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QUASICONFORMAL MAPPINGS AND APPLICATIONS TO SOME LINEAR AND NONLINEAR ELLIPTIC PROBLEMS

TESI DI DOTTORATO DI RICERCA

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Introduction

The concept of quasiconformal mapping can be considered not only as a technical tool in complex analysis but actually as an independent topic with applications in various mathematical contexts. Let Ω be a domain in $\mathbb{R}^n$; recall that $f : \Omega \to \mathbb{R}^n$ is a K–quasiconformal mapping for some $K \geq 1$ if $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ and

$$|Df(x)|^n \leq K J_f(x) \qquad \text{a.e. } x \in \Omega.$$

In two dimensions, quasiconformal mappings have a natural connection with partial differential equations in divergence form. This fact has been evident for at least 70 years, beginning with M.A. Lavrentiev [48], C.B. Morrey [63], R. Caccioppoli [13], B. Bojarski [9] and Serrin [76] among many others.

The present thesis brings together several different topics related to quasiconformal mappings and elliptic PDE's. It is organized as follows.

In Chapter 1 we review some of the standard facts in the theory of planar quasiconformal mappings and second order elliptic partial differential equations of the type

$$\mathrm{div} A(x)\nabla u = 0. \tag{1}$$

Here $A = A(x)$ is symmetric matrix which belongs to $L^\infty(\Omega, \mathbb{R}^{2\times 2})$ which satisfies $\det A(x) = 1$ a.e. in Ω and

$$\frac{|\xi|^2}{K} \leq \langle A(x)\xi, \xi\rangle \leq K|\xi|^2 \qquad \text{a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^2,$$

for some $K \geq 1$. The connection between the class of quasiconformal mappings and the elliptic equations in divergence form is established by means of the *Laplace–Beltrami* operator of f, namely

$$\mathcal{L}_f u = \mathrm{div}\,(A_f(x)\nabla u)\,,$$

where A_f is the inverse of the distortion tensor G_f of f defined as

$$G_f(x) = \frac{{}^tDf(x)Df(x)}{J_f(x)}.$$

Conversely, every solution of (1) is the composition of some harmonic function and a K–quasiconformal mapping.

Chapter 2 is devoted to the Hölder regularity estimates for the solutions to linear elliptic equations in divergence form. Let Ω be a bounded domain of $\mathbb{R}^2$ and let A be a positive definite matrix-valued function, with coefficient in $L^\infty(\Omega)$ and satisfying, for some $0 < \lambda \leq \Lambda$

$$\lambda|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq \Lambda|\xi|^2 \qquad \text{for a.e. } x \in \Omega, \ \ \forall \xi \in \mathbb{R}^2.$$

In their seminal article [69], Piccinini–Spagnolo proved that the best Hölder exponent for weak solutions to the elliptic equation (1) is given by

$$\alpha = \frac{1}{\sqrt{L}},$$

where $L = \Lambda/\lambda$ denotes the ellipticity constant of A. Furthermore, they showed that if A takes the isotropic form $A(x) = a(x)I$ for some real measurable function a satisfying $1 \leq a(x) \leq L$ the best Hölder exponent improves and takes the value

$$\alpha = \arctan \frac{1}{\sqrt{L}}.$$

A key ingredient used in the proof of the second result of Piccinini and Spagnolo is the knowledge of the explicit value of the best constant C in the inequality of Wirtinger type

$$\int_0^{2\pi} a(t)u(t)^2 dt \leq C \int_0^{2\pi} a(t)u'(t)^2 dt, \tag{2}$$

where $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R})$ is 2π–periodic and satisfies

$$\int_0^{2\pi} a(t)u(t)dt = 0,$$

and the weight function $a \in L^\infty(\mathbb{R})$ is 2π–periodic and satisfies $1 \leq a(t) \leq L$, for some $L \geq 1$. In this direction, our main results are given by Theorem 2.18 and Theorem 2.19. Our aim is to give an extension of inequality (2) to the vectorial case. More precisely, our result is concerned with the inequality

$$\int_0^T a(t)|u(t)|^p dt \leq C \int_0^T a(t)|u'(t)|^p dt,$$

for any exponent $p > 1$. Here $u : [0, T] \to \mathbb{R}^N$ is a function in $W_0^{1,p}([0, T], \mathbb{R}^N)$ ($N \geq 1$) and the weight function a satisfies the bounds $1 \leq a(t) \leq L$. We provide the best constant C in the inequality, as well as all extremals. More precisely, we prove that the best constant is achieved if and only if the weight function a is a particular piecewise constant function $\tilde{a}$; moreover, by a delicate gluing (see Lemma 2.22) we construct the extremals $\tilde{u}$ in terms of generalized trigonometric function (defined in Section 2.2). It is worth to point out that our result may be also seen as a generalization of inequalities involving vector valued functions considered in [53] and [54] because of the presence of a weight function a in the inequality. For related results see also [16, 18, 23]. We conclude the chapter with the construction of a solution of a degenerate nonlinear equation. Some of these results can be found in [27].

In Chapter 3 we are concerned with G-convergence and the theory of homogenization for linear operators in divergence form. Our main results are Theorem 3.5 and Theorem 3.6. We assume that A^ε is a sequence of matrices (not necessary symmetric) satisfying the conditions

$$\begin{aligned} \langle A(x)\xi, \xi\rangle &\geq \alpha|\xi|^2 \qquad \text{a.e. } x \in \Omega \qquad \forall \xi \in \mathbb{R}^2, \\ \langle A^{-1}(x)\zeta, \zeta\rangle &\geq \beta^{-1}|\zeta|^2 \qquad \text{a.e. } x \in \Omega \qquad \forall \zeta \in \mathbb{R}^2, \end{aligned}$$

and such that

$$\det A^\varepsilon \to c^0 \text{ a.e.},$$

for some bounded measurable function c^0. We prove that if A^ε is assumed to H–converge to some A^0 then necessarily

$$\det A^0 = c^0.$$

In order to state our results precisely, we review some fundamental properties of G–convergence as considered by De Giorgi and Spagnolo in [22] and [79]. We also define H–convergence as a generalization of G–convergence to non–symmetric matrices as considered by Murat–Tartar [66]. We note that Theorem 3.5 may be seen as an extension of the classical result in the theory of bidimensional homogenization which states that the class of matrices with unit determinant is closed with respect to the H–convergence. Theorem 3.5 and Theorem 3.6 can be found in [26].

In Chapter 4 we consider some problems related to the variational formulation of equation (1). We provide examples of functionals which are weakly lower semicontinuous on $W_0^{1,p}(\Omega)$ for every $p > 2$ but not weakly lower semicontinuous on $W_0^{1,2}(\Omega)$, see Theorem 4.2, Theorem 4.3 and Theorem 4.4.

Our functionals are constructed by a careful use of the sharp Hardy–Sobolev inequalities, as obtained by [6, 11, 40]. The results of this chapter can be found in [28].

In Chapter 5 we analyze some properties of the Orlicz space EXP of exponentially integrable functions. Such a space play a key role in the study of the continuity properties of mappings of finite distortion (see for instance [4, 19, 45]). We introduce the notion of composition operator $T_g : u \longmapsto u \circ g$, induced by a homeomorphism $g \in \mathrm{Hom}(\Omega, \Omega')$ between domains Ω, Ω' of $\mathbb{R}^n$. The main result of the chapter is given by Theorem 5.6. We prove that a principal K–quasiconformal mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$, which is conformal outside the unit disk and which maps the unit disc $\mathbb{D}$ onto itself preserves the space $\mathrm{EXP}(\mathbb{D})$ of exponentially integrable functions over $\mathbb{D}$, in the sense that $u \in \mathrm{EXP}(\mathbb{D})$ if and only if $u \circ f^{-1} \in \mathrm{EXP}(\mathbb{D})$. We prove that

$$\frac{1}{1 + K\log K} \|u\|_{\mathrm{EXP}(\mathbb{D})} \le \left\|u \circ f^{-1}\right\|_{\mathrm{EXP}(\mathbb{D})} \le (1 + K\log K) \|u\|_{\mathrm{EXP}(\mathbb{D})},$$

for every $u \in \mathrm{EXP}\,(\mathbb{D})$.

Our results are in the direction of the one of Reimann [71] which proves that $T_{f^{-1}}$ is a bounded linear operator which maps $\mathrm{BMO}\,(\Omega)$ into $\mathrm{BMO}\,(\Omega')$. The starting point of our study will be Lemma 5.5 where we will establish that $u \in \mathrm{EXP}\,(G)$ if and only if $u \circ f^{-1} \in \mathrm{EXP}\,(f(G))$. Moreover, in Theorem 5.7 we will also prove that if $f : \mathbb{D} \to \mathbb{D}$ is a K–quasiconformal mapping then

$$\mathrm{dist}_{\mathrm{EXP(G')}}\left(u \circ f^{-1}, L^\infty(G')\right) \le K \mathrm{dist}_{\mathrm{EXP(G)}}\left(u, L^\infty(G)\right),$$

and

$$\frac{1}{K}\mathrm{dist}_{\mathrm{EXP(G)}}\left(u, L^\infty(G)\right) \le \mathrm{dist}_{\mathrm{EXP(G')}}\left(u \circ f^{-1}, L^\infty(G')\right),$$

for every open subset G of $\mathbb{D}$ and for every $u \in \mathrm{EXP}\,(G)$, with $G' = f(G)$. We recall that $\mathrm{dist}_{\mathrm{EXP(G)}}\left(u, L^\infty(G)\right)$ denotes the distance from L^∞ with respect to the Luxemburg norm (see Section 5.1). Theorem 5.6 and Theorem 5.7 can be found in [29].

Chapter 1

Quasiconformality, PDE's and related results

In this chapter we introduce the basic properties and definitions in the theory of *quasiconformal mappings*. Furthermore, we focus our attention on the second order elliptic linear equation

$$\mathrm{div} A(x)\nabla u = 0,$$

where $A = A(x)$ is a symmetric matrix satisfying $\det A(x) = 1$ a.e. and the ellipticity condition

$$\frac{|\xi|^2}{K} \leq \langle A(x)\xi, \xi\rangle \leq K|\xi|^2 \qquad \text{a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^n,$$

for some constant $K \geq 1$.

In dimension $n = 2$, such a class of equations naturally arises in connection with quasiconformal mappings.

1.1 Basic properties and definitions

Let $f : \Omega \to \Omega'$ be a homeomorphism from the domain $\Omega \subset \mathbb{R}^n$ onto the domain $\Omega' \subset \mathbb{R}^n$. If f belongs to $W^{1,1}_{\mathrm{loc}}(\Omega, \Omega')$ we denote by $Df(x)$ the *differential matrix* of f at the point $x \in \Omega$ and by $J_f(x)$ the *jacobian determinant* of f

$$J_f(x) = \det Df(x).$$

The norm of $Df(x)$ is defined as follows

$$|Df(x)| = \sup_{\xi \in \mathbb{R}^n,\ |\xi|=1} |Df(x)\xi|.$$

Our starting point is the following definition.

Definition 1.1. Let Ω and Ω' be domains of $\mathbb{R}^n$. A homeomorphism $f : \Omega \to \Omega'$ is a K-quasiconformal mapping for a constant $K \geq 1$ if

$$f \in W^{1,n}_{\text{loc}}(\Omega, \Omega'),$$

and

$$|Df(x)|^n \leq K J_f(x) \qquad \text{a.e. } x \in \Omega.$$

From now on, we deal with the case of dimension $n = 2$. We review some of the standard facts on quasiconformal mappings in the plane by means of the following proposition.

Proposition 1.1. *Let $\Omega, \Omega', \Omega''$ be domains of $\mathbb{R}^2$. Let $f : \Omega \to \Omega'$ be a K–quasiconformal mapping and let $g : \Omega' \to \Omega''$ be a K'–quasiconformal mapping.*

(i) The composition $g \circ f$ is KK'–quasiconformal. For a.e. $x \in \Omega$ it results that

$$D(g \circ f)(x) = Dg(f(x))Df(x), \qquad J_{g \circ f}(x) = J_g(f(x))J_f(x).$$

(ii) The inverse f^{-1} is K–quasiconformal.

(iii) For every measurable set E of Ω

$$|E| = 0 \qquad \text{if and only if} \qquad |f(E)| = 0.$$

(iv) $J_f(x) > 0$ for a.e. $x \in \Omega$.

(v) If $w \in L^1(\Omega')$ then $(w \circ f)\, J_f \in L^1(\Omega)$ and

$$\int_\Omega w(f(z))J_f(z)dz = \int_{\Omega'} w(y)dy.$$

We also recall that every quasiconformal mapping is differentiable a.e., as a consequence of the following result due to Gehring–Letho [34].

Theorem 1.2. *Let $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous open mapping. Then f is differentiable a.e. in the classical sense in Ω if and only if f has finite first partial derivatives a.e.*

1.2 Beltrami equation and the existence of principal quasiconformal mappings

We denote by $\mathbb{C}$ the complex plane. For later use let us identify a point $x = (x_1, x_2) \in \mathbb{R}^2$ with a point $z \in \mathbb{C}$ through the relation $z = x_1 + ix_2$. Therefore, a mapping $f = (u, v) : \Omega \to \mathbb{R}^2$ defined in a domain $\Omega \subset \mathbb{R}^2$ is regarded as the function $f = u + iv$.

Let us introduce the *Cauchy–Riemann operators*

$$f_z = \frac{1}{2}\left(f_{x_1} - i f_{x_2}\right), \qquad f_{\bar{z}} = \frac{1}{2}\left(f_{x_1} + i f_{x_2}\right).$$

Next classical result relates quasiconformal mappings in the plane to the solution of a partial differential in the complex plane.

Theorem 1.3. *Let $f : \Omega \to \Omega'$ be a homeomorphism of the domain $\Omega \subset \mathbb{C}$ onto the domain $\Omega' \subset \mathbb{C}$ and let $f \in W^{1,2}_{\mathrm{loc}}(\Omega, \Omega')$. Then f is K-quasiconformal for some $K \geq 1$ if and only if for a.e. $z \in \Omega$ it results that*

$$f_{\bar{z}} = \mu(z) f_z, \tag{1.1}$$

for some function $\mu \in L^\infty(\Omega)$ such that

$$\|\mu\|_\infty = k = \frac{K-1}{K+1} < 1.$$

The differential equation (1.1) is called *Beltrami equation*, while the coefficient μ in (1.1) is called *complex dilatation* of f, often denoted by μ_f.

We want to point out that if $K = 1$ or equivalently $\mu_f \equiv 0$, the Beltrami equation reduces to

$$f_{\bar{z}} = 0$$

which represent the *Cauchy–Riemann sistem.* Therefore, the class of 1–quasiconformal mappings coincedes with the one of *conformal mappings.* Hence, f is conformal if it is injective and holomorphic.

It should be mentioned that the result of the existence and uniqueness for the solution of (1.1) goes under the name of *Riemann mapping Theorem* and can be found for instance in [5, 49, 75]. We recall here the case of compactly supported dilatation, for instance we consider the case

$$|\mu(z)| \leq k\chi_{\mathbb{D}}(z) \tag{1.2}$$

where $\chi_{\mathbb{D}}(\cdot)$ denotes the characteristic function of the unit disk $\mathbb{D}$ and $0 \leq k < 1$.

Theorem 1.4. *Let μ be a measurable function satisfying* (1.2) *for some* $0 \leq k < 1$. *There exists a unique solution $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{C}, \mathbb{C})$ of the Beltrami equation* (1.1) *satisfying the normalization*

$$f(z) = z + \mathcal{O}\left(\frac{1}{z}\right) \qquad \text{if } |z| \geq 1. \tag{1.3}$$

Any homeomorphism which is a solution of the Beltrami equation with complex dilatation μ satisfying (1.2) for some $0 \leq k < 1$ and satisfying the normalization (1.3) is called *principal* quasiconformal mapping.

1.3 Linear and quasilinear elliptic equations in divergence form

Let Ω be a bounded open subset of $\mathbb{R}^2$. For every constant $K \geq 1$ we consider the class $\mathcal{M}(K, \Omega)$ of measurable matrix field $A : \Omega \to \mathbb{R}^{2\times 2}$ such that $A = A(x) \in L^\infty(\Omega, \mathbb{R}^{2\times 2})$, A is symmetric and satisfies the condition

$$\frac{|\xi|^2}{K} \leq \langle A(x)\xi, \xi\rangle \leq K|\xi|^2 \qquad \text{a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^2. \tag{1.4}$$

If (1.4) holds, we say that the matrix A satisfies a *uniform ellipticity condition.* We also remark that the bounds in (1.4) are equivalent to the following single inequality

$$|\xi|^2 + |A(x)\xi|^2 \leq \left(K + \frac{1}{K}\right) \langle A(x)\xi, \xi\rangle \qquad \text{a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^2.$$

Let us denote by a_{ij}, $i, j = 1, \ldots, n$, the entries of A, we consider the second order elliptic differential operator

$$\mathcal{L} = \operatorname{div}(A(x)\nabla) = \sum_{i,j=1}^{n} D_i \left(a_{ij}(x) D_j\right),$$

for each $A \in \mathcal{M}(K, \Omega)$. The divergence operator is understood in the sense of distribution, according with the following definition.

Definition 1.2. Let Ω be a bounded open subset of $\mathbb{R}^2$ and let $A \in \mathcal{M}(K,\Omega)$. We say that a function $u \in W^{1,2}_{\rm loc}(\Omega)$ is a *weak solution* of the equation

$$\mathrm{div} A(x)\nabla u = 0, \tag{1.5}$$

if

$$\int_\Omega \langle A(x)\nabla u(x), \nabla\varphi(x)\rangle \, dx = 0 \qquad \forall \varphi \in C_0^\infty(\Omega).$$

We will give here a brief review of the classical results for the equation (1.5). A general reference here will be [35]. First, we discuss the local regularity for the weak solution. To this aim, we recall that a function $u : \Omega \to \mathbb{R}$ is locally Hölder continuous with exponent $0 < \alpha \leq 1$ if, for every compact subset $E \subset\subset \Omega$ there holds

$$\sup_{x,y\in E,\, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty.$$

It is well known that every weak solution $u \in W^{1,2}_{\rm loc}(\Omega)$ of the equation (1.5) is locally Hölder continuous in Ω, as stated by the following result (see e.g. [21], [64] and [67]).

Theorem 1.5. *Let Ω be an open subset of $\mathbb{R}^n$. Let $A \in \mathcal{M}(K,\Omega)$ and let $u \in W^{1,2}_{\rm loc}(\Omega)$ is a weak solution of the equation* (1.5)*. Then for every compact set $E \subset\subset \Omega$ there exist $C > 0$ and $0 < \alpha \leq 1$ depending only on K and* dist $(E, \partial\Omega)$ *such that*

$$|u(x)-u(y)| \leq C|x-y|^\alpha \left(\int_\Omega |u|^2 dx\right)^{\frac{1}{2}} \qquad a.e.\ x, y \in E.$$

Next result shows that every weak solution of (1.5) belongs to $W^{1,p}_{\rm loc}(\Omega)$ for some $p = p(n,K) > 2$.

Theorem 1.6. *Let u be a weak solution of* (1.5) *and let $R > 0$ such that $B_{2R}(x_0) \subset \Omega$. Then, there exists $p > 2$ depending only on n and K such that*

$$\left(\fint_{B_R(x_0)} |\nabla u(x)|^p \, dx\right)^{\frac{1}{p}} \leq C \left(\fint_{B_{2R}(x_0)} |\nabla u(x)|^2 \, dx\right)^{\frac{1}{2}}.$$

The classical Harnack's principle, which holds for positive harmonic function, also holds for the case of equation (1.5), as proved by Moser in [65].

Theorem 1.7. *Let u be a positive weak solution of* (1.5)*. Then, for every compact set $E \subset\subset \Omega$ the inequality*

$$\max_E u \leq C \min_E u,$$

holds for some constant $C > 1$ depending only on E and Ω.

The maximum principle holds, in the sense of the following result.

Theorem 1.8. *Let u be a weak solution of* (1.5) *in Ω which is continuous in a neighborhood of $\partial\Omega$. Then,*

$$\sup_\Omega u \leq \sup_{\partial\Omega} u.$$

We recall that some of the results above for the equation (1.5) can be proved without assuming that A is a symmetric matrix, as observed by Morrey in [63].

A generalization of (1.5) is the quasilinear equation

$$\mathrm{div} A(x, \nabla u) = 0, \tag{1.6}$$

called *Leray–Lions equation.* Here and in what follows $A : \Omega \times \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ is a function such that such that

$$A(\cdot, \xi) \text{ is a measurable function for every } \xi \in \mathbb{R}^2, \tag{1.7}$$

and

$$A(x, \cdot) \text{ is a continuous function for a.e. } x \in \Omega. \tag{1.8}$$

For every Ω bounded open subset of $\mathbb{R}^2$ and for every constant $K \geq 1$ we consider the class $\mathcal{N}(K, \Omega)$ of functions $A : \Omega \times \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ satisfying (1.7), (1.8) and the condition

$$|\xi|^2 + |A(x,\xi)|^2 \leq \left(K + \frac{1}{K}\right) \langle A(x,\xi), \xi\rangle \qquad \text{a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^2.$$

To each $A \in \mathcal{N}(K, \Omega)$ we associate the nonlinear equation

$$\mathrm{div} A(x, \nabla u) = 0,$$

called *Leray–Lions equation.* One should immediately check that the Leray–Lions equation (1.6) reduces to the linear equation (1.5) if

$$A(x, \xi) = A(x)\xi,$$

for some $A \in \mathcal{M}(K, \Omega)$. Equations (1.5) and (1.6) are strongly related by the following result, which can be found in [78].

Theorem 1.9. *Let Ω be a bounded open set in $\mathbb{R}^2$, $A = A(x,\xi) \in \mathcal{N}(K,\Omega)$ and $u \in W^{1,2}_{\rm loc}(\Omega)$ be a weak solution of the quasilinear equation* (1.6). *Then there exists a unique symmetric matrix $\mathcal{A} \in \mathcal{M}(K,\Omega)$, with* $\det \mathcal{A}(x) = 1$ *a. e. in Ω such that*

$$\operatorname{div}(\mathcal{A}(x)\nabla u) = 0.$$

Therefore, every equation of the type (1.6) reduces in a certain sense to a linear equation with the same ellipticity bounds as the original one. We indicate that $\mathcal{A}$ depends on A and u, by writing $\mathcal{A} = \mathcal{A}[A,u]$. Finally, we remark that in the linear case $A(x,\xi) = A(x)\xi$ with $A \in \mathcal{M}(K,\Omega)$, the new matrix $\mathcal{A}$ is different from A, unless $\det A(x) = 1$ a.e. in Ω.

