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Abstract social choice and implementation in finite and infinite societies

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Contents

1 Introduction

The main objective of this work is twofold. First, we provide a unified analytical framework within wich impossibility results in abstract social choice can be stated and proved. Second, we hint at some links between impossibility results on abstract preference aggregation and implementation of allocation rules for pure exchange economies.

In the abstract preference aggregation section we will present the classical impossibility results by Arrow and Gibbard Satthertwaite in a unified way, to point at the similarities between the two. It is quite interesting that both theorems can be proved through the ultrafilter property of some subsets of agents, suggesting the existence of a common mathematical structure behind the two results.

This type of characterization clarifies the main difference between finite and infinite societies. In particular, aggregation procedures that satisfy the axioms in the respective settings are proven to exist when society is formed by an infinite number of individuals. That is, there is a discontinuity going from the finite to infinite case. However, even though non -dictatorial aggregation procedures can be found by appealing to the existence of free-ultrafilters on infinite sets, they admit arbitrarily small coalitions of agents that are in some sense dictatorial, and converge to a limit outside the space of agents: *the invisible dictator*.

Strictly speaking, the implementation problem in exchange economies is a particular case of a social choice problem. However, the results in the abstract setting do not directly carry over, because the domain of the social choice function is considerably restricted by the classical assumptions on continuity convexity and smoothness of preferences. It is therefore interesting that results that have the flavor of dictatorial results hold for finite societies in this restricted setting as well. In particular, it is the case that Pareto efficiency when coupled with an incentive compatible requirement produces allocation that leave some consumers with consumptions level that is arbitrarily small. This surprising result represents the parallel to the impossibility theorems in the abstract setting.

As it is well known, in exchange economies a prominent role is played by the walrasian mechanism, that is characterized by linear prices and price taking behavior. This last assumption has been justified by an argument that relies on agents that have zero measure in economies with a continuum of agents (e.g. Aumann 1964). However, it was the pioneering article by Hurwicz(1972) that made explicit the link between preference manipulation and incentives to manipulate equilibrium prices by showing that walrasian equilibrium is not incentive compatible in finite economies. It is therefore the objective of the models that assume pure exchange economies with infinite agents to study whether price taking behavior (and therefore walrasian equilibrium) can be justified by a zero measure assumption. In this respect, there seems to be a discontinuity as well. If one assumes a continuum, it is quite easy to find incentive compatible walrasian equilibria. However, the results on sequences of large but finite economies are much more limited, suggesting a parallel with the invisible dictator result.

The sections are organized as follows. Part I deals with abstract social choice. In section 2 I briefly summarize the notation and properties of abstract aggregation procedures. In section 3 the classical impossibility results for social choice and social welfare function are presented, and in section 4 the corresponding possibility results with infinitely many agents. Section 5 is devoted to the proofs, in particular the common proof of impossibility based on ultrafilters.Part II deals with incentive compatible allocation mechanism. Section 6 describe the implementation problem in pure exchange economies with a finite number of agents and section 7 treats the infinite agents counterpart. Section 8 concludes.

Part I

Abstract social choice

2 Definitions and properties of aggregation functions

In this sectionwe we summarize the notation and main definitions , with the aim at providing a unifying framework for social choice and social welfare functions.

A preference aggregation problem consists of a triple $((U, v), A, \mathcal{R})$ and an aggregation function F, whose domain consists of individual information to be aggregated into an outcome, the value in the codomain, that represents the social aggregation. U is the set of individuals, A is the set of feasible alternatives and $\mathcal{R}$ is the set of binary preference relations on A. The function $v : U \to \mathcal{R}' \subseteq \mathcal{R}$ associates each individual with a preference relation over A, and the set of all possible situations is the set of functions $V = \mathcal{R}'^U$. Notice in particular that if U is finite, V is simply the $|U| - fold$ Cartesian product of $\mathcal{R}$. The advantage of this general formulation is that it allows to deal with the case of U not finite (countably or even uncountably) easily.

While the domain of the aggregation function is always V, the codomain may be either A or $\mathcal{R}'$, with the following definitions:

Definition 1 *A social welfare function is a function* $F_W : V \to \mathcal{R}$. *A social choice function is a function* $F_C : V \to A$

The interpretation is straightforward. The social welfare function, first considered by Arrow(1963) aggregates individual preference relations in a social preference relation, while the social choice function aggregates individual preferences into a social choice, that might be interpreted as being the alternative that society prefers when individuals have preferences $v \in V$.

The basic question every aggregation theory wants to answer is the following: suppose we believe in certain properties of an aggregation procedure, then does it exists a social welfare or social choice function for which these properties are mutually compatible?

The literature on social aggregation has focused on institutional and ethical properties of the aggregation problem.

2.1 Institutional properties

We will name institutional all those properties that refer to the positive environment to which we apply the aggregation rule. First, the cardinality of A. As we will see in the following section, the impossibility result depends crucially on the assumption that society has the choice among more than 3 alternatives. Second, and more important, the cardinality of the set of individuals U. Here, the key assumption is whether this set is finite or infinite.

Another important set of assumptions is related to the subset of preferences that constitutes the domain of the aggregation rule $F : \mathcal{R}'$.A minimal common assumption is that $\mathcal{R}'$ is the set of weak orders over A (with the added technical requirement that the upper and lower contour sets are closed when A is an infinite set). If these are the only restrictions put on the possible preferences profiles, the domain is *rich.* In general, richness of the domain implies more restrictions on the aggregation rules that can be implemented, while the opposite is true if we restrict the domain to include only some particular types of preferences. [1] Since the main focus will be on the consequences of allowing for infinitely large societies, we shall not go into the details of other types of domain restrictions[2].

Last, a key institutional assumption is that an hypothetical social planner does not know the profile of individual preferences, implying that he has to rely on aggregation rules that are *incentive compatible* . The aggregation rule is *individually manipulable* if there exists an individual u, a profile v and another profile v' differing from v only for preferences of u (that is to say $v(u) \neq v'(u)$ and $v(U\backslash\{u\}) = v'(U\backslash\{u\})$ such that $F_C(w)v(u)F_C(v)$. The interpretation of this condition is clear: if a social choice

[1]One much studied restriction that applies when A is an ordered set is *single-peakdness:*$[\exists a \in A$ s.t. (b, c) with the property $a < b < c$ (or with inequalities reversed) $\Longrightarrow aPbPc]$, where P is the asymmetric part of R.

[2]For some recent characterization results, see Nehring Puppe(2002) and Aswal Chatterji Sen(2003)

function is individually manipulable, then at some profile there is an individual who finds profitable to hide his preferences because the social choice resulting from this strategic behavior is preferred, according to his true preferences, to the social choice resulting from sincere revelation of his preference relation. An aggregation rule is *individually incentive compatible* if it is not *individually manipulable.*

2.2 Ethical properties

I will name ethical all those properties that concern the normative implications of an aggregation rule.

Starting with a social welfare function, we first can identify assumptions on the social preference relation (that is, the relation $F_W(v)$). Arrow first required $F_W(v)$ to be coherent with individual preferences, being itself transitive and complete: $F_W(v) \subseteq \mathcal{R}'$.. Another important arrovian assumption is *independence of irrelevant alternatives (IIA).* F_W satisfies IIA iff $[\forall a\&b \in A, v\&w \in V$, $(v = w$ on $(a,b)) \Longrightarrow F_W(v) = F_W(w)]$.

An important set of ethical properties are those related to some form of efficiency of the aggregation rule. A minimal efficiency requirement is *unanimity:* $[\forall a, b \in A, av(U)b \Longrightarrow aF_W(v)b]$.For a social choice function, the analogous definition requires unanimity in the case in which every individual has the same alternative on top of his preference relation: $[av(U)b\forall b \in A \Rightarrow F(v) = a]$. For a social choice function a stronger property is *Pareto-efficiency* $[av(U)b \Rightarrow F(v) \neq b]$

Finally, a social aggregation rule might give decision power to a unique individual. F_W is *dictatorial* if $\exists u_0$ s.t. $\forall v \in V$, $av(u_0)b \Rightarrow aF_W(v)b.$ similarly, F_C is dictatorial if $\exists u_0$ s.t. $\forall v \in V, av(u_0)b$ for each $b \in A) \Rightarrow a = F_C(v)$

3 Impossibility results with finitely many agents

In the previous section,we have summarized properties that is reasonable to impose on aggregation rules. Unfortunately, starting with the famous Arrow's theorem, many of these properties have been proved to be mutually inconsistent when society is composed by a finite number of individuals. In order to underline the role of finiteness,we will describe some quite general formal results, from which Arrow and Gibbard-Satthertwaite theorems will follow as direct corollaries in the case of a finite society.

In first place, we need a formal concept of "large" sets (Brown 1975). Let U be a set. A *filter* $\mathcal{F}$ on U is a collection of subsets such that:

(F1) $S \subseteq \mathcal{F}\,, S \subseteq S' \Rightarrow S' \in \mathcal{F}$

(F2)$S, S' \subseteq \mathcal{F} \Rightarrow (S \cap S') \subseteq \mathcal{F}$

(F3) $\emptyset \notin \mathcal{F}$

An *ultrafilter* $\mathcal{U}$ is a filter that is not strictly contained in another filter. For an ultrafilter, we have the property:

(UF) $\forall S \subseteq U$ either $S \subseteq \mathcal{U}$ or its complement $S^c \subseteq \mathcal{U}$.

If a filter has the empty intersection property, $\cap_{S \in \mathcal{F}} S = \emptyset$, it is a *free filter.* Filters that are not free are called *fixed*.From these definitions, the properties in the following lemma follow easily (Aliprantis et al. 1999):

Lemma 1 *i)If $\mathcal{U}$ is an ultrafilter on U and $\{S_i\}_{i=1}^n$ a finite partition of U then*

$S_i \in \mathcal{U}$ for some i.

ii) If $\mathcal{U}$ is a free ultrafilter on U, then it contains no finite elements of U. In particular, only infinite sets admit free ultrafilters

iii) Every fixed ultrafilter on U has the form: $\mathcal{U}_x = \{S \subseteq U \text{ s.t.} x \in S\}$ for a unique $x \in U$

When we consider a social aggregation environment, it turns out that some particular subsets of the space of individuals have the ultrafilter property. For a social welfare function, we can define the set of *decisive coalitions:*

Definition 2 *The set of decisive coalitions associated with the social welfare function F_W is $\mathcal{U}_D = \{S \subseteq U \text{ s.t. } av(S)b \Rightarrow aF_w(v)b\}$*

In words, whenever individuals in a decisive coalition prefer an alternative over another, the aggregate preference relation agrees with this ordering, independently of the preferences of other members of the society. An important remark is that if there is an individual belonging to every decisive coalition, then he is a dictator.

In the context of a social choice function, we have an analogous definition of *families of preventing sets* (Batteau et al. 1981)

Definition 3 *The family of preventing sets for an alternative b by an alternative a associated with a social choice function F_C is $\mathcal{U}_C^{ab} = \{S \subseteq U \text{ s.t. } av(S)b \Rightarrow F_C(v) \neq b\}$*

Agents in a preventing set have veto power. They are able to block the choice of an alternative if they all prefer another alternative over it.

The main results of this section are the following theorems, showing that the ultrafilter property of decisive coalitions (Kirman Sondermann 1972) and of (the unique) family of preventing set (Batteau et al. 1981) descends directly from the assumptions described in the previous discussion.

