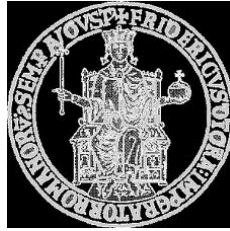Dipartimento di Scienze Físiche

Facoltá di Scienze Matematiche, Fisiche e Naturali

Universitá Degli Studi di Napoli Federico II

Select Topics in Quantum Gravity :
A Maiden Voyage

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PhD Thesis

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totally irrelevant in any scientific way, but nonetheless very entertaining. Rather than attempting to give an exhaustive list and perhaps offending some by forgetting to mention them, I will only name one person explicitly : the one who had almost acted as my local guardian safeguarding me from all sorts of maze of petty Italian bureaucracy, even helping me consult a doctor, or find a cozier apartment and in all sorts of daily life amenities, without you all these would have been impossible. This honour goes without a doubt to Guido Celentano. *Salute!*

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Whenever a theory appears to you as the only possible one, take this as a sign that you have neither understood the theory nor the problem which it was intended to solve.

Karl R. Popper

Contents

Abstract

We study selected aspects of Theoretical Physics confronting some key issues related to the fundamental interactions along the line of Black Holes (BHs) and Attractors and its thread may be found in the three concepts of Supersymmetry, Supergravity and Holography which encompass all of String theory and Quantum gravity. Then we also had an encounter with maximally symmetric spaces in General Relativity such as de Sitter and we did some significant computation in this backdrop which is tempting to pursue keeping in mind the recent observational data in favor of inflationary picture of the Universe.

We present a simple model for studying the effect of quantum perturbative correction to the $\mathcal{N} = 2$ prepotential function. In [1], we compute the effective black hole potential V_{BH} of the most general $\mathcal{N} = 2$, $d = 4$ (*local*) special Kähler geometry with quantum perturbative corrections, consistent with axion-shift Peccei-Quinn symmetry and with cubic leading order behavior.

We also determine the charge configurations supporting axion-free attractors, and explain the differences among various configurations in relations to the presence of "flat" directions of V_{BH} at its critical points. Furthermore, we elucidate the role of the sectional curvature at the non-supersymmetric critical points of V_{BH}, and compute the Riemann tensor (and related quantities), as well as the so-called *E*-tensor, which expresses the non-symmetricity of the considered quantum perturbative special Kähler geometry.

Then in a follow-up paper [1], we reconsider the sub-leading quantum perturbative corrections to $\mathcal{N} = 2$ cubic special Kähler geometries. Imposing the invariance under axion-shifts, all such corrections (but the imaginary constant one) can be introduced or removed through suitable, lower unitriangular symplectic transformations, dubbed Peccei-Quinn (PQ) transformations. Since PQ transformations do not belong to the $d = 4$ *U*-duality group G_4, in symmetric cases they generally have a non-trivial action on the unique quartic invariant polynomial $\mathcal{I}_4$ of the charge representation **R** of G_4. This leads to interesting phenomena in relation to theory of

extremal black hole attractors; i.e., the possibility to make transitions between different charge orbits of **R**, with corresponding change of the supersymmetry properties of the supported attractor solutions. Furthermore, a suitable action of PQ transformations can also set $\mathcal{I}_4$ to zero, or vice versa it can generate a non-vanishing $\mathcal{I}_4$: this corresponds to transitions between "large" and "small" charge orbits, which we classify in some detail within the "special coordinates" symplectic frame.

Finally, after a brief account of the action of PQ transformations on the recently established correspondence between Cayley's hyperdeterminant and elliptic curves, we derive an equivalent, alternative expression of $\mathcal{I}_4$, with relevant application to black hole entropy. It is worth pointing out here that, In the case of target manifolds for scalars (of $\mathcal{N} = 2$ vector multiplets) which are "symmetric spaces", the theory of attractors displays a beautiful connection with Group Theory and Differential Geometry. And quite recently there had been a flurry of activities in the interplay between two different disciplines for e.g. Black Hole Physics and Quantum Information Theory, giving birth to new intriguing ideas such as "Black Hole/Qubit correspondence". We can hope for many such surprising results to come along the way from Black Holes related research in near future.

We also studied, in [1] spinor two-point functions for spin-1/2 and spin-3/2 fields in maximally symmetric spaces such as de Sitter(dS) spacetime, by using intrinsic geometric objects. The Feynman, positive- and negative-frequency Green functions are then obtained for these cases, from which we eventually display the supercommutator and the Peierls bracket under such a setting in two-component-spinor language. In a follow-up paper [2], we complete, the explicit representation of the massive gravitino propagator in four-dimensional de Sitter space with the help of the general theory of the Heun equation. As a result of our original contribution, all weight functions which multiply the geometric invariants in the gravitino propagator are expressed through Heun functions, and the resulting plots are displayed and discussed after resorting to a suitable truncation in the series expansion of the Heun function. It turns out that there exist two ranges of values of the independent variable in which the weight functions can be divided into dominant and sub-dominant family.

As a note, Chapters 6, 7, 8 contain original results obtained during the PhD years.

List of Publications

- G.Esposito, R. Roychowdhury, *The Eastwood-Singer Gauge in Einstein Spaces*, Int. J. Geom. Meth. Mod. Phys. **6** 583, (2009) `arXiv:0805.1182 [hep-th]`.

- * G.Esposito, R. Roychowdhury, *Spinor Two-Point Functions and Peierls Bracket in de-Sitter Space*, Int. J. Geom. Meth. Mod. Phys. **6** 1069, (2009) `arXiv:0902.2098 [hep-th]`.

- * G.Esposito, R. Roychowdhury, *On the Complete Analytic Structure of the Massive Gravitino Propagator in Four-Dimensional de-Sitter Space*, Gen. Rel. Grav. **42** 1221, `arXiv:0907.3634 [hep-th]`.

- R. Roychowdhury, *Attractor Flows in st^2 Black Holes*, `arXiv:0910.2609 [hep-th]`.

- * S. Bellucci, A. Marrani and R. Roychowdhury, *On Quantum Special Kähler Geometry*, Int. J. Mod. Phys. **A25**, 1891 (2010), `arXiv:0910.4249 [hep-th]`.

- G. Esposito, R. Roychowdhury, C. Rubano and P. Scudellaro *Transition from complex to real scalar fields in modern cosmology*, `arXiv:1009.2887 [hep-th]`.

- * S. Bellucci, A. Marrani and R. Roychowdhury, *Topics in Cubic Special Geometry*, `arXiv:1011.0705 [hep-th]`.

- V. Parente and R. Roychowdhury, *A Study on Charged Neutron Star in AdS_5*, `arXiv:1011.5362 [hep-th]`.

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[1]* indicates the papers on which this thesis is based.

Chapter 1

Prolegomena

The first part of the thesis considers various aspects of black holes, which are objects with their mass compressed to a tiny volume. Their salient feature is that nothing, not even light, can escape from within a certain vicinity of them. Since we can't look beyond the boundary of this vicinity, it is rightly called the "horizon". The extraordinary features of black holes raise a variety of puzzles which are worth studying. Many different branches of Physics come together with their own proper description of black holes. The attractive force of black holes on other objects ranges typically over macroscopic, and even astronomical, length scales. Therefore, the classical branches of Physics such as mechanics, gravity, electromagnetism and thermodynamics are relevant for studying these massive objects. However, since the mass of such objects is confined to an extremely small volume, the proper understanding of black holes demands for a framework combining consistently physics of very large and small length scales. Such an elusive framework is yet to be found and the most promising candidate for this is String theory or M theory at present.

The first notion of a black hole appeared in a letter by the Rev. John Michell to Henry Cavendish back in 27th of November, 1783. It read as follows [1] :

[...] If there should really exist in nature any bodies whose density is not less than that of the Sun, and whose diameters are more than 500 times the diameter of the Sun, since their light could not arrive at us, or of there should exist any other bodies of a somewhat smaller size which are not naturally luminous; of the existence of bodies under either of these circumstances, we could have no information from sight; yet, if any luminous bodies infer their existence of the central ones with some degree of probability, as this might afford a clue to some of the apparent irregularities of the revolving bodies, which would not be easily explicable on any other hypothesis; but as the consequences of such a supposition are very obvious, I shall not prosecute them any further. [...]

Michell's predictions, though deeply routed in 18th century concepts about gravity and light and fallen into long oblivion, were well ahead of his time. Not only did he envisage objects whose escape velocity exceeds the speed of light, rendering them completely dark, but also proposed an indirect method to detect them, which is essentially one of those few currently employed. Presently there is empirical evidence for black holes to be regarded as ubiquitous in the universe, occupying centers of most galaxies [2]. A decade after Michell's discovery, in 1795, Laplace independently suggested the existence of black holes. The most striking thing about the history of black holes is that although the name "black hole" seems apt enough for such an object which does not emit light, the name itself was coined by J. A. Wheeler nearly 200 years later, in 1967 only.

The notion of black hole was made more precise after General Relativity (GR) superseded Newtonian gravity in 1915-16. Einstein's GR provides a new revolutionary vision on gravity in general and it was the first time in the history of physics an equation was written with geometry on one side and matter on the other. Space and time are no longer static, but are deformed by the presence of massive objects. The attractive forces between massive objects is a consequence of the deformation of space-time precisely given by the so-called Einstein's equations. The first black hole solution was discovered by Schwarzschild in 1915. In this solution, all mass of the universe was concentrated in a point, which is surrounded by the horizon. GR predicts that the horizon of a black hole with the mass of the Earth is $1cm^2$. Another interesting quantity found was the surface gravity, the strength of gravitational force at the horizon.

Soon after, more complicated black hole geometries came into being, for example, solution of a theory which captures both GR and Electromagnetism. The black holes of this Einstein-Maxwell theory carry electromagnetic charges and were first described by Reissner in 1916 and Nördstrom in 1918. It turns out that the mass of these black holes must be larger than a certain minimum value, as predicted by cosmic censorship conjecture, and is determined in terms of the charges. At the minimum value of the mass, the surface gravity vanishes and then the black holes are called extremal black holes. Such "extremal" black holes are considered in this thesis for reasons to be explained later.

During the 1970's, Bardeen, Bekenstein, Carter, Gibbons, Hawking and others studied the laws that are satisfied by the black hole quantities, such as their mass, horizon area and surface gravity. Interestingly enough, these studies revealed a closed resemblance with the laws of classical thermodynamics. In fact, the three laws of thermodynamics could be naturally extended to incorporate GR. This is

more than an analogy, indeed there is a one-to-one correspondence between horizon area with entropy and surface gravity with temperature.With these identifications, the three laws of thermodynamics also hold good for Black Hole physics. In 1975 Hawking performed a semi-classical analysis [3] to show that black holes radiate as objects with a certain temperature like in Planck's black-body radiation.

After this little exposition of black holes, we hope it is clear that, not only the astrophysical significance, but their unusual and counter intuitive properties make them interesting in their own right to a relativist, and there are sufficiently good reasons to study them in detail. In modern parlance black holes have acquired the status of "the hydrogen atom of quantum gravity" as aptly put by Juan Maldacena in his thesis [4], for it is in black holes that the urge to reconcile GR and Quantum Mechanics becomes most apparent : Black Holes do not conform to the laws of thermodynamics unless the quantum effect - the Hawking Radiation is taken into account, but if the quantization is restricted only to electromagnetic radiation and does not include gravity itself, the purely thermal Hawking radiation violates unitary evolution of states in quantum mechanics, thus irrecoverably destroying all the informations that have entered the black hole, this famous puzzle i.e. "information paradox", has a hope for resolution only in a quantum gravity set up [5].

The crucial step in any attempted quantum description of black hole consists of identifying their micro-states. A proposed model in this direction can then be tested by verifying whether the statistical Boltzmann's entropy agrees with the entropy computed from the macroscopic properties of black hole. In theories involving scalar fields the latter will in general depend on the horizon values of the scalar. For charged extremal black holes this poses a potential problem regardless of the detail of the model, because the microscopic entropy is fully determined in terms of the quantized charges and should not depend on any continuously varying parameters. It turns out, however, that a phenomenon called "Attractor Mechanism" ensures that the horizon values of the scalars are not arbitrary, rather they are determined in terms of the dyonic black hole charges.

The Attractor Mechanism was first invented in the context of Supersymmetric black holes [6–9] and later was extended to non-supersymmetric extremal counterparts [10; 11] in four dimensions. In the absence of higher-curvature corrections to the bosonic part of the action, the attractor equations constraining the moduli scalars at the horizon to be functions of the moduli field, arise as the extremization condition for the effective potential, known as the black hole effective potential [8; 10–12]. It is intuitively understood as the electromagnetic energy of the vector fields in a scalar medium.

A different way to describe the attractor mechanism is the entropy function formalism [13; 14]. In this approach one defines an entropy function, whose extremization determines the values of the scalar fields at the horizon. The entropy of the black hole is then given by the value of the entropy function at the extremum. The original calculation defines the entropy function as the Legendre transform with respect to the electric field of the Lagrangian density integrated over the event horizon and applies to spherically symmetric black holes in a broader class of theories rather than being confined to theories with an effective black hole potential, i.e. arbitrary theories of gravity including possible higher order curvature corrections coupled to Abelian gauge fields and neutral scalars, provided that the gauge potentials appear in the Lagrangian solely through field strengths or are trivial for a given solution.

The attractor mechanism reduces the problem of finding the horizon values of the scalar field to solving a set of equations, but to obtain full solutions interpolating between the asymptotic values of field at infinity and that at the horizon, one still needs to solve the second-order differential equation of motion. A subset of solutions can however be derived by rewriting the action as a sum of squares of the first order flow equations [10; 15; 16]. The interpolating solutions are then given in terms of harmonic functions [6; 8; 17–20]. This is always the case for supersymmetric solutions, but there are examples where harmonic functions have also been found for the non-supersymmetric case [21] as well and the authors of [22; 23] demonstrated a class of non-supersymmetric solutions described by the first-order equations.

In this thesis we shall concern ourselves with the black hole attractor mechanism in four- dimensional $\mathcal{N} = 2$ supergravity (SUGRA). The amount of supersymmetry in this theory (8 supercharges in four dimension) already permits non-trivial dynamics, but is simultaneously restrictive enough to substantially simplify the analysis, as the theory is completely specified in terms of a single function called the prepotential function. In a broader context, since $\mathcal{N} = 2$ SUGRA provide low-energy field-theoretic description of Calabi- Yau compactifications in string and M theory, the results obtained in the SUGRA regime might be directly employed to test the string theoretical microscopic models of these black holes.

1.1   Timeline of Black Hole Physics updated till 2005

- 1640 - Ismael Bullialdus suggests an inverse-square gravitational force law.

- 1684 - Isaac Newton writes down inverse-square Law of universal gravitation.

- 1758 - Rudjer Josip Boscovich develops his Theory of forces, where gravity

can be repulsive on small distances. So according to him such strange classical bodies, similar to white holes, can exist, which won't let other bodies to reach their surfaces.

- 1784 - John Michell discusses classical bodies which have escape velocities greater than the speed of light.

- 1795 - Pierre Laplace discusses classical bodies which have escape velocities greater than the speed of light.

- 1798 - Henry Cavendish measures the gravitational constant G.

- 1876 - William Clifford suggests that the motion of matter may be due to changes in the geometry of space.

- 1909 - Albert Einstein together with Marcel Grossmann starts to develop a theory which would bind metric tensor g_{ik}, which defines a space geometry, with a source of gravity, i.e. with mass.

- 1910 - Hans Reissner and Gunnar Nordström defines Reissner-Nordström singularity and Hermann Weyl solves special case for a point-body source.

- 1916 - Karl Schwarzschild solves the Einstein vacuum field equations for uncharged spherically-symmetric non-rotating systems.

- 1917 - Paul Ehrenfest gives conditional principle a three-dimensional space.

- 1918 - Hans Reissner and Gunnar Nordstróm solve the Einstein-Maxwell field equations for charged spherically-symmetric non-rotating systems.

- 1918 - Friedrich Kottler gets Schwarzschild solution without Einstein vacuum field equations.

- 1923 - George Birkhoff proves that the Schwarzschild spacetime geometry is the unique spherically symmetric solution of the vacuum Einstein equations.

- 1939 - Robert Oppenheimer and Hartland Snyder calculate the gravitational collapse of a pressure-free homogeneous fluid sphere and find that it cuts itself off from communication with the rest of the Universe.

- 1963 - Roy Kerr solves the vacuum Einstein equations for uncharged symmetric rotating systems.

- 1964 - Roger Penrose proves that an imploding star will necessarily produce a singularity once it has formed an event horizon.

- 1965 - Ezra Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash, and Robert Torrence solve the Einstein-Maxwell field equations for charged rotating systems.

- 1967 - Werner Israel presented the proof of the no hair theorem at Kings College in London.

- 1967 - John Wheeler coins the term "black hole".

- 1968 - Brandon Carter uses Hamilton-Jacobi theory to derive first-order equations of motion for a charged particle moving in the external fields of a Kerr-Newman black hole.

- 1969 - Roger Penrose discusses the Penrose process for the extraction of the spin energy from a Kerr black hole.

- 1969 - Roger Penrose proposes the cosmic censorship hypothesis.

- 1971 - Identification of Cygnus X-1/HDE 226868 as a binary black hole candidate system.

- 1972 - Stephen Hawking proves that the area of a classical black hole's event horizon cannot decrease.

- 1972 - James Bardeen, Brandon Carter, and Stephen Hawking propose four laws of black hole mechanics in analogy with the laws of thermodynamics.

- 1972 - Jacob Bekenstein suggests that black holes have an entropy proportional to their surface area due to information loss effects.

- 1974 - Stephen Hawking applies quantum field theory to black hole spacetimes and shows that black holes will radiate particles with a blackbody spectrum which can cause black hole evaporation.

- 1989 - Identification of GS2023+338/V404 Cygni as a binary black hole candidate system.

- 1994 - Robert Wald and Vivek Iyer give a proposal for dynamical black hole entropy.This is known as Wald entropy in the literature and this generalization implies an elegant formal expression for the black hole entropy given a general action including higher derivative terms.

- 1996 - Sergio Ferrara, Gary Gibbons, Renata Kallosh and Andrew Strominger together invoked Attractor Mechanism for $\mathcal{N} = 2$ extremal Black Holes in Maxwell-Einstein Supergravity theory.

- 1996 - Andrew Strominger and Cumrun Vafa explains the microscopic origin of the black hole entropy, originally calculated thermodynamically by Stephen Hawking and Jacob Bekenstein, from string theory.

- 2002 - Astronomers present evidence for the hypothesis that Sagittarius A* is a supermassive black hole at the centre of the Milky Way galaxy.

- 2002 - NASA's Chandra X-ray Observatory identifies double galactic black holes system in merging galaxies NGC 6240.

- 2004 - Further observations by a team from UCLA present even stronger evidence supporting Sagittarius A* as a black hole.

- 2005 - Ashoke Sen proposes Black Hole entropy function and Attractor Mechanism for higher derivative gravity theories.

1.2 A de-Sitter Space Odyssey

1.2.1 Mathematical Formulation

De Sitter n-space or dS_n is the maximally symmetric n-dimensional spacetime with positive cosmological constant Λ. Its symmetry group is $SO(1, n)$. If we introduce variables $x_0, x_1, \ldots, x_n$ obeying $x_0^2 - \sum_{1=1}^{n} x_i^2$, the de Sitter metric is simply (up to a constant factor)

$$ds^2 = -dx_0^2 + \sum_{i=1}^{n} dx_i^2. \tag{1.2.1.1}$$

Alternatively, one can write the metric as

$$ds^2 = -dt^2 + \cosh^2 t \, d\Omega^2, \tag{1.2.1.2}$$

where $d\Omega^2$ is the metric on a unit round $(n-1)$-sphere. This spacetime has compact spatial sections (such as $t = 0$), so when we speak of *asymptotically* de Sitter space – as we should in the presence of gravity, since the metric fluctuates – the asymptopia in question is in the past and future. There is no notion of spatial infinity. This is in sharp contrast with Anti de Sitter space, the maximally symmetric spacetime of negative cosmological constant, where, as we have come to know well in the last

few years, asymptopia is at spatial infinity. It also contrasts with Minkowski space, which from a conformal point of view has a natural null infinity.

In de Sitter space, there is no positive conserved energy. In fact, no matter what generator we pick for $SO(1, n)$, the corresponding Killing vector field, though perhaps timelike in some region of de Sitter space, is spacelike in some other region. For example, a typical Lorentz generator in de Sitter space is

$$K = x_1 \frac{\partial}{\partial x_0} + x_0 \frac{\partial}{\partial x_1}. \tag{1.2.1.3}$$

Whether this generator moves us forwards or backwards in time (towards increasing or decreasing x_0) depends on the sign of x_1. The conserved charge associated with K is positive for excitations supported at positive x_1 and negative for those at negative x_1. This is the best we can do: there is no positive conserved energy in de Sitter space.

Consequently, there cannot be unbroken supersymmetry in de Sitter space. If there is a nonzero supercharge Q, we can (possibly after replacing Q by $Q + Q^\dagger$ or $i(Q - Q^\dagger)$) assume that Q is Hermitian. Then Q^2 cannot be zero, and is a nonnegative bosonic conserved quantity; but there is no such object.

We can rotate de Sitter space to Euclidean signature by setting $x_0 \rightarrow ix_0$ (or equivalently, set $t = i\tau$ and take $\tau = \pi/2 - \theta$). The Euclidean continuation is a standard n-sphere $\mathcal{S}^n$, with symmetry group $SO(n + 1)$. After the continuation, the operator K becomes the generator of a rotation, and obeys

$$\exp(2\pi K) = 1. \tag{1.2.1.4}$$

Because of this, the Euclidean de Sitter path integral can be interpreted in terms of a thermal ensemble. This leads to the notion of a de Sitter temperature [24] and the associated entropy [25] Like the Bekenstein-Hawking entropy of a black hole, the de Sitter entropy can be written

$$S = \frac{A}{4G}, \tag{1.2.1.5}$$

where G is Newton's constant, and A is the area of a horizon. In this case, however, the horizon is observer-dependent, and because of this it is not entirely clear which concepts about black holes carry over to de Sitter space.

An observer in de Sitter space can only see part of the space. This is because of the exponential inflation that occurs in the future: the space expands so fast that light rays do not manage to propagate all the way around it. To make the causal structure of de Sitter space clear, one can introduce a new "time" coordinate u by

$$u = 2\tan^{-1} e^t, \tag{1.2.1.6}$$

so that for $-\infty < t < \infty$, u ranges over $0 < u < \pi$. The metric becomes

$$ds^2 = \frac{1}{\sin^2 u}\left(-du^2 + d\Omega^2\right). \tag{1.2.1.7}$$

The asymptotic past $\mathcal{I}_-$ consists of a copy of $\mathcal{S}^{n-1}$ at $u = 0$, and the asymptotic future $\mathcal{I}_+$ consists of a copy of $\mathcal{S}^{n-1}$ at $u = \pi$. Any trajectory in de Sitter space begins at some point P in $\mathcal{I}_-$ and ends at some point Q in $\mathcal{I}_+$. From a causal point of view, in a sense considered by Bousso [26] any observer can be identified with the pair (P, Q). The region of de Sitter space that one can influence, and likewise the region that one can see, depend only on P and Q, and not on the details of one's trajectory in spacetime. What one can see is determined only by Q, and the region that one can influence depends only on P.

To describe in detail the horizon of an observer, let us write $d\Omega^2 = d\chi^2 + \sin^2 \chi d\tilde{\Omega}^2$, where χ is a polar angle, ranging over $0 \leq \chi \leq \pi$, and $d\tilde{\Omega}^2$ is the round metric on an $(n-2)$-sphere. The de Sitter metric then becomes

$$ds^2 = \frac{1}{\sin^2 u}\left(-du^2 + d\chi^2 + \sin^2 \chi \, d\tilde{\Omega}^2\right). \tag{1.2.1.8}$$

Consider now an observer who sits at the "north pole" of the sphere, that is, at $\chi = 0$. (In fact, any geodesic in de Sitter space is equivalent to $\chi = 0$ by the action of the de Sitter group.) From the form of the metric, we see that the propagation of light rays is bounded by $|d\chi/du| \leq 1$. Since the spacetime "ends" in this coordinate system at $u = \pi$, a light ray emitted at $\chi > \pi - u$ will never reach the observer at $\chi = 0$. So the boundary of the region that this observer can see is given by

$$\chi = \pi - u. \tag{1.2.1.9}$$

This is the horizon. In general, the $(n-2)$-sphere of given χ and u has metric $(\sin \chi / \sin u)^2 d\tilde{\Omega}^2$, and its area is proportional to $(\sin \chi / \sin u)^{n-2}$. Relating χ to u by 1.2.1.9, we see that the horizon area is time-independent. This is in keeping with general theorems saying that the area of the past horizon of an observer cannot decrease in time. For de Sitter space, this horizon area is precisely constant, and for a generic perturbation of de Sitter space, it is an increasing function of time.

By studies of D-branes and in a variety of other ways, we have learned in the last few years to interpret the Bekenstein-Hawking entropy of a black hole like every other entropy in statistical mechanics: it is the logarithm of the number of quantum states of the black hole. It has been argued [27] that the same holds for de Sitter space, more precisely that the Hilbert space of quantum gravity in asymptotically de Sitter space time has a finite dimension N, and that the entropy of de Sitter space is

$$S = \ln N. \tag{1.2.1.10}$$

1.2.2 Physical Consideration

The recent observational data in favor of the presence of a positive cosmological constant in the universe make it especially important to understand how to formulate consistent theory of all interactions in de Sitter (dS) space. This is highly nontrivial: Quantum field theory in dS presents us with a lot of puzzles [28–31], and whether and how they could be resolved in the underlying fundamental theory is not at all clear, as is still not known how to obtain a stable de Sitter solution in the best-so-far candidate for such a unifying theory – string theory.[1] A key ingredient in the final picture may be holography, which is believed to be an essential feature for any consistent theory of quantum gravity [34; 35]. One realization of this idea is the AdS/CFT correspondence [36–38], which has been studied in a huge number of cases during the last few years (for a review see [39]). Another is the recently proposed dS/CFT correspondence [40]. Although it is hoped that it may shed light on quantum gravity in de Sitter space, the lack of a clear relation to string theory is hindering an explicit realization of this proposal, and it is largely modeled on analogy with AdS/CFT (see, for example, the prescription for computation of scalar field correlation functions in the boundary theory [41]). There have even been papers arguing that dS/CFT is merely an analytic continuation of AdS/CFT [42; 43]. One should not forget, however, the fundamental differences between physics in dS and AdS. For example, the analytic continuation of the vacuum state in AdS space does not coincide with any of the vacua of de Sitter space[2]. Also, unlike AdS, dS has two boundaries, posing the (as yet unsettled) question whether the dual theory should be thought of as a single CFT [40; 57] or two entangled CFT's [58].

Field theory in de Sitter space was studied extensively in the '80s due to interest sparked by inflationary cosmology. A technique of calculation of propagators in maximally symmetric spaces was developed in a series of papers [59–61]. The main idea is the following: One chooses a basis of bitensors which are invariant under the symmetry group of the space under consideration, and makes an Ansatz for the propagator in terms of these bitensors multiplied by coefficients that are functions only of the geodesic distance. The coefficient functions are then determined so that

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[1]Recently there has been a progress in that direction: Fré, Trigiante and Van Proeyen [32] found stable de Sitter vacua in $N = 2$ supergravity in 4 dimesnions. However, their embedding in string theory is still an open problem. Another exciting recent development is [33], where metastable dS vacua were found in type IIB string theory.

[2]As known since [44–46], there is an infinite one parameter family of de Sitter invariant vacua in dS. Their possible role in the cosmology of the early universe has been explored in [47–51] and references therein. In addition the question which one is the most reasonable (physical) vacuum state in dS is still unsettled [50; 52–56].

the propagator satisfies the appropriate field equations and constraints. The original papers considered spins 0 and 1 in arbitrary dimension and also spins 1/2 and 2 in four dimensions. Subsequently these methods were used to find the antisymmetric tensor propagator in dS [62] and also the propagators of various $p-$forms of interest in supergravity/string theory in AdS [63–65].[3] However, only quite recently was this method extended to spin 1/2 field in arbitrary dimension [66], and the spin 3/2 field has not been treated so far. [4]

Since dS is not a supersymmetric background, it may seem uninteresting to consider the superpartner of the graviton in it. However, if dS is to be reconciled with the current lore of a fundamental theory, i.e. string theory, then the lack of supersymmetry in de Sitter space should be understood as a symmetry which is present in the theory but broken in the specific vacuum state. Given that superymmetry breaking in supergravity leads to massive gravitinos, massive spin 3/2 fields are essential for understanding the effective description of quantum gravity processes in de Sitter space. Motivated by this, we will find in this thesis the propagator of massive gravitino in four space-time dimension.

1.3 Synopsis

The physics of Black Holes is the main theme of this thesis. Black Holes (BHs) can be studied either macroscopically in terms of geometrical quantities related to their thermodynamics or microscopically by microstate counting, a prescription provided by Statistical Mechanics. The Attractor Mechanism due to Ferrara, Kallosh and Strominger connects the entropy of extremal Black Holes to the extrema of a certain effective potential, in a way which is reminiscent of the moduli stabilization in flux compactifications. Moreover, Attractors and their entropy formula seems deeply connected to the topological string partition functions that appear in counting problems for instantons and other non-perturbative phenomena. The attractive nature of four and higher-dimensional Extremal Black Holes and extended objects (p-branes) toward universal horizon geometries is also at the heart of the holographic (AdS/CFT) Correspondence between (super)conformal Yang-Mills theories and (super)gravity theories in Anti de Sitter spaces as was conjectured by Malda-

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[3]The basis of bitensor structures used in the AdS/CFT literature differs from the original basis of [59]. We comment more on that in a subsequent footnote.

[4]Aspects of spin-3/2 propagation were studied already by Lichnerowicz [67]; more recent interesting studies on higher spin fields in constant curvature spaces (e.g. the discovery of partial masslessness and new local gauge invariances) can be found in [68] and references therein.

cena. Despite many attempts, however, a satisfactory microscopic explanation of Black Hole entropy is still missing. The best results obtained so far show a precise agreement between the entropy determined by the microstates describing the degrees of freedom of special configurations of D-branes wrapped on Calabi-Yau manifolds and the macroscopic semiclassical result, obtained using the Bekenstein-Hawking formula (or its generalization given by Wald) for supersymmetric extremal charged Black Holes in supergravity theories, in the large charge limit. This result relies heavily on the Attractor Mechanism, which also explains why the entropy does not depend on any continuous parameters, even though a large number of massless scalar fields enter the low-energy supergravity models. Although this mechanism was first found in supersymmetric configurations, there are by now good reasons to believe that it also extends to other non-supersymmetric, albeit extremal, configurations. The thesis aims to extend the study of Extremal Black Holes to more general situations, and study in detail, the mathematical structure of the moduli space of the scalars of the vector multiplets. It also aims to explore quantum corrections to the classical formulae for the pre-potential function for $\mathcal{N} = 2$ which are quite important for backgrounds with enhanced supersymmetry.

State-of-the-art

Supersymmetry, a deep and elegant space-time symmetry relating fermions, with half-odd spin, and bosons, with integer spin, to one another, led to major advances in Quantum Field Theory and accounts for the construction of a consistent candidate for a unified theory encompassing Quantum Gravity and Standard Model of Particle Physics. When combined with local gauge invariance, global supersymmetry yields Supersymmetric Yang Mills Theories (SYM). Thanks to remarkable cancellations between bosons and fermions in their quantum corrections, SYM's can be reliably studied beyond perturbation theory, so that certain holomorphic observables can be fully determined, and provide a possible solution of the Hierarchy problem, a natural candidate for Dark Matter and a conceptual framework for addressing the Dark Energy problem. When combined with general covariance, supersymmetry becomes a local symmetry. The resulting Supergravity theories provide a low-energy effective description of more fundamental theories such as Superstrings and M-theory and play a crucial role in the analysis of Supersymmetry Breaking, a necessary ingredient for all realistic elaborations beyond the Standard Model. The gravity part of the theory reduces to the Einstein-Hilbert action coupled to a certain number of matter fields whose specific nature depend on the particular low-energy effective theory. Typically these fields are massless scalars, called moduli, spin 1/2

fermions, spin 1 gauge fields and spin 3/2 fermions and gravitinos. The letter, $\mathcal{N}$ in $\mathcal{N}$-extended supergravity, are the gauge fields of local supersymmetry. In the past ten years or so, (local) supersymmetry proved to be an unprecedented tool also in the study of Black Holes (BHs), the endpoints of gravitational collapse whereby a horizon surface prevents the possible formation of a space-time singularity. BHs are classically inaccessible, but are known to possess rich thermodynamical properties and emit Hawking Radiation due to quantum fluctuations. Both supersymmetric and other extremal BHs are subject to an Attractor Mechanism that allows one to understand better, if not to fully prove, the Bekenstein-Hawking formula relating Entropy to Horizon area. Indeed BHs can be treated as thermodynamical objects, so that a characteristic entropy, proportional to the area of their Horizon, can be associated to each and every one of them. The physical explanation of this quantity is deeply linked to quantum gravity effects. Hence, BHs present them to be an important test candidate for the quantum theories of gravity such as Superstring Theory or M-Theory, for which Supergravity represents a universal low-energy limit. In situations where higher curvature effects may be neglected, asymptotically flat charged BH solutions, with a static and spherically symmetric ansatz can be identified. These solutions generalize the famous Schwarzschild BH, but the presence of additional quantum numbers (such as charges and scalar hair) make their properties drastically different and brings about new phenomena. A new important feature of electrically (and/or magnetically) charged BHs, as well as of rotating BHs, is the unconventional thermodynamic behavior called Extremality. Extremal BHs are possibly stable gravitational objects with finite entropy but vanishing temperature, where the contribution to the gravitational energy comes from their electromagnetic and rotational (angular momentum-spin) attributes. The extremal situation entails a particular relation between entropy, charges and spin, since in this case the gravitational mass is not an independent quantity. For four-dimensional stationary and spherically symmetric BHs, in an environment of scalar fields, typically described by a non linear sigma-model, BHs have scalar 'hair' (scalar charges) which correspond to values of the moduli fields at (asymptotically flat-space) infinity. These values may vary continuously, since they represent the coordinates of an arbitrary point in the moduli space of the theory or, in a more geometrical language, a point in the target manifold of the scalar non-linear Lagrangian. Nevertheless the BH entropy, as given by the Bekenstein-Hawking entropy-area formula, remains independent of the scalar charges (no hair), and depends only on the asymptotic (generally dyonic) charges. The apparent puzzle is resolved by the "Attractor Mechanism", a fascinating phenomenon that combines Supersymmetry, BHs, Dynamical system, Algebraic geometry and even Number theory. For instance, in Type II superstring compactifications

to four dimension on Calabi-Yau manifolds, the low-energy dynamics is governed by (ungauged) $\mathcal{N} = 2$ supergravity coupled to vector and hyper-multiplets with no scalar potential. The corresponding dyonic BH solutions expose two different behaviors: the hyper-scalars can take arbitrary constant values while the radial evolution of the vector multiplet scalars is described by a dynamical system. Under some mild assumptions the scalar trajectory flows to a fixed point, located at the BH horizon radius, in the target (moduli) space. The attractive nature of the "fixed point", a point of vanishing phase velocity that represents the system being in equilibrium, is due to supersymmetry. Hence supersymmetric attractors, also known as BPS after Bogomolny-Prasad-Sommerfeld, are somehow reminiscent of the dynamical flows of dissipative systems. In approaching the fixed point, the orbit looses practically all memory of initial conditions (scalar hair), even though the dynamics is fully deterministic. As a result, the scalar fields at the BH horizon depend only on the dyonic asymptotic charges. For all BPS, $\mathcal{N} = 2$ critical points the scalars are fixed, and the resulting attractor varieties are of interest for both Algebraic Geometry and Number Theory. For "large BHs", the Einstein approximation is valid, a non-vanishing horizon area emerges (no naked singularities), and the entropy can be shown to depend solely on the BH gravitational mass (ADM mass) computed at the critical point, which is satisfied at the horizon. The horizon geometry has in this case a universal form, described by the Bertotti-Robinson metric, which is the product of a 2-dimensional Anti de-Stter space and a 2-sphere. Non-BPS extremal BHs exist as well, and in some specific cases they also show an attractor behavior. However in this case not all scalars of the vector multiplets flow to a fixed point, but some of them remain, at least in the classical approximation, as flat directions as is always the case with hypermultiplet scalars. In spite of this, the entropy of non BPS BHs enjoys the same property met in the supersymmetric case: it only depends on the dyonic charges and not on the continuous values of the moduli fields. In the case of target manifolds for scalars which are "symmetric spaces", the theory of attractors displays a beautiful connection with Group Theory and Differential Geometry. In this case the BPS or non-BPS nature of the BH attractors can be related to the nature of the orbits for the dyonic (asymptotic) charge vector. Different orbits correspond to fixed points of different BPS types. All non-flat directions are attractive, which mean the Hessian matrix of a certain BH effective potential function is semi-positive definite. A microscopic account of the BH entropy and other thermodynamic properties may require a more fundamental description in terms of the branes that naturally appear both in Supergravity and in String Theory.

The structure of this thesis is then the following. In chapter 2, we discuss some general preliminaries on Black Holes in Einstein General Relativity and then in Su-

pergravity theories. We give a general introduction to Supersymmetric Black Holes, BPS bound, Extremality and the horizon geometry before giving a pictorial representation of Attractor Mechanism at work for extreme RN BHs in $\mathcal{N} = 2$, $d = 4$ Maxwell Einstein Supergravity Theory. Finally we present a realization of AdS/CFT conjecture from low energy supergravity point of view by considering asymptotically flat D3-"black brane" as a solitonic solution of $\mathcal{N} = 2$, $d = 10$ Type IIB SUGRA.

In chapter 3, we recall the so-called thermodynamic properties of Black Holes, and give a general introduction to the Laws of Black Hole thermodynamics. We also invoke the concept of Hawking temperature and the semi-classical Bekenstein-Hawking Entropy Area formula. Then we present a physical picture of the Attractor Mechanism and towards the end of the chapter give an illustrative toy example sketching the Attractor Mechanism at work for $\mathcal{N} = 2$, $d = 4$ dilatonic BH of the heterotic string theory.

In chapter 4, we first present the field contents of $\mathcal{N} = 2$, $d = 4$ Maxwell Einstein Supergravity theory and show that the scalar fields z^i of the vector multiplets are solely responsible for the Attractor behavior and they can be regarded as arbitrary coordinates on a complex manifold M_{n_V} ($dim_{\mathbb{C}} M_{n_V} = n_V$), which is actually a special Kähler manifold. We present a fairly detailed introduction to Special Kähler Hodge geometry and the symplectic structure of the Moduli space. Then we make a digression and update the reader with nitty-gritty details of the mathematical formulation of the Electric Magnetic duality, Central Charge and Attractor Mechanism in this setting.

The structure of chapter 5 is the following. After a general introduction on Black holes and Constrained geodesic motion we reconsider Extreme Black Holes in the previously introduced n_V-fold $\mathcal{N} = 2$, $d = 4$ Maxwell-Einstein supergravity theory (Maxwell Einstein Supergravity Theory), i.e. a $\mathcal{N} = 2$, $d = 4$ supergravity theory in which the gravity multiplet is coupled to n_V Abelian vector supermultiplets, and therefore the overall gauge group is $(U(1))^{n_V+1}$. We then show how the (regular) Special Kähler geometry (SKG) of the moduli space of such a theory allows one to simplify the investigation of the critical points of the Effective Black Hole potential V_{BH}. We apply the formalism not only to Supersymmetric Attractors, but also give a flavor of the same in the non-supersymmetric case.

Chapter 6 studies the effective black hole potential V_{BH} of the most general $\mathcal{N} = 2$, $d = 4$ (*local*) special Kähler geometry with quantum perturbative corrections, consistent with axion-shift Peccei-Quinn symmetry and with cubic leading order behavior. Then we determine the charge configurations supporting axion-free attractor, and explain the differences among various configurations in relations

to the presence of "flat" directions of V_{BH} at its critical points and also elucidate the role of the sectional curvature at the non-supersymmetric critical points of V_{BH}, and compute the Riemann tensor (and related quantities), as well as the so-called E-tensor, which expresses the non-symmetricity of the considered quantum perturbative special Kähler geometry.

Chapter 7 discusses the sub-leading quantum perturbative corrections to $\mathcal{N} = 2$ cubic special Kähler geometries. We prove that imposing the invariance under axion-shifts, all such corrections (but the imaginary constant one) can be introduced or removed through suitable, lower unitriangular symplectic transformations, called Peccei-Quinn (PQ) transformations. Finally we stressed the important fact that since PQ transformations do not belong to the $d = 4$ U-duality group G_4, in symmetric cases they generally have a non-trivial action on the unique quartic invariant polynomial $\mathcal{I}_4$ of the charge representation **R** of G_4 which leads to interesting phenomena in relation to theory of extremal black hole attractors; i.e., the possibility to make transitions between different charge orbits of **R**, with corresponding change of the supersymmetry properties of the supported attractor solutions. Toward the very end of the chapter, after a brief account of the action of PQ transformations, we explain some new results on the recently established correspondence between Cayley's hyperdeterminant and elliptic curves, we derive an equivalent, alternative expression of $\mathcal{I}_4$, with relevant application to black hole entropy.

In **Chapter** 8 we switch gears and treat at length topics related to the analytic structure of massive gravitino propagator in four-dimensional de Sitter space. Incidentally this chapter seems altogether different from the rest of the thesis and is independent of considerations made in other chapters and so can be read as a separate chapter. Here we present a self-consistent analysis of spinor two-point functions for spin-1/2 and spin-3/2 fields in maximally symmetric spaces such as de Sitter(dS) spacetime, by using intrinsic geometric objects. We then obtain the Feynman, positive- and negative-frequency Green functions and eventually display the supercommutator and the Peierls bracket under such a setting in two- component-spinor language. We also complete, the hitherto unknown, explicit representation of the massive gravitino propagator in four-dimensional de Sitter space with the help of the general theory of the Heun equation. We find that, all weight functions multiplying the geometric invariants in the gravitino propagator can be expressed through Heun functions, and there exist two ranges of values of the independent variable in which the weight functions can be divided into dominant and sub-dominant families.

We round off in chapter 9 with an extensive general summary and outlook.

Bibliography

[1] J. Michell, *On the means of discovering distance, magnitude and c. of the velocity of light, in case of such diminution should be found to take place in any of them, and such other data should be procured from observations, as would be further necessary for that purpose*, Phil. Trans. R. Soc. London **74** 35 (1784).

[2] A. Müller, "Experimental evidence of black holes", in *School on Particle Physics, Gravity and Cosmology*, L. Bonora et. al eds., vol. PoS(P2GC)017. `astro-ph/0701228`

[3] S. W. Hawking, Comm. Math. Phys. 43, 199 (1975).

[4] J.M. Maldacena, *Black Holes in string theory.* PhD thesis, Princeton University,1996. `hep-th/9607235`

[5] Bryce DeWitt, *Quantum Gravity Yesterday and Today*, Gen.Rel.Grav.**41** 413, 2009, `arXiv:0805.2935`

[6] S. Ferrara, R. Kallosh and A. Strominger, "$\mathcal{N} = 2$ *Extremal Black Holes*", Phys. Rev. **D52**, 5412 (1995), `hep-th/9508072`.

[7] A. Strominger, " *Macroscopic Entropy of $\mathcal{N} = 2$ Extremal Black Holes*", Phys. Lett. **B383**, 39 (1996), `hep-th/9602111`.

[8] S. Ferrara and R. Kallosh, " *Supersymmetry and Attractors*", Phys. Rev. **D54**, 1514 (1996), `hep-th/9602136`.

[9] S. Ferrara and R. Kallosh, " *Universality of supersymmetric attractors*", Phys. Rev. **D54**, 1525 (1996), `hep-th/9603090`.

[10] S. Ferrara, G. W. Gibbons, and R. Kallosh, "Black holes and critical points in moduli space," Nucl. Phys. B500 (1997) 75-93, hep-th/9702103.

[11] G. W. Gibbons, "Supergravity vacua and solitons," in Duality and Supersymmetric Theories, D. I. Olive and P. C. West, eds., Publications of the Newton Institute, pp. 267-296. CUP, 1999. Prepared for A Newton Institute Euroconference on Duality and Supersymmetric Theories, Cambridge, England, Apr 1997

[12] K. Goldstein, N. Iizuka, R. P. Jena, and S. P. Trivedi, "Non-supersymmetric attractors," Phys. Rev. D72 (2005) 124021, hep-th/0507096.

[13] A. Sen, "Black hole entropy function and the attractor mechanism in higher derivative gravity," JHEP 09 (2005) 038, hep-th/0506177.

[14] A. Sen, "Entropy function for heterotic black holes," JHEP 03 (2006) 008, hep-th/0508042.

[15] F. Denef, "Supergravity flows and D-brane stability," JHEP 08 (2000) 050, hep-th/0005049.

[16] G. W. Moore, "Arithmetic and attractors," hep-th/9807087.

[17] K. Behrndt, G. L. Cardoso, B. de Wit, R. Kallosh, D. Lüst, and T. Mohaupt, "Classical and quantum N = 2 supersymmetric black holes," Nucl. Phys. B488 (1997) 236-260, hep-th/9610105.

[18] K. Behrndt, D. Lüst, and W. A. Sabra, "Stationary solutions of N = 2 supergravity," Nucl. Phys. B510 (1998) 264-288, hep-th/9705169.

[19] W. A. Sabra, "General static N = 2 black holes," Mod. Phys. Lett. A12 (1997) 2585-2590, hep-th/9703101.

[20] W. A. Sabra, "Black holes in N = 2 supergravity theories and harmonic functions," Nucl. Phys. B510 (1998) 247-263, hep-th/9704147.

[21] R. Kallosh, N. Sivanandam, and M. Soroush, "Exact attractive non-BPS STU black holes," Phys. Rev. D74 (2006) 065008, hep-th/0606263.

[22] L. Andrianopoli, R. D'Auria, E. Orazi, and M. Trigiante, "First order description of black holes in moduli space," arXiv:0706.0712 [hep-th].

[23] A. Ceresole and G. Dall'Agata, "Flow equations for non-BPS extremal black holes," JHEP 03 (2007) 110, hep-th/0702088.

[24] R. Figari, R. Hoegh-Krohn, and C. R. Nappi, "Interacting Relativistic Boson Fields in The De Sitter Universe with Two Spacetime Dimensions," Commun. Math. Phys. **44** 265.

[25] G. W. Gibbons and S. W. Hawking, "Cosmological Event Horizons, Thermodynamics, and Particle Creation," Phys. Rev. **D15** 2738 (1977).

[26] R. Bousso, "A Covariant Entropy Conjecture," JHEP **9907** (1999) 004, hep-th/9905177, "Holography In General Spacetimes," JHEP **9906** (1999) 028, hep-th/9906022, "Positive Vacuum Energy And The N-Bound," hep-th/0010252.

[27] T. Banks, "Cosmological Breaking Of Supersymmetry, or Little Lambda Goes Back To The Future II," hep-th/0007146.

[28] E. Witten, *Quantum Gravity in de Sitter space*, hep-th/0106109

[29] T. Banks, W. Fischler, S. Paban, *Recurrent Nightmares?: Measurement Theory in de Sitter Space*, JHEP 0212 (2002) 062, hep-th/0210160

[30] N. Goheer, M. Kleban, L. Susskind, *The trouble with de Sitter Space*, hep-th/0212209

[31] L. Dyson, M. Kleban, L. Susskind, *Disturbing Implications of a Cosmological Constant*, JHEP 0210 (2002) 011, hep-th/0208013

[32] P. Fre, M. Trigiante, A. Van Proeyen, *Stable de Sitter Vacua from $N = 2$ Supergravity*, Class. Quant. Grav. 19 (2002) 4167-4194

[33] S. Kachru, R. Kallosh, A. Linde, S. Triverdi, *de Sitter Vacua in String Theory*, hep-th/0301240

[34] G. 't Hooft, *Dimensional Reduction in Quantum Gravity*, gr-qc/9310006

[35] L. Susskind, *The World as a Hologram*, J. Math. Phys. 36 (1995) 6377-6396 , hep-th/9409089

[36] J. Maldacena, *The Large N Limit of Superconformal Field Theories and Supergravity*, Adv. Math. Phys. **2** 231, hep-th/9711200

[37] S. Gubser, I. Klebanov, A. Polyakov, *Gauge Theory Correlators from Non-critical String Theory*, Phys. Lett. B 428, (1998) 105, hep-th/9802109

[38] E. Witten, *Anti-de Sitter Space and Holography*, Adv. Theor. Math. Phys. 2 (1998) 253, hep-th/9802150

[39] O. Aharony, S. Gubser, J. Maldacena, H. Ooguri, Y. Oz, *Large N Field Theories, String Theory and Gravity*, Phys. Rept. 323 (2000) 183-386, hep-th/9905111

[40] A. Strominger, *The dS/CFT Correspondence*, JHEP 0110 (2001) 034, hep-th/0106113

[41] M. Spradlin, A. Volovich, *Vacuum States and the S-Matrix in dS/CFT*, Phys. Rev. **D65** (2002) 104037, hep-th/0112223

[42] K. Skenderis, *Lectures Notes on Holographic Renormalization*, Class. Quant. Grav. 19 (2002) 5849, hep-th/0204054

[43] M. Cadoni, P. Carta, *Tachyons in de Sitter Space and Analytical Continuation from dS/CFT to AdS/CFT*, hep-th/0211018

[44] N. Chernikov, E. Tagirov, *Quantum Theory of Fields in de Sitter Space-time*, Annales Poincare Phys. Theor. A9 (1968) 109; E. Tagirov, *Consequences of Field Quantization in de Sitter Type Cosmological Models*, Ann. Phys. 76 (1973) 561

[45] B. Allen, *Vacuum States in de Sitter Space*, Phys. Rev. D32 (1985) 3136

[46] E. Mottola, *Particle Creation in de Sitter Space*, Phys. Rev. D31 (1985) 754

[47] U. Danielsson, *A Note on Inflation and Transplanckian Physics*, Phys. Rev. D66 (2002) 023511, hep-th/0203198

[48] U. Danielsson, *Inflation, Holography and the Choice of Vacuum in de Sitter Space*, JHEP 0207 (2002) 040, hep-th/0205227

[49] N. Kaloper, M. Kleban, A. Lawrence, S. Shenker, L. Susskind, *Initial Conditions for Inflation*, JHEP 0211 (2002) 037, hep-th/0209231

[50] U. Danielsson, *On the Consistency of de Sitter Vacua*, JHEP 0212 (2002) 025, hep-th/0210058

[51] U. Danielsson, D. Dombert, M. Olsson, *Miracles and Complementarity in de Sitter Space*, hep-th/0210198

[52] T. Banks, L. Manneli, *de Sitter Vacua, Renormalization and Locality*, hep-th/0209113

[53] M. Parikh, I. Savonije, E. Verlinde, *Elliptic de Sitter Space: dS/Z_2*, hep-th/0209120

[54] M. Einhorn, F. Larsen, *Interacting Quantum Field Theory in de Sitter Vacua*, Phys. Rev. D67 (2003) 024001, hep-th/0209159

[55] A. Casher, P. Mazur, A. Staruszkiewicz, *De Sitter Invariant Vacuum States, Vertex Operators and Conformal Field Theory Correlators*, hep-th/0301023

[56] K. Goldstein, D. Lowe, *A Note on α-vacua and Interacting Field Theory in de Sitter Space*, hep-th/0302050

[57] F. Leblond, D. Marolf, R. Myers, *Tall Tales from de Sitter Space. I: Renormalization Group Flows*, JHEP 0206 (2001) 052, hep-th/0202094

[58] V. Balasubramanian, J. de Boer, D. Minic, *Exploring de Sitter Space and Holography*, hep-th/0207245

[59] B. Allen, T. Jacobson, *Vector Two-Point Functions in Maximally Symmetric Spaces*, Commun. Math. Phys. 103 (1986) 669-692

[60] B. Allen, C. Lütken, *Spinor Two-Point Functions in Maximally Symmetric Spaces*, Commun. Math. Phys. 106 (1986) 201-210

[61] B. Allen, M. Turyn, *An Evaluation of the Graviton Propagator in de Sitter Space*, Nucl. Phys. B292 (1987) 813; B. Allen, *The Graviton Propagator in Homogeneous and Isotropic Space-Times*, Nucl. Phys. B287 (1987) 743

[62] C. Pathinayake, A. Vilenkin, B. Allen, *Massless Scalar and Antisymmetric Tensor Fields in de Sitter Space*, Phys. Rev. D35 (1987) 3771

[63] I. Bena, *The Antisymmetric Tensor Propagator in AdS*, Phys. Rev. D62 (2000) 127901, hep-th/9910059

[64] I. Bena, *The Propagator for a General Form Field in AdS_{d+1}*, Phys. Rev. D62 (2000) 126008, hep-th/9911073

[65] I. Bena, H. Nastase, D. Vaman, *Propagators for p-forms in AdS_{2p+1} and Correlation Functions in the $AdS_7/(2,0)$ CFT Correspondence*, Phys. Rev. D64 (2001) 106009

[66] W. Mück, *Spinor Parallel Propagator and Green Function in Maximally Symmetric Spaces*, J. Phys. A33 (2000) 3021-3026, hep-th/9912059

[67] A. Lichnerowicz, *Propagateurs et Commutateurs En Relativité Générale*, Inst. Haut. Et. Sc. **10** (1961) 293; A. Lichnerowicz, *Champs Spinoriels et Propagateurs en Relativité Générale*, Bull. Soc. Math. Fr. **92** (1964) 11

[68] S. Deser, *Gauge Invariance with Mass: Higher Spins in Cosmological Spaces*, gr-qc/0202053; S. Deser, A. Waldron, *Inconsistencies of Massive Charged Gravitating Higher Spins*, Nucl. Phys. B631 (2002) 369, hep-th/0112182

Chapter 2

Black Holes in Supergravity

2.1 General considerations

The aim of the present chapter is to deal with black holes (BHs) in different space time dimensions and find their relations to supersymmetry (SUSY). On the same footing of monopoles, massless point-particles, massive charged particles and so on in Quantum Field theory (QFT), BHs are indeed at the heart of any theory of quantum gravity and play a central role for testing the correctness of these theories such as String theory and Loop Quantum Gravity for example.

