Università degli Studi di Napoli FEDERICO II

Dottorato di Ricerca in Scienze Matematiche

A.A. 2009–2010

Regularity for non-autonomous convex Functionals with non-standard Growth conditions.

Tesi di Dottorato

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Dottorato in Scienze Matematiche XXIII Ciclo

Contents

Introduction

In this thesis we shall investigate the $C^{1,\gamma}$ partial regularity of minimizers of non-autonomous integral functionals of the Calculus of Variations, with non-standard growth conditions.

In the first chapter, we briefly outline the history of the regularity problem in the Calculus of Variations. Moreover we introduce various examples of functionals exhibiting non-standard growth conditions, and we analyze the main points in which the regularity theory of minimizers of functionals with non-standard growth differs from the standard growth situation.

In the second chapter, we fix the notations and collect several definitions and results needed to develop the theory.

In the third chapter, we begin studying the the $C^{1,\gamma}$ partial regularity of minimizers of functionals with (p, q)-growth conditions. We consider the functional

$$\mathcal{F}(u; \Omega) := \int_\Omega f(x, Du)\, dx \tag{0.0.1}$$

with

$$\frac{1}{L}|\xi|^p \leq f(x,\xi) \leq L(1 + |\xi|^q) \qquad \text{for some} \qquad L \geq 1, \tag{F1}$$

where $2 \leq p \leq q < +\infty$, $u : \Omega \to \mathbb{R}^N$ and Ω is a bounded open set in $\mathbb{R}^n$.

Here we shall assume that there exist constants $C, \nu > 0$ and an exponent $\alpha \in (0, 1)$ such that $f(x, \xi)$ is a $C^2(\Omega, \mathbb{R}^{n\times N})$ function fulfilling (F1) and whose derivatives satisfy the following assumptions:

$$|D_\xi f(x_1, \xi) - D_\xi f(x_2, \xi)| \leq C|x_1 - x_2|^\alpha\ (1 + |\xi|^{q-1}); \tag{F2}$$

$$\nu(1 + |\xi|^2)^{\frac{p-2}{2}}|\zeta|^2 \leq \langle D_{\xi\xi} f(x, \xi)\zeta, \zeta\rangle; \tag{F3}$$

for any $\xi \in \mathbb{R}^{nN}$ and for any $x, x_1, x_2 \in \Omega$.

It is well known that condition (F3), which is a strict uniform ellipticity condition on $D^2 f$, is equivalent to the strict uniform convexity of f. We stress that no control on the growth of the second derivatives of f from above will be assumed.

As it will be showed in the first chapter, from the very beginning it has been clear that, even in the scalar case, no regularity can be expected if the exponents p and q are too far apart. On the other hand, if the ratio

$$\frac{q}{p} \leq c(n) \to 1 \tag{0.0.2}$$

as $n \to +\infty$, many regularity results are available both in the scalar and in the vectorial setting. The starting issue in treating the regularity of minimizers is to show the higher integrability of the gradient. In this direction we quote [29, 30, 31, 44, 69]. We stress that, in this setting, this kind of regularity is crucial; indeed, since many apriori estimates depend on the L^q norm because of the right hand side of (F1), the first step in the analysis of the regularity of minimizers is just to improve the integrability of Du from L^p to L^q. Moreover the higher integrability of the gradient of minimizers has revealed to be crucial when one try to argue approximating the integrand with a sequence of functions having standard growth conditions. In fact, the useful apriori estimates depend on the L^q norm of the gradient of minimizer because of the right hand side of (F1) (for a self contained treatment we refer to [9] and the references therein).

On the other hand, $C^{1,\mu}$ partial regularity results have been established (see [5], [77]), without using higher integrability of the gradient, by means of a blow up argument. It is worth pointing out that all the quoted results concern autonomous functionals.

Only recently, the study of the regularity of non autonomous functionals with non standard growth produced both higher integrability and $C^{1,\mu}$ partial regularity. In particular, we quote the paper [31] by Esposito, Leonetti and Mingione where, it has been proved that if f satisfies assumptions (F1), (F2) and (F3), with p, q such that

$$1 < p \le q < p\frac{n+\alpha}{n} \tag{0.0.3}$$

and if there is no Lavrentiev Phenomenon for the functional, then a $W^{1,p}$ local minimizer of $\mathcal{F}$ actually belongs to $W^{1,q}$.

The combination of the facts that f both depends on x and exhibits a gap could determine the occurrence of the Lavrentiev Phenomenon, that translates into the impossibility of approximate in energy a $W^{1,p}$ function with $W^{1,q}$ functions.

We shall prove $C^{1,\gamma}$ partial regularity of minimizers of $\mathcal{F}$ under the assumptions (F1), (F2) and (F3), and provided that no Lavrentiev Phenomenon occurs. We shall assume that

$$2 \le p \le q < p\frac{n+\alpha}{n}, \tag{0.0.4}$$

that is condition (0.0.3) with $p \ge 2$. We need the right hand bound of (0.0.4) because we first establish an higher integrability property of the minimizers following [31], and afterwards we perform a blow-up procedure. Moreover, we also confine ourselves to the case $p \ge 2$, because the usual finite difference quotient method, used to prove higher integrability, led us to heavy technical difficulties in the case $1 < p < 2$. Indeed, even if the result of Esposito, Leonetti and Mingione [31] is proved for every $p > 1$, we need an higher integrability result which had to be uniform with respect the rescaling procedure necessary for the blow-up method. However, the results of this chapter sensibly improve the outcome of Bildhauer and Fuchs' work [11], where Df was assumed to be Lipschitz continuous with respect the x variable and $D^2 f$ had controlled growth from above. Moreover, we are also able to give a bound for the Hausdorff dimension of the singular set of minimizers, as it is usual when higher integrability/differentiability results are available.

In the fourth chapter we prove a new $C^{1,\gamma}$ partial regularity result for minimizers of the functional (0.0.1), under the assumptions (F1)–(F3), with the following gap between growth and coercivity exponent:

$$1 < p \leq q < \min\left\{p\frac{n}{n-1}, p+1\right\}. \tag{0.0.5}$$

This is somehow surprising, since the condition (0.0.5) is independent of the exponent α, which is produced by the α-Hölder continuity dependence of Df with respect to the x variable. Moreover the new range in (0.0.5) is wider than the one given by (0.0.3). We also would like to stress that even in the case $\alpha = 1$, which is the situation considered by Bildhauer and Fuchs in [11], our new range (0.0.5) is still better than (0.0.3) .

In this context, we present a completely new proof, which allows us to improve the result on partial regularity proved in the third chapter, and directly treat the case $p > 1$. The higher integrability step, which entailed the bound (0.0.3), is replaced by the proof of a Caccioppoli type inequality for the minimizers of a suitable perturbation of the rescaled functionals. The Caccioppoli type estimate will present some extra terms that won't effect the blow-up procedure. The main difficulty in studying the regularity properties of minimizers of integrals with non-standard growth is that the usual test functions, whose gradient is essentially proportional to the gradient of the minimizers, don't have the right degree of integrability. A gluing Lemma due to Fonseca and Maly ([36]), used to connect in an annulus two $W^{1,p}$ functions with a $W^{1,q}$ function, will play a key role to overcome this difficulty.

We also point out that regularity for minimizers of non autonomous functionals with standard growth conditions is usually achieved via the Ekeland principle after a comparison between the minimizer of the original functional and the minimizer of a suitable "frozen" one (see [2, 43]).

However, owing to the anisotropic growth of the functional, it seems that the comparison method cannot work in our context.

In the fifth chapter, we shall study the case in which the functional (0.0.1) is not too far from being linear in $|\xi|$, that is

$$\lim_{|\xi|\to+\infty}\frac{|f(x,\xi)|}{|\xi|}=+\infty,\qquad \lim_{|\xi|\to+\infty}\frac{|f(x,\xi)|}{|\xi|^p}=0\qquad \forall p>1. \tag{0.0.6}$$

It is worth mentioning that many regularity results have been established for integrals with nearly linear growth in case they do not depend on the x variable.

The earliest paper on this subject is due to Greco, Iwaniec and Sbordone (see [53]), in which the higher integrability of the minimizers has been proved in the scale of Orlics spaces for a large class of autonomous functionals satisfying (0.0.6).

After that, Fuchs and Seregin in [42] proved the $C^{1,\gamma}$ partial regularity for minimizers of

$$J(u)=\int_\Omega |Du|\log(1+|Du|)\,dx$$

under the assumption $n\le 4$. Such result has been extended to any dimension n by Esposito and Mingione in [32] and later on the full $C^{1,\gamma}$ regularity has been established in [33]. All the quoted papers concern the autonomous case.

Note that functionals with nearly linear growth have features in common with ones satisfying (p,q)-growth conditions since, by virtue of (0.0.6), we have that

$$|\xi|\le f(x,\xi)\le|\xi|^p,\qquad \forall p>1.$$

However, the regularity properties of the minimizers of such functionals cannot be deduced from the those of minimizers of functionals with (p,q)-growth conditions, because of the linear growth that the integrand function f exhibits on the left hand side of the previous inequality. Indeed, the regularity results for minimizers in the context of (p,q)-growth require that p, the growth exponent from below of the integrand f, is strictly greater than one. Therefore, functionals with almost linear growth have to be treated by means of different techniques.

We will establish $C^{1,\gamma}$ partial regularity of minimizers of (0.0.1) with an integrand f satisfying the assumption

$$c_0 h(|\xi|) - c_1 \le f(x,\xi) \le c_2 h(|\xi|) + c_3 \tag{L1}$$

where c_i are positive constants, $\xi \in \mathbb{R}^{nN}$, Ω is a bounded open set in $\mathbb{R}^n$ and

$$h(t) = t\log(1+t),$$

with $t \ge 0$.

Here we shall assume that there exist constants $c_4, c_5, \nu > 0$ and an exponent $\alpha \in (0,1)$ such that f is a function fulfilling (L1) and whose derivatives satisfy the following assumptions:

$$|D_\xi f(x,\xi)| \le c_4 \log(1+|\xi|); \tag{L2}$$

$$|D_\xi f(x_1,\xi) - D_\xi f(x_2,\xi)| \le c_5 |x_1 - x_2|^\alpha \log(1+|\xi|); \tag{L3}$$

$$\nu(1+|\xi_1|+|\xi_2|)^{-1}|\xi_1-\xi_2|^2 \le \big\langle D_\xi f(x,\xi_1) - D_\xi f(x,\xi_2), \xi_1 - \xi_2 \big\rangle; \tag{L4}$$

for any $\xi_1, \xi_2 \in \mathbb{R}^{nN}$ and for any $x, x_1, x_2 \in \Omega$. Moreover to perform the blow up procedure we shall need $D_{\xi\xi} f \in C^0(\Omega \times \mathbb{R}^{nN})$ and satisfying the following assumptions

$$\nu(1+|\xi|)^{-1}|\zeta|^2 \le \big\langle D_{\xi\xi} f(x,\xi)\zeta, \zeta \big\rangle \le c_6 \frac{\log(1+|\xi|)}{|\xi|} |\zeta|^2, \tag{L5}$$

with a positive constant c_6.

The first result of the chapter is an higher integrability property of minimizers of the functional (0.0.1), that will be useful to prove $C^{1,\gamma}$ partial regularity and it is also of interest by itself. Thanks to these two results, we can give an estimate of the Hausdorff dimension of the singular set of the minimizers. Afterwards, under further assumptions, we obtaing full $C^{1,\gamma}$ regularity of minimizers (i.e. everywhere $C^{1,\gamma}$).

In the sixth and last chapter, we shall estimate the Hausdorff dimension of the singular set of minimizers of functionals of the type (0.0.1) with standard growth conditions. We shall obtain bounds which improve the ones that can be obtained as a particular case from the paper [31], where the anisotropic growth conditions were examined. A more careful analysis in the case of the functional (0.0.1) in the case of standard growth conditions, allows to improve the estimate obtained in [31].

Our proofs of $C^{1,\gamma}$ partial regularity of minimizers are based on a decay estimate for the excess function. This function, roughly speaking, measures how the gradient of minimizers is far from being constant in a ball $B_R(x_0)$. It turns out that in the setting of Chapter III, where the functional is studied under the growth assumptions (F1)–(F3) with $p \geq 2$, a suitable excess function can be defined as

$$E(x,r) = \fint_{B_r(x)} |Du - (Du)_r|^2 + |Du - (Du)_r|^p + r^\beta, \tag{0.0.7}$$

where u is a minimizer of (0.0.1), while in the context of Chapter IV and V, where also the subquadratic and almost linear growth are treated, the excess will be defined as

$$\tilde{E}(x,r) = \fint_{B_r(x)} |V_p(Du - (Du)_r)|^2 + r^\beta, \tag{0.0.8}$$

where V is a function defined as a suitable power of Du. When $p \geq 2$ it turns out that the two functions E and $\tilde{E}$ are equivalent.

Chapter I

HISTORICAL NOTE

1.1 Standard growth conditions

One of the most studied problem of the calculus of variations consists of the research of a function $u(x)$ minimizing integral functionals of the form

$$\mathcal{I}(u) := \int_{\Omega} f(x, u(x), D(x))\, dx \tag{1.1.1}$$

among all the functions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ taking a prescribed value $u_0(x)$ on $\partial\Omega$. Here Ω is a bounded open set and $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is a given function. Solving the minimization problem

$$(P) \qquad m := \inf\{I(u) : u \in X\}$$

associated to the functional $\mathcal{I}$, means that we wish to find $\bar{u} \in X$ such that

$$m = I(\bar{u}) \leq I(u) \qquad \text{for every} \qquad u \in X,$$

where X is the space of admissible functions.

The problem (P), that bears the name of Dirichlet, emerges in the study of variational models of mathematical physics, where the integral represents the energy of a physical system which is minimized by the equilibrium configurations of the system itself.

The existence of a solution to the problem (P) strongly depends on the choice of the space X of admissible functions. A natural choice for X would be a subspace of $C^1(\Omega, \mathbb{R}^N)$, since the integrand function f in (1.1.1) depends on the gradient of the function u. The original approach to the *Dirichlet problem*, implemented by Euler and Lagrange, the founders of the calculus of variations, who first faced the *scalar case* $n = N = 1$, was based on the solution of a suitable second order ordinary differential equation associated to the functional (1.1.1). Hence, the space X should be a subspace of $C^2(\Omega, \mathbb{R}^N)$. Of course, when we pass to the *vectorial case*, i.e. $N > 1$, we are lead to systems of partial differential equations and the Euler-Lagrange method becomes very difficult to apply.

Riemann marked a turning point in the theory, introducing the so called *Direct Methods*. The essence of these methods is to split the problem into two parts. First to enlarge the space of admissible function X so as to get a general existence theorem. Afterwards, one tries to prove some regularity results for the minimizer of (P).

The importance of adopting this new point of view was well realized by David Hilbert, who attracted the attention of the international mathematical community on the solution, in a suitable sense, of the Dirichlet problem, formulating the 19th and 20th problem of his famous list of twenty problems for the mathematicians in the 20th century.

The existence problem is treated considering the functional (1.1.1) as a map

$$\mathcal{I} : u \in X \mapsto \mathcal{I}(u) \in \mathbb{R},$$

and trying to prove a suitable generalization of the Weierstrass theorem, assuring the existence of the minimum (and of the maximum) of any semicontinuous function, for the map $\mathcal{I}$ in this general setting.

In this way one can prove the existence of the minimum of the integral functional (1.1.1) without passing through the Euler-Lagrange equation, but deducing it directly from the properties of the functional considered as a map from the manifold X to $\mathbb{R}$.

The right setting in order to implement this method is the one of *Sobolev spaces*, and the existence of minimizers in these spaces relies on the fundamental property of (sequential) *weak lower semicontinuity*, meaning that

$$u_n \rightharpoonup \bar{u} \quad \text{in} \quad W^{1,p} \Rightarrow \liminf_{n\to\infty} I(u_n) \geq I(\bar{u}). \tag{1.1.2}$$

In fact, the key role in this strategy is played by the following

Theorem 1.1.1. *Let X be a metric space. Let $g : X \to \mathbb{R}\cup+\infty$ be a (sequentially) lower semicontinuous function and let K be a sequentially compact subset of X. Then there exists a minimum point for g in X.*

It turns out that, in the scalar case, the property (1.1.2) is intimately related to the *convexity* of the function $\xi \to f(x, u, \xi)$:

$$f(x, u, t\xi + (1-t)\zeta) \leq tf(x, u, \xi) + (1-t)f(\zeta),$$

for every $\xi, \zeta \in \mathbb{R}^{n\times N}$, and every $t \in [0, 1]$.

In the vectorial case, the convexity of the same function can be replaced by a more general notion, the *quasiconvexity* (in the sense of Morrey), i.e., $\xi \to f(x, u, \xi)$ is a Borel measurable and locally bounded function such that

$$f(x, u, \xi) \leq \frac{1}{|D|}\int_D f(x, u, \xi + D\varphi(x))\, dx,$$

for every bounded open set $D \subset \mathbb{R}^n$, for every $(x, u) \in \Omega\times\mathbb{R}^N$, for every $\xi \in \mathbb{R}^{n\times N}$ and for every $\varphi \in W_0^{1,\infty}$ ($|\cdot|$ denotes the usual Lebesgue measure).

It is now clear that convexity and coercivity, or quasiconvexity and coercivity, ensure the existence of minimizers. We will not be concerned with the problem of existence of minimizers for which we refer to Dacorogna [20] and Giusti [51].

Solving the existence problem in the Sobolev spaces opens up the problem of *regularity of minimizers*, that is, the attempt to prove that minimizers belonging to

Sobolev spaces actually are minimizers in the classical sense. Indeed, the Sobolev functions have derivatives only in a weak sense and, in general, are not even continuous.

In the first half of the 20th century, thanks to the contributions of various authors such as Bernstein [8], Schauder [78], Caccioppoli [15], Morrey [70, 71, 72], it was established that, in the case of f depending only on Du, the so-called *autonomous case*, under the growth assumptions

$$|\xi|^p \leq f(\xi) \leq L(|\xi|^p + 1) \tag{1.1.3}$$

with $p = 2$, and non-degenerate ellipticity,

$$\left\langle D_{\xi\xi} f(\xi)\lambda, \lambda \right\rangle \geq \nu |\lambda|^2 \tag{1.1.4}$$

the essence of regularity theory was to prove that minimizers are functions of class $C^{1,\alpha}$, i.e., functions which are Hölder continuous together with their gradient. But in general we are only able to find minimizers in Sobolev spaces. Therefore the regularity problem is exactly the problem of filling this gap. Once $C^{1,\alpha}$ regularity has been achieved, the higher regularity can be obtained by boot-straps methods based on the Schauder estimates. This problem was solved, in the scalar case, by Morrey when $n = 2$ (and $N = 1$) and in [21] by De Giorgi and, independently, in [74] by Nash in the general (scalar) case, at the end of fifties.

The growth conditions (1.1.3) are called *standard growth conditions* and represent, in some sense, the natural request to ensure the well-posedness of the minimum problem for the functional (1.1.1).

The regularity theorem of De Giorgi and Nash does not extend to the vectorial case ($N > 1$), as shown by De Giorgi himself, who constructed a linear elliptic system (see [22]) with bounded measurable coefficients having discontinuous solutions.

In the vectorial case, the only global regularity property that survives is the *higher integrability* of the gradient of minimizers, proved by Giaquinta and Giusti in [48]. Hence, the $C^{1,\alpha}$ regularity of minimizers can be achieved only in a suitable sense. More precisely, one can try to prove the *partial regularity* or *almost everywhere* regularity of minimizers. This means that minimizers are $C^{1,\alpha}$ regular outside of a closed set, called the *singular set*, which has zero Lebesgue measure. This kind of study was started by Morrey in [73]. Next, one attempts to estimate the Hausdorff dimension of the singular set.

Of course, it is often necessary to consider integral functionals more general than the type (1.1.1), such as functionals with integrand functions $f(x,u,z)$ also depending on both (or simply on one) of the other variables x and u. These are the so-called *non-autonomous* functionals. Then, the growth conditions (1.1.3) have to be reformulated and the usual assumptions in the non-autonomous case are the following:

$$\xi \to f(x,u,\xi) \quad \text{is} \quad C^2; \tag{1.1.5}$$

$$\nu|\xi|^p \le f(x,u,\xi) \le L(1+|\xi|^p); \tag{1.1.6}$$

$$\nu(\mu^2+|\xi|^2)^{\frac{p-2}{2}}|\lambda|^2 \le \big\langle D_{\xi\xi} f(x,u,\xi)\lambda,\lambda\big\rangle; \tag{1.1.7}$$

$$|f(x,u,\xi)-f(y,v,\xi)| \le L\omega(|x-y|+|u-v|)(1+|\xi|^p), \tag{1.1.8}$$

for every $x,y \in \Omega$, $u,v \in \mathbb{R}$ and $\xi,\lambda \in \mathbb{R}^n$, where $\mu \in [0,1]$ is a fixed constant and $\omega : \mathbb{R}^+ \to (0,1)$ is a continuous, non-decreasing modulus of continuity, such that for some $\alpha \in (0,1)$:

$$\omega(s) \le s^\alpha.$$

Autonomous functionals for which the sole dependence on the gradient occurs through the modulus $|Du|$, represent the only known structure, up to now, ensuring everywhere $C^{1,\alpha}$-regularity of minimizers in the vectorial case. This special structure was first identified in the fundamental work of K. Uhlenbeck [83]. It prescribes that

$$f(x, u, \xi) \equiv f(\xi) = g(|\xi|),$$

for a suitable function $g : [0, \infty] \to [0, \infty]$, such that (1.1.5)–(1.1.8) are still satisfied.

The regularity theory for non-autonomous functionals is much more delicate with respect to the autonomous one. The Uhlenbeck structure, which up to now is the only general condition which can ensure everywhere regularity in the vectorial case, prescribes that the functional must be autonomous and, moreover, the integrand can only depend on the modulus of the gradient.

Nowadays the partial regularity theory for non-autonomous functionals with standard growth conditions is rather complete. Giusti's book is an exhaustive reference on regularity theory for minimizers of functional with standard growth conditions. For counterexamples to regularity in the interior of Ω, even the L^∞ one, of minimizers of regular variational integrals in the vectorial case, and solutions to nonlinear elliptic systems, see for istance [22, 56, 75, 81, 82]. For a general survey on regularity theory in the calculus of variations, the interested reader may look at [69].

In the next chapter we shall begin to focus our attention on the theory of functionals with non-standard growth conditions, which is the core of this thesis.

1.2 On the methods of proof in the case of standard growth conditions

$C^{1,\alpha}$ partial regularity results are usually obtained using a linearization technique and a comparison procedure of the original minimizer with the smooth solution of the linear elliptic system with constant coefficients coming out of the linearization procedure. To implement this scheme, different methods are nowadays available. Throughout this thesis we will use the so-called *indirect methods* via blow-up techniques, introduced in this context by Morrey [73] and Giusti and Miranda [52] and then recovered directly for the quasiconvex case by Evans, Acerbi, Fusco, Hutchinson and Hamburger [34, 2, 43, 54].

There are three essentials elements in the proof of partial regularity. The first element is a *Caccioppoli type inequality*, or *reverse Poincaré inequality*, which allows to control the L^p norm of the derivative of a minimizer on a ball in terms of the structure constants, by the L^p norm of the solution and the averaged mean square deviation of the derivative on a ball of larger radius. The second element of the proof can be roughly described as a suitable improvement of the Caccioppoli inequality. The third step consists in showing that the *excess* function, which measures how the gradient of the minimizers is far from being constant in a ball, decays faster than the square ratio of the radii. This is straightforward for solutions to equations with constant coefficients. Therefore the idea is to reduce the problem to that case, using a suitable *freezing procedure.* With the expression indirect methods, we essentially refer to the method of proof employed in the second step described above.

The so called *direct methods* consist in proving a suitable reverse Hölder type inequality for the gradient of minimizers. These methods are very technical but they have the advantage of generating explicit information on the sensitivity of the various estimates to changes in the structure parameters. In [51] all the proofs of regularity results, which are contained in the Chapters 6–10, are of direct type, so we refer to that book and the references therein for these methods.

In the indirect type of proof, one proves the desired estimate by contradiction: if the desired inequality were false, one could construct a particular sequence of minimizers, each of wich fails to satisfy the inequality but which, when appropriately rescaled, or *blown-up*, form a sequence which converges to a solution of a linear systems of equations for which the inequality holds. Compactness arguments then allow to reach the contradiction. For an introduction to regularity results obtained in this way, the reader may look at the book [46] (in particular Chapter 4) by Giaquinta.

We conclude by mentioning that there is also a third way to perform the second step in the regularity proof, the so called *A-harmonic approximation method*. In some casis, this technique allows to directly obtain optimal results in a simpler way. For example, the case of nonlinear elliptic systems is treated in [26] (see also [79, 28]).

1.3 Non-standard growth conditions

As we have seen in the previous chapter, all the regularity results follow assuming at least one common, main condition, that is a growth conditions of the type

$$|\xi|^p \leq f(x,\xi) \leq L(|\xi|^p + 1). \tag{1.3.1}$$

But there are many important physical situations in which the conditions (1.3.1) are too restrictive, and therefore more general growth conditions must be introduced. For example the study of variational models in nonlinear elasticity leads to functionals whose integrand functions f often have the form:

$$f(Du) = g(Du, Adj Du, det Du),$$

where $Adj Du$ is the vector whose components are all the minors of the matrix Du having order smaller than n, and the function g satisfies growth assumptions of the type (1.3.1), such as, for istance:

$$g(Du, Adj Du, det Du) = |Du|^p + |Adj Du|^p + |det Du|^p.$$

Since for every square matrix $[a_{ij}]$ of order n the following Hadamard's inequality holds

$$det^2(a_{ij}) \leq \prod_{j=i}^{n} \left(\sum_{i=1}^{n} a_{ij}^2 \right),$$

the function f satisfies the growth conditions

$$|\xi|^p \leq |f(\xi)| \leq c(1 + |\xi|^{np}).$$

Therefore, the function f does not have the same behavior from above and from below and it cannot satisfy the growth conditions (1.3.1).

Moreover, in the theory of elasticity it is also necessary to consider medium composed by different materials. The simpler case it that of two different materials, that is:

$$\Omega = \Omega_1 \cup \Omega_2,$$

with $\Omega_1 \cap \Omega_2 = \emptyset$. In this case the functional which models the problem is of the type:

$$\int_\Omega |Du|^{p(x)}\, dx$$

where

$$p(x) = \begin{cases} p_1 & if \quad x \in \Omega_1 \\ p_2 & if \quad x \in \Omega_2 \end{cases}$$

If the non-homogeneity of the material is modelled at a mesoscopic level, the function $p(x)$ can also be a function varying from point to point, instead of being a piecewise constant function as in the previous case.

From nonlinear elasticity theory the so-called *anisotropic functionals* also come out. These functionals have a structure of the type:

$$\sum_{i=1}^{n} |\xi_i|^{p_i} \leq f(\xi) \leq L \left(\sum_{i=1}^{n} |\xi_i|^{p_i} + 1 \right),$$

so they exhibits a variable growth.

There are many other possible examples of integral functionals which don't satisfy (1.3.1). To see this, it is enough to fix two real numbers $1 < p < q$ and consider, for istance, the following functionals:

$$\mathcal{F}_1 := \int_\Omega |Du|^p \log(1 + |Du|)\, dx;$$

$$\mathcal{F}_2 := \int_\Omega |Du|^p + a(x)|Du|^q\, dx, \qquad 0 \leq a(x) \leq L;$$

$$\mathcal{F}_3 := \int_\Omega \sum_{i=1}^{n} a_i(x)|D_i u|^{p_i}\, dx,$$

$$0 \leq a_i(x) \leq L \qquad 1 < p := p_1 \leq p_2 \leq \ldots \leq p_n =: q;$$

$$\mathcal{F}_4 := \int_\Omega |Du|^{p(x)}\, dx, \qquad 1 < p \leq p(x) \leq q.$$

It is also possible to consider functionals with oscillating growth such as

$$\mathcal{F}_5 := \int_\Omega |Du|^{2+\sin\log\log(1+|Du|)}\, dx.$$

None of the integrands corresponding to the functionals $\mathcal{F}_1 - \mathcal{F}_5$ satisfies (1.3.1), for any possible choice of the exponent $p \geq 1$. But all of them satisfy, for a couple of numbers (p, q), and suitable ν and L, the more general growth conditions

$$\nu|\xi|^p - L \leq f(x, \xi) \leq L(1 + |\xi|^q), \qquad 1 < p < q. \tag{1.3.2}$$

Functionals satisfying conditions (1.3.2), and not meeting the ones in (1.3.1), are called functionals with (p, q)-*growth conditions*, according to the terminology of Marcellini, who was the first to initiate a systematic study of these integrals in a series of seminal papers [61, 62, 63, 64, 65, 66].

The usual assumptions to study the regularity of minimizers of functionals of the type (1.3.2) are the following (p, q) version of (1.1.5):

$$\xi \rightarrow f(x, u, \xi) \quad \text{is} \quad C^2; \tag{1.3.3}$$

$$\nu(1 + |\xi|^2)^{\frac{p-2}{2}} \leq f(x, u, \xi) \leq L(1 + |\xi|^q); \tag{1.3.4}$$

$$\nu|\xi|^{p-2}|\lambda|^2 \leq \left\langle D_{\xi\xi} f(x,u,\xi)\lambda, \lambda \right\rangle \leq L(1+|\xi|^2)^{\frac{q-2}{2}}|\lambda|^2; \tag{1.3.5}$$

$$|f(x,\xi) - f(y,\xi)| \leq L|x-y|^{\alpha}(1+|\xi|^p), \tag{1.3.6}$$

It is worth noting that the conditions (1.3.2) are of polynomial type, so that they are of the same type of (1.3.1). In the literature we also find the study of the regularity under more general growth conditions. For istance, when $p = 1$ the functional $\mathcal{F}_1$ doesn't satisfy (1.3.2), because the integrand grows too slowly in the gradient variable and it fails to be polynomially super-linear. This kind of functionals also appears in mathematical physics, for example in the study of plasticity problems with logarithmic hardening, and they are called functionals with *almost linear* growth. The functional $\mathcal{F}_1$ with $p = 1$, which is often called $L \log L$ functional, is the model for this family of functionals. Using the theory of duality in Orlicz space (see, for example, [6]), it is possible to define the "dual" functional of $L \log L$, that turns out to be

$$\int_\Omega \exp(|Du|)\, dx.$$

Also this functional doesn't satisfy (1.3.2), because in this case the integrand grows faster than any power.

1.4 Autonomous functionals

The study of regularity of minima of functionals with growth of (p, q)-type (we will also use the terminology (p, q)-growth) was initiated by Marcellini (see references in Section 2.3), who first identified a condition which, under suitable smoothness assumptions on the integrand f, ensures the regularity of minima. This condition prescribes that the *gap ratio of the integrand* f, or simply the *gap*, defined as $q/p > 1$, cannot differ too much from 1, in other words the number p and q cannot bee too far apart.

This restriction on the gap, even in the autonomous case, is suggested by a first series of examples constructed in [63] by Marcellini himself which show the existence of unbounded minimizers if the gap differs too much from 1. Even though Marcellini's examples concern with degenerate integrals, it is possible to observe the same phenomenon for the regular, non-degenerate elliptic functional

$$\int_\Omega |Du|^2 + \frac{1}{2}|D_n u|^4 \, dx,$$

having a regular integrand, and exhibiting the following minimizer for $n \geq 6$:

$$u(x) = \sqrt{\frac{n-4}{24}} \frac{x_n^2}{\sqrt{\sum_{i=1}^{n-1} x_i{}^2}} - \frac{2}{n-2}\sqrt{\frac{n-4}{24}}\sqrt{\sum_{i=1}^{n-1} x_i{}^2}.$$

This example, considered in [55] by Hong, confirms the intuitive fact that, for functionals with (p,q)-growth, problems mainly come from the behaviour of the integrand $f(z)$ for large value of $|z|$. Indeed, unless we are not dealing with degenerate problems, the behaviour of a non-standard growth functional differs from that of the standard one only for the growth conditions in the gradient variable z, and therefore for large values of $|z|$.

When dealing with (p,q)-growth conditions, autonomous functionals are more simple to study for two principal reasons. The first one is the non occurrence of the Lavrentiev phenomenon (see section 2.6 below). The second one is that the higher integrability theory for non-autonomous functionals strongly depends on the interaction between the q/p gap and the regularity of the function $x \to f(x,\xi)$, as it will be showed in the next section.

1.5 Non-autonomous functionals

Throghout this thesis we shall restrict our attention to non-autonomous functionals of the type

$$\mathcal{F}(u) := \int_\Omega f(x, Du)\, dx \tag{1.5.1}$$

where Ω is an open bounded set in $\mathbb{R}^n$, $f : \Omega \times \mathbb{R}^{nN} \to [0, +\infty)$ is a function whose regularity properties will be specified according to the problem we are going to treat, and $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ is a function belonging to a suitable Sobolev class.

In the case of non-autonomous functionals with (p, q)-growth, the presence of the x drastically changes the regularity theory, as far as concern the higher integrability properties of the gradient of the minimizers. A first theorem which shows the effect of the presence of the x dependence in (1.5.1), has been proved by Fonseca, Malý and Mingione in [37].

Theorem 1.5.1. *For every choice of the parameters*

$$2 \leq n, \qquad \alpha \in (0, \infty), \qquad 1 < p < n < n + \alpha < q < \infty, \qquad \varepsilon > 0,$$

there exists a functional

$$\mathcal{G} : u \in W^{1,p}(\Omega) \mapsto \int_\Omega \left[\left(1 + |Du|^2\right)^{\frac{p}{2}} + a(x) \left(1 + |Du|^2\right)^{\frac{q}{2}} \right] dx,$$

with $\Omega \subset \mathbb{R}^n$ being a bounded Lipschitz domain, $a \in C^\alpha(\Omega)$, $a \geq 0$, a local minimizer $u \in W^{1,p}(\Omega)$ of $\mathcal{F}$ and a closed set $\Sigma \subset \Omega$ with

$$dim_{\mathcal{H}}(\Sigma) > n - p - \varepsilon,$$

such that all the points of Σ are non-Lebesgue points of the precise representative of u.

In other words, the authors show that, provided p and q are far enough, depending on the dimension n and the regularity of $x \mapsto f(x, z)$, the set of non-Lebesgue points of minimizers can be nearly as bad as that of any other $W^{1,p}$-function. Indeed, a well known measure theory result states that the set of non-Lebesgue point of the precise representative of a $W^{1,p}$-function has Hausdorff dimension not larger than the maximal dimension $n - p$ (see, for istance, [51], Chapter 2). Therefore, in the non-autonomous case, if the ratio q/p is too far apart from 1, it is possible to find a convex, regular and scalar variational integral whose minimizers have a singular set of nearly maximal dimension.

