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Lee-Carter Mortality Forecasting: Methodological and Computational Issues

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To my sister Andreina

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Introduction

The primary objective of this thesis is to implement the Lee-Carter methodology to analyse and forecast mortality and other vital rates of the Italian population. In particular we focus on two alternative approaches to forecasting life expectancies at birth.

In the 20^{th} century, the human mortality has declined globally. Such trends in mortality reduction present risks for insurers which have planned on the basis of tables that do not take these trends into account. To face this risk, it is necessary to resort to lifetables that includes forecasts of the future trends of mortality: the projected tables. Thus, reasonable mortality forecasting techniques have to be used to consistently predict the trends.

Over the years a number of approaches have been developed for forecasting mortality using stochastic models. Lee and Carter (1992) proposed a model for describing the secular change in mortality as a function of a single index. The method describes a time series of age-specific log-mortality rates as the sum of an age-specific component that is independent on time and a bilinear term, in which one component is a time-varying parameter reflecting general change in mortality and the second one is an age-specific factor, describing the pattern of deviations from the age profile.

Recently the Lee-Carter model has been widely discussed in the actuarial literature [Haberman & Renshaw (1996), Sithole, Haberman, Verrall (2000), Renshaw & Haberman (2003a) and Brouhns, Denuit & Vermunt (2002a,b)].

This thesis aims at contributing to this research area by analysing the empirical implications of applying the Lee-Carter methodology to the Italian population. We analyse, in a Lee Carter mortality context, the standard endowment policy under a fair value approach. In order to determine an actuarial model for the fair valuation at time t of the stochastic stream of cash flows, we base our demographic assumptions on the life and death probabilities extracted from the tables constructed using the Lee Carter model. The main contributions of this work are as follows. Through the Lee-Carter model we generate forecasts both of the level and of the age distribution of Italian mortality from 2001 to 2025. On the basis of these results we construct a valuation model that fully captures the interest and mortality rate dynamics. So far the emphasis has been on financial markets; the primary feature of our model is its focus on the demographic reference system.

The outline of the thesis is as follows:

Chapter 1: The Historical Review. We describe relevant aspects of the development of survival modelling in actuarial mathematics. In particular we analyse discrete-time vs continuous-time modelling, single decrements vs multiple decrements models and population homogeneity vs population heterogeneity.

We initially present the early actuarial models, proposed in the latter half of the 17^{th} century and we discuss their relevant features. This remaining survey is divided in two parts. In the first part we simply focus on specific scientific contribution that we consider as landmarks in the evolution of survival modelling. In particular, we discuss concepts and tools used in the actuarial field that pertain to the area of survival modelling.

In the second part, we concentrate our attention on recent literature representative of the current trend in actuarial research to also account for problems arising in life insurance and pension practice. We also offer guidelines in the field of mortality forecasts for consideration. We then introduce some recent mortality projections models and research dealing with uncertainty in future trends and the relevant actuarial evidence.

Chapter 2: Lee-Carter mortality forecasting: application to the Italian population. We investigate the feasibility of using the Lee-Carter methodology to construct mortality forecasts for the Italian population. We fit the model to the matrix of Italian death rates for each gender from 1950 to 2000. A time-varying index of mortality is forecasted in an ARIMA framework and is used to generate projected life tables. In particular we focus on life expectancies at birth and, for the purpose of comparison, we introduce an alternative approach for forecasting life expectancies on a period basis. The first method allows us to compute life expectancies from forecasted mortality rates. In this approach we find an appropriate ARIMA time series model for the mortality index k_t and then we use that mortality model to generate forecasts of the mortality rates. From the forecasts of mortality rates it is straight forward to calculate life tables and life expectancy at birth. Next we introduce an alternative approach by modelling and forecasting life expectancy directly; we perform a time series analysis of the annual life expectancies at age x to generate forecasts directly. The resulting forecasts generated by the two methods are then compared showing that the forecasts based on the LC model are dominated by the forecasts obtained under the direct time series approach (for both genders), thus bearing out the conservative nature of the life expectancy under the LC approach. Our results are consistent with the findings of Lee and Carter (1992) and Renshaw and Haberman (2003a), in their forecasting of life expectancies in the USA and in England and Wales, respectively. The results are interesting; the *a priori* assumption would be that they would be different, and this is what we find in our analysis. The modelling of the underlying mortality rates is a superior method in theoretical terms, yet employing the alternative allows us to examine the effect of a different approach. Moreover, the difference in results is evident for both genders.

Chapter 3: An application of the Lee-Carter model within the Fair Valuation context. Over the last few years the International Accounting Standards Board (IASB) in Europe and the Financial Accounting Standards Board (FASB) in the US have been considering fair value as an approach to valuing insurance contracts. Despite of a number of advantages, the fair value of insurance liabilities raises a number of issues. As we know, the dependence of payment of benefits on human life means that no regular market exists for such liabilities. Thus the market value of these liabilities is not readily available and must be estimated. The problem is that the demographic valuation is not supported by the hypothesis of the completeness of the market as for the financial valuation. It will also typically not be possible to find traded securities with a sufficiently close similarity to the life insurance and pension obligations such that fair value estimates can be obtained. Having seen the problems from not having markets for trading insurance liabilities, we need to construct a mathematical model of a pricing system that coherently represents the insurance realm. Thus we determine an actuarial model for the fair valuation of the stochastic stream of cash flows and we apply it to the case of an endowment policy with unitary benefits for a male policyholder. For the purpose of comparison we determine the value of the policy at time 0, subdividing the analysis into two stages. Firstly, we examine the case of a policy for an insured aged 40 at issue with a time to maturity of 15 years. Secondly we also apply the model to the case of a policy for an insured aged 65 at issue, with the same time to maturity. We present a comparison between the expected present value of the endowment policy in two cases: when the mortality rates are derived from the life table obtained with the Lee Carter methodology and when the mortality rates are derived from the life table SIM'92. What we find is a difference in the results due to the capturing of the improvements in mortality rates by the Lee Carter model, which determines a stronger projection.

Chapter 1

The Historical Review

1.1. Introduction

In 1762 Equitable Life Assurance Society was established as the first mutual life assurance company, based on the inspiration of a man ahead of his time – James Dodson.

It was not until 1750 that James Dodson, Fellow of the Royal Society, revolutionised the way that life assurance worked by developing his scientific basis for calculating premiums. Dodson used mortality tables and probability studies to calculate tables of fair annual premiums. The great advantage of these was that the policyholder's premium was fixed throughout the term of the policy and the amount paid on death was guaranteed. Although Dodson died before Equitable was founded, his ideas formed the basis of modern life assurance upon which all life assurance schemes were subsequently based.

In recent years the Society has undergone an exceptionally difficult period. During 1999 and the first half of 2000 a legal test case was fought to clarify the Society's approach to the Guaranteed Annuity Rates (GAR) offered by some with-profits pension policies sold up to the late 1980s. In July 2000 the House of Lords ruled that the Society's approach was inappropriate. As a result the then Board decided that it was in the best interest of members to put the Society up for sale.

After much initial interest in the Society, each potential purchaser withdrew. Without the proceeds of a sale to restore the capital strength of the with-profits fund, it was clear that the investment freedom, and so the performance of the fund, would be constrained. The former Board decided on 8 December 2000 to stop selling new business.

1.2. Landmarks in the history of actuarial models

1.2.1. The early actuarial models

In the latter half of the 17th century the early actuarial models were proposed. In 1671, Jan de Witt, in a report to the States of Holland, showed the first attempt to determine scientifically the purchase price of annuities, using mortality tables. De Witt's life table was hypothetical, although his report refers to some investigations of mortality of annuitants. He considered an immediate life annuity of 1 unit per annum payable in arrears; with x he denoted the present age of the annuitant and a_x the expected present value (i.e. the actuarial value) of a whole life annuity-immediate, such that:

$$a_x = a_{\overline{1}|}\ {}_{1/1}q_x + a_{\overline{2}|}\ {}_{2/1}q_x + a_{\overline{3}|}\ {}_{3/1}q_x + \ldots\ldots \qquad (1)$$

where ${}_{h/1}q_x = \frac{l_{x+h} - l_{x+h+1}}{l_x} = \frac{d_{x+h}}{l_x}$ and $\{l_x\}$ denoting the (expected) number of survivors at age x in a given life table, assumed as a survival model.

De Witt's report was forgotten until Hendriks (1852) rediscovered it and provided an English translation and commentary.

In 1693 Edmund Halley, the famous astronomer, constructed a life table from observations of the yearly number of deaths in Breslau (where the parish registers were among the first to contain age at death). He calculated the first table of values of annuities as a function of the nominee's age and developed formulae for calculating the value of joint life annuities (for two and three lives; with geometrical diagrams by way of explanation) and emphasised the benefit of using logarithms to reduce the volume of calculation. His approach to calculating the present value of annuities was through the distribution of the number of survivors, that is via the formula:

$$a_x = v\,{}_1p_x + v^2\,{}_2p_x + v^3\,{}_3p_x + \ldots\ldots \qquad (2)$$

where ${}_h p_x = \frac{l_{x+h}}{l_x}$.

It is worth noting that this formula is algebraically equivalent to de Witt's although computationally more straightforward, whereas de Witt's formula is much more interesting for further developments.

Halley remarked that the government was selling annuities too cheaply and at a price independent of the age of the annuitant: his advice was ignored. As many commentators have noted, the life table function tabulated by Halley was what we would call L_{x-1} rather than l_x. It is noteworthy that the Breslau table was reproduced in the updated version of 1737 of the abstract of the Amicable Society's charter and by laws.

1.2.2. Survival models: some features

In a modern perspective, the survival model used for evaluating life annuities was: (a) deterministic; (b) time-discrete; (c) single decrement; (d) (implicitly) assuming homogeneity; (e) (implicitly) static.

Early actuarial models for insurance products other than life annuities had analogous features. This was the case, for instance, of the model proposed by James Dodson in 1755, for calculating level premiums in whole life assurance (see Haberman, 1996). Some comments about these aspects follow.

(a) Although de Witt's formula refers to the expected value of a random variable, the only language available in the latter half of the 17th century for describing probability models was the language from games of chance, as pointed out by Hald (1987). Actuarial models for life insurance have been explicitly proposed in terms of random variables just in the 1950's. De Finetti (1950, 1957) and Sverdrup (1952) first defined the random present value, Y, of insurance benefits as a function of the random residual lifetime T_x.

Regardless of terminology, it is important to remark that the early survival models, albeit referring to random variables, did not allow for the riskiness inherent in insurance contracts, and hence can be considered as "deterministic".

(b) Halley's formula for the evaluation of life annuities constitutes one of the implementations of his life table, constructed from observed numbers of deaths in Breslau, whereas de Witt's life table was hypothetical. In both cases, since the proposed formulae explicitly refer to survival tables, it is quite natural that the adopted model is a time-discrete one. An important step towards time-continuous modelling follows from the early mortality "laws" originated from the fitting of mathematical formulae to mortality data.

(c) The type of benefits concerned in the early actuarial models, i.e. life annuity benefits (and assurances as well), naturally lead to a single-decrement setting. In the actuarial field, resorting to multiple decrement models follows the need to evaluate benefits depending on health status.

(d) Heterogeneity in respect of mortality is one of the most important issues in both survival modelling and actuarial practice. Although the early actuarial models did not allow for heterogeneity in populations, the problem of adverse selection was carefully considered at that time. As pointed out by Hald (1987), de Witt stressed that the nominee of an annuity contract usually is a person in "full health, and with a manifest likelihood of prolonged existence", thus a low mortality follows, at least in the initial annuity period.

(e) It was not until the construction of a long series of mortality observations that trends in mortality clearly emerged and hence the concept of dynamic mortality was achieved, namely at the beginning of the 20th century. At present, allowing for mortality trends is one of the most important issues in actuarial modelling, especially when life annuities and other living benefits are concerned.

The contributions underpinning the early survival models were progressed further, and actuarial models as well. Development of survival modelling required a lot of work, involving actuarial science, probability theory, demography, medical statistics, etc. In recent times, numerical approaches to actuarial problems gained effectiveness thanks to the availability of high speed computers, so paving the way to a new "computational" actuarial mathematics, also based on stochastic simulation procedures. It is worth noting that, unfortunately, many interesting results were ignored for decades and practically forgotten, before being rediscovered and finally implemented. Moreover, a number of the results of demographers were ignored by actuaries and vice versa.

1.3. Mortality: Old versus Modern Assumptions

1.3.1. Some basic ideas

Actuarial calculation in life insurance and pension funds involves the use of mortality assumptions, commonly expressed by the annual probabilities of death q_x, or the force of mortality μ_x. Within a traditional framework, these quantities are usually determined from period mortality observations. From the q_x's, a survival table is then derived as follows:

$$(1) \qquad l_{x+1} = l_x(1-q_x), \quad x = 0,1,....,\omega-1,$$

where ω is the assumed maximum age (105 or 110, say) and, for example, $l_0 = 100.000$, or $l_0 = 1$ as we assume in what follows. Using the force of mortality, the survival function is given, for $x > 0$, by

$$(2) \qquad l_x = l_0 e^{-\int_0^x \mu_t d_t}.$$

Formulae (1) and (2) implicitly assume that the q_x and the function μ_x can provide an appropriate representation of the age pattern of mortality over a period of, say, 110 years (namely, the maximal life span of humans).

In many countries, statistical evidence shows that human mortality declined over the 20th century, and in particular over its last decades. So, an hypothesis of "static" mortality, as implicitly involved by (1) and (2), cannot be assumed in principle. In actuarial practice, however, it is worth distinguishing between different calculation purposes. When mortality assumptions are required for pricing and reserving death benefits, the period-based q_x (or the μ_x) are on the safe-side for the insurer. Moreover, where term insurance is concerned, a short period (5–10 years) is usually involved. Conversely, when life annuities and other insurance living benefits are dealt with, calculations using period-based assumptions induce underestimation of insurer's or pension fund's liabilities because of mortality improvements.

1.3.2. Rectangularisation and expansion phenomena

Recent changes in mortality contribute in defining a moving scenario which clearly affects life insurance covers and annuities. Mortality trends at adult ages reveal two different features: at old ages probabilities of death are decreasing, whilst at young ages probabilities of death higher than in the past are observed, in particular, in the range 20–40. As far as life insurance valuations are concerned, the former aspect mainly affects living benefits, whilst the latter affects death benefits. In both cases, the calculation of expected present values (needed in pricing and reserving) requires an appropriate mortality projection in order to avoid underestimation of future costs. However, the projection itself is affected by uncertainty, since future changes in mortality are not known at the time of valuation; this uncertainty should be specifically considered in the appraisal.

