state term:

$$\psi_0 = \varphi + q(0)\psi_{\alpha(0)}, \; \varphi \in H^2(\mathbb{R}^3), \; \varphi(\underline{0}) = 0, \; q \in \mathbb{C} \tag{2.56}$$

Then the source term in (2.54) may be splitted in two components:

$$\int_0^t \frac{U_s\psi_0(\underline{0})}{\sqrt{t-s}} ds = \int_0^t \frac{U_s\varphi(\underline{0})}{\sqrt{t-s}} ds + q(0) \int_0^t \frac{U_s\psi_{\alpha(0)}(\underline{0})}{\sqrt{t-s}} ds \tag{2.57}$$

We want to prove that (2.57) defines a bounded function on finite time intervals. To this aim we consider the two contributions of (2.57) separately.

The first term at second member is the one half integral of $U_t\varphi(\underline{0})$. Making use of the representation (2.36) we have:

$$U_t\varphi(\underline{0}) = U_t\varphi_r(\underline{0})$$

where $\varphi_r \in H^2(\mathbb{R}^3)$ denotes the radial part of φ. The Fourier integral of $U_t\varphi$ evaluated in $\underline{x} = 0$ is:

$$U_t\varphi(\underline{0}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \mathcal{F}\varphi_r(k) \, e^{-i\,k^2\,t} \, d\underline{k} \tag{2.58}$$

Next observe that the Fourier transform of a radial function $\varphi_r \in H^2(\mathbb{R}^3)$ has the following characterization:

$$\mathcal{F}\varphi_r \in L^2(\mathbb{R}^3) : k^2\mathcal{F}\varphi_r(k) \in L^2(\mathbb{R}^3) \tag{2.59}$$

which implies:

$$\left\{ \begin{array}{l} \mathcal{F}\varphi_r \in L^1(\Omega) \quad \forall\,\Omega \subset \mathbb{R}^3, \; |\Omega| < \infty \\ \lim_{k\to+\infty} \frac{\mathcal{F}\varphi_r(k)}{k^{\frac{7}{2}}} = 0 \end{array} \right. \Rightarrow \mathcal{F}\varphi_r \in L^1(\mathbb{R}^3) \tag{2.60}$$

Relations (2.58) and (2.60) allow us to conclude that $U_t\varphi(\underline{0})$ as well as the first source term $\int_0^t \frac{U_s\varphi(\underline{0})}{\sqrt{t-s}} ds$ are continuous functions of time.

The second source term of (2.57) may be evaluated explicitely; from (2.30) we get:

$$U_t\psi_{\alpha(0)}(\underline{0}) = \mathcal{L}^{-1}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{-iC}{\left(\sqrt{ip} - i4\pi\alpha(0)\right)} = \\ = C'\left[\frac{1}{\sqrt{\pi t}} + (4\pi\alpha(0)\sqrt{i})e^{i(4\pi\alpha(0))^2 t}\, erfc(4\pi\alpha(0)\sqrt{it})\right] \tag{2.61}$$

where C and C' are a suitable complex constants. Then the second contribution of (2.57) is:

$$q(0)\int_0^t \frac{U_s\psi_{\alpha(0)}(\underline{0})}{\sqrt{t-s}}ds = q(0)C'\left[\sqrt{\pi} + (4\pi\alpha(0)\sqrt{i})\int_0^t \frac{e^{i(4\pi\alpha(0))^2 t}\, erfc(4\pi\alpha(0)\sqrt{it})}{\sqrt{t-s}}ds\right] \tag{2.62}$$

From the boundedness of the function $e^{i(4\pi\alpha(0))^2 t}\, erfc(4\pi\alpha(0)\sqrt{it})$, we conclude that $q(0)\int_0^t \frac{U_s\psi_{\alpha(0)}(\underline{0})}{\sqrt{t-s}}ds$ is continuous.

The continuity of the nonhomogeneous term (2.57), allows us to apply Theorem (1) to equation (2.54) for any finite time interval.

□

Chapter 3

Energy Transfer Control via Point Interaction

In this Chapter we build up a point interaction model of a time dipendent Schrödinger operator; this interaction will be used as a control for the energy transfer between continuous and discrete spectrum of a one body quantum system. We will investigate the possibility of finding a time dipendence profile such that part of the energy of a particle, initially placed in a scattering state, moves on the bound state in finite time.

3.1 Introduction

Any state belonging to the domain of a Schrödinger operator H may be represented as the sum of two ortogonal contributions: the first one, ψ^B, is given by the projection of ψ on the stationary states of H; the other one, ψ^S, is a scattering state simply defined as the rimainder: $\psi^S = \psi - \psi^B$. Then, consider a quantum particle subject to the action of the hamiltonian H and assume that the initial state of this system is given by

$$\psi_0 = \psi_0^B + \psi_0^S; \ \left(\psi_0^B, \psi_0^S\right)_{L^2} = 0$$

Due to the ortogonality relation between ψ^B and ψ^S, the unitary evolution e^{-itH} preserves indipendently the L^2 norms of these vectors, and the mass conservation law for the system can be written in the following form:

$$\begin{cases} \left\|\psi_t^B\right\|_2 = \left\|\psi_0^B\right\|_2 \\ \left\|\psi_t^S\right\|_2 = \left\|\psi_0^S\right\|_2 \end{cases} \quad \forall t \in \mathbb{R} \tag{3.1}$$

where ψ_t^B and ψ_t^S are the bound state and scattering state components of $e^{-itH}\psi_0$ respectively.

This relation does not hold anymore if a time dependent Schrödinger operator is considered. In this case - altough the total mass is still preserved - the spectrum of $H(t)$ may change at any time and it is not possible, in principle, to identify state components of constant mass.

In what follows we consider a quantum system defined by an hamiltonian operator H which depends on time through a real parameter α:

$$\begin{cases} H = H(\alpha) \\ \alpha \in \mathbb{R} \times [0,T] \text{ (control function)} \\ \alpha(0) = \alpha(T) = \bar{\alpha} \in \mathbb{R} \text{ (boundary condition)} \end{cases} \tag{3.2}$$

Let us suppose that, for any possible choice of α in a suitable space of control functions B, is well defined the propagator associated to $H(\alpha)$, given at a formal level by:

$$U_{H(\alpha),t} = e^{-i\int_0^t H(\alpha)(s)ds}$$

This is equivalent to assume that exists an unique solution for the Schrodinger equation:

$$\begin{cases} i\frac{d}{dt}\psi = H(\alpha)\psi \\ \psi(0) = \psi_0 \in D(H(\bar{\alpha})) \end{cases}$$

in the strong sense.

By virtue of the boundary condition in (3.2), the propagator associated to $H(\alpha)$ at time T maps the initial operator domain into itself:

$$U_{H(\alpha),T} : D(H(\bar{\alpha})) \to D(H(\bar{\alpha}))$$

For the space $D(H(\bar{\alpha}))$ we may use the decomposition already introduced:

$$\psi \in D(H(\bar{\alpha})) \Rightarrow \psi = \psi^B + \psi^S \tag{3.3}$$

Let $\psi_0, \psi_T \in D(H(\bar{\alpha}))$ be the initial state and its evolution at time T; using (3.3), they can be written in the form:

$$\begin{cases} \psi_0 = \psi_0^B + \psi_0^S \\ \psi_T = \psi_T^B + \psi_T^S \end{cases}$$

Then, for any $\alpha \in B$, it results to be defined an application $F(\alpha,\cdot,\cdot)$ defined on the sector $\mathbb{R}^+ \times \mathbb{R}^+$ which maps the couple $\left(\left\|\psi_0^B\right\|_2, \left\|\psi_0^S\right\|_2\right)$ into $\left(\left\|\psi_T^B\right\|_2, \left\|\psi_T^S\right\|_2\right)$.

Definition 1 *$F(\alpha,\cdot,\cdot)$ is* ***controllable*** *in the point (x_1,x_2) if for any:*

$$(y_1,y_2) \in \mathbb{R}^+ \times \mathbb{R}^+ : \left\|\underline{y}\right\| = \left\|\underline{x}\right\|$$

does exist $\alpha \in B$ such that:

$$F\left(\alpha, \left\|\psi_0^B\right\|_2, \left\|\psi_0^S\right\|_2\right) = (y_1,y_2)$$

This propertie implies the possibility of transfer mass between stationary states and scattering states of the operator $H(\bar{\alpha})$, by using a monodimensional time dependent control α.

In what follows we will make use of the weaker concept of **local controllability**:

Definition 2 *Set* $(x_1, x_2) \in \mathbb{R}^+ \times \mathbb{R}^+$ *and let* (y_1, y_2) *be he value of* F *in* $\underline{x}$ *for a fixed control* $\bar{\alpha} \in B$*:*

$$F(\bar{\alpha}, x_1, x_2) = (y_1, y_2)$$

Then F *is* ***locally controllable*** *in* (x_1, x_2) *if exists a neighborhood* $I_{(y_1,y_2)}$ *of* (y_1, y_2) *such that:*

$$\begin{aligned} I_{(y_1,y_2)} &\subset C \\ \forall\, (y_1', y_2') &\in I_{(y_1,y_2)} \Rightarrow \exists\, \alpha \in B : F(\alpha, x_1, x_2) = (y_1', y_2') \end{aligned}$$

with:

$$C = \left\{ \underline{z} \in \mathbb{R}^2 : \|\underline{z}\| = \|x\| \right\}$$

The local controllability of F allows to reach points in a neighborhood of (y_1, y_2) by choosing a suitable control.