If we look at the case $p = q$ we see that the degree of Hölder continuity of $f(x, z)$ with respect to x only influences the degree of Hölder continuity of Du but not the fact that Du is Hölder continuous or not. In other words, any degree of Hölder continuity of $x \mapsto f(x, z)$ suffices in order to get a Hölder continuous gradient. The modest influence of the presence of the x-variable in the integrand is also clear when looking at the techniques of proof of regularity theorems when $p = q$, because the x is treated essentially using local perturbation methods. This is not possible when dealing with (p, q)-growth conditions, when the presence of the x cannot be treated as a perturbation anymore. Look, for istance, at the functional $\mathcal{F}_2$ at the beginning of this chapter. If we keep x fixed and let z vary, the integrand exhibits standard growth conditions: p growth if $a(x) \equiv 0$, q growth otherwise. But if the x-variable is varying simultaneuosly with z it globally exhibits (p, q)-growth conditions. This tells us that the presence of the x can be itself responsible for the (p, q)-growth conditions to appear.

However, we want to stress that the regularity of the integrand with respect the x variable only affects the higher integrability properties of the gradient of minimizers and then the estimate of the dimension of the singualar set, while it is irrelevant to obtain $C^{1,\alpha}$ partial regularity of the minimizers.

Note that functionals of the type (1.5.1) with almost linear growth, that is functionals whose integrands which are not too far from being linear in $|z|$, i.e.,

$$\lim_{|z| \to +\infty} \frac{|f(x, z)|}{|z|} = +\infty, \qquad \lim_{|z| \to +\infty} \frac{|f(x, z)|}{|z|^p} = 0 \qquad \forall p > 1.$$

have features in common with ones satisfying (p, q)-growth conditions since, from the previous relations, we have that

$$|z| \leq f(x,z) \leq |z|^p, \qquad \forall p > 1.$$

1.6 The Lavrentiev phenomenon

The Lavrentiev phenomenon is a feature that functionals of the type (1.5.1) typically exhibit when they satisfy (p,q)-growth conditions. It occurs at a map $v \in W^{1,p}(\Omega, \mathbb{R}^N)$ when it is not possible to find a sequence of more regular maps $v_n \in W^{1,q}_{loc}(\Omega, \mathbb{R}^N)$ such that v_n weakly converges to v in $W^{1,p}_{loc}(\Omega, \mathbb{R}^N)$ and the following approximation in energy takes place

$$\int_A f(x, Dv_n)\, dx \to \int_A f(x, Dv)\, dx,$$

for every $A \Subset \Omega$. The phenomenon plays an important role in non-linear elasticity, see [7, 39, 40].

When a Lavrentiev phenomenon occurs at a local minimizer u it then follows, in particular, that it is not possible to realize locally minimizing sequences $\{u_n\}$ for $\mathcal{F}$ with more regular maps $u_n \in W^{1,q}_{loc}(\Omega, \mathbb{R}^N)$. If u is a minimizer such that $u \in W^{1,q}_{loc}(\Omega, \mathbb{R}^N)$, then by definition there is no Lavrentiev phenomenon at u.

The Lavrentiev phenomenon is a clear obstruction to the existence of minimizers in a specified class of admissible functions. For this reason, the usual finite element methods (by taking piecewise affine functions, which are in $W^{1,\infty}$) in numerical analysis will then not be able to detect a unique and well defined minimum of integrals such as the one in the Theorem 1.6.1 below.

It is interesting, and significant, to see that $\mathcal{F}$ never exhibits Lavrentiev phenomenon either when $p = q$ or when $f(x,z) \equiv f(z)$. Therefore the phenomenon results from the coupling of (p,q)-growth conditions with dependence on x in the integrand.

It is important to observe that the Lavrentiev phenomenon can also occur in the scalar case. In fact, examples of this phenomenon can be constructed even in the case of one dimensional integrals, thus showing that a careful choice of the space of admissible functions is necessary in order to get both existence and regularity results. The following Theorem, which shows an example of the Lavrentiev phenomenon for one dimensional integrals, is due to Mania (see [60]).

Theorem 1.6.1. *Let* $f(x,u,\xi) := (x-u^3)^2\xi^6$ *and*

$$I(u) := \int_0^1 f(x, u(x), u'(x))\, dx.$$

Let

$$\mathcal{W}_\infty := \left\{ u \in W^{1,\infty}(0,1) : u(0)=0, \quad u(1)=1 \right\},$$

$$\mathcal{W}_1 := \left\{ u \in W^{1,1}(0,1) : u(0)=0, \quad u(1)=1 \right\}.$$

Then

$$\inf\{I(u) : u \in \mathcal{W}_\infty\} > \inf\{I(u) : u \in \mathcal{W}_1\}.$$

Moreover, $\bar{u}(x) = x^{1/3}$ *is a minimizer of* I *over* $\mathcal{W}_1$*.*

We refer to [20] for the proof and further details. It is also interesting to note that a result similar to that of the Theorem 1.6.1 holds for a function such as

$$f(x,u,\xi) = (x^4 - u^6)|\xi|^s + \varepsilon|\xi|^2,$$

with $\varepsilon > 0$ and $s \geq 27$. This last example has the advantage of leading to an integral which grows from below as $|\xi|^2$, i.e., it is *coercive* in $W^{1,2}$, while this is not the case in the previous theorem.

For a nice survey on the Lavrentiev phenomenon see, for istance, [14], while for several examples of this phenomenon in the setting of functionals with (p,q)-growth see [84, 85].

Chapter II

PRELIMINARY MATERIAL

In this chapter we recall some standard definitions and collect several Lemmas that we shall need to establish the main results of this thesis. We shall indicate with $B_R(x_0)$ the ball centered at the point $x_0 \in \mathbb{R}^n$ and having radius $R > 0$. We shall omit the center of the ball when no confusion arises. All the balls considered will be concentric unless differently specified.

As usual $\{e_s\}_{1 \le s \le n}$ is the standard basis in $\mathbb{R}^n$ and if $u, v \in \mathbb{R}^k$ the tensor product $u \otimes v \in \mathbb{R}^{k^2}$ of u and v is defined by $(u \otimes v)_{i,j} := v_i w_j$.

In the estimates c will denote a constant, depending on the data of the problem, that may change from line to line.

2.1 Function spaces

We first recall the definition of the Sobolev space $W^{1,p}$.

Definition 2.1.1. If A is a smooth, bounded open subset of $\mathbb{R}^n$ and $1 \le p \le +\infty$, a function u belongs to the Sobolev space $W^{1,p}(A;\mathbb{R}^N)$, if and only if $u \in L^p(A;\mathbb{R}^N)$ and the weak partial derivatives $\frac{\partial u}{\partial x_i}$ exist and belong to $L^p(A;\mathbb{R})$, $i \in \{1,\dots,n\}$.

If we define

$$||u||_{W^{1,p}} := \left(\int_A (|u(x)|^p + |Du(x)|^p) \, dx \right)^{\frac{1}{p}}$$

for $1 \le p < +\infty$, and

$$||u||_{W^{1,\infty}} := ess \sup_A (|u| + |Du|),$$

we obtain a norm making $W^{1,p}(A;\mathbb{R}^N)$ a Banach space.

In order to prove regularity of minimizers for functional with (p,q)-growth conditions, in the sense of higher integrability and higher differentiability properties of minimizers, we shall need the machinery of fractional order Sobolev spaces.

Regularity results of this kind are important to reduce the dimension of the singular set or minimizers and they often are useful in the proof of $C^{1,\alpha}$ partial regularity results too.

Today the thoeory of fractional Sobolev spaces is well developed and the reader may look at Chapter 7 of [6]. In particular, the fractional Sobolev spaces we are going to use extensively, named Nikol'skii spaces, are treated in the Section 7.7.3. These spaces are defined as follows.

Definition 2.1.2. If A is a smooth, bounded open subset of $\mathbb{R}^n$ and $\theta \in (0,1)$, $1 \leq p < +\infty$ a function u belongs to the fractional order Sobolev space $W^{\theta,p}(A;\mathbb{R}^N)$ if and only if

$$||u||_{W^{\theta,p}} := \left(\int_A |u(x)|^p\,dx\right)^{\frac{1}{p}} + \left(\int_A\int_A \frac{|u(x)-u(y)|^p}{|x-y|^{n+p\theta}}\,dx\,dy\right)^{\frac{1}{p}} < \infty.$$

This quantity is a norm making $W^{\theta,p}(A;\mathbb{R}^N)$ a Banach space.

In the context of fractional order Sobolev spaces we have to use fractional difference quotients. Therefore we recall the finite difference operator.

Definition 2.1.3. For every vector valued function $F : \mathbb{R}^n \to \mathbb{R}^N$ the finite difference operator is defined by

$$\tau_{s,h}F(x) = F(x + he_s) - F(x)$$

where $h \in \mathbb{R}$, e_s is the unit vector in the x_s direction and $s \in \{1,\ldots,n\}$.

The difference quotient is defined for $h \in \mathbb{R}\setminus\{0\}$ as

$$\Delta_{s,h}F(x) = \frac{\tau_{s,h}F(x)}{h}.$$

The following proposition describes some elementary properties of the finite difference operator and can be found, for example, in the Section 8.1 of the book [51] by Giusti.

Proposition 2.1.4. *Let F and G be two functions such that $F, G \in W^{1,p}(\Omega)$, with $p \geq 1$, and let us consider the set*

$$\Omega_{|h|} := \{x \in \Omega : dist(x, \partial\Omega) > |h|\}.$$

Then

(d1) $\tau_{s,h}F \in W^{1,p}(\Omega)$ *and*

$$D_i(\tau_{s,h}F) = \tau_{s,h}(D_i F).$$

(d2) If at least one of the functions F or G has support contained in $\Omega_{|h|}$ then

$$\int_\Omega F\,\tau_{s,h}G\,dx = -\int_\Omega G\,\tau_{s,-h}F\,dx.$$

(d3) We have

$$\tau_{s,h}(FG)(x) = F(x + he_s)\tau_{s,h}G(x) + G(x)\tau_{s,h}F(x).$$

Next Lemma was proved in [3]. (See Lemma 2.2).

Lemma 2.1.5. *For every $\gamma \in (-1/2, 0)$ and $\mu \geq 0$ we have*

$$(2\gamma+1)|\xi-\eta| \leq \frac{|(\mu^2+|\xi|^2)^\gamma \xi - (\mu^2+|\eta|^2)^\gamma \eta|}{(\mu^2+|\xi|^2+|\eta|^2)^\gamma} \leq \frac{c(k)}{2\gamma+1}|\xi-\eta|$$

for every $\xi, \eta \in \mathbb{R}^k$.

The next result about finite difference operator is a kind of integral version of Lagrange Theorem (see Lemma 8.1 in [51]).

Lemma 2.1.6. *If* $0 < \rho < R$, $|h| < \frac{R-\rho}{2}$, $1 < p < +\infty$, $s \in \{1, \ldots, n\}$ *and* $F, D_s F \in L^p(B_R)$ *then*

$$\int_{B_\rho} |\tau_{s,h} F(x)|^p \, dx \le |h|^p \int_{B_R} |D_s F(x)|^p \, dx. \tag{2.1.1}$$

Moreover

$$\int_{B_\rho} |F(x + he_s)|^p \, dx \le c(n,p) \int_{B_R} |F(x)|^p \, dx. \tag{2.1.2}$$

Next Lemma, useful to estimate the different quotient of a function, is of particular interest for us.

Lemma 2.1.7. *For every* $p > 1$ *and* $G : B_R \to \mathbb{R}^k$ *there exists a positive constant* $c \equiv c(k,p)$ *such that*

$$|\tau_{s,h}((1 + |G(x)|^2)^{(p-2)/4} G(x))|^2 \le c(1 + |G(x)|^2 + |G(x + he_s)|^2)^{(p-2)/2} |\tau_{s,h} G(x)|^2$$

for every $x \in B_\rho$, *with* $|h| < \frac{R-\rho}{2}$ *and every* $s \in \{1, \ldots, n\}$.

Now we recall the fundamental embedding properties for fractional Sobolev spaces. (For the proof we refer to Chapter 7 of [6]).

Lemma 2.1.8. *If* $F : \mathbb{R}^n \to \mathbb{R}^N$, $F \in L^2(B_R)$ *and for some* $\rho \in (0, R)$, $\beta \in (0, 1]$, $M > 0$,

$$\sum_{s=1}^{n} \int_{B_\rho} |\tau_{s,h} F(x)|^2 \, dx \le M^2 |h|^{2\beta},$$

for every h *with* $|h| < \frac{R-\rho}{2}$, *then*

$$F \in W^{k,2}(B_\rho; \mathbb{R}^N) \cap L^{\frac{2n}{n-2k}}(B_\rho; \mathbb{R}^N),$$

for every $k \in (0, \beta)$ *and*

$$||F||_{L^{\frac{2n}{n-2k}}(B_\rho)} \le c \left(M + ||F||_{L^2(B_R)} \right),$$

with $c \equiv c(n, N, R, \rho, \beta, k)$.

Previous Lemma can be reformulated as follows

Lemma 2.1.9. *If $F : \mathbb{R}^n \to \mathbb{R}^N$, $F \in L^p(B_R)$ with $1 < p < +\infty$ and for some $\rho \in (0, R)$, $\beta \in (0, 1]$, $M > 0$,*

$$\sum_{s=1}^{n} \int_{B_\rho} |\tau_{s,h} F(x)|^p \, dx \le M^p |h|^{p\beta}$$

for every h with $|h| < \frac{R-\rho}{2}$, then

$$F \in W^{k,p}(B_\rho; \mathbb{R}^N) \cap L^{\frac{np}{n-kp}}(B_\rho; \mathbb{R}^N),$$

for every $k \in (0, \beta)$ and

$$||F||_{L^{\frac{np}{n-kp}}(B_\rho)} \le c \left(M + ||F||_{L^p(B_R)} \right),$$

with $c \equiv c(n, N, R, \rho, \beta, k)$.

To study regularity for functionals with almost linear growth, the main tool are the Orlicz-Sobolev spaces (for more details on this topic we refer to [6], Chapter 8).

Definition 2.1.10. a) A function $\varphi : [0, \infty) \to [0, \infty)$ is called a Young function, if φ is strictly increasing, convex and satisfies

$$\lim_{t \to 0} \frac{\varphi(t)}{t} = \lim_{t \to \infty} \frac{t}{\varphi(t)} = 0.$$

b) If φ satisfies in addition a global (Δ_2)-condition, i.e.

$$\varphi(2t) \le c\varphi(t) \quad \text{for all } t \ge 0,$$

then we define

$$L_\varphi(\Omega, \mathbb{R}^N) := \left\{ u \in L_1(\Omega, \mathbb{R}^N) : \int_\Omega \varphi(|u|) \, dx < \infty \right\},$$

which is a Banach-space together with the Luxemburg norm

$$\|u\|_\varphi := \inf \left\{ k \ge 0 : \int_\Omega \varphi \left(\frac{|u|}{k} \right) \, dx \le 1 \right\}.$$

c) A function $u : \Omega \to \mathbb{R}^N$ belongs to the space $W^{1,\varphi}(\Omega, \mathbb{R}^N)$ if $u \in L_\varphi(\Omega, \mathbb{R}^N)$ and its distributional gradient $Du \in L_\varphi(\Omega, \mathbb{R}^{nN})$. $W^{1,\varphi}(\Omega, \mathbb{R}^N)$ is a Banach-space together with the norm

$$\|u\|_{1,\varphi} := \|u\|_\varphi + \|Du\|_\varphi.$$

d) We define $W_0^{1,\varphi}(\Omega, \mathbb{R}^N)$ as the closure of $C_0^\infty(\Omega, \mathbb{R}^N)$ with respect to the $W^{1,\varphi}(\Omega, \mathbb{R}^N)$-norm.

2.2 Blow-up tools

In this section, we collect several tools which we shall extensively use in the blow-up procedure we shall perfom in the next chapters. As we saw in the first chapter, the proof of regularity results essentially consists in three steps, and the first two steps rely on the proof of a Caccioppoli type inequality and in the improvement of such an inequality.

Next Lemma finds an important application in the so called hole-filling method. Its proof can be found in [51] (See Lemma 6.1). This method is often used to obtain Caccioppoli type inequalities.

Lemma 2.2.1. *Let $h : [\rho, R_0] \to \mathbb{R}$ be a non-negative bounded function and $0 < \theta < 1$, $0 \le A$, $0 \le B$ and $0 < \beta$. Assume that*

$$h(r) \le \frac{A}{(d-r)^\beta} + B + \theta h(d)$$

for $\rho \le r < d \le R_0$. Then

$$h(\rho) \le \frac{cA}{(R_0-\rho)^\beta} + B,$$

where $c = c(\theta, \beta) > 0$.

Since we treat functionals with growth exponents $p > 1$ and even almost linear growth, we shall need the following Poincaré-Sobolev inequality, whose proof can be found in [27] (for other versions of this inequality we refer to [17, 16]).

Lemma 2.2.2. *Assume* $1 < p < 2$ *and let* $u \in W^{1,p}(\Omega, \mathbb{R}^N)$*. Then there exists a positive constant* $c \equiv c(n, N, p)$ *such that*

$$\left(\fint_{B_\rho(x_0)} \left| V_p \left(\frac{u - (u)_\rho}{\rho} \right) \right|^{\frac{2n}{n-p}} dx \right)^{\frac{n-p}{2n}} \leq c \left(\fint_{B_\rho(x_0)} |V(Du)|^2 \, dx \right)^{\frac{1}{2}}.$$

Next result is important in the comparison of the minimizer of the functional with the smooth solution of the linear elliptic system with constant coefficients coming out of the linearization procedure required by the blow-up method. It is a simple consequence of the a priori estimates for solutions to linear elliptic systems with constant coefficients.

Proposition 2.2.3. *Let* $u \in W^{1,p}(\Omega; \mathbb{R}^N)$*,* $p \geq 1$ *be such that*

$$\int_\Omega A^{ij}_{\alpha\beta} D_\alpha u^i D_\beta \varphi^j \, dx = 0$$

for every $\varphi \in C^\infty_0(\Omega; \mathbb{R}^N)$*, where* $A^{ij}_{\alpha\beta}$ *is a constant matrix satisfying the strong Legendre Hadamard condition*

$$A^{ij}_{\alpha\beta} \lambda^i \lambda^j \mu_\alpha \mu_\beta \geq \nu |\lambda|^2 |\mu|^2 \qquad \forall \lambda \in \mathbb{R}^N \,, \; \mu \in \mathbb{R}^n.$$

Then $u \in C^\infty$ *and for any ball* $B_R(x_0) \Subset \Omega$ *we have*

$$\sup_{B_{\frac{R}{2}}(x_0)} |Du| \leq \frac{c}{R^n} \int_{B_R} |Du| \, dx$$

For the proof see [46, 51] in case $p \geq 2$ and see [16, 17] in case $1 \leq p < 2$.

2.3 An important auxiliary function

We shall use the following auxiliary function, which is a common tool in treating functionals with subquadratic growth. For $\xi \in \mathbb{R}^k$, we define

$$V_\beta(\xi) = (1 + |\xi|^2)^{\frac{\beta-2}{4}}\xi,$$

for any exponent $\beta \geq 1$. Recall that for $\beta > 1$

$$|V_\beta(\xi)| \text{ is a non-decreasing function of } |\xi|; \tag{2.3.1}$$

$$|V_\beta(\xi + \eta)| \leq c(\beta)(|V_\beta(\xi)| + |V_\beta(\eta)|); \tag{2.3.2}$$

$$\min\{t^2, t^\beta\}|V_\beta(\xi)|^2 \leq |V_\beta(t\xi)|^2 \leq \max\{t^2, t^\beta\}|V_\beta(\xi)|^2; \tag{2.3.3}$$

$$(1 + |\xi|^2 + |\eta|^2)^{\frac{\beta}{2}} \leq 1 + (1 + |\xi|^2 + |\eta|^2)^{\frac{\beta-2}{2}}(|\xi|^2 + |\eta|^2) \qquad \text{if} \quad \beta \leq 2; \tag{2.3.4}$$

$$c(\beta)(|\xi|^2 + |\xi|^\beta) \leq |V_\beta(\xi)|^2 \leq C(\beta)(|\xi|^2 + |\xi|^\beta) \qquad \text{if} \quad \beta \geq 2; \tag{2.3.5}$$

$$|V_\beta(\xi)|^2 \qquad \text{is convex if} \qquad 1 < \beta < 2. \tag{2.3.6}$$

Many of the previous properties of the function V_β can be easily checked and they have been successfully employed in the study of the regularity of minimizers of convex and quasiconvex integrals under subquadratic growth conditions ([3, 17, 16, 79]).

Next Lemma can be found in a slightly different form in [36] (Lemma 2.2), see also [77] and [79], and it will be crucial in the following chapters. In fact it will allow us to construct admissible test functions needed to establish Caccioppoli type inequalities.

Lemma 2.3.1. *Let $0 < r < s < 1$ and let $v \in W^{1,p}(B_1(0); \mathbb{R}^N)$. If $1 < p \leq q < \frac{pn}{n-1}$ there exist a function $w \in W^{1,p}(B_1(0); \mathbb{R}^N)$ and two radii $0 < r < r' < s' <$*

$s < 1$ depending on v such that

$$w = \begin{cases} v & in \quad B_{r'} \\ \\ v & in \quad B_1 \setminus B_{s'} \end{cases} \tag{2.3.7}$$

$$\frac{s-r}{3} \leq s' - r' \leq s - r$$

and

$$\int_{B_s \setminus B_r} |w|^p \, dx \leq c(n,p) \int_{B_s \setminus B_r} |v|^p \, dx; \tag{2.3.8}$$

$$\int_{B_s \setminus B_r} |Dw|^p \, dx \leq c(n,p) \int_{B_s \setminus B_r} |Dv|^p \, dx. \tag{2.3.9}$$

Moreover if $p \geq 2$ we have

$$\int_{B_{s'} \setminus B_{r'}} |w|^q \, dx \leq c(n,p,q)(s-r)^{n\left(1-\frac{q}{p}\right)} \left(\int_{B_s \setminus B_r} |v|^p \, dx \right)^{\frac{q}{p}}; \tag{2.3.10}$$

$$\int_{B_{s'} \setminus B_{r'}} |Dw|^q \, dx \leq c(n,p,q)(s-r)^{n\left(1-\frac{q}{p}\right)} \left(\int_{B_s \setminus B_r} |Dv|^p \, dx \right)^{\frac{q}{p}}. \tag{2.3.11}$$

While, in case $1 < p < 2$, we have that

$$\int_{B_s \setminus B_r} |V_p(w)|^2 \, dx \leq c(n,p) \int_{B_s \setminus B_r} |V_p(v)|^2 \, dx. \tag{2.3.12}$$

$$\int_{B_s \setminus B_r} |V_p(Dw)|^2 \, dx \leq c(n,p) \int_{B_s \setminus B_r} |V_p(Dv)|^2 \, dx. \tag{2.3.13}$$

$$\int_{B_{s'} \setminus B_{r'}} |V_p(w)|^{\frac{2q}{p}} \, dx \leq c(n,p,q)(s-r)^{n\left(1-\frac{q}{p}\right)} \left(\int_{B_s \setminus B_r} |V_p(v)|^2 \, dx \right)^{\frac{q}{p}}; \tag{2.3.14}$$

$$\int_{B_{s'} \setminus B_{r'}} |V_p(Dw)|^{\frac{2q}{p}} \, dx \leq c(n,p,q)(s-r)^{n\left(1-\frac{q}{p}\right)} \left(\int_{B_s \setminus B_r} |V_p(Dv)|^2 \, dx \right)^{\frac{q}{p}}. \tag{2.3.15}$$

In the study of functionals with subquadratic growth, the following inequality will also be useful. It is standard a standard result if $p \geq 2$ and can be inferred from [3] (Lemma 2.2) in the case $1 < p < 2$.

Lemma 2.3.2. *For $\beta > 1$ and $\eta, \xi \in \mathbb{R}^{N\times n}$ there holds*

$$C_1(1+|\eta|^2+|\xi|^2)^{\frac{\beta-2}{2}} \le \int_0^1 (1+|\eta+t\xi|^2)^{\frac{\beta-2}{2}}\,dt \le C_2(1+|\eta|^2+|\xi|^2)^{\frac{\beta-2}{2}}$$

with some positive constants C_1, C_2 depending only on β.

When we shall treat functionals with almost linear growth, the following two elementary inequalities will also be useful.

Lemma 2.3.3. *Set*

$$V_p(\xi) = (1+|\xi|^2)^{\frac{p-2}{4}}\xi.$$

Then for every $\rho > 0$ and function v with the suitable integrability degree, we have

$$\int_{B_\rho} |V_p(Dv)|^2\,dx \le c\int_{B_\rho} |V_1(Dv)|^2\,dx + c\int_{B_\rho} |V_1(Dv)|^{2p}\,dx.$$

Proof. We start by noting that

$$(1+|\xi|^2)^{\frac{1}{2}} \le c[1+(1+|\xi|^2)^{-\frac{1}{2}}|\xi|^2]. \tag{2.3.16}$$

Indeed if $|\xi| \le 1$ we have

$$(1+|\xi|^2)^{\frac{1}{2}} \le \sqrt{2},$$

while, if $|\xi| > 1$ we have

$$(1+|\xi|^2)^{\frac{1}{2}} = \frac{1+|\xi|^2}{(1+|\xi|^2)^{\frac{1}{2}}} \le \frac{2|\xi|^2}{(1+|\xi|^2)^{\frac{1}{2}}}.$$

Hence, recalling that $p > 1$, we can conclude that

$$\begin{aligned}
\int_{B_\rho} |V_p(Dv)|^2\,dx &= \int_{B_\rho} |Dv|^2(1+|Dv|^2)^{\frac{p-2}{2}}\,dx \\
&= \int_{B_\rho} |Dv|^2(1+|Dv|^2)^{-\frac{1}{2}}(1+|Dv|^2)^{\frac{p-1}{2}}\,dx \\
&\le \int_{B_\rho} |Dv|^2(1+|Dv|^2)^{-\frac{1}{2}}\left[1+|Dv|^2(1+|Dv|^2)^{-\frac{1}{2}}\right]^{p-1}\,dx \\
&\le \int_{B_\rho} |Dv|^2(1+|Dv|^2)^{-\frac{1}{2}}\,dx
\end{aligned}$$

$$+ c\int_{B_\rho} \left(\left(|Dv|^2(1+|Dv|^2) \right)^{-\frac{1}{2}} \right)^p \, dx$$

where we also used (2.3.16). □

Lemma 2.3.4. *For every $x \geq 0$ and $1 < p < 2$ we have*

$$\log(1+x) \leq Mx(1+x^2)^{\frac{p-2}{2}}.$$

Proof. The function

$$\varphi(x) = \frac{\log(1+x)}{x}(1+x^2)^{\frac{p-2}{2}}$$

is nonnegative for every $x > 0$ and

$$\lim_{x \to 0^+} \varphi(x) = 1.$$

Moreover, since $p < 2$, we have

$$\lim_{x \to +\infty} \varphi(x) = 0.$$

Since φ is continuous, there exists $M \geq 0$ such that $\varphi(x) \leq M$ for every $x \in [0, +\infty]$. Hence the conclusion follows. □

2.4 The singular set of minimizers

Let us recall that the singular set Σ of a local minimizer u of the functional $\mathcal{F}$ is included in the set of non-Lebesgue points of Du.

In the sequel we shall obtain estimates on the Hausdorff dimension of the singular set of minimizers, by applying the following proposition that can be found, for example, in [59] (see also Section 4 in [68] for a simple proof).

Lemma 2.4.1. *Let $v \in W^{\theta,p}(\Omega, \mathbb{R}^N)$ where $\theta \in (0,1)$, $p > 1$ and set*

$$A := \left\{ x \in \Omega : \limsup_{\rho \to 0^+} \fint_{B(x,\rho)} |v(y) - (v)_{x,\rho}|^p \, dy > 0 \right\},$$

$$B := \left\{ x \in \Omega : \limsup_{\rho \to 0^+} |(v)_{x,\rho}| = +\infty \right\}.$$

Then

$$dim_{\mathcal{H}}(A) \leq n - \theta p \quad and \quad dim_{\mathcal{H}}(B) \leq n - \theta p.$$

As it is clear from the statement of the previous proposition and as we already observed at the beginning of this chapter, a key role in the estimate the Hausdorff dimension of the singular set is played by higher integrability/differentiability property of minimizers. In this context the best way to prove such regularity results is to work in fractional order Sobolev spaces.

Chapter III

PARTIAL REGULARITY THROUGH HIGHER INTEGRABILITY: THE DIFFERENCE QUOTIENT TECHNIQUE

The results of this chapter have been obtained in [24]. We prove $C^{1,\alpha}$ partial regularity of minimizers of the functional (0.0.1), passing through higher integrability of minimizers. The proof of the higher integrability heavily relies on the difference quotient technique.

Throughout the chapter we shall assume that the integrand f is a $C^2(\Omega\times\mathbb{R}^{n\times N})$ function satisfying the non standard growth conditions (F1)–(F3).

We recall that in the paper [31] by Esposito, Leonetti and Mingione, under the above assumptions on f, it has been proved that a minimizer $u \in W^{1,p}_{loc}(\Omega)$ of $\mathcal{F}$ actually belongs to $W^{1,q}_{loc}(\Omega)$ if $\frac{q}{p} < \frac{n+\alpha}{n}$, provided that for the functional $\mathcal{F}$ does not occur the Lavrentiev Phenomenon. More precisely, introducing for a fixed ball $B_R \subset\subset \Omega$ and for every $u \in W^{1,p}(B_R)$ the gap functional relative to $\mathcal{F}$:

$$\mathcal{L}(u, B_R) := \bar{\mathcal{F}}(u) - \mathcal{F}(u), \qquad \mathcal{L}(u, B_R) := 0 \quad \text{if} \quad \mathcal{F}(u) = +\infty$$

where $\bar{\mathcal{F}}$ is the sequentially lower semicontinuous (s.l.s.) envelope of $\mathcal{F}$:

$$\bar{\mathcal{F}} := \sup\big\{\mathcal{G} : W^{1,p}(B_R) \to [0, +\infty] : \mathcal{G} \quad \text{is} \quad \text{s.l.s.}, \quad \mathcal{G} \le \mathcal{F} \quad \text{on} \\ W^{1,p}(B_R) \cap W^{1,q}(B_R)\big\},$$

the requirement is that:

$$\mathcal{L}(u, B_R) = 0, \quad \text{for any} \quad B_R \subset\subset \Omega. \tag{F4}$$

When the dependence on x is allowed, it is clear that a bound similar to (0.0.2) has to be assumed with $c(n)$ replaced by $c(n, \alpha)$ where α is the Hölder continuity exponent appearing in $(F3)$. More precisely Esposito, Leonetti and Mingione proved in [31] that a sufficient condition in order to have that a $W^{1,p}$ local minimizer of $\mathcal{F}$ belongs to $W^{1,q}$ is

$$\frac{q}{p} < \frac{n+\alpha}{n}. \tag{3.0.1}$$

Actually, by mean of a counterexample, in [31] the authors showed that (3.0.1) cannot be avoided in order to prove higher integrability of minimizers. In fact, if $\frac{q}{p} > \frac{n+\alpha}{n}$ there are local minimizers $u \in W^{1,p}_{loc}$ of suitable functionals such that $u \notin W^{1,q}_{loc}$.

In [11], assuming (3.0.1), Bildhauer and Fuchs proved $C^{1,\mu}$ partial regularity assuming that $D_\xi f$ is Lipschitz continuous with respect to x and that the second derivative of f with respect to ξ have a $(q-2)$-power type growth. These assumptions are stronger than the usual when one tries to establish $C^{1,\mu}$ partial regularity results.

The aim of the present chapter is to remove these stronger assumptions on f showing that $C^{1,\mu}$ partial regularity still hold for minimizers. In fact we are able to prove the following

Theorem 3.0.2. *Let $f \in C^2(\Omega \times \mathbb{R}^{n\times N})$ satisfy the assumptions (F1), (F2), (F3), (F4) and let $u \in W^{1,p}_{loc}(\Omega; \mathbb{R}^N)$ be a local minimizer of $\mathcal{F}$. Assume that $p \geq 2$ and* (3.0.1) *holds. Then there exists an open subset Ω_0 of Ω such that*

$$|\Omega \setminus \Omega_0| = 0$$

and $u \in C^{1,\mu}(\Omega_0; \mathbb{R}^N)$ for some $\mu \in (0,1)$.

Our proof is based on a decay estimate for the excess function defined in (0.0.7), and that we recall for the reader's convenience:

$$E(x,r) = \fint_{B_r(x)} |Du - (Du)_r|^2 + |Du - (Du)_r|^p + r^\beta,$$

with $\beta < \alpha$, where α is the Hölder continuity exponent appearing in (F3).

We shall prove the decay estimate by using a standard argument consisting in blowing up the solution in small balls and reducing the problem to the study of convergence of minimizers of a suitable rescaled functionals in the unit ball. A useful tool in order to let this argument work is the higher integrability of the minimizers of the rescaled functionals. Note that we need an higher integrability result which is uniform with respect to the rescaling procedure. Hence we cannot use the result in [31] and the higher integrability result will be proved in Proposition 3.1.1.

We also mention that by the method introduced in [59] we are able to estimate the Hausdorff dimension of the singular set. In fact we have the following

Theorem 3.0.3. *Under the same assumptions on f, p and q as in Theorem 3.0.2, if $u \in W^{1,p}_{loc}(\Omega;\mathbb{R}^N)$ is a local minimizer of $\mathcal{F}$ then*

$$dim_{\mathcal{H}}(\Omega \setminus \Omega_0) < n - \frac{\alpha}{2}p \tag{3.0.2}$$

where α is the exponent appearing in (F3).

We recall the definition of local minimizer for a functional with nonstandard growth conditions.

Definition 3.0.4. A function $u \in W^{1,1}_{loc}(\Omega,\mathbb{R}^N)$ is a local minimizer of $\mathcal{F}$ if $x \to f(x, Du(x)) \in L^1_{loc}(\Omega)$ and

$$\int_{supp\,\varphi} f(x, Du)\,dx \le \int_{supp\,\varphi} f(x, Du + D\varphi)\,dx,$$

for any $\varphi \in W^{1,1}_{loc}(\Omega,\mathbb{R}^N)$ with $supp\,\varphi \subset \Omega$.

Now, for our future needs, we introduce the rescaled functional on the unit ball $B \equiv B_1(0)$

$$\mathcal{I}(v) := \int_B g(y, Dv)\, dy$$

where

$$g(y,\xi) = \frac{f(x_0 + r_0 y, A + \lambda\xi) - f(x_0 + r_0 y, A) - D_\xi f(x_0 + r_0 y, A)\lambda\xi}{\lambda^2}. \quad (3.0.3)$$

Here A is a matrix such that $|A|$ is uniformly bounded by a positive constant M and λ is a parameter such that $0 < \lambda < 1$. Next Lemma contains the growth conditions on g.