In Einstein's GR a BH is nothing but a singular metric satisfying the Einstein equations. The simplest example of such a metric is given by the four-dimensional Schwarzschild metric solution. This is a spherically symmetric, static solution of the vacuum Einstein's equation $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = 0$ that follow from the Einstein Hilbert action action:

This metric is expected to describe the spacetime outside a gravitationally collapsed non-spinning star with zero charge. The solution for the line element is given by

$$ds^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu = -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{r}\right)} + r^2 d\Omega^2. \qquad (2.1.1)$$

where t is time, r is the radial coordinate and Ω is the solid angle on a 2-sphere. This metric appears to be singular at $r = 2GM$ as some of its components vanish or diverge. However, this is a well known fact that the singularity at $r = 2GM$ is not a real one but it is a coordinate artifact. The Riemann-Christoffel(RC) curvature tensor is well-behaved here. The surface $r = 2M$ (in natural units) is called the Event Horizon(EH) of BH. The EH is special in the sense that it is a quite particular sub-manifold of the 4-dim Schwarzschild background which is a null hypersurface i.e.

a codimension-1 surface locally tangent to the light cone structure. the normal four-vector n_μ to such a hypersurface is lightlike. If dx^μ is the set of tangent directions to the EH, then the covariant one-tensor n_μ satisfies the following relation:

$$n_\mu dx^\mu = 0 = n_\mu n^\mu = g^{\mu\nu} n_\mu n_\nu \qquad (2.1.2)$$

Thus n_μ is both normal and tangent to the EH and represent the direction along which the local light-cone structure is tangent to the EH, thus characterizing it as the boundary submanifold, topologically separating the "outer" part, where the light rays escape to infinity, from the "inner" part, where it is trapped.

The real singularity of the Schwarzschild metric appears at $r = 0$ where the RC tensor diverges as follows:

$$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{48G^2 M^2}{r^6} \qquad (2.1.3)$$

There is a strong principle in Black Hole physics called "Cosmic Censorship Principle (CCP)" that tells that for every point of the space-time continuum having a singular RC tensor, it should be "covered" by a surface named event horizon (EH) with the property of being an asymptotical locus for the dynamics of particle probes falling toward the singularity, and preventing any information from evading away from beneath the singularity to the rest of the universe through the horizon, thus forbidding existence of any "naked singularity". From this point of view, black holes are the solutions of Einstein equations exhibiting an EH in the Penrose diagram.

Two salient features related to the EH are its area A_H and the surface gravity κ_s. A_H is simply the area of the 2-sphere S^2 defined by the EH, while the surface gravity is constant on the horizon and is related to the force measured at spatial infinity holding a unit test charge in place or equivalently the red-shifted acceleration felt by a particle staying on the EH. Formally, κ_s is defined to be the coefficient that relates the Riemann-covariant directional derivative of the horizon normal four-vector n^μ along itself to n^μ, i.e.

$$n^\nu \nabla_\nu n^\mu = \kappa_s n^\mu \qquad (2.1.4)$$

At this point a question naturally crops in our mind : "How do we incorporate SUSY in such a framework?" and the answer is not difficult. As is well known, that GR can be made supersymmetric by adding a spin $s = \frac{3}{2}$ Rarita-Schwinger (RS) field, i.e. the Gravitino, to the field content of the GR theory, we consider. The resulting theory is the $\mathcal{N} = 1$ supergravity (SUGRA) theory. Clearly, setting the gravitino field to zero, the Schwarzschild metric is still a singular solution of the $\mathcal{N} = 1, d = 4$ SUGRA, as it is nothing but the bosonic sector of such a theory.

Nevertheless, it breaks SUSY, indeed no fermionic Killing symmetries are preserved by the Schwarzschild BH metric background. Mathematically speaking,

$$\delta_{\varepsilon(x)} \Psi_\mu \big|_{Schw.BH} = 0 \tag{2.1.5}$$

has no solutions with $\varepsilon(x)$ being the fermionic local SUSY transformation parameter, and Ψ_μ denoting the gravitino RS field. In general the Riemann flat metric backgrounds preserve SUSY. For example, 4-dim Minkowski space preserve four SUSYs related to four constant spinors which are the components of the 4-dim Majorana spinor, thus allowing one to include the fermionic Killing symmetries in the isometries of the manifold under consideration.

In summary, while the 4-dim Minkowski space preserves the four SUSYs corresponding to the constant spinors, the Schw. metric background does not have any fermionic isometry, and therefore, it breaks all the SUSY degrees of freedom (d.o.f.s). Of course, due to the asymptotically Minkowskian nature of the Schw. singular metric, such SUSY d.o.f.s are restored in the limit $r \to \infty$. This feature will characterize all singular spherically symmetric, static, asymptotically Minkowskian solutions to SUGRA field equations, that will be considered in the following section.

2.2 Supersymmetric Black Holes

The most general static, spherically symmetric, charged solution of the Einstein Maxwell theory given by the Lagrangian

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[R - F_{\mu\nu} F^{\mu\nu} \right] \tag{2.2.1}$$

gives the Reissner-Nordstrom (RN) black hole whose line element is given by

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)} + r^2 d\Omega^2 \tag{2.2.2}$$

This 4-dim RN BH metric reduces to the Schw. BH metric when the total charge Q is set to zero. Now 2.2.2 can be written as

$$ds^2 = -\frac{\Delta}{r^2} dt^2 + \frac{r^2}{\Delta} dr^2 + r^2 d\Omega^2 \tag{2.2.3}$$

where

$$\Delta = r^2 - 2Mr + Q^2 = (r - r_+)(r - r_-) \tag{2.2.4}$$

where $r_\pm$ are not necessarily real

$$r_\pm = M \pm \sqrt{M^2 - Q^2} \tag{2.2.5}$$

So in this case beside the real space-time singularity at $r = 0$ there are two other distinct "coordinate-singular" surfaces at $r_\pm$. The outer one placed at r_+, is called the "Cauchy horizon", while the one at r_- is called proper EH. A general comment here is that both Schw. and RN BHs belong to the large family of spherically symmetric, static, asymptotically flat 4-dim singular metric backgrounds of Maxwell-Einstein theory and thus they may be re-obtained from the stationary Kerr-Newman solution describing the most general asymptotically-flat *stationary* and *axi-symmetric* vacuum spacetime that is non-singular on and outside an event horizon, and obeys 'vacuum' Einstein-Maxwell equations.

Now the KN metric in *Boyer-Linquist coordinates* can be written as

$$
\begin{aligned}
ds^2 &= -\frac{\left(\Delta - a^2 \sin^2\theta\right)}{\Sigma} dt^2 - 2a\sin^2\theta \frac{\left(r^2 + a^2 - \Delta\right)}{\Sigma} dt\,d\phi \\
&\quad + \left(\frac{\left(r^2 + a^2\right)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2
\end{aligned} \tag{2.2.6}
$$

where

$$\Sigma = r^2 + a^2 \cos^2\theta \tag{2.2.7}$$
$$\Delta = r^2 - 2Mr + a^2 + e^2 \tag{2.2.8}$$

The three parameters are M, a, and e. It can be shown that

$$a = \frac{J}{M} \tag{2.2.9}$$

where J is the total angular momentum, while

$$e = \sqrt{Q^2 + P^2} \tag{2.2.10}$$

where Q and P are the electric and magnetic (monopole) charges, respectively. Now setting the parameter a to be zero we get back our favorite non-rotating dilatonic RN metric written as 2.2.3 and setting both the parameters a and e to zero we recover the Schw. metric given by 2.1.1.

Clearly, the reality of the radii 2.2.5 crucially depends on the ratio between the mass and the total electric charge of the RN-BH. Indeed $M^2 < Q^2$ implies non-existence of BH horizons and presence of space-time RN "naked" singularity, which

is thus prohibited by CCP as discussed before. The physical and reality condition on the radii thus, as dictated by CCP, is equivalent to the constraint

$$M^2 \geqslant Q^2 \tag{2.2.11}$$

which is stunningly similar to the Bogomol'ny-Prasad-Sommerfeld (BPS) bound for the stability of monopole solutions in spontaneously broken gauge theories, formulated in natural units as adopted here.

When the BPS-like condition arising from the CCP is "saturated", i.e. when

$$M^2 = Q^2, \tag{2.2.12}$$

the EH and the Cauchy Horizon coincide; and the resulting RN BH is said to become "extremal"[1] (or "extreme"), acquiring an extra feature of $\frac{1}{2}$-BPS SUSY-enhancement. Indeed, it may be rigorously proved that an extremal RN BH preserves 4 supersymmetries out of the total 8 related to the asymptotical $\mathcal{N} = 2$ Minkowski background[2]. Though a surprise, yet the appearance of the BPS-saturated bound (2.2.12) stems from the fact that the extremal RN BH metric background is a solitonic stationary solution of field equations in $\mathcal{N} = 2$, $d = 4$ Maxwell-Einstein supergravity theory (Maxwell Einstein Supergravity Theory).

For a generic RN BH, the surface gravity reads

$$\kappa_s = \frac{1}{2}\frac{r_+ - r_-}{r_+^2} = \frac{\sqrt{M^2 - Q^2}}{r_+^2}. \tag{2.2.13}$$

It is worth pointing out that in the case of a Schw. BH ($Q^2 = 0$, $r_+ = 2M$), the usual expression for the surface gravity of a massive star is recovered

$$\kappa_s = \frac{1}{4M}. \tag{2.2.14}$$

---

[1]In Sect. 5 we will give a general, equivalent characterization of extreme (and non-extreme) BHs, pointing out that extreme RN BHs are only a particular subset of the class of 4-d. static, spherically symmetric, asymptotically flat extreme BHs.

[2]A generalization to electrically and magnetically charged static BHs yields a BPS-like saturated bound of the kind

$$M^2 = Q^2 + P^2,$$

allowing one to interpret the considered s-t singularity as a Schwinger dyonic massive particle with electric charge Q and magnetic charge P (related by the Dirac-Schwinger quantization relation).

This is the first example of electric-magnetic duality, due to the $U(1)$-invariance of the classical Maxwell Eqs., corresponding to $SL(2, \mathbb{R})$-duality rotational covariance on the Abelian field strength F and its Hodge dual $*F$. In presence of n electric and n magnetic charges, the electric-magnetic duality group is enlarged to $Sp(2n, \mathbb{R})$ ([1], [2]). As it will be seen later, the existence of dyons is strictly related to the number of s-t dimensions being considered.

In what follows we will not explicitly consider magnetic charges, but such a fact will not touch the core and the generality of the whole treatment.

But the most interesting consequence of Eq. (2.2.13) is that the saturation (2.2.12) of the BPS bound implies the vanishing of the surface gravity. Actually, the extreme RN BH is just a particular example of 4-d. static, spherically symmetric and asymptotically flat extreme BHs, which, within such fundamental structural features, may be characterized as the most general $(U(1))^n$-charged class of singular Riemann backgrounds with vanishing surface gravity (with $n \in \mathbb{N}$).

As is well known, the $\mathcal{N} = 2$, $d = 4$ Maxwell Einstein Supergravity Theory may be obtained from the classical, non-SUSY, 4-d. Maxwell-Einstein theory (whose field content is given by the Riemann metric $g_{\mu\nu}$ and the Maxwell vector potential A_μ) just by adding two $s = \frac{3}{2}$ RS gravitino fields $\Psi^A_{\mu a}(x)$ ($A = 1, 2$ is the SUSY index, while μ and a are the Lorentz vector and spinor indices, respectively). Notice that, in such an approach to supersymmetrization, no extra bosonic fields are introduced; consequently, all non-SUSY solutions of Maxwell-Einstein theory (including RN BH) are also solutions of $\mathcal{N} = 2$ Maxwell Einstein Supergravity Theory, provided that fermions are set to zero.[3]

For generic values of the couple of parameters (M, Q), neither does the RN BH possess a regular Horizon geometry, nor does it preserve any of the 8 supersymmetries of the local maximal $\mathcal{N} = 2$, $d = 4$ SUSY algebra. The necessary condition to obtain a minimal regularity of the geometric structure in proximity of the Horizon(s) is expressed by the CCP BPS-like constraint (2.2.11).

The 8 supersymmetries related to the asymptotical maximally-SUSY Minkowski background in $\mathcal{N} = 2$, $d = 4$ Maxwell Einstein Supergravity Theory simply come from the existence of two Majorana spinors, each with 4 real components. Moreover, in $\mathcal{N} = 2$, $d = 4$ Maxwell Einstein Supergravity Theory it is possible to prove the CCP by using the local SUSY algebra in the same way the Positive Energy Theorem can be proved in GR with the use of SUSY. Roughly speaking, we may obtain the condition $M^2 \geqslant Q^2$ from the requirement of positivity of the operators appearing on the right-hand sides (r.h.s.'s) of the anticommutator of two supercharges in the RN BH metric background. The saturation of the CCP BPS-like bound (2.2.11) makes the RN BH "extremal", and allows one to obtain 4 independent solutions to the Killing spinor Eqs.

$$\delta_{\varepsilon(x)} \Psi^A_\mu \Big|_{extreme\ RN\ BH} = 0. \qquad (2.2.15)$$

---

[3]Such an argument is very powerful and versatile; for instance, it may be applied to disentangle some symmetry structures of ordinary pure QCD. In fact, such a theory (containing only gluons) may be supersymmetrized just by adding some $s = \frac{3}{2}$ fermionic fields; such an additive procedure makes nothing but explicit some hidden SUSY properties of the starting theory. For instance, this has been used in literature in the calculation of tree-level gluonic amplitude in pure QCD.

Thus, BPS-saturated RN BHs can be actually described in terms of massive charged particles, corresponding to (M, Q^2)-parameterized, pointlike representations of the $\mathcal{N} = 2, d = 4$ SUSY algebra. BPS-saturation implies nothing but the extreme RN BH solution to preserve one half of the supersymmetries related to 4-d. asymptotical Minkowski background.

Another fundamental feature of the $\mathcal{N} = 2$ ($d = 4$) extreme RN BHs is the restoration of maximal SUSY at the EH.

Denoting with $r_H \equiv r_+ = r_-$ the radius of the EH, for an arbitrary value $r > r_H$ of the radius the spherically-symmetric solutions of $\mathcal{N} = 2, d = 4$ Maxwell Einstein Supergravity Theory represented by extreme RN BHs preserve only one half of the 8 supersymmetries related to their asymptotical limit, i.e. to the 4-d. Minkowski space, and therefore to the associated $\mathcal{N} = 2, d = 4$ superPoincaré algebra. Going towards the EH, i.e. performing the limit $r \rightarrow r_H^+$, one gets a restoration of the previously lost 4 additional supersymmetries, reobtaining a maximally-symmetric $\mathcal{N} = 2$ metric background, for e.g. the 4-d. Bertotti-Robinson (BR) $AdS_2 \times S^2$ BH metric[4] ([3]-[5]).

It is instructive to explicitly show that the "near-Horizon" limit of the extreme RN BH metric in $d = 4$ is the BR metric $AdS_2 \times S^2$. First of all, let us BPS-saturate the 4-d. RN BH metric given by Eq. (2.2.2), by simply putting $M^2 = Q^2$

$$
\begin{aligned}
ds^2_{RN,extreme}(M) \quad &\equiv \quad ds^2_{RN}\left(M, Q^2\right)\Big|_{Q^2 = M^2} = \\
&= \quad -\left(1 - \frac{r_g(M)}{2r}\right)^2 dt^2 + \left(1 - \frac{r_g(M)}{2r}\right)^{-2} dr^2 + r^2 d\Omega.
\end{aligned}
$$

(2.2.16)

Eq. (2.2.16) describes a 1-parameter family of static, spherically-symmetric, asymptotically flat, charged singular metrics in $d = 4$. The metric functions diverge at two points, i.e. at $r = 0$ (real s-t singularity) and at $r_H \equiv r_g(M)/2$ (EH), where $r_g(M) \equiv 2M$ is the Schwarzchild radius. It is worth noting that the charged nature of the extreme RN BH decreases the radial coordinate of the EH, which is now at one half of the value related to the corresponding uncharged Schw. BH with same mass.

---

[4]Actually, the BR metric provides the first example of the celebrated Maldacena's AdS/CFT conjecture, i.e. the $AdS_2/CFT1$ case. Indeed, the dynamics of superstring theories in the bulk of AdS_2 may be associated with a supersymmetric conformal field theory on the 1-d. boundary of such a space, i.e. with the superconformal (quantum) mechanics (see e.g. [6], [7] and [8]).

Redefining $r_H \equiv r'_g \equiv r_g(M)/2$, and dropping the prime and the notation of the dependence on M, we get

$$
\begin{aligned}
ds^2_{RN,extreme}(M) &= -\left(1 - \frac{r_g}{r}\right)^2 dt^2 + \left(1 - \frac{r_g}{r}\right)^{-2} dr^2 + r^2 d\Omega = \\
&= -\frac{1}{r^2}(r - r_g)^2 dt^2 + r^2(r - r_g)^{-2} dr^2 + r^2 d\Omega.
\end{aligned}
$$
(2.2.17)

By performing the limit $r \to r_g^+$ and considering only the leading order, we therefore obtain

$$
lim_{r \to r_g^+}\left[ds^2_{RN,extreme}(M)\right] = -\frac{1}{r_g^2}(r - r_g)^2 dt^2 + r_g^2(r - r_g)^{-2} dr^2 + r_g^2 d\Omega. \quad (2.2.18)
$$

The mass of the spherically-symmetric BR geometry is related to the area $A_H = 4\pi r_g^2$ of its EH by the simple relation

$$
M^2_{BR} = \frac{A_H}{4\pi} = r_g^2; \quad (2.2.19)
$$

by substituting such a relation in Eq. (2.2.18), we get

$$
lim_{r \to r_g^+}\left[ds^2_{RN,extreme}(M)\right] = -\frac{1}{M^2_{BR}}(r - r_g)^2 dt^2 + M^2_{BR}(r - r_g)^{-2} dr^2 + M^2_{BR} d\Omega. \quad (2.2.20)
$$

Now, by performing the change of radial variable

$$
r' \equiv r - r_g \quad (2.2.21)
$$

and dropping out the prime once again, we get the following expression:

$$
lim_{r \to 0^+}\left[ds^2_{RN,extreme}(M)\Big|_{r^{(\prime)} \equiv r - r_g}\right] = -\frac{r^2}{M^2_{BR}} dt^2 + \frac{M^2_{BR}}{r^2}\left(dr^2 + r^2 d\Omega\right). \quad (2.2.22)
$$

It is easy to recognize that this is nothing but the BR metric $AdS_2 \times S^2$, with opposite scalar curvatures for AdS_2 and S^2. Indeed, the metric given by Eq. (2.2.22) belongs to the general class of static 4-d. black hole metrics of the kind

$$
ds^2 = -e^{2U(\underline{x})} dt^2 + e^{-2U(\underline{x})} d\underline{x}^2, \quad (2.2.23)
$$

with $U(\underline{x})$ satisfying the 3-d. D'Alembert equation

$$
\Delta e^{-U(\underline{x})} = 0. \quad (2.2.24)
$$

In particular, the spherically-symmetric BR metric corresponds to the choice

$$e^{-2U(\underline{x})} = \frac{A_H}{4\pi \, |\underline{x}|^2} = \frac{M_{BR}^2}{r^2},$$ (2.2.25)

which consequently relates $U(\underline{x})$ to the Newtonian gravitational potential (see Subsects. 5.1 and **??**).

Notice that the change of radial coordinate specified by Eq. (2.2.21) encodes the very relationship between the extremal RN BH and the BR metric background: indeed Eq. (2.2.21) yields that the real s-t singularity of the BR geometry is on the EH of the extreme RN BH, which, as previously observed, is at one half of the gravitational radius of the Schw. BH of the same mass. Consequently, the BR geometry may be seen as the "near-Horizon" asymptotical metric structure of the extreme RN BH[5]; the r.h.s. of Eq. (2.2.22) should always be considered for small values of the radius (i.e. for $r \to 0^+$), physically corresponding to the proximity to the EH of the extreme RN BH.

The BR metric $AdS_2 \times S^2$ yielded by Eq. (2.2.22) corresponds to the direct product of two spaces of constant (and opposite) Riemann-Christoffel scalar curvature. Consequently, it is R-flat, and it may be also shown that it is conformally-flat, i.e. that all components of the related Weyl tensor vanish. Such a peculiar feature may be made manifest by choosing a suitable system of coordinates, called "conformal coordinates", defined as follows:

$$\rho \equiv \frac{M_{BR}^2}{r} \Leftrightarrow \left|\underline{y}\right| \equiv \frac{M_{BR}^2}{|\underline{x}|}.$$ (2.2.26)

By exploiting such a change of coordinates, we finally get

$$
\begin{aligned}
lim_{\rho \to \infty} \left[ds_{RN,extreme}^2 (M) \Big|_{\rho \equiv \frac{M_{BR}^2}{r}} \right] &= -\frac{M_{BR}^2}{\rho^2} dt^2 + \frac{M_{BR}^2}{\rho^2} \left(d\rho^2 + \rho^2 d\Omega \right) = \\
&= \frac{M_{BR}^2}{\left|\underline{y}\right|^2} \left(-dt^2 + d\underline{y}^2 \right),
\end{aligned}
$$ (2.2.27)

which is manifestly conformally flat, as it can be also seen by explicitly checking that the Weyl tensor vanishes

$$C_{\mu\nu\lambda\delta} = 0.$$ (2.2.28)

---

[5] In Subsects. 5.1 and **??** we will see that such a result may be extended to a generic (4-d., static, spherically symmetric and asymptotically flat) extreme BH.

Notice that the conformal coordinates make the conformal flatness of the BR metric manifest by giving a stereographic treatment of the singularity, because they map the real s-t singularity at $r = 0$ to the point at the infinity $\rho \to \infty$.

The phenomenon of the doubling of the SUSY near the EH was discovered for the first time in Maxwell Einstein Supergravity Theory in [9] (see [10] for an introductory report and further References). As we will see later, it is related to the appearance of a covariantly-constant on-shell superfield of $\mathcal{N} = 2$ ($d = 4$) SUGRA [11]. In presence of a dilaton such a mechanism was studied in [12]. In the context of exact 4-d. BHs, string theory and conformal theories on the world-sheet, the BR metric has been studied in [13]. Finally, the idea of extremal, singular p-branes metric configurations interpolating between maximally-symmetric asymptotical backgrounds has been developed in [14].

Therefore, for what concerns the SUSY-preserving features of the considered extreme RN BHs, there is a strong similarity between the asymptotical ($r \to \infty$) and "near-Horizon" ($r \to r_H^+$) limits. They share the identical property corresponding to maximally-SUSY metric backgrounds in 4 dimensions, thus preserving 8 different supersymmetries, although they do deeply differ on the algebraic side. The asymptotical 4-d. Minkowski flat background is associated to the $\mathcal{N} = 2$, $d = 4$ super-Poincaré algebra (rigid SUSY asymptotical algebra). Instead, the Horizon geometry has an $AdS_2 \times S^2$ structure of direct product of two spaces with non-vanishing, constant (and opposite) curvature, and it is associated to another 4-d. maximal $\mathcal{N} = 2$ SUSY algebra, i.e. to $\mathfrak{psu}(1,1|2)$.

$\mathfrak{psu}(1,1|2)$ is an interesting example of superalgebra containing neither Poincaré nor semisimple groups, but (direct products of) simple groups as maximal bosonic subalgebra (m.b.s.). Indeed, in this case the m.b.s. is $\mathfrak{so}(1,2) \oplus \mathfrak{su}(2)$, with related maximal spin bosonic subalgebra (m.s.b.s.) $\mathfrak{su}(1,1) \oplus \mathfrak{su}(2)$. This perfectly matches the corresponding bosonic isometry group of the BR metric, which is nothing but the direct product of a 2-d. hyperboloid and a 2-sphere

$$AdS_2 \times S^2 = \frac{SO(1,2)}{SO(1,1)} \times \frac{SO(3)}{SO(2)}. \tag{2.2.29}$$

Summarizing, it may be shown that the $\mathcal{N} = 2$, $d = 4$ extreme RN BH is a $\frac{1}{2}$-BPS SUSY-preserving soliton solution in $\mathcal{N} = 2$, $d = 4$ Maxwell Einstein Supergravity Theory. It interpolates between two maximally supersymmetric metric backgrounds, i.e. Minkowski for $r \to \infty$ and BR for $r \to r_H^+$, related to two different 4-dim. $\mathcal{N} = 2$ superalgebras, i.e. respectively to the rigid $\mathcal{N} = 2$, $d = 4$ SUSY algebra
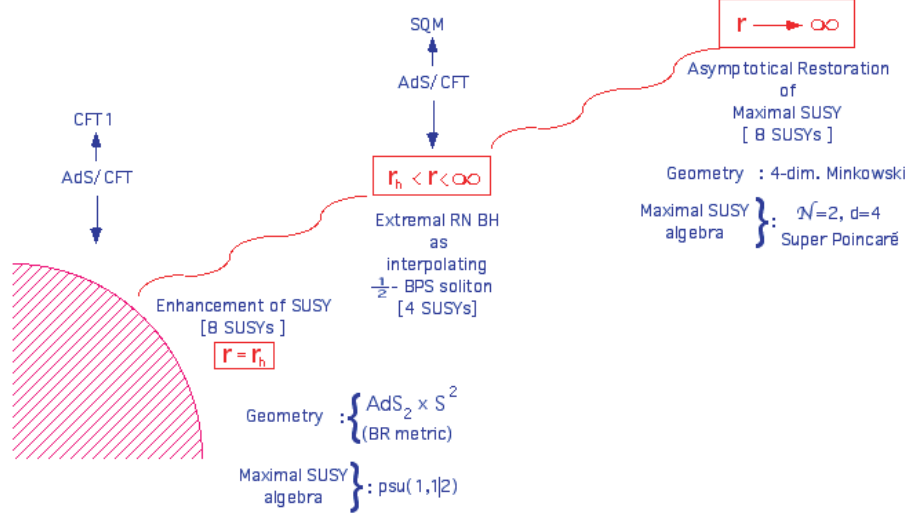
Figure 2.1: The $d = 4$ extreme RN BH as a $\frac{1}{2}$-BPS SUSY-preserving soliton solution in $\mathcal{N} = 2, d = 4$ Maxwell Einstein Supergravity Theory. It interpolates between two maximally supersymmetric metric backgrounds, i.e. Minkowski (related to the rigid $\mathcal{N} = 2, d = 4$ superPoincaré algebra) for $r \to \infty$ and Bertotti-Robinson (related to the $\mathfrak{psu}(1, 1 \,|2)$ superalgebra) for $r \to r_H^+$. SQM stands for supersymmetric (but not superconformal) quantum mechanics, related by ADS/CFT correspondence to the interpolating regime of the considered RN extremal BH.

given by the superPoincaré algebra and to the $\mathfrak{psu}(1, 1 \,|2)$ superalgebra[6]. See Fig. 2.1 for a graphical synthesis.

There exists an interesting connection with the statistical mechanics of dynamical systems, which will be amply treated in the following Sections; here we anticipate that the radius r_H of the EH of the extreme RN BH may be considered as an "attractor" for the evolution dynamics of the (scalar fields of the) physical system being considered, corresponding to the restoration of maximal SUSY.

---

[6]$\mathcal{N} = 2, d = 4$ superPoincarè and $\mathfrak{psu}(1, 1 \,|2)$ are the only superalgebras compatible with the constraint of asymptotically flat metric background in the considered case.

The situation drastically changes when one removes such a constraint (i.e. when generic, asymptotically Riemann geometries are considered). For example, asymptotical maximally symmetric metric configurations could be considered; among the Riemann manifolds with non-zero constant Riemann-Christoffel intrinsic scalar curvature, one of the most studied in such a framework is the anti De Sitter (AdS) space. When endowing the AdS background with some local SUSY, one obtains a particular case of the so-called "gauged" SUGRAs.

Generalizations of the previous treatment to the case of p-dim. objects in d s-t dimensions are also possible. Nevertheless, as we will discuss later, it may be shown that for $d \geqslant 6$ it is not possible to have regular (generalized) Horizon geometries, and the calculations of the entropy of the considered (possibly extended) s-t singularities always give vanishing (or unphysical constant) results. The aforementioned case of the extreme RN BH is a particular example of $p = 0$-dim. brane in $d = 4$ s-t dimensions, and, as shown by Gibbons and Townsend in [14], BR geometry is nothing but a $p = 0$-"black brane".

In general, a p-dim. extreme "black brane" in d s-t dimensions is an extended p-dim. object saturating a suitable generalization of the BPS bound (2.2.12), for which the $(p+1)$-dim. generalization of EH may be construed, together with a dimensionally extended version of the CCP. Also notice that in this case the real s-t singularity extends over a p-dim. (hyper)volume in s-t. The "near-Horizon" asymptotical geometry of a p-dim. "black brane" is given by the (p, d)-generalization of BR metric, i.e. by the direct product

$$AdS_{p+2} \times S^{d-p-2}. \tag{2.2.30}$$

In general, the request of asymptotically Minkowski d-dim. s-t geometry in presence of a p-brane implies the consistency condition [15]

$$p < d - 3. \tag{2.2.31}$$

Moreover, in d s-t dimensions an electric p-brane has a $(d - p - 4)$-brane as magnetic dual. In the particular case in which the dimensions of an electric brane and of its magnetic dual coincide, i.e. when the pair (p, d) satisfies the condition

$$\frac{d}{2} = p + 2, \tag{2.2.32}$$

the considered p-brane can be dyonic, i.e. it may have both electric and magnetic charge, respectively denoted with e and m. Finally, when the p satisfying the dyonic condition (2.2.32) is odd, the related p-brane may be self- (or anti-self-)dual, i.e. with $e = \pm m$, depending on the projective (or anti-projective) nature of the Hodge $*$-operator

$$(*)^2 = \pm \mathbb{I}. \tag{2.2.33}$$

Therefore, in $d = 4$ the only possible choice is $p = 0$, corresponding to the extreme BHs. Moreover, the couple $(p, d) = (0, 4)$ satisfies the dyonic condition

(2.2.32), but p is not odd. Consequently, in $d = 4$ the 0-brane may be dyonic, but not self-(or anti-self-)dual. In other words, the extreme BH, such as the extreme RN one, may have simultaneously electric and magnetic charge, but they will not be related by the simple relation $e = \pm m$.

For $d = 5$ the condition (2.2.31) yields $p = 0, 1$ as allowed values. The relation (2.2.32) is never satisfied, therefore 5-dim. dyons do not exist.

1) $p = 0$ corresponds to the Tangherlini extreme BH ([21], [22]); its "near-Horizon" geometry corresponds to $AdS_2 \times S^3$, admitting two Killing spinors. Moreover, by AdS/CFT it corresponds to completely solvable superconformal field theory (*SCFT*2) on the 2-d. Minkowski manifold corresponding to the boundary of AdS_3.

2) $p = 1$ corresponds to a "black-string" in 5 dimensions, which is the magnetic dual of the Tangherlini extreme BH. It has an $AdS_3 \times S^2$ "near-Horizon" geometry and, by application of the AdS/CFT correspondence, it yields a completely solvable superconformal quantum mechanics (*SCFT*1).

2.3 A Prelude to AdS/CFT

The most popular realization of Maldacena's AdS/CFT correspondence is given by the 10-dim. manifold $AdS_5 \times S^5$. By the previous reasonings, this may correspond to the "near-Horizon" geometry of a 3-"black-brane" in a 10-dim. s-t. It is worth noticing that, by the previous analysis, in $d = 10$ the asymptotical flatness implies $0 \leqslant p \leqslant 6$, and the dyonic condition (2.2.32) holds true for the odd value $p = 3$. Therefore, a 3-"black-brane" in $d = 10$ may be dyonic, with $e = \pm m$, depending on the projectivity (or anti-projectivity) of the 10-dim. Hodge $*$-operator.

Actually, $AdS_5 \times S^5$ describes the "near-Horizon" geometry of a D3-brane in $\mathcal{N} = 2$, $d = 10$ Type IIB SUGRA[7]. In such a context, the flat asymptotical ($r \to \infty$) geometry is the 10-d. Minkowski space with the associated maximally symmetric $\mathcal{N} = 2$, $d = 10$ rigid superPoincaré algebra (32 supersymmetries, related to the existence of two Majorana-Weyl spinors, each with 16 real components). On the other side, also $AdS_5 \times S^5$ is maximally supersymmetric, being related to the $\mathfrak{psu}(2, 2\,|4)$ superalgebra[8] (with 32 real fermionic generators).

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[7]We do not consider Type IIA SUGRA simply because it does not admit D3-"black-branes" as solutions. In general, the p-dim. "black-brane" solutions have p even in IIA and p odd in IIB theories.

[8]The considered Lie superalgebras $\mathfrak{psu}(1, 1\,|2)$ and $\mathfrak{psu}(2, 2\,|4)$ belong to the so-called unitary series of superalgebras $\mathfrak{psu}(n_1, n_2|m)$, admitting $\mathfrak{su}(n_1, n_2) \oplus \mathfrak{su}(m) \oplus (1 - \delta_{n_1+n_2,m})\,\mathfrak{u}(1)$ as m.s.b.s..

$\mathfrak{psu}(2,2\,|4)$ is another example of superalgebra containing neither Poincaré nor semisimple groups, but (direct products of) simple groups as m.b.s.; indeed, in this case the m.b.s. and m.s.b.s. are respectively $\mathfrak{so}(4,2) \oplus \mathfrak{so}(6)$ and $\mathfrak{su}(2,2) \oplus \mathfrak{su}(4)$, and there is a perfect match with the corresponding bosonic isometry group of $AdS_5 \times S^5$, which is nothing but the direct product of a 5-d. hyperboloid and a 5-sphere

$$AdS_5 \times S^5 = \frac{SO(4,2)}{SO(4,1)} \times \frac{SO(6)}{SO(5)}. \tag{2.3.1}$$

Notice that the isometry group $SO(4,2)$ of AdS_5 is nothing but the conformal group in 4 dimensions, i.e. the symmetry group of the $\mathcal{N} = 4$ Super Yang-Mills (SYM) gauge theory on the 4-dim. Minkowski space corresponding to the boundary of the 5-dim. hyperboloid AdS_5. Thus, the conformally invariant 4-dim. $\mathcal{N} = 4$ SYM gauge theory stands to the embedding of a D3-"black brane" in a 10-dim. (asymptotically flat) s-t, as the superconformal quantum mechanics ($SC\,(Q)\,M\,=\,CFT1$) stands to an extreme BH, eventually of the extremal RN type treated above, in 4-d. (asymptotically flat) space-time.

Such cases are different realizations of the AdS/CFT, which conjectures a close (holographic) duality between gravity theories (superstrings and their low-energy limit given by SUGRA theories) in the bulk of AdS manifolds and strongly coupled, conformally invariant gauge theories on the flat Minkowskian boundaries of such spaces.

Thus, as shown in Fig. 2.2, the considered asymptotically flat D3-"black brane" is a solitonic solution of $\mathcal{N} = 2, d = 10$ Type IIB SUGRA, which interpolates between two maximally supersymmetric metric backgrounds, i.e. Minkowski at $r \to \infty$ (by construction) and $AdS_5 \times S^5$ (which may be seen as an higher-dim. generalization of BR metric) in the "near-Horizon" limit. It corresponds to a consistent $\frac{1}{2}$-BPS solution, therefore preserving 16 supersymmetries out of the 32 related to the maximally SUSY backgrounds.

It is worth noticing that such a $\frac{1}{2}$-BPS solution can still be interpreted in terms of a $\mathcal{N} = 4$ SYM gauge theory, but the conformal invariance is lost (or better, spontaneously broken) for a generic value of[9] $r_H < r < \infty$. This is due to the fact that

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In general, Lie SUSY algebras admit a classification similar to their non-supersymmetric counterparts (see e.g. [16]-[20]). For instance, beside the exceptional cases, another infinite series of Lie superalgebras is the orthosymplectic one, i.e. $\mathfrak{osp}\,(n_1,n_2|m)$, admitting $\mathfrak{so}(n_1,n_2) \oplus \mathfrak{sp}(2m)$ as m.s.b.s..

In general, the fermionic generators are in the bi-fundamental representation of the corresponding superalgebra, e.g. in $(n_1 + n_2, m)$-repr. for both $\mathfrak{psu}\,(n_1,n_2|m)$ and $\mathfrak{osp}\,(n_1,n_2|m)$.

[9]r_H now stands for (the set of parameters specifying) the suitable generalization of the EH in the case of spatially-extended s-t singularities embedded in higher dimensions.

not conf.-inv.
$\mathcal{N}$=4 SYM

$\mathcal{N}$=4 SYM

AdS/CFT

AdS/CFT

$r \longrightarrow \infty$

Asymptotical Restoration
of
Maximal SUSY
[32 SUSYs]

Geometry : 10-dim. Minkowski

$r_h < r < \infty$

Maximal SUSY
algebra

$\mathcal{N}$=2, d =10
Super Poincaré

Extremal D3"black-brane"
as
interpolating
$\frac{1}{2}$ - BPS soliton
[16 SUSYs]

Enhancement of SUSY
[32 SUSYs]

$r = r_h$

Geometry :
$AdS_5 \times S^5$
[(p,d) =(3,10)-generalization]
of BR metric

Maximal SUSY
algebra
$:$ psu(2,2|4)
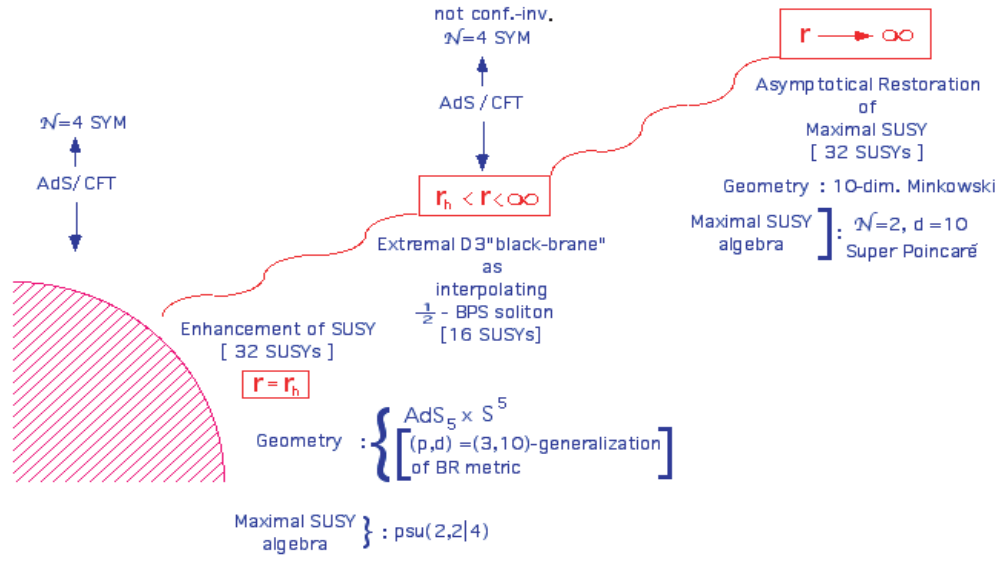
Figure 2.2: The asymptotically flat D3-"black brane" as a $\frac{1}{2}$-BPS SUSY-preserving soliton solution in $\mathcal{N} = 2$, $d = 10$ Type IIB SUGRA. It interpolates between two maximally supersymmetric metric backgrounds, i.e. 10-d. Minkowski (related to the rigid $\mathcal{N} = 2$, $d = 10$ superPoincaré algebra) for $r \to \infty$ and $AdS_5 \times S^5$ (related to the $\mathfrak{psu}(2,2\,|4)$ superalgebra) for $r \to r_H^+$.

for $r_H < r < \infty$ the $\mathcal{N} = 4$ SYM gauge theory "living" on the boundary may be approximately described in terms of a Born-Infeld action, containing higher-order derivative terms which (spontaneously) break conformal invariance. The conformal invariance of the 4-d. $\mathcal{N} = 4$ SYM gauge theory defined on the boundary manifold is restored only in the "near-Horizon" limit, i.e. when $r \to r_H^+$, and therefore when the bulk tends to a direct product structure $AdS_5 \times S^5$. The restoration of the maximal supersymmetry of the metric background at the (generalized) EH (from 16 to 32 preserved supersymmetries) yields an enhancement of the symmetry features exhibited by the (holographically) related "boundary" (strongly-coupled) $\mathcal{N} = 4$ SYM gauge theory, which correspondingly becomes conformally invariant.

Concluding, in d-dim. $\mathcal{N}$-extended SUGRAs there exist stable (i.e. BPS-saturated), static, spherically symmetric, asymptotically flat $p \, (< d - 3)$-dim. solitonic metric background solutions. They interpolate between two maximally supersymmetric backgrounds, i.e. the d-dim. flat Minkowski space in the limit $r \to \infty$, and the d-dim. generalized BR geometry. The latter is obtained in the "near-Horizon" limit $r \to r_H^+$, and it may be expressed as the direct product of a constant, (strictly) negative-curvature space (the $(p + 2)$-dim. hyperboloid, or anti de Sitter space $AdS_{p+2} = \frac{SO(p+1,2)}{SO(p+1,1)}$) and of a constant, (strictly) positive-curvature space (the $(d - p - 2)$-dim. sphere $S^{d-p-2} = \frac{SO(d-p-1)}{SO(d-p-2)}$).

Depending on the number of (real) supersymmetries preserved by the maximal backgrounds (and therefore depending on d and $\mathcal{N}$), the interpolating solitonic solutions may have different BPS SUSY-preserving features. Despite being extremal (i.e. saturating - a suitable generalization of - the BPS-like bound (2.2.11)), they may also be non-BPS, i.e. they may also not preserve any of the supersymmetries of the two regimes considered above. For example, in 4-dim. $\mathcal{N} = 8$-extended SUGRA (having 32 real fermionic generators) we may have $\frac{1}{2}$-BPS, $\frac{1}{4}$-BPS, $\frac{1}{8}$-BPS and non-BPS stable (i.e. BPS-saturated) singular solitonic metric backgrounds, with 16, 8, 4 and 0 supersymmetries preserved out of 32, respectively.

Lastly, it is possible to classify the BPS-preserving features of such solutions in an invariant way, using the lowest-order invariants and the orbits of the U-duality symmetry groups of the starting SUGRA theory. Such groups are Lie non-compact exceptional groups of various ranks and correspond to the isometry groups of the manifold of the non-linear sigma model related to the relevant set of scalars. Such a manifold is nothing but a moduli space of the considered SUGRA theory. The process of restoration of maximal SUSY in the "near-Horizon" dynamics of the considered system is deeply related to the "Attractor Mechanism" in the moduli space.

Chapter 3

Black Hole Entropy and Attractors

One of the remarkable properties of black holes is that one can derive a set of laws of black hole mechanics which bear a very close resemblance to the laws of thermodynamics. This is quite surprising because a priori there is no reason to expect that the spacetime geometry of black holes has anything to do with thermal physics.

3.1   Laws of Black Hole Mechanics and the concept of Hawking Temperature

(1) Zeroth Law : In thermal physics, the zeroth law states that the temperature T of a body at thermal equilibrium is constant throughout the body. Otherwise heat will flow from hot spots to cold spots. Correspondingly for black holes one can show that the surface gravity κ is constant on the event horizon. This is obvious for spherically symmetric horizons but is also true for more general non-spherical horizons of spinning black holes.

(2) First Law : As is well known, the first law of thermodynamics reads

$$\delta E = T\delta S - p\delta V, \tag{3.1.1}$$

and expresses the total variation of the energy E as equal to the temperature T times the variation of the entropy S, plus other work terms, such as a term proportional (through the pressure p) to the change of the volume V of the considered system. The corresponding formula for BHs is [23]

$$\delta M = \frac{\kappa_s}{4\pi}\frac{\delta A_H}{4} + \phi\delta q + \omega\delta J. \tag{3.1.2}$$

It states that the variation of the mass M of the BH is related to the variation of the EH area A_H, with two kind of additional terms: a work term proportional (through the rotational angular frequency ω) to the variation of the total angular momentum J, and another term proportional (through the electric/magnetic potential ϕ evaluated at the Horizon) to the variation of the charge q.

Second Law : In a physical process the total entropy S never decreases, $\Delta S \geq 0$. Correspondingly for black holes one can prove the area theorem that the net area never decreases, $\Delta A \geq 0$. For example, two Schwarzschild black holes with masses M_1 and M_2 can coalesce to form a bigger black hole of mass M. This is consistent with the area theorem since the area is proportional to the square of the mass and $(M_1 + M_2)^2 \geq M_1^2 + M_2^2$. The opposite process where a bigger black hole fragments is however disallowed by this law.

The formal analogy is actually much more than what it seems. Bekenstein and Hawking discovered that there is a deep connection between black hole geometry, thermodynamics and quantum mechanics.

Hawking ([24], [25], [26]) has shown that $\frac{\kappa_s}{4\pi}$ can be interpreted precisely as the temperature of the BH

$$T_{BH} = \frac{\kappa_s}{4\pi}. \tag{3.1.3}$$

Therefore, by comparing Eqs. (3.1.1) and (3.1.2), one obtains the Bekenstein-Hawking entropy-area (BHEA) formula, relating the entropy S of a s-t singularity with the area A_H of its EH (that should be always there, if one forbids the existence of "naked" singularities by advocating the CCP)

$$S = \frac{A_H}{4}. \tag{3.1.4}$$

In Eqs. (3.1.2) and (3.1.4) the various quantities have been defined in Planck units, i.e. they have been made dimensionless by multiplication with an appropriate power of Newton's constant G_0 (recall we set $\hbar = c = G_0 = 1$). By recalling that such a constant appears in the Einstein-Hilbert Lagrangian density

$$\mathcal{L}_{EH} = \frac{1}{16\pi G_0} \sqrt{|g|} R, \tag{3.1.5}$$

it is clear that the chosen normalization makes all quantities appearing in the first law of BH mechanics independent of the scale of the metric.

In the case of extreme BHs in SUGRA theories, the formula (3.1.4) may be macroscopically determined by using the U-duality symmetries of string theories encoded

in the SUGRA low-energy actions. More specifically, the classical Einstein-Maxwell theory may be naturally embedded into $\mathcal{N} = 2$ Maxwell Einstein Supergravity Theory, leading to extensions involving a number of Abelian gauge fields and a related variety of massless scalar moduli fields. The BH mass M will, in general, depend on the values taken by the moduli at the spatial infinity, and therefore additional terms on the r.h.s. of Eq. (3.1.2) will appear.

For Schw. BHs the only relevant parameter is clearly the mass M, and, beside Eq. (2.2.14), we get the relation[1]

$$A_H = 16\pi M^2 = 4\pi r_{H,Schw.}^2, \tag{3.1.6}$$

where $r_{H,Schw.} \equiv r_g(M) \equiv 2M$. By differentiation, Eq. (3.1.6) is consistent with Eq. (3.1.2) constrained by $(\delta)\,q = 0 = (\delta)\,J$.

For the RN BH, the situation is more involved, due to the previously performed classification based on the ratio between M and q. As previously pointed out, for extreme RN BHs (i.e. with $M = |q|$), the surface gravity vanishes; the other relevant relations read

$$A_H = 4\pi M^2 = 4\pi r_{H,extreme\ RN}^2, \quad \phi = \sqrt{\frac{4\pi}{A_H}}\,q = \frac{q}{r_{H,extreme\ RN}}, \tag{3.1.7}$$

where $r_{H,extreme\ RN} \equiv M = \frac{r_{H,Schw.}}{2}$. As it has to be, by differentiating, we obtain consistency with Eq. (3.1.2) constrained in the subspace of static, extreme RN BHs (i.e. with $\delta J = 0$ and $\delta M = \delta q$). Since in this case $\kappa_s = 0$, and therefore the extreme RN BHs, as all extreme BHs, have $T_{BH} = 0$, by the "BH counterpart" of the third law of thermodynamics one would expect that the entropy vanishes. Clearly, this is not the case, because Eq. (3.1.7) yields that the area of the Horizon remains finite for zero surface gravity (and thus, by Eq. (3.1.3), for $T_{BH} = 0$), and the BHEA (3.1.4) still holds, yielding

$$S_{BH} = \pi M^2 = \pi r_{H,extreme\ RN}^2. \tag{3.1.8}$$

3.2   Attractor Mechanism : A Propaedeutic Introduction

This part of the thesis deals with a general dynamical principle, named "Attractor Mechanism" (AM), which governs the dynamics inside the moduli space, and therefore allows one to determine the BH entropy through the special role that the moduli of the theory have in (generalized) BR geometries. In such a framework,

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[1]In all spherically symmetric 4-d. BHs $A_H = 4\pi r_H^2$, where r_H is the radius of the EH of the BH.

SUSY is related to dynamical systems with fixed points, describing the equilibrium state and the stability features of the system[2]. When the AM holds, the particular property of the long-range behavior of the dynamical flows in the considered (dissipative) systems is the following: in approaching the fixed points, properly named "attractors", the orbits of the dynamical evolution lose all memory of their initial conditions, but however the overall dynamics remains completely deterministic.

The first example of AM in supersymmetric systems was discovered in the theory of extreme BHs in $\mathcal{N} = 2$, $d = 4$ and 5 Maxwell Einstein Supergravity Theorys coupled with matter multiplets (i.e., Abelian vector multiplets and hypermultiplets) ([27], [28]). The corresponding dynamical system to be considered in this case is the one related to the radial evolution of the configurations of the relevant set of scalar fields of such theories (in this case, as it will be explained later, only the scalars from the vector multiplets have to be taken into account for the dynamics in the "near-Horizon" limit).

Otherwise speaking, we have to consider the behavior of the moduli fields of the theory as they approach the core of the s-t singularity. When reaching the proximity of the EH, they dynamically run into fixed points, getting some fixed values which are only function (of the ratios) of the electric and magnetic charges of the configuration of Abelian Maxwell vector potentials being considered.

The inverse distance to the Horizon is the fundamental evolution parameter in the dynamics towards the fixed points represented by the "attractor configurations" of the moduli. Such "near-Horizon" configurations of the moduli, which "attract" the dynamical evolutive flows in the moduli space, are completely independent of the initial data of such an evolution, i.e. on the asymptotical ($r \to \infty$) configurations of the moduli. Therefore, for what concerns the dynamics of the moduli, the system completely loses memory of its initial data, because the dynamical evolution will be "attracted" by some fixed configuration points, exclusively depending on the electric and magnetic charges of the Maxwell vector field content of the theory.