It is trivial notice that controllability properties are strongly connected to the specific form of the interaction taken into account. The problem we're going to study is the partial transfer on the discrete spectrum of the energy of a quantum particle which at time $t = 0$ is placed in a scattering state and which evolves under the action of a time dependent 3-D point interaction.

3.2 The Model

Consider a quantum particle subject to the action of a point interaction placed into the origin of the 3-D space[1]; its state, ψ, at time t is defined by the equations (2.52) that we recall here:

$$\begin{cases} \psi(t) = e^{it\Delta}\psi_0 + i\int_0^t U(t-s, \underline{x})q(s)ds \\ q(t) + 4\sqrt{\pi i}\int_0^t \frac{\alpha(s)q(s)}{\sqrt{t-s}}ds = 4\sqrt{\pi i}\int_0^t \frac{U_s\psi_0(0)}{\sqrt{t-s}}ds \end{cases} \tag{3.4}$$

where $U(t, \underline{x})$ is the kernel of the free propagator for the Schrödinger equation, ψ_0 is the initial state and $\alpha(t)$ is a real valued function which characterizes the time dependence of the point ineraction Hamiltonian $H_{\alpha(t)}$. We refer to

[1] We adress to the Albeverio's book [7] for an extended treatement of this subject.

the previous chapter for an introduction to time dependent point interactions in particular for what concern the properties of the charge equation in (3.4).

As already metioned in system (3.4) defines properly a quantum dynamics if the condition $\alpha \in C^2_{loc}(0,+\infty)$ is satisfied. In what follows we will always assume, for the function $\alpha(t)$, regularity hypothesis strong enough to garantee the validity of this result.

Fix a real value $\bar{\alpha} < 0$ of the parameter α; the discrete spectrum of the point interaction operator $H_{\bar{\alpha}}$ contains a single point, given by: $\lambda_{\bar{\alpha}} = -16\pi^2\bar{\alpha}^2$. The eigenstate related to $\lambda_{\bar{\alpha}}$ is:

$$\psi_{\bar{\alpha}} = \frac{\sqrt{2\,|\bar{\alpha}|}e^{4\pi\bar{\alpha}|\underline{x}|}}{|\underline{x}|} \tag{3.5}$$

Moreover $H_{\bar{\alpha}}$ has an absolute continuous spectrum which coincides with the set of nonnegative real numbers.

Our purpose is to analyze the possibility of an energy transfer from continuous to point spectrum of $H_{\bar{\alpha}}$ by the use of a control interaction of the type $H_{\alpha(t)}$.

Let us suppose that at time $t = 0$ the particle is placed in some scattering state of the Hamiltonian $H_{\bar{\alpha}}$:

$$\psi_0 = \phi \in D(H_{\bar{\alpha}}) : (\phi, \psi_{\bar{\alpha}})_{L^2} = 0 \tag{3.6}$$

Then, choosing a suitable $\alpha(t)$ in a control function space B, we look for a control interaction able to stress the system into a state, with non null component along $\psi_{\bar{\alpha}}$ for $t \geq T$.

We will use a control interaction of the form: $H_{\alpha+\bar{\alpha}}$, where $\bar{\alpha}$ is fixed and α belongs to the space:

$$B = \big\{\alpha \in H^3(0,T;\mathbb{R})\,|\,\alpha(0) = \alpha(T) = 0\big\} \tag{3.7}$$

the boundary conditions on α guarantee that at the inital and final times the Hamiltonian is $H_{\bar{\alpha}}$.

The projection on the bound state, $\psi_{\bar{\alpha}}$, of the solution at time T of equations (3.4) for a system evolving under the action of the operator $H_{\alpha+\bar{\alpha}}$, is given by:

$$\begin{cases} F(\alpha) := \Big(e^{iT\Delta}\phi + i\int_0^T U(T-s,\underline{x})V(\alpha)(s)ds,\ \psi_{\bar{\alpha}}(\underline{x})\Big)_{L^2(\mathbb{R}^3)} \\ V(\alpha) = q(t) : q(t) + 4\sqrt{\pi i}\int_0^t \frac{[\alpha(s)+\bar{\alpha}]\ q(s)}{\sqrt{t-s}}ds = 4\sqrt{\pi i}\int_0^t \frac{U_s\phi(0)}{\sqrt{t-s}}ds \end{cases} \tag{3.8}$$

This is a complex valued map on B such that[2]: $F(0) = 0$. In order to move some energy on the discrete spectrum, or, as it is equivalent, a part of

[2]For $\alpha(t) = 0$, our system evolves under the action of $H_{\bar{\alpha}}$, remaining in a scattering state.

the mass on the bound state, it is necessary (and in fact sufficient) to have $F(\alpha) \neq 0$ for a suitable choice of $\alpha \in B$. Then, a sufficient condition for the solvability of our problem results to be the local surjectivity of the map:

$$\begin{cases} F(\alpha) = z \in \mathbb{C} \\ \alpha \in B \end{cases} \tag{3.9}$$

around the point $\alpha = 0$.

The main goal of this Chapter is to demonstrate the following:

Theorem 21 *Let ϕ be a scattering state of the Hamiltonian $H_{\bar{\alpha}}$ fulfilling the condition:*

$$\phi \in \left\{ \phi \in L_2(\mathbb{R}^3) \,|\, \phi = \gamma - (\gamma, \psi_{\bar{\alpha}})_{L^2}\, \psi_{\bar{\alpha}};\ \gamma = \gamma(|\underline{x}|) \in C_c^{\infty}\,(0, +\infty) \right\} \tag{3.10}$$

Then the functional $F : B \to \mathbb{C}$, defined by (3.8), (3.7), is a locally surjective map around the point $\alpha = 0$.

We will adopt here a standard procedure in the analysis of nonlinear systems. First we prove the surjectivity of the linearized map d_0F. To this aim we will study a *non controllability condition* for the linearized system; it will be shown that, under the hypothesis (3.10) on the initial state, this condition is never satisfied, obtaining, in this way, a controllability result. Then we conclude using a Rank Theorem for functional defined on Banach spaces.

3.3 The Linearized System

We're intrested into study the controllability properties of the system:

$$\begin{cases} d_0F(u) = z \\ u \in B \end{cases} \tag{3.11}$$

where d_0F is the Frechét derivative of functional F evalueted in $\alpha = 0$. Our aim is to prove the following result:

Theorem 22 *In the assumption of Theorem 21, the map defined by (3.11), (3.7) is surjective.*

The proof of Theorem 22 will be given in Sections 3-6 following an *ad absurdo* procedure.

First we set the functional (3.11) into an explicit form:

$$\begin{cases} d_0F(u) = i\left(\int_0^T U(T-s,\underline{x})\, d_0V(u)(s)ds,\ \psi_{\bar\alpha}(\underline{x})\right)_{L^2(\mathbb{R}^3)} \\ d_0V(u) = q(t) : q(t) + 4\sqrt{\pi i}\,\bar\alpha \int_0^t \frac{q(s)}{\sqrt{t-s}}ds = -4\sqrt{\pi i}\int_0^t \frac{u(s)\,V(0)(s)}{\sqrt{t-s}}ds \end{cases} \tag{3.12}$$

The dependence of $d_0F(u)$ on the charge of the umperturbed system, $V(0)(t)$, may be obtained by observing that the function $d_0V(u)$ satisfie an equation of type (2.24), whose solution may be expressed through (2.26) in the form:

$$d_0V(u)(t) = 4\pi\sqrt{i}\int_0^t G(t-s)\,u(s)\,V(0)(s)\,ds \tag{3.13}$$

Making use of (3.13) plus the Fubini Theorem - to invert time and space integrations in (3.12) - we get:

$$d_0F(u) = i\int_0^T (U_{T-s},\psi_{\bar\alpha})_{L^2(\mathbb{R}^3)}\ d_0V(u)(s)ds =$$
$$= -4\pi i^{\frac{3}{2}}\int_0^T (U_{T-s},\psi_{\bar\alpha})_{L^2(\mathbb{R}^3)}\int_0^s G(s-s')\,V(0)(s')\,u(s')\,ds'ds$$

and applying Dirichlet formula to the double integral we finally obtain:

$$d_0F(u) = -4\pi i^{\frac{3}{2}}\int_0^T ds'\,V(0)(s')\,u(s')\int_{s'}^T (U_{T-s},\psi_{\bar\alpha})_{L^2(\mathbb{R}^3)}\,G(s-s')ds \tag{3.14}$$

Let us suppose that $d_0F : B \to \mathbb{C}$ is not a surjective map; then $d_0F(u)$ should have a constant direction in the complex plane for any $u \in B$. We can express this as a **non-controllability condition:**

$$\exists\, C\in\mathbb{C} : \begin{cases} C \neq 0 \\ C\cdot\int_0^T ds'\,V(0)(s')\,u(s')\int_{s'}^T (U_{T-s},\psi_{\bar\alpha})_{L^2(\mathbb{R}^3)}\,G(s-s')ds = 0 \end{cases} \quad \forall u\in B$$

where '$\cdot$' indicates the scalar product in $\mathbb{C}$. In particular, being this condition true for any real valued $u \in C_0^\infty(0,T)$, it results equivalent to:

$$\exists\, C\in\mathbb{C} : \begin{cases} C \neq 0 \\ C\cdot V(0)(t)\int_t^T (U_{T-s},\psi_{\bar\alpha})_{L^2(\mathbb{R}^3)}\,G(s-t)ds = 0 \end{cases} \quad \forall t\in[0,T] \tag{3.15}$$

