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OPTIMALITY OF EQUILIBRIA IN DIFFERENTIAL INFORMATION ECONOMIES WITH RESTRICTED COALITIONS

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Introduction

It is well known that the set of unblocked allocations, the core, and the set of Walras allocations coincide in an atomless economy when all measurable coalitions are allowed to form. Clearly, if only a subset of the set of all coalitions is allowed to form, the set of unblocked allocations enlarges, and generally we can merely say that this larger set contains the set of Walrasian allocations.

In reality the lack of communication restricts the set of coalitions that can be formed. The purpose of our work is investigate the Core-Walras equivalence by imposing to the set of all coalitions some restrictions.

The notion of the core is based on the premise that any group of agents can cooperate and agree upon a coordinated set of actions which can then be enforced. In the context of a differential information economy, an allocation should be seen as a state-contingent allocation satisfying physical resource constraints in each information state. A central role is played by the information set of each agent. Agents which enter into a coalition contract at the ex-ante stage, i.e. before any agents receives private information, or at the interim, i.e. after each agent has received her private information. It is well known that the ex-post stage, i.e. decisions are made after the information state is known, is no different from a model with complete information. An appropriate notion of the core must take into account of whether the coalition decisions stage is ex-ante or interim.

The definition of cooperative solution concept, the core, differs with the information sharing rule used by agents in a coalition. In an economy with differential information, the set of allocation that a coalition can block depends upon the initial information and the communication opportunities of the members of a coalition.

In the main literature, there are three kinds of information sharing rule in a coalition:

i) pooling information, introduced by Wilson [50], that is an alternative allocation that a coalition prefer must be enforced in an event which they all can discern;

ii) private information, introduced first by Yannelis [51], such that the set of feasible allocations for a blocking coalition must involve a net trade of each member of the coalition that is measurable with respect to his information partition;

iii) common knowledge information, that is net trades are measurable with respect to the joint partition of all members of the coalition and agents can discern only the events in the *fine* field.

Both with the coarse and fine core problems arises associated with existence and incentive compatibility. On the contrary, it has been shown that if the economy has a finite number of traders, the private core has some interesting properties: it exists under standard continuity and concavity assumptions on utility functions, it is coalitionally incentive, i.e. there is truthful revelation of information in each coalition, and it takes into account the information superiority of traders.

From the non cooperative side, we deal with two main equilibrium concepts:

iv) the rational expectations equilibrium which is an interim concept in which prices are referred to as signals reflecting and transferring information;

v) the competitive private equilibrium, which is closer to the Walrasian equilibrium notion in the deterministic case: this non-cooperative solution concept presumes that agents maximize their ex-ante expected utility subject to their budget constraint in which information constraints, besides the classical ones, are considered.

In Rational expectations equilibria prices do not reflect the information asymmetries among agents. More precisely, assuming that each agent makes her consumption decisions according to rules which use whatever private information is available when the market takes place, one would expect equilibrium prices to depend on the state of the world and to reflect some or all of the prior private information possessed by agents.

On the contrary, competitive private equilibria exist for finite economies under the standard assumptions which guarantee the existence of Walrasian equilibria.

We will consider a differential information exchange economy obtained by introducing in the classical Arrow-Debreu model both uncertainty and asymmetries in information. In these model, uncertainty is exogenous and is represented by a measure space $(\Omega, \mathcal{F})$ where Ω denotes the finite set of all possible states of nature and $\mathcal{F}$ is the set of all possible events.

We will assume that agents make coalitional decisions at the ex-ante stage but each agent receives private information which is not publicly verifiable before consumption takes place. In particular, agents trades with the anonymous market rather than with other agents directly and it becomes very natural to require that agents' trade be measurable with respect to their private information. This notion of the core, the *Private Core*, was first introduced and studied by Allen [4], Yannelis [51], Koutsougheras and Yannelis [36].

In Section 3.4, we briefly consider the interim stage of coalitions formation. We will refer to Wilson [50] model, in which the utility function is the conditional expected utility function. We will consider the restriction that objections be coordinated on a common knowledge event. In this case, we will refer to the concept of *Fine Core*.

We characterize in terms of decentralizing prices several notions of core allocations resulting from different possible restrictions imposed to the set of blocking coalitions. Reciprocal relations among cores are also studied.

It is shown by Scmeidler [45] that if in an atomless pure exchange economy, for arbitrary $\epsilon > 0$, only coalitions with measure less than ϵ are allowed to form, we still have the identity between the set of unblocked allocations and the set of Walras allocation.

Furthermore, Grodal [27] has shown that if the coalitions which are allowed to form consists of relatively "'few"' agents, and that agents in the coalition are ϵ similar, then the core-Walras equivalence still holds.

Finally, in Vind [49] shows that if the allocation is determined by a vote among the traders, the only allocations for which we will not have few agents suggesting and voting for some other allocation are the competitive allocations.

Mas Colell [38] has attempted to give some economic interpretation of this results. Specifically, he argued that we need not tether (link) ourselves to credulity-stretching informational requirements of the idealized notion of free Edgeworthian recontract. If whatever can be done by a coalition, can be done by any arbitrarily small coalition, then one only needs a few well informed people to take us to Walrasian equilibrium. He also suggests that we think of these few as arbitrageurs. With the rest of people in the economy remaining passive, it is enough for this small, profit seeking group to do their duty and take us to equilibrium.

Another issue we have investigated in our work, starting from the reality restrictions of coalitions due by various rules imposed over the society (i.e. information, transportation, legal and institutional constraints), is the number and composition of the set of blocking coalitions.

The more general treatment of the problem under smoothness assumption for large finite pure exchange economies has been provided by Mas-Colell [37], who showed that nearly half of all coalitions block a Pareto optimal allocation which is "bounded away from being Walrasian". Since Pareto optimality of a given allocation rules out the possibility of being blocked by a coalition as well as its complement, the 50% of all coalitions represents an upper bound on the number of blocking coalitions.

Thus, the result of Mas-Colell [37], implies that if the number of individuals in an economy rises, the proportion of blocking coalitions within the entire set of coalitions would approach this upper bound.

The result of Mas-Colell [37] can be restated in an alternative manner: if one puts the uniform distribution on the space of two coalition partitions of the set of all individuals and chooses such a partition at random, then with probability one it contains a blocking coalition. This conclusion has been generalized by Greenberg and Weber [25] who considered partitions of all individuals into several coalitions. Under the uniform distribution on the set of partitions that contain a given number $J \geq 2$ of coalitions, Greenberg and Weber [25] show that the probability of such a partition to contain a coalition that blocks a given non-Walrasian allocation, is arbitrarily close to one. The extension of this result is shown in Section 3.5

Shitovitz [47] initiated the study of the number or the measure of blocking coalitions in atomless economies. He analyzed economies with a finite number of types and identified a coalition with its profile. By considering profiles that represents coalitions with the same proportion of types as in the whole economy, Shitovitz [47] proved a local result, that for every equal treatment Pareto optimal allocation which is not Walrasian, there is a ball in the type profile space around the given type profile so that nearly half of the profiles in the ball are blocking. Following the Shitovitz's approach, Graziano [23] investigated this problem in atomless economies with a continuum of commodities.

The work is organized as follows.

In Section 2.1, we formulate and extend results of Okuda and Shitovitz [39] in a differential information framework. Then, we can classifying core allocations with respect to the family of all coalitions that include one of the members of partition. Specifically, for a given coalition R, we consider the allocation that cannot be blocked by any coalition that includes (or exclude) R.

Starting from a finite partition $\mathcal{P}$ of the whole set of agents, we classify the core allocations with respect to the family of all coalitions that include one of the members of the partition.

The blocking mechanism we consider in our results depends on the measure space of agents. We start, Section 3.1, considering continuum atomless economies in which only a finite number of characteristics can be observed. For such economies, the set of traders is partitioned into a finite number of coalitions such that individuals belonging to the same coalition have identical densities of initial bundles and final bundles, the same random utilities, the same private information and priors. We define the profile of a coalition as the finite dimensional vector that valuates the weight of each type in the coalition. Then, starting from the private blocking mechanism, we define the set of all blocking profile for a fixed Pareto optimal allocation that is not a Radner equilibrium. We show that for every profile π in which the proportion of different types in the same as in the whole society, almost half of the profiles around π are privately blocking. In particular, we extend to economies with asymmetries results proved in Shitovitz [47].

In the case of finite differential information economies, Section 3.3, the cooperative characterization of Radner equilibria via private core notions is possible enlarging the coalition formation mechanism. The notion of *generalized (or fuzzy) coalition* introduced by [7] allows to show private core equivalence theorems even in finite and atomic case (see [24], [31]). In this framework, we show that for a Pareto optimal allocation of a finite differential information economy that is not a Radner equilibrium, to any symmetric fuzzy coalition there corresponds a ball centered in the coalition such that “almost half” of the coalitions it contains are privately blocking. Mainly the result follows from a suitable correspondence between blocking coalitions of the finite economy and blocking profiles of a continuum associated economy.

Finally, Section 3.4, we underly the rule played by information sharing inside a blocking coalition. An appropriate notion of the core must take account of whether the coalition decision stage is *ex-ante*, i.e., before the agents learn their types, *interim*, i.e., when every agent only knows his own type, or *ex-post*, i.e., when all types are revealed publicly.

In Section 3.5, the analogous of Theorems 3.2.1 and 3.3.4 are investigated in connection with the notion of *social coalition structure*.

Following [23], we introduce for a finite differential information economy a social coalition structure in the form of a finite set of generalized coalitions. A coalition can be formed if and only if it belongs to the given structure. Moreover, any trader is required to redistribute is initial endowment among the coalitions in the given structure. The need of imposing a social coalition structure on the society is motivated by the fact that, although many coalitions can block an allocation that is not in the private core, it is not true that such coalitions will really formed. In particular, in economies with differential information, the interest in such structures is connected with costs of communication and information that may reduce the possibility of free coalition formation.

Finally, in Chapter 4 we have found a sufficient condition for the equivalence between the following concept of supporting price:

* $x_i \succ_i x_i^*$ implies $p \cdot x_i \geq p \cdot x_i^*$,
* $x_i \succ_i x_i^*$ implies $p \cdot x_i > p \cdot x_i^*$.

where x_i^* is the optimal demand for all i. Moreover, we prove an equivalence result between:

* $z_i \succeq_i x_i$ implies $p \cdot z_i \geq p \cdot x_i$,
* $z_i \succ_i x_i$ implies $p \cdot z_i > p \cdot x_i$.

Chapter 1

The Model and Main Definitions

In this chapter we introduce the theoretical framework for studying restriction on coalition formation. By one side, we will extend some consolidated results in an economy with differential information related to Core - Walras equivalence. By the other side, we will deal with equilibria concepts for the economy $\mathcal{E}$ used throughout our study.

The organization of Chapter 1 is as follows. In Section 1.1 we set the basic economic model, describing its component. In Section 1.2 we furnish the basic definitions of the main part of the solution concepts used throughout the work. Other concepts will be introduced when required.

1.1 Economic Model

We consider a Radner-type exchange economy $\mathcal{E}$ with differential information that takes place over two time periods. At time $t = 0$ there is uncertainty about the state of nature that is going to be realized. At this period agents make contracts that may be available on the realized state of nature at time $t = 1$. At the second period consumption takes place. It is modeled by the following set:

$$\mathcal{E} = \{(\omega, \mathcal{F}); (T, \sigma, \mu); I\!\!B_+; (\Pi_t, q_t, u_t, e_t)_{t \in T}\}$$

Following [40], the exogenous uncertainty is modeled by a measurable space $(\Omega, \mathcal{F})$, where Ω denotes a finite set of states of nature and the field $\mathcal{F}$ represents the set of all events. The space of traders is described by a measure space (T, Σ, μ), where T is the set of all traders, Σ is a σ-field of all coalitions, the measure μ defines the weight of each coalition on the market. With respect to the traders space, the situations that will be significant in the sequel are those of finite and continuum economies. The former will be characterize by μ to be the counting measure over a finite set T of traders. The latter will be given by a finite atomless measure space, typically the unit interval $[0, 1]$ with its Lebesgue measure.

The physical commodity space will be represented, in each state, by an ordered separable Banach space $I\!\!B$ whose positive cone $I\!\!B_+$ is assumed to have non empty norm interior. The dual space of $I\!\!B$, denoted by $I\!\!B'$, will represent the price space.

The *initial information* of traders $t \in T$ is described by a measurable partition $\Pi_t \in \Omega$. We denote by $\mathcal{F}_t$ the field generated by Π_t. If ω_0 is the true state of nature that is going to be realized, trader t observes the member of Π_t which contains ω_0. Every trader $t \in T$ has a probability measure q_t on $\mathcal{F}$ representing his *prior beliefs*, i.e. probability conditioned by his information set.

The preference of a trader $t \in T$ is represented by a *state dependent utility function*, $u_t : \Omega \times I\!\!B_+ \to I\!\!R$. In each state $\omega \in \Omega$ and for all $t \in T$, the function $u_t(\omega, \cdot) : I\!\!B_+ \to I\!\!R$ is assumed to be continuous, concave and strictly monotone. Moreover, for all $\omega \in \Omega$ the mapping $(t, x) \to u_t(\omega, x)$ is $\Sigma \times \mathcal{B}$-measurable, where $\mathcal{B}$ is the σ-field of Borel subsets of $I\!B_+$.

The *initial endowment* of physical resources of a trader $t \in T$ is a specification of the quantity of physical commodities in each state of nature. It is represented by a function $e : T \times \Omega \to I\!B_+$ such that $e(\cdot, \omega)$ is μ-integrable in each state $\omega \in \Omega$. By integral of the function $e(\cdot, \omega) : T \to I\!B$ with respect to μ, we mean the Bochner integral as defined in [14]. To represent the fact that traders do not acquire any new information from their initial endowments, the function $e(t, .) : \Omega \to I\!B_+$ is assumed to be $\mathcal{F}_t$-measurable for μ-almost all $t \in T$. This assumption implies, since Ω is a finite set, that it is a constant function on each element of $\mathcal{F}_t$. Finally, we shall assume that $e(t, \omega) \gg 0$, for μ-almost all $t \in T$ and for all $\omega \in \Omega$. We remind that a vector $v \in I\!B_+$ is strictly positive ($v \gg 0$) if for any non zero $p \in I\!B'_+$, $p \cdot v > 0$.

For any function $x : \Omega \to I\!B_+$, we will denote by

$$h_t(x) = \sum_{\omega \in \Omega} q_t(\omega) u_t(\omega, x(\omega))$$

the *ex-ante expected utility* from x of trader t. It will represent the agent's utility function in the complete information economy associated with $\mathcal{E}$.