Thus, there is a substantial (and irreversible) loss of physical information in the

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[2]We recall that a point x_{fix} where the phase velocity $v\left(x_{fix}\right)$ vanishes is called a fixed point, and it gives a representation of the considered dynamical system in its equilibrium state,

$$v\left(x_{fix}\right) = 0.$$

The fixed point is said to be an *attractor* of some motion $x\left(t\right)$ if

$$lim_{t\to\infty}x(t) = x_{fix}.$$

motion of moduli configurations towards the EH of the extreme BHs, which therefore may be considered as dissipative dynamical systems from an information theory perspective (for recent developments along this line, see e.g. [29]).

Now, it should be reminded that there exists an interesting phenomenon in the physics of BHs, described by the so-called No-Hair Theorem: there is a limited number of parameters describing (geo)metric structures and physical fields far away from the s-t singularity represented by the BH, i.e. in the $r \to \infty$ limit. In other words, the spatial asymptotical configurations of BH metric are finitely-determined.

In the framework of SUGRA theories extreme BHs may be interpreted as BPS-saturated interpolating metric singularities in the low-energy effective limit of higher-dim. superstring or M theory. Their asymptotically relevant parameters include the mass, the (conserved, quantized) electrical and magnetic charges (defined by integrating the fluxes of related field strengths over 2-spheres at the infinity), and the asymptotical values of the (dynamically relevant set of) scalar fields.

From what shortly mentioned above, we may generalize and strengthen the No-Hair Theorem for extreme BHs in SUGRA theories, stating that such BHs lose all their "scalar hair" near the EH[3]. This means that the extreme BH metric solutions, in the "near-Horizon" limit in which they approach the BR metric, are characterized only by those discrete (quantized) parameters which correspond to the conserved charges associated with the gauge symmetries of the theory, but not by the continuously-varying asymptotical values of the (dynamically relevant set of) scalar fields.

Thence, it appears evident that our ability to make (microscopic) sense of the entropy of a BPS-saturated BH in SUGRA is deeply based on the AM.

Indeed, by such a general dynamical principle, starting from unconstrained, continuously-varying scalar field configurations, in the "near-Horizon" limit $r \to r_H^+$ we obtain some discrete, "attractor" field configurations, completely independent of the initial data of the evolution, but instead totally determined by the conserved charges of the system.

The change of the nature (continuous→discrete, quantized) of the scalar field configurations approaching the EH allows one to consistently define the concept of entropy of an extreme s-t singularity, at least in a microscopic approach. Indeed, being the moduli some continuous parameters which can be freely specified in the asymptotical Minkowskian metric background of the theory, in general one could

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[3]As it will be shown in Subsect. 4.2, such a phenomenon holds, under certain conditions, also in generic, non (necessarily) supersymmetric frameworks.

think that the entropy might depend on such values. Such a dependence on unconstrained values of the moduli would presumably lead to a possible violation of the Second Law of Thermodynamics, because, due to the functional moduli-dependence exhibited by the entropy, one could be allowed to quasi-statically decrease it by performing infinitesimal transformations in the moduli space. Thanks to the AM, the entropy actually depends only on the values of the moduli at the EH of the BH, and such "attractor configurations" of the moduli turn out to be insensitive to the asymptotical continuous moduli configurations. Therefore, the BH entropy ends being a function purely of the (quantized) conserved charges of the system.

At this point, one could (and should) ask the following question: *how the initial-data-independent "attractor" moduli configurations are fixed?*

A priori, one can expect that the answer would be completely model-dependent, i.e. that such fixed, quantized values of the "near-Horizon" moduli configurations would (strictly) depend on the features of the dynamical dissipative system given by the evolution of the (dynamically relevant set of) scalar fields in the moduli space. In other words, one would expect that such an answer would (heavily) rely on the (geo)metrical structure of the moduli space of the considered SUGRA theory.

But actually this is *not* the story. Indeed, at least in supersymmetric frameworks, the AM characterizes the "attractors" as stable fixed points corresponding to the critical points of the absolute value of the "central charge function" Z in the moduli space. This is an universal, model-independent feature of the "attractors"[4]. The area A_H of the EH is proportional to the square of such an absolute value, computed at the point where it is extremized in the moduli space [30].

Let us denote with $\{\varphi\}$ a configuration of the relevant set of scalar fields of the considered SUGRA theory. $\{\varphi\}$ will correspond to a point in the moduli space M and, in general, it will depend on the continuously varying, unconstrained initial configuration $\{\varphi_\infty\}$, i.e. on the initial point of the dynamical flow in M corresponding to the radial evolution of the moduli (which is the only relevant in the considered class of static, spherically-symmetric SUGRA solutions)

$$\{\varphi\} = \{\varphi\left(\varphi_\infty\right)\} . \tag{3.2.1}$$

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[4]Strictly speaking, this holds only for supersymmetric extreme BH attractors, i.e. for attractor configurations which preserve $\frac{1}{2}$ of the original supersymmetries of the $\mathcal{N} = 2$, $d = 4$ Maxwell Einstein Supergravity Theory being considered.

But non-supersymmetric extreme BH attractors may exist, too. Such a class of attractor configurations, which has been recently pointed out to be "discretely disjoint" from the class of supersymmetric attractors (at least in the one-modulus case, see [72]), is defined as the class of critical points of a suitably defined "BH effective potential" function V_{BH}, which are not also critical points of $|Z|$. For a detailed teatment, see Sect. 5, and in particular the Subsubsects. 5.3.1 and 5.3.2.

The AM states that the "near-Horizon" asymptotical moduli configurations $\{\varphi_H\} \equiv lim_{r \to r_H^+}\{\varphi\}$ will be *independent* of $\{\varphi\}$. Moreover, at least at the quantum level, it will be discrete, since it exclusively depends on the (quantized) asymptotical values of the electric charges $\{q\}$ and magnetic charges $\{p\}$ of the system

$$AM : \begin{cases} \{\varphi_H\} \neq \{\varphi_H(\varphi_\infty)\}, \\[2mm] \{\varphi_H\} = \{\varphi_H(p,q)\}. \end{cases} \tag{3.2.2}$$

Such a functional dependence on the charges may be determined by solving the general, model-independent "Attractor" or "Extremal" Equations (AEs)

$$\left. \frac{\partial |Z(\varphi;p,q)|}{\partial \varphi} \right|_{\varphi = \varphi_H(q,p)} = 0, \tag{3.2.3}$$

where Z is the "central charge" function[5] of the SUSY algebra in $\mathcal{N} = 2$ SUGRAs, or

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[5]Usually, what is initially known is the central charge Z, which is the asymptotical ($r \to \infty$) value of the "central charge" function

$$Z(\varphi_\infty;p,q) \equiv lim_{r \to \infty} Z(\varphi(r);p,q),$$

for a given BH charge configuration (p,q).

Clearly, if no other informations are available, the extrapolation of $Z(\varphi(r);p,q)$ from $Z(\varphi_\infty;p,q)$ is simply obtained by substituting φ_∞ with $\varphi(r)$. Consequently, such an operation relies on the assumption (implying a certain loss of generality) that the limit $r \to \infty$ is "smooth", in the sense that there are no functional structures vanishing for $r \to \infty$ (they potentially would contribute in determining the criticality conditions (3.2.3), and thus they would eventually modify the form of the AEs).

Let us analyze this issue a bit more in depth. Let us consider the function $Z(\varphi(r);p,q)$, assuming that

$$\forall a : \begin{cases} \exists \, lim_{r \to \infty} \varphi^a(r) \equiv \varphi_\infty^a; \\[2mm] |\varphi_\infty^a| < \infty. \end{cases}$$

First of all, one should assume that, at least for the considered BH charge configuration, the following limit exists:

$$lim_{r \to \infty} Z(\varphi(r);p,q) \equiv \Im(\varphi_\infty;p,q), \ |\Im(\varphi_\infty;p,q)| < \infty.$$

Now, in general, it holds that

$$\Im(\varphi(r);p,q) \neq Z(\varphi(r);p,q),$$

where

$$\Im(\varphi(r);p,q) \equiv \Im(\varphi_\infty;p,q)|_{\varphi_\infty = \varphi(r)}.$$

In other words, in general,

$$lim_{r \to \infty} Z(\varphi(r);p,q) \neq Z(lim_{r \to \infty} \varphi(r);p,q) = Z(\varphi_\infty;p,q) :$$

the highest absolute-valued eigenvalue of the complex antisymmetric central charge matrix in $\mathcal{N} > 2$-extended SUGRAs (see Sect. 6 for explanations).

Eq. (3.2.3) has the following meaning. The (charge-dependences of the) "near-Horizon" moduli configurations $\{\varphi_H\}$ are such that, when substituted in the function $Z(q, p, \varphi)$, they give an extremum value of Z with respect to (w.r.t.) its functional dependence on $\{\varphi\}$. Otherwise speaking, the "near-Horizon" value (independent of $\{\varphi_\infty\}$)

$$Z_H(q, p) \equiv Z(q, p, \varphi_\infty = \varphi_H(q, p)) \tag{3.2.4}$$

the asymptotical limit of a function is, in general, different from the function of the asymptotical limit(s). Clearly, this yields that, in general,

$$\frac{\partial Z(\varphi(r); p, q)}{\partial \varphi^a(r)} \neq \left. \frac{\partial \Im(\varphi_\infty; p, q)}{\partial \varphi^a_\infty} \right|_{\varphi_\infty = \varphi(r)}, \ \forall a.$$

Now, if one assumes the asymptotical limit $r \to \infty$ to be "smooth", i.e. that it holds true that

$$lim_{r \to \infty} Z(\varphi(r); p, q) = Z(\varphi_\infty; p, q),$$

it is thence clear that

$$\frac{\partial Z(\varphi(r); p, q)}{\partial \varphi^a(r)} = \left. \frac{\partial Z(\varphi_\infty; p, q)}{\partial \varphi^a_\infty} \right|_{\varphi_\infty = \varphi(r)}, \ \forall a.$$

Thus, by also assuming that

$$\forall a : \begin{cases} \exists \, lim_{r \to r_H^+} \varphi^a(r) \equiv \varphi^a_H; \\ \\ \left| \varphi^a_H \right| < \infty. \end{cases} ,$$

and that the Horizon limit $r \to r_H^+$ is "smooth":

$$lim_{r \to r_H^+} \frac{\partial Z(\varphi(r); p, q)}{\partial \varphi^a(r)} = \left. \frac{\partial Z}{\partial \varphi^a}(\varphi(r); p, q) \right|_{\varphi(r) = \varphi_H},$$

one finally gets

$$\begin{aligned} lim_{r \to r_H^+} \frac{\partial Z(\varphi(r); p, q)}{\partial \varphi^a(r)} &= \left. \frac{\partial Z}{\partial \varphi^a}(\varphi(r); p, q) \right|_{\varphi(r) = \varphi_H} = \\ &= \left. \frac{\partial Z(\varphi_\infty; p, q)}{\partial \varphi^a_\infty} \right|_{\varphi_\infty = lim_{r \to r_H^+} \varphi(r) \equiv \varphi_H}, \ \forall a. \end{aligned}$$

Therefore, by such assumptions the general criticality conditions of the function $Z(\varphi(r); p, q)$ at the "attractor" points read

$$\left. \frac{\partial Z}{\partial \varphi^a}(\varphi(r); p, q) \right|_{\varphi(r) = \varphi_H(p,q)} = \left. \frac{\partial Z(\varphi_\infty; p, q)}{\partial \varphi^a_\infty} \right|_{\varphi_\infty = \varphi_H(p,q)} = 0, \ \forall a,$$

corresponding to a more precise formulation of Eq. (3.2.3).

We will assume all aforementioned hypotheses to hold throughout this thesis.

is an extremum[6] value in the functional dependence of Z on $\{\varphi\}$ at given BH charges (p, q).

---

[6]It is worth noticing that usually such an extremum is assumed to be a (local, not necessarily global) minimum, as it can be explicitly verified in some models.

However, for the time being it is not possible to exclude situations with different extrema (such as local or global maxima, flex or cusp points), or also cases in which Eq. (3.2.3) does *not* admit solutions.

By the way, due to positive definiteness of the potential in SUSY theories, for sure a minimum will exist, but a priori nothing forbids the existence of an entire, discrete or (not necessarily countable) continuous family of minima. If this happens, the Horizon geometry of a p-dim. "black brane" in a d-dim. s-t will still be given by the (p, d)-generalization of BR metric, i.e. by the direct product $AdS_{p+2} \times S^{d-p-2}$, but such a limit geometry will now be realized by *each one* of the "near-Horizon" moduli configurations belonging to the considered family.

Also, given the set of moduli $\left\{\varphi^i\right\}_{i \in I}$, it could happen that a subset J of the discrete index range I exists, such that

$$\nexists \lim_{r \to r_H^+} \varphi^j, \ \ \forall j \in J \subseteq I, \tag{$*$}$$

i.e., that a certain subset of the moduli does not admit a "near-Horizon" limit.

Consequently, in order to preserve the core of the AM in such a particular case, *a priori* a number of possible assumptions may be made:

1) actually $Z = Z\left(q, p, \left\{\varphi^k\right\}_{k \in K}\right)$, where K is the complementary set of J with respect to I;

or 2) AEs should be slightly generalized as

$$\frac{\partial \left|Z\left(\varphi; p, q\right)\right|}{\partial \varphi^k}\Bigg|_{\varphi^k = \varphi_H^k(p,q)} = 0, \ \ \forall k \in K,$$

meaning that the "near-Horizon" extremization of the central charge function happens only w.r.t. the moduli well-defined at the EH. Thus, in the limit $r \to r_H^+$ the central charge function, extremized w.r.t. to its functional dependence on $\left\{\varphi^k\right\}_{k \in K}$, might still possibly depend on the subset of unconstrained, continuously-varying asymptotical configurations of moduli $\left\{\varphi_\infty^j\right\}_{j \in J}$:

$$Z_H\left(\left\{\varphi_\infty^j\right\}_{j \in J}; p, q\right) \equiv Z\left(\left\{\varphi^k\right\}_{k \in K} = \left\{\varphi_H^k(p, q)\right\}_{k \in K}, \left\{\varphi_\infty^j\right\}_{j \in J}; p, q\right).$$

Such a possibility, however, should be disregarded, because, in general, it should lead to a violation of the Second Principle of Thermodynamics in the BH physics;

or 3) in general, Eq. ($*$) corresponds to a vanishing Horizon value of the central charge function

$$Z_H(p, q) \equiv Z\left(\varphi_\infty = \varphi_H(p, q); p, q\right) = 0,$$

and therefore the BHEA (and ADM mass -see a bit further below in the main text-) formulae become inconsistent and inapplicable, leading to a non-regular Horizon geometry. As we will see later, this happens for all non-minimal BPS SUSY-preserving extremal solutions in $\mathcal{N} > 2$-extended, $d = 4, 5$-dim. SUGRAs, and also in $\mathcal{N} \geqslant 2$-extended, $d \geqslant 6$-dim. SUGRAs (where the BHEA formula may also give unphysical, constant non-zero results).

3.3   An Illustration

A simple example illustrating the AM at work may be given by the $\mathcal{N} = 2$, $d = 4$ dilatonic BH of the heterotic string theory. In this case the BPS-saturation condition fixes the so-called Arnowitt-Deser-Misner (ADM) mass of the BH to be equal to the absolute value of the central charge function, which in turn will be a function of the electric charge q and magnetic charge p of the BH, and of the asymptotical value ϕ_∞ of the dilaton

$$M_{ADM}(q, p, \phi_\infty) = |Z(q, p, \phi_\infty)| = \tfrac{1}{2}\left(e^{-\phi_\infty}|p| + e^{\phi_\infty}|q|\right),$$

$$\phi_\infty \in \mathbb{R}, \quad (q, p) \in \mathbb{Z}^2 \text{ (in suitable units).}$$

$$(3.3.1)$$

The general theory based on the AM, when applied to the present case, gives the following "four-step recipe" to obtain the entropy of the dilatonic BH:

i) Write down the extremization condition for the absolute value of the central charge function depending on the dilatonic function $g(\phi) \equiv e^{\phi}$, at fixed values of the charges (p, q) (see Footnote 6)

$$\frac{\partial|Z(\phi(g); p, q)|}{\partial g} = \frac{1}{2}\frac{\partial}{\partial g}\left(\frac{1}{g}|p| + g|q|\right) = -\frac{1}{g^2}|p| + |q| = 0. \qquad (3.3.2)$$

ii) Solve such a condition, obtaining the fixed value of the dilatonic function

$$\frac{\partial|Z(\phi(g); p, q)|}{\partial g} = 0 \Leftrightarrow g = g_H(p, q) = \left|\frac{p}{q}\right|^{\frac{1}{2}}, \qquad (3.3.3)$$

In the present pedagogical treatment we will implicitly assume, for simplicity's sake, that the AEs admit, at least in relation to the minimal BPS SUSY-preserving extremal backgrounds, (at least) one regular solution, corresponding to a purely charge-dependent "near-Horizon" moduli configuration.

Finally, it should be mentioned that for an arbitrary geometry of the moduli space the form of the relevant central charge function $Z(\varphi; p, q)$ may be also very complicated. For instance, this is what happens for the $\mathcal{N} = 2$, $d = 4$ SUGRA obtained by the compactification of $\mathcal{N} = 2$, $d = 10$ type *IIB* SUGRA on Calabi-Yau threefolds.

Nevertheless, despite this fact, the extremization procedure expressed by the AEs allow one to consistently compute the entropy of the corresponding extremal singular metric backgrounds following a model-independent, universal procedure.

As far as we know, no Existence and/or Uniqueness Theorems have been proved for Eq. (3.2.3), even though substantial progress has been made in the study of the topological properties of the moduli spaces as "attractor varieties" (see e.g. [31], [32] and [33]).
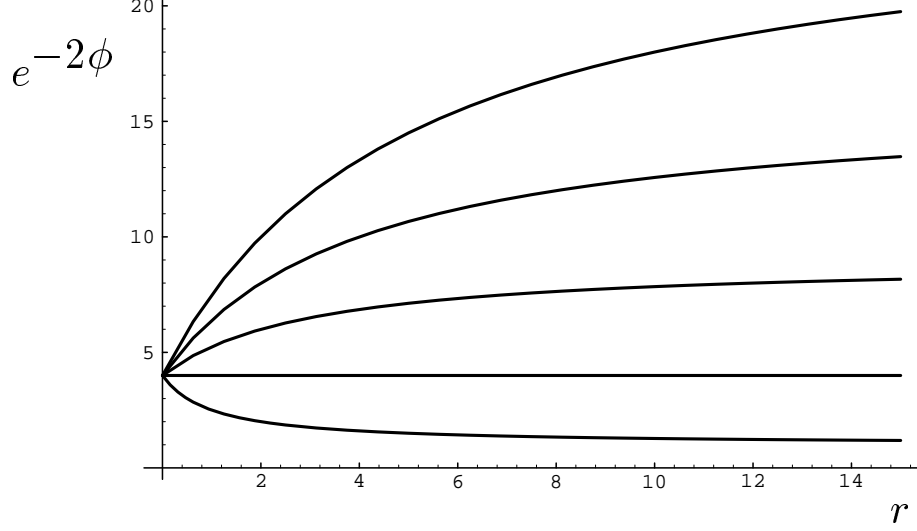
Figure 3.1: Realization of the Attractor Mechanism in the $\mathcal{N} = 2$, $d = 4$ extremal $\frac{1}{2}$-BPS dilatonic BH. Independently of the set of initial (asymptotical $r \to \infty$) moduli configurations (corresponding to the initial data of the dynamical flow inside the moduli space), the "near-Horizon" ($r \to 0^+$, with r denoting the radial distance from the Event Horizon) evolution of the moduli-dependent dilatonic function $g^{-2}(\phi) \equiv e^{-2\phi}$ converges towards a fixed "attractor" value, which is purely dependent on the (ratio of the) quantized conserved charges of the BH. Such a purely charge-dependent phenomenon of "attraction" of the moduli field configurations encodes the intrinsic loss of information in the (equilibrium) thermodynamics of the extremal dilatonic BH.

and therefore of the dilatonic moduli at the EH

$$\phi_H(g) \equiv \phi(g_H(p,q)) = ln[g_H(p,q)] = \frac{1}{2}ln\left|\frac{p}{q}\right|. \tag{3.3.4}$$

An example of the evolution of the moduli-dependent dilatonic function $g^{-2}(\phi) \equiv e^{-2\phi}$ towards a purely charge-dependent value at the EH of the $\mathcal{N} = 2$, $d = 4$ dilatonic BH is shown in Fig. 3.1.

iii) Insert such a fixed value into the expression of the central charge function, by putting $\phi(g) = \phi_H(g)$. In such a way, one gets the fixed value $|Z_H(p,q)|$ of the absolute value of the central charge function at the EH of the dilatonic BH; clearly, due to the saturation of the BPS bound, it equals the value of the ADM mass of the

EH, too (see Eq. (3.3.1))

$$
\begin{aligned}
M_{ADM,H}(p,q) &= M_{ADM}\left(\phi(g) = \phi_H(g) = \frac{1}{2}\ln\left|\frac{p}{q}\right|; p,q\right) = \\
&= |Z_H(p,q)| = \left|Z\left(\phi(g) = \phi_H(g) = \frac{1}{2}\ln\left|\frac{p}{q}\right|; p,q\right)\right| = \\
&= |pq|^{\frac{1}{2}}.
\end{aligned}
\tag{3.3.5}
$$

iv) Use the BHEA formula to get the (semiclassical, leading-order) entropy of the $\mathcal{N}=2$, $d=4$ dilatonic BH

$$
S_{BH} = \frac{A_{Horizon}}{4} = \pi M_{ADM,H}^2(p,q) = \pi |Z_H(p,q)|^2 = \pi |pq|,
\tag{3.3.6}
$$

where we used the definition of the ADM mass at the EH of the BH

$$
M_{ADM,H}^2 \equiv \frac{A_{Horizon}}{4\pi}.
\tag{3.3.7}
$$

Notice that the BH entropy given by Eq. (3.3.6) is purely charge-dependent, and it may be checked that it coincides with the result obtained by completely different (model-dependent, microscopic) methods.

In the $d=4$ (5)-dim. $\mathcal{N}=2$ SUGRAs coupled to n_V Abelian vector multiplets (named $\mathcal{N}=2$ n_V-fold Maxwell Einstein Supergravity Theorys), the extremization of the central charge function Z through Eq. (3.2.3) may be made "coordinate-free" in the moduli space M_{n_V}, by using the fact that such a n_V-dim. complex manifold has actually a (real) special Kähler metric structure. The geometric properties of the moduli space and the overall symplectic structure of such $\mathcal{N}=2$ SUGRAs will be considered in the next Section.

The final result of the AM in such theories is the macroscopic, model-independent derivation of BHEA formula, yielding

$$
S_{BH} = \frac{A}{4} = \pi |Z_H(p,q)|^2
\tag{3.3.8}
$$

and

$$
S_{BH} = \frac{A}{4} \sim |Z_H(p,q)|^{\frac{3}{2}},
\tag{3.3.9}
$$

in $d=4$ and $d=5$, respectively.

Recently, many applications of the above ideas have been worked out, especially in the case of string theory compactified on 3-dim. Calabi-Yau manifolds. Also, by

using some properly formulated *D*-brane techniques, the topological entropy formula of BH has been obtained, by counting the related microstates in string theory. The results of such a procedure, whenever obtainable, are in agreement with the model-independent determination of the entropy which uses the Attractor Mechanism. The four-step algorithm given by Eqs. (3.3.2)-(3.3.6) is just one of the possible realizations of such a model-independent approach to the equilibrium thermodynamics of BHs.

It should be also mentioned that several properties of the fixed "attractor" moduli configurations have been investigated. In particular, it has been shown that the Attractor Mechanism is also relevant in the discussion of the BH thermodynamics out of the extremality (i.e. when the BPS-like bound (2.2.11) is not saturated).

In the remaining part of these introductory notes we will see how the AM works in the relevant context, for e.g. in the so-called $\mathcal{N} = 2$, $d = 4$, n_V-fold Maxwell Einstein Supergravity Theorys.

Chapter 4

Attractor Mechanism in $\mathcal{N} = 2, d = 4$ Maxwell-Einstein Supergravity

The multiplet content of a completely general $\mathcal{N} = 2, d = 4$ supergravity (SUGRA) theory is the following (see e.g. [34] and [35]):

1. the *gravitational multiplet*

$$\left(V_\mu^a, \psi^A, \psi_A, A^0 \right), \tag{4.0.1}$$

described by the Vielbein one-form V^a ($a = 0, 1, 2, 3$) (together with the spin-connection one-form ω^{ab}), the $SU(2)$ doublet of gravitino one-forms ψ^A, ψ_A ($A = 1, 2$, with the upper and lower indices respectively denoting right and left chirality, i.e. $\gamma_5 \psi_A = -\gamma_5 \psi^A = 1$), and the graviphoton one-form A^0;

2. n_V *vector supermultiplets*

$$\left(A^I, \lambda^{iA}, \overline{\lambda}_A^{\bar{i}}, z^i \right), \tag{4.0.2}$$

each containing a gauge boson one-form A^I ($I = 1, ..., n_V$), a doublet of gauginos (zero-form spinors) $\lambda^{iA}, \overline{\lambda}_A^{\bar{i}}$, and a complex scalar field (zero-form) z^i ($i = 1, ..., n_V$). The scalar fields z^i can be regarded as arbitrary coordinates on a complex manifold M_{n_V} ($dim_{\mathbb{C}} M_{n_V} = n_V$), which is actually a special Kähler manifold;

3. n_H *hypermultiplets*

$$(\zeta_\alpha, \zeta^\alpha, q^u), \tag{4.0.3}$$

each formed by a doublet of zero-form spinors, that is the hyperinos $\zeta_\alpha, \zeta^\alpha$ ($\alpha = 1, ..., 2n_H$), and four real scalar fields q^u ($u = 1, ..., 4n_H$), which can be considered as arbitrary coordinates of a quaternionic manifold $\mathcal{Q}_{n_H}$ ($dim_{\mathbb{R}} \mathcal{Q}_{n_H} = 4n_H$).

In this Section we will sketch the formulation of the $\mathcal{N} = 2, d = 4$ SUGRA coupled to n_V Abelian vector multiplets in presence of electric and magnetic charges, i.e. of the so-called $\mathcal{N} = 2, d = 4$ n_V-fold Maxwell Einstein Supergravity Theory. We will then show how the Attraction Mechanism explicitly works, in relation to the special Kähler geometry of the manifold M_{n_V} of the scalars z^i's from the Abelian vector supermultiplets, finally specializing the AE (3.2.3) for such a framework[1].

---

[1]Here we will not deal with the n_H hypermultiplets. Indeed, in the $\mathcal{N} = 2, d = 4$ n_V-fold Maxwell Einstein Supergravity Theory the symplectic special Kähler geometry is completely determined by the n_V complex scalar fields coming from the considered n_V Abelian vector supermultiplets.

Such a fact may be understood by looking at the transformation properties of the Fermi fields: the hyperinos $\zeta_\alpha, \zeta^\alpha$'s transform independently of the vector fields, whereas the gauginos' SUSY transformations depend on the Maxwell vector fields.

Consequently, the contribution of the hypermultiplets may be dynamically decoupled from the rest of the physical system. Thus, it is also completely independent of the evolution dynamics of the complex scalars z^i's coming from the vector multiplets (i.e. from the evolution flow in the moduli space M_{n_V}).

Disregarding for simplicity's sake the fermionic and gauging terms, the SUSY transformations of hyperinos (see Eq. (4.2.1) further below) read

$$\delta\zeta_\alpha = i\mathcal{U}_u^{B\beta}\partial_\mu q^u \gamma^\mu \varepsilon^A \epsilon_{AB} \mathbb{C}_{\alpha\beta}. \tag{$**$}$$

Eq. ($**$) does not constrain the asymptotical configurations of the quaternionic scalars of the hypermultiplets, which therefore may continuously vary in the manifold $\mathcal{Q}_{n_H}$ of the related quaternionic non-linear sigma model.

In the gauged $\mathcal{N}$-extended SUGRA (generally corresponding to asymptotically non-flat backgrounds), and consequently also in the $\mathcal{N} = 2, d = 4$, (n_V, n_H)-fold gauged Maxwell Einstein Supergravity Theory, the situation is much more complicated.

Of course, the geometry of the scalar sigma models remains the same, since it is completely fixed by the internal metric structure of the kinetic terms of the scalars. For a generic value of $(n_V, n_H) \in \mathbb{N}^2$, it is given by the direct product

$$M_{n_V} \times \mathcal{Q}_{n_H}$$

of the special Kähler-Hodge manifold of the complex scalars from the Maxwell vector supermultiplets and of the quaternionic manifold of the scalar fields from the hypermultiplets, respectively.

But, unlike the "ungauged", asymptotically flat case which will be treated in the following pages, some interaction terms between the above two different sets of scalars will arise in the bosonic part of the gauged SUGRA Lagrangian. Such terms are generated by the Killing vectors coming from the introduction of covariant derivatives w.r.t. the gauging of (some of) the isometries of $\mathcal{Q}_{n_H}$, and they do *not* allow one to dynamically decouple the hypermultiplets any more.

4.1 Special Kähler-Hodge Geometry and the Symplectic structure of Moduli Space

Let us start by considering the moduli space M_{n_V} of the $\mathcal{N} = 2$, $d = 4$ n_V-fold Maxwell Einstein Supergravity Theory; it is a complex n_V-dim. manifold having the n_V scalar complex fields z^i ($i = 1, ..., n_V$) as local coordinates; such fields come from the vector multiplets coupling to $\mathcal{N} = 2, d = 4$ SUGRA.

The key feature is that M_{n_V} is a Kähler-Hodge manifold with special Kähler structure, i.e. a n_V-dim. special Kähler-Hodge manifold with symplectic structure.

Firstly, M_{n_V} is a Kähler manifold, i.e. a complex Hermitian manifold with the metric

$$G_{i\bar{j}}(z, \bar{z}) \equiv \bar{\partial}_{\bar{j}} \partial_i K(z, \bar{z}), \tag{4.1.1}$$

where $K(z, \bar{z})$ is the so-called (real) "Kähler potential" scalar function. The Hermiticity of the metric directly follows from the reality of K (and from the fact that such a function is assumed to satisfy the Schwarz Lemma about partial derivatives on M_{n_V})

$$\overline{G_{i\bar{j}}} = \overline{\partial_{\bar{j}} \partial_i K} = \partial_j \bar{\partial}_{\bar{i}} K = \bar{\partial}_{\bar{i}} \partial_j K = G_{j\bar{i}}. \tag{4.1.2}$$

Secondly, since M_{n_V} is a special Kähler manifold, its Riemann-Christoffel curvature tensor satisfies the so-called "special Kähler geometry (SKG) constraints"

$$R_{i\bar{j}k\bar{l}} = G_{i\bar{j}}G_{k\bar{l}} + G_{i\bar{l}}G_{k\bar{j}} - C_{ikp}\overline{C}_{\bar{j}\bar{l}\bar{p}}G^{p\bar{p}}, \tag{4.1.3}$$

where C_{ijk} is the rank-3, completely symmetric, Kähler-covariantly holomorphic tensor of SKG

$$\begin{cases} C_{ijk} = C_{(ijk)}; \\ \\ \overline{D}_{\bar{l}}C_{ijk} = 0. \end{cases} \tag{4.1.4}$$

It is also immediate to show that [56]

$$D_{[l}C_{i]jk} = 0, \tag{4.1.5}$$

where square brackets denote antisymmetrization w.r.t. the enclosed indices. Indeed, the (differential) Bianchi identities for the Riemann-Christoffel tensor read

$$D_{[l}R_{i]\bar{k}j\bar{p}} = 0; \tag{4.1.6}$$

by using the SKG constraints (7.2.1.39) and recalling the Kähler-covariant holomorphicity of C_{ijk} ($\overline{D}_{\bar{l}}C_{ijk} = 0$) and the validity of the Metric Postulate in M_{n_V} ($D_k G_{i\bar{j}} = 0$), one immediately gets

$$\left(D_{[l}C_{i]jn}\right)\overline{C}_{\bar{k}\bar{p}\bar{n}}G^{n\bar{n}} = 0, \tag{4.1.7}$$

and Eq. (4.1.5) follows from the observation that Eq. (4.1.7) holds for any (non-vanishing) $\overline{C}_{\overline{k}\overline{p}}{}^{n} = \overline{C}_{\overline{k}\overline{p}\overline{n}} G^{n\overline{n}}$.

Since in a (commutative) Kähler manifold the completely covariant Riemann-Christoffel tensor $R_{i\overline{j}l\overline{m}}$ may be rewritten as[2]

$$R_{i\overline{j}k\overline{l}} = -G^{m\overline{n}} \left(\overline{\partial}_{\overline{l}}\overline{\partial}_{\overline{j}}\partial_m K\right) \partial_i \overline{\partial}_{\overline{n}} \partial_k K + \overline{\partial}_{\overline{l}}\partial_i \overline{\partial}_{\overline{j}}\partial_k K, \qquad (4.1.8)$$

the SKG constraints (7.2.1.39) may be reformulated as follows:

$$-G^{m\overline{n}} \left(\overline{\partial}_{\overline{l}}\overline{\partial}_{\overline{j}}\partial_m K\right) \partial_i \overline{\partial}_{\overline{n}} \partial_k K + \overline{\partial}_{\overline{l}}\partial_i \overline{\partial}_{\overline{j}}\partial_k K = \qquad (4.1.9)$$

$$= \left(\overline{\partial}_{\overline{j}}\partial_i K\right) \overline{\partial}_{\overline{l}}\partial_k K + \left(\overline{\partial}_{\overline{l}}\partial_i K\right) \overline{\partial}_{\overline{j}}\partial_k K - C_{ikp}\overline{C}_{\overline{j}\overline{l}\overline{p}} G^{p\overline{p}};$$

$$\Updownarrow$$

$$G^{m\overline{n}} \left[\left(\overline{\partial}_{\overline{l}}\overline{\partial}_{\overline{j}}\partial_m K\right) \left(\partial_i \overline{\partial}_{\overline{n}} \partial_k K\right) - C_{ikm}\overline{C}_{\overline{j}\overline{l}\overline{n}}\right] = \qquad (4.1.10)$$

$$= \overline{\partial}_{\overline{l}}\partial_i \overline{\partial}_{\overline{j}}\partial_k K - \left(\overline{\partial}_{\overline{j}}\partial_i K\right) \overline{\partial}_{\overline{l}}\partial_k K - \left(\overline{\partial}_{\overline{l}}\partial_i K\right) \overline{\partial}_{\overline{j}}\partial_k K;$$

$$\Updownarrow$$

$$C^{\overline{n}}{}_{km}\overline{C}_{\overline{n}\overline{j}\overline{l}} =$$

$$= \left(\overline{\partial}_{\overline{j}}\partial_i K\right) \overline{\partial}_{\overline{l}}\partial_k K + \left(\overline{\partial}_{\overline{l}}\partial_i K\right) \overline{\partial}_{\overline{j}}\partial_k K +$$

$$-\overline{\partial}_{\overline{l}}\partial_i \overline{\partial}_{\overline{j}}\partial_k K + G^{m\overline{n}} \left(\overline{\partial}_{\overline{l}}\overline{\partial}_{\overline{j}}\partial_m K\right) \left(\partial_i \overline{\partial}_{\overline{n}} \partial_k K\right),$$

$$(4.1.11)$$

where, as usual, the contravariant and covariant metric tensors are related by the orthonormality condition

$$G^{i\overline{j}}G_{l\overline{j}} = G^{i\overline{j}}\overline{\partial}_{\overline{j}}\partial_l K = \delta_l^i. \qquad (4.1.12)$$

Thirdly, since M_{n_V} is a Kähler-Hodge manifold, it admits a $U(1)$ line (Hodge)

---

[2]It should also be recalled that in a Kählerian manifold the (completely covariant) Riemann-Christoffel tensor, as well as its contractions $R_{i\overline{j}}$ and R and the (completely covariant) Weyl conformal curvature tensor $C_{i\overline{j}k\overline{l}}$, are all real [37]

$$\overline{R_{i\overline{j}k\overline{l}}} = R_{i\overline{j}k\overline{l}}; \qquad \overline{R_{i\overline{j}}} = R_{i\overline{j}};$$

$$\overline{R} = R; \qquad \overline{C_{i\overline{j}k\overline{l}}} = C_{i\overline{j}k\overline{l}}.$$

bundle $\Im$, whose first Chern class coincides with the Kähler class of M_{n_V}

$$c_1[\Im] = \mathcal{K}[M_{n_V}]. \tag{4.1.13}$$

Such a property allows one to locally write the $U(1)$ connection Q as

$$Q = -\frac{i}{2}\left[(\partial_i K)\, dz^i - (\bar{\partial}_{\bar{i}} K)\, d\bar{z}^{\bar{i}}\right]. \tag{4.1.14}$$

Let us now consider a Kähler transformation

$$K(z,\bar{z}) \to K(z,\bar{z}) + f(z) + \bar{f}(\bar{z}), \tag{4.1.15}$$

where f is an arbitrary holomorphic function. Clearly, due to definition (4.1.1), such a transformation does not affect the Kähler metric structure, and thus it actually expresses an intrinsic gauge metric degree of freedom of the considered manifold. Consequently, beside the usual Hermitian covariance, one will have to take it into account when writing down the Kähler-covariant derivatives of any tensor quantity. In a general (commutative) Kähler geometry, a generic vector $\mathcal{V}^i$ which under (4.1.15) transforms as

$$\mathcal{V}^i(z,\bar{z}) \to exp\left\{-\frac{1}{2}\left[pf(z) + \bar{p}\bar{f}(\bar{z})\right]\right\}\mathcal{V}^i(z,\bar{z}), \quad (p,\bar{p}) \in \mathbb{R}^2, \tag{4.1.16}$$

is said to have Kähler weights[3] $(p,\bar{p})$. Its Kähler-covariant derivatives are defined as follows:

$$\begin{cases} D_j\mathcal{V}^i(z,\bar{z}) = \partial_j\mathcal{V}^i(z,\bar{z}) + \Gamma_{jk}{}^i(z,\bar{z})\,\mathcal{V}^k(z,\bar{z}) + \frac{p}{2}\left(\partial_j K(z,\bar{z})\right)\mathcal{V}^i(z,\bar{z})\,; \\[2mm] \overline{D}_{\bar{j}}\mathcal{V}^i(z,\bar{z}) = \bar{\partial}_{\bar{j}}\mathcal{V}^i(z,\bar{z}) + \frac{\bar{p}}{2}\left(\bar{\partial}_{\bar{j}}K(z,\bar{z})\right)\mathcal{V}^i(z,\bar{z})\,, \end{cases} \tag{4.1.17}$$

where $\Gamma_{jk}{}^i(z,\bar{z})$ denotes the symmetric connection given by the Christoffel symbols of the second kind of the Kähler metric

$$\Gamma_{jk}{}^i(z,\bar{z}) \equiv \left\{{}^i_{jk}\right\}(z,\bar{z}) = G^{i\bar{l}}(z,\bar{z})\,\partial_j G_{k\bar{l}}(z,\bar{z}) =$$

$$= G^{i\bar{l}}(z,\bar{z})\,\partial_j\bar{\partial}_{\bar{l}}\partial_k K(z,\bar{z}) = \Gamma_{(jk)}{}^i(z,\bar{z}). \tag{4.1.18}$$

The Kähler transformation property (4.1.16) may be rewritten as follows:

$$\mathcal{V}^i(z,\bar{z}) \to exp\left\{-\frac{1}{2}(p+\bar{p})\,Re\,(f(z))\right\}exp\left\{-\frac{i}{2}(p-\bar{p})\,Im\,(f(z))\right\}\mathcal{V}^i(z,\bar{z})\,; \tag{4.1.19}$$

---

[3] The Kähler weights are real. Notice that $\bar{p}$ is not the complex conjugate of the holomorphic Kähler weight p, but it rather simply stands for the anti-holomorphic Kähler weight.

it is then immediate to realize that a generic Kähler transformation may always be decomposed in a $U(1)$ phase transformation (singled out by $p = -\overline{p}$) and in a proper Kähler transformation (singled out by $p = \overline{p}$). Due to the reality of the Kähler weights, the complex conjugation of Eq. (4.1.16) yields

$$\overline{\mathcal{V}}^{\overline{i}}\left(z,\overline{z}\right) \rightarrow exp\left\{-\frac{1}{2}\left[\overline{p}f(z) + p\overline{f}(\overline{z})\right]\right\}\overline{\mathcal{V}}^{\overline{i}}\left(z,\overline{z}\right), \qquad (4.1.20)$$

and thus one gets that the complex conjugation simply exchanges the Kähler weights: if $\mathcal{V}^i$ has Kähler weights $(p,\overline{p})$, then $\overline{\mathcal{V}}^{\overline{i}}$ has Kähler weights $(\overline{p},p)$.

Since we are considering a $U(1)$ line bundle $\Im$ over the moduli space M_{n_V}, only the quantities with Kähler weights constrained by $p = -\overline{p}$ will properly belong to the related $U(1)$ ring. Clearly, real or (anti)holomorphic quantities will not belong to such a $U(1)$ ring, unless they are Kähler gauge-invariant, i.e. they have $(p,\overline{p}) = (0,0)$. An example of tensor belonging to the $U(1)$ ring is the completely symmetric, Kähler-covariantly holomorphic rank-3 tensor $C_{ijk}\left(z,\overline{z}\right)$, having Kähler weights $(2,-2)$; as a consequence of the general formulae (4.1.17), its Kähler-covariant derivatives read

$$\begin{cases} D_l C_{ijk}\left(z,\overline{z}\right) = \\[2mm] = \partial_l C_{ijk}\left(z,\overline{z}\right) - \Gamma_{li}^{\,m}\left(z,\overline{z}\right) C_{mjk}\left(z,\overline{z}\right) - \Gamma_{lj}^{\,m}\left(z,\overline{z}\right) C_{imk}\left(z,\overline{z}\right) + \\[2mm] -\Gamma_{lk}^{\,m}\left(z,\overline{z}\right) C_{ijm}\left(z,\overline{z}\right) + \left(\partial_l K\left(z,\overline{z}\right)\right) C_{ijk}\left(z,\overline{z}\right); \\[4mm] \overline{D}_{\overline{l}} C_{ijk}\left(z,\overline{z}\right) = \overline{\partial}_{\overline{l}} C_{ijk}\left(z,\overline{z}\right) - \left(\overline{\partial}_{\overline{l}} K\left(z,\overline{z}\right)\right) C_{ijk}\left(z,\overline{z}\right) = 0. \end{cases}$$

$$(4.1.21)$$

Therefore, the integrability condition (4.1.5) may be rewritten as follows:

$$\partial_{[l}C_{i]jk} - \Gamma_{[l|i]}{}^{m}C_{mjk} - \Gamma_{[l|j}{}^{m}C_{|i]mk} - \Gamma_{[l|k}{}^{m}C_{|i]jm} + \left(\partial_{[l}K\right)C_{i]jk} = 0;$$

(4.1.22)

$$\Updownarrow$$

$$\partial_l C_{ijk} - \partial_i C_{ljk} - G^{m\bar{k}}\left[\begin{array}{c}\left(\partial_l\bar{\partial}_{\bar{k}}\partial_j K\right)C_{imk} - \left(\partial_i\bar{\partial}_{\bar{k}}\partial_j K\right)C_{lmk} + \\[2mm] + \left(\partial_l\bar{\partial}_{\bar{k}}\partial_k K\right)C_{ijm} - \left(\partial_i\bar{\partial}_{\bar{k}}\partial_k K\right)C_{ljm}\end{array}\right]$$

$$+ \left(\partial_l K\right)C_{ijk} - \left(\partial_i K\right)C_{ljk} = 0.$$

(4.1.23)

A more intrinsic characterization of M_{n_V}, which makes its $Sp\left(2n_V + 2\right)$-covariance manifest, is the following one.

Let us start by defining the (Kähler-covariantly holomorphic with Kähler weights $(1, -1)$) symplectic sections of the Hodge bundle $\Im$ on M_{n_V} ($\Lambda = 0, 1, ..., n_V$)

$$V\left(z, \bar{z}\right) \equiv \begin{pmatrix} L^{\Lambda}\left(z, \bar{z}\right) \\[2mm] M_{\Lambda}\left(z, \bar{z}\right) \end{pmatrix}, \text{ with } \overline{D}_{\bar{i}}V = \left(\bar{\partial}_{\bar{i}} - \frac{1}{2}\bar{\partial}_{\bar{i}}K\right)V = 0.$$

(4.1.24)

Notice that such sections may be arranged in a $Sp\left(2n_V + 2\right)$-covariant vector V. By defining a scalar product in the related representation space using the $(2n_V + 2)$-dim. symplectic metric

$$\epsilon \equiv \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$$

(4.1.25)

($\mathbb{I}$ stands for the $(n_V + 1)$-dim. identity), the symplectic sections may be normalized as follows:

$$\left\langle V, \overline{V} \right\rangle \equiv V^T \epsilon \overline{V} = \overline{L}^{\Lambda}M_{\Lambda} - \overline{M}_{\Lambda}L^{\Lambda} \equiv -i.$$

(4.1.26)

Therefore, it is natural to introduce the $(2n_V + 2)$-dim. vector of the holomorphic Kähler-covariant derivatives of the sections[4] of $\Im$

$$U_i \equiv D_i V = \left(\partial_i + \frac{1}{2}\partial_i K\right)\begin{pmatrix} L^{\Lambda} \\ M_{\Lambda} \end{pmatrix} \equiv \begin{pmatrix} f_i^{\Lambda} \\ h_{i\Lambda} \end{pmatrix};$$

(4.1.27)

---

[4]In general, f_i^{Λ} and $h_{i\Lambda}$ are functions defined in M_{n_V}, with a local index i and a global index Λ. As the n-bein allows one to transform local Poincaré-covariant indices in global diffeomorphism-covariant indices (and viceversa), similarly such quantities allow one to switch between global

consequently, the Kähler-covariant holomorphicity of V implies $U_{\bar{\imath}} \equiv \overline{D}_{\bar{\imath}} V = 0 = \overline{U}_i \equiv D_i \overline{V}$. It may be then shown that in SKG

$$D_i U_j \;\; = \;\; i C_{ijk} G^{k\bar{k}} \overline{U}_{\bar{k}}, \tag{4.1.28}$$

$$D_i \overline{U}_{\bar{\jmath}} \;\; = \;\; G_{i\bar{\jmath}} \overline{V}; \tag{4.1.29}$$

here C_{ijk} may be defined to be the $(2, -2)$-Kähler-weighted section of $(T^*)^3 \otimes \Im^2$, totally symmetric in its indices and Kähler-covariantly holomorphic[5]. In [36] it was shown that Eqs. (7.2.1.34)-(4.1.29) (with the properties (4.1.4) and the constraints (7.2.1.39) - or equivalently the integrability condition (4.1.5) for C_{ijk}) define a flat symplectic connection; thus, the symplectic-covariant derivatives always coincide with the ordinary, flat derivatives.

It is worth mentioning that, while Eq. (7.2.1.38) is typical of SKG, Eq. (4.1.29) holds in contexts more general than SKG. To clarify such a point, let us derive it, by considering, without any loss of generality, the section L^Λ. As stated above, this is a Kähler-covariantly holomorphic symplectic section with Kähler weights $(1, -1)$; thus, it holds that

$$
\begin{aligned}
D_i \overline{D}_{\bar{\jmath}} \overline{L}^\Lambda \;\; &= \;\; \left(\partial_i - \frac{1}{2} \left(\partial_i K \right) \right) \left(\overline{\partial}_{\bar{\jmath}} + \frac{1}{2} \left(\overline{\partial}_{\bar{\jmath}} K \right) \right) \overline{L}^\Lambda = \\[2mm]
&= \;\; \partial_i \overline{\partial}_{\bar{\jmath}} \overline{L}^\Lambda + \frac{1}{2} \left(\partial_i \overline{\partial}_{\bar{\jmath}} K \right) \overline{L}^\Lambda + \frac{1}{2} \left(\overline{\partial}_{\bar{\jmath}} K \right) \partial_i \overline{L}^\Lambda + \\[2mm]
&\quad - \frac{1}{2} \left(\partial_i K \right) \overline{\partial}_{\bar{\jmath}} \overline{L}^\Lambda - \frac{1}{4} \left(\partial_i K \right) \left(\overline{\partial}_{\bar{\jmath}} K \right) \overline{L}^\Lambda;
\end{aligned} \tag{4.1.30}
$$

now, by recalling Eq. (4.1.1) and using the fact that the Kähler-covariant holomorphicity of L^Λ implies

$$\partial_i \overline{L}^\Lambda = \frac{1}{2} \left(\partial_i K \right) \overline{L}^\Lambda, \tag{4.1.31}$$

$Sp\,(2n_V + 2)$-covariant indices and local indices in the Kähler-Hodge manifold M_{n_V} associated to the non-linear σ-model of the complex scalars coming from the n_V considered vector multiplets.

It is also worth noticing that, in the particular cases in which such a manifold is a symmetric space of the kind G/H (as it happens for all $\mathcal{N} \geqslant 3, d = 4$ SUGRAs, and in particular for the maximal $\mathcal{N} = 8, d = 4$ SUGRA with non-compact $E_{7(7)}$ symmetry: see Subsect. 6.2), the functions f_i^Λ and $h_{i\Lambda}$ are nothing but the representative cosets of such a space.

[5]In an alternative defining approach, Eqs. (7.2.1.34), (4.1.27), (7.2.1.38) and (4.1.29) may be also considered as the fundamental differential constraints defining the local special Kähler geometry of M_{n_V}. Indeed, it may be shown that they yield the SKG constraints (7.2.1.39) (see e.g. [38]). For a thorough analysis of the various approaches to the definitions of (global and local) SKG, see e.g. [40] and [41].

one gets

$$D_i \overline{D}_{\overline{j}} \overline{L}^\Lambda = \partial_i \overline{\partial}_{\overline{j}} \overline{L}^\Lambda + \frac{1}{2} G_{i\overline{j}} \overline{L}^\Lambda - \frac{1}{2} (\partial_i K) \overline{\partial}_{\overline{j}} \overline{L}^\Lambda; \tag{4.1.32}$$

since $\overline{L}^\Lambda$ satisfies the Schwarz lemma on (flat) partial derivatives in M_{n_V}, by reusing Eq. (4.1.31), this implies

$$D_i \overline{D}_{\overline{j}} \overline{L}^\Lambda = G_{i\overline{j}} \overline{L}^\Lambda. \tag{4.1.33}$$

By repeating the same procedure for $\overline{M}_\Lambda$, one obtains the result (4.1.29), which therefore relies only on the Kähler-covariant holomorphicity of the vector V with Kähler weights $(1, -1)$ (and, rigorously, on the commutation of flat partial derivatives acting on K and V).

The SG constraints (7.2.1.39) (or (4.1.9)-(4.1.10)) may be solved by formulating the following fundamental Ansätze:

$$M_\Lambda (z, \overline{z}) = \mathcal{N}_{\Lambda\Sigma} (z, \overline{z}) L^\Sigma (z, \overline{z}), \tag{4.1.34}$$

$$h_{i\Lambda} (z, \overline{z}) = \overline{\mathcal{N}}_{\Lambda\Sigma} (z, \overline{z}) f_i^\Sigma (z, \overline{z}). \tag{4.1.35}$$

where $\mathcal{N}_{\Lambda\Sigma}$ is a complex symmetric matrix. Such Ansätze are the fundamental relations on which the symplectic special Kähler geometry of the $\mathcal{N} = 2$, $d = 4$ n_V-fold Maxwell Einstein Supergravity Theory is founded. They express the $Sp(2n_V + 2)$ symmetry acting on the special Kähler geometry of the moduli space M_{n_V}.