Lemma 3.0.5. *Let $p \geq 2$ and let $f \in C^2(\Omega \times \mathbb{R}^{n\times N})$ be a function satisfying the assumptions (F1), (F2) and (F3). Let $g(y,\xi)$ be defined by (3.0.3) then we have*

$$c(|\xi|^2 + \lambda^{p-2}|\xi|^p) \leq g(y,\xi) \leq c(|\xi|^2 + \lambda^{q-2}|\xi|^q); \quad \text{(I1)}$$

$$|D_\xi g(y,\xi)| \leq c(|\xi| + \lambda^{q-2}|\xi|^{q-1}); \quad \text{(I2)}$$

$$|D_\xi g(y_1,\xi) - D_\xi g(y_2,\xi)| \leq c\frac{r_0^\alpha}{\lambda}\,(1 + \lambda^{q-1}|\xi|^{q-1})|y_1 - y_2|^\alpha; \quad \text{(I3)}$$

$$c(1 + \lambda^2|\xi|^2)^{\frac{p-2}{2}}|\zeta|^2 \leq \left\langle D_{\xi\xi} g(y,\xi)\zeta, \zeta\right\rangle \quad \text{(I4)}$$

where the constant c depends on M and on q.

Proof. The (I1) can be proved as in Lemma 2.3 of [4] and the (I2) is an immediate consequence of the convexity of g.

Now we prove (I3). Thanks to the definition of g we have that

$$D_\xi g(y,\xi) = \frac{1}{\lambda}[D_\xi f(x_0 + r_0 y, A + \lambda\xi) - D_\xi f(x_0 + r_0 y, A)].$$

So by (F3) we get

$$\begin{aligned}|D_\xi g(y_1,\xi)-D_\xi g(y_2,\xi)| \leq & \frac{1}{\lambda}|D_\xi f(x_0+r_0y_1,A+\lambda\xi)-D_\xi f(x_0+r_0y_2,A+\lambda\xi)| \\ & +\frac{1}{\lambda}|D_\xi f(x_0+r_0y_1,A)-D_\xi f(x_0+r_0y_2,A)| \\ \leq & \frac{r_0^\alpha}{\lambda}|y_1-y_2|^\alpha[(1+|A+\lambda\xi|^{q-1})+(1+|A|^{q-1})] \\ \leq & \frac{r_0^\alpha}{\lambda}|y_1-y_2|^\alpha(c(M)+\lambda^{q-1}|\xi|^{q-1}) \\ \leq & c\frac{r_0^\alpha}{\lambda}|y_1-y_2|^\alpha(1+\lambda^{q-1}|\xi|^{q-1}).\end{aligned}$$

where the constant c depends on M and on q.

To prove the (I4) it is enough to develop the second derivatives of g with respect to ξ and to observe that

$$D_{\xi\xi}g(y,\xi)=D_{\xi\xi}f(x_0+r_0y,A+\lambda\xi).$$

So we are led to the ellipticity condition (F2) on f. □

We shall denote by MF the Hardy-Littlewood maximal function of a function $F\in L^1_{loc}(\mathbb{R}^n)$, which is defined as

$$MF(x)=\sup_{x\in Q}⨍_Q |F(y)|\,dy,$$

where the supremum is taken over all cubes $Q\subset\mathbb{R}^n$, with sides parallel to coordinate axes.

The following Lemma can be found in [1].

Lemma 3.0.6. *Let $u\in W^{1,p}(\mathbb{R}^n,\mathbb{R}^N)$ and $p\geq 1$. For every $K>0$, if we set*

$$H_K=\{x\in\mathbb{R}^n\,:\,M(|Du|)\leq K\}\,,$$

then there exists $v\in W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^N)$ such that $||Dv||_\infty\leq c\,K$, $v=u$ on H_K, and

$$meas(\mathbb{R}^n\setminus H_K)\leq\frac{c\,||Dv||^p_{L^p}}{K^p}.$$

3.1 Higher integrability

The first step in the proof of Theorem 3.0.2 is to obtain an higher integrability result for minimizers of the rescaled functional $\mathcal{I}$. To be more precise we need this result for the following perturbation or $\mathcal{I}$

$$\mathcal{J}(v) := \int_{B_{\tilde{R}}} g(y, Dv)\, dy + \int_{B_{\tilde{R}}} \frac{D_\xi f(x_0 + r_0 y, A)}{\lambda}(Dw - Dv)\, dy.$$

where $v \in w + W_0^{1,q}(B_{\tilde{R}})$ and x_0, r_0, A are the same appearing in the definition of g, therefore $|A| \leq M$.

We obtain the higher integrability with the following

Proposition 3.1.1. *Let us suppose that $g \in C^2(B_1(0), \mathbb{R}^{n\times N})$ satisfies the assumptions (I1), (I2), (I3) and (I4) with $2 \leq p \leq q < p\left(\frac{n+\alpha}{n}\right)$. If the function $v \in W^{1,q}(\Omega; \mathbb{R}^N)$ is a local minimizer of $\mathcal{J}$ then there exist $\delta > 0$, $\sigma > 0$ such that*

$$\int_{B_\rho} (|Dv(y)|^2 + \lambda^{q-2}|Dv(y)|^q)^{1+\delta}\, dy \leq c\left(\int_{B_R} (1 + |Dv(y)|^2 + \lambda^{p-2}|Dv(y)|^p)\, dy\right)^\sigma \tag{3.1.1}$$

for every $B_R \subset\subset \Omega$, $\rho < R$ and for a positive constant c which depends on ρ and R but does not depend on v and it is also independent of the parameters λ, r_0 and of the point x_0 appearing in the definition of $g(y, \xi)$.

Proof. Let us fix a ball $B_{\tilde{R}} \subset\subset \Omega$; by the minimality of $v \in W^{1,q}(\Omega; \mathbb{R}^N)$ we have

$$\int_{B_{\tilde{R}}} g(y, Dv)\, dy \leq \int_{B_{\tilde{R}}} g(y, Dv + D\varphi)\, dy + \int_{B_{\tilde{R}}} \frac{D_\xi f(x_0 + r_0 y, A)}{\lambda} D\varphi\, dy. \tag{3.1.2}$$

for every $\varphi \in W_0^{1,q}(B_{\tilde{R}}; \mathbb{R}^N)$. For a fixed $\varepsilon \in (0, 1)$ we can write (3.1.2) as follows

$$\int_{B_{\tilde{R}}} [g(y, Dv + \varepsilon\, D\varphi) - g(y, Dv)]\, dy + \int_{\tilde{R}} \frac{D_\xi f(x_0 + r_0 y, A)}{\lambda} \varepsilon D\varphi\, dy \geq 0$$

which is equivalent to

$$\int_{B_{\tilde{R}}} \int_0^1 D_\xi g(y, Dv + \varepsilon t D\varphi)\, \varepsilon\, D\varphi\, dt\, dy + \int_{B_{\tilde{R}}} \frac{D_\xi f(x_0 + r_0 y, A)}{\lambda} \varepsilon D\varphi\, dy \geq 0.$$

Dividing the previous inequality by ε, changing φ in $-\varphi$ and taking the limit as $\varepsilon \to 0^+$, thanks to the assumption of continuity of the function $D_\xi g$, we get the Euler-Lagrange equations

$$\int_{B_{\tilde{R}}} D_\xi g(y, Dv)\, D\varphi\, dy + \int_{B_{\tilde{R}}} \frac{D_\xi f(x_0 + r_0 y, A)}{\lambda} D\varphi\, dy = 0. \tag{3.1.3}$$

Let us pick $0 < \rho \le r < d \le \tilde{R} \le 1$ and let η be a cut-off function in $C_0^\infty(B_{\frac{d+r}{2}})$ with $0 \le \eta \le 1$, $\eta \equiv 1$ on B_r and $|D\eta| < 4/(d-r)$. Let us consider the function $\varphi = \tau_{s,-h}(\eta^2 \tau_{s,h} v)$ with s fixed in $\{1, \dots, n\}$ (which from now on we shall omit for the sake of simplicity) and $0 \le |h| < (d-r)/4$. Now we plug such function φ into (3.1.3) and use $(d1)$ and $(d2)$ of Propostion 2.1.4 to get

$$\begin{aligned} &-\int_{B_{\tilde{R}}} \tau_h \left(D_\xi g(y, Dv)\right) D(\eta^2 \tau_h v)\, dy \\ &\qquad - \frac{1}{\lambda} \int_{B_{\tilde{R}}} \left[D_\xi f(x_0 + r_0(y + h e_s), A) - D_\xi f(x_0 + r_0 y, A)\right] \cdot D(\eta^2 \tau_h v)\ dy = 0. \end{aligned}$$

We develop the derivatives inside the first integral and use the Hölder continuity condition (F3) and the bound $|A| \le M$ into the second one obtaining the estimate

$$\begin{aligned} \int_{B_{\tilde{R}}} \eta^2\, \tau_h \left(D_\xi g(y, Dv)\right)\, \tau_h Dv\, dy \le& -2 \int_{B_{\tilde{R}}} \eta\, \tau_h (D_\xi g(y, Dv))\, D\eta \otimes \tau_h v\, dy \\ &+ c\, \frac{r_0^\alpha}{\lambda}\, |h|^\alpha \int_{B_{\tilde{R}}} |D(\eta^2 \tau_h v)|\ dy \end{aligned} \tag{3.1.4}$$

with the constant c depending on M. Observing that

$$\begin{aligned} &\int_{B_{\tilde{R}}} \eta^2\, \tau_h \left(D_\xi g(y, Dv)\right)\, \tau_h Dv\, dy \\ &\quad = \int_{B_{\tilde{R}}} \eta^2\ \left[D_\xi g(y + h e_s, Dv(y + h e_s)) - D_\xi g(y, Dv(y))\right]\, \tau_h Dv\, dy \\ &\quad = \int_{B_{\tilde{R}}} \eta^2\ \left[D_\xi g(y + h e_s, Dv(y + h e_s)) - D_\xi g(y, Dv(y + h e_s))\right]\, \tau_h Dv\, dy \\ &\qquad + \int_{B_{\tilde{R}}} \eta^2\ \left[D_\xi g(y, Dv(y + h e_s)) - D_\xi g(y, Dv(y))\right]\, \tau_h Dv\, dy \end{aligned}$$

we can write (3.1.4) as

$$\int_{B_R}\int_0^1 \eta^2 [D_{\xi\xi} g(y, Dv + t\tau_h Dv)] D(\tau_h v) D(\tau_h v)\, dt\, dy$$

$$\leq -\int_{B_{\tilde R}} \eta^2\; [D_\xi g(y+he_s, Dv(y+he_s)) - D_\xi g(y, Dv(y+he_s))]\cdot \tau_h Dv\, dy$$

$$-\,2\int_{B_{\tilde R}} \eta\, \tau_h (D_\xi g(y, Dv))\, D\eta \otimes \tau_h v\, dy + c\,\frac{r_0^\alpha}{\lambda}\,|h|^\alpha \int_{B_{\tilde R}} |D(\eta^2 \tau_h v)|\; dy.$$

Now we use ellipticity condition (I4) in the left hand side and the growth conditions (I2) and (I3) in the right hand side. Thus the following estimate holds:

$$\int_{B_{\tilde R}} \eta^2 (1 + \lambda^2 |Dv(y)|^2 + \lambda^2 |Dv(y+he_s)|^2)^{\frac{p-2}{2}} |\tau_h Dv|^2\, dy$$

$$\leq c\,|h|^\alpha\, \frac{r_0^\alpha}{\lambda} \int_{B_{\tilde R}} \eta^2 (1 + \lambda^{q-1} |Dv(y+he_s)|^{q-1})\, |\tau_h Dv|\, dy$$

$$+\, c\int_{B_{\tilde R}} \eta\, |D\eta| (|Dv(y)| + |Dv(y+he_s)| +$$

$$\lambda^{q-2} |Dv(y)|^{q-1} + \lambda^{q-2} |Dv(y+he_s)|^{q-1})\, |\tau_h v|\, dy$$

$$+\, c\,\frac{r_0^\alpha}{\lambda}\,|h|^\alpha \int_{B_{\frac{d+r}{2}}} |D(\eta^2 \tau_h v)|\; dy$$

$$:= (I) + (II) + (III), \tag{3.1.5}$$

with $c \equiv c(n, N, p, q, L, \nu, M)$.

The use of Lemma 2.1.7 in the left hand side of (3.1.5) yields

$$\int_{B_{\tilde R}} \eta^2 |\tau_h ((1 + \lambda^2 |Dv|^2)^{\frac{p-2}{4}} Dv)|^2\; dy \leq (I) + (II) + (III). \tag{3.1.6}$$

We have to estimate the integrals (I), (II) and (III). For (I) we simply use the definition of $\tau_h Dv$ and remember how we choose $|h|$ so that we can apply (2.1.2) of Lemma 2.1.6 as follows

$$(I) \leq c\,|h|^\alpha\, \frac{r_0^\alpha}{\lambda} \int_{B_d} (|Dv(y)| + \lambda^{q-1} |Dv(y)|^q)\, dy$$

where $c \equiv c(n, N, p, q, L, \nu, M)$.

To estimate (II) we remember the assumptions on $|D\eta|$ and use triangle inequality which yields

$$
\begin{aligned}
(II) \leq & \frac{c}{(d-r)} \int_{B_{\frac{d+r}{2}}} (|Dv(y)| + |Dv(y+he_s)|) \, |\tau_h v| \, dy \\
& + \frac{c}{(d-r)} \lambda^{q-2} \int_{B_{\frac{d+r}{2}}} (|Dv(y)|^{q-1} + |Dv(y+he_s)|^{q-1}) \, |\tau_h v| \, dy
\end{aligned}
$$

then we apply Hölder inequality to each integral and we use (2.1.1) of Lemma 2.1.6 in each of the resulting addend, thus getting:

$$
\begin{aligned}
(II) \leq & \frac{c}{(d-r)} \, |h| \left(\int_{B_d} |Dv(y)|^2 \, dy \right)^{\frac{1}{2}} \left(\int_{B_d} |Dv(y)|^2 \, dy \right)^{\frac{1}{2}} \\
& + \frac{c}{(d-r)} \, |h| \, \lambda^{q-2} \left(\int_{B_d} |Dv(y)|^q \, dy \right)^{1-\frac{1}{q}} \left(\int_{B_d} |Dv(y)|^q \, dy \right)^{\frac{1}{q}}
\end{aligned}
$$

that is

$$
(II) \leq \frac{c}{(d-r)} \, |h| \left(\int_{B_d} (|Dv(y)|^2 + \lambda^{q-2} |Dv(y)|^q) \, dy \right)
$$

To estimate (III) we develop the derivative inside the integral and use triangle inequality, the assumptions on η and $|D\eta|$ and (2.1.1) of Lemma (2.1.6):

$$
\begin{aligned}
(III) & \leq c \, |h|^\alpha \, \frac{r_0^\alpha}{\lambda} \int_{B_{\frac{d+r}{2}}} |\tau_h Dv(y)| \, dy + \frac{c}{(d-r)} \, |h|^\alpha \frac{r_0^\alpha}{\lambda} \int_{B_{\frac{d+r}{2}}} |\tau_h v(y)| \, dy \\
& \leq c \, |h|^\alpha \, \frac{r_0^\alpha}{\lambda} \int_{B_d} |Dv(y)| \, dy
\end{aligned}
$$

where we also used the assumption $|h| < \frac{d-r}{4}$.

Collecting the estimates for (I), (II) and (III) and summing up on $s \in \{1, \dots, n\}$ we get, in place of (3.1.6), the following estimate

$$
\begin{aligned}
& \int_{B_{\tilde{R}}} \sum_{s=1}^n \eta^2 |\tau_{s,h}((1 + \lambda^2 |Dv|^2)^{\frac{p-2}{4}} Dv)|^2 \, dy \\
& \qquad \leq c \, |h|^\alpha \, \frac{r_0^\alpha}{\lambda} \int_{B_d} (|Dv(y)| + \lambda^{q-1} |Dv(y)|^q) \, dy \\
& \qquad\qquad + \frac{c}{(d-r)} \, |h| \int_{B_d} (|Dv(y)|^2 + \lambda^{q-2} |Dv|^q) \, dy,
\end{aligned}
$$

with $c \equiv c(n, N, r, d, p, q, L, \nu, M)$ independent of v, λ, r_0 and x_0. Notice that, in what follows, we shall have $\frac{r_0^\alpha}{\lambda} < 1$. Now we apply Lemma 2.1.8 and find that

$$(1 + \lambda^2 |Dv|^2)^{\frac{p-2}{4}} Dv \in L^{\frac{2n}{n-2\theta}}(B_r), \qquad \forall\, \theta \in \left(0, \frac{\alpha}{2}\right) \tag{3.1.7}$$

and

$$\int_{B_r} ((1 + \lambda^2 |Dv|^2)^{\frac{p-2}{2}} |Dv|^2)^{\frac{n}{n-2\theta}} dy \leq c \left(\int_{B_d} (1 + |Dv(y)|^2 + \lambda^{q-2} |Dv(y)|^q) dy \right)^{\frac{n}{n-2\theta}},$$

where

$$\frac{n}{n - 2\theta} > 1.$$

But, since $2 \leq p \leq q$ we have

$$\begin{aligned} (J) &:= \int_{B_r} (|Dv(y)|^2 + \lambda^{p-2} |Dv(y)|^p)^{\frac{n}{n-2\theta}}\, dy \\ &\leq c \left(\int_{B_d} (1 + |Dv(y)|^2 + \lambda^{q-2} |Dv(y)|^q)\, dy \right)^{\frac{n}{n-2\theta}}. \end{aligned} \tag{3.1.8}$$

Now we are going to estimate (J) from below in order to have an inequality which can be used to perform the same iteration procedure of [31]. We have

$$\begin{aligned} (J) &= \int_{B_r} (|Dv(y)|^{2\frac{q}{p}\frac{p}{q}} + \lambda^{(p-2)\frac{q}{p}\frac{p}{q}} |Dv(y)|^{p\frac{p}{q}\frac{q}{p}})^{\frac{n}{n-2\theta}}\, dy \\ &\geq c(p,q) \int_{B_r} (|Dv(y)|^{2\frac{q}{p}} + \lambda^{(p-2)\frac{q}{p}} |Dv(y)|^q)^{\frac{p}{q}\frac{n}{n-2\theta}}\, dy \end{aligned} \tag{3.1.9}$$

where we used the elementary inequality

$$(a^p + b^p) \geq c(p)(a+b)^p, \qquad \forall\, p > 0.$$

Thus we have

$$(J') := \int_{B_r} (1 + |Dv(y)|^{2\frac{q}{p}} + \lambda^{(p-2)\frac{q}{p}} |Dv(y)|^q)^{\frac{p}{q}\frac{n}{n-2\theta}}\, dy$$

$$\leq \; c\left(\int_{B_d}(1+|Dv(y)|^2+\lambda^{q-2}|Dv(y)|^q)\,dy\right)^{\frac{n}{n-2\theta}} \qquad (3.1.10)$$

But we also have

$$(J') \;\geq\; c\int_{B_r}((1+|Dv(y)|^2)^{\frac{q}{p}}+\lambda^{(p-2)\frac{q}{p}}|Dv(y)|^q)^{\frac{p}{q}\frac{n}{n-2\theta}}\,dy$$

$$\geq\; c\int_{B_r}(1+|Dv(y)|^2+\lambda^{(p-2)\frac{q}{p}}|Dv(y)|^q)^{\frac{p}{q}\frac{n}{n-2\theta}}\,dy \qquad (3.1.11)$$

since

$$(1+|Dv(y)|^2)^{\frac{q}{p}} \geq 1+|Dv(y)|^2.$$

Now we remember that $0<\lambda<1$ and observe that

$$(p-2)\frac{q}{p} \leq q-2$$

since $2\leq p\leq q$, so that

$$\lambda^{q-2} < \lambda^{(p-2)\frac{q}{p}}.$$

Hence we conclude that

$$c\int_{B_r}(1+|Dv(y)|^2+\lambda^{(p-2)\frac{q}{p}}|Dv(y)|^q)^{\frac{p}{q}\frac{n}{n-2\theta}}\,dy$$
$$\geq c\int_{B_r}(1+|Dv(y)|^2+\lambda^{q-2}|Dv(y)|^q)^{\frac{p}{q}\frac{n}{n-2\theta}}\,dy. \qquad (3.1.12)$$

Collecting (3.1.8), (3.1.9), (3.1.10), (3.1.11) and (3.1.12) we can conclude that

$$\int_{B_r}(1+|Dv(y)|^2+\lambda^{q-2}|Dv(y)|^q)^{\frac{p}{q}\frac{n}{n-2\theta}}\,dy$$
$$\leq c\left(\int_{B_d}(1+|Dv(y)|^2+\lambda^{q-2}|Dv(y)|^q)\,dy\right)^{\frac{n}{n-2\theta}}.$$

From here we can complete the proof using exactly the same iteration scheme of [31] with the same exponents. □

3.2 Decay estimate

Let $u \in W^{1,p}_{loc}(\Omega)$ be a local minimizer of $\mathcal{F}$ under the assumptions (F1), (F2), (F3), (F4) and recall that its excess function is defined as in (0.0.7):

$$E(x,r) = \fint_{B_r(x)} |Du - (Du)_r|^2 + |Du - (Du)_r|^p + r^\beta \tag{3.2.1}$$

with $\beta < \alpha$.

As usual the proof of Theorem 3.0.2 relies on a blow up argument which is contained in the following

Proposition 3.2.1. *Fix $M > 0$. There exists a constant $C(M) > 0$ such that, for every $0 < \tau < \frac{1}{4}$, there exists $\varepsilon = \varepsilon(\tau, M)$ such that, if*

$$|(Du)_{x_0,r}| \leq M \qquad \textit{and} \qquad E(x_0, r) \leq \varepsilon,$$

then

$$E(x_0, \tau r) \leq C(M)\, \tau^\beta\, E(x_0, r).$$

Proof. Step 1. Blow up

Fix $M > 0$. Assume by contradiction that there exists a sequence of balls $B_{r_j}(x_j) \subset\subset \Omega$ such that

$$|(Du)_{x_j,r_j}| \leq M \qquad \text{and} \qquad \lambda_j^2 = E(x_j, r_j) \to 0 \tag{3.2.2}$$

but

$$\frac{E(x_j, \tau r_j)}{\lambda_j^2} > \tilde{C}(M)\tau^\beta \tag{3.2.3}$$

where $\tilde{C}(M)$ will be determined later. Setting $A_j = (Du)_{x_j,r_j}$, $a_j = (u)_{x_j,r_j}$ and

$$v_j(y) = \frac{u(x_j + r_j y) - a_j - r_j A_j y}{\lambda_j r_j} \tag{3.2.4}$$

for all $y \in B_1(0)$, one can easily check that $(Dv_j)_{0,1} = 0$ and $(v_j)_{0,1} = 0$. By the definition of λ_j at (3.2.2), we get

$$\fint_{B_1(0)} |Dv_j|^2 + \lambda_j^{p-2}|Dv_j|^p \, dy + \frac{r_j^\beta}{\lambda_j^2} = 1 \tag{3.2.5}$$

Therefore passing possibly to not relabeled sequences

$$v_j \rightharpoonup v \qquad \text{weakly in } W^{1,2}(B_1(0);\mathbb{R}^N)$$

$$A_j \longrightarrow A$$

$$r_j \longrightarrow 0 \qquad \frac{r_j^\gamma}{\lambda_h^2} \longrightarrow 0, \qquad \gamma > \beta. \tag{3.2.6}$$

Step 2. Minimality of v_j

We normalize f around A_j as follows

$$f_j(y,\xi) = \frac{f(x_j + r_j y, A_j + \lambda_j \xi) - f(x_j + r_j y, A_j) - D_\xi f(x_j + r_j y, A_j)\lambda_j \xi}{\lambda_j^2} \tag{3.2.7}$$

and we consider the corresponding rescaled functionals

$$\mathcal{I}_j(w) = \int_{B_1(0)} [f_j(y, Dw)]dy. \tag{3.2.8}$$

Observe that Lemma 3.0.5 applies to each f_j thus having that (I1), (I2), (I3), (I4) hold for f_j. The minimality of u yields that

$$\int_{B_1(0)} f(x_j+r_j y, Du(x_j+r_j y))\, dy \le \int_{B_1(0)} f(x_j+r_j y, Du(x_j+r_j y)+D\varphi(x_j+r_j y))\, dy$$

for every $\varphi \in W^{1,q}(B_{r_j}(x_j);\mathbb{R}^N)$ that is

$$\int_{B_1(0)} f(x_j + r_j y, A_j + \lambda_j Dv_j(y))\, dy$$
$$\le \int_{B_1(0)} f(x_j + r_j y, A_j + \lambda_j Dv_j(y) + D\varphi(x_j + r_j y))\, dy$$

for every $\varphi \in W^{1,q}(B_{r_j}(x_j);\mathbb{R}^N)$. Thus by the definition of the rescaled functionals, we have

$$\mathcal{I}_j(v_j) \leq \mathcal{I}_j(v_j+\varphi) + \int_{B_1(0)} \frac{D_\xi f(x_j + r_j y, A_j) D\varphi}{\lambda_j}\, dy. \tag{3.2.9}$$

Hence using (I3)

$$\begin{aligned} \mathcal{I}_j(v_j) &\leq \mathcal{I}_j(v_j+\varphi) + \int_{B_1(0)} \frac{[D_\xi f(x_j + r_j y, A_j) - D_\xi f(x_j, A_j)] D\varphi}{\lambda_j}\, dy \\ &\leq \mathcal{I}_j(v_j+\varphi) + c(M)\frac{r_j^\alpha}{\lambda_j} \int_{B_1(0)} |D\varphi|\, dy. \end{aligned} \tag{3.2.10}$$

Step 3. Higher integrability

Since $u \in W^{1,p}_{loc}(\Omega)$ is a local minimizer of $\mathcal{F}$ under the assumptions (F1), (F2), (F3), (F4), by Theorem 4 in [31], $u \in W^{1,q}(B_{r_j}(x_j))$. Therefore, by a simple change of variables, we also have that each $v_j \in W^{1,q}(B_1)$. Moreover, since v_j satisfy (3.2.9) and f_j satisfy (I1), (I2), (I3) and (I4), we are legitimate to apply Theorem 3.1.1. Hence there exist $\delta > 0$ and $\sigma > 0$ such that for all $\rho < 1$

$$\begin{aligned} &\int_{B_\rho} (|Dv_j(y)|^2 + \lambda^{q-2}|Dv_j(y)|^q)^{1+\delta}\, dy \\ &\qquad \leq c\left(\int_{B_1} (1 + |Dv_j(y)|^2 + \lambda^{p-2}|Dv_j(y)|^p)\, dy\right)^\sigma \end{aligned} \tag{3.2.11}$$

with c depending on M and ρ. But (3.2.5) yields

$$\int_{B_\rho} (|Dv_j(y)|^2 + \lambda^{q-2}|Dv_j(y)|^q)^{1+\delta}\, dy \leq c,$$

for every ball B_ρ contained in B_1. ¿From that we obtain

$$v_j \rightharpoonup v \qquad \text{weakly in } W^{1,2(1+\delta)}_{loc}(B_1(0);\mathbb{R}^N).$$

Step 4. v solves a linear system

Using that v_j satisfies inequality (3.2.10), we conclude that

$$0 \leq \frac{c}{\lambda_j} \int_{B_1(0)} [D_\xi f(x_j + r_j y, A_j + \lambda_j Dv_j) - D_\xi f(x_j + r_j y, A_j)] D\varphi\, dy$$

$$+\frac{c(M)r_j^\alpha}{\lambda_j}\int_{B_1(0)}|D\varphi|dy. \tag{3.2.12}$$

Following the argument in [2, 77], let us split

$$B_1(0)=E_j^+\cup E_j^-=\{y\in B_1:\ \lambda_j|Dv_j|>1\}\cup\{y\in B_1:\ \lambda_j|Dv_j|\le 1\}$$

By (3.2.5) we get

$$|E_j^+|\le\int_{E_j^+}\lambda_j^2|Dv_j|^2\,dy\le\lambda_j^2\int_{E_j^+}|Dv_j|^2\,dy\le\lambda_j^2. \tag{3.2.13}$$

By assumption (F1) and the convexity of f, applying Hölder's inequality we obtain

$$\begin{aligned}&\frac{1}{\lambda_j}\left|\int_{E_j^+}[D_\xi f(x_j+r_jy,A_j+\lambda_jDv_j)-D_\xi f(x_j+r_jy,A_j)]D\varphi\,dy\right|\\ \le&\frac{c}{\lambda_j}|E_j^+|+c\lambda_j^{q-2}\int_{E_j^+}|Dv_j|^{q-1}\,dy\le\frac{c}{\lambda_j}|E_j^+|+c\lambda_j^{q-2}\left(\int_{E_j^+}|Dv_j|^q\,dy\right)^{\frac{q-1}{q}}|E_j^+|^{\frac{1}{q}}\\ \le&c\lambda_j\left(1+\left(\lambda_j^{q-2}\int_{E_j^+}|Dv_j|^q\,dy\right)^{\frac{q-1}{q}}\right).\end{aligned} \tag{3.2.14}$$

The last term in (3.2.14) vanishes as $j\to\infty$. In fact, the higher integrability at (3.2.11) implies that

$$\lambda_j^{q-2}\int_{E_j^+}|Dv_j|^q\,dy\le c.$$

Hence we infer that

$$\lim_{j\to\infty}\frac{c}{\lambda_j}\left|\int_{E_j^+}[D_\xi f(x_j+r_jy,A_j+\lambda_jDv_j)-D_\xi f(x_j+r_jy,A_j)]D\varphi\,dy\right|=0. \tag{3.2.15}$$

On E_j^- we have

$$\begin{aligned}&\frac{1}{\lambda_j}\int_{E_j^-}[D_\xi f(x_j+r_jy,A_j+\lambda_jDv_j)-D_\xi f(x_j+r_jy,A_j)]D\varphi\,dy\\ =&\int_{E_j^-}\int_0^1 D_{\xi\xi}f(x_j+r_jy,A_j+t\lambda_jDv_j)\,dtDv_jD\varphi\,dy\end{aligned} \tag{3.2.16}$$

Note that (3.2.13) yields that $\chi_{E_j^-} \to \chi_{B_1}$ in L^r, for every $r < \infty$. Moreover by (3.2.6) we have, at least for subsequences, that

$$\lambda_j Dv_j \to 0 \quad \text{a.e. in } B_1, \quad r_j \to 0 \quad \text{and} \quad x_j \to x_0.$$

Hence the uniform continuity of $D_{\xi\xi}f$ on bounded sets implies

$$\lim_j \frac{1}{\lambda_j} \int_{E_j^-} [D_\xi f(x_j + r_j y, A_j + \lambda_j Dv_j) - D_\xi f(x_j + r_j y, A_j)] D\varphi \, dy$$
$$= \int_{B_1} D_{\xi\xi} f(x_0, A) Dv D\varphi \, dy. \tag{3.2.17}$$

Observe that by (3.2.6)

$$\lim_j \frac{r_j^\alpha}{\lambda_j} = 0. \tag{3.2.18}$$

By estimates (3.2.15), (3.2.17) and (3.2.18), passing to the limit as $j \to \infty$ in (3.2.12) yields

$$0 \le \int_{B_1} D_{\xi\xi} f(x_0, A) Dv D\varphi \, dy$$

Changing φ in $-\varphi$ we finally get

$$\int_{B_1} D_{\xi\xi} f(x_0, A) Dv D\varphi \, dy = 0,$$

that is v solves a linear system which is elliptic thank to the convexity of f. Classical regularity results (see [45], [51]) imply that $v \in C^\infty(B_1)$ and for any $0 < \tau < 1$

$$\fint_{B_\tau} |Dv - (Dv)_\tau|^2 \, dy \le c\tau^2 \fint_{B_1} |Dv - (Dv)_1|^2 \, dy \le c\tau^2, \tag{3.2.19}$$

for a constant c depending on M.

Step 5. Upper bound

Let us fix $r < \frac{1}{4}$. Passing to a subsequence, it is not restrictive to assume that

$$\lim_j [\mathcal{I}_{j,r}(v_j) - \mathcal{I}_{j,r}(v)]$$

exists. We shall prove that

$$\lim_j [\mathcal{I}_{j,r}(v_j) - \mathcal{I}_{j,r}(v)] \le 0 \tag{3.2.20}$$

Let us choose $s < r$ and a cut-off function $\eta \in C_0^1(B_r)$ such that $\eta = 1$ on B_s, $0 \le \eta \le 1$ and $|D\eta| \le \frac{c}{r-s}$. Using in (3.2.10) as test function $\varphi_j = \eta(v - v_j)$, we get

$$\mathcal{I}_{j,r}(v_j) - \mathcal{I}_{j,r}(v) \le \mathcal{I}_{j,r}(v_j + \varphi_j) - \mathcal{I}_{j,r}(v) + \frac{c(M) r_j^\alpha}{\lambda_j} \int_{B_r} |D\varphi_j| dy$$

$$\begin{aligned}
&\le \int_{B_r \setminus B_s} [f_j(y, Dv_j + D\varphi_j) - f_j(y, Dv)]\, dy + \frac{c(M) r_j^\alpha}{\lambda_j} \int_{B_r} |D\varphi_j| dy \\
&\le c \int_{B_r \setminus B_s} (|Dv_j|^2 + \lambda_j^{q-2} |Dv_j|^q)\, dy + c \int_{B_r \setminus B_s} (|Dv|^2 + \lambda_j^{q-2} |Dv|^q)\, dy \\
&+ c \int_{B_r \setminus B_s} \left(\frac{|v_j - v|^2}{(r-s)^2} + \lambda_j^{q-2} \frac{|v_j - v)|^q}{(r-s)^2} \right) dy + \frac{c(M) r_j^\alpha}{\lambda_j} \int_{B_r} |Dv_j - Dv|\, dy \\
&+ \frac{c(M) r_j^\alpha}{\lambda_j (r-s)} \int_{B_r \setminus B_s} |v_j - v|\, dy,
\end{aligned} \tag{3.2.21}$$

thanks to the growth conditions on f_j. Now, we use (3.2.5) and (3.2.11) in order to have

$$\begin{aligned}
\int_{B_r \setminus B_s} (|Dv_j|^2 + \lambda_j^{q-2} |Dv_j|^q) dy &\le \left(\int_{B_r \setminus B_s} (|Dv_j|^2 + \lambda_j^{q-2} |Dv_j|^q)^{(1+\delta)} dy \right)^{\frac{1}{1+\delta}} |B_r \setminus B_s|^{\frac{\delta}{1+\delta}} \\
&\le c(r-s)^{\frac{\delta}{1+\delta}}.
\end{aligned} \tag{3.2.22}$$

Moreover, since $v \in C^\infty(B_1)$, we get

$$\int_{B_r \setminus B_s} (|Dv|^2 + \lambda_j^{q-2} |Dv|^q)\, dy \le c \left[1 + \sup_{B_r} |Dv|^2 \right] (r-s). \tag{3.2.23}$$

For the third integral in (3.2.21) we have that

$$c \left(\int_{B_r \setminus B_s} \frac{|v_j - v|^2}{(r-s)^2}\, dy + \lambda_j^{q-2} \int_{B_r \setminus B_s} \frac{|v_j - v|^q}{r-s)^q\, dy} \right) = I_j + II_j. \tag{3.2.24}$$

Note that, by (3.2.6), $v_j \to v$ strongly in $L^2(B_1)$, hence

$$\lim_j I_j = 0. \tag{3.2.25}$$

Moreover denoting by

$$q^* = \begin{cases} \frac{nq}{n-q} & if\ q < n \\ \\ r > q & if\ q \geq n \end{cases}$$

there exists $\mu \in (0,1)$ such that $\frac{1}{q} = \frac{\mu}{q^*} + \frac{1-\mu}{2}$. Using Hölder and Sobolev Poincaré inequalities we get

$$\begin{aligned} II_j &\leq \lambda_j^{q-2} \left(\int_{B_1} |v_j - v|^2 \, dy \right)^{\frac{q(1-\mu)}{2}} \left(\int_{B_1} |v_j - v|^{q^*} \, dy \right)^{\frac{q\mu}{q^*}} \\ &\leq c\lambda_j^{q-2} \left(\int_{B_1} |v_j - v - (v_j - v)_{B_1}|^{q^*} dy \right)^{\frac{q\mu}{q^*}} + c\lambda_j^{q-2} \left(\int_{B_1} |(v_j - v)_{B_1}|^{q^*} dy \right)^{\frac{q\mu}{q^*}} \\ &\leq c\lambda_j^{q-2} \left(\int_{B_1} |Dv_j - Dv|^q \, dy \right)^{\mu} + c\lambda_j^{q-2} \leq c\lambda_j^{q-2} \left(\int_{B_1} |Dv_j|^q \, dy \right)^{\mu} + c\lambda_j^{q-2} \\ &\leq c\lambda_j^{(q-2)(1-\mu)}. \end{aligned} \tag{3.2.26}$$

Since $0 < \mu < 1$ we obtain

$$\lim_j II_j = 0 \tag{3.2.27}$$

Moreover we have that

$$\begin{aligned} & \frac{c(M) r_j^\alpha}{\lambda_j} \int_{B_r} |Dv_j - Dv| dy + \frac{c(M) r_j^\alpha}{\lambda_j (r-s)} \int_{B_r \setminus B_s} |v_j - v| \, dy \\ \leq\ & \frac{c(M) r_j^\alpha}{\lambda_j} \left(\int_{B_1} |Dv_j|^2 dy \right)^{\frac{1}{2}} + \frac{c(M) r_j^\alpha}{\lambda_j} \left(\int_{B_1} |Dv|^2 dy \right)^{\frac{1}{2}} \\ +\ & \frac{c(M) r_j^\alpha}{\lambda_j (r-s)} \left(\int_{B_r \setminus B_s} |v_j - v|^2 \, dy \right)^{\frac{1}{2}} (r-s)^{\frac{1}{2}}. \end{aligned} \tag{3.2.28}$$

Hence, using that $\lim_j \frac{r_j^\alpha}{\lambda_j} = 0$, the fact that $v \in C^\infty(B_1)$ and (3.2.5) we get that the right hand side of (3.2.28) vanishes as $j \to \infty$. Therefore we conclude with (3.2.20), taking first the limit as $j \to \infty$ and then as $s \to r$ in (3.2.21).