1.2 Definitions

In this section we furnish the definitions of the main concept analyzed in our study for the economy $\mathcal{E}$.

Specifically, we introduce here the two main equilibria concept used throughout the paper: from a cooperative point of view, the concept of private core and from the non cooperative one, the concept of competitive private equilibria. All the definition stated at this level are presented in the unified framework represented by the basic model showed in Section 1.1.

Definition 1.2.1 *A feasible private allocation for the economy $\mathcal{E}$ is a function*

$$x : T \times \Omega \to \mathbb{B}_+$$

such that

i) $x(\cdot, \omega)$ *is μ-integrable over T, for all $\omega \in \Omega$;*

ii) $x(t, \cdot)$ *is $\mathcal{F}_t$-measurable, for μ-a.e. $t \in T$;*

iii) $\displaystyle\int_T x(t, \omega)\, d\mu \leq \int_T e(t, \omega)\, d\mu$, *for all $\omega \in \Omega$.*

Condition *ii)* above is interpreted as informational feasibility of the allocation x while condition iii) refers to its physical feasibility (see [31]). Any function $x : T \times \Omega \to \mathbb{B}_+$ that satisfies conditions $i) - ii)$ is said to be a *(private) allocation*. When conditions $i) - ii)$ refer to a coalition S, we say that x is a private allocation over the coalition S.

The free disposability requirement contained in *iii)*, is usually required to ensure the existence of Radner equilibrium allocations supported by non-negative prices (see [17]). This assumption is, generally, not replaced in the main literature. See, for example, Yannelis [51], Koutsougeras and Yannelis [36], Allen and Yannelis [5]. From a technical view point and economic mean, the assumption of free disposal is required for the positiveness of prices for Radner equilibria, as showed by Glycopantis, Muir and Yannelis [21] and Einy and Shitovitz [18].

Now, we turn to give the definition of cooperative solution concept, the core, which differ with the information sharing rule used by agents in a coalition. In an economy with differential information, the set of allocation that a coalition can block depends upon the initial information and the communication opportunities of the members of a coalition. In the main literature, there are three kinds of information sharing rule in a coalition:

i) pooling information, introduced by Wilson [50], that is an alternative allocation that a coalition prefer must be enforced in an event which they all can discern;

ii) private information, introduced first by Yannelis [51], such that the set of feasible allocations for a blocking coalition must involve a net trade of each member of the coalition that is measurable with respect to his information partition;

iii) common knowledge information, that is net trades are measurable with respect to the joint partition of all members of the coalition and agents can discern only the events in the *fine* field.

Throughout our study, we have focused attention on the notion of private core, which is non-empty under appropriate assumptions. Moreover, if there is a finite number of traders, the private core is coalitionally incentive compatible. In sections following, we will briefly show some results related to the different cooperative concept discussed above.

Definition 1.2.2 *A coalition $S \in \Sigma$ with $\mu(S) > 0$ privately blocks an allocation $x : T \times \Omega \to I\!B_+$, if there exists a private allocation y over S such that:*

i) $\displaystyle\int_S y(t,\omega) \leq \int_S e(t,\omega)$, *for all $\omega \in \Omega$;*

ii) $h_t(y(t,\cdot)) > h_t(x(t,\cdot))$, *for μ-a.e. $t \in S$.*

The *private core* of the economy $\mathcal{E}$, $\mathbf{C}_p(\mathcal{E})$, is accordingly defined as the set of all feasible private allocations that are not privately blocked by any coalition. In other words, if it is not possible for agents to join a coalition, redistribute their endowment among themselves and using his own private information to obtain a strictly preferred allocation for each member of the coalition The notion of private core, introduced in [51], is the most appropriate when traders have no access to any communication system, and are not able to share their own informations.

Definition 1.2.3 *The feasible private allocation* $x : T \times \Omega \to I\!B_+$ *is Pareto Optimal if it cannot be privately blocked by the full coalition of traders.*

It is clear that allocations in the private core are Pareto optimal. The characterization of private core and Pareto optimal allocations in terms of supporting prices is possible via the notion of efficient and competitive prices.

Definition 1.2.4 *A price system is a non-zero function* $p : \Omega \to I\!B'_+$.

Let us introduce for any trader $t \in T$ the set M_t formed by all assignments reflecting his private information, that is:

$$M_t = \{x : \Omega \to I\!B_+ : x \text{ is } \mathcal{F}_t\text{-measurable}\}.$$

Definition 1.2.5 *A non-zero price system p is an efficient price vector for the allocation* $x : T \times \Omega \to I\!B_+$ *if:*

i) μ*-a.e. in* T *the function* $x(t, \cdot)$ *is the maximal element of* h_t *in the efficiency set*

$$B^*_t(p) = \left\{ z : z \in M_t \text{ and } \sum_{\omega \in \Omega} p(\omega) \cdot z(\omega) \leq \sum_{\omega \in \Omega} p(\omega) \cdot x(t, \omega) \right\}.$$

After the definition of price system for the economy $\mathcal{E}$ we can proceed to the non-cooperative context by furnishing the notion of competitive private equilibria

Definition 1.2.6 *Let p be a non-zero price system and x be a feasible private allocation. The pair* (x, p) *is said to be a Radner equilibrium if*

i) *μ-a.e. in T the function $x(t, \cdot)$ is the maximal element of h_t in the budget set*

$$B_t(p) = \left\{ z : z \in M_t \text{ and } \sum_{\omega \in \Omega} p(\omega) \cdot z(\omega) \leq \sum_{\omega \in \Omega} p(\omega) \cdot e(t, \omega) \right\};$$

ii) $\sum_{\omega \in \Omega} p(\omega) \cdot \int_T x(t, \omega)\, d\mu = \sum_{\omega \in \Omega} p(\omega) \cdot \int_T e(t, \omega)\, d\mu.$

Usually condition *ii)* is not used in the definition of supporting prices. We add it because the definition of feasible private allocation allows free disposability.

Clearly, under an efficient or a competitive price system, agents maximize ex-ante their expected utilities over their budget sets independently one to each other. Moreover, given the structure of the sets $B_t^*(p)$ and $B_t(p)$, the notion of supporting price system takes into account the better information of an agent. Indeed, agents that are better informed will be in general better off.

Definition 1.2.7 *We say that a feasible private allocation $x : T \times \Omega \to I\!B_+$ satisfies the smoothness assumption if, aside from scalar multiples, there exists a unique efficient price vector p for x.*

Throughout the paper there will be some technical modification of this model, justified by a deeper analysis of an economy with differential information and its equilibria concepts. In particular, in Chapter 2, we consider a finite and measurable partition $\mathcal{P} = (R_1, ..., R_k)$ of the grand coalition, with k large enough. Starting from this partition we define a new cooperative concept, which is more general.

In Chapter 3, we shall limit consideration to continuum atomless economy in which it is possible to distinguish only finitely many different traders' types [47]. Precisely, we will assume that the set T can be partitioned into a finite number of coalitions, $S_1, ..., S_m$, such that

- $e(t, \omega) = e_i(\omega), \forall \omega \in \Omega$ and $t \in S_i$;
- $u_t(\omega, x) = u_i(\omega, x), \forall t \in S_i$, $x \in I\!B_+$ and $\omega \in \Omega$;

- $\mathcal{F}_t = \mathcal{F}_i, \ \forall t \in S_i$.

Moreover, we will use different information sharing rules which inflects consequences over equilibria measurability.

It is very clear, that the partition considered in Chapter 2 is more general, and it includes the partition considered in Chapter 3.

Chapter 2

The Restricted Mechanism of Coalition Formation

The restriction of coalition formation is inflated by incomplete information. In an economy with N people, a person will only know the preferences and endowments of a subset $K \subseteq N$ of people and can decide only to form coalitions with people from this group. There is an upper maximum to the size of possible coalitions in the economy. Moreover, another interpretation can be the presence of transaction costs to coalition formation.

We want investigate how people can aggregate and form a coalition. It becomes more real to think on a limit of the size of a coalition. In other word, we take in account of some difficulties to join a coalition. We consider an ϵ-core concept. We think about an economy with uncertainty and differential information.

There are some consequences of placing an upper limit on the size of possible coalitions. Intuitively the core will be larger. We call a core with an upper maximum a *restricted core*.

The first study on this direction were made by Schmeidler [45], Vind [49] and Grodal [27].
Schmeidler's theorem [45] says that if we have a coalition S which blocks an allocation $x(t, \omega)$ with an allocation $y(t, \omega)$, then we can find an arbitrarily small sub-coalition E that can also blocks allocation x with y. Thus, in an atomless economy, $\mathrm{C}(\mathcal{E}) = \mathrm{C}_{\epsilon}(\mathcal{E})$ for any ϵ.

Schmeidler's theorem is surprising for it says, in a continuum economy, that the work that is done by huge coalitions can be also done by very small coalitions.

Schmeidler's original results were strengthened by Vind [49], he demonstrated that if there is a coalition S which can block allocation x with the allocation y, then not only there is a smaller coalition which blocks that allocation, but we can find a smaller coalition of any size to block it, i.e. for any δ where $\delta \leq \mu(S)$, we can find a coalition T of size $\mu(T) = \delta$ that can block allocation x with allocation y.

Brigit Grodal [27] imposing a different type of restriction: she not only restricted the size but also the composition of the coalition. She restricted coalitions to a radius of *neighboring* agents, i.e. people with similar preferences and endowments. She showed that we obtain the same result for a continuum economy: any allocation x that can be blocked by a coalition S can be blocked by a smaller coalition of less diverse people.

Mas Colell [38] has attempted to give some economic interpretation of this results. Specifically, he argued that we need not tether (link) ourselves to credulity-stretching informational requirements of the idealized notion of free Edgeworthian recontract. If whatever can be done by a coalition, can be done by any arbitrarily small coalition, then one only needs a few well informed people to take us to Walrasian equilibrium. He also suggests that we think of these few as arbitrageurs. With the rest of people in the economy remaining passive, it is enough for this small, profit seeking group to do their duty and take us to equilibrium.

In this section, we formulate and extend results of Okuda and Shitovitz [39] in a differential information framework. Then, we can classifying core allocations with respect to the family of all coalitions that include one of the members of partition.

2.1 Restricted Process

We have remarked that the formation of a coalition may imply some theoretical difficulties. It is not suffice to say that a coalition can be formed by several agents. We must take into account all limits imposed by society to the aggregation in coalition. It is very simple to thing that agents are not free to form any coalition, especially in our framework. In fact, it is usually argued that the costs, which arise from forming a coalition, are not all negligible. Moreover, traders will form a coalition only if they know each other. Incompatibilities among different agents may arise and a big amount of information an communication might be needed to form a coalition. Thus, it will be not enough to say merely that several agents form a coalition.

We define a set of all possible coalition as the set of those coalition that can be formed and joint by any agent. There exists, in this way, a rule imposed over coalition formation. We assume that only a subset $\mathcal{S}$ of Σ are allowed to be formed. In such way, we fix over the set of agents a rule of aggregation for which the coalitions can be formed only if belonging to this subset. We have restricted the set of coalitions that can be joined by traders.

A coalition S is a measurable subset of T, such that $\mu(S) > 0$ which represents the size of coalition S. In the case of atomless economy, the size of a coalition S can be interpreted, following [45], as the amount of information and communication, or costs, needed in order to form the coalition S. Then, may be meaningfully to consider those coalitions whose size converges to zero or, symmetrically, to one; that is, the coalitions that do not involve high costs can be formed.

The difficulty to argue that coalition formation is costless leads to consider a restricted mechanism. That is, we restrict the set of coalitions considering a subset $\mathcal{S} \in \Sigma$ of all admissible coalitions. Following [10], we introduce a new concept of core solution in a private framework that we call $\mathcal{S}$-private core.

Definition 2.1.1 *Let $\mathcal{S} \in \Sigma$ be the subset of all admissible coalitions,* *with $\mu(S) > 0$ for every $S \in \mathcal{S}$. A feasible allocation $x(t,\omega)$ belongs to the $\mathcal{S}$-private core of $\mathcal{E}$ if it is not privately blocked by any coalition $S \in \mathcal{S}$.*

We denote this core as $\mathcal{S}$-$\mathbf{C}_p(\mathcal{E})$. This core concept is a generalization of the core defined in definition 3.4.1. In particular, if $\mathcal{S} = \Sigma$ then two concepts coincide.

In each coalition S belonging to the subset $\mathcal{S}$ agents do not share their information, accordingly with definition of private allocation. Traders joint a coalition which belongs to $\mathcal{S}$, and they choose a private allocation over S which improves upon the allocation x.

From the definition of $\mathcal{S}$-core given $\mathcal{S}_1$, $\mathcal{S}_2 \subseteq \Sigma$ we can easily infer the following properties:

i) if $\mathcal{S}_1 \subseteq \mathcal{S}_2$ then $\mathcal{S}_2$-$\mathbf{C}_p(\mathcal{E})\subseteq\mathcal{S}_1$-$\mathbf{C}_p(\mathcal{E})$;

ii) $\mathcal{S}_1$-$\mathbf{C}_p(\mathcal{E})\cap\mathcal{S}_2$-$\mathbf{C}_p(\mathcal{E}) = (\mathcal{S}_1 \cup \mathcal{S}_2)$-$\mathbf{C}_p(\mathcal{E})$

From the property i) it is deduced that if the private core is non-empty, then so is the $\mathcal{S}$-private core. The property ii) implies that if $\Sigma = \bigcup_i \mathcal{S}_i$, then $\bigcap_i(\mathcal{S}_i - \mathbf{C}_p(\mathcal{E})) = \mathbf{C}_p(\mathcal{E})$. That is, for any partition P of the whole coalition set Σ the allocations belonging to the private core are those allocations that belong to every $\mathcal{S}$-private core, with $\mathcal{S} \in P$, and the intersection of the $\mathcal{S}$-private cores of a partition P does not depend on P.

In this framework, we can replace results of Schmeidler, Vind and Grodal [1972]. The restricted mechanism we have defined above, in fact, allows to formalize Schmeidler [45] result in terms of $\mathcal{S}$-private core. Vind's result can also be formulated in terms of $\mathcal{S}$- private core. Precisely, if for almost all $t \in T$ the preference relation $\succeq_t$ is continuous, monotone and measurable, then $\mathcal{S}_\epsilon$-$\mathbf{C}_p(\mathcal{E}) = \mathbf{C}(\mathcal{E})$.