1.4 The connection between PDE's and quasiconformal mappings in the plane

In the case of dimension $n = 2$ there is a precise interplay between the theory of quasiconformal mappings and the elliptic PDE's of the type (1.5) . Indeed, for $f \in W^{1,1}_{\rm loc}(\Omega,\Omega')$ we define a matrix field $G_f : \Omega \to \mathbb{R}^{2\times 2}$ given by

$$G_f(x) = \begin{cases} \frac{{}^tDf(x)Df(x)}{J_f(x)} & \text{if } J_f(x) > 0, \\ I & \text{otherwise,} \end{cases}$$

here ${}^tDf(z)$ denotes the transpose of the differential matrix of f and I denotes the identity matrix. The matrix field G_f is called *distortion tensor* of f. It is easy to check that G_f is a symmetric matrix with

$$\det G_f(x) = 1 \qquad \text{for a.e. } x \in \mathbb{R}^2.$$

Moreover, if we assume that f is a K–quasiconformal mapping, then the distortion inequality for f is equivalent to the condition

$$\frac{|\xi|^2}{K} \leq \langle G_f(x)\xi,\xi\rangle \leq K|\xi|^2 \qquad \text{for a.e. } x \in \mathbb{R}^2 \qquad \forall \xi \in \mathbb{R}^2.$$

Let A_f be the inverse matrix of the G_f, namely $A_f = G_f^{-1}$. Clearly A_f is a symmetric matrix field which satisfies

$$\det A_f(x) = 1 \qquad \text{for a.e. } x \in \Omega,$$

while the distortion inequality for f easily give us

$$\frac{|\xi|^2}{K} \leq \langle A_f(x)\xi, \xi\rangle \leq K|\xi|^2 \qquad \text{for a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}^2.$$

Let us define the *Laplace-Beltrami operator* as

$$\mathcal{L}_f = \operatorname{div}\left(A_f(x)\nabla\right). \tag{1.9}$$

The following fundamental result holds.

Theorem 1.10. *Let Ω, Ω' be open subsets of $\mathbb{R}^2$ and let $f = (u, v) : \Omega \to \Omega'$ be a K–quasiconformal mapping. Then, the components u and v of f are weak solution of the equations*

$$\mathcal{L}_f u = 0 \qquad \textit{and} \qquad \mathcal{L}_f v = 0,$$

where $\mathcal{L}_f$ is the Laplace–Beltrami operator defined in (1.9).

On the other hand, elliptic equations generate quasiconformal mappings, in the sense of the following result.

Theorem 1.11. *Let Ω be a bounded open subset of $\mathbb{R}^2$. For each non-constant solution $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ of the elliptic equation*

$$\operatorname{div}\left(A(x)\nabla u\right) = 0,$$

where ${}^tA = A$ also satisfies the uniform elliptic bound

$$\frac{|\xi|^2}{K} \leq \langle A(x)\xi, \xi\rangle \leq K|\xi|^2 \qquad \textit{for a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}^2,$$

for some $K \geq 1$, there exists a K-quasiconformal mapping $g : \Omega \to \mathbb{D}$, where $\mathbb{D}$ denotes the unit disc of $\mathbb{R}^2$, and a real valued harmonic function $h : \mathbb{D} \to \mathbb{R}$ such that

$$u = h \circ g \quad \textit{in } \Omega.$$

1.5 Area distortion estimates

The aim of this section is to provide the exact degree of integrability for the differential of a planar quasiconformal mapping. More precisely, what we want to point out is that if $f:\Omega \to \Omega'$ is a K–quasiconformal mapping defined in a domain Ω of $\mathbb{R}^2$ then $|Df| \in W^{1,p}_{\mathrm{loc}}(\Omega)$ for an exponent $p = p(K)$ strictly larger than 2 and depending only on K.

This result is a direct consequence of the area distortion estimate, established by Astala [2].

Theorem 1.12 ([2]). *Let f be a K–quasiconformal mapping which maps the unit disk $\mathbb{D}$ onto itself and such that $f(0) = 0$. Then*

$$|f(E)| \leq M|E|^{\frac{1}{K}},$$

for some constant $M = M(K)$ depending only on K.

We give here version of Theorem 1.12 given by Erëmenko and Hamilton [25], where the optimal value of the constant $M(K)$ is computed.

Theorem 1.13 ([25]). *Let $f:\mathbb{R}^2 \to \mathbb{R}^2$ be a K–quasiconformal principal mapping which is conformal outside the unit disk $\mathbb{D}$.*

(i) If f is conformal outside a measurable set $E \subset \mathbb{D}$, then

$$|f(E)| \leq K|E|.$$

(ii) If f is conformal in a measurable set $E \subset \mathbb{D}$, then

$$|f(E)| \leq \pi^{1-\frac{1}{K}}|E|^{\frac{1}{K}}.$$

(iii) For every measurable set $E \subset \mathbb{D}$

$$|f(E)| \leq K\pi^{1-\frac{1}{K}}|E|^{\frac{1}{K}}.$$

As mentioned before, Theorem 1.12 has the following fundamental consequence.

Corollary 1.14. *If f is a K–quasiconformal mapping defined in a planar domain Ω then*

$$f \in W^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^2) \qquad \text{if} \quad p < \frac{2K}{K-1},$$

and the exponent $p(K) = 2K/(K-1)$ is the best possible one, in the sense that for each $K > 1$ there are K–quasiconformal mappings f such that $f \notin W^{1,\frac{2K}{K-1}}_{\mathrm{loc}}(\Omega,\mathbb{R}^2)$.

For the last statement of Corollary 1.14 it is sufficient to consider the K–quasiconformal mapping

$$f(x) = \frac{x}{|x|^{1-\frac{1}{K}}} \qquad \forall x \in \mathbb{R}^2.$$

Observe that the result above implies that

$$J_f \in L^p_{\mathrm{loc}}(\Omega) \qquad \text{if} \quad p < \frac{K}{K-1}, \tag{1.10}$$

if f is a K–quasiconformal mapping defined in a planar domain Ω.

1.6 A generalization of quasiconformality: mappings of finite distortion

We recall that quasiconformal homeomorphism are a special kind of mapping on finite distortion.

Definition 1.3. A mapping $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ is said to have finite distortion if $J_f \in L^1_{\mathrm{loc}}(\Omega)$ and if there exists a measurable function $K : \Omega \to [1, \infty]$ such that

$$|Df(x)|^2 \leq K(x) J_f(x) \qquad \text{a.e. } x \in \Omega. \tag{1.11}$$

In the case of a homeomorphism the assumption on local integrability of the Jacobian determinant is redundant.

The existence of a measurable function $K : \Omega \to [1, \infty]$ finite a.e. satisfying (1.11) holds allow us to say define the function

$$K_f(x) = \begin{cases} \frac{|Df(x)|^2}{J_f(x)} & \text{if } J_f(x) > 0, \\ 1 & \text{otherwise.} \end{cases} \tag{1.12}$$

In other words, K_f is the smallest function greater or equal to 1 for which (1.11) holds.

We remark that a K–quasiconformal mapping f is a finite distortion homeomorphism with $K_f \leq K$ a.e. in Ω.

Moreover, in [42] is proved that, in the planar case, there is the equivalence between the class of the bi–Sobolev homeomorphism and the class of homeomorphism with finite distortion.

Definition 1.4. A homeomorphism $f : \Omega \xrightarrow{\text{onto}} \Omega'$ is called bi–Sobolev mapping if both $f \in W^{1,p}_{\text{loc}}(\Omega, \Omega')$ and its inverse $f^{-1} \in W^{1,p}_{\text{loc}}(\Omega', \Omega)$, for some $1 \le p \le \infty$.

The case of a matrix which satisfies a bound of the type

$$\frac{|\xi|^2}{K(x)} \le \langle A(x)\xi, \xi\rangle \le K(x)|\xi|^2 \qquad \text{a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^2,$$

for some measurable function $K : \Omega \to [1, \infty]$, naturally arises in connection with the mapping of finite distortion.

Theorem 1.15. *Let Ω, Ω' be open subsets of $\mathbb{R}^2$. Then, to each bi–Sobolev mapping $f : \Omega \to \Omega'$, $f = (u, v)$, there corresponds a measurable function $A_f = A_f(x)$ valued in symmetric matrices with*

$$\det A_f(x) = 1 \qquad \textit{for a.e. } x \in \Omega,$$

such that

$$\frac{|\xi|^2}{K_f(x)} \le \langle A_f(x)\xi, \xi\rangle \le K_f(x)|\xi|^2 \qquad \textit{for a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}^2,$$

where K_f denotes the distortion function of f defined in (1.12)*. The components u and v of f are very weak solution of an elliptic equation of the type* (1.5)*, i.e.*

$$\operatorname{div}(A(x)\nabla u) = 0 \qquad \textit{and} \qquad \operatorname{div}(A(x)\nabla v) = 0, \tag{1.13}$$

with finite energy, i.e.

$$\int_\Omega \langle A(x)\nabla u, \nabla u\rangle\, dx < \infty \qquad \textit{and} \qquad \int_\Omega \langle A(x)\nabla v, \nabla v\rangle\, dx < \infty.$$

Dealing with the last statement, we recall that u and v are *very weak solutions* of the equations (1.13) if u and v belongs $W^{1,1}_{\text{loc}}(\Omega)$ and satisfies (1.13) in the sense of the distributions.

Chapter 2

Sharp Hölder estimates

This chapter is concerned with the Hölder regularity results for weak solutions to the elliptic equation in divergence form

$$\mathrm{div} A(x)\nabla u = 0,$$

where $A = (a_{ij}),\ i, j = 1, 2$ is a 2×2 symmetric, positive definite matrix satisfying the uniform elliptic bound $\lambda|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq \Lambda|\xi|^2$ for all $\xi \in \mathbb{R}^2$, for a.e. $x \in \Omega$ where $\Omega \subset \mathbb{R}^2$ is a bounded domain and for some $0 < \lambda \leq \Lambda$.

2.1 Explicit values of the best Hölder exponent

Let Ω be a bounded open subset of $\mathbb{R}^2$ and let $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a weak solution of the equation in divergence form

$$\mathrm{div} A(x)\nabla u = 0, \tag{2.1}$$

where $A = A(x) \in L^\infty(\Omega; \mathbb{R}^{2\times 2})$ is a symmetric matrix, i.e.

$$^tA = A, \tag{2.2}$$

satisfying the uniform elliptic bound

$$\lambda|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq \Lambda|\xi|^2 \qquad \text{a.e. } x \in \Omega \quad \forall \xi \in \mathbb{R}^2, \tag{2.3}$$

for some constants λ and Λ such that $0 < \lambda \leq \Lambda$. In this context we say that the quantity

$$L = \frac{\Lambda}{\lambda},$$

is the *ellipticity coefficient* of the matrix A.

We have already observed in Chapter 1 that every weak solution $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ of the equation (2.1) is locally Hölder continuous in Ω. In [69] Piccinini and Spagnolo computed the best Hölder exponent for weak solutions to the elliptic equation (2.1). Their result states as follow.

Theorem 2.1. *Let Ω be an open subset of $\mathbb{R}^2$. Assume that $A = A(x) \in L^\infty(\Omega; \mathbb{R}^{2\times 2})$ is a matrix satisfying* (2.2) *and* (2.3) *and that $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a weak solution of the equation* (2.1)*. Then u is locally Hölder continuous with exponent α given by*

$$\alpha = \frac{1}{\sqrt{L}},$$

where $L = \Lambda/\lambda$.

The fact that $\alpha = 1/\sqrt{L}$ is the best possible Hölder exponent is proved by means of the following example (see Meyers [58]).

Example 2.1. Let $L \geq 1$ and let us define a matrix $A = A(x)$ whose entries a_{ij} are defined by

$$\begin{aligned} a_{11} &= \left(Lx_1^2 + x_2^2\right)|x|^{-2}, \\ a_{12} &= (L-1)x_1x_2|x|^{-2} = a_{21}, \\ a_{22} &= \left(x_1^2 + Lx_2^2\right)|x|^{-2}. \end{aligned}$$

The ellipticity coefficient of A is L. Let

$$u(x) = \frac{x_1}{|x|^{1-\frac{1}{\sqrt{L}}}}.$$

Then u is a Hölder continuous function of exponent is $\alpha = 1/\sqrt{L}$ and is a solution of (2.1) with this choice of A above. It should be observed that the equation is given in polar coordinates by

$$L\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho}^2\frac{\partial^2 u}{\partial\theta^2} = 0,$$

and that the solution u may be rewritten as

$$u(\rho,\theta) = \rho^{1/\sqrt{L}}\cos\theta.$$

When in (2.3) one has

$$\Lambda = K, \qquad \lambda = \frac{1}{K},$$

for some $K \geq 1$ then the ellipticity coefficient $L = K^2$ and therefore the best Hölder exponent is given by

$$\alpha = \frac{1}{K}.$$

This is in agreement with the following result concerning with quasiconformal mappings, see [1, 60].

Theorem 2.2. *Every K–quasiconformal mapping $f : \Omega \to \Omega'$, where Ω and Ω' are planar domains, is locally Holder continuous with exponent $\alpha = 1/K$. The K–quasiconformal mapping $f : \mathbb{D} \to \mathbb{D}$ defined as*

$$f(z) = \frac{z}{|z|^{1-\frac{1}{K}}},$$

shows that the exponent is the best possible one.

A key ingredient for the proof of Theorem 2.1 is the sharp Wirtinger inequality.

Lemma 2.3. *Let w be a function in $W^{1,2}_{\mathrm{loc}}(\mathbb{R})$ periodic of period 2π such that*

$$\int_0^{2\pi} w(t)dt = 0,$$

Then the following inequality holds

$$\int_0^{2\pi} |w(t)|^2 dt \leq \int_0^{2\pi} |w'(t)|^2 dt. \tag{2.4}$$

A second result proved by Piccinini and Spagnolo in [69] states that the best Hölder exponent for weak solutions to the elliptic equation (2.1) improves for isotropic matrices.

We recall that a matrix A is said to be *isotropic* if A is of the type

$$A(x) = a(x)I, \tag{2.5}$$

where I is the identity matrix and $a : \Omega \to \mathbb{R}$ is a measurable function such that

$$\lambda \leq a(x) \leq \Lambda \qquad \text{for a.e. } x \in \Omega, \tag{2.6}$$

for some constants λ and Λ such that $0 < \lambda \leq \Lambda$.

Theorem 2.4. *Let Ω be an open subset of $\mathbb{R}^2$. Assume that $a : \Omega \to \mathbb{R}$ is a measurable function satisfying (2.6) and that $u \in W^{1,2}_{\text{loc}}(\Omega)$ is a weak solution of the equation (2.1) where A takes the form (2.5). Then u is locally Hölder continuous with exponent α given by*

$$\alpha = \arctan \frac{1}{\sqrt{L}},$$

where $L = \Lambda/\lambda$.

A key ingredient for the proof of Theorem 2.4 is the following sharp weighted Wirtinger inequality.

Lemma 2.5. *Let a be a real measurable function periodic of period 2π such that $1 \le a(t) \le L$. Let w be a function in $W^{1,2}_{\text{loc}}(\mathbb{R})$ periodic of period 2π such that*

$$\int_0^{2\pi} a(t)w(t)dt = 0,$$

Then the following inequality holds

$$\int_0^{2\pi} a(t)|w(t)|^2 dt \le \left(\frac{4}{\pi}\arctan\frac{1}{\sqrt{L}}\right)^{-2} \int_0^{2\pi} a(t)|w'(t)|^2 dt. \tag{2.7}$$

Inequality (2.7) reduces to an equality if and only if $a(t) = \tilde{a}(t+\Phi)$, $w(t) = C\tilde{w}(t+\delta)$, where C and δ are real constants and $\tilde{a}$ and $\tilde{w}$ are defined by

$$\tilde{a}(t) = \begin{cases} 1 & \text{for} \quad 0 \le t < \frac{\pi}{2},\ \ \pi \le t \le \frac{3}{2}\pi, \\ \\ L & \text{for} \quad \frac{\pi}{2} \le t < \pi,\ \ \frac{3}{2}\pi \le t < 2\pi. \end{cases} \tag{2.8}$$

$$\tilde{w}(t) = \begin{cases} \sin\left[\sqrt{\lambda}\left(t - \frac{\pi}{4}\right)\right] & \text{for} \quad 0 \le t \le \frac{\pi}{2}, \\ \\ \frac{1}{\sqrt{L}}\cos\left[\sqrt{\lambda}\left(t - \frac{3}{4}\pi\right)\right] & \text{for} \quad \frac{\pi}{2} \le t \le \pi, \\ \\ -\sin\left[\sqrt{\lambda}\left(t - \frac{5}{4}\pi\right)\right] & \text{for} \quad \pi \le t \le \frac{3}{2}\pi, \\ -\frac{1}{\sqrt{L}}\cos\left[\sqrt{\lambda}\left(t - \frac{7}{4}\pi\right)\right] & \text{for} \quad \frac{3}{2}\pi \le t \le 2\pi, \end{cases} \tag{2.9}$$

where $\lambda = \left(\frac{4}{\pi}\arctan\frac{1}{\sqrt{L}}\right)^2$.

The fact that $\alpha = \arctan\left(1/\sqrt{L}\right)$ is the best possible exponent in the isotropic case is proved by means of the following example.

Example 2.2. Let us define a matrix $A(x) = \tilde{a}(\theta)I$ where $\theta = \theta(x) = \arctan\frac{x_2}{x_1}$ and $\tilde{a}$ is defined in (2.8). The corresponding differential equation is given by

$$\operatorname{div}(\tilde{a}(\theta)I\nabla u) = 0. \tag{2.10}$$

The ellipticity coefficient of A is equal to L and the function

$$u(x) = |x|^{\frac{4}{\pi}\arctan\frac{1}{\sqrt{L}}}\tilde{w}(\theta),$$

with $\tilde{w}$ defined in (2.9) satisfies (2.10) and is Hölder continuous with exponent $\alpha = \frac{4}{\pi}\arctan\frac{1}{\sqrt{L}}$.

If we additionally assume that the matrix A has unit determinant, namely

$$\det A(x) = 1 \qquad \text{a.e. } x \in \Omega, \tag{2.11}$$

the following estimate holds, see [72] (and [74, 73] for related results).

Theorem 2.6. *Let $A = (a_{ij})$ satisfy (2.2), (2.3) and (2.11) and let $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a weak solution of (2.1). Then u is locally Hölder continuous in Ω with α given by*

$$\alpha = 2\pi\left(\sup_{x_0\in\Omega}\ \operatorname*{ess\,sup}_{0<r<\operatorname{dist}(x_0,\partial\Omega)}\int_{|\xi|=1}\langle A(x_0+r\xi)\xi,\xi\rangle d\sigma(\xi)\right)^{-1}.$$

Corollary 2.7. *Let $A = (a_{ij})$ satisfy (2.2), (2.3) and (2.11) and let $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a weak solution of (2.1). Then the least upper bound for the admissible values of the Hölder exponent of u is given by*

$$\bar{\alpha} = 2\pi\left(\sup_{x_0\in\Omega}\ \inf_{0<r_0<\operatorname{dist}(x_0,\partial\Omega)}\ \operatorname*{ess\,sup}_{0<r<r_0}\int_{|\xi|=1}\langle A(x_0+r\xi)\xi,\xi\rangle d\sigma(\xi)\right)^{-1}.$$

Theorem 2.6 is sharp in the sense of the following example.

Example 2.3. Hereafter if $x = (x_1, x_2) \in \mathbb{R}^2$ the notation $x \otimes x$ stands for the matrix

$$x \otimes x = \begin{pmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{pmatrix}.$$

Let $\Omega = \mathbb{D}$ be the unit disc in $\mathbb{R}^2$, let $\theta(x) = \arctan\frac{x_2}{x_1}$ and let

$$A(x) = \frac{1}{k(\theta)}I + \left(k(\theta) - \frac{1}{k(\theta)}\right)\frac{x\otimes x}{|x|^2}, \tag{2.12}$$

where $k = k(\theta) : \mathbb{R} \to \mathbb{R}^+$ is a 2π-periodic function bounded from above and away from zero. Then $\det A(x) = 1$. By suitable choice of k we may obtain

$$\bar{\alpha} = 2\pi \left(\int_0^{2\pi} k \right)^{-1}.$$

On the other hand the function $u \in W^{1,2}(\mathbb{D})$ defined by

$$u(x) = |x|^{\bar{\alpha}} \cos\left(\bar{\alpha} \int_0^{\theta(x)} k \right),$$

satisfies the equation with A given by (2.12). Clearly its Hölder exponent is exactly $\bar{\alpha}$.

A key ingredient in the proof of Theorem 2.6 is the following sharp weighted Wirtinger inequality.

Theorem 2.8. *Let a be a real measurable function periodic of period 2π bounded from above and away from zero. Let w be a function in $W^{1,2}_{\mathrm{loc}}(\mathbb{R})$ periodic of period 2π such that*

$$\int_0^{2\pi} a(t)w(t)dt = 0.$$

Then the following inequality holds

$$\int_0^{2\pi} a(t)|w(t)|^2 dt \le \left(\frac{1}{2\pi} \int_0^{2\pi} a \right)^2 \int_0^{2\pi} \frac{1}{a(t)} |w'(t)|^2 dt. \tag{2.13}$$

Inequality (2.13) *is attained if and only if w is of the form*

$$w(\theta) = C \cos\left(\frac{2\pi}{\int_0^{2\pi} a} \int_0^{\theta} a + \delta \right),$$

for some $C \in \mathbb{R} \setminus \{0\}$ and $\delta \in \mathbb{R}$.

2.2 Wirtinger–Poincaré type inequalities

Motivated by the regularity results considered in Section 2.1 and also by various problems in analysis and geometry, several extensions and variations of (2.4) of the type

$$\left(\int_0^T |u|^q \right)^{1/q} \le C \left(\int_0^T |u'|^p \right)^{1/p},$$

have been obtained. Here and in what follows $N \geq 1$, $T > 0$, $p, q > 1$ and the function $u : [0,T] \to \mathbb{R}^N$ is subjected to various boundary conditions or integral constrains.

For later use, we briefly define the generalized trigonometric functions and outline their main properties (for details see e.g. [43, 50, 51, 53]).