Making use of (2.50), we get:

$$\exists\, C\in\mathbb{C} : \begin{cases} C \neq 0 \\ C\cdot V(0)(t)e^{-i(T-t)\lambda_{\bar\alpha}} = 0 \end{cases} \quad \forall t\in[0,T] \tag{3.16}$$

Next we will investigate the small time asymptotic properties of condition (3.16)

3.4 Small Time Asymptotics for the Charge

The charge of unperturbed system, $V(0)$, satisfies an Abel equation of the second kind:

$$q(t)+\frac{4\pi\bar{\alpha}\sqrt{i}}{\Gamma(\frac{1}{2})}\int_0^t\frac{q(s)}{\sqrt{t-s}}ds=4\sqrt{\pi i}\int_0^t\frac{U_s\phi(\underline{0})}{\sqrt{t-s}}ds$$

where the source term - according with our hypothesis on the initial state (3.10) and with relation (2.33) - is given by:

$$4\sqrt{\pi i}\int_0^t\frac{U_s\phi(\underline{0})}{\sqrt{t-s}}ds=4\sqrt{\pi i}\int_0^t\frac{U_s\gamma(\underline{0})}{\sqrt{t-s}}ds-(\gamma,\psi_{\bar{\alpha}})_{L^2}\,q_{\bar{\alpha}}(0)\,E_{\frac{1}{2}}(4\pi\bar{\alpha}\sqrt{i}\,t^{\frac{1}{2}})$$

Then, the solution $V(0)(t)$ may be written (see (2.27) and (2.34)) in the following form:

$$V(0)(t)=4\pi\sqrt{i}\int_0^t K_{\bar{\alpha}}(t-s)U_s\gamma(\underline{0})\,ds-(\gamma,\psi_{\bar{\alpha}})_{L^2}\,q_{\bar{\alpha}}(0)\,e^{i16\,\bar{\alpha}^2\,\pi^2 t} \quad (3.17)$$

and its small time behaviour is connected to the limiting behaviour of $K_{\bar{\alpha}}(t)$ and $U_t\gamma(\underline{0})$ for $t\to 0$. To study this problem we need the following Lemma:

Lemma 23 *Let γ belong to the space of functions of rapid decrease $S(\mathbb{R}^3)$. If we assume that:*

$$D_0=\left\{n\in\mathbb{N}\cup\{0\}:\ \frac{d^n}{dt^n}U_t\gamma(\underline{0})\bigg|_{t=0}\neq 0\right\} \quad (3.18)$$

is a non empty set, then the function $\int_0^t\frac{U_s\gamma(\underline{0})}{\sqrt{t-s}}ds$ admit the expansion:

$$\int_0^t\frac{U_s\gamma(\underline{0})}{\sqrt{t-s}}ds=a_m\,t^{m+\frac{1}{2}}+o(t^{m+\frac{3}{2}});\ \ a_m\neq 0 \quad (3.19)$$

with

$$\left|o(t^{m+1})\right|\leq c_2\,t^{m+1}\quad c_2\in\mathbb{R}^+ \quad (3.20)$$

for $t\in[0,\delta)$, $\delta\in\mathbb{R}^+$, $m\in\mathbb{N}$.

Proof First recall that the Fourier transform operator, that we shall indicate with $\mathcal{F}$, is an homeomorfism of the space S into itself (e.g. [9]). It acts on $U_t\gamma(\underline{0})$ as follows:

$$\mathcal{F}U_t\gamma(\underline{k})=\mathcal{F}\gamma(\underline{k})\,e^{-ik^2t}$$

Then we may represent $U_t\gamma(\underline{0})$ in the integral form:

$$U_t\gamma(\underline{0}) = \frac{4\pi}{(2\pi)^{\frac{3}{2}}} \int_0^{+\infty} k^2 \mathcal{F}\gamma(\underline{k})\, e^{-ik^2 t}\, dk \tag{3.21}$$

From the regularity assumptions on γ, we have $\mathcal{F}\gamma(\underline{k}) \in S(\mathbb{R}^3)$ and $U_t\gamma(\underline{0}) \in C^\infty(0, +\infty)$. Now, setting $m = \min D_0$ - whose existance is assured by our hypothesis - $\frac{d^m}{dt^m} U_t\gamma(\underline{0})\big|_{t=0}$ is the first derivative of $U_t\gamma(\underline{0})$ which is different from zero in the origin and the Taylor's expansion of $U_t\gamma(\underline{0})$ up to order m in a right neighborhood of the origin, $t \in [0, \delta)$, is:

$$U_t\gamma(\underline{0}) = r_m\, t^m + o(t^{m+1})$$

with a_m explicitely given by $\frac{1}{m!}\frac{d^m}{dt^m} U_t\gamma(\underline{0})\big|_{t=0}$, which, from (3.21), is:

$$r_m = \frac{4\pi}{(2\pi)^{\frac{3}{2}}} \frac{(-i)^m}{m!} \int_0^{+\infty} k^{2m+2}\, \mathcal{F}\gamma(\underline{k})\, dk$$

Moreover, the reminder term, $o(t^{m+1})$, can be evaluated using Taylor's formula:

$$o(t^{m+1}) = \frac{1}{(m+1)!} \frac{d^{m+1}}{dt^{m+1}} U_t\gamma(\underline{0})\bigg|_{t=\bar{t}} t^{m+1}; \quad \bar{t} \in (0, t)$$

from which follows the estimate:

$$\exists\, c_1 \in \mathbb{R}^+ : \left|o(t^{m+1})\right| \leq c_1\, t^{m+1} \quad \forall\, t \in [0, \delta)$$

The one-half integral of the function $U_t\gamma(\underline{0})$ is given for any $t \in [0, \delta)$ by the following relation:

$$\int_0^t \frac{U_s\gamma(\underline{0})}{\sqrt{t-s}} ds = r_m \int_0^t \frac{t^m}{\sqrt{t-s}} ds + \int_0^t \frac{o(t^{m+1})}{\sqrt{t-s}} ds$$

from which a simple calculation lead us to the result:

$$\int_0^t \frac{U_s\gamma(\underline{0})}{\sqrt{t-s}} ds = a_m t^{m+\frac{1}{2}} + o(t^{m+\frac{3}{2}})$$

with:

$$\left|o(t^{m+\frac{3}{2}})\right| \leq c_2 t^{m+\frac{3}{2}},\ c_2 > 0 \quad \forall\, t \in [0, \delta)$$

and:

$$a_m = \sqrt{\pi}\, r_m \frac{\Gamma(m+1)}{\Gamma(m+\frac{3}{2})}$$

□

We will use this result to get an expansion in power of $t^{\frac{1}{2}}$ for the charge (3.17). First we notice that assumptions (3.10) on the regular part, γ, of the initial state are consistent with the hypothesis: $\gamma \in S(\mathbb{R}^3)$, of Lemma 23. If we assume condition (3.18) to hold, then relation (3.19) may be applied to our case (with the only restriction $m \neq 0$ due to the boundary condition: $\gamma(\underline{0}) = 0$ of definition (2.4)): this allows us to use the formula (1.42) with $\lambda = 4\pi\bar{\alpha}\sqrt{i}$ in order to obtain a power expansion of the solution $V(0)$ in a right neighbourhood of the origin:

$$V(0)(t) + (\gamma, \psi_{\bar{\alpha}})_{L^2}\, q_{\bar{\alpha}}(0)\, e^{-i\lambda_{\bar{\alpha}} t} = \\ -\frac{\left(4\pi\bar{\alpha}\sqrt{i}\right)\, a_m}{\Gamma(\frac{1}{2})} \frac{2^{m+1} m!}{(2m+1)!!} t^{m+\frac{1}{2}} + \left(4\pi\bar{\alpha}\sqrt{i}\right)^2 a_m \frac{t^{m+1}}{m+1} + o(t^{m+\frac{3}{2}}) \tag{3.22}$$

with:

$$\left| o(t^{m+\frac{3}{2}}) \right| \leq c_3\, t^{m+\frac{3}{2}} \quad c_3 \in \mathbb{R}^+ \tag{3.23}$$

3.5 The Non Controllability Condition in the limit $t \to 0$

Here we study condition (3.16) in a neighborhood $[0, \delta)$ of the origin with $\delta < T$. First, we set the problem in the form:

$$\exists\, K \in [0, 2\pi) : \arg V(0)(t) e^{-i(T-t)\lambda_{\bar{\alpha}}} = K \quad \forall t \in [0, T] \tag{3.24}$$

which is equivalent to (3.16). Making use of (3.17), we see that:

$$\arg V(0)(t) e^{-i(T-t)\lambda_{\bar{\alpha}}} = \\ = \arg\left\{ \left[4\pi\sqrt{i} \int_0^t K_{\bar{\alpha}}(t-s) U_s \gamma(\underline{0})\, ds - (\gamma, \psi_{\bar{\alpha}})_{L^2}\, q_{\bar{\alpha}}(0) e^{-it\lambda_{\bar{\alpha}}} \right] e^{-i(T-t)\lambda_{\bar{\alpha}}} \right\} = \\ = \arg\left[4\pi\sqrt{i} \int_0^t K_{\bar{\alpha}}(t-s) U_s \gamma(\underline{0})\, ds\, e^{-i(T-t)\lambda_{\bar{\alpha}}} - (\gamma, \psi_{\bar{\alpha}})_{L^2}\, q_{\bar{\alpha}}(0) e^{-iT\lambda_{\bar{\alpha}}} \right]$$

Then, condition (3.24) implies also:

$$\exists\, K' \in [0, 2\pi) : \arg 4\pi\sqrt{i} \int_0^t K_{\bar{\alpha}}(t-s) U_s \gamma(\underline{0})\, ds\, e^{-i(T-t)\lambda_{\bar{\alpha}}} = K' \quad \forall\, t \in [0, T] \tag{3.25}$$

In order to analyze (3.25), we prove the following:

Lemma 24 *In the assumptions of Lemma23 the function:*

$$\arg 4\pi\sqrt{i}\int_0^t K_{\bar\alpha}(t-s)U_s\gamma(\underline{0})\,ds\,e^{-i(T-t)\lambda_{\bar\alpha}}$$

admit the following expansion:

$$\begin{aligned}&\arg 4\pi\sqrt{i}\int_0^t K_{\bar\alpha}(t-s)U_s\gamma(\underline{0})\,ds\,e^{-i(T-t)\lambda_{\bar\alpha}} = \\ &= \frac{\pi}{4} + \arg a_m + c\sin(\arg b_0)\,t^{\frac{1}{2}} - c^2\frac{1}{2}\sin\left(2\arg b_0\right)\,t - (T-t)\lambda_{\bar\alpha} + o(t^{\frac{3}{2}})\end{aligned} \tag{3.26}$$

for $t\in[0,\delta)$.