2.2 Some new definitions

Given a fixed coalition $R \in \Sigma$, let

$$\mathcal{Q}_R = \{S \in \Sigma : R \subseteq S\}$$

be the set of all coalitions which contain R. This structure define the only coalitions that can be formed as those containing R.
Define with $T \backslash \mathcal{Q}_R = \{S \in \Sigma : R \cap S = \phi\}$.

If $\mathcal{P}$ is any partition of the whole set Σ, then the allocation belonging to the core are those that belong to every $\mathcal{C}(\mathcal{S})$ with $\mathcal{S} \subseteq \mathcal{P}$.

Now we define the appropriate core concept for these information structure:

Definition 2.2.1 *Let R be a fixed coalition. An allocation $x(t,\omega)$ is said to belong to the R-inclusive private core if it cannot be privately improved upon by any coalition $S \in \mathcal{Q}_R$; i.e. if there is no coalition S and an assignment y, $\mathcal{F}_t$-measurable, $y : S \times \Omega \to I\!\!B_+$ such that $R \subseteq S$, $\mu(S) > 0$, $\int_S y(t,.)d\mu \leq \int_S e(t,.)d\mu$ and $h_t(y(t,.)) > h_t(x(t,.))$ for almost every t in S.*

Definition 2.2.2 *A feasible allocation $x(t,\omega)$ is individually rational if $h_t(x) \geq h_t(e)$ for almost every t in T.*

Definition 2.2.3 *A non-zero vector $p : \Omega \to I\!\!B'_+$ is an efficient price vector for the allocation $x(t,\omega)$ if μ a.e. in T, $x(t,\omega)$ is the maximal element of h_t over the efficiency set*

$$B^*_t(p) = \left\{ z \in M_t \mid \sum_{\omega\in\Omega} p(\omega)\cdot z(\omega) \leq \sum_{\omega\in\Omega} p(\omega)\cdot x(t,\omega) \right\}.$$

We denote the cone of all efficiency price vectors for an allocation $x(t,\omega)$ by

$$P(x, \succ_t) = \left\{ p \in I\!\!B'_+ : x \succ_t y \Rightarrow \sum_{\omega\in\Omega} p(\omega)\cdot x(t,\omega) \geq \sum_{\omega\in\Omega} p(\omega)\cdot y(t,\omega) \right\}$$

and its linear dimension by $r = \dim P$[1].

[1] As shown in Grodal [27], it is always true that the linear dimension of the cone P of the efficiency price vectors $r \leq l$, where l is the number of commodities in the market, and that under classical assumption of differentiability and interiority $r = 1$

We consider a finite and measurable partition $\mathcal{P} = (R_1, ..., R_k)$ of the grand coalition, with k large enough[2]. We have defined in the previous section, the concept of R-inclusive core, or $\mathcal{S}$-inclusive core. We prove that an optimal allocation x belongs to the core if and only if it cannot be improved upon by any coalition of the subset $\mathcal{S}$ that includes at least one of the R_i.

Lemma 2.2.4 *Let $x(t,\omega)$ be a strictly positive allocation for almost all $t \in T$ and for all $\omega \in \Omega$ and let p be a non negative price, $p \in I\!B'_+$. Then (p,x) is an efficient equilibrium if and only if $p \cdot G^*(t) \geq 0$ for almost all traders t.*

PROOF: The first implication is trivial: if (p,x) is an efficient equilibrium, necessarily $p \cdot G^*(t) \geq 0$.

Conversely, suppose that there exists a supporting price for the set $G^*(t)$. We want to show that $x(t,\omega)$ is the maximal element of the efficiency budget set $B_t^*(p)$: i.e, all $z(t,\omega)$ such that $h_t(z) > h_t(x)$ for almost all $t \in T$ does not belong to the efficiency budget set $B_t^*(p)$.

Suppose that $z \in B_t^*(p)$, then $\sum\limits_{\omega\in\Omega} p(\omega)\cdot z(\omega) \leq \sum\limits_{\omega\in\Omega} p(\omega)\cdot x(t,\omega)$. By continuity, there exists $\alpha < 1$ such that $h_t(\alpha z) > h_t(x)$ for almost all $t \in T$, therefore, $\sum\limits_{\omega\in\Omega} p(\omega)\cdot \alpha z(\omega) > \sum\limits_{\omega\in\Omega} p(\omega)\cdot x(t,\omega) \geq \sum\limits_{\omega\in\Omega} p(\omega)\cdot z(\omega) \geq \sum\limits_{\omega\in\Omega} p(\omega)\cdot \alpha z(\omega)$. Since $\alpha > 1$, $\sum\limits_{\omega\in\Omega} p(\omega)\cdot z(\omega) = \sum\limits_{\omega\in\Omega} p(\omega)\cdot x(t,\omega) = 0$. But $x(t,\omega) \gg 0$ for almost all $t \in T$, then $p = 0$ for almost all $t \in T$, which is a contradiction. Then, x is the maximal element of the efficient budget set. □

Lemma 2.2.5 *For a given allocation $x(t,\omega)$, let F be a set-valued function such that $G^*(t) \subseteq F(t)$ for almost all traders t. If p is a non negative price such that $p \cdot \int F \geq 0$, then*

i) *(p,x) is an efficiency equilibrium,*

ii) *$p \cdot f(t) \geq 0$ for all integrable selection f and almost all $t \in T$.*

[2] We refer to Okuda and Shitovitz [39]

PROOF: For each $z(t,\omega) \in ...$, let $G^{*-1}(z) = \{t : z \in G^*(t)\}$ be the set of all agents t for which the allocation z belongs to the preferred set. Define with $G^{*-1}(x) = \{t : h_t(z+x) > h_t(x)\}$, this set is measurable for each x. Let N be the set of all rational points $r \in \Re^l$ for which $G^{*-1}(r)$ is null. Obviously, N is denumerable. Define with $S = \bigcup_{r\in\Re^l} G^{*-1}(r)$. Then S is a null coalition. Suppose that for some $t \notin S$, there is a bundle $z(t,\omega) \in G^*(t)$ with $\sum_{\omega\in\Omega} p(\omega)\cdot[z(t,\omega) - x(t,\omega)] < 0$. By continuity, we may find a rational point $r \in G^*(t)$ sufficiently close to z,so that we still have $\sum_{\omega\in\Omega} p(\omega)\cdot r < 0$. Defined with $A = G^{*-1}(r)$ then $\mu(A) > 0$. By desirability, for each $\epsilon > 0$, we have an integrable selection $f = r\chi_A + \epsilon e\chi_{T\setminus A}$ from $G^*(t)$. Hence, $f \in F(t)$. Therefore

$$0 \leq \sum_{\omega\in\Omega} p(\omega)\cdot \int f = \sum_{\omega\in\Omega} p(\omega)\cdot r\mu(A) + \epsilon \sum_{\omega\in\Omega} p(\omega)\cdot e(t,\omega)\mu(T\setminus A) \longrightarrow_{\epsilon\to 0} \sum_{\omega\in\Omega} p(\omega)\cdot r\mu(A) < 0$$

a contradiction. Therefore, $\sum_{\omega\in\Omega} p(\omega)\cdot G^*(t) \geq 0$ for almost all traders t, and by Lemma 2.2.4, (p,x) is an efficiency equilibrium.

Let f be an integrable selection from $F(t)$. Define with $A = \left\{t : \sum_{\omega\in\Omega} p(\omega)\cdot f(t,\omega) > 0\right\}$, then, for each $\epsilon > 0$, the integrable function $f = r\chi_A + \epsilon e\chi_{T\setminus A}$ belongs to $F(t)$. Therefore $0 \leq \sum_{\omega\in\Omega} p(\omega)\cdot \int f = \sum_{\omega\in\Omega} p(\omega)\cdot \int_A f + \epsilon \sum_{\omega\in\Omega} p(\omega)\cdot e(t,\omega)\mu(T\setminus A) \longrightarrow_{\epsilon\to 0} \sum_{\omega\in\Omega} p(\omega)\cdot \int_A f$. Therefore, $\sum_{\omega\in\Omega} p(\omega)\cdot \int_A f \geq 0$, which implies by the definition of A that $\mu(A) > 0$. This completes the proof of the Lemma.

$\square$

2.3 The equivalence $\mathcal{C}_p(\mathcal{E}) = \mathcal{S} - \mathcal{C}_p(\mathcal{E})$

Proposition 2.3.1 *Let $x(t,\omega)$ be an individually rational allocation. Then x is supported by a price p ($p \neq 0$) if and only if x is a Pareto optimal allocation.*

PROOF: By contrary, suppose that x is not a Pareto optimal allocation. Then there exists an allocation $y : T \times \Omega \to \mathbb{B}_+$, with $y(t,\omega) \in M_t$ such that $\int_T y(t,.) \leq \int_T e(t,.)$ and $h_t(y) > h_t(x)$ for almost all $t \in T$. For assumption, there exists a supporting price $p : \Omega \to \mathbb{B}'_+$ such that $\sum\limits_{\omega\in\Omega} p(\omega) \cdot y(t,\omega) > \sum\limits_{\omega\in\Omega} p(\omega) \cdot x(t,\omega)$. By integrability, $\int_T p(.) \cdot y(t,.) > \int_T p(.) \cdot x(t,.)$ and y is feasible. The contradiction follows.

For the converse, we define a correspondence G from T in $\mathbb{B}_+$ by setting for all $t \in R_i$, $G(t) = \{z \in M_t : h_t(z(\cdot)) > h_t(x(t,\cdot))\}$ and we denote by $G^*(t)$ the correspondence defined by $G^*(t) = G(t) - x(t,\cdot)\ \forall t \in R_i$. Under classical assumption these sets are convex. We can define the integral $\int_T G^*(t)$ which is convex, and by Pareto optimal assumption, we know that $0 \notin \int_T G^*(t)$. Therefore, by Separation hyperplane Theorem, there exists a price $p \neq 0$ such that $p \cdot \int_T G^* \geq 0$, i.e. (p,x) is an efficient equilibrium. □

Theorem 2.3.2 *Let $x(t,\omega)$ be an individual rational allocation with $r = dimP$ and let $\mathcal{P} = (R_1, ..., R_k)$ be a measurable partition of T. If $k \geq r+1$, then x belongs to the core if and only if x belongs to each R_i-inclusive core for all i, $i = 1, ..., k$.*

The proof of our results needs the following result:

Theorem 2.3.3 *Let $x(t,\omega)$ be an individual rational allocation and let R be a fixed coalition whose complement $T \backslash R$ is atomless. Then x belongs to the R-inclusive core if and only if there exists an efficiency price vector $p(\omega)$ such that $\sum\limits_{\omega\in\Omega} p(\omega) \cdot x(t,\omega) \leq \sum\limits_{\omega\in\Omega} p(\omega) \cdot e(t,\omega)$ for almost each t in $T \backslash R$.*

PROOF: For the proposition 2.3.1 x is a Pareto optimal allocation, than x is in the R-inclusive core. Let us looking at the "only if" part. For the proposition 2.3.1 and for hypothesis x is a Pareto optimal allocation and there exists a price p such that (p, x) is an efficient equilibrium on T.

Define with $F(t)$ the correspondence:

$$F(t) = \begin{cases} G(t) \text{ for } t \in R \\ G(t) \cup [e(t,\omega) - x(t,\omega)] \text{ otherwise} \end{cases}$$

where $G^*(t) = \{z(\omega) - x(t,\omega) | z(\omega) \in M_t \text{ and } h_t(z(\omega)) > h_t(x(t,\omega))\}$, $\forall\ t \in T$. By Pareto optimality $0 \notin \int F(t)$.

From supporting Theorem there exists a price $p : \Omega \to \mathbb{B}'_+$ such that $\sum\limits_{\omega\in\Omega} p(\omega) \cdot \int F(t) \geq 0$ and (p, x) is an efficient equilibrium. By monotonicity , there exist a measurable and integrable selection $f(t, .) = (e(t, .) - x(t, .))_{\chi_{T\setminus R}} + z(.)_{\chi_R}$, with $f(t, .) \in F(t)$ for almost all $t \in T$. Therefore, by lemma 2.2.5 $0 \leq p \cdot f(t, .) = p \cdot e(t, .) - p \cdot x(t, .)$ for almost all $t \in T \setminus R$ □

Let us try to give an interpretation. If we consider a partition of T into two sets, namely R and its complement, non atomic, we will say that an individually rational allocation, and in particular a core allocation, belongs to the R-inclusive core if and only if it can be possible for individuals belonging to $T \setminus R$ to chose the efficiency price vector $p(\omega)$, in each state of nature, so that the value of their bundle is less than or equal to the value of initial bundle. So that, despite of the measure of the fixed coalition R, agents in R are not willing to leave this coalition to join its complement and to gain.

Now we can show the demonstration of the main theorem:

PROOF: (Theorem 2.3.2)

Suppose that x belongs to each R_i-inclusive core. By theorem 2.3.3 there are efficient price vectors $p_i \geq 0$ for x, one for each R_i such that:

$$\sum_{\omega\in\Omega} p_i(\omega) \cdot x(t,\omega) \leq \sum_{\omega\in\Omega} p_i(\omega) \cdot e(t,\omega)$$

$\forall\ \ i = 1, ...k$ and for almost all $t \in T \setminus R_i$. Such $p_i(\omega)$ are linearly dependent for all $\omega \in \Omega$, i.e., there exist $\alpha_1(\omega), ...\alpha_k(\omega)$ not all vanishing, with $\sum_{i=1}^{k} \alpha_i(\omega)p_i(\omega) = 0$ for all $\omega \in \Omega$. Let $I^+ = \{j : \alpha_j(\omega) > 0\}$ and $I^- = \{j : \alpha_j(\omega) < 0\}$. Since $p_i \geq 0$ for all $i = 1, ..., k$, I^+ and $I^-$are both nonempty. Let us define P by

$$P(.) = \sum_{i \in I^+} \alpha_i(.)p_i(.) = \sum_{i \in I^-} (-\alpha_i)(.)p_i(.)$$

P is the competitive price vector for x. Indeed,

i) P is an efficient price vector for x since by definition P is a convex cone.

ii) $\sum_{\omega \in \Omega} P(\omega) \cdot x(t, \omega) \leq \sum_{\omega \in \Omega} P(\omega) \cdot e(t, \omega)$ for almost each $t \in T$. In fact, let t be in T. Since $(R_1, ..., R_k)$ is a partition of T, there exists i_0 such that $t \in R_{i_0}$. Assume, w.l.o.g., that $i_0 \notin I^+$. Therefore, for every $j \in I^+$, we have $j \neq i_0$, in particular $t \notin R_j$ and therefore, by definition of the $p_j(.)$, we have $\sum_{\omega \in \Omega} p_j(\omega) \cdot x(t, \omega) \leq \sum_{\omega \in \Omega} p_j(\omega) \cdot e(t, \omega)$. Since $\alpha_j(\omega) > 0$ for $j \in I^+$, we have $\sum_{\omega \in \Omega} \alpha_j(\omega)p_j(\omega) \cdot x(t, \omega) \leq \sum_{\omega \in \Omega} \alpha_j(\omega)p_j(\omega) \cdot e(t, \omega)$. Summing over I^+, we obtain the inequality

$$\sum_{\omega \in \Omega} P(\omega) \cdot x(t, \omega) = \sum_{\omega \in \Omega} \sum_{j \in I^+} \alpha_j(\omega)p_j(\omega) \cdot x(t, \omega) \leq$$
$$\sum_{\omega \in \Omega} \sum_{j \in I^+} \alpha_j(\omega)p_j(\omega) \cdot e(t, \omega) = \sum_{\omega \in \Omega} P(\omega) \cdot e(t, \omega).$$

for almost each $t \in T$.