Theorem 1 *Let* $\mid A \mid \geq 3$*,* $\mathcal{R}'$ *rich,* F_W *a social welfare function such that* $F_W(V) = \mathcal{R}'$ *and satisfying Unanimity and IIA , then the set of decisive coalitions* $\mathcal{U}_D$ *is an ultrafilter.*

The analogous theorem for a social choice function is slightly more involved, since we have defined a family of preventing set for an alternative over another. In principle, these families may vary with the considered alternatives. However, in the case of an incentive compatible rule, the following theorem tells us that there is a unique family of preventing sets, and it has the ultrafilter property.

Theorem 2 *Let* $\mid A \mid \geq 3$*,* $\mathcal{R}'$ *rich,* F_C *a social welfare function such that* $F_C(V) = A$*. Then:*

i)F_C *is incentive compatible in dominant strategies if and only if for each* $v, w \in V$ *and* $a \neq b$ *s.t.* $F_C(v) = a$ *and* $av(u)b \Rightarrow aw(u)b$*, it is true that* $F_C(w) \neq b$ *(property of monotonicity)*

ii)If F_C *is incentive compatible in dominant strategies then it satisfies unanimity and Pareto efficiency.*

iii)If F_C *is incentive compatible in dominant strategies then for each pair of alternatives* (a, b) *the families of preventing sets are identical:* $\mathcal{U}_C^{ab} = \mathcal{U}_C^{ba} \equiv \mathcal{U}_C$*..Moreover,* $\mathcal{U}_C$ *is an ultrafilter.*

Property i) is a result of independent interest. The monotonicity property says that, if going from a profile v to a profile w the set of individuals who prefer alternative a to alternative b does not shrink, and if alternative a was chosen for the profile v, then alternative b cannot be chosen for profile w.

From these two theorems, the famous Arrow and Gibbard-Satterthwaite results descend as direct corollaries in the case of a finite society.

Corollary 1 *(Arrow 1963) Under the assumptions of Theorem 1, if U is finite F_W is dictatorial.*

The logic is straightforward. We know from Lemma 1 that every ultrafilter on a finite set is fixed. Moreover, fixed ultrafilters on the space of individuals U have the property that there is a unique individual belonging to every set of the ultrafilter. Since the set of decisive coalitions is an ultrafilter, there is a unique individual belonging to each decisive coalition. Therefore, he is a dictator.

The analogous result for a social choice function is the Gibbard-Satthertwaite theorem:

Corollary 2 *(Gibbard 1973 Satterthwaite 1975) Under the assumptions of Theorem 2, if F_C is incentive compatible, then it is dictatorial.*

Again, from the ultrafilter property we know that there is a unique individual that belongs to every set of the unique family of preventing sets. This individual is able to veto every alternative, in particular every alternative different from his most preferred one. Therefore, he is able to impose his most preferred alternative no matter what the preferences of other members of society are. This is precisely the relevant definition of a dictator for a social choice function.[3]

4 Infinitely many agents and the invisible dictator

In finite societies, the ultrafilter structure of decisive coalitions produces necessarily a dictatorial result. The scenario changes dramatically in the case of an infinite society (Fishburn 1970, Kirman Sondermann 1972) . The reason is that a free ultrafilter always exists over infinite sets.

Lemma 2 *Assume Zorn's Lemma (or equivalently the Axiom of Choice). Every filter is included in at least one ultrafilter. Every infinite set has at least one ultrafilter.*

Since it will be useful in giving an example of a non dictatorial aggregation rules over infinite societies, I briefly describe the logic of this mathematical result. If one considers a generic filter $\mathcal{F}$ over U, the collection $\mathcal{C} = \{\mathcal{G} \text{ s.t. } \mathcal{G} \text{ is a filter and } \mathcal{F} \subset \mathcal{G}\}$ is partially ordered by inclusion. Therefore if one considers a chain in $\mathcal{C}$ it will have an upper bound in $\mathcal{C}$ and, by Zorn's Lemma, a maximal element, that, by definition, is an ultrafilter including $\mathcal{F}$. When the set U is infinite we can consider the collection of subsets that are complement of finite sets $\mathcal{F}_{cofinite} = \{S \subseteq U \text{ s.t. } \mid S^c \mid < \infty\}$,

[3] The close relationship between the two theorems has been recently studied by Reny(2001), who was able to provide a word-by-word unique proof adapting the procedure introduced by Geanakoplos(1996) to find a *pivotal* individual, who turns out to be a dictator.Interestingly, that procedure works only in the finite case.

that is easily seen to be a free filter over U.By the preceding argument, this will be included in at least one ultrafilter.

Going back to the aggregation setting, by Lemma 2, there exists a free ultrafilter $\mathcal{U}_{D^*}$ over the infinite set of individuals U. Then define a social preference relation by:

$$aF_W^*(v)b \Longleftrightarrow S_0 \equiv \{u_0 \in U \text{ s.t. } av(u_0)b\} \in \mathcal{U}_{D^*}$$

According to this aggregation rule, society prefers a over b if and only if the coalition of individuals who prefer a over b belongs to the free ultrafilter $\mathcal{U}_{D^*}$. Intuitively, F_W^* gives decision power only to large coalitions[4]. By construction, it is clearly non dictatorial and satisfies unanimity. The relevance of the following theorem is then the assertion that F_W^* is actually a preference relation and satisfies IIA.

Theorem 3 *Assume Zorn's Lemma. Then when U is infinite there exists a non dictatorial social welfare function F_W^* , such that $F_W^*(V) = \mathcal{R}'$ that satisfies Unanimity and IIA.*

Again, for a social choice function an analogous result, holds, where the construction of a non-dictatorial rule follows the same logic of the previous case. However, it will be important to assume that the set of alternatives is finite.

It is quite easy to construct a non dictatorial *individually* incentive compatible social choice function. It will suffice to define for each alternative $a_j \in A = \{a_1, .., a_m\}$ and profile of preferences $v \in V$ the set of individuals for which alternative a_j is the

[4]This interpretation is reinforced considering a measure theoretic representation of ultrafilters. It is a fact that to every ultrafilter corresponds a $\{0,1\}$-measure over the set on which the ultrafilter is defined, and takes value 1 on the elements of the ultrafilter. The social welfare function F_W^* therefore assigns decisive power only to coalitions of full measure, giving a precise meaning to what a large set is.

most preferred one:$D(a_j, v)$ and set the social choice $F^*_C(\ v) = a_{j^*}$ where $_{j^*}$ is the first index for which $D(a_j, v)$ is an infinite set. Since no individual can alter the finiteness of the sets D, it will clearly be individually incentive compatible, and it will satisfy unanimity by definition.

Through a free ultrafilter we can go even further and construct a *coalitionally* incentive compatible social choice function(Pazner Wesley 1977). A social choice function is coalitionally incentive compatible if it is not manipulable by a subset of agents that jointly hide their preferences. In first place, notice that the sets $\{D(a_j, v)\}_{j=1}^m$ form a finite partition of the infinite set U, that admits a free ultrafilter $\mathcal{U}_{C^*}$[5]. By part i) of Lemma 1 one of the elements of the partition is an element of the free ultrafilter. So $D(a_{j_0}, v) \in \mathcal{U}_{C^*}$ for some a_{j_0}.Let this alternative be the social choice. The social choice function we have constructed is therefore:

$$F^*_C(v) = a_{j_0} \Longleftrightarrow [D(a_{j_0},\, v) \in \mathcal{U}_C \text{ for a free ultrafilter } \mathcal{U}_{C^*}]$$

F^*_C is by construction non dictatorial, since an element of the free ultrafilter cannot be finite, and it satisfies unanimity. Pazner Wesley(1977) prove that it is also coalitionally incentive compatible.

Theorem 4 *Let U infinite, A finite*[6] *and assume Zorn's Lemma. Then there exists a*

[5] Clearly, as in the previous case, Zorn's Lemma needs to be assumed. For a constructive proof without the use of the Axiom of Choice, but restricting the set of admissible profiles to be a Boolean algebra, see Mijara(2001)

[6] When A is not finite, Pazner Wesley(1977) show that the theorem continues to hold, however the social choice function is not Pareto efficient. Moreover, if U is countably infinite and A infinite, there exists no coalitionally incentive compatible social choice function that are Pareto efficient and non dictatorial. Quite interestingly, the problem of whether there exists a set of individuals (with higher cardinality than $\mathbb{N}$) for which a Pareto efficient coalitionally incentive compatible non dictatorial social choice function exists is linked to the existence of a set of measurable cardinality.

coalitionally incentive compatible social choice function that satisfies unanimity and is non dictatorial.

Up to this point, it may seem that going from a finite society to an infinite society solves the problem of aggregation nicely. However, some further insight in the structure of the rules we have constructed for the infinite case casts some doubt on what non dictatorship really means for an infinite society.

Let us specialize our discussion and consider a social welfare function setting assuming that we have an atomless measure space of agents (U, Ω, μ), where Ω is a $\sigma - algerba$ of coalitions and μ a non negative measure over Ω.By the properties of an atomless measure, the set U can be partitioned in a finite collection of sets each of measure less than an arbitrarily small $\varepsilon > 0$. Using property i) of Lemma 1 again, one of the element of this partition will belong to the ultrafilter of decisive coalitions. Therefore, even though there is no individual dictator, it is true that:

Proposition 5 *A social welfare function that satisfies Unanimity and IIA over an atomless measure space of agents admits an arbitrarily small coalition of decisive individuals.*

For a finite set of alternatives, some topological considerations allow us to say more about the nature of dictatorship in an infinite society. In that case the set of preference relations $\mathcal{R}$ is a finite set, endowed with the discrete topology. Recall also that the set of possible profiles of preferences for the society is the set of of functions $V = \{v : U \to \mathcal{R}\}$. The Stone-Cech compactification of U is a compact set '$\tilde{U}$ such that U is dense in $\tilde{U}$ and (what is important for the present discussion) every $v \in V$ can be uniquely extended to a continuos function from $\tilde{U}$ into $\mathcal{R}$. The key observation is that the ultrafilter of decisive coalitions for the infinite set U has a *unique* limit $\tilde{u}_0$in $\tilde{U}$ whose preferences are represented by the unique extension $v(\tilde{u}_0) = lim_{u,\mathcal{U}_D} v(u)$.Since $\mathcal{R}$ is discrete, it will exists a set S in the ultrafilter of decisive coalitions whose preferences are exactly the preference of the limit point $\tilde{u}_0 : v(\tilde{u}_0) = v(S)$.This discussion implies that there will be an element in $\tilde{U}$ that has the properties of a dictator:

Theorem 6 *(Kirman Sondermann "The invisible dictator" 1972) A social welfare function that satisfies Unanimity and IIA over an infinite set of agents U and a finite set of alternatives admits an "invisible dictator" $\tilde{u}_0$ in the Stone-Cech compactification of U, whose preferences are the limit preferences of an arbitrarily small set of decisive individuals in U.*

This beautiful and elegant theorem clarifies the nature of dictatorship in the infinite case. Even though in large societies the weight of each individual is small enough (actually zero), there exists arbitrarily small hierarchies of agents that are as decisive as a traditional dictator, and are represented by an agent "behind the scenes" (the invisible dictator).

5 Proofs

In this section we prove th main results stated in the previous discussion on abstract aggregation. The first part is devoted to the proof of the impossibility results in a finite society. The objective is to prove Arrow and Gibbard Satthertwaite theorem in a unified fashion, where the unifying theme is represented by the ultrafilter property of decisive coalitions and of preventing sets. To that end, we need two preliminary steps. The first, provided by theorem 2- i) allows us to substitute the incentive compatibility property with the equivalent property of monotonicity. The second step, theorem 2-ii), shows that Pareto efficiency and unanimity are implied by incentive compatibility and the onto assumption.