Proof We make use of the following relations, regarding the argument of a sum of complex numbers, whose proof is straghtforward: set $z_1 = \rho_1 e^{i\varphi_1}$and $z_2 = \rho_2 e^{i\varphi_2}$; Taylor expansions of $\arg(z_1+z_2)$ w.r.t. the ratio $\varepsilon = \frac{\rho_2}{\rho_1}$ up to first order are given by:

$$\arg(z_1+z_2) = \varphi_1 + R_1(\bar\varepsilon)\,\varepsilon \tag{3.27}$$

$$R_1(x) = \frac{\sin(\varphi_2-\varphi_1)}{1+x^2+2x\cos(\varphi_2-\varphi_1)} \tag{3.28}$$

$$\arg(z_1+z_2) = \varphi_1 + \sin(\varphi_2-\varphi_1)\,\varepsilon - \frac{1}{2}\sin\left(2(\varphi_2-\varphi_1)\right)\,\varepsilon^2 + R_3(\bar\varepsilon)\varepsilon^3 \tag{3.29}$$

$$\begin{aligned}R_3(x) &= -\frac{1}{6}\sin(\varphi_2-\varphi_1)\frac{(1+x^2+2x\cos(\varphi_2-\varphi_1))}{\left(1+x^2+2x\cos(\varphi_2-\varphi_1)\right)^3} + \\ &+\frac{1}{6}\sin(\varphi_2-\varphi_1)\frac{4\left(x+\cos(\varphi_2-\varphi_1)\right)^2}{\left(1+x^2+2x\cos(\varphi_2-\varphi_1)\right)^3}\end{aligned} \tag{3.30}$$

where the remainders R_i are evaluated in a suitable point $\bar\varepsilon\in[0,\varepsilon)$.
Now, setting $A_m = -\frac{4\pi\bar\alpha}{\Gamma(\frac{1}{2})}\frac{2^{m+1}m!}{(2m+1)!!}$ and $B_m = \frac{16\pi^2\bar\alpha^2}{m+1}$ in (3.22), and using (3.17) we have:

$$\begin{aligned}&\arg 4\pi\sqrt{i}\int_0^t K_{\bar\alpha}(t-s)U_s\gamma(\underline{0})\,ds\,e^{-i(T-t)\lambda_{\bar\alpha}} = \\ &= \arg\left[\left(A_m a_m\sqrt{i}t^{m+\frac{1}{2}} + B_m a_m\,i\,t^{m+1} + o(t^{m+\frac{3}{2}})\right)\,e^{-i(T-t)\lambda_{\bar\alpha}}\right];\ t\in[0,\delta)\end{aligned} \tag{3.31}$$

The right hand side of (3.31) can be expanded again using (3.27) and (3.28); first, we apply (3.27) and with $\varepsilon = \frac{\mathrm{mod}\,o(t^{m+\frac{3}{2}})}{\mathrm{mod}(A_m a_m\sqrt{i}t^{m+\frac{1}{2}}+B_m a_m\,i\,t^{m+1})}$, obtaining:

$$
\begin{aligned}
&\arg 4\pi\sqrt{i}\int_0^t K_{\bar{\alpha}}(t-s)U_s\gamma(\underline{0})\,ds\,e^{-i(T-t)\lambda_{\bar{\alpha}}} = \\
&= \arg\left[\left(A_m a_m\sqrt{i}t^{m+\frac{1}{2}} + B_m a_m\, i\, t^{m+1}\right)\, e^{-i(T-t)\lambda_{\bar{\alpha}}}\right] + \\
&\qquad + R_1(x)\frac{\operatorname{mod} o(t^{m+\frac{3}{2}})}{\operatorname{mod}(A_m a_m\sqrt{i}t^{m+\frac{1}{2}} + B_m a_m\, i\, t^{m+1})}
\end{aligned}
\tag{3.32}
$$

with $x \in \left(0, \frac{\operatorname{mod} o(t^{m+\frac{3}{2}})}{\operatorname{mod}(A_m a_m\sqrt{i}t^{m+\frac{1}{2}}+B_m a_m\, i\, t^{m+1})}\right)$; then expand by (3.28) the first term at second member of (3.32) w.r.t $\varepsilon = \operatorname{mod}\frac{B_m}{A_m}t^{\frac{1}{2}}$:

$$
\begin{aligned}
&\arg\left(A_m a_m\sqrt{i}t^{m+\frac{1}{2}} + B_m a_m\sqrt{i}b_0 t^{m+1}\right) = \\
&= \frac{\pi}{4} + \arg a_m + c\sin(\frac{\pi}{4})\, t^{\frac{1}{2}} - c^2\frac{1}{2}\sin\left(\frac{\pi}{2}\right)\, t + R_3(x')\, c^3 t^{\frac{3}{2}}
\end{aligned}
\tag{3.33}
$$

with $c = \operatorname{mod}\frac{B_m}{A_m}$ and $x' \in \left(0, c^3 t^{\frac{3}{2}}\right)$. Equation (3.26) is a straightforward consequence of (3.32) and (3.33).

□

Lemma 24 leads us to an asymptotic formulation of the non controllability condition; from relations (3.26) and (3.25), indeed, we have:

$$
\frac{c}{\sqrt{2}}\, t^{\frac{1}{2}} + \left(\lambda_{\bar{\alpha}} - \frac{c^2}{2}\right) t + o(t^{\frac{3}{2}}) = 0 \quad \forall\, t \in [0, \bar{t}) \tag{3.34}
$$

Here we note that c is a real positive constant different from zero, given explicitely by:

$$
c = \Gamma(\frac{1}{2})4\pi\,|\bar{\alpha}|\,\frac{(2m+1)!!}{2^{m+1}\,(m+1)!}
$$

while, by definition, $\lambda_{\bar{\alpha}}$ is real and negative defined. Then the difference $\left(\lambda_{\bar{\alpha}} - \frac{c^2}{2}\right)$ is always different from zero.

Relation (3.34) is an evident contaddiction we approached to by supposing system (3.11) to be not surjective. This concludes the proof of Theorem 22 for all choices of initial states satisfying condition (3.18) of Lemma 23.

In the next section we will study an extension of the proof to those cases in which Lemma 23 does not hold.

3.6 Finite Time Asymptotics for the Charge and Proof of Theorem 22

If condition (3.18) does not hold, we may still recover our results just by changing the point in which perform the expansions of (3.17) and (3.24).We claim the following:

Lemma 25 *Let $\gamma \in S(\mathbb{R}^3)$ - S being the space of functions with rapid decrease - and $\gamma \neq 0$. There exists a $t_0 \in [0,T]$ such that the expansion:*

$$\int_0^t \frac{U_{t_0+s}\gamma(\underline{0})}{\sqrt{t-s}} ds = a_m\,(t-t_0)^m + o((t-t_0)^{m+1});\ a_m \neq 0 \tag{3.35}$$

with:

$$\left|o((t-t_0)^{m+1})\right| \leq c_4\, t^{m+1} \quad c_4 \in \mathbb{R}^+$$

holds for $t \in [t_0, t_0+\delta)$, $\delta \in \mathbb{R}^+$, $m \in \mathbb{N}$.

Proof Define the set:

$$D_t = \left\{ n \in \mathbb{N} \cup \{0\} : \left. \frac{d^n}{dt^n} U_t \gamma(\underline{0}) \right|_t \neq 0 \right\}$$

It is trivial noticing that:

$$D_t = \varnothing \quad \forall\, t \in [0,T] \Rightarrow \gamma = 0$$

Then, if $\gamma \neq 0$ as we suppose, it is always possible to find a $t_0 \in [0,T]$ such that $D_{t_0} \neq \varnothing$.

The proof can be concluded following the same line of the proof of Lemma 23.