By conjugating Eq. (4.1.34), the symmetry of $\mathcal{N}_{\Lambda\Sigma}$ and the conditions of normalization of sections given by Eq. (4.1.26) imply

$$-i \equiv \langle V, \overline{V} \rangle = \overline{L}^\Lambda M_\Lambda - \overline{M}_\Lambda L^\Lambda =$$

$$= \overline{L}^\Lambda \mathcal{N}_{\Lambda\Sigma} L^\Sigma - \overline{\mathcal{N}}_{\Lambda\Sigma} \overline{L}^\Sigma L^\Lambda =$$

$$= (\mathcal{N}_{\Lambda\Sigma} - \overline{\mathcal{N}}_{\Lambda\Sigma}) L^\Lambda \overline{L}^\Sigma = 2i Im (\mathcal{N}_{\Lambda\Sigma}) L^\Lambda \overline{L}^\Sigma;$$

$$\Updownarrow$$

$$Im (\mathcal{N}_{\Lambda\Sigma}) L^\Lambda \overline{L}^\Sigma = -\frac{1}{2}. \tag{4.1.36}$$

Thence, by using Eqs. (4.1.26), (7.2.1.38), (4.1.29), (4.1.34) and (4.1.35), it may be

explicitly calculated that

$$
\begin{cases}
I. \ \langle V, U_{\bar{i}} \rangle = 0 \Leftrightarrow \langle \overline{V}, \overline{U}_i \rangle = 0; \\[4mm]
II. \ G_{i\bar{j}} = -i \left\langle U_i, \overline{U}_{\bar{j}} \right\rangle; \\[4mm]
III. \ C_{ijk} = \left\langle D_i U_j, U_k \right\rangle.
\end{cases}
\tag{4.1.37}
$$

Notice that the first result, i.e. $\langle V, U_{\bar{i}} \rangle = 0$, is trivial because $U_{\bar{i}} \equiv \overline{D}_{\bar{i}} V = 0$ by construction.

Moreover, it can also be proved that

$$
\langle V, U_i \rangle = 0 \Longleftrightarrow \langle \overline{V}, \overline{U}_{\bar{i}} \rangle = 0.
\tag{4.1.38}
$$

Indeed, by exploiting the distributivity of the Kähler-covariant derivative w.r.t. the symplectic scalar product $\langle \cdot, \cdot \rangle$ and the Kähler-covariant holomorphicity of V, and using Eq. (7.2.1.38), one gets

$$
D_i \left\langle \overline{V}, U_j \right\rangle = \left\langle D_i \overline{V}, U_j \right\rangle + \left\langle \overline{V}, D_i U_j \right\rangle = i C_{ijk} G^{k\bar{k}} \left\langle \overline{V}, \overline{U}_{\bar{k}} \right\rangle.
\tag{4.1.39}
$$

Now, by also recalling the normalization (4.1.26), it holds that

$$
0 = D_i \left\langle V, \overline{V} \right\rangle = \left\langle D_i V, \overline{V} \right\rangle + \left\langle V, D_i \overline{V} \right\rangle = \left\langle U_i, \overline{V} \right\rangle = - \left\langle \overline{V}, U_i \right\rangle.
\tag{4.1.40}
$$

By substituting such a result back into Eq. (4.1.39), one gets

$$
C_{ijk} (z, \bar{z}) \, G^{k\bar{k}} (z, \bar{z}) \left\langle \overline{V} (z, \bar{z}), \overline{U}_{\bar{k}} (z, \bar{z}) \right\rangle = 0, \ \forall (z, \bar{z}) \in M_{n_V}
$$
$$
\Updownarrow
$$
$$
\left\langle \overline{V}, \overline{U}_{\bar{i}} \right\rangle = 0 \Longleftrightarrow \langle V, U_i \rangle = 0,
\tag{4.1.41}
$$

q.e.d. .

Moreover, it is straightforward to calculate

$$
\langle V, \overline{U}_{\bar{i}} \rangle \ \equiv \ V^T \epsilon \overline{U}_{\bar{i}} = -L^\Lambda \overline{h}_{\bar{i}\Lambda} + M_\Lambda \overline{f}_{\bar{i}}^\Lambda =
$$

$$
= \ -L^\Lambda \mathcal{N}_{\Lambda\Sigma} f_{\bar{i}}^\Sigma + \mathcal{N}_{\Lambda\Sigma} L^\Sigma f_{\bar{i}}^\Lambda = 0,
\tag{4.1.42}
$$

where in the second line we used the Ansätze (4.1.34) and (4.1.35) and the symme-try of $\mathcal{N}_{\Lambda\Sigma}$. Summarizing, in the SKG framework the vector V is symplectically

orthogonal to all its Kähler-covariant derivatives

$$
\begin{cases}
\langle V, U_i \rangle = 0; \\[2ex]
\langle V, U_{\bar{i}} \rangle = 0; \\[2ex]
\langle V, \overline{U}_i \rangle = 0; \\[2ex]
\langle V, \overline{U}_{\bar{i}} \rangle = 0.
\end{cases}
\tag{4.1.43}
$$

Notice that Eq. (7.2.1.38) and the last relation of Eq. (4.1.43) yield

$$
\langle V, D_i U_j \rangle = \langle V, D_i D_j V \rangle = i C_{ijk} G^{k\bar{k}} \langle V, \overline{D}_{\bar{k}} \overline{V} \rangle = 0.
\tag{4.1.44}
$$

By applying the Kähler-covariant holomorphic derivative to $\langle V, \overline{U}_{\bar{i}} \rangle = 0$ and using Eqs. (4.1.26) and (4.1.29), it is immediate to prove the result II of Eq. (4.1.37); indeed

$$
\begin{aligned}
0 &= D_j \langle V, \overline{U}_{\bar{i}} \rangle = D_j \langle V, \overline{D}_{\bar{i}} \overline{V} \rangle = \langle D_j V, \overline{D}_{\bar{i}} \overline{V} \rangle + \langle V, D_j \overline{D}_{\bar{i}} \overline{V} \rangle = \\[2ex]
&= \langle D_j V, \overline{D}_{\bar{i}} \overline{V} \rangle + G_{j\bar{i}} \langle V, \overline{V} \rangle \iff \langle D_j V, \overline{D}_{\bar{i}} \overline{V} \rangle = i G_{j\bar{i}}.
\end{aligned}
\tag{4.1.45}
$$

Now, by complex conjugating the Ansatz (4.1.34) and considering the Ansatz (4.1.35), one gets

$$
\begin{cases}
\overline{M}_\Lambda = \overline{\mathcal{N}}_{\Lambda\Sigma} \overline{L}^\Sigma, \\[2ex]
h_{i\Lambda} = \overline{\mathcal{N}}_{\Lambda\Sigma} f_i^\Sigma.
\end{cases}
\tag{4.1.46}
$$

It then appears natural to define some square matrices with $(n_V + 1)^2$ complex entries, corresponding to completing the $(n_V + 1) \times n_V$ complex matrices f_i^Λ and $h_{i\Lambda}$ to a square form as follows:

$$
f_I^\Lambda \equiv \left(f_i^\Lambda, \overline{L}^\Lambda \right), \quad h_{I\Lambda} \equiv \left(h_{i\Lambda}, \overline{M}_\Lambda \right);
\tag{4.1.47}
$$

consequently, $\overline{\mathcal{N}}_{\Lambda\Sigma}$ may be written as

$$
\overline{\mathcal{N}}_{\Lambda\Sigma} = h_{I\Lambda} \left(f^{-1} \right)^I_\Sigma.
\tag{4.1.48}
$$

It is clear that Eq. (4.1.48), through the definitions given by Eq. (4.1.47), is completely

equivalent to the set of Ansätze (4.1.34) and (4.1.35)

$$\overline{\mathcal{N}}_{\Lambda\Sigma}(z,\bar{z}) = h_{I\Lambda}(z,\bar{z})\left(f^{-1}\right)^{I}_{\Sigma}(z,\bar{z}) \overset{Eq.\,(4.1.47)}{\Longleftrightarrow} \begin{cases} M_{\Lambda}(z,\bar{z}) = \mathcal{N}_{\Lambda\Sigma}(z,\bar{z})\, L^{\Sigma}(z,\bar{z})\,, \\[2ex] h_{i\Lambda}(z,\bar{z}) = \overline{\mathcal{N}}_{\Lambda\Sigma}(z,\bar{z})\, f^{\Sigma}_{i}(z,\bar{z})\,. \end{cases}$$

$$(4.1.49)$$

Moreover, by Kähler-covariantly differentiating Eq. (4.1.34) and using Eq. (4.1.35) and (7.2.1.34), we may obtain the following results:

$$\left(\mathcal{N}_{\Lambda\Sigma} - \overline{\mathcal{N}}_{\Lambda\Sigma}\right) f^{\Sigma}_{i} = -\left(D_{i}\mathcal{N}_{\Lambda\Sigma}\right) L^{\Sigma}. \qquad (4.1.50)$$

$$\left(\overline{D}_{\bar{i}}\mathcal{N}_{\Lambda\Sigma}\right) L^{\Sigma} = 0. \qquad (4.1.51)$$

Clearly, the very definition of $\mathcal{N}_{\Lambda\Sigma}$ by Eq. (4.1.34) implies that such a matrix has vanishing Kähler weights, because M_{Λ} and L^{Λ} are components of the same $(2n_V + 2)$-tet in the vector representation of the symplectic group $Sp(2n_V + 2)$. If $\mathcal{N}_{\Lambda\Sigma}$ were not Kähler gauge-invariant, it would violate the symplectic inner structure of the special Kähler-Hodge geometry of M_{n_V} (such a feature of $\mathcal{N}_{\Lambda\Sigma}$ is clear also by looking at Eq. (4.1.36), by simply noticing that a quantity and its complex conjugate have always opposite Kähler weights: see Eqs. (4.1.16) and (4.1.20)).

Therefore, Eqs. (4.1.50) and (4.1.51) may actually be rewritten as follows:

$$\left(\mathcal{N}_{\Lambda\Sigma} - \overline{\mathcal{N}}_{\Lambda\Sigma}\right) f^{\Sigma}_{i} = -\left(\partial_{i}\mathcal{N}_{\Lambda\Sigma}\right) L^{\Sigma}$$
$$\Updownarrow$$
$$2i\left(Im\left(\mathcal{N}\right)\right)_{\Lambda\Sigma} f^{\Sigma}_{i} = -\left(\partial_{i}\mathcal{N}_{\Lambda\Sigma}\right) L^{\Sigma}; \qquad (4.1.52)$$

$$\left(\bar{\partial}_{\bar{i}}\mathcal{N}_{\Lambda\Sigma}\right) L^{\Sigma} = 0. \qquad (4.1.53)$$

It is worth mentioning that Eq. (4.1.53) does not imply the holomorphicity of $\mathcal{N}_{\Lambda\Sigma}$, as it will be clear further below.

Now, due to the Kähler-covariant holomorphicity of the sections L^{Λ}'s and M_{Λ}'s of the Hodge bundle $\mathfrak{I}$ over M_{n_V}, we may define some symplectic-indexed holomorphic functions $X^{\Lambda}(z)$ and $F_{\Lambda}(z)$ in the moduli space M_{n_V} by using the related Kähler potential

$$L^{\Lambda}(z,\bar{z}) \equiv exp\left(\tfrac{1}{2}K(z,\bar{z})\right) X^{\Lambda}(z)\,;$$

$$(4.1.54)$$

$$M_{\Lambda}(z,\bar{z}) \equiv exp\left(\tfrac{1}{2}K(z,\bar{z})\right) F_{\Lambda}(z)\,;$$

we may then arrange them in the holomorphic $(2n_V + 2)$-dim. symplectic vector

$$\Phi(z) \equiv \begin{pmatrix} X^\Lambda(z) \\ \\ F_\Lambda(z) \end{pmatrix} = exp\left(-\frac{1}{2}K(z,\bar{z})\right) V(z,\bar{z}). \tag{4.1.55}$$

It is also easy to realize that Eqs. (4.1.54) define nothing but sections of an holomorphic line bundle over M_{n_V}, and all previous formulae may be rewritten in terms of such sections. First of all, we may obtain a simple symplectic-invariant expression of the Kähler potential in the moduli space by recalling the normalization of the sections given by Eq. (4.1.26)

$$-i = \langle V, \overline{V} \rangle = exp(K(z,\bar{z})) \langle \Phi(z), \overline{\Phi}(\bar{z}) \rangle \tag{4.1.56}$$

$$\Updownarrow$$

$$K(z,\bar{z}) = -ln\left[i\langle \Phi(z), \overline{\Phi}(\bar{z}) \rangle\right] \equiv -ln\left[i\Phi^T(z)\,\epsilon\overline{\Phi}(\bar{z})\right] =$$

$$= -ln\left[i\left(X^\Lambda(z),\ F_\Lambda(z)\right)\begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}\begin{pmatrix} \overline{X}^\Lambda(\bar{z}) \\ \\ \overline{F}_\Lambda(\bar{z}) \end{pmatrix}\right] =$$

$$\tag{4.1.57}$$

$$= -ln\left[i\left(X^\Lambda(z),\ F_\Lambda(z)\right)\begin{pmatrix} -\overline{F}_\Lambda(\bar{z}) \\ \\ \overline{X}^\Lambda(\bar{z}) \end{pmatrix}\right] =$$

$$= -ln\left\{i\left[\overline{X}^\Lambda(\bar{z}) F_\Lambda(z) - X^\Lambda(z)\overline{F}_\Lambda(\bar{z})\right]\right\}$$

$$\Updownarrow$$

$$exp[-K(z,\bar{z})] = i\left[\overline{X}^\Lambda(\bar{z}) F_\Lambda(z) - X^\Lambda(z)\overline{F}_\Lambda(\bar{z})\right]. \tag{4.1.58}$$

Eq. (4.1.55) trivially yields

$$|V(z,\bar{z})\rangle = exp\left(\tfrac{1}{2}K(z,\bar{z})\right) \begin{pmatrix} X^\Lambda(z) \\ \\ F_\Lambda(z) \end{pmatrix}$$

$$\Downarrow$$

$$|U_i(z,\bar{z})\rangle = \exp\left(\tfrac{1}{2}K(z,\bar{z})\right) \begin{pmatrix} (\partial_i K)\, X^\Lambda(z) + \partial_i X^\Lambda(z) \\ \\ (\partial_i K)\, F_\Lambda(z) + \partial_i F_\Lambda(z) \end{pmatrix},$$

$$(4.1.59)$$

and therefore, using Eq. (4.1.41), we get that

$$\langle V, U_i\rangle = 0 \Leftrightarrow X^\Lambda(z)\,\partial_i F_\Lambda(z) - \left(\partial_i X^\Lambda(z)\right) F_\Lambda(z) = 0, \quad \forall i = 1, ..., n_V. \quad (4.1.60)$$

Notice that the symplectic holomorphic vector Φ has Kähler weights $(2,0)$, i.e. under a Kähler gauge transformation (4.1.15) it transforms as $\Phi(z) \to \Phi(z)\, e^{-f(z)}$. This is clearly due to the fact that the symplectic holomorphic sections $X^\Lambda(z)$ and $F_\Lambda(z)$ have Kähler weights $(2,0)$, and therefore under a Kähler gauge transformation (4.1.15) they respectively transform as

$$\begin{cases} X^\Lambda(z) \longrightarrow e^{-f(z)} X^\Lambda(z)\,; \\ \\ F_\Lambda(z) \longrightarrow e^{-f(z)} F_\Lambda(z)\,. \end{cases}$$

$$(4.1.61)$$

Thus, $X^\Lambda(z)$ and $F_\Lambda(z)$ may be considered as symplectic sections of the holomorphic line bundle over M_{n_V}. Due to its Kähler transformation properties (4.1.61), the set $\left\{X^\Lambda\right\}_{\Lambda=0,1,...,n_V}$ may be regarded, at least locally, as a set of homogeneous coordinates in the Kähler-Hodge manifold M_{n_V}, provided that the $n_V \times n_V$ holomorphic matrix of change between the Kähler gauge-invariant sets of coordinates $\left\{z^i\right\}_{i=1,...,n_V}$ and

$$\{t^a(z)\}_{a=1,...,n_V} \equiv \left\{\frac{X^a(z)}{X^0(z)}\right\}_{a=1,...,n_V}, \quad (4.1.62)$$

i.e.

$$e_i^a(z) \equiv \partial_i\left(\frac{X^a(z)}{X^0(z)}\right), \quad (4.1.63)$$

is invertible.

If, as we suppose, this is the case, then

$$F_\Lambda(z) = F_\Lambda(z(X)) = F_\Lambda(X). \tag{4.1.64}$$

By using the relation (4.1.60) and the homogeneity of degree 1 of $F_\Lambda(X)$

$$X^\Sigma \partial_\Sigma F_\Lambda = F_\Lambda \tag{4.1.65}$$

(where $\partial_\Sigma \equiv \partial/\partial X^\Sigma$), it is thus possible to state that a symplectic coordinate frame $\{X^\Lambda\}$ always exists such that $F_\Lambda(X)$ may be written in terms of a scalar potential F, holomorphic and homogeneous of degree 2 in the X^Λ's

$$F_\Lambda(X) = \partial_\Lambda F(X),$$

$$\tag{4.1.66}$$

$$X^\Sigma \partial_\Sigma F = 2F.$$

The function $F(X) = F(X(z))$ is the (holomorphic) prepotential of vector multiplet couplings ([39], [42], [43], [44]) in the considered $\mathcal{N} = 2$ ($d = 4$) Maxwell Einstein Supergravity Theory. Due to the additivity of the Kähler weights, by definition the prepotential F has Kähler weights $(4, 0)$.

From the definition (4.1.62) it follows that the t^a's are Kähler gauge-invariant coordinates, i.e. they have Kähler weights $(0, 0)$. It is also possible to choose a particular set of homogeneous coordinates in M_{n_V}, named "special coordinates" ([45], [46], [47], [36], [48]), corresponding to the position

$$e_i^a(z) \equiv \partial_i \left(\frac{X^a(z)}{X^0(z)} \right) = \delta_i^a, \tag{4.1.67}$$

i.e. to

$$\begin{cases} X^0 = 1; \\ \\ X^i = t^i = z^i. \end{cases} \implies f_i^\Lambda \equiv D_i L^\Lambda = e^{\frac{1}{2}K} D_i X^\Lambda = e^{\frac{1}{2}K} \delta_i^\Lambda.$$

$$\tag{4.1.68}$$

By such considerations, it is then clear that the coordinates z^i's and $\bar{z}^{\bar{i}}$'s and the related partial differential operators ∂_i's and $\bar{\partial}_{\bar{i}}$'s have vanishing Kähler weights.

By using the definitions (4.1.54), the lower-boundedness of the Kähler potential allows one to rewrite Eq. (4.1.53) as follows:

$$\left(\bar{\partial}_{\bar{i}} \mathcal{N}_{\Lambda\Sigma} \right) X^\Sigma = 0. \tag{4.1.69}$$

By considering the set of local homogenous coordinates $\{t^a(z)\}_{a=1,...,n_V}$ previously defined, the above result may be recast in the following form:

$$\left[\frac{\partial}{\partial \bar{z}^i}\mathcal{N}_{\Lambda\Sigma}(z,\bar{z})\right] X^\Sigma(z) = 0$$

$$\Updownarrow$$

$$\overline{\partial}_{\bar{i}}\left[\overline{\left(\frac{X^a}{X^0}\right)}(\bar{z})\right]\left[\frac{\partial}{\partial\left(\frac{\overline{X^a}}{\overline{X^0}}\right)}\mathcal{N}_{\Lambda\Sigma}(X,\overline{X})\right] X^\Sigma = 0$$

$$\Updownarrow$$

$$\bar{e}_{\bar{i}}^{\bar{a}}(\bar{z})\left[\frac{\partial}{\partial\left(\frac{\overline{X^a}}{\overline{X^0}}\right)}\mathcal{N}_{\Lambda\Sigma}(X,\overline{X})\right] X^\Sigma = 0,$$

$$(4.1.70)$$

where in the last line we introduced $\bar{e}_{\bar{i}}^{\bar{a}}(\bar{z}) \equiv \overline{e_i^a(z)}$. By specializing Eq. (4.1.70) to "special coordinates", we may rewrite it as

$$\frac{\partial}{\partial\overline{X}^{\bar{i}}}\mathcal{N}_{\Lambda 0}(X,\overline{X}) + \left[\frac{\partial}{\partial\overline{X}^{\bar{i}}}\mathcal{N}_{\Lambda j}(X,\overline{X})\right] X^j = 0. \tag{4.1.71}$$

Thus, provided that the matrix e_i^a exists (and it is invertible), due to the generally non-trivial dependence of $\mathcal{N}_{\Lambda\Sigma}$ on the (eventually "special") homogenous coordinates of M_{n_V}, it is clear that

$$\left(\overline{\partial}_{\bar{i}}\mathcal{N}_{\Lambda\Sigma}\right) L^\Sigma = 0 \nRightarrow \overline{\partial}_{\bar{i}}\mathcal{N}_{\Lambda\Sigma} = 0, \tag{4.1.72}$$

as previously pointed out.

At this point, in order to investigate more in depth the differential properties of the complex symmetric matrix $\mathcal{N}_{\Lambda\Sigma}$, let us consider the non-trivial orthogonal relation given by Eq. (4.1.41), and let us use the Ansätze (4.1.34) and (4.1.35)

$$0 = \langle V, U_i \rangle \equiv V^T \epsilon U_i;$$

$$\Updownarrow$$

$$M_\Lambda f_i^\Lambda - L^\Lambda h_{i\Lambda} = 2i\left(Im\left(\mathcal{N}\right)\right)_{\Lambda\Sigma} L^\Lambda f_i^\Sigma = 0. \tag{4.1.73}$$

Thence, Eqs. (4.1.52) and (4.1.73) imply (for lower-bounded Kähler potential)

$$\left(\partial_i\mathcal{N}_{\Lambda\Sigma}\right) L^\Lambda L^\Sigma = 0 \Leftrightarrow \left(\partial_i\mathcal{N}_{\Lambda\Sigma}\right) X^\Lambda X^\Sigma = 0. \tag{4.1.74}$$

It is interesting to notice that, despite the symmetry of $\partial_i \mathcal{N}_{\Lambda\Sigma}$ and $X^\Lambda X^\Sigma$ in the symplectic indices, the dipendence of $\mathcal{N}_{\Lambda\Sigma}$ on the X's is such to make the product $(\partial_i \mathcal{N}_{\Lambda\Sigma}) X^\Lambda X^\Sigma$ vanish. Thus, the differential properties of $\mathcal{N}_{\Lambda\Sigma}$ may be summarized as follows:

$$\begin{cases} \left(\bar{\partial}_{\bar{i}} \mathcal{N}_{\Lambda\Sigma}\right) X^\Sigma = 0; \\ \\ (\partial_i \mathcal{N}_{\Lambda\Sigma}) X^\Lambda X^\Sigma = 0. \end{cases} \tag{4.1.75}$$

It should also be noticed that under coordinate transformations the holomorphic symplectic vector of sections $\Phi(z)$ transforms as

$$\widetilde{\Phi}(z) = e^{-f_S(z)} \mathcal{S}(z)\Phi(z), \tag{4.1.76}$$

where the holomorphic $(2n_V + 2) \times (2n_V + 2)$ matrix $\mathcal{S}(z)$ has a symplectic real structure, i.e. it is an element of $Sp(2n_V + 2, \mathbb{R})$, preserving the $(2n_V + 2)$-dim. symplectic metric defined in Eq. (4.1.25); the $\mathcal{S}$-dependent factor $e^{-f_S(z)}$ corresponds to (an holomorphic) Kähler transformation. We may naturally divide $\mathcal{S}(z)$ in $(n_V + 1)$-dim. sub-blocks

$$\mathcal{S}(z) = \begin{pmatrix} A(z) & B(z) \\ \\ C(z) & D(z) \end{pmatrix}. \tag{4.1.77}$$

The symplecticity condition $\mathcal{S}^T(z)\epsilon\mathcal{S}(z) = \epsilon$ then implies the following relations among the sub-blocks:

$$\begin{cases} A^T D - C^T B = \mathbb{I}, \\ \\ A^T C - C^T A = B^T D - D^T B = 0. \end{cases} \tag{4.1.78}$$

Now, by differentiating both sides of the degree 2 homogeneity property of $F(X)$ (with Kähler weights $(4, 0)$)

$$F(X) = \frac{1}{2} X^\Lambda F_\Lambda, \tag{4.1.79}$$

we trivially reobtain that F_Λ (having Kähler weights $(2, 0)$) is homogeneous of degree 1 in the X^Λ's (see Eq. (4.1.65))

$$F_\Sigma = X^\Lambda F_{\Lambda\Sigma}, \tag{4.1.80}$$

where we defined the Kähler gauge-invariant rank-2 symmetric tensor $F_{\Lambda\Sigma} \equiv \frac{\partial^2 F}{\partial X^\Lambda \partial X^\Sigma}$, denoted with $\mathcal{F}(z)$ in symplectic matrix notation[6]. By iterating the differentiation,

---

[6]Attention should be paid to carefully distinguish between:

we get

$$X^\Lambda F_{\Lambda\Sigma\Xi} = 0, \tag{4.1.81}$$

simply meaning that the completely symmetric, rank-3 symplectic tensor $F_{\Lambda\Sigma\Xi} \equiv \frac{\partial^2 F}{\partial X^\Lambda \partial X^\Sigma \partial X^\Xi}$ is homogeneous of degree 0 in the coordinates X^Λ's; moreover, by a simple counting, such a tensor turns out to have Kähler weights $(-2, 0)$.

By recalling the definitions (4.1.27) and (4.1.54), the Kähler covariant derivatives of $F_{\Lambda\Sigma}$ read

$$\begin{aligned}
D_i F_{\Lambda\Sigma} &= \partial_i F_{\Lambda\Sigma} = \frac{\partial F_{\Lambda\Sigma}(z)}{\partial z^i} = D_i X^\Xi(z) \frac{\partial F_{\Lambda\Sigma}(z)}{\partial X^\Xi} = \\
&= e^{-\frac{1}{2}K(z,\bar{z})} D_i L^\Xi(z,\bar{z}) \frac{\partial F_{\Lambda\Sigma}(z)}{\partial X^\Xi} = e^{-\frac{1}{2}K(z,\bar{z})} f_i^\Xi(z,\bar{z}) F_{\Lambda\Sigma\Xi}(z).
\end{aligned} \tag{4.1.82}$$

Consequently, by using such a result we may write

$$\begin{aligned}
h_{i\Lambda} &\equiv D_i M_\Lambda = e^{\frac{1}{2}K} D_i F_\Lambda = e^{\frac{1}{2}K} D_i \left(X^\Sigma F_{\Lambda\Sigma} \right) = \\
&= e^{\frac{1}{2}K} \left(D_i X^\Sigma \right) F_{\Lambda\Sigma} + e^{\frac{1}{2}K} X^\Sigma D_i F_{\Lambda\Sigma} = \left(D_i L^\Sigma \right) F_{\Lambda\Sigma} + f_i^\Xi X^\Sigma F_{\Lambda\Sigma\Xi} = \\
&= F_{\Lambda\Sigma} f_i^\Sigma.
\end{aligned} \tag{4.1.83}$$

By recalling the Ansatz (4.1.35), we thus obtain that the two following formulae are equivalent:

$$h_{i\Lambda} = \overline{\mathcal{N}}_{\Lambda\Sigma} f_i^\Sigma; \tag{4.1.84}$$

$$h_{i\Lambda} = F_{\Lambda\Sigma} f_i^\Sigma. \tag{4.1.85}$$

Nevertheless, it is worth pointing out that, whereas Eq. (4.1.84) always holds, Eq. (4.1.85) is meaningful only in the cases in which the prepotential F may be defined[7].

---

1) the quantities F, F_Λ, $F_{\Lambda\Sigma} \equiv \frac{\partial^2 F}{\partial X^\Lambda \partial X^\Sigma} \equiv \mathcal{F}$

and

2) the quantities $\mathcal{F}^{-\Lambda}$, $\mathcal{F}^{+\Lambda}$, $\mathcal{F}^\Lambda$ and $^*\mathcal{F}^\Lambda$, which are related to the Abelian vector field strengths in the $\mathcal{N} = 2$, $d = 4$ n_V-fold Maxwell Einstein Supergravity Theory; they will be introduced in Subsect. 3.2.

[7] For a discussion of some relevant cases in which F does not exist (such as the low energy effective action of $\mathcal{N} = 2$ heterotic string theory), see e.g. [38].

Now, by using the definition (4.1.27) and Eqs. (4.1.82) and (4.1.83), from the third of Eqs. (4.1.37) we get

$$C_{ijk} = \langle D_i U_j, U_k \rangle \equiv (D_i U_j)^T \epsilon U_k =$$

$$= \begin{pmatrix} D_i D_j L^\Lambda, & D_i D_j M_\Lambda \end{pmatrix} \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \begin{pmatrix} D_k L^\Lambda \\ D_k M_\Lambda \end{pmatrix} =$$

$$= \begin{pmatrix} D_i D_j L^\Lambda, & D_i D_j M_\Lambda \end{pmatrix} \begin{pmatrix} -D_k M_\Lambda \\ D_k L^\Lambda \end{pmatrix} =$$

$$= - \left(D_i D_j L^\Lambda \right) D_k M_\Lambda + (D_i D_j M_\Lambda) D_k L^\Lambda =$$

$$= -h_{k\Lambda} D_i f_j^\Lambda + f_k^\Lambda D_i h_{j\Lambda} = -\overline{\mathcal{N}}_{\Lambda\Sigma} f_k^\Sigma D_i f_j^\Lambda + f_k^\Lambda D_i \left(\overline{\mathcal{N}}_{\Lambda\Sigma} f_j^\Sigma \right) =$$

$$= -\overline{\mathcal{N}}_{\Lambda\Sigma} f_k^\Sigma D_i f_j^\Lambda + f_k^\Lambda \left(D_i \overline{\mathcal{N}}_{\Lambda\Sigma} \right) f_j^\Sigma + f_k^\Lambda \overline{\mathcal{N}}_{\Lambda\Sigma} D_i f_j^\Sigma =$$

$$= f_k^\Lambda \left(\partial_i \overline{\mathcal{N}}_{\Lambda\Sigma} \right) f_j^\Sigma = f_k^\Lambda \left(\partial_i F_{\Lambda\Sigma} \right) f_j^\Sigma =$$

$$= e^{-\frac{1}{2}K} f_k^\Lambda f_i^\Xi f_j^\Sigma F_{\Lambda\Sigma\Xi} = e^{-\frac{1}{2}K} f_i^\Lambda f_j^\Sigma f_k^\Xi F_{\Lambda\Sigma\Xi}, \tag{4.1.86}$$

where we also used the Kähler gauge-invariance of the complex matrix $\mathcal{N}_{\Lambda\Sigma}$ and the symmetry of the tensor C_{ijk}. The symplectic-invariant and Kähler-covariant expression (4.1.86) for C_{ijk} may be further elaborated (at the price of losing the manifest Kähler covariance) by expliciting the Kähler-covariant derivative encoded in f_i^Λ and using Eq. (4.1.81)

$$C_{ijk} = e^{-\frac{1}{2}K} f_i^\Lambda f_j^\Sigma f_k^\Xi F_{\Lambda\Sigma\Xi} = e^{-\frac{1}{2}K} \left(D_i L^\Lambda \right) \left(D_j L^\Sigma \right) \left(D_k L^\Xi \right) F_{\Lambda\Sigma\Xi} =$$

$$= e^K \left(D_i X^\Lambda \right) \left(D_j X^\Sigma \right) \left(D_k X^\Xi \right) F_{\Lambda\Sigma\Xi} =$$

$$= e^K \left[\partial_i X^\Lambda + (\partial_i K) X^\Lambda \right] \left[\partial_j X^\Sigma + (\partial_j K) X^\Sigma \right] \left[\partial_k X^\Xi + (\partial_k K) X^\Xi \right] F_{\Lambda\Sigma\Xi} =$$

$$= e^K \left(\partial_i X^\Lambda \right) \left(\partial_j X^\Sigma \right) \left(\partial_k X^\Xi \right) F_{\Lambda\Sigma\Xi}. \tag{4.1.87}$$

By further specializing such a result in the symplectic frame (4.1.68) of "special co-

ordinates", for which $f_i^\Lambda = e^{\frac{1}{2}K}\delta_i^\Lambda$, one finally gets (see also Eq. (4.1.62))

$$C_{ijk} \quad = \quad e^K \delta_i^\Lambda \delta_j^\Sigma \delta_k^\Xi F_{\Lambda\Sigma\Xi}(t) = e^K F_{ijk}(t) = e^K \partial_i \partial_j \partial_k F(t),$$

$$(4.1.88)$$

which is symplectic-invariant, but manifestly Kähler-non-covariant.

It is easy to see that, in the case in which the prepotential F exists, the symplectic-orthogonality relation (4.1.41) between the $Sp(2n_V + 2)$-covariant vectors V and U_i reduces to nothing but an integrability condition in the "special coordinates" (4.1.68) of M_{n_V}. In order to show this, let us firstly explicit the relation (4.1.41), by writing

$$0 \quad = \quad \langle V, U_i \rangle \equiv V^T \epsilon U_i = \begin{pmatrix} L^\Lambda, & M_\Lambda \end{pmatrix} \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \begin{pmatrix} D_i L^\Lambda \\ D_i M_\Lambda \end{pmatrix} =$$

$$= \quad \begin{pmatrix} L^\Lambda, & M_\Lambda \end{pmatrix} \begin{pmatrix} -D_i M_\Lambda \\ D_i L^\Lambda \end{pmatrix} = M_\Lambda D_i L^\Lambda - L^\Lambda D_i M_\Lambda =$$

$$= \quad -e^K \left(X^\Lambda D_i F_\Lambda - F_\Lambda D_i X^\Lambda \right) =$$

$$= \quad -e^K \left\{ X^\Lambda \left[\partial_i F_\Lambda + (\partial_i K) F_\Lambda \right] - F_\Lambda \left[\partial_i X^\Lambda + (\partial_i K) X^\Lambda \right] \right\} =$$

$$= \quad -e^K \left(X^\Lambda \partial_i F_\Lambda - F_\Lambda \partial_i X^\Lambda \right) = e^K \left[\partial_i \left(X^\Lambda F_\Lambda \right) - 2 F_\Lambda \partial_i X^\Lambda \right] =$$

$$= \quad e^K \left[\partial_i \left(X^\Lambda F_\Lambda \right) - 2 \partial_i F \right] = e^K \partial_i \left(X^\Lambda F_\Lambda - 2F \right). \qquad (4.1.89)$$

If we now specify the result (4.1.89) to the "special coordinates" (4.1.68) and recall the property (4.1.81) of homogeneity of degree 0 of the function $F_{\Lambda\Sigma\Xi}$, we get

$$\langle V, U_i \rangle = 0 \Longleftrightarrow \partial_i F - X^j \partial_i \partial_j F = 0;$$

$$\Downarrow$$

$$\partial_k \partial_i F - \partial_i \partial_k F - X^j \partial_k \partial_i \partial_j F = 0;$$

$$\Updownarrow$$

$$\partial_k \partial_i F(t) = \partial_i \partial_k F(t), \; \forall (i,k) \in \{1,...,n_V\}^2, \qquad (4.1.90)$$

which is satisfied iff the function $F(t)$ satisfies the Schwarz Lemma on partial derivatives, i.e. if it is integrable in the "special coordinates" of M_{n_V}. Thus, we may say

that, whenever the prepotential F exists, its generalized, symplectic-invariant integrability condition in the moduli space M_{n_V} is given by the orthogonality relation $\langle V, U_i \rangle = 0$.

Now, by recalling Eqs. (4.1.76)-(4.1.78) and using the relation (4.1.79), we may rewrite the transformation law of X^Λ and $F(X)$ under symplectic transformations (disregarding the Kähler transformation factors) respectively as follows:

$$\widetilde{X}(z) = [A(z) + B(z)\mathcal{F}(z)]X(z); \tag{4.1.91}$$

$$\widetilde{F}\left(\widetilde{X}\right) = \widetilde{F}((A + B\mathcal{F})X) = \tfrac{1}{2}\widetilde{X}^\Lambda \widetilde{F}_\Lambda =$$

$$= \left[F(X) + X^\Lambda \left(C^T B\right)^\Sigma_\Lambda F_\Sigma + \tfrac{1}{2}X^\Lambda \left(C^T A\right)_{\Lambda\Sigma} X^\Sigma + \tfrac{1}{2}F_\Lambda \left(D^T B\right)^{\Lambda\Sigma} F_\Sigma\right]. \tag{4.1.92}$$

Analogously, by using the Ansatz (4.1.34), the transformation property (4.1.76) yields the following transformation law for the matrix $\mathcal{N}$:

$$\widetilde{\mathcal{N}}_{\Lambda\Sigma}\left(\widetilde{X}, \widetilde{F}\right) = (C + D\mathcal{N}(X, F))(A + B\mathcal{N}(X, F))^{-1}. \tag{4.1.93}$$

Eq. (4.1.91) shows that the transformation $X \to \widetilde{X}$ can eventually be singular, thus implying the non existence of the prepotential $F(X)$, depending on the choice of the symplectic gauge ([49], [38]). On the other hand, some physically interesting cases, such as the $\mathcal{N} = 2 \longrightarrow \mathcal{N} = 0$ SUSY breaking [50], correspond to situations in which $F(X)$ does not exist. Therefore, the tensor calculus constructions of the $\mathcal{N} = 2$ theories actually turn out to be not completely general, because they use special coordinates from the very beginning, and they are essentially founded on the existence of the prepotential $F(X)$.

By considering the low-energy $\mathcal{N} = 2$, $d = 4$ Maxwell Einstein Supergravity Theory Lagrangian density, we may observe that $Im(\mathcal{N}_{\Lambda\Sigma})$ and $Re(\mathcal{N}_{\Lambda\Sigma})$ are respectively related to the kinetic and topological terms $\mathcal{F}^2$ and $\mathcal{F}\widetilde{\mathcal{F}}$ of the (field strenghts of the) Maxwell vector fields; for this reason, usually the matrix $\mathcal{N}_{\Lambda\Sigma}$ is referred to as the "kinetic matrix" of the $\mathcal{N} = 2$, $d = 4$ Maxwell Einstein Supergravity Theory.

Furthermore, from the Ansatz (4.1.35) and the second result of (4.1.37) we obtain an interesting relation, relating the metric $G_{i\bar{j}}$ of the Kähler-Hodge manifold M_{n_V} to

the symplectic vector functions f_i^Λ and $h_{i\Lambda}$ through $Im\,(\mathcal{N}_{\Lambda\Sigma})$

$$
\begin{aligned}
G_{i\bar{j}} &= -i\left\langle U_i, \overline{U}_{\bar{j}}\right\rangle \equiv -iU_i^T \epsilon \overline{U}_{\bar{j}} = \\[2mm]
&= -i\left[-\left(D_i L^\Lambda\right)\overline{D}_{\bar{j}}\overline{M}_\Lambda + \left(D_i M_\Lambda\right)\overline{D}_{\bar{j}}\overline{L}^\Lambda\right] = \\[2mm]
&= -i\left[-\left(D_i L^\Lambda\right)\mathcal{N}_{\Lambda\Sigma}\overline{D}_{\bar{j}}\overline{L}^\Sigma + \overline{\mathcal{N}}_{\Lambda\Sigma}\left(D_i L^\Sigma\right)\overline{D}_{\bar{j}}\overline{L}^\Lambda\right] = \\[2mm]
&= -2Im\,(\mathcal{N}_{\Lambda\Sigma})\left(D_i L^\Lambda\right)\overline{D}_{\bar{j}}\overline{L}^\Sigma = -2Im\,(\mathcal{N}_{\Lambda\Sigma})\,f_i^\Lambda \overline{f}_{\bar{j}}^\Sigma.
\end{aligned}
\tag{4.1.94}
$$

Whenever the prepotential F may be defined, by using Eq. (4.1.85) it may be analogously obtained that

$$
G_{i\bar{j}} = 2Im\,(F_{\Lambda\Sigma})\,f_i^\Lambda \overline{f}_{\bar{j}}^\Sigma.
\tag{4.1.95}
$$

As previously noticed, the function f_i^Λ, endowed with a local index in the SKG of M_{n_V} and with a global index in $Sp(2n_V + 2)$ (symplectic symmetry), plays a key role in intertwining such two different levels of symmetry, revealing the inner special Kähler-Hodge symplectic structure of the $\mathcal{N} = 2$, $d = 4$ Maxwell Einstein Supergravity Theory.

In regular SKG, the Kähler metric $G_{i\bar{j}}$ is (strictly) positive definite in all M_{n_v}. By using Eq. (4.1.94), this implies the (strictly) negative definiteness of the real $(n_V + 1) \times (n_V + 1)$ matrix $Im\,(\mathcal{N}_{\Lambda\Sigma})$; a shorthand notation for such a condition, encoding the regularity of the SKG of M_{n_V}, reads

$$
Im\,(\mathcal{N}_{\Lambda\Sigma}) < 0,
\tag{4.1.96}
$$

which also follows from the position of such term in the low-energy $\mathcal{N} = 2$ $(d = 4)$ Maxwell Einstein Supergravity Theory Lagrangian density.

At this point, whenever the Jacobian matrix $e_i^a(z)$ exists and is invertible, a number of useful formulae may be obtained, relating the two main symplectic matrices introduced so far, i.e. the "kinetic" one ($\mathcal{N}_{\Lambda\Sigma}$) and the one given by the double symplectic derivatives of the prepotential ($F_{\Lambda\Sigma}$, also denoted with $\mathcal{F}$). The main result is[8]

$$
\mathcal{N}_{\Lambda\Sigma} = \overline{F}_{\Lambda\Sigma} - 2i\overline{T}_\Lambda \overline{T}_\Sigma \left(LIm\,(\mathcal{F})\,L\right),
\tag{4.1.97}
$$

---

[8]Provided that the holomorphic prepotential $F(X)$ satisfies the Schwarz lemma in the moduli space, the symmetry of the $(n_V + 1) \times (n_V + 1)$ complex matrix $\mathcal{N}_{\Lambda\Sigma}$ is evident from Eq. (4.1.97), which anyway holds true only whenever the holomorphic Jacobian matrix $e_i^a(z)$, defined by Eq. (4.1.63), exists and is invertible.

In general, the fundamental Ansätze, expressed by Eqs. (4.1.34) and (4.1.35) and formulated in

where the symplectic vector T_Λ is defined as follows:

$$T_\Lambda \equiv -i\frac{\left(Im\,(\mathcal{F})\,\overline{L}\right)_\Lambda}{\overline{L}Im\,(\mathcal{F})\,\overline{L}} = 2i\,(Im\,(\mathcal{N})\,L)_\Lambda\,, \tag{4.1.98}$$

and the following relations hold:

$$\begin{cases} LIm\,(\mathcal{F})\,\overline{L} = -\frac{1}{2}; \\[2mm] T_\Lambda \overline{L}^\Lambda = -i; \\[2mm] 4\overline{L}Im\,(\mathcal{F})\,\overline{L} = (LIm\,(\mathcal{N})\,L)^{-1}. \end{cases} \tag{4.1.99}$$

Now, instead of saturating the symplectic indices of the product $f_i^\Lambda \overline{f}_{\overline{j}}^\Sigma$, as made in (4.1.94) and (4.1.95), we may instead saturate the Kähler ones, and the obvious choice is to use $G^{i\overline{j}}$; by doing this, we introduce the symplectic tensor

$$U^{\Lambda\Sigma} \equiv G^{i\overline{j}}f_i^\Lambda \overline{f}_{\overline{j}}^\Sigma = -\frac{1}{2}\left((Im\,(\mathcal{N}))^{-1}\right)^{\Lambda\Sigma} - \overline{L}^\Lambda L^\Sigma, \tag{4.1.100}$$

where in the last passage we used Eqs. (4.1.37) and (4.1.94). Notice that in our notation $\mathcal{N}^{\Lambda\Sigma}$ is nothing but the inverse of $\mathcal{N}_{\Lambda\Sigma}$

$$\mathcal{N}^{\Lambda\Sigma}\mathcal{N}_{\Sigma\Delta} \equiv \delta_\Delta^\Lambda, \tag{4.1.101}$$

and moreover it holds that

$$\mathcal{N}_{\Lambda\Sigma} = \mathcal{N}_{\Sigma\Lambda} \Longrightarrow \left((Im\mathcal{N})^{-1}\right)^{\Lambda\Sigma} = \left((Im\mathcal{N})^{-1}\right)^{\Sigma\Lambda}. \tag{4.1.102}$$

By considering Eqs. (4.1.94), (4.1.95) and (4.1.100), we finally get

$$\begin{aligned} U^{\Lambda\Sigma} &= \frac{1}{2}\left((Im\,(\mathcal{F}))^{-1}\right)^{\Lambda\Sigma} + L^\Lambda \overline{L}^\Sigma \equiv \\ &\equiv T_I^\Lambda G^{IJ}\overline{T}_J^\Sigma. \end{aligned} \tag{4.1.103}$$

order to solve the so-called "Special Geometry constraints" given by Eq. (7.2.1.39), does not imply the symmetry of $\mathcal{N}_{\Lambda\Sigma}$. Therefore, assuming such a property, which is then largely used, would seem to imply some loss of generality.

Actually, also in the particular cases in which it is not possible to define a local system of homogenous coordinates in the moduli space (i.e. when the matrix $e_i^a\,(z)$ does not exist or it is not invertible), it may be shown that Eqs. (4.1.34) and (4.1.35) are always solved by a symmetric matrix $\mathcal{N}_{\Lambda\Sigma}$.

Thus,

$$\mathcal{N}_{\Lambda\Sigma} = \mathcal{N}_{\Sigma\Lambda}$$

does not yield any loss of generality in the study of the symplectic special Kähler structure (of the moduli space) of the $\mathcal{N} = 2$ ($d = 4$) n_V-fold Maxwell Einstein Supergravity Theory.

In the second line we defined the $(n_V + 1)$-dim. square matrix

$$\mathcal{T}_I{}^\Lambda \equiv \left(\mathcal{T}_i{}^\Lambda, \mathcal{T}_0{}^\Lambda \equiv L^\Lambda \right) \tag{4.1.104}$$

and, similarly to what previously done for the f's and h's, we extended the Kähler metric to a $(n_V + 1)$-dim. square form

$$G^{IJ} \equiv \left(G^{ij} \equiv G^{i\bar{j}}, G^{i0} = 0, G^{00} = -1 \right). \tag{4.1.105}$$

Because of Eqs. (4.1.94), (4.1.95) and (4.1.96), we obtain that $Im\,(\mathcal{F})$ is a $(n_V + 1)$-dim. square symplectic matrix, with n_V positive and one negative eigenvalues. $U^{\Lambda\Sigma}$ is a $(n_V + 1)$-dim. square symplectic matrix, too, but instead it has rank n_V because, as it may be explicitly shown, it annihilates the vector[9] T_Λ and its conjugate $\overline{T}_\Lambda$

$$T_\Lambda U^{\Lambda\Sigma} = U^{\Lambda\Sigma} \overline{T}_\Sigma = 0. \tag{4.1.106}$$

From Eq. (4.1.103) we can further compute

$$[det\,(2Im\,(\mathcal{F}))]^{-1} = det\left(U^{\Lambda\Sigma} - L^\Lambda \overline{L}^\Sigma \right), \tag{4.1.107}$$

and using the $(n_V + 1)$-dim. square matrices $\mathcal{T}_I{}^\Lambda$ and G^{IJ}, we obtain

$$det\,(2Im\,(\mathcal{F})) = -det\left(G_{i\bar{j}} \right) |det\,(\mathcal{T})|^{-2}. \tag{4.1.108}$$

By means of simple properties of the determinants, such relations yield the following result:

$$det\,(\mathcal{T}) = exp\left[(n_V + 1)\frac{1}{2}K \right] (det\,(e)) \left(X^0 \right)^{n_V + 1} \tag{4.1.109}$$

$$\Downarrow$$

$$|det\,(\mathcal{T})|^2 = exp\left[(n_V + 1)K \right] |det\,(e)|^2 \left(X^0 \overline{X}^0 \right)^{n_V + 1}, \tag{4.1.110}$$

where $det\,(e)$ is the determinant of the $(n_V + 1)$-dim. square matrix defined in (4.1.63), i.e. it is nothing but the Jacobian of the change of basis of coordinates

$$\left\{ z^i \right\}_{i=1,...,n_V} \longleftrightarrow \left\{ t^a\,(z) \right\}_{a=1,...,n_V} \equiv \left\{ \frac{X^a\,(z)}{X^0\,(z)} \right\}_{a=1,...,n_V} \tag{4.1.111}$$

in the Kähler-Hodge moduli space M_{n_V} of the considered theory.

---

[9]In the next Subsection we will see that T_Λ is nothing but the graviphoton projector in the $\mathcal{N} = 2$, $d = 4$ Maxwell Einstein Supergravity Theory.

It then follows that ([40], [41], [43], [44])

$$\partial_i \bar{\partial}_{\bar{j}} ln \left(det \left(Im \left(\mathcal{F} \right) \right) \right) = \partial_i \bar{\partial}_{\bar{j}} ln \left(det \left(G_{k\bar{l}} \right) \right) - \left(n_V + 1 \right) G_{i\bar{j}}. \tag{4.1.112}$$

At this point, by using the SG constraints (7.2.1.39) satisfied by the RC tensor in the moduli space, we may compute the corresponding Ricci tensor as follows:

$$R_{i\bar{j}} \equiv R_{i\bar{j}l\bar{l}} G^{l\bar{l}} = \left(n_V + 1 \right) G_{i\bar{j}} - C_{ilp} \overline{C}_{\bar{j}\bar{l}\bar{p}} G^{l\bar{l}} G^{p\bar{p}}. \tag{4.1.113}$$

By using Eq. (4.1.112), and recalling that on a Kähler manifold the Ricci tensor can always be written as [37]

$$R_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} ln \left(det \left(G_{k\bar{l}} \right) \right), \tag{4.1.114}$$

we finally get

$$\partial_i \bar{\partial}_{\bar{j}} ln \left(det \left(Im \mathcal{F} \right) \right) = -C_{ilp} \overline{C}_{\bar{j}\bar{l}\bar{p}} G^{l\bar{l}} G^{p\bar{p}}. \tag{4.1.115}$$

Notice that such a result generally characterizes every special Kähler-Hodge symplectic manifold which admits local coordinates defined by means of the (ratios of the) sections of the related holomorphic line bundle, i.e. for which the $(n_V + 1)$-dim. square matrix defined in (4.1.63) exists and it is invertible.

4.2 Electro Magnetic Duality, Central Charge and Attractor Mechanism : A Primer

In this Subsection we will briefly report how, in $\mathcal{N} = 2$, $d = 4$ SUGRA coupled with n_V Abelian vector multiplets (and n_H hypermultiplets), the phenomenon of the doubling of preserved supersymmetries (and therefore of the restoration of maximal supersymmetry) occurs near the EH of the $\frac{1}{2}$-BPS stable soliton metric solution, whose simplest example is represented by the previously considered extremal RN BH. Furthermore, we will see the AM at work in the dynamical evolution of the relevant set of scalars, i.e. in the on-shell dynamics of the manifold M_{n_V}.