Now, by Theorem 2.3.3, x is a core allocation. $\square$

Chapter 3

The Measure of the Set of Blocking Private Coalition

It is well known that in the case of finite economies that are "large enough" or in the case of atomless exchange economies, the set of allocations for which blocking coalitions do not exist (core allocations) coincides with the set of Walras (or competitive) allocations. As a natural consequence, many authors have investigated the interesting problem of valuating the number or the "proportion" of coalitions potentially blocking a non competitive allocation. Mas-Colell in his paper [37] showed that any Pareto optimal allocation which is "bounded away from being competitive" in a differentiable pure exchange economy can be blocked by a number of coalitions which is arbitrarily closed to one half of the total number of coalitions. Related results in large finite economies have been proved in Greenber and Weber [25] and Graziano [23]. In the case of atomless economies, the problem of the measure of blocking coalitions is investigated in Shitovitz [47] and Grodal [28].

Private core equivalence results proved in Einy et al. [17], Hervés-Beloso et al. [31] and [32], Graziano and Meo [24], show that in economies with differential information the set of Radner equilibrium allocations is equivalent to some private core notion. Consequently, for any allocation that is not a Radner equilibrium, there exists a privately blocking coalition. Going further, we investigate the problem of the measure of coalitions that privately block a non-competitive allocation. Related results in the case of economies with complete information are covered by papers of [47], [25], [28], [23]. They all originate from the question first raised in Mas-Colell [37], in which the author asked for the number of coalition that blocks a certain given Pareto optimal allocation which is "bounded away from being competitive". The starting point there is the equivalence for finite economies between the core and the set of competitive equilibria when the number of agents is large enough.

The blocking mechanism we consider in our results depends on the measure space of agents. We start, Section 3.1, considering continuum atomless economies in which only a finite number of characteristics can be observed. For such economies, the set of traders is partitioned into a finite number of coalitions such that individuals belonging to the same coalition have identical densities of initial bundles and final bundles, the same random utilities, the same private information and priors. We define the profile of a coalition as the finite dimensional vector that valuates the weight of each type in the coalition. Then, starting from the private blocking mechanism, we define the set of all blocking profile for a fixed Pareto optimal allocation that is not a Radner equilibrium. We show that for every profile π in which the proportion of different types in the same as in the whole society, almost half of the profiles around π are privately blocking. In particular, we extend to economies with asymmetries results proved in Shitovitz [47].

In the case of finite differential information economies, Section 3.3, the cooperative characterization of Radner equilibria via private core notions is possible enlarging the coalition formation mechanism. The notion of *generalized (or fuzzy) coalition* introduced by [7] allows to show private core equivalence theorems even in finite and atomic case (see [24], [31]). In this framework, we show that for a Pareto optimal allocation of a finite differential information economy that is not a Radner equilibrium, to any symmetric fuzzy coalition there corresponds a ball centered in the coalition such that "almost half" of the coalitions it contains are privately blocking. Mainly the result follows from a suitable correspondence between blocking coalitions of the finite economy and blocking profiles of a continuum associated economy.

Finally, Section 3.4, we underly the rule played by information sharing inside a blocking coalition. An appropriate notion of the core must take account of whether the coalition decision stage is *ex-ante*, i.e., before the agents learn their types, *interim*, i.e., when every agent only knows his own type, or *ex-post*, i.e., when all types are revealed publicly.

3.1 The setting

The equivalence theorems for the set of Radner equilibrium allocations and the private or the Aubin private core are the starting point of this section. Given a Pareto optimal allocation that is not a Radner equilibrium allocation, equivalence theorems ensure that there exists for this allocation a privately blocking coalition (or a generalized coalition in the case of finite economy). Our aim is to evaluate the measure of the set of coalitions privately blocking the given allocation.

As we have defined in previous sections, we shall limit consideration to continuum atomless economies in which it is possible to distinguish only finitely many different traders' types. For this purpose, we need the following

Assumption 3.1.1 *A private allocation $x : T \times \Omega \to \mathbb{B}_+$ satisfies the finiteness assumption, if there exist the measurable functions $x_i \in M_i$, $i = 1 \ldots m$, such that $x(t, \omega) = x_i(\omega)$ for each $t \in S_i$ and $\omega \in \Omega$.*

The *profile* of a coalition $S \subseteq T$ is defined as the vector

$$\pi\,(S) \equiv (\pi_i(S))_{i=1}^m = (\mu(S \cap S_i))_{i=1}^m$$

that evaluates the weights of the different types in the coalition S.

Let us denote by $f = (f_1., ..., f_m)$, with $f_i = \mu(S_i) > 0$, the profile of the full coalition T. Due to the non-atomicity of the Lebesgue measure μ, the set of all the profiles of coalitions in T is the closed interval $\Pi \equiv [0, f] \subseteq \mathbb{R}_+^m$. We say that a coalition S is *symmetric* if there exists $\alpha \in (0, 1)$, such that $\pi(S) = \alpha f$. Finally, we call the support of a profile $\pi \in \Pi$ the set $supp\,\pi = \{i \in \{1, \ldots, m\} : \pi_i > 0\}$.

For a given profile $\pi \in \Pi$, let S be any coalition with $\pi\,(S) = \pi$. Given a feasible private allocation x with the finiteness assumption, we denote by E and X the functions defined on $\Pi \times \Omega$ with values in $\mathbb{B}_+$ defined by:

$$E(\pi, \omega) \equiv \sum_{i=1}^{m} \pi_i \cdot e_i(\omega) = \sum_{i=1}^{m} \pi_i(S) \cdot e_i(\omega) = \int_S e(t, \omega)\, d\mu,$$

$$X(\pi,\omega) \equiv \sum_{i=1}^{m} \pi_i \cdot x_i(\omega) = \sum_{i=1}^{m} \pi_i(S) \cdot x_i(\omega) = \int_S x(t,\omega)\, d\mu.$$

We define a correspondence G from T in $I\!\!B_+^\Omega$ by setting for all $t \in S_i$,

$$G(t) = \{z \in M_i : h_i(z(\cdot)) > h_i(x_i(\cdot))\}$$

and denote by $G^*(t)$ the correspondence defined by

$$G^*(t) = G(t) - x_i(\cdot) \quad \forall t \in S_i.$$

Note that $G(t)$, $G^*(t)$ e $G^*(t)+I\!\!B_+^\Omega$ are all convex sets. Denote by $G(S)$ and $G^*(S)$ the Aumann integrals of the correspondences G and G^* over the coalition S (see [14]). For any profile $\pi \equiv (\pi_1, ..., \pi_m)$, denote by $G(\pi)$ the convex set

$$G(\pi) = \sum_{i=1}^{m} \pi_i \cdot \{z \in M_i : h_i(z(\cdot)) > h_i(x_i(\cdot))\}.$$

3.1.1 Some preliminary "technical" results

Before establishing the main Theorems of the Chapter 3, we need some preliminary results.

Lemma 3.1.2 *Let $S \subseteq S_i$ be a coalition with $\mu(S) > 0$, $h : S \times \Omega \to I\!\!B_+$ be a function such that $h_i(h(t,\cdot)) > h_i(x_i(\cdot))$ for every $t \in S$. Then the function*

$$\bar{h}(\cdot) = \frac{1}{\mu(S)} \int_S h(t,\cdot)\, d\mu$$

satisfies $h_i(\bar{h}(\cdot)) > h_i(x_i(\cdot))$.

PROOF: The statement follows from [31, Lemma 3.1]. Note that this Lemma is stated for $I\!\!B$ finite dimensional but its proof is valid for the general commodity space considered here. □

We list in the following Lemma some properties of the correspondence $G(\pi)$.

Lemma 3.1.3 *The correspondence $G : [0, f] \to I\!\!B_+^{\Omega}$ satisfies the properties:*

i) $G(\pi) = G(S)$, *for any coalition S such that $\pi(S) = \pi$;*

ii) $G(\alpha\pi) = \alpha G(\pi)$, *for any $\pi \in [0, f]$ and $\alpha \in (0, 1)$;*

iii) $G(\pi_1) + G(\pi_2) = G(\pi_1 + \pi_2)$, *for any π_1 and $\pi_2 \in [0, f]$ such that $\pi_1 + \pi_2 \in [0, f]$.*

PROOF: It is clear that $G(\pi) \subseteq G(S)$. To show *i*), consider an integrable selection h of the correspondence G over the coalition S. Then $h(t,\cdot) \in M_t$ for μ-almost all $t \in S$ and $h_i(h(t,\cdot)) > h_i(x_i(\cdot))$, for μ-almost all $t \in S \cap S_i$, $i = 1, \ldots, m$. Denote by A the support of π. Then

$$\int_S h(t,\cdot)\, d\mu = \sum_{i \in A} \int_{S \cap S_i} h(t,\cdot)\, d\mu = \sum_{i \in A} \pi_i \cdot \left(\frac{1}{\pi} \int_{S \cap S_i} h(t,\cdot)\, d\mu \right) \in G(\pi),$$

where the last inclusion follows from Lemma 3.1.2.

The property $ii)$ is obvious. Property $iii)$ immediately follows from the linearity of the sum, the concavity of the expected values and the convexity of set M_i. $\square$

Lemma 3.1.4 *Let $S \subseteq R$ be coalitions such that supp $\pi(S) =$ supp $\pi(R \setminus S)$. Then $G^*(S) \subseteq G^*(R)$.*

PROOF: Clearly the claim is proved if the support of S is empty. Assume that $\mu(S) > 0$. Let a be an element of $G^*(S)$ and denote by A the set *supp* $\pi(S) =$ *supp* $\pi(R \setminus S)$. Lemma 3.1.3 allows us to write

$$a(\cdot) = \sum_{i \in A} \pi \cdot h_i(\cdot) - \sum_{i \in A} \pi \cdot x_i(\cdot)$$

where $h_i(\cdot) \in M_i$ and $V_i(h_i(\cdot)) > V_i(x_i(\cdot))$, for all $i \in A$. By continuity assumption, there exists $\varepsilon \in (0,1)$ such that $V_i(\varepsilon \cdot h_i(\cdot)) > V_i(x_i(\cdot))$, for all $i \in A$. Define an assignment $k : R \times \Omega \to I\!B_+$ by means of

$$k(t,\cdot) = \begin{cases} \varepsilon \cdot h_i(\cdot) & t \in S \cap S_i, \ i \in A \\ \\ x_i(\cdot) + w_i & t \in (R \setminus S) \cap S_i, \ i \in A \end{cases}$$

where

$$w_i = \frac{(1-\varepsilon) \cdot \mu(S \cap S_i)}{\mu\left((R \setminus S) \cap S_i\right)} h_i(\cdot).$$

Then k is a private allocation over R and, by monotonicity, $V_i(k(t,\cdot)) > V_i(x_i(\cdot))$ for μ-almost all $t \in R \cap S_i$, $i \in A$. Moreover,

$$\int_R k(t,\cdot)\, d\mu - \int_R x(t,\cdot)\, d\mu = \int_{R \setminus S} k(t,\cdot)\, d\mu + \int_S k(t,\cdot)\, d\mu - \int_{R \setminus S} x(t,\cdot)\, d\mu - \int_S x(t,\cdot)\, d\mu =$$

$$= \sum_{i \in A} \int_{(R \setminus S) \cap S_i} k(t,\cdot)\, d\mu + \sum_{i \in A} \int_{S \cap S_i} \varepsilon \cdot h_i(\cdot)\, d\mu - \sum_{i \in A} \int_{(R \setminus S) \cap S_i} x_i(\cdot)\, d\mu - \sum_{i \in A} \int_{S \cap S_i} x_i(\cdot)\, d\mu =$$

$$= \sum_{i \in A} \int_{S \cap S_i} h_i(t,\cdot)\, d\mu - \sum_{i \in A} \int_{S \cap S_i} x_i(\cdot)\, d\mu = a$$

that proves the desired inclusion. $\square$

We say that *a profile $\pi \in \Pi$ privately blocks* an allocation x, if there exists a coalition S with the given profile that privately blocks the allocation. By Lemma 3.1.3, a profile π privately blocks an allocation x if and only if the function $E(\pi, \cdot) : \Omega \to I\!\!B_+$ belongs to $G(\pi) + B_+^\Omega$. This inclusion depends only on the profile π. Therefore, the notion of blocking profile is well posed since two coalitions with the same weights in the different types behave in the same way.

We denote by K the subset of $[0, f]$ formed by all blocking profiles for a given feasible private allocation x.

Proposition 3.1.5 *K is a convex subset of* $[0, f]$.

PROOF: First assume that $\pi \in K$ and $\alpha \in [0, 1]$. Then, by Lemma 3.1.3, *ii*), $E(\alpha\pi, \cdot) = \alpha E(\pi, \cdot) \in \alpha G(\pi) + I\!\!B_+^\Omega = G(\alpha\pi) + I\!\!B_+^\Omega$ and $\alpha\pi \in K$. Again Lemma 3.1.3, *iii*) ensures that if π_1 and π_2 are blocking profiles and $\pi_1 + \pi_2 \in [0, f]$, then $E(\pi_1, \cdot) + E(\pi_2, \cdot) \in G(\pi_1) + G(\pi_2) + I\!\!B_+^\Omega = G(\pi_1 + \pi_2) + I\!\!B_+^\Omega$. Then $\pi_1 + \pi_2$ is a blocking profile and the claim is proved. □

We say that the feasible private allocation x with the finiteness assumption is strictly positive, or $x \gg 0$, if $x_i(\omega) \gg 0$ for each $i = 1, \ldots, m$ and for each $\omega \in \Omega$. The next result furnishes a direct proof of the Second Welfare Theorem for a Pareto optimal allocation in our differential information economy with a finite number of types. Note that an indirect proof would follow from [24, Theorem 5.2]. As usual, the strict positivity assumption on the allocation x could be replaced by irreducibility assumptions like those formulated in Einy et al. [17].