Let $p, q > 1$. The function $\arcsin_{pq} : [0,1] \to \mathbb{R}$ is defined by

$$\arcsin_{pq}(\sigma) = \int_0^\sigma \frac{ds}{(1-s^p)^{1/q^*}}.$$

Let us define

$$\frac{\pi_{pq}}{2} = \arcsin_{pq}(1) = \frac{1}{p} B\left(\frac{1}{p}, \frac{1}{q}\right).$$

where $B(\cdot,\cdot)$ denotes the Beta function defined by

$$B(h,k) = \int_0^1 t^{h-1}(1-t)^{k-1} dt = B(k,h),$$

for every $h, k > 0$. The function $\arcsin_{pq} : [0,1] \to [0, \frac{\pi_{pq}}{2}]$ is strictly increasing and its inverse function is denoted by $\sin_{pq}$. The function $\sin_{pq}$ is extended as an odd function to the interval $[-\pi_{pq}, \pi_{pq}]$ by setting $\sin_{pq}(t) = \sin_{pq}(\pi_{pq} - t)$ in $[\pi_{pq}/2, \pi_{pq}]$, $\sin_{pq}(t) = -\sin_{pq}(-t)$ in $[-\pi_{pq}, 0]$, and to the whole real axis as a $2\pi_{pq}-$periodic function. The function $w(t) = \sin_{qp^*}(\pi_{qp^*} t)$ is the unique solution of the initial value problem

$$\begin{cases} (\phi_p(w'))' + \frac{q}{p^*}\phi_q(w) = 0, \\ w(0) = 0, \qquad w'(0) = 1. \end{cases} \tag{2.14}$$

Here and in what follows we define the function $\phi_p : \mathbb{R} \to \mathbb{R}$ by

$$\phi_p(s) = \begin{cases} |s|^{p-2}s & \text{if } s \in \mathbb{R} \setminus \{0\}, \\ 0 & \text{if } s = 0. \end{cases} \tag{2.15}$$

The existence and uniqueness of the solution of a general kind of initial value problem of the type (2.14) is established for instance in [23, 43].

Lemma 2.9. *Let $a, b, t_0 \in \mathbb{R}$, $\lambda > 0$ and $p, q > 1$. Then the problem*

$$\begin{cases} (\phi_p(w'))' + \lambda\phi_q(w) = 0, \\ w(t_0) = a, \qquad w'(t_0) = b. \end{cases}$$

has a unique solution defined in $\mathbb{R}$. Moreover, every solution of (2.14) satisfies

$$\frac{|w'(t)|^p}{p} + \lambda\frac{|w(t)|^q}{q^*} = \frac{|b|^p}{p} + \lambda\frac{|a|^q}{q^*}$$

In what follows we shorten our notation by defining

$$p^* = \frac{p}{p-1},$$

$$\sin_p(t) = \sin_{pp^*}(t)$$

and

$$\pi_p = \pi_{pp^*}.$$

We define the function $\cos_p$ is defined by

$$\cos_p(t) = \phi_p(\sin_p'(t)). \tag{2.16}$$

It is $2\pi_p$-periodic and satisfies:

$$\cos_p(-t) = \cos_p(t),$$

$$\cos_p(\pi_p - t) = -\cos_p(t),$$

$$\cos_p(\pi_p + t) = -\cos_p(t).$$

The following identity holds, which generalizes the fundamental identity for trigonometric functions:

$$|\cos_p(t)|^{p^*} + |\sin_p(t)|^p \equiv 1. \tag{2.17}$$

For later purposes, we also note the following identity:

$$\cos_p\left(\frac{\pi_p}{2} - t\right) = \sin_{p^*}\left(\frac{p}{p^*}t\right). \tag{2.18}$$

The derivative of $\cos_p$ satisfies

$$\cos_p'(t) = -\frac{p}{p^*}\phi_p(\sin_p(t)). \tag{2.19}$$

On the other hand, from (2.16) we have:

$$\sin_p'(t) = \phi_{p^*}(\cos_p(t)). \tag{2.20}$$

Finally, we define $\tan_p$ as follows:

$$\tan_p(t) = \frac{\sin_p(t)}{\phi_{p^*}(\cos_p(t))}.$$

The function $\tan_p$ is π_p-periodic, with singularities at the zeros of $\cos_p$. The inverse of $\tan_p$ restricted to the interval $[-\pi_p/2, \pi_p/2]$, denoted by $\arctan_p$, is given by

$$\arctan_p(\sigma) = \int_0^\sigma \frac{dy}{1+|y|^p},$$

for every $\sigma \in \mathbb{R}$. It results that

$$\lim_{\sigma\to+\infty} \arctan_p(\sigma) = \frac{\pi_p}{2}. \tag{2.21}$$

The next lemma generalizes to the case $p \neq 2$ a well known identity.

Lemma 2.10. *For every $p > 1$ and for every $\sigma > 0$ the following identity holds*

$$\arctan_p(\sigma^{-p^*/p}) + \frac{p^*}{p} \arctan_{p^*}(\sigma) = \frac{\pi_p}{2}. \tag{2.22}$$

Proof. In view of (2.21) we have

$$\frac{\pi_p}{2} = \int_0^{+\infty} \frac{dy}{1+y^p} = \arctan_p(\sigma^{-p^*/p}) + \int_{\sigma^{-p^*/p}}^{+\infty} \frac{dy}{1+y^p}.$$

Performing the change of variables $y = z^{-p^*/p}$ we obtain

$$\int_{\sigma^{-p^*/p}}^{+\infty} \frac{dy}{1+y^p} = \frac{p^*}{p} \int_0^\sigma \frac{dz}{1+z^{p^*}}.$$

Hence, the asserted identity follows. $\square$

The space of functions which satisfy the periodic boundary condition will be denoted by $W^{1,p}_{\text{per}}([0,T],\mathbb{R}^N)$, namely

$$W^{1,p}_{\text{per}}([0,T],\mathbb{R}^N) = \left\{u \in W^{1,p}([0,T],\mathbb{R}^N) : u(0) = u(T)\right\}, \tag{2.23}$$

where $N \geq 1$, $T > 0$, $p > 1$. In what follows $W^{1,p}_{\text{per}}(0,T)$ stands for the space defined in (2.23) when $N = 1$.

A general result which holds in the case $N = 1$ is the following (see [16] and [18]).

Theorem 2.11. *Consider the minimization problem*

(2.24)

$$\lambda_\#(p,q,r) = \inf\left\{ \frac{\left(\int_0^T |u'|^p\right)^{1/p}}{\left(\int_0^T |u|^q\right)^{1/q}} : u \in W^{1,p}_{\text{per}}(0,T) \setminus \{0\},\ \int_0^T |u|^{r-2}u = 0 \right\},$$

where $p > 1$, $q \geq r - 1 \geq 1$. *Then*

$$\lambda_{\#}(p,q,r) = \lambda_{\#}(p,q,q) \quad \text{if} \quad q \leq rp + r - 1,$$
$$\lambda_{\#}(p,q,r) < \lambda_{\#}(p,q,q) \quad \text{if} \quad q > (2r-1)p.$$

Furthermore

$$\lambda_{\#}(p,q,q) = \frac{4}{T^{\frac{1}{p^*}+\frac{1}{q}}} \left(\frac{1}{p^*}\right)^{\frac{1}{q}} \left(\frac{1}{q}\right)^{\frac{1}{p^*}} \left(\frac{1}{p^*+q}\right)^{\frac{1}{p}-\frac{1}{q}} B\left(\frac{1}{p^*}, \frac{1}{q}\right). \tag{2.25}$$

The above formula is also valid when $q = r > 1$, $q = 1$ *(*$p > 1$ *and* $r = 2$*) and* $p = \infty$ *(*$q \geq r - 1 \geq 1$*).*

Observe that the constant (2.25) in may be also written in the following way

$$\lambda_{\#}(p,q) \equiv \lambda_{\#}(p,q,q) = 2\left(\frac{1}{p^*}\right)^{1/q} q^{1/p} \left(\frac{1}{p^*+q}\right)^{\frac{1}{p}-\frac{1}{q}} \frac{\pi_{qp^*}}{T^{\frac{1}{p^*}+\frac{1}{q}}}.$$

Therefore, in the homogeneous case $p = q$

$$\lambda_{\#}(p) \equiv \lambda_{\#}(p,p,p) = 2\,(p-1)^{1/p}\,\frac{\pi_p}{T}.$$

When $r = q$ the result above also can be found in [23] where extremals are characterized.

Theorem 2.12. *The extremals for problem* (2.24) *are the functions*

$$u(t) = C \sin_{qp^*}\left(\frac{2\pi_{qp^*}}{T}t + \delta\right),$$

for some $C \in \mathbb{R} \setminus \{0\}$ *and for some* $\delta \in \mathbb{R}$.

In [53, 54] the vectorial case of (2.24) is treated when $p = q = r$. Namely, the following problem is considered

$$\mu_{\#}(p,N) = \inf\left\{\frac{\int_0^T |u'|^p}{\int_0^T |u|^p} : u \in W^{1,p}_{\text{per}}([0,T],\mathbb{R}^N) \setminus \{0\},\ \int_0^T |u|^{p-2}u = 0\right\},$$

Here and in what follows let $\psi_p : \mathbb{R}^N \to \mathbb{R}^N$ be the continuous function defined by

$$\psi_p(x) = \begin{cases} |x|^{p-2}x & \text{if } x \in \mathbb{R}^N \setminus \{0\}, \\ 0 & \text{if } x = 0. \end{cases}$$

Observe that ϕ_p in (2.15) is the function ψ_p when $N = 1$. As for the scalar case, the following result existence and uniqueness result can be proved (see [53]).

Lemma 2.13. *Let $\xi, \eta \in \mathbb{R}^N$, $t_0 \in \mathbb{R}$, $\lambda > 0$ and $p > 1$. Then the problem*

$$\begin{cases} (\psi_p(u'))' + \lambda \psi_p(u) = 0, \\ u(t_0) = \xi, \qquad u'(t_0) = \eta. \end{cases}$$

has a unique solution defined in $\mathbb{R}$.

Theorem 2.14. *For each $p > 1$, $N \geq 1$ and $u \in W^{1,p}_{\mathrm{per}}\left([0,T], \mathbb{R}^N\right)$ such that*

$$\int_0^T |u|^{p-2} u = 0,$$

one has

$$\mu_\#(p,N) \int_0^T |u|^p \leq \int_0^T |u'|^p.$$

Moreover, if $\lambda_{\#,1}(p,N)$ is the smallest possible eigenvalue of the nonlinear eigenvalue problem

$$\begin{cases} (\psi_p(u'))' + \lambda \psi_p(u) = 0, \\ u(0) = u(T), \qquad \int_0^T |u|^{r-2} u = 0, \end{cases} .$$

with $\lambda > 0$, then $\mu_\#(p,N)$ satisfies the identity

$$\lambda_{\#,1}(p,N) = \mu_\# \left(\frac{p}{\min\{p-1, 1\}}, N \right)^{\min\{p-1,1\}}.$$

We remark here that $\lambda_{\#,1}(p,N)$ is computed explicitly in [54] in the special case $N = 1$ and satisfies the identity

$$\lambda^{\frac{1}{p}}_{\#,1}(p,1) = 2\,(p-1)^{1/p} \frac{\pi_p}{T}.$$

The space of functions which satisfy the Dirichlet boundary condition will be denoted by $W^{1,p}_0([0,T], \mathbb{R}^N)$, namely

$$W^{1,p}_0([0,T], \mathbb{R}^N) = \left\{ u \in W^{1,p}([0,T], \mathbb{R}^N) : u(0) = 0 \text{ and } u(T) = 0 \right\}, \tag{2.26}$$

where $N \geq 1$, $T > 0$, $p > 1$. In what follows $W^{1,p}_0(0,T)$ stands for the space defined in (2.26) above when $N = 1$.

First, we concern with the minimization problem

$$\lambda_0(p,q) = \inf \left\{ \frac{\left(\int_0^T |u'|^p \right)^{1/p}}{\left(\int_0^T |u|^q \right)^{1/q}} : u \in W^{1,p}_0(0,T) \setminus \{0\} \right\},$$

where $p, q > 1$.

Theorem 2.15. *For each $p, q > 1$, and $u \in W_0^{1,p}\left([0,T], \mathbb{R}^N\right)$ one has*

$$\lambda_0(p,q)\left(\int_0^T |u|^q\right)^{1/q} \leq \left(\int_0^T |u'|^p\right)^{1/p}, \tag{2.27}$$

where

$$\lambda_0(p,q) = \left(\frac{1}{p^*}\right)^{1/q} q^{1/p}\left(\frac{1}{p^*+q}\right)^{\frac{1}{p}-\frac{1}{q}} \frac{\pi_{qp^*}}{T^{\frac{1}{p^*}+\frac{1}{q}}},$$

The inequality (2.27) *holds with equal sign if and only if*

$$u(t) = C \sin_{qp^*}\left(\frac{\pi_{qp^*}}{T} t\right),$$

for some $C \in \mathbb{R} \setminus \{0\}$.

It should be mentioned that, in [53, 54] the vectorial case of (2.27) is treated when $p = q$. Namely, the following problem is considered

$$\mu_{\#}(p, N) = \inf\left\{\frac{\int_0^T |u'|^p}{\int_0^T |u|^p} : u \in W_0^{1,p}([0,T], \mathbb{R}^N) \setminus \{0\}\right\},$$

Theorem 2.16. *For each $p > 1$, $N \geq 1$ and $u \in W_0^{1,p}\left([0,T], \mathbb{R}^N\right)$ one has*

$$\frac{(p-1)^{\frac{1}{p}}\pi_p}{T}\left(\int_0^T |u|^p\right)^{1/p} \leq \left(\int_0^T |u'|^p\right)^{1/p}. \tag{2.28}$$

The inequality (2.28) *holds with equal sign if and only if*

$$u(t) = \sin_p\left(\frac{\pi_p}{T} t\right) d,$$

for some $d \in \mathbb{R}^N \setminus \{0\}$.

We consider now the case of weighted inequalities and we are interested in a generalization of the inequalities (2.13) and (2.7) (for references see [38, 73, 72, 37, 36].)

Theorem 2.17. *Let $a \in L^1(0,T)$, $a \geq 0$. Let w be a function in $W^{1,2}_{\text{loc}}(\mathbb{R})$ periodic of period T such that*

$$\int_0^T a(t)|w(t)|^{q-2} w(t) dt = 0$$

For every $p, q > 1$ the following inequality holds

$$\left(\int_0^T a(t)|w(t)|^q dt\right)^{\frac{1}{q}} \leq C(p,q)\left(\frac{1}{T}\int_0^T a\right)^{\frac{1}{p^*}+\frac{1}{q}}\left(\int_0^T \frac{1}{a(t)}|w'(t)|^p dt\right)^{\frac{1}{p}}. \tag{2.29}$$

where

$$C(p,q)=\left[2\left(\frac{1}{p^*}\right)^{\frac{1}{q}}\left(\frac{1}{q}\right)^{\frac{1}{p^*}}\left(\frac{2}{p^*+q}\right)^{\frac{1}{p}-\frac{1}{q}}B\left(\frac{1}{p^*},\frac{1}{q}\right)\right]^{-1}.$$

Inequality (2.29) *is attained if and only if* w *is of the form*

$$w(t)=C\sin_{qp^*}\left(\frac{2\pi_{qp^*}}{\int_0^T a(\theta)d\theta}\int_0^t a(\theta)d\theta+\delta\right),$$

for some $C\in\mathbb{R}\neq\{0\}$ *and* $\delta\in\mathbb{R}$.

It should be observd that the previous result is a generalization of Theorem 2.8 to general powers of $|u|$ under the more natural assumption $a\in L^1$.

The result that we want to prove is concerned with the following weighted vector inequality of Poincaré type

$$\int_0^T a|u|^p\le C\int_0^T a|u'|^p, \tag{2.30}$$

where u belongs to the space $W_0^{1,p}([0,T],\mathbb{R}^N)$. The function $a\in L^\infty(0,T)$ satisfies $1\le a\le L$ for some $L\ge 1$. Our aim is to estimate the best constant C in (2.30). Let

$$\mathcal{A}=\{a\in L^\infty(0,T):\inf a=1 \text{ and } \sup a=L\},$$

and let

$$\frac{1}{C_p(a)}=\inf\left\{\frac{\int_0^T a(t)|u'|^p}{\int_0^T a(t)|u|^p}:u\in W_0^{1,p}([0,T],\mathbb{R}^N)\setminus\{0\}\right\}, \tag{2.31}$$

for every given function $a\in\mathcal{A}$. By standard arguments it follows that the infimum in (2.31) is achieved for some $u\in W_0^{1,p}([0,T],\mathbb{R}^N)\setminus\{0\}$. We prove that if

$$\frac{1}{C_p}=\inf_{a\in\mathcal{A}}\frac{1}{C_p(a)},$$

then the infimum is achieved for a unique piecewise constant function $\tilde{a}\in\mathcal{A}$. It is convenient to define:

$$\beta(L)=\left[\frac{L^{p^*/p}(L-1)}{L^{p^*/p}-1}\right]^{1/p^*}. \tag{2.32}$$

With this notation, we have:

Theorem 2.18. *Let $N \geq 1$, $p > 1$ and $T > 0$. Let $a : [0,T] \to \mathbb{R}$ be a measurable function such that $1 \leq a(t) \leq L$. Then, the following inequality holds:*

$$\int_0^T a(t)|u(t)|^p dt \leq C_p \int_0^T a(t)|u'(t)|^p dt \tag{2.33}$$

for every $u \in W_0^{1,p}([0,T], \mathbb{R}^N)$, where

$$C_p = \left(\frac{T}{2}\right)^p \frac{(p/p^*)^{p/p^*}}{\left[\frac{\pi_{p^*}}{2} - \arctan_{p^*} \beta(L) + \arctan_{p^*} \frac{\beta(L)}{L}\right]^p}.$$

We note that in view of identity (2.22) we may write:

$$\begin{aligned}\frac{\pi_{p^*}}{2} &- \arctan_{p^*} \beta(L) + \arctan_{p^*} \frac{\beta(L)}{L} \\ &= \frac{p}{p^*} \arctan_p \left[L^{-p^*/p^2} \left(\frac{L^{p^*/p} - 1}{L - 1}\right)^{1/p}\right] + \arctan_{p^*} \left[L^{-1/p^*} \left(\frac{L - 1}{L^{p^*/p} - 1}\right)^{1/p^*}\right].\end{aligned}$$

Therefore, in the special case $p = 2$ and $T = \pi$, the best constant C_p takes the value

$$C_2 = \left(\frac{\pi}{4 \arctan L^{-1/2}}\right)^2,$$

in agreement with Piccinini and Spagnolo's result [69].

Our next result shows that Theorem 2.18 is sharp, and characterizes all extremals.

Theorem 2.19. *Inequality* (2.33) *reduces to an equality if and only if $a = \tilde{a}$, where $\tilde{a}$ is defined by*

$$\tilde{a}(t) = \begin{cases} 1 & \text{for} \quad 0 \leq t < \tilde{\tau}, \;\; T - \tilde{\tau} \leq t \leq T, \\ \\ L & \text{for} \quad \tilde{\tau} \leq t < T - \tilde{\tau}, \end{cases}$$

with

$$\tilde{\tau} = \frac{T}{2}\left(1 - \frac{\arctan_{p^*} \frac{\beta(L)}{L}}{\frac{\pi_{p^*}}{2} - \arctan_{p^*} \beta(L) + \arctan_{p^*} \frac{\beta(L)}{L}}\right), \tag{2.34}$$

and $u = \tilde{u} = \tilde{w} d$ for some $d \in \mathbb{R}^N$, where $\tilde{w}$ is the scalar function defined by

(2.35)

$$\tilde{w}(t) = \begin{cases} \left(\frac{\tilde{\lambda} p^*}{p}\right)^{-1/p} \sin_p \left[\left(\frac{\tilde{\lambda} p^*}{p}\right)^{1/p} t\right] & \text{for } 0 \leq t \leq \tilde{\tau}, \\ \left(\frac{\tilde{\lambda} p^*}{p}\right)^{-1/p} L^{-1/p} \cos_{p^*} \left[\left(\frac{p}{p^*}\right)^{1/p^*} \tilde{\lambda}^{1/p} \left(t - \frac{T}{2}\right)\right] & \text{for } \tilde{\tau} \leq t \leq T - \tilde{\tau}, \\ \left(\frac{\tilde{\lambda} p^*}{p}\right)^{-1/p} \sin_p \left[\left(\frac{\tilde{\lambda} p^*}{p}\right)^{1/p} (T - t)\right] & \text{for } T - \tilde{\tau} \leq t \leq T. \end{cases}$$

with

$$\tilde{\lambda} = C_p^{-1} = \left(\frac{2}{T}\right)^p \left(\frac{p^*}{p}\right)^{p/p^*} \left[\frac{\pi_{p^*}}{2} - \arctan_{p^*} \beta(L) + \arctan_{p^*} \frac{\beta(L)}{L}\right]^p .$$

We consider the nonlinear eigenvalue problem:

$$\begin{cases} (a(t)\psi_p(u'))' + \lambda a(t)\psi_p(u) = 0, \\ u(0) = 0, \qquad u(T) = 0 \end{cases} \tag{2.36}$$

corresponding to the Euler-Lagrange equation for (2.31). Our aim is to show that if a is smooth, then solutions to (2.36) are necessarily one-dimensional. We shall need the following uniqueness result, see [32].

Proposition 2.20. *Suppose that $\beta \in L^1_{\text{loc}}(\mathbb{R})$ with $\beta > 0$ a.e. Then, for any $\xi, \eta \in \mathbb{R}^N$ and $s_0 \in \mathbb{R}$, the problem*

$$\begin{cases} (\psi_p(v'))' + \beta(s)\psi_p(v) = 0, \\ v(s_0) = \xi, \qquad v'(s_0) = \eta \end{cases} \tag{2.37}$$

has a unique C^1 solution globally defined on $\mathbb{R}$.

The existence of a local solution is a direct application of Schauder's fixed point theorem. The main idea to prove the uniqueness is to write the equation in (2.37) in the equivalent form

$$v'(s) = \psi_{p^*}\left[\psi_p(\eta) - \int_{s_0}^{s} \beta(\theta)\psi_p(v(\theta))d\theta\right].$$

Then, a careful use of the properties of β allows to overcome the possible lack of Lipschitz continuity of the function ψ_p.

Proposition 2.21. *Let $a : [0,T] \to \mathbb{R}$ be a smooth function such that $1 \leq a(t) \leq L$ for any $t \in [0,T]$. If $u \in W_0^{1,p}([0,T], \mathbb{R}^N)$ is a weak solution of the vector eigenvalue problem*

$$(a(t)\psi_p(u'))' + \lambda a(t)\psi_p(u) = 0, \tag{2.38}$$

then $u \in C^1$ and it follows that

$$u(t) = w(t)d, \tag{2.39}$$

where $d = u'(0)$ and w is a solution of the scalar eigenvalue problem

$$\begin{cases} (a(t)\phi_p(w'))' + \lambda a(t)\phi_p(w) = 0, \\ w(0) = 0, \qquad w(T) = 0 \end{cases} \tag{2.40}$$

satisfying $w'(0) = 1$.