The analysis of mortality over the last decades (see for example Benjamin and Soliman, 1993; Macdonald, 1997; Macdonald et al., 1998) shows various aspects which affect the shape of curves such as the curve of deaths and the graph of the survival function. In particular:

1. an increasing concentration of deaths around the mode (at adult and old ages) of the curve of deaths[1] is evident; so the graph of the survival function moves towards a rectangular shape, whence the term “rectangularization” to denote this aspect;
2. the mode of the curve of deaths (which, owing to the rectangularization, tends to coincide with the maximum age ω) moves towards very old ages, originating the so-called “expansion” of the survival function.
3. More recently, a further aspect, called young mortality hump, has been observed: higher levels and a larger dispersion of accidental deaths at young ages (primarily due to AIDS and drugs).

The above mentioned mortality trends clearly affect claim frequencies in life insurance. In particular, aspects 1 and 2 in the above list affect living benefits, whilst aspect 3 affects death benefits. In order to avoid underestimation of future costs, mortality projections are required in discounting the benefits.

Further aspects of mortality trends can be captured looking at the behaviour, for each integer age x, of the annual probability of death q_x drawn from a sequence of life tables pertaining to the same kind of population (e.g. males living in a given country). The graph constructed plotting the q_x’s against time is usually called “mortality profile”. Mortality profiles are often decreasing, in particular at adult and old ages.

1.3.3. *Mortality in a dynamic context*

Mortality improvements could induce underestimation of liabilities related to life annuities and other living benefits. So, trends in mortality imply the

[1] The graph of the probability density function of the random life, in an age-continuous setting

use of "projected" survival models for several actuarial purposes, e.g. for pricing and reserving as well as for assessing solvency in life offices and pension plans. A projected survival model aims at describing future age patterns of mortality, on the basis of the experienced mortality trend.

In actuarial practice, a common approach to mortality projections consists of choosing a model and estimating the relevant parameters simply aiming at extrapolating recent trends, as far as these can be perceived from mortality statistics.

In a different approach, models are adopted which allow one to express the basic characteristics of the evolving scenario in which mortality changes take place. For this purpose, analytical laws should be used, with parameters assumed to be functions of the calendar year.

A dynamic approach to mortality underpins projected survival models. When working in a dynamic context (in particular when projecting mortality), the basic idea is to express mortality as a function of the (future) calendar year y. Where a single-figure representation of mortality is concerned, a dynamic model is a real-valued function $\Psi(y)$. For example, the expected lifetime for a newborn, denoted by $\mathring{e}_0$ in a non dynamic context, is represented by $\mathring{e}_0(y)$, a function of the calendar year y (namely the year of birth), when the mortality trend is allowed for. Similarly, the general death rate in a given population can be represented by a function $q(y)$, where y denotes the calendar year in which the population is considered.

In actuarial calculations, age-specific measures of mortality are usually needed. Then in a dynamic context, mortality is assumed to be a function of both the age x and the calendar year y. In a rather general setting, a dynamic survival model is a function $\Gamma(x, y)$, usually with real values.

However, a vector-valued function is concerned if, for example, causes of death are allowed for.

In concrete terms, a real-valued function $\Gamma(x,y)$ may represent mortality rates, mortality odds, a force of mortality, a survival function, some transform of the survival function, etc. The projected survival model is given by the restriction $\Gamma(x,y)\setminus y > y'$, where y' denotes the current calendar year, or possibly the year for which the latest (reliable) period life table is available. The projected survival model is constructed (and, in particular, the relevant parameters are estimated) by applying appropriate statistical procedures to past mortality experience.

Although age-specific functions are needed in actuarial calculations, the interest of single-figure indices as functions of the calendar year should not be underestimated. In particular, important features of past mortality trends can be singled out focussing on the behaviour of some indices meant as "markers" of the probability distribution of the random lifetime at birth, T_0 (or at some given age x, T_x). Examples of markers providing a "location" measure are as follows (the notation refers to a non-dynamic context):

(1) the expected lifetime for a newborn, $\overset{\circ}{e}_0$;

(2) the expected lifetime at some fixed age x_0, $\overset{\circ}{e}_0$;

(3) the mode (at adult ages) of the curve of deaths, also called the Lexis point.

Turning back to age-specific functions, assume now that both age and calendar year are discrete variables. Hence, $\Gamma(x,y)$ can be represented by a matrix whose rows correspond to ages and columns to calendar years. For example, let $\Gamma(x,y) = q_x(y)$. Then, the annual probabilities of death in the matrix can be read according to three arrangements:

(a) a "vertical" arrangement (i.e. by columns),

$$(3) \qquad q_0(y), q_1(y), \ldots\ldots\ldots, q_x(y), \ldots.$$

corresponding to a sequence of period life tables, each table referring to a given calendar year y;

(b) a "diagonal" arrangement,

$$(4) \qquad q_0(y), q_1(y+1), \ldots\ldots, q_x(y+x), \ldots.$$

corresponding to a sequence of cohort life tables, each table referring to the cohort born in year y;

(c) a "horizontal" arrangement (i.e. by rows),

$$(5) \qquad \ldots\ldots, q_x(y-1), q_x(y), q_x(y+1), \ldots.$$

yielding the mortality profiles, each profile referring to a given age x.

As will emerge from the discussion of some of the contributions, thinking in terms of the various arrangements can also help in understanding different approaches to the interpolation of mortality data.

1.4. Mortality forecasts: seminal contributions

1.4.1. The forerunners

As noted by Cramér and Wold (1935), the earliest attempt to project mortality is probably due to the Swedish astronomer H. Gyldén. In a work presented to the Swedish Assurance Association in 1875, he fitted a straight line to the sequence of general death rates of the Swedish population during the years 1750 to 1870. A similar graphical interpolation was proposed in 1901 by T. Richardt for sequences of the annuity values a_{60} and a_{65}, calculated according to various Norwegian life tables, and then projected via extrapolation for application to pension plan calculations. Note that, as in the proposal by Gyldén, this case also concerned the projection of a single-figure index.

Mortality trends and the relevant effects on life assurance and pension annuities were clearly perceived at the beginning of the 20th century, as witnessed by various initiatives in the actuarial field. In particular, it is worth noting that the subject “Mortality tables for annuitants” was one of the topics discussed at the 5th International Congress of Actuaries, held in Berlin in 1906. Nordenmark (1906), for instance, points out that improvements in mortality must be carefully considered when pricing life annuities and, in particular, cohort mortality should be addressed to avoid underestimation of the related liabilities.

The 7th International Congress of Actuaries, held in Amsterdam in 1912, included the subject “The course, since 1800, of the mortality of assured persons”. Here a “dynamic” approach to mortality analysis was established.

As Cramér and Wold (1935) note, a life table for annuities was constructed in 1912 by A. Lindstedt, who used data from Swedish population experience and, for each age x, extrapolated the sequence of annual probability of death, namely the mortality profile $q_x(y)$, hence adopting a “horizontal” approach. Probably, this work constitutes the earliest projection of age-specific functions.

1.4.2. Mortality forecast: the earliest models

Blaschke (1923) proposed a Makeham-based projected survival model. A dynamic Makeham's law was defined as follows:

$$\mu_x(y) = \alpha(y) + \beta(y)\gamma(y)^x \qquad (6)$$

Hence, the three parameters are functions of the calendar year y. For the projection, a “vertical” method was proposed, consisting in the estimation of the constants for each period table (or “cross sectional” table) based on the experienced mortality, and then in fitting the estimated values; projected values of the three parameters are obtained via extrapolation.

As Cramér and Wold (1935) note, in 1924 the Institute of Actuaries in London proposed an “horizontal” method for mortality projection, assuming for the annual probability of death the following expression:

$$(7)\qquad q_x(y) = a_x + b_x c_x^y$$

thus, $q_x(y)$ is an exponential function of the calendar year y, from which the name “exponential formula” is commonly used to denote this approach to mortality projections. Parameters a_x, b_x and c_x are estimated on the basis of observed mortality profiles.

It is worth noting that projection formulae currently used by UK actuaries for annuitants and pensioners tables are particular cases of formula (8). For instance, with $a_x = 0$, $b_x = q_x(y')r_x^{-y'}$, $c_x = r_x$, where y' denotes the current year and r_x represents the annual rate of mortality improvement (if $r_x < 1$) at age x, the so-called “reduction factor”, we obtain

$$(8)\qquad q_x(y) = q_x(y')r_x^{y-y'}$$

Moreover, with $a_x = \lambda_x q_x(y')$, $b_x = (1-\lambda_x)q_x(y')r^{-y'}$, $c_x = r$, we find

$$(9)\qquad q_x(y) = q_x(y')\left[\lambda_x + (1-\lambda_x)r^{y-y'}\right]$$

where $\lambda_x q_x(y')$ represents (if $r < 1$) the asymptotic mortality at age x; in this case the speed of convergence, and hence r, is assumed to be independent of age. CMIR10 (1990) and CMIR17 (1999) can be referred to for more details. The formula proposed in 1929 by the German actuary C. W. Sachs also represents a particular case of (8), being as follows:

$$(10)\qquad q_x(y) = q_x(y')a^{\frac{y-y'}{x+b}}$$

where a and b are constants.

Let us turn to the “diagonal” approach. In 1927 A. R. Davidson and A. R. Reid proposed a Makeham-based model, with a dynamic Makeham's law defined as follows:

$$(11)\qquad \mu_x(y) = \delta(\tau) + \varphi(\tau)\psi(\tau)^x$$

where $\tau = y - x$ denotes the year of birth. In the implementation, $\psi(\tau) = \psi$ was assumed for all τ, whereas the functions $\delta(\tau)$ and $\varphi(\tau)$ were estimated via a cohort graduation (see Davidson and Reid, 1927).

The use of Makeham-based projected survival models is discussed by Cramér and Wold (1935), dealing with graduation and extrapolation of Swedish mortality. In particular, the diagonal and the vertical approach are compared.

The assumption formulated in 1934 by Kermack, McKendrick and McKinlay constitutes another example of the diagonal approach to mortality projections.

As Pollard (1949) notes, these authors showed that, for some countries, it was reasonable to assume that the force of mortality depended on the attained age x and the year of birth τ, and they deduced that

$$\mu_x(y) = Q(x)R(\tau) \tag{12}$$

where $y = \tau + x$, $Q(x)$ is a function of age only and $R(\tau)$ is a function of the year of birth only.

1.4.3. Some contributions from demography

Seminal contributions to survival modelling and mortality projections have been produced by demographers throughout the latter half of the 20th century. The "optimal" table, model tables and relational methods probably constitute three of the most influential proposals in recent times, in the framework of survival analysis.

As aforementioned, it clearly emerges that a number of projection methods are based on the extrapolation of observed mortality trends. Important examples are provided by formulae (6), (7) and (11). Albeit it seems quite natural that mortality forecasts are based on past mortality observations, different approaches to the construction of projected tables can be adopted.

Let us suppose that the existence of an "optimal" life table is assumed. The relevant age pattern of mortality must be meant as the limit to mortality improvements. Let q_x^* denote the limit probability of death at age x, whereas $q_x(y')$ denotes the current mortality. Assume then that the projected mortality $q_x(y)$ is expressed as follows:

$$(13) \qquad q_x(y) = F\left[q_x^*, q_x(y')\right]$$

where the symbol F denotes some interpolation formula. In particular, an exponential interpolation can be adopted, leading for example to:

$$(14) \qquad q_x(y) = q_x^* + \left(q_x(y') - q_x^*\right) r^{y-y'}$$

with $r < 1$. Note that formula (9) can be easily linked to (14), choosing λ_x such that $q_x(y')\lambda_x = q_x^*$.

The idea of an "optimal" table was proposed by Bourgeois-Pichat (1952). The question was: "can mortality decline indefinitely or is there a limit, and if so, what is this limit?" Determining a limit table requires a number of assumptions about the trend in various mortality causes, so that an analysis of mortality by causes of death is required.

When a mortality law is used to fit observed data, the age pattern of mortality is summarised by parameters (two or three, for Gompertz's law and Makeham's law respectively). Then, the projection procedure can be applied to the set of parameters (instead of the set of age-specific mortality rates). This results in a dramatic reduction in the "dimension" of the forecasting problem – namely in the number of "degrees of freedom". However, the age pattern of mortality can be summarized without resorting to mathematical laws (and hence avoiding the choice of appropriate laws).

In particular, some typical values, or "markers", of the mortality pattern can be used to this purpose, as mentioned before.

The possibility of summarising the age pattern of mortality by using some markers underpins the use of "model tables" in mortality projections. The first set of model tables was constructed in 1955 by the United Nations. The set was indexed on the expectation of life at birth, $\mathring{e}_0$, so that each table was summarized by the relevant value of this marker.

Model tables can be used for mortality forecasts as follows. A set of model tables is chosen, representing the mortality in a given population at several epochs, and assumed to also represent future mortality for that population. Trends in some markers are analysed and then projected, possibly using some mathematical formula, to predict their future values. Projected age-specific mortality rates are then obtained entering the system of model life tables for the various projected values of the markers.

A new way to mortality forecasts was paved by the "relational method" proposed by W. Brass (see Brass, 1974), who focussed on the logit transform of the survival function, namely

$$(15) \qquad \Lambda_x = \frac{1}{2}\ln\left(\frac{1-S(x)}{S(x)}\right)$$

Brass noted empirically that Λ_x can be expressed in the terms of the logit, $\Lambda_x^{s\tan d}$, pertaining to a " standard" population via a linear relation, i.e.

$$(16) \qquad \Lambda_x = \alpha + \beta\Lambda_x^{s\tan d}$$

whose parameters are (almost) independent of age.

For the purpose of forecasting mortality, equation (16) can be used in a dynamic sense. In a dynamic survival modelling context, the Brass logit transformation is particularly interesting when applied to cohort data, as the logits pertaining to successive birth-year cohorts seem to be linearly related (see Pollard, 1987). Hence, denoting $\Lambda_x(\tau)$ as the logit of the survival function for the cohort born in the calendar year τ, $S(x,\tau)$, we have:

$$(17) \qquad \Lambda_x(\tau) = \frac{1}{2}\ln\left(\frac{1-S(x,\tau)}{S(x,\tau)}\right)$$

Referring to a couple of birth years, τ_κ and τ_{k+1}, assume

$$(18) \qquad \Lambda_x(\tau_{k+1}) = \alpha_k + \beta_k\Lambda_x(\tau_k)$$

So, the problem of projecting mortality reduces to the problem of extrapolating the two series α_k and β_k. Projected values of various life table functions can be derived from the inverse logit transformation:

$$(19) \qquad S(x,\tau) = \frac{1}{1 + \exp[2\Lambda_x(\tau)]}$$

A different transform of the survival function S(x) has been addressed by Petrioli and Berti. The proposed transform is the "resistance function" (see Petrioli and Berti, 1979; Keyfitz, 1982), defined as follows:

$$(20) \qquad r(x) = \frac{\dfrac{S(x)}{\omega - x}}{\dfrac{1 - S(x)}{x}}$$

where ω denotes the maximum age. Thus, the transform is the ratio of the average annual probability of death beyond age x to the average annual probability of death prior to age x. The resistance function has been graduated with the curve:

$$(21) \qquad r(x) = x^{\alpha}(\omega - x)^{\beta} e^{Ax^2 + Bx + C}$$

and, in particular, with the three-parameter curve:

$$(22) \qquad r(x) = kx^{\alpha}(\omega - x)^{\beta}$$

Model tables have been constructed on combinations of the three parameters, focussing on the values of some markers.