Proof of Theorem 2 i)-ii)

i)Suppose F_C is not incentive compatible. Then there exists an individual u_0 and profiles v and w differing only in the preferences of u (that is $v(u_0) \neq w(u_0)$ and $v(U \backslash \{u_0\}) = w(U \backslash \{u_0\})$) such that $F_C(v) \neq F_C(w)$ and $F_C(w)v(u_0)F_C(w)$. Since w and v differ only on u_0, and at v this agent prefers $F_C(w)$ to $F_C(v)$ we have $[F_C(v)v(u)F_C(w) \Longrightarrow F_C(v)w(u)F_C(v)]$.Since $F_C(v) \neq F_C(w)$ this social choice function is not monotonic. In the other direction, suppose F_C is incentive compatible. We want to prove that it is monotonic. That is: for each $v, v' \in V$ and $a \neq b$ s.t. $F_C(v) = a$ and $av(u)b \Rightarrow av'(u)b$, it is true that $F_C(v') \neq b$.Pick v,w satisfying $[av(u)b \Rightarrow av'(u)b]$ and $F_C(v) = a$. Define the following sets:

$S_1 \equiv \{u \in U : av(u)b\}$

$S_2 \equiv \{u \in U : bv'(u)a\}$

Form a new profile v''according to the following:

S_1	S_2	$S_1 \cap S_2$		S_1	S_2	$S_1 \cap S_2$
a	b	a		a	b	b
b	a	b	or	b	a	a
.	.	.		.	.	.
.	.	.		.	.	.
.	.	.		.	.	.

In profile v'' we have moved $\{a, b\}$ to the top of the each preference profile in v and v' while preserving their relative order. First notice that $F_C(v'') \in \{a, b\}$ otherwise individuals in S_1 would gain by misrepresenting their preferences.

ii) Let v be a profile for which the set of individuals who prefer b to a is empty and suppose $F_C(v) = b$. Since for any other profile v' we would have that the set of individuals who prefer b to a contains the empty set, monotonicity would require $F_C(v') \neq a$, thus excluding a from the range of F_C, contradicting the onto assumption. Therefore $F_C(v) \neq b$, and Pareto efficiency is proved. Now if every individual prefers a to all every other alternative b by unanimity we must have $F_C(v) \neq b$ for each $b \neq a$ .Therefore $F_C(v) = a$ and unanimity is proved.

A direct corollary of Pareto efficiency is the following *tops only* property:

Corollary 3 *(Tops only) Let $B \subseteq A$, F_C satisfies the assumptions of theorem (???). If $[b \notin B \ \&a \in B \Longrightarrow av(U)b]$ then $F_C(v) \in B$.*

In words, if a subset of alternatives is on top of each agent's profile, then the social choice must belong to this subset, if the aggregation function is to be incentive compatible and onto.

Proof of theorem 1 and theorem 2-*iii*) (*Ultrafilter property)*

The two previous steps allow us to use in the proof the fact that F_C satisfies monotonicity, efficiency and tops only. Let us define the following system of subsets:

$\mathcal{U}_D^{ab} = \{S \subseteq U : \forall v \in V \ av(S)b\&bv(S^c)a \Longrightarrow aF_W(v)b\}$

$\mathcal{U}_D'^{ab} = \{S \subseteq U : \exists v \in V \text{ s.t. } av(S)b\&bv(S^c)a\&aF_W(v)b\}$

$\mathcal{U}_C^{ab} = \{S \subseteq U : \forall v \in V \ . \ av(S)b \Rightarrow F_C(v) \neq b\}$

$\mathcal{U}_C'^{ab} = \{S \subseteq U : \exists v \in V \text{ s.t. } av(S)b\&F_C(v) = a\}$

First, let's prove that $\mathcal{U}_D^{ab} = \mathcal{U}_D'^{ab}$ and $\mathcal{U}_C^{ab} = \mathcal{U}_C'^{ab}$. The inclusion $\mathcal{U}_D'^{ab} \subseteq \mathcal{U}_D^{ab}$ follows easily from IIA and $\mathcal{U}_C'^{ab} \subseteq \mathcal{U}_C^{ab}$ from monotonicity. In the other direction, the inclusion $\mathcal{U}_D^{ab} \subseteq \mathcal{U}_D'^{ab}$ is trivial, while the inclusion $\mathcal{U}_C^{ab} \subseteq \mathcal{U}_C'^{ab}$ follows from the observation that we can always find a profile

S	S^c
a	b
b	a
.	.
.	.
.	.
.	.

with $\{a, b\}$ on top of each agent preference relation and $S \in \mathcal{U}_C^{ab}$, implying $F_C(v) \in \{a, b\}$ and $F_C(v) \neq b$, therefore $F_C(v) = a$ and $S \in \mathcal{U}_C'^{ab}$.

Next we show that both the families of decisive coalitions,$\mathcal{U}_D^{ab}$, and the family of preventing sets, $\mathcal{U}_C^{ab}$,do not depend on the specific alternative, which implies, in both settings, the existence of a unique family of agents with decisive power. We want to prove that $\mathcal{U}_D^{ab} = \mathcal{U}_D^{ba}$ and $\mathcal{U}_C^{ab} = \mathcal{U}_C^{ba}$.

We begin with the inclusion $\mathcal{U}^{ab} \subseteq \mathcal{U}^{ac}$ for $c \notin \{a, b\}$. Let $S \in \mathcal{U}_D^{ab}, T \in \mathcal{U}_D^{bc}$ and $S \in \mathcal{U}_C^{ab}$, $T \in \mathcal{U}_D^{bc}$ and define the following profile v:

$S - T$	$T \cap S$	$T - S$	$(T \cup S)^c$
a	a	b	c
c	b	c	b
b	c	a	a
.	.	.	.
.	.	.	.
.	.	.	.

Now $S \in \mathcal{U}_D^{ab}$ implies $aF_W(v)b$ and $T \in \mathcal{U}_D^{bc}$ implies $bF_W(v)c$, and by transitivity of the social preference relation it follows that $aF_W(v)c$. We have then found a profile for which $av(S)c\&cv(S^c)a\&aF_W(v)c$, so $S \in \mathcal{U}_D'^{ac} = \mathcal{U}_D^{ac}$. For the social choice function, tops only implies $F_C(v) \in \{a, b, c\}$, $S \in \mathcal{U}_C^{ab}$ implies $F_C(v) \neq b$ and $T \in \mathcal{U}_D^{bc}$ implies $F_C(v) \neq c$, therefore $F_C(v) = a$. Again we have found a profile for which $av(S)c\&F_C(v) = a$, then $S \in \mathcal{U}_C'^{ac} = \mathcal{U}_C^{ab}$.

By swapping b and c we obtain $\mathcal{U}^{ac} \subseteq \mathcal{U}^{ab}$, which implies $\mathcal{U}^{ab} = \mathcal{U}^{ac}$ for each $c \neq a$. Moreover by fixing the second alternative the same proof can be used to show that $\mathcal{U}^{ab} = \mathcal{U}^{eb}$ for each $e \neq b$. Therefore $\mathcal{U}^{ac} = \mathcal{U}^{eb}$ for $c \neq a$ and $e \neq b$. Letting $c = b$ and $e = a$ we obtain $\mathcal{U}^{ab} = \mathcal{U}^{ba}$, therefore there exists a unique family of decisive coalitions. Notice that the result holds both for a social choice function and for a social welfare function.

We can now prove that the systems $\mathcal{U}_D^{ab}$ and $\mathcal{U}_C^{ab}$ form an ultrafilter on V.

- $\forall\ S \subseteq V$, either $S \in \mathcal{U}^{ab}$ or $S^c \in \mathcal{U}^{ba}$. Consider the following profile

$$\begin{array}{cc} S & S^c \\ b & a \\ a & b \\ . & . \\ . & . \\ . & . \end{array}$$

If $S \in \mathcal{U}_C^{ba}$ then $F_C(v) \neq a$, by tops only $F_C(v) = b$ and this implies $S^c \notin \mathcal{U}_C^{ab}$. Also, if $S \in \mathcal{U}_D^{ba}$ then $bF_W(v)a$ which implies $S^c \notin \mathcal{U}_D^{ab}$.

- $V \in \mathcal{U}^{ab}$ .This is simply a consequence of unanimity, which is an assumption for the social welfare function, while its a property of an incentive compatible social choice function by $ii)$ in theorem 2. . Consequently, $\varnothing \notin \mathcal{U}^{ab}$, since $\varnothing = V^c$.

- $S, T \in \mathcal{U}^{ab} \Longrightarrow S \cap T \in \mathcal{U}^{ab}$.Consider the following profile v:

$S-T$	$T\cap S$	$T-S$	$(T\cup S)^c$
a	c	b	b
b	a	c	a
c	b	a	c
.	.	.	.
.	.	.	.
.	.	.	.

- $S\in\mathcal{U}_C^{ab}$ implies $F_C(v)\neq b$, $T\in\mathcal{U}_C^{ab}=\mathcal{U}_C^{ca}$(by the fact that there exists a unique family of preventing sets) implies $F_C(v)\neq a$, tops only then requires $F_C(v)=c$. We have found a profile for which $cv(T\cap S)b\&F(v)=c$, so $T\cap S\in\mathcal{U}_C'^{cb}=\mathcal{U}_C^{cb}=\mathcal{U}_C^{ab}$. For the social welfare function, $S\in\mathcal{U}_D^{ab}$ implies $aF_W(c)b$ and $T\in\mathcal{U}_D^{ab}=\mathcal{U}_D^{ca}$ implies $cF_W(v)a$, then by transitivity $cF_W(v)b$. We have found a profile such that $cv(T\cap S)b\&bv((T\cap S)^c)c\&cF_W(v)b$, that is $T\cap S\in\mathcal{U}_D'^{cb}=\mathcal{U}_D^{cb}=\mathcal{U}_D^{ab}$.

We have thus proved that for the social choice function the family $\mathcal{U}_C^{ab}$ does not depend on (a,b), and it is an ultrafilter on the space of agents. The last step is to prove that for the social welfare function if a coalition belongs to the system $\mathcal{U}_D^{ab}$, which does not depend on (a,b) and has the ultrafilter property,then it is a decisive coalition.That is, $S\in\mathcal{U}_D^{ab}\Longrightarrow[av(S)b\Longrightarrow aF_W(v)b]$.Consider the following partition for a given profile v and alternatives (a,b):

$S_1\equiv\{u\in U:av(u)b\}$

$S_2\equiv\{u\in U:bv(u)a\}$

$S_3\equiv(S_1\cup S_2)^c$

Define the following profile w for $c \neq \{a, b\}$:

S_1	S_2	S_3
a	b	$a \ b$
c	c	c
b	a	
.	.	.
.	.	.
.	.	.

Suppose $S \in \mathcal{U}_D^{ab}$. It is true that $S \subseteq S_1$ and $S \subseteq S_1 \cup S_3$, implying that $S_1 \in \mathcal{U}_D^{ab} = \mathcal{U}_D^{cb}$and $S_1 \cup S_3 \in \mathcal{U}_D^{ab} = \mathcal{U}_D^{ca}$ by the properties of an ultrafilter and the fact that the system does not depend on (a, b). Therefore, $cF_W(w)b$ and $aF_W(w)c$, thus by transitivity $aF_W(w)b$. Since v and w agree on $\{a, b\}$, by IIA $aF_W(v)b$: S is a decisive coalition.