□

Next we observe that, starting from definition (2.27), a simple change of variables provide us an equation for the charge $V(0)$ when the initial time $t = t_0$ is assigned:

$$q(t) + \frac{4\pi\bar{\alpha}\sqrt{i}}{\Gamma(\frac{1}{2})} \int_0^t \frac{q(s)}{\sqrt{t-s}} ds = 4\sqrt{\pi i} \int_0^t \frac{U_{t_0+s}\phi(\underline{0})}{\sqrt{t-s}} ds$$

where the source term - according with our hypothesis on the initial state (3.10) and with relation (2.33) - is given by:

$$4\sqrt{\pi i} \int_0^t \frac{U_{t_0+s}\phi(\underline{0})}{\sqrt{t-s}} ds = 4\sqrt{\pi i} \int_0^t \frac{U_{t_0+s}\gamma(\underline{0})}{\sqrt{t-s}} ds - (\gamma, \psi_{\bar{\alpha}})_{L^2}\, q_{\bar{\alpha}}(t_0)\, E_{\frac{1}{2}}(4\pi\bar{\alpha}\sqrt{i}\ (t+t_0)^{\frac{1}{2}})$$

Once more, according to the relation (2.35), the charge $V(0)$ may be written in the following form:

$$V(0)(\tau) = 4\pi\sqrt{i}\int_0^\tau K_{\bar\alpha}(\tau - s)U_{t_0+s}\gamma(\underline{0})\,ds - (\gamma,\psi_{\bar\alpha})_{L^2}\,q_{\bar\alpha}(0)\,e^{-i(\tau+t_0)\lambda_{\bar\alpha}} \tag{3.36}$$

and, taking into account the results of Lemma 25 and the formula (1.42), an expansion of the charge in half integer power of time around the point $t = t_0$ is obtained:

$$\begin{aligned} &V(0)(\tau) + (\gamma,\psi_{\bar\alpha})_{L^2}\,q_{\bar\alpha}(0)e^{-i(\tau+t_0)\lambda_{\bar\alpha}} = \\ &= -\frac{\lambda\,a_m}{\Gamma(\frac{1}{2})}\frac{2^{m+1}m!}{(2m+1)!!}t^{m+\frac{1}{2}} + \lambda^2\,a_m\frac{t^{m+1}}{m+1} + o(t^{m+\frac{3}{2}});\ \tau\in[t_0,t_0+\delta) \end{aligned} \tag{3.37}$$

with:

$$\left|o(\tau^{m+\frac{3}{2}})\right| \le c_5\,\tau^{m+\frac{3}{2}} \quad c_5\in\mathbb{R}^+ \tag{3.38}$$

Now, proceeding on the same line of Section 5, it is easy to proof the following:

Lemma 26 *In the assumptions of Lemma 25 the function* $\arg 4\pi\sqrt{i}\int_0^t K_{\bar\alpha}(t-s)U_s\gamma(\underline{0})\,ds\,e^{-i(T-t)\lambda_{\bar\alpha}}$ *admit the following expansion:*

$$\begin{aligned} &\arg 4\pi\sqrt{i}\int_0^t K_{\bar\alpha}(t-s)U_s\gamma(\underline{0})\,ds\,e^{-i(T-t)\lambda_{\bar\alpha}} = \\ &= \frac{\pi}{4} + \arg a_m + c\sin(\frac{\pi}{4})\ (t-t_0)^{\frac{1}{2}} - c^2\frac{1}{2}\sin\left(\frac{\pi}{2}\right)\ (t-t_0) - (T-t)\lambda_{\bar\alpha} + o(t^{\frac{3}{2}}) \end{aligned} \tag{3.39}$$

for $t\in[t_0,t_0+\delta)$.

This concludes the proof of Theorem 22

3.7 The Nonlinear System

Here we study the regularity properties of the nonlinear functional (3.9). To this aim, we introduce the following lemma:

Lemma 27 *Let* $V : B \to L^\infty(0,T)$ *be the map such that* $\forall\alpha\in B \Rightarrow V(\alpha) = q$ *:*

$$q(t) + 4\sqrt{\pi i}\int_0^t \frac{[\alpha(s)+\bar\alpha]\ q(s)}{\sqrt{t-s}}ds = 4\sqrt{\pi i}\int_0^t \frac{U_s\phi(\underline{0})}{\sqrt{t-s}}ds$$

with $\bar\alpha$ *real constant.*

Then V *is a functional of class* C^1.

Proof Let $\alpha, \beta \in B$; the difference $V(\alpha)(t) - V(\beta)(t)$ integrates the equation:

$$q(t) + 4\sqrt{\pi i} \int_0^t \frac{[\alpha(s) + \bar{\alpha}]\ q(s)}{\sqrt{t-s}} ds = 4\sqrt{\pi i} \int_0^t \frac{[\beta(s) - \alpha(s)]\ V(\beta)(s)}{\sqrt{t-s}} ds$$

whose solution satisfies the estimate (see (2.55)):

$$\|q\|_\infty \leq \left|4\sqrt{\pi i}\right| \|V(\beta)\|_\infty \|\beta - \alpha\|_\infty 2T^{\frac{1}{2}} \left[1 + \sum_{n=1}^{+\infty} \left|4\sqrt{\pi i}\right|^n \|\alpha + \bar{\alpha}\|_\infty^n A_n \pi^{\frac{n}{2}} T^{\frac{n}{2}}\right] \tag{3.40}$$

Taking into account the Sobolev inequality (see for instance [11]):

$$\|\alpha\|_{L^\infty(0,T)} \leq C \|\alpha\|_{H^1(0,T)} \tag{3.41}$$

and the trivial bound:

$$\|\alpha\|_{H^1(0,T)} \leq \|\alpha\|_{H^3(0,T)} \tag{3.42}$$

we recover from (3.40) the continuity of the functional on the space B in the topology of $H^3(0,T)$.

The Frechét derivative of V in the point α is the map $d_\alpha V$ whose action on $u \in B$ is given by the solution of the integral equation:

$$q(t) + 4\sqrt{\pi i} \int_0^t \frac{[\alpha(s) + \bar{\alpha}]\ q(s)}{\sqrt{t-s}} ds = -4\sqrt{\pi i} \int_0^t \frac{u(s)\, V(\alpha)(s)}{\sqrt{t-s}} ds$$

The continuity of $d_\alpha V$ on B is, once again, a consquence of the estimates (2.55), (3.41).

□

From Lemma27 and definition (3.8) easily follows:

Theorem 28 *The functional (3.9) defined by (3.8) is a C^1 class map: $B \to \mathbb{C}$*

Proof Let α and β be a couple of points in B; from the definition of F (3.8) we have tha the difference:

$$F(\alpha) - F(\beta) = i \int_0^T U_{T-s} \psi_{\bar{\alpha}}(\underline{0})\ (V(\alpha)(s) - V(\beta)(s))\ ds$$

satisfies the estimate:

$$|F(\alpha) - F(\beta)| \leq \|V(\alpha)(s) - V(\beta)(s)\|_\infty \int_0^T |U_{T-s}\psi_{\bar{\alpha}}(\underline{0})|\ ds \tag{3.43}$$

The continuity of F then follows directely form Lemma 27.

The same proof allows for $d_\alpha F$.

□

3.8 Proof of the Main Result

In previous sections we have shown that the functional F, (3.9), is a C^1 map between a Banach space B and a finite dimensional linear space $\mathbb{C}$ (Theorem 28), whose differential evaluated in the point $\alpha = 0$ is surjective. Then, using Rank Theorem (e.g. [12]), we prove the existence of neighborhood I_0 of $z = 0$ in $\mathbb{C}$ and C^1 class map $G : I_0 \to B$ such that:

$$F(G(z)) = z \quad \forall z \in I_0 \tag{3.44}$$

This conclude the proof of Theorem 21.

3.9 Remarks and Conclusions

Our main remark is about the assumptions (3.10) on the initial state. It is clear that, by taking functions with radial symmetry, we exclude all scattering state which are of p type (see section 2.4.3):

$$\phi_P(r, \vartheta, \varphi) = \sum_{\substack{l=1 \\ l \neq 0}}^{+\infty} \sum_{m=-l}^{l} f_{lm}(r) Y_l^m(\vartheta, \varphi)$$

This fact does not represent merely a thecnical restricion. Infact we already observed that, for these states, the following relation holds:

$$U_t \phi_P(\underline{0}) = 0 \ \forall t$$

Then from the definition (3.4) and the uniqueness of solution of the charge equation, follows that a particle, initially placed in the state ϕ_P and subjected to the action of any hamiltonian of type $H_{\alpha(t)}$, results to have a null charge and evolves under the action of the free propagator: it means that, in these conditions, the particle doesn't feel the interaction at all. In this case any transfer of energy is physically impossible.

On the other hand, we stress out that our proof, although not taking into account all scattering state of radial type, has been performed on the set:

$$\{\phi \in C_c^\infty(0, +\infty) \oplus \{\psi_{\bar{\alpha}}\} : (\phi, \psi_{\bar{\alpha}}) = 0\}$$

which is dense in the space of scattering radial functions.

In conclusion, we have proved the local controllability of a process of energy-mass transfer, from scattering to bound states, for a one body quantum system under the action of a time dependent point interaction. Further development of this studies may regard the global controllability of the same process, as well as the inverse problem of finite time ionization.

Chapter 4

Ionization for Three Dimensional Time-dependent Point Interactions

4.1 Introduction

We shall study the time evolution of a three dimensional system with time-dependent Hamiltonian given by

$$H(t) = H_0 + H_I(t)$$

where the "perturbation" $H_I(t)$ is a zero-range interaction with time-dependent (periodic) "strength". In particular we are interested in proving complete ionization of the system as $t \to \infty$, starting from an initial condition at $t = 0$ given by a bound state of the system. By complete ionization one can mean two different statements. The weaker one is that the survival probability of the bound state, i.e. the square modulus of the scalar product of the state at time t with the bound state, goes to zero as $t \to \infty$. The stronger one is that every state Ψ in the Hilbert space of the system is a scattering state (see for example [21, 24]) of $H(t)$, i.e. for every compact set $S \subset \mathbb{R}^3$,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t d\tau \int_S d^3\vec{x}\, |\Psi_\tau(\vec{x})|^2 = 0$$

Ψ_t denoting the time evolution of the state Ψ. The last statement is related to the absence of eigenvalues of the Floquet operator associated to $H(t)$ (see [25, 23, 29]).