In a dynamic context, the mortality trend is calculated assuming that a number of the parameters of the resistance function depend on the calendar year *y*.

Experienced mortality trends lead to parameters fitting through time, so that, referring to equation (22), we have:

$$(23) \qquad r(x,y) = k(y) x^{\alpha(y)} (\omega - x)^{\beta(y)}$$

Note that, assuming a model for the resistance function (see (21) and (22)) means that the resulting projection model can be classified as an analytical model, even though it does not directly address the survival function.

The Petrioli-Berti model has been used to project the mortality of the Italian population, and has thus been adopted by the Italian Association of Insurers to build projected mortality tables for annuity business.

1.4.4. Modern contributions to mortality forecast

In the last decades of the 1900's, various mortality law-based projection models have been proposed. In 1980 Heligman and Pollard proposed the following law to model mortality odds:

$$\frac{q_x}{p_x} = A^{(x+B)^C} + De^{-E(\ln x - \ln F)^2} + GH^x \qquad (24)$$

where the $\frac{q_x}{p_x}$'s are the so-called "odds". As far as the meaning of the law is concerned, the first term in (24), $A^{(x+B)^C}$, describes infant mortality, the second term, $De^{-E(\ln x - \ln F)^2}$, mortality at young ages and the third term, GH^x, mortality at old ages.

Forfar and Smith (1988) have performed mortality projections using the Heligman-Pollard law, assuming that various relevant parameters are functions of the calendar year: $A(y), B(y), \ldots$ (see also Benjamin and Soliman, 1993). Poulin (1980) has proposed a Makeham-based projection formula, whereas Wetterstrand (1981) has used Gompertz's law.

In the 1990's, a new method for forecasting the age pattern of mortality was proposed and then extended by L. Carter and R.D. Lee (see Lee and Carter, 1992; Lee, 2000). The Lee-Carter (LC) method used the central death rate to represent the age-specific mortality. Let $m_x(y)$ be the central death rate for age x at time y. The model is as follows:

$$\ln m_x(y) = a_x + b_x k_y + e_{x,y} \qquad (25)$$

where the a_x's describes the age pattern of mortality averaged over time, whereas the b_x's describes the deviations from the averaged pattern when the coefficient k_y varies. The variation in the level of mortality with y is described by k_y. Finally, the quantity $e_{x,y}$ denotes the error term.

Parameters a_x, b_x and k_y are estimated from experienced mortality, obtaining the estimates $\hat{a}_x$, $\hat{b}_x$, $\hat{k}_y$ (see also Renshaw and Haberman, 2002). Forecasts follow by modelling the values of k_y as a time series, e.g. a random walk with drift.

Starting from a given year y', forecasts of mortality rates are then computed, for $s = 1,2,....,$ as follows:

$$m_x(y'+s) = \exp\left(\hat{a}_x + \hat{b}_x \hat{k}_{y'+s}\right) = m_x(y')\exp\left[b_x\left(\hat{k}_{y'+s} - \hat{k}_{y'}\right)\right] \tag{26}$$

An important feature of the LC methodology should be stressed. Traditional projections models provide the forecaster with point estimates of future mortality rates (or other age-specific quantities). On the contrary, the LC method allows for uncertainty in forecasts. In fact, $m_x(y)$ is modelled as a stochastic process driven by the stochastic process k_y, whence interval estimates can be computed for the projected values of mortality rates.

The LC methodology represents one of the most influential proposals of mortality forecasting models in recent times. Much of the most recent research and many applications to actuarial problems are directly related to this methodology.

In 2002, Brouhns, Denuit and Vermunt, inspired from a comment made by Alho (2000) on Lee (2000), proposed an improvement of the LC method, using Poisson random variation for the number of deaths, instead of using the error term. They kept the Lee-Carter log-bilinear form for the force of mortality but replaced ordinary least-squares regression with Poisson regression for the death counts. In order to circumvent the problems associated with the OLS method, they considered that the random number of deaths was given by

$$(27) \qquad D_x(y) \cong Poisson(E_x(y)\mu_x(y))$$

with $E_x(y)$ the central number of exposed to risk, and $\mu_x(y)$ the force of mortality. The force of mortality is thus assumed to have the log-bilinear form

$$(28) \qquad \ln \mu_x(y) = \alpha_x + \beta_x \kappa_t \,.$$

The meaning of the α_x, β_x, and κ_t parameters is essentially the same as in the classical Lee-Carter model.

Renshaw and Haberman (2003) suggested ways in which the Lee-Carter methodology of fitting and forecasting mortality trends might be adopted for the construction of mortality reduction factors.

Whereas the LC mortality forecasting approach had a homoscedastic additive Gaussian error structure, Renshaw and Haberman (2003) and Brouhns et al. (2002), had each implemented similar alternative approaches to mortality forecasting, based on heteroscedastic Poisson (non additive) error structures. A key difference between Renshaw and Haberman (2003) and the method proposed by Brouhns et al. (2002) centres on the interpretation of time, which, in the Lee-Carter and Brouhns et al. (2002) approach is modelled as a factor and estimated by the singular value decomposition (SVD), and under the approach proposed by Renshaw and Haberman (2003), is modelled as a known covariate.

1.5. Concluding remarks

Methods for mortality projections can be classified according to various points of view. For brevity, we only focus on two criteria. Whatever the approach may be, mortality forecasts are obviously based on observed data, which usually consists of (cross-sectional) mortality tables. As regards the "use" of data in extrapolating observed trends, the following classification seems to be interesting.

I. Age-specific data can be directly used for mortality forecasts. Thus, the projection procedure is applied to quantities such as mortality rates q_x (see, for example, the exponential formula (7)), central mortality rates m_x (see the LC methodology), mortality odds $\frac{q_x}{p_x}$, etc.

II. Data can be "summarized" in several ways. Important examples are provided by the use of mortality laws (for instance Makeham's law), by model tables and by the Brass relational method. In these cases, the projection procedure is applied to the parameters of the law, or the markers associated to the model tables, or the parameters of the Brass linear relation.

As far as the link between experience data (i.e. mortality tables) and projected mortality is concerned, it is worth noting that:

- mortality tables provide estimates of random mortality in a (past) population;
- mortality in a future population is random, also because of its unknown trend.

The stochastic nature of mortality should not be disregarded, particularly when forecasts are considered. As regards the allowance for stochastic mortality, we can note what follows.

1) Traditional projection methods disregard the stochastic nature of (observed) mortality and provide the forecaster only with "point" estimates of future mortality.
2) The stochastic model underlying the LC methodology recognises the observed mortality rates as estimates, and allows for interval estimation of future mortality rates.

3) Uncertainty in future mortality rates is first attributable to random fluctuations around the relevant point estimates, namely to “process risk”. Moreover, deviations may also be attributed to the choice of the projection model, because the relevant parameters or the structure of the model itself do not reflect the actual mortality trend. Hence, “parameter risk” and “model risk” should both be allowed for when projecting mortality.

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Chapter 2

Lee-Carter mortality forecasting: application to the Italian population

2.1. Introduction and motivation

2.1.1. *Mortality on the move*

During the 20th century life expectancy has increased dramatically. The Human Mortality Database shows that Italian life expectancy at birth from 1900 to 1999 rose from 41.90 to 82.26 years for females and from 41.65 to 76.12 for males. Moreover, the trends in mortality rates for many industrialised countries have also been downwards for several years. Usually we view such mortality improvements in an optimistic way: according to the statistics we live longer than our ancestors. But these changes clearly affect pricing and reserve allocation for life annuities and represent one of the major threats to a social security system that has been planned on the basis of more modest life expectancy. Even when using updated mortality tables, these trends in mortality reduction present risks for insurers. This is because these tables do not take these trends into account. Put otherwise, the risk is of underestimating the survival probability, thus determining inappropriate premiums. This risk, is known in the actuarial literature as Longevity Risk, that being the risk derived from a future mortality rate which, ex post, does not reflect the forecasted one: see Brouhns, Denuit, Vermunt (2002b). To face this risk, it is necessary to build projected tables including this trend. Thus, reasonable mortality forecasting techniques have to be used to consistently predict the trends (see Brouhns, Denuit, Vermunt (2002a).

2.1.2. *Previous literature*

Over previous years a rapidly increasing body of literature has dealt with the issue of uncertainty in population forecasting and a number of approaches have been developed for forecasting mortality using stochastic models (Alho (1990), Alho & Spencer (1985), Bell & Monsell (1991)). In their article Alho and Spencer developed measures of uncertainty for forecasts of the national population for up to 15 years in the future when the forecasts were made by a popular projection method, the linear ("Leslie") model for population growth, adjusted for migration. Errors in population forecasts arise from errors in the jump-off population and errors in the predictions of future vital rates. Alho and Spencer studied the propagation of these errors through the linear growth model and developed prediction intervals for future populations. They compared the prediction intervals for U.S. national forecasts with the U.S. Census Bureau's high-low intervals. To assess the accuracy of the predictions of future vital rates, they derived the predictions from a parametric statistical model and estimated the extent of model misspecification and errors in parameter estimates. A novel aspect of their research was the incorporation of "expert opinion" into their statistical modelling. The mechanism they used to incorporate expert opinion was the "mixed estimation" regression model, which used expert opinion as a future observation distributed independently of all the other past and future observations.

Alho (1990) in his paper presented a stochastic version of the demographic cohort-component method of forecasting future population. The cohort-component method of population forecasting typically projects future numbers of annual births, deaths and migration by one-or five-year age-sex groups, adds them to form a new population vector, and repeats the calculations for each forecast year. The stochastic version of the cohort-component method treated the vital rates as realisations of random processes. This yields high-low intervals that have a given probability of covering the true size of an age-sex group in a given future year.

Alho compared the use of expert opinion in mortality forecasting with simple extrapolation techniques to see how useful each approach has been in the past.

An interesting alternative for forecasting mortality was proposed in 1992 by Lee and Carter, who published a new method extrapolating long-run forecasts of the level and age pattern of mortality, based on a combination of statistical time series methods and parametric approach. Recently the Lee-Carter model has been widely discussed in the actuarial literature. Haberman & Renshaw (1996), Sithole, Haberman, Verrall (2000) and Renshaw & Haberman (2003a) have implemented an alternative approach to mortality forecasting based on generalised linear models and heteroscedastic Poisson error structures. Brouhns, Denuit & Vermunt (2002a,b) kept the Lee-Carter log-bilinear form for the force of mortality, but replaced the ordinary least-squares regression structure with a Poisson regression model for the death counts.

2.1.3. Our motivations

There were two reasons for selecting the Lee-Carter model in our work. Firstly, this model represents one of the most influential recent developments in the field of mortality forecasts. Secondly, the important feature of this model is that for a precise value of the time index k, we can define a complete set of death probabilities that allow us to calculate all of the life table. Once we estimate the parameters, depending on age $\{\alpha_x, \beta_x\}$, they stay constant and invariant through time. Hence, when we know k, we can use the parameters for any year of interest. Another important feature that drove us to choose this model is that traditional projection models provide the forecaster with point estimates of future mortality rates. On the contrary, the LC method allows for uncertainty in forecasts (the so-called longevity risk).

2.2. Lee-Carter mortality forecasting methodology

2.2.1. The model

The Lee-Carter method is a powerful approach to mortality projections which describes the log of a time series of age-specific death rates $m_{x,t}$ as the sum of an age-specific component α_x, that is independent on time and another component that is the product of a time-varying parameter k_t, reflecting the general level of mortality, and an age-specific component β_x, that represents how rapidly or slowly mortality at each age varies when the general level of mortality changes.

In this contribution we consider the Lee-Carter model, which represents mortality level by a single index and we fit this demographic model to the matrix of Italian death rates, from year 1950 to 2000. We follow the methodology of Renshaw and Haberman (2003a), which is the inspiration for this chapter. Then we use the forecasts of this single parameter to generate forecasts both of the level and of the age distribution of mortality for the next 25 years. In particular we focus on life expectancies at birth and, for the purpose of comparison, we introduce an alternative approach for forecasting life expectancies on a period basis.

2.2.2. Notation, data and model fitting

The data for the Italian population, supplied by the Human Mortality Database, is divided by gender (Wilmoth et Al., 2000). Rather than using the entire dataset, we consider a subgroup of death rates for five-year age groups under 105 years old, so as to only cover five-year groups with a sample size significant enough for our analysis. The same is repeated for the corresponding exposure to risk.

To estimate the model for a given matrix of rates $m_{x,t}$, we seek the least squares solution to the equation:

$$(1) \qquad \ln(m_{x,t}) = \alpha_x + \beta_x k_t + \varepsilon_{x,t}$$

The model cannot be fitted by ordinary regression methods, because there are no given regressors; on the right side of the equation there are only parameters to be estimated and the unknown index k_t.

The data is comprised of the "Number of deaths" and the "Exposure to risk" denoted by two 5×1 matrices supplied by The Human Mortality Database. In this notation, the first number refers to the age interval, and the second number refers to the time interval (Elandt-Johnson and Johnson, 1980). For each gender and for each calendar year: $t = t_1, t_1+1, ..., t_1+h-1 = t_n$, where $h = t_n - t_1 + 1$, we consider all the ages $x = x_1, x_2, ..., x_k$, grouped in classes as $[0,1-4,5-9,10-14,.........,95-99,100-104]$. From these data we construct an array of crude rates of deaths $m_{x,t} = \dfrac{d_{x,t}}{e_{x,t}}$.

In order to find a least squares solution we use a close approximation, suggested by Lee and Carter (1992), to the singular value decomposition (SVD) method, assuming that the errors are homoschedastic. To obtain a unique solution, we impose that the sum of the β_x coefficients is equal to 1.0, and that the sum of the k_t parameters is equal to zero.

Under these assumptions, it can be seen that the α_x coefficients must be simply the average values over time of the $\ln(m_{x,t})$ values for each x.

We estimate α_x as the logarithm of the geometric mean of the crude mortality rates, averaged over all t, for each x:

$$(2) \qquad \alpha_x = \tfrac{1}{h}\sum_{t=t1}^{tn} \ln m_{xt} = \ln\left[\prod_{t=t1}^{tn} m_{xt}^{\frac{1}{h}}\right]$$

Furthermore, k_t must equal the sum over age of $(\ln(m_{x,t})-\alpha_x)$. All that remains, is to estimate the $\beta_x s$. We found each β_x by regressing $(\ln(m_{x,t})-\alpha_x)$ on k_t, without a constant term, separately for each age group x. More precisely, we estimate β_x from $(\ln m_{xt} - a_x) = b_x k_t^{(1)} + \varepsilon'_{xt}$ (where $k_t^{(1)}$ refers to k_t above) using the least squares estimation, i.e. choosing b_x to minimize $\sum_{x,t}(\ln m_{xt} - \alpha_x - \beta_x k_t^{(1)})^2 \Rightarrow \beta_x = \frac{\sum_{t=t1}^{tn} k_t^{(1)}(\ln m_{xt}-\alpha_x)}{\sum_{t=t1}^{tn} k_t^{(1)2}}$. The raw estimates of α_x, β_x and k_t are inserted in the Appendix A.