Proof of theorem 3

By lemma 2 we know there always exists a free ultrafilter, $\mathcal{U}_{D^*}$,on U in the infinite case. Since $U \in \mathcal{U}_{D^*}$ by property of an ultrafilter (therefore $F_W^*(v)$ satisfies unanimity), IIA is satisfied by construction and a free ultrafilter does not contain finite sets, that is $F_W^*(v)$ is non -dictatorial, we are left to prove that $aF_W^*(v)b \Longleftrightarrow S_0 \equiv \{u_0 \in U \text{ s.t. } av(u_0)b\} \in \mathcal{U}_{D^*}$ is a weak order. If $S \in \mathcal{U}_{D^*}$ then $S^c \notin \mathcal{U}_{D^*}$, therefore $aF_W^*(v)b$ implies *not* $bF_W^*(v)a$ so $F_W^*(v)$ satisfies asymmetry. Now suppose *not* $aF_W^*(v)b$ and *not* $bF_W^*(v)c$. By the definition of the social relation this implies $S_1 = \{u_0 \in U \text{ s.t. } av(u_0)b\} \notin \mathcal{U}_{D^*}$ and $S_2 = \{u_0 \in U \text{ s.t. } bv(u_0)c\} \notin \mathcal{U}_{D^*}$. Therefore

$S_1^c \in \mathcal{U}_{D^*}$ and $S_2^c \in \mathcal{U}_{D^*}$, so that $S_1^c \cap S_2^c = (S_1 \cup S_2)^c \in \mathcal{U}_{D^*}$ and $S_1 \cup S_2 \notin \mathcal{U}_{D^*}$.Since v is a weak order, $av(u)c$ implies for each $b \neq \{a, c\}$ either $av(u)b$ or $bv(u)c$ so that $\{u_0 \in U$ s.t. $av(u_0)c\} \subseteq S_1 \cup S_2 \notin \mathcal{U}_{D^*}$ implying *not* $aF_{W*}(v)c$ and proving negative transitivity.

Proof of theorem 4

F_C^* satisfies Pareto efficiency since $U \in \mathcal{U}_{C^*}$ and it is non dictatorial since a free ultrafilter does not contain a finite set. We then need to prove that it is coalitionally incentive compatible. We will prove the proposition for profiles of strict preferences. Suppose $F_C^*$is not coalitionally incentive compatible. Then there exists a coalition $S \subseteq V$ and profiles v ,v' such that $v(S) \neq v'(S), v(U\backslash S) = v'(U\backslash S)$ and $F_C^*(v')v(S)F_C^*(v)$.Now consider the set $D(F_C^*(v), v)$ defined in the above section, which by definition belongs to the ultrafilter $\mathcal{U}_{C^*}$. Suppose $D(F_C^*(v), v) \cap S \neq \varnothing$. Then for $u_0 \in D(F_C^*(v), v) \cap S$ we have both $F_C^*(v')v(u_0)F_C^*(v)$ and $F_C^*(v)$ maximal for $v(u_0)$, a contradiction. Suppose instead that $D(F_C^*(v), v) \cap S = \varnothing$. $D(F_C^*(v), v)$ and $D(F_C^*(v'), v')$ both belong to $\mathcal{U}_{C^*}$ and since the empty set does not belong to the ultrafilter, there exists $u_0 \in D(F_C^*(v), v) \cap D(F_C^*(v'), v')$. Since $u_0 \in D(F_C^*(v'), v')$, $F_C^*(v')$ is maximal for $v'(u_0)$.In particular $F_C^*(v')v'(u_0)F_C^*(v)$.On the other hand, since $D(F_C^*(v), v) \cap S = \varnothing$, $u_0 \notin S$ implying $v(u_0) = v'(u_0)$, so $F_C^*(v')v(u_0)F_C^*(v)$ which is a contradiction since $u_0 \in D(F_C^*(v), v)$ and we are considering only strict orders.

Part II

Implementation in economic domains

6 Economic domains: characterization and impossibility results with finitely many agents

The discussion so far has involved a generic aggregation setting, where the nature of the set of alternatives and the preferences of individuals were not specified. In this and the following section, I specialize the discussion to an environment pertaining resource allocation of economic goods. To be concrete, suppose a society formed by economic agents, who privately own endowments ,and have preferences over consumption, of bundles of pure private goods. The object of choice is the allocation of consumption among consumers. A mechanism of resource allocation associates to each economy a final feasible allocation of consumption . Suppose, in addition, that a wise social engineer suggests that, given preferences and endowments, the final allocation must have some efficiency property, say the Pareto property. The first question is whether there exists a mechanism of resource allocation whose set of equilibria, however defined, coincides with the set of Pareto efficient allocations. The answer, in the form of the two welfare theorems, is well known: under relatively mild[7] assumptions, the Walrasian price system does the job. However, the previous discussion on incentive compatibility should warn us that something is missing. In particular, suppose agents' endowments and preferences are private information. The mechanism then has to rely on messages sent by agents on their own private characteristics, therefore adding a new constraint in the form of incentive compatibility. The question then becomes whether there exists a mechanism of resource allocation that is Pareto efficient and incentive compatible. Hurwicz(1972) was the first to notice, through a counterexample, that the walrasian mechanism performs badly in this respect, i.e. it is Pareto efficient but it is not incentive compatible.

Example 1 *(Hurwicz 1972 in Jackson 2001). Suppose an economy with 2 goods and 2 agents , with endowments* $(e^A, e^B) = ((1,0),(0,1))$ *Agent A might have preferences* $v^A(x_1, x_2) = x_1^A x_2^{`A}$ *or* $\tilde{v}^A(x_1, x_2) = x_1^A - \frac{1}{1+x_2^A}$*, while agent B has preferences* $v^B(x_1, x_2) = x_1^B x_2^{`B}$*. At* (v^A, v^B) *the Walrasian equilibrium allocation is* $x^A = x^B = (\frac{1}{2}, \frac{1}{2})$*, while at* $(\tilde{v}^A, v^B)$ *it is* $\tilde{x}^A = (\frac{1}{2}, \frac{7}{9})$ *and* $\tilde{x}^B = (\frac{1}{2}, \frac{2}{9})$*. Therefore at* (v^A, v^B) *agent A would misreport his preferences.*

From the informal description given above, it is clear that, in addition to preferences, agents have another piece of private information, namely their own endowment. This implies another possibility of manipulation: agents can withhold their endowment from the market (Yi 1991, Postlewaite 1979). The following example

[7] Here the adjective is attributed to standard *explicit* assumptions, such as monotonicity and convexity of preferences. The result depends also on several more or less *hidden* assumptions, such as absence of externalities or complete markets (in economies with uncertainty), that are for sure *not mild*.

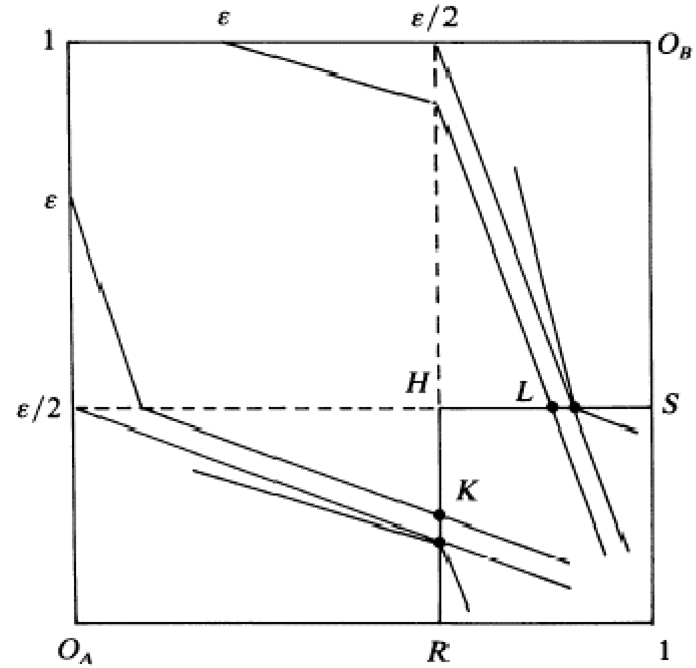
shows that every mechanism that yields Pareto efficient allocations is subject to manipulation via endowment withholding:

Example 2 *Suppose an economy with two goods and two agents, A and B. Let* $0 < \varepsilon \leq 1$*, utility functions are*

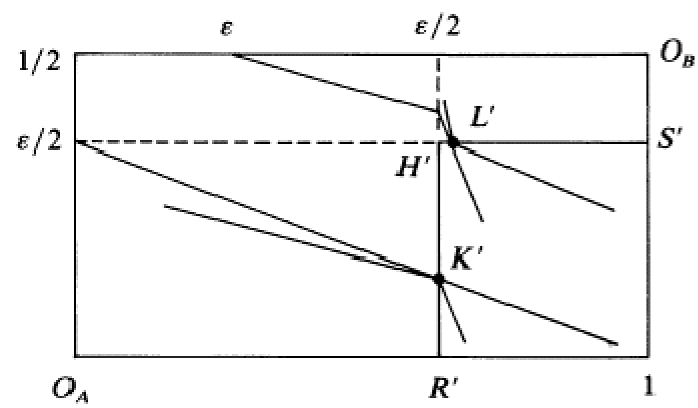
$$v^A(x_1, x_2) = \begin{cases} 3x_1 + \varepsilon x_2 & \text{if } x_2 \geq \frac{\varepsilon}{2} \\ 3x_1 + \frac{6}{\varepsilon}x_2 + (\frac{\varepsilon}{2} - 3) & \text{if } x_2 < \frac{\varepsilon}{2} \end{cases}$$

$$v^B(x_1, x_2) = \begin{cases} \varepsilon x_1 + 3x_2 & \text{if } x_1 \geq \frac{\varepsilon}{2} \\ \frac{6}{\varepsilon}x_1 + 3x_2 + (\frac{\varepsilon}{2} - 3) & \text{if } x_1 < \frac{\varepsilon}{2} \end{cases}$$

and endowments $\omega^A = (1, 0)$ $\omega^B = (0, 1)$.*Suppose the allocation rule assigns to each individual a bundle on the intersection of the contract curve with the set of* $\varepsilon-$*individually rational allocations, that is allocations such that* $v^i(x) \geq v^i(\varepsilon\omega^i)$ *for* $i = A, B$. *This set is the segment* $\overline{KHL}$ *in the following figure:*



Now suppose the allocation is in the subset $\overline{KH}$ and consider the possibility of agent A of withholding $\frac{1}{2}$ of his endowment. This new economy is depicted in the following figure



Now agent A can obtain an allocation on the segment $\overline{K\prime H\prime}$. *If he can consume the endowment he withheld from the market, his utility will be higher than the utility he can obtain in the truthful economy, so he has incentive to manipulate via endowment withholding. A parallel argument applies for agent 2. This example shows that every allocation mechanism that yields Pareto efficient and* $\varepsilon-$*individually rational allocations is subject to manipulation via endowment withholding. This is also true in the particular case in which* $\varepsilon = 1$ *(standard individual rationality)*

The first of these examples illustrate the logic of how manipulation affects walrasian equilibria. The idea behind "walrasian behavior" is that individuals express demands in the market taking prices as given, and the market, through some form of virtual adjustment process, expresses equilibrium prices on the basis of expressed demands. The key assumption is that individuals take prices as given, or, stated differently, they behave as if they were not able to modify their final consumption of goods through some sort of strategic behavior. However, there is a simple way they can do that, as the example shows: they can behave in the market as if they had different preferences or a different endowment. This kind of strategic behavior will be profitable *if* it implies a different final allocation of consumption that gives them a higher utility. The bottom line of this brief informal discussion is that incentive compatibility embodies a more stringent strategic rationality requirement than walrasian behavior.

Let us turn now to the more general question of whether there exists mechanisms, different from the walrasian one, that yield Pareto efficient allocations, are incentive compatible (with respect to preference manipulation) and satisfy some form of non dictatorship property. It turns out that in economies with finitely many agents the answer seems to be negative, although the literature has come just short of proving it. To illustrate formally the results , let us describe more precisely the environment of the analysis.