Step 6. Lower bound

We claim that for $t < r < \frac{1}{4}$ we have

$$\limsup_j \int_{B_t} |Dv_j - Dv|^2 + \lambda_j^{p-2}|Dv_j - Dv|^p \, dy \le c \limsup_j [\mathcal{I}_{j,r}(v_j) - \mathcal{I}_{j,r}(v)].$$

Let us choose a cut-off function $\phi \in C_0^1(B_{\frac{1}{2}})$ such that $\phi = 1$ on $B_{\frac{1}{4}}$, $0 \le \phi \le 1$ and $|D\phi| \le c$. Set

$$\tilde{v}_j = \phi v_j \qquad\qquad \tilde{v} = \phi v.$$

We can always suppose that the higher integrability exponent δ of (3.2.11) is such that $2(1+\delta) < q^*$, so we may apply Sobolev-Poincaré inequality to have that

$$\int_{\mathbb{R}^n} (|D\tilde{v}_j|^2 + \lambda_j^{q-2}|D\tilde{v}_j|^q)^{1+\delta} \, dy \le c. \tag{3.2.29}$$

Fix $k > 0$. By Lemma 3.0.6 we can find a sequence $(w_j) \in W^{1,\infty}(\mathbb{R}^n; \mathbb{R}^N)$ such that if $S_{j,k} = \{y \in \mathbb{R}^n : \; M(|D\tilde{v}_j|) > k\}$ then

$$w_j = \tilde{v}_j \qquad \text{on } \mathbb{R}^n \setminus S_{j,k} \tag{3.2.30}$$

and

$$||Dw_j||_\infty \le c(n)k. \tag{3.2.31}$$

Passing to a subsequence we may suppose that

$$w_j \rightharpoonup w \qquad\qquad \text{weakly}^* \text{ in } W^{1,\infty}(\mathbb{R}^n; \mathbb{R}^N). \tag{3.2.32}$$

By the maximal theorem and (3.2.29) we deduce that

$$\int_{\mathbb{R}^n} (M(|D\tilde{v}_j|)^2 + \lambda_j^{q-2} M(|D\tilde{v}_j|)^q)^{1+\delta} \, dy \le c, \tag{3.2.33}$$

hence the sequences

$$\left\{(|D\tilde{v}_j|^2+\lambda_j^{q-2}|D\tilde{v}_j|^q)\right\}, \qquad \left\{(M(|D\tilde{v}_j|)^2+\lambda_j^{q-2}M(|D\tilde{v}_j|)^q)\right\}$$

are uniformly bounded in $L^{1+\delta}(\mathbb{R}^n)$ and therefore also equiabsolutely continuous in $L^1(\mathbb{R}^n)$. Then

$$\lim_{k\to\infty}\int_{S_{j,k}}(|D\tilde{v}_j|^2+\lambda_j^{q-2}|D\tilde{v}_j|^q)\,dy=\lim_{k\to\infty}\int_{S_{j,k}}(M(|D\tilde{v}_j|)^2+\lambda_j^{q-2}M(|D\tilde{v}_j|)^q)\,dy=0.$$

Fix $\varepsilon>0$ and observe that

$$\exists k_\varepsilon:\ \text{if } k\geq k_\varepsilon,\ \forall j\quad \int_{S_{j,k}}(M(|D\tilde{v}_j|)^2+\lambda_j^{q-2}M(|D\tilde{v}_j|)^q)\,dy<\varepsilon. \tag{3.2.34}$$

Therefore, from the definition of $S_{j,k}$, for k sufficiently large we get

$$|S_{j,k}|k^2\leq\int_{S_{j,k}}M(|D\tilde{v}_j|)^2\leq\varepsilon$$

and so

$$|S_{j,k}|<\frac{\varepsilon}{k^2}. \tag{3.2.35}$$

Let us write

$$\begin{aligned}\mathcal{I}_{j,r}(v_j)-\mathcal{I}_{j,r}(v)=&[\mathcal{I}_{j,r}(\tilde{v}_j)-\mathcal{I}_{j,r}(w_j)]+[\mathcal{I}_{j,r}(w_j)-\mathcal{I}_{j,r}(w)]+[\mathcal{I}_{j,r}(w)-\mathcal{I}_{j,r}(v)]\\=&R_j^1+R_j^2+R_j^3.\end{aligned} \tag{3.2.36}$$

Now, by (3.2.30) and (3.2.31), we have

$$\begin{aligned}|R_j^1|\leq\int_{S_{j,k}\cap B_r}|f_j(y,D\tilde{v}_j)-f_j(y,Dw_j)|\,dy\leq&\int_{S_{j,k}\cap B_r}(|D\tilde{v}_j|^2+\lambda_j^{q-2}|D\tilde{v}_j|^q)\,dy\\&+\int_{S_{j,k}\cap B_r}(|Dw_j|^2+\lambda_j^{q-2}|Dw_j|^q)\,dy\\\leq\int_{S_{j,k}\cap B_r}(|D\tilde{v}_j|^2+\lambda_j^{q-2}|D\tilde{v}_j|^q)\,dy+ck^2|S_{j,k}|&\end{aligned} \tag{3.2.37}$$

since for every $k>k_\varepsilon$ there exists $j_0=j_0(\varepsilon)$ such that

$$j>j_0\ \Rightarrow\ |Dw_j|^2+\lambda_j^{q-2}|Dw_j|^q\leq 2k^2.$$

Therefore, by (3.2.34) and (3.2.35) we get

$$\lim_{k\to\infty} \sup_j |R_j^1| \leq \varepsilon. \tag{3.2.38}$$

Choose $s < r$ and take ζ a cut-off function between B_s and B_r. Define

$$\psi_j = \zeta(w_j - w)$$

and split R_j^2 as follows:

$$\begin{aligned} R_j^2 &= [\mathcal{I}_{j,r}(w_j) - \mathcal{I}_{j,r}(w + \psi_j)] + [\mathcal{I}_{j,r}(w + \psi_j) - \mathcal{I}_{j,r}(w) - \mathcal{I}_{j,r}(\psi_j)] + \mathcal{I}_{j,r}(\psi_j) \\ &= R_j^4 + R_j^5 + R_j^6. \end{aligned} \tag{3.2.39}$$

Then, by (3.2.31), (3.2.32) and the growth conditions on f_j, we have

$$\begin{aligned} |R_j^4| &\leq \int_{B_r \setminus B_s} |f_j(y, Dw_j) - f_j(y, Dw + D\psi_j)|\, dy \leq \int_{B_r \setminus B_s} (|Dw_j|^2 + \lambda_j^{q-2} |Dw_j|^q)\, dy \\ &+ \int_{B_r \setminus B_s} (|Dw|^2 + \lambda_j^{q-2} |Dw|^q)\, dy + \int_{B_r \setminus B_s} (|w - w_j|^2 + \lambda_j^{q-2} |w - w_j|^q)\, dy \\ &\leq c(k)|B_r \setminus B_s| + \int_{B_r \setminus B_s} (|w - w_j|^2 + \lambda_j^{q-2} |w - w_j|^q)\, dy. \end{aligned} \tag{3.2.40}$$

Using (3.2.32), we conclude that

$$\limsup_j |R_j^4| \leq c(k)|B_r \setminus B_s|. \tag{3.2.41}$$

To bound R_j^5, we use the definition of f_j in order to have

$$|R_j^5| = \int_{B_r} dy \int_0^1 \int_0^1 D^2 f(x_j + r_j y, A_j + s\lambda_j Dw + t\lambda_j D\psi_j) Dw D\psi_j \, ds\, dt. \tag{3.2.42}$$

Hence

$$\limsup_j |R_j^5| = 0 \tag{3.2.43}$$

thank to (3.2.32), since $D^2 f(x_j + r_j y, A_j + s\lambda_j Dw + t\lambda_j D\psi_j)$ uniformly converges to $D^2 f(x_0, A)$.

On the other hand, by (I1), we get

$$|R_j^6| = \mathcal{I}_{j,r}(\psi_j) = \int_{B_r} f_j(y, D\psi_j)\, dy \geq \int_{B_s} (|Dw_j - Dw|^2 + \lambda_j^{p-2}|Dw_j - Dw|^p)\, dy. \tag{3.2.44}$$

Therefore, passing possibly to a subsequence, we may suppose that $\lim_j R_j^2$ exists and collecting estimates (3.2.41), (3.2.43) and (3.2.44), we obtain

$$\lim_j R_j^2 \geq \limsup_j \int_{B_s} (|Dw_j - Dw|^2 + \lambda_j^{p-2}|Dw_j - Dw|^p)\, dy - c(k)(r-s). \tag{3.2.45}$$

Setting $S = \{y \in B_r : \ v(y) \neq w(y)\}$ and $\tilde{S} = S \cap \{y \in B_r : \ v(y) \neq \lim_j v_j(y)\}$ we have $|S| = |\tilde{S}|$. We claim that

$$|S| \leq \frac{2\varepsilon}{k^2}. \tag{3.2.46}$$

In fact, suppose by contradiction that $|S| > \frac{2\varepsilon}{k^2}$. Then by (3.2.35) for j large enough we would have

$$|\tilde{S} \setminus S_{j,k}| > \frac{\varepsilon}{k^2}.$$

But by Lemma 3.0.6 there exists $\bar{y} \in B_r$ such that $\bar{y} \in \tilde{S} \setminus S_{j,k}$ for infinitely many j and hence

$$v(\bar{y}) = w(\bar{y})$$

and this is a contradiction. Since $Dv = Dw$ in $B_r \setminus S$, we have

$$\begin{aligned} |R_j^3| &\leq \int_{B_r \cap S} |f_j(y, Dw) - f_j(y, Dv)|\, dy \leq \int_{B_r \cap S} (|Dw|^2 + \lambda_j^{q-2}|Dw|^q)\, dy \\ &\quad + \int_{B_r \cap S} (|Dv|^2 + \lambda_j^{q-2}|Dv|^q)\, dy \leq c|S| \leq \frac{c\varepsilon}{k^2}. \end{aligned} \tag{3.2.47}$$

Estimates (3.2.38), (3.2.45) and (3.2.47) leads us to

$$\begin{aligned} \lim_j [\mathcal{I}_{j,r}(v_j) - \mathcal{I}_{j,r}(v)] &\geq -\frac{c\varepsilon}{k^2} - c(k)(r-s) \\ &\quad + \limsup_j \int_{B_s} (|Dw_j - Dw|^2 + \lambda_j^{p-2}|Dw_j - Dw|^p)\, dy. \end{aligned} \tag{3.2.48}$$

Now, if $t < s < r$ we have that

$$
\begin{aligned}
\int_{B_t} (|Dv_j - Dv|^2 + &\lambda_j^{p-2}|Dv_j - Dv|^p)\, dy \\
&\leq \int_{B_s} (|Dw_j - Dw|^2 + \lambda_j^{p-2}|Dw_j - Dw|^p)\, dy \\
&\quad + \int_{B_s} (|Dw_j - Dv_j|^2 + \lambda_j^{p-2}|Dw_j - Dv_j|^p)\, dy \\
&\quad + \int_{B_s} (|Dw - Dv|^2 + \lambda_j^{p-2}|Dw - Dv|^p)\, dy. \qquad (3.2.49)
\end{aligned}
$$

Last two integrals in (3.2.49) can be treated exactly as R_j^1 and R_j^3 thus leading to

$$
\begin{aligned}
\lim_j [\mathcal{I}_{j,r}(v_j) - \mathcal{I}_{j,r}(v)] \geq &- \frac{c\varepsilon}{k^2} - c(k)(r - s) \\
&+ \limsup_j \int_{B_t} (|Dv_j - Dv|^2 + \lambda_j^{p-2}|Dv_j - Dv|^p)\, dy. \qquad (3.2.50)
\end{aligned}
$$

The desired estimate follows letting first $s \to r$ and then $k \to \infty$ in (3.2.50).

Step 7. Conclusion

From previous two steps we can conclude that

$$
\lim_j \int_{B_r} |Dv - Dv_j|^2 + \lambda_j^{p-2}|Dv - Dv_j|^p = 0. \qquad (3.2.51)
$$

The conclusion follows observing that

$$
\begin{aligned}
\lim_j \frac{E(x_j, \tau r_j)}{\lambda_j^2} &= \lim_j \frac{1}{\lambda_j^2} \fint_{B_{\tau r_j}(x)} (|Du - (Du)_{\tau r_j}|^2 + |Du - (Du)_{\tau r_j}|^p)\, dy + \lim_j \frac{\tau^\beta r_j^\beta}{\lambda_j^2} \\
&\leq \lim_j \fint_{B_\tau(0)} (|Dv_j - (Dv_j)_\tau|^2 + \lambda_j^{p-2}|Dv_j - (Dv_j)_\tau|^p)\, dy + \tau^\beta \\
&= \lim_j \fint_{B_\tau(0)} (|Dv_j - Dv|^2 + \lambda_j^{p-2}|Dv_j - Dv|^p)\, dy \\
&\quad + \lim_j \fint_{B_\tau(0)} (|(Dv_j)_\tau - (Dv)_\tau|^2 + \lambda_j^{p-2}|(Dv_j)_\tau - (Dv)_\tau|^p)\, dy \\
&\quad + \lim_j \fint_{B_\tau(0)} (|Dv - (Dv)_\tau|^2 + \lambda_j^{p-2}|Dv - (Dv)_\tau|^p)\, dy + \tau^\beta
\end{aligned}
$$

$$\leq \fint_{B_\tau(0)} |Dv - (Dv)_\tau|^2 \, dy \leq c_M \tau^2 + \tau^\beta \leq c_M \tau^\beta, \tag{3.2.52}$$

since the first integral vanishes as $j \to +\infty$ thanks to (3.2.51), the second one vanishes since $(Dv_j)_\tau \to (Dv)_\tau$ as $j \to +\infty$,

$$\lambda_j^{p-2} |Dv - (Dv)_\tau|^p \leq c\lambda^{p-2}$$

vanishes as $j \to +\infty$ and thanks to (3.2.5)

$$\lim_{j \to +\infty} \frac{\tau^\beta r_j^\beta}{\lambda_j^2} \leq \tau^\beta.$$

Estimate (3.2.52) is a contradiction if we choose $\tilde{c}(M) > c_M$ and this concludes the proof. □

The proof of Theorem 3.0.2 now follows by a standard iteration procedure, see [46]. The following proof of Theorem 3.0.3 is an immediate corollary of the higher differentiability result for the gradient of minimizers of $\mathcal{F}$ that can be inferred from the proof of the Proposition 3.1.1 (see (3.1.7)) or from the proof of Theorem 4 in [31].

Proof. (of Theorem 3.0.3) The singular set Σ of minimizers of $\mathcal{F}$ turns out to be contained in the set

$$\Sigma_0 := \left\{ x \in \Omega : \limsup_{\rho \to 0^+} \fint_{B(x,\rho)} |Du(y) - (Du)_{x,\rho}|^p \, dy > 0 \right\}$$
$$\cup \left\{ x \in \Omega : \limsup_{\rho \to 0^+} |(Du)_{x,\rho}| = +\infty \right\}.$$

Hence Lemma 2.4.1 applies in order to conclude the proof. □

Chapter IV

PARTIAL REGULARITY THROUGH FONSECA-MALY EXTENSION LEMMA

The results of this chapter have been obtained in [25].

As in the previous chapter, we consider the functional (0.0.1), satisfying the assumptions (F1)–(F3), and we shall prove $C^{1,\gamma}$ partial regularity of minimizers, with the gap between growth and coercivity exponent defined in (0.0.5), and that we recall here for the reader's convenience:

$$1 < p \leq q < \min\left\{p\frac{n}{n-1}, p+1\right\}.$$

As we already observed, the main difficulty in studying the regularity properties of minimizers of integrals with non-standard growth is the costruction of test functions having the right degree of integrability. The gluing Lemma 2.3.1, due to Fonseca and Maly ([36]), will play a key role to overcome this difficulty and partly provide the bound (0.0.5). In fact Lemma 2.3.1 holds if

$$q < p\frac{n}{n-1}.$$

To be more precise we could allow $q \leq p+1$ if

$$p+1 < p\frac{n}{n-1},$$

that is when $p > n-1$. This restriction on q is explained in the following remark, taken from [77].

Remark 4.0.2. [The Euler-Lagrange system for $q \leq p+1$.] If u is a local minimizer of the functional $\mathcal{F}$ and $\phi \in C^1_c(\Omega, \mathbb{R}^N)$ we get by the minimality condition that for any $\varepsilon > 0$:

$$0 \leq \int_\Omega [F(Du + \varepsilon D\phi) - F(Du)]\,\mathrm{d}x = \varepsilon \int_\Omega \int_0^1 \frac{\partial F}{\partial \xi^i_\alpha}(Du + \varepsilon t D\phi) D_\alpha \phi^i \,\mathrm{d}t\,\mathrm{d}x\,,$$

where the usual summation convention is in force. Dividing this inequality by ε, and letting $\varepsilon \searrow 0$, we infer from the growth assumptions and since $q \leq p+1$, that

$$\int_\Omega \frac{\partial F}{\partial \xi^i_\alpha}(Du) D_\alpha \phi^i \,\mathrm{d}x \geq 0.$$

Consequently, u is a weak solution to the Euler-Lagrange system for $\mathcal{I}$:

$$\int_\Omega \frac{\partial F}{\partial \xi^i_\alpha}(Du) D_\alpha \phi^i \,\mathrm{d}x = 0 \qquad \forall \phi \in C^1_c(\Omega, \mathbb{R}^N).$$

After having established the Caccioppoli type estimate, the blow-up argument, aimed to establish a decay estimate for the excess function of a minimizer, can be started up. Moreover, by skipping the higher integrability step, it is not necessary to assume the non occurrence of the Lavrentiev Phenomenon (see [31]).

The main result of this chapter is the following.

Theorem 4.0.3. *Let f be a $C^2(\Omega, \mathbb{R}^{n\times N})$ integrand satisfying the assumptions (F1), (F2) and (F3) with growth exponents p, q such that*

$$1 < p \leq q < \min\left\{p\frac{n}{n-1}, p+1\right\}. \tag{4.0.53}$$

If $u \in W^{1,p}_{loc}(\Omega, \mathbb{R}^N)$ is a local minimizer of the functional $\mathcal{F}$, then there exists an open subset Ω_0 of Ω such that

$$meas(\Omega \setminus \Omega_0)$$

and

$$u \in C^{1,\gamma}_{loc}(\Omega_0, \mathbb{R}^N) \qquad \textit{for every} \qquad \gamma < \frac{\alpha}{2},$$

where α is the exponent appearing in (F2).

Since our regularity result is only partial, we are not in contradiction with the counterexample of [31], which shows that (0.0.3) is unavoidable to boost the integrability of the $W^{1,p}$-minimizers up to $W^{1,q}$.

As we saw in Chapter II, partial regularity results are a common feature when treating vectorial minimizers, because everywhere regularity cannot be proved in this case. Hence, the next issue is trying to estimate the Hausdorff dimension of the singular set. In the case of functionals with standard growth conditions, these estimates have been established in [59] (see also [23]). But in our setting, this kind of result cannot be achieved. In fact, the example constructed in [37] (see Chapter II) shows that if p and q are far enough, depending on the dimension n and the regularity of $x \mapsto f(x, Du)$, then the set of non-Lebesgue points of a minimizer can be nearly as bad as that of any other $W^{1,p}$ function.

We recall once again, for the reader's convenience, the definition of local minimizer for a functional with nonstandard growth conditions.

Definition 4.0.4. A function $u \in W^{1,1}_{loc}(\Omega, \mathbb{R}^N)$ is a local minimizer of $\mathcal{F}$ if $f(x, Du(x)) \in L^1_{loc}(\Omega)$ and

$$\int_{supp\,\varphi} f(x, Du)\,dx \leq \int_{supp\,\varphi} f(x, Du + D\varphi)\,dx,$$

for any $\varphi \in W^{1,1}_{loc}(\Omega, \mathbb{R}^N)$ with $supp\,\varphi \subset \Omega$.

In the linearization procedure we shall use the translated functional of $\mathcal{F}$ on the unit ball $B \equiv B_1(0)$

$$\mathcal{I}(v) := \int_B g(y, Dv)\;dy$$

defined by setting

$$g(y, \xi) = f(x_0 + r_0 y, A + \xi) - f(x_0 + r_0 y, A) - D_\xi f(x_0 + r_0 y, A)\xi, \qquad (4.0.54)$$

where A is a matrix such that $|A|$ is uniformly bounded by a positive constant M. Next Lemma, whose proof is given in [24], contains the growth conditions on g.

Lemma 4.0.5. *Let $f \in C^2(\Omega \times \mathbb{R}^{n\times N})$ be a function satisfying the assumptions (F1), (F2) and (F3) and let $g(y,\xi)$ be the function defined by (4.0.54). Then we have*

$$c_1|V_p(\xi)|^2 \leq g(y,\xi) \leq c_2|V_q(\xi)|^2; \tag{I1}$$

$$|D_\xi g(y,\xi)| \leq c(1+|\xi|^2)^{\frac{q-2}{2}}|\xi|; \tag{I2}$$

$$|D_\xi g(y_1,\xi) - D_\xi g(y_2,\xi)| \leq cr_0^\alpha \ |y_1-y_2|^\alpha \ (1+|\xi|^{q-1}); \tag{I3}$$

$$c(1+|\xi|^2)^{\frac{p-2}{2}}|\zeta|^2 \leq \left\langle D_{\xi\xi}g(y,\xi)\zeta,\zeta\right\rangle \tag{I4}$$

where the constant c, c_1 and c_2 depend on M, p and q.

4.1 A Caccioppoli type inequality

In order to perform the blow up procedure, it will be convenient to introduce suitable translations of minimizers of the functional $\mathcal{F}$. More precisely, if u is a local minimizer of $\mathcal{F}$ we shall consider the function

$$v(y) = \frac{u(x_0+r_0y) - r_0Ay - (u)_{B_1(0)}}{r_0}.$$

The minimality of u implies that

$$\int_{B_1(0)} f(x_0+r_0y, Du(x_0+r_0y))\,dy \leq \int_{B_1(0)} f(x_0+r_0y, Du(x_0+r_0y)+D\varphi(x_0+r_0y))\,dy$$

that is

$$\int_{B_1(0)} f(x_0+r_0y, Dv(y)+A)\,dy \leq \int_{B_1(0)} f(x_0+r_0y, Dv(y)+A+D\varphi(x_0+r_0y))\,dy$$

and hence

$$\int_{B_1(0)} g(y,Dv)\,dy \leq \int_{B_1(0)} g(y,Dv+D\varphi)\,dy + cr_0^\alpha \int_{B_1(0)} |D\varphi|\,dy, \tag{4.1.1}$$

for every $\varphi \in W^{1,1}(B_1(0);\mathbb{R}^N)$ with compact support, where g is the function defined at (4.0.54).

Therefore, the first step in the proof of Theorem 4.0.3 is to obtain a Caccioppoli type inequality for every function $v \in W^{1,p}(B_1(0);\mathbb{R}^N)$ which satisfies the minimality inequality (4.1.1).

Proposition 4.1.1. *Let us suppose that $g(y,\xi) \in C^2(B_1(0);\mathbb{R}^{nN})$ satisfies the assumptions $(I1)$, $(I2)$, $(I3)$ with*

$$1 < p \le q < p\left(\frac{n}{n-1}\right) \tag{4.1.2}$$

and set $t = \min\{2,p\}$. If the function $v \in W^{1,p}(B_1(0);\mathbb{R}^N)$ satisfies the inequality (4.1.1) *then, for every $\rho < 1$, we have*

$$\begin{aligned}\fint_{B_{\frac{\rho}{2}}} |V_p(Dv)|^2\,dy \le & c\fint_{B_\rho}\left|V_p\left(\frac{v}{\rho}\right)\right|^2\,dy + c\left(\fint_{B_\rho}|V_p(Dv)|^2 + \left|V_p\left(\frac{v}{\rho}\right)\right|^2\,dy\right)^{\frac{q}{p}}\\ &+ cr_0^\alpha\left(\fint_{B_\rho}|Dv|^t\,dy\right)^{\frac{1}{t}} + cr_0^\alpha\left(\fint_{B_\rho}\frac{|v|^t}{\rho^t}\,dy\right)^{\frac{1}{t}},\end{aligned} \tag{4.1.3}$$

for a positive constant c independent of the parameter r_0 and of the point x_0 appearing in the definition of $g(y,\xi)$.

Proof. Let us fix two radii $\frac{\rho}{2} < r < s < \rho$. Lemma 2.3.1 implies that there exist $\psi \in W^{1,p}(B_1(0))$ and $r < r' < s' < s$ such that

$$\psi = v \quad \text{on} \quad B_{r'} \qquad\qquad \psi = v \quad \text{on} \quad B_1 \setminus B_{s'},$$

$$\frac{s-r}{3} \le s' - r' \le s - r. \tag{4.1.4}$$

Thanks to the assumption (4.1.2), the function ψ satisfies the estimates (2.3.8)–(2.3.11) in case $p \ge 2$ and (2.3.12)–(2.3.15) in case $1 < p < 2$.

Fix now a cut-off function $\eta \in C_0^\infty(B_{s'})$ such that $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_{r'}$ and $|D\eta| \le \frac{c}{s'-r'}$ and set

$$\varphi = (1-\eta)\psi \qquad\qquad \tilde{\varphi} = \eta\psi.$$

By the left hand inequality in assumption $(I1)$, we get

$$\begin{aligned}
&\int_{B_{r'}} (1+|Dv|^2)^{\frac{p-2}{2}}|Dv|^2\,dy \le c\int_{B_{s'}} g(y,D\tilde{\varphi})\,dy \\
=&\int_{B_{s'}\setminus B_{r'}} [g(y,D\tilde{\varphi})-g(y,Dv)]\,dy + \int_{B_{s'}} [g(y,Dv)-g(y,D\varphi)]\,dy \\
&+\int_{B_{s'}\setminus B_{r'}} [g(y,D\varphi)]\,dy = I+II+III, \qquad (4.1.5)
\end{aligned}$$

where we used that in $B_{r'}$ one has $\tilde{\varphi}=v$ and $\varphi=0$. By the minimality inequality (4.1.1) for v we have that

$$II \le cr_0^\alpha\left(\int_{B_{s'}} |Dv-D\varphi|\,dy\right), \qquad (4.1.6)$$

since $v-\varphi\in W_0^{1,p}(B_{s'})$ Moreover, since $g(y,\xi)\ge 0$ for all $y\in B_1$ and all $\xi\in\mathbb{R}^{n\times N}$, we have that

$$I \le \int_{B_{s'}\setminus B_{r'}} [g(y,D\tilde{\varphi})]\,dy. \qquad (4.1.7)$$

Hence inserting (4.1.6) and (4.1.7) in (4.1.5) we get

$$\begin{aligned}
&\int_{B_{r'}} (1+|Dv|^2)^{\frac{p-2}{2}}|Dv|^2\,dy \\
\le\ & c\int_{B_{s'}\setminus B_{r'}} [g(y,D\tilde{\varphi})]\,dy + \int_{B_{s'}\setminus B_{r'}} [g(y,D\varphi)]\,dy + cr_0^\alpha\left(\int_{B_{s'}} |Dv-D\varphi|\,dy\right) \\
=\ & J+JJ+JJJ. \qquad (4.1.8)
\end{aligned}$$

Now we treat the cases $1<p\le 2$ and $p>2$ separately.

- *The case* $1<p\le 2$.

In order to estimate J, we use the right inequality in assumption $(I1)$ thus getting

$$J \le c\int_{B_{s'}\setminus B_{r'}} (1+|D\tilde{\varphi}|^2)^{\frac{q-2}{2}}|D\tilde{\varphi}|^2\,dy = c\int_{B_{s'}\setminus B_{r'}} (1+|D\tilde{\varphi}|^2)^{\frac{p-2}{2}+\frac{q-p}{2}}|D\tilde{\varphi}|^2\,dy$$

$$
\begin{aligned}
&= c\int_{B_{s'}\setminus B_{r'}} (1+|D\tilde{\varphi}|^2)^{\frac{p-2}{2}}(1+|D\tilde{\varphi}|^2)^{\frac{p}{2}\frac{q-p}{p}}|D\tilde{\varphi}|^2\,dy \\
&\le c\int_{B_{s'}\setminus B_{r'}} (1+|D\tilde{\varphi}|^2)^{\frac{p-2}{2}}|D\tilde{\varphi}|^2\Big[1+|D\tilde{\varphi}|^2(1+|D\tilde{\varphi}|^2)^{\frac{p-2}{2}}\Big]^{\frac{q-p}{p}}\,dy. \qquad (4.1.9)
\end{aligned}
$$

where we used (2.3.4) in the last line. Hence

$$
\begin{aligned}
J &\le c\int_{B_{s'}\setminus B_{r'}} (1+|D\tilde{\varphi}|^2)^{\frac{p-2}{2}}|D\tilde{\varphi}|^2\,dy + c\int_{B_{s'}\setminus B_{r'}} \Big(|D\tilde{\varphi}|^2(1+|D\tilde{\varphi}|^2)^{\frac{p-2}{2}}\Big)^{\frac{q}{p}}\,dy \\
&\le c\int_{B_{s'}\setminus B_{r'}} |V_p(D\tilde{\varphi})|^2\,dy + c\int_{B_{s'}\setminus B_{r'}} |V_p(D\tilde{\varphi})|^{\frac{2q}{p}}\,dy. \qquad (4.1.10)
\end{aligned}
$$

Arguing exactly in the same way we have

$$
JJ \le c\int_{B_{s'}\setminus B_{r'}} |V_p(D\varphi)|^2\,dy + c\int_{B_{s'}\setminus B_{r'}} |V_p(D\varphi)|^{\frac{2q}{p}}\,dy. \qquad (4.1.11)
$$

¿From (4.1.10) and (4.1.11), using the properties of the function V_p and the definition of $\tilde{\varphi}$ and φ we obtain

$$
\begin{aligned}
J+JJ &\le c\int_{B_{s'}\setminus B_{r'}} |V_p(D\tilde{\varphi})|^2\,dy + c\int_{B_{s'}\setminus B_{r'}} |V_p(D\tilde{\varphi})|^{\frac{2q}{p}}\,dy \\
&+ c\int_{B_{s'}\setminus B_{r'}} |V_p(D\varphi)|^2\,dy + c\int_{B_{s'}\setminus B_{r'}} |V_p(D\varphi)|^{\frac{2q}{p}}\,dy \\
&= c\int_{B_{s'}\setminus B_{r'}} |V_p(D(1-\eta)\psi)|^2\,dy + c\int_{B_{s'}\setminus B_{r'}} |V_p(D(1-\eta)\psi)|^{\frac{2q}{p}}\,dy \\
&+ c\int_{B_{s'}\setminus B_{r'}} |V_p(D(\eta\psi))|^2\,dy + c\int_{B_{s'}\setminus B_{r'}} |V_p(D(\eta\psi))|^{\frac{2q}{p}}\,dy \\
&\le c\int_{B_{s'}\setminus B_{r'}} |V_p(D\psi)|^2\,dy + c\int_{B_{s'}\setminus B_{r'}} \left|V_p\left(\frac{\psi}{s'-r'}\right)\right|^2\,dy \\
&+ c\int_{B_{s'}\setminus B_{r'}} |V_p(D\psi)|^{\frac{2q}{p}}\,dy + c\int_{B_{s'}\setminus B_{r'}} \left|V_p\left(\frac{\psi}{s'-r'}\right)\right|^{\frac{2q}{p}}\,dy, \qquad (4.1.12)
\end{aligned}
$$

where we also used the properties of η. Therefore, using (2.3.12)–(2.3.15) and

(4.1.4), we get

$$J + JJ \leq c\int_{B_s\setminus B_r} |V_p(Dv)|^2\, dy + c\int_{B_s\setminus B_r} \left|V_p\left(\frac{v}{s-r}\right)\right|^2\, dy$$
$$+ \quad c(s-r)^n \left(\frac{1}{(s-r)^n}\int_{B_s\setminus B_r} |V_p(Dv)|^2 + \left|V_p\left(\frac{v}{s-r}\right)\right|^2 dy\right)^{\frac{q}{p}} \tag{4.1.13}$$

Concerning JJJ, recalling that $\varphi = 0$ on $B_{r'}$, using Hölder's inequality and Lemma 2.3.1 we have

$$\begin{aligned}
JJJ =& cr_0^\alpha \left[\int_{B_{s'}} |Dv|\, dy + \int_{B_{s'}\setminus B_{r'}} |D\psi|\, dy\right]\\
\leq& cr_0^\alpha \left[\int_{B_\rho} |Dv|\, dy + \int_{B_{s'}\setminus B_{r'}} |D\psi|\, dy + \int_{B_{s'}\setminus B_{r'}} \frac{|\psi|}{(s'-r')}\, dy\right]\\
\leq& cr_0^\alpha \rho^{\frac{n}{p'}} \left[\int_{B_\rho} |Dv|^p\, dy\right]^{\frac{1}{p}} + cr_0^\alpha \rho^{\frac{n}{p'}} \left[\int_{B_{s'}\setminus B_{r'}} |D\psi|^p\, dy + \int_{B_{s'}\setminus B_{r'}} \frac{|\psi|^p}{(s'-r')^p}\, dy\right]^{\frac{1}{p}}\\
\leq& cr_0^\alpha \rho^{\frac{n}{p'}} \left[\int_{B_\rho} |Dv|^p\, dy\right]^{\frac{1}{p}} + cr_0^\alpha \rho^{\frac{n}{p'}} \left[\int_{B_\rho} \frac{|v|^p}{(s-r)^p}\, dy\right]^{\frac{1}{p}},
\end{aligned} \tag{4.1.14}$$

where p' is the Hölder conjugate of p and we used again (2.3.8), (2.3.9) and (4.1.4).