Proof. We first prove that if u is a solution of (2.38) then $u \in C^1$. By continuity of a, ψ_p, u and using equation (2.38), we have that $(a(t)\psi_p(u'))'$ is continuous. Therefore, $h(t) = a(t)\psi_p(u')$ belongs to $C^1([0,T], \mathbb{R}^N)$ and $\psi_p(u') = a(t)^{-1}h(t)$ is continuous. Now the claim follows by continuity of $\psi_{p^*} = \psi_p^{-1}$.
By a change of variables, we first reduce the equation in (2.38) to an equation of the form (2.37). Let us first consider the function $G : [0,T] \to [0,T]$ defined by

$$G(t) = \frac{T}{\int_0^T a^{-\frac{1}{p-1}}} \int_0^t a^{-\frac{1}{p-1}}.$$

Since $1 \le a(t) \le L$ the function G is well defined. It is easily seen that G is a nondecreasing differentiable function whose derivative is given by

$$G'(t) = \frac{T}{\int_0^T a^{-\frac{1}{p-1}}} a(t)^{-\frac{1}{p-1}}.$$

Now, suppose that u is a solution of (2.38) with $u(0) = 0$ and $u'(0) = d$; we claim that the function $v : [0,T] \to \mathbb{R}^N$ defined by

$$v(s) = u(G^{-1}(s)),$$

is a C^1 solution of the initial value problem

$$\begin{cases} (\psi_p(v'))' + \mu\alpha(s)\psi_p(v) = 0, \\ \\ v(0) = 0, \qquad v'(0) = \gamma a(0)^{\frac{1}{p-1}} d, \end{cases} \tag{2.41}$$

where

$$\alpha(s) = a(G^{-1}(s))^{p^*}, \qquad \mu = \gamma^p \lambda \qquad \gamma = \frac{1}{T}\int_0^T a^{-\frac{1}{p-1}}. \tag{2.42}$$

Indeed, it results that $u(t) = v(G(t))$ and consequently the derivative of u is given by

$$\frac{du}{dt}(t) = \gamma^{-1} a(t)^{-\frac{1}{p-1}} \frac{dv}{ds}(G(t)). \tag{2.43}$$

From (2.43) it follows that

$$\frac{d}{dt}\Big[a(t)\psi_p(u'(t))\Big] = \gamma^{-p} a(t)^{-\frac{1}{p-1}} \Big[\frac{d}{ds}\psi_p(v'(s))\Big]_{s=G(t)},$$

and therefore we obtain

$$\frac{d}{dt}\Big[a(t)\psi_p(u'(t))\Big] + \lambda a(t)\psi_p(u(t)) =$$
$$= \gamma^{-p} a(t)^{-\frac{1}{p-1}}\left[\frac{d}{ds}\psi_p(v'(s)) + \mu\alpha(s)\psi_p(v(s))\right]_{s=G(t)},$$

with α, γ and μ given by (2.42). On the other hand, the function $s \in [0,T] \mapsto \gamma a(0)^{\frac{1}{p-1}} g(s)d \in \mathbb{R}^N$, where g is the unique solution of the scalar initial value problem (see again Proposition 2.20 for $N = 1$)

$$\begin{cases} (\phi_p(g'))' + \mu\alpha(s)\phi_p(g) = 0, \\ g(0) = 0, \qquad g'(0) = 1, \end{cases}$$

is a solution of the problem (2.41). Therefore, $v(s) = \gamma a(0)^{\frac{1}{p-1}} g(s)d$. Consequently, the vector initial value problem

$$\begin{cases} (a(t)\psi_p(u'))' + \lambda a(t)\psi_p(u) = 0, \\ u(0) = 0, \qquad u'(0) = d. \end{cases}$$

has a unique C^1 solution given by $u(t) = v(G(t)) = w(t)d$ where $w(t) = \gamma a(0)^{\frac{1}{p-1}} g(G(t))$. Moreover, w is the unique C^1 solution of the scalar initial value problem

$$\begin{cases} (a(t)\phi_p(w'))' + \lambda a(t)\phi_p(w) = 0, \\ w(0) = 0, \qquad w'(0) = 1. \end{cases}$$

Since u in (2.39) also satisfies $u(T) = 0$ it must be that $w(T) = 0$; thus w is a solution to the scalar eigenvalue problem (2.40) and this completes the proof. $\square$

Remark 2.1. *Proposition 2.21 shows that the problems* (2.38) *and* (2.40) *share the same eigenvalues; moreover, it is possible to prove that they form a sequence* $\lambda_n = \lambda_n(a)$ *such that* $0 < \lambda_1(a) < \lambda_2(a) < \cdots < \lambda_n(a) < \cdots$. *Indeed, we recall that (see Section 3 in [32] when* $N \geq 1$ *and Section 2 in [84] when* $N = 1$*) for any* $\alpha \in L^1(0,T)$ *with* $\alpha > 0$ *a.e. and for any* $\mu > 0$, *a problem of the type*

$$\begin{cases} (\psi_p(v'))' + \mu\alpha(s)\psi_p(v) = 0, \\ v(0) = 0, \qquad v(T) = 0, \end{cases} \tag{2.44}$$

has a strictly monotone sequence of eigenvalues. On the other hand, the proof of Proposition 2.21 implies that λ *is an eigenvalue of* (2.38) *if and only if*

$\mu = \gamma^p \lambda$ *is an eigenvalue of* (2.44) *with* α *and* γ *as in* (2.42). *This proves the asserted property.*

Let us turn to the proofs of Theorem 2.18 and Theorem 2.19.

Proof of Theorem 2.18. By a standard approximation argument it is sufficient to prove Theorem 2.18 in the special case where $a \in \mathcal{A}$ is a smooth function. It is well known that $C_p^{-1}(a) = \lambda_1(a)$, hence the following estimate holds

$$\int_0^T a(t)|u(t)|^p dt \leq \frac{1}{\lambda_1(a)} \int_0^T a(t)|u'(t)|^p dt,$$

for every u. Therefore, in order to prove (2.33) it is sufficient to show that, if $\lambda \neq 0$ and $u \not\equiv 0$ satisfy (2.38), then necessarily

$$\lambda \geq \left(\frac{2}{T}\right)^p \left(\frac{p^*}{p}\right)^{p/p^*} \left[\frac{\pi_{p^*}}{2} - \arctan_{p^*} \beta(L) + \arctan_{p^*} \frac{\beta(L)}{L}\right]^p.$$

In view of Proposition 2.21 there exists a vector $d \in \mathbb{R}^N$ such that $u(t) = w(t)d$ where w is a solution of the scalar problem (2.40). Now we apply the arguments of Piccinini and Spagnolo [69], as extended in [36], to problem (2.40). By standard properties of eigenfunctions any solution w of (2.40) in $[0, T]$ has at least two zeros, and between any pair of zeros of w there is exactly one zero of its derivative w'. Let t_0 and t_2 be two consecutive zeros of w and let t_1 be a zero of w' in such a way that $t_0 < t_1 < t_2$. Without loss of generality we may suppose that $w(t_1) > 0$. It is obvious that

$$t_2 - t_0 \leq T. \tag{2.45}$$

We define, for $t_0 < t \leq t_1$, the function

$$f(t) = \frac{a(t)\phi_p(w'(t))}{\phi_p(w(t))}.$$

In view of (2.40) it results that f satisfies the following first order differential equation

$$f'(t) = -\lambda a(t) - \frac{p}{p^*} \frac{|f(t)|^{p^*}}{a(t)^{p^*/p}}.$$

We remark that f is strictly decreasing, since $f'(t) < 0$. Furthermore $\lim_{t \to t_0^+} f(t) = +\infty$, $f(t_1) = 0$. Hence, there is exactly one point, say τ, in the interval (t_0, t_1)

such that $f(\tau)=(\lambda p^*/p)^{1/p^*}\beta(L)$, where $\beta(L)$ is defined in (2.32). Now we prove that the following inequalities hold:

$$\text{(2.46)} \qquad \begin{cases} -\lambda a(t) - \frac{p}{p^*}\frac{|f|^{p^*}}{a^{p^*/p}} \geq -\lambda - \frac{p}{p^*}|f|^{p^*} & \text{for} \quad t_0 < t \leq \tau \\ \\ -\lambda a(t) - \frac{p}{p^*}\frac{|f|^{p^*}}{a^{p^*/p}} \geq -\lambda L - \frac{p}{p^*}\frac{|f|^{p^*}}{L^{p^*/p}} & \text{for} \quad \tau \leq t \leq t_1. \end{cases}$$

Indeed, it is readily checked that the first inequality in (2.46) is equivalent to

$$f(t)^{p^*} \geq \frac{\lambda p^*}{p}\beta^{p^*}(a(t)) \qquad \text{for} \quad t_0 < t \leq \tau$$

where the function β is defined in (2.32). Since f is decreasing and β is increasing in $(1, L)$, for $t \leq \tau$ we obtain

$$f(t)^{p^*} \geq f(\tau)^{p^*} = \frac{\lambda p^*}{p}\beta^{p^*}(L) \geq \frac{\lambda p^*}{p}\beta^{p^*}(a(t)).$$

Hence, the first inequality in (2.46) is established. On the other hand, the second inequality in (2.46) is equivalent to

$$f(t)^{p^*} \leq \frac{\lambda p^*}{p}L^{p^*/p}\gamma(a(t)) \qquad \text{for} \quad \tau \leq t \leq t_1$$

where γ is the function defined for $1 \leq a \leq L$ by

$$\gamma(a) = \frac{a^{p^*/p}(L-a)}{L^{p^*/p} - a^{p^*/p}}.$$

Since f is decreasing and γ is increasing, we have for $t \geq \tau$:

$$f(t)^{p^*} \leq f(\tau)^{p^*} = \frac{\lambda p^*}{p}\beta^{p^*}(L) = \frac{\lambda p^*}{p}L^{p^*/p}\gamma(1) \leq \frac{\lambda p^*}{p}L^{p^*/p}\gamma(a(t)).$$

Hence, the second inequality in (2.46) is also established.

Now, we prove that the Cauchy problem

$$\text{(2.47)} \qquad \begin{cases} f_0'(t) = \begin{cases} -\lambda - \frac{p}{p^*}|f_0|^{p^*} & \text{for} \quad t_0 < t \leq \tau \\ \\ -\lambda L - \frac{p}{p^*}\frac{|f_0|^{p^*}}{L^{p^*/p}} & \text{for} \quad \tau \leq t < t_1 \end{cases} \\ \\ f_0(\tau) = (\lambda p^*/p)^{1/p^*}\beta(L) \end{cases}$$

has a unique solution. Indeed, note that f_0 is strictly decreasing. Denoting by g_0 its inverse, it results that

$$(2.48)\quad \begin{cases} g_0'(s) & = \begin{cases} -\left(\lambda L + \frac{p}{p^*} L^{-p^*/p} s^{p^*}\right)^{-1} & \text{for} \quad f_0(t_1) < s \leq f_0(\tau) \\ -\left(\lambda + \frac{p}{p^*} s^{p^*}\right)^{-1} & \text{for} \quad f_0(\tau) \leq s < f_0(t_0) \end{cases} \\ g_0(f_0(\tau)) & = \tau. \end{cases}$$

Hence, there exists a unique solution for (2.48). It follows that uniqueness holds for (2.47) and that f_0 is given by:

$$(2.49)\quad f_0(t) = \begin{cases} (\frac{\lambda p^*}{p})^{1/p^*} \tan_{p^*}\left[\lambda^{1/p}\left(\frac{p}{p^*}\right)^{1/p^*}(\tau - t) + \arctan_{p^*}\beta(L)\right] \\ \qquad \text{for} \quad t_0 < t \leq \tau \\ L(\frac{\lambda p^*}{p})^{1/p^*} \tan_{p^*}\left[\lambda^{1/p}\left(\frac{p}{p^*}\right)^{1/p^*}(\tau - t) + \arctan_{p^*}\frac{\beta(L)}{L}\right] \\ \qquad \text{for} \quad \tau \leq t \leq t_1. \end{cases}$$

In particular, we obtain

$$(2.50)\quad \begin{cases} f_0(t) \geq f(t) & \text{for} \quad t \leq \tau \\ f_0(t) \leq f(t) & \text{for} \quad t \geq \tau. \end{cases}$$

Since

$$\lim_{t \to \frac{\pi_{p^*}}{2}} \tan_{p^*}(t) = +\infty$$

we have that

$$f_0(t) \to +\infty \text{ as } t \to \tau - \frac{1}{\lambda^{1/p}\left(\frac{p}{p^*}\right)^{1/p^*}}\left(\frac{\pi_{p^*}}{2} - \arctan_{p^*}\beta(L)\right)$$

and vanishes for $t = \tau + \frac{1}{\lambda^{1/p}\left(\frac{p}{p^*}\right)^{1/p^*}} \arctan_{p^*}\frac{\beta(L)}{L}$. It follows:

$$t_1 - t_0 \geq \frac{1}{\lambda^{1/p}\left(\frac{p}{p^*}\right)^{1/p^*}}\left[\frac{\pi_{p^*}}{2} - \arctan_{p^*}\beta(L) + \arctan_{p^*}\frac{\beta(L)}{L}\right].$$

In a similar way we can prove that

$$t_2 - t_1 \geq \frac{1}{\lambda^{1/p}\left(\frac{p}{p^*}\right)^{1/p^*}}\left[\frac{\pi_{p^*}}{2} - \arctan_{p^*}\beta(L) + \arctan_{p^*}\frac{\beta(L)}{L}\right];$$

hence by the relations above we derive

$$t_2 - t_0 \geq \frac{2}{\lambda^{1/p}\left(\frac{p}{p^*}\right)^{1/p^*}}\left[\frac{\pi_{p^*}}{2} - \arctan_{p^*}\beta(L) + \arctan_{p^*}\frac{\beta(L)}{L}\right].$$

So recalling (2.45), we can state

$$T \geq \frac{2}{\lambda^{1/p}\left(\frac{p}{p^*}\right)^{1/p^*}}\left[\frac{\pi_{p^*}}{2} - \arctan_{p^*}\beta(L) + \arctan_{p^*}\frac{\beta(L)}{L}\right],$$

that is

$$\lambda \geq \left\{\frac{2}{T\left(\frac{p}{p^*}\right)^{1/p^*}}\left[\frac{\pi_{p^*}}{2} - \arctan_{p^*}\beta(L) + \arctan_{p^*}\frac{\beta(L)}{L}\right]\right\}^p. \tag{2.51}$$

The proof of Theorem 2.18 is complete. □

In order to characterize the extremals as in Theorem 2.19 we shall need the following.

Lemma 2.22. *Let $u \in W_0^{1,p}([0,T],\mathbb{R}^N)$ be a weak solution of the equation*

$$(\tilde{a}(t)\psi_p(u'))' + \tilde{\lambda}\tilde{a}(t)\psi_p(u) = 0. \tag{2.52}$$

with $\tilde{\lambda}$ and $\tilde{a}$ as in Theorem 2.19. Let

$$\lim_{t\to\tilde{\tau}^-} u'(t) = u'(\tilde{\tau}^-), \qquad \lim_{t\to\tilde{\tau}^+} u'(t) = u'(\tilde{\tau}^+).$$

Then

$$u'(\tilde{\tau}^-) = L^{p^*-1}u'(\tilde{\tau}^+). \tag{2.53}$$

Proof. Since $\tilde{a}(t) \equiv 1$ in $[0,\tilde{\tau}]$ and $\tilde{a}(t) \equiv L$ in $[\tilde{\tau}, T/2]$, from (2.52) we conclude that the restrictions of u respectively to the intervals $[0,\tilde{\tau}]$ and $[\tilde{\tau}, T/2]$ are both C^1 functions. Now, we prove that $u'(\tilde{\tau}^+)$ is completely determined by $u'(\tilde{\tau}^-)$. Since u is a weak solution of (2.52) we have, for any function $\varphi \in W^{1,p}([0,T/2],\mathbb{R}^N)$

$$-\int_0^{T/2} \tilde{a}(t)\langle\psi_p(u');\varphi'\rangle = \tilde{\lambda}\int_0^{T/2} \tilde{a}(t)\langle\psi_p(u);\varphi\rangle, \tag{2.54}$$

Let $1 \le j \le N$ and $\varepsilon > 0$. In (2.54) we first choose vector valued piecewise linear test function $\varphi(t) = (\varphi_1(t), \varphi_2(t), \dots, \varphi_N(t))$ defined by

$$\varphi_k(t) = 0 \qquad \text{if } k \neq j, \quad \varphi_j(t) = \begin{cases} 0 & \text{if } 0 \le t \le \tilde{\tau} - \varepsilon, \\ \frac{1}{\varepsilon}(t - \tilde{\tau} + \varepsilon) & \text{if } \tilde{\tau} - \varepsilon \le t \le \tilde{\tau}, \\ 1, & \text{if } \tilde{\tau} \le t \le T/2. \end{cases}$$

The derivative of φ_j is given by

$$\varphi_j'(t) = \begin{cases} \frac{1}{\varepsilon}, & \text{if } \tilde{\tau} - \varepsilon \le t \le \tilde{\tau}, \\ 0, & \text{if } 0 \le t \le \tilde{\tau} - \varepsilon, \ \tilde{\tau} \le t \le T/2. \end{cases}$$

Let us set for every $1 \le j \le N$

$$\psi_{p,j}(x) = \begin{cases} |x|^{p-2} x_j & \text{if } x \in \mathbb{R}^N \setminus \{0\}, \\ 0 & \text{if } x = 0. \end{cases}$$

Hence,

$$\int_0^{T/2} \tilde{a}(t) \langle \psi_p(u'); \varphi' \rangle = \frac{1}{\varepsilon} \int_{\tilde{\tau}-\varepsilon}^{\tilde{\tau}} \psi_{p,j}(u'), \tag{2.55}$$

and in a similar way

$$\int_0^{T/2} \tilde{a}(t) \langle \psi_p(u); \varphi \rangle = \frac{1}{\varepsilon} \int_{\tilde{\tau}-\varepsilon}^{\tilde{\tau}} (t - \tilde{\tau} + \varepsilon) \psi_{p,j}(u), + L \int_{\tilde{\tau}}^{T/2} \psi_{p,j}(u). \tag{2.56}$$

By substituting (2.55) and (2.56) in (2.54) and letting $\varepsilon \to 0^+$ we obtain

$$-|u'(\tilde{\tau}^-)|^{p-2} u_j'(\tilde{\tau}^-) = \tilde{\lambda} L \int_{\tilde{\tau}}^{T/2} \psi_{p,j}(u), dt, \tag{2.57}$$

A second choice of φ, namely

$$\varphi_k(t) = 0 \qquad \text{if } k \neq j, \quad \varphi_j(t) = \begin{cases} 0 & \text{if } 0 \le t \le \tilde{\tau}, \\ \frac{1}{\varepsilon}(t - \tilde{\tau}) & \text{if } \tilde{\tau} \le t \le \tilde{\tau} + \varepsilon, \\ 1, & \text{if } \tilde{\tau} + \varepsilon \le t \le T/2 \end{cases}$$

and an argument similar to the one that yields (2.57) leads to

$$-|u'(\tilde{\tau}^+)|^{p-2} u_j'(\tilde{\tau}^+) = \tilde{\lambda} \int_{\tilde{\tau}}^{T/2} \psi_{p,j}(u), dt. \tag{2.58}$$

Thus, from (2.57) and (2.58) we have, for every $1 \le j \le N$

$$-L |u'(\tilde{\tau}^+)|^{p-2} u_j'(\tilde{\tau}^+) = -|u'(\tilde{\tau}^-)|^{p-2} u_j'(\tilde{\tau}^-),$$

and therefore

$$L \psi_p(u'(\tilde{\tau}^+)) = \psi_p(u'(\tilde{\tau}^-)).$$

From the above and from the fact that $\psi_p^{-1} = \psi_{p^*}$ we obtain (2.53). $\square$

Proof of Theorem 2.19. The inequalities (2.45), (2.46), (2.50), (2.51) in the proof of Theorem 2.18 hold strictly unless $t_0 = 0$, $t_2 = T$, $f(t) = f_0(t)$ and $a(t) = \tilde{a}(t)$. In this case the function f_0 satisfies

$$\lim_{t \to 0^+} f_0(t) = +\infty. \tag{2.59}$$

Since $\tan_{p^*}(\theta) \to +\infty$ as $\theta \to (\pi_{p^*}/2)^-$, in view of (2.49) there is a unique value of τ, denoted by $\tilde{\tau}$, such that (2.59) holds. Thus $\tilde{\tau}$ satisfies

$$\left(\frac{p}{p^*}\right)^{1/p^*} \tilde{\lambda}^{1/p} \tilde{\tau} + \arctan_{p^*} \beta(L) = \frac{\pi_{p^*}}{2}, \tag{2.60}$$

and this yields (2.34). By requiring that $f_0(t_1) = 0$ we obtain

$$\left(\frac{p}{p^*}\right)^{1/p^*} \tilde{\lambda}^{1/p} (\tilde{\tau} - t_1) + \arctan_{p^*} \frac{\beta(L)}{L} = 0, \tag{2.61}$$

and this implies $t_1 = T/2$. It remains to prove that all extremals of inequality (2.33) with $a = \tilde{a}$ are of the form $u = \tilde{u} = \tilde{w}\, d$, where $\tilde{w}$ is defined by (2.35). Hence, we seek all non-trivial solutions of the equation

$$(\tilde{a}(t)\psi_p(u'))' + \tilde{\lambda} \tilde{a}(t) \psi_p(u) = 0, \tag{2.62}$$

such that $u(0) = 0$ and $u(T) = 0$. Since $\tilde{a}(t) \equiv 1$ in $[0, \tilde{\tau}]$, in view of Proposition 2.20 (see also Lemma 3.1 in [54]) we have that, for any given $d \in \mathbb{R}^N$, there exists a unique solution $\tilde{u}$ defined in the interval $[0, \tilde{\tau}]$ of equation (2.62) satisfying the initial conditions

$$u(0) = 0, \qquad u'(0) = d.$$

Recalling the definition of $\sin_p$, we may write $\tilde{u}$ in the form

$$\tilde{u}(t) = \left(\frac{\tilde{\lambda} p^*}{p}\right)^{-1/p} \sin_p \left[\left(\frac{\tilde{\lambda} p^*}{p}\right)^{1/p} t \right] d \qquad \forall t \in [0, \tilde{\tau}].$$