For simplicity's sake, let us set the fermionic bilinears and the $\mathcal{N} = 2$ generalization of the Fayet-Iliopoulos term to zero (i.e. let us disregard the fermionic contributions and the presence of supersymmetric gaugings). We may then write the local supersymmetry transformations for the gravitino, for the gaugino and for the hyperino in a manifestly symplectic covariant way as follows (see [34] and [35],

where the complete, general SUGRA transformations may be found, too):

$$\begin{cases} \delta\psi_{A\mu} = \mathcal{D}_\mu\varepsilon_A + \epsilon_{AB}\varepsilon^B T^-_{\mu\nu}\gamma^\nu; \\[2mm] \delta\lambda^{iA} = i\varepsilon^A\gamma^\mu\partial_\mu z^i + \epsilon^{AB}\varepsilon_B\mathcal{F}^{i-}_{\mu\nu}\gamma^{\mu\nu}; \\[2mm] \delta\zeta_\alpha = i\epsilon_{AB}\varepsilon^A\mathcal{U}^{B\beta}_u\gamma^\mu\mathbb{C}_{\alpha\beta}\partial_\mu q^u, \end{cases} \qquad (4.2.1)$$

where we recall once again that λ^{iA}, $\psi_{A\mu}$ and ζ_α respectively are the chiral gaugino, gravitino and hyperino fields. Moreover, ε_A and ε^A respectively denote the chiral and antichiral local supersymmetry parameters, and ϵ^{AB} is the $SO(2)$ Ricci tensor

$$\epsilon^{AB} = -\epsilon^{BA}, \epsilon^2 = -\mathbb{I}, \qquad (4.2.2)$$

i.e. the 2-dim. contravariant counterpart of the symplectic metric defined by Eq. (4.1.25). The moduli-dependent, symplectic-invariant quantities $T^-_{\mu\nu}$ and $\mathcal{F}^{i-}_{\mu\nu}$ respectively are the (imaginary self-dual) graviphoton and vector field strenghts. For what concerns the gravitino field $\psi_{A\mu}$, apart from being a spinor-valued one-form on space-time, it behaves as a section of the bundle $\Im \otimes \mathcal{SU}$, where $\mathcal{SU}$ is an $SU(2)$ principal bundle on the quaternionic scalar manifold M_{n_H} ($dim_\mathbb{R} M_{n_H} = 4n_H$) related to the considered n_H hypermultiplets. Consequently, the derivative $\mathcal{D}_\mu$ appearing in the first line of the transformations (4.2.1) is the covariant derivative w.r.t. the (ungauged) $U(1)$-bundle $\Im$ on M_{n_V} (with connection Q given by Eq. (4.1.14)) and the (ungauged) $SU(2)$ principal bundle $\mathcal{SU}$ on M_{n_H} (with connection ω^x, where x is an $SU(2)$ index). In the formalism of forms, we thus have

$$\mathcal{D}\varepsilon_A = d\varepsilon_A - \frac{1}{4}\gamma_{ab}\omega^{ab} \wedge \varepsilon_A + \frac{i}{2}Q \wedge \varepsilon_A + \frac{i}{2}\omega^x(\sigma_x)^B_A \wedge \varepsilon_B, \qquad (4.2.3)$$

where d is the flat space-time differential, ω^{ab} is the spin-connection and σ_x denotes the vector of Pauli matrices. Finally, $\mathcal{U}^{B\beta}_u$ and $\mathbb{C}_{\alpha\beta}$ respectively stand for the quaternionic Vielbein 1-form and the $Sp(2n_H)$-invariant flat metric

$$\mathbb{C}_{\alpha\beta} = -\mathbb{C}_{\beta\alpha}, \mathbb{C}^2 = -\mathbb{I}. \qquad (4.2.4)$$

In order to describe the restoration of the maximal supersymmetry of the metric background of the $\mathcal{N} = 2, d = 4, n_V$-fold Maxwell Einstein Supergravity Theory, i.e. the doubling of the number of preserved supersymmetries with respect to the four ones preserved by the $\frac{1}{2}$-BPS stable solitonic solution represented by the extremal (eventually RN) BH, we have to find solutions with unbroken $\mathcal{N} = 2, d = 4$ local SUSY.

In the present case, the relevant Killing spinor conditions to be solved are those given by Eqs. (4.2.1), with the r.h.s.'s set to zero, i.e.

$$\delta\psi_{A\mu} = \delta\lambda^{iA} = \delta\zeta_\alpha = 0. \tag{4.2.5}$$

I. The first solution to Eqs. (4.2.5) is the one corresponding to the standard flat vacuum, which is the asymptotical limit ($r \to \infty$) of the spherically symmetric, static extremal RN BH metric background. The corresponding unbroken, maximal $\mathcal{N} = 2$, $d = 4$ SUSY algebra is the $\mathcal{N} = 2$, $d = 4$ superPoincaré one (*asymptotical rigid $\mathcal{N} = 2$ SUSY*).

Concerning the field content of the theory in such a case, the 4-d. metric is the flat, Minkowski $\eta_{\mu\nu}$, there are no vector fields, and all complex scalar fields in the considered n_V Abelian vector supermultiplets, as well as the quaternionic scalars in the n_H hypermultiplets, take arbitrary constant values

$$\begin{cases} g_{\mu\nu} = \eta_{\mu\nu}, \\[1.5em] T^-_{\mu\nu} = 0 = \mathcal{F}^{i-}_{\mu\nu}, \\[1.5em] \partial_\mu z^i = 0 \Leftrightarrow z^i = z^i_\infty \in \mathbb{C}, \\[1.5em] \partial_\mu q^u = 0 \Leftrightarrow q^u = q^u_\infty \in \mathbb{H}. \end{cases} \tag{4.2.6}$$

z^i_∞ is an unconstrained scalar field configuration in the n_V-dim. Kähler-Hodge complex moduli space M_{n_V} of the $\mathcal{N} = 2$, $d = 4$, n_V-fold Maxwell Einstein Supergravity Theory. The positions (4.2.6) do provide solutions for the unbroken $\mathcal{N} = 2$, $d = 4$ SUGRA Killing spinor equations with constant, unconstrained values of the SUSY parameter ε_A, which therefore makes the local SUSY structure "rigid", i.e. global.

Thus, the unbroken SUSY manifests itself in the fact that each non-vanishing scalar field is the first component of a covariantly-constant $\mathcal{N} = 2$ superfield for the vector and/or the hypermultiplet, but the supergravity superfield vanishes.

II. The second solution to Eqs. (4.2.5) is much more sophisticated; as we will see by solving the related consistency conditions, it corresponds to the 4-d. BR metric, which is the "near-Horizon" limit ($r \to r^+_H$) of the spherically symmetric, static extremal RN BH metric background.

Firstly, it is possible to solve the Killing conditions for the gaugino and the hy-

perino just by using a suitable part of the previous Ansätze (4.2.6), i.e.

$$
\begin{cases}
\mathcal{F}^{i-}_{\mu\nu} = 0, \\[2ex]
\partial_\mu z^i = 0 \Leftrightarrow z^i = z^i_\infty \in \mathbb{C}, \\[2ex]
\partial_\mu q^u = 0 \Leftrightarrow q^u = q^u_\infty \in \mathbb{H}.
\end{cases}
\tag{4.2.7}
$$

Secondly, we observe that the Killing equation for the gravitino

$$
\delta\psi_{A\mu} = \mathcal{D}_\mu \varepsilon_A + \epsilon_{AB} T^-_{\mu\nu} \gamma^\nu \varepsilon^B = 0
\tag{4.2.8}
$$

is not gauge-invariant. Consequently, without loss of generality we may consider variation of the gravitino field strength in a particular, suitable way, as shown in [11] and [12].

For what concerns the s-t metric, we may consider the geometry of the background with vanishing Riemann-Christoffel intrinsic scalar curvature R, vanishing Weyl tensor $C_{\mu\nu\lambda\delta}$ and covariantly constant graviphoton field strength $T^-_{\mu\nu}$

$$
\begin{cases}
R = 0, \\[2ex]
C_{\mu\nu\lambda\delta} = 0, \\[2ex]
\mathcal{D}_\lambda \left(T^-_{\mu\nu} \right) = 0.
\end{cases}
\tag{4.2.9}
$$

While the first solution had a vanishing supergravity superfield, it may be shown that such a configuration corresponds to a covariantly constant superfield of $\mathcal{N} = 2$, $d = 4$, n_V-fold Maxwell Einstein Supergravity Theory $W_{\alpha\beta}(x,\theta)$, whose first component is given by a two-component graviphoton field strength $T_{\alpha\beta}$.

The phenomenon of the doubling of preserved supersymmetries near the EH of the extremal RN BH may be qualitatively explained as follows.

It may be shown that the algebraic condition for the choice of broken versus unbroken $\mathcal{N} = 2$, $d = 4$ local SUSY is given in terms of a combination of the Weyl tensor and of the Riemann-covariant derivative of the graviphoton field strength. However, by the set (4.2.9) of Ansätze on the structure of the "near-Horizon" metric background, both the Weyl tensor $C_{\mu\nu\lambda\delta}$ and the Riemann-covariant derivative of the graviphoton field strength $\mathcal{D}_\lambda \left(T^-_{\mu\nu} \right)$ separately vanish in proximity of the EH. Thus, all supersymmetries are restored in this limit, and one gets a covariantly constant superfield of $\mathcal{N} = 2$, $d = 4$, n_V-fold Maxwell Einstein Supergravity Theory $W_{\alpha\beta}(x,\theta)$.

Considering a generic configuration of such a theory, in which the supergravity multiplet interacts with n_V Abelian vector supermultiplets and n_H hypermultiplets, we obtain that, beside the $\mathcal{N} = 2$ supergravity superfield $W_{\alpha\beta}(x, \theta)$, we also have covariantly-constant $\mathcal{N} = 2$ superfields, whose first component is given, similarly to what happened for the first solution, by the scalars of the corresponding multiplets.

However, whereas the flat vacuum given by the first solution admitted any value of the scalars, in the present case the non-trivial geometry of the metric background (which will then reveal to be the 4-d. BR metric[10]), defined by the positions (4.2.9), imposes two consistency conditions for this second solution, i.e.

1. The Riemann-Christoffel tensor must match the product of two graviphoton field strengths

$$R_{\alpha\beta\alpha'\beta'} = T_{\alpha\beta}T_{\alpha'\beta'}. \tag{4.2.10}$$

2. The vector field strength must vanish (as given by the first position of Ansätze (4.2.7), too)

$$\mathcal{F}^{i-}_{\mu\nu} = 0. \tag{4.2.11}$$

Later on, we will analyze the consistency conditions (4.2.10) and (4.2.11) more in depth. Now we move to deal with some noteworthy symplectic features of the special geometry of the n_V-dim. Kähler-Hodge complex moduli space M_{n_V} of such a theory. The additional symplectic structure allows one to introduce a central extension operator (and the related Kähler-covariant condensate) by purely geometric reasonings and in a completely symplectic-invariant way.

Considering the low-energy effective action of the $\mathcal{N} = 2$, $d = 4$ Maxwell Einstein Supergravity Theory, the Kähler metric of M_{n_V} appears in the kinetic term of the complex scalars coming from the considered n_V Maxwell vector multiplets; it reads

$$G_{i\bar{j}}\partial_\mu z^i \partial_\nu \bar{z}^{\bar{j}} g^{\mu\nu} \sqrt{-g}. \tag{4.2.12}$$

As previously mentioned, the symmetric matrix $\mathcal{N}_{\Lambda\Sigma}$ appears in the vector part of the action, which reads (setting the fermionic contributions to zero)

$$-2Im\left(\mathcal{F}^{-\Lambda}_{\mu\nu}\overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Sigma\mu\nu}\right) = -2Im\left(\mathcal{F}^{-\Lambda}_{\mu\nu}\mathcal{G}^{-\mu\nu}_{\Lambda}\right), \tag{4.2.13}$$

---

[10]Notice indeed that the conditions of vanishing Weyl tensor and zero (overall) scalar curvature, respectively expressed by the second and first position of the Ansätze (4.2.9), are compatible with the properties of the BR metric (see Section 1).

where $\mathcal{F}_{\mu\nu}^{-\Lambda}$ is the complex, imaginary self-dual Maxwell field strength (see below). Instead, in general $\mathcal{G}_{\Lambda}^{-}$ is the Legendre transform of $\mathcal{F}^{-\Lambda}$

$$\mathcal{G}_{\Lambda}^{-\mu\nu} \equiv \frac{\delta\mathcal{L}}{\delta\mathcal{F}_{\mu\nu}^{-\Lambda}}. \tag{4.2.14}$$

As yielded by Eq. (4.2.13), in the symplectic structure of the $\mathcal{N} = 2$, $d = 4$ n_V-fold Maxwell Einstein Supergravity Theory, the above functional derivative may be equivalently re-expressed as the following linear combination:

$$\mathcal{G}_{\Lambda}^{-} \equiv \overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Sigma}. \tag{4.2.15}$$

$\mathcal{F}^{-\Lambda}$ is clearly moduli-independent

$$\partial_i\mathcal{F}^{-\Lambda} = 0 = \overline{\partial}_{\bar{i}}\mathcal{F}^{-\Lambda}. \tag{4.2.16}$$

Through the functional derivative of $\mathcal{L}$ given by Eq. (4.2.14), instead $\mathcal{G}_{\Lambda}^{-}$ depends on the moduli purely through the matrix $\overline{\mathcal{N}}_{\Lambda\Sigma}$ which however, as previously pointed out, has vanishing Kähler weights, because otherwise the Kähler structure of M_{n_V} would clash with the $Sp(2n_V + 2)$-covariance of electric–magnetic duality of the theory. In general, the differential properties of $\mathcal{G}_{\Lambda}^{-}$ are the following:

$$\begin{cases} D_i\mathcal{G}_{\Lambda}^{-} = \partial_i\mathcal{G}_{\Lambda}^{-} = \left(\partial_i\overline{\mathcal{N}}_{\Lambda\Sigma}\right)\mathcal{F}^{-\Sigma} = \left(D_i\overline{\mathcal{N}}_{\Lambda\Sigma}\right)\mathcal{F}^{-\Sigma} \neq 0, \\ \\ \overline{D}_{\bar{i}}\mathcal{G}_{\Lambda}^{-} = \overline{\partial}_{\bar{i}}\mathcal{G}_{\Lambda}^{-} = \left(\overline{\partial}_{\bar{i}}\overline{\mathcal{N}}_{\Lambda\Sigma}\right)\mathcal{F}^{-\Sigma} = \left(\overline{D}_{\bar{i}}\overline{\mathcal{N}}_{\Lambda\Sigma}\right)\mathcal{F}^{-\Sigma} \neq 0. \end{cases} \tag{4.2.17}$$

The upperscript "$-$" in $\mathcal{F}^{-\Lambda}$ and $\mathcal{G}_{\Lambda}^{-}$ denotes the (imaginary) self-duality of such complex symplectic vectors. In order to clarify such a point, let us now briefly address the issue of the general structure of an Abelian theory of vectors endowed with Hodge duality (for more details, see e.g. [34], [1] and [2]).

In general, in the considered context we may introduce a formal operator $\mathcal{H}$ that maps an Abelian field strength into its Hodge dual

$$\left(\mathcal{H}\mathcal{F}^{\Lambda}\right)_{\mu\nu} \equiv \left({}^{*}\mathcal{F}^{\Lambda}\right)_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\Lambda\rho\sigma} = \frac{1}{2}g^{\rho\lambda}g^{\sigma\tau}\epsilon_{\mu\nu\rho\sigma}\mathcal{F}_{\lambda\tau}^{\Lambda}. \tag{4.2.18}$$

It is immediate to check that such an operator is anti-projective

$$\begin{aligned} \left(\mathcal{H}^2\mathcal{F}^{\Lambda}\right)_{\mu\nu} &= \left({}^{**}\mathcal{F}^{\Lambda}\right)_{\mu\nu} = \frac{1}{2}g^{\alpha\gamma}g^{\beta\delta}\epsilon_{\mu\nu\gamma\delta}\left(\mathcal{H}\mathcal{F}^{\Lambda}\right)_{\alpha\beta} = \\ \\ &= \frac{1}{4}g^{\alpha\gamma}g^{\beta\delta}g^{\lambda\sigma}g^{\rho\tau}\epsilon_{\mu\nu\gamma\delta}\epsilon_{\alpha\beta\sigma\tau}\mathcal{F}_{\lambda\rho}^{\Lambda} = \\ \\ &= -\frac{1}{2}\left(\mathcal{F}_{\mu\nu}^{\Lambda} - \mathcal{F}_{\nu\mu}^{\Lambda}\right) = -\mathcal{F}_{\mu\nu}^{\Lambda}, \end{aligned} \tag{4.2.19}$$

where we used the result

$$g^{\alpha\gamma}g^{\beta\delta}g^{\lambda\sigma}g^{\rho\tau}\epsilon_{\mu\nu\gamma\delta}\epsilon_{\alpha\beta\sigma\tau} = \epsilon_{\mu\nu}{}^{\alpha\beta}\epsilon_{\alpha\beta}{}^{\lambda\rho} = -2\left(\delta_\mu^\lambda\delta_\nu^\rho - \delta_\mu^\rho\delta_\nu^\lambda\right). \tag{4.2.20}$$

Thus, since $\mathcal{H}^2 = -\mathbb{I}$, its eigenvalues are $\pm i$, and out of the real Abelian field strengths $\mathcal{F}_{\mu\nu}^\Lambda$ we can introduce imaginary anti-self-dual and imaginary self-dual complex combinations, respectively as follows:

$$\mathcal{F}_{\mu\nu}^{\pm\Lambda} \equiv \mathcal{F}_{\mu\nu}^\Lambda \pm i\left(\mathcal{H}\mathcal{F}^\Lambda\right)_{\mu\nu} = \mathcal{F}_{\mu\nu}^\Lambda \pm \frac{i}{2}g^{\rho\lambda}g^{\sigma\tau}\epsilon_{\mu\nu\rho\sigma}\mathcal{F}_{\lambda\tau}^\Lambda, \tag{4.2.21}$$

such that

$$\left(\mathcal{H}\mathcal{F}^{\pm\Lambda}\right)_{\mu\nu} = \left(\mathcal{H}\mathcal{F}^\Lambda\right)_{\mu\nu} \pm i\left(\mathcal{H}^2\mathcal{F}^\Lambda\right)_{\mu\nu} =$$

$$= \mp i\mathcal{F}_{\mu\nu}^\Lambda + \left(\mathcal{H}\mathcal{F}^\Lambda\right)_{\mu\nu} = \mp i\mathcal{F}_{\mu\nu}^{\pm\Lambda}. \tag{4.2.22}$$

Notice also that

$$\overline{\mathcal{F}_{\mu\nu}^{\pm\Lambda}} = \mathcal{F}_{\mu\nu}^{\mp\Lambda}. \tag{4.2.23}$$

In $\mathcal{N} = 2$, $d = 4$ n_V-fold Maxwell Einstein Supergravity Theory the symplectic symmetry underlying the geometric structure of the equations of motion becomes elegantly manifest by considering the following four different kinds of vectors:

1) the $(2n_V + 2) \times 1$, $Sp\,(2n_V + 2)$-covariant complex symplectic vector of imaginary self-dual Abelian field strengths

$$\mathcal{Z}^- \equiv \begin{pmatrix} \mathcal{F}^{-\Lambda} \\ \mathcal{G}_\Lambda^- \end{pmatrix} \equiv \begin{pmatrix} \mathcal{F}^\Lambda - i\mathcal{H}\mathcal{F}^\Lambda \\ \mathcal{G}_\Lambda - i\mathcal{H}\mathcal{G}_\Lambda \end{pmatrix}; \tag{4.2.24}$$

recall that $Sp\,(2n_V + 2, \mathbb{R})$ is the generalized electric-magnetic duality symmetry group, i.e. the U-duality symmetry group[11], in the $\mathcal{N} = 2$, $d = 4$ n_V-fold Maxwell Einstein Supergravity Theory.

---

[11]More correctly, it should be said that $Sp\,(2n_V + 2, \mathbb{R})$ is the "classical supergravity limit" of the U-duality group of the corresponding quantum theory, i.e. of the "discrete" version $Sp\,(2n_V + 2, \mathbb{Z})$.

Indeed, the quantization of the conserved charges (related to the Abelian gauge-invariance exhibited by the Maxwell Einstein Supergravity Theory) leads to the "discretization" of the numeric field of definition of the group classifying the electric-magnetic transformations. In the case at hand, this yields

$$Sp\,(2n_V + 2, \mathbb{R}) \rightarrow Sp\,(2n_V + 2, \mathbb{Z}).$$

The classical formulation of the theories is recovered in the (semiclassical) limit of large values of the integer quantized charges.

For simplicity's sake and with a slight abuse of language, in the following treatment we will simply talk about "discrete" and "continuous" versions of the same U-group.

2) By complex-conjugating $\mathcal{Z}^-$, we get the $(2n_V + 2) \times 1$, $Sp\,(2n_V + 2)$-covariant complex symplectic vector of imaginary anti-self-dual Abelian field strengths

$$\mathcal{Z}^+ \equiv \overline{(\mathcal{Z}^-)} = \begin{pmatrix} \overline{\mathcal{F}^{-\Lambda}} \\ \overline{\mathcal{G}_\Lambda^-} \end{pmatrix} = \begin{pmatrix} \mathcal{F}^{+\Lambda} \\ \mathcal{G}_\Lambda^+ \end{pmatrix} \equiv \begin{pmatrix} \mathcal{F}^\Lambda + i\mathcal{H}\mathcal{F}^\Lambda \\ \mathcal{G}_\Lambda + i\mathcal{H}\mathcal{G}_\Lambda \end{pmatrix}. \qquad (4.2.25)$$

By definition, the real and imaginary parts of $\mathcal{Z}^-$ and its complex conjugate $\mathcal{Z}^+$ are the real Abelian field strengths of the theory and their Hodge-duals, respectively reading

3)

$$\mathcal{Z} \equiv Re\,(\mathcal{Z}^-) = \frac{1}{2}\,(\mathcal{Z}^- + \mathcal{Z}^+) = \begin{pmatrix} \mathcal{F}^\Lambda \\ \mathcal{G}_\Lambda \end{pmatrix}. \qquad (4.2.26)$$

4)

$$^*\mathcal{Z} \equiv \mathcal{H}\mathcal{Z} = \mathcal{H}\left[\frac{1}{2}\,(\mathcal{Z}^- + \mathcal{Z}^+)\right] = \frac{i}{2}\,(\mathcal{Z}^- - \mathcal{Z}^+) = -Im\,(\mathcal{Z}^-) = \begin{pmatrix} \mathcal{H}\mathcal{F}^\Lambda \\ \mathcal{H}\mathcal{G}_\Lambda \end{pmatrix}. \qquad (4.2.27)$$

Thus, we may summarize Eqs. (4.2.24)-(4.2.27) as follows:

$$\mathcal{Z}^\pm = \begin{pmatrix} \mathcal{F}^{\pm\Lambda} \\ \mathcal{G}_\Lambda^\pm \end{pmatrix} = \begin{pmatrix} \mathcal{F}^\Lambda \pm i\mathcal{H}\mathcal{F}^\Lambda \\ \mathcal{G}_\Lambda \pm i\mathcal{H}\mathcal{G}_\Lambda \end{pmatrix}, \qquad (4.2.28)$$

with

$$\overline{\mathcal{Z}^\pm} = \mathcal{Z}^\mp, \quad \mathcal{H}\mathcal{Z}^\pm = \mp i\mathcal{Z}^\pm. \qquad (4.2.29)$$

Since $\mathcal{Z}^-$, $\mathcal{Z}^+$, $\mathcal{Z}$ and $^*\mathcal{Z}$ are all $Sp\,(2n_V + 2, \mathbb{R})$-covariant vectors, it is clear that

$$[\mathcal{C}, Sp\,(2n_V + 2, \mathbb{R})] = 0 = [\mathcal{H}, Sp\,(2n_V + 2, \mathbb{R})], \qquad (4.2.30)$$

where $\mathcal{C}$ is the complex conjugation operator, $\mathcal{H}$ stands for the Hodge duality operator, and "$Sp\,(2n_V + 2, \mathbb{R})$" denotes the covariance w.r.t. the action of such a group. Otherwise speaking, the complex coniugation and/or the Hodge Abelian dualization do not have any effect on the symplectic covariance.

Using the summarizing relations (4.2.28), it is therefore possible to decompose Eq. (4.2.15) in a real and imaginary part

$$
\begin{aligned}
\mathcal{G}_{\Lambda}^{-} &\equiv \overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Sigma} = [Re\,(\mathcal{N}_{\Lambda\Sigma}) - iIm\,(\mathcal{N}_{\Lambda\Sigma})]\left(\mathcal{F}^{\Sigma} - i\,{}^{*}\mathcal{F}^{\Sigma}\right) = \\[2mm]
&= \left[Re\,(\mathcal{N}_{\Lambda\Sigma})\,\mathcal{F}^{\Sigma} - Im\,(\mathcal{N}_{\Lambda\Sigma})\,{}^{*}\mathcal{F}^{\Sigma}\right] - i\left[Re\,(\mathcal{N}_{\Lambda\Sigma})\,{}^{*}\mathcal{F}^{\Sigma} + Im\,(\mathcal{N}_{\Lambda\Sigma})\,\mathcal{F}^{\Sigma}\right],
\end{aligned}
$$

$$(4.2.31)$$

implying, for instance

$$
\mathcal{G}_{\Lambda} \equiv Re\left(\mathcal{G}_{\Lambda}^{-}\right) = Re\,(\mathcal{N}_{\Lambda\Sigma})\,\mathcal{F}^{\Sigma} - Im\,(\mathcal{N}_{\Lambda\Sigma})\,{}^{*}\mathcal{F}^{\Sigma}. \tag{4.2.32}
$$

Thus, in a source-free theory we may write, in the differential form language

$$
d\left[Re\left(\mathcal{Z}^{-}\right)\right] = 0, \tag{4.2.33}
$$

Instead, in presence of electric and magnetic sources with non-vanishing fluxes, we obtain the following "space-dressing" of the components of $Re\,(\mathcal{Z}^{-})$:

$$
\int_{S_{\infty}^{2}} \mathcal{F}^{\Lambda} \equiv n_{m}^{\Lambda},
$$

$$(4.2.34)$$

$$
\int_{S_{\infty}^{2}} \mathcal{G}_{\Lambda} \equiv n_{\Lambda}^{e},
$$

where the integration is performed in the physical space, and S_{∞}^{2} is the 2-sphere at the infinity.

The integration of $\mathcal{F}^{\Lambda}$ and its Legendre transform $\mathcal{G}_{\Lambda}$ performed in (4.2.34) may respectively be considered as the definition, in a suitable system of units, of the asymptotical values of the magnetic and electric charges characterizing the charge configuration of the $n_{V} + 1$ Maxwell vector fields of the theory (indeed we get a vector potential from the gravity multiplet plus another one for each considered vector multiplet).

Clearly, the quantization of such conserved charges (related to the $(U(1))^{n_{V}+1}$ gauge invariance of the $\mathcal{N} = 2$, $d = 4$ n_{V}-fold Maxwell Einstein Supergravity Theory) implies a discrete range for the quantities on the r.h.s.'s of (4.2.34), and therefore a "discretization" of the symplectic covariance. Consequently, since for a fixed Λ $\left(n_{m}^{\Lambda}, n_{\Lambda}^{e}\right) \in \mathbb{Z}^{2}$, it is clear that the "dressings" (4.2.34) will be covariant only under $Sp\,(2n_{V} + 2, \mathbb{Z})$, which is the "discrete" counterpart of the symplectic symmetry group $Sp\,(2n_{V} + 2, \mathbb{R})$.

Therefore, the defining Eqs. (4.2.34) allow one to introduce the $(2n_V + 2)$-dim. symplectic vector of the electric and magnetic charges of the system as the (asymptotical) "space dressing" of $Re\left(\mathcal{Z}^{-}\right)$

$$n \equiv \int_{S^2_\infty} Re\left(\mathcal{Z}^{-}\right) = \begin{pmatrix} n_m^\Lambda \\ n_\Lambda^e \end{pmatrix}. \qquad (4.2.35)$$

Once again, due to the quantization of the electric and magnetic charges, such a vector is actually $Sp\left(2n_V + 2, \mathbb{Z}\right)$-covariant.

Particular attention should be paid to the issue of moduli dependence. As is clear from Eqs. (4.2.16) and (4.2.17), $\mathcal{Z}^{-}$ is composed by a moduli-independent term $\mathcal{F}^{-\Lambda}$ and a moduli-dependent Kähler-scalar $\mathcal{G}_\Lambda^{-}$. Of course, the same holds for its real part $Re\left(\mathcal{Z}^{-}\right)$. The subtle, key point is that n, which, as defined in Eq. (4.2.35), is nothing but the (asymptotical) "space dressing" of $Re\left(\mathcal{Z}^{-}\right)$, is completely moduli-independent

$$\partial_i n = \partial_i \left(\int_{S^2_\infty} Re\left(\mathcal{Z}^{-}\right)\right) = 0 = \bar{\partial}_{\bar{i}} \left(\int_{S^2_\infty} Re\left(\mathcal{Z}^{-}\right)\right) = \bar{\partial}_{\bar{i}} n. \qquad (4.2.36)$$

In particular

i)

$$\int_{S^2_\infty} \mathcal{F}^\Lambda \equiv n_m^\Lambda \qquad (4.2.37)$$

defines (in suitable units) the magnetic charges of the system; we have moduli independence both at the "pre-dressing" and "post-dressing" stages.

ii) By recalling Eqs. (4.2.14) and (4.2.32) we obtain that

$$\int_{S^2_\infty} \mathcal{G}_\Lambda = \int_{S^2_\infty} Re\left(\frac{\delta\mathcal{L}}{\delta\mathcal{F}^{-\Lambda}}\right) =$$

$$= \int_{S^2_\infty} \left[Re\left(\mathcal{N}_{\Lambda\Sigma}\left(z, \bar{z}\right)\right)\mathcal{F}^\Sigma - Im\left(\mathcal{N}_{\Lambda\Sigma}\left(z, \bar{z}\right)\right){}^*\mathcal{F}^\Sigma\right] \equiv n_\Lambda^e$$

$$(4.2.38)$$

defines (in suitable units) the electric charges of the system; while at the "pre-dressing" stage there is non-trivial moduli-dependence through $\mathcal{N}_{\Lambda\Sigma}\left(z, \bar{z}\right)$, the (asymptotical) "space-dressing" of $\mathcal{G}_\Lambda$ is such that in the "post-dressing" stage there is no moduli dependence.

By using the previously introduced symplectic-invariant scalar product, we can now define two symplectic-invariant combinations of the symplectic field strength vector $\mathcal{Z}^{-}$.

The first one is

$$T^- \equiv - \langle \mathcal{Z}^-, V \rangle = M_\Lambda \mathcal{F}^{-\Lambda} - L^\Lambda \mathcal{G}_\Lambda^- =$$

$$= \mathcal{N}_{\Lambda\Sigma} L^\Sigma \mathcal{F}^{-\Lambda} - L^\Lambda \overline{\mathcal{N}}_{\Lambda\Sigma} \mathcal{F}^{-\Sigma} =$$

$$= 2i \left(Im \left(\mathcal{N} \right) L \right)_\Lambda \mathcal{F}^{-\Lambda} =$$

$$= T_\Lambda \mathcal{F}^{-\Lambda},$$

(4.2.39)

where use of the symmetry of $\mathcal{N}_{\Lambda\Sigma}$ and of Eqs. (4.1.34), (4.1.98) and (4.2.15) has been made. T_Λ may be considered the symplectic vector counterpart of the graviphoton field strength $T_{\mu\nu}^-$ (or, more rigorously, the graviphoton projector).

In general, since the U-duality group $Sp \left(2n_V + 2, \mathbb{R} \right)$ is defined over the real numbers, a complex symplectic invariant will yield two distinct real symplectic invariants, given by its real and imaginary parts, or by (linear) combination of them. In such a "decomposition" the symplectic invariance is mantained simply due to the saturation of symplectic, uppercase Greek indices. Further below, we will see that the two fundamental SKG Ansätze (4.1.34) and (4.1.35) will always determine the vanishing of one of the two real symplectic invariants obtained by some kind of "decomposition" of a complex $Sp \left(2n_V + 2, \mathbb{R} \right)$-invariant quantity.

Let us start by considering the complex $Sp \left(2n_V + 2 \right)$-invariant T^- defined in Eq. (4.2.39). By using Eq. (4.2.26), we obtain

$$T^- \equiv - \langle \mathcal{Z}^-, V \rangle = -2 \langle Re \left(\mathcal{Z}^- \right), V \rangle + \langle \mathcal{Z}^+, V \rangle. \qquad (4.2.40)$$

Moreover, Eqs. (4.1.34), (4.2.29) and (4.2.15) yield

$$\langle \mathcal{Z}^+, V \rangle = L^\Lambda \mathcal{G}_\Lambda^+ - M_\Lambda \mathcal{F}^{+\Lambda} = \mathcal{N}_{\Lambda\Sigma} L^\Lambda \mathcal{F}^{+\Sigma} - M_\Lambda \mathcal{F}^{+\Lambda} =$$

$$= M_\Lambda \mathcal{F}^{+\Lambda} - M_\Lambda \mathcal{F}^{+\Lambda} = 0. \qquad (4.2.41)$$

Thus, the first complex $Sp \left(2n_V + 2 \right)$-invariant may be written as

$$T^- \equiv - \langle \mathcal{Z}^-, V \rangle = -2 \langle Re \left(\mathcal{Z}^- \right), V \rangle = 2M_\Lambda \mathcal{F}^\Lambda - 2L^\Lambda \mathcal{G}_\Lambda. \qquad (4.2.42)$$

On the other hand, the second complex symplectic-invariant combination which may be considered reads (recall that $\overline{D}_{\bar{\jmath}}$ denotes the antiholomorphic Kähler-covariant derivative in the moduli space)

$$\mathcal{F}^{-i} \equiv -G^{i\bar{\jmath}} \left\langle \mathcal{Z}^-, \overline{D}_{\bar{\jmath}} \overline{V} \right\rangle = G^{i\bar{\jmath}} \left[\left(\overline{D}_{\bar{\jmath}} \overline{M}_\Lambda \right) \mathcal{F}^{-\Lambda} - \left(\overline{D}_{\bar{\jmath}} \overline{L}^\Lambda \right) \mathcal{G}_\Lambda^- \right]. \qquad (4.2.43)$$

By complex-conjugating it, we get

$$\mathcal{F}^{+\bar{j}} \equiv \overline{\mathcal{F}^{-j}} = -\overline{G^{j\bar{i}} \left\langle \mathcal{Z}^-, \overline{D_{\bar{i}}\overline{V}} \right\rangle} = -G^{\bar{i}j} \left\langle \mathcal{Z}^+, D_i V \right\rangle =$$

$$= G^{\bar{i}j} \left[(D_i M_\Lambda) \mathcal{F}^{+\Lambda} - \left(D_i L^\Lambda \right) \mathcal{G}_\Lambda^+ \right]. \tag{4.2.44}$$

By using Eq. (4.2.26), we obtain

$$\mathcal{F}^{+\bar{j}} = -G^{\bar{i}j} \left\langle \mathcal{Z}^+, D_i V \right\rangle = -2G^{\bar{i}j} \left\langle Re\left(\mathcal{Z}^- \right), D_i V \right\rangle + G^{\bar{i}j} \left\langle \mathcal{Z}^-, D_i V \right\rangle. \tag{4.2.45}$$

As before, Eqs. (4.1.34), (4.2.29) and (4.2.15) yield

$$\left\langle \mathcal{Z}^-, D_i V \right\rangle = \left(D_i L^\Lambda \right) \mathcal{G}_\Lambda^- - (D_i M_\Lambda) \mathcal{F}^{-\Lambda} =$$

$$= \overline{\mathcal{N}}_{\Lambda\Sigma} \left(D_i L^\Lambda \right) \mathcal{F}^{-\Sigma} - \overline{\mathcal{N}}_{\Lambda\Sigma} \left(D_i L^\Sigma \right) \mathcal{F}^{-\Lambda} = 0. \tag{4.2.46}$$

Thus, the second complex $Sp\left(2n_V + 2\right)$-invariant may be written as

$$\mathcal{F}^{+\bar{j}} = -G^{\bar{i}j} \left\langle \mathcal{Z}^+, D_i V \right\rangle = -2G^{\bar{i}j} \left\langle Re\left(\mathcal{Z}^- \right), D_i V \right\rangle =$$

$$= 2G^{\bar{i}j} \left[(D_i M_\Lambda) \mathcal{F}^\Lambda - \left(D_i L^\Lambda \right) \mathcal{G}_\Lambda \right]. \tag{4.2.47}$$

By complex-conjugating Eqs. (4.2.41) and (4.2.46), we may summarize the obtained symplectic-orthogonality relations as follows:

$$\left\langle \mathcal{Z}^-, \overline{V} \right\rangle = 0 \Leftrightarrow \left\langle \mathcal{Z}^+, V \right\rangle = 0,$$

$$\left\langle \mathcal{Z}^-, D_i V \right\rangle = 0 \Leftrightarrow \left\langle \mathcal{Z}^+, \overline{D_{\bar{i}}\overline{V}} \right\rangle = 0. \tag{4.2.48}$$

Let us now consider the "space-dressing" of $-\frac{1}{2}T^-$ in the case of staticity and spherical symmetry of the moduli configurations (which therefore will at most be radially dependent $z^i = z^i\left(r\right)$). Eqs. (4.2.37), (4.2.38) and (4.2.42) yield

$$-\frac{1}{2} \int_{S^2_\infty} T^- = \int_{S^2_\infty} L^\Lambda\left(z\left(r\right), \bar{z}\left(r\right)\right) \mathcal{G}_\Lambda - \int_{S^2_\infty} M_\Lambda\left(z\left(r\right), \bar{z}\left(r\right)\right) \mathcal{F}^\Lambda =$$

$$= L^\Lambda_\infty \int_{S^2_\infty} \mathcal{G}_\Lambda - M_{\Lambda,\infty} \int_{S^2_\infty} \mathcal{F}^\Lambda =$$

$$= L^\Lambda_\infty n^e_\Lambda - M_{\Lambda,\infty} n^\Lambda_m \equiv$$

$$\equiv Z\left(z_\infty, \bar{z}_\infty; n_m, n^e\right), \tag{4.2.49}$$

where z_∞, L_∞^Λ and $M_{\Lambda,\infty}$ respectively stand for the asymptotical values of the moduli and of the symplectic sections[12]

$$z_\infty^i \equiv lim_{r\to\infty} z^i(r); \qquad (4.2.50)$$

$$L_\infty^\Lambda \equiv L^\Lambda(z_\infty, \bar{z}_\infty) = lim_{r\to\infty} L^\Lambda(z(r), \bar{z}(r)); \qquad (4.2.51)$$

$$M_{\Lambda,\infty} \equiv M_\Lambda(z_\infty, \bar{z}_\infty) = lim_{r\to\infty} M_\Lambda(z(r), \bar{z}(r)). \qquad (4.2.52)$$

Rigorously, the Z defined by Eq. (6.3.0.6) should be denoted[13] by Z_∞: the central charge of the asymptotical supersymmetry algebra is the asymptotical value of the so-called "central charge" function (see Footnote 6 of Sect. 2)

$$Z(z(r), \bar{z}(r); n_m, n^e) \equiv L^\Lambda(z(r), \bar{z}(r)) n_\Lambda^e - M_\Lambda(z(r), \bar{z}(r)) n_m^\Lambda. \qquad (4.2.53)$$

In the considered static and spherically symmetric case[14], Eqs. (6.3.0.6) and (4.2.53) yield

$$Z_\infty(z_\infty, \bar{z}_\infty; n_m, n^e) = lim_{r\to\infty} Z(z(r), \bar{z}(r); n_m, n^e). \qquad (4.2.54)$$

From the above definitions, it follows that both the central charge Z_∞ and the "central charge" function $Z(z(r), \bar{z}(r); n_m, n^e)$ are symplectic invariant and they have the same Kähler weights as the symplectic, Kähler-covariantly holomorphic sections L^Λ and M_Λ, i.e. $(1, -1)$. Here and in the following treatment, unless otherwise noted, we will formulate the hypotheses of staticity and spherical symmetry.

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[12]Eqs. (4.2.51) and (4.2.52) clearly yield the assumption that the asymptotical limit $r \to \infty$ is "smooth" for the symplectic sections $L^\Lambda(z(r), \bar{z}(r))$ and $M_\Lambda(z(r), \bar{z}(r))$, in the sense specified in Footnote 6 of Sect. 2.

[13]As for the symplectic sections $L^\Lambda(z(r), \bar{z}(r))$ and $M_\Lambda(z(r), \bar{z}(r))$, the asymptotical limit $r \to \infty$ is assumed to be "smooth" also for $Z(z(r), \bar{z}(r); n_m, n^e)$, in the sense specified in Footnote 6 of Sect. 2.

In what follows, we will mainly deal with "central charge" function $Z(z(r), \bar{z}(r); n_m, n^e)$. The distinction from the central charge Z_∞ of the asymptotical SUSY algebra will usually be clear from the context, thus we will sometimes omit the subscript "∞".

[14]When considering the most general case in which the hypotheses of spherical symmetry and staticity are both removed, at least the central charge Z_∞ may still be defined as follows:

$$Z_\infty \equiv -\frac{1}{2} \int_{S_\infty^2} T^- =$$

$$= \int_{S_\infty^2} L^\Lambda(z(t,r,\theta,\varphi), \bar{z}(t,r,\theta,\varphi)) \mathcal{G}_\Lambda - \int_{S_\infty^2} M_\Lambda(z(t,r,\theta,\varphi), \bar{z}(t,r,\theta,\varphi)) \mathcal{F}^\Lambda,$$

where (r, θ, φ) denotes the usual spherical spatial coordinates. Clearly, in this case Z_∞ will generally be a non-trivial function of the time t and of the asymptotical configurations $(z_\infty, \bar{z}_\infty)$ of the moduli.

Let us now "space-dress" $-\frac{1}{2}\mathcal{F}^{+\bar{j}}G_{i\bar{j}}$; by recalling Eq. (4.2.47), we obtain

$$-\frac{1}{2}\int_{S^2_\infty} \mathcal{F}^{+\bar{j}}G_{i\bar{j}} = \int_{S^2_\infty} G_{i\bar{j}}G^{l\bar{j}}\left[\left(D_l L^\Lambda\right)\mathcal{G}_\Lambda - (D_l M_\Lambda)\mathcal{F}^\Lambda\right] =$$

$$= \int_{S^2_\infty}\left[\left(D_i L^\Lambda\right)\mathcal{G}_\Lambda - (D_i M_\Lambda)\mathcal{F}^\Lambda\right]. \tag{4.2.55}$$

Here the following subtlety arises. For what concerns the first term, by using Eq. (4.2.32) we get

$$\int_{S^2_\infty}\left(D_i L^\Lambda\right)\mathcal{G}_\Lambda = \int_{S^2_\infty}\left[D_i\left(L^\Lambda \mathcal{G}_\Lambda\right) - L^\Lambda D_i \mathcal{G}_\Lambda\right] =$$

$$= \int_{S^2_\infty}\left\{D_i\left(L^\Lambda \mathcal{G}_\Lambda\right) - L^\Lambda D_i\left[Re\left(\mathcal{N}_{\Lambda\Sigma}\right)\mathcal{F}^\Sigma - Im\left(\mathcal{N}_{\Lambda\Sigma}\right){}^*\mathcal{F}^\Sigma\right]\right\} =$$

$$= D_{i,\infty}\int_{S^2_\infty} L^\Lambda \mathcal{G}_\Lambda$$

$$- \int_{S^2_\infty} L^\Lambda\left[\partial_i\left(Re\left(\mathcal{N}_{\Lambda\Sigma}\right)\right)\mathcal{F}^\Sigma - \partial_i\left(Im\left(\mathcal{N}_{\Lambda\Sigma}\right)\right){}^*\mathcal{F}^\Sigma\right],$$

$$\tag{4.2.56}$$

where $D_{i,\infty}$ denotes the Kähler-covariant derivative w.r.t. the asymptotical configurations of the moduli defined by Eq. (4.2.50). Therefore the holomorphic Kähler-covariant derivative cannot be moved outside the "space-dressing" integral, because $\mathcal{G}_\Lambda$ is a moduli-dependent Kähler-scalar.

Nevertheless, it should be recalled that the asymptotical "dressing" of such a Kähler-scalar, i.e. the electric charge (see Eqs. (4.2.34) and (4.2.38)), is by definition moduli-independent. Therefore it holds that

$$\int_{S^2_\infty}\left(D_i L^\Lambda\right)\mathcal{G}_\Lambda = \left(D_i L^\Lambda\right)_\infty\int_{S^2_\infty}\mathcal{G}_\Lambda = D_{i,\infty}\left[L^\Lambda_\infty\left(\int_{S^2_\infty}\mathcal{G}_\Lambda\right)\right]. \tag{4.2.57}$$

For what concerns the second term, no problems arise, because $\mathcal{F}^\Lambda$ is moduli-independent, and therefore we may move D_i outside the spatial integral over S^2_∞ after collecting the term $D_i\left(M_\Lambda \mathcal{F}^\Lambda\right)$ inside it

$$\int_{S^2_\infty}(D_i M_\Lambda)\mathcal{F}^\Lambda = \int_{S^2_\infty} D_i\left(M_\Lambda \mathcal{F}^\Lambda\right) =$$

$$= D_{i,\infty}\int_{S^2_\infty} M_\Lambda \mathcal{F}^\Lambda = D_{i,\infty}\left(M_{\Lambda,\infty}\int_{S^2_\infty}\mathcal{F}^\Lambda\right). \tag{4.2.58}$$

Thus, by collecting Eqs. (4.2.57) and (4.2.58), we finally get

$$
-\frac{1}{2} \int_{S^2_\infty} \mathcal{F}^{+\bar{j}} G_{i\bar{j}} = \int_{S^2_\infty} \left[\left(D_i L^\Lambda \right) \mathcal{G}_\Lambda - (D_i M_\Lambda) \mathcal{F}^\Lambda \right] =
$$

$$
= D_{i,\infty} \left[L^\Lambda_\infty \left(\int_{S^2_\infty} \mathcal{G}_\Lambda \right) \right] - D_{i,\infty} \left(M_{\Lambda,\infty} \int_{S^2_\infty} \mathcal{F}^\Lambda \right) =
$$

$$
= D_{i,\infty} \left[L^\Lambda_\infty \int_{S^2_\infty} \mathcal{G}_\Lambda - M_{\Lambda,\infty} \int_{S^2_\infty} \mathcal{F}^\Lambda \right] =
$$

$$
= D_{i,\infty} Z_\infty \left(z_\infty, \bar{z}_\infty; n_m, n^e \right) \equiv Z_{i,\infty} \left(z_\infty, \bar{z}_\infty; n_m, n^e \right),
$$

$$(4.2.59)$$

where in the last line we recalled the definition of the central charge Z_∞ given by Eq. (6.3.0.6). Once again, the quantity[15] $Z_{i,\infty} \left(z_\infty, \bar{z}_\infty; n_m, n^e \right)$, defined by (4.2.59) and called *Kähler condensate of the asymptotical SUSY algebra*, may be seen as the asymptotical limit[16] of the so-called "Kähler condensate" function

$$
Z_i \left(z(r), \bar{z}(r); n_m, n^e \right) \equiv (D_i Z) \left(z(r), \bar{z}(r); n_m, n^e \right). \qquad (4.2.60)
$$

Such a function is the Kähler-covariant derivative (w.r.t. the r-dependent moduli) of the "central charge" function defined by Eq. (4.2.53). Thus, in the considered static and spherically symmetric case, Eqs. (4.2.59) and (4.2.60) yield

$$
Z_{i,\infty} \left(z_\infty, \bar{z}_\infty; n_m, n^e \right) = \lim_{r \to \infty} (D_i Z) \left(z(r), \bar{z}(r); n_m, n^e \right). \qquad (4.2.61)
$$

Summarizing, in the assumed hypotheses of staticity and spherical symmetry, Eqs. (6.3.0.6) and (4.2.59) respectively are the definitions of the central charge of the asymptotical $\mathcal{N} = 2$, $d = 4$ SUSY algebra and of the so-called Kähler condensate

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[15]In what follows, we will mainly consider the "Kähler condensate" function $(D_i Z) \left(z(r), \bar{z}(r); n_m, n^e \right)$. The distiction from the Kähler condensate $Z_{i,\infty} \left(z_\infty, \bar{z}_\infty; n_m, n^e \right)$ of the asymptotical SUSY algebra will usually be clear from the context, thus we will sometimes omit the subscript "∞".

[16]As for the "central charge" function $Z \left(z(r), \bar{z}(r); n_m, n^e \right)$ and for the symplectic sections $L^\Lambda \left(z(r), \bar{z}(r) \right)$ and $M_\Lambda \left(z(r), \bar{z}(r) \right)$, the asymptotical limit $r \to \infty$ is assumed to be "smooth" also for the functions $Z_i \left(z(r), \bar{z}(r); n_m, n^e \right)$, in the sense specified in Footnote 6 of Sect. 2 (the reasoning made in such a Footnote for Z and Z_∞ may be repeated in a completely analogous fashion for Z_i and $Z_{i,\infty}$).

of such a central extension operator. Such Eqs. are nothing but the asymptotical limit of the definitions of the "central charge" function $Z\left(z\left(r\right),\bar{z}\left(r\right);n_m,n^e\right)$ and of the "Kähler condensate" function $\left(D_iZ\right)\left(z\left(r\right),\bar{z}\left(r\right);n_m,n^e\right)$, respectively given by Eqs. (4.2.53) and (4.2.60).

A number of equivalent expressions for the "central charge" function and the related "Kähler condensate" function[17] may be easily obtained. First of all, we may rewrite Eq. (4.2.53) by recalling Eqs. (4.1.34) and (4.1.54)

$$
\begin{aligned}
Z\left(z,\bar{z};n_m,n^e\right) \;\; & = \;\; L^\Lambda\left(z,\bar{z}\right)n^e_\Lambda - M_\Lambda\left(z,\bar{z}\right)n^\Lambda_m = \\[2mm]
& = \;\; L^\Lambda\left(z,\bar{z}\right)n^e_\Lambda - \mathcal{N}_{\Lambda\Sigma}\left(z,\bar{z}\right)L^\Sigma\left(z,\bar{z}\right)n^\Lambda_m = \\[2mm]
& = \;\; \left(n^e_\Sigma - \mathcal{N}_{\Lambda\Sigma}\left(z,\bar{z}\right)n^\Lambda_m\right)L^\Sigma\left(z,\bar{z}\right) = \\[2mm]
& = \;\; \left[\exp\left(\frac{1}{2}K\left(z,\bar{z}\right)\right)\right]\left[X^\Lambda\left(z\right)n^e_\Lambda - F_\Lambda\left(z\right)n^\Lambda_m\right] = \\[2mm]
& = \;\; \left[\exp\left(\frac{1}{2}K\left(z,\bar{z}\right)\right)\right]\left(n^e_\Sigma - \mathcal{N}_{\Lambda\Sigma}\left(z,\bar{z}\right)n^\Lambda_m\right)X^\Sigma\left(z\right).
\end{aligned}
$$

$$(4.2.62)$$

Eqs. (7.2.1.34) and (4.2.53) directly yield the Kähler-covariant holomorphicity of the central charge $Z\left(z,\bar{z};n_m,n^e\right)$ of $\mathcal{N} = 2$, $d = 4$, n_V-fold Maxwell Einstein Supergravity Theory

$$
\overline{D}_{\bar{i}}Z\left(z,\bar{z};n_m,n^e\right) = 0 \Leftrightarrow D_i\overline{Z}\left(z,\bar{z};n_m,n^e\right) = 0. \tag{4.2.63}
$$

By definition, a Kähler-covariantly holomorphic scalar function f with antiholomorphic Kähler weight -1 satisfies

$$
\overline{D}_{\bar{i}}f\left(z,\bar{z}\right) = \left(\bar{\partial}_{\bar{i}} - \tfrac{1}{2}\bar{\partial}_{\bar{i}}K\left(z,\bar{z}\right)\right)f\left(z,\bar{z}\right) = 0
$$

$$
\Updownarrow
$$

$$
f\left(z,\bar{z}\right) = \left[exp\left(\tfrac{1}{2}K\left(z,\bar{z}\right)\right)\right]g\left(z\right), \;\; \bar{\partial}_{\bar{i}}g\left(z\right) = 0.
$$

$$(4.2.64)$$

---

[17]Since we will always be dealing with functions in the r-dependent moduli space, in the following treatment we will omit to say "function".