- *The case* $p \geq 2$.

In this case we use the right inequality in assumption $(I1)$, property (2.3.5) and the definition of φ and $\tilde{\varphi}$ as follows

$$\begin{aligned}
J + JJ \leq& c\int_{B_{s'}\setminus B_{r'}} (1+|D\tilde{\varphi}|^2)^{\frac{q-2}{2}}|D\tilde{\varphi}|^2\, dy + c\int_{B_{s'}\setminus B_{r'}} (1+|D\varphi|^2)^{\frac{q-2}{2}}|D\varphi|^2\, dy\\
\leq& c\int_{B_{s'}\setminus B_{r'}} |D\tilde{\varphi}|^2 + |D\tilde{\varphi}|^q\, dy + c\int_{B_{s'}\setminus B_{r'}} |D\varphi|^2 + |D\varphi|^q\, dy\\
\leq& c\int_{B_{s'}\setminus B_{r'}} |D\psi|^2 + |D\psi|^q\, dy + c\int_{B_{s'}\setminus B_{r'}} \left|\frac{\psi}{s'-r'}\right|^2 + \left|\frac{\psi}{s'-r'}\right|^q dy
\end{aligned} \tag{4.1.15}$$

Hence, by Lemma 2.3.1, we get

$$J + JJ \leq c\int_{B_s\setminus B_r} |Dv|^2 + c(s-r)^{n\left(1-\frac{q}{p}\right)} \left(\int_{B_s\setminus B_r} |Dv|^p\, dy\right)^{\frac{q}{p}}$$

$$+ \quad c\int_{B_s\setminus B_r}\left|\frac{v}{s-r}\right|^2 + c(s-r)^{n\left(1-\frac{q}{p}\right)}\left(\int_{B_s\setminus B_r}\left|\frac{v}{s-r}\right|^p dy\right)^{\frac{q}{p}}$$

$$\leq \quad c\int_{B_s\setminus B_r}|V_p(Dv)|^2\, dy + c\int_{B_s\setminus B_r}\left|V_p\left(\frac{v}{s-r}\right)\right|^2\, dy$$

$$+ \quad c(s-r)^n\left(\frac{1}{(s-r)^n}\int_{B_s\setminus B_r}|V_p(Dv)|^2 + \left|V_p\left(\frac{v}{s-r}\right)\right|^2 dy\right)^{\frac{q}{p}}, \qquad (4.1.16)$$

where we used again (4.1.4).

Now we argue exactly as in (4.1.14) and obtain that

$$\begin{aligned} JJJ =& cr_0^\alpha\left[\int_{B_{s'}}|Dv|\, dy + \int_{B_{s'}\setminus B_{r'}}|D\psi|\, dy\right] \\ \leq& cr_0^\alpha\left[\int_{B_\rho}|Dv|\, dy + \int_{B_{s'}\setminus B_{r'}}|D\psi|\, dy + \int_{B_{s'}\setminus B_{r'}}\frac{|\psi|}{(s'-r')}\, dy\right] \\ \leq& cr_0^\alpha\rho^{\frac{n}{2}}\left[\int_{B_\rho}|Dv|^2\, dy\right]^{\frac{1}{2}} + cr_0^\alpha\rho^{\frac{n}{2}}\left[\int_{B_{s'}\setminus B_{r'}}|D\psi|^2\, dy + \int_{B_{s'}\setminus B_{r'}}\frac{|\psi|^2}{(s'-r')^2}\, dy\right]^{\frac{1}{2}} \\ \leq& cr_0^\alpha\rho^{\frac{n}{2}}\left[\int_{B_\rho}|Dv|^2\, dy\right]^{\frac{1}{2}} + cr_0^\alpha\rho^{\frac{n}{2}}\left[\int_{B_\rho}\frac{|v|^2}{(s-r)^2}\, dy\right]^{\frac{1}{2}}. \end{aligned} \qquad (4.1.17)$$

Hence we can write a final estimate for JJJ as follows:

$$JJJ \leq cr_0^\alpha\rho^{\frac{n}{t'}}\left(\int_{B_\rho}|Dv|^t\, dy\right)^{\frac{1}{t}} + cr_0^\alpha\rho^{\frac{n}{t'}}\left(\int_{B_\rho}\frac{|v|^t}{\rho^t}\, dy\right)^{\frac{1}{t}}. \qquad (4.1.18)$$

where $t = \min\{2, p\}$ and t' is the Hölder conjugate of t.

Inserting (4.1.13) and (4.1.18) or (4.1.16) and (4.1.18) in (4.1.8) in case $1 < p \leq 2$ and $p \geq 2$ respectively, we obtain

$$\int_{B_r}|V_p(Dv)|^2\, dy \leq c\int_{B_s\setminus B_r}|V_p(Dv)|^2\, dy + c\int_{B_s\setminus B_r}\left|V_p\left(\frac{v}{s-r}\right)\right|^2\, dy$$

$$+ \quad c(s-r)^n\left(\frac{1}{(s-r)^n}\int_{B_s\setminus B_r}|V_p(Dv)|^2 + \left|V_p\left(\frac{v}{s-r}\right)\right|^2 dy\right)^{\frac{q}{p}}$$

$$+ \quad c r_0^{\alpha} \rho^{\frac{n}{t'}} \left(\int_{B_\rho} |Dv|^t \, dy \right)^{\frac{1}{t}} + c r_0^{\alpha} \rho^{\frac{n}{t'}} \left(\int_{B_\rho} \frac{|v|^t}{\rho^t} \, dy \right)^{\frac{1}{t}}, \tag{4.1.19}$$

where $t = \min\{2, p\}$.

Now, we fill the hole by adding the quantity

$$c \int_{B_r} |V_p(Dv)|^2 \, dy$$

to both sides of (4.1.19) and use the iteration Lemma 2.2.1 to obtain that

$$\int_{B_{\frac{\rho}{2}}} |V_p(Dv)|^2 dy \leq c \int_{B_\rho} \left| V_p \left(\frac{v}{\rho} \right) \right|^2 dy + c\rho^n \left(\frac{1}{\rho^n} \int_{B_\rho} |V_p(Dv)|^2 + \left| V_p \left(\frac{v}{\rho} \right) \right|^2 dy \right)^{\frac{q}{p}}$$

$$+ c r_0^{\alpha} \rho^{\frac{n}{t'}} \left(\int_{B_\rho} |Dv|^t \, dy \right)^{\frac{1}{t}} + c r_0^{\alpha} \rho^{\frac{n}{t'}} \left(\int_{B_\rho} \frac{|v|^t}{\rho^t} \, dy \right)^{\frac{1}{t}}. \tag{4.1.20}$$

The conclusion follows dividing both sides by ρ^n. □

4.2 Decay estimate

As usual the proof of Theorem 4.0.3 relies on a blow up argument aimed to establish a decay estimate for the excess function of the minimizer, which is defined as in (0.0.8)

$$\tilde{E}(x, r) = \fint_{B_r(x)} |V_p(Du - (Du)_r)|^2 + r^{\beta} \tag{4.2.1}$$

with $\beta < \alpha$. The blow up argument for a local minimizer $u \in W^{1,p}_{\mathrm{loc}}$ of $\mathcal{F}$ with an integrand function $f(x, \xi) \in C^2(\Omega, \mathbb{R}^{n \times N})$ fulfilling assumptions (F1), (F2) and (F3) for a couple of exponents satisfying (0.0.5), is contained in the following

Proposition 4.2.1. *Fix $M > 0$. There exists a constant $C(M) > 0$ such that, for every $0 < \tau < \frac{1}{4}$, there exists $\varepsilon = \varepsilon(\tau, M)$ such that, if*

$$|(Du)_{x_0,r}| \leq M \qquad \textit{and} \qquad \tilde{E}(x_0, r) \leq \varepsilon,$$

then

$$\tilde{E}(x_0, \tau r) \leq C(M)\, \tau^{\beta}\, \tilde{E}(x_0, r).$$

Proof. Step 1. Blow up

Fix $M > 0$. Assume by contradiction that there exists a sequence of balls $B_{r_j}(x_j) \subset\subset \Omega$ such that

$$|(Du)_{x_j,r_j}| \le M \qquad \text{and} \qquad \lambda_j^2 = \tilde{E}(x_j, r_j) \to 0 \tag{4.2.2}$$

but

$$\frac{\tilde{E}(x_j, \tau r_j)}{\lambda_j^2} > \tilde{C}(M)\tau^\beta \tag{4.2.3}$$

where $\tilde{C}(M)$ will be determined later. Setting $A_j = (Du)_{x_j,r_j}$, $a_j = (u)_{x_j,r_j}$ and

$$v_j(y) = \frac{u(x_j + r_j y) - a_j - r_j A_j y}{\lambda_j r_j} \tag{4.2.4}$$

for all $y \in B_1(0)$, one can easily check that $(Dv_j)_{0,1} = 0$ and $(v_j)_{0,1} = 0$. By the definition of λ_j at (4.2.2), we get

$$\fint_{B_1(0)} \frac{|V(\lambda_j Dv_j)|^2}{\lambda_j^2}\, dy + \frac{r_j^\beta}{\lambda_j^2} = 1, \tag{4.2.5}$$

and hence

$$\fint_{B_1(0)} |Dv_j|^p\, dy \le C \qquad\qquad 1 < p < 2 \tag{4.2.6}$$

$$\fint_{B_1(0)} |Dv_j|^2 + \lambda_j^{p-2}|Dv_j|^p\, dy \le C \qquad\qquad p \ge 2. \tag{4.2.7}$$

Therefore passing possibly to not relabeled sequences

$$v_j \rightharpoonup v \qquad \text{weakly in } W^{1,p}(B_1(0);\mathbb{R}^N) \qquad 1 < p < 2;$$

$$v_j \rightharpoonup v \qquad \text{weakly in } W^{1,2}(B_1(0);\mathbb{R}^N) \qquad p \ge 2;$$

$$A_j \longrightarrow A$$

$$r_j \longrightarrow 0; \qquad \frac{r_j^\gamma}{\lambda_h^2} \longrightarrow 0, \qquad \forall\, \gamma > \beta. \tag{4.2.8}$$

Step 2. Minimality of v_j

We normalize f around A_j as follows

$$f_j(y,\xi) = \frac{f(x_j + r_j y, A_j + \lambda_j \xi) - f(x_j + r_j y, A_j) - D_\xi f(x_j + r_j y, A_j)\lambda_j \xi}{\lambda_j^2} \tag{4.2.9}$$

and we consider the corresponding rescaled functionals

$$\mathcal{I}_j(w) = \int_{B_1(0)} [f_j(y, Dw)]dy. \tag{4.2.10}$$

The minimality of u yields that

$$\int_{B_1(0)} f(x_j+r_j y, Du(x_j+r_j y))\, dy \le \int_{B_1(0)} f(x_j+r_j y, Du(x_j+r_j y)+D\varphi(x_j+r_j y))\, dy$$

for every $\varphi \in W_0^{1,1}(B_{r_j}(x_j);\mathbb{R}^N)$, that is

$$\begin{aligned}\int_{B_1(0)} & f(x_j + r_j y, A_j + \lambda_j Dv_j(y))\, dy \\ & \le \int_{B_1(0)} f(x_j + r_j y, A_j + \lambda_j Dv_j(y) + D\varphi(x_j + r_j y))\, dy,\end{aligned}$$

for every $\varphi \in W_0^{1,1}(B_{r_j}(x_j);\mathbb{R}^N)$. Thus, by the definition of the rescaled functionals, we have

$$\mathcal{I}_j(v_j) \le \mathcal{I}_j(v_j + \varphi) + \int_{B_1(0)} \frac{D_\xi f(x_j + r_j y, A_j) D\varphi}{\lambda_j}\, dy. \tag{4.2.11}$$

Using (F2) we conclude that

$$\begin{aligned}\mathcal{I}_j(v_j) &\le \mathcal{I}_j(v_j + \varphi) + \int_{B_1(0)} \frac{[D_\xi f(x_j + r_j y, A_j) - D_\xi f(x_j, A_j)] D\varphi}{\lambda_j}\, dy \\ &\le \mathcal{I}_j(v_j + \varphi) + c(M)\frac{r_j^\alpha}{\lambda_j} \int_{B_1(0)} |D\varphi|\, dy. \end{aligned} \tag{4.2.12}$$

Step 3. v solves a linear system

Since v_j satisfies inequality (4.2.12) we have that

$$0 \le \mathcal{I}_j(v_j + s\varphi) - \mathcal{I}_j(v_j) + c(M)\frac{r_j^\alpha}{\lambda_j} \int_{B_1(0)} |sD\varphi|\, dy, \tag{4.2.13}$$

for every $\varphi \in C^1_0(B)$ and for every $s \in (0,1)$. Now, by the definition of the rescaled functionals we get

$$\mathcal{I}_j(v_j + s\varphi) - \mathcal{I}_j(v_j) = \int_{B_1(0)} \int_0^1 [D_\xi f_j(x_j + r_j y, A_j + \lambda_j(Dv_j + tsD\varphi))] sD\varphi \, dt \, dy$$

$$= \frac{c}{\lambda_j} \int_{B_1(0)} [D_\xi f(x_j + r_j y, A_j + \lambda_j(Dv_j + sD\varphi)) - D_\xi f(x_j + r_j y, A_j)] sD\varphi \, dy. \tag{4.2.14}$$

Inserting (4.2.14) in (4.2.13), dividing by s and taking the limit as $s \to 0$, we conclude that

$$\begin{aligned} 0 \;&\leq\; \frac{c}{\lambda_j} \int_{B_1(0)} [D_\xi f(x_j + r_j y, A_j + \lambda_j Dv_j) - D_\xi f(x_j + r_j y, A_j)] D\varphi \, dy \\ &+\; \frac{c(M) r_j^\alpha}{\lambda_j} \int_{B_1(0)} |D\varphi| \, dy. \end{aligned} \tag{4.2.15}$$

Let us split

$$B_1(0) = E_j^+ \cup E_j^- = \{y \in B_1 : \; \lambda_j |Dv_j| > 1\} \cup \{y \in B_1 : \; \lambda_j |Dv_j| \leq 1\}.$$

By (4.2.6), in case $1 < p < 2$, we get

$$|E_j^+| \leq \int_{E_j^+} \lambda_j^p |Dv_j|^p \, dy \leq \lambda_j^p \int_{E_j^+} |Dv_j|^p \, dy \leq c\lambda_j^p. \tag{4.2.16}$$

By assumption (F1) and the convexity of f we have that

$$|D_\xi f(x, \xi)| \leq c(1 + |\xi|^{q-1})$$

Since $q < p + 1$, we can apply Hölder's inequality thus obtaining

$$\begin{aligned} &\frac{1}{\lambda_j} \left| \int_{E_j^+} [D_\xi f(x_j + r_j y, A_j + \lambda_j Dv_j) - D_\xi f(x_j + r_j y, A_j)] D\varphi \, dy \right| \\ \leq\;& \frac{c}{\lambda_j} |E_j^+| + c\lambda_j^{q-2} \int_{E_j^+} |Dv_j|^{q-1} \, dy \end{aligned}$$

$$\leq \; c\lambda_j^{p-1} + c\lambda_j^{q-2}\left(\int_{E_j^+}|Dv_j|^p\,dy\right)^{\frac{q-1}{p}}|E_j^+|^{\frac{p-q+1}{p}}$$

$$\leq \; c\lambda_j^{p-1}. \tag{4.2.17}$$

In case $p \geq 2$, by (4.2.7) we get

$$|E_j^+| \leq \int_{E_j^+}\lambda_j^2|Dv_j|^2\,dy \leq \lambda_j^2\int_{E_j^+}|Dv_j|^2\,dy \leq c\lambda_j^2. \tag{4.2.18}$$

Arguing as before, we have

$$\frac{1}{\lambda_j}\left|\int_{E_j^+}[D_\xi f(x_j+r_jy,A_j+\lambda_jDv_j)-D_\xi f(x_j+r_jy,A_j)]D\varphi\,dy\right|$$

$$\leq \; \frac{c}{\lambda_j}|E_j^+| + c\lambda_j^{q-2}\int_{E_j^+}|Dv_j|^{q-1}\,dy$$

$$\leq \; c\lambda_j + c\lambda_j^{\frac{2q-p-2}{p}}\left(\int_{E_j^+}\lambda_j^{p-2}|Dv_j|^p\,dy\right)^{\frac{q-1}{p}}|E_j^+|^{\frac{p-q+1}{p}}$$

$$\leq \; c\lambda_j. \tag{4.2.19}$$

Hence, for every $p > 1$, we infer that

$$\lim_{j\to\infty}\frac{c}{\lambda_j}\left|\int_{E_j^+}[D_\xi f(x_j+r_jy,A_j+\lambda_jDv_j)-D_\xi f(x_j+r_jy,A_j)]D\varphi\,dy\right| = 0. \tag{4.2.20}$$

On E_j^- we have

$$\frac{1}{\lambda_j}\int_{E_j^-}[D_\xi f(x_j+r_jy,A_j+\lambda_jDv_j)-D_\xi f(x_j+r_jy,A_j)]D\varphi\,dy$$

$$= \; \int_{E_j^-}\int_0^1 D_{\xi\xi}f(x_j+r_jy,A_j+t\lambda_jDv_j)\,dtDv_jD\varphi\,dy. \tag{4.2.21}$$

Note that (4.2.16) yields that $\chi_{E_j^-} \to \chi_{B_1}$ in L^r, for every $r < \infty$. Moreover by (4.2.8) we have, at least for subsequences, that

$$\lambda_jDv_j \to 0 \qquad \text{a.e. in } B_1$$

$$r_j \to 0$$

and

$$x_j \to x_0.$$

Hence the uniform continuity of $D_{\xi\xi}f$ on bounded sets implies

$$\lim_j \frac{1}{\lambda_j} \int_{E_j^-} [D_\xi f(x_j + r_j y, A_j + \lambda_j Dv_j) - D_\xi f(x_j + r_j y, A_j)] D\varphi \, dy$$

$$= \int_{B_1} D_{\xi\xi} f(x_0, A) Dv D\varphi \, dy. \tag{4.2.22}$$

Since $\beta < \alpha$, by (4.2.8) we deduce that

$$\lim_j \frac{r_j^\alpha}{\lambda_j} = 0. \tag{4.2.23}$$

By estimates (4.2.20), (4.2.22) and (4.2.23), passing to the limit as $j \to \infty$ in (4.2.15) yields

$$0 \leq \int_{B_1} D_{\xi\xi} f(x_0, A) Dv D\varphi \, dy$$

Changing φ in $-\varphi$ we finally get

$$\int_{B_1} D_{\xi\xi} f(x_0, A) Dv D\varphi \, dy = 0$$

i.e. v solves a linear system which is uniformly elliptic thanks to the uniform convexity of f. The regularity result stated in Proposition 2.2.3 implies that $v \in C^\infty(B_1)$ and for any $0 < \tau < 1$

$$\fint_{B_\tau} |Dv - (Dv)_\tau|^2 \, dy \leq c\tau^2 \fint_{B_1} |Dv - (Dv)_1|^2 \, dy \leq c\tau^2, \tag{4.2.24}$$

for a constant c depending on M.

Step 4. Conclusion

Fix $\tau \in (0, \frac{1}{4})$, set $b_j = (v_j)_{B_{2\tau}}$, $B_j = (Dv_j)_{B_\tau}$ and define

$$w_j(y) = v_j(y) - b_j - B_j y.$$

After rescaling, we note that $\lambda_j w_j$ satisfies the following integral inequality

$$\int_{B_1(0)} g_j(y, \lambda_j Dw_j)\, dy \leq \int_{B_1(0)} g_j(y, \lambda_j Dw_j + D\varphi)\, dy + c r_j^\alpha \int_{B_1(0)} |D\varphi|\, dy,$$

for every $\varphi \in W_0^{1,1}(B_1(0))$ where

$$g_j(y,\xi) = f(x_j + r_j y, A_j + \lambda_j B_j + \xi) - f(x_j + r_j y, A_j + \lambda_j B_j) - D_\xi f(x_j + r_j y, A_j + \lambda_j B_j)\xi.$$

It is easy to check that Lemma 4.0.5 applies to each g_j, for some constants that could depend on τ through $|\lambda_j B_j|$. But, given τ, we may always choose j large enough to have $|\lambda_j B_j| < \frac{\lambda_j}{\tau^{\frac{n}{t}}} < 1$, where $t = \min\{2, p\}$. Hence we can apply Proposition 4.1.1 to each $\lambda_j w_j$. In case $1 < p < 2$ we have that

$$\begin{aligned}
\lim_j \frac{\tilde{E}(x_j, \tau r_j)}{\lambda_j^2} &= \lim_j \frac{1}{\lambda_j^2} \fint_{B_{\tau r_j}(x)} |V_p(Du - (Du)_{\tau r_j})|^2\, dy + \lim_j \frac{\tau^\beta r_j^\beta}{\lambda_j^2} \\
&\leq \lim_j \frac{1}{\lambda_j^2} \fint_{B_\tau} |V_p(\lambda_j Dw_j)|^2\, dy + \tau^\beta \\
&\leq c \lim_j \fint_{B_{2\tau}} \frac{1}{\lambda_j^2} \left| V_p\left(\frac{\lambda_j w_j}{\tau}\right)\right|^2\, dy \\
&\quad + c \lim_j \lambda_j^{\frac{2(q-p)}{p}} \left(\fint_{B_{2\tau}} \frac{|V_p(\lambda_j Dw_j)|^2}{\lambda_j^2} + \frac{1}{\lambda_j^2} \left| V_p\left(\frac{\lambda_j w_j}{\tau}\right)\right|^2 dy \right)^{\frac{q}{p}} \\
&\quad + c \lim_j \frac{r_j^\alpha}{\lambda_j^2} \left(\fint_{B_\tau} \lambda_j^p |Dw_j|^p\, dy \right)^{\frac{1}{p}} + c \lim_j \frac{r_j^\alpha}{\lambda_j^2} \left(\fint_{B_\tau} \lambda_j^p \frac{|w_j|^p}{\tau^p}\, dy \right)^{\frac{1}{p}} + \tau^\beta \\
&\leq c \lim_j \fint_{B_{2\tau}} \frac{1}{\lambda_j^2} \left| V_p\left(\frac{\lambda_j w_j}{\tau}\right)\right|^2\, dy + \tau^\beta
\end{aligned}$$

since

$$\lim_j \lambda_j^{\frac{2(q-p)}{p}} = 0, \qquad \lim_j \frac{r_j^\alpha}{\lambda_j^2} = 0$$

and the integrals appearing as their factors are bounded as $j \to \infty$. Now, since $v_j \to v$ strongly in $L^p(B_1(0))$, using the Sobolev-Poincaré inequality stated in Lemma 2.2.2, one can easily check that

$$\lim_{j \to +\infty} \int_{B_{\frac{1}{2}}} \frac{|V_p(\lambda_j (v_j - v))|^2}{\lambda_j^2}\, dy = 0. \tag{4.2.25}$$

In fact, for every $\vartheta \in (0, \frac{p}{2})$ we can use Hölder's inequality of exponents $\frac{p}{2\vartheta}$ and $\frac{p}{p-2\vartheta}$ as follows

$$\int_{B_{\frac{1}{2}}} \frac{|V_p(\lambda_j(v_j - v))|^2}{\lambda_j^2}\, dy = \int_{B_{\frac{1}{2}}} |v_j - v|^2 (1 + \lambda_j^2 |v_j - v|^2)^{\frac{p-2}{2}}\, dy$$

$$\leq \left(\int_{B_{\frac{1}{2}}} |v_j - v|^p (1 + \lambda_j^2 |v_j - v|^2)^{\frac{p(p-2)}{4}}\, dy \right)^{\frac{2\vartheta}{p}}$$

$$\times \left(\int_{B_{\frac{1}{2}}} |v_j - v|^{\frac{2p(1-\vartheta)}{p-2\vartheta}} (1 + \lambda_j^2 |v_j - v|^2)^{\frac{p(p-2)(1-\vartheta)}{2(p-2\vartheta)}}\, dy \right)^{\frac{p-2\vartheta}{p}}$$

$$\leq \left(\int_{B_{\frac{1}{2}}} |v_j - v|^p\, dy \right)^{\frac{2\vartheta}{p}} \left(\int_{B_{\frac{1}{2}}} \left(\frac{|V_p(\lambda_j(v_j - v))|^2}{\lambda_j^2} \right)^{\frac{p(1-\vartheta)}{p-2\vartheta}} dy \right)^{\frac{p-2\vartheta}{p}}$$

$$\leq \left(\int_{B_{\frac{1}{2}}} |v_j - v|^p\, dy \right)^{\frac{2\vartheta}{p}} \left(\int_{B_{\frac{1}{2}}} \frac{|V_p(\lambda_j(Dv_j - Dv))|^2}{\lambda_j^2}\, dy \right)^{1-\vartheta}.$$

Last inequality is obtained applying Lemma 2.2.2 to the second integral, choosing $\vartheta \in (0, \frac{p}{2})$ such that $\frac{p(1-\vartheta)}{p-2\vartheta} = \frac{n}{n-p}$. Hence (4.2.25) follows noticing that the first integral vanishes as j goes to infinity and second one stays bounded thanks to (4.2.7), since $v \in C_0^\infty(B_1(0))$.

Since $b_j \to (v)_{2\tau}$ and $B_j \to (Dv)_\tau$, using (4.2.25) and the definition of w_j we get

$$\lim_j \frac{\tilde{E}(x_j, \tau r_j)}{\lambda_j^2} \leq c \lim_j ⨍_{B_{2\tau}} \frac{1}{\lambda_j^2} \left| V_p \left(\frac{\lambda_j(w_j - v + v)}{\tau} \right) \right|^2 dy + \tau^\beta$$

$$= c \lim_j ⨍_{B_{2\tau}} \frac{1}{\lambda_j^2} \left| V_p \left(\frac{\lambda_j(v_j - v + v - b_j - B_j y)}{\tau} \right) \right|^2 dy + \tau^\beta$$

$$\leq c ⨍_{B_{2\tau}} \frac{|v - (v)_{2\tau} - (Dv)_\tau y|^2}{\tau^2}\, dy + \tau^\beta$$

$$\leq c ⨍_{B_{2\tau}} \frac{|v - (v)_{2\tau} - (Dv)_{2\tau} y|^2}{\tau^2}\, dy + c ⨍_{B_{2\tau}} \frac{|(Dv)_\tau y - (Dv)_{2\tau} y|^2}{\tau^2}\, dy + \tau^\beta$$

$$\leq c ⨍_{B_{2\tau}} |Dv - (Dv)_{2\tau}|^2\, dy + c|(Dv)_\tau - (Dv)_{2\tau}|^2 + \tau^\beta$$

$$\leq c\tau^2 + c\tau^\beta \leq c_M^\star \tau^\beta.$$

The contradiction follows, if $1 < p < 2$, by choosing $c_M^\star > \tilde{C}(M)$.

Now we face the case $p \geq 2$. Arguing as we did for the case $1 < p < 2$ and using property (2.3.5) we get

$$\begin{aligned}
\lim_j \frac{\tilde{E}(x_j, \tau r_j)}{\lambda_j^2} &\leq c \lim_j \fint_{B_\tau} (|Dw_j|^2 + \lambda_j^{p-2}|Dw_j|^p)\, dy + \tau^\beta \\
&\leq c \lim_j \fint_{B_{2\tau}} \left(\frac{|w_j|^2}{\tau^2} + \lambda_j^{p-2}\frac{|w_j|^p}{\tau^p} \right) dy \\
&+ c \lim_j \lambda_j^{\frac{2(q-p)}{p}} \left(\fint_{B_{2\tau}} \left(|Dw_j|^2 + \lambda_j^{p-2}|Dw_j|^p\right) dy \right)^{\frac{q}{p}} \\
&+ c \lim_j \frac{r_j^\alpha}{\lambda_j^2} \left(\int_{B_{2\tau}} \lambda_j^2 |Dw_j|^2\, dy \right)^{\frac{1}{2}} + c \lim_j \frac{r_j^\alpha}{\lambda_j^2} \left(\int_{B_{2\tau}} \lambda_j^2 \frac{|w_j|^2}{\tau^2}\, dy \right)^{\frac{1}{2}} + \tau^\beta \\
&\leq c \fint_{B_{2\tau}} \frac{|v - (v)_{2\tau} - (Dv)_\tau y|^2}{\tau^2}\, dy + \tau^\beta \\
&\leq c \fint_{B_{2\tau}} \frac{|v - (v)_{2\tau} - (Dv)_{2\tau} y|^2}{\tau^2}\, dy + c \fint_{B_{2\tau}} \frac{|(Dv)_\tau y - (Dv)_{2\tau} y|^2}{\tau^2}\, dy \\
&\quad + \tau^\beta \\
&\leq c \fint_{B_{2\tau}} |Dv - (Dv)_{2\tau}|^2 \, dy + c|(Dv)_\tau - (Dv)_{2\tau}|^2 + \tau^\beta \\
&\leq c\tau^2 + c\tau^\beta \leq c_M^\star \tau^\beta.
\end{aligned}$$

The contradiction follows, if $p \geq 2$, by choosing $c_M^\star > \tilde{C}(M)$. □

4.3 Proof of Theorem 4.0.3

The proof of our regularity result follows from the decay estimate of Proposition 4.2.1 by a standard iteration argument. We sketch it here for the reader's convenience.

Proof of Theorem 4.0.3. Following the arguments used in Section 6 of [43], from Proposition 4.2.1 we deduce that for every $M > 0$ there exist $0 < \tau < \frac{1}{4}$ and $\eta > 0$

such that if

$$|(Du)_{x_0,R}| \leq M \qquad \text{and} \qquad \tilde{E}(x_0,R) < \eta \tag{4.3.1}$$

then

$$|(Du)_{x_0,\tau^k R}| \leq 2M \qquad \text{and} \qquad \tilde{E}(x_0,\tau^k R) < c(M)\tau^{\beta k}\tilde{E}(x_0,R) \tag{4.3.2}$$

for every $k \in \mathbb{N}$. Estimate (4.3.2) yields that if (4.3.1) holds for any $\rho \in (0,R)$ we have

$$|(Du)_{x_0,\rho}| \leq c(M) \qquad \text{and} \qquad \tilde{E}(x_0,\rho) < c(M)\left(\frac{\rho}{R}\right)^{\beta}\tilde{E}(x_0,R)$$

Therefore, in case $1 < p < 2$, using (2.3.3) we obtain

$$\begin{aligned}
-\!\!\!\!\!\!\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|\,dx &= -\!\!\!\!\!\!\int_{B_\rho(x_0)\cap\{x:\,|Du-(Du)_{x_0,\rho}|\leq 1\}} |Du - (Du)_{x_0,\rho}|\,dx \\
&\quad + -\!\!\!\!\!\!\int_{B_\rho(x_0)\cap\{x:\,|Du-(Du)_{x_0,\rho}|> 1\}} |Du - (Du)_{x_0,\rho}|\,dx \\
&\leq c -\!\!\!\!\!\!\int_{B_\rho(x_0)} |V_p(Du - (Du)_{x_0,\rho})|\,dx + \left(-\!\!\!\!\!\!\int_{B_\rho(x_0)} |V_p(Du - (Du)_{x_0,\rho})|^2\,dx\right)^{\frac{1}{p}} \\
&\leq c\tilde{E}^{\frac{1}{2}}(x_0,\rho) + c\tilde{E}^{\frac{1}{p}}(x_0,\rho) \leq c(M,R)\rho^{\frac{\beta}{2}} \qquad (4.3.3)
\end{aligned}$$

while in case $p \geq 2$ we use (2.3.5) thus getting

$$\begin{aligned}
-\!\!\!\!\!\!\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|\,dx &\leq \left(-\!\!\!\!\!\!\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2\,dx\right)^{\frac{1}{2}} \\
&\leq \left(-\!\!\!\!\!\!\int_{B_\rho(x_0)} |V_p(Du - (Du)_{x_0,\rho})|^2\,dx\right)^{\frac{1}{2}} \\
&= cE^{\frac{1}{2}}(x_0,\rho) \leq c(M,R)\rho^{\frac{\beta}{2}} \qquad (4.3.4)
\end{aligned}$$

From estimates (4.3.3) and (4.3.4) it is clear that, setting

$$\Omega_0 = \{x \in \Omega : \sup_{r>0} |(Du)_{x_0,r}| < \infty \text{ and } \lim_{r\to 0}\tilde{E}(x_0,r) = 0\},$$

Ω_0 is an open subset of Ω of full measure and $u \in C^{1,\gamma}(\Omega_0)$ for every $\gamma < \frac{\beta}{2}$, and the conclusion follows since β is any number less than α. $\square$

Chapter V

REGULARITY FOR NON-AUTONOMOUS FUNCTIONALS WITH ALMOST LINEAR GROWTH

The results of this chapter have been obtained in [13]. In this chapter we will focus our attention on integrands of (0.0.1) which are not too far from being linear in $|\xi|$, in the sense of conditions (0.0.6). We will establish $C^{1,\gamma}$ partial regularity of minimizers of (0.0.1) with an integrand f satisfying the assumptions (L1)–(L5).

The first result of this chapter is the following higher integrability property of minimizers of the functional $\mathcal{F}$. It will be proved under weaker assumptions than the ones needed to prove regularity.

Theorem 5.0.1. *Let $u \in W^{1,h}_{loc}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}$, with an integrand function f satisfying (L1) – (L4). Then we have*

$$Du \in L^s_{loc}(\Omega), \qquad \forall\, s < \frac{n}{n-\alpha},$$

and

$$||\,(V_1(Du))^2\,||_{L^{\frac{n}{n-2b}}(B_\rho)} \leq c \int_{B_{2R}} |Du| \log(1+|Du|)\,dx + c \int_{B_{2R}} |V_1(Du)|^2\,dx,$$

for every $\rho < 2R$ and every $b \in (0, \frac{\alpha}{2})$, where α is the exponent appearing in (L3), where we denoted by $V_1(\xi) = (1+|\xi|)^{-\frac{1}{4}}\xi$.