Proposition 3.1.6 *Let x be a strictly positive Pareto optimal allocation with the finiteness assumption. Then there exists an efficient price vector for the allocation x satisfying the additional condition*

$$(\star) \quad \sum_{\omega \in \Omega} p(\omega) \cdot \sum_{i=1}^{m} f_i x_i(\omega) = \sum_{\omega \in \Omega} p(\omega) \cdot \sum_{i=1}^{m} f_i e_i(\omega).$$

PROOF: Define the convex set $F \subseteq I\!\!B^\Omega$ by

$$F = G(f) - E(f, \cdot).$$

By Pareto optimality of allocation x, the intersection $F \cap I\!B_-^\Omega$ is empty. Applying one of the infinite dimensional versions of the separation theorem (note that both sets are convex and $I\!B_-^\Omega$ has non-empty norm interior), we find a non-zero function $p : \Omega \to I\!B'$ such that $p \cdot F \geq p \cdot I\!B_-^\Omega$. Since $I\!B_-^\Omega$ is a cone, p is non negative. Moreover, $p \cdot F \geq 0$. By continuity assumption,

$$p \cdot \sum_{i=1}^{m} f_i \cdot x_i(\cdot) \geq p \cdot \sum_{i=1}^{m} f_i \cdot e_i(\cdot).$$

Then feasibility of allocation x ensures that condition $(\star)$ is satisfied.

Consider now a function $z \in M_k$ such that $h_k\left(z(\cdot)\right) > h_k\left(x_k(\cdot)\right)$. By monotonicity, for any $i \neq k$ we find a function $z_i \in M_i$ such that $h_i\left(z_i(\cdot)\right) > h_i\left(x_i(\cdot)\right)$.

By concavity, for any $\alpha \in (0,1)$, the function of M_k defined by $\overline{z}_k(\cdot) = \alpha z_k(\cdot) + (1-\alpha) z(\cdot)$ satisfies the inequality $h_k(\overline{z}_k(\cdot)) > h_k(x_k(\cdot))$. Concavity and monotonicity assumptions ensure that the functions $\overline{z}_i(\cdot) = \alpha z_i(\cdot) + (1-\alpha) x_i(\cdot) + \alpha e_i(\cdot)$, for $i \neq k$, satisfy $h_i(\overline{z}_i(\cdot)) > h_i(x_i(\cdot)$.

Then,

$$p \cdot \left(\sum_{i=1}^{m} f_i \cdot \overline{z}_i(\cdot) - \sum_{i=1}^{m} f_i \cdot x_i(\cdot) \right) \geq 0.$$

The last inequality implies that for any $\alpha \in (0,1)$

$$\alpha \cdot \sum_{i \neq k} f_i \cdot p \cdot (\overline{z}_i(\cdot)) + (1-\alpha) \cdot \sum_{i \neq k} f_i \cdot p \cdot x_i(\cdot) + \alpha \cdot \sum_{i \neq k} f_i \cdot p \cdot e_i(\cdot) + \alpha \cdot f_k \cdot p \cdot z_k + (1-\alpha) f_k \cdot p \cdot z \geq$$

$$\sum_{i \neq k} f_i \cdot p \cdot x_i(\cdot) + f_k \cdot p \cdot x_k.$$

Letting α goes to zero, we find that $p \cdot z \geq p \cdot x_k$. Assume now that $p \cdot z = p \cdot x_k$. Choose $\alpha \in (0,1)$ such that $h_k\left(\alpha z(\cdot) + (1-\alpha) x_k(\cdot)\right) > h_k\left(x_k(\cdot)\right)$. By strict positivity of allocation x, we get

$$p \cdot (\alpha z(\cdot) + (1-\alpha) x_k(\cdot)) < p \cdot (\alpha x(\cdot)) + p \cdot ((1-\alpha) x_k(\cdot)) = p \cdot x_k(\cdot)$$

and a contradiction. $\square$

In a similar manner we can prove the next Proposition.

Proposition 3.1.7 *Let x be a strictly positive Pareto optimal allocation satisfying the finiteness assumptions. Then a non-zero function $p : \Omega \to \mathbb{B}^{\Omega}$ is an efficient price vector for the allocation x if and only if $p \cdot \left(G^*(I) + \mathbb{B}^{\Omega}_{+}\right) \geq 0$.*

PROOF: One implication is clear. Conversely, consider a non-zero function $p : \Omega \to \mathbb{B}'$ such that $p \cdot \left(G^*(I) + \mathbb{B}^{\Omega}_{+}\right) \geq 0$. Since $\mathbb{B}^{\Omega}_{+}$ is a cone, p is non-negative and $p \cdot (G^*(I)) \geq 0$. Now, with the same arguments of Proposition 3.1.6, one shows that p is an efficient system of prices for the allocation x. $\square$

3.2 Private core: the measure of the set K

In this section we state the main results concerning the measure of blocking coalitions in a continuum economy with finitely many types.

Denote by $B(a, \epsilon)$ the ball of $\mathbb{R}^m$ with a as its center and radius ϵ. By λ the Lebesgue measure on $\mathbb{R}^m$.

Theorem 3.2.1 *Let $x : T \times \Omega \to \mathbb{B}^+$ be a strictly positive Pareto optimal allocation that is not a Radner equilibrium allocation. Assume that x satisfies the finiteness and the smoothness assumptions. Then for any symmetric profile $\alpha_o f \in [0, f]$*

$$\lim_{\epsilon \to 0} \frac{\lambda \left(K \ \cap \ B(\alpha_0 f, \epsilon)\right)}{\lambda \left(B(\alpha_0 f, \epsilon)\right)} = \frac{1}{2}.$$

We show Theorem 3.2.1 via a series of Lemmas.

Lemma 3.2.2 *Let π be a profile such that $\sum\limits_{\omega \in \Omega} p(\omega) \cdot X(\pi, \omega) > \sum\limits_{\omega \in \Omega} p(\omega) \cdot E(\pi, \omega)$, where p is the unique normalized efficiency price vector for the allocation x. Then there exists $\varepsilon_0 \in (0, 1)$ such that for any $\varepsilon \in (0, \varepsilon_o]$*

$$\varepsilon \left(X(\pi, \cdot) - E(\pi, \cdot)\right) \in G^*(I) + \mathbb{B}_+^{\Omega}.$$

PROOF: Define the convex set $B = \{\epsilon \left(X(\pi, \cdot) - E(\pi, \cdot)\right) \ : \varepsilon \in (0, 1)\}$. We claim that B has a non-empty intersection with the convex set $G^*(I) + \mathbb{B}_+^{\Omega}$. Assume on the contrary that $B \cap \left(G^*(I) + \mathbb{B}_+^{\Omega}\right) = \emptyset$. By separation theorem, there exists a $q : \Omega \to \mathbb{B}'$, $\|q\| = 1$, such that $q \cdot B \leq q \cdot \left(G^*(I) + \mathbb{B}_+^{\Omega}\right)$.

Since 0 belongs to the closure of B, then $q \cdot \left(G^*(I) + \mathbb{B}_+^{\Omega}\right) \geq 0$, i.e., by Proposition 3.1.6, q is a normalized efficiency price vector for x. Consequently, by smoothness assumption, $q = p$. Since 0 belongs to the closure of $G^*(I) + \mathbb{B}_+^{\Omega}$, it is true that $p \cdot B \leq 0$, that contradicts hypothesis. Then there exists $\varepsilon_o \in (0, 1)$ such that $\varepsilon_o \left(X(\pi, \cdot) - E(\pi, \cdot)\right) \in G^*(I) + \mathbb{B}_+^{\Omega}$. Finally, Lemma 3.1.4 ensures that $\alpha \left(G^*(I) + \mathbb{B}_+^{\Omega}\right) = G^*(\alpha I) + \mathbb{B}_+^{\Omega} \subseteq G^*(I) + \mathbb{B}_+^{\Omega}$ and the desired conclusion. □

Lemma 3.2.3 *Let π be a profile such that*

$$\varepsilon\left(X(\pi,\cdot)-E(\pi,\cdot)\right)\in G^*(I)+I\!B_+^{\Omega}$$

for $\varepsilon\in(0,\varepsilon_o]$, where $\varepsilon_o\in(0,1)$. Then $f-\frac{1}{2}\varepsilon\pi\in K$.

PROOF: We show the statement for $\varepsilon=\varepsilon_o$.
It follows from

$$\varepsilon_o\left(X(\pi,\cdot)-E(\pi,\cdot)\right)\in G^*(I)+I\!B_+^{\Omega}$$

and Lemma 3.1.4, that

$$\frac{1}{2}\varepsilon_o\left(X(\pi,\cdot)-E(\pi,\cdot)\right)\in\frac{1}{2}\left(G^*(f)+I\!B_+^{\Omega}\right)\subseteq G^*\left(\frac{1}{2}f\right)+I\!B_+^{\Omega}\subseteq$$

$$G^*\left(\frac{1}{2}f+\frac{1}{2}f-\frac{1}{2}\varepsilon_o\pi\right)+I\!B_+^{\Omega}=G^*\left(f-\frac{1}{2}\varepsilon_o\pi\right)+I\!B_+^{\Omega}.$$

By feasibility, $X(f,\cdot)\leq E(f,\cdot)$. Hence the inequality

$$E\left(f-\frac{1}{2}\varepsilon_o\pi,\cdot\right)-X\left(f-\frac{1}{2}\varepsilon_o\pi,\cdot\right)\geq X\left(\frac{1}{2}\varepsilon_o\pi,\cdot\right)-E\left(\frac{1}{2}\varepsilon_o\pi,\cdot\right)\in G^*\left(f-\frac{1}{2}\varepsilon_o\pi\right)+I\!B_+^{\Omega}$$

implies that

$$E\left(f-\frac{1}{2}\varepsilon_o\pi,\cdot\right)-X\left(f-\frac{1}{2}\varepsilon_o\pi,\cdot\right)\in G^*\left(f-\frac{1}{2}\varepsilon_o\pi\right)+I\!B_+^{\Omega}$$

and the conclusion. □

Let C be a convex cone in $I\!R^m$ with full dimension and vertex c_o. We say that a convex subset $K\subseteq C$ satisfies the contraction property with respect to C and c_o, if for any $c\in C$ there exists $\delta_c>0$ such that $(1-\delta)c_o+\delta c\in K$ for $\delta\in(0,\delta_c]$.

Lemma 3.2.4 *The set K of blocking profiles satisfies the contraction property with respect to the symmetric profile $\alpha_o f$ and the cone C defined by*

$$C=\left\{\pi\in I\!R_+^m : p\cdot X(\pi,\cdot)<p\cdot E(\pi,\cdot)\right\},$$

where p is the unique normalized efficiency price vector for the allocation x.

PROOF: Clearly the convex set K is contained in C. Let $\pi \in C$ be a profile. By smoothness assumption and Proposition 3.1.6, $p \cdot X(f, \cdot) = p \cdot E(f, \cdot)$ and hence $p \cdot X(f - \pi, \cdot) > p \cdot E(f - \pi, \cdot)$. By Lemma 3.2.2 and Lemma 3.2.3, there exists $\varepsilon_o \in (0, 1)$ such that $f - \frac{1}{2}\varepsilon f + \frac{1}{2}\pi \in K$, for each $\varepsilon \in (0, \varepsilon_o]$. Then, by a suitable $\varepsilon_o \in (0, 1)$, $(1 - \varepsilon)f + \varepsilon\pi \in K$, for each $\varepsilon \in (0, \varepsilon_o]$. As in [47, Corollary 3] one can use this fact to show that $(1 - \delta)\alpha_o f + \delta_o\pi \in K$ for any $\delta \in (0, \delta_o]$ and then the Lemma is proved. $\square$

PROOF: (of Theorem 3.2.1) By Lemma 3.2.4 and [47, Lemma page 254], we can write that

$$\lim_{\epsilon \to 0} \frac{\lambda\left(K \ \cap \ B(\alpha_0 f, \epsilon)\right)}{\lambda\left(B(\alpha_0 f, \epsilon)\right)} = 1.$$

On the other hand, the set $I\!R^m_+ \setminus C$ can be written as

$$I\!R^m_+ \setminus C = \left\{\pi \in I\!R^m_+ : p \cdot X(\pi, \cdot) = p \cdot E(\pi, \cdot)\right\} \cup \left\{\pi \in I\!R^m_+ : p \cdot X(\pi, \cdot) > p \cdot E(\pi, \cdot)\right\}$$

where the first set is an hyperplane and then has measure zero. By smoothness and Proposition 3.1.6, $p \cdot X(\alpha_o f, \cdot) = p \cdot E(\alpha_o f, \cdot)$ and consequently $p \cdot X(\pi, \cdot) < p \cdot E(\pi, \cdot)$ if and only if $p \cdot X(\alpha_o f - \pi, \cdot) > p \cdot E(\alpha_o f - \pi, \cdot)$. Since the Lebesgue measure λ is translation invariant, we find that

$$\lambda\left(\left\{\pi \in I\!R^m_+ : p \cdot X(\pi, \cdot) = p \cdot E(\pi, \cdot)\right\}\right) = \lambda\left(\left\{\pi \in I\!R^m_+ : p \cdot X(\pi, \cdot) > p \cdot E(\pi, \cdot)\right\}\right)$$

and then

$$\lambda\left(B(\alpha_o f, \varepsilon)\right) = 2\lambda\left(C \cap B(\alpha_o f, \varepsilon)\right)$$

that gives the desired conclusion. $\square$

3.3 The case of finite economies

The standard relations between core allocations and competitive equilibria in the framework of complete information economies, can be generalized to private core allocations and Radner equilibria of differential information economies. It is easy to show that every Radner equilibrium is in the private core. Actually, it is possible to show that in the case of continuum atomless economies, the private core coincides with the set of Radner equilibria (see [17], [31], [24]). When the differential information economy $\mathcal{E}$ has a finite number of traders (or more generally, when it admits atoms) private core allocations may not be decentralized by prices. The notion of Aubin private core allows to restore the equivalence. It is based on the following generalized notion of coalition.