Here α_x describes the general age shape of the age specific death rates $m_{x,t}$, while k_t is an index that describes the variation in the level of mortality to t. The β_x coefficients describe the tendency of mortality at age x to change when the general level of mortality (k_t) changes. When β_x is large for some x, then the death rate at age x varies substantially when the general level of mortality changes (as with $x = 0$ for infant mortality, for example) and when β_x is small, then the death rates for that age vary little when the general level of mortality changes (as is often the case with mortality at older ages).

The Lee Carter model also assumes that all the age specific death rates move up or down together, although not necessarily by the same amounts, since all are driven by the same period index, k_t. Although not all occurrences of β_x need to have the same sign, in practice all the β_x do have the same sign, at least when the model is fit over fairly long periods. As shown in the Appendix A, the $\beta_x s$ for both females and males have the same sign, which is positive. In Fig. 1, the values of β_x, as determined with the SVD, are plotted against x, for each case separately i.e. by gender.

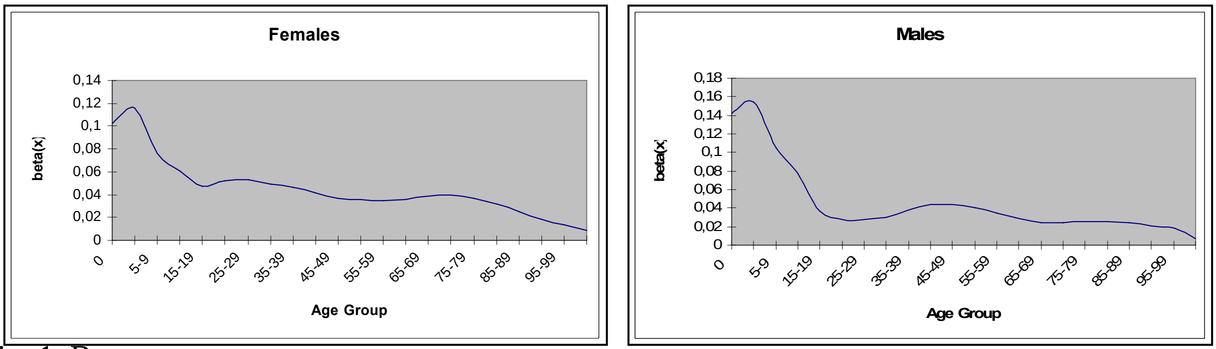

Fig. 1. Beta versus age

From Fig.1 we can see that when β_x is large for some x, then the death rate at age x varies significantly when the general level of mortality changes (again, as with $x = 0$ for infant mortality) and when β_x is small, then the death rate at that age varies little when the general level of mortality changes. This often the case with mortality at older ages.

2.2.3. Reestimating kt

Because the first stage estimation is based on logs of death rates rather than the death rates themselves, sizable discrepancies can occur between predicted and actual deaths. To guarantee that the fitted death rates will lead to the actual numbers of deaths, when applied to given population age distribution, we have reestimated k_t in a second step, taking the α_x and β_x estimates from the first step. To correct for this, we apply the methodology from Section 3 of Lee and Carter (1992). We thereby find a new estimate for k by an iterative search, adjusting the estimated k_t so that the actual total observed deaths $\sum_{x=x1}^{xk} d_{xt}$ equal the total expected deaths $\sum_{x=x1}^{xk} e_{xt} e^{(\alpha_x + \beta_x k_t)}$, for each year t.

The iterative method proceeds as follows:

1) We compare the total expected deaths $\sum_{x=x1}^{xk} e_{xt} e^{(\alpha_x + \beta_x k_t^{(1)})}$ to the actual total observed deaths $\sum_{x=x1}^{xk} d_{xt}$ in each period.
2) This comparison reveals one of three possible states:
   (i) If $\sum_{x=x1}^{xk} e_{xt} e^{(\alpha_x + \beta_x k_t^{(1)})} > \sum_{x=x1}^{xk} d_{xt}$, we need to decrease the expected deaths, adjusting the estimated k_t so that the new estimate of k_t, say $k_t^{(2)}$, will be: $k_t^{(2)} = k_t^{(1)}(1-d)$, if $k_t^{(1)} > 0$ (where $k_t^{(1)}$ is the first estimate of k_t) ; $k_t^{(2)} = k_t^{(1)}(1+d)$, if $k_t^{(1)} < 0$, where d is a small number.
   (ii) If $\sum_{x=x1}^{xk} e_{xt} e^{(\alpha_x + \beta_x k_t^{(1)})} = \sum_{x=x1}^{xk} d_{xt}$, we stop here the iterations.
   (iii) If $\sum_{x=x1}^{xk} e_{xt} e^{(\alpha_x + \beta_x k_t^{(1)})} < \sum_{x=x1}^{xk} d_{xt}$, we need to increase the expected deaths adjusting the estimated k_t so that : $k_t^{(2)} = k_t^{(1)}(1+d)$, if $k_t^{(1)} > 0$; $k_t^{(2)} = k_t^{(1)}(1-d)$, , if $k_t^{(1)} < 0$.
3) Go back to Step 1.

As Lee and Carter (1992) point out, this approach differs from the direct SVD estimates. This is because the low death rates of youth contribute far less to the total deaths, yet when fitting the log-transformed rates they are weighted equivalently to the high death rates of the older ages. It is also worth noting that differences in population age group sizes also results in different weights in the second-stage estimation of k.

2.2.4. First application and comments

We have run this iterative process 1000 times using a VBA macro and Microsoft Excel to find the new estimate of k, shown in the Appendix B.

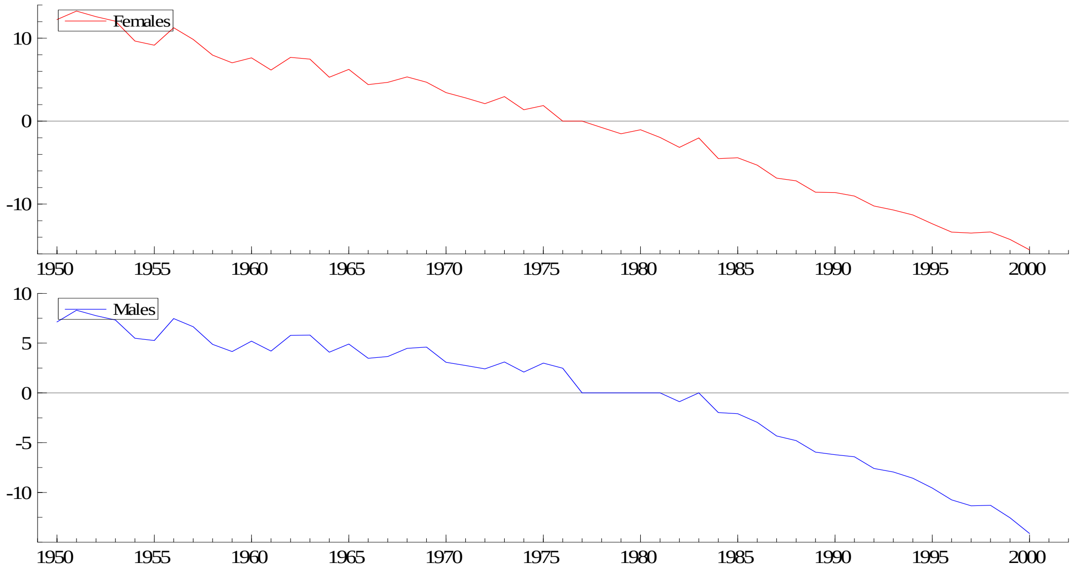

Fig.2 Re-estimates of k

Fig. 2 plots estimates of k, for females and males; as shown, k declines roughly linearly from 1950-2000, more for females than for males. If we look at the values of k, shown in Appendix B, k declines at about the same pace during the first half of the period as it does during the second half. It also is striking that short-run fluctuations in k do not appear much greater in the first part of the period than they do in the second, with the exception of the male series in the first years. Both these features of k (its linear decline and its relatively constant variance) are very convenient for forecasting purposes.

We can see from the re-estimated k_t that mortality improved in Italy. For the purposes of comparison with other countries, for example Britain (as presented in Renshaw and Haberman, 2003), we can see that the Italian improvement is more pronounced. This is probably due to the fact that mortality was initially higher in Italy than in Britain, making the relative improvement greater and therefore more apparent. If we compare male to female mortality we might expect to see the same effect. Male mortality is higher than female mortality, thus possible improvements in male mortality could again be more evident than improvements in female mortality in an analogous way to the country comparison.

2.3. ARIMA methodology

2.3.1. Modelling mortality index

The estimated time-dependent parameter k_t can be modelled as a stochastic process; we thus used the standard Box and Jenkins methodology (identification-estimation-diagnosis) to generate an appropriate ARIMA (p,d,q) model for the mortality index k_t (Box and Jenkins; Hamilton, 1994).

Considering the time series given by the reestimated k_t, we need to identify a correct model, for our series, among the general class of ARIMA models. The procedure to construct the model goes through different iterative phases to arrive at a model that fits our data well (Francis X. Diebold, 2004; Makridakis, Wheelwright, Hyndman, 1998). The phases are the following:

1) Preliminary analysis of the series and possible transformation.
2) Identification of the order of the model.
3) Parameter estimation.
4) Evaluation of the model.

In the first step, we analyse the general pattern of the time series, as is illustrated in Fig. 2. A clear, almost linear, trend emerges, indicating that mortality enjoyed a steady erosion over the years.

The input series for an ARIMA needs to be stationary, that is, it should have a constant mean, variance, and autocorrelation through time. Therefore, the series usually needs to be differenced first until it is stationary. The number of times the series needs to be differenced to achieve stationarity is reflected in the d parameter. In order to determine the necessary level of differencing, one should examine the plot of the data and autocorrelogram, that displays graphically and numerically the autocorrelation function (ACF). We examine the ACF of the series and choose the value of d that gives rise to a rapid decrease of the ACF towards zero.

2.3.2. Identification phase

In the Identification phase, after we made the series stationary, we also need to decide how many autoregressive parameters (p) and/or moving average parameters (q) are necessary to yield an effective, but still parsimonious model of the process. In practice, the numbers of the p or q parameters very rarely need to be greater than 2.

The major tools used in the Identification phase are plots of the series, correlograms of autocorrelation (ACF) and partial autocorrelation (PACF). The decision is not straightforward, and in less typical cases requires not only experience but also a good deal of experimentation with alternative models (as well as the technical parameters of ARIMA models). We experimented with twelve models, based on combinations of the p and q parameters varying between zero and two. The sample autocorrelations and partial autocorrelations, together with related diagnostics, provided graphical aids to model selection. This complemented our automatic identification criteria, the Akaike and Schwarz information criterion per model. To guide model selection we use these two criterions even though the SIC usually selects more parsimonious models due to its greater concern over the number of parameters to be estimated. Using a model selection strategy involving not just examination of AIC and SIC, but also examination of autocorrelations and partial autocorrelations, we are led to choose the ARIMA (0,1,0) for males and an ARIMA (0,1,1) for female. For males a model with an ar(1) term added could be marginally superior, but we preferred a random walk with drift on grounds of parsimony. We examine the general pattern of the time series for both genders in Fig.2, and we saw that a clear, decreasing trend emerges for each, indicating that the series are not stationary in mean. We are led to the same conclusions if we look at the autocorrelation function or the partial autocorrelation functions in Fig.3 (females) and 4 (males).

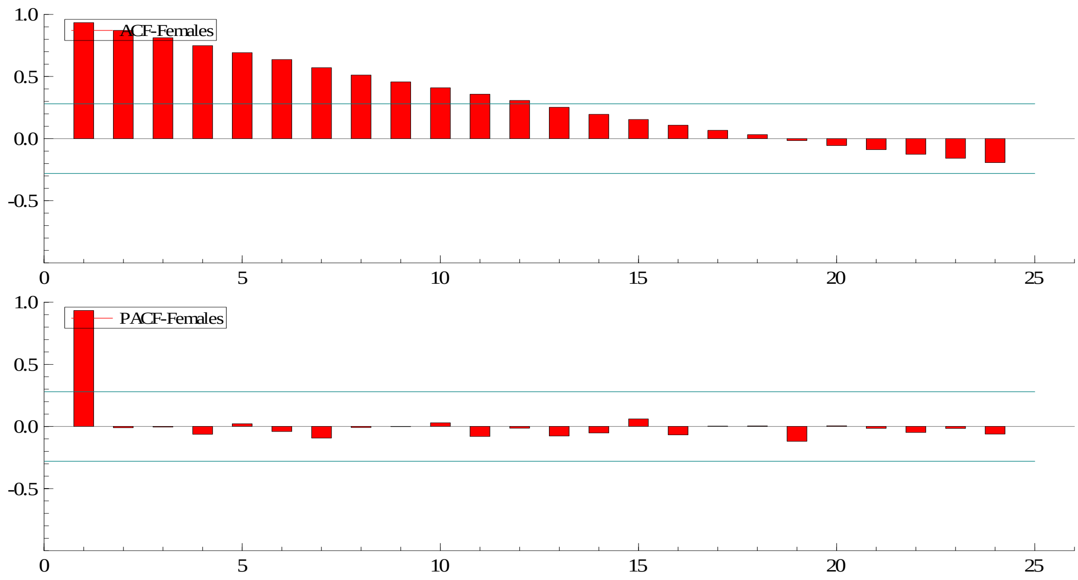

Fig.3 Female autocorrelation and partial autocorrelation function

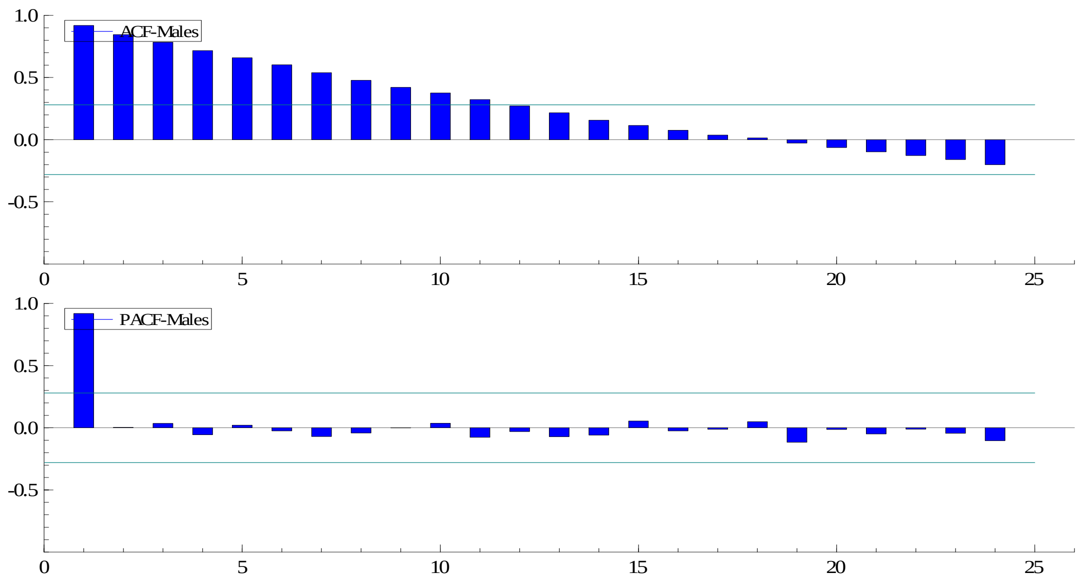

Fig.4 Male autocorrelation and partial autocorrelation function

As we can see, if we look at the graph of the autocorrelation function (ACF), this approaches zero gradually rather than abruptly. On the contrary, the partial autocorrelation function (PACF) cuts off abruptly; specifically, at displacement 1, the partial autocorrelations are significant while coefficients on all longer lags are zero. This is a clear sign of a nonstationary series.