Corollary 5.0.2. *Under the same assumptions of the previous theorem, if $u \in W^{1,h}_{loc}(\Omega, \mathbb{R}^N)$ is a local minimizer of the functional $\mathcal{F}$, then we have*

$$Du \in W^{k,p}_{loc}(\Omega, \mathbb{R}^{nN}), \tag{5.0.5}$$

for every $k \in (0, \frac{\alpha}{2})$ and for every $1 < p < \frac{n}{n-\frac{\alpha}{2}}$.

The higher integrability of Theorem 5.0.1 allows us to prove an $C^{1,\gamma}$-partial regularity result which is formulated in the following

Theorem 5.0.3. *Let f be a $C^2(\Omega, \mathbb{R}^{n\times N})$-integrand satisfying the assumptions (L1) and (L3) – (L5). If $u \in W^{1,h}_{loc}(\Omega, \mathbb{R}^N)$ is a local minimizer of the functional $\mathcal{F}$, then there exists an open subset Ω_0 of Ω such that*

$$meas(\Omega \setminus \Omega_0) = 0$$

and

$$u \in C^{1,\gamma}_{loc}(\Omega_0, \mathbb{R}^N) \qquad \text{for every} \qquad \gamma < \frac{\alpha}{2},$$

where α is the exponent appearing in (L3).

Our proof is based on a blow up argument aimed to establish a decay estimate for the excess function of the minimizer. The proof has features in common with [32], since we use the higher integrability Theorem 5.0.1 in order to define the excess function as

$$\tilde{E}(x, r) = \fint_{B_r(x)} |V_p(Du) - V_p((Du)_r)|^2 + r^\beta$$

with

$$V_p(\xi) = (1 + |\xi|^2)^{\frac{p-2}{4}}\xi.$$

The main difference with [32] is that, in order to perform the blow up procedure, we use a Caccioppoli type inequality for minimizers of a suitable perturbation of the rescaled functional, as done in [24].

The main difficulty in order to prove the Caccioppoli type inequality is a uniform higher integrability result for the minimizers of the rescaled functionals. We have to combine the difference quotient method with properties of Orlicz-Sobolev classes generated by an Orlicz function which grows almost linearly. We also use the properties of the function $V_p(\xi)$ which is an useful tool to deal with subquadratic setting.

In order to improve this to everywhere regularity additional assumptions are necessary. The first is the modulus dependence, i.e.,

$$f(x,\xi) = \widehat{f}(x,|\xi|) \tag{F6}$$

for a function $\widehat{f} : \Omega \times [0,\infty) \to \mathbb{R}$ which is strictly increasing in the real variable. According to counterexamples of DeGiorgi (compare [22]) when dealing with vectorial minimizers, i.e. $N > 1$, it is well-known that without this assumption there is no hope for full regularity. On the other hand we need a Caccioppoli-type inequality in order to apply DeGiorgi arguments, hence we assume for every $s \in \{1,...,n\}$

$$\partial_s D_\xi f \in C^0(\Omega \times \mathbb{R}^{nN}, \mathbb{R}^{nN}) \quad \text{and} \quad |\partial_s D_\xi f(x,\xi)| \leq c(1+|\xi|)^{p-1} \tag{L7}$$

for an exponent $1 < p < \frac{n-\frac{\alpha}{2}}{n-\alpha}$. Finally we suppose that

$$|D^2_{\xi\xi}(x,\xi_1) - D^2_{\xi\xi}(x,\xi_2)| \leq c(1+|\xi_1|+|\xi_2|)^{p-2-\mu}|\xi_1-\xi_2|^\mu \tag{L8}$$

for all $x \in \Omega$, $\xi_1, \xi_2 \in \mathbb{R}^{nN}$ and for an exponent $\mu \in (0,1)$. Of course (L7) and (L8) are true in the autonomous case for $f(x,\xi) = |\xi|\log^\theta(1+|\xi|)$, $\theta > 0$, for every choice of $p > 1$. The full regularity result of this paper is the following

Theorem 5.0.4. *Let $u \in W^{1,h}_{loc}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}$, with an integrand function f satisfying (L1) and (L3) – (L8). Then we have*

$$u \in C^{1,\gamma}_{loc}(\Omega, \mathbb{R}^{nN}), \quad \textit{for all } \gamma < 1.$$

Thanks to Theorem 5.0.3 we have a nonempty set of regular points for every minimizer of the functional $\mathcal{F}$ with a general integrand function f. Therefore Corollary 5.0.2 allows us to apply Lemma 2.4.1 (stated in Chapter I) to give an estimate of the Hausdorff dimension of the singular set of minimizers of $\mathcal{F}$.

Corollary 5.0.5. *If f is a C^2 function satisfying the assumptions (L1) and (L3)–(L5) and the function $u \in W^{1,h}(\Omega;\mathbb{R}^N)$ is a local minimizer of $\mathcal{F}$ in Ω, then for the Hausdorff dimension of the singular set Σ of the function u the following estimate hold*

$$dim_{\mathcal{H}}(\Sigma) \leq n - \frac{\alpha}{2}q$$

where $q = \frac{n}{n-\frac{\alpha}{2}}$.

As usual, the proof of $C^{1,\alpha}$ partial regularity will be achieved through the blow up method. In the linearization procedure we shall use the rescaled functional of $\mathcal{F}$ on the unit ball $B \equiv B_1(0)$

$$\mathcal{I}(v) := \int_B g(y, Dv)\, dy$$

defined by setting

$$g(y,\xi) = \lambda^{-2}[f(x_0 + r_0 y, A + \lambda\xi) - f(x_0 + r_0 y, A) - D_\xi f(x_0 + r_0 y, A)\lambda\xi], \quad (5.0.6)$$

where A is a matrix such that $|A|$ is uniformly bounded by a positive constant M. Next Lemma contains the growth conditions on g.

Lemma 5.0.6. *Let $f \in C^2(\Omega \times \mathbb{R}^{n\times N})$ be a function satisfying the assumptions (L1)–(L5) and let $g(y,\xi)$ be the function defined by (5.0.6). Then we have*

$$\widetilde{\nu}\frac{|\xi|^2}{1+|\lambda\xi|} \leq |g(y,\xi)| \leq c\frac{\log(1+|\lambda\xi|)}{|\lambda\xi|}|\xi|^2; \quad \text{(I1)}$$

$$|D_\xi g(y,\xi)| \leq c\frac{\log(e+|\lambda\xi|)}{\lambda}; \quad \text{(I2)}$$

$$|D_\xi g(y_1,\xi) - D_\xi g(y_2,\xi)| \leq \frac{cr_0^\alpha}{\lambda}|y_1 - y_2|^\alpha\ (\log(e+|\xi|)); \quad \text{(I3)}$$

$$\widetilde{\nu}\frac{|\zeta|^2}{1+|\lambda\xi|} \le \langle D_{\xi\xi}g(y,\xi)\zeta,\zeta\rangle \tag{I4}$$

where the constant c depends on M in all statements.

Proof. (I2), (I3) and (I4) can be proven as in [24] (Lemma 2.9) using the growth conditions of f. The lower bound in (I1) is a consequence of the representation

$$g(y,\xi) = \int_0^1 \int_0^t D_{\xi\xi}f(x_0 + r_0 y, A + s\lambda\xi)(\xi,\xi)\, ds\, dt$$

since we have by (L4)

$$\begin{aligned} D_{\xi\xi}f(x_0 + r_0 y, A + s\lambda\xi)(\xi,\xi) &\ge \mu\frac{|\xi|^2}{1+|A+s\lambda\xi|} \\ &\ge \widetilde{\nu}\frac{|\zeta|^2}{1+|\lambda\xi|}. \end{aligned}$$

The upper bound is an immediate consequence of (L5). □

5.1 Higher integrability

In this section we prove Theorem 5.0.1.

Proof. Let $u \in W^{1,h}_{loc}(\Omega,\mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}$, with an integrand function f satisfying (L1) – (L4). Then u satysfies the Euler system related to the functional $\mathcal{F}$:

$$\int_\Omega D_\xi f(x,Du)D\varphi\, dx = 0 \tag{5.1.1}$$

for every $\varphi \in W^{1,h}_0(\Omega)$. Let η be a cut-off function in $C^1_0(B_{3R/2})$ with $0 \le \eta \le 1$, $\eta \equiv 1$ on B_R and $|D\eta| < c/R$. Let us consider the function $\varphi = \tau_{s,-h}(\eta^2(x)\tau_{s,h}u)$ with s fixed in $\{1,\dots,n\}$ (which from now on we shall omit for the sake of simplicity) and $|h| < R/10$. Substituting in (5.1.1) the function φ and using $(d2)$ of Propostion 2.1.4 we get

$$\int_{B_{2R}} \tau_h(D_\xi f(x,Du))\, D(\eta^2\tau_h u)\, dx = 0.$$

This equality can be written as

$$
\begin{aligned}
I =& \int_{B_{2R}} \eta^2 \left[D_\xi f(x+he_s, Du(x+he_s)) - D_\xi f(x+he_s, Du(x)) \right] \tau_h Du \, dx \\
=& - \int_{B_{2R}} \eta^2 [D_\xi f(x+he_s, Du(x)) - D_\xi f(x, Du(x))] \; \tau_h Du \, dx \\
& - 2 \int_{B_{2R}} \eta \; \left(D_\xi f(x+he_s, Du(x+he_s)) - D_\xi f(x, Du)\right) D\eta \otimes \tau_h u \, dx \\
=& - II - III
\end{aligned}
\tag{5.1.2}
$$

where we used $(d1)$ of Propostion 2.1.4. Assumption $(L4)$ yields that

$$
\nu \int_{B_{2R}} \eta^2 \left(1 + |Du(x+he_s)| + |Du(x)|\right)^{-1} |\tau_h Du|^2 \, dx \le I. \tag{5.1.3}
$$

Using assumption $(L3)$ we obtain:

$$
|II| \le c|h|^\alpha \int_{B_{2R}} \log(1+|Du|)|\tau_h Du| \, dx
$$

and hence, by Young's Inequality for Young functions and properties of η, it follows that

$$
\begin{aligned}
|II| \le& c|h|^\alpha \left(\int_{B_{3R/2}} |Du| \log(1+|Du|) \, dx + \int_{B_{3R/2}} |\tau_h Du| \log(1+|\tau_h Du|) \, dx \right) \\
\le& c|h|^\alpha \int_{B_{2R}} |Du| \log(1+|Du|) \, dx.
\end{aligned}
\tag{5.1.4}
$$

To estimate III we use assumption $(L2)$ and Young's Inequality as follows

$$
\begin{aligned}
|III| \le& c|h| \int_{B_{2R}} \eta |D\eta| \log(1+|Du(x+he_s)|)|\Delta_h u| \, dx \\
& + c|h| \int_{B_{2R}} \eta |D\eta| \log(1+|Du(x)|)|\Delta_h u| \, dx \\
\le& c|h| \int_{B_{3R/2}} \log(1+|Du(x+he_s)|)|Du(x+he_s)| \, dx \\
& + c|h| \int_{B_{3R/2}} \log(1+|\Delta_h u|)|\Delta_h u| \, dx + c|h| \int_{B_{3R/2}} \log(1+|Du|)|Du| \, dx \\
\le& c|h|^\alpha \int_{B_{2R}} \log(1+|Du|)|Du| \, dx.
\end{aligned}
\tag{5.1.5}
$$

In order to estimate the $\Delta_h u$ integral in the last step, we used the following inequality which is valid for each convex function φ according to Jensen's Inequality:

$$\begin{aligned}\int_{B_{3R/2}} \varphi(|\Delta_h u|)\,dx &= \int_{B_{3R/2}} \varphi\left(\left|\int_0^1 \frac{du}{ds}(x+the_s)\,dt\right|\right)\,dx \\ &\leq \int_{B_{3R/2}} \int_0^1 \varphi\left(\left|\frac{du}{ds}(x+the_s)\right|\right)\,dt\,dx \\ &\leq \int_{2R} \varphi\left(|Du|\right)\,dx. \qquad (5.1.6)\end{aligned}$$

Inserting estimates (5.1.3), (5.1.4) and (5.1.5) into (5.1.2) we get

$$\begin{aligned}\nu \int_{B_{2R}} &\eta^2 \left(1+|Du(x+he_s)|+|Du(x)|\right)^{-1} |\tau_h Du|^2\,dx \\ &\leq c|h|^\alpha \int_{B_{2R}} \log(1+|Du|)|Du|\,dx. \qquad (5.1.7)\end{aligned}$$

The left hand side of (5.1.7) can be controlled from below as follows

$$\begin{aligned}\nu \int_{B_{2R}} &\eta^2 \frac{|\tau_h Du|^2}{1+|Du(x+he_s)|+|Du(x)|}dx \\ &\geq c\int_{B_{2R}} \eta^2 \frac{|\tau_h Du|^2}{(1+|Du(x+he_s)|^2+|Du(x)|^2)^{\frac{1}{2}}}\,dx \\ &= c\int_{B_{2R}} \eta^2 \left(\frac{|Du(x+he_s)-Du(x)|}{(1+|Du(x+he_s)|^2+|Du(x)|^2)^{\frac{1}{4}}}\right)^2\,dx.\end{aligned}$$

Lemma 2.1.5 applied for $\gamma = -\frac{1}{4}$ implies that

$$\begin{aligned}&\frac{|Du(x+he_s)-Du(x)|}{(1+|Du(x+he_s)|^2+|Du(x)|^2)^{\frac{1}{4}}} \\ &\geq c|(1+|Du(x+he_s)|^2)^{-\frac{1}{4}}Du(x+he_s)-(1+|Du(x)|^2)^{-\frac{1}{4}}Du(x)| \\ &= c|\tau_{s,h}V_1(Du(x))|.\end{aligned}$$

Hence

$$\nu \int_{B_{2R}} \eta^2 \frac{|\tau_h Du|^2}{1+|Du(x+he_s)|+|Du(x)|}dx \geq c\int_{B_{2R}} \eta^2 |\tau_{s,h}(V_1(Du))|^2\,dx.$$

Plugging this estimate in (5.1.7) we get

$$\int_{B_{2R}} \eta^2 |\tau_{s,h}(V_1(Du))|^2\,dx \le c|h|^\alpha \int_{B_{2R}} |Du|\log(1+|Du|).$$

Lemma 2.1.8 implies that

$$V_1(Du) \in W^{b,2} \cap L^{\frac{2n}{n-2b}} \qquad \forall\, b \in \left(0, \frac{\alpha}{2}\right),$$

and

$$||V_1(Du)||_{L^{\frac{2n}{n-2b}}(B_\rho)} \le c\left(\int_{B_{2R}} |Du|\log(1+|Du|)\,dx\right)^{\frac{1}{2}} + c\left(\int_{B_{2R}} |V_1(Du)|^2\,dx\right)^{\frac{1}{2}},$$

for every $\rho < 2R$. Hence we get the claim and the final estimate:

$$||\,(V_1(Du))^2\,||_{L^{\frac{n}{n-2b}}(B_\rho)} \le c\int_{B_{2R}} |Du|\log(1+|Du|)\,dx + c\int_{B_{2R}} |V_1(Du)|^2\,dx,$$

for every $\rho < 2R$. □

The proof of Corollary 5.0.5 can be immediately obtained by applying Hölder's inequality with exponents $2/p$ and $2/(2-p)$ to the right hand side of the following equality

$$\int_\Omega \eta^p |\tau_{h,s}Du|^p\,dx = \int_\Omega \eta^p (1+|Du(x+he_s)|+|Du(x)|)^{-\frac{p}{2}} |\tau_{h,s}Du|^p \cdot (1+|Du(x+he_s)|+|Du(x)|)^{\frac{p}{2}}\,dx,$$

where η is a suitable cut-off function. Then we use estimate (5.1.7), Theorem 5.0.1 and Lemma 2.1.8.

5.2 Decay estimate

Define the excess function, as in (0.0.8), in accordance to [32] as

$$\tilde{E}(x,r) = \fint_{B_r(x)} |V_p(Du) - V_p((Du)_r)|^2 + r^\beta \tag{5.2.1}$$

with $\beta < \alpha$ and $p < \frac{n}{n-\alpha}$. We remark that the higher integrability stated in Theorem 5.0.1 allows us to give sense to $\tilde{E}(x,r)$ when $p < \frac{n}{n-\alpha}$ and therefore we may use a blow-up technique similar to the one used for functionals with p-growth, when $p < 2$.

The blow-up argument we need to prove Theorem 5.0.3 is contained in the following

Proposition 5.2.1. *Fix $M > 0$. There exists a constant $C(M) > 0$ such that, for every $0 < \tau < \frac{1}{4}$, there exists $\varepsilon = \varepsilon(\tau, M)$ such that, if*

$$|(Du)_{x_0,r}| \leq M \qquad \textit{and} \qquad \tilde{E}(x_0, r) \leq \varepsilon,$$

then

$$\tilde{E}(x_0, \tau r) \leq C(M)\, \tau^{\beta}\, \tilde{E}(x_0, r).$$

Proof. Step 1. Blow up

Fix $M > 0$. Assume by contradiction that there exists a sequence of balls $B_{r_j}(x_j) \Subset \Omega$ such that

$$|(Du)_{x_j,r_j}| \leq M \qquad \text{and} \qquad \lambda_j^2 = \tilde{E}(x_j, r_j) \to 0 \tag{5.2.2}$$

but

$$\frac{\tilde{E}(x_j, \tau r_j)}{\lambda_j^2} > \tilde{C}(M)\tau^{\beta} \tag{5.2.3}$$

where $\tilde{C}(M)$ will be determined later. Setting $A_j = (Du)_{x_j,r_j}$, $a_j = (u)_{x_j,r_j}$ and

$$v_j(y) = \frac{u(x_j + r_j y) - a_j - r_j A_j y}{\lambda_j r_j} \tag{5.2.4}$$

for all $y \in B_1(0)$, one can easily check that $(Dv_j)_{0,1} = 0$ and $(v_j)_{0,1} = 0$. By the definition of λ_j it follows that

$$⨍_{B_1(0)} \frac{|V_p(\lambda_j D u_j)|^2}{\lambda_j^2}\, dy + \frac{r_j^{\beta}}{\lambda_j^2} = 1. \tag{5.2.5}$$

Therefore passing possibly to not relabeled sequences (note that we obtain by (5.2.5) uniform L^p-bounds on Du_j)

$$v_j \rightharpoonup v \qquad \text{weakly in } W^{1,p}(B_1(0);\mathbb{R}^N)$$

$$\lambda_j v_j \to 0 \qquad \text{strongly in } W^{1,p}(B_1(0);\mathbb{R}^N)$$

$$v_j \to v \qquad \text{strongly in } L^p(B_1(0);\mathbb{R}^N)$$

$$A_j \longrightarrow A$$

$$r_j \longrightarrow 0 \qquad \frac{r_j^\vartheta}{\lambda_j^2} \longrightarrow 0, \qquad \vartheta > \beta. \tag{5.2.6}$$

Step 2. Minimality of v_j

We normalize f around A_j as follows

$$f_j(y,\xi) = \frac{f(x_j + r_j y, A_j + \lambda_j \xi) - f(x_j + r_j y, A_j) - D_\xi f(x_j + r_j y, A_j)\lambda_j \xi}{\lambda_j^2} \tag{5.2.7}$$

and we consider the corresponding rescaled functionals

$$\mathcal{I}_j(w) = \int_{B_1(0)} [f_j(y, Dw)]dy. \tag{5.2.8}$$

The minimality of u and a simple change of variable yield that

$$\int_{B_1(0)} f(x_j+r_j y, Du(x_j+r_j y))\,dy \le \int_{B_1(0)} f(x_j+r_j y, Du(x_j+r_j y)+D\varphi(x_j+r_j y))\,dy$$

for every $\varphi \in W_0^{1,h}(B_{r_j}(x_j);\mathbb{R}^N)$, that is

$$\int_{B_1(0)} f(x_j + r_j y, A_j + \lambda_j Dv_j(y))\,dy \le \int_{B_1(0)} f(x_j + r_j y, A_j + \lambda_j Dv_j(y) + D\varphi(x_j + r_j y))\,dy,$$

for every $\varphi \in W_0^{1,h}(B_{r_j}(x_j);\mathbb{R}^N)$. Thus, by the definition of the rescaled functionals, we have

$$\mathcal{I}_j(v_j) \le \mathcal{I}_j(v_j+\varphi) + \int_{B_1(0)} \frac{D_\xi f(x_j + r_j y, A_j) D\varphi}{\lambda_j}\, dy. \tag{5.2.9}$$

Using (L3) we conclude that

$$\begin{aligned}\mathcal{I}_j(v_j) &\le \mathcal{I}_j(v_j+\varphi) + \int_{B_1(0)} \frac{[D_\xi f(x_j + r_j y, A_j) - D_\xi f(x_j, A_j)] D\varphi}{\lambda_j}\, dy \\ &\le \mathcal{I}_j(v_j+\varphi) + c(M)\frac{r_j^\alpha}{\lambda_j}\int_{B_1(0)} |D\varphi|\, dy. \end{aligned}\tag{5.2.10}$$

Step 3. v solves a linear system

Using that v_j satisfies inequality (5.2.9), we have that

$$0 \le \mathcal{I}_j(v_j+s\varphi) - \mathcal{I}_j(v_j) + c(M)\frac{r_j^\alpha}{\lambda_j}\int_{B_1(0)} |sD\varphi|\, dy, \tag{5.2.11}$$

for every $\varphi \in C_0^1(B)$ and for every $s \in (0,1)$. Now, using again the definition of the rescaled functionals, we observe that

$$\begin{aligned}\mathcal{I}_j(v_j+s\varphi) - \mathcal{I}_j(v_j) &= \int_{B_1(0)}\int_0^1 [D_\xi f_j(x_j + r_j y, A_j + \lambda_j(Dv_j + tsD\varphi))] sD\varphi\, dt\, dy \\ = \frac{c}{\lambda_j}\int_{B_1(0)} &[D_\xi f(x_j + r_j y, A_j + \lambda_j(Dv_j + sD\varphi)) - D_\xi f(x_j + r_j y, A_j)] sD\varphi\, dy. \end{aligned}\tag{5.2.12}$$

Inserting (5.2.12) in (5.2.11), dividing by s and taking the limit as $s \to 0$, we conclude that

$$\begin{aligned} 0 &\le \frac{c}{\lambda_j}\int_{B_1(0)} [D_\xi f(x_j + r_j y, A_j + \lambda_j Dv_j) - D_\xi f(x_j + r_j y, A_j)] D\varphi\, dy \\ &+ \frac{c(M) r_j^\alpha}{\lambda_j}\int_{B_1(0)} |D\varphi|\, dy. \end{aligned}\tag{5.2.13}$$

Let us split

$$B_1(0) = E_j^+ \cup E_j^- = \{y \in B_1 :\ \lambda_j |Dv_j| > 1\} \cup \{y \in B_1 :\ \lambda_j |Dv_j| \le 1\}.$$

Using (5.2.5) we get

$$|E_j^+| \leq \int_{E_j^+} \lambda_j^p |Dv_j|^p \, dy \leq \lambda_j^p \int_{E_j^+} |Dv_j|^p \, dy \leq c\lambda_j^p. \tag{5.2.14}$$

Using (5.2.5), (L2) and the elementary inequality $\log(1+t) \leq ct^p$, we obtain

$$\begin{aligned}
&\frac{1}{\lambda_j} \left| \int_{E_j^+} [D_\xi f(x_j + r_j y, A_j + \lambda_j Dv_j) - D_\xi f(x_j + r_j y, A_j)] D\varphi \, dy \right| \\
\leq \;& \frac{1}{\lambda_j} \int_{E_j^+} \left(\log(1 + |A_j + \lambda_j Dv_j|) + \log(1 + |A_j|) \right) \, dy \\
\leq \;& c\frac{|E_j^+|}{\lambda_j} + \frac{1}{\lambda_j} \int_{E_j^+} \left(|A_j|^p + |\lambda_j Dv_j|^p \right) \, dy \\
\leq \;& c\lambda_j^{p-1}.
\end{aligned} \tag{5.2.15}$$

Hence, we infer that

$$\lim_{j\to\infty} \frac{c}{\lambda_j} \left| \int_{E_j^+} [D_\xi f(x_j + r_j y, A_j + \lambda_j Dv_j) - D_\xi f(x_j + r_j y, A_j)] D\varphi \, dy \right| = 0. \tag{5.2.16}$$

On E_j^- we have

$$\begin{aligned}
&\frac{1}{\lambda_j} \int_{E_j^-} [D_\xi f(x_j + r_j y, A_j + \lambda_j Dv_j) - D_\xi f(x_j + r_j y, A_j)] D\varphi \, dy \\
= \;& \int_{E_j^-} \int_0^1 D_{\xi\xi} f(x_j + r_j y, A_j + t\lambda_j Dv_j) \, dt Dv_j D\varphi \, dy.
\end{aligned} \tag{5.2.17}$$

Note that (5.2.14) yields that $\chi_{E_j^-} \to \chi_{B_1}$ in L^r, for every $r < \infty$. Moreover by (5.2.6) we have, at least for subsequences, that

$$\lambda_j Dv_j \to 0 \qquad \text{a.e. in } B_1$$

$$r_j \to 0$$

and

$$x_j \to x_0.$$

Hence the uniform continuity of $D_{\xi\xi}f$ on bounded sets implies

$$\lim_j \frac{1}{\lambda_j}\int_{E_j^-}[D_\xi f(x_j+r_jy, A_j+\lambda_j Dv_j) - D_\xi f(x_j+r_jy, A_j)]D\varphi\, dy$$

$$= \int_{B_1} D_{\xi\xi}f(x_0, A)DvD\varphi\, dy. \tag{5.2.18}$$

Since $\beta < \alpha$, by (5.2.6) we deduce that

$$\lim_j \frac{r_j^\alpha}{\lambda_j^2} = 0. \tag{5.2.19}$$

By estimates (5.2.16), (5.2.18) and (5.2.19), passing to the limit as $j \to \infty$ in (5.2.13) yields

$$0 \le \int_{B_1} D_{\xi\xi}f(x_0, A)DvD\varphi\, dy$$

Changing φ in $-\varphi$ we finally get

$$\int_{B_1} D_{\xi\xi}f(x_0, A)DvD\varphi\, dy = 0$$

that is v solves a linear system which is uniformly elliptic thanks to the uniform convexity of f. The regularity result stated in Proposition 2.2.3 implies that $v \in C^\infty(B_1)$ and for any $0 < \tau < 1$

$$⨍_{B_\tau} |Dv - (Dv)_\tau|^2\, dy \le c\tau^2 ⨍_{B_1} |Dv - (Dv)_1|^2\, dy \le c\tau^2, \tag{5.2.20}$$

for a constant c depending on M.

Step 4. Higher integrability of v_j

In this step we will prove a higher integrability result for Dv_j which is uniform with respect to the rescaling procedure. We will drop the index j for simplicity.

Lemma 5.2.2. *Let g be a function satisfying (I1)-(I4) and $v \in W^{1,h}(B;\mathbb{R}^N)$ a solution of*

$$\mathcal{I}(v) \;\le\; \mathcal{I}(v+\varphi) + c(M)\frac{r_0^\alpha}{\lambda}\int_{B_1(0)} D_\xi f(x_0+r_0y, A)D\varphi\, dy$$

for every $\varphi \in W_0^{1,h}(B_1(0);\mathbb{R}^N)$. Then we have

$$\left(\int_{B_{\frac{\rho}{2}}} \left|\left\{\lambda^{-1}V_1(\lambda Dv)\right\}\right|^{\frac{2n}{n-2k}}\,dy\right)^{\frac{n-2k}{2n}}$$

$$\leq \frac{c}{\lambda}\left(\int_{B_\rho}\left\{|V_p(\lambda Dv)|^2\right\}\,dy\right)^{\frac{1}{2}} + c\frac{r_0^{\frac{\alpha}{2}}}{\lambda}\left(\int_{B_\rho}\left\{|\lambda Dv| + \log(1+|\lambda Dv|)|\lambda Dv|\right\}\,dy\right)^{\frac{1}{2}}$$

$$+\left(\int_{B_\rho}\left|\left\{\lambda^{-1}V_1(\lambda Dv)\right\}\right|^2\,dy\right)^{\frac{1}{2}}$$

Here c does not depend on r_0, λ and v.

Proof. Let us fix two radii $\frac{\rho}{2} < r < s < \rho$ and a cut-off function $\eta \in C_0^\infty(B_s)$ such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ on B_r and $|\nabla\eta| \leq \frac{c}{s-r}$. As in [24] using $\varphi = \tau_{s,-h}(\eta^2\tau_{s,h}v)$ we obtain

$$\int_{B_\rho}\int_0^1 \eta^2 D_{\xi\xi}g(y, Dv + t\tau_h Dv)(\tau_h Dv, \tau_h Dv)\,dt\,dy$$
$$\leq -\int_{B_\rho}\eta^2[D_\xi g(y+he_s, Dv(y+he_s)) - D_\xi g(y, Dv(y+he_s))]\tau_h Dv\,dy$$
$$-2\int_{B_\rho}\eta\tau_h\left\{D_\xi g(y,Dv)\right\}D\eta\otimes\tau_h v\,dy + c\frac{r_0^\alpha}{\lambda}|h|^\alpha\int_B |D(\eta^2\tau_h v)|\,dy. \qquad (5.2.21)$$

By the definition of g, we can write the second integral in the previous inequality as follows

$$-2\int_{B_\rho}\eta\tau_h\left\{D_\xi g(y,Dv)\right\}D\eta\otimes\tau_h v\,dy =$$
$$= -\frac{2}{\lambda}\int_{B_\rho}\eta\tau_h\left\{D_\xi f(x_0+r_0y, A+\lambda Dv(y)) - D_\xi f(x_0+r_0y, A)\right\}D\eta\otimes\tau_h v\,dy$$
$$= -\frac{2}{\lambda}\int_{B_\rho}\eta\{D_\xi f(x_0+r_0(y+h), A+\lambda Dv(y+h)) - D_\xi f(x_0+r_0(y+h), A)$$
$$- D_\xi f(x_0+r_0y, A+\lambda Dv(y)) + D_\xi f(x_0+r_0y, A)\}D\eta\otimes\tau_h v\,dy$$
$$= -\frac{2}{\lambda}\int_{B_\rho}\eta\Big\{D_\xi f(x_0+r_0(y+h), A+\lambda Dv(y+h)) - D_\xi f(x_0+r_0y, A+\lambda Dv(y+h))$$
$$+ D_\xi f(x_0+r_0y, A+\lambda Dv(y+h)) - D_\xi f(x_0+r_0y, A+\lambda Dv(y))$$

$$- D_\xi f(x_0 + r_0(y+h), A) + D_\xi f(x_0 + r_0 y, A)\Big\} D\eta \otimes \tau_h v \, dy. \tag{5.2.22}$$

By (I4) and the argumentation at the end of the previous section the l.h.s. in (5.2.21) is bounded from below by

$$c \int_{B_\rho} \eta^2 (1 + |\lambda Dv| + |\lambda Dv(y + he_s)|)^{-1} |\tau_h Dv|^2 \, dy$$
$$\geq c \int_{B_\rho} \eta^2 \left| \tau_h \left\{ \lambda^{-1} V_1(\lambda Dv) \right\} \right|^2 \, dy. \tag{5.2.23}$$

Whereas on the r.h.s. of (5.2.21), taking into account (5.2.22), using (I3) and (L3) we are led to

$$\begin{aligned}
T_1 &= c\frac{r_0^\alpha}{\lambda} |h|^\alpha \int_{B_\rho} \eta^2 (1 + \log(1 + |\lambda Dv(y + he_s)|)) |\tau_h Dv| \, dy; \\
T_2 &= c\frac{r_0^\alpha}{\lambda} |h|^\alpha \int_{B_\rho} \eta |\nabla \eta| \log(1 + |A| + |\lambda Dv(y + he_s)|) |\tau_h v| \, dy \\
&\quad + \frac{c}{\lambda} \int_{B_\rho} \eta |\nabla \eta| \left| \int_0^1 D_{\xi\xi} f(x_0 + r_0 y, A + s\lambda \tau_h(Dv))) ds \right| |\lambda \tau_h(Dv)| |\tau_h v| \, dy; \\
&= T_{2,1} + T_{2,2} \\
T_3 &= c\frac{r_0^\alpha}{\lambda} |h|^\alpha \int_{B_\rho} |D(\eta^2 \tau_h v)| \, dy.
\end{aligned}$$

Using Young's inequality for $h(t) = t \log(1 + t)$ we get

$$\begin{aligned}
T_1 &\leq c\frac{r_0^\alpha}{\lambda^2} |h|^\alpha \int_{B_\rho} \{ |\lambda Dv| + \log(1 + |\lambda Dv|) |\lambda Dv| \} \, dy; \\
T_{2,1} &\leq c\frac{r_0^\alpha}{\lambda^2} |h|^\alpha \int_{B_\rho} \{ |\lambda Dv| + \log(1 + |\lambda Dv|) |\lambda Dv| \} \, dy; \\
T_3 &\leq c\frac{r_0^\alpha}{\lambda^2} |h|^\alpha \int_{B_\rho} |\lambda Dv| \, dy,
\end{aligned}$$

In order to estimate the integral $T_{2,2}$ we use (L5) and Young's Inequality as follows

$$\begin{aligned}
\left| \int_0^1 D_{\xi\xi} f(x_0 + r_0 y, A + s\lambda \tau_h(Dv))) ds \right| &\leq c \int_0^1 \frac{\log(1 + |A + s\lambda \tau_h(Dv)|)}{|A + s\lambda \tau_h(Dv)|} \, ds \\
&\leq c \int_0^1 (1 + |A + s\lambda \tau_h(Dv)|^2)^{\frac{p-2}{2}} \, ds
\end{aligned}$$

$$\leq c(1+|\lambda\tau_h(Dv)|^2)^{\frac{p-2}{2}},$$

where we used Lemma 2.3.4 and [2] Lemma 2.1. Hence

$$\begin{aligned}
T_{2,2} &\leq \frac{c}{\lambda}\int_{B_s}(1+|\lambda\tau_h(Dv)|^2)^{\frac{p-2}{2}}|\lambda\tau_h(Dv)||\tau_h v| \\
&= \frac{c|h|}{\lambda^2}\int_{B_s}(1+|\lambda\tau_h(Dv)|^2)^{\frac{p-2}{2}}|\lambda\tau_h(Dv)||\lambda\Delta_h v|.
\end{aligned}$$

We observe that for the Young function $\varphi(t) := (1+t^2)^{\frac{p-2}{2}}t^2$ we have

$$\varphi'(t) \approx (1+t^2)^{\frac{p-2}{2}}t; \quad \varphi^*(\varphi'(t)) \approx \varphi(t). \tag{5.2.24}$$

Here $\varphi*$ denotes the conjugate Young function. The second statement in (5.2.24) is a consequence of

$$\varphi^*(\varphi'(t)) = \int_0^{\varphi'(t)}(\varphi')^{-1}(s)\,ds = \int_0^t s\varphi''(s)\,ds \approx \int_0^t \varphi'(s)\,ds = \varphi(t).$$