There are N agents in the economy, each having consumption set $\mathbb{R}_{+}^{M}$ and utility function $v^i : \mathbb{R}_{++}^{M} \to \mathbb{R}$. Denote the set of all possible utility functions that satisfy the standard assumptions of continuity, monotonicity and quasi-concavity by V. Notice that, compared with the abstract social choice setting of the previous section, we are restricting in a sensible way the domain of admissible preferences. There is a total endowment of the M goods in the economy given by $\omega \in \mathbb{R}_{++}^{M}$, that might be privately owned, in which case $\omega = \Sigma_{i=1}^{N} \omega^i$. Denote the set of feasible bundles by $A = \{x \in \mathbb{R}_{+}^{MN} : \Sigma_{i=1}^{N} x^i \leq \omega\}$.The object of social choice is allocation of consumption among consumers. Therefore a mechanism is a function $F : V \to \dot{A}$. Notice that the same notation of section 3 is used here, to stress the fact that we are in a particular "concrete" case of social choice. This implies that the definition of an incentive compatible aggregation function is the same as before, with the slight adaptation due to the fact that now agents have utility indexes. Moreover, F is Pareto efficient if $F(v)$ is a Pareto efficient allocation for each v.

Recall that a dictator is an agent who always obtains his most preferred alternative, no matter what the preferences of other agents are.Therefore, in a pure exchange economy with monotonic preferences a dictator is an agent who always obtains the whole endowment. F is dictatorial if there exists an i such that, for each

v, $F^i(v) = \omega$.

In the case of privately owned endowments, dictatorship is related to the concept of individual rationality (Serizawa 2002). F is individually rational if $v^i(F^i(v)) \geq v^i(\omega^i)$ for each i and v. Suppose that each agent has a strictly positive endowment of at least some good. Then an individually rational social choice function is also non dictatorial, while the converse is not true. The first impossibility result assumes individual rationality:

Theorem 7 *Suppose the economy has N individuals, M goods and endowments are privately owned. Let V contain all utility functions that are continuos, strictly monotonic, strictly quasi-concave, smooth and homotetic*[8]*. Then there is no social choice rule F on V that is Pareto efficient, incentive compatible and individually rational.*

Sketch of proof: The proof of the theorem goes by showing that the conclusion is true for the restricted domains of utilities with the stated characteristics, while the theorem is in general true for any domain that contains those utilities. This easy observation comes from the fact that if it was not true for the containing superset, then it would be false also for the restriction of F on the subdomain.

To illustrate the role of homothetic preferences, and how they interact with strategy-proofness, I will sketch the proof for only one case, that is when there

[8]Smooth preferences are such that at each point there is a unique vector generating the supporting hyperplane of the upper contour set at that point.

Homothetic preferences are such that the preference relation is preserved along proportional bundles.

These definitions are standard.

is some individual i that has endowment vector ω^i not proportional to the total endowment $\dot{\omega}$. Suppose there exists an F that is Pareto efficient, incentive compatible and individually rational. Suppose first that at profile v all individuals have the same utility function $v(x;\rho) = [\Sigma_{m=1}^{M}(\omega_m)^{1-\rho}(x_m)^{\rho}]^{\rho^{-1}}$ It is quite easy to see that, by Pareto efficiency, the social choice function, on this profile, must assign to individuals bundles along the diagonal $[0,\omega]$, that is proportional to $\omega : F(v(x;\rho)) \in [0,\omega]$.. Now consider another bundle w proportional to ω such that $\Sigma_{i=1}^{N}\ w^i = \Sigma_{i=1}^{N}\omega^i$ Since w is proportional to ω and preferences are homothetic, the pair $\{w,(1,1,..,1)\}$ constitute a walrasian equilibrium for utility functions v. Also, since there exists some individual that has endowment vector not proportional to ω, for this individual the set $M(i) = \{m \in M \text{ s.t. } w_m^i < \omega_m^i\}$ will be non empty. Assign a different "price" vector for each individual so that for goods $m \in M(i)$, $p_m^i = 2$ while the other prices stay at 1. Denote each of these vectors by p^i, and let a new utility function be $\hat{v}^i(x;\sigma) = [\Sigma_{m=1}^{M}p_m^i(\omega_m)^{1-\sigma}(x_m)^{\sigma}]^{\sigma^{-1}}$ Last, let $\bar{w}^i$ be another vector proportional to ω and such that $p^i \cdot \bar{w}^i = p^i \cdot \omega^i$ Now we can derive some conclusions and the desired contradiction from these definitions.

First notice that as $\sigma \to 1$, $\hat{v}^i$ converges to the linear preference relation, with normal vector p^i, while as $\rho \to -\infty$, v converges to Leontief preferences. Together, these facts imply that if individual i has preferences $\hat{v}^i(x,\sigma)$ and all other individuals have common preference v, the set of Pareto efficient allocations, as $\sigma \to 1$ and $\rho \to -\infty$, converges to the segment $[0,\omega]$.[9]Moreover, since F is Pareto efficient and

[9]This assertion would require a proof. It is directly true if there were only 2 agents. In the $N \geq 2$ case, a lemma in the paper shows that the set of Pareto efficient allocations when all other individuals have common homothetic preferences, from $i's$ point of view, is equivalent to the set of Pareto efficient allocations of an economy in which there are only 2 agents, in which the agent

individually rational, $F^i(\hat{v}^i(x,\sigma), v^{-i}(x,\rho))$ will converge to the segment $[\bar{w}^i, \omega] = [0,\omega] \cap \{x \in \mathbb{R}_+^M \text{ s.t. } p^i \cdot x \geq p^i \omega^i\}$..Notice that set at the right of the intersection symbol is the requirement for individual rationality for the limiting linear utility.

Now notice that for an individual i such that $M(i) \neq \emptyset$, we have $p^i \cdot (\omega^i - w^i) = 2\Sigma_{m\in M(i)}\ p^i \cdot (\omega^i_m - w^i_m) + \Sigma_{m \notin M(i)}\ p^i \cdot (\omega^i_m - w^i_m) > \Sigma_{m=1}^M (\omega^i_m - w^i_m) = 0$ Therefore $p^i \cdot \omega^i > p^i \cdot w^i$.But since w^i and $\bar{w}^i$ are both proportional to ω, and $p^i \cdot w^i = p^i \cdot \omega^i$ this implies that $\bar{w}^i_m > w^i_m$ for each m. Since $F^i(\hat{v}^i(x,\sigma), v^{-i}(x,\rho))$ converges to $\bar{w}^i$, there will be (σ_0, ρ_0) such that $F^i(\hat{v}^i(\cdot,\sigma_0), v^{-i}(\cdot,\rho_0)) > w^i$ and by monotonicity, $v(F^i(\hat{v}^i(\cdot,\sigma_0), v^{-i}(\cdot,\rho_0)); \rho_0) > v(w^i;\rho_0)$.Finally, individual rationality implies[10] that $F^i(v(\cdot,\rho_0)) \geq w^i$ for each i. But then, since there is some $i^{'}$ whose endowment is not proportional to ω, this implies $F^{i^{'}}(v(\cdot,\rho_0)) \leq w^{i'}$.Putting these observations together, we get the following inequalities: $v(F^i(\hat{v}^i(\cdot,\sigma_0), v^{-i}(\cdot,\rho_0)); \rho_0) > v(w^i;\rho_0) \geq v(F^{i^{'}}(v(\cdot,\rho_0)); \rho_0)$ for an individual i for some (σ_0, ρ_0) In words this means that i, when he has preferences $v(\cdot,\rho_0)$, finds profitable to deviate and reveal preferences $\hat{v}^i(\cdot,\sigma_0)$ therefore contradicting incentive compatibility.

As already mentioned, individual rationality is related to non dictatorship. In fact, the dictator is someone who always imposes his most preferred alternative, among all the alternative in the feasible set. If preferences are monotonic the dictator will always receive the entire aggregate endowment of the economy. If endowments are privately owned, and each agent has a strictly positive endowment of at least some good, individual rationality immediately implies non dictatorship. The natural

$j \neq i$ has that common preference.

[10] Again, this assertion would require a proof, through a lemma in the paper.

question to ask is whether weakening the assumption from individual rationality to non dictatorship restores some sort of possibility result. Notice that non dictatorship represents a weaker assumption in a slightly different sense as well. In fact, if we interpret the social choice problem as a (re)distribution problem, individual rationality takes into account, at least to a minimal extent, private ownership of *initial* endowments. Instead, requiring non-dictatorship and taking into account only aggregate endowments concerns exclusively *final* allocations, regardless, we might say, of individual starting points.

Unfortunately, even with non dictatorship there might seem to be little hope for possibility Zhou(1991). The following result shows impossibility for an economy with an arbitrary number of goods and 2 agents:

Theorem 8 *Suppose an economy with M goods and 2 agents. Suppose V contains continuous monotonic strictly-quasi concave and smooth utility functions. There is no social choice function $F : V \rightarrow A$ that is Pareto-efficient, incentive compatible and non-dictatorial.*

Sketch of proof. Let us first notice that in the 2 agents case, the social choice function, by feasibility, needs to specify only the allocation of one of the agents, the other being the residual. Consequently, wlog, let $F^1 \equiv F$, and $F^2 \equiv \omega - F$.and a dictatorial social choice function is such that either $F \equiv 0$ or $F \equiv \omega$ for each v.A useful device to track how the social choice must change (in an incentive compatible way) as preferences change is by allowing transformations of utility functions that imply a "shrinking" of the upper contour set at a particular point. This type of transformations are usually referred to in the literature as *Monotonic.* Formally, at a point a and utility function v^1, the transformed utility $\hat{v}^1$ is such that $\hat{v}^1(x) \geq \hat{v}^1(a)$ and $x \neq a$ implies $v^1(x) > v^1(a)$, Clearly, a dictatorial social choice function is trivially incentive compatible (being constant). However, if F is not constant, then it can never give the zero endowment to one of the agents without violating incentive compatibility, because otherwise the agent can manipulate via a monotonic transformation of his preferences. Therefore, $0 << F(v^1, v^2) << \omega$ for each (v^1, v^2). A quite intuitive fact, although requiring a separate proof, omitted here, is that each strictly increasing curve (in $\mathbb{R}^M$) going from 0 to the aggregate endowment ω is the contract curve for some pair of utility functions. Now consider first the segment $\overline{0\omega}$. There is a unique bundle d that belongs both to the range of F and to $\overline{0\omega}$.Existence can be proven by efficiency of F and considering the vector of utilities $v^1(x) = v^2(x) = \Sigma p^m(x^m + \omega^m)$ where $p^m > 0 \vee m$. Uniqueness derives from the following observation. Suppose $d = F(v^1, v^2)$ and $0 << d \leq b << \omega$, and by contradiction, suppose that $b = F(u^1, u^2)$ for some pair of utilities (u^1, u^2). Consider a monotonic transformation (in the sense outlined above) $\hat{v}^1$ of v^1 Incentive compatibility then requires $v^1(d) = v^1(F(v^1, v^2)) \geq v^1(F(\hat{v}^1, v^2))$ and $\hat{v}^1(F(\hat{v}^1, v^2)) \geq \hat{v}^1(F(v^1, v^2)) = v^1(F(v^1, v^2)) = v^1(d)$ (by definition of monotonic transformation).