The usual way to deal with problems of this kind is by means of time-dependent perturbation theory and Fermi's golden rule, which gives for the

survival probability the well known exponential decay for each order n in the perturbative expansion. On the other hand simple examples of regular perturbations show that the survival probability decays to zero as a power-law (i.e. the limits $t \to \infty$ and $n \to \infty$ can not be interchanged). When the perturbation is not small, it is in general very difficult to solve the problem and find the law of decay. Therefore it is interesting to find models in which a non-perturbative solution exists and study the survival probability. In this paper we study one such model, in which $H_I(t)$ is given by a three dimensional point interaction. We shall see that it is possible to prove asymptotic complete ionization and find a power law decay for the survival probability, under generic condition on the scattering length[1].
The one-dimensional version of the same problem has been widely analyzed in [14][15][16][17], where complete ionization is proved under a suitable and very weak condition on the Fourier coefficients of the strength of the interaction. We shall see that the same *genericity* condition is also sufficient in three dimensions to have complete ionization of the system.
From a physical point of view, the model we are going to study is related to the strong laser ionization of Rydberg atoms[2], showing many features of experimental data. Indeed, despite of the simplicity of the model, as in the one-dimensional case, it is possible to reproduce many effects of multiphoton ionization of excited hydrogen atoms by microwave field, with a good agreement with experiments (see [18]).

4.2 The Model

The model we are going to study is a quantum particle subjected to a time-dependent point interaction fixed at the origin in three dimensions, namely a system defined by the time-dependent self-adjoint Hamiltonian $H_{\alpha(t)}$,

$$\begin{aligned} \mathcal{D}(H_{\alpha(t)}) = \{\Psi \in L^2(\mathbb{R}^3) : \exists\, q_\lambda(t) \in \mathbb{C}, \big(\Psi(\vec{x}) - q_\lambda(t)\mathcal{G}^\lambda(\vec{x})\big) \in H^2(\mathbb{R}^3), \\ \big(\Psi - q_\lambda(t)\mathcal{G}^\lambda\big)\big|_{\underline{x}=0} = \left(\alpha(t) + \frac{\sqrt{\lambda}}{4\pi}\right) q_\lambda(t)\} \\ \big(H_{\alpha(t)} + \lambda\big)\,\Psi = (H_0 + \lambda)\big(\Psi - q_\lambda(t)\mathcal{G}^\lambda\big) \end{aligned} \tag{4.1}$$

where $\lambda \in \mathbb{R}$, $\lambda > 0$ and

$$\mathcal{G}^\lambda(\vec{x} - \vec{x}') = \frac{e^{-\sqrt{\lambda}|\vec{x}-\vec{x}'|}}{4\pi|\vec{x} - \vec{x}'|}$$

[1] In three dimensions the parameter $\alpha(t)$ is proportional to the inverse of the scattering length.

[2] See the discussion contained in [14, 18] and references therein.

is the Green function of the free Hamiltonian $H_0 = -\Delta$.
As already mentioned in Chapter 2, it is well known (see [19, 20, 22, 28, 8]) that, under suitable hypothesis of regularity on $\alpha(t)$, the operator (4.1) defines a time propagation $U(t,s)$ given by a two-parameters unitary family, solving the time-dependent Shrödinger equation (2.51); the state at time t may be also defined as the solution of equations (2.52), that we recall here:

$$\Psi_t(\vec{x}) = U(t,s)\Psi_s(\vec{x}) = U_0(t-s)\Psi_s(\vec{x}) + i\int_s^t d\tau\, q(\tau)U_0(t-\tau;\vec{x}) \tag{4.2}$$

$$q(t) + 4\sqrt{\pi i}\int_s^t d\tau\, \frac{\alpha(\tau)q(\tau)}{\sqrt{t-\tau}} = 4\sqrt{\pi i}\int_s^t d\tau\, \frac{\left(U_0(\tau)\Psi_s\right)(\underline{0})}{\sqrt{t-\tau}} \tag{4.3}$$

where $U_0(t) = \exp(-iH_0 t)$, $U_0(t;\vec{x})$ is the kernel associated to the free propagator and the charge $q(t)$ satisfies the usual Volterra integral equation for $t \geq s$. We are interested in studying complete ionization of the system defined by (4.1) and (2.51), starting from initial conditions

$$\Psi_0(\vec{x}) = \varphi_{\alpha(0)}(\vec{x}) \tag{4.4}$$

$\varphi_{\alpha(0)}(\vec{x})$ being the bound state[3] of $H_{\alpha(0)}$.

We shall assume that $\alpha(t)$ is a real periodic continuous function with period T.

The meaningful parameter of the system is the negative lower bound of $\alpha(t)$. Indeed, if $\inf(\alpha(t)) \geq 0$, the wave operator associated to $(H_0, H_{\alpha(t)})$ is unitary (see [8]) so that any initial state evolves into a scattering state. Hence we require for $\alpha(t)$ the validity of assumptions (1.47), (1.48) and (1.54) and adopted in Section 1.3-1.4.

4.3 Complete Ionization in the Generic Case

In what follows we shall prove asymptotic complete ionization of the system (4.2)-(4.3) under generic conditions on $\alpha(t)$.

A straightforward consequence of Theorem 12 (and of the analogous results for the case $\alpha_0 \geq 0$) is that the scalar product (and thus the survival probability of the bound state)

$$\theta(t) = \left(\varphi_{\alpha(0)}, \Psi_t\right)_{L^2(\mathbb{R}^3)}$$

tends to 0 when $t \to \infty$:

[3] In order to do this analysis we shall require that $\alpha(0) < 0$.

Corollary 29 *If* $\{\alpha_n\}$ *satisfies (1.48) and the genericity condition (1.56) with respect to* $\mathcal{T}$*, the system shows asymptotic complete ionization and, as* $t \to \infty$*,*

$$|\theta(t)| \leq D\, t^{-\frac{3}{2}} + E(t)$$

where $D \in \mathbb{R}$ *and* $E(t)$ *has an exponential decay.*

Proof: Using the decomposition of the wave function at time t defined by (4.2), we can write the survival probability in the following way:

$$\theta(t) \equiv \left(\varphi_{\alpha(0)}, \Psi_t\right)_{L^2(\mathbb{R}^3)} = \left(\varphi_{\alpha(0)}, e^{-iH_0 t}\varphi_{\alpha(0)}\right)_{L^2(\mathbb{R}^3)} + \tag{4.5}$$

$$+ i\left(\varphi_{\alpha(0)}(\vec{x}), \int_0^t d\tau\, q(\tau) U_0(t-\tau;\vec{x})\right)_{L^2(\mathbb{R}^3)}$$

Let us define

$$Z_1(t) \equiv \left(\varphi_{\alpha(0)}, e^{-iH_0 t}\varphi_{\alpha(0)}\right)_{L^2(\mathbb{R}^3)}$$

By the usual dissipative estimate for the free propagator, one has

$$|Z_1(t)| \leq c_1 t^{-\frac{3}{2}}$$

as $t \to \infty$ for some constant $c_1 \in \mathbb{R}$. Hence $Z_1(t)$ belongs to $L^1(\mathbb{R}^+)$ and then its Laplace transform $\tilde{Z}_1(p)$ is analytic at least for $\Re(p) \geq 0$.
The second piece of the scalar product is given by

$$Z(t) \equiv i\left(\varphi_{\alpha(0)}(\vec{x}), \int_0^t d\tau\, q(\tau) U_0(t-\tau;\vec{x})\right)_{L^2(\mathbb{R}^3)} =$$

$$= i\int_0^t d\tau\, q(\tau)\left(e^{-iH_0(t-\tau)}\varphi_{\alpha(0)}\right)(0)$$

and taking the Laplace transform of $Z(t)$, we have

$$\tilde{Z}(p) = \tilde{Z}_2(p)\,\tilde{q}(p)$$

where

$$\tilde{Z}_2(p) \equiv -\frac{4\sqrt{2\pi|\alpha(0)|}}{4\pi\alpha(0) - \sqrt{-ip}}$$

is analytic for $\Re(p) > 0$ and never equal to 0, because of condition (1.47).
Hence the Laplace transform of $\theta(t)$ is given by

$$\tilde{\theta}(p) = \tilde{Z}_1(p) + \tilde{Z}_2(p)\,\tilde{q}(p)$$

where $\tilde{Z}_1(p)$ is analytic on the closed right half plane and $\tilde{Z}_2(p)$ has only a branch point at the origin of the form $a_1 + a_2\sqrt{p}$.
Hence $\tilde{\theta}(p)$ has the same singularities as $\tilde{q}(p)$ and then its asymptotic behavior coincides with that of $q(t)$, i.e.

$$|\theta(t)| \leq D\, t^{-\frac{3}{2}} + E(t)$$

for some constant $D \in \mathbb{R}$ and for a bounded function $E(t)$ with exponential decay.

□

In the following we shall prove a stronger result about complete ionization of the system, namely that every state $\Psi \in L^2(\mathbb{R}^3)$ is a scattering state[4] for the operator $H_{\alpha(t)}$, i.e.