By considering $f(z, \bar{z}) = Z(z, \bar{z}; n_m, n^e)$, clearly Eq. (4.2.62) implies

$$g(z; n_m, n^e) = \left(n_\Sigma^e - \mathcal{N}_{\Lambda\Sigma}(z, \bar{z}) n_m^\Lambda\right) X^\Sigma(z).$$

(4.2.65)

Such a function, despite the presence of $\mathcal{N}_{\Lambda\Sigma}(z, \bar{z})$, is holomorphic due to the differential property of $\mathcal{N}_{\Lambda\Sigma}(z, \bar{z})$ expressed by Eq. (4.1.69); indeed

$$\bar{\partial}_{\bar{i}} g(z; n_m, n^e) = \bar{\partial}_{\bar{i}} \left[\left(n_\Sigma^e - \mathcal{N}_{\Lambda\Sigma}(z, \bar{z}) n_m^\Lambda\right) X^\Sigma(z)\right] =$$

$$= -n_m^\Lambda \left(\bar{\partial}_{\bar{i}} \mathcal{N}_{\Lambda\Sigma}(z, \bar{z})\right) X^\Sigma(z) = 0.$$

(4.2.66)

Now, in order to explicit the Kähler condensate, we must apply the holomorphic Kähler-covariant derivative to the central charge; by using Eqs. (4.1.27), (4.1.35), (4.2.36) and (4.2.62), we obtain

$$Z_i(z, \bar{z}; n_m, n^e) \equiv D_i Z(z, \bar{z}; n_m, n^e) = D_i \left[L^\Lambda(z, \bar{z}) n_\Lambda^e - M_\Lambda(z, \bar{z}) n_m^\Lambda\right] =$$

$$= \left(D_i L^\Lambda(z, \bar{z})\right) n_\Lambda^e - \left(D_i M_\Lambda(z, \bar{z})\right) n_m^\Lambda =$$

$$= \left(D_i L^\Lambda(z, \bar{z})\right) n_\Lambda^e - \overline{\mathcal{N}}_{\Lambda\Sigma}(z, \bar{z}) \left(D_i L^\Sigma(z, \bar{z})\right) n_m^\Lambda =$$

$$= \left(n_\Sigma^e - \overline{\mathcal{N}}_{\Lambda\Sigma}(z, \bar{z}) n_m^\Lambda\right) D_i L^\Sigma(z, \bar{z}) =$$

$$= \left(n_\Sigma^e - \overline{\mathcal{N}}_{\Lambda\Sigma}(z, \bar{z}) n_m^\Lambda\right) f_i^\Sigma(z, \bar{z}) =$$

$$= \left(n_\Sigma^e - \overline{\mathcal{N}}_{\Lambda\Sigma}(z, \bar{z}) n_m^\Lambda\right) \left(\partial_i + \tfrac{1}{2}\partial_i K\right) \left[\exp\left(\tfrac{1}{2}K(z, \bar{z})\right) X^\Sigma(z)\right] =$$

$$= \left(n_\Sigma^e - \overline{\mathcal{N}}_{\Lambda\Sigma}(z, \bar{z}) n_m^\Lambda\right) \left(\partial_i K\right) \left[\exp\left(\tfrac{1}{2}K(z, \bar{z})\right) X^\Sigma(z)\right] +$$

$$+ \left(n_\Sigma^e - \overline{\mathcal{N}}_{\Lambda\Sigma}(z, \bar{z}) n_m^\Lambda\right) \exp\left(\tfrac{1}{2}K(z, \bar{z})\right) \left(\partial_i X^\Sigma(z)\right) =$$

$$= \left(\partial_i K\right) Z(n_m, n^e, z, \bar{z})\big|_{\mathcal{N}_{\Lambda\Sigma} \to \overline{\mathcal{N}}_{\Lambda\Sigma}} +$$

$$+ \left(n_\Sigma^e - \overline{\mathcal{N}}_{\Lambda\Sigma}(z, \bar{z}) n_m^\Lambda\right) \exp\left(\tfrac{1}{2}K(z, \bar{z})\right) \left(\partial_i X^\Sigma(z)\right).$$

(4.2.67)

Attention should be paid to the complex conjugation of the Kähler-condensate. Indeed

$$Z_{\bar{i}} \equiv \overline{D}_{\bar{i}} Z = 0 \neq \overline{Z_{\bar{i}}} = \overline{(D_i Z)} = \overline{D}_{\bar{i}} \overline{Z}.$$

(4.2.68)

On the other hand, $\overline{Z}_{\overline{i}} = 0 \Leftrightarrow Z_i = 0$.

By using the second relation of Eqs. (4.1.99), the expression (4.2.39) of the symplectic-invariant quantity T^- may be rewritten as follows:

$$T^- = 2i\left(Im\left(\mathcal{N}\right)L\right)_{\Lambda}\mathcal{F}^{-\Lambda} = T_{\Lambda}\mathcal{F}^{-\Lambda} = i\overline{L}^{\Lambda}T_{\Sigma}\mathcal{F}^{-\Sigma}T_{\Lambda}.$$

(4.2.69)

On the other hand, by using the Ansatz (4.1.35) and Eqs. (4.1.27) and (4.2.15), the expression (4.2.43) of the other symplectic-invariant quantity $\mathcal{F}^{-i}$ yields

$$\mathcal{F}^{-i} = G^{i\overline{j}}\left[\left(\overline{D}_{\overline{j}}\overline{M}_{\Lambda}\right)\mathcal{F}^{-\Lambda} - \left(\overline{D}_{\overline{j}}\overline{L}^{\Lambda}\right)\mathcal{G}_{\Lambda}^-\right] = 2iG^{i\overline{j}}\left(Im\left(\mathcal{N}\right)\right)_{\Lambda\Sigma}\overline{f}_{\overline{j}}^{\Lambda}\mathcal{F}^{-\Sigma}.$$

(4.2.70)

Now, we can introduce $\widehat{\mathcal{F}}^{-\Lambda}$ as the component of the imaginary self-dual Maxwell field strength $\mathcal{F}^{-\Lambda}$ orthogonal to the graviphoton projector T_{Λ}

$$\widehat{\mathcal{F}}^{-\Lambda}T_{\Lambda} \equiv 0.$$

(4.2.71)

By putting

$$\widehat{\mathcal{F}}^{-\Lambda} \equiv \mathcal{F}^{-\Lambda} + \check{\mathcal{F}}^{-\Lambda},$$

(4.2.72)

Eq. (4.2.69) yields

$$\check{\mathcal{F}}^{-\Lambda} = -i\overline{L}^{\Lambda}T_{\Sigma}\mathcal{F}^{-\Sigma},$$

(4.2.73)

and therefore

$$\widehat{\mathcal{F}}^{-\Lambda} \equiv \mathcal{F}^{-\Lambda} + \check{\mathcal{F}}^{-\Lambda} = \mathcal{F}^{-\Lambda} - i\overline{L}^{\Lambda}T_{\Sigma}\mathcal{F}^{-\Sigma} =$$

$$= \left(\delta_{\Sigma}^{\Lambda} - i\overline{L}^{\Lambda}T_{\Sigma}\right)\mathcal{F}^{-\Sigma}.$$

(4.2.74)

Let us now apply the antiholomorphic Kähler-covariant derivative to Eq. (4.1.36); by using Eqs. (7.2.1.34), (4.1.27) and (4.1.51), we get

$$\overline{D}_{\overline{i}}\left(Im\left(\mathcal{N}_{\Lambda\Sigma}\right)\overline{L}^{\Lambda}L^{\Sigma}\right) = 0 \Leftrightarrow Im\mathcal{N}_{\Lambda\Sigma}\overline{f}_{\overline{i}}^{\Lambda}L^{\Sigma} = 0.$$

(4.2.75)

Notice that such a result cannot be obtained by complex conjugating Eq. (4.1.73). By adding Eq. (4.1.73) to Eq. (4.2.75), one gets

$$\left(Im\mathcal{N}_{\Lambda\Sigma}\right)L^{\Lambda}\left(Ref_i^{\Sigma}\right) = 0 \Leftrightarrow \begin{cases} \left(Im\mathcal{N}_{\Lambda\Sigma}\right)\left(ReL^{\Lambda}\right)\left(Ref_i^{\Sigma}\right) = 0; \\ \\ \left(Im\mathcal{N}_{\Lambda\Sigma}\right)\left(ImL^{\Lambda}\right)\left(Ref_i^{\Sigma}\right) = 0. \end{cases}$$

(4.2.76)

On the other hand, by subtracting Eq. (4.1.73) to Eq. (4.2.75), one instead obtains

$$(Im\mathcal{N}_{\Lambda\Sigma}) L^{\Lambda} \left(Im f_i^{\Sigma}\right) = 0 \Leftrightarrow \begin{cases} (Im\mathcal{N}_{\Lambda\Sigma}) \left(ReL^{\Lambda}\right) \left(Im f_i^{\Sigma}\right) = 0; \\ \\ (Im\mathcal{N}_{\Lambda\Sigma}) \left(ImL^{\Lambda}\right) \left(Im f_i^{\Sigma}\right) = 0. \end{cases}$$

$$(4.2.77)$$

Now, due to Eq. (4.1.73), in Eq. (4.2.70) we may substitute $\mathcal{F}^{-\Lambda}$ with $\widehat{\mathcal{F}}^{-\Lambda}$ given by (4.2.74), because the extra term is zero

$$Im\mathcal{N}_{\Lambda\Sigma}\overline{f}_{\bar{j}}^{\Lambda}\widehat{\mathcal{F}}^{-\Sigma} = Im\mathcal{N}_{\Lambda\Sigma}\overline{f}_{\bar{j}}^{\Lambda}\mathcal{F}^{-\Sigma} - iIm\mathcal{N}_{\Lambda\Sigma}\overline{f}_{\bar{j}}^{\Lambda}\overline{L}^{\Sigma}T_{\Delta}\mathcal{F}^{-\Delta} =$$

$$= Im\mathcal{N}_{\Lambda\Sigma}\overline{f}_{\bar{j}}^{\Lambda}\mathcal{F}^{-\Sigma}. \tag{4.2.78}$$

Consequently, it holds that

$$\mathcal{F}^{-i} = 2iG^{i\bar{j}} \left(Im\left(\mathcal{N}\right)\right)_{\Lambda\Sigma} \overline{f}_{\bar{j}}^{\Lambda}\widehat{\mathcal{F}}^{-\Sigma}, \tag{4.2.79}$$

and the symplectic-invariant quantity $\mathcal{F}^{-i}$ is orthogonal to the graviphoton projector T_{Λ}, too.

This result allows us to interpret Eq. (4.2.59) as the geometrization of the fluxes of those Maxwell field strengths which are orthogonal to the graviphoton projector T_{Λ}.

It is also worth noticing that actually, by the previous construction, the "charge operators" (Z, Z_i) are in correspondence with the integer conserved charges $\left(n_m^{\Lambda}, n_{\Lambda}^e\right)$, but they refer to the eigenstates of the vector supermultiplets, and therefore, in general, they exhibit a non-trivial functional dependence on the moduli.

In a generic point of the n_V-dim. Kähler-Hodge complex moduli space of the $\mathcal{N} = 2$, $d = 4$, n_V-fold Maxwell Einstein Supergravity Theory there exist, in general, two independent $Sp\left(2n_V + 2\right)$-invariants homogeneous of degree two in the (quantized) electric and magnetic charges of the system. Such invariants may be expressed in a model-independent way as follows [51] :

$$I_1\left(z, \overline{z}; n_m, n^e\right) \equiv |Z|^2\left(z, \overline{z}; n_m, n^e\right) + G^{i\bar{j}}\left(z, \overline{z}\right) Z_i\left(z, \overline{z}; n_m, n^e\right) \overline{Z}_{\bar{j}}\left(z, \overline{z}; n_m, n^e\right),$$

$$I_2\left(z, \overline{z}; n_m, n^e\right) \equiv |Z|^2\left(z, \overline{z}; n_m, n^e\right) - G^{i\bar{j}}\left(z, \overline{z}\right) Z_i\left(z, \overline{z}; n_m, n^e\right) \overline{Z}_{\bar{j}}\left(z, \overline{z}; n_m, n^e\right).$$

$$(4.2.80)$$

At this point it is useful to introduce the real symplectic $(2n_V + 2)$-dim. square matrix

$$\mathcal{M}\left(Re\left(\mathcal{N}\right), Im\left(\mathcal{N}\right)\right) \equiv \mathcal{R}^T\left(Re\left(\mathcal{N}\right)\right)\mathcal{D}\left(Im\left(\mathcal{N}\right)\right)\mathcal{R}\left(Re\left(\mathcal{N}\right)\right),$$

$$(4.2.81)$$

where

$$\mathcal{R}\left(Re\left(\mathcal{N}\right)\right) \equiv \begin{pmatrix} \mathbb{I} & 0 \\ -Re\left(\mathcal{N}\right) & \mathbb{I} \end{pmatrix}, \quad \mathcal{D}\left(Im\left(\mathcal{N}\right)\right) \equiv \begin{pmatrix} Im\left(\mathcal{N}\right) & 0 \\ 0 & \left(Im\left(\mathcal{N}\right)\right)^{-1} \end{pmatrix};$$

$$(4.2.82)$$

consequently

$$\mathcal{M}\left(Re\left(\mathcal{N}\right), Im\left(\mathcal{N}\right)\right) =$$

$$= \begin{pmatrix} Im\left(\mathcal{N}\right) + Re\left(\mathcal{N}\right)\left(Im\left(\mathcal{N}\right)\right)^{-1} Re\left(\mathcal{N}\right) & -Re\left(\mathcal{N}\right)\left(Im\left(\mathcal{N}\right)\right)^{-1} \\ -\left(Im\left(\mathcal{N}\right)\right)^{-1} Re\left(\mathcal{N}\right) & \left(Im\left(\mathcal{N}\right)\right)^{-1} \end{pmatrix}.$$

$$(4.2.83)$$

Notice that

$$\left[Re\left(\mathcal{N}\right), \left(Im\left(\mathcal{N}\right)\right)^{-1}\right] \neq 0 \qquad (4.2.84)$$

but, since $\mathcal{N}_{\Lambda\Sigma} = \mathcal{N}_{(\Lambda\Sigma)}$ (see Footnote 8) and

$$\left[Re\left(\mathcal{N}\right)\left(Im\left(\mathcal{N}\right)\right)^{-1}\right]^T = \left(Im\left(\mathcal{N}\right)\right)^{-1} Re\left(\mathcal{N}\right), \qquad (4.2.85)$$

the real matrix $\mathcal{M}\left(Re\left(\mathcal{N}\right), Im\left(\mathcal{N}\right)\right)$ is symmetric.

By using Eqs. (4.1.97)-(4.1.99), (4.2.35) and (4.2.83) and by recalling the definition of the $(n_V + 1)$-dim. complex symmetric square matrix $F_{\Lambda\Sigma} \equiv \frac{\partial^2 F}{\partial X^\Lambda \partial X^\Sigma}$, denoted with $\mathcal{F}\left(z\right)$ in matrix notation, we can rewrite the two symplectic-invariants of degree two as follows (recall Footnote 6, too):

$$I_1\left(z, \bar{z}; n_m, n^e\right) = -\frac{1}{2} n^T \mathcal{M}\left(Re\left(\mathcal{N}\right), Im\left(\mathcal{N}\right)\right) n =$$

$$= -\frac{1}{2}\left(n^e_\Lambda - \overline{\mathcal{N}}_{\Lambda\Sigma} n^\Sigma_m\right)\left(\left(Im\left(\mathcal{N}\right)\right)^{-1}\right)^{\Lambda\Delta}\left(n^e_\Delta - \mathcal{N}_{\Delta\Gamma} n^\Gamma_m\right),$$

$$(4.2.86)$$

$$I_2\left(z,\bar{z};n_m,n^e\right) = -\frac{1}{2}n^T\mathcal{M}\left(Re\left(\mathcal{F}\right),Im\left(\mathcal{F}\right)\right)n =$$

$$= -\frac{1}{2}\left(n^e_\Lambda - \overline{F}_{\Lambda\Sigma}n^\Sigma_m\right)\left(\left(Im\left(\mathcal{F}\right)\right)^{-1}\right)^{\Lambda\Delta}\left(n^e_\Delta - F_{\Delta\Gamma}n^\Gamma_m\right);$$

$$(4.2.87)$$

As is evident, Eqs. (4.2.86) and (4.2.87) are simply related by the matrix interchange $\mathcal{N} \leftrightarrow \mathcal{F}$.

By considering Eqs. (4.2.81)-(4.2.83), it is easy to realize that $Re\left(\mathcal{N}\right)$ and $Re\left(\mathcal{F}\right)$ do not play any role in the expressions (4.2.86) and (4.2.87), because the matrix function $\mathcal{R}$ can be included in the symplectic vector n by a simple redefinition. Indeed, defining $n_{\mathcal{R}(\mathcal{K})} \equiv \mathcal{R}\left(\mathcal{K}\right)n$ (where $\mathcal{K} = \mathcal{N},\mathcal{F}$ in this case), one immediately gets

$$n^T\mathcal{M}\left(Re\left(\mathcal{K}\right),Im\left(\mathcal{K}\right)\right)n = n^T\mathcal{R}^T\left(Re\left(\mathcal{K}\right)\right)D\left(Im\left(\mathcal{K}\right)\right)\mathcal{R}\left(Re\left(\mathcal{K}\right)\right)n =$$

$$= n^T_{\mathcal{R}(\mathcal{K})}\mathcal{D}\left(Im\left(\mathcal{K}\right)\right)n_{\mathcal{R}(\mathcal{K})}. \qquad (4.2.88)$$

Therefore, by looking at the signatures of the quadratic forms appearing on the r.h.s.'s of Eqs. (4.2.86) and (4.2.87), we get that $n^T_{\mathcal{R}(\mathcal{K})}\mathcal{D}\left(Im\left(\mathcal{K}\right)\right)n_{\mathcal{R}(\mathcal{K})}$ is manifestly a quadratic form with negative signature for $\mathcal{K} = \mathcal{N}$, and with n_V positive and one negative eigenvalues for $\mathcal{K} = \mathcal{F}$. Summarizing, Eqs. (4.2.86) and (4.2.87) reflect the fact that $Im\left(\mathcal{N}\right)$ is negative definite and that, as previously mentioned, $Im\left(\mathcal{F}\right)$ has an $(n_V,1)$ signature (i.e. has n_V positive and one negative eigenvalues).

We will now explicitly derive some important identities of the SKG of M_{n_V}, which generalize the calculations of Ferrara and Kallosh in [30]. Further below, we will see that, when evaluated at some particular points in M_{n_V}, the obtained identities will yield the so-called *non(-BPS)-SUSY extreme BH Attractor Equations*, recently rediscovered by Kallosh [52] (and explicitly checked in some examples in [53]), but which had actually already been written in a slightly different fashion in [54].

Let us start by considering $\overline{D}_{\bar{i}}\overline{Z}$; by recalling the definition (4.2.53), we may write

$$\overline{D}_{\bar{i}}\overline{Z} = n^e_\Lambda\overline{D}_{\bar{i}}\overline{L}^\Lambda - n^\Lambda_m\overline{D}_{\bar{i}}\overline{M}_\Lambda; \qquad (4.2.89)$$

by using the Ansatz (4.1.35) we thus get

$$\overline{D}_{\bar{i}}\overline{Z} = n^e_\Lambda\overline{D}_{\bar{i}}\overline{L}^\Lambda - n^\Lambda_m\mathcal{N}_{\Lambda\Delta}\overline{D}_{\bar{i}}\overline{L}^\Delta. \qquad (4.2.90)$$

The contraction of both sides with $G^{i\bar{i}}D_iL^\Sigma$ then yields

$$G^{i\bar{i}}\left(D_iL^\Sigma\right)\overline{D}_{\bar{i}}\overline{Z} = n^e_\Lambda G^{i\bar{i}}\left(D_iL^\Sigma\right)\overline{D}_{\bar{i}}\overline{L}^\Lambda - n^\Lambda_m\mathcal{N}_{\Lambda\Delta}G^{i\bar{i}}\left(D_iL^\Sigma\right)\overline{D}_{\bar{i}}\overline{L}^\Delta;$$

$$(4.2.91)$$

now, by using Eqs. (4.1.27) and (4.1.100) and the symmetry of $\mathcal{N}_{\Lambda\Sigma}$ and its inverse $\mathcal{N}^{\Lambda\Sigma}$ (see Eq. (4.1.102)), such an expression may be further elaborated as

$$G^{i\bar{i}} \left(D_i L^\Sigma \right) \overline{D}_{\bar{i}} \overline{Z} =$$

$$= n^e_\Lambda \left[-\tfrac{1}{2} \left((Im\mathcal{N})^{-1} \right)^{\Sigma\Lambda} - \overline{L}^\Sigma L^\Lambda \right] - n^\Lambda_m \mathcal{N}_{\Lambda\Delta} \left[-\tfrac{1}{2} \left((Im\mathcal{N})^{-1} \right)^{\Sigma\Delta} - \overline{L}^\Sigma L^\Delta \right] =$$

$$= -\tfrac{1}{2} \left((Im\mathcal{N})^{-1} \right)^{\Sigma\Lambda} n^e_\Lambda - \overline{L}^\Sigma L^\Lambda n^e_\Lambda +$$

$$+ \tfrac{1}{2} \left((Im\mathcal{N})^{-1} \right)^{\Sigma\Delta} \left[(Re\mathcal{N})_{\Delta\Lambda} + i \left(Im\mathcal{N} \right)_{\Delta\Lambda} \right] n^\Lambda_m + \overline{L}^\Sigma \mathcal{N}_{\Lambda\Delta} L^\Delta n^\Lambda_m =$$

$$= -\tfrac{1}{2} \left((Im\mathcal{N})^{-1} \right)^{\Sigma\Lambda} n^e_\Lambda - \overline{L}^\Sigma \left(L^\Lambda n^e_\Lambda - M_\Lambda n^\Lambda_m \right) +$$

$$+ \tfrac{1}{2} \left((Im\mathcal{N})^{-1} \right)^{\Sigma\Delta} (Re\mathcal{N})_{\Delta\Lambda} n^\Lambda_m + \tfrac{i}{2} n^\Sigma_m =$$

$$= \tfrac{i}{2} n^\Sigma_m - \overline{L}^\Sigma Z + \tfrac{1}{2} \left((Im\mathcal{N})^{-1} \right)^{\Sigma\Delta} (Re\mathcal{N})_{\Delta\Lambda} n^\Lambda_m - \tfrac{1}{2} \left((Im\mathcal{N})^{-1} \right)^{\Sigma\Lambda} n^e_\Lambda,$$

$$(4.2.92)$$

where in the last two lines we used the Ansatz (4.1.34) and the definition (4.2.53). Now, by subtracting from the expression (4.2.92) its complex conjugate, one gets

$$n^\Lambda_m = 2Re \left[i\overline{Z}L^\Lambda + iG^{i\bar{i}} \left(\overline{D}_{\bar{i}} \overline{L}^\Lambda \right) D_i Z \right] = -2Im \left[\overline{Z}L^\Lambda + G^{i\bar{i}} \left(\overline{D}_{\bar{i}} \overline{L}^\Lambda \right) D_i Z \right].$$

$$(4.2.93)$$

On the other hand, the contraction of both sides of Eq. (4.2.90) with $G^{i\bar{i}} D_i M_\Sigma$ yields

$$G^{i\bar{i}} \left(D_i M_\Sigma \right) \overline{D}_{\bar{i}} \overline{Z} = n^e_\Lambda G^{i\bar{i}} \left(D_i M_\Sigma \right) \overline{D}_{\bar{i}} \overline{L}^\Lambda - n^\Lambda_m \mathcal{N}_{\Lambda\Delta} G^{i\bar{i}} \left(D_i M_\Sigma \right) \overline{D}_{\bar{i}} \overline{L}^\Delta =$$

$$= n^e_\Lambda G^{i\bar{i}} \overline{\mathcal{N}}_{\Sigma\Delta} \left(D_i L^\Delta \right) \overline{D}_{\bar{i}} \overline{L}^\Lambda - n^\Lambda_m \mathcal{N}_{\Lambda\Delta} G^{i\bar{i}} \overline{\mathcal{N}}_{\Sigma\Xi} \left(D_i L^\Xi \right) \overline{D}_{\bar{i}} \overline{L}^\Delta,$$

$$(4.2.94)$$

where in the last line we used the Ansatz (4.1.35). Once again, by using Eqs. (4.1.100) and the symmetry of $\mathcal{N}_{\Lambda\Sigma}$ and its inverse $\mathcal{N}^{\Lambda\Sigma}$, the above expression may be further

elaborated as follows:

$$G^{i\bar{i}}\left(D_i M_\Sigma\right)\overline{D}_{\bar{i}}\overline{Z} =$$

$$= n^e_\Lambda \overline{\mathcal{N}}_{\Sigma\Delta}\left[-\tfrac{1}{2}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Delta\Lambda}-\overline{L}^\Delta L^\Lambda\right]+$$

$$-n^\Lambda_m \mathcal{N}_{\Lambda\Delta}\overline{\mathcal{N}}_{\Sigma\Xi}\left[-\tfrac{1}{2}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Xi\Delta}-\overline{L}^\Xi L^\Delta\right] =$$

$$= \left[-\tfrac{1}{2}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Delta\Lambda}-\overline{L}^\Delta L^\Lambda\right]\left[\left(Re\mathcal{N}\right)_{\Sigma\Delta}-i\left(Im\mathcal{N}\right)_{\Sigma\Delta}\right]n^e_\Lambda +$$

$$+ \left[\tfrac{1}{2}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Xi\Delta}+\overline{L}^\Xi L^\Delta\right]\left[\left(Re\mathcal{N}\right)_{\Sigma\Xi}-i\left(Im\mathcal{N}\right)_{\Sigma\Xi}\right]\mathcal{N}_{\Lambda\Delta}n^\Lambda_m =$$

$$= -\tfrac{1}{2}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Delta\Lambda}\left(Re\mathcal{N}\right)_{\Sigma\Delta}n^e_\Lambda+\tfrac{i}{2}n^e_\Sigma-\overline{L}^\Delta L^\Lambda \overline{\mathcal{N}}_{\Sigma\Delta}n^e_\Lambda+$$

$$+\tfrac{1}{2}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Xi\Delta}\left(Re\mathcal{N}\right)_{\Sigma\Xi}\mathcal{N}_{\Lambda\Delta}n^\Lambda_m-\tfrac{i}{2}\mathcal{N}_{\Lambda\Sigma}n^\Lambda_m+\overline{L}^\Xi L^\Delta \overline{\mathcal{N}}_{\Sigma\Xi}\mathcal{N}_{\Lambda\Delta}n^\Lambda_m =$$

$$= -\tfrac{1}{2}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Delta\Lambda}\left(Re\mathcal{N}\right)_{\Sigma\Delta}n^e_\Lambda+\tfrac{i}{2}n^e_\Sigma-\overline{M}_\Sigma L^\Lambda n^e_\Lambda+$$

$$+\tfrac{1}{2}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Xi\Delta}\left(Re\mathcal{N}\right)_{\Sigma\Xi}\left[\left(Re\mathcal{N}\right)_{\Lambda\Delta}+i\left(Im\mathcal{N}\right)_{\Lambda\Delta}\right]n^\Lambda_m+$$

$$-\tfrac{i}{2}\mathcal{N}_{\Lambda\Sigma}n^\Lambda_m+\overline{M}_\Sigma \mathcal{M}_\Lambda n^\Lambda_m =$$

$$= -\tfrac{1}{2}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Delta\Lambda}\left(Re\mathcal{N}\right)_{\Sigma\Delta}n^e_\Lambda+\tfrac{i}{2}n^e_\Sigma-\overline{M}_\Sigma Z+$$

$$+\tfrac{1}{2}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Xi\Delta}\left(Re\mathcal{N}\right)_{\Sigma\Xi}\left(Re\mathcal{N}\right)_{\Lambda\Delta}n^\Lambda_m+\tfrac{i}{2}\left(Re\mathcal{N}\right)_{\Sigma\Lambda}n^\Lambda_m+$$

$$-\tfrac{i}{2}\left[\left(Re\mathcal{N}\right)_{\Lambda\Sigma}+i\left(Im\mathcal{N}\right)_{\Lambda\Sigma}\right]n^\Lambda_m =$$

$$= -\tfrac{1}{2}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Delta\Lambda}\left(Re\mathcal{N}\right)_{\Sigma\Delta}n^e_\Lambda+\tfrac{i}{2}n^e_\Sigma-\overline{M}_\Sigma Z+$$

$$+\tfrac{1}{2}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Xi\Delta}\left(Re\mathcal{N}\right)_{\Sigma\Xi}\left(Re\mathcal{N}\right)_{\Lambda\Delta}n^\Lambda_m+\tfrac{1}{2}\left(Im\mathcal{N}\right)_{\Lambda\Sigma}n^\Lambda_m,$$

$$(4.2.95)$$

where in the last lines we used the Ansatz (4.1.34) and the definition (4.2.53). Thence, by subtracting to the expression (4.2.95) its complex conjugate, one gets

$$n_\Lambda^e = 2Re\left[i\overline{Z}M_\Lambda + iG^{i\overline{i}}\left(\overline{D_{\overline{i}}M}_\Lambda\right)D_iZ\right] = -2Im\left[\overline{Z}M_\Lambda + G^{i\overline{i}}\left(\overline{D_{\overline{i}}M}_\Lambda\right)D_iZ\right].$$
(4.2.96)

By expressing the identities (4.2.93) and (4.2.96) in a vector $Sp\left(2n_V + 2\right)$-covariant notation, one finally gets

$$\begin{pmatrix} n_m^\Lambda \\ \\ n_\Lambda^e \end{pmatrix} = 2Re\left[i\overline{Z}\begin{pmatrix} L^\Lambda \\ \\ M_\Lambda \end{pmatrix} + iG^{i\overline{i}}\begin{pmatrix} \left(\overline{D_{\overline{i}}L}^\Lambda\right) \\ \\ \left(\overline{D_{\overline{i}}M}_\Lambda\right) \end{pmatrix} D_iZ\right] =$$

$$= -2Im\left[\overline{Z}\begin{pmatrix} L^\Lambda \\ \\ M_\Lambda \end{pmatrix} + G^{i\overline{i}}\begin{pmatrix} \left(\overline{D_{\overline{i}}L}^\Lambda\right) \\ \\ \left(\overline{D_{\overline{i}}M}_\Lambda\right) \end{pmatrix} D_iZ\right], \qquad (4.2.97)$$

or in compact form

$$n = 2Re\left[i\overline{Z}V + iG^{i\overline{i}}\left(\overline{D_{\overline{i}}V}\right)D_iZ\right] = -2Im\left[\overline{Z}V + G^{i\overline{i}}\left(\overline{D_{\overline{i}}V}\right)D_iZ\right],$$
(4.2.98)

where we recalled the definitions (7.2.1.34) and (4.2.35) of the $(2n_V + 2) \times 1$ vectors V and n, respectively. It is worth pointing out that the vector identity (4.2.98) has been obtained only by using the properties of the SKG of M_{n_V}. Such an identity expresses nothing but a change of basis in the lattice Γ of BH charge configurations, between the real basis $\left(n_m^\Lambda, n_\Lambda^e\right)_{\Lambda=0,1,...,n_V}$ and the complex basis $(Z, D_iZ)_{i=1,...,n_V}$. Such a change of basis also introduces a non-trivial dependence on the moduli $\left(z^i, \overline{z}^{\overline{i}}\right)$, since the complex charges $(Z, D_iZ)_{i=1,...,n_V}$ refer to the supermultiplet eigenstates, and they thus are moduli-dependent. The relations yielded by the identity (4.2.98) are $2n_V + 2$ real ones, but they have been obtained by starting from an expression for $\overline{D_{\overline{i}}Z}$, corresponding to n_V complex, and therefore $2n_V$ real, degrees of freedom. The two redundant real degrees of freedom are encoded in the homogeneity (of degree 1) of the identity (4.2.98) under complex rescalings of the symplectic BH charge vector n; indeed, by recalling the definition (4.2.53) it is immediate to check that the r.h.s. of identity (4.2.98) acquires an overall factor λ under a global rescaling of n of the kind

$$n \longrightarrow \lambda n, \ \lambda \in \mathbb{C}. \qquad (4.2.99)$$

The summation of the expressions (4.2.92) and (4.2.95) with their complex conjugates respectively yields

$$
\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Delta\Lambda}\left(Re\mathcal{N}\right)_{\Delta\Sigma} n_m^{\Sigma} - \left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Lambda\Sigma} n_{\Sigma}^e =
$$

$$
= 2Re\left[\overline{Z}L^{\Lambda} + G^{i\bar{i}}\left(\overline{D}_{\bar{i}}\overline{L}^{\Lambda}\right)D_i Z\right];
$$

(4.2.100)

$$
\left[Im\mathcal{N}_{\Lambda\Sigma} + \left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Xi\Delta}\left(Re\mathcal{N}\right)_{\Lambda\Xi}\left(Re\mathcal{N}\right)_{\Sigma\Delta}\right]n_m^{\Sigma} +
$$

$$
- \left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Delta\Sigma}\left(Re\mathcal{N}\right)_{\Lambda\Delta} n_{\Sigma}^e =
$$

$$
= 2Re\left[\overline{Z}M_{\Lambda} + G^{i\bar{i}}\left(\overline{D}_{\bar{i}}\overline{M}_{\Lambda}\right)D_i Z\right].
$$

(4.2.101)

In order to elaborate a shorthand notation for the obtained SKG identities (4.2.93), (4.2.96) and (4.2.100), (4.2.101), let us now reconsider the starting expressions (4.2.92) and (4.2.95), respectively reading

$$
\left[\delta_{\Sigma}^{\Lambda} - i\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Lambda\Delta}\left(Re\mathcal{N}\right)_{\Delta\Sigma}\right]n_m^{\Sigma} + i\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Lambda\Sigma} n_{\Sigma}^e =
$$

$$
= -2i\overline{L}^{\Lambda}Z - 2iG^{i\bar{i}}\left(D_i L^{\Lambda}\right)\overline{D}_{\bar{i}}\overline{Z};
$$

(4.2.102)

$$
-i\left[\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Xi\Delta}\left(Re\mathcal{N}\right)_{\Lambda\Xi}\left(Re\mathcal{N}\right)_{\Sigma\Delta} + \left(Im\mathcal{N}\right)_{\Lambda\Sigma}\right]n_m^{\Sigma} +
$$

$$
+ \left[\delta_{\Lambda}^{\Sigma} + i\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Delta\Sigma}\left(Re\mathcal{N}\right)_{\Lambda\Delta}\right]n_{\Sigma}^e =
$$

$$
= -2i\overline{M}_{\Lambda}Z - 2iG^{i\bar{i}}\left(D_i M_{\Lambda}\right)\overline{D}_{\bar{i}}\overline{Z}.
$$

(4.2.103)

By recalling the definitions (7.2.1.34), (4.1.25) and (4.2.35) and Eq. (4.2.83), the identities (4.2.102) and (4.2.103) may be synthesized in vector notation as follows:

$$
n - i\epsilon\mathcal{M}\left(\mathcal{N}\right)n = -2i\overline{V}Z - 2iG^{i\bar{i}}\left(D_i V\right)\overline{D}_{\bar{i}}\overline{Z},
$$

(4.2.104)

where $\mathcal{M}(\mathcal{N})$ denotes   the $(2n_V + 2) \times (2n_V + 2)$ real   matrix $\mathcal{M}(Re(\mathcal{N}), Im(\mathcal{N}))$ given by Eq. (4.2.83). By using the symplectic-orthogonality relations given by Eqs. (4.1.26), II of (4.1.37), (4.1.44) and first and fourth of (4.1.43), the SKG identity (4.2.104) yields the following relations:

$$\begin{cases} \langle V, n - i\epsilon \mathcal{M}(\mathcal{N})n \rangle = -2Z; \\[2ex] \langle \overline{V}, n - i\epsilon \mathcal{M}(\mathcal{N})n \rangle = 0; \\[2ex] \langle D_i V, n - i\epsilon \mathcal{M}(\mathcal{N})n \rangle = 0; \\[2ex] \langle \overline{D}_{\bar{i}} V, n - i\epsilon \mathcal{M}(\mathcal{N})n \rangle = -2\overline{D}_{\bar{i}}\overline{Z}. \end{cases} \tag{4.2.105}$$

The real part of the general, fundamental SKG vector identity (4.2.104) yields

$$\begin{aligned} n &= -2Re\left[i\overline{V}Z + iG^{i\bar{i}}(D_i V)\overline{D}_{\bar{i}}\overline{Z}\right] = 2Re\left[iV\overline{Z} + iG^{i\bar{i}}(\overline{D}_{\bar{i}}\overline{V})D_i Z\right] = \\[2ex] &= 2Im\left[\overline{V}Z + G^{i\bar{i}}(D_i V)\overline{D}_{\bar{i}}\overline{Z}\right] = -2Im\left[V\overline{Z} + G^{i\bar{i}}(\overline{D}_{\bar{i}}\overline{V})D_i Z\right], \end{aligned} \tag{4.2.106}$$

which is nothing but the SKG vector identity (4.2.98), which in turn summarizes the identities (4.2.93) and (4.2.96). On the other hand, the imaginary part of (4.2.104) yields

$$\begin{aligned} \epsilon\mathcal{M}(\mathcal{N})n &= 2Im\left[i\overline{V}Z + iG^{i\bar{i}}(D_i V)\overline{D}_{\bar{i}}\overline{Z}\right] = -2Im\left[-iV\overline{Z} - iG^{i\bar{i}}(\overline{D}_{\bar{i}}\overline{V})D_i Z\right] = \\[2ex] &= 2Re\left[\overline{V}Z + G^{i\bar{i}}(D_i V)\overline{D}_{\bar{i}}\overline{Z}\right] = 2Re\left[V\overline{Z} + G^{i\bar{i}}(\overline{D}_{\bar{i}}\overline{V})D_i Z\right], \end{aligned} \tag{4.2.107}$$

and it summarizes the identities (4.2.100) and (4.2.101). Notice that the imaginary and real parts of the SKG identity (4.2.104) are linearly related by the $(2n_V + 2) \times (2n_V + 2)$ real matrix

$$\epsilon\mathcal{M}(\mathcal{N}) =$$

$$= \begin{pmatrix} (Im(\mathcal{N}))^{-1} Re(\mathcal{N}) & -(Im(\mathcal{N}))^{-1} \\[3ex] Im(\mathcal{N}) + Re(\mathcal{N})(Im(\mathcal{N}))^{-1} Re(\mathcal{N}) & -Re(\mathcal{N})(Im(\mathcal{N}))^{-1} \end{pmatrix}.$$

$$\tag{4.2.108}$$

By transporting such a relation to the r.h.s.'s of the identities (4.2.106) and (4.2.107), one obtains

$$Re\left[\overline{V}Z + G^{i\bar{i}}\left(D_i V\right)\overline{D_{\bar{i}}Z}\right] = \epsilon\mathcal{M}\left(\mathcal{N}\right) Im\left[\overline{V}Z + G^{i\bar{i}}\left(D_i V\right)\overline{D_{\bar{i}}Z}\right] :$$

(4.2.109)

the real and imaginary parts of the symplectic-invariant quantity $\overline{V}Z + G^{i\bar{i}}\left(D_i V\right)\overline{D_{\bar{i}}Z}$ are simply related through a "symplectic rotation" given by the matrix $\epsilon\mathcal{M}\left(\mathcal{N}\right)$, explicited in Eq. (4.2.108). Clearly, all this is consistent with the previously performed counting of the real degrees of freedom, since there are only $2n_V$ real independent relations.

In Sect. 5 we will see that the algebraic Attractor Equations, both for the $\frac{1}{2}$-*BPS-SUSY extreme BH "attractor(s)"* and for the *non(-BPS)-SUSY extreme BH "attractor(s)"*, are given by nothing but the evaluation of the SKG identity (4.2.106) at some peculiar points in the moduli space M_{n_V}, i.e. at the critical points of a suitably defined *"BH effective potential"* function $V_{BH}\left(z, \bar{z}; n_m, n^e\right)$.

At this point, we may come back and reconsider the consistency conditions (4.2.10) and (4.2.11) for the second solution of the unbroken $\mathcal{N} = 2$, $d = 4$, n_V-fold Maxwell Einstein Supergravity Theory Killing spinor Eqs. (4.2.5).

In particular, the condition (4.2.11) expresses the vanishing of the Abelian vector field strengths of the vector supermultiplets. It may be shown that it is nothing but an extremum condition for the radial dependence of the moduli of the theory; i.e. we may equivalently reformulate condition (4.2.11) as follows ($\forall i = 1, ..., n_V$ understood throughout):

$$\frac{d}{dr}z^i\left(r\right) = 0,$$

(4.2.110)

where r is the radial distance from the surface of the EH. It should be recalled that the radial dependence is the only relevant in this framework, due to the spherical symmetry of the (geo)metric structures involved. Let us remind also that the moduli of the considered $\mathcal{N} = 2$, $d = 4$, n_V-fold Maxwell Einstein Supergravity Theory are the n_V complex scalar fields coming from the n_V Abelian vector supermultiplets coupled to the supergravity one.

Notice that Eq. (4.2.110), even though not resembling the previously considered AEs, is the first case in which some extremizing equation arises in the dynamics of extremal supersymmetric BHs.

By using the whole formal-geometrical machinery reported above, it may be proved that Eq. (4.2.110) implies the vanishing of the holomorphic Kähler-covariant derivative of the central charge, i.e. of the so-called Kähler condensate of the local

$\mathcal{N} = 2, d = 4$ SUSY algebra

$$Z_i \equiv D_i Z = \left(\partial_i + \frac{1}{2} \partial_i K \right) Z \left(z, \bar{z}; n_m, n^e \right) = 0. \tag{4.2.111}$$

As explained in Sect. 2, the fixed values of the moduli at the EH of the extremal RN BH will be obtained by solving Eqs. (4.2.111), provided that such equations do have (at least one) solution, i.e. provided that the n_V-dim. Kähler-Hodge complex moduli space M_{n_V} of the $\mathcal{N} = 2, d = 4$, n_V-fold Maxwell Einstein Supergravity Theory may be characterized as an "attractor variety" with at least one "attractor" point ([31], [32], [33]). When existing, such "attractor" solutions will be independent of the asymptotical values of the moduli, i.e. on the initial data of their dynamical evolution flow inside the moduli space, and instead will depend only on the conserved, quantized electric and magnetic charges of the considered system.

Thus, Eq. (4.2.111) should be more precisely specified at the "attractor" points

$$Z_i|_{(z,\bar{z})=(z_H,\bar{z}_H)} \equiv (D_i Z)_{(z,\bar{z})=(z_H,\bar{z}_H)} =$$

$$\tag{4.2.112}$$

$$= \left[\left(\partial_i + \tfrac{1}{2} \partial_i K \right) Z \left(z, \bar{z}; n_m, n^e \right) \right]_{(z,\bar{z})=(z_H,\bar{z}_H)} = 0,$$

where $(z_H, \bar{z}_H) = (z_H (n_m, n^e), \bar{z}_H (n_m, n^e))$ determines the position of the "attractor" point in M_{n_V}. As already pointed out, such a point is independent of the set of continuously varying, unconstrained initial (asymptotical $r \to \infty$) data $(z_\infty, \bar{z}_\infty) \equiv lim_{r \to \infty} (z(r), \bar{z}(r)) \in M_{n_V}$, but instead depends only on the set of quantized electric and magnetic charges $(n_m, n^e) \in \Gamma$ of the system. Consequently, $(z_H, \bar{z}_H)$ generally corresponds to a discrete set of quantized "attractor" fixed points.

Therefore, beside being always a Kähler-covariantly holomorphic function (see Eq. (4.2.63)), in correspondence of the "attractor" point(s) the central charge becomes a Kähler-covariantly anti-holomorphic function, too. Otherwise speaking, the set of "attractor" point(s) in M_{n_V} could be characterized as follows:

$$M_{n_V} \ni \left\{ (z_H (n_m, n^e), \bar{z}_H (n_m, n^e)) \right\} : \begin{cases} (D_i Z) (z_H, \bar{z}_H; n_m, n^e) = 0, \\ \\ (\overline{D_{\bar{i}}} Z) (z_H, \bar{z}_H; n_m, n^e) = 0. \end{cases}$$

$$\tag{4.2.113}$$

Such a set of Kähler-covariant differential conditions may be seen as the realization of the Attractor Mechanism in the moduli space, or equivalently as the Kähler-covariant extremization of the central extension operator of the considered superal-

gebra. The AM selects the configurations of the moduli at the EH as the ones that make the central charge Kähler-covariantly anti-holomorphic.

Indeed, we will show that for non-vanishing Z Eq. (4.2.113) is the Kähler-covariant form of the general, model-independent "Attractor" or "Extremal" Eqs. (3.2.3), the so-called $\frac{1}{2}$-*BPS extreme BH Attractor Eqs.*.

Before doing this, let us briefly comment on the Kähler weights of the central charge Z. As previously mentioned, from its very definition (4.2.53) it follows that Z is a Kähler-scalar function in the moduli space M_{n_V} with Kähler weights $(1, -1)$. Therefore, as largely used above, its Kähler-covariant derivatives read

$$
\begin{cases}
D_i Z = \left(\partial_i + \frac{1}{2}\partial_i K \right) Z, \\[2mm]
\overline{D}_{\bar{i}} Z = \left(\overline{\partial}_{\bar{i}} - \frac{1}{2}\overline{\partial}_{\bar{i}} K \right) Z.
\end{cases}
$$

$$(4.2.114)$$

As, in general, it follows from Eqs. (4.1.16) and (4.1.20), the complex conjugation acts as a parity on the Kähler weights. Thus, $\overline{Z}$ is a Kähler-scalar function in M_{n_V} with Kähler weights $(-1, 1)$, and its Kähler-covariant derivatives read

$$
\begin{cases}
D_i \overline{Z} = \left(\partial_i - \frac{1}{2}\partial_i K \right) \overline{Z} = \overline{\overline{D}_{\bar{i}} Z}, \\[2mm]
\overline{D}_{\bar{i}} \overline{Z} = \left(\overline{\partial}_{\bar{i}} + \frac{1}{2}\overline{\partial}_{\bar{i}} K \right) \overline{Z} = \overline{D_i Z}.
\end{cases}
$$

$$(4.2.115)$$

Since the Kähler weights are additive under multiplication, it is clear that the square absolute value of Z, i.e. $|Z|^2 \equiv Z\overline{Z}$, is a Kähler gauge-invariant quantity, i.e. it has Kähler weights $(0,0)$. Consequently, the Kähler-covariant derivatives of such a Kähler-scalar trivially correspond to the ordinary, flat ones; this can be seen also by explicitly calculating that the terms of Kähler connections $\partial_i K$ cancel each other

$$
\begin{aligned}
D_i \left(|Z|^2 \right) &= D_i \left(Z\overline{Z} \right) = \left(D_i Z \right) \overline{Z} + Z \left(D_i \overline{Z} \right) = \\[2mm]
&= \left[\left(\partial_i + \frac{1}{2}\partial_i K \right) Z \right] \overline{Z} + Z \left[\left(\partial_i - \frac{1}{2}\partial_i K \right) \overline{Z} \right] = \\[2mm]
&= \partial_i \left(Z\overline{Z} \right) = \partial_i \left(|Z|^2 \right) = 2\, |Z|\, \partial_i \, |Z| \,.
\end{aligned}
$$

$$(4.2.116)$$

Let us now calculate[18]

$$\partial_i \left| Z \right| \;=\; \partial_i \sqrt{Z\overline{Z}} = \frac{1}{2 \left| Z \right|} \left[(\partial_i Z) \overline{Z} + Z \left(\partial_i \overline{Z} \right) \right] =$$

$$= \; \frac{1}{2 \left| Z \right|} \left[(\partial_i Z) \overline{Z} + \frac{1}{2} (\partial_i K) Z \overline{Z} \right] = \frac{\overline{Z}}{2 \left| Z \right|} D_i Z, \qquad (4.2.117)$$

where in the second line we used the Kähler-covariant anti-holomorphicity of $\overline{Z}$ expressed by Eq. (4.2.63), recalling Eq. (6.3.0.7). Thus, we showed that

$$\partial_i \left| Z \right| = \frac{\overline{Z}}{2 \left| Z \right|} D_i Z \Leftrightarrow D_i \overline{Z} = 0, \qquad (4.2.118)$$

or, by complex conjugating, that

$$\overline{\partial}_{\bar{i}} \left| Z \right| = \frac{Z}{2 \left| Z \right|} \overline{D}_{\bar{i}} \overline{Z} \Leftrightarrow \overline{D}_{\bar{i}} Z = 0. \qquad (4.2.119)$$

Eq. (4.2.118) yields

$$\partial_i \left| Z \right| = 0 \Leftrightarrow D_i Z = 0. \qquad (4.2.120)$$

This means that, when considering a Kähler-covariant holomorphic Z, its Kähler-covariant extremization is equivalent to the ordinary extremization of its absolute value. Thus, we may complete Eq. (4.2.113), obtaining Eq. (3.2.3), i.e. the general form of the $\frac{1}{2}$-*BPS extreme BH Attractor Eqs.*

$$Z_i|_{(z,\bar{z})=(z_H,\bar{z}_H)} \equiv (D_i Z)_{(z,\bar{z})=(z_H,\bar{z}_H)} =$$

$$= \left[\left(\partial_i + \tfrac{1}{2}\partial_i K \right) Z\left(z,\bar{z}; n_m, n^e\right) \right]_{(z,\bar{z})=(z_H,\bar{z}_H)} = 0$$

$$\Updownarrow$$

$$\left[\partial_i \left| Z\left(z,\bar{z}; n_m, n^e\right) \right| \right]_{(z,\bar{z})=(z_H,\bar{z}_H)} = 0$$

$$\Updownarrow \qquad\qquad (4.2.121)$$

$$\left[\overline{\partial}_{\bar{i}} \left| Z\left(z,\bar{z}; n_m, n^e\right) \right| \right]_{(z,\bar{z})=(z_H,\bar{z}_H)} = 0.$$

---

[18]Throughout this thesis we will, in general, assume the non-vanishing of the central charge

$$\left| Z \right| \neq 0 \Leftrightarrow Z \neq 0 \Leftrightarrow \overline{Z} \neq 0.$$

Thus, in a generic supergravity theory (having a Kähler moduli space) with a non-vanishing and Kähler-covariantly holomorphic central charge Z, we explicitly showed that the Kähler-covariant extremization of such a function (expressed by Eq. (4.2.112)) is equivalent to the ordinary, flat extremization of its absolute value (given by Eq. (4.2.121)).

Now, we can specialize the general form (4.2.121) of the $\frac{1}{2}$-*BPS extreme BH Attractor Eqs.* to $\mathcal{N} = 2$, $d = 4$, n_V-fold Maxwell Einstein Supergravity Theory. Such a theory has a complex moduli space M_{n_V} endowed with SKG, and the explicit form of the central charge is given by Eq. (4.2.62). Thus, by also using Eq. (4.2.67), we get a more explicit (model-dependent) form of the Kähler-covariant extremization of Z at the Event Horizon

$$(D_i Z)_{(z,\bar{z})=(z_H,\bar{z}_H)} = 0,$$

$$\Updownarrow$$

$$\left[\left(n_\Sigma^e - \overline{\mathcal{N}}_{\Lambda\Sigma} (z,\bar{z}) \, n_m^\Lambda \right) f_i^\Sigma (z,\bar{z}) \right]_{(z,\bar{z})=(z_H,\bar{z}_H)} = 0,$$

$$\Updownarrow$$

$$\left[\begin{array}{l} \left(n_\Sigma^e - \overline{\mathcal{N}}_{\Lambda\Sigma} (z,\bar{z}) \, n_m^\Lambda \right) (\partial_i K) \left[\exp \left(\tfrac{1}{2} K (z,\bar{z}) \right) X^\Sigma (z) \right] + \\ \\ + \left(n_\Sigma^e - \overline{\mathcal{N}}_{\Lambda\Sigma} (z,\bar{z}) \, n_m^\Lambda \right) \exp \left(\tfrac{1}{2} K (z,\bar{z}) \right) \left(\partial_i X^\Sigma (z) \right) \end{array} \right]_{(z,\bar{z})=(z_H,\bar{z}_H)} = 0,$$

$$\Updownarrow$$

$$\left[\begin{array}{l} (\partial_i K) \, Z \, (n_m, n^e, z, \bar{z}) |_{\mathcal{N}_{\Lambda\Sigma} \to \overline{\mathcal{N}}_{\Lambda\Sigma}} + \\ \\ + \left(n_\Sigma^e - \overline{\mathcal{N}}_{\Lambda\Sigma} (z,\bar{z}) \, n_m^\Lambda \right) \exp \left(\tfrac{1}{2} K (z,\bar{z}) \right) \left(\partial_i X^\Sigma (z) \right) \end{array} \right]_{(z,\bar{z})=(z_H,\bar{z}_H)} = 0.$$

$$(4.2.122)$$

By using Eq. (4.2.121) and recalling Eq. (4.2.53), we finally get

$$\left[\begin{array}{l} (\partial_i K) \, Z \, (n_m, n^e, z, \bar{z})|_{\mathcal{N}_{\Lambda\Sigma} \to \overline{\mathcal{N}}_{\Lambda\Sigma}} + \\[3ex] + \left(n^e_\Sigma - \overline{\mathcal{N}}_{\Lambda\Sigma} \, (z, \bar{z}) \, n^\Lambda_m \right) \exp\left(\tfrac{1}{2} K \, (z, \bar{z}) \right) \left(\partial_i X^\Sigma \, (z) \right) \end{array} \right]_{(z, \bar{z}) = (z_H, \bar{z}_H)} = 0$$

$$\Updownarrow$$

$$\left[\partial_i \left| \left(n^e_\Sigma - \mathcal{N}_{\Lambda\Sigma} \, (z, \bar{z}) \, n^\Lambda_m \right) L^\Sigma \, (z, \bar{z}) \right| \right]_{(z, \bar{z}) = (z_H, \bar{z}_H)} = 0$$

$$\Updownarrow$$

$$\left\{ \partial_i \left| \left[\exp\left(\frac{1}{2} K \, (z, \bar{z}) \right) \right] \left(n^e_\Sigma - \mathcal{N}_{\Lambda\Sigma} \, (z, \bar{z}) \, n^\Lambda_m \right) X^\Sigma \, (z) \right| \right\}_{(z, \bar{z}) = (z_H, \bar{z}_H)} = 0$$

$$\Updownarrow$$

$$\left\{ \bar{\partial}_{\bar{i}} \left| \left[\exp\left(\frac{1}{2} K \, (z, \bar{z}) \right) \right] \left(n^e_\Sigma - \overline{\mathcal{N}}_{\Lambda\Sigma} \, (z, \bar{z}) \, n^\Lambda_m \right) \overline{X}^\Sigma \, (\bar{z}) \right| \right\}_{(z, \bar{z}) = (z_H, \bar{z}_H)} = 0,$$

$$(4.2.123)$$

where in the last three lines the flat derivatives may be substituted by the Kähler-covariant ones, due to the vanishing of the Kähler weights of the absolute value of the central charge Z. Eqs. (4.2) and (4.2) are the $\frac{1}{2}$-*BPS extreme BH Attractor Eqs.* of $\mathcal{N} = 2, d = 4, n_V$-fold Maxwell Einstein Supergravity Theory.