Observe that

$$\tilde{u}(\tilde{\tau}^-) = \left(\frac{\tilde{\lambda} p^*}{p}\right)^{-1/p} \sin_p \left[\left(\frac{\tilde{\lambda} p^*}{p}\right)^{1/p} \tilde{\tau} \right] d. \tag{2.63}$$

In order to simplify the above expression for $\tilde{u}(\tilde{\tau}^-)$ we note that, using identity (2.18), we may write

$$\begin{aligned}\sin_p\left[\left(\frac{\tilde{\lambda}p^*}{p}\right)^{1/p}\tilde{\tau}\right] &= \sin_p\left[\frac{p^*}{p}\left(\frac{p}{p^*}\right)^{1/p^*}\tilde{\lambda}^{1/p}\tilde{\tau}\right]\\ &= \cos_{p^*}\left(\frac{\pi_{p^*}}{2}-\left(\frac{p}{p^*}\right)^{1/p^*}\tilde{\lambda}^{1/p}\tilde{\tau}\right) = \cos_{p^*}(\arctan_{p^*}\beta(L))\end{aligned}$$

where we used (2.60) in order to derive the last equality. In turn, from identity (2.17) we derive

$$|\cos_{p^*}(t)|^p = \frac{1}{1+|\tan_{p^*}(t)|^{p^*}}$$

and therefore we may write

$$\cos_{p^*}(\arctan_{p^*}\beta(L)) = \left(\frac{1}{1+\beta^{p^*}(L)}\right)^{1/p} = \left[\frac{L^{p^*/p}-1}{L^{p^*}-1}\right]^{1/p}.$$

We conclude from (2.63) and the arguments above that

$$\tilde{u}(\tilde{\tau}^-) = \left(\frac{\tilde{\lambda}p^*}{p}\right)^{-1/p}\left[\frac{L^{p^*/p}-1}{L^{p^*}-1}\right]^{1/p} d.$$

We still denote by $\tilde{u}$ the restriction of the solution of equation (2.62) to the interval $[\tilde{\tau}, T-\tilde{\tau}]$. By continuity of $\tilde{u}$,

$$\tilde{u}(\tilde{\tau}^+) = \tilde{u}(\tilde{\tau}^-) = \left(\frac{\tilde{\lambda}p^*}{p}\right)^{-1/p}\left[\frac{L^{p^*/p}-1}{L^{p^*}-1}\right]^{1/p} d. \tag{2.64}$$

Now we compute derivatives. Using (2.20), we have

$$\tilde{u}'(\tilde{\tau}^-) = \phi_{p^*}\left(\cos_p\left[\left(\frac{\tilde{\lambda}p^*}{p}\right)^{1/p}\tilde{\tau}\right]\right). \tag{2.65}$$

On the other hand, similarly as before, using (2.18) and (2.60) we compute:

$$\begin{aligned}\cos_p\left[\left(\frac{\tilde{\lambda}p^*}{p}\right)^{1/p}\tilde{\tau}\right] &= \cos_p\left[\frac{p^*}{p}\left(\frac{p}{p^*}\right)^{1/p^*}\tilde{\lambda}^{1/p}\tilde{\tau}\right]\\ &= \sin_{p^*}\left(\frac{\pi_{p^*}}{2}-\left(\frac{p}{p^*}\right)^{1/p^*}\tilde{\lambda}^{1/p}\tilde{\tau}\right) = \sin_{p^*}(\arctan_{p^*}\beta(L)).\end{aligned}$$

From the basic identity (2.17) we derive

$$|\sin_p(t)|^p = \frac{|\tan_p(t)|^p}{1+|\tan_p(t)|^p}$$

and consequently

$$\sin_{p^*}(\arctan_{p^*}\beta(L)) = \left[\frac{L^{p^*}-L^{p^*/p}}{L^{p^*-1}}\right]^{1/p^*}.$$

We conclude from (2.65) and the arguments above that

$$\tilde{u}'(\tilde{\tau}^-) = \left[\frac{L^{p^*}-L^{p^*/p}}{L^{p^*}-1}\right]^{1/p} d.$$

Now, in view of Lemma 2.22 we have

$$\tilde{u}'(\tilde{\tau}^+) = L^{-p^*/p}\left[\frac{L^{p^*}-L^{p^*/p}}{L^{p^*}-1}\right]^{1/p} d = \left[\frac{L-1}{L(L^{p^*}-1)}\right]^{1/p} d. \tag{2.66}$$

Since $\tilde{a}(t) \equiv L$ in $[\tilde{\tau}, T-\tilde{\tau}]$, again by Proposition 2.20, $\tilde{u}$ coincides in $[\tilde{\tau}, T-\tilde{\tau}]$ with the unique solution of (2.62) satisfying the initial conditions

$$u(\tilde{\tau}) = \left(\frac{\tilde{\lambda}p^*}{p}\right)^{-1/p}\left[\frac{L^{p^*/p}-1}{L^{p^*}-1}\right]^{1/p} d, \tag{2.67}$$

$$u'(\tilde{\tau}) = \left[\frac{L-1}{L(L^{p^*}-1)}\right]^{1/p} d. \tag{2.68}$$

according to (2.64) and (2.66). We claim that

$$\tilde{u}(t) = \left(\frac{\tilde{\lambda}p^*}{p}\right)^{-1/p}\frac{1}{L^{1/p}}\cos_{p^*}\left[\left(\frac{p}{p^*}\right)^{1/p^*}\tilde{\lambda}^{1/p}\left(t-\frac{T}{2}\right)\right] d \ \ \forall t \in [\tilde{\tau}, T-\tilde{\tau}].$$

Indeed, using (2.61) it follows that $\tilde{u}$ satisfies (2.67). Moreover, recalling that (see (2.19)) $p\cos'_{p^*}(t) = -p^*\phi_{p^*}(\sin_{p^*}(t))$ we have

$$\tilde{u}'(t) = -\frac{1}{L^{1/p}}\phi_{p^*}\left\{\sin_{p^*}\left[\left(\frac{p}{p^*}\right)^{1/p^*}\tilde{\lambda}^{1/p}\left(t-\frac{T}{2}\right)\right]\right\} d. \tag{2.69}$$

By similar arguments as above, we compute

$$\sin_{p^*}(\arctan_{p^*}\frac{\beta(L)}{L}) = \left(\frac{L-1}{L^{p^*}-1}\right)^{1/p^*}.$$

Hence, $\tilde{u}$ satisfies (2.68). From (2.69) we have

$$\phi_p\left(\tilde{u}'(t)\right) = -\frac{1}{L^{1/p^*}}\sin_{p^*}\left[\left(\frac{p}{p^*}\right)^{1/p^*}\tilde{\lambda}^{1/p}\left(t-\frac{T}{2}\right)\right] d. \tag{2.70}$$

Differentiating (2.70) we obtain

$$\left(\phi_p\left(\tilde{u}'(t)\right)\right)' = -\tilde{\lambda}\phi_p\left(\tilde{u}(t)\right),$$

and thus we check that $\tilde{u}$ solves (2.62) in $[\tilde{\tau}, T-\tilde{\tau}]$. By similar arguments we evaluate $\tilde{u}$ in the interval $[T-\tilde{\tau}, T]$. The proof is complete. $\square$

2.3 A concrete example

The goal of this section is to give an example of explicit non-trivial degenerate elliptic equation of its own interest. In view of this example we cannot expect to extend the Piccinini and Spagnolo argument to the case of the a p–laplacian type equation. Namely, let (ρ, θ) be the usual polar coordinates

$$\rho = \sqrt{x_1^2 + x_2^2}, \qquad \theta = \arctan\frac{x_2}{x_1}.$$

Consider the following partial differential equation

$$\frac{1}{r}\frac{\partial}{\partial\theta}\left(a(\theta)\left|\frac{1}{r}\frac{\partial u}{\partial\theta}\right|^{p-2}\frac{1}{r}\frac{\partial u}{\partial\theta}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(ra(\theta)\left|\frac{\partial u}{\partial r}\right|^{p-2}\frac{\partial u}{\partial r}\right) = 0, \tag{2.71}$$

where $p \geq 2$. Our aim is to provide a solution of (2.71) of the form

$$u(r,\theta) = r^{\alpha}w(\theta). \tag{2.72}$$

We may refer to a function of the type (2.72) as an *angular stretching* or a *quasi radial function*.

We also recall, from the result of [37], that if $\lambda > 0$ is a eigenvalue of the nonlinear problem

$$\begin{cases} (a(\theta)|w'|^{p-2}w')' + \lambda a(\theta)|w|^{p-2}w = 0 \\ w(0) = w(2\pi) \end{cases} \tag{2.73}$$

then necessarily

$$\lambda \geq \lambda_p(L) \equiv \left\{\frac{2}{\pi}\left(\frac{p^*}{p}\right)^{\frac{1}{p^*}}\left[\frac{\pi_{p^*}}{2} - \arctan_{p^*}\beta(L) + \arctan_{p^*}\frac{\beta(L)}{L}\right]\right\}^{p}. \tag{2.74}$$

Before we give the main result of this section, we need to prove the following lemma.

Lemma 2.23. *Let $p \geq 2$. The function $F : [0, \infty) \to \mathbb{R}$ defined by*

$$F(\alpha) = \alpha^{p-1} \left[(\alpha - 1)(p-1) + 1 \right],$$

is continuous and increasing for every $\alpha \geq \frac{p-2}{p-1}$. Moreover

$$F\left(\frac{p-2}{p-1}\right) = 0.$$

Proof. We compute the derivative of F, which is given by

$$F'(\alpha) = (p-1)\alpha^{p-2}(p\alpha - p + 2)$$

We deduce that $F'(\alpha) \geq 0$ if $\alpha \geq (p-2)/p$. The result follows from the fact that $(p-2)/p < (p-2)/(p-1)$. □

Proposition 2.24. *Let $p \geq 2$ and let $a = a(t)$ be a 2π–periodic measurable function such that $1 \leq a(t) \leq L$. For every λ satisfying* (2.74) *there exists a unique $\alpha \geq \frac{p-2}{p-1}$ such that the function $u(r, \theta) = r^\alpha w(\theta)$ is a solution of* (2.71)*, where w is a solution to the problem* (2.73)*. Moreover, λ and α are related by the following condition*

$$\lambda = \alpha^{p-1} \left[(\alpha - 1)(p-1) + 1 \right]. \tag{2.75}$$

Proof. The existence of a solution w to the problem (2.73) is a direct consequence of the estimate (2.74). Let $u(r, \theta) = r^\alpha w(\theta)$ be a solution of (2.71). Then, substituting the function u in (2.71) and recalling that

$$\frac{\partial u}{\partial r} = \alpha r^{\alpha - 1} w(\theta)$$

and

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = r^{\alpha - 1} w'(\theta)$$

then necessarily (2.75) holds. □

Chapter 3

Convergences for sequences of elliptic operators

In this Chapter we will discuss the G-convergence and H-convergence (in the general case of matrices not necessarily symmetric) of the operators in divergence form.

3.1 Introduction and definitions

Let Ω be a bounded open subset of $\mathbb{R}^n$. We consider the class $\mathcal{M}(K,\Omega)$ for each constant $K \geq 1$ of measurable matrix field $A : \Omega \to \mathbb{R}^{n\times n}$ such that $A = A(x) \in L^\infty(\Omega, \mathbb{R}^{n\times n})$, A is symmetric and satisfies the condition

$$(3.1) \qquad \frac{|\xi|^2}{K} \leq \langle A(x)\xi, \xi\rangle \leq K|\xi|^2 \qquad \text{a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^2$$

Let A^ε be a sequence of matrices of $M(K,\Omega)$, namely A^ε satisfies (3.1) uniformly in ε. Assume that u^ε is the unique solution of the Dirichlet boundary problem

$$\begin{cases} -\mathrm{div} A^\varepsilon \nabla u^\varepsilon = f & \text{in } \mathcal{D}'(\Omega), \\ u^\varepsilon \in W_0^{1,2}(\Omega), \end{cases}$$

with right hand side $f \in H^{-1}(\Omega)$. Here and in what follows we denote by $H^{-1}(\Omega)$ the dual space of $W_0^{1,2}(\Omega)$. It is not difficult to see that

$$\frac{1}{K}\|u^\varepsilon\|_{W_0^{1,2}(\Omega)} \leq \|f\|_{H^{-1}(\Omega)},$$

hence, up to a subsequence, we may assume that

$$u^\varepsilon \rightharpoonup u^0 \qquad \text{in } W_0^{1,2}(\Omega) \text{ weakly,}$$

for some $u^0 \in W_0^{1,2}(\Omega)$. One may ask if u^0 satisfies an equation of the same type of u^ε. In order to answer this question, the notion of *G-convergence* was introduced by De Giorgi and Spagnolo (see for [22] and [79]).

Definition 3.1. A sequence of matrices A^ε of $M(K,\Omega)$ is said to G-converge to a matrix A^0 of $M(K,\Omega)$ if, for every $f \in H^{-1}(\Omega)$, the solution u^ε of the problem

$$\begin{cases} -\mathrm{div} A^\varepsilon \nabla u^\varepsilon = f & \text{in } \mathcal{D}'(\Omega), \\ u^\varepsilon \in W_0^{1,2}(\Omega), \end{cases}$$

satisfies

$$u^\varepsilon \rightharpoonup u^0 \qquad \text{in } W_0^{1,2}(\Omega) \text{ weakly,}$$

where u^0 is the solution of the problem

$$\begin{cases} -\mathrm{div} A^0 \nabla u^0 = f & \text{in } \mathcal{D}'(\Omega), \\ u^0 \in W_0^{1,2}(\Omega). \end{cases}$$

In this case one writes

$$A^\varepsilon \xrightarrow{G} A^0.$$

One of the properties of the G–convergence, which explains the interest of Definition 3.1, is following fundamental compactness result, which can be found in [79].

Theorem 3.1. *Any sequence of matrices A^ε of $M(K,\Omega)$ admits a subsequence which G-converges to a matrix A^0 of $M(K,\Omega)$.*

We recall here some well known facts for the G-convergence.

Lemma 3.2. *Let Ω be a bounded open subset of $\mathbb{R}^n$, let $A^\varepsilon = (a_{ij}^\varepsilon)$ and $A^0 = (a_{ij}^0)$ be matrices in $\mathcal{M}(K,\Omega)$.*

(i) If for every $i,j = 1,\dots,n$

$$a_{ij}^\varepsilon \to a_{ij}^0 \qquad \textit{in } L^1_{\mathrm{loc}}(\Omega) \textit{ strongly,}$$

then $A^\varepsilon \xrightarrow{G} A^0$.

(ii) Let $n = 1$ and let Ω be an open interval of $\mathbb{R}$. Then

$$A^\varepsilon \xrightarrow{G} A^0 \qquad \text{if and only if} \qquad \frac{1}{A^\varepsilon} \rightharpoonup \frac{1}{A^0} \qquad \text{in } L^\infty(\Omega) \text{ weakly } *.$$

(iii) Let $A^\varepsilon \xrightarrow{G} A^0$ and $f^\varepsilon \to f$ in $H^{-1}(\Omega)$ strongly. If u^ε and u^0 satisfy

$$\begin{cases} -\mathrm{div} A^\varepsilon \nabla u^\varepsilon = f^\varepsilon & \text{in } \mathcal{D}'(\Omega), \\ u^\varepsilon \in W_0^{1,2}(\Omega), \end{cases}$$

and

$$\begin{cases} -\mathrm{div} A^0 \nabla u^0 = f & \text{in } \mathcal{D}'(\Omega), \\ u^0 \in W_0^{1,2}(\Omega). \end{cases}$$

then

$$u^\varepsilon \rightharpoonup u^0 \qquad \text{in } W_0^{1,2}(\Omega) \text{ weakly.}$$

The notion of G-convergence has been extended to the non–symmetric case by Murat and Tartar under the name of H*-convergence* (see [66]). Before we give the definition, we introduce the class of matrices $M(\alpha, \beta, \Omega)$, where $0 < \alpha \le \beta < +\infty$ of 2×2 matrices A which belongs to $A \in (L^\infty(\Omega))^{n \times n}$ and satisfies

$$\langle A(x)\xi, \xi\rangle \ge \alpha |\xi|^2 \qquad \text{a.e. } x \in \Omega \qquad \forall \xi \in \mathbb{R}^n, \tag{3.2}$$

$$\langle A^{-1}(x)\zeta, \zeta\rangle \ge \beta^{-1} |\zeta|^2 \qquad \text{a.e. } x \in \Omega \qquad \forall \zeta \in \mathbb{R}^n. \tag{3.3}$$

Observe that, in view of (3.2), the matrix $A(x)$ is invertible a.e. so that $A^{-1}(x)$ exists and is measurable. Observe also that taking $\zeta = A(x)\xi$ in (3.3) one has

$$|A(x)\xi| \le \beta |\xi| \qquad \text{a.e. } x \in \Omega \qquad \forall \xi \in \mathbb{R}^n. \tag{3.4}$$

Moreover, we are making no symmetric assumption on the elements of $M(\alpha, \beta, \Omega)$.

Definition 3.2. Let α and β be real numbers such that $0 < \alpha \le \beta < +\infty$ and let Ω be a bounded open subset of $\mathbb{R}^2$. A sequence of matrices A^ε of $M(\alpha, \beta, \Omega)$ is said to H-converge to a matrix A of $M(\alpha, \beta, \Omega)$ if, for every $f \in H^{-1}(\Omega)$, the solution u^ε of the problem

$$\begin{cases} -\mathrm{div} A^\varepsilon \nabla u^\varepsilon = f & \text{in } \mathcal{D}'(\Omega), \\ u^\varepsilon \in W_0^{1,2}(\Omega), \end{cases}$$

satisfies

$$\begin{cases} u^\varepsilon \rightharpoonup u^0 & \text{in } W_0^{1,2}(\Omega) \text{ weakly,} \\ A^\varepsilon \nabla u^\varepsilon \rightharpoonup A^0 \nabla u^0 & \text{in } (L^2(\Omega))^n \text{ weakly,} \end{cases}$$

where u^0 is the solution of the problem

$$\begin{cases} -\mathrm{div} A^0 \nabla u^0 = f & \text{in } \mathcal{D}'(\Omega), \\ u^0 \in W_0^{1,2}(\Omega). \end{cases}$$

In this case one writes

$$A^\varepsilon \xrightarrow{H} A^0.$$

The class $M(\alpha, \beta, \Omega)$ is sequentially compact with respect to the H–convergence.

Theorem 3.3. *Let α and β be real numbers such that $0 < \alpha \leq \beta < +\infty$ and let Ω be a bounded open subset of $\mathbb{R}^n$, with $n \geq 1$. Any sequence of matrices A^ε of $M(\alpha, \beta, \Omega)$ admits a subsequence which H-converges to a matrix A^0 of $M(\alpha, \beta, \Omega)$.*

3.2 The class of matrices with unit determinant

We denote by $M_1(\alpha, \beta, \Omega)$ whose elements are the matrices $A \in M(\alpha, \beta, \Omega)$ which satisfies the condition

$$\det A(x) = 1 \qquad \text{a.e. } x \in \Omega$$

This class is stable under H-convergence. This is a consequence of the following result, whose proof can be found in [24, 47, 30, 57, 59, 81].

Theorem 3.4. *Let Ω be a bounded open subset of $\mathbb{R}^2$ and let A^ε be a sequence of matrices of $M(\alpha, \beta, \Omega)$ which H-converges to a matrix A^0. Then*

$$\frac{A^\varepsilon}{\det A^\varepsilon} \xrightarrow{H} \frac{A}{\det A}.$$

It should be mentioned that the previous result is true only in dimension $n = 2$.

We would like to prove here a result strictly related to Theorem 3.4, which can be found in [26].

Theorem 3.5. *Let Ω be a bounded open subset of $\mathbb{R}^2$ and let A^ε be a sequence of matrices of $M(\alpha, \beta, \Omega)$ which H-converges to a matrix A^0. Assume that*

$$\det A^\varepsilon \to c^0 \quad a.e. \ in \ \Omega, \tag{3.5}$$

where c^0 is a function in $L^\infty(\Omega)$. Then

$$\det A^0 = c^0. \tag{3.6}$$

One of the key ingredients of the proof of Theorem 3.5 is following result.

Theorem 3.6. *Let Ω be a bounded open subset of $\mathbb{R}^N$ with $N \geq 1$ and let A^ε be a sequence of matrices of $M(\alpha, \beta, \Omega)$ which H-converges to a matrix A^0. Assume that b^ε is a sequence of measurable functions such that*

$$m \leq b^\varepsilon(x) \leq M \qquad a.e. \ x \in \Omega, \tag{3.7}$$

where $0 < m \leq M < +\infty$ and

$$b^\varepsilon \to b^0 \quad a.e. \ in \ \Omega. \tag{3.8}$$

Then

$$b^\varepsilon A^\varepsilon \xrightarrow{H} b^0 A^0. \tag{3.9}$$

Proof. We divide the proof in two steps.
Step 1. Assume first that, further to (3.7) and (3.8), one has

$$b^\varepsilon \in C^1(\overline{\Omega}), \ b^0 \in C^1(\overline{\Omega}), \quad b^\varepsilon \to b^0 \quad \text{in } C^1(\overline{\Omega}) \text{ strongly.} \tag{3.10}$$

We claim that in this case the sequence $b^\varepsilon A^\varepsilon$ H-converges to $b^0 A^0$, i.e. that for every $f \in H^{-1}(\Omega)$, the solution u^ε of the problem

$$\begin{cases} -\operatorname{div}(b^\varepsilon A^\varepsilon \nabla u^\varepsilon) = f & \text{in } \mathcal{D}'(\Omega), \\ u^\varepsilon \in W_0^{1,2}(\Omega), \end{cases} \tag{3.11}$$

satisfies

$$\begin{cases} u^\varepsilon \rightharpoonup u^0 & \text{in } W_0^{1,2}(\Omega) \ , \\ b^\varepsilon A^\varepsilon \nabla u^\varepsilon \rightharpoonup b^0 A^0 \nabla u^0 & \text{in } (L^2(\Omega))^N, \end{cases} \tag{3.12}$$

where u^0 is the solution of the problem

$$(3.13)\qquad \begin{cases} -\mathrm{div}(b^0 A^0 \nabla u^0) = f \quad \text{in } \mathcal{D}'(\Omega), \\ u^0 \in W_0^{1,2}(\Omega). \end{cases}$$

Actually it is sufficient to prove this result for $f \in L^2(\Omega)$.