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t d\tau \, \|F(|\vec{x}| \leq R) U(\tau, 0)\Psi\|^2 = 0 \tag{4.6}$$

where $F(S)$ is the multiplication operator by the characteristic function of the set $S \subset \mathbb{R}^3$ and $U(t,s)$ the unitary two-parameters family associated to $H_{\alpha(t)}$ (see (2.51)).
In order to prove (4.6), we first need to study the evolution of a generic initial datum in a suitable dense subset of $L^2(\mathbb{R}^3)$ and then we shall extend the result to every state using the unitarity of the evolution defined by (2.51) (see e.g. [19]).

Proposition 30 *Let $\Psi \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ a smooth radial function compactly supported away from 0 and $q(t)$ be the solution of equation (4.3) with initial condition $\Psi_0 = \Psi$. If $\{\alpha_n\}$ satisfies (1.48) and the genericity condition (1.56) with respect to $\mathcal{T}$, as $t \to \infty$,*

$$|q(t)| \leq A\, t^{-\frac{3}{2}} + R(t) \tag{4.7}$$

where $A \in \mathbb{R}$ and $R(t)$ has an exponential decay, $R(t) \sim Ce^{-Bt}$ for some $B > 0$.

Proof: First we notice that, using the decomposition of the wave function at time t defined by (4.2), we can write the survival probability in the following way:

$$\theta(t) \equiv (\Psi, \Psi_t)_{L^2(\mathbb{R}^3)} = \left(\Psi, e^{-iH_0 t}\Psi\right)_{L^2(\mathbb{R}^3)} + \tag{4.8}$$

[4] For the definition of scattering states of a time-dependent operator see e.g. [21, 24].

$$+i\left(\Psi(\vec{x}),\int_0^t d\tau\, q(\tau)U_0(t-\tau;\vec{x})\right)_{L^2(\mathbb{R}^3)}$$

Let us define

$$Z_1(t)\equiv\left(\Psi,e^{-iH_0t}\Psi\right)_{L^2(\mathbb{R}^3)}$$

By the usual dissipative estimate for the free propagator, one has

$$|Z_1(t)|\leq c_1t^{-\frac{3}{2}}$$

as $t\to\infty$ for some constant $c_1\in\mathbb{R}$. Hence $Z_1(t)$ belongs to $L^1(\mathbb{R}^+)$ and then its Laplace transform $\tilde{Z}_1(p)$ is analytic at least for $\Re(p)\geq 0$.
The second piece of the scalar product is given by

$$Z(t)\equiv i\left(\Psi(\vec{x}),\int_0^t d\tau\, q(\tau)U_0(t-\tau;\vec{x})\right)_{L^2(\mathbb{R}^3)}=$$

$$=i\int_0^t d\tau\, q(\tau)\left(e^{-iH_0(t-\tau)}\Psi\right)(\underline{0})$$

and taking the Laplace transform of $Z(t)$, we have

$$\tilde{Z}(p)=\tilde{Z}_2(p)\,\tilde{q}(p)$$

where

$$\tilde{Z}_2(p)\equiv\mathcal{L}\left[\left(e^{-iH_0t}\Psi\right)(0)\right](p)$$

The function of time $\left(e^{-iH_0t}\Psi\right)(\underline{0})$ may be represented as a fourier integral:

$$\left(e^{-iH_0t}\Psi\right)(\underline{0})=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}e^{-ik^2t}\hat{\Psi}(k)\,d\underline{k}$$

which is continuous in the variable t, due to the regularity of $\hat{\Psi}\in\mathcal{S}$ (class of Schwartz functions). The dispersive estimate:

$$\left\|e^{-iH_0t}\Psi\right\|_\infty\underset{t\to\infty}{\leq}t^{-\frac{3}{2}}\left\|\Psi\right\|_1$$

assures once more that $e^{-iH_0t}\Psi\in L^1(\mathbb{R}^+)$. This implies that $\tilde{Z}_2(p)$ is analytic for $\Re(p)>0$.

Furthermore we notice that:

$$\mathcal{L}\left[\left(e^{-iH_0t}\Psi\right)(\underline{0})\right](p)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}\frac{\hat{\Psi}(\underline{k})}{p+ik^2}\,d\underline{k}$$

If we suppose that this function has a zero in the point $\bar{p}$ of the complex plane, then it should results also that:

$$\bar{p}^n \int_{\mathbb{R}^3} \frac{\hat{\Psi}(\underline{k})}{\bar{p}+ik^2}\, d\underline{k} = 0 \qquad \forall n \in \mathbb{N}_0 \tag{4.9}$$

But, from our hypothesis on Ψ we know that:

$$\mathcal{L}\left[\frac{d^n}{dt^n}\left(e^{-iH_0t}\Psi\right)(\underline{0})\right](p) = \frac{p^n}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}\frac{\hat{\Psi}(\underline{k})}{p+ik^2}\,d\underline{k} + \frac{1}{(2\pi)^{\frac{3}{2}}}\sum_{j=0}^{n-1}\int_{\mathbb{R}^3}\left(-ik^2\right)^j \hat{\Psi}(\underline{k})\,d\underline{k} =$$

$$= \frac{p^n}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}\frac{\hat{\Psi}(\underline{k})}{p+ik^2}\,d\underline{k} + \frac{1}{(2\pi)^{\frac{3}{2}}}\sum_{j=1}^{n-1}(i)^j\,\Delta^j\Psi(0) = \frac{p^n}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}\frac{\hat{\Psi}(\underline{k})}{p+ik^2}\,d\underline{k}$$

Replacing this relation in (4.9) we get:

$$\mathcal{L}\left[\frac{d^n}{dt^n}\left(e^{-iH_0t}\Psi\right)(0)\right](\bar{p}) = 0$$

From Schrödinger equation this is equivalent to:

$$\mathcal{L}\left[(-iH_0)^n\left(e^{-iH_0t}\Psi\right)(\underline{0})\right](\bar{p}) = 0 \Rightarrow \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}\left(k^2\right)^n\frac{\hat{\Psi}(\underline{k})}{\bar{p}+ik^2}\,d\underline{k} = 0$$

Next, recalling that the fourier transform of a radial function is still radial, we have:

$$\int_0^{+\infty} k^{2n+2}\frac{\hat{\Psi}(k)}{\bar{p}+ik^2}\,dk = 0 \qquad \forall n \in \mathbb{N}_0$$

which implies, from the unicity of solutions of the Hamburger moment equation,

$$\tilde{\Psi} = 0$$

Then, if $\Psi \neq 0$, as we suppose, the function $\tilde{Z}_2(p)$ has no finite zeros in the complex plane.

The Laplace transform of $\theta(t)$ is given by

$$\tilde{\theta}(p) = \tilde{Z}_1(p) + \tilde{Z}_2(p)\,\tilde{q}(p)$$

But $\theta(t)$ is a bounded function[5], because of unitarity of the evolution (2.51), and then its Laplace transform is analytic on the open right half plane. From analyticity of $\tilde{Z}_1(p)$, $\tilde{Z}_2(p)$ and absence of zeros of $\tilde{Z}_2(p)$ follows that the Laplace transform of $q(t)$, solution of (4.3), is analytic at least for $\Re(p) > 0$.

[5] Actually $|\theta(t)| \leq 1$, since the initial state is normalized.

Consider the Laplace transform of equation (4.3), which has the form (1.52) with

$$f(p) = \sqrt{\frac{2}{\pi}}\sqrt{\frac{i}{p}}\int_0^\infty dt\, e^{-pt}\int_{\mathbb{R}^3} d^3\vec{k}\,\hat{\Psi}(\vec{k})\,e^{-ik^2t}$$

where $\hat{\Psi}(\vec{k})$ is the Fourier transform of $\Psi(\vec{x})$.
The equation for $\tilde{q}(p)$ is then given by

$$\tilde{q}(p) = -\frac{4\pi}{4\pi\alpha_0+\sqrt{-ip}}\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}}\alpha_k\,\tilde{q}(p+i\omega k) + \frac{g(p)}{4\pi\alpha_0+\sqrt{-ip}}$$

where

$$g(p) = \sqrt{\frac{2}{\pi}}\int_0^\infty dt\, e^{-pt}\int_{\mathbb{R}^3} d^3\vec{k}\,\hat{\Psi}(\vec{k})\,e^{-ik^2t}$$

It is now sufficient to show that the solution $\tilde{q}(p)$ is also analytic on the imaginary axis except at most square root branch points at $p = i\omega n$ as in the discussion of section 3.3.2 and 3.3.3.
For every smooth function Ψ with compact support, $\hat{\Psi}(\vec{k})$ is a smooth function with an exponential decay as $k \to \infty$, so that

$$g(is) = \lim_{r\to 0^+}\sqrt{\frac{2}{\pi}}\int_{\mathbb{R}^3} d^3\vec{k}\,\frac{\hat{\Psi}(\vec{k})}{r+(s+k^2)i} = -i\sqrt{\frac{2}{\pi}}\int_{\mathbb{R}^3} d^3\vec{k}\,\frac{\hat{\Psi}(\vec{k})}{s+k^2}$$

is a bounded function for $s > 0$. Hence the function $g(p)$ has no pole for $\Im(p) \in (0,\omega)$ and therefore the result contained in Proposition 8 still holds.
Moreover

$$g(0) = \sqrt{\frac{2}{\pi}}\int_{\mathbb{R}^3} d^3\vec{k}\,\hat{\Psi}(\vec{k})\int_0^\infty dt\, e^{-ik^2t} = -i\sqrt{\frac{2}{\pi}}\int_{\mathbb{R}^3} d^3\vec{k}\,\frac{\hat{\Psi}(\vec{k})}{k^2}$$

which is again bounded, so that $g(p)$ has at the origin at most a branch point singularity of the form $a(p)+b(p)\sqrt{p}$: following the proofs of Proposition 9 and 11, we can show that $\tilde{q}(p)$ has the same behavior at the origin.
In conclusion the solution is analytic on the closed right half plane except branch points at $p = i\omega n$, $n \in \mathbb{Z}$, of the form $a(p)+b(p)\sqrt{p-i\omega n}$. The proof of Theorem 12 then implies that $q(t)$ has the prescribed behavior as $t \to \infty$.