Now, it should be recalled that in ($\mathcal{N} = 2$) supersymmetric theories the saturation of the BPS bound fixes the ADM mass of the BH to be equal to the absolute value of the central charge

$$M_{ADM} \, (z_\infty, \bar{z}_\infty; n_m, n^e) = |Z| \, (z_\infty, \bar{z}_\infty; n_m, n^e) . \qquad (4.2.124)$$

By admitting an extension of such a saturated bound to the r-dependent moduli space M_{n_V}, one gets[19]

$$M_{ADM} \, (z \, (r), \bar{z} \, (r) ; n_m, n^e) = |Z| \, (z \, (r), \bar{z} \, (r) ; n_m, n^e) , \qquad (4.2.125)$$

---

[19]Otherwise speaking, we move to consider the "ADM mass" function in M_{n_V}; moreover, we assume the limits $r \to \infty$ and $r \to r_H^+$ to be "smooth" (in the sense specified by Footnote 6 of Sect. 2) also for such a function.

such that Eq. (4.2.124) is the asymptotical limit $r \to \infty$ of Eq. (4.2.125). Thus, in the considered case of $\frac{1}{2}$-BPS extremal BHs we may directly translate the previous results in terms of the "ADM mass" function, obtaining

$$\left[\partial_i M_{ADM}\left(z, \bar{z}; n_m, n^e\right)\right]_{(z,\bar{z})=(z_H,\bar{z}_H)} = 0 \tag{4.2.126}$$

$$\Updownarrow$$

$$\left[\bar{\partial}_{\bar{i}} M_{ADM}\left(z, \bar{z}; n_m, n^e\right)\right]_{(z,\bar{z})=(z_H,\bar{z}_H)} = 0. \tag{4.2.127}$$

Moreover, at the EH it holds that

$$M_{ADM}\left(z = z_H\left(n_m, n^e\right), \bar{z} = \bar{z}_H\left(n_m, n^e\right); n_m, n^e\right) =$$

$$\tag{4.2.128}$$

$$= M_{ADM,H}\left(n_m, n^e\right) = M_{BR}\left(n_m, n^e\right),$$

where we recalled that the "near-Horizon" geometry is described by the BR metric. Consequently, the extremum value of the "ADM mass" function of the BPS solutions at the EH is equal to the mass of the BR geometry.

Thus, we may conclude that the AM for $\frac{1}{2}$-BPS extremal BHs, encoded in the condition of Kähler-covariant anti-holomorphicity of the central charge (see Eq. (4.2.113)), also implies the extremization of the "ADM mass" function w.r.t. its dependence on z and $\bar{z}$.

More in particular, by considering Eq. (4.2.62), we get the explicit expression of the "ADM mass" function of the $\frac{1}{2}$-BPS extremal BHs in the framework of the $\mathcal{N} = 2, d = 4, n_V$-fold Maxwell Einstein Supergravity Theory

$$M_{ADM}\left(z, \bar{z}; n_m, n^e\right) = \left|\left(n^e_\Sigma - \mathcal{N}_{\Lambda\Sigma}\left(z, \bar{z}\right) n^\Lambda_m\right) L^\Sigma\left(z, \bar{z}\right)\right| =$$

$$= \left[\exp\left(\frac{1}{2} K\left(z, \bar{z}\right)\right)\right] \left|\left(n^e_\Sigma - \mathcal{N}_{\Lambda\Sigma}\left(z, \bar{z}\right) n^\Lambda_m\right) X^\Sigma\left(z\right)\right|. \tag{4.2.129}$$

An example of the extremization[20] of the absolute value of the "central charge" function Z of the local SUSY algebra (or equivalently for BPS extremal BHs, of the "ADM mass" function) in the Kähler-Hodge moduli space M_{n_V} of the $\mathcal{N} = 2, d = 4$, n_V-fold Maxwell Einstein Supergravity Theory is shown in Fig. 4.1.

---

[20]In Subsect. 4.4 it will be shown that in $\mathcal{N} = 2, d = 4, n_V$-fold Maxwell Einstein Supergravity Theory with strictly positive definite metric of the moduli space and with a single continuous branch of the function $|Z|\,(z, \bar{z}; n_m, n^e)$, at most only one extremum point exists, and it is a minimum. Clearly, the situation completely changes if the hypotheses of strictly positive definiteness of the metric and/or single continuous branch for $|Z|$ are removed. See e.g. [55].
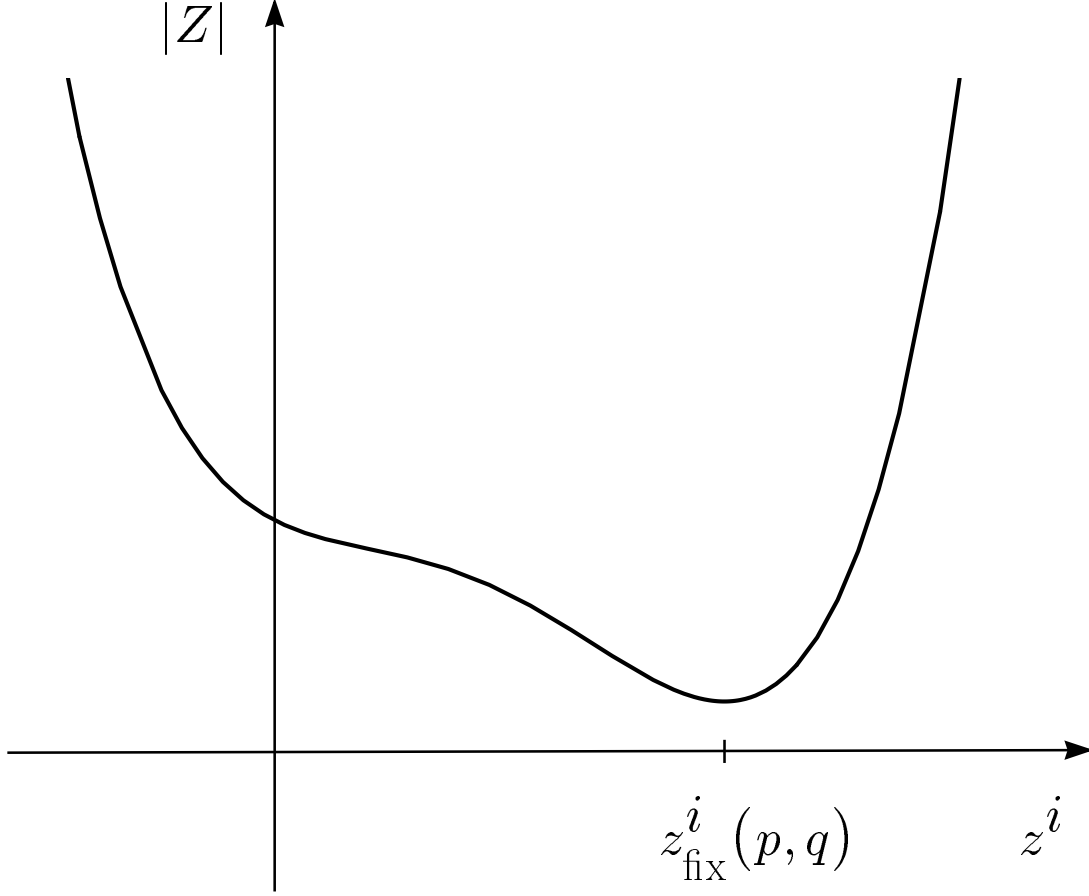
Figure 4.1: Minimization of the absolute value of the "central charge" function $|Z|(z, \overline{z}; n_m, n^e)$ of the local SUSY algebra in the (holomorphic part of the) Kähler-Hodge complex moduli space M_{n_V} of the $\mathcal{N} = 2$, $d = 4$, n_V-fold Maxwell Einstein Supergravity Theory. In the picture $z^i_{FIX}(p, q)$ stands for $z_H(n_m, n^e)$, i.e. for the "attractor", purely charge-dependent value of the moduli at the Event Horizon of the considered $\frac{1}{2}$-BPS extremal (eventually RN) BH. The Attractor Mechanism fixes the extrema of the central charge to correspond to the discrete "fixed" points of the "attractor variety" ([31], [32], [33]) M_{n_V}. Of course, the moduli-dependence of the central charge is shown at a fixed charge configuration of the system, i.e. for a fixed $(2n_V + 2)$-dim. symplectic-covariant vector n defined in Eq. (4.2.35).

Moreover, at the "attractor" point(s) corresponding to the radius $r = r_H$, the two independent $Sp(2n_V + 2)$-invariants I_1 and I_2 homogeneous of degree two in the (quantized) electric and magnetic charges (defined in Eqs. (4.2.80) and then explicited in Eqs. (4.2.86) and (4.2.87)) coincide one with the other, "degenerating" in one unique value[21]

$$I_H(n_m, n^e) \equiv I_1(z_H(n_m, n^e), \bar{z}_H(n_m, n^e); n_m, n^e) =$$

$$= I_2(z_H(n_m, n^e), \bar{z}_H(n_m, n^e); n_m, n^e) =$$

$$= |Z(z_H(n_m, n^e), \bar{z}_H(n_m, n^e); n_m, n^e)|^2 \equiv |Z_h(n_m, n^e)|^2 =$$

$$= M^2_{ADM,H}(n_m, n^e) = M^2_{BR}(n_m, n^e);$$

$$(4.2.130)$$

$$\Updownarrow$$

$$n^T \mathcal{M}(\mathcal{N}(z_H(n_m, n^e), \bar{z}_H(n_m, n^e))) n = n^T \mathcal{M}(\mathcal{F}(z_H(n_m, n^e), \bar{z}_H(n_m, n^e))) n =$$

$$= \left[n^e_\Lambda - \overline{\mathcal{N}}_{\Lambda\Sigma}(z_H(n_m, n^e), \bar{z}_H(n_m, n^e)) n^\Sigma_m\right] \left((Im((z_H(n_m, n^e), \bar{z}_H(n_m, n^e))))^{-1}\right)^{\Lambda\Delta} \cdot$$

$$\cdot \left[n^e_\Delta - \mathcal{N}_{\Delta\Gamma}(z_H(n_m, n^e), \bar{z}_H(n_m, n^e)) n^\Gamma_m\right] =$$

$$= \left[n^e_\Lambda - \overline{F}_{\Lambda\Sigma}(z_H(n_m, n^e), \bar{z}_H(n_m, n^e)) n^\Sigma_m\right] \left((Im(\mathcal{F}(z_H(n_m, n^e), \bar{z}_H(n_m, n^e))))^{-1}\right)^{\Lambda\Delta} \cdot$$

$$\cdot \left[n^e_\Delta - F_{\Delta\Gamma}(z_H(n_m, n^e), \bar{z}_H(n_m, n^e)) n^\Gamma_m\right] =$$

$$= M^2_{ADM,H}(n_m, n^e) = M^2_{BR}(n_m, n^e) = |Z|^2_H(n_m, n^e).$$

$$(4.2.131)$$

$|Z|_H(n_m, n^e)$ is the purely charge-dependent extremized value of the absolute value of the "central charge" function of the local $\mathcal{N} = 2$, $d = 4$ SUSY algebra, reached at the EH of the BPS extremal (RN) BH.

Now, by recalling the relation between the Horizon area and the BR mass

$$M^2_{BR}(n_m, n^e) = \frac{A_H}{4\pi} \tag{4.2.132}$$

---

[21]As done above, we assume that the limit $r \to r^+_H$ is "smooth" (in the sense specified by Footnote 6 of Sect. 2) for all considered functions in the moduli space.

and by using the BHEA formula, we may relate the entropy of the extremal BPS (RN) BH to the area of its EH, and therefore to its "ADM mass" function, whose "near-Horizon" limit coincides with the BR mass.

Thus, the final result is the expression of the entropy of the extremal BPS (RN) BH in terms of the extremized (minimized) square absolute value of the "central charge" function of the local $\mathcal{N} = 2, d = 4$ SUSY algebra, reached in correspondence of the discrete "attractor" moduli configuration(s) at the EH

$$
\begin{aligned}
S_{BH} &= \frac{A_H}{4} = \pi M_{BR}^2 \left(n_m, n^e \right) = \\
\\
&= \pi M_{ADM}^2 \left(z_H \left(n_m, n^e \right), \bar{z}_H \left(n_m, n^e \right); n_m, n^e \right) = \\
\\
&= \pi M_{ADM,H}^2 \left(\left(n_m, n^e \right) \right) = \\
\\
&= \pi \left| Z \right|_H^2 \left(n_m, n^e \right).
\end{aligned}
\tag{4.2.133}
$$

As mentioned above, a key feature of the $d = 4$ and 5, $\mathcal{N} = 2$ SUGRAs coupled to n_V Abelian vector supermultiplets is the fact that the extremization of the "central charge" function Z through the AEs may be made "coordinate-free" in the moduli space M_{n_V}, by using the fact that such a n_V-dim. complex manifold is endowed with a special Kähler metric structure, on which we reported above for the $d = 4$ case.

Clearly, the *U*-duality-invariant, i.e. symplectic-invariant, (re)formulation of the BHEA in the case of $d = 4$ and 5, $\mathcal{N} = 2$ Maxwell Einstein Supergravity Theorys has various advantages, coming from its manifest symmetry.

Finally, one can also check the first consistency condition (4.2.10) for unbroken $\mathcal{N} = 2$ SUSY at the EH; such a relation relates the Riemann-Christoffel tensor of the metric background to the graviphoton field strength. By using the definition of the "central charge" function, and by evaluating it at the "attractor" fixed point(s), it is possible to show that one obtains nothing but the BPS-saturation condition for the BR metric, expressing the validity of the Cosmic Censorship Principle, and consequently yielding the existence of an EH with a regular geometry covering the inner s-t singularity

$$
M_{BR}^2(n_m, n^e) = \left| Z \right|_H^2 \left(n_m, n^e \right).
\tag{4.2.134}
$$

while, by recalling Eq. (4.2.132), one reobtains the main result given by Eq. (4.2.133).

Chapter 5

Black Holes and Critical Points in Moduli Space

As we have seen, the $d = 4$, $\mathcal{N} = 2$ ungauged SUGRAs have two types of geometries: the space-time geometry and the moduli space geometry. In this Section, mainly following the seminal paper [55] of Ferrara, Gibbons and Kallosh, we will consider the fundamental interplay between these two geometries, especially in relation with the Attractor Mechanism.

5.1   Black Holes and Constrained Geodesic Motion

Let us start by considering the 4-d. Lagrangian density of a system of real scalars and Abelian gauge fields coupled to gravity [55]

$$\mathcal{L}_4 = -\frac{R}{2} + \frac{1}{2}G_{ab}\partial_\mu\phi^a\partial_\nu\phi^b g^{\mu\nu} - \frac{1}{4}\mu_{\Lambda\Sigma}\mathcal{F}_{\mu\nu}^\Lambda \mathcal{F}_{\lambda\rho}^\Sigma g^{\mu\lambda}g^{\nu\rho} - \frac{1}{4}\nu_{\Lambda\Sigma}\mathcal{F}_{\mu\nu}^\Lambda {}^*\mathcal{F}_{\lambda\rho}^\Sigma g^{\mu\lambda}g^{\nu\rho},$$

$$(5.1.1)$$

with space-time lower Greek indices running from 0 through 3, moduli lower Latin indices running $1, ..., m_\phi$, and symplectic capital Greek indices running $1, ..., n_V + 1$. $g^{\mu\nu}(x)$ and $G_{ab}(\phi)$ are the 4-d. space-time metric and the m_ϕ-dim. moduli space metric, respectively. $\mu_{\Lambda\Sigma}(\phi)$ and $\nu_{\Lambda\Sigma}(\phi)$ respectively are the real, strictly[1] posi-

---

[1]We may disregard the possibility to have vanishing eigenvalues for the matrices $\mu_{\Lambda\Sigma}(\phi)$ and $\nu_{\Lambda\Sigma}(\phi)$. Indeed, such zero modes would correspond to Abelian gauge fields with vanishing kinetic term, which can be thus omitted from the considered Lagrangian density (5.1.1). Consequently, since the matrices $\mu_{\Lambda\Sigma}$ and $\nu_{\Lambda\Sigma}$ are real, symmetric and without zero modes, they are always invertible by an orthogonal transformation. By the way, as it will be evident by looking at Eq. (6.2.0.8), only $\mu_{\Lambda\Sigma}$ needs to be invertible in order for V_{BH} to be consistently defined.

tive definite, moduli-dependent matrices of dilatonic and axionic couplings of the Abelian gauge fields (they may be considered symmetric without loss of generality). Finally, $^*\mathcal{F}^\Sigma_{\lambda\rho}$ denotes the usual Hodge $*$-dual

$$^*\mathcal{F}^\Sigma_{\lambda\rho} \equiv \frac{1}{2}\epsilon_{\lambda\rho\sigma\varphi}\mathcal{F}^{\Sigma\sigma\varphi}, \qquad (5.1.2)$$

where $\epsilon_{\lambda\rho\sigma\varphi}$ is the 4-d. completely antisymmetric Ricci-Levi-Civita tensor.

We restrict our attention to static (i.e. time-independent) metric backgrounds, described by the metric Ansatz[2] (remind that, unless otherwise indicated, we put $c = \hbar = G_0 = 1$, and $i, j = 1, 2, 3$)

$$ds^2 = e^{2U(\underline{x})}dt^2 - e^{-2U(\underline{x})}\gamma_{ij}(\underline{x})\,dx^i dx^j. \qquad (5.1.3)$$

Such an Ansatz is a generalization (with non necessarily Euclidean spatial sections) of the previously considered 4-d. BH metric given by Eq. (2.2.23). The assumption of staticity allows one to get a 3-d. effective Lagrangian density, from which the field Eqs. may be derived

$$\mathcal{L}_3 = \frac{R\left[\gamma_{ij}\right]}{2} - \frac{1}{2}\gamma^{ij}\partial_i\widehat{\phi}^{\widehat{a}}\partial_j\widehat{\phi}^{\widehat{b}}\widehat{G}_{\widehat{a}\widehat{b}}, \qquad (5.1.4)$$

where $R\left[\gamma_{ij}\right]$ denotes the intrinsic scalar curvature related to the 3-d. spatial metric $\gamma_{ij}(\underline{x})$. Moreover, the "hatted" scalar fields include, beside the scalar fields ϕ^a of the 4-d. theory, also the function $U(\underline{x})$ defining the space-time metric and the electrostatic ψ^Λ and magnetostatic χ_Λ potentials related to the $U(1)$ gauge fields

$$\widehat{\phi}^{\widehat{a}} \equiv \left(U, \phi^a, \psi^\Lambda, \chi_\Lambda\right), \qquad (5.1.5)$$

with the "hatted" indices $\widehat{a}$ ranging in a set of cardinality $m_\phi + 2n_V + 3$.

In other words, in the passage from the 4-d. theory to the related effective 3-d. theory, it is convenient to enlarge the scalar manifold $\mathcal{M}_\phi$ as follows:

$$\left(\mathcal{M}_\phi, \{\phi^a\}, G_{ab}(\phi)\right) \longrightarrow \left(\mathcal{M}_{\widehat{\phi}}, \{\widehat{\phi}^{\widehat{a}}\}, \widehat{G}_{\widehat{a}\widehat{b}}(\widehat{\phi})\right), \qquad (5.1.6)$$

where it should be noted that the $U(1)^{n_V+1}$ gauge invariance implies that $\widehat{G}_{\widehat{a}\widehat{b}}$ is independent of the e.m. potentials

$$\widehat{G}_{\widehat{a}\widehat{b}}(\widehat{\phi}) = \widehat{G}_{\widehat{a}\widehat{b}}(U, \phi). \qquad (5.1.7)$$

---

[2]It has been shown by Tod [81] that in $\mathcal{N} = 2$ supergravity theories the general form of static metrics admitting supersymmetries is given by Eq. (5.1.3).

We further increase the symmetry of the considered s-t metric background, by formulating the hypothesis of spherical symmetry corresponding to the Ansatz [55]

$$\gamma_{ij}\left(\underline{x}\right)dx^i dx^j = \frac{\mathbf{c}^4 d\tau^2}{sinh^4\left(\mathbf{c}\tau\right)} + \frac{\mathbf{c}^2}{sinh^2\left(\mathbf{c}\tau\right)}\left(d\theta^2 + sin^2\theta d\varphi^2\right), \qquad (5.1.8)$$

where

$$\tau \equiv \frac{1}{r_H - r}. \qquad (5.1.9)$$

Therefore, since $r \in [r_H, +\infty)$, it follows that τ runs from $-\infty$ (BH Event Horizon) to 0^- (spatial infinity). Moreover,

$$\mathbf{c}^2 \equiv \frac{\kappa_s A_H}{8\pi} = (2S_{BH}T_{BH})^2, \qquad (5.1.10)$$

where in the last passage we recalled Eqs. (3.1.3) and (3.1.4) (S_{BH} and T_{BH} respectively denote the entropy and the temperature of the BH).

Summarizing, we are considering the following 4-d. static, spherically symmetric BH metrics:

$$ds^2 = e^{2U(\tau)}dt^2 - e^{-2U(\tau)}\left[\frac{\mathbf{c}^4 d\tau^2}{sinh^4\left(\mathbf{c}\tau\right)} + \frac{\mathbf{c}^2}{sinh^2\left(\mathbf{c}\tau\right)}\left(d\theta^2 + sin^2\theta d\varphi^2\right)\right], \qquad (5.1.11)$$

where τ is the 1-dim. effective evolution parameter defined in Eq. (5.1.9), and we introduced $U'(\tau) = U(r)$ and dropped the prime out. By further using the spherical symmetry (i.e. the (θ, φ)-independence) of the BH metric (5.1.11), one obtains a 1-dim., τ-dependent effective theory.

It can be shown that the 1-dim. effective Lagrangian from which the radial Eqs. of motion may be derived has the purely geodesic form [57]

$$\mathcal{L}_1 = \widehat{G}_{\widehat{a}\widehat{b}}\left(U, \phi\right)\frac{d\widehat{\phi}^{\widehat{a}}\left(\tau\right)}{d\tau}\frac{d\widehat{\phi}^{\widehat{b}}\left(\tau\right)}{d\tau} \qquad (5.1.12)$$

constrained by the condition

$$\widehat{G}_{\widehat{a}\widehat{b}}\left(U, \phi\right)\frac{d\widehat{\phi}^{\widehat{a}}\left(\tau\right)}{d\tau}\frac{d\widehat{\phi}^{\widehat{b}}\left(\tau\right)}{d\tau} = \mathbf{c}^2, \qquad (5.1.13)$$

which characterizes τ as a "generalized proper time" for the enlarged scalar manifold $\mathcal{M}_{\widehat{\phi}}$.

Consequently, by assuming the space-time symmetries expressed by Eqs. (5.1.3) and (5.1.8), the dynamics related to the starting 4-d. Lagrangian (5.1.1) may be

shown to reduce to a geodesic, constrained dynamics described by Eqs. (5.1.12) and (5.1.13).

In order to further explicit $\mathcal{L}_1$, we may formulate the following "block-diagonal" Ansatz[3] for $\widehat{G}_{\widehat{a}\widehat{b}}$

$$\widehat{G}_{\widehat{a}\widehat{b}}(U,\phi) = \begin{pmatrix} 1 & & & \\ & \frac{1}{2}G_{ab}(\phi) & & \\ & & \widehat{G}_{\Lambda\Sigma}(U,\phi) & \\ & & & \widehat{G}^{\Lambda\Sigma}(U,\phi) \end{pmatrix}, \qquad (5.1.14)$$

where as usual

$$\widehat{G}^{\Lambda\Sigma}(U,\phi)\,\widehat{G}_{\Sigma\Xi}(U,\phi) = \delta^{\Lambda}_{\Xi}, \quad \forall U,\phi, \qquad (5.1.15)$$

and the unwritten components vanish. Therefore, $\mathcal{L}_1$ read

$$\mathcal{L}_1 = \left(\frac{dU(\tau)}{d\tau}\right)^2 + \frac{1}{2}G_{ab}(\phi)\frac{d\phi^a(\tau)}{d\tau}\frac{d\phi^b(\tau)}{d\tau} +$$

$$+\widehat{G}_{\Lambda\Sigma}(U,\phi)\frac{d\psi^\Lambda(\tau)}{d\tau}\frac{d\psi^\Sigma(\tau)}{d\tau} + \widehat{G}^{\Lambda\Sigma}(U,\phi)\frac{d\chi_\Lambda(\tau)}{d\tau}\frac{d\chi_\Sigma(\tau)}{d\tau}.$$
$$(5.1.16)$$

Now, since $\widehat{G}_{\widehat{a}\widehat{b}}$ is independent of ψ^Λ and χ_Λ, we obtain that

$$\frac{dp^\Lambda}{d\tau} = 0, \qquad \frac{dq_\Lambda}{d\tau} = 0, \qquad (5.1.17)$$

where

$$\begin{cases} p^\Lambda \equiv \frac{1}{2}\frac{\delta\mathcal{L}_1}{\delta\left(\frac{d\chi_\Lambda}{d\tau}\right)} = \widehat{G}^{\Lambda\Sigma}\frac{d\chi_\Sigma}{d\tau}; \\[4mm] q_\Lambda \equiv \frac{1}{2}\frac{\delta\mathcal{L}_1}{\delta\left(\frac{d\psi^\Lambda}{d\tau}\right)} = \widehat{G}_{\Lambda\Sigma}\frac{d\psi^\Sigma}{d\tau}, \end{cases}$$
$$(5.1.18)$$

---

[3]A particular(ly simple) formulation of the "block-diagonal" Ansatz (5.1.14) reads

$$\widehat{G}_{\widehat{a}\widehat{b}}(U,\phi) = \begin{pmatrix} 1 & & \\ & \frac{1}{2}G_{ab}(\phi) & \\ & & \epsilon_{n_V+1} \end{pmatrix},$$

where ϵ_{n_V+1} is the $(2n_V+2)$-dim. symplectic metric given by Eq. (4.1.25).
   The factor $\frac{1}{2}$ in front of $G_{ab}(\phi)$ is introduced for later convenience.

are identified with the magnetic and electric charges of the BH, respectively ($p^\Lambda \equiv n_m^\Lambda, q_\Lambda \equiv n_\Lambda^e$). Thus, by using the definitions (5.1.18), Eq. (5.1.16) can be further elaborated as

$$\mathcal{L}_1 = \left(\frac{dU(\tau)}{d\tau}\right)^2 + \frac{1}{2}G_{ab}(\phi)\frac{d\phi^a(\tau)}{d\tau}\frac{d\phi^b(\tau)}{d\tau} + q_\Lambda\frac{d\psi^\Lambda(\tau)}{d\tau} + p^\Lambda\frac{d\chi_\Lambda(\tau)}{d\tau}.$$
(5.1.19)

Now, it can be shown ([58], [57], [59] and [60]; see also [61]) that

$$q_\Lambda\frac{d\psi^\Lambda(\tau)}{d\tau} + p^\Lambda\frac{d\chi_\Lambda(\tau)}{d\tau} = e^{2U}V_{BH}(\phi; p, q),$$
(5.1.20)

where $V_{BH}(\phi; p, q)$ is the so-called "BH effective potential", i.e. a particular, positive function of the scalars ϕ's and of the BH charges, constructed from the (strictly) positive definite couplings $\mu_{\Lambda\Sigma}(\phi)$ and $\nu_{\Lambda\Sigma}(\phi)$ as follows:

$$V_{BH}(\phi, p; q) \equiv \frac{1}{2}\left(p^\Lambda, q_\Lambda\right) \mathbf{M}(\phi) \begin{pmatrix} p^\Sigma \\ q_\Sigma \end{pmatrix},$$
(5.1.21)

where the $(2n_V + 2) \times (2n_V + 2)$, ϕ-dependent matrix $\mathbf{M}(\phi)$ is defined as

$$\mathbf{M}(\phi) \equiv \begin{pmatrix} \mu_{\Lambda\Sigma}(\phi) + \nu_{\Lambda\Delta}(\phi)\left(\mu^{-1}(\phi)\right)^{\Delta\Xi}\nu_{\Xi\Sigma}(\phi) & \nu_{\Lambda\Xi}(\phi)\left(\mu^{-1}(\phi)\right)^{\Xi\Sigma} \\ \\ \left(\mu^{-1}(\phi)\right)^{\Lambda\Xi}\nu_{\Xi\Sigma}(\phi) & \left(\mu^{-1}(\phi)\right)^{\Lambda\Sigma} \end{pmatrix}.$$
(5.1.22)

The reality, symmetry and (strict) positive definiteness[4] of $\mu_{\Lambda\Sigma}(\phi)$ and $\nu_{\Lambda\Sigma}(\phi)$ imply the reality, symmetry and (strict) positive definiteness of the matrix $\mathbf{M}(\phi)$, and consequently the positivity of $V_{BH}(\phi, p; q)$ in all $\mathcal{M}_\phi \times \Gamma$.

By substituting Eq. (5.1.20) in Eq. (5.1.19), we can finally write the 1-dim. effective Lagrangian density as

$$\mathcal{L}_1[U(\tau), \phi(\tau); p, q] =$$

$$= \left(\frac{dU(\tau)}{d\tau}\right)^2 + \frac{1}{2}G_{ab}(\phi(\tau))\frac{d\phi^a(\tau)}{d\tau}\frac{d\phi^b(\tau)}{d\tau} + e^{2U(\tau)}V_{BH}(\phi(\tau); p, q).$$
(5.1.23)

---

[4]It is worth pointing out once again that, in order for $\mathbf{M}(\phi)$ to be well-defined, at least $\mu_{\Lambda\Sigma}(\phi)$ must be strictly positive definite on the whole moduli space $\mathcal{M}_\phi$.

Analogously, it may be shown that the constraint (5.1.13) is equivalent to

$$\left(\frac{dU(\tau)}{d\tau}\right)^2 + \tfrac{1}{2}G_{ab}\left(\phi\left(\tau\right)\right)\frac{d\phi^a(\tau)}{d\tau}\frac{d\phi^b(\tau)}{d\tau} - e^{2U(\tau)}V_{BH}\left(\phi\left(\tau\right);p,q\right) =$$

$$= \mathbf{c}^2 = (2S_{BH}T_{BH})^2.$$

(5.1.24)

The general formalism described above, which allows one to treat 4-d. static, spherically symmetric, $\mathbf{c}^2$-parameterized BHs with "scalar hairs" coupled to Abelian vector fields, essentially relies on the metric $G_{ab}\left(\phi\right)$ of the moduli space $\mathcal{M}_\phi$ and on the "effective BH potential" function $V_{BH}\left(\phi;p,q\right)$.

To a certain extent, the presented geodesic formulation is the most symmetrical one, in which the "hatted" fields $\widehat{\phi}$ comprise the real scalars ϕ^a, as well as the electromagnetic potentials $\psi^\Lambda, \chi_\Lambda$ and the Newtonian gravitational potential U. The enlargement of the scalar manifold is related to the performed dimensional reduction procedure ($d = 4 \rightarrow d = 1$), which allows one to put U, ϕ^a and $\psi^\Lambda, \chi_\Lambda$ all on the same footing.

Physically, by exploiting the $(U(1))^{n_V+1}$ gauge invariance of $\widehat{G}_{\widehat{a}\widehat{b}}$, it is more convenient to eliminate the potentials $\psi^\Lambda, \chi_\Lambda$ by introducing their canonically conjugate variables q_Λ, p^Λ, corresponding to the BH electric and magnetic charges. Such a procedure allows one to define a "BH effective potential" function $V_{BH}\left(\phi;p,q\right)$, whereas the real scalars ϕ^a's and the Newtonian potential U remain on the same footing, and they are described by a simple dynamical model (5.1.23) in the (U,ϕ)-space, with a potential $V_{BH}\left(\phi;p,q\right)$, and constrained and $\mathbf{c}^2$-parameterized by Eq. (5.1.24).

5.2   Extreme Black Holes and Special Kähler Geometry

We now reconsider the previously introduced n_V-fold $\mathcal{N} = 2$, $d = 4$ Maxwell-Einstein supergravity theory (Maxwell Einstein Supergravity Theory), i.e. a $\mathcal{N} = 2$, $d = 4$ supergravity theory in which the gravity multiplet is coupled to n_V Abelian vector supermultiplets, and therefore the overall gauge group is $(U(1))^{n_V+1}$. We will see how the (regular) Special Kähler geometry (SKG) of the moduli space of such a theory allows one to simplify the investigation of the critical points of the function V_{BH}. In this and in the next Subsection we will refer to and complete the treatment presented in Sect. 3. We will denote the BH charges as follows: $n_\Lambda^e \equiv q_\Lambda$, $n_m^\Lambda \equiv p^\Lambda$.

Let us start by switching to a complex parametrization of the moduli space: in order to do this, we assume m_ϕ to be even, i.e. $m_\phi = 2n_\phi$, $n_\phi \in \mathbb{N}$. Therefore, by complexifying the $2n_\phi$-dim. real Riemann manifold $\mathcal{M}_\phi$ (with local coordinates $\{\phi^a\}$, $a = 1, ..., m_\phi$), we obtain a n_ϕ-dim. complex Hermitian manifold $\mathcal{M}_{z,\bar{z}}$ with local coordinates $\{z^i, \bar{z}^{\bar{i}}\}$ ($i, \bar{i} = 1, ..., n_\phi$) [72]

$$G_{ab}(\phi)\, d\phi^a d\phi^b \;=\; 2G_{i\bar{j}}(z,\bar{z})\, dz^i d\bar{z}^{\bar{j}}, \;\; \overline{G_{i\bar{j}}} = G_{\bar{j}i}. \tag{5.2.1}$$

In particular, as it pertains to the framework of n_V-fold $\mathcal{N} = 2$, $d = 4$ Maxwell Einstein Supergravity Theory, we assume that such an Hermitian geometry is a Kählerian one, regular (i.e. with the metric tensor strictly positive definite everywhere) and of the special type; i.e., we assume that

$$G_{ab}(\phi)\, d\phi^a d\phi^b \;=\; 2\frac{\partial^2 K(z,\bar{z})}{\partial \bar{z}^{\bar{j}} \partial z^i}\, dz^i d\bar{z}^{\bar{j}}, \; K(z,\bar{z}) = \overline{K(z,\bar{z})}; \tag{5.2.2}$$

$$G_{i\bar{j}}(z,\bar{z}) \text{ strictly positive definite } \forall\, (z,\bar{z}) \;\in\; \mathcal{M}_{z,\bar{z}}; \tag{5.2.3}$$

$$R_{i\bar{j}l\bar{m}} \;=\; G_{i\bar{j}}G_{l\bar{m}} + G_{i\bar{m}}G_{l\bar{j}} - C_{ilp}\overline{C}_{\bar{j}\bar{m}\bar{p}}G^{p\bar{p}}, \tag{5.2.4}$$

where the real function $K(z,\bar{z})$ (satisfying the Schwarz Lemma in $\mathcal{M}_{z,\bar{z}}$) is called Kähler potential, $R_{i\bar{j}l\bar{m}}$ is the Kähler Riemann-Christoffel curvature tensor and C_{ilm} is the rank-3, completely symmetric, Kähler-covariantly holomorphic tensor of SKG (with Kähler weights $(2,2)$).

Now, in order to study the "BH effective potential" function $V_{BH}(z,\bar{z};p,q)$ in (regular) SKG, we need to identify it with a symplectic-invariant, Kähler gauge-invariant, real positive function in such a geometric context. The natural and immediate choice is given by the first invariant $I_1(z,\bar{z};p,q)$ of the SKG, defined as [51]

$$I_1(z,\bar{z};p,q) \;\equiv\; |Z|^2(z,\bar{z};p,q) + G^{i\bar{i}}(z,\bar{z})(D_iZ)(z,\bar{z};p,q)(\overline{D}_{\bar{i}}\overline{Z})(z,\bar{z};p,q), \tag{5.2.5}$$

where $Z(z,\bar{z};p,q)$ is the central charge function of n_V-fold $\mathcal{N} = 2$, $d = 4$ Maxwell Einstein Supergravity Theory; let us also recall that Eqs. (4.1.54) and (4.2.53) yield

$$Z(z,\bar{z};p,q) \;=\; L^\Lambda(z,\bar{z})\, q_\Lambda - M_\Lambda(z,\bar{z})\, p^\Lambda = e^{\frac{1}{2}K(z,\bar{z})}\left[X^\Lambda(z)\, q_\Lambda - F_\Lambda(z)\, p^\Lambda\right]. \tag{5.2.6}$$

By recalling Eq. (4.2.86), I_1 may also be defined as

$$I_1\left(z,\bar{z};p,q\right) \equiv -\frac{1}{2}\left(p^\Lambda,q_\Lambda\right)\mathcal{M}\left(Re\left(\mathcal{N}\right),Im\left(\mathcal{N}\right)\right)\begin{pmatrix}p^\Sigma \\ \\ q_\Sigma\end{pmatrix}, \qquad (5.2.7)$$

with $\mathcal{M}\left(Re\left(\mathcal{N}\right),Im\left(\mathcal{N}\right)\right)$ defined by Eqs. (4.2.81)-(4.2.83) to be the real $(2n_V+2)\times(2n_V+2)$, $(z,\bar{z})$-dependent symmetric matrix

$$\mathcal{M}\left(Re\left(\mathcal{N}\left(z,\bar{z}\right)\right),Im\left(\mathcal{N}\left(z,\bar{z}\right)\right)\right)\equiv$$

$$\equiv\begin{pmatrix} \begin{array}{c} Im\left(\mathcal{N}\left(z,\bar{z}\right)\right)_{\Lambda\Sigma}+ \\[6pt] +Re\left(\mathcal{N}\left(z,\bar{z}\right)\right)_{\Lambda\Delta} \\[6pt] \left(\left(Im\mathcal{N}\left(z,\bar{z}\right)\right)^{-1}\right)^{\Delta\Xi} \\[6pt] Re\left(\mathcal{N}\left(z,\bar{z}\right)\right)_{\Xi\Sigma} \end{array} & -Re\left(\mathcal{N}\left(z,\bar{z}\right)\right)_{\Lambda\Xi}\left(\left(Im\mathcal{N}\left(z,\bar{z}\right)\right)^{-1}\right)^{\Xi\Sigma} \\[40pt] -\left(\left(Im\mathcal{N}\left(z,\bar{z}\right)\right)^{-1}\right)^{\Lambda\Xi}Re\left(\mathcal{N}\left(z,\bar{z}\right)\right)_{\Xi\Sigma} & \left(\left(Im\mathcal{N}\left(z,\bar{z}\right)\right)^{-1}\right)^{\Lambda\Sigma} \end{pmatrix}.$$

$$(5.2.8)$$

Consequently, by performing the fundamental identification

$$V_{BH}\left(z,\bar{z};p,q\right) = -I_1\left(z,\bar{z};p,q\right), \qquad (5.2.9)$$

the comparison of Eqs. (6.2.0.7)-(6.2.0.8) with Eqs. (5.2.7)-(5.2) yields

$$\left.\begin{array}{l} Re\left(\mathcal{N}\left(z,\bar{z}\right)\right)_{\Lambda\Sigma}=-\nu_{\Lambda\Sigma}\left(z,\bar{z}\right) \\[10pt] Im\left(\mathcal{N}\left(z,\bar{z}\right)\right)_{\Lambda\Sigma}=-\mu_{\Lambda\Sigma}\left(z,\bar{z}\right) \end{array}\right\} \implies \mathcal{N}_{\Lambda\Sigma}\left(z,\bar{z}\right)=-\nu_{\Lambda\Sigma}\left(z,\bar{z}\right)-i\mu_{\Lambda\Sigma}\left(z,\bar{z}\right).$$

$$(5.2.10)$$

The reality, symmetry and (strict) positive definiteness of the matrices $\mu_{\Lambda\Sigma}\left(z,\bar{z}\right)$ and $\nu_{\Lambda\Sigma}\left(z,\bar{z}\right)$ imply the reality, symmetry and (strict) negative definiteness of the

matrix $\mathcal{N}_{\Lambda\Sigma}(z,\bar{z})$, and thence of its real and imaginary parts separately (concerning its imaginary part, this was already noted in Eq. (4.1.96)). Consequently, the matrix $\mathcal{M}\left(Re\left(\mathcal{N}\right), Im\left(\mathcal{N}\right)\right)$ is (strictly) negative definite, and Eq. (5.2.7) yields that $I_1(z,\bar{z};p,q)$ (and thus, by the identification (5.2.9), the "BH effective potential" function $V_{BH}(z,\bar{z};p,q)$) is (real and) positive in all $\mathcal{M}_{z,\bar{z}} \times \Gamma$. The (strict) negative definiteness of the quadratic form of BH charges appearing in the r.h.s. of Eq. (5.2.7) implies that I_1 and V_{BH} vanish iff the fluxes of the $n_V + 1$ Abelian vector field strengths all vanish

$$I_1(z,\bar{z};p,q) = 0 = V_{BH}(z,\bar{z};p,q)$$

$$\Updownarrow$$

$$p^\Lambda = 0 = q_\Lambda, \ \forall \Lambda = 0,1,...,n_V. \tag{5.2.11}$$

By using Eqs. (5.2.1) ($G_{i\bar{j}} = \bar{\partial}_{\bar{j}}\partial_i K$ understood throughout) and (5.2.10), we may rewrite the 4-d. Lagrangian density (5.1.1) as follows:

$$\mathcal{L}_4 = -\frac{R}{2} + G_{i\bar{j}}\partial_\mu z^i \partial_\nu \bar{z}^{\bar{j}} g^{\mu\nu} +$$

$$+\frac{1}{2}\left(Im\mathcal{N}_{\Lambda\Sigma}\right)\mathcal{F}_{\mu\nu}^\Lambda \mathcal{F}_{\lambda\rho}^\Sigma g^{\mu\lambda}g^{\nu\rho} + \frac{1}{2}\left(Re\mathcal{N}_{\Lambda\Sigma}\right)\mathcal{F}_{\mu\nu}^\Lambda \, {}^*\mathcal{F}_{\lambda\rho}^\Sigma g^{\mu\lambda}g^{\nu\rho}.$$

$$\tag{5.2.12}$$

Now $\mathcal{L}_4$ denotes the purely bosonic part of the Lagrangian density of n_V-fold $\mathcal{N} = 2$, $d = 4$ Maxwell Einstein Supergravity Theory, with $i,\bar{i} \in \{1,...,n_V\}$ and $\Lambda, \Sigma \in \{0,1,...,n_V\}$.

Let us now consider the infinitesimal Kählerian metric interval in $\mathcal{M}_{z,\bar{z}}$; by using Eq. (5.2.2) we get

$$|dz|^2 = G_{i\bar{j}}dz^i d\bar{z}^{\bar{j}} = \left(\bar{\partial}_{\bar{j}}\partial_i K\right)dz^i d\bar{z}^{\bar{j}} = \frac{1}{2}G_{ab}d\phi^a d\phi^b; \tag{5.2.13}$$

$$\Updownarrow$$

$$\left|\frac{dz}{d\tau}\right|^2 = G_{i\bar{j}}\frac{dz^i}{d\tau}\frac{d\bar{z}^{\bar{j}}}{d\tau} = \left(\bar{\partial}_{\bar{j}}\partial_i K\right)\frac{dz^i}{d\tau}\frac{d\bar{z}^{\bar{j}}}{d\tau} = \frac{1}{2}G_{ab}\frac{d\phi^a}{d\tau}\frac{d\phi^b}{d\tau}. \tag{5.2.14}$$

Thus, by recalling Eqs. (5.2.5) and (5.2.9), Eqs. (5.1.23) and (5.1.24) may be respec-

tively rewritten as

$$\mathcal{L}_1\left[U(\tau), z(\tau), \bar{z}(\tau); p, q\right] = \left(\frac{dU(\tau)}{d\tau}\right)^2 + \left|\frac{dz(\tau)}{d\tau}\right|^2 +$$

$$+ e^{2U(\tau)} \left[\begin{array}{l} |Z|^2 \left(z(\tau), \bar{z}(\tau); p, q\right) + G^{i\bar{i}}\left(z(\tau), \bar{z}(\tau)\right) \cdot \\ \cdot (D_i Z)\left(z(\tau), \bar{z}(\tau); p, q\right)\left(\overline{D_{\bar{i}}\overline{Z}}\right)\left(z(\tau), \bar{z}(\tau); p, q\right) \end{array} \right].$$
(5.2.15)

$$(2S_{BH}T_{BH})^2 = \mathbf{c}^2 = \left(\frac{dU(\tau)}{d\tau}\right)^2 + \left|\frac{dz(\tau)}{d\tau}\right|^2 +$$

$$- e^{2U(\tau)} \left[\begin{array}{l} |Z|^2 \left(z(\tau), \bar{z}(\tau); p, q\right) + G^{i\bar{i}}\left(z(\tau), \bar{z}(\tau)\right) \cdot \\ \cdot (D_i Z)\left(z(\tau), \bar{z}(\tau); p, q\right)\left(\overline{D_{\bar{i}}\overline{Z}}\right)\left(z(\tau), \bar{z}(\tau); p, q\right) \end{array} \right].$$
(5.2.16)

5.3   Critical Points of Black Hole Effective Potential in Special Kähler Geometry

We will now study the critical points of the "BH effective potential" function V_{BH} in the (regular) Special Kähler Geometry (SKG) of the vector supermultiplets' moduli space $\mathcal{M}_{z,\bar{z}}$ of the n_V-fold $\mathcal{N}=2$, $d=4$ Maxwell Einstein Supergravity Theory. As previously pointed out, such critical points are "attractors" in the dynamical system describing the radial evolution of the moduli from $r \to \infty$ to $r \to r_H^+$. In order to perform such an analysis, we need to recall a few results from SKG[5]; beside the Kähler-covariant holomorphicity of Z, i.e. (see Eq. (4.2.63))

$$\overline{D}_{\bar{i}}Z = 0 \Leftrightarrow D_i\overline{Z} = 0,$$
(5.3.1)

we will largely use Eqs. (7.2.1.38) and (4.1.29) which, by definition (4.2.53), yield

$$D_i D_j Z = iC_{ijk}G^{k\bar{k}}\overline{D}_{\bar{k}}\overline{Z};$$
(5.3.2)

$$D_i\overline{D}_{\bar{j}}\overline{Z} = G_{i\bar{j}}\overline{Z} \Leftrightarrow \overline{D}_{\bar{i}}D_j Z = G_{\bar{j}i}Z.$$
(5.3.3)

---

[5]Beside Sect. 3, see e.g. [36], [44], [45], [46], [47], [56], [73], [74], [46], [75] and [76] for further insights on SKG and moduli space geometries of $\mathcal{N}=2$ SUGRA more in general.

Let us start from the fundamental identification (5.2.9)

$$
\begin{aligned}
V_{BH}\left(z,\overline{z};p,q\right) &= I_1\left(z,\overline{z};p,q\right) \equiv \\
&\equiv \; |Z|^2\left(z,\overline{z};p,q\right) + G^{i\overline{i}}\left(z,\overline{z}\right)\left(D_i Z\right)\left(z,\overline{z};p,q\right)\left(\overline{D}_{\overline{i}}\overline{Z}\right)\left(z,\overline{z};p,q\right).
\end{aligned}
$$

(5.3.4)

Thence, by recalling that V_{BH} and $|Z\left(z,\overline{z};p,q\right)|$ are Kähler-gauge invariant scalars in $\mathcal{M}_{z,\overline{z}}$, by using Eqs. (4.2.117), (5.3.2) and (5.3.3) we can calculate (also remind that in $\mathcal{M}_{z,\overline{z}}$ the Metric Postulate holds)

$$
\begin{aligned}
D_i V_{BH} &= \partial_i V_{BH} = \partial_i\left[|Z|^2 + G^{j\overline{k}}\left(D_j Z\right)\left(\overline{D}_{\overline{k}}\overline{Z}\right)\right] = \\
&= 2\overline{Z}D_i Z + iC_{ijk}G^{j\overline{m}}G^{k\overline{k}}\left(\overline{D}_{\overline{m}}\overline{Z}\right)\left(\overline{D}_{\overline{k}}\overline{Z}\right).
\end{aligned}
$$

(5.3.5)

Therefore, we get that the critical points of $|Z|$ are critical points also for V_{BH}; indeed, by assuming that $Z \neq 0$ (everywhere in $\mathcal{M}_{z,\overline{z}}$, and in particular at the Horizon, critical "attractor" points) and using Eq. (4.2.117), Eq. (5.3.5) yields

$$
\partial_i |Z| = 0 \Leftrightarrow D_i Z = 0 \Longrightarrow \partial_i V_{BH} = 0.
$$

(5.3.6)

It should be stressed that the opposite, in general, is not true

$$
\partial_i V_{BH} = 0 \nRightarrow \partial_i |Z| = 0 \Leftrightarrow D_i Z = 0.
$$

(5.3.7)

Thus, in the framework of the n_V-fold $\mathcal{N} = 2, d = 4$ Maxwell Einstein Supergravity Theory with (regular) SKG of $\mathcal{M}_{z,\overline{z}}$, the Horizon, "attractor" points for the considered extreme BH, i.e. the critical points of the "BH effective potential" function $V_{BH}\left(z,\overline{z};p,q\right)$ in $\mathcal{M}_{z,\overline{z}}$, may be divided in two disjoint classes:

1] the "attractors" which are critical points also of the absolute value of the central charge $|Z|\left(z,\overline{z};p,q\right)$ (the so-called $\frac{1}{2}$-*BPS-SUSY preserving* extreme BH "attractors", treated in Subsubsect. 4.3.1)

and

2] those that are *not* critical points of $|Z|\left(z,\overline{z};p,q\right)$ in $\mathcal{M}_{z,\overline{z}}$ (the so-called *NON-(BPS-)SUSY* extreme BH "attractors", treated in Subsubsect. 4.3.2).

Clearly, such a distinction (and the whole treatment given below) is parametrically dependent on the BH charge configuration, i.e. it is parameterized by the $Sp\left(2n_V + 2\right)$-covariant vector $\left(p^\Lambda, q_\Lambda\right)$, with the group $Sp\left(2n_V + 2\right)$ defined on $\mathbb{R}$ at classical level and on $\mathbb{Z}$ when the charge quantization is taken into account.