Definition 3.3.1 *Define the set*

$$\mathcal{A} = \{\gamma : T \to [0,1] : \gamma \textit{ is simple, measurable and } \mu(\{t \in T : \gamma(t) > 0\}) > 0\}\,.$$

We call any element γ in the set $\mathcal{A}$ a generalized (or fuzzy) coalition and the set $\{t \in T : \gamma(t) > 0\}$ the support of γ.

The set $\mathcal{A}$ can be interpreted as a generalized coalitions in the sense that $\gamma(t)$ represents the share of resources employed by agent t in the coalition γ. Ordinary coalitions form a subset of $\mathcal{A}$ since they can be identified with their characteristic functions. In the case the set T is formed by m traders, a generalized coalition is a vector $(\gamma_1, \ldots, \gamma_m))$ of $[0,1]^m$.

Definition 3.3.2 *The coalition $\gamma \in \mathcal{A}$ of support A privately blocks an allocation $x : T \times \Omega \to I\!B_+$, if there exists a private allocation $y : A \times \Omega \to I\!B_+$ over A s.t.*

i) $\displaystyle\int_A \gamma(t)y(t,\omega)\,d\mu \le \int_A \gamma(t)e(t,\omega)\,d\mu, \quad$ *for all $\omega \in \Omega$;*

ii) $h_t(y(t,\cdot)) > h_t(x(t,\cdot)), \quad$ *μ-a.e. on A.*

The *Aubin private core* of the economy $\mathcal{E}$ is accordingly defined as the set of all feasible private allocations which are not privately blocked by a fuzzy coalition (see [24], [32]). Since the set of coalitions has been enlarged with respect to the field Σ, the private core contains the Aubin private core. It is easy to show that Radner equilibrium allocations are in the Aubin private core. Conversely, [24, Theorem 3.1] ensures the equivalence between the Aubin private core and Radner equilibria in the case of the general measure space of traders treated here and hence, in particular, in the case of finite differential information economies.

The main aim of this section is to extend the result of Mas-Colell on the measure of blocking coalitions in a large economy with complete information, to generalized coalitions of finite economies with differential information.

Let us consider a finite differential information economy $\mathcal{E}$ with m traders as described in Section 1.1. Given a generalized coalition $\gamma : \{1, \ldots, m\} \to [0,1]$ with non-empty support, we say that γ is *symmetric* if γ is a constant function, that is all traders employ the same non-zero share of their initial endowments in the given coalition. We denote by K_f the subset of $[0,1]^m$ formed by all generalized coalitions that privately blocks a feasible private allocation $x \equiv (x_1, \ldots, x_m)$.

By a standard procedure, we shall associate to the finite differential information economy $\mathcal{E}$, a continuum differential information economy E^c with a finite number of types.

Let us consider the m consecutive disjoint sub-intervals $\{S_1, \ldots, S_m\}$ of the real unit interval $[0,1]$ of equal length $\frac{1}{m}$, that is:

$$I_i = \left[\frac{i-1}{m}, \frac{i}{m}\right) \text{ , if } i \neq m \text{ and } I_m = \left[\frac{m-1}{m}, 1\right].$$

We consider the continuum economy $\mathcal{E}^c$ by assuming $[0,1]$ as the agents space and:

$$e(t, \cdot) = e_i,\ \mathcal{F}_t = \mathcal{F}_i,\ u_t = u_i,\ q_t = q_i,\ \forall\, t \in S_i.$$

To a feasible private allocation $x \equiv (x_1, \ldots, x_m)$ of $\mathcal{E}$ we associate the feasible private allocation $\hat{x} : T \times \Omega \to I\!\!B_+$ with the finiteness assumption defined by

$$\hat{x}(t,\omega) = x_i(\omega), \quad t \in S_i.$$

Denote by K the set of blocking profiles for the allocation $\hat{x}$.

Lemma 3.3.3 *Let the set of blocking coalitions K_f be non-empty. Then $K_f = mK$.*

PROOF: Assume that the fuzzy coalition $\gamma : \{1,\ldots,m\} \to [0,1]$ privately blocks the allocation x. Let us denote by A its support. According to Definition 3.3.2, there exist functions $z_i \in M_i$ such that

$$\sum_{i\in A} \gamma_i \cdot z_i(\omega) \le \sum_{i\in A} \gamma_i \cdot e_i(\omega), \quad \forall\, \omega \in \Omega$$

and

$$h_i\left(z_i(\cdot)\right) > h_i\left(x_i(\cdot)\right), \quad \forall\, i \in A.$$

Consider the coalition S of $\mathcal{E}^c$ defined by $S = \bigcup_{i\in A} \left[\frac{i}{m} - \frac{\gamma(i)}{m}, \frac{i}{m}\right]$. Then $\pi(S) = \frac{1}{m}\gamma$ and $\pi(S) \in K$. Indeed, denoted by $\hat{z}$ the allocation of $\mathcal{E}^c$ with the finiteness assumption corresponding to z, we have that

$$\hat{Z}(\frac{1}{m}\gamma,\omega) = \sum_{i\in A} \frac{\gamma_i}{m} \cdot z_i(\omega) \le \sum_{i\in A} \frac{\gamma_i}{m} \cdot e_i(\omega) = E(\frac{1}{m}\gamma,\cdot), \quad \forall\, \omega \in \Omega$$

and

$$h_i\left(\hat{z}_i(\cdot)\right) > h_i\left(\hat{x}_i(\cdot)\right), \quad \forall\, i \in A.$$

Conversely, consider a profile $\pi \in K$, a coalition S with $\pi(S) = \pi$ and a feasible private assignment $\hat{z}$ over S such that $\hat{z}$ privately blocks the allocation $\bar{x}$. Define a feasible private allocation of the finite economy $\mathcal{E}$ by means of

$$z_i(\cdot) = \frac{1}{\pi_i} \int_{S\cap S_i} \hat{z}(t,\cdot)\, d\mu, \quad i \in supp\ \pi.$$

Then

$$\sum_{i\in A} \pi_i \cdot z_i(\omega) \le \sum_{i\in A} \pi_i \cdot e_i(\omega), \quad \forall\, \omega \in \Omega$$

and, by Lemma 3.1.2,

$$h_i\left(z_i(\cdot)\right) > h_i\left(x_i(\cdot)\right), \quad \forall\ i \in supp\ \pi.$$

It follows that $\pi \in K_f$ and then $m\pi \in K_f$. $\square$

If an allocation x of the finite economy $\mathcal{E}$ is not a Radner equilibrium, by [24, Theorem 3.1], it is not in the Aubin private core. Then the set K_f of blocking fuzzy coalitions is non-empty.

Theorem 3.3.4 *Let $x : \{1, \ldots, m\} \times \Omega \to I\!B^+$ be a strictly positive Pareto optimal allocation that is not a Radner equilibrium allocation. Assume that x satisfies the smoothness assumptions. Then for any symmetric fuzzy coalition $\gamma \in [0,1]^m$*

$$\lim_{\epsilon \to 0} \frac{\lambda\left(K_f\ \cap\ B(\gamma, \epsilon)\right)}{\lambda\left(B(\gamma, \epsilon)\right)} = \frac{1}{2}.$$

PROOF: The result follows from Theorem 3.2.1. Indeed, once we have observed that the allocation $\hat{x}$ is Pareto optimal and non competitive, the result follows from Lemma 3.3.3 and the fact that the Lebesgue measure λ is translation invariant, that is:

$$\lim_{\epsilon \to 0} \frac{\lambda\left(K_f\ \cap\ B(\gamma, \epsilon)\right)}{\lambda\left(B(\gamma, \epsilon)\right)} = \lim_{\epsilon \to 0} \frac{\lambda\left(mK\ \cap\ mB(\frac{1}{m}\gamma, \frac{\epsilon}{m})\right)}{\lambda\left(mB(\frac{1}{m}\gamma, \frac{\epsilon}{m})\right)} =$$

$$= \lim_{\epsilon \to 0} \frac{\lambda\left(K\ \cap\ B(\frac{1}{m}\gamma, \frac{\epsilon}{m})\right)}{\lambda\left(B(\frac{1}{m}\gamma, \frac{\epsilon}{m})\right)} = \frac{1}{2}.$$

$\square$

3.4 Fine Core and Ex-post Core

In previous section we have analyzed the case of ex-ante decision rule. Now, we briefly focus our attention on a new information sharing rule. An appropriate (interim) notion of the core for an economy with incomplete information depends on the amount of information that coalitions can share. In particular we study a notion of a core that involving an arbitrary information sharing.

We assume that by pooling their information, agents could discern the events in the *fine* field. That is, when different agents in a coalition have different information their opportunities to take blocking actions jointly are necessarily contingent upon events which they can all discern.

The ex-post stage, where decisions are made after the information state is known, is no different from a model with complete information. In fact, in Einy et al. [16], the Ex-post Core of an economy $\mathcal{E}$ is defined as all the selections from the core correspondence of the associated family of complete information economies $\{\mathcal{E}(\omega)\}_{\omega\in\Omega}$.

Einy, Moreno and Shitovitz [16] provide conditions for the convergence of the ex- post core to the set of fully revealing rational expectations equilibrium allocations. Starting from a theorem of Vind [49] that establishes that if an allocation is not in the core of an atomless economies with full information, then it can be blocked by an arbitrarily large coalition, they assume that full information corresponds to joint information of traders in an economy, then a sufficiently large coalition can discern any state of nature.

3.4.1 Ex-post blocking mechanism and equilibria

Remind that, as described above, $\mathcal{F}$ is a field of subsets of Ω. The information of trader $t \in T$ is described by a measurable partition Π_t of Ω. We denote by $\mathcal{F}_t$ the field generated by Π_t. Since Ω is finite, there is a finite subfamily $(\mathcal{F}_i)_{i=1}^m$ of $(\mathcal{F}_t)_{t\in T}$ such that for all $t \in T$ there is $i \in \{1, ..., m\}$ with $\mathcal{F}_t = \mathcal{F}_i$.
For all $i \in \{1, ..., m\}$, we assume that the set

$$T_i = \{t \in T | \mathcal{F}_t = \mathcal{F}_i\}$$

is measurable and $\mu(T_i) > 0$. In such way, the set $\{T_1, ..., T_m\}$ is a measurable partition of the agents set T. Throughout this section we will assume that

i) for all $\omega \in \Omega$, $\int_T e(t, \omega) \gg 0$, which ensures that each commodity is present;

ii) for all $t \in T$ and $\omega \in \Omega$ the function $u_t(t, \cdot)$ is continuous and strictly increasing on $I\!B_+$;

iii) $\mathcal{F} = \bigvee_{i=1}^m \mathcal{F}_i$, which ensures that $\mathcal{F}$ contains no superfluous events about which no trader has information and therefore cannot affect anyone's consumption decision.

In the rest of this section, $\mathcal{E}$ is an economy with asymmetric information as described above. For any economy $\mathcal{E}$ and a state of nature $\omega \in \Omega$, we will denote by $\mathcal{E}(\omega)$ the complete information economy in which the commodity space is $I\!B_+$, the space of traders is (T, Σ, μ), and for every trader $t \in T$, his initial endowment is $e(t, \omega)$ and his utility function is $u_t(\cdot, \omega)$.

Definition 3.4.1 *Let x be an allocation,let $S \in \Sigma$ be a coalition and let $\omega_0 \in \Omega$. We say that an assignment y is an ex-post improvement of S upon x at the state ω_0 if*

i) $\mu(S) > 0$*;*

ii) $\int_S y(\omega_0, t) d\mu \geq \int_S e(\omega_0, t) d\mu$*;*

iii) $u_t(y(\omega_0, t), \omega_0) > u_t(x(\omega_0, t), \omega_0)$ *μ-a.e. in S.*

The *Ex-post Core* of $\mathcal{E}$, denoted by Ex-PC($\mathcal{E}$), is the set of all feasible allocation that are not blocked by any coalition in each state of nature $\omega \in \Omega$ the ex-post allocation of $\mathcal{E}$.

Theorem 3.1 in Einy et al. [16] shows the non-emptiness of the Ex-post Core under assumption of our model, and moreover that

$$\text{Ex-PC}(\mathcal{E}) = \{x \in M_t | x(\cdot, .\omega) \in C(\mathcal{E}(\omega)) \forall \omega \in \Omega\}.$$

If $p : \Omega \to I\!B'_+$ is a price system defined in 1.2.4, we denote by $\sigma(p)$ the smallest subfield $\mathcal{G}$ of $\mathcal{F}$ for which p is $\mathcal{G}$-measurable. The atoms of $\sigma(p)$ are the elements of the partition of Ω generated by the function p. The ex-post budget set is defined state by state

Definition 3.4.2 *The budget set of a trader $t \in T$ at the state $\omega \in \Omega$ with the price system p is given by*

$$B_t(\omega, p) = \{z \in I\!B_+ |\ p(\omega) \cdot z \leq p(\omega) \cdot e(\omega, t)\}$$

Note that is the ex-post budget set. We can also introduce the efficient budget set state by state

Definition 3.4.3 *A non-zero price system p is an efficient price vector for the allocation $x : T \times \Omega \to I\!B_+$ at the state $\omega \in \Omega$ if:*

i) *μ-a.e. in T the function $x(t, \cdot)$ is the maximal element of h_t in the efficiency set*

$$B^*_t(\omega, p) = \left\{ z : z \in M_t \text{ and } \sum_{\omega \in \Omega} p(\omega) \cdot z(\omega) \leq \sum_{\omega \in \Omega} p(\omega) \cdot x(t, \omega) \right\}.$$

If $\mathcal{G}$ is a subfield of $\mathcal{F}$, $f : \Omega \to I\!B_+$ is an $\mathcal{F}$-measurable function, and $t \in T$, we denote by $E_t(f|\mathcal{G})$ the conditional expectation of f with respect to q_t. As stated in Theorem 4.5 by Einy et al. [16], we consider the *Rational expectation equilibria* as the competitive equilibrium concept coinciding with the Ex-post Core.

Definition 3.4.4 *Let p be a non-zero price system and x be a feasible allocation. The pair (x, p) is said to be a Rational Expectation Equilibrium (REE) if*

i) *for almost every $t \in T$, $x(t, \cdot)$ is $\sigma(p) \vee \mathcal{F}_t$-measurable;*

ii) *for all $\omega \in \Omega$ and almost all $t \in T$, $x(t, \omega) \in B_t(\omega, p)$;*

iii) *for almost all $t \in T$, if $y : \Omega \to \mathbb{B}_+$ is $\sigma(p) \vee \mathcal{F}_t$-measurable and satisfies $y(\omega) \in B_t(\omega, p) \forall \omega \in \Omega$, then*

$$E_t\left(u_t(., x(., t))\right| \sigma(p) \vee \mathcal{F}_t\right)(\omega) \geq E_t\left(u_t(., x(.))\right| \sigma(p) \vee \mathcal{F}_t\right)(\omega),$$

pointwise on Ω.