But then $d = F(\hat{v}^1, v^2)$. Similarly, at profile (u^1, u^2) , we must have $b = F(\hat{u}^1, u^2)$ for a monotonic transformation $\hat{u}^1$ of u^1Normalize the two transformations such that $v^1(d) = \hat{v}^1(d)$ and $u^1(b) = \hat{u}^1(b)$ and take $\tilde{v}^1 = min\{\ \hat{u}^1, \hat{v}^1\}$.Clearly $\tilde{v}^1$ is a monotonic transformation of both v^1 at d and $\hat{v}^1$ at b. Therefore, $F(\tilde{v}^1, v^2) = d$ and $F(\tilde{v}^1, u^2) = b$ and so $F(\tilde{v}^1, v^2) \leq .F(\tilde{v}^1, u^2)$. But this implies $u^2(\omega - F(\tilde{v}^1, u^2)) < u^2(F(\tilde{v}^1, v^2))$, violating incentive compatibility. Since along the segment $\overline{0\omega}$ we have either $x \leq y$ or $y \leq x$, we have uniqueness.

Take now any other allocation $0 << e << \omega$ and consider the curve $\overline{0e\omega}$. This will be the contract curve for some vector of utilities. Take any point $b \in \overline{0e\omega}$. If $b \leq d$ then the previous discussion shows that we cannot have b belonging to the range of F. If $b \nleq d$ it can also be proved that b is not in the range of F by finding a particular utility function such that $\tilde{u}^1(b) > \tilde{u}^1(d)$ and through which agent 1 can manipulate. But together these observations imply that for the contract curve $\overline{0e\omega}$ there is no allocation in the range of F, contradicting efficiency of F.

Although it might seem a reasonable generalization,a proof of dictatorial result for an arbitrary number of agents has not been established yet. When one has more than 2 agents, Zhou proposed the property of *inverse dictatorship:* an agent is an inverse dictator if he always receives the zero bundle. Notice that in the 2 agents case, dictatorship and inverse dictatorship are equivalent. He conjectured that with $N \geq 2$ agents there is no social choice function for a pure exchange economy that is efficient, incentive compatible and non-inversely dictatorial. However, this conjecture has been proved wrong when there are at least 4 agents, by the following non inversely dictatorial mechanism, proposed by Kato Ohseto(2002):

Kato Ohseto mechanism: Suppose $\#N \geq 4$ and consider a partition of the preferences into 2 subsets V_A and V_B. Let the social choice function be $F^i(v) = \omega$ for all $v \in V$ if there exists an i such that $v^j \in V_A$ for all $j \in N \backslash \{i-1, i, n\}$ (the n in parenthesis refers to the Nth agent) and $v^{i-1} \in V_B$, and $F^n(v) = \omega$ otherwise.

This mechanism is efficient and incentive compatible. However, as noted by the authors in the paper, it has some quite non desirable properties. First, even though it is non inversely dictatorial, there is always an agent who receives the zero bundle, but his identity is profile-dependent. Second, it is *bossy,* that is there is an agent that can change the allocation of consumption to other agents without changing his own. Third, there is a *dummy agent*, someone who can never affect the final allocation of consumption.

The last result in this section gives a final hint on the kind of impossibility theorem that one obtains in the general $N \times M$ case. In the previous paragraph, I have noticed that the Kato Ohseto mechanism always has some agent consuming the zero bundle. One natural question is whether there are mechanisms that are Pareto efficient , incentive compatible and can always guarantee a minimum level of consumption to each individual for each profile of preferences. Serizawa Weymark (2002) show, using techniques similar to Serizawa(2002), that such a mechanism does not exist:

Theorem 9 *Let all the assumptions of theorem 7 hold. There is no social choice function* $F : V \rightarrow A$ *that is Pareto efficient, incentive compatible, and such that, for* $\varepsilon > 0$, $\|F^i(v)\| \geq \varepsilon$ *for each* $i \in N$ *and* $v \in V$.

This last theorem gives a representation of the so long recognized (and too often neglected) conflict between the concept of Pareto efficiency and some minimal requirement of distributional justice.

7 Economies with infinitely many agents

In this final section, which is the parallel of section 4,we will present some results on incentive compatibility in pure exchange economies when the number of agents is arbitrarily large. The reason for studying large economies rests on the intuitive idea that incentives to misrepresent preferences might vanish as the number of agents increases, as the influence each agent has on the final allocation of resources becomes arbitrarily small. As we remarked in the previous section, walrasian equilibrium is a *specific* mechanism to allocate resources, and we have seen that it performs poorly in terms of incentives when there are a finite number of agents. The possibility of manipulating walrasian equilibria rests on the ability to influence equilibrium prices. Studying incentive compatibility of walrasian equilibrium with many agents is, therefore, a check on the "folk justification" of price taking behavior in terms of individual agents having negligible size. In this respect, on one hand, economies with a continuum of agents satisfy incentive compatibility almost by definition, each agent having zero measure. On the other hand, the real significance of the continuum hypothesis rests on the possibility of it being the limit of a sequence of economies with a finite, but arbitrarly large, number of agents. A remark here is in order to informally clarify the difference between convergence for an incentive compatible mechanism to the set of walrasian equilibria and the concept of core convergence . It is clear that both approaches define an equilibrium concept restricting the set of attainable allocations to be stable with respect to deviations. The difference rests on the kind of deviations that are allowed. For the core, individuals are allowed to block a particular allocation only consuming their own endowment, while (individual) incentive compatibility permits deviations to arbitrary preferences, or, in the specific case of walsarian equilibrium, deviations to arbitrary demand functions.

In the following subsections we will first describe results that have the flavor of possibility results in economies with a continuum of agents. We will see that in an infinite society the walrasian mechanism is the only symmetric, Pareto efficient (obviously) and incentive compatible, if it involves no lump sum transfers.When we pass to the asymptotics, matters are much less straightforward. First, if we require the mechanism to be walrasian along the sequence of economies, then incentive compatibility is obtained only for large enough economies . Second, we will describe a mechanism that is in the spirit of Vickrey-Clarke-Groves, therefore incentive compatible even in finite ordinal economies.

7.1 Continuum

In the following we will refer to this basic model of an economy with a continuum of agents. The set of agents is assumed to be a measurable space (I, Ω, μ) with $\mu(I) = 1$. As in the finite model, $\omega \in \mathbb{R}^M_{++}$ represents the total endowment of goods in the economy. Each agent is characterized by a vector of individual characteristics $\theta_i \in \Theta$, which determines the consumption set $X(\theta_i)$ and the utility index $U(x, \theta_i)$.

In this context an allocation mechanism is a function $F : I \times \Theta \longrightarrow \mathbb{R}^M$ satisfying the feasibility conditions $F_i(\theta) \in X(\theta_i)$ a.e. and $\int_I F_i(\theta) d\mu \leq \omega$ where θ represents the vector of agents' characteristics.

An incentive compatible mechanism is such that there does not exist an agent i and characteristics θ_i, θ'_i such that $U(F_i(\theta'_i, \theta_{-i}), \theta_i)) > U(F_i(\theta_i, \theta_{-i}), \theta_i)$.

A Pareto efficient mechanism is such that $F(\theta)$ is a Pareto efficient allocation.

From the definition of the space of agents we can derive a distribution δ on the space of characteristics Θ by defining $\delta(\theta, A) = \mu(\{i \in I : \theta \in A\})$ .Notice that with an atomless space of agents the distribution on the space of characteristics does not depend on the characteristics of a single agent. By requiring the mechanism to be symmetric (the allocation does not depend on the name of the agent, but only on his characteristic) , we can let it depend only on the distribution of characteristics(Hammond 1979, 1987). with the obvious feasibility conditions $F(\delta, \theta) \in X(\theta)$ and $\int_\Theta F d\delta \leq \omega$. It is true then:

Theorem 10 *i) if F is symmetric then it is incentive compatible if and only if there exist a set $B(\delta)$ such that , for each θ and i , $F(\delta, \theta) \in \arg\max_{x \in B(\delta) \cap X_i(\theta)} U(x, \theta_i)$*

ii)Suppose that Θ is path-connected, U is strictly quasi-concave, monotone, C^1 in (x, θ), agent's' demand function is C^1 in (x, θ),agents' marginal utility of money is strictly positive and $F \in intX$ then F is incentive compatible and Pareto efficient only if the set $B(\delta)$ is a walrasian budget set with no lump sum transfers.

The first of these statements is obtained from a simple argument. In one direction, if $F(\delta, \theta)$ maximizes $U(x, \theta_i)$ over $B(\delta) \cap X_i(\theta)$ then, since the set $B(\delta)$ does not depend on individual characteristics, it is also true that by misrepresenting preferences agents cannot improve upon $F(\delta, \theta)$, obtaining in this way incentive compatibility. In the other direction, if the mechanism is incentive compatible it suffices to define the set $B(\delta) \equiv \{F(\delta, \theta) : \theta \in \Theta\}$ and notice that $U(F(\delta, \theta), \theta) \geq U(F(\delta, \theta), \theta')$ for each $\theta' \in \Theta$, by incentive compatibility and the fact that δ does not change when a singe agent deviates and misreports. Therefore $F(\delta, \theta) \in \arg\max_{x \in B(\delta) \cap X_i(\theta)} U(x, \theta_i)$.

The second statement says two things. First, by a standard argument based on the second welfare theorem with a continuum of agents, Pareto efficiency implies the existence of a supporting price vector $p(\delta)$ and transfer system $T(\delta,\theta)$ such that the set $B(\delta)=\{x\in\mathbb{R}^M : p(\delta)x\leq T(\delta,\theta)\}$ is a walrasian competitive budget set. Second, if we insist on requiring incentive compatibility then the tranfers T must be identically equal to zero. In fact, fix the distribution δ so that we can drop the dependence of equilibrium variables on it, and let $x(p,T(\theta),\theta)$ be the demand function and $V(p,T(\theta),\theta)$ the indirect utility function. Incentive compatibility requires for $\theta'\neq\theta$:

$$U(x(p,T(\theta'),\theta'),\theta)\leq V(p,T(\theta),\theta)$$
$$U(x(p,T(\theta),\theta),\theta')\leq V(p,T(\theta'),\theta')$$

On the other hand, since V is differentiable, by the mean value theorem there exists $(T_1(\theta'),T_2(\theta'))\in[T(\theta),T(\theta')]^2$ such that:

$$V(p,T(\theta'),\theta)-V(p,T(\theta),\theta)=[T(\theta')-T(\theta)]\tfrac{\partial}{\partial T}V(p,T_1(\theta'),\theta)$$
$$V(p,T(\theta'),\theta')-V(p,T(\theta),\theta')=[T(\theta')-T(\theta)]\tfrac{\partial}{\partial T}V(p,T_2(\theta'),\theta')$$

Therefore we obtain:

$$\frac{U(x(p,T(\theta),\theta),\theta')-U(x(p,T(\theta),\theta'),\theta')}{\frac{\partial}{\partial T}V(p,T_2(\theta'),\theta')}\leq[T(\theta')-T(\theta)]\leq\frac{U(x(p,T(\theta'),\theta),\theta)-U(x(p,T(\theta'),\theta'),\theta)}{\frac{\partial}{\partial T}V(p,T_1(\theta'),\theta')}$$

Dividing all terms by $\theta'-\theta$, we obtain

$$\lim_{\theta'\longrightarrow\theta}\frac{T(\theta')-T(\theta)}{\theta'-\theta}=-\frac{[\nabla_x U(x(p,T(\theta),\theta),\theta)]^T D_\theta x(p,T(\theta),\theta)}{\frac{\partial}{\partial T}V(p,T(\theta),\theta)}.$$

Since $[\nabla_x U(x(p,T(\theta),\theta),\theta)] = p\frac{\partial}{\partial T}V(p,T(\theta),\theta)$ by the properties of the value function of consumer maximization we finally obtain $\frac{\partial}{\partial\theta}T(\theta) = -p^T D_\theta x(p,T(\theta),\theta)$. Now by the budget constraint $T(\theta) = p^T x(p,T(\theta),\theta)$, so $\frac{\partial}{\partial\theta}T(\theta) = p^T D_\theta x(p,T(\theta),\theta) = -p^T D_\theta x(p,T(\theta),\theta) = 0$. Clearly, transfers can be indipendent of characteristic only if they are everywhere identically equal to zero , $T(\theta) = 0$.