Thus, following the Box and Jenkins methodology, we considered the differenced series, which we show in Fig.5

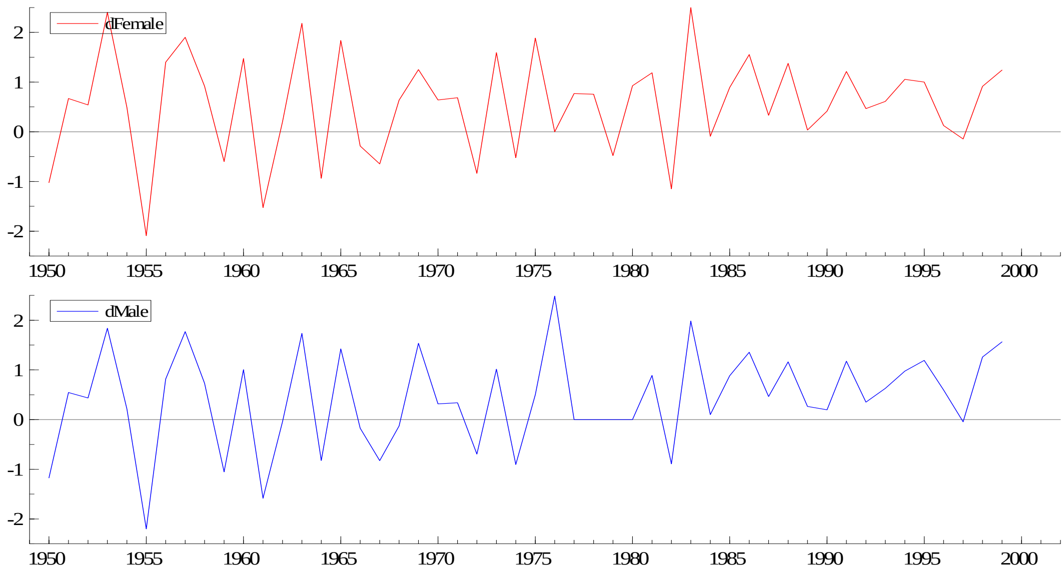

Fig.5 Differenced females and males series

After differencing the series, the nonstationarity in mean seems to be eliminated. Also the autocorrelation and partial autocorrelation functions (Fig.6), become consistent with the hypothesis of a stationary series. Because of the decreasing trend, when we estimated our model we also took a constant into consideration.

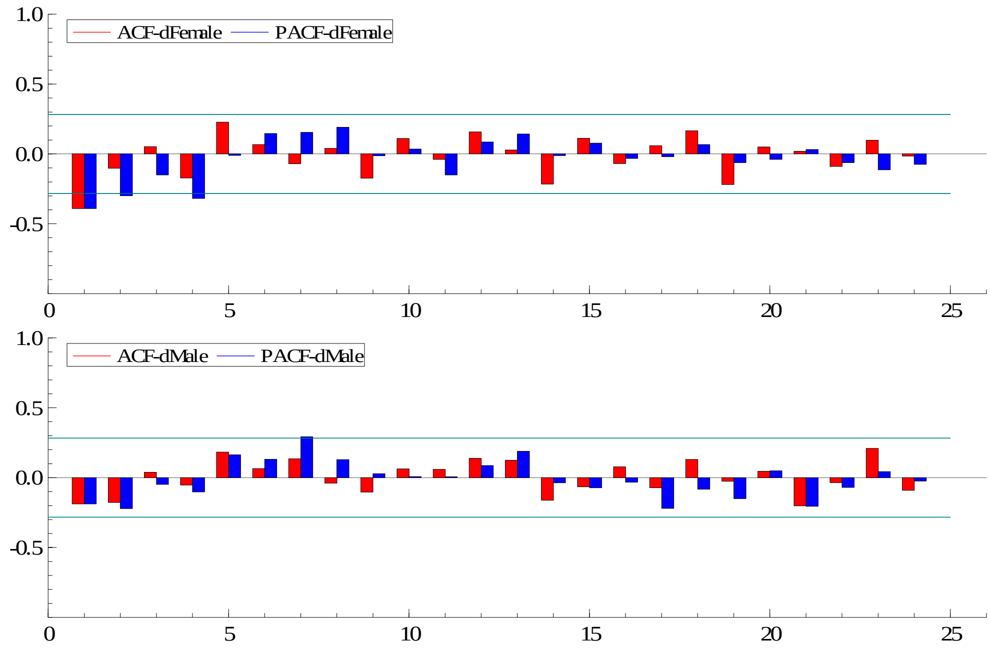

Fig.6 Autocorrelation and partial autocorrelation functions after differencing the series

2.3.3. Parameters estimation

Concerning the third phase, there are several different methods for estimating parameters. All of them should produce very similar estimates, but may be more or less efficient for any given model. Model parameters are estimated using statistical software, in our case time series estimation was performed by EViews using a least squares procedure. The k_t index for males was modelled as an ARIMA (0,1,0) process, i.e.:

$$K_t = K_{t-1} + \lambda + \varepsilon_t$$

and for females as an ARIMA (0,1,1) process, i.e.:

$$K_t = K_{t-1} + \lambda + \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

The constant terms λ indicate the average annual change of k_t. It is this change that drives the forecasts of the long-run change in mortality. θ represents the moving average term.

The estimated parameters for both genders, and their standard errors, appear in the table below:

Male ARIMA (0,1,0)

Variable	Coefficient	Std. Error	t-Statistic	Prob.
λ	-0.424882	0.137488	-3.090321	0.0033

Female ARIMA (0,1,1)

Variable	Coefficient	Std. Error	t-Statistic	Prob.
λ	-0.566485	0.045168	-12.54168	0.0000
θ	-0.644956	0.108801	-5.927839	0.0000

The autoregressive parameter φ is equal to zero in both cases; as we see from the t-statistics, the parameters are significant. Furthermore, the Ljung-Box test and the residual plot guides us towards retaining the chosen model due to its good fit to the data.

For comparison, we note that Renshaw and Haberman (2003a), fitted the same ARIMA (1,1,0) process for males and females using the LC model, obtaining a parameters estimates of $\varphi = -0{,}532$ and $\lambda = -0{,}3041$ for males and of $\varphi = -0{,}572$ and $\lambda = -0{,}3525$ for females. This was based on data for England and Wales over the period 1950-1998, and results in parameters which are comparable with our above estimates.

2.3.4. Evaluation of the model

The evaluation of the model aims at verifying that the model identified and estimated in the previous phases is adequate. If it is not, we have to suggest an alternative model. The objective of diagnostic checking is to ascertain whether the model "fits" the historical data well enough. This diagnostic checking is undertaken analysing the residuals of the estimated model: if the model is adequate, the residuals should reflect the features of *white noise*. One way to check for adequacy is to use the model to forecast all of the known values of the data series, compute the differences (i.e., the residuals) between the known and forecasted values, and generate the simple autocorrelation correlograms for the residuals. If none of the residuals autocorrelations is significantly different from zero, the model may be judged adequate. Another approach to diagnostic checking is to estimate a model with higher-ordered autoregressive and moving average terms, then observe (i.e., draw an inference from the t-statistic) whether the regression coefficients of the additional terms are statistically significant. Yet another approach to diagnostic checking is to employ the Chi-square statistic as a diagnostic criterion. We may compute a test statistic employing the equation $Q=(n-d)*\sum R^2$, where n is the number of observations in the series, d is the degree of differencing, R^2 is the square of the autocorrelation coefficient, and the sum is taken over the range of 1 to k, the order of autocorrelation. The appropriate number of degrees of freedom is $k-d-1$. If the computed value of Q is less than the Chi-square statistic for $k-d-1$ degrees of freedom, the model is judged adequate.

To verify that the model we have previously identified and estimated fits the historical data well, we perform a number of analyses. We fit different models to the matrix of Italian death rates from 1950 to 1985, thereby using a 35 years in-sample period, to generate out-of-sample forecasts for the next 15 years. After fitting a range of models in-sample, we compute the Root of Mean Square Error (RMSE) for each ARIMA model and we find that the models we have chosen (ARIMA (0,1,0) for males and ARIMA (0,1,1) for females) are the ones with the lowest RMSE. This indicates that these are the models which best approximate the historical data.

2.4. Projecting lifetables

2.4.1. Traditional method

Now we can use the ARIMA (0,1,1) and ARIMA (0,1,0) models to generate the forecasts, for the next 25 years based on the period 1950-2000, of the index of mortality k_t. Appendix C lists these values for both genders.

Figure 7 and 8, instead, plot the past values of k along with the forecasts based on the time series model and the associated confidence intervals, for females and males respectively. It is worth noticing that we have used the Lee-Carter method for calculating the prediction intervals that concentrates just on variability due to kappa. The other sources of variability could be allowed for by using a bootstrap method: see Brouhns, N., Denuit, M., Van Keilegom (2005).

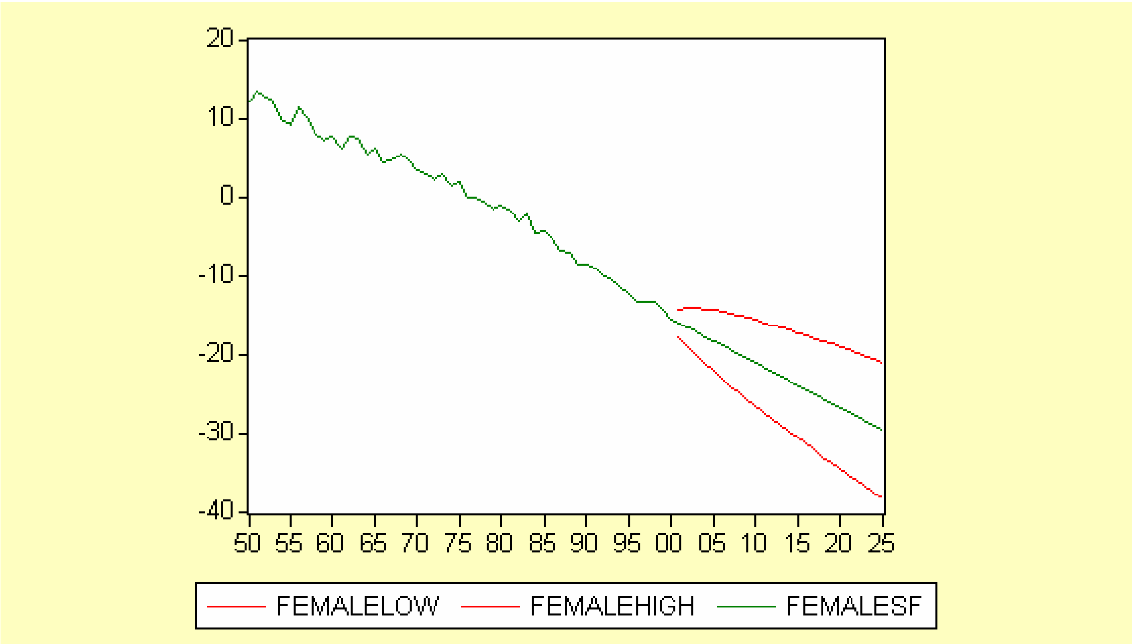

Fig. 7 Forecasts of Female Mortality Index k with confidence interval

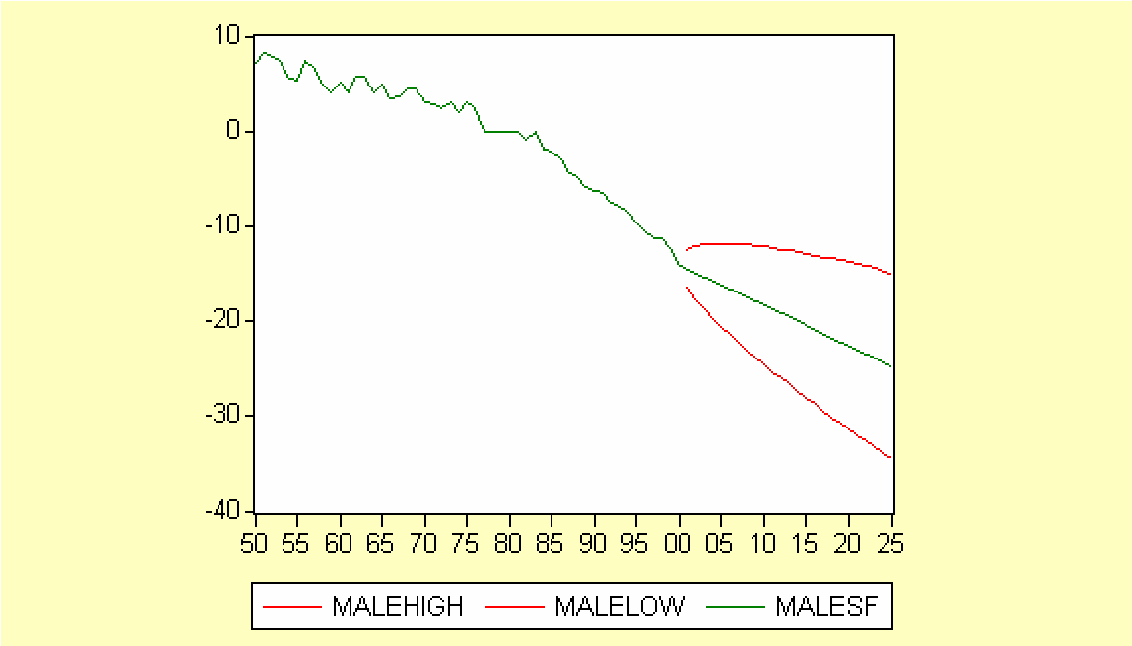

Fig. 8 Forecasts of Male Mortality Index k with confidence interval

Once we have forecasted the index of mortality, we can generate associated life table values at five-years intervals. First we insert the projected k_{2000+s}, $s = 1,2,.....,25,$ into the formulas

$$(3) \qquad \overset{\circ}{m}_{x,2000+s} = \hat{m}_{x,2000} \exp\left\{ \hat{\beta}_x \left(\overset{\circ}{k}_{2000+s} - \hat{k}_{2000} \right) \right\}$$

to compute forecast mortality rates by alignment to the latest available empirical mortality rates $\hat{m}_{x,2000}$.