A rational expectation equilibrium (p, x) is fully revealing if $\sigma(p) = \mathcal{F}$[1].

We want to extend the results of Shitovitz [47], as in the previous section, to our model with this new concept of competitive and cooperative equilibria. It is very clear that in the definition of blocking concept, the ex-post one is an extension state by state of the case of an economy without uncertainty.

We replace the correspondence G as in the previous section by setting for all $t \in T_i$. The properties of the correspondence are the same, noting that $G(t)$ is defined over the preferences of all agents. Moreover, we define the profile π for a coalition S in he same way and the support for such coalition as $I(S) = \{i \in \{1, ..., m\} \,|\mu(S \cap T_i) > 0\}$. We say that a profile $\pi \in \Pi$ blocks ex-post an allocation x, if there exists a coalition S with the profile π that blocks the allocation at the state $\omega_0 \in \Omega$. In the same way, by Lemma 3.1.3, we will say that a profile π blocks ex-post an allocation x if and only if the function $E(\pi,)$ belongs to $G(\pi) + \mathbb{B}_+$
We denote by $K(\omega_0)$ the subset of $[0, f]$ formed by all blocking profiles at the state $\omega_0 \in \Omega$, for a given feasible allocation. It is very easy

[1]The assumption of fully revealing equilibrium is needed for the equivalence result, [16]. Moreover, it is also required that each trader knows his state-dependent utility function, i.e. his utility function is measurable with respect to his information field.

to recognize in $K(\omega_0)$ for all $\omega_0 \in \Omega$ the set defined in [47], i.e. the set of all blocking profiles in a complete information economy. In such way, the results is proved without many difficulties. It will be more interesting to focus our attention on a new convex set. We consider the convex hull of all $K(\cdot)$ as $C = co\left\{\bigcup_{\omega\in\Omega} K(\omega)\right\}$. This set is convex and non empty. The propositions 3.1.6, 3.1.7 and lemmas 3.2.2, 3.2.3 hold with the set C.
Note that $K(\omega_0) \subseteq C \Rightarrow K(\omega_0) \cap B(\alpha_0 f, \epsilon) \subseteq C \cap B(\alpha_0 f, \epsilon)$ and $\mu[K(\omega_0) \cap B(\alpha_0 f, \epsilon)] \leq \mu[C \cap B(\alpha_0 f, \epsilon)]$, then $\lim_{\epsilon\to 0} \frac{\mu(C \,\cap\, B(\alpha_0 f,\epsilon))}{\mu(B(\alpha_0 f,\epsilon)} \geq \frac{1}{2}$.

3.4.2 Fine blocking mechanism and equilibria

The difficulty that economies with asymmetric information raise stems from the fact that agents evaluate bundles after they have received their information and that not all of them have the same information. Wilson [50] notes that when information is asymmetric it is not enough for each member of a coalition to know that he prefers one allocation to another in order for a coalition to improve upon the latter. It must be commonly known by all members of the coalition that this is so. The requirement that the improvement be common knowledge is needed because agents necessarily learn that they are improving upon an allocation when they are doing so and they must be willing to transact after they have learned everything they learn.

When opportunities for communication are allowed on the other hand the relevant information should be the initial information refined by the information transmission that has taken place. Therefore a coalition improves upon an allocation when it becomes common knowledge among its members that they can enforce something better after the permitted communication has taken place. Wilson's Fine Core takes this into account and allows for unlimited communication among agents. As Wilson [50] notes the opportunities for communication may disrupt arrangements for mutual insurance causing the emptiness of the Core.

The definition of Fine Core presumes that traders can share their information. A coalition blocks if it has a feasible allocation that is preferred by every member of the coalition in an event which the coalition can jointly discern. In Einy et al. [16] it is established that the Fine Core is a subset of the Ex-post Core of an economy with differential information, it can be applied to any economy in which the state can be identified by pooling the information of agents in some coalition with an ex post objection. Then, the Fine Core is also related to the core of a full information economy and to the rational expectation equilibrium. Einy et al. [16] show that this is generally the case in an atomless economy with a finite number of states. The proof is based on the argument that in an atomless economy, if there is an objection in a certain state, there exists an objection by an arbitrarily large coalition. With a finite number of states it is then possible to construct such a coalition in which the state can be discerned by pooling the private information in the large coalition. The Fine Core allowing for arbitrary forms of information pooling. In our framework, there are only finitely many different information fields, and since we assume that *full information* corresponds to joint information of traders in the economy.

For an allocation y to be a fine improvement upon x it should be possible to redistribute the information initially held by the members of the coalition in a way that nobody learns more than what can be learned by pooling all the information nobody forgets what he knows and that makes it common knowledge that y is strictly preferred to x.

Consider the economy $\mathcal{E}$ defined in the previous section. For every $S \in \Sigma$ let

$$I(S) = \{i \in \{1, ..., m\} \,|\mu(S \cap T_i) > 0\}$$

be the support of a coalition S, where the set $T_i = \{t \in T | \mathcal{F}_t = \mathcal{F}_i\}$ is measurable for all $i \in \{1, ..., m\}$, and $\mu(T_i) > 0$. Note that, $\bigcup_{i=1}^{m} T_i = T$, i.e. $(T_1, ..., T_m)$ is a measurable partition of the agents set T.

Definition 3.4.5 *An information structure for a coalition $S \in \Sigma$ is a family $(\mathcal{H}_t)_{t \in S}$ of subfield of $\mathcal{F}$ such that for every subfield $\mathcal{G}$ of $\mathcal{F}$ the set $\{t \in S | \mathcal{H}_t = \mathcal{G}\}$ is in Σ.*

We can define a scheme for sharing information among the members of a coalition

Definition 3.4.6 *A communication system for a coalition $S \in \Sigma$ is an information structure $(\mathcal{H}_t)_{t \in S}$ for S such that, for all $t \in S$, $\mathcal{F}_t \subseteq \mathcal{H}_t \subseteq \bigvee_{i \in I(S)} \mathcal{F}_t$.*

In particular we consider a *full communication system* for a coalition S, i.e. $\mathcal{H}_t = \bigvee_{i \in I(S)} \mathcal{F}_t$.

Let S be a coalition and let $(\mathcal{H}_t)_{t\in S}$ be a communication system for S. Since Ω is finite, there is a finite subfamily $(\mathcal{H}_i)_{i=1}^k$ of $(\mathcal{H}_t)_{t\in S}$ such that for every $t \in S$ there is $1 \le i \le k$ with $\mathcal{H}_t = \mathcal{H}_i$ and for all $1 \le i \le k$ we have $\mu(\{t \in S | \mathcal{H}_t = \mathcal{H}_i\}) > 0$

Definition 3.4.7 *An event $A \in \mathcal{F}$ is called common knowledge for S with respect to the communication system $(\mathcal{H}_t)_{t\in S}$ if $A \in \bigwedge_{i=1}^{k}(\mathcal{H}_i)$.*

In this framework we can give the definition of the cooperative equilibrium concept.

Definition 3.4.8 *An allocation x is a Fine Core allocation for the economy $\mathcal{E}$ if there does not exist a coalition S, $\mu(S) > 0$, an assignment $y : T \times \Omega \to I\!\!B_+$, a communication system $(\mathcal{H}_t)_{t\in S}$ for S and a non-empty event A which is common knowledge for S with respect to the communication system, such that:*

i) $\int_S y(t,\omega)d\mu = \int_S e(t,\omega)d\mu$ *for all $\omega \in A$;*

ii) $E_t[u_t(\cdot, y(t,\cdot))|H_t] > E_t[u_t(\cdot, x(t,\cdot))|H_t]$, *on A for almost all $t \in S$.*

We can observe that a coalition S can block an allocation x only in an event A which is common knowledge for each member with respect to an admissible communication system.

We say that a profile $\pi \in \Pi$ blocks an allocation x, if there exists a coalition S with the given profile that blocks the allocation, with the communication system$(\mathcal{H}_t)_{t\in S}$ in the common knowledge event A.

By lemma 3.1.3, a profile π blocks an allocation x for the event $A \in \bigwedge_S \mathcal{H}_t$ if and only if the function $E(\pi, \cdot)$ belongs to $G(\pi) + I\!\!B_+$ for all $\omega \in A$. We can note that this results is the same related to the different framework defined because it depends only on the definition of profile of a fixed coalition.

Now we can denote with $K(A)$ the set of all blocking profile in the *fine* sense with respect an event $A \in \bigwedge_S \mathcal{H}_t$, which is a convex set. Now we can use the definition 1.2.3 for a Pareto optimal allocation, and it is very simple to replace the results. Moreover, because the relation between the Fine ore and the Ex-post Core is known, it is clear that

$$\lim_{\epsilon\to 0}\frac{\mu(K(A))\ \cap\ B(\alpha_0 f,\epsilon))}{\mu(B(\alpha_0 f,\epsilon)}\geq \lim_{\epsilon\to 0}\frac{\mu(K(\omega_0))\ \cap\ B(\alpha_0 f,\epsilon))}{\mu(B(\alpha_0 f,\epsilon)}=\frac{1}{2}$$

for all $\omega_0 \in A$ and $A \in \bigwedge\limits_{t\in T} \mathcal{H}_t$.

Wilson [50] proposed a new definition for a *fine* efficient allocation, that is an allocation is fine efficient if and only if in each event of a full communication system there is no allocation which each agent prefers given his own information. Fine objections are based on events that can be discerned by pooling the information of the members of the coalition. The agents are cooperating on their own in Wilsons theory. Objections emerge from coalitions. We can give this new

Definition 3.4.9 *An allocation $x : \Omega \times R \to \mathbb{B}_+$ is fine efficient if there does not exist, for each event $E \in \bigvee\limits_{t\in T} \mathcal{F}_t$, an allocation y such that*

i) $E_t\left(u_t(\cdot, y(\cdot))|\bigvee\limits_{t\in T}\mathcal{F}_t\right) > E_t\left(u_t(\cdot, x(t,\cdot))|\bigvee\limits_{t\in T}\mathcal{F}_t\right)$

for μ-almost all $t \in T$.

A fine efficient allocation is obtained from $\bigvee\limits_{t\in T}\mathcal{F}_t$-measurable weights, not all zero on any event, i.e. constant over the whole set of states. We must define a fine efficient price system in such way

Definition 3.4.10 *A non-zero price system p is a fine efficient price vector for the allocation x on the event $A \in \bigvee\limits_{t\in T}\mathcal{F}_t$ if μ-a.e. in T the function $x(t,\cdot)$ is the maximal element of h_t in the efficiency set*

$$B_t^*(\omega,p)=\left\{z : z\in M_t \text{ and } \sum_{\omega\in A} p(\omega)\cdot z(\omega)\leq \sum_{\omega\in A} p(\omega)\cdot x(t,\omega)\right\}$$

3.5 Social Coalition Structure and Differential Information

In this section the analogous of Theorems 3.2.1 and 3.3.4 are investigated in connection with the notion of *social coalition structure.*

Following [23], we introduce for a finite differential information economy a social coalition structure in the form of a finite set of generalized coalitions. A coalition can be formed if and only if it belongs to the given structure. Moreover, any trader is required to redistribute is initial endowment among the coalitions in the given structure. The need of imposing a social coalition structure on the society is motivated by the fact that, although many coalitions can block an allocation that is not in the private core, it is not true that such coalitions will really formed. In particular, in economies with differential information, the interest in such structures is connected with costs of communication and information that may reduce the possibility of free coalition formation.

Definition 3.5.1 *In a finite differential information economy E with m traders and for any integer $j \geq 2$, we call a social coalition structure any non-negative matrix $\Gamma = (\gamma_h^k)$ of dimension $j \times m$ such that $\sum_{h=1}^{j} \gamma_h^k =$ 1, for all $k = 1, ..., m$.*

Each row of Γ represents a generalized coalition. The elements γ_h^k describe the levels of participations of trader k in any sub-coalition. Then in a social coalition structure any trader is required to redistribute in full his initial endowment.

Extending the notion of blocking social coalition structure introduced in [23], we will say that the social coalition structure Γ privately blocks a private allocation x if there exists at least one subcoalition of Γ that blocks x.

For a fixed feasible private allocation x that is not competitive, we denote with Π_f^j the set of social coalition structure of Γ and with K_f^j the subset of Π_f^j formed by blocking social structures. For any integer

$j \geq 2$, we denote by Γ^j the symmetric social structure defined by $\gamma_k^h = \frac{1}{j}$ for all h,k.

In order to valuate the relative measure of social coalition structures that privately blocks a non competitive allocation, we define a correspondence between social coalition structures of the finite economy and ordered partitions of an associated continuum economy. We extend in particular [47, Result 2] to an atomless economy with a finite number of types and differential information.

Let us consider an atomless differential information economy $\mathcal{E}^c$ whit a finite number m of types, and a feasible private allocation $x(t,\omega)$ with the finiteness assumption that is not a Radner equilibrium. For an integer $j \geq 2$ and for any ordered partition $B \equiv (B_1, ..., B_j)$ of T into j coalitions, we define the profile of B as the vector of profiles $\Pi(B) \equiv (\Pi(B_1), ..., \Pi(B_j))$.

Definition 3.5.2 *The profile $\Pi(B)$ privately blocks the allocation $x(t,\omega)$ if at least one of the profiles $\Pi(B_h)$ $(h = 1, ..., j)$ is blocking.*

Let Π^j be the set of all possible profiles of the ordered partitions of T into j coalitions, and denote with K^j the subset of Π^J formed by blocking profiles. Clearly, Π^j and K^j can be considered as subsets of $I\!R^{(j-1)m}$. Denote by $f^j \equiv (f, ..., f) \in I\!R^{(j-1)m}$ the vector of profiles of the full coalition. Then $\Pi^j \subseteq [0, f^j]$. Following [23, Theorem 5.2] we find

Theorem 3.5.3 *Let x be a feasible private allocation that is Pareto optimal and non-competitive. Assume that x satisfies the finiteness and smoothness assumptions. Then, for all $\alpha_0 \in \left(0, \frac{1}{j-1}\right)$,*

$$\lim_{\epsilon \to 0^+} \frac{\mu\left(K_j \bigcap B(\alpha_0 f^j, \epsilon)\right)}{\mu\left(B(\alpha_0 f^j, \epsilon)\right)} = 1.$$

PROOF: As in [47, Result 2], one defines for $h = 1, \ldots, j$, the sets

$$K_h = \left\{\pi \in \Pi^j : p \cdot X(\pi_r, \cdot) > p \cdot E(\pi_r, \cdot),\ r = 1, \ldots, h-1 \text{ and } \pi_h \in K\right\},$$

$$C_h = \left\{\pi \in \Pi^j : p \cdot X(\pi_r, \cdot) > p \cdot E(\pi_r, \cdot),\ r = 1, \dots, h-1 \textit{ and } p \cdot X(\pi_h, \cdot) < p \cdot E(\pi_h, \cdot)\right\}.$$

The set K_h (and C_h) are pairwise disjoint. Moreover any K_h is a subset of the corresponding C_h. With the same arguments of Lemma 3.2.4, it is possible to show that K_h has the contraction property with respect to C_h and $\alpha_o f^j$. Then

$$\lim_{\epsilon \to 0^+} \frac{\mu\left(K_h \bigcap B(\alpha_0 f^j, \epsilon)\right)}{\mu\left(C_h \bigcap B(\alpha_0 f^j, \epsilon)\right)} = 1.$$

Since $\bigcup_h K_h = K_j$ and the sets $\bigcup_h C_h$ and Π_j has the same measure (see [47, Result 2]) the conclusion follows. $\square$

Let us state the result for a social coalition structure of a finite economy.