The first part of this theorem sheds light on the role of the number of agents for incentive compatibility. As it is clear, by assuming that each agent is of negiglible size a mechanism can be found that does not depend on individual actions, being based only on the distribution of characteristics in the economy. Notice here the link with the kind of aggregation rule we were able to find in section 4. There, the mechanism gave decision power only to coalitions belonging to an ultrafilter, or equivalently to coalitions of full measure, thereby ruling out the possibility that individual might manipulate the social outcome. Here, in a similar fashion, the mechanism forces agents to maximize over a set that does not depend on their announced characteristics, thereby excluding the possibility of manipulation. There is however a restriction, represented by the the second part of the theorem. In this pure exchange economy setting, since the outcome of the mechanism in Pareto efficient, it will be walrasian with tranfers. However, transfers must depend on individual characteristics, and the possibility of transfers kills incentive compatibility. The bottom line is that the set of allocations that can be implemented is restricted to the walrasian ones *without* tranfers.

7.2 Restricted domain: the "magnification principle"

This subsection tackles informally the problem of incentive compatibility from a slightly different perspective. As we have seen, incentive compatibility for the walrasian mechanism is related to the ability to influence equilibrium prices. In the continuum, this is obtained easily since each agent, having zero measure, does not influence the distribution of characteristics in the economy. Asymptotically, since we are considering deviations by single agents, the true and apparent economy converge to the continuum, again an argument based on the weight of individual consumers (and relying on the continuity of the price correspondece). In this subsection we take a different view. We start from a price selection of walrasian equilibria, and we ask ourself whether at an equilibrium, single person deviations might change equilibrium prices. If that is not the case (and if there are no lump sum transfers) it is natural to call the corresponding economy (measure) a *perfectly competitive economy* (Makowsky Ostroy and Segal 1999). In other words, it is an economy for which single agent deviations do not change the hyperplane containing the equilibrium price. Almost by definition, a perfectly competitive economy is incentive compatible (in addition of being individually rational and Pareto efficient). The interesting result is that by putting a restriction on the domain of preferences to which the mechanism applies, the converse is also true. The restriction is quite a simple one: if a point (preference profile) is in the domain then all profiles for which utilities have the same marginal rate of substitution of the original preference profile must be in the domain as well. Moreover, the domain must contain all possible one agent deviations. Provided these two conditions are met, and provided there is a finite number of equilibria, we have a characterization of incentive compatibility in terms of perfect competition, that is, no ability to influence equilibrium prices.

The perfect competition characterization of incentive compatible mechanisms holds in finite and continuum economies. However, it should be intuitive why the existence of perfectly competitive economies is non generic in finite economies, therefore incentive compatible mechanisms are non generic. If an agent is not to modify equilibrium prices by deviating, it has to be true that, from his perspective, the allocation he can obtain must lie on a flat segment. In finite economies, this is the case only if other agents have a flat segment in their indifference curve, an instance which is rare. In continuum, things change dramatically. Due to smoothness, at least locally, the range of possible allocation that an agent can obtain can generically be approximated by a flat surface, implying genericity for perfectly competitive economies. Rather than the argument based of negigible size, here the focus is on the smoothness properties of continuum economies. This is called by Makowski, Ostroy and Segal the maginification principle: the flat segments required by perfect competition in finite economy magnify the economy how the economy looks like from a perfect competitor perspective in a continuum.

7.3 Asymptotics

The interest for limiting behavior stems from the fact that in continuum economies,as we have seen in the previous section, incentive compatibility is somewhat trivially satisfied if one assumes that each agent is of measure zero. A natural question is therefore if there is a sequence of finite economies approaching the continuum one such that incentives to misrepresent get arbitrarily small as the economy grows large Assume that strategies for each agent are correspondences from prices into net trades that satisfy the budget constraint. To each profile of strategies corresponds a vector of equilibrium prices (not necessarily deriving from truthful revelation of individual information) that clear the market, inducing an equilibrium correspondence between economies and price vectors, Therefore, walrasian demand correspondence is a particular strategy (i.e. net trade that maximizes the individual preference relation on the budget set induced by linear prices). A competitive equilibrium is incentive compatible if each agent best response to walrasian demand of others is the walrasian correspondence, and prices clear the market. It turns out that the limiting incentive properties of competitive mechanism depend in a crucial way on the way the sequence of finite economies approaches the limit one.

7.3.1 Limiting Incentive compatibility

Let us consider first replica economies, that is economies in which there is a finite number of M agents' characteristics and the number of agents is kM, with k approaching infinity. In this case, if the consumption set of each agent is convex and bounded below, if a price is attainable (possibly through misrepresentation) in k it is also attainable in $k^* < k$, where aggregate demand is a fraction of the larger economy. This leads to the key property that the set of prices that appear in equilibrium form a decreasing sequence whose intersection is the set of "true" competitive prices. Assume in addition that these sets are closed, then for each sequence of equilibrium prices there exists a subsequence p^k that converges to the true competitive equilibrium price $\bar{p}$.But then suppose that for each true competitive allocation x to an agent i there exists another allocation x^k that i strictly prefers and is able to induce by misrepresenting his preferences : $U_\theta(x^k) > U_\theta(x) + \varepsilon$, $\varepsilon > 0$.Denoting V_θ his indirect utility function we would have $V_\theta(p^k) \geq U_\theta(x^k)$ Appealing to the continuity of the V function we have $lim\ supU_\theta(x^k) \leq V_\theta(\bar{p}) \equiv U_\theta(x)$, in contradiction with the definition of x^k. Therefore this discussion establishes the following:

Theorem 11 *Let E^k a sequence of economies, with corresponding equilibrium prices p^k belonging to a closed and nested family of sets. If agent' i has preferences that are represented by continuous direct and indirect utility functions then for each equilibrium allocation x^k in E^k for i and each $\varepsilon > 0$, there exists a competitive allocation y for i and a k^* such that for $k > k^*$, $U(y) > U(x^k) - \varepsilon$.*

. Unfortunately this *limiting incentive compatibility* (Roberts Postlewaite 1976) property of the competitive mechanism cannot be generalized to arbitrary sequence of economies, as it can be shown by example. The key property needed to restore the result is a form of continuity of the correspondence that maps economies into equilibrium prices. To be more precise, let $\mathcal{S}$ be the set of all correspondences from prices P into $\mathbb{R}^N$, where each correspondence $S(\theta, p)$ represents a possible (excess demand)response by an agent θ to price p. Each economy can be represented by a measure μ on the Borel subsets F of $\mathcal{S}$, e.g. if the economy has M agents, then $\mu(F) = \frac{|F \cap \{S_1,..,S_M\}|}{M}$. Equilibrium prices P (not necessarily deriving from truthful representation of individual characteristics) associated with an economy μ satisfy the market clearing condition:

$$P(\mu) = \{p \in \Delta^N : 0 \in \int_{\mathcal{S}} S(p) d\mu\}$$

where Δ^N is the N-dimensional simplex and S are intended as excess demands.

Therefore we can construct a correspondence from measures (i.e. economies) into equilibrium prices $\mu \rightrightarrows Q(\mu) \subset P$. Endowing the set of measures with the topology of weak convergence we can speak of continuity of this map. Since it will be needed in the sequel, a precise definition of continuity of a correspondence is given:

Definition 4 *A correspondence $\varphi : X \rightrightarrows Y$ between topological spaces is continuous at x if the following two conditions hold:*

i) for any open set U s.t.$\varphi(x) \subset U$ there is a neighborhood V_x of x with the property $[x' \in V_x \Longrightarrow \varphi(x') \subset V_x]$ (upper hemi-continuity)

ii)for any open set U s.t. $U \cap \varphi(x) \neq \varnothing$ there is a neighborhood V_x of x with the property $[x' \in V_x \Longrightarrow \varphi(x') \cap V_x \neq \varnothing]$ (lower hemi-continuity)

A correspondence $\varphi : X \rightrightarrows Y$, on the other hand, can be seen as a *function* $f : X \rightarrow 2^Y$.The argument below will be based on the fact that continuity implies equilibrium prices getting "close". We then need a metric on 2^Y. The Hausdorff distance is of common use in economics since Hildenbrand(1970). Suppose (Y, d) is a metric space, the distance between a point x and a set $E \subset Y$ is given by $d(x, E) = \inf_{y \in E} d(x, y)$ and an $\varepsilon - neighborhood$ of a subset E is given by $B_\varepsilon(E) = \{x \in Y$ s.t. $d(x, E) < \varepsilon\}$. Then:

Definition 5 *The Hausdorff distance is the function $\delta(P, P') = \inf\{\varepsilon \in (0, \infty] : P \subset B_\varepsilon(P')$ and $P' \subset B_\varepsilon(P)\}$*

By means of the Hausdorff distance, we can define a topology on the closed subsets of 2^Y by defining the $\varepsilon - ball$ around $E \in 2^Y : H\varepsilon(E) = \{C \in 2^Y : \delta(C, E) < \varepsilon\}$ and let the collection of them be the base of a topology. It is then true that:

Lemma 3 *i) The collection of* $\varepsilon -$ *balls* $H\varepsilon(E)$ *with* $E \in 2^Y$ *and* $\varepsilon \in (0, \infty]$ *forms the base of a first countable Hausdorff topology on* 2^Y *(Hausdorff metric topology)*

ii) Let $\varphi : X \rightrightarrows Y$ *be a non-empty compact-valued correspondence. Let* $f : X \longrightarrow K_Y$*, where* K_Y *are the compact subsets of* Y *endowed with the Hausdorff metric topology and* $f(x) = \varphi(x)$*, then* φ *is continuous according to definition 4 if and only if* f *is continuous as a function between metric spaces.* [11]

Now let μ^k be the sequence of true economies (associated with response correspondences $\{S_1^k, .., S_M^k\}$ for each finite economy k) converging to the limit economy μ, and consider a deviation by a single agent through a correspondence S'^k . This defines a new apparent economy whose simple measure is given by:

$$v^k(F) = \frac{|F \cap [(support\ \mu_k) \cup \{S'^k\} \char`~ \{S^k\}|}{|(support\ \mu_k)|}$$

With this construction it is now easy to see why continuity is sufficient for limiting incentive compatibility. The apparent sequence v^k converges to the true measure μ, since by assumption we are considering only deviation by single agents, whose measure goes to zero.Therefore the true and apparent economy get arbitrarily close. On the other hand, continuity of the equilibrium price correspondence implies that, as the apparent economy converges to the true one, equilibrium prices get arbitrarly close in the Hausdorff distance, In fact, given $\varepsilon > 0$,for true economies $\delta(Q(\mu^k), Q(\mu)) < \frac{\varepsilon}{2}$ and for apparent economies $\delta(Q(v^k), Q(\mu)) < \frac{\varepsilon}{2}$, for k large enough. Therefore

[11] Aliprantis et al (1999).

the triangle inequality implies that $\delta(Q(v^k), Q(\mu^k)) < \varepsilon$. So the true and apparent economies prices are close to each other and within a neighborhood of the true limiting economy. Then the same limiting argument used to establish the previous theorem can be applied here to obtain:

Theorem 12 *Suppose $\{E^k\}$ is a sequence of economies such that $\mid E^K \mid \longrightarrow \infty$, with corresponding sequence of simple measures $\mu^k \longrightarrow \mu$. If the equilibrium price correspondence Q is continuous at μ and the inverse utility functions are continuous then for each equilibrium allocation x^k in E^k for i and each $\varepsilon > 0$, there exists a competitive allocation y for i and a k^* such that for $k > k^*$, $U(y) > U(x^k) - \varepsilon$.*