Hence we obtain with the help of Young's Inequality for Young functions, (5.1.6) and Lemma 2.3.4

$$\begin{aligned}
T_{2,2} &\leq \frac{c|h|}{\lambda^2}\left\{\int_{B_s}\varphi^*\left((1+|\lambda\tau_h(Dv)|^2)^{\frac{p-2}{2}}|\lambda\tau_h(Dv)|\right)\,dy + \int_{B_s}\varphi(|\lambda\Delta_h v|)\,dy\right\} \\
&\leq \frac{c|h|}{\lambda^2}\left\{\int_{B_s}\varphi\left(|\lambda\tau_h(Dv)|\right)\,dy + \int_{B_s}\varphi(|\lambda\Delta_h v|)\,dy\right\} \\
&\leq c\frac{c|h|}{\lambda^2}\int_{B_\rho}|V_p(\lambda Dv)|^2\,dy.
\end{aligned}$$

Inserting the estimates for T_i in (5.2.21) and using (5.2.23), we finally get

$$\begin{aligned}
&\int_{B_\rho}\eta^2\left|\tau_h\left\{\lambda^{-1}V_1(\lambda Dv)\right\}\right|^2\,dy \\
\leq\ & c\frac{r_0^\alpha}{\lambda^2}|h|^\alpha\int_{B_\rho}\{|\lambda Dv| + \log(1+|\lambda Dv|)|\lambda Dv|\}\,dy \\
+\ & \frac{c|h|}{\lambda^2}\int_{B_\rho}|V_p(\lambda Dv)|^2\,dy
\end{aligned} \tag{5.2.25}$$

The conclusion follows applying Lemma 2.1.8. □

Step 5. A Caccioppoli type inequality

The higher integrability of the previous step allows us to prove a Caccioppoli type inequality for minimizers of the rescaled functional, which is contained in the following

Proposition 5.2.3. *Let g be a function satisfying (I1)-(I4) and $v \in W^{1,h}(B;\mathbb{R}^N)$ a solution of*

$$\mathcal{I}(v) \;\; \leq \;\; \mathcal{I}(v+\varphi) + c(M)\frac{r_0^\alpha}{\lambda}\int_{B_1(0)} |D\varphi|\,dy \tag{5.2.26}$$

for every $\varphi \in W_0^{1,h}(B_1(0);\mathbb{R}^N)$. Then we have

$$\int_{B_{\frac{\tau}{2}}} \left|\frac{V_1(\lambda Dv)}{\lambda}\right|^2 \leq \frac{c}{\lambda^2}\int_{B_\tau} \left|V_p\left(\lambda\frac{|v-v_\tau|}{\tau}\right)\right|^2 dy + c\lambda^{2p-2}\left(\int_{B_{2\tau}} \frac{|V_p(\lambda Dv)|^2}{\lambda^2}\,dy\right)^p$$
$$+ c\lambda^{2p-2}\left(\int_{B_{2\tau}} \frac{|V_1(\lambda Dv)|^2}{\lambda^2}\,dy\right)^p + c\frac{r_0^{\alpha p}}{\lambda^2}\left(\int_{B_{2\tau}} |\lambda Dv|\,dy\right)^p + c\frac{r_0^\alpha}{\lambda^2}\int_{B_\tau} \lambda|Dv|. \tag{5.2.27}$$

Proof. Let us fix two radii $\frac{\tau}{2} < r < s < \tau$ and a cut-off function $\eta \in C_0^\infty(B_s)$ such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ on B_r and $|\nabla\eta| \leq \frac{c}{s-r}$. Using $\varphi = \eta(v_\tau - v)$ as a test function in (5.2.26), by virtue of the left inequality at (I1), we get

$$\begin{aligned}
\int_{B_r} \left|\frac{V_1(\lambda Dv)}{\lambda}\right|^2 \;\; &\leq \;\; \int_{B_1} g(y,Dv)\,dy \\
&\leq \;\; \int_{B_1} g(y,Dv+D\varphi) + c(M)\frac{r_0^\alpha}{\lambda}\int_{B_1(0)} |D\varphi| \\
&= \;\; \int_{B_s\setminus B_r} g(y,Dv+D(\eta(v_\tau-v))) + c(M)\frac{r_0^\alpha}{\lambda}\int_{B_s} |D(\eta(v_\tau-v))| \\
&= \;\; \int_{B_s\setminus B_r} g(y,(1-\eta)Dv+D\eta(v_\tau-v)) + c(M)\frac{r_0^\alpha}{\lambda}\int_{B_s} |Dv| \\
&+ \;\; c(M)\frac{r_0^\alpha}{\lambda(s-r)}\int_{B_s} |v-v_\tau|.
\end{aligned} \tag{5.2.28}$$

The first integral in the right hand side of (5.2.28) can be estimated by the right inequality at (I1) and the properties of η as follows

$$\begin{aligned}
& \int_{B_s\setminus B_r} g(y,(1-\eta)Dv + D\eta(v_\tau - v)) \\
\leq\ & \frac{c}{\lambda}\int_{B_s\setminus B_r} \log(1+\lambda|Dv|+\lambda|D\eta||v-v_\tau|)(|Dv|+|D\eta||v-v_\tau|) \\
\leq\ & \frac{c}{\lambda}\int_{B_s\setminus B_r} \log\left(1+\lambda|Dv|+\lambda\frac{|v-v_\tau|}{s-r}\right)\left(|Dv|+\frac{|v-v_\tau|}{s-r}\right). \qquad (5.2.29)
\end{aligned}$$

By (I1), Lemma 2.3.4 and Lemma 2.3.3 we obtain

$$\begin{aligned}
& \int_{B_s\setminus B_r} g(y,(1-\eta)Dv + D\eta(v_\tau - v)) \\
\leq\ & \frac{c}{\lambda^2}\int_{B_s\setminus B_r} \left|V_p\left(\lambda|Dv|+\lambda\frac{|v-v_\tau|}{s-r}\right)\right|^2 dy \\
\leq\ & \frac{c}{\lambda^2}\int_{B_s\setminus B_r} |V_p(\lambda|Dv|)|^2\,dy + \frac{c}{\lambda^2}\int_{B_s\setminus B_r} \left|V_p\left(\lambda\frac{|v-v_\tau|}{s-r}\right)\right|^2 dy \\
\leq\ & \frac{c}{\lambda^2}\int_{B_s\setminus B_r} |V_1(\lambda|Dv|)|^2\,dy + \frac{c}{\lambda^2}\int_{B_s\setminus B_r} |V_1(\lambda|Dv|)|^{2p}\,dy \\
+\ & \frac{c}{\lambda^2}\int_{B_s\setminus B_r} \left|V_p\left(\lambda\frac{|v-v_\tau|}{s-r}\right)\right|^2 dy. \qquad (5.2.30)
\end{aligned}$$

Inserting (5.2.30) in (5.2.28), we get

$$\begin{aligned}
\int_{B_r}\left|\frac{V_1(\lambda Dv)}{\lambda}\right|^2 \ \leq\ & \frac{c}{\lambda^2}\int_{B_s\setminus B_r} |V_1(\lambda|Dv|)|^2\,dy \\
+\ & \frac{c}{\lambda^2}\int_{B_s\setminus B_r} |V_1(\lambda|Dv|)|^{2p}\,dy \\
+\ & \frac{c}{\lambda^2}\int_{B_s\setminus B_r} \left|V_p\left(\lambda\frac{|v-v_\tau|}{s-r}\right)\right|^2 dy
\end{aligned}$$

$$+ \quad c(M)\frac{r_0^\alpha}{\lambda}\int_{B_s}|Dv|$$

$$+ \quad c(M)\frac{r_0^\alpha \tau}{\lambda(s-r)}\int_{B_\tau}|Dv|, \tag{5.2.31}$$

where we also used Poincaré's Inequality. Now we fill the hole by adding to both sides of (5.2.31) the quantity

$$\int_{B_r}\left|\frac{V_1(\lambda Dv)}{\lambda}\right|^2$$

and use the iteration Lemma 2.2.1 to obtain

$$\int_{B_{\frac{\tau}{2}}}\left|\frac{V_1(\lambda Dv)}{\lambda}\right|^2 \le \frac{c}{\lambda^2}\int_{B_\tau}|V_1(\lambda|Dv|)|^{2p}\,dy$$

$$+ \quad \frac{c}{\lambda^2}\int_{B_\tau}\left|V_p\left(\lambda\frac{|v-v_\tau|}{\tau}\right)\right|^2 dy + c(M)\frac{r_0^\alpha}{\lambda}\int_{B_\tau}|Dv|. \tag{5.2.32}$$

Now we apply to the first integral in the right hand side of (5.2.32) the estimate of Lemma 5.2.2 with $p = \frac{n}{n-2k}$, thus having

$$\int_{B_\tau}|\{V_1(\lambda Dv)\}|^{2p}\,dy \le c\left(\int_{B_{2\tau}}\{|V_p(\lambda Dv)|^2\}\,dy\right)^p$$
$$+ cr_0^{\alpha p}\left(\int_{B_{2\tau}}\{|\lambda Dv| + \log(1+|\lambda Dv|)|\lambda Dv|\}\,dy\right)^p + \left(\int_{B_{2\tau}}|\{V_1(\lambda Dv)\}|^2\,dy\right)^p. \tag{5.2.33}$$

Inserting (5.2.33) in (5.2.32) and using again Lemma 2.3.4, we have

$$\int_{B_{\frac{\tau}{2}}}\left|\frac{V_1(\lambda Dv)}{\lambda}\right|^2 \le \frac{c}{\lambda^2}\int_{B_\tau}\left|V_p\left(\lambda\frac{|v-v_\tau|}{\tau}\right)\right|^2 dy$$
$$+ \frac{c}{\lambda^2}\left(\int_{B_{2\tau}}|V_p(\lambda Dv)|^2\,dy\right)^p + c\frac{r_0^{\alpha p}}{\lambda^2}\left(\int_{B_{2\tau}}\{|\lambda Dv| + \log(1+|\lambda Dv|)|\lambda Dv|\}\,dy\right)^p$$
$$+ \frac{c}{\lambda^2}\left(\int_{B_{2\tau}}|V_1(\lambda Dv)|^2\,dy\right)^p + c(M)\frac{r_0^\alpha}{\lambda^2}\int_{B_\tau}\lambda|Dv|$$
$$\le \frac{c}{\lambda^2}\int_{B_\tau}\left|V_p\left(\lambda\frac{|v-v_\tau|}{\tau}\right)\right|^2 dy + c\lambda^{2p-2}\left(\int_{B_{2\tau}}\frac{|V_p(\lambda Dv)|^2}{\lambda^2}\,dy\right)^p$$

$$+ c\lambda^{2p-2}\left(\int_{B_{2\tau}} \frac{|V_1(\lambda Dv)|^2}{\lambda^2}\,dy\right)^p + c\frac{r_0^{\alpha p}}{\lambda^2}\left(\int_{B_{2\tau}} |\lambda Dv|\,dy\right)^p$$

$$+ c(M)\frac{r_0^{\alpha}}{\lambda^2}\int_{B_\tau} \lambda|Dv| \tag{5.2.34}$$

which is the conclusion. □

Step 6. Conclusion

Fix $\tau \in (0, \frac{1}{4})$, set $b_j = (v_j)_{B_{2\tau}}$, $B_j = (Dv_j)_{B_\tau}$ and define

$$w_j(y) = v_j(y) - b_j - B_j y.$$

After rescaling, we note that $\lambda_j w_j$ satisfies the following integral inequality

$$\int_{B_1(0)} g_j(y, \lambda_j Dw_j)\,dy \le \int_{B_1(0)} g_j(y, \lambda_j Dw_j + D\varphi)\,dy + c\frac{r_j^{\alpha}}{\lambda_j}\int_{B_1(0)} |D\varphi|\,dy,$$

for every $\varphi \in W_0^{1,h}(B_1(0))$ where

$$g_j(y,\xi) = \frac{1}{\lambda_j^2}\left[f(x_j + r_j y, A_j + \lambda_j B_j + \xi) - f(x_j + r_j y, A_j + \lambda_j B_j)\right.$$
$$\left. - D_\xi f(x_j + r_j y, A_j + \lambda_j B_j)\xi\right].$$

It is easy to check that Lemma 5.0.6 applies to each g_j, for some constants that could depend on τ through $|\lambda_j B_j|$. But, given τ, we may always choose j large enough to have $|\lambda_j B_j| \le \frac{\lambda_j}{\tau^{\frac{n}{p}}} < 1$ (remember (5.2.6)). Hence we can apply Proposition 5.2.3 to each $\lambda_j w_j$ obtaining

$$\begin{aligned}
\lim_j \frac{\tilde{E}(x_j, \tau r_j)}{\lambda_j^2} &= \lim_j \frac{1}{\lambda_j^2} \fint_{B_{\tau r_j}(x)} |V_p(Du - (Du)_{\tau r_j})|^2\,dy + \lim_j \frac{\tau^\beta r_j^\beta}{\lambda_j^2}\\
&\le \lim_j \frac{1}{\lambda_j^2}\fint_{B_\tau} |V_p(\lambda_j Dw_j)|^2\,dy + \tau^\beta\\
&\le c\lim_j \fint_{B_{2\tau}} \frac{1}{\lambda_j^2}\left|V_p\left(\frac{\lambda_j(w_j - (w_j)_{2\tau})}{\tau}\right)\right|^2\,dy\\
&\quad + c\lim_j \lambda_j^{2p-2}\left(\fint_{B_{2\tau}} \frac{|V_p(\lambda_j Dw_j)|^2}{\lambda_j^2}\,dy\right)^p
\end{aligned}$$

$$
\begin{aligned}
&+ c \lim_j \lambda_j^{2p-2} \left(\fint_{B_{2\tau}} \frac{|V_1(\lambda_j Dw_j)|^2}{\lambda_j^2}\, dy \right)^p \\
&+ c \lim_j \frac{r_j^{\alpha p}}{\lambda_j^2} \left(\fint_{B_\tau} \lambda_j |Dw_j|\, dy \right)^p + c \lim_j \frac{r_j^{\alpha}}{\lambda_j^2} \left(\fint_{B_\tau} \lambda_j |Dw_j|\, dy \right) + \tau^\beta \\
&\le c \lim_j \fint_{B_{2\tau}} \frac{1}{\lambda_j^2} \left| V_p\left(\frac{\lambda_j (w_j - (w_j)_{2\tau})}{\tau} \right) \right|^2 dy + \tau^\beta,
\end{aligned}
$$

since $\lim_j \lambda_j^{2p-2} = 0$, $\lim_j \frac{r_j^\alpha}{\lambda_j^2} = 0$, $\lim_j \frac{r_j^{\alpha p}}{\lambda_j^2} = 0$ and the integrals appearing as their factors are bounded as $j \to \infty$. Now, since $v_j \to v$ strongly in $L^p(B_1(0))$, using the Sobolev-Poincaré inequality stated in Lemma 2.2.2, one can easily check that

$$
\lim_{j \to +\infty} \int_{B_{\frac{1}{2}}} \frac{|V_p(\lambda_j (v_j - v))|^2}{\lambda_j^2}\, dy = 0. \tag{5.2.35}
$$

In fact, for every $\vartheta \in (0, \frac{p}{2})$ we can use Hölder's inequality of exponents $\frac{p}{2\vartheta}$ and $\frac{p}{p-2\vartheta}$ as follows

$$
\begin{aligned}
&\int_{B_{\frac{1}{2}}} \frac{|V_p(\lambda_j (v_j - v))|^2}{\lambda_j^2}\, dy = \int_{B_{\frac{1}{2}}} |v_j - v|^2 (1 + \lambda_j^2 |v_j - v|^2)^{\frac{p-2}{2}}\, dy \\
&\le \left(\int_{B_{\frac{1}{2}}} |v_j - v|^p (1 + \lambda_j^2 |v_j - v|^2)^{\frac{p(p-2)}{4}}\, dy \right)^{\frac{2\vartheta}{p}} \\
&\qquad \times \left(\int_{B_{\frac{1}{2}}} |v_j - v|^{\frac{2p(1-\vartheta)}{p-2\vartheta}} (1 + \lambda_j^2 |v_j - v|^2)^{\frac{p(p-2)(1-\vartheta)}{2(p-2\vartheta)}}\, dy \right)^{\frac{p-2\vartheta}{p}} \\
&\le \left(\int_{B_{\frac{1}{2}}} |v_j - v|^p\, dy \right)^{\frac{2\vartheta}{p}} \left(\int_{B_{\frac{1}{2}}} \left(\frac{|V_p(\lambda_j (v_j - v))|^2}{\lambda_j^2} \right)^{\frac{p(1-\vartheta)}{p-2\vartheta}} dy \right)^{\frac{p-2\vartheta}{p}} \\
&\le \left(\int_{B_{\frac{1}{2}}} |v_j - v|^p\, dy \right)^{\frac{2\vartheta}{p}} \left(\int_{B_{\frac{1}{2}}} \frac{|V_p(\lambda_j (Dv_j - Dv))|^2}{\lambda_j^2}\, dy \right)^{1-\vartheta}.
\end{aligned}
$$

Last inequality is obtained applying Lemma 2.2.2 to the second integral, choosing $\vartheta \in (0, \frac{p}{2})$ such that $\frac{p(1-\vartheta)}{p-2\vartheta} = \frac{n}{n-p}$. Hence (5.2.35) follows noticing that the first integral vanishes as j goes to infinity and second one stays bounded thanks to (5.2.5), since $v \in C_0^\infty(B_1(0))$.

Since $b_j \to (v)_{2\tau}$ and $B_j \to (Dv)_\tau$, using (5.2.35) and the definition of w_j we get

$$
\begin{aligned}
\lim_j \frac{\tilde{E}(x_j, \tau r_j)}{\lambda_j^2} &\le c \lim_j \fint_{B_{2\tau}} \frac{1}{\lambda_j^2} \left| V_p \left(\frac{\lambda_j (w_j - v + v)}{\tau} \right) \right|^2 \, dy + \tau^\beta \\
&= c \lim_j \fint_{B_{2\tau}} \frac{1}{\lambda_j^2} \left| V_p \left(\frac{\lambda_j (v_j - v + v - b_j - B_j y)}{\tau} \right) \right|^2 \, dy + \tau^\beta \\
&\le c \fint_{B_{2\tau}} \frac{|v - (v)_{2\tau} - (Dv)_\tau y|^2}{\tau^2} \, dy + \tau^\beta \\
&\le c \fint_{B_{2\tau}} \frac{|v - (v)_{2\tau} - (Dv)_{2\tau} y|^2}{\tau^2} \, dy + c \fint_{B_{2\tau}} \frac{|(Dv)_\tau y - (Dv)_{2\tau} y|^2}{\tau^2} \, dy \\
&\qquad + \tau^\beta \\
&\le c \fint_{B_{2\tau}} |Dv - (Dv)_{2\tau}|^2 \, dy + c |(Dv)_\tau - (Dv)_{2\tau}|^2 + \tau^\beta \\
&\le c\tau^2 + c\tau^\beta \le c_M^\star \tau^\beta.
\end{aligned}
$$

The contradiction follows by choosing $c_M^\star > \tilde{C}(M)$. □

5.3 Full regularity

In this section we will prove that the minimizer u belongs to the space $C^{1,\gamma}(\Omega, \mathbb{R}^N)$ for every $\gamma < 1$ if we assume (L1) and (L3)–(L8). We follow the lines of [12] (section 4) and use the fact that the range of anisotropy in the almost linear growth situation is arbitrary small. Note that in [12] Breit studies (p,q)-elliptic integrands. Here we just clarify the main differences. The first step is to regularize the problem. Here we consider the standard regularization (compare, for example, [11] and the references therein): u_δ is defined as the unique minimizer of

$$
\mathcal{F}_\delta(u, B) := \int_B \left\{ f(x, Du) + \delta (1 + |Du|^2)^{\frac{q}{2}} \right\} dx
$$

in $(u)_\varepsilon + W_0^{1,q}(B)$ for $B \Subset \Omega$ and $1 < p < q < \frac{n - \frac{\alpha}{2}}{n - \alpha}$ (p is defined in (L7)). Thereby $(u)_\varepsilon$ is the mollification of u with parameter ε and

$$
\delta = \delta(\varepsilon) := \frac{1}{1 + \varepsilon^{-1} + \|D(u)_\varepsilon\|_{L^q(B)}^{2q}}.
$$

For u_δ we obtain:

Lemma 5.3.1. • *As $\varepsilon \to 0$ we have: $u_\delta \rightharpoondown u$ in $W^{1,1}(B, \mathbb{R}^N)$,*

$$\delta \int_B \left(1 + |Du_\delta|^2\right)^{\frac{q}{2}} dx \to 0; \quad \int_B F(Du_\delta)dx \to \int_B F(\nabla u)dx;$$

- *$Du_\delta \in W^{1,2}_{loc} \cap L^\infty_{loc}(\Omega, \mathbb{R}^N)$.*

For the last statement we can refer to [11] (Lemma 2.7), since u_δ is the minimizer of a isotropic problem and the second derivatives $D_{\xi\xi} f_\delta$ fulfills a Hölder-condition by (L8) ($f_\delta(x, \xi) := f(x, \xi) + \delta(1 + |\xi|^2)^{\frac{q}{2}}$). The rest can be quoted from [11], Lemma 2.1. Only the week convergence needs a comment: Following the ideas of [11] one easily sees that Du_δ in bounded in $L_h(B)$. According to the Poincaré-inequality in Orlicz spaces (see [41]) and the uniform boundedness of u_ε in $W^{1,h}_{loc}(\Omega)$ (remember $u \in W^{1,h}(\Omega)$) we obtain $\sup_\delta \|u_\delta\|_{W^{1,h}(B)} < \infty$. By the De La Valée Poussin Lemma we can select a subsequence such that

$$u_\delta \rightharpoondown: v \in W^{1,1}(B), \quad v = u \quad \text{on } \partial B$$

and v minimizes $\mathcal{F}(\cdot, B)$ with respect to boundary data u which means $v = u$.

Next we prove higher integrability with respect to the parameter δ, i.e.,

$$Du_\delta \in L^t_{loc}(B) \quad \text{uniformly in } \delta \text{ for all } t < \frac{n}{n - \alpha}. \tag{5.3.1}$$

Here we proceed exactly as in section 3, observing that our bounds are now independent of δ. We only have to calculate the additional integral ($F_(Z) := (1 + |Z|^2)^{\frac{q}{2}}$)

$$\begin{aligned}
&\delta \int_B D_\xi F_0(Du_\delta) D\tau_{-h}(\eta^2 \tau_h u_\delta)\, dx \\
&\qquad = -\delta \int_B \tau_h D_\xi F_0(Du_\delta) D(\eta^2 \tau_h u_\delta)\, dx \\
&\qquad = -\delta \int_B \eta^2 \int_0^1 D_{\xi\xi} F_0(Du_\delta + t\tau_h Du_\delta)(\tau_h Du_\delta, \tau_h Du_\delta)\, dx
\end{aligned}$$

$$- 2\delta \int_B \eta \tau_h D_\xi F_0(Du_\delta) D\eta \otimes \tau_h u_\delta \, dx$$

on the r.h.s. Here the first integral on the last calculation is nonnegative, so we can drop it. The last one can be estimated by (using Lemma 5.3.1)

$$c(\eta) h \int_B \left(1 + |Du_\delta|^2\right)^{\frac{q}{2}} dx \le c(\eta) h.$$

Hence we obtain (5.3.1) if we apply the arguments of section 3 (remember the uniform $W^{1,h}(B)$-bounds on u_δ).

In order to prove Lipschitz-regularity of the solution u we have to show a growth condition for the function

$$\tau(k,r) := \int_{A(k,r)} \Gamma_\delta^{q-\frac{1}{2}} (\omega_\delta - k)^2 \, dx$$

where we abbreviated $\Gamma_\delta := 1 + |Du_\delta|^2$, $\omega_\delta := \log \Gamma_\delta$ and $A(k,r) := B_r \cap [|Du_\delta| > k]$. We want to show

$$\tau(h,r) \le \frac{c}{(\widehat{r} - r)^\kappa (h-k)^\Theta} \tau(k, \widehat{r})^\mu \tag{5.3.2}$$

for $0 < h < k$, $0 < r < \widehat{r} < R_0$ with exponents $\kappa, \Theta > 0$ and $\mu > 1$. From (5.3.2) we arrive at uniform L^∞_{loc}-bounds on Du_δ using Stampacchia's Lemma ([80], Lemma 5.1, p. 219), details are given in [9]. Note that uniform bounds for τ (which are necessary) follows from (5.3.1) and

$$q < \frac{n - \frac{\alpha}{2}}{n - \alpha}.$$

Hence we have $u_\delta \in W^{1,\infty}_{loc}(B)$ uniformly in δ (remember Lemma 5.3.1). It follows with the help of Arzel -Ascoli's theorem that $u \in W^{1,\infty}_{loc}(B)$ and since B is arbitrary $u \in W^{1,\infty}_{loc}(\Omega)$. This means that

$$\int f(x, Du) \, dx \longrightarrow \min$$

is a problem with quadratic growth (at least locally, compare (L5)) and the claim follows from [11], Lemma 2.7.

In order to prove (5.3.2) we have to notice that the integrand satisfies the growth conditions

$$\begin{aligned}\nu(1+|\xi|^2)^{-\frac{1}{2}}|Z|^2 \le D^2_{\xi\xi}f(x,\xi)(Z,Z) \le& \Lambda(1+|\xi|^2)^{\frac{q-2}{2}}|Z|^2,\\ |\partial_s D_\xi f(x,\xi)| \le& \Lambda(1+|\xi|^2)^{\frac{q-1}{2}}.\end{aligned}$$

Since the exponent from above ($p = 1$) and below are close enough, we can exactly argue as in [12] (section 4) and obtain (5.3.2). Note that in this part of [12] the condition $p > 1$ is not used.

Chapter VI

BOUNDS FOR THE SINGULAR SET FOR FUNCTIONALS WITH STANDARD GROWTH CONDITIONS

The results of this chapter have been obtained in [23].

In this chapter we get bounds on the Hausdorff dimension of the singular set for local minimizers of integral functionals of the Calculus of Variations of the type (0.0.1) defined for Sobolev maps $u \in W^{1,p}(\Omega;\mathbb{R}^N)$, $p > 1$. Here for $n \geq 2$ and $N \geq 1$, Ω is a bounded open set in $\mathbb{R}^n$ and $f : \Omega \times \mathbb{R}^{nN} \to \mathbb{R}$ is a continuous function. The assumptions we are going to consider are the following.

There exist positive constants $L, \nu, c > 0$ such that for every $p > 1$

$$(1 + |\xi|^2)^{\frac{p}{2}} \leq f(x, \xi) \leq L(1 + |\xi|^2)^{\frac{p}{2}}; \tag{H1}$$

$$f\left(x, \frac{\xi_1 + \xi_2}{2}\right) \leq \frac{1}{2} f(x, \xi_1) + \frac{1}{2} f(x, \xi_2) - \nu(1 + |\xi_1|^2 + |\xi_2|^2)^{\frac{p-2}{2}} |\xi_1 - \xi_2|^2; \tag{H2}$$

$$|f_\xi(x_1, \xi) - f_\xi(x_2, \xi)| \leq c|x_1 - x_2|^\alpha \ (1 + |\xi|^2)^{\frac{p-1}{2}}; \tag{H3}$$

for any $\xi, \xi_1, \xi_2 \in \mathbb{R}^{nN}$, $x, x_1, x_2 \in \Omega$ and $\alpha \in (0, 1)$.

Bounds for the singular set of minimizers of functionals of the type (0.0.1) can be obtained as a particular case from the paper, [31], where the anisotropic growth conditions $1 < p \leq q < p\left(\frac{n+\alpha}{n}\right)$ were examined. But a more careful analysis in the case of the functional (0.0.1) with standard growth conditions, allows to improve the bounds obtained in [31].

Since the functional (0.0.1) is convex there is no difference between minimizers and critical points, i.e. minimizers are precisely the weak solutions to the Euler system div $f_\xi(x, Du) = 0$. Moreover if we require $f(\cdot, \xi) \in C^2$, so that (see Lemma 6.0.5 in the next section) $(H2)$ is equivalent to the following ellipticity condition

$$\left\langle f_{\xi\xi}(x,\xi)\lambda, \lambda\right\rangle \geq \nu\,|\lambda|^2\,(1+|\xi|^2)^{\frac{p-2}{2}} \quad \forall x \in \Omega, \forall \xi, \lambda \in \mathbb{R}^{nN}, \tag{6.0.3}$$

the Euler system turns out to be elliptic. To show that the gradient of u belongs to some suitable fractional order Sobolev space, the usual method consists, once again, in exploiting the nice embedding properties of these spaces. This technique has been used, for example, in [68] where the weak solutions to elliptic systems like

$$\operatorname{div} a(x, Du) = 0 \tag{6.0.4}$$

with $a : \Omega \times \mathbb{M}^{N\times n} \to \mathbb{M}^{N\times n}$, have been studied under the following growth, ellipticity and continuity assumptions:

$$|D_\xi a(x,\xi)| \leq L(1+|\xi|^2)^{\frac{p-2}{2}}, \tag{6.0.5a}$$

$$L^{-1}|\lambda|^2(1+|\xi|^2)^{\frac{p-2}{2}} \leq \frac{\partial a_i^k}{\partial \xi_j^h}(x,\xi)\lambda_i^k\lambda_j^h, \tag{6.0.5b}$$

$$|a(x,\xi) - a(x_0,\xi)| \leq L|x-x_0|^\alpha(1+|\xi|^2)^{\frac{p-1}{2}} \tag{6.0.5c}$$

for any $z, \lambda \in \mathbb{M}^{n\times N}$ and $x, x_0 \in \Omega$, where $p \geq 2$, $L \in (1, +\infty)$ and $\alpha \in (0, 1)$ (see also [67]). As usual, a key role in the proof of the existence of fractional derivatives is played by assumption (6.0.5a) that in the case $a(x, \xi) = f_\xi(x, \xi)$ becomes in turn an assumption on the growth of second derivatives of f,

$$|D^2 f(x,\xi)| \leq L(1+|\xi|^2)^{\frac{p-2}{2}}. \tag{6.0.6}$$

The main purpose of this chapter is to provide a regularity result without any assumption on the growth of $D^2 f$, and then to get the bounds for the singular set of minimizers from the regularity property. This result relies essentially on a fundamental approximation procedure first introduced in [35].

In the case $p \geq 2$, our main result is the following.

Theorem 6.0.2. *Let f satisfy the assumptions $(H1)$, $(H2)$ and $(H3)$, with $p \geq 2$. If the function $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ is a local minimizer of $\mathcal{F}$ in Ω then for every $B_\rho \subset B_R \subset\subset \Omega$ we have that*

$$Du \in W^{\frac{k}{p-1},p}(B_\rho; \mathbb{R}^{nN}) \cap L^{\frac{np}{n-kq}}(B_\rho; \mathbb{R}^{nN})$$

for every $k \in (0, \alpha)$, where $q \equiv \frac{p}{p-1}$ and

$$||Du||_{L^{\frac{np}{n-kq}}(B_\rho)} \leq c \left(\int_{B_R} (1 + |Du(x)|^p) \ dx \right)^{\frac{1}{2}}, \tag{6.0.7}$$

with $c \equiv c(n, N, L, \nu, R, \rho, \alpha, k, p)$.

As far as we know, also in [31] the authors use the difference quotient method without the assumption (6.0.6) (or (6.0.5a)) but our higher integrability exponent is greater than the one found in [31] where the anisotropic growth conditions $1 < p \leq q < p\left(\frac{n+\alpha}{n}\right)$ were examined. Here, the improved regularity stated in Theorem 6.0.2 depends on the assumption $p \geq 2$. In fact, in the case $1 < p < 2$ our higher integrability exponent is slightly greater than the one obtained in [31] in case $\alpha \geq \frac{1}{2}$ while is again better if $\alpha < \frac{1}{2}$. More precisely we have

Theorem 6.0.3. *Let f satisfy the assumptions $(H1)$, $(H2)$ and $(H3)$, with $1 < p < 2$. If the function $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ is a local minimizer of $\mathcal{F}$ in Ω then for every $B_\rho \subset B_R \subset\subset \Omega$ we have that*

$$(1 + |Du(x)|^2)^{\frac{p-2}{4}} Du(x) \in W^{k,2}(B_\rho; \mathbb{R}^{nN}) \cap L^{\frac{2n}{n-2k}}(B_\rho; \mathbb{R}^{nN}) \tag{6.0.8}$$

for every $0 < k < \min\left\{\alpha, \frac{1}{2}\right\}$. As a consequence

$$Du \in W^{k,p}(B_\rho; \mathbb{R}^{nN}) \cap L^{\frac{np}{n-2k}}(B_\rho; \mathbb{R}^{nN}) \tag{6.0.9}$$

for every $0 < k < \min\left\{\alpha, \frac{1}{2}\right\}$ and

$$||Du||_{L^{\frac{np}{n-2k}}(B_\rho)} \leq c \left(\int_{B_R} (1 + |Du(x)|^p) \ dx \right)^{\frac{1}{2}}, \tag{6.0.10}$$

with $c \equiv c(n, N, L, \nu, R, \rho, \alpha, k, p)$.

The proof of these two theorems is divided in two steps. In the first step we assume that $f(\cdot,\xi)\in C^2$ but we are able to establish the estimates (6.0.7) and (6.0.10) independently of the C^2 norm of the integrand f, by adopting an argument first used in [35]. In the second step we remove the assumption $f(\cdot,\xi)\in C^2$ using an approximation procedure introduced in [35] and developed in [19], [38] and [30]. More precisely we approximate f by a sequence $\{f_h\}$ of C^2 functions which are strictly elliptic (and the ellipticity constant is precisely the ν appearing in $(H2)$). The minimizers $\{u_h\}$ of $\{f_h\}$ all satisfy estimates (6.0.7) and (6.0.10). More important the estimates are independent of the C^2 norm of $\{f_h\}$ and thus are preserved in passing to the limit. Hence a control of the type

$$|D^2 f(x,\xi)| \le c\,(1+|\xi|^2)^{\frac{p-2}{2}}, \quad \forall\ (x,\xi)\in\Omega\times\mathbb{R}^n,$$

on the growth of second derivatives of f never enters into play.

We remark that the cases $1<p<2$ and $p\ge 2$ have different technical difficulties and therefore they have to be treated separetely. The subquadratic case has been treated in [17] for the first time, in the quasiconvex setting; however the paper [17] does not deal with the full case $1<p<2$ but only with $2n/(n+2)<p<2$. The extension to the full interval $1<p<2$ has been achieved in the subsequent paper [16].

Moreover we would like to notice that in the case $p=2$ we recover the same regularity of [68] without the growth assumption on the second derivatives (6.0.5a).

Let us recall the following definition of local minimizer for the functional $\mathcal{F}$.

Definition 6.0.4. A map $u\in W^{1,p}(\Omega;\mathbb{R}^N)$ is a local minimizer of the functional $\mathcal{F}$ if

$$\mathcal{F}(u;A)\le\mathcal{F}(v;A)$$

whenever $A\subset\subset\Omega$ and $u-v\in W^{1,p}_0(A;\mathbb{R}^N)$.

The following statement has been proved in [38]. It states that the condition of uniform convexity of the functional $\mathcal{F}$ is equivalent to the ellipticity condition for the Euler system of $\mathcal{F}$.