Figure 9 shows the shapes of the mortality rates that we forecast for the females generations born in years 2001 and 2025. It is worth noticing that the mortality rates for age groups 1 - 4 and 5 - 9 become virtually identical by 2001 and 2025.

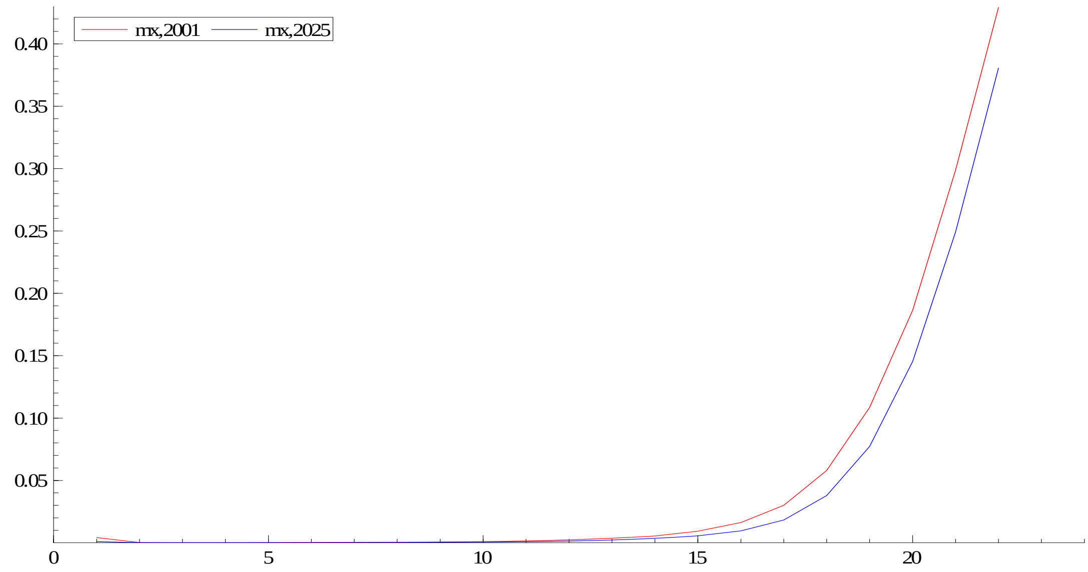

Fig. 9 Forecasted mortality rates for the female generations born in years 2001 and 2025

From these projected mortality rates, we can build projected life tables and compute life expectancy at birth: see Keyfitz N. (1977).

Thus, we convert the life table death rates, m_x, into probabilities of death, q_x. Let f_x be the average number of years lived within the age interval $[x, x+1)$ for people dying at that age. As in Renshaw and Haberman (2003a), we assume that $f_x = \frac{1}{2}$ for all age group except age 0 (for $x=0$ we fix $f_x = 0{,}15$ for males and $f_x = 0{,}16$ for females). We then compute q_x from m_x and f_x according to the formula,

$$(4) \qquad q_x \cong \frac{w_x m_x}{1 + f'_x\, w_x m_x}, \qquad\qquad x = x_0, x_1, \ldots., x_{k-2},$$

for $x = 0, 1-4, 5-9, \ldots\ldots, 100-104$, $w_{xi} = x_{i+1} - x_i$, $k = 22$ and $f'_x = 1 - f_x$.

To complete the life table calculation, let p_x be the probability of surviving from age x to $x+1$.

Therefore,

$$(5) \qquad p_x = 1 - q_x,$$

for all five-year age groups up the age of 104.

From q_x calculated by (4) and an arbitrary l_0 (in our case we make it equal to 100000) the life table is constructed by working down the column of l's and d's, applying the recurrence equations

$$(6) \quad l_{x+wx} = l_x(1-q_x), \qquad x = x_0, x_1, \ldots, x_{k-2},$$

$$(7) \quad d_x = l_x - l_{x+w_x} = l_x q_x, \qquad x = x_0, x_1, \ldots, x_{k-2},$$

where l_x indicates the number of survivors and d_x is the distribution of deaths by age in the life table population.

The person-years lived by the life-table population in the age interval $[x, x+1)$ are

$$(8) \quad L_x = w_x(l_x - f'_x d_x), \qquad x = x_0, x_1, \ldots, x_{k-2}.$$

The person-years remaining for individuals of age x equal

$$(9) \quad T_{x_i} = \sum_{x=x_i}^{x_{k-1}} L_x$$

imply that life expectancy is given by

$$(10) \quad e_{x_i} = T_{x_i} / l_{x_i}.$$

Appendix D lists forecasts of life expectancy at birth obtained using the Lee-Carter model and also shows forecasts obtained with the alternative method which will be discussed later.

2.4.2. The alternative approach to forecast life expectancy

The method seen above allowed us to compute life expectancies from forecasted mortality rates. In that approach we found an appropriate ARIMA time series model for the mortality index k_t and then we used that mortality model to generate forecasts of the mortality rates. From the forecasts of mortality rates it was straight forward to calculate life tables and life expectancy at birth.

Now we introduce an alternative approach by modelling and forecasting life expectancy directly; we perform a time series analysis of the annual life expectancies at age x to generate forecasts directly. In particular, we consider annual life expectancies at birth for the Italian population, supplied by the Human Mortality Database and divided by gender, from 1950 to 2000. As before, we use the standard Box and Jenkins methodology to generate an appropriate ARIMA (p,d,q) model for our time series, represented in this case by the males and females life expectancies at birth.

In this case the life expectancies are intrinsically viewed as a stochastic process and are estimated and forecasted within an ARIMA time series model. We find that an appropriate model for males and females is ARIMA (1,1,1):

$$\nabla e_t = \varphi_1 \nabla e_{t-1} + \lambda + \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

where ∇ is the differencing operator and $\{\varepsilon_t\}$ denotes white noise.

The fitted ARIMA (1,1,1) model generates sex-specific life expectancy forecasts directly. Appendix D shows forecasts of life expectancy at birth, comparing the results obtained using the Lee-Carter methodology and the alternative approach. Both approaches are illustrated in Figure 10, which shows life expectancy at birth from 1950 to 2000 and forecasts from 2001 to 2025. As shown the forecasts based on the LC model are dominated by the forecasts obtained under the direct time series approach (for both genders), thus bearing out the conservative nature of the life expectancy under the LC approach. We want to stress that our results are consistent with the findings of Lee and Carter (1992) and Renshaw and Haberman (2003a), in their forecasting of life expectancies in the USA and in England and Wales, respectively.

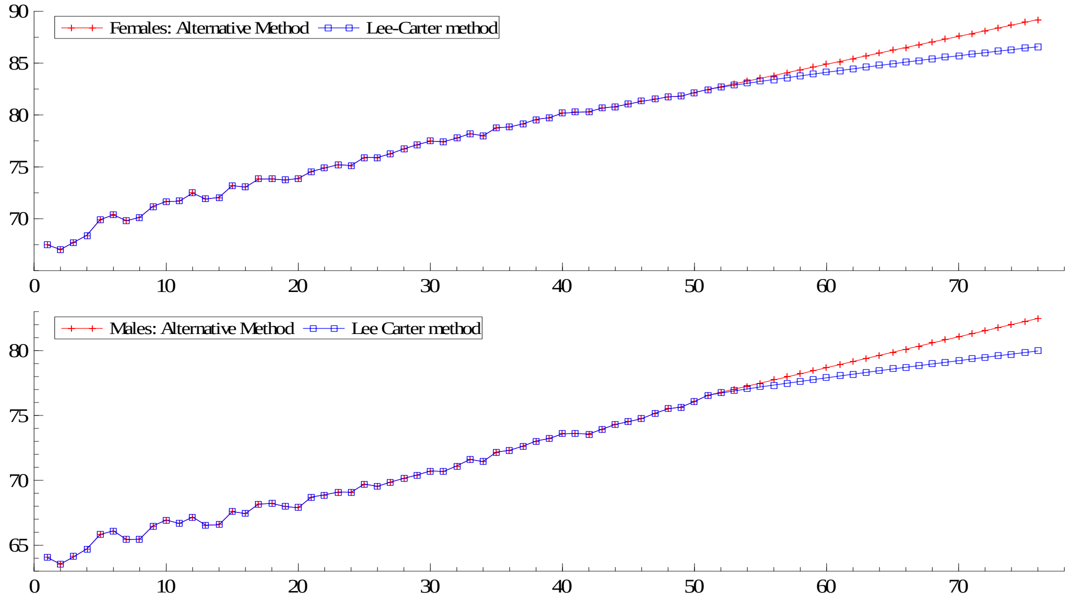

Fig. 10 Life Expectancy at birth and Forecasts

2.5. Conclusions

We have presented an application of the model underpinning the Lee-Carter methodology for forecasting vital rates. In particular we have focused on forecasting life expectancies on a period basis and we have compared the life expectancies forecasted under the LC model, with the time-series-based forecast. The results are interesting; the *a priori* assumption would be that they would be different, and this is what we find in our analysis. The modelling of the underlying mortality rates is a superior method in theoretical terms, yet employing the alternative allows us to examine the effect of a different approach. Moreover, the difference in results is evident for both genders.

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APPENDICES

Appendix A: raw estimates of α_x, β_x and k_t

Estimation αx

Age Group	Females	Males
0	-4,033699707	-3,835790179
1-4	-7,213929985	-7,10874839
5-9	-8,160779498	-7,8680444
10-14	-8,26407312	-7,813319463
15-19	-7,864148005	-6,945887141
20-24	-7,651584535	-6,717847306
25-29	-7,452283749	-6,677807228
30-34	-7,176244489	-6,52591061
35-39	-6,82668437	-6,267172662
40-44	-6,42665121	-5,857718365
45-49	-5,97721047	-5,367224209
50-54	-5,5362239	-4,85898577
55-59	-5,099981417	-4,371698852
60-64	-4,618943106	-3,908334419
65-69	-4,091446245	-3,46120974
70-74	-3,513642443	-3,004627826
75-79	-2,91609241	-2,533438599
80-84	-2,340328469	-2,054049223
85-89	-1,816543952	-1,608759955
90-94	-1,360558507	-1,204260676
95-99	-0,98275526	-0,858826013
100-104	-0,683975682	-0,571001792

Estimation βx

Age Group	Females	Males
0	0,102499919	0,141392134
1-4	0,115756234	0,154637924
5-9	0,076369591	0,1048845
10-14	0,06054872	0,077513092
15-19	0,046862446	0,036496079
20-24	0,052411099	0,027122682
25-29	0,052634309	0,028254762
30-34	0,049035161	0,029940744
35-39	0,046391497	0,03824621
40-44	0,041574381	0,043840993
45-49	0,0371411	0,043890003
50-54	0,035471203	0,040208161
55-59	0,034728713	0,034730071
60-64	0,036185567	0,029289642
65-69	0,038141047	0,024775806
70-74	0,03928069	0,024092927
75-79	0,03702842	0,024840021
80-84	0,031747846	0,024819726
85-89	0,025296883	0,023897241
90-94	0,018481342	0,021030037
95-99	0,013483991	0,018642631
100-104	0,00892984	0,007454613

Raw kt

Year	Females	Males
1950	14,76794409	8,583743518
1951	14,89155126	9,913426279
1952	13,408219	8,394651321
1953	12,52239962	7,846443414
1954	10,62992411	6,157210056
1955	9,842540872	6,678494859
1956	10,87206043	7,63722919
1957	9,952250993	6,926523736
1958	8,333794839	5,760670293
1959	7,666052493	4,684290325
1960	7,514274804	5,502144527
1961	6,452023934	4,67497325
1962	7,350708893	5,924349376
1963	7,437428484	6,008580608
1964	5,035484367	4,149811148
1965	5,574608009	4,440954214
1966	3,765069333	3,160189717
1967	4,097046229	2,997962203
1968	4,145444986	3,741762929
1969	3,538285116	3,239875422
1970	2,803734674	2,594174948
1971	2,15831635	2,1059274
1972	1,746190265	1,687278316
1973	1,709463933	2,05402866
1974	0,043476874	0,49638264
1975	0,085074077	1,008839268
1976	-0,48958581	0,302168793
1977	-1,406965414	-0,064006487
1978	-2,454922639	-0,523079984
1979	-2,794406103	-0,942828408
1980	-1,942370504	-0,584121479
1981	-3,91343144	-1,793326397
1982	-4,657875266	-2,739795273
1983	-4,248820047	-2,475348897
1984	-5,901708476	-3,960446015
1985	-5,964310903	-4,129592882
1986	-6,714897273	-4,911413514
1987	-7,42132546	-5,417138884
1988	-7,575054175	-5,531328024
1989	-8,556732355	-6,055195481
1990	-8,365313614	-5,705874419
1991	-8,247115251	-5,245437366
1992	-8,664645546	-5,975670242
1993	-8,610286143	-6,494699583
1994	-9,306065233	-7,242041819
1995	-9,841560985	-7,062980989
1996	-10,14304994	-8,138229592
1997	-11,10687368	-9,046047702
1998	-11,81185036	-9,821817576
1999	-12,90587746	-11,02581041
2000	-13,29832396	-11,78585499

Appendix B: k_t re-estimated

Reestimated kt

Year	Females	Males
1950	12,239065	7,127597
1951	13,261274	8,301183
1952	12,594144	7,754879
1953	12,055052	7,318683
1954	9,651698	5,478459
1955	9,163218	5,265724
1956	11,254295	7,463961
1957	9,853836	6,647819
1958	7,950729	4,876913
1959	7,034265	4,145711
1960	7,632881	5,196660
1961	6,159950	4,190676
1962	7,683678	5,772402
1963	7,480289	5,813922
1964	5,299921	4,079973
1965	6,234911	4,901331
1966	4,399276	3,478168
1967	4,684472	3,650243
1968	5,330330	4,475201
1969	4,693498	4,603788
1970	3,442278	3,068314
1971	2,802146	2,750540
1972	2,117816	2,410300
1973	2,955119	3,103914
1974	1,363817	2,088754
1975	1,885872	2,989996
1976	0,000000	2,486175
1977	0,000000	0,000000
1978	-0,767902	0,000000
1979	-1,524324	0,000000
1980	-1,043566	0,000000
1981	-1,971115	0,000000
1982	-3,159363	-0,890456
1983	-2,010931	0,000000
1984	-4,507087	-1,983047
1985	-4,416243	-2,085719
1986	-5,308517	-2,968874
1987	-6,862385	-4,323954
1988	-7,192844	-4,789860
1989	-8,570573	-5,953422
1990	-8,606526	-6,217906
1991	-9,022171	-6,417020
1992	-10,235814	-7,592074
1993	-10,703506	-7,944450
1994	-11,313666	-8,573662
1995	-12,367972	-9,549911
1996	-13,367315	-10,741046
1997	-13,494783	-11,336797
1998	-13,349057	-11,294018
1999	-14,263166	-12,552274
2000	-15,503808	-14,116502