Theorem 3.5.4 *Let $x : \{1, \dots, m\} \times \Omega \to I\!\!B_+$ be a feasible private allocation that is Pareto optimal and non-competitive. Assume that x satisfies the smoothness assumption. Then, for all $\alpha_0 \in (0,1)$,*

$$\lim_{\epsilon \to 0^+} \frac{\mu\left(K_f^j \bigcap B(\alpha_0 \Gamma^j, \epsilon)\right)}{\mu\left(B(\alpha_0 \Gamma^j, \epsilon)\right)} = 1.$$

PROOF: Let us denote by $\hat{x}$ the private allocation with the finiteness assumption defined by x in the continuum economy $\mathcal{E}^c$ associated with $\mathcal{E}$. We claim that $K_f^j = mK^j$. Indeed, for any coalition structure $\Gamma \in K_f^j$ let us denote by A_1 the support of coalition γ_1 contained in Γ and by A_h the set $\left\{i : \sum_{r=1}^{h} \gamma_r^i \neq 0\right\}$, $h = 2, \dots j$. Then, $\frac{1}{m}\Gamma$ is the profile of the ordered partition

$$B_1 = \bigcup_{i \in A_1} \left(\frac{i}{m} - \frac{\gamma_1^i}{m}, \frac{i}{m}\right],\ B_h = \bigcup_{i \in A_h} \left(\frac{i}{m} - \frac{\sum_{r=1}^{h} \gamma_r^i}{m}, \frac{i}{m} - \frac{\sum_{r=1}^{h-1} \gamma_r^i}{m}\right],\ h = 2, \dots j$$

and, as in the proof of Theorem 3.3.4, this profile blocks the allocation $\hat{x}$. Conversely, it is clear that any profile $\Pi(B) \in K^j$ gives rise to a social coalition structure $m\Pi(B) \in K_f^j$. Then by Theorem ... and property of the Lebesgue measure we have that

$$\lim_{\epsilon\to 0^+} \frac{\mu\left(K_f^j \bigcap B(\alpha_0\Gamma^j,\epsilon)\right)}{\mu\left(B(\alpha_0\Gamma^j,\epsilon)\right)} = \lim_{\epsilon\to 0^+} \frac{\mu\left(mK^j \bigcap mB(\frac{\alpha_0}{j}f^j,\frac{\epsilon}{mj})\right)}{\mu\left(mB(\frac{\alpha_0}{j}f^j,\frac{\epsilon}{mj})\right)} =$$

$$= \lim_{\epsilon\to 0^+} \frac{\mu\left(K^j \bigcap B(\frac{\alpha_0}{j}f^j,\frac{\epsilon}{mj})\right)}{\mu\left(B(\frac{\alpha_0}{j}f^j,\frac{\epsilon}{mj})\right)} = 1.$$

Remark 3.5.5 The assumption of smoothness on the Pareto optimal allocation x that we made in the main Theorems is standard if we look to economies with complete information. The uniqueness (aside of scalar multiples) of the supporting price is guaranteed if at least one preference relations is represented by a differentiable utility function. The uniqueness of supporting prices in infinite dimensional setting again follows from differentiability of at least one utility function (see [8] for a discussion of this topic). In the case of economies with differential information, the smoothness assumption is satisfied if $u_i(\cdot, \omega)$ is differentiable in each state $\omega \in \Omega$, given the form of the expected value.

Remark 3.5.6 Notice that, differently from results proved in [23] for complete information economies, we assume in the paper that the commodity space has non-empty norm interior. Indeed, in presence of differential information, properness assumptions on preferences used in [23] to compensate for the lack of interior points, do not work. This is true in general for private core equivalence results (see [24, Remark 5.6].

$\square$

Chapter 4

Some Remarks on the Second Welfare Theorem

The second fundamental welfare theorem (SWT) gives conditions under which a Pareto optimal allocation can be supported by an equilibrium price. It tells us that we can achieve any desired Pareto optimal allocation as a market equilibrium. In an Economy with a finite number of agents, characterized by a consumption set that coincides with the positive orthant, it's sufficient the convexity of the preference to guarantee the theorem. The convexity assumption plays a central role in this theorem. But, the interpretation of the second welfare theorem is strongest when the number of economic agents become large, because the price-taking assumption is most realistic. We now observe that, the second welfare theorem can be viewed as a special case of the existence of a Walrasian equilibrium for economies in which endowments are distributed in a particular manner.

Proposition 4.0.7 *If*

$\forall\ i\ X_i \subset R^l_+$ *is convex;*

$\forall\ i\ x_i, x_i', x_i'' \in X_i, x_i'' \succ_i x_i' \Rightarrow (1-t)x_i'' + tx_i \succ_i x_i'\ \forall\ 0 < t < 1;$

$\forall\ \succeq_i$ *are continuous,*

then, any Pareto optimal allocation $(x_i^0)_{i=1}^m$ *is supported by a price* p *s.t:* $\forall\ i, x_i \in X_i,\ x_i \succeq_i x_i^0\ \Rightarrow p \cdot x_i \geq p \cdot x_i^0.$

Under the classical assumption on the agents' preferences (continuous, convex, strongly monotone, and locally non satiated), and if the initial endowments are strictly positive for all the agents ($e_i \gg 0$) the existence of walrasian equilibrium is guaranteed. So, now we can show that the SWT is a particular case of this existence result. To see this, suppose that $x = (x_1, ..., x_m)$ is a Pareto optimal allocation of a pure exchange economy, then a Walrasian equilibrium $(p, \hat{x})$ existS for the economy in which endowments are $\forall\ i\ e_i = x_i$. In fact, $\hat{x} \succeq_i x_i$, and x_i is affordable at price p for every consumer i. It follows, from the Pareto optimality of x, that $\hat{x} \sim_i x_i\ \forall i$. But since $\hat{x}_i$ is the optimal demand given prices p, $p \cdot e_i = p \cdot \hat{x}_i = p \cdot x_i$, x_i must be an optimal demand for consumer i for price p. Hence, the price vector p support the allocation x . The SWT insures us on the existence of a supporting hyperplane for a given Pareto optimal allocation. From the First welfare theorem we know that if $x_i \succ_i x_i^*$ then $p \cdot x_i > p \cdot x_i^*$, and it's easy to verify that, under the same assumption, $p \cdot x_i \geq p \cdot x_i^*$. Now we provides a sufficient condition under which the condition "$x_i \succ_i x_i^*$ implies $p \cdot x_i \geq p \cdot x_i^*$" is equivalent to preference maximization condition "$x_i \succ_i x_i^*$ implies $p \cdot x_i > p \cdot x_i^*$".

Proposition 4.0.8 *Assume that X_i is convex and $\geq_i$ is continuous. Suppose, also, that the consumption vector $x_i^* \in X_i$, the price vector p are such that $x_i \succ_i x_i^*$ implies $p \cdot x_i \geq p \cdot e_i$. Then, if there is a consumption vector $x_i' \in X_i$ such that $p \cdot x_i' < p \cdot e_i$, it follows that $x_i \succ_i x_i^*$ implies $p \cdot x_i > p \cdot e_i$.*

PROOF: Assume that $p \cdot x_i^* = p \cdot e_i$. Suppose by the contrary that there is an $x_i \succ_i x_i^*$ with $p \cdot x_i = p \cdot e_i$. By the cheaper point assumption, there exist an $x_i' \in X_i$ such that $p \cdot x_i' < p \cdot e_i$. Then, for all $\alpha \in [0, 1($, we have $\alpha x_i + (1 - \alpha)x_i' \in X_i$ and $p \cdot \alpha x_i + (1 - \alpha)x_i' < p \cdot e_i$. But, if α is close enough to 1, the continuity of $\succeq_i$ implies that $\alpha x_i + (1 - \alpha)x_i' \succ_i x_i^*$, which is a contradiction because we have found a consumption bundle that is preferred to x_i^* and costs less.

Moreover, if $x_i^* = e_i$, then the proposition 4.0.8 gives a sufficient conditions for the equivalence between this two problems:

$$x_i^* \text{ minimizes the expenditure in the set } \{x_i \in X_i : x_i \succeq x_i^*\}$$
$$\text{and}$$
$$x_i^* \text{ maximizes } \succeq_i \text{ in the budget set } \{x_i \in X_i : p \cdot x_i \leq p \cdot x_i^*\}$$

□

Turn now to the case of an economy with a large number of consumers. It's well known the equivalence result between the core allocations and the walrasian equilibrium. In particular, the asserting that the core allocations are Walrasian constitutes a version of the SWT.

Our interest is in economy with differential information. In particular, we give our attention to the characterization of Radner equilibrium allocations as those random consumption plans which are not blocked by grand coalition. In fact, if the initial endowments is such that $e_i = x_i \ \forall i$, the SWT is a particular case of the equivalence theorem. We can observe that, with this hypothesis, the economy $\mathcal{E}(a, x)$ is equivalent to the economy $\mathcal{E}(0, x)$, and x_i is its endowments for all agents. So x_i is a Pareto optimal allocation such that $\sum_i x_i = \sum_i e_i$, and it is not blocked by grand coalition. It is not so hardly to prove that an equivalent result is true for economies with a continuum of agents.

4.1 Supporting Price

In the previous section, we have found a sufficient condition for the equivalence between the following concept of supporting price:

* $x_i \succ_i x_i^*$ implies $p \cdot x_i \geq p \cdot x_i^*$,
* $x_i \succ_i x_i^*$ implies $p \cdot x_i > p \cdot x_i^*$.

where x_i^* is the optimal demand for all i. Now we want to prove an equivalence result between:

* $z_i \succeq_i x_i$ implies $p \cdot z_i \geq p \cdot x_i$,
* $z_i \succ_i x_i$ implies $p \cdot z_i > p \cdot x_i$.

We focus our attention on two cases: the first (a), the consumption set $X_i \equiv R_+^l$; the second (b), instead, the consumption set, X_i, differs among the agents.

For our proof, is necessary to look at the result present in Debreu 1954 on these propositions:

Proposition 4.1.1 *Under classical assumption on preferences and on consumption sets, with every Pareto optimum $(x_i^0)_{i=1}^m$, where some x_i^0 is not a saturation point, is associated a continuous linear price p such that, for every i*

$$x_i \in X_i, \;\; x_i \succeq_i x_i^0 \;\Rightarrow p \cdot x_i \geq p \cdot x_i^0$$

Definition 4.1.2 *Let p a linear continuous function. $(x_i^0)_{i=1}^m$ is an equilibrium with respect to p if:*

i) $(x_i^0)_{i=1}^m$ is attainable;

ii) For every i, $x_i \in X_i, \quad p \cdot x_i \geq p \cdot x_i^0 \;\Rightarrow x_i \succeq_i x_i^0$

We want to show that, under the classical assumption on convexity of X_i and of the preference relation, if there is, for every i, an $x_i' \in X_i$ such that $p \cdot x_i' < p \cdot x_i^0$, then 4.1.1 implies 4.1.2

PROOF: Consider an $x_i \in X_i$ s.t. $p{\cdot}x_i \leq p{\cdot}x_i^0$. Let $x_i(t) = (1-t)x_i + tx_i^0$. For all t, $0 < t < 1$, $p \cdot x_i(t) \leq p \cdot x_i^0$, and thus, by 4.1.1, $x_i^0 \succ x_i(t)$. For t converging to 0, follows that $x_i^0 \succeq x_i(t)$.

With these results in mind, we can prove:

Case a

Consider $z, x \in \Re^l_+$, $z \succ_i x_i \ \forall\ i \Rightarrow\ p \cdot z > p \cdot x_i$. For the convexity, for all $0 < t < 1$, $tz + (1-t)x_i \succ_i x_i{}^1 \ \Rightarrow\ tp \cdot z + (1-t)p \cdot x_i > p \cdot x_i$. As $t \to 0$, we have the assert.

Consider, now, $z, x \in \Re^l_+$, $z \succeq_i x_i \ \Rightarrow\ p{\cdot}z \geq p{\cdot}x_i$. For the convexity and for the locally non satiation, there always exists a strictly preferred bundle. Thus, given $y \neq 0 \in \Re^l_+$, $z + y \succ_i x_i$ and $p \cdot z + p \cdot y > p \cdot x_i$.

Case b

Consider $z, x \in X_i$, X_i convex and closed set. From continuity of preferences and from convexity of X_i, if $z \succ_i x_i \ \forall\ i$, there exists an $\alpha \in (0, 1]$ such that $\alpha z + (1-\alpha)x_i \succ_i x_i$. As $\alpha \to 0$, we have the assert.

Consider $z, x \in X_i$, $z \succeq_i x_i$ implies $p \cdot z \geq p \cdot x_i$. Inlight proposition 4.0.8, if there exists a $g_i \in X_i$, $p \cdot g < p \cdot x_i$, with $x_i = e_i$, we can write that $g_i \succeq x_i \ \forall i$. By this, the contradiction, because if $g_i \succeq x_i$, then $p \cdot g \leq p \cdot x_i$. □

[1] For the completness of the preferences, we know that if $x \succ x'$ then $x \succeq x'$.

4.1.1 Considerations on Cheaper Consumption

Finally, we want to appoint some consideration on the existence of cheaper bundle. First, we can note that closeness, convex assumptions on the consumption set X_i, and the locally nonsatiation of the preferences ensure us on the nonemptyness of X_i. But, for the existence of a cheaper bundle is not suffice.

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