Lemma 6.0.5. *Let $f : \mathbb{R}^{nN} \to [0, +\infty)$ be a C^2 function and $p > 1$. Then f satisfies $(H2)$ if and only if there exists a constant c_0 such that for all $\xi \in \mathbb{R}^{nN}$*

$$\left\langle f_{\xi\xi}(x,\xi)\lambda, \lambda \right\rangle \geq c_0 \nu \left(1 + |\xi|^2\right)^{\frac{p-2}{2}} |\lambda|^2 \quad \forall\, \lambda \in \mathbb{R}^{nN}.$$

where the constant ν is the same constant appearing in $(H2)$.

6.1 A priori estimates

In this section we prove the estimates (6.0.7) and (6.0.10) assuming that f satisfies the growth assumption $(H1)$, the Hölder condition $(H3)$ and $f(\cdot,\xi) \in C^2$ for every $\xi \in \mathbb{R}^{nN}$. Recall (see 6.0.5) that under this last assumption $(H2)$ is equivalent to the ellipticity condition (6.0.3) with the same ν appearing in $(H2)$. Then in the next section we use the fundamental approximation procedure of [35], to prove Theorems 6.0.2 and 6.0.3. In any case we explicitly point out that in this section we establish the estimates (6.0.7) and (6.0.10) independently of the C^2 norm of f.

Now we observe that the convexity assumption $(H2)$ together with $(H1)$ implies the estimate

$$|f_\xi(x,\xi)| \leq c(1 + |\xi|^2)^{\frac{p-1}{2}} \quad \forall\, (x,\xi) \in \Omega \times \mathbb{R}^{nN}, \tag{6.1.1}$$

where $c \equiv c(n, N, p, L)$.

We start proving (6.0.7).

Lemma 6.1.1. *Suppose f satisfies $(H1)$, $(H3)$ for a $p \geq 2$ and $f(\cdot,\xi) \in C^2$ for every $\xi \in \mathbb{R}^{nN}$. If $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ is a local minimizer of $\mathcal{F}$ then the estimate* (6.0.7) *holds.*

Proof. We assumed $f(\cdot,\xi) \in C^2$ so f satisfies the ellipticity condition (6.0.3) by Lemma 6.0.5. Let $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}$ and let us take $0 < R < 1$ such that $B_{2R} \subset\subset \Omega$; then u is a solution of the Euler system

$$\int_\Omega f_\xi(x, Du)\, D\varphi\, dx = 0, \tag{6.1.2}$$

for every $\varphi \in W^{1,p}(\Omega;\mathbb{R}^N)$ such that supp$\varphi \subset\subset \Omega$.

Let η be a cut-off function in $C_0^1(B_{3R/2})$ with $0 \le \eta \le 1$, $\eta \equiv 1$ on B_R and $|D\eta| < c/R$. Let us consider the function $\varphi = \eta^2(x)\tau_{s,-h}(\tau_{s,h}u)$ with s fixed in $\{1,\dots,n\}$ (which from now on we shall omit for the sake of simplicity) and $|h| < R/10$. Substituting in (6.1.2) the function φ we get

$$\int_{B_{2R}} \eta^2(x) f_\xi(x, Du)\, D(\tau_{-h}(\tau_h u))\, dx = -2\int_{B_{2R}} f_\xi(x, Du)\eta(x)D\eta \otimes \tau_{-h}(\tau_h u)\, dx$$

and thanks to $(d1)$ and $(d2)$ of Propostion 2.1.4 we get

$$\begin{aligned}
&\int_{B_{2R}} \eta^2(x+he_s)\,[f_\xi(x+he_s, Du(x+he_s)) - f_\xi(x+he_s, Du(x))]\, D(\tau_h u)\, dx \\
&\quad + \int_{B_{2R}} \eta^2(x+he_s)[f_\xi(x+he_s, Du(x)) - f_\xi(x, Du(x))]\; D(\tau_h u)\, dx \\
&\quad + \int_{B_{2R}} [\eta^2(x+he_s) - \eta^2(x)]\; f_\xi(x, Du) D(\tau_h u)\, dx \\
&\quad = 2\int_{B_{2R}} f_\xi(x, Du)\; \eta(x)\, D\eta \otimes \tau_{-h}(\tau_h u)\, dx.
\end{aligned}$$

Assumption $(H3)$ and inequality (6.1.1) yield

$$\begin{aligned}
&\int_{B_{2R}} \eta^2(x+he_s)\,[f_\xi(x+he_s, Du(x+he_s)) - f_\xi(x+he_s, Du(x))] D(\tau_h u)\, dx \\
&\quad \le c\,|h|^\alpha \int_{B_{2R}} \eta^2(x+he_s)|D(\tau_h u)|(1+|Du|^2)^{\frac{p-1}{2}}\, dx \\
&\qquad + c\int_{B_{2R}} |\eta^2(x+he_s) - \eta^2(x)|\,|D(\tau_h u)|\,(1+|Du|^2)^{\frac{p-1}{2}}\, dx \\
&\qquad + c\int_{B_{2R}} \eta(x)\,|D\eta|\,|\tau_{-h}(\tau_h u)|\,(1+|Du|^2)^{\frac{p-1}{2}}\, dx,
\end{aligned} \tag{6.1.3}$$

with $c \equiv c(n, N, p, L)$.

Now we can use the ellipticity condition (6.0.3) in the left hand side of (6.1.3) as follows

$$\begin{aligned}&\int_{B_{2R}} \eta^2(x+he_s)(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p-2}{2}}|\tau_h Du|^2\,dx\\&\quad\leq \int_{B_{2R}}\int_0^1 [f_{\xi\xi}(x+he_s, Du+t\tau_h Du)]\eta^2(x+he_s)D(\tau_h u)D(\tau_h u)\,dt\,dx\end{aligned}$$

and, since

$$|\tau_h Du|^p = |\tau_h Du|^{p-2}|\tau_h Du|^2 \leq c(n,p)(1+|Du(x)|^2+|Du(x+he_s|^2)^{\frac{p-2}{2}}|\tau_h Du|^2$$

and $p \geq 2$, we get the estimate

$$\begin{aligned}&\int_{B_{2R}} \eta^2(x+he_s)|\tau_h Du|^p\,dx\\&\quad\leq c\,|h|^\alpha \int_{B_{2R}} \eta^2(x+he_s)|\tau_h Du|\;(1+|Du|^2)^{\frac{p-1}{2}}\;dx\\&\quad\quad + c\int_{B_{2R}} |\eta^2(x+he_s)-\eta^2(x)||\tau_h Du|\,(1+|Du|^2)^{\frac{p-1}{2}}\,dx\\&\quad\quad + c\int_{B_{2R}} \eta(x)|D\eta|\,|\tau_{-h}(\tau_h u)|\,(1+|Du|^2)^{\frac{p-1}{2}}\,dx\\&\quad := (I)+(II)+(III),\end{aligned} \tag{6.1.4}$$

with $c \equiv c(n, N, p, L, \nu)$. We have to estimate the integrals appearing in the right hand side of (6.1.4). In what follows ε is a real number such that $0 < \varepsilon < 1$ to be chosen later.

Let us begin from (I). We can apply Young's inequality with the exponents p and $q \equiv \frac{p}{p-1}$ so we have

$$(I) \leq c\,|h|^{q\alpha}\int_{B_{2R}} \eta^2(x+he_s)(1+|Du(x)|^2)^{\frac{p}{2}}\,dx + c\,\varepsilon\int_{B_{2R}} \eta^2(x+he_s)|\tau_h Du|^p\,dx$$

where $c \equiv c(n, N, p, L, \nu)$.

Let us estimate (II); we can apply Lagrange's theorem to estimate $|\eta^2(x+he_s)-\eta^2(x)|$, the assumptions on $|D\eta|$ and again Young's inequality with exponents p and q obtaining

$$
\begin{aligned}
(II) \leq & c\,\frac{|h|}{R}\int_{B_{2R}} |\eta(x+he_s)+\eta(x)|(1+|Du(x)|^2)^{\frac{p-1}{2}}|\tau_h Du|\; dx \\
\leq & c\,\frac{|h|^q}{R^q}\int_{B_{2R}} (1+|Du(x)|^2)^{\frac{p}{2}}\; dx + c\,\varepsilon\int_{B_{2R}} |\eta^p(x+he_s)+\eta^p(x)||\tau_h Du|^p\; dx \\
\leq & c\,\frac{|h|^q}{R^q}\int_{B_{2R}} (1+|Du(x)|^2)^{\frac{p}{2}}\, dx + c\,\varepsilon\int_{B_{2R}} \eta^p(x+he_s)|\tau_h Du|^p\; dx \\
& + c\,\varepsilon\int_{B_{2R}} \eta^p(x)|\tau_h Du|^p\; dx \\
\leq & c\,\frac{|h|^q}{R^q}\int_{B_{2R}} (1+|Du(x)|^2)^{\frac{p}{2}}\, dx + c\,\varepsilon\int_{B_{2R}} \eta^2(x+he_s)|\tau_h Du|^p\; dx \\
& + c\,\varepsilon\int_{B_{2R}} \eta^2(x)|\tau_h Du|^p\; dx,
\end{aligned}
$$

where we used the assumptions $p \geq 2$ and $0 \leq \eta \leq 1$ to get the last estimate.

To estimate (III) we use, once again, Young's inequality and the properties of η obtaining

$$
(III) \leq c\,\frac{|h|^q}{R^q}\int_{B_{2R}} (1+|Du(x)|^2)^{\frac{p}{2}}\, dx + \frac{c\,\varepsilon}{|h|^p}\int_{B_{2R}} \eta^p(x)|\tau_{-h}(\tau_h u)|^p\, dx. \tag{6.1.5}
$$

Now using the definition of $\tau_h u$ we can write the last integral in (6.1.5) as follows

$$
\begin{aligned}
& \frac{c\,\varepsilon}{|h|^p}\int_{B_{2R}} |\eta(x-he_s)(\tau_h u)(x-he_s)-\eta(x)(\tau_h u)(x) \\
& \qquad\qquad + (\eta(x)-\eta(x-he_s))(\tau_h u)(x-he_s)|^p\, dx \\
\leq & \frac{c\,\varepsilon}{|h|^p}\int_{B_{2R}} |\tau_{-h}(\eta\tau_h u)(x)|^p dx + \frac{c}{|h|^p}\int_{B_{2R}} |\eta(x)-\eta(x-he_s)|^p|(\tau_h u)(x-he_s)|^p dx \\
\leq & c\,\varepsilon\int_{B_{2R}} |D(\eta\tau_h u)(x)|^p\, dx + \frac{c\,\varepsilon}{R^p}\int_{B_{2R}} |(\tau_h u)(x-he_s)|^p\, dx,
\end{aligned} \tag{6.1.6}
$$

where we used Lemma 2.1.6 and Lagrange's theorem to get the last estimate.

Recalling how we chose η and $|h|$ at the beginning we can estimate the last sum in (6.1.6) with

$$
c\,\varepsilon\int_{B_{2R}} |D(\eta\tau_h u)(x)|^p\; dx + \frac{c\,\varepsilon}{R^p}\int_{B_{2R}} |(\tau_h u)(x)|^p\, dx
$$

$$
\begin{aligned}
\leq & c\,\varepsilon \int_{B_{2R}} \eta^p(x)|\tau_h(Du)(x)|^p\,dx + c\,\varepsilon \int_{B_{2R}} |D\eta|^p|(\tau_h u)(x)|^p\,dx \\
& + \frac{c\,\varepsilon}{R^p} \int_{B_{2R}} |(\tau_h u)(x)|^p\,dx \\
\leq & c\,\varepsilon \int_{B_{2R}} \eta^2(x)|\tau_h(Du)(x)|^p\,dx + \frac{c\,\varepsilon}{R^p} \int_{B_{2R}} |(\tau_h u)(x)|^p\,dx,
\end{aligned}
$$

where we used the assumptions on p and η again. So we have

$$
\begin{aligned}
(III) \leq & c\,\frac{|h|^q}{R^q} \int_{B_{2R}} (1+|Du(x)|^2)^{\frac{p}{2}}\,dx + c\,\varepsilon \int_{B_{2R}} \eta^2(x)|\tau_h(Du)(x)|^p\,dx \\
& + \frac{c\,\varepsilon}{R^p} \int_{B_{2R}} |(\tau_h u)(x)|^p\,dx.
\end{aligned}
$$

Since $\tau_h Du(x) = Du(x+he_s) - Du(x)$ and noting that

$$
\begin{aligned}
c\,\varepsilon \int_{B_{2R}} \eta^p(x)|\tau_h Du|^p\,dx \leq & c \int_{B_{2R}} |\eta(x)-\eta(x+he_s)|^p|\tau_h(Du)(x)|^p\ dx \\
& + c\varepsilon \int_{B_{2R}} \eta^p(x+he_s)|\tau_h Du(x)|^p\ dx \\
\leq & c\frac{|h|^p}{R^p} \int_{B_{2R}} |\tau_h Du(x)|^p\ dx + c\varepsilon \int_{B_{2R}} \eta^2(x+he_s)|\tau_h Du|^p\,dx \\
\leq & c\frac{|h|^p}{R^p} \int_{B_{2R}} (1+|Du|^2)^{\frac{p}{2}} dx + c\,\varepsilon \int_{B_{2R}} \eta^2(x+he_s)|\tau_h Du|^p dx
\end{aligned}
$$

we obtain

$$
\begin{aligned}
(III) \leq & c\,\frac{|h|^p}{R^p} \int_{B_{2R}} (1+|Du(x)|^2)^{\frac{p}{2}}\,dx + c\,\varepsilon \int_{B_{2R}} \eta^2(x+he_s)|\tau_h Du|^p\,dx \\
& + c\,\frac{|h|^q}{R^q} \int_{B_{2R}} (1+|Du(x)|^2)^{\frac{p}{2}}\,dx.
\end{aligned}
$$

Collecting the estimates for (I), (II) and (III) we get

$$
\begin{aligned}
\int_{B_{2R}} \eta^2(x+he_s)|\tau_h Du|^p\,dx \leq & c\,\varepsilon \int_{B_{2R}} \eta^2(x+he_s)|\tau_h Du|^p\,dx \\
& + c\,\frac{|h|^p}{R^p} \int_{B_{2R}} (1+|Du(x)|^2)^{\frac{p}{2}}\,dx \\
& + c\,|h|^{q\alpha} \int_{B_{2R}} (1+|Du(x)|^2)^{\frac{p}{2}}\,dx \\
& + c\frac{|h|^q}{R^q} \int_{B_{2R}} (1+|Du(x)|^2)^{\frac{p}{2}}\,dx.
\end{aligned}
$$

Now choosing $\varepsilon > 0$ small enough we get

$$\begin{aligned}\int_{B_{2R}} \eta^2(x+he_s)|\tau_h Du|^p \, dx \leq & c\,|h|^{q\alpha} \int_{B_{2R}} (1+|Du(x)|^2)^{\frac{p}{2}} \, dx \\ & + c\,\frac{|h|^q}{R^q} \int_{B_{2R}} (1+|Du(x)|^2)^{\frac{p}{2}} \, dx \\ & + c\,\frac{|h|^p}{R^p} \int_{B_{2R}} (1+|Du(x)|^2)^{\frac{p}{2}} \, dx,\end{aligned}$$

but since $q\alpha < q$, $q \leq 2$ and recalling that $R < 1$ the following estimate easily follows

$$\int_{B_{2R}} \eta^2(x+he_s)|\tau_h Du|^p \, dx \leq c\,|h|^{p(\frac{\alpha}{p-1})} \int_{B_{2R}} (1+|Du(x)|^p) \, dx, \tag{6.1.7}$$

with $c \equiv c(n, N, L, \nu, p, R)$. We can conclude applying Lemma 2.1.9 and performing a standard covering procedure. □

Now we prove (6.0.10) again under the C^2 regularity assumption on the integrand f.

Lemma 6.1.2. *Suppose f satisfies (H1), (H3), for a $1 < p < 2$ and $f(\cdot, \xi) \in C^2$ for every $\xi \in \mathbb{R}^{nN}$. If $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ is a local minimizer of $\mathcal{F}$ then the estimate* (6.0.10) *holds.*

Proof. Let η be a cut-off function in $C_0^1(B_{3R/2})$ with $0 \leq \eta \leq 1$, $\eta \equiv 1$ on B_R and $|D\eta| < c/R$. Let us consider the function $\varphi = \eta^2(x)\tau_{s,-h}(\tau_{s,h}u)$ with s fixed in $\{1, \ldots, n\}$ (which from now on we shall omit for the sake of simplicity) and $|h| < R/10$. Substituting in (6.1.2) the function φ and arguing as in the first part of the proof of Lemma 6.1.1 we get the estimate

$$\begin{aligned}&\int_{B_{2R}} \eta^2(x+he_s)(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p-2}{2}}|\tau_h Du|^2 \, dx \\ &\quad \leq c\,|h|^\alpha \int_{B_{2R}} \eta^2(x+he_s)|\tau_h Du| \, (1+|Du|^2)^{\frac{p-1}{2}} \, dx\end{aligned}$$

$$
\begin{aligned}
&+ c\int_{B_{2R}} |\eta^2(x+he_s)-\eta^2(x)||\tau_h Du|\,(1+|Du|^2)^{\frac{p-1}{2}}\,dx\\
&+ c\int_{B_{2R}} \eta(x)|D\eta|\,|\tau_{-h}(\tau_h u)|\,(1+|Du|^2)^{\frac{p-1}{2}}\,dx\\
&:= (I)+(II)+(III),
\end{aligned}
\tag{6.1.8}
$$

with $c \equiv c(n, N, p, L, \nu)$. We have to estimate the integrals appearing in the right hand side of (6.1.8). In what follows ε is a real number such that $0 < \varepsilon < 1$ to be chosen later.

Let us begin from (I). Observing that

$$
\frac{p-1}{2} = \frac{p-2}{4} + \frac{p}{4},
$$

we can apply Young's inequality with the exponent 2, so we have

$$
\begin{aligned}
(I) \le& c\,|h|^{\alpha}\int_{B_{2R}} \eta^2(x+he_s)(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p-1}{2}}|\tau_h Du|\,dx\\
\le& c\,\varepsilon\,|h|^{2\alpha}\int_{B_{2R}} \eta^2(x+he_s)(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p}{2}}\,dx\\
&+\varepsilon\int_{B_{2R}} \eta^2(x+he_s)(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p-2}{2}}|\tau_h Du|^2\,dx\\
\le& c\,\varepsilon\,|h|^{2\alpha}\int_{B_{2R}} (1+|Du(x)|^2)^{\frac{p}{2}}\,dx\\
&+\varepsilon\int_{B_{2R}} \eta^2(x+he_s)(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p-2}{2}}|\tau_h Du|^2\,dx
\end{aligned}
\tag{6.1.9}
$$

with $c \equiv c(n, N, p, L, \nu)$, where in the last inequality we used Lemma 2.1.6.

Now let us estimate (II). We can apply Lagrange's theorem to estimate $|\eta^2(x+he_s)-\eta^2(x)|$, the assumption on $|D\eta|$ and again Young's inequality with exponent 2 obtaining

$$
\begin{aligned}
(II) \le& c\,\frac{|h|}{R}\int_{B_{2R}} |\eta(x+he_s)+\eta(x)|(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p-1}{2}}|\tau_h Du|\;dx\\
\le& c\,\varepsilon\,\frac{|h|^2}{R^2}\int_{B_{2R}} (1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p}{2}}\;dx
\end{aligned}
$$

$$+\varepsilon\int_{B_{2R}}|\eta^2(x+he_s)+\eta^2(x)|(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p-2}{2}}|\tau_h Du|^2dx$$
$$\leq c\varepsilon\frac{|h|^2}{R^2}\int_{B_{2R}}(1+|Du(x)|^2)^{\frac{p}{2}}dx$$
$$+\varepsilon\int_{B_{2R}}\eta^2(x+he_s)(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p-2}{2}}|\tau_h Du|^2dx$$
$$+\varepsilon\int_{B_{2R}}\eta^2(x)(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p-2}{2}}|\tau_h Du|^2\,dx \tag{6.1.10}$$

where we used Lemma 2.1.6.

To estimate (III) we use Hölder's inequality, the definition of $\tau_h u$ and the properties of η and h obtaining

$$(III)\leq\frac{c}{R}\left(\int_{B_{2R}}(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p}{2}}\right)^{1-\frac{1}{p}}\left(\int_{B_{2R}}|\tau_{-h}(\tau_h u)|^p\,dx\right)^{\frac{1}{p}}$$
$$\leq|h|\,\frac{c}{R}\int_{B_{2R}}(1+|Du(x)|^p)\,dx, \tag{6.1.11}$$

where we also used Lemma 2.1.6. Now set

$$W_h=W_h(Du)=1+|Du(x)|^2+|Du(x+he_s)|^2$$

and, since $\tau_h Du(x)=Du(x+he_s)-Du(x)$, we get

$$c\int_{B_{2R}}W_h^{\frac{p-2}{2}}\eta^2(x)|\tau_h Du|^2\,dx\leq c\varepsilon\int_{B_{2R}}W_h^{\frac{p-2}{2}}|\eta(x)-\eta(x+he_s)|^2|\tau_h(Du)(x)|^2\,dx$$
$$+c\varepsilon\int_{B_{2R}}W_h^{\frac{p-2}{2}}\eta^2(x+he_s)|\tau_h(Du)(x)|^2\,dx$$
$$\leq c\frac{|h|^2}{R^2}\int_{B_{2R}}W_h^{\frac{p-2}{2}}|\tau_h Du(x)|^2\,dx+\varepsilon\int_{B_{2R}}W_h^{\frac{p-2}{2}}\eta^2(x+he_s)|\tau_h Du|^2\,dx$$
$$\leq c\frac{|h|^2}{R^2}\int_{B_{2R}}W_h^{\frac{p-2}{2}}|Du(x)|^2\,dx+c\,\varepsilon\int_{B_{2R}}W_h^{\frac{p-2}{2}}\eta^2(x+he_s)|\tau_h Du|^2\,dx$$
$$\leq c\,\frac{|h|^2}{R^2}\int_{B_{2R}}(1+|Du(x)|^2)^{\frac{p}{2}}\,dx+c\,\varepsilon\int_{B_{2R}}W_h^{\frac{p-2}{2}}\eta^2(x+he_s)|\tau_h Du|^2\,dx$$

Collecting the estimates for (I), (II) and (III), we get

$$\int_{B_{2R}}\eta^2(x+he_s)(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p-2}{2}}|\tau_h Du|^2\,dx$$

$$\begin{aligned}
\le & c\,\varepsilon \int_{B_{2R}} W_h^{\frac{p-2}{2}} \eta^2(x+he_s)|\tau_h Du|^2\,dx + c\,\frac{|h|^2}{R^2}\int_{B_{2R}} (1+|Du(x)|^p)\,dx \\
& + c\,\varepsilon\,|h|^{2\alpha}\int_{B_{2R}} (1+|Du(x)|^p)\,dx + |h|\,\frac{c}{R}\int_{B_{2R}} (1+|Du(x)|^p)\,dx.
\end{aligned}$$

From this estimate, choosing $\varepsilon > 0$ small enough, we get

$$\begin{aligned}
\int_{B_{2R}} & \eta^2(x+he_s)(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p-2}{2}}|\tau_h Du|^2\,dx \\
\le & c\,\frac{|h|^2}{R^2}\int_{B_{2R}} (1+|Du(x)|^p)\,dx + c\,\varepsilon\,|h|^{2\alpha}\int_{B_{2R}} (1+|Du(x)|^p)\,dx \\
& + |h|\,\frac{c}{R}\int_{B_{2R}} (1+|Du(x)|^p)\,dx,
\end{aligned}$$

but since $2\alpha < 2$ and $R < 1$ the following estimate easily follows

$$\begin{aligned}
\int_{B_{2R}} & \eta^2(x+he_s)(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p-2}{2}}|\tau_h Du|^2\,dx \\
& \le c|h|^{2\alpha}\int_{B_{2R}} (1+|Du(x)|^p)\,dx + c|h|\int_{B_{2R}} (1+|Du(x)|^p)\,dx,
\end{aligned} \tag{6.1.12}$$

with $c \equiv c(n, N, L, \nu, p, R)$.

Now if $2\alpha < 1$, that is $\alpha < \frac{1}{2}$, we have

$$\begin{aligned}
\int_{B_{2R}} & \eta^2(x+he_s)(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p-2}{2}}|\tau_h Du|^2\,dx \\
& \le c|h|^{2\alpha}\int_{B_{2R}} (1+|Du(x)|^p)\,dx
\end{aligned}$$

while, if $\alpha \ge \frac{1}{2}$ we have

$$\begin{aligned}
\int_{B_{2R}} & \eta^2(x+he_s)(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p-2}{2}}|\tau_h Du|^2\,dx \\
& \le c|h|\int_{B_{2R}} (1+|Du(x)|^p)\,dx.
\end{aligned}$$

We can get the final estimate applying Lemma 2.1.7 which yelds

$$\int_{B_R} |\tau_{s,h}((1+|Du(x)|^2)^{(p-2)/4}Du(x))|^2\,dx \le c|h|^{\beta}\int_{B_{2R}} (1+|Du(x)|^p)\,dx,$$

where

$$\beta = 2\alpha \quad \text{if} \quad \alpha < \frac{1}{2},$$
$$\beta = 1 \quad \text{if} \quad \alpha \geq \frac{1}{2}.$$

Now we can conclude applying Lemma 2.1.8 and performing a standard covering procedure. □

6.2 The approximation

We need now the following fundamental result that can be obtained with a procedure first introduced in [35, 38] and then developed in [19] and [30], that plays a key role in the completion of the proof of our theorems.

Lemma 6.2.1. *Let us suppose that the function f satisfies assuptions $(H1)$, $(H2)$ and $(H3)$. Then there exist a sequence $f_h(x,\cdot) \in C^2(\mathbb{R}^{nN})$ and a constant $c > 1$ independent of h such that, for every $x \in \Omega$ and $\lambda, \xi \in \mathbb{R}^{nN}$, we have*

$$(i) \qquad \frac{1}{c}(1 + \frac{1}{h^2} + |\xi|^2)^{\frac{p}{2}} \leq f_h(x,\xi) \leq cL\left(1 + \frac{1}{h^2} + |\xi|^2\right)^{\frac{p}{2}};$$

$$(ii) \qquad \frac{\nu}{c}|\lambda|^2\,(1 + |\xi|^2)^{\frac{p-2}{2}} \leq \left\langle D^2_{ij} f_h(x,\xi)\lambda_i\lambda_j\right\rangle;$$

$$(iii) \qquad |D_\xi f_h(x_1,\xi) - D_\xi f_h(x_2,\xi)| \leq c|x_1 - x_2|^\alpha \left(1 + \frac{1}{h^2} + |\xi|^2\right)^{\frac{p-1}{2}};$$

$$(iv) \qquad f_h \to f \qquad \text{uniformly on compact subsets of} \quad B_R \times \mathbb{R}^{nN},$$

with $\alpha \in (0,1)$, and where the number ν is the same appearing in $(H1)$, so it is independent of h.

Let us observe that since every $f_h(x,\cdot) \in C^2(\mathbb{R}^{nN})$ condition (ii) turns out to be equivalent to $(H2)$ thanks to Lemma 6.0.5.

Now we can complete the proof of Theorem 6.0.2 and Theorem 6.0.3 at once.

Proof. (Theorem 6.0.2 and Theorem 6.0.3.) Let us consider, for every h, the solution u_h of the Dirichlet problem

$$\min\left\{\int_{B_R} f_h(x,Dv)\ dx : v\in u+W_0^{1,p}(B_R;\mathbb{R}^N)\right\}.$$

Thanks to Lemma 6.0.2 and Lemma 6.0.3, the sequence $\{u_h\}$ turns out to be locally bounded in $W^{1,\gamma}(B_R;\mathbb{R}^N)$ for every k in $(0,\alpha)$, where

$$\gamma=\frac{np}{n-kq} \qquad \text{if } p\geq 2;$$
$$\gamma=\frac{np}{n-2k} \qquad \text{if } 1<p<2.$$

Therefore, up to a subsequence, $\{u_h\}$ converges weakly to some u_∞ in $W^{1,\gamma}_{loc}(B_R;\mathbb{R}^N)$; let us prove that u_∞ verifies the estimates (6.0.7) and (6.0.10). From (i) we have

$$\begin{aligned}||Du_\infty||_{L^\gamma(B_\rho)} &\leq \liminf_h ||Du_h||_{L^\gamma(B_\rho)} \leq c\liminf_h\left(\int_{B_R}(1+|Du_h|^2)^{\frac{p}{2}}dx\right)^{\frac{1}{2}}\\ &\leq c\liminf_h\left(\int_{B_R} f_h(x,Du_h)dx\right)^{\frac{1}{2}} \leq c\liminf_h\left(\int_{B_R} f_h(x,Du)\ dx\right)^{\frac{1}{2}}\\ &\leq c\left(\int_{B_R}(1+|Du|^2)^{\frac{p}{2}}\ dx\right)^{\frac{1}{2}}\end{aligned}$$

where we also used the minimality of u_h.

Now, exploiting the local higher equi-integrability of $\{u_h\}$ which follows from the estimates provided by Lemma 6.0.2 and Lemma 6.0.3, we shall prove that $u_\infty\equiv u$. Fixed $M\in\mathbb{N}$ we can consider for every $\rho<R$

$$\begin{aligned}\int_{B_\rho} f(x,Du_h)\ dx &= \int_{B_\rho\cap\{|Du_h|\leq M\}} f(x,Du_h)\ dx + \int_{B_\rho\cap\{|Du_h|>M\}} f(x,Du_h)\ dx\\ &\leq \int_{B_\rho\cap\{|Du_h|\leq M\}} [f(x,Du_h)-f_h(x,Du_h)]\ dx + \int_{B_\rho} f_h(x,Du_h)\ dx\\ &\quad + \int_{B_\rho\cap\{|Du_h|>M\}} f(x,Du_h)\ dx.\end{aligned}$$

Remembering that f_h converges uniformly to f on compact subset (see (iii)) we have

$$\lim_h \int_{B_\rho\cap\{|Du_h|\leq M\}} [f(x,Du_h)-f_h(x,Du_h)]\ dx = 0,$$

therefore

$$\liminf_h \int_{B_\rho} f(x, Du_h)\ dx \le \limsup_h \int_{B_\rho} f_h(x, Du_h)\ dx$$
$$+ \limsup_h \int_{B_\rho \cap \{|Du_h| > M\}} f(x, Du_h)\ dx. \qquad (6.2.1)$$

Moreover, since

$$\limsup_h \int_{B_\rho \cap \{|Du| \le M\}} [f_h(x, Du) - f(x, Du)]\ dx = 0,$$

by the minimality of u_h we can control the right hand side of (6.2.1) by

$$\int_{B_\rho} f(x, Du)\ dx + \limsup_h \int_{B_\rho \cap \{|Du| > M\}} f_h(x, Du)\ dx + \limsup_h \int_{B_\rho \cap \{|Du_h| > M\}} f(x, Du_h)\ dx.$$

Using the growth conditions (i) on f and f_h we have

$$\liminf_h \int_{B_\rho} f(x, Du_h)\ dx \le \int_{B_\rho} f(x, Du)\ dx + cL \int_{B_\rho \cap \{|Du| > M\}} (1 + |Du|^2)^{\frac{p}{2}}\ dx$$
$$+ cL \limsup_h \int_{B_\rho \cap \{|Du_h| > M\}} (1 + |Du_h|^2)^{\frac{p}{2}}\ dx, \quad (6.2.2)$$

where $L > 0$ is the same growth constant appearing in the assumption $(H1)$. Applying Hölder's inequality we can estimate the last integral in (6.2.2) with

$$\limsup_h cL \left[\left(\int_{B_\rho} \left(1 + |Du_h|^2\right)^{\frac{\gamma p}{2}}\ dx \right)^{\frac{1}{\gamma}} \cdot \quad |\{|Du_h| > M\} \cap B_\rho|^{1 - \frac{1}{\gamma}} \right]$$

where the first factor is finite and independent of h. So we get the estimate

$$\liminf_h \int_{B_\rho} f(x, Du_h) dx \le \int_{B_\rho} f(x, Du) dx + cL \left(\limsup_h |\{|Du_h| > M\} \cap B_\rho| \right)^{1 - \frac{1}{\gamma}}$$
$$+ cL \int_{B_\rho \cap \{|Du| > M\}} (1 + |Du|^2)^{\frac{p}{2}}\ dx.$$

Note that

$$\limsup_{M \to +\infty} \int_{B_\rho \cap \{|Du| > M\}} (1 + |Du|^2)^{\frac{p}{2}}\ dx = 0$$

and

$$|\{|Du_h| > M\} \cap B_\rho| \, M^\gamma \leq \int_{B_\rho} |Du_h|^\gamma \, dx \leq C,$$

where the constant C does not depend on h. Therefore

$$|\{|Du_h| > M\} \cap B_\rho| \leq \frac{C}{M^\gamma}$$

for every h and

$$\limsup_h |\{|Du_h| > M\} \cap B_\rho| = 0$$

when $M \to +\infty$. Then we have

$$\int_{B_\rho} f(x, Du_\infty) \, dx \leq \liminf_h \int_{B_\rho} f(x, Du_h) \, dx \leq \int_{B_\rho} f(x, Du) \, dx.$$

Passing to the limit as $\rho \uparrow R$ we can conclude

$$\int_{B_R} f(x, Du_\infty) \, dx \leq \int_{B_R} f(x, Du) \, dx.$$

Since u is a local minimizer of the functional $\mathcal{F}$ and $u_\infty \equiv u$ on the boundary of B_R, the strict convexity of f implies $u_\infty \equiv u$. □

Now we can apply Theorems 6.0.2 and 6.0.3 to get bounds on the Hausdorff dimension of the singular set Σ of minimizers of functional $\mathcal{F}$.

In our case we have the following result.

Corollary 6.2.2. *If f is a C^2 function satisfying the assumptions $(H1)$, $(H2)$, $(H3)$ and the function $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ is a local minimizer of $\mathcal{F}$ in Ω, then for the Hausdorff dimension of the singular set Σ of the function u the following estimates hold*

$$dim_{\mathcal{H}}(\Sigma) \leq n - \alpha q \qquad \text{if } p \geq 2;$$

$$dim_{\mathcal{H}}(\Sigma) \leq n - \beta p \qquad \text{if } 1 < p < 2;$$

where $q = \frac{p}{p-1}$ and $\beta := \min\left\{\alpha, \frac{1}{2}\right\}$.

Proof. If $u \in W^{1,p}$, $p > 1$, is a local minimizer of the functional $\mathcal{F}$ as a consequence of the Theorems 6.0.2 and 6.0.3 we have in particular that

$$Du \in W^{\frac{k}{p-1},p} \qquad \text{if } p \geq 2;$$

for every $k \in (0, \alpha)$ and

$$Du \in W^{k,p} \qquad \text{if } 1 < p < 2;$$

for every $k \in (0, \beta)$ where $\beta := \min\left\{\alpha, \frac{1}{2}\right\}$. Therefore applying Lemma 2.4.1 we immediately conclude the proof. □

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