Appendix C: Forecasted k_t

Forecasted kt		
Years	**Kt_Females**	**Kt_Males**
2001	-16,07029342	-14,54138354
2002	-16,63677847	-14,96626551
2003	-17,20326352	-15,39114747
2004	-17,76974857	-15,81602944
2005	-18,33623361	-16,2409114
2006	-18,90271866	-16,66579337
2007	-19,46920371	-17,09067533
2008	-20,03568876	-17,5155573
2009	-20,60217381	-17,94043926
2010	-21,16865886	-18,36532123
2011	-21,73514391	-18,79020319
2012	-22,30162895	-19,21508515
2013	-22,868114	-19,63996712
2014	-23,43459905	-20,06484908
2015	-24,0010841	-20,48973105
2016	-24,56756915	-20,91461301
2017	-25,1340542	-21,33949498
2018	-25,70053924	-21,76437694
2019	-26,26702429	-22,18925891
2020	-26,83350934	-22,61414087
2021	-27,39999439	-23,03902284
2022	-27,96647944	-23,4639048
2023	-28,53296449	-23,88878677
2024	-29,09944953	-24,31366873
2025	-29,66593458	-24,7385507

Appendix D: Comparison between the two different approaches

	Alternative Method	Lee-Carter method
Years	Female(1,1,1)	Female(0,1,1)
2001	82,72917824	82,67351256
2002	82,99835649	82,86201097
2003	83,26753473	83,04787938
2004	83,53671298	83,23118379
2005	83,80589122	83,41198693
2006	84,07506946	83,59034841
2007	84,34424771	83,76632497
2008	84,61342595	83,93997054
2009	84,8826042	84,11133649
2010	85,15178244	84,28047172
2011	85,42096068	84,44742281
2012	85,69013893	84,61223418
2013	85,95931717	84,77494818
2014	86,22849542	84,93560524
2015	86,49767366	85,09424397
2016	86,7668519	85,25090125
2017	87,03603015	85,40561236
2018	87,30520839	85,55841107
2019	87,57438664	85,70932971
2020	87,84356488	85,85839928
2021	88,11274312	86,00564951
2022	88,38192137	86,15110895
2023	88,65109961	86,29480504
2024	88,92027786	86,43676419
2025	89,1894561	86,57701179
	Alternative Method	Lee Carter method
Years	Males (1,1,1)	Males (0,1,0)
2001	76,78736631	76,74722129
2002	77,02473263	76,8981853
2003	77,26209894	77,04728176
2004	77,49946526	77,19458884
2005	77,73683157	77,34018047
2006	77,97419788	77,48412656
2007	78,2115642	77,62649319
2008	78,44893051	77,76734287
2009	78,68629683	77,90673464
2010	78,92366314	78,04472435
2011	79,16102945	78,18136473
2012	79,39839577	78,31670564
2013	79,63576208	78,45079417
2014	79,8731284	78,58367483
2015	80,11049471	78,71538963
2016	80,34786102	78,84597827
2017	80,58522734	78,97547824
2018	80,82259365	79,10392494
2019	81,05995997	79,23135178
2020	81,29732628	79,35779032
2021	81,53469259	79,48327033
2022	81,77205891	79,60781993
2023	82,00942522	79,73146564
2024	82,24679154	79,85423249
2025	82,48415785	79,9761441

Chapter 3

An application of the Lee-Carter model within the Fair Valuation context

3.1. Introduction

Placing a value on life insurance liabilities is not easy. In the twenty-first century there are still debates on how to value the liabilities arising from life insurance policies and the International Accounting Standards Board is making efforts to design an international financial reporting standard for insurance contracts, to be used in insurers' accounts.

If we look back a decade or so, we can see that financial accounting was an area not regarded as critical by managers of life insurance and pension companies (L&P companies) to their business. This has changed drastically in the new millennium. Today the attention of the top management of L&P companies is focused primarily on the new international accounting standards for insurance, some of which have been defined during the first half of 2004 and some of which are still undergoing much disputed revision.

3.2. Accounting standards and Fair value

3.2.1. Background on Accounting Standards

Accounting standards are rules and guidelines which should be followed by those who prepare financial statements of companies. The firm's accounts are valued according to specific methods depending on the rules applicable to accounts in the jurisdiction concerned. There are national accounting standard-setting bodies that issue standards on the preparation of accounts along side the International Accounting Standards Board (IASB). These standards are sometimes referred to as "general purpose" financial statements; in any case, the common purpose of the accounting standards is to define what is meant by a true and fair view in various contexts and circumstances. There is also the objective of narrowing the differences that exist between countries. Some of these bodies have developed standards specific to insurance, although an international standard that applies to insurance contracts is now being developed by IASB. Indeed, with increasing internationalisation of business and the globalisation of capital markets has come the need for international harmonisation of accounting rules. In April 2001 the IASB was formed as a new standard setting authority, the successor body to the International Accounting Standard Committee (IASC), which was initiated as a private organisation back in 1973.

The IASB has adopted and will continue to promote and improve the International Accounting Standards (IASs) issued by IASC and has already issued new ones called International Financial Reporting Standards (IFRS).

According to the current plans, all listed companies in the EU must prepare their consolidated financial statements in accordance with international accounting standards from 2005 onwards.

In 1997 the IASC launched an Insurance Project, the only accounting standards entirely devoted to insurance and pensions: IAS 19, IAS 32 and IAS 39. These were the only ones of particular relevance to pension funds and life insurers. The first phase of the Insurance Project has been completed by the issuance of the International Financial Reporting Standards 4 Insurance Contracts (IFRS 4) at the end of March 2004.

The IFRS 4 provided guidance on accounting for insurance contracts for the first time. This also marked the first step in the IASB's project to achieve the convergence of widely varying insurance industry accounting practices around the world.

The basic idea emerging from the new guidelines is to depict the firm's economic profile as realistically as possible. This simple insight has generated increased interest in the concept of fair value in relation to financial reporting in all areas of business.

3.2.2. *The fair valuation problem*

Over the last decade, there has been a gradual reformation of accounting conventions from being largely based on historical cost to being increasingly based on fair value. Applying fair value in L&P companies' balance sheets means that assets and liabilities will have to be marked to market. Nevertheless, as the American Academy of Actuaries (2003) clearly states, market valuations do not exist for many items on the insurance balance sheet; this leads to the reliance on entity specific measurements for determining insurance contracts and asset fair values.

Several definitions of fair value have been proposed; one common definition of fair value in relation to financial instruments has been "*the amount of the consideration that would be agreed upon in an arm's length transaction between knowledgeable, willing parties who are under no compulsion to act*".

This definition has been recently revised by the FASB now defining fair value as ([FASB, 2004]) "*the price at which an asset or liability could be exchanged in a current transaction between knowledgeable unrelated willing parties*".

In the new IFRS 4 the definition of fair value is essentially identical and reads ([IASB, 2004]) "*the amount for which an asset could be exchanged, or a liability settled, between knowledgeable, willing parties in an arm's length transaction*". What is new in the FASB and IASB definitions of fair value, compared to previously, is that both organisations now explicitly state that the definition is intended to apply for all assets and liabilities and not just financial instruments. Yet what is really an innovation is the introduction of a fair value hierarchy for the valuation techniques [(FASB, 2004]). This hierarchy states that valuation techniques used to estimate fair values should maximise the use of market inputs and it prioritises the market inputs that should be used. In general, quoted prices in active markets are preferred and should be used whenever available. Fair value estimates are classified by three quality levels:

- The first one where estimates are obtained using quoted prices for *identical* assets or liabilities in active markets to which an entity has immediate access;
- the second level using quoted prices for *similar* assets or liabilities in active markets, adjusted as appropriate for differences.
- The third level will include estimates based on valuation models, where there are products with no appropriate market to help define prices.

In practice, we do not have markets for trading insurance liabilities where we can readily observe prices, particularly because of the dependence of payments on human life. It is in these situations that company accountants will have to use Level 3 estimates for insurance and pension liabilities. This inevitably will involve some kind of estimation process based on financial valuation models.

The fair value of life insurance liabilities has been a recent point of discussion. However we still have debates on how a fair value system might actually function in practice.

Key issues in this debate are: should the fair value framework allow for the possibility of gain or loss at contract issue? What are the estimated cash flows from a life policy? What discount rate should be applied to the cash flows to derive a present value? Under what circumstances should assumptions about future cash flows liabilities be updated?

For marketable securities, the definition of fair value is unambiguous. However, for some assets and virtually all insurance and annuity liabilities, the definition is less than clear, especially after acquisition of the asset or liability. This is because these assets and liabilities are thinly traded post acquisition, so there are few or no examples of actual transactions. Price are therefore difficult to define unambiguously.

3.2.3. Valuation of Life insurance liabilities

Skerman (1966) suggested five principles for the valuation of actuarial liabilities, forming a suitable underlying basis for a solvency standard.
According to these principles:

- The liabilities should be valued by a net premium method.
- An appropriate reserve would be acceptable in order to allow for initial expenses.
- Adequate margins over the current rate of expenses should be kept in the valuation of liabilities, in order to provide for future renewal expenses.
- Appropriate recognised tables of mortality should be employed.
- The valuation of liabilities should be at rates of interest lower than implicit in the valuation of the assets.

The Skerman principles have been influential; they have been taken into account by regulators, and have been used to put forward practical proposals for valuations. There is no uniform set of valuation rules made by regulators. However, the International Association of Insurance Supervisors has issued principles on prudential regulation, the first of which is that the technical provisions of an insurer have to be adequate, reliable, objective, and allow comparison across insurers.
As we do not yet have an international standard setting out how to value liabilities from insurance contracts in accounts, it is not surprising that there is a wide variety of practices on how this is done. Actuaries need to understand accounting so that they can appreciate what is needed in life insurers' accounts to meet accounting standards.
In carrying out the valuation, the actuary needs to choose a method of

valuation and also a valuation basis, that is a set of assumptions to be used.
Fair value liability valuation methods can be divided into two primary families: direct and indirect methods.
Under the former, the fair value of the liability is the discounted value of future liabilities. Under the latter, the fair value of liabilities is calculated as the fair value of assets supporting the liabilities less the actuarial appraisal value of the block of business.
The fundamental premise of the indirect method is that the appraisal value and the fair value of equity are equal. This is really a definition of fair value of equity. But the fair value of equity is also the fair value of assets less liabilities, so that the fair value of liabilities is a derived value under the indirect approach.
According to the direct method the liability value is the value of the cash flows to the reporting entity, so that it is possible to value the liabilities directly without first determining the fair value of equity.
The indirect approach to liability fair value is inconsistent with accounting principles, because the approach confuses the value of a business to an investor with the separately determinable values of the asset and liability cash flows. Thus the direct approach seems to be the appropriate one for financial reporting purpose.
As far as the fair value allowing for the possibility of gain or loss at contract issue is concerned, some believe that this possibility is fundamentally inconsistent with fair value. Insurance and annuity contracts are sold in a free market, and policyholders choose to purchase them in "arm's length transaction". The price of a contract in an unforced sale is the best indication of the fair value of the contract at the point of sale.
Others believe that only coincidentally will the fair value of a contract at issue equal its price. They believe that liabilities should be valued based on discounted cash flows, possibly with adjustments for risk, and that a gain or

loss at issue will emerge depending on the specific facts and circumstances. Contracts with different expected cash flows but the same price would have different fair values.

An important part of the valuation process is forecasting the future cash flows to be valued. This can require assumptions on matters such as the contingency on which payment of the benefits under the policy depends (death, sickness, etc.); factors that determine the amount of payment (such as bonuses added under participating policies); expenses, investment returns, tax, the rate of early discontinuances, and so on.

A number of issues arise and are discussed in the literature: if the assumptions that should be used are those used when the product was priced, or those current at the date of valuation; if the cash flows should be best estimates, should they be prudent or should they allow for risk.

There are two principal approaches to projecting future cash flows. According to the first approach we can project interest rates using the implied forward curve, that is the set of future interest rates consistent with today's yield curve. Then we can project expected cash flows consistent with this scenario. This also happens also under the second scenario, but in this case a large number of future interest rate scenarios are projected in such a way that the collection of scenarios is arbitrage-free. This type of projection has the advantage of capturing the optionality in the cash flows and is consistent with established asset valuation.

It must be pointed out that the expected cash flows produced by a stochastically generated set of interest rate scenarios are not equal to the expected cash flows in the single scenario. The stochastic approach is complex and in practice it can lead to somewhat erratic earnings patterns.

Another key question in the valuation of life insurance liabilities is which rate should the actuary use to discount future cash flows. Numerous possibilities exist for defining the discount rate. In some cases the rate used to discount liabilities is based on the Treasury rates. They are objectively determinable and reflect the certain nature of the obligations. Others believe that the asset earnings rate should be used as the discount rate. The calculated fair value would vary when asset earnings rates change even if there were no change in future liability cash flows or in the economic environment. Additionally, this approach does not lend itself well to valuing stochastically generated cash flows. Many would say that the value of a liability is independent of the assets held to back it, which is the approach taken when assessing fair value.

A third approach is to calculate a liability spread at issue and lock it in for the life of the contract. One way of determining the liability spread is to calculate it in such a way that there is no gain or loss at issue. This approach has several advantages. Firstly, it gives the correct value at issue. Secondly, it implicitly incorporates the company's evaluation of risk. Finally, the approach is well suited to use with stochastically generated cash flows for pathwise discounting.

In determining the fair values, best-estimate assumptions as to future experience must be made. Those best estimates are inherently imprecise, because they will change over time with mortality. Under a fair value approach, these changes in best estimates should be reflected in the fair values. Given the inherent uncertainty and imprecision in selecting best estimate assumptions for insurance liabilities it is important to ensure consistency in approach over time. If there are differences or inconsistencies in the approach used to develop best estimates, the entire present value of the impact of those differences can flow into earnings for the period in which that change in best estimates is made.

3.2.4. Choice of valuation method: background hypotheses

The stochastic nature of life policies cash flows is such that a valuation method that merely derives one value from deterministic assumptions will be inappropriate in some cases. A number of authors have used stochastic models to value liabilities. This need for stochastic methods has led to the development of models to use in such situations.

We need some way of using the probability distribution of liabilities from a stochastic projection to determine the liabilities to use in the valuation.

To this end and aforementioned, the IASB has been considering fair value as an approach to valuing insurance contracts.

Despite of a number of advantages, the fair value of insurance liabilities raises a number of issues, both whether using market values or models. In particular, while our concern is the value in a transaction to exchange liabilities, what type of transaction are we envisaging? It could be, for example, the price a policyholder would pay for the benefits being offered by the policy; this may imply that, at the outset, the value of the liabilities would equal the present value of the premiums. Or it might be the value payable if the policyholder wished to surrender the policy, or that he could obtain by selling the policy in the open market. We must note that only certain contracts can be sold in the open market. Alternatively, it could be the amount that the insurer would have to pay to a third party to take over the liabilities under the policy.