To this end, we observe that

$$(3.14)\qquad -\mathrm{div}(b^\varepsilon A^\varepsilon \nabla u^\varepsilon) = -b^\varepsilon \mathrm{div}(A^\varepsilon \nabla u^\varepsilon) - \langle A^\varepsilon \nabla u^\varepsilon, \nabla b^\varepsilon \rangle,$$

where $b^\varepsilon \mathrm{div}(A^\varepsilon \nabla u^\varepsilon) \in H^{-1}(\Omega)$ is defined by

$$\langle b^\varepsilon \mathrm{div}(A^\varepsilon \nabla u^\varepsilon), v \rangle = \langle \mathrm{div}(A^\varepsilon \nabla u^\varepsilon), b^\varepsilon v \rangle \qquad \forall v \in W_0^{1,2}(\Omega).$$

(Note that $b^\varepsilon v \in W_0^{1,2}(\Omega)$ for every $v \in H_0^1(\Omega)$ when $b^\varepsilon \in C^1(\overline{\Omega})$; this proves that the distribution $b^\varepsilon \mathrm{div}(A^\varepsilon D u^\varepsilon)$ is well-defined as an element of $H^{-1}(\Omega)$.)

Set

$$g^\varepsilon = \frac{f + \langle A^\varepsilon \nabla u^\varepsilon, \nabla b^\varepsilon \rangle}{b^\varepsilon}.$$

Since u^ε is the solution of the problem (3.11), the sequence u^ε is bounded in $W_0^{1,2}(\Omega)$. We can assume that (up to a subsequence)

$$u^\varepsilon \rightharpoonup u \text{ in } W_0^{1,2}(\Omega) \text{ weakly},$$

for some $u \in W_0^{1,2}(\Omega)$. Since $A^\varepsilon \in M(\alpha, \beta, \Omega)$, from (3.4) it follows that $A^\varepsilon \nabla u^\varepsilon$ is bounded in $L^2(\Omega)$. This proves that g^ε is bounded in $L^2(\Omega)$ and that (up to a subsequence)

$$g^\varepsilon \rightharpoonup g \ \text{ in } L^2(\Omega),$$

for some $g \in L^2(\Omega)$. We now observe that u^ε is the solution of the problem

$$\begin{cases} -\mathrm{div}(A^\varepsilon \nabla u^\varepsilon) = g^\varepsilon \quad \text{in } \mathcal{D}'(\Omega), \\ u^\varepsilon \in W_0^{1,2}(\Omega). \end{cases}$$

Since A^ε is assumed to H-converges to A^0 and since g^ε converges to g in $L^2(\Omega)$ weakly (and therefore in $H^{-1}(\Omega)$ strongly), we deduce that (up to a subsequence)

$$(3.15)\qquad A^\varepsilon \nabla u^\varepsilon \rightharpoonup A^0 \nabla u \ \text{ in } (L^2(\Omega))^N \text{ weakly},$$

where u is the solution of the problem

$$\text{(3.16)} \qquad \begin{cases} -\mathrm{div}(A^0 \nabla u) = g & \text{in } \mathcal{D}'(\Omega), \\ u \in W_0^{1,2}(\Omega). \end{cases}$$

In view of (3.15) and of the strong convergence (3.10), we have

$$\text{(3.17)} \qquad g = \frac{f + \langle A^0 \nabla u, \nabla b^0 \rangle}{b^0}.$$

Similarly to (3.14) we have, since $b^0 \in C^1(\overline{\Omega})$,

$$-\mathrm{div}(b^0 A^0 \nabla u) = -b^0 \mathrm{div}(A^0 \nabla u) - \langle A^0 \nabla u, \nabla b^0 \rangle,$$

so that (3.16) and (3.17) imply that u is the solution of the problem

$$\begin{cases} -\mathrm{div}(b^0 A^0 \nabla u) = f & \text{in } \mathcal{D}'(\Omega), \\ u \in W_0^{1,2}(\Omega). \end{cases}$$

This implies that u coincides with u^0 defined by (3.13) and that the convergences (3.12) hold for the whole sequence ε; indeed, we do not have to extract any subsequence since the limits $u, A^0 Du$ and g are uniquely defined.

We have proved the result of Theorem 3.6 when hypothesis (3.10) holds true.

Step 2. We now prove the assertion in the general case, i.e. when only (3.7) and (3.8) hold true. In view of Theorem 3.3 we assume that (up to a subsequence) the sequence of matrices $b^\varepsilon A^\varepsilon$ of $M(\alpha m, \beta M, \Omega)$ satisfies

$$\text{(3.18)} \qquad b^\varepsilon A^\varepsilon \xrightarrow{H} B^0,$$

for some B^0 of $M(\alpha m, \beta M, \Omega)$.

Extend b^ε and b^0 to the whole of $\mathbb{R}^N$ by

$$b^\varepsilon(x) = b^0(x) = m \qquad \forall x \in \mathbb{R}^N \setminus \Omega.$$

Let ϱ_δ be a mollifier and let $b^\varepsilon * \varrho_\delta$ be the convolution of b^ε and ϱ_δ. Since for $\delta > 0$ fixed we have

$$b^\varepsilon * \varrho_\delta \to b^0 * \varrho_\delta \quad \text{in } C^1(\overline{\Omega}) \text{ strongly},$$

the result of the first step proves that for every $\delta > 0$ fixed

$$(b^\varepsilon * \varrho_\delta)A^\varepsilon \xrightarrow{H} (b^0 * \varrho_\delta)A^0. \tag{3.19}$$

On the other hand, since the sequence A^ε is equi-bounded in $L^\infty(\Omega)$ (see (3.4)) we have

$$|b^\varepsilon A^\varepsilon - (b^\varepsilon * \varrho_\delta)A^\varepsilon| \leq \gamma_\delta^\varepsilon, \tag{3.20}$$

where $\gamma_\delta^\varepsilon$ is the function defined by

$$\gamma_\delta^\varepsilon = \beta \left| b^\varepsilon - (b^\varepsilon * \varrho_\delta) \right|,$$

for every $\delta > 0$ fixed. Hypothesis (3.8) implies that

$$\gamma_\delta^\varepsilon \to \gamma_\delta^0 \quad \text{a.e. in } \Omega, \tag{3.21}$$

where γ_δ^0 is the function defined by

$$\gamma_\delta^0 = \beta |b^0 - (b^0 * \varrho_\delta)|, \tag{3.22}$$

for every $\delta > 0$ fixed. Then (3.18), (3.19), (3.20), (3.21) and Theorem 3.1 in [8] imply for every $\delta > 0$ fixed

$$|B^0 - (b^0 * \varrho_\delta)A^0| \leq \gamma_\delta^0. \tag{3.23}$$

The fact that $b^0 * \varrho_\delta$ tends to b^0 a.e. as δ tends to zero, (3.22) and (3.23) imply then that $B^0 = b^0 A^0$.

This concludes the proof of Theorem 3.6. □

Proof of Theorem 3.5. Define

$$b^\varepsilon = \frac{1}{\det A^\varepsilon}.$$

In view of hypothesis (3.5) we have

$$b^\varepsilon \to b^0 = \frac{1}{c^0} \quad \text{a.e. in } \Omega.$$

Applying the result of Theorem 3.6, the sequence $b^\varepsilon A^\varepsilon$ H-converges to $b^0 A^0 = \frac{A^0}{c^0}$. Since here the dimension is $N = 2$, Theorem 3.4 implies that

$$\frac{A^\varepsilon}{\det A^\varepsilon} \xrightarrow{H} \frac{A^0}{\det A^0}.$$

Since the H-limit is unique, it results that

$$\frac{A^0}{c^0} = \frac{A^0}{\det A^0},$$

and therefore

$$c^0 = \det A^0.$$

This proves Theorem 3.5. $\square$

3.3 Quasiconformal mappings and approximation of the inverse matrix

Let us suppose that A^ε is a sequence in $\mathcal{M}(K, \mathbb{R}^2)$ and that $\det A^\varepsilon = 1$ a.e. in Ω. Up to a subsequence, we may assume that both

$$A^\varepsilon \xrightarrow{G} A$$

and

$$(A^\varepsilon)^{-1} \xrightarrow{G} B$$

G–converges to some A and B in $\mathcal{M}(K, \mathbb{R}^2)$. In general, B is different from A. However, the following result can be obtained, performing a suitable change of variables (see [61]).

Theorem 3.7. *Let A^ε be a sequence of matrices in $\mathcal{M}(K, \mathbb{R}^2)$ such that*

$$\det A^\varepsilon = 1.$$

Assume that $A^\varepsilon \xrightarrow{G} A$ for some $A \in \mathcal{M}(K, \mathbb{R}^2)$. Let B be any open ball in $\mathbb{R}^2$ and let $\hat{A}^\varepsilon$ and $\hat{A}$ be the matrices defined as

$$\hat{A}^\varepsilon(x) = \begin{cases} A^\varepsilon(x) & \textit{if } x \in B \\ I & \textit{otherwise.} \end{cases} \qquad \hat{A}(x) = \begin{cases} A(x) & \textit{if } x \in B \\ I & \textit{otherwise.} \end{cases}$$

where I denotes the identity matrix. Then, there exists a sequence of K–quasiconformal mappings $f^\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$ which converges locally uniformly to a K–quasiconformal mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$(\hat{A}^\varepsilon)^{-1} \circ (f^\varepsilon)^{-1} \xrightarrow{G} (A)^{-1} \circ f^{-1}.$$

We want to point out here that the result in specific of dimension $n = 2$.

3.4 Beltrami operators

For any fixed $K \geq 1$, let $\mathcal{F}(K)$ of the class of operators of the type

$$\mathcal{B} = \frac{\partial}{\partial \bar{z}} - \mu \frac{\partial}{\partial z} - \nu \overline{\frac{\partial}{\partial z}},$$

where μ and ν are function in $L^\infty(\mathbb{C})$ such that

$$|\mu(z)| + |\nu(z)| \leq k = \frac{K-1}{K+1}.$$

We say that an element of $\mathcal{F}(K)$ is a *Beltrami operators.* We follow [5] and give the notion of G–convergence for the Beltrami operators.

Definition 3.3. The sequence of differential operator $\mathcal{B}^\varepsilon$ in $\mathcal{F}(K)$ is said to G-converge to a operator $\mathcal{B}$ in $\mathcal{F}(K)$ if for any sequence $f^\varepsilon \in L^2(\mathbb{C})$ which converges strongly to $f \in L^2(\mathbb{C})$ and such that $\mathcal{B}^\varepsilon f^\varepsilon$ converges strongly $L^2(\Omega, \mathbb{R}^2)$ one has

$$(\mathcal{B}^\varepsilon)^{-1} f^\varepsilon \rightharpoonup \mathcal{B}^{-1} f \qquad \text{weakly in } L^2(\mathbb{C}).$$

The following compactness result is proved in [44].

Theorem 3.8. *For every $1 \leq K < 3$ the class $\mathcal{F}(K)$ is G–compact, in the sense that any sequence of operators $\mathcal{B}^\varepsilon$ in $\mathcal{F}(K)$ has a subsequence which G–converges to some $\mathcal{B}$ in $\mathcal{F}(K)$.*

3.5 Examples of G–dense classes

We dedicate this section to fundamental examples of classes that are compact or dense with respect to the G-convergence.

Now, we mention the result of Marino and Spagnolo [56] which is true in every dimension n. They prove that every elliptic matrix $A \in \mathcal{M}(K, \Omega)$ is the G-limit of a sequence of isotropic matrices of the type

$$A^\varepsilon(x) = \begin{pmatrix} \beta^\varepsilon(x) & 0 \\ 0 & \beta^\varepsilon(x) \end{pmatrix}$$

Theorem 3.9. *Let Ω be a bounded open subset of $\mathbb{R}^n$ and $K \geq 1$. If $A = A(x) \in \mathcal{M}(K, \Omega)$ then there exists a constant c depending only on n and a sequence of coefficients $\beta^\varepsilon = \beta^\varepsilon(x)$ satisfying*

$$\frac{1}{cK} \leq \beta^\varepsilon(x) \leq cK,$$

such that

$$\beta^\varepsilon I \xrightarrow{G} A,$$

where I is the $n \times n$ identity matrix.

Every 2×2 matrix A which satisifies the additional assumption $\det A(x) = 1$ can be approximated in the sense of the G–convergence by a sequence of anisotropic matrices

$$A^\varepsilon = \begin{pmatrix} \gamma^\varepsilon(x) & 0 \\ 0 & \frac{1}{\gamma^\varepsilon(x)} \end{pmatrix}$$

provided some elliptic bound is satisfied.

Theorem 3.10 ([62]). *Let Ω be a bounded open subset of $\mathbb{R}^n$ and $K \geq 1$. assume that $A = A(x) \in \mathcal{M}(K, \Omega)$ and*

$$\det A(x) = 1 \quad a.e. \ in\ \Omega$$

There exists a sequence γ^ε satisfying

$$\frac{1}{K} \leq \gamma^\varepsilon(x) \leq K$$

such that

$$\begin{pmatrix} \gamma^\varepsilon(x) & 0 \\ 0 & \frac{1}{\gamma^\varepsilon(x)} \end{pmatrix} \xrightarrow{G} A$$

if and only if A satisfies

$$\frac{|\xi|^2}{\frac{1}{2}\left(K + \frac{1}{K}\right)} \leq \langle A(x)\xi, \xi \rangle \leq \frac{1}{2}\left(K + \frac{1}{K}\right)|\xi|^2.$$

Chapter 4

Variational integrals

4.1 Classical semicontinuity result

Let Ω be a bounded open subset of $\mathbb{R}^n$. In this section we consider functionals J of the integral form

$$J(v) = \int_\Omega F(x, v, \nabla v)dx \quad \forall v \in W^{1,p}(\Omega), \tag{4.1}$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function satisfying

$$a_0(x) + c_0|\xi|^p \leq F(x, s, \xi) \leq a_1(x) + b_1|s|^p + c_1|\xi|^p, \tag{4.2}$$

with $p > 1$, $c_0 > 0$ and $a_0, a_1 \in L^1(\Omega)$.

Observe that, for $p = 2$, an example of a functional which satisfies condition (4.2) is the *A–harmonic energy*

$$\mathcal{E}_A(u) = \int_\Omega \langle A(x)\nabla u, \nabla u\rangle dx \tag{4.3}$$

where $A = A(x) \in L^\infty(\Omega, \mathbb{R}^{n\times n})$ is a symmetric matrix satisfying, for some $K \geq 1$, the usual bounds

$$\frac{|\xi|^2}{K} \leq \langle A(x)\xi, \xi\rangle \leq K|\xi|^2 \tag{4.4}$$

Such an example is relevant in connection with the Dirichlet problem

$$\begin{cases} -\mathrm{div}(A\nabla u) = 0 \quad \text{in } \mathcal{D}'(\Omega), \\ u \in W_0^{1,2}(\Omega), \end{cases} \tag{4.5}$$

because any minimizer $u \in W_0^{1,2}(\Omega)$ of (4.3) is the unique solution of (4.5)

Since condition (4.4) is fulfilled, the energy (4.3) is equivalent to the quantity

$$\|u\|^2_{W_0^{1,2}(\Omega)} = \int_\Omega |\nabla u(x)|^2 dx$$

in the sense of the following estimates

$$\frac{1}{K}\int_\Omega |\nabla u(x)|^2 \le \mathcal{E}_A(u) \le K\int_\Omega |\nabla u(x)|^2. \tag{4.6}$$

The functional $\mathcal{E}_A$ has indeed quadratic growth with respect to $\|\nabla u\|_2 = \|\nabla u\|_{(L^2(\Omega))^2}$. For general functionals of the type (4.1), the following classical result in the Calculus of Variations holds (see for instance [17],[55]).

Theorem 4.1. *The functional J in* (4.1) *where $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying* (4.2) *is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega)$ if and only if $F(x, s, \cdot)$ is a convex function for a.e. $x \in \Omega$ and for every $s \in \mathbb{R}$.*

We recall that a functional J defined in $W^{1,p}(\Omega)$ is said to be weakly lower semicontinuous on $W^{1,p}(\Omega)$ if

$$J(v) \le \liminf_{k\to\infty} J(v_k) \qquad \text{if } v_k \rightharpoonup v \text{ in weakly in } W^{1,p}(\Omega).$$

4.2 Some examples

Let Ω be a bounded open subset of $\mathbb{R}^N$, with $0 \in \Omega$ if $N \ge 2$, and $\Omega = (0, R_0)$ if $N = 1$. In this section, we give an example of functional which is defined and coercive on $W_0^{1,2}(\Omega)$, which has quadratic growth with respect to $\|\nabla v\|_2 = \|\nabla v\|_{(L^2(\Omega))^n}$, which is sequentially weakly lower semicontinuous on $W_0^{1,p}(\Omega)$ for every $p > 2$, but which is not sequentially weakly lower semicontinuous on $W_0^{1,2}(\Omega)$.

More precisely, when $n \ge 3$, we recall the Hardy-Sobolev inequality (see e.g. Theorems 21.7 and 21.8 in [68], Lemma 17.1 in [81])

$$m_n^2 \int_{\mathbb{R}^n} \frac{|v|^2}{|x|^2} dx \le \int_{\mathbb{R}^n} |\nabla v|^2 dx \qquad \forall v \in W_0^{1,2}(\mathbb{R}^n), \tag{4.7}$$

where m_n^2 denotes the best possible constant in the inequality, i.e.

$$m_n^2 = \inf_{v \in W_0^{1,2}(\mathbb{R}^n)} \frac{\displaystyle\int_{\mathbb{R}^n} |\nabla v|^2 dx}{\displaystyle\int_{\mathbb{R}^n} \frac{|v|^2}{|x|^2} dx}. \tag{4.8}$$

It is well known that m_n^2 is given by (see the references above)

$$m_n^2 = \frac{(n-2)^2}{4}.$$

We consider a function φ which is defined and continuous on $[0, \infty]$, which is non negative and decreasing and which satisfies

$$\varphi(0) > m_n^2 \text{ and } \varphi(\infty) < \frac{m_n^2}{2}. \tag{4.9}$$

Finally we define the functional J by

$$J(v) = \int_\Omega |\nabla v|^2 dx - \varphi(\|\nabla v\|_2^2) \int_\Omega \frac{|v|^2}{|x|^2} dx \qquad \forall v \in W_0^{1,2}(\Omega). \tag{4.10}$$

Our result is the following

Theorem 4.2. *Let $n \geq 3$ and let Ω be a bounded open subset of $\mathbb{R}^n$, with $0 \in \Omega$. Assume that φ is a continuous, non negative and decreasing function on $[0, \infty]$ satisfying (4.9), where m_n^2 is given by (4.8). Then the functional J defined by (4.10) satisfies*

(i) *there exists a constant $C > 0$ such that*

$$-C + \frac{1}{2} \int_\Omega |\nabla v|^2 dx \leq J(v) \leq \int_\Omega |\nabla v|^2 dx \qquad \forall v \in W_0^{1,2}(\Omega); \tag{4.11}$$

(ii) *the functional J is sequentially weakly lower semicontinuous on $W_0^{1,p}(\Omega)$ for every $p > 2$, i.e.*

$$J(v) \leq \liminf_{k \to \infty} J(v_k) \text{ if } v_k \rightharpoonup v \text{ in } W_0^{1,p}(\Omega) \text{ weakly}; \tag{4.12}$$

(iii) *the functional J is not sequentially weakly lower semicontinuous on $W_0^{1,2}(\Omega)$; more precisely, there exists a sequence of functions $w_k \in W_0^{1,2}(\Omega)$ such that $w_k \rightharpoonup 0$ in $W_0^{1,2}(\Omega)$ weakly and*

$$\liminf_{k \to \infty} J(w_k) < J(0). \tag{4.13}$$

Proof. We start by proving (i). By the definition of $J(v)$ we have

$$J(v) \leq \int_\Omega |\nabla v|^2 dx,$$

since φ is non negative.

It remains to prove the first inequality of (4.11). Since φ is continuous and satisfies (4.9), there exists $t_0 > 0$ such that $\varphi(t_0) = m_n^2/2$.

If $\|\nabla v\|_2^2 \geq t_0$ then $\varphi(\|\nabla v\|_2^2) \leq m_n^2/2$. Therefore

$$\begin{aligned} J(v) &\geq \int_\Omega |\nabla v|^2 dx - \frac{m_n^2}{2} \int_\Omega \frac{|v|^2}{|x|^2} dx \\ &\geq \int_\Omega |\nabla v|^2 dx - \frac{1}{2} \int_\Omega |\nabla v|^2 dx \\ &\geq \frac{1}{2} \int_\Omega |\nabla v|^2 dx, \end{aligned}$$

and the first inequality of (4.11) holds.

On the other hand, if $\|Dv\|_2^2 \leq t_0$, then

$$\begin{aligned} J(v) &\geq \int_\Omega |\nabla v|^2 dx - \varphi(0) \int_\Omega \frac{|v|^2}{|x|^2} dx \\ &\geq \int_\Omega |\nabla v|^2 dx - \frac{\varphi(0)}{m_n^2} \int_\Omega |\nabla v|^2 dx \\ &\geq \left(1 - \frac{\varphi(0)}{m_n^2} \right) \int_\Omega |\nabla v|^2 dx \\ &\geq \left(1 - \frac{\varphi(0)}{m_n^2} \right) t_0, \end{aligned}$$

in view of (4.9). If we choose a constant C such that

$$\frac{\varphi(0)}{m_N^2} t_0 \leq C,$$

we have

$$J(v) \geq t_0 - \frac{\varphi(0)}{m_n^2} t_0 \geq \int_\Omega |\nabla v|^2 dx - C,$$

and the first inequality of (4.11) is again proved. This proves (i).

Now we prove (ii). Let $p > 2$. Assume that $v_k \rightharpoonup v$ in $W_0^{1,p}(\Omega)$ weakly. Since Ω is bounded, $v_k \rightharpoonup v$ in $W_0^{1,2}(\Omega)$ weakly and there exists $\alpha \geq 0$ such that

$$\liminf_{k \to \infty} \|\nabla v_k\|_2^2 = \|\nabla v\|_2^2 + \alpha. \tag{4.14}$$

Since φ is continuous and decreasing, there exists some $\beta \geq 0$ such that

$$\liminf_{k\to\infty} -\varphi(\|\nabla v_k\|_2^2) = -\varphi(\|\nabla v\|_2^2) + \beta. \tag{4.15}$$

Moreover, by the compactness of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^2\left(\Omega; \frac{1}{|x|^2}dx\right)$ for $p > 2$ we get

$$\lim_{k\to\infty} \int_\Omega \frac{|v_k|^2}{|x|^2} dx = \int_\Omega \frac{|v|^2}{|x|^2} dx. \tag{4.16}$$

Combining (4.14), (4.15) and (4.16), we obtain

$$\liminf_{k\to\infty} J(v_k) \geq J(v) + \alpha + \beta \int_\Omega \frac{|v|^2}{|x|^2} dx \geq J(v),$$

which proves (ii).