This result establishes limiting incentive compatibility of the walrasian mechanism. Before commenting on the restrictions imposed by the continuity of the equilibrium price correspondence, it is worth mentioning a complementary limiting result. Limiting incentive compatibility asserts that the utility gain of deviating from competitive behavior becomes arbitrarly small as the number of agents goes to infinity. A related question is whether by allowing deviations along the expanding sequence of economies the allocation itself becomes nonetheless close to the walrasian competitive allocation (Jackson 1992). It is then useful to distinguish between the competitive demand of each agent when taking prices as given, that is $x_\theta(p) \in \arg\max_{x\varepsilon X(\theta)\cap B(p)} U(x,\theta)$, and the (individually) feasible deviations $d_\theta(p) \in D_\theta = \{x : \Delta^N \rightarrow X(\theta) \cap B(p)\}$, where $B(p)$ denotes the walrasian budget set. Assuming for simplicity continuity and strict quasi-concavity, we can work with functions[12]. In order to define closeness, the space of demands $D = \cup_{\theta \in \Theta} D_\theta$ has to be endowed with a metric $\rho : D \times D \to \mathbb{R}_+$. As before, each economy is defined by a measure μ on the Borel subsets of D, and the resulting space of measures can be endowed with the topology of weak convergence. We let a sequence of true economies μ^k approach the limiting continuum economy μ. In order to stress the fact that an agent deviates by using a demand function d, we denote the apparent economy induced by such a deviation by v_d^k. A key hypothesis, which will play the same role played the previous theorem by the continuity of the equilibrium price correspondence , is the finiteness of the set of equiliubrium prices arising in the true limiting economy. The following definition is standard:

Definition 6 *An economy E, represented by a measure μ, is regular if there exists a neighborhood of μ where the set $\{p_i(\mu) : p_i(\mu) \in P(\mu)\}$ of equilibrium prices is finite and each $p_i(\cdot)$ is continuous.*

We let a sequence of finite economies μ^k converge to the continuum economy μ. We wish to prove that, when k is large enough, for each agent θ and for each possible deviation $d \in D$ there exists a demand function d'^* arbitrarily close to the competitive demand function x_θ that the agent prefers. First notice that the same argument used for the previous theorem establishes that the true and apparent economy get arbitarily close together (we might add, in the Prohorov metric), while both converging to the continuum economy μ. In particular, this implies that for k larger than a finite integer K, v_d^k lies, for each $d \in D$, within the neighborhood of μ

[12] Notice here strict concavity implies single-valued competitive demand, while single-valued deviations is a further restriction.

where economies are regular.Fix $\varepsilon > 0$ and define the desired function d^* as follows:

$$d^*(p) = \begin{cases} d(p) & \text{if } p \text{ is an equilibrium price at } v_d^k \text{ and } \|d(p(v_d^k) - d(p_i(\mu_{x_\theta}^k))\| < \varepsilon \\ x_\theta(p) & \text{otherwise} \end{cases}$$

Because v_d^k and $\mu_{x_\theta}^k$ are both regular , the set of equilibrium prices is finite and each equilibrium price function is continuous. Therefore, for each $\varphi_i > 0$ we can find a K' large enough so that $k > K'$ implies $\|p_i(v_d^k) - p_i(\mu_{x_\theta}^k))\| < \varphi_i$ for all $d \in D$ and all equilibrium price functions p_i. On the other hand, because of continuity and strict quasi concavity of preferences, we can choose φ_i such that if $\|d(p(v_d^k) - d(p_i(\mu_{x_\theta}^k))\| \geq \varepsilon$, then $d(p(v_d^k)$ satisfies the budget constraint at $p(v_d^k)$ and $U_\theta(x(p(\mu_{x_\theta}^k)) > U_\theta(d(p(v_d^k))$. But then , if $k > \max\{K, K'\}$,agent θ weakly prefers the allocation d^* over the deviation d. FInally observe that by choosing the metric $\rho(d, d') = \sup_{p \in \Delta_{++}} \|d(p) - d'(p)\|$, the definition of $d^*(p)$ implies that $\rho(x_\theta, d^*) < \varepsilon$, so d^* is arbitrarily close to the competitive demand function. This proves the following result:

Theorem 13 *Suppose $\{E^k\}$ is a sequence of economies such that $| E^K | \longrightarrow \infty$, with corresponding sequence of simple measures $\mu^k \longrightarrow \mu$, where μ is regular. Suppose preferences are strictly convex and continuos. Let d be a possible deviation by an agent θ belonging to each economy of the sequence. Then for each $\varepsilon > 0$ there exists a demand function d^* such that $\rho(x_\theta, d^*) < \varepsilon$ and $U_\theta(d^*) \geq U_\theta(d)$.*

It is worth commenting on the restrictiveness of the assumption of regularity of the limiting measure (or the continuity of the equilibrium price correspondence). In particular, is the set of regular economies a "large" subset of the universe of possible economies? The answer turns out to be positive, whenever the heterogeneity among consumers is not too large. As we described before, an economy can be seen as a measure on the space of all possible demand functions. By assuming differentiability, this space can be made into a topological space by considering the topology of uniform convergence on compacta. In turn, suppose we restrict attention to a *compact* subset of this space , and the relative space of Borel measures on it, endowed with weak convergence topology. With these assumptions, the measures (i.e. economies) that posses the regularity property form an open and dense subset (Hildebrand 1974). In this sense, regularity is quite an attractive assumption, and moreover, an assumption that cannot be easily dismissed.

On the other hand, limiting incentive compatibility is not quite incentive compatibility. As we have seen in the previous theorems, the walrasian mechanism is incentive compatible for sufficiently large economies, while there is still room for gain from misrepresentation in the "small". The natural question we now approach is whether there exists mechanisms different from the walrasian one that are incentive compatible and efficient along the entire sequence of economies, while approaching the walrasian one in the limit.

7.3.2 A Vickrey-Clarke-Groves type mechanism

As it is well known in the literature on incentive compatibility in quasi-linear environments, there exists a particularly attractive mechanism for this class of economies that guarantees incentive compatibility and some form of efficiency: the Vickrey - Clarke- Groves mechanism (VCG for brevity). It is therefore natural to look for a mechanism in the spirit of VCG, applied to a general class of non transferable utility economies, while working through some form of price system (Kovalenkov 2002). In our pure exchange economy setting, the message each consumer sends is a continuous demand function $d_\theta(p)$, while the measure $\mu(d)$ represents the fraction of consumer reporting demand function d. Also, from the walrasian equilibrium price correspondence, fix a continuous selection $p^*(\mu)$, that is $p^*(\cdot)$ is single valued, continuous and such that $p^*(\mu) \in Q(\mu)$ for each economy μ. Consumer θ is then allocated the consumption vector $d^{'}_\theta(p^*(\mu^{-\theta}))$, where $\mu^{-\theta}$ is the economy *without* consumer θ, with the relative equilibrium price $p^*(\mu^{-\theta})$, and $d^{'}_\theta$ is the demand function he or she declared. The resemblance with VCG is clear: each consumer's allocation is "insulated" from his message , thereby ensuring incentive compatibility in dominant strategies.

Let μ_k be a sequence of economies converging to the continuum economy μ. Along the sequence each consumer will truthfully report his demand function, but nevertheless the mechanism is not walrasian, since in general $p^*(\mu_k^{-\theta}) \neq p^*(\mu_k)$. However, $\mu_k^{-\theta}$ and μ_k get close together (again, this is basically the same argument used in the previous subsection on limiting incentive compatibility). Since we have chosen the price selection to be continuos, we obtain that the price used by the mechanism converges to walrasian equilibrium price:

$$\lim_k \sup_\theta \|p^*(\mu_k^{-\theta}) - p^*(\mu_k)\| = 0$$

By basically the same reasoning, the allocation appearing in equilibrium converge to the walrasian one:

$$\lim_k \sup_\theta \|d(p^*(\mu_k^{-\theta})) - d(p^*(\mu_k))\| = 0$$

However, and here is the main drawback of the mechanism, since we are not requiring any market clearing condition along the sequence, the mechanism creates an imbalance of some of the goods, just like VCG mechanism requires either outside funding of the numeraire good or a balancing agent. One might think that imbalance will vanish in the limit, however this is not the case. The limited result that can be obtained is that *per capita* imbalance, $\frac{1}{|E_k|}\| \sum_\theta d_\theta(p^*(\mu_k^{-\theta}))\|$, vanishes in the limit. In fact, $\frac{1}{|E_k|}\| \sum_\theta d_\theta(p^*(\mu_k^{-\theta}))\| = \frac{1}{|E_k|}\| \sum_\theta [d_\theta(p^*(\mu_k^{-\theta})) - d(p^*(\mu_k))]\|$, since $\sum_\theta d(p^*(\mu_k)) = 0$,

by the fact that $p^*(\mu_k)$ is walrasian at μ_k. But $\frac{1}{|E_k|}\| \sum_\theta [d_\theta(p^*(\mu_k^{-\theta})) - d(p^*(\mu_k))]\| \leq \sup_\theta \|d(p^*(\mu_k^{-\theta})) - d(p^*(\mu_k))\|$, and the right end side of this inequality converges to zero by the previous argument.

Compared to the limiting incentive compatibility result, the VCG mechanism studied in this section has the desirable property of being incentive compatible for each finite economy. The drawback is its lack of balancedness. In fact, although per capita imbalance converges to zero, there is still room for large total imbalance as the economy grows large. Last, notice that we have not gained anything in terms of genericity. In fact, the mechanism requires the existence of a continuous selection from the equilibrium price correspondence. Existence of such a selection can be proved only for an open and dense subset of the domain of regular measures (Mas Colell 1985).

8 Conclusions

In this work, we have attempted to establish some common threads in two contiguous strands of literature: social aggregation and implementation of allocation rules in pure exchange economies.

First, social aggregation impossibility/possibility results for social welfare functions and social choice functions can be stated in a strikingly similar fashion. Gibbard's proof of the Gibbard-Satthertwaite theorem relies on Arrow's theorem. Recently, Reny(2001) provided a word-by-word unique proof for impossibility (i.e. the finite case) in the two settings. Here , we have provided a unique proof via ultrafilter property of some subset of agents, because it allows to treat the finite and infinite case in a unified way. All this suggests the intriguing possibility of finding a unique common mathematical structure from which the results would follow as corollaries in the respective specialized settings.

Second, in both abstract social aggregation and aggregation in economic domains, there is a conflict between some form of efficiency and some form of justice, when one takes into account incentive compatibility in finite societies. It is quite interesting to notice that the abstract concept of democratic justice violated in the abstract aggregation setting corresponds to the violation of a more concrete requirement of minimal distributive justice in the allocation of resources.

Third, a folk justification for competitive behavior rests on large economies. At a first glance, it might seem that atomless spaces of agents restore possibility in both setting. However, the invisible dictator result should warn us that there is a real difficulty in extending this result to large but finite societies (sequences converging to the continuum,). In this work we have provided hints that this difficulty carries over to the economic domain. In particular, if we insist on the mechanism be walrasian, then incentive compatibility can be obtained only for large enough economies, while leaving ample space for incentives to misrepresent for small numbers. If we abandon the realm of walrasian mechanisms, we can An interesting topic for future research is to build a model for the economic domain that allows the study of the finite and infinite case in a unified way, much in the spirit of the Kirman Sondermann paper. This would give an exact meaning to the discontinuity between the finite and infinite case, just like the invisible dictator result in the abstract social choice setting.

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