As we know, the dependence of payment of benefits on human life means that no regular market exists for such liabilities. Thus the market value of these liabilities is not readily available and must be estimated. The problem is that the demographic valuation is not supported by the hypothesis of the completeness of the market as for the financial valuation. As suggested in [De Felice and Morriconi, 2004] the problem of the hypothesis of market completeness in the demographic framework can be overcome by constructing an appropriate probability measure, in order to guarantee the relevant properties of the price function.

It will also typically not be possible to find traded securities with a sufficiently close similarity to the life insurance and pension obligations such that fair value estimates can be obtained. Having seen the problems from not having markets for trading insurance liabilities, we can see that we need to construct a mathematical model of a pricing system that coherently represents the insurance realm.

So far the emphasis has been on financial markets; in this contribution we construct a valuation model that fully captures the interest and mortality rate dynamics. The primary feature of our model is its focus on the demographic reference system. We base our demographic assumptions on the results of chapter 2, where we obtain the forecasts to construct life tables using the Lee Carter model.

Because reserves cannot be set within a reference market, we consider it reasonable to express their current valuations by means of the expectation framed within the *best prediction* of the demographic scenario. For this reason, referring to evaluations consistent with the model, we obtain only proxies of the reserve market value. Hence, a market-based valuation for them would produce irrelevant information; thus we will no longer refer to a "marked to market" valuation of the outstanding liabilities, but to a "marked to model" valuation.

3.3. The model

In this contribution we analyse, in a Lee Carter mortality context, the standard endowment policy. This has been one of the most common life insurance policies sold in Italy during the last two decades. This policy is priced in a standard way, given an interest rate and a mortality table from which the life and death probabilities are extracted. In order to determine an actuarial model for the fair valuation at time t of the stochastic stream of cash flows, that is the stochastic loss at time t, we maintain the standard assumptions of market efficiency. We assume perfectly competitive, frictionless and arbitrage-free securities market, populated by rational agents, all sharing the same information, without restrictions on borrowing or short-sales and with zero-coupon bonds and stocks both infinitely divisible.

Let us define $\{r_t; t = 1,....,T\}$ and $\{m_{x+t}; t = 1,....,T\}$ the random spot rate and the mortality process measurable by means of the filtrations F^r and F^m respectively. We assume that the randomness in mortality is independent of fluctuations of interest rates. These random processes are defined by a unique probability space $(\Omega, F^{r,m}, P)$, such that the σ-algebra $F^{r,m} = F^r \cup F^m$ contains both the information about mortality and financial history.

Let us denote by N_j the number of claims (the survivors or the dead according to the kind of life contract) at time j within a portfolio of identical policies and by $X^t = N_{t+1}X_{t+1}, N_{t+2}X_{t+2},...., N_n X_n$ the stochastic stream of cash flows (Coppola, Di Lorenzo, Sibillo, 2005).

Applying risk-neutral valuation, the fair value at time t of the stochastic loss, can be calculated as:

$$(1) \qquad \Re_t = \mathrm{E}\left[\sum_{j>t} N_j X_j v(t,j) \middle| F_t\right]$$

where E denotes the expectation under some risk-neutral probability measure, F_t is the information flow at time t and $v(t,j)$ represents the present value at time t of one monetary unit due at time j.

If we indicate by c the number of policies at time 0 from equation (1), that computes the stochastic loss at time t for a portfolio, we can define a formulation for each policy. For simplicity, we consider the specific case of a single policy $(c = 1)$, which provides for payments of a life annuity as long as the beneficiary lives or it provides for payments at the end of the death year.

Consequently, equation (1) can be rewritten as follow:

$$(2) \qquad \Re_t = \mathrm{E}\left[\sum_{j>t}\left(X_j^s 1_{\{k_{x,t}>j\}} + X_j^d 1_{\{k_{x,t}=j\}}\right)v(t,j)\middle| F_t\right]$$

where X_j^s and X_j^d are the cash flows at time j in the case of survival and in the case of death respectively. The indicator functions $1_{\{k_{x,t}>j\}}$ and $1_{\{k_{x,t}=j\}}$ are Boolean; they take the value of 1 if the curtate future lifetime of the insured, aged x at issue, is equal to or greater than $t+j$ $(j=1,2,.....)$, respectively, or 0 otherwise.

We follow this line of reasoning and extend the above to the case of an endowment policy issued at time 0 and maturing at time n, for a life aged x at inception (time 0). This contract assumes that the sum insured is payable at the end of the year of death, if this occurs within the first n years, otherwise at the end of the nth year.

On the basis of the information the insurer has at time t, we can write the fair value of the loss at t of the endowment policy as:

$$(3) \qquad \Re_t = \mathrm{E}\left[\sum_{j>t}\left(X_j^d 1_{\{k_{x,t}=j-1\}} v(t,j) + X_n^s 1_{\{k_{x,t}>n-t\}} v(t,n)\right)\middle| F_t\right]$$

where the indicator functions $1_{\{k_{x,t}=j-1\}}$ and $1_{\{k_{x,t}>n-t\}}$ take the value of 1 if $k_{x,t}=j-1$ or $k_{x,t}>n-t$, that is that the insured aged $x+t$ dies within the time $t+j$ or survives to the time n respectively, or 0 otherwise.

By virtue of the basic assumptions on the risk sources, we get:

$$(4) \qquad \Re_t = \sum_{j>t} \left\{ X_j^d \mathrm{E}\left[1_{\{k_{x,t}=j-1\}} \middle| F_t\right] \mathrm{E}\left[v(t,j) \middle| F_t\right] + X_n^s \mathrm{E}\left[1_{\{k_{x,t}>n-t\}} \middle| F_t\right] \mathrm{E}\left[v(t,n) \middle| F_t\right] \right\}$$

$$= \sum_{j>t} \left\{ X_j^d \;{}_{j-1|1}q_{x+t} \mathrm{E}\left[v(t,j) \middle| F_t\right] + X_n^s \;{}_{n-t}p_{x+t} \mathrm{E}\left[v(t,n) \middle| F_t\right] \right\}$$

where ${}_t p_x$ denotes the probability that a life aged x will survive at least t years and ${}_{s|t}q_x$ denotes the probability that the life aged x will survive s years and subsequently die within t years.

The terms on the right side of the equation show that the expected discounted value of the stochastic stream can be regarded as the valuation of zero coupon bonds with maturities in j.

In a fair valuation approach, we can regard the price of the endowment policy at time t as the market price of the zero coupon bonds. We specifically use market prices for determining the current value.

3.3.1. Determining mortality risk stochastically

As far as the dynamics of the process $\{m_{x+t}; t = 1,2,......\}$ is concerned, we choose a model based on the Lee Carter methodology. This method allows us to extrapolate long-run forecasts of the level and age pattern of mortality, using a combination of statistical time series methods and a parametric approach.

Using the traditional actuarial approach, we define T_x to be a random variable which represents the remaining lifetime of a life aged x at time t. Under the probability measure P, we can express the survival function of the random variable T_x as:

$$(5) \qquad {}_y p_x = \mathrm{P}\left(T_x > y \middle| F^m\right)$$

where F^m contains the information flow about mortality.
If we observe $m_{x+t:t}$ in year t, namely the stochastic mortality for a life aged $x+t$, and we explicitly allow for the hypothesis of time dependence in mortality, the formulation (5) can be re-written as:

$$(6) \qquad {}_y p_x = \mathrm{E}\left[e^{-\int_0^y \mu_{x+t:t} dt} \middle| F^m \right]$$

To project mortality rates we consider the Lee-Carter model which has been widely used in actuarial literature. The reason behind our choice is that traditional projection models provide the forecaster with point estimates of future mortality rates. On the contrary, the LC method allows for uncertainty in forecasts, the so-called longevity risk.
This model has traditionally been formulated as:

$$(7) \qquad \ln(m_{x,t}) = a_x + b_x k_t + \varepsilon_{x,t}$$

where the log of a time series of age-specific death rates $m_{x,t}$ is the sum of:

- a_x, an age-specific component that is independent of time

  and another component that is the product of:
- k_t, a time-varying parameter reflecting the general level of mortality and
- b_x, an age-specific component, that represents how rapidly or slowly mortality at each age varies when the general level of mortality changes.

The error term $\varepsilon_{x,t}$, with mean 0 and variance σ_ε^2 reflects particular age-specific historical influences not captured in the model.

Usually for uniqueness of the model specification the following constrains are imposed:

$$\sum_t k_t = 0 \text{ and } \sum_x b_x = 1$$

We fit this model to historical data (see chapter 2). The resulting estimates of the time-varying parameter is then modelled and forecast as a stochastic time series using the standard Box-Jenkins methodology. From this forecast of the general level of mortality, the actual age specific rates are derived using the estimated age effects.

Denoting the resulting forecasts by $\{k_{T+s} : s > 0\}$, we use the following model for the evolution of the forecast mortality rates:

$$\text{(8)} \qquad \mathring{m}_{x,T+s} = \hat{m}_{x,T} \exp\left\{ \hat{\beta}_x \left(\mathring{k}_{T+s} - \hat{k}_T \right) \right\}$$

This model allows us to compute forecasted mortality rates $\mathring{m}_{x,T+s}$ by alignment to the latest available empirical rates $\hat{m}_{x,T}$.

If the latest empirical mortality rates appear atypical, an alternative would be to average across a few years at the end of the observed period.

While the last method was the one used from Lee and Carter (1992), the method of forecast alignment reflects Lee's current thinking (Lee, 2000).

3.3.2. Financial risk modelling

We provide a closed analytical formula for equation (4) modelling the financial risk making use of the Cox-Ingersoll-Ross model for the term structure rates.

The spot rate r_t is a diffusion process described by the stochastic differential equation:

$$(9) \qquad dr_t = f^r(r_t,t)dt + g^r(r_t,t)dZ_t^r$$

where Z_t^r is a standard Brownian motion; $f^r(r_t,t)$ is the drift function from the CIR (Cox-Ingersoll-Ross) model:

$$f^r(r_t,t) = \alpha(\gamma - r_t), \qquad \alpha, \gamma > 0$$

and the diffusion function is defined by:

$$g^r(r_t,t) = \sigma\sqrt{r_t}, \qquad \sigma > 0$$

We estimate the term structure on the basis of a Cox-Ingersoll-Ross square root model according to a simple discretisation (Chan et al., 1992; Deelstra et al., 1995), in which the continuous centred interest rate is defined by the stochastic differential equation

$$(10) \qquad dr_t = -\alpha(r_t - \gamma)dt + \sigma\sqrt{r_t}dB_t \quad \text{with} \quad \alpha, \sigma > 0,$$

where it is assumed that we have mean-reverting drift, with a long term mean of γ and speed of adjustment of α, and "square root" diffusion with a volatility parameter of σ.

The interest rate behaviour implied by this structure has some empirically relevant properties: negative interest rates are precluded; if the interest rate reaches zero, it can subsequently become positive; when the interest rate increases, the absolute variance of the interest rate itself increases.

Even if the Vasicek model is more simple than the CIR model, for our application we use the latter. This is because it offers a useful trade-off between economic consistency and mathematical tractability. It must be pointed out that despite its wide use for pricing interest rate derivatives, the Vasicek model appears inadequate to life insurance applications. It assigns positive probabilities to negative values of the spot rate. This produces relevant effects for long maturities, such as discount factors greater than one.

3.4. Applications of the model

In this section, we consider equation (4) and apply it to the case of an endowment policy with unitary benefits for a male policyholder. For the purpose of comparison we determine the value of the policy at time 0, subdividing the analysis into two stages.

Firstly, we examine the case of a policy for an insured aged 40 at issue with a time to maturity of 15 years. Secondly we also apply the model to the case of a policy for an insured aged 65 at issue, with the same time to maturity.

The term structure is estimated on the basis of a Cox-Ingersoll-Ross square root process, using the equation (cf. Deelstra et al., 1995):

$$E\left[e^{\int_o^t r_u du}\right] = \frac{e^{-\frac{x}{\sigma^2} w \frac{1+\frac{k}{w}\coth\left(\frac{wt}{2}\right)}{\coth\left(\frac{wt}{2}\right)+\frac{k}{w}} + \frac{xk}{\sigma^2} + \frac{k^2\gamma t}{\sigma^2}}}{\left[\cosh\left(\frac{wt}{2}\right) + \frac{k}{w}\sinh\left(\frac{wt}{2}\right)\right]^{\frac{2k\gamma}{\sigma^2}}}$$

in which $x = r_0$ and $w = \sqrt{k^2 + 2\sigma^2}$.

As far as the data is concerned, we assume a long term mean of $\gamma = 0{,}0452$, a volatility parameter of $\sigma = 0{,}0053$ and an initial value of $r_0 = 0{,}01724$.

We use the interest rates from 3-month Treasury Bills from January 1996 until January 2004, extracted from Bank of Italy official statistics.

Mortality effects are taken into consideration through the survival probabilities which have been derived from the life table, calculated using the Lee-Carter model (cf. chapter 2) and considering the data for the Italian population, between the years 1950 and 2000.

In order to express the range of possible values the contract can assume around each age, we consider three sets of probabilities according to different age groups. More particularly, we split our analysis between x=35-39, x=40-44 and x=45-49. We then consider x=60-64, x=65-69 and x=70-74.

On the basis of equation (4), we ascertain the fair price of the endowment policy for both cases and for each group separately.

In Table 1 we report the results obtained for the ages around 40 and in Table 2 the results for the ages around 65. We compare the results obtained using a stochastic rate (CIR+LC), with another two cases both calculated with a contractual annual rate of 0,04. It should be noted that in the first case the mortality rates are derived from the life table obtained with the Lee Carter methodology, and in the latter the mortality rates are derived from the life table SIM'92.

3.5. Conclusions

Focusing on the demographic scenario, it is worth noticing the general increase of the expected present value of the endowment policy as age increases. This is due to a higher probability of paying out at the end of the year of death, before the policy expires. Moreover we can observe this phenomenon both horizontally, that is as the age increases, and comparing the results obtained for the age group around 40 with the ones around 65.

Furthermore this rise in the expected present value is evident both in the case of a stochastic rate (CIR+LC) and when the rate is deterministic.

Further examination of the Tables 1 and 2 reveals that the values obtained by "FIX RATE+LC" are always larger than the corresponding values calculated on the basis of the life table SIM'92. This is due to the capturing of the improvements in mortality rates by the Lee Carter model, which determines the stronger projection.

Table 1

Endowment policy x=40, n=15

	X=35-39	*X=40-44*	*X=45-49*
CIR+LC	0,73006	0,733434	0,741662
FIX RATE(4%)+LC	0,565025	0,569842	0,581584
FIX RATE(4%)+SIM'92	0,56268		

Table 2

Endowment policy x=65, n=15

	X=60-64	*X=65-69*	*X=70-74*
CIR+LC	0,805376	0,845429	0,887555
FIX RATE(4%)+LC	0,673369	0,732329	0,796162
FIX RATE(4%)+SIM'92	0,634246		

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