Finally we prove (iii). Let λ be such that $m_n^2 < \lambda < \varphi(0)$ (such a λ exists in view of (4.9)). Recalling the definition (4.8) of m_n^2, there exists a function $\psi \in C_0^\infty(\mathbb{R}^n)$ such that

$$\lambda \int_{\mathbb{R}^n} \frac{|\psi|^2}{|x|^2} dx > \int_{\mathbb{R}^n} |\nabla \psi|^2 dx.$$

Since φ is continuous and satisfies (4.9), there exists $t_1 > 0$ such that $\varphi(t_1) = \lambda$. Take s such that $0 < s^2 \|\nabla\psi\|_2^2 \leq t_1$. The function $w = s\psi$ belongs to $C_0^\infty(\mathbb{R}^n)$ and satisfies

$$\varphi(\|\nabla w\|_2^2) \geq \lambda, \tag{4.17}$$

as well as

$$\lambda \int_{\mathbb{R}^n} \frac{|w|^2}{|x|^2} dx > \int_{\mathbb{R}^n} |\nabla w|^2 dx. \tag{4.18}$$

Define the sequence w_k by

$$w_k(x) = k^{\frac{n-2}{2}} w(kx);$$

then

$$\nabla w_k(x) = k^{\frac{n}{2}} \nabla w(kx).$$

For k sufficiently large, the function w_k belongs to $W_0^{1,2}(\Omega)$ and

$$\int_\Omega |\nabla w_k|^2 dx = \int_{\mathbb{R}^n} |\nabla w|^2 dx \quad \text{and} \quad \int_\Omega \frac{|w_k|^2}{|x|^2} dx = \int_{\mathbb{R}^n} \frac{|w|^2}{|x|^2} dx.$$

Therefore, for k sufficiently large, the sequence w_k is bounded in $W_0^{1,2}(\Omega)$ with $w_k \rightharpoonup 0$ in $W_0^{1,2}(\Omega)$ weakly, and

$$J(w_k) = \int_{\mathbb{R}^n} |\nabla w|^2 dx - \varphi(\|\nabla w\|_2^2) \int_{\mathbb{R}^n} \frac{|w|^2}{|x|^2} dx.$$

Therefore $J(w_k) < 0$ in view of (4.17) and (4.18). This proves (iii). $\square$

On the other hand, when $n = 2$ we consider a bounded open subset Ω of $\mathbb{R}^2$, with $0 \in \Omega$ and some R_0 for which $\overline{\Omega} \subset B_{R_0}$. We recall the Hardy-Sobolev inequality (see e.g. Theorems 4.2 and 5.4 in [6] and Lemma 17.4 in [81])

$$m_2^2 \int_\Omega \frac{|v|^2}{|x|^2 \log^2 \frac{|x|}{R_0}} dx \leq \int_\Omega |\nabla v|^2 dx \qquad \forall v \in W_0^{1,2}(\Omega), \tag{4.19}$$

where m_2^2 denotes the best possible constant in the inequality, i.e.

$$m_2^2 = \inf_{v \in W_0^{1,2}(\Omega)} \frac{\displaystyle\int_\Omega |\nabla v|^2 dx}{\displaystyle\int_\Omega \frac{|v|^2}{|x|^2 \log^2 \frac{|x|}{R_0}} dx}. \tag{4.20}$$

It is well known that m_2^2 is given by (see the references above)

$$m_2^2 = \frac{1}{4}.$$

We consider a function φ which is defined and continuous on $[0, \infty]$, which is non negative and decreasing and which satisfies

$$\varphi(0) > m_2^2 \text{ and } \varphi(\infty) < \frac{m_2^2}{2}, \tag{4.21}$$

and we define the functional J by

$$J(v) = \int_\Omega |\nabla v|^2 dx - \varphi(\|\nabla v\|_2^2) \int_\Omega \frac{|v|^2}{|x|^2 \log^2 \frac{|x|}{R_0}} dx \qquad \forall v \in W_0^{1,2}(\Omega). \tag{4.22}$$

In this case, we prove the following

Theorem 4.3. *Let $n = 2$ and let Ω be a bounded open subset of $\mathbb{R}^2$, with $0 \in \Omega$ and $\overline{\Omega} \subset B_{R_0}$. Assume that φ is a continuous, non negative and decreasing function on $[0, \infty]$ satisfying* (4.21)*, where m_2^2 is given by* (4.20)*. Then the functional J defined by* (4.22) *satisfies the conditions* (i), (ii) *and* (iii) *of Theorem 4.2.*

Proof. Condition (4.11) is proved exactly as in the proof of Theorem 4.2. Now we prove (ii). Let $p > 2$. Assume that $v_k \rightharpoonup v$ in $W_0^{1,p}(\Omega)$ weakly. Then, as in the proof of Theorem 4.2 , we have, for some $\alpha \geq 0$ and $\beta \geq 0$,

$$\liminf_{k\to\infty} \|\nabla v_k\|_2^2 = \|\nabla v\|_2^2 + \alpha, \tag{4.23}$$

$$\liminf_{k\to\infty} \big\{ -\varphi(\|\nabla v_k\|_2^2)\big\} = -\varphi(\|\nabla v\|_2^2) + \beta. \tag{4.24}$$

Moreover, since $p > n = 2$, we have that $v_k \to v$ uniformly in Ω, and, since

$$\frac{1}{|x|^2 \log^2 \frac{|x|}{R_0}} \in L^1(\Omega),$$

we have

$$\lim_{k\to\infty} \int_\Omega \frac{|v_k|^2}{|x|^2 \log^2 \frac{|x|}{R_0}} dx = \int_\Omega \frac{|v|^2}{|x|^2 \log^2 \frac{|x|}{R_0}} dx. \tag{4.25}$$

Combining (4.23), (4.24) and (4.25), we obtain

$$\liminf_{k\to\infty} J(v_k) = J(v) + \alpha + \beta \int_\Omega \frac{v^2}{|x|^2 \log^2 \frac{|x|}{R_0}} dx \geq J(v),$$

which proves (ii).

Now we prove (iii). Let λ be such that $m_1^2 < \lambda$, where m_1^2 is the best constant (defined by (4.29)) in the one-dimensional Hardy-Sobolev inequality (see (4.28) below). Then there exists $\psi \in C_0^\infty(0,\infty)$ such that

$$\lambda \int_0^\infty \frac{|\psi(t)|^2}{t^2} dt > \int_0^\infty |\psi'(t)|^2 dt.$$

Since φ is continuous and satisfies (4.21), and since the best constant m_2^2 (defined by (4.20)) in the two-dimensional Hardy-Sobolev inequality (4.19) coincides with m_1^2, we can choose λ such that $m_2^2 = m_1^2 < \lambda < \varphi(0)$ (if we do not want to use the property $m_2^2 = m_1^2$, it would be sufficient to assume in (4.21) that $\varphi(0) > m_1^2$ in place of $\varphi(0) > m_2^2$). Then, there exists $t_1 > 0$ such that $\varphi(t_1) = \lambda$. Take s such that $0 < 2\pi s^2 \|\psi'\|_2^2 \leq t_1$. The function $w = s\psi$ belongs to $C_0^\infty(0,\infty)$ and satisfies

$$\varphi\Big(2\pi \int_0^\infty |w'(t)|^2 dt\Big) \geq \lambda, \tag{4.26}$$

as well as

$$\lambda \int_0^\infty \frac{|w(t)|^2}{t^2} dt > \int_0^\infty |w'(t)|^2 dt. \tag{4.27}$$

Define the sequence w_k by

$$w_k(x) = \begin{cases} \frac{1}{\sqrt{k}} w\left(-k \log \frac{|x|}{R_0}\right) & \text{if } |x| \le R_0, \\ 0 & \text{if } |x| \ge R_0, \end{cases}$$

then

$$Dw_k(x) = \begin{cases} -\sqrt{k} w'\left(-k \log \frac{|x|}{R_0}\right) \frac{x}{|x|^2} & \text{if } |x| < R_0, \\ 0 & \text{if } |x| > R_0. \end{cases}$$

For k sufficiently large, the function w_k belongs to $W_0^{1,2}(\Omega)$ and

$$\int_\Omega |\nabla w_k|^2 dx = 2\pi \int_0^{R_0} \left| w'\left(-k \log \frac{r}{R_0}\right)\right|^2 \frac{k}{r} dr = 2\pi \int_0^\infty |w'(t)|^2 dt,$$

while

$$\int_\Omega \frac{|w_k|^2}{|x|^2 \log^2 \frac{|x|}{R_0}} dx = 2\pi \int_0^{R_0} \frac{\left| w\left(-k \log \frac{r}{R_0}\right)\right|^2}{kr \log^2 \frac{r}{R_0}} dr = 2\pi \int_0^\infty \frac{|w(t)|^2}{t^2} dt.$$

Therefore, for k sufficiently large, the sequence w_k is bounded in $W_0^{1,2}(\Omega)$ with $w_k \rightharpoonup 0$ in $W_0^{1,2}(\Omega)$ weakly, and

$$J(w_k) = 2\pi \int_0^\infty |w'(t)|^2 dt - 2\pi \varphi\left(2\pi \int_0^\infty |w'(t)|^2 dt\right) \int_0^\infty \frac{|w(t)|^2}{t^2} dt.$$

Therefore $J(w_k) < 0$ in view of (4.26) and (4.27). This proves (iii). $\square$

Finally, in the one-dimensional case, let Ω be the interval $\Omega = (0, R_0)$. We recall the Hardy-Sobolev inequality (see e.g. Theorem 327 in [40] and Lemma 1.3 in [68])

$$m_1^2 \int_0^\infty \frac{|v|^2}{|x|^2} dx \le \int_0^\infty |v'|^2 dx \quad \forall v \in W_0^{1,2}(0,\infty), \tag{4.28}$$

where m_1^2 denotes the best possible constant in the inequality, i.e.

$$m_1^2 = \inf_{v \in W_0^{1,2}(0,\infty)} \frac{\int_0^\infty |v'|^2 dx}{\int_0^\infty \frac{|v|^2}{|x|^2} dx}. \tag{4.29}$$

It is well known that m_1^2 is given by (see the references above)

$$m_1^2 = \frac{1}{4}.$$

We consider a function φ which is defined and continuous on $[0, \infty]$, which is non negative, decreasing and which satisfies

$$\varphi(0) > m_1^2 \text{ and } \varphi(\infty) < \frac{m_1^2}{2}, \tag{4.30}$$

and we define the functional J by

$$J(v) = \int_0^{R_0} |v'|^2 dx - \varphi(\|v'\|_2^2) \int_0^{R_0} \frac{|v|^2}{|x|^2} dx \qquad \forall v \in H_0^1(0, R_0). \tag{4.31}$$

In this case we prove the following

Theorem 4.4. *Let $n = 1$ and let Ω be the interval $\Omega = (0, R_0)$. Assume that φ is a continuous, non negative and decreasing function on $[0, \infty]$ satisfying* (4.30)*, where m_1^2 is given by* (4.29)*. Then the functional J defined by* (4.31) *satisfies the conditions* (i), (ii) *and* (iii) *of Theorem 4.2.*

The proof of Theorem 4.4 follows along the lines of Theorem 4.2 and will not be given here. Observe that, in contrast with the case $n \geq 2$, the functions $v \in H_0^1(0, R_0)$ vanish in 0 in the one-dimensional case.

We want to point out that, when $0 \in \Omega$, the embedding $H_0^1(\Omega) \hookrightarrow L^2\left(\Omega; \frac{1}{|x|^2} dx\right)$ is not compact.

Example 4.1. Consider the functions

$$u_k(x) = \frac{1}{\sqrt{k}} T_k\left(G_{R_0}(x)\right), \tag{4.32}$$

where $G_{R_0} : \mathbb{R}^n \to \mathbb{R}$ is the function defined by

$$G_{R_0}(x) = \begin{cases} \dfrac{1}{|x|^{n-2}} - \dfrac{1}{R_0^{n-2}} & \text{if } |x| \leq R_0, \\ 0 & \text{if } |x| \geq R_0, \end{cases} \tag{4.33}$$

with $R_0 > 0$ such that the ball $B_{R_0} \subset \Omega$, and where $T_k : \mathbb{R} \to \mathbb{R}$ is the truncation at height k, i.e.

$$T_k(t) = \begin{cases} t & \text{if } |t| \leq k, \\ k\dfrac{t}{|t|} & \text{if } |t| \geq k. \end{cases}$$

Then

$$\int_\Omega |Du_k|^2 dx = \int_{B_{R_0}} |Du_k|^2 dx = \frac{(n-2)^2 S_{n-1}}{k} \int_{r_k}^{R_0} \frac{1}{r^{n-1}} dr,$$

where S_{n-1} is the area of the unit sphere of $\mathbb{R}^n$ and where r_k is defined by

$$\frac{1}{r_k^{n-2}} - \frac{1}{R_0^{n-2}} = k.$$

Therefore
$$\int_\Omega |Du_k|^2 dx = (n-2)S_{n-1},$$
and $u_k \rightharpoonup 0$ in $W_0^{1,2}(\Omega)$ weakly. On the other hand, one has
$$\int_\Omega \frac{|u_k|^2}{|x|^2}dx \geq \int_{B_{r_k}} \frac{|u_k|^2}{|x|^2}dx = S_{n-1}k\int_0^{r_k} r^{n-3}dr = \frac{S_{n-1}}{n-2}kr_k^{n-2},$$
and then
$$\lim_{k\to\infty}\int_\Omega \frac{|u_k|^2}{|x|^2}dx \geq \frac{S_{n-1}}{n-2}.$$
This proves that the embedding $W_0^{1,2}(\Omega) \hookrightarrow L^2\left(\Omega; \frac{1}{|x|^2}dx\right)$ is not compact.

In dimension $n = 2$, this counterexample continues to hold if one replaces the function G_{R_0} defined in (4.33) by the function $G_{R_0}(x) = -\log\frac{|x|}{R_0}$ if $|x| \leq R_0$. In dimension $n = 1$, one uses the continuous piecewise affine functions u_k such that $u_k(0) = 0$, $u_k(R_0/k) = 1/\sqrt{k}$ and $u_k(R_0) = 0$.

Moreover, it should be observed that, when

$$u_k \rightharpoonup u \text{ in } W_0^{1,2}(\Omega) \text{ weakly with } |Du_k| \text{ equi-integrable in } L^2(\Omega), \tag{4.34}$$

then $u_k \to u$ in $L^2\left(\Omega; \frac{1}{|x|^2}dx\right)$. Note that every sequence satisfying $u_k \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ weakly, with $p > 2$, satisfies (4.34) since Ω is bounded; therefore this claim implies that the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^2\left(\Omega; \frac{1}{|x|^2}dx\right)$ is compact for $p > 2$.

Let $\delta > 0$ be small. We write

$$\int_\Omega \frac{|u_k - u|^2}{|x|^2}dx = \int_{\Omega\setminus B_\delta} \frac{|u_k - u|^2}{|x|^2}dx + \int_{B_\delta} \frac{|u_k - u|^2}{|x|^2}dx, \tag{4.35}$$

where B_δ is the ball of radius δ. Since $\frac{1}{|x|^2} \in L^\infty(\Omega \setminus B_\delta)$ and since the embedding
$W_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ is compact for Ω bounded, the first term of (4.35) tends to zero when $k \to \infty$.

Let ψ_δ be the radial function defined by

$$\psi_\delta(x) = \begin{cases} 1 & \text{if } |x| \leq \delta, \\ 2 - \frac{|x|}{\delta} & \text{if } \delta \leq |x| \leq 2\delta, \\ 0 & \text{if } |x| \geq 2\delta. \end{cases}$$

For δ sufficiently small, ψ_δ has compact support in Ω, and using Hardy-Sobolev inequality (4.7) we have

$$\begin{aligned} m_n^2 \int_{B_\delta} \frac{|u_k-u|^2}{|x|^2}dx &\le m_n^2 \int_\Omega \frac{|\psi_\delta(u_k-u)|^2}{|x|^2}dx \\ &\le \int_\Omega |\nabla(\psi_\delta(u_k-u))|^2 dx \\ &\le 2\int_\Omega |\nabla\psi_\delta|^2|u_k-u|^2dx + 2\int_\Omega |\nabla(u_k-u)|^2|\psi_\delta|^2dx \\ &\le 2\int_\Omega |\nabla\psi_\delta|^2|u_k-u|^2dx + 2\int_{B_{2\delta}} |\nabla(u_k-u)|^2dx. \end{aligned}$$

For δ fixed, the first term tends to zero when $k \to \infty$ (still because the embedding $W_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ is compact), while the second term is small uniformly in n when δ is small in view of the equi-integrability assumption (4.34). This proves the claim. This proves the following assertion: if $n \ge 3$ then

$$J(v) \le \liminf_{k\to\infty} J(v_k) \quad \text{if } v_k \rightharpoonup v \text{ in } W_0^{1,2}(\Omega) \text{ weakly with } |Dv_k| \text{ equi-integrable in } L^2(\Omega).$$

The same result continues to hold for $n = 1$ and $n = 2$. Assertion (ii) of Theorems 4.2, 4.3 and 4.4 is a special case of this assertion since Ω is assumed to be bounded.

Remark 4.1. *Actually in dimension $n \ge 3$, Theorem 4.2 continues to hold (with the same proof) if the Hardy-Sobolev inequality* (4.7) *is replaced by the Sobolev inequality*

$$m^2\left(\int_{\mathbb{R}^n}|v|^{2^*}dx\right)^{\frac{2}{2^*}} \le \int_{\mathbb{R}^n}|\nabla v|^2dx \quad \forall v \in W_0^{1,2}(\mathbb{R}^n), \tag{4.36}$$

where 2^* is the Sobolev's exponent defined by $2^* = 2n/(n-2)$ and where m^2 is the best possible constant in (4.36)*, and if in the definition* (4.10) *of the functional J the integral $\int_\Omega \frac{|v|^2}{|x|^2}dx$ is replaced by $\left(\int_\Omega |v|^{2^*}dx\right)^{\frac{2}{2^*}}$. More than that, Theorem 4.2, Theorem 4.3 and Theorem 4.4 still continue to hold (with the same proof) if the inequalities* (4.7), (4.19), (4.28) *and* (4.36) *are replaced by an inequality of the type*

$$m_{X(\Omega)}\|v\|_{X(\Omega)} \le \|\nabla v\|_2,$$

where $X(\Omega)$ is a Banach space such that the embedding $W_0^{1,2}(\Omega) \hookrightarrow X(\Omega)$ is not compact while the embedding $W_0^{1,p}(\Omega) \hookrightarrow X(\Omega)$ is compact for any $p > 2$.

The non compactness of the embedding $W_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega; \omega(x)dx)$ *and the compactness of the embedding* $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega; \omega(x)dx)$ *for* $p > 2$*, where*

$$\omega(x) = \begin{cases} \frac{1}{|x|^2} & \text{if } n = 1 \text{ or } n \geq 3, \\ \\ \frac{1}{|x|^2 \log^2 \frac{|x|}{R_0}} & \text{if } n = 2, \end{cases}$$

are indeed at the root of the proofs of (iii) and (ii). This explains why Theorem 4.2 continues to hold by replacing the Hardy-Sobolev inequality by the Sobolev inequality.

In contrast, if the embedding $W_0^{1,2}(\Omega) \hookrightarrow X(\Omega)$ *is compact (e.g. in the case* $X(\Omega) = L^2(\Omega)$ *for* Ω *bounded), it is straightforward to prove that the functional*

$$J(v) = \int_\Omega |\nabla v|^2 dx - \varphi(\|\nabla v\|_2^2)\|v\|_{X(\Omega)}^2 \quad \forall v \in W_0^{1,2}(\Omega)$$

is sequentially weakly lower semicontinuous on $W_0^{1,2}(\Omega)$ *whenever* φ *is decreasing: just take a sequence* v_k *such that* $v_k \rightharpoonup v$ *in* $W_0^{1,2}(\Omega)$ *weakly, and observe that in this framework*

$$\int_\Omega |\nabla v|^2 dx \leq \liminf_{k\to\infty} \int_\Omega |\nabla v_n|^2 dx,$$

$$-\varphi(\|\nabla v\|_2^2) \leq \liminf_{k\to\infty} -\varphi(\|\nabla v_k\|_2^2),$$

$$\lim_{k\to\infty} \|v_k\|_{X(\Omega)}^2 = \|v\|_{X(\Omega)}^2.$$

Chapter 5

Function spaces related to quasiconformal mappings

The space of functions of bounded mean oscillation, introduced by John and Nirenberg in [46] naturally arises in connection with function theory and PDE's. Similarly, the space of exponentially integrable functions plays a key role in the study of continuity for mappings of finite distortion. The aim of this chapter is to report several results in connection with such a functional spaces. Moreover, we will prove, in dimension $n = 2$, that the composition operator $T_{f^{-1}} : u \longmapsto u \circ f^{-1}$ maps $\mathrm{EXP}(G)$ into $\mathrm{EXP}(f(G))$.

5.1 Functions of bounded mean oscillation: logarithm of the jacobian and composition results

Before we describe the results of this section, we recall the definition of function of *bounded mean oscillation*.

Definition 5.1. Let Ω be a domain of $\mathbb{R}^n$. A locally integrable function $u : \Omega \to \mathbb{R}$ has bounded mean oscillation, $u \in \mathrm{BMO}\,(G)$, if

$$\|u\|_{\mathrm{BMO}(G)} = \sup_Q \fint_Q |u(x) - u_Q| \, dx < \infty. \tag{5.1}$$

The supremum in (5.1) is taken over all open cubes Q of Ω with sides parallel to the axes and u_Q denotes the mean value of u over the cube Q, namely

$$u_Q = \fint_Q u(x)\, dx = \frac{1}{|Q|}\int_Q u(x)dx.$$

We recall that the space BMO was originally introduced in [46] by John and Nirenberg; their fundamental result states that the distribution function which corresponds to a function of bounded mean oscillation, is exponentially decreasing. More precisely, if $u \in \mathrm{BMO}\,(\Omega)$ then for every cube $Q \subset \Omega$ and for every $\sigma > 0$ it results that

$$|\{x \in Q : |u(x) - u_Q| > \sigma\}| \le A|Q|e^{-\frac{B\sigma}{\|u\|_{\mathrm{BMO}(G)}}},$$

for some constants A, B depending only on n.

We want to point out that the concept of bounded mean oscillation is extremely significative in connection with quasiconformal mappings. More precisely, the first result that we mention is the one of Reimann [71], which proves that the logarithm of the jacobian of a quasiconformal mapping is a function of bounded mean oscillation.

Theorem 5.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a K quasiconformal mapping. Then* $\log J_f \in \mathrm{BMO}\,(\mathbb{R}^n)$.