□

Theorem 31 *If $\{\alpha_n\}$ satisfies (1.48) and the genericity condition (1.56) with respect to $\mathcal{T}$, every $\Psi \in L^2(\mathbb{R}^3)$ is a scattering state of $H_{\alpha(t)}$, i.e.*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t d\tau \, \|F(|\vec{x}| \leq R)U(\tau,0)\Psi\|^2 = 0$$

Proof: We shall restrict the proof to the dense subset of $L^2(\mathbb{R}^3)$ given by smooth functions with compact support and then we shall extend the result to every state using the unitarity of the evolution defined by (4.2) (see e.g. [19]). Actually we are going to prove an equivalent but slightly different statement, i.e. $\forall \varepsilon > 0$, there exists t_0 such that $\forall t > t_0$,

$$\|F(|\vec{x}| \leq R)U(t,0)\Psi\| \leq \varepsilon$$

The evolution of an initial state Ψ according to (4.2) is given by

$$\Psi_t(\vec{x}) = U(t,s)\Psi_s(\vec{x}) = U_0(t-s)\Psi_s(\vec{x}) + i\int_s^t d\tau \, q(\tau)\, U_0(t-\tau;\vec{x}) \tag{4.10}$$

Moreover, since $\Psi_t \in \mathcal{D}(H_{\alpha(t)})$, the following decomposition holds

$$\Psi_t(\vec{x}) = \varphi_t(\vec{x}) + \frac{q(t)}{4\pi|\vec{x}|} \tag{4.11}$$

where $q(t)$ is the solution of (4.3), $\varphi_t \in H^2_{\text{loc}}(\mathbb{R}^3)$ and

$$\varphi_t(0) = \alpha(t)q(t)$$

We are going to show that, if $q(t) \in L^1(\mathbb{R}^+)$, Ψ_t satisfies the required property. Let us start analyzing the second term in (4.10): imposing the unitarity condition of the evolution we have

$$\|\Psi_s\|^2 = \|\Psi_t\|^2 = \left\| U_0(t-s)\Psi_s(\vec{x}) + i\int_s^t d\tau\, q(\tau)\, U_0(t-\tau;\vec{x}) \right\|^2$$

and then

$$\left\| \int_s^t d\tau\, q(\tau) U_0(t-\tau;\vec{x}) \right\|^2 = 2\Im \left(\int_s^t d\tau\, q(\tau) U_0(t-\tau;\vec{x}), U_0(t-s)\Psi_s(\vec{x}) \right) =$$

$$= 2\Im \left[\int_s^t d\tau\, q^*(\tau) \left(e^{-iH_0(\tau-s)}\Psi_s \right)(0) \right]$$

but, using the decomposition (4.11),

$$\left(e^{-iH_0(s-\tau)}\Psi_s \right)(0) = \left(e^{-iH_0(s-\tau)}\varphi_s \right)(0) + \int_{\mathbb{R}^3} d^3\vec{k}\, e^{-ik^2(\tau-s)} \frac{q(s)}{(2\pi)^3 k^2} =$$

$$= \left(e^{-iH_0(s-\tau)}\varphi_s\right)(0) + \frac{q(s)}{4\pi\sqrt{\pi i}\sqrt{\tau - s}}$$

Since $\varphi_s \in H^2_{\text{loc}}(\mathbb{R}^3)$, the absolute value of the first term on the right hand side is bounded by a constant $c(\tau, s) < \infty$ such that $c(s, s) = q(s)$ and

$$\lim_{\tau \to \infty} c(\tau, s) = 0$$

Hence there exists $s_1(\varepsilon) > 0$ such that, $\forall s > s_1$,

$$2\left|\int_s^t d\tau\, q^*(\tau)\left(e^{-iH_0(s-\tau)}\varphi_s\right)(0)\right| \leq \frac{2\varepsilon^2}{9}$$

if $q(t) \in L^1(\mathbb{R}^+)$. Moreover by the same reason there exists $s_2(\varepsilon) > 0$ such that $\forall s > s_2$,

$$2\left|\int_s^t d\tau\, q^*(\tau)\frac{q(s)}{4\pi\sqrt{\pi i}\sqrt{\tau - s}}\right| \leq \frac{2\varepsilon^2}{9}$$

Setting $s_0(\varepsilon) = \max(s_1(\varepsilon), s_2(\varepsilon))$, one has $\forall s > s_0$

$$\left\|\int_s^t d\tau\, q(\tau)\, U_0(t-\tau; \vec{x})\right\| \leq \frac{2\varepsilon}{3} \tag{4.12}$$

so that the whole L^2-norm of the second term in decomposition (4.10) is suitably small for $s > s_0$.
On the other hand the first term in (4.10) is the free evolution of a L^2-function and hence there exists $\delta(\varepsilon) > 0$ such that $\forall t > s + \delta$ and $\forall R < \infty$,

$$\|F(|\vec{x}| \leq R)U(t-s)\Psi_s\| \leq \frac{\varepsilon}{3} \tag{4.13}$$

Setting $t_0(\varepsilon) = s_0(\varepsilon) + \delta(\varepsilon)$, from (4.10), (4.12) and (4.13) one has

$$\|F(|\vec{x}| \leq R)\Psi_t\| \leq \varepsilon$$

$\forall t > t_0$, if $q(t) \in L^1(\mathbb{R}^+)$.
By Proposition 30 the inequality is then satisfied by every $\Psi \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$: unitarity of the family $U(t, s)$ allows to extend the result to the whole Hilbert space $L^2(\mathbb{R}^3)$.

□

Corollary 32 *If $\{\alpha_n\}$ satisfies (1.48) and the genericity condition with respect to $\mathcal{T}$ (1.56), the discrete spectrum of the Floquet operator associated to $H_{\alpha(t)}$,*

$$K \equiv -i\frac{\partial}{\partial t} + H_{\alpha(t)}$$

is empty.

Proof: The result is a straightforward consequence of Theorem 31: every eigenvector of K differs from a periodic function by a phase factor and hence can not satisfy (4.6).

□

4.4 Further Remarks

We have proved that, under the genericity condition on $\alpha(t)$, the system defined in Section 2 shows asymptotic complete ionization.
If $\inf(\alpha(t)) < 0$, the genericity condition may be a necessary condition to have complete ionization: for example, in one dimension, it is possible to exhibit (see [14]) explicit functions $\alpha(t)$ for which the genericity condition fails[6] and the ionization is not complete. On the other hand, also in one dimension, it is not known whether the condition is necessary. It would be interesting to check if non generic $\alpha(t)$ give rise to asymptotic partial ionization in three dimensions.
A possible way to investigate this problem is the analysis of the discrete spectrum of the Floquet operator. If one can find an explicit relation between existence of eigenvalues of the Floquet operator and the genericity condition, it would be probably easy to check if the condition is truly necessary.
On the other hand, as we expected, if $\alpha(t)$ is positive at any time, no further condition on $\alpha(t)$ is required to prove complete ionization.
Two interesting future applications of these methods can be the problem of complete ionization for moving point interactions and for N time-dependent point interactions.

[6] A simple example of $\alpha(t)$, for which the genericity condition is not satisfied is the geometric series, $\alpha_n = \lambda^{|n|}$ for some $\lambda < 1$.

Conclusions and Perspectives

The asymptotic properties of solutions of fractional integral equations - as it also emerges from this work - have a great relevance in the applications perspective. Unfortunately most of the results obtainable in this direction are closely connected with the tools of the Laplace transform analysis (nevertheless see the work of D.R. Yafaev [30]: "On the asymptotics of solutions of Volterra integral equations", and the book of V. Kiryakova ([31]) for Erdélyi-Kober fractional integral operators). As a matter of fact, in many relevant cases the application of Laplace transform technique does not work due to the presence of nonconstant coefficiets or nonlinearities in the integral operator, leaving the problem of the asymptotic analysis as an open question.

In this work we studied the large time asymptotic properties of equation (1.44) with $\alpha(t)$ periodic; this may be considered, to some extent, as a boundary case for the utilization of the Laplace transform. Our results have shown an interesting lack of continuity in the behavior of solutions between the "generic" and the "non generic" case[7]. For $\alpha(t)$ generic, indeed, we proved that the solution of (1.44) goes to zero as $t \to \infty$ with a negative power law. On the other hand, the same proof does not work anymore for periodic coefficients of non generic kind; in this last case, there are no general results regarding the large time behavior of the solutions, although simple examples of this type[8] are known in which equation (1.44) is explicitely solvable and the solution exhibit a non vanishing limit for $t \to \infty$.

In order to extend the analysis presented in this work to more general cases - i.e. Abel equations with time dependent bounded coefficients or of non linear type - it seems to be necessary the developing of a new investigation strategy based upon different tools such as operator analysis.

[7] In the sense expressed by relation (1.56).

[8] See the discussion for the case $\alpha = cost.$ in Section 2.4 of this work. See also [14] for a less trivial example of non generic periodic function.

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