The result that follows proves that the composition operator $T_{f^{-1}}$ maps $\mathrm{BMO}\,(\Omega)$ into $\mathrm{BMO}\,(\Omega')$ provided $f : \Omega \to \Omega'$ is a quasiconformal mapping. Again, this result is due to Reimann [71].

Theorem 5.2. *Let Ω and Ω' be domains in $\mathbb{R}^n$. If $f : \Omega \to \Omega'$ be a $K-$quasiconformal mapping. Then there exists a constant C which depends only on n and K such that*

$$\frac{1}{C}\|u\|_{\mathrm{BMO}(\Omega)} \le \left\|u \circ f^{-1}\right\|_{\mathrm{BMO}(\Omega')} \le C\,\|u\|_{\mathrm{BMO}(\Omega)}\,, \tag{5.2}$$

for every $u \in \mathrm{BMO}\,(\Omega)$, with $\Omega' = f(\Omega)$.

Conversely, let $f : \Omega \to \Omega'$ be a orientation preserving homeomorphism such that

(i) $f \in \mathrm{ACL}$ and differentiable a.e.,

(ii) there exists a constant C for which (5.2) holds for every $u \in \mathrm{BMO}\,(\Omega)$.

Then f is a K-quasiconformal mapping for some $K \geq 1$ depending only on n and C.

In the result above $f \in \mathrm{ACL}$ means that f is absolutely continuous on the lines (see [85]), that is to say that f is continuous everywhere and absolutely continuous on almost all line segments parallel to one of the axes which are contained in the domain of f.

Actually, Theorem 5.2 provides a characterization of quasiconformality. Furthermore, we report that [3] the second part of Theorem 5.2 is proved dropping the regularity assumption (i) and assuming that the following inequality holds

$$\frac{1}{C} \|u\|_{\mathrm{BMO}(G)} \leq \left\|u \circ f^{-1}\right\|_{\mathrm{BMO}(G')} \leq C \|u\|_{\mathrm{BMO}(G)},$$

for every subdomain $G \subset \Omega$ and for every $u \in \mathrm{BMO}\,(G)$, with $G' = f(G)$.

It is worth noting that BMO can be considered as the appropriate substitute of L^∞ in many different cases. This seems to be the case of the mappings of BMO–*bounded distortion*.

Definition 5.2. A mapping $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ of finite distortion is said to have BMO–*bounded distortion* $K_f = K_f(x)$ if there exists a function $M \in \mathrm{BMO}\,(\mathbb{R}^n)$ such that

$$K_f(x) \leq M(x) \quad \text{a.e. } x \in \Omega.$$

Such a mappings were considered for instance in [45] and in [4]. Estimates of moduli of continuity are obtained, we give here an example, see [19].

Theorem 5.3. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a mapping of* BMO*–bounded distortion with $f(0,0) = 0$, $f(1,0) = (1,0)$. Then, there are positive constants A and b such that the modulus of continuity estimate holds*

$$|f(x) - f(y)| \leq A \left|\log|x-y|\right|^{-\frac{b}{\|M\|_{\mathrm{BMO}}}},$$

for x, y in the ball $B_2(0)$.

A quasiconformal f is a homeomorphism of BMO-bounded distortion since the distortion K_f is bounded and we may choose M to be a constant function in the definition above. Moreover, mappings of BMO–bounded distortion are clearly invariant under quasiconformal change of variables. This is a consequence of Theorem 5.2.

Moreover, functions which are bounded by a function in BMO can be characterized by means of the following fundamental lemma.

Lemma 5.4 ([45]). *Let G be an open subset of $\mathbb{R}^n$ and let $u : G \to \mathbb{R}$ be a measurable function. There exists a $\lambda > 0$ such that*

$$\int_\Omega \frac{e^{\frac{|u(x)|}{\lambda}}}{1+|x|^{n+1}} dx < \infty \tag{5.3}$$

if and only if there exists $v \in \mathrm{BMO}\,(G)$ such that

$$|u(x)| \le v(x) \qquad a.e.\ in\ G.$$

Moreover, there exists a constant C which depends only on n such that

$$\|v\|_{\mathrm{BMO}(G)} \le C\lambda.$$

5.2 Exponentially integrable functions

If G is a bounded open subset of $\mathbb{R}^n$ with measure $|G|$ the space $\mathrm{EXP}\,(G)$ is the set of measurable functions $u : G \to \mathbb{R}$ such that there exists $\lambda > 0$ for which

$$⨍_G \exp \frac{|u(x)|}{\lambda}\, dx < \infty.$$

We recall (see e.g. [7]) that $\mathrm{EXP}\,(G)$ is a Banach space equipped with the norm

$$\|u\|_{\mathrm{EXP}(G)} = \sup_{0<t<|G|} \left(1 + \log \frac{|G|}{t}\right)^{-1} u^*(t), \tag{5.4}$$

where u^* is the non–increasing rearrangement of u

$$u^*(t) = \sup\{\tau \ge 0 : \mu_u(\tau) > t\} \qquad \forall t \in (0, |G|)\,, \tag{5.5}$$

and μ_u is the distribution function of u

$$\mu_u(\tau) = |\{x \in G : |u(x)| > \tau\}| \qquad \forall \tau \ge 0.$$

Lemma 5.4 gives a precise characterization of the space of exponentially integrable functions, proving that a function u belongs to $\mathrm{EXP}\,(G)$ if and only if there exists a $v \in \mathrm{BMO}\,(G)$ such that $|u| \le v$ a.e. in G. Therefore, by means of Theorem 5.2 we are able to prove the following result, which is the starting point of our study.

Lemma 5.5. *Let Ω be an open subset of $\mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}^n$ be a quasiconformal mapping. Let G be any bounded open subset of Ω and let $G' = f(G)$. Then $u \in \mathrm{EXP}\,(G)$ if and only if $u \circ f^{-1} \in \mathrm{EXP}\,(G')$.*

Proof. Since both f and f^{-1} are quasiconformal mappings it is sufficient to prove that $u \circ f^{-1} \in \mathrm{EXP}\,(G')$ if $u \in \mathrm{EXP}\,(G)$. Since G is a bounded open subset of $\mathbb{R}^n$, from Lemma 5.4 to the function $u \in \mathrm{EXP}\,(G)$ there corresponds a function $v \in \mathrm{BMO}\,(G)$ such that $|u(x)| \le v(x)$ for a.e. $x \in G$. As a consequence of Theorem 5.2 $v \circ f^{-1}$ belongs to $\mathrm{BMO}\,(G')$. Clearly $|u\,(f^{-1}(y))\,| \le v\,(f^{-1}(y))$ for a.e. $y \in G'$. The result immediately follows from Lemma 5.4. □

Let us turn to the problem of composing functions in $\mathrm{EXP}(G)$ with quasiconformal mappings and we deal with the case of dimension $n = 2$.
We denote by $\mathbb{D}$ the unit disc $\mathbb{D} = \{x \in \mathbb{R}^2 : |x| < 1\}$ and we prove the following result.

Theorem 5.6. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a K–quasiconformal principal mapping that is conformal outside $\mathbb{D}$ and maps $\mathbb{D}$ onto itself. Then*

$$\frac{1}{1 + K\log K}\,\|u\|_{\mathrm{EXP}(\mathbb{D})} \le \left\|u \circ f^{-1}\right\|_{\mathrm{EXP}(\mathbb{D})} \le (1 + K\log K)\,\|u\|_{\mathrm{EXP}(\mathbb{D})}\,, \tag{5.6}$$

for every $u \in \mathrm{EXP}\,(\mathbb{D})$.

Recall that a quasiconformal mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ is called *principal* if it is conformal outside a compact set and the following normalization holds

$$|f(x) - x| = \mathcal{O}\left(\frac{1}{|x|}\right) \qquad \text{as } |x| \to \infty.$$

Observe that our result actually gives that if f is a conformal, then (5.6) reduces to the equality

$$\left\|u \circ f^{-1}\right\|_{\mathrm{EXP}(\mathbb{D})} = \|u\|_{\mathrm{EXP}(\mathbb{D})}\,,$$

for every $u \in \mathrm{EXP}\,(\mathbb{D})$.

Proof of Theorem 5.6. The proof is based on Theorem 1.13. Let $u \in \mathrm{EXP}(\mathbb{D})$. First, we notice that for every $\tau > 0$

$$\left\{y \in \mathbb{D} : \left|u\left(f^{-1}(y)\right)\right| > \tau\right\} = f\left(\{x \in \mathbb{D} : |u(x)| > \tau\}\right).$$

We compare the distribution function of u and $u \circ f^{-1}$ by means of the area distortion estimates in Theorem 1.13 and we obtain

$$\begin{aligned}\mu_{u\circ f^{-1}}(\tau) &= \left|\left\{y \in \mathbb{D} : \left|u\left(f^{-1}(y)\right)\right| > \tau\right\}\right| \\ &= \left|f\left(\{x \in \mathbb{D} : |u(x)| > \tau\}\right)\right| \\ &\leq K\pi^{1-\frac{1}{K}}\mu_u(\tau)^{\frac{1}{K}}.\end{aligned}$$

Since for every $t \in (0, \pi)$

$$\{\tau \geq 0 : \mu_{u\circ f^{-1}}(\tau) > t\} \subset \left\{\tau \geq 0 : \mu_u(\tau) > \frac{t^K}{K^K\pi^{K-1}}\right\},$$

it follows from the definition of non–increasing rearrangement (5.5) that

$$\left(u \circ f^{-1}\right)^*(t) \leq u^*\left(\frac{t^K}{K^K\pi^{K-1}}\right). \tag{5.7}$$

We deduce directly from the definition of the norm (5.4) that

$$\begin{aligned}u^*\left(\frac{t^K}{K^K\pi^{K-1}}\right) &\leq \|u\|_{\mathrm{EXP}(\mathbb{D})}\left(1 + \log\frac{\pi}{\frac{t^K}{K^K\pi^{K-1}}}\right) \\ &= \|u\|_{\mathrm{EXP}(\mathbb{D})}\left(1 + K\log K\frac{\pi}{t}\right) \\ &= \|u\|_{\mathrm{EXP}(\mathbb{D})}\left(1 + K\log K + K\log\frac{\pi}{t}\right).\end{aligned}$$

Thus, from (5.7) we get

$$\left(u \circ f^{-1}\right)^*(t) \leq \|u\|_{\mathrm{EXP}(\mathbb{D})}\left(1 + K\log K + K\log\frac{\pi}{t}\right).$$

Our aim is to prove that there exists a constant $c = c(K)$ which depends on K such that

$$1 + K\log K + K\log\frac{\pi}{t} \leq c(K)\left(1 + \log\frac{\pi}{t}\right) \qquad \forall t \in (0, \pi). \tag{5.8}$$

It will be sufficient to prove that the function

$$\gamma(t) = \frac{1 + K\log K + K\log\frac{\pi}{t}}{1 + \log\frac{\pi}{t}} \qquad \forall t \in (0, \pi).$$

is bounded in the interval $(0,\pi)$ by some constant which only depends on K. To this aim, we observe that

$$\gamma'(t) = \frac{1+K\log K - K}{t\left(1+\log\frac{\pi}{t}\right)^2} \qquad \forall t\in(0,\pi).$$

We define

$$\psi(K) = 1+K\log K - K \qquad \forall K\in[1,\infty).$$

Since

$$\psi'(K) = \log K \geq 0 \qquad \forall K\in[1,\infty),$$

we have

$$\psi(K) \geq \psi(1) = 0 \qquad \forall K\in[1,\infty),$$

and therefore γ is increasing in $(0,\pi)$. Then

$$\gamma(t) \leq \gamma(\pi) = 1+K\log K \qquad \forall t\in(0,\pi),$$

and inequality (5.8) holds with

$$c(K) = 1+K\log K.$$

Therefore (5.7) gives

$$\left(u\circ f^{-1}\right)^*(t) \leq (1+K\log K)\,\|u\|_{\mathrm{EXP}(\mathbb{D})}\left(1+\log\frac{\pi}{t}\right),$$

that is

$$\left(1+\log\frac{\pi}{t}\right)^{-1}\left(u\circ f^{-1}\right)^*(t) \leq (1+K\log K)\,\|u\|_{\mathrm{EXP}(\mathbb{D})}.$$

Hence, the inequality

$$\left\|u\circ f^{-1}\right\|_{\mathrm{EXP}(\mathbb{D})} \leq (1+K\log K)\,\|u\|_{\mathrm{EXP}(\mathbb{D})} \qquad \forall u\in\mathrm{EXP}\,(\mathbb{D}) \tag{5.9}$$

is proved when f is a K–quasiconformal mapping. Recalling that the inverse of a K–quasiconformal mapping is also a K–quasiconformal mapping, it follows that

$$\|v\circ f\|_{\mathrm{EXP}(\mathbb{D})} \leq (1+K\log K)\,\|v\|_{\mathrm{EXP}(\mathbb{D})} \qquad \forall v\in\mathrm{EXP}\,(\mathbb{D}) \tag{5.10}$$

If we substitute $v=u\circ f^{-1}$ with $u\in\mathrm{EXP}\,(\mathbb{D})$ into (5.10), we have

$$\|u\|_{\mathrm{EXP}(\mathbb{D})} \leq (1+K\log K)\left\|u\circ f^{-1}\right\|_{\mathrm{EXP}(\mathbb{D})} \qquad \forall u\in\mathrm{EXP}\,(\mathbb{D}) \tag{5.11}$$

Inequalities (5.9) and (5.11) prove that (5.6) holds and this complete the proof. $\square$

The Luxemburg norm of a function $u \in \mathrm{EXP}\,(G)$ is defined as

$$\|u\|'_{\mathrm{EXP}(G)} = \inf\left\{\lambda > 0\,:\, \fint_G \exp\frac{|u(x)|}{\lambda}\,dx \le 2\right\}. \tag{5.12}$$

We recall that (see e.g. [7] and [70]) the Luxemburg norm is equivalent to the norm defined in (5.4). We also remark that $L^\infty(G)$ is not a dense subspace of $\mathrm{EXP}\,(G)$ (see e.g. [70]) and that the distance to $L^\infty(G)$ in $\mathrm{EXP}\,(G)$ evaluated with respect to the Luxemburg norm (5.12) is defined as

$$\mathrm{dist}_{\mathrm{EXP}(G)}\,(u, L^\infty(G)) = \inf_{\varphi\in L^\infty(G)} \|u-\varphi\|'_{\mathrm{EXP}(G)} \tag{5.13}$$

In [14] and [31] is proved that the distance (5.13) is given by

$$\mathrm{dist}_{\mathrm{EXP}(G)}\,(u, L^\infty(G)) = \inf\left\{\lambda > 0\,:\, \fint_G \exp\frac{|u(x)|}{\lambda}\,dx < \infty\right\}.$$

Our next result compares the distances from L^∞ of u and $u\circ f^{-1}$. We address that the estimates that we prove are sharp (see Example 5.1 below).

Theorem 5.7. *Let $f : \mathbb{D} \to \mathbb{D}$ be a K–quasiconformal mapping. Then*

$$\mathrm{dist}_{\mathrm{EXP(G')}}\left(u\circ f^{-1}, L^\infty(G')\right) \le K\mathrm{dist}_{\mathrm{EXP(G)}}\,(u, L^\infty(G))\,, \tag{5.14}$$

$$\frac{1}{K}\mathrm{dist}_{\mathrm{EXP(G)}}\,(u, L^\infty(G)) \le \mathrm{dist}_{\mathrm{EXP(G')}}\left(u\circ f^{-1}, L^\infty(G')\right), \tag{5.15}$$

for every open subset G of $\mathbb{D}$ and for every $u \in \mathrm{EXP}\,(G)$, with $G' = f(G)$.

As for Theorem 5.6, the result above gives that if f is a conformal mapping then (5.14) and (5.15) reduce to the equality

$$\mathrm{dist}_{\mathrm{EXP(G')}}\left(u\circ f^{-1}, L^\infty(G')\right) = \mathrm{dist}_{\mathrm{EXP(G)}}\,(u, L^\infty(G))\,,$$

for every $u \in \mathrm{EXP}\,(G)$.

Proof of Theorem 5.7. Let λ be such that

$$\lambda > q\mathrm{dist}_{\mathrm{EXP(G)}}\,(u, L^\infty(G)) \tag{5.16}$$

where

$$q = \frac{p}{p-1} \qquad \text{and} \qquad 1 < p < \frac{K}{K-1}.$$

Since

$$\left(\exp\frac{|u(x)|}{\lambda}\right)^q = \exp\frac{|u(x)|}{\lambda/q}$$

from (5.16) it follows that

$$\exp\frac{|u|}{\lambda} \in L^q(G). \tag{5.17}$$

Recalling that $J_f \in L^p(G)$ (see (1.10)), we deduce from (5.17) that

$$\exp\frac{|u|}{\lambda}J_f \in L^1(G).$$

It follows directly from the change of variables formula that

$$\int_{G'} \exp\frac{|u(f^{-1}(y))|}{\lambda}dy = \int_G \exp\frac{|u(x)|}{\lambda}J_f(x)dx < \infty.$$

Therefore

$$\mathrm{dist}_{\mathrm{EXP}(G')}\left(u\circ f^{-1}, L^\infty(G')\right) \leq q\mathrm{dist}_{\mathrm{EXP}(G)}\left(u, L^\infty(G)\right). \tag{5.18}$$

Passing to the limit in (5.18) for p approaching to $K/(K-1)$ we finally prove (5.14). Recalling that the inverse of a K–quasiconformal mapping is also a K–quasiconformal mapping, it follows that

$$\mathrm{dist}_{\mathrm{EXP}(G)}\left(v\circ f, L^\infty\left(G\right)\right) \leq K\mathrm{dist}_{\mathrm{EXP}(G')}\left(v, L^\infty\left(G'\right)\right) \quad \forall v \in \mathrm{EXP}\left(G'\right) \tag{5.19}$$

If we substitute $v = u\circ f^{-1}$ with $u \in \mathrm{EXP}\left(G'\right)$ into (5.19), we have

$$\mathrm{dist}_{\mathrm{EXP}(G)}\left(u, L^\infty\left(G\right)\right) \leq K\mathrm{dist}_{\mathrm{EXP}(G')}\left(u\circ f^{-1}, L^\infty\left(G'\right)\right) \quad \forall u \in \mathrm{EXP}\left(G\right)$$

and this proves (5.15). □

Now we prove, by means of an example, that inequality (5.14) can be attained as an equality.

Example 5.1. Here and in what follows let $0 < R \leq 1$ and $\mathbb{D}_R = \{x \in \mathbb{R}^2 : |x| < R\}$. For every $K \geq 1$ we show that there exist a K–quasiconformal mapping $f : \mathbb{D} \to \mathbb{D}$ and a function $u \in \mathrm{EXP}\left(\mathbb{D}_R\right)$ such that

$$\mathrm{dist}_{\mathrm{EXP}(f(\mathbb{D}_R))}\left(u\circ f^{-1}, L^\infty(f(\mathbb{D}_R))\right) = K\mathrm{dist}_{\mathrm{EXP}(\mathbb{D}_R)}\left(u, L^\infty(\mathbb{D}_R)\right). \tag{5.20}$$

Let $f : \mathbb{D} \to \mathbb{D}$ be the K–quasiconformal mapping defined as

$$f(z) = \frac{z}{|z|^{1-\frac{1}{K}}}$$

and let

$$u(x) = -2 \log |x|.$$

Then $u \in \mathrm{EXP}\,(\mathbb{D}_R)$ and

$$\mathrm{dist}_{\mathrm{EXP}(\mathbb{D}_R)}\,(u, L^\infty(\mathbb{D}_R)) = 1.$$

This follows from the fact that if $\lambda > 1$ then

$$\fint_{\mathbb{D}_R} e^{\frac{|u(x)|}{\lambda}}\, dx = \frac{\lambda}{(\lambda-1)R^{\frac{2}{\lambda}}} < \infty$$

while $e^{\frac{|u|}{\lambda}} \notin L^1(\mathbb{D}_R)$ for $\lambda \leq 1$. We notice that the inverse of f is given by

$$f^{-1}(y) = |y|^{K-1} y.$$

Therefore, the function $v = u \circ f^{-1}$ is given by

$$v(y) = -2K \log |y|.$$

Then $v \in \mathrm{EXP}\,(\mathbb{D}_R)$ and arguing as for u one has

$$\mathrm{dist}_{\mathrm{EXP}(\mathbb{D}_R)}\,(v, L^\infty(\mathbb{D}_R)) = K.$$

This proves (5.20).

5.3 Invariance of $W^{1,n}_{\mathrm{loc}}$ under quasiconformal change of variables

In this section we concern with the composition operator between Sobolev spaces. The first result which we recall is a classical one in the theory of quasiconformal mappings. More precisely, the composition operator $T_g : u \mapsto u \circ g$ maps $W^{1,n}_{\mathrm{loc}}(\Omega')$ into $W^{1,n}_{\mathrm{loc}}(\Omega)$ if $g : \Omega \to \Omega'$ is a quasiconformal mapping. We refer to [10, 49, 75, 85] for a proof.

Theorem 5.8. *Let Ω and Ω' be bounded open subsets of $\mathbb{R}^n$ and let $g : \Omega \to \Omega'$ be a K–quasiconformal mapping. Then, there exists a constant C depending only on K and n such that*

$$\frac{1}{C}\|\nabla u\|_{L^n(G')} \leq \|\nabla (u \circ g)\|_{L^n(G)} \leq C\|\nabla u\|_{L^n(G')},$$

for every open subset G of Ω and for every $u \in W^{1,n}_{\mathrm{loc}}(\Omega')$, with $G' = g(G)$.

It is worthwhile noting that similar results can be obtained when g is a homeomorphism of finite distortion and if we made some precise integrability assumption on the distortion function of g. The following result holds.

Theorem 5.9. *Let $g : \Omega \to \Omega'$ be a homeomorphism of finite distortion K_g between the bounded domains Ω and Ω' of $\mathbb{R}^n$ and let $1 \leq p \leq n$. Suppose that $K_g \in L^{\frac{p}{n-p}}(\Omega)$. If $u \in W^{1,n}_{\mathrm{loc}}(\Omega')$ then $u \circ g \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and the following estimate holds*

$$\|\nabla(u \circ g)\|_{L^p(G)} \leq \|K_g\|_{L^{\frac{p}{n-p}}(G)}\|\nabla u\|_{L^n(g(G))}.$$

Theorem 5.9 was proved first in [82]; recently, Hencl and Koskela gives in [41] a new proof of the result above. Furthermore, they prove that the integrability condition $K_g \in L^{\frac{p}{n-p}}(\Omega)$ is optimal. For further reference see [39, 82].

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