5.3.1   Supersymmetric Attractors

Let us start by considering the $\frac{1}{2}$-*BPS-SUSY preserving* extreme BH "attractors", i.e. the points $\left(z_{susy}\left(p,q\right) ,\overline{z}_{susy}\left(p,q\right) \right)$ in $\mathcal{M}_{z,\overline{z}}$ defined by

$$
\forall i \; \in \; \{1,...,n_V\} : \begin{cases} \left(\partial_i \left| Z \right| \right)_{\left(z_{susy},\overline{z}_{susy}\right)} = 0 \Leftrightarrow \left(D_i Z \right)_{\left(z_{susy},\overline{z}_{susy}\right)} = 0; \\[2mm] \qquad\qquad\qquad \Updownarrow \\[2mm] \left\{ \left[\partial_i + \tfrac{1}{2}\partial_i K\left(z,\overline{z}\right)\right] Z\left(z,\overline{z};p,q\right) \right\}_{\left(z_{susy},\overline{z}_{susy}\right)} = 0; \\[2mm] \qquad\qquad\qquad \Downarrow \\[2mm] \left(\partial_i V_{BH}\right)_{\left(z_{susy},\overline{z}_{susy}\right)} = 0. \end{cases} \tag{5.3.1.1}
$$

In order to eventually characterize such points as maxima or minima of the function $\left| Z \right|\left(z,\overline{z};p,q\right)$ in $\mathcal{M}_{z,\overline{z}}$, we have at least to calculate the Kähler-covariant second derivatives of $\left| Z \right|$, and then evaluate them at $\left(z_{susy},\overline{z}_{susy}\right)$. By using Eqs. (5.3.1), (4.2.117), (5.3.2) and (5.3.3), we obtain

$$
\begin{aligned}
D_i D_j \left| Z \right| &= D_i \partial_j \left| Z \right| = D_i \left(\frac{\overline{Z}}{2\left| Z \right|}D_j Z \right) = \\[2mm]
&= i\frac{\overline{Z}}{2\left| Z \right|}\left[i\frac{\overline{Z}}{2\left| Z \right|^2}\left(D_i Z\right) D_j Z + C_{ijk}G^{k\overline{k}}\overline{D}_{\overline{k}}\overline{Z}\right];
\end{aligned} \tag{5.3.1.2}
$$

$$
\begin{aligned}
\overline{D}_{\overline{i}}D_j \left| Z \right| &= \overline{D}_{\overline{i}}\partial_j \left| Z \right| = \overline{D}_{\overline{i}}\left(\frac{\overline{Z}}{2\left| Z \right|}D_j Z \right) = \\[2mm]
&= \frac{1}{4\left| Z \right|}\left(\overline{D}_{\overline{i}}\overline{Z}\right) D_j Z + \frac{1}{2}\left| Z \right| G_{j\overline{i}}.
\end{aligned} \tag{5.3.1.3}
$$

On the other hand, by recalling the general properties of Hermitian and Kählerian manifolds [37], one gets

$$
\begin{aligned}
D_i D_j \left| Z \right| &= D_i \partial_j \left| Z \right| = \partial_i \partial_j \left| Z \right| - \Gamma_{ij}{}^k \partial_k \left| Z \right| = \\[2mm]
&= \partial_i \partial_j \left| Z \right| - G^{k\overline{i}}\left(\partial_i \overline{\partial}_{\overline{i}}\partial_j K\right)\partial_k \left| Z \right|;
\end{aligned} \tag{5.3.1.4}
$$

$$
\overline{D}_{\overline{i}}D_j \left| Z \right| = \overline{D}_{\overline{i}}\partial_j \left| Z \right| = \overline{\partial}_{\overline{i}}\partial_j \left| Z \right|. \tag{5.3.1.5}
$$

Since the Kähler potential K and the central charge $\left| Z \right|$ are both assumed to satisfy the Schwarz Lemma in $\mathcal{M}_{z,\overline{z}}$, such Eqs. respectively yield

$$
D_i D_j \left| Z \right| = D_j D_i \left| Z \right|; \tag{5.3.1.6}
$$

$$\overline{D}_{\bar{\imath}} D_j |Z| = D_j \overline{D}_{\bar{\imath}} |Z| , \qquad (5.3.1.7)$$

as it can be checked by looking at the explicit expressions (5.3.1.2) and (5.3.1.3). Consequently, by evaluating at the point(s) $\left(z_{susy}, \overline{z}_{susy}\right)$ in $\mathcal{M}_{z,\bar{z}}$ defined by Eq. (5.3.1.2), Eqs. (5.3.1.2)-(5.3.1.5) yield

$$\left(D_i D_j |Z|\right)_{\left(z_{susy}, \overline{z}_{susy}\right)} = \left(\partial_i \partial_j |Z|\right)_{\left(z_{susy}, \overline{z}_{susy}\right)} = 0; \qquad (5.3.1.8)$$

$$
\begin{aligned}
\left(\overline{D}_{\bar{\imath}} D_j |Z|\right)_{\left(z_{susy}, \overline{z}_{susy}\right)} &= \left(\overline{\partial}_{\bar{\imath}} \partial_j |Z|\right)_{\left(z_{susy}, \overline{z}_{susy}\right)} = \\
&= \frac{1}{2} |Z| \left(z_{susy}, \overline{z}_{susy}; p, q\right) G_{j\bar{\imath}} \left(z_{susy}, \overline{z}_{susy}\right). \quad (5.3.1.9)
\end{aligned}
$$

It is now possible to introduce the $2n_V \times 2n_V$ complex Hessian matrix $H_{\hat{\imath}\hat{\jmath}}^{|Z|}$ of the function $|Z| (z, \overline{z}; p, q)$ in $\mathcal{M}_{z,\bar{z}}$, as follows:

$$
H_{\hat{\imath}\hat{\jmath}}^{|Z|} (z, \overline{z}; p, q) = \begin{pmatrix} H_{ij}^{|Z|} & H_{i\bar{\jmath}}^{|Z|} \\[2ex] H_{\bar{\jmath}i}^{|Z|} & H_{\bar{\imath}\bar{\jmath}}^{|Z|} \end{pmatrix} \equiv
$$

$$
\equiv \begin{pmatrix} D_i D_j |Z| & D_i \overline{D}_{\bar{\jmath}} |Z| \\[2ex] D_j \overline{D}_{\bar{\imath}} |Z| & \overline{D}_{\bar{\imath}} \overline{D}_{\bar{\jmath}} |Z| \end{pmatrix} = \begin{pmatrix} D_i D_j |Z| & \overline{\overline{D}_{\bar{\imath}} D_j |Z|} \\[2ex] \overline{D}_{\bar{\imath}} D_j |Z| & \overline{D_i D_j |Z|} \end{pmatrix} =
$$

$$\qquad\qquad (5.3.1.10)$$

$$
= \begin{pmatrix} i\frac{\overline{Z}}{2|Z|} \left[\begin{array}{l} i\frac{\overline{Z}}{2|Z|^2} \left(D_i Z\right) D_j Z + \\[1ex] + C_{ijk} G^{k\bar{k}} \overline{D}_{\bar{k}} \overline{Z} \end{array} \right] & \frac{1}{4|Z|} \left(D_i Z\right) \overline{D}_{\bar{\jmath}} \overline{Z} + \frac{1}{2} |Z| G_{i\bar{\jmath}} \\[6ex] \frac{1}{4|Z|} \left(\overline{D}_{\bar{\imath}} \overline{Z}\right) D_j Z + \frac{1}{2} |Z| G_{j\bar{\imath}} & -i\frac{Z}{2|Z|} \left[\begin{array}{l} -i\frac{Z}{2|Z|^2} \left(\overline{D}_{\bar{\imath}} \overline{Z}\right) \overline{D}_{\bar{\jmath}} \overline{Z} + \\[1ex] + \overline{C}_{\bar{\imath}\bar{\jmath}\bar{k}} G^{k\bar{k}} D_k Z \end{array} \right] \end{pmatrix} ,
$$

where the "hatted" indices $\hat{\imath}$ and $\hat{\jmath}$ may be holomorphic or anti-holomorphic ($n_\phi = n_V$) Thus, by evaluating at the point(s) $\left(z_{susy} (p, q), \overline{z}_{susy} (p, q)\right)$ in $\mathcal{M}_{z,\bar{z}}$ defined by

Eq. (5.3.1.2), the Hessian becomes

$$H_{\hat{i}\hat{j}}^{|Z|}\left(z_{susy}\left(p,q\right),\overline{z}_{susy}\left(p,q\right);p,q\right) =$$

$$= \tfrac{1}{2}\,|Z|\left(z_{susy}\left(p,q\right),\overline{z}_{susy}\left(p,q\right);p,q\right)\cdot$$

$$\cdot\begin{pmatrix} 0 & G_{i\bar{j}}\left(z_{susy}\left(p,q\right),\overline{z}_{susy}\left(p,q\right)\right) \\ \\ G_{\bar{j}i}\left(z_{susy}\left(p,q\right),\overline{z}_{susy}\left(p,q\right)\right) & 0 \end{pmatrix};$$

(5.3.1.11)

since $\overline{G_{i\bar{j}}} = G_{\bar{j}i}$, we obtain

$$\left(H_{\hat{i}\hat{j}}^{|Z|}\left(z_{susy}\left(p,q\right),\overline{z}_{susy}\left(p,q\right);p,q\right)\right)^{\dagger} = H_{\hat{i}\hat{j}}^{|Z|}\left(z_{susy}\left(p,q\right),\overline{z}_{susy}\left(p,q\right);p,q\right).$$

(5.3.1.12)

Eq. (5.3.1.12) means that the $2n_V \times 2n_V$ complex Hessian matrix $H_{\hat{i}\hat{j}}^{|Z|}$ evaluated at the $\tfrac{1}{2}$-*BPS-SUSY preserving* extreme BH "attractor" point(s) $(z_{susy}, \overline{z}_{susy})$ in $\mathcal{M}_{z,\overline{z}}$ is Hermitian for any BH charge configuration. Consequently, $H_{\hat{i}\hat{j}}^{|Z|}(z_{susy}, \overline{z}_{susy}; p, q)$ is always diagonalizable by a unitary transformation, and it has $2n_V$ real eigenvalues; from Eq. (5.3.1.11) and well known Theorems of mathematical analysis, it then follows that, for an arbitrary but fixed BH charge configuration $(p^\Lambda, q_\Lambda) \in \Gamma$

$$G_{i\bar{j}}\left(z_{susy}, \overline{z}_{susy}\right) \text{ strictly positive (negative) definite}$$

$$\Updownarrow$$

$(z_{susy}, \overline{z}_{susy})$ *at least local* miminum (maximum) of $|Z|\left(z, \overline{z}; p, q\right)$ in $\mathcal{M}_{z,\overline{z}}$. (5.3.1.13)

Since we assume that the SKG of $\mathcal{M}_{z,\overline{z}}$ is regular, i.e., that the metric $G_{i\bar{j}}$ is strictly positive definite everywhere, we obtain at least a local minimum of $|Z|$ at the $\tfrac{1}{2}$-*BPS-SUSY preserving* extreme BH "attractor" point(s). However, if we go beyond the regular regime of SKG, $G_{i\bar{j}}$ may be singular (i.e. not invertible) and/or without a well-defined definiteness (i.e. with some positive as well as negative eigenvalues); in such a case, Eq. (5.3.1) yields that the eventually existing (at least local) maxima of $|Z|$ are reached out of the regular SKG of $\mathcal{M}_{z,\overline{z}}$. In general, going beyond the regular regime of SKG, some "phase transitions" may happen in $\mathcal{M}_{z,\overline{z}}$, corresponding to a breakdown of the 1-dim. effective Lagrangian picture[6] of 4-d. (extreme) BHs presented in Subsects. 4.1-4.3, unless new massless states appear [55].

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[6] Such a 1-dim. effective framework should be understood as being obtained by integrating all massive states of the theory out.

Moreover, by recalling Eq. (5.3.4) and using the very definition (5.3.1.2), the value of the function V_{BH} at the $\frac{1}{2}$-*BPS-SUSY preserving* extreme BH "attractor" point(s) reads

$$V_{BH}\left(z_{susy}, \overline{z}_{susy}; p, q\right) = |Z|^2 \left(z_{susy}, \overline{z}_{susy}; p, q\right),\qquad(5.3.1.14)$$

implying that the (semiclassical, leading order) entropy at such $\frac{1}{2}$-*BPS-SUSY preserving* extreme BH "attractor(s)" is

$$S_{BH,susy} = \pi |Z|^2 \left(z_{susy}, \overline{z}_{susy}; p, q\right).\qquad(5.3.1.15)$$

Now, in order to establish if the points $\left(z_{susy}, \overline{z}_{susy}\right)$ are eventually maxima or minima of $V_{BH}\left(z, \overline{z}; p, q\right)$ in $\mathcal{M}_{z,\overline{z}}$, we have at least to calculate the Kähler-covariant second derivatives of V_{BH}, and then evaluate them at $\left(z_{susy}, \overline{z}_{susy}\right)$. By using Eqs. (4.1.4), (5.3.1), (5.3.2), (5.3.3) and (5.3.5) and exploiting the validity of the Metric Postulate in $\mathcal{M}_{z,\overline{z}}$, we obtain

$$D_i D_j V_{BH} = D_i \left[2\overline{Z}D_j Z + iC_{jkl}G^{k\overline{m}}G^{l\overline{l}}\left(\overline{D_{\overline{m}}Z}\right)\left(\overline{D_{\overline{l}}Z}\right)\right] =$$

$$= 2i \left[2C_{(ij)k}G^{k\overline{k}}\overline{Z}D_{\overline{k}}\overline{Z} + \tfrac{1}{2}\left(D_{(i}C_{j)kl}\right)G^{k\overline{m}}G^{l\overline{l}}\left(\overline{D_{\overline{m}}Z}\right)\overline{D_{\overline{l}}Z}\right];\qquad(5.3.1.16)$$

$$\overline{D_{\overline{i}}}D_j V_{BH} = \overline{D_{\overline{i}}}\left[2\overline{Z}D_j Z + iC_{jkl}G^{k\overline{m}}G^{l\overline{l}}\left(\overline{D_{\overline{m}}Z}\right)\left(\overline{D_{\overline{l}}Z}\right)\right] =$$

$$= 2\left[\left(\overline{D_{\overline{i}}}\overline{Z}\right)D_j Z + G_{j\overline{i}}|Z|^2 + C_{jkl}\overline{C}_{\overline{i}\overline{m}\overline{k}}G^{k\overline{m}}G^{l\overline{l}}G^{n\overline{k}}\left(D_n Z\right)\overline{D_{\overline{l}}Z}\right],\qquad(5.3.1.17)$$

where in the last lines of both Eqs. we used the symmetry of the rank-3 tensor C_{jkl}: $C_{jkl} = C_{(jkl)}$, and in the last line of Eq. (5.3.1.16) also the symmetry of the Kähler-covariant derivative of such a tensor ($D_{[i}C_{j]kl} = 0$, see Eq. (4.1.5)).

By using Eqs. (7.2.1.39), (4.1.8) and (4.1.9)-(4.1.11), the expression (5.3.1.17) can be further written as follows:

$$\overline{D_{\overline{i}}}D_j V_{BH} = 2\left[\left(\overline{D_{\overline{i}}}\overline{Z}\right)D_j Z + G_{j\overline{i}}|Z|^2 + C_{jkl}\overline{C}_{\overline{i}\overline{m}\overline{k}}G^{k\overline{m}}G^{l\overline{l}}G^{n\overline{k}}\left(D_n Z\right)\overline{D_{\overline{l}}Z}\right] =$$

$$= 2\left\{ \begin{aligned} &\left(\overline{\partial}_{\overline{i}}\partial_j K\right)|Z|^2 + \\ &+ \left[\begin{aligned} &2\delta^n_j\delta^{\overline{l}}_{\overline{i}} + G^{n\overline{l}}\left(\overline{\partial}_{\overline{i}}\partial_j K\right) + \\ &+ G^{l\overline{l}}G^{n\overline{k}}G^{m\overline{m}}\left(\overline{\partial}_{\overline{k}}\overline{\partial}_{\overline{i}}\partial_m K\right)\partial_j\overline{\partial}_{\overline{m}}\partial_l K - G^{l\overline{l}}G^{n\overline{k}}\overline{\partial}_{\overline{k}}\partial_l\overline{\partial}_{\overline{i}}\partial_j K \end{aligned}\right]\cdot \\ &\cdot\left[\left(\partial_n + \tfrac{1}{2}\partial_n K\right)Z\right]\left[\left(\overline{\partial}_{\overline{l}} + \tfrac{1}{2}\overline{\partial}_{\overline{l}}K\right)\overline{Z}\right]. \end{aligned}\right\}.$$

$$(5.3.1.18)$$

On the other hand, by recalling the general properties of Hermitian and Kählerian manifolds [37], one gets

$$D_i D_j V_{BH} = D_i \partial_j V_{BH} = \partial_i \partial_j V_{BH} + \Gamma_{ij}^k \partial_k V_{BH} = \partial_i \partial_j V_{BH} + G^{k\bar{i}} \partial_i \bar{\partial}_{\bar{i}} \partial_j K \partial_k V_{BH};$$

$$\tag{5.3.1.19}$$

$$\overline{D}_{\bar{i}} D_j V_{BH} = \overline{D}_{\bar{i}} \partial_j V_{BH} = \bar{\partial}_{\bar{i}} \partial_j V_{BH}. \tag{5.3.1.20}$$

Since the Kähler potential K and "BH effective potential" V_{BH} are both assumed to satisfy the Schwarz Lemma in $\mathcal{M}_{z,\bar{z}}$, such Eqs. respectively yield

$$D_i D_j V_{BH} = D_j D_i V_{BH}; \tag{5.3.1.21}$$

$$\overline{D}_{\bar{i}} D_j V_{BH} = D_j \overline{D}_{\bar{i}} V_{BH}, \tag{5.3.1.22}$$

as it can be checked by looking at the expressions (5.3.1.16), (5.3.1.17) and (5.3.1.18).

For completeness, since $\overline{D}_{\bar{k}} \overline{Z} = \left(\bar{\partial}_{\bar{k}} + \frac{1}{2} \bar{\partial}_{\bar{k}} K \right) \overline{Z}$ and C_{jkl} is a rank-3 completely symmetric, Kähler-covariantly holomorphic tensor with Kähler weights $(2, -2)$ for which then (see Eq. (4.1.21))

$$D_i C_{jkl} = \partial_i C_{jkl} + (\partial_i K) C_{jkl} - \Gamma_{ij}^m C_{mkl} - \Gamma_{ik}^m C_{jml} - \Gamma_{il}^m C_{jkm}. \tag{5.3.1.23}$$

Eq. (5.3.1.16) may be further elaborated as follows :

$$D_i D_j V_{BH} = 4i C_{ijk} G^{k\bar{k}} \overline{Z} \overline{D}_{\bar{k}} \overline{Z} + i \left(D_{(i} C_{j)kl} \right) G^{k\bar{m}} G^{l\bar{l}} \left(\overline{D}_{\bar{m}} \overline{Z} \right) \overline{D}_{\bar{l}} \overline{Z} =$$

$$= i G^{k\bar{m}} \left[\left(\bar{\partial}_{\bar{m}} + \frac{1}{2} \bar{\partial}_{\bar{m}} K \right) \overline{Z} \right] \cdot$$

$$\tag{5.3.1.24}$$

$$\cdot \left\{ \begin{array}{l} 4\overline{Z} C_{ijk} + G^{l\bar{l}} \left[\left(\bar{\partial}_{\bar{l}} + \frac{1}{2} \bar{\partial}_{\bar{l}} K \right) \overline{Z} \right] \cdot \\[2mm] \cdot \left[\partial_{(i} C_{j)kl} + \left(\partial_{(i} K \right) C_{j)kl} - G^{m\bar{k}} C_{mkl} \bar{\partial}_{\bar{k}} \partial_i \partial_j K - 2 G^{m\bar{k}} C_{ml(j|} \bar{\partial}_{\bar{k}} \partial_{|i)} \partial_k K \right] \end{array} \right\}.$$

Now, by evaluating at the point(s) $\left(z_{susy} \left(p, q \right), \bar{z}_{susy} \left(p, q \right) \right)$ in $\mathcal{M}_{z,\bar{z}}$ defined by Eq. (5.3.1.2), Eqs. (5.3.1.16), (5.3.1.24), (5.3.1.17) and (5.3.1.18) yield

$$\left(D_i D_j V_{BH} \right)_{\left(z_{susy}, \bar{z}_{susy} \right)} = \left(\partial_i \partial_j V_{BH} \right)_{\left(z_{susy}, \bar{z}_{susy} \right)} = 0; \tag{5.3.1.25}$$

$$\left(\overline{D}_{\bar{i}} D_j V_{BH} \right)_{\left(z_{susy}, \bar{z}_{susy} \right)} = \left(\bar{\partial}_{\bar{i}} \partial_j V_{BH} \right)_{\left(z_{susy}, \bar{z}_{susy} \right)} =$$

$$= 2 |Z|^2 \left(z_{susy}, \bar{z}_{susy}; p, q \right) G_{j\bar{i}} \left(z_{susy}, \bar{z}_{susy} \right). \tag{5.3.1.26}$$

As previously done for the function $|Z|$, it is now possible to introduce the $2n_V \times 2n_V$ complex Hessian matrix $H_{\hat{i}\hat{j}}^{V_{BH}}$ of the function $V_{BH}(z, \bar{z}; p, q)$ in $\mathcal{M}_{z,\bar{z}}$, as follows:

$$
H_{\hat{i}\hat{j}}^{V_{BH}}(z, \bar{z}; p, q) = \begin{pmatrix} H_{ij}^{V_{BH}} & H_{i\bar{j}}^{V_{BH}} \\[2em] H_{\bar{j}i}^{V_{BH}} & H_{\bar{i}\bar{j}}^{V_{BH}} \end{pmatrix} \equiv
$$

$$
= 2 \begin{pmatrix} i \begin{bmatrix} 2C_{ijk}G^{k\bar{k}}\overline{Z}D_{\bar{k}}\overline{Z}+ \\[0.5em] +\frac{1}{2}D_{(i}C_{j)kl} \cdot \\[0.5em] \cdot G^{k\overline{m}}G^{l\bar{l}}\left(\overline{D_{\overline{m}}Z}\right)\overline{D_{\bar{l}}Z} \end{bmatrix} & \begin{bmatrix} (D_iZ)\,\overline{D_{\bar{j}}\overline{Z}} + G_{i\bar{j}}\,|Z|^2 + \\[0.5em] +\overline{C}_{\overline{jkl}}C_{imk}G^{m\bar{k}}G^{l\bar{l}}G^{k\bar{n}} \cdot \\[0.5em] \cdot \left(\overline{D_{\bar{n}}Z}\right)D_lZ \end{bmatrix} \\[4em] \begin{bmatrix} \left(\overline{D_{\bar{i}}\overline{Z}}\right)D_jZ + G_{\bar{j}i}\,|Z|^2 + \\[0.5em] +C_{jkl}\overline{C}_{\overline{imk}}G^{k\overline{m}}G^{l\bar{l}}G^{n\bar{k}} \cdot \\[0.5em] \cdot (D_nZ)\,\overline{D_{\bar{l}}\overline{Z}} \end{bmatrix} & -i \begin{bmatrix} 2\overline{C}_{\overline{ijk}}G^{k\bar{k}}ZD_kZ+ \\[0.5em] +\frac{1}{2}\overline{D}_{(\bar{i}}\overline{C}_{\bar{j})kl} \cdot \\[0.5em] \cdot G^{m\bar{k}}G^{l\bar{l}}(D_mZ)\,D_lZ \end{bmatrix} \end{pmatrix}.
$$

$$
\text{(5.3.1.27)}
$$

Thus, by evaluating at the point(s) $\left(z_{susy}(p,q), \bar{z}_{susy}(p,q)\right)$ in $\mathcal{M}_{z,\bar{z}}$ defined by Eq. (5.3.1.2) and recalling Eq. (5.3.1.14), the Hessian becomes

$$
H_{\hat{i}\hat{j}}^{V_{BH}}\left(z_{susy}(p,q), \bar{z}_{susy}(p,q); p, q\right) =
$$

$$
= 2V_{BH}\left(z_{susy}(p,q), \bar{z}_{susy}(p,q); p, q\right) \cdot
$$

$$
\cdot \begin{pmatrix} 0 & G_{i\bar{j}}\left(z_{susy}(p,q), \bar{z}_{susy}(p,q)\right) \\[1.5em] G_{\bar{j}i}\left(z_{susy}(p,q), \bar{z}_{susy}(p,q)\right) & 0 \end{pmatrix} \cdot
$$

$$
\text{(5.3.1.28)}
$$

Since $\overline{G_{i\bar{j}}} = G_{j\bar{i}}$, also in this case we obtain

$$\left(H_{\hat{i}\hat{j}}^{V_{BH}} \left(z_{susy}, \overline{z}_{susy}; p, q \right) \right)^{\dagger} = H_{\hat{i}\hat{j}}^{V_{BH}} \left(z_{susy}, \overline{z}_{susy}; p, q \right), \tag{5.3.1.29}$$

i.e. the $2n_V \times 2n_V$ complex Hessian matrix $H_{\hat{i}\hat{j}}^{V_{BH}}$ evaluated at the $\frac{1}{2}$-BPS-SUSY *preserving* extreme BH "attractor", point(s) $\left(z_{susy}, \overline{z}_{susy} \right)$ in $\mathcal{M}_{z,\overline{z}}$ is Hermitian for any BH charge configuration. Thus, $H_{\hat{i}\hat{j}}^{V_{BH}} \left(z_{susy}, \overline{z}_{susy}; p, q \right)$ is always diagonalizable by a unitary transformation, and it has $2n_V$ real eigenvalues; from Eq. (5.3.1.28) it then follows that, for an arbitrary but fixed BH charge configuration $\left(p^{\Lambda}, q_{\Lambda} \right) \in \Gamma$

$$G_{i\bar{j}} \left(z_{susy}, \overline{z}_{susy} \right) \text{ strictly positive (negative) definite}$$

$$\Updownarrow$$

$$\left(z_{susy}, \overline{z}_{susy} \right) \text{ at least local miminum (maximum) of } V_{BH} \left(z, \overline{z}; p, q \right) \text{ in } \mathcal{M}_{z,\overline{z}}.$$
$$\tag{5.3.1.30}$$

Such a result also follows from the comparison of $H_{\hat{i}\hat{j}}^{|Z|} \left(z_{susy}, \overline{z}_{susy}; p, q \right)$ (given by Eq. (5.3.1.11)) with $H_{\hat{i}\hat{j}}^{V_{BH}} \left(z_{susy}, \overline{z}_{susy}; p, q \right)$ (given by Eq. (5.3.1.28)), yielding

$$H_{\hat{i}\hat{j}}^{V_{BH}} \left(z_{susy}, \overline{z}_{susy}; p, q \right) = 4 |Z| \left(z_{susy}, \overline{z}_{susy}; p, q \right) H_{\hat{i}\hat{j}}^{|Z|} \left(z_{susy}, \overline{z}_{susy}; p, q \right) =$$

$$= 4 \left(V_{BH} \left(z_{susy}, \overline{z}_{susy}; p, q \right) \right)^{1/2} H_{\hat{i}\hat{j}}^{|Z|} \left(z_{susy}, \overline{z}_{susy}; p, q \right), \tag{5.3.1.31}$$

where in the last line we recalled Eq. (5.3.1.14).

As mentioned above, since we assume that the SKG of $\mathcal{M}_{z,\overline{z}}$ is regular, we obtain at least a local minimum of V_{BH} at the $\frac{1}{2}$-BPS-SUSY *preserving* extreme BH "attractor" point(s). However, different situations may arise if we go beyond the regular regime of SKG; in such a case, Eq. (5.3.1) yields that the eventually existing (at least local) maxima of V_{BH} are reached out of the regular SKG of $\mathcal{M}_{z,\overline{z}}$.

Summarizing, in the context of regular SKG of $\mathcal{M}_{z,\overline{z}}$, all $\frac{1}{2}$-BPS-SUSY *preserving* extreme BH "attractor" points, defined by the differential Eq. (5.3.1.2) ($\forall i = 1, ..., n_V$)

$$\left(\partial_i |Z| \right) \left(z_{susy}, \overline{z}_{susy}; p, q \right) = 0 \tag{5.3.1.32}$$

$$\Updownarrow$$

$$\left(D_i Z \right) \left(z_{susy}, \overline{z}_{susy}; p, q \right) =$$
$$\tag{5.3.1.33}$$
$$= \left[\left(\partial_i Z \right) \left(z, \overline{z}; p, q \right) + \tfrac{1}{2} \left(\partial_i K \right) \left(z, \overline{z} \right) Z \left(z, \overline{z}; p, q \right) \right]_{\left(z_{susy}, \overline{z}_{susy} \right)} = 0$$

$$\Downarrow$$

$$\left(\partial_i V_{BH} \right) \left(z_{susy}, \overline{z}_{susy}; p, q \right) = 0, \tag{5.3.1.34}$$

are (at least local) minima[7] of both the real, positive functions $V_{BH}(z, \bar{z}; p, q)$ and $|Z|(z, \bar{z}; p, q)$, for the arbitrary but fixed BH charge configuration being considered.

However, if one considers only one (p, q)-parameterized continuous branch of $V_{BH}(z, \bar{z}; p, q)$ and $|Z|(z, \bar{z}; p, q)$ in $\mathcal{M}_{z, \bar{z}}$, then just one critical point

$$\left(z_{susy}(p, q), \bar{z}_{susy}(p, q)\right) = lim_{r \to r_H^+}\left(z(r), \bar{z}(r)\right) \tag{5.3.1.35}$$

exists as solution of the set of n_V complex differential Eqs. (5.3.1.32)-(5.3.1.33), and it is a global minimum for the (p, q)-parameterized continuous branch of $V_{BH}(z, \bar{z}; p, q)$ and $|Z|(z, \bar{z}; p, q)$ in $\mathcal{M}_{z, \bar{z}}$.

Clearly, the situation changes if, for the considered $Sp(2n_V + 2)$-covariant BH charge configuration $(p^\Lambda, q_\Lambda) \in \Gamma$, more than one continuous branch of $V_{BH}(z, \bar{z}; p, q)$ and $|Z|(z, \bar{z}; p, q)$ may exist in $\mathcal{M}_{z, \bar{z}}$, or also if one considers not only the continuous branch(es) of V_{BH} and/or $|Z|$. In such cases, one would obtain that a variety of critical points may exist, corresponding to (at least local) minima of V_{BH} and $|Z|$ in $\mathcal{M}_{z, \bar{z}}$, in 1 : 1 correspondence with possibly existing disconnected continuous branches of such functions, or in (not necessarily 1 : 1) correspondence with eventually existing disconnected, non-continuous branches of V_{BH} and $|Z|$.

Furthermore, by going beyond the regular SKG of $\mathcal{M}_{z, \bar{z}}$, and thus by admitting changes of definiteness of the Kählerian metric $G_{i\bar{j}}$, one would obtain various possi-

---

[7]Since the real functions $|Z|$ and V_{BH} are (strictly) positive in $\mathcal{M}_{z, \bar{z}}$, their stable critical points are (at least local) minima. Since the "attractor" points are generally defined as stable critical points, it follows that in the considered framework of regular SKG of $\mathcal{M}_{z, \bar{z}}$ all critical points of $|Z|$ and V_{BH} are actually "attractors".

In general, this does not continue to hold when the assumption of regularity of SKG is removed.

ble cases[8]: 1)

$$G_{i\bar{j}}\left(z_{susy}\left(p,q\right),\bar{z}_{susy}\left(p,q\right)\right)\ \text{strictly positive definite} \qquad (5.3.1.36)$$

$$\Updownarrow$$

$$\left.\begin{array}{l} H^{V_{BH}}_{\hat{i}\hat{j}}\left(z_{susy},\bar{z}_{susy};p,q\right) \\[2.5ex] H^{|Z|}_{\hat{i}\hat{j}}\left(z_{susy},\bar{z}_{susy};p,q\right) \end{array}\right\} \text{strictly positive definite} \qquad (5.3.1.37)$$

$$\Updownarrow$$

$\left(z_{susy}\left(p,q\right),\bar{z}_{susy}\left(p,q\right)\right) = lim_{r\to r_H^+}\left(z\left(r\right),\bar{z}\left(r\right)\right)$
(at least local) *minimum* for both V_{BH} and $|Z|$ in $\mathcal{M}_{z,\bar{z}}$
(for the considered $\left(p^{\Lambda},q_{\Lambda}\right) \in \Gamma$)
[*proper $\frac{1}{2}$-BPS supersymmetric extreme BH "attractor"*]; $\qquad (5.3.1.38)$

2)

$$G_{i\bar{j}}\left(z_{susy}\left(p,q\right),\bar{z}_{susy}\left(p,q\right)\right)\ \text{strictly negative definite} \qquad (5.3.1.39)$$

$$\Updownarrow$$

$$\left.\begin{array}{l} H^{V_{BH}}_{\hat{i}\hat{j}}\left(z_{susy},\bar{z}_{susy};p,q\right) \\[2.5ex] H^{|Z|}_{\hat{i}\hat{j}}\left(z_{susy},\bar{z}_{susy};p,q\right) \end{array}\right\} \text{strictly negative definite} \qquad (5.3.1.40)$$

$$\Updownarrow$$

$\left(z_{susy}\left(p,q\right),\bar{z}_{susy}\left(p,q\right)\right) = lim_{r\to r_H^+}\left(z\left(r\right),\bar{z}\left(r\right)\right)$
(at least local) *maximum* for both V_{BH} and $|Z|$ in $\mathcal{M}_{z,\bar{z}}$
(for the considered $\left(p^{\Lambda},q_{\Lambda}\right) \in \Gamma$); $\qquad (5.3.1.41)$

---

[8]Notice that the not strict positive (negative) definiteness of $G_{i\bar{j}}$ at the critical points $\left(z_{susy}\left(p,q\right),\bar{z}_{susy}\left(p,q\right)\right)$ is only a necessary, but not necessarily a sufficient, condition for them to be (at least local) minima (maxima) for the functions V_{BH} and $|Z|$ in $\mathcal{M}_{z,\bar{z}}$.

Indeed, when the positive (negative) definiteness of $G_{i\bar{j}}$ is not strict, explicit counterexamples may be considered in which $\left(z_{susy}\left(p,q\right),\bar{z}_{susy}\left(p,q\right)\right)$ is a saddle point for V_{BH} and $|Z|$. Thus, when the definiteness of $G_{i\bar{j}}$ is not strict, in order to discriminate between the different possibilities a more detailed investigation is needed, for instance consisting in the study of the function V_{BH} and/or $|Z|$ in a neighbourhood of the considered critical point(s) $\left(z_{susy}\left(p,q\right),\bar{z}_{susy}\left(p,q\right)\right)$.

3)

$$G_{i\bar{j}}\left(z_{susy}\left(p,q\right),\bar{z}_{susy}\left(p,q\right)\right)$$
neither positive nor negative definite
(i.e. it has some positive, some negative,
and possibly some vanishing, eigenvalues)

(5.3.1.42)

$$\Updownarrow$$

$$\left.\begin{array}{l} H_{\hat{i}\hat{j}}^{V_{BH}}\left(z_{susy},\bar{z}_{susy};p,q\right) \\ \\ H_{\hat{i}\hat{j}}^{|Z|}\left(z_{susy},\bar{z}_{susy};p,q\right) \end{array}\right\}$$ neither positive nor negative definite   (5.3.1.43)

$$\Updownarrow$$

$$\left(z_{susy}\left(p,q\right),\bar{z}_{susy}\left(p,q\right)\right)=lim_{r\to r_{H}^{+}}\left(z\left(r\right),\bar{z}\left(r\right)\right)$$
saddle point for both V_{BH} and $|Z|$ in $\mathcal{M}_{z,\bar{z}}$ (5.3.1.44)
(for the considered $\left(p^{\Lambda},q_{\Lambda}\right)\in\Gamma$).

Thus, when going beyond the regular SKG of the vector supermultiplets' moduli space $\mathcal{M}_{z,\bar{z}}$, one gets a much richer casistics, for example consisting in the possibility to have different maxima and minima, together with saddle points, also for only one (p,q)-parameterized continuous branch of the functions $V_{BH}\left(z,\bar{z};p,q\right)$ and $|Z|\left(z,\bar{z};p,q\right)$. In such a non-regular geometric framework, also disconnected and/or non-continuous branch(es) of V_{BH} and $|Z|$ might be considered.

In [77] Kallosh *et al.* performed a detailed analysis of the issue of the uniqueness of the critical points of both V_{BH} and $|Z|$, not necessarily relying on the regularity of the Kähler geometry. They worked in the framework of $\mathcal{N}=2$, $d=5$ Maxwell Einstein Supergravity Theory, whose moduli space is endowed with a "very special" (or "real special") Kähler geometry. An analogous approach in the corresponding 4-d. framework of SKG of the vector supermultiplets' moduli space $\mathcal{M}_{z,\bar{z}}$ of n_V-fold $\mathcal{N}=2$, $d=4$ Maxwell Einstein Supergravity Theory was sketchily outlined in [55].

5.3.2 Non-supersymmetric Attractors

Let us now consider the case of the non-supersymmetric, non-BPS (*NON-(BPS-)SUSY*) extreme BH "attractors". They are stable critical points of $V_{BH}\left(z,\bar{z};p,q\right)$, but *not* of $|Z|\left(z,\bar{z};p,q\right)$, in $\mathcal{M}_{z,\bar{z}}$. In the considered context of n_V-fold $\mathcal{N}=2$, $d=4$ Maxwell Einstein Supergravity Theory, their existence has been firstly pointed out

in [55]; recently, they have been rediscovered and studied in a number of papers, also in not necessarily supersymmetric frameworks ([61], [66], [71], [72], [78], [79], [80]).

The elements of such a particular class of critical points of $V_{BH}(z, \bar{z}; p, q)$ will be denoted as $(z_{non-susy}(p, q), \bar{z}_{non-susy}(p, q))$. They are Horizon, "attractor" vector supermultiplets' scalar configurations which do *not* preserve any supersymmetric degree of freedom out of the ones of the underlying n_V-fold $\mathcal{N} = 2$, $d = 4$ Maxwell Einstein Supergravity Theory. By recalling Eq. (5.3.5), they are defined by the following set of differential conditions (also remind that we assume $Z(z_{non-susy}(p, q), \bar{z}_{non-susy}(p, q); p, q) \neq 0$):

$$
\begin{cases}
\begin{cases}
(D_i V_{BH})(z_{non-susy}(p, q), \bar{z}_{non-susy}(p, q); p, q) = \\[2mm]
= \left[2\bar{Z}D_i Z + iC_{ijk}G^{j\bar{m}}G^{k\bar{k}}(\overline{D_{\bar{m}}Z})(\overline{D_{\bar{k}}Z}) \right]_{(z_{non-susy}(p,q), \bar{z}_{non-susy}(p,q))} = 0, \\[2mm]
\forall i \in \{1, ..., n_V\};
\end{cases} \\[10mm]
\begin{cases}
(D_i Z)(z_{non-susy}(p, q), \bar{z}_{non-susy}(p, q); p, q) \neq 0 \\
\quad i \in \mathcal{I} \subseteq \{1, ..., n_V\}, \mathcal{I} \neq \emptyset,
\end{cases}
\end{cases}
\tag{5.3.2.1}
$$

which may be written explicitly as follows:

$$
\begin{cases}
\begin{cases}
\begin{cases}
2\bar{Z}(z, \bar{z}; p, q) \left[\left(\partial_i + \tfrac{1}{2}\partial_i K(z, \bar{z})\right) Z(z, \bar{z}; p, q) \right] + \\[2mm]
+iC_{ijk}(z, \bar{z}) G^{j\bar{m}}(z, \bar{z}) G^{k\bar{k}}(z, \bar{z}) \cdot \\[2mm]
\cdot \left[\left(\bar{\partial}_{\bar{m}} + \tfrac{1}{2}\bar{\partial}_{\bar{m}} K(z, \bar{z})\right) \bar{Z}(z, \bar{z}; p, q) \right] \cdot \\[2mm]
\cdot \left[\left(\bar{\partial}_{\bar{k}} + \tfrac{1}{2}\bar{\partial}_{\bar{k}} K(z, \bar{z})\right) \bar{Z}(z, \bar{z}; p, q) \right]
\end{cases} = 0, \\[2mm]
\hspace{6cm} (z_{non-susy}(p,q), \bar{z}_{non-susy}(p,q)) \\[2mm]
\forall i \in \{1, ..., n_V\}; \\[4mm]
\begin{cases}
\left\{ \left(\partial_i + \tfrac{1}{2}\partial_i K(z, \bar{z})\right) Z(z, \bar{z}; p, q) \right\}_{(z_{non-susy}(p,q), \bar{z}_{non-susy}(p,q))} \neq 0; \\[2mm]
i \in \mathcal{I} \subseteq \{1, ..., n_V\}, \mathcal{I} \neq \emptyset.
\end{cases}
\end{cases}
\end{cases}
\tag{5.3.2.2}
$$

Thus, by inserting the explicit expressions of $K(z, \bar{z})$, $C_{ijk}(z, \bar{z})$ and $Z(z, \bar{z}; p, q)$ as

input[9], the set of differential conditions (5.3.2.1)-(5.3.2.2) should give, as output, the purely charge-dependent *NON-(BPS-)SUSY* extreme BH "attractors"

$$\left(z_{non-susy}\left(p,q\right),\overline{z}_{non-susy}\left(p,q\right)\right)=lim_{r\to r_H^+}\left(z\left(r\right),\overline{z}\left(r\right)\right). \tag{5.3.2.3}$$

Let us now reconsider the condition of criticality for $V_{BH}\left(z,\overline{z};p,q\right)$ in $\mathcal{M}_{z,\overline{z}}$. From Eq. (5.3.5) it reads ($\forall i \in \{1,...,n_V\}$)

$$2\overline{Z}D_iZ=-iC_{ijk}G^{j\overline{m}}G^{k\overline{k}}\left(\overline{D}_{\overline{m}}\overline{Z}\right)\left(\overline{D}_{\overline{k}}\overline{Z}\right); \tag{5.3.2.4}$$

$$\Updownarrow$$

$$D_iZ=-\frac{i}{2}\frac{Z}{|Z|^2}C_{ijk}G^{j\overline{m}}G^{k\overline{k}}\left(\overline{D}_{\overline{m}}\overline{Z}\right)\left(\overline{D}_{\overline{k}}\overline{Z}\right); \tag{5.3.2.5}$$

$$\Updownarrow$$

$$\overline{D}_{\overline{i}}\overline{Z}=\frac{i}{2}\frac{\overline{Z}}{|Z|^2}\overline{C}_{\overline{ijk}}G^{m\overline{j}}G^{k\overline{k}}\left(D_mZ\right)\left(D_kZ\right). \tag{5.3.2.6}$$

By using Eqs. (5.3.2.4)-(5.3.2.6), the evaluation of Eq. (5.3.1.16) at the critical points of $V_{BH}\left(z,\overline{z};p,q\right)$ in $\mathcal{M}_{z,\overline{z}}$ yields

$$D_iD_jV_{BH}\big|_{\partial_r V_{BH}=0,\,\forall r\in\{1,...,n_V\}}=2i\left[\begin{array}{l}2C_{ijk}G^{k\overline{k}}\overline{Z}D_{\overline{k}}\overline{Z}+\\[2mm]+\frac{1}{2}\left(D_{(i}C_{j)kl}\right)G^{k\overline{m}}G^{l\overline{l}}\left(\overline{D}_{\overline{m}}\overline{Z}\right)\overline{D}_{\overline{l}}\overline{Z}\end{array}\right]_{\partial_r V_{BH}=0,\,\forall r}$$

$$=2i\left\{\frac{\overline{Z}^2}{|Z|^2}\left(D_nZ\right)\left(D_pZ\right)\cdot\right. \tag{5.3.2.7}$$

$$\cdot\left[\begin{array}{l}i\left(\delta_i^n\delta_j^p+\delta_i^p\delta_j^n-G^{n\overline{n}}G^{p\overline{p}}R_{i\overline{n}j\overline{p}}\right)+\\[3mm]+\frac{1}{8|Z|^2}\left[D_{(i|}\left(G^{l\overline{l}}G^{n\overline{n}}G^{p\overline{p}}G^{q\overline{q}}G^{r\overline{r}}R_{|j)\overline{n}l\overline{p}}\overline{C}_{\overline{lqr}}\right)\right]\left(D_qZ\right)D_rZ\end{array}\right]_{\partial_r V_{BH}=0,\,\forall r\in\{1,...,n_V\}}\right\}.$$

In the last line of Eq. (5.3.2.7) we used the result

$$\left(D_iC_{jkl}\right)\overline{C}_{\overline{mnp}}\overline{C}_{\overline{lqr}}G^{k\overline{m}}G^{l\overline{l}}G^{n\overline{n}}G^{p\overline{p}}G^{q\overline{q}}G^{r\overline{r}}=-D_{(i|}\left(G^{l\overline{l}}G^{n\overline{n}}G^{p\overline{p}}G^{q\overline{q}}G^{r\overline{r}}R_{|j)\overline{n}l\overline{p}}\overline{C}_{\overline{lqr}}\right), \tag{5.3.2.8}$$

---

[9]It is worth recalling once again that, beside $G_{i\overline{j}}=\partial_{\overline{j}}\partial_i K$, the Kähler potential K also determines the contravariant metric by the orthonormality condition

$$G^{i\overline{j}}\partial_{\overline{j}}\partial_l K=\delta_l^i.$$

following from the Metric Postulate in $\mathcal{M}_{z,\bar{z}}$, from the SKG constraints (7.2.1.39) and from the Bianchi identities (4.1.6) for the Riemann-Christoffel tensor.

On the other hand, by means of the criticality conditions (5.3.2.4)-(5.3.2.6) of V_{BH} (holding for non-vanishing Z, as we assumed) and recalling Eq. (5.3.1.18), we can express $\overline{D}_{\bar{i}}D_j V_{BH}$ at the critical points by using either holomorphic or anti-holomorphic Kähler-covariant derivatives of Z, respectively as it follows:

$$
\overline{D}_{\bar{i}}D_j V_{BH}\big|_{\partial_r V_{BH}=0,\ \forall r\in\{1,\dots,n_V\}} =
$$

$$
= 2 \left[\begin{array}{l} G_{j\bar{i}}\,|Z|^2 + \\[2mm] + \left(2\delta_j^n \delta_{\bar{i}}^{\bar{l}} + G^{n\bar{l}}G_{j\bar{i}} - G^{l\bar{l}}G^{n\bar{k}}R_{\bar{j}i l\bar{k}}\right)(D_n Z)\,\overline{D}_{\bar{l}}\overline{Z} \end{array} \right]_{\partial_r V_{BH}=0,\ \forall r\in\{1,\dots,n_V\}} =
$$

$$(5.3.2.9)$$

$$
= 2 \left[\begin{array}{l} G_{j\bar{i}}\,|Z|^2 + \\[2mm] -\frac{i}{2}\frac{Z}{|Z|^2}\left(2\delta_j^n \delta_{\bar{i}}^{\bar{l}} + G^{n\bar{l}}G_{j\bar{i}} - G^{l\bar{l}}G^{n\bar{k}}R_{\bar{j}i l\bar{k}}\right)\cdot \\[2mm] \cdot C_{nrs} G^{r\bar{r}}G^{s\bar{s}}\left(\overline{D}_{\bar{l}}\overline{Z}\right)\left(\overline{D}_{\bar{r}}\overline{Z}\right)\overline{D}_{\bar{s}}\overline{Z} \end{array} \right]_{\partial_r V_{BH}=0,\ \forall r\in\{1,\dots,n_V\}}.
$$

Notice that the expressions (5.3.2.7) and (5.3.2.9) are manifestly symmetric, as, in general, it holds true for Eqs. (5.3.1.16), (5.3.1.24) and (5.3.1.17), (5.3.1.18).

In general, Eqs. (5.3.2.7) and (5.3.2.9) hold for every critical point of the function $V_{BH}(z,\bar{z};p,q)$ in $\mathcal{M}_{z,\bar{z}}$. In the case of $\frac{1}{2}$-*BPS-SUSY preserving* extreme BH "attractor" point(s) (which, by definition (5.3.1.2) are also critical points of the function $|Z|(z,\bar{z};p,q)$), such Eqs. reduce to the much simpler expressions (5.3.1.25) and (5.3.1.26), respectively. Thus, since we already treated the $\frac{1}{2}$-*BPS-SUSY preserving* extreme BH "attractors" in Subsubsect. 4.4.1, we will here understand Eqs. (5.3.2.7) and (5.3.2.9) in their non-trivial form in n_V-fold $\mathcal{N}=2$, $d=4$ Maxwell Einstein Supergravity Theory, i.e. evaluated at the *NON-(BPS-)SUSY* extreme BH "attractor(s)" which, by the definitions (5.3.2.1)-(5.3.2.2), are critical points of V_{BH}, but *not* of $|Z|$.

Thus, by evaluating the Hessian $H_{\hat{i}\hat{j}}^{V_{BH}}(z,\bar{z};p,q)$ at the point(s) $\left(z_{non-susy}(p,q),\bar{z}_{non-susy}(p,q)\right)$ in $\mathcal{M}_{z,\bar{z}}$ defined by the differential conditions (5.3.2.1)-

(5.3.2.2), we get

$$H_{\hat{i}\hat{j}}^{V_{BH}}\left(z_{non-susy}\left(p,q\right),\bar{z}_{non-susy}\left(p,q\right);p,q\right) =$$

$$=\begin{pmatrix} 2i\frac{\bar{Z}^2}{|Z|^2}\left(D_nZ\right)\left(D_pZ\right)\cdot \begin{bmatrix} i\begin{pmatrix} \delta_i^n\delta_j^p+\delta_i^p\delta_j^n+ \\ -G^{n\bar{n}}G^{p\bar{p}}R_{i\bar{n}j\bar{p}} \end{pmatrix}+ \\ -\frac{1}{8|Z|^2}\left(D_{(i}C_{j)kl}\right)\overline{C_{\bar{m}n\bar{p}}}\overline{C_{\bar{l}\bar{q}\bar{r}}}\cdot \\ \cdot G^{k\bar{m}}G^{l\bar{l}}G^{n\bar{n}}G^{p\bar{p}}\cdot \\ \cdot G^{q\bar{q}}G^{r\bar{r}}\left(D_qZ\right)D_rZ \end{bmatrix} & 2\begin{bmatrix} G_{i\bar{j}}|Z|^2+ \\ -\frac{i}{2}\frac{Z}{|Z|^2}\cdot \\ \begin{pmatrix} 2\delta_{\bar{j}}^{\bar{n}}\delta_i^l+G^{l\bar{n}}G_{i\bar{j}}+ \\ -G^{l\bar{l}}G^{k\bar{n}}R_{i\bar{j}k\bar{l}} \end{pmatrix}\cdot \\ \cdot C_{lmp}G^{m\bar{m}}G^{p\bar{p}}\cdot \\ \cdot\left(\overline{D_{\bar{n}}\bar{Z}}\right)\left(\overline{D_{\bar{m}}\bar{Z}}\right)\overline{D_{\bar{p}}\bar{Z}} \end{bmatrix} \\ \\ 2\begin{bmatrix} G_{j\bar{i}}|Z|^2+ \\ +\frac{i}{2}\frac{\bar{Z}}{|Z|^2}\cdot \\ \begin{pmatrix} 2\delta_j^n\delta_{\bar{i}}^{\bar{l}}+G^{n\bar{l}}G_{j\bar{i}}+ \\ -G^{l\bar{l}}G^{n\bar{k}}R_{j\bar{i}l\bar{k}} \end{pmatrix}\cdot \\ \cdot\overline{C_{\bar{l}\bar{m}\bar{p}}}G^{m\bar{m}}G^{p\bar{p}}\cdot \\ \cdot\left(D_nZ\right)\left(D_mZ\right)D_pZ \end{bmatrix} & -2i\frac{Z^2}{|Z|^2}\left(\overline{D_{\bar{n}}\bar{Z}}\right)\left(\overline{D_{\bar{p}}\bar{Z}}\right)\cdot \begin{bmatrix} -i\begin{pmatrix} \delta_{\bar{i}}^{\bar{n}}\delta_{\bar{j}}^{\bar{p}}+\delta_{\bar{i}}^{\bar{p}}\delta_{\bar{j}}^{\bar{n}}+ \\ -G^{n\bar{n}}G^{p\bar{p}}R_{n\bar{i}p\bar{j}} \end{pmatrix}+ \\ -\frac{1}{8|Z|^2}\left(\overline{D_{(\bar{i}}\bar{C}_{\bar{j})\bar{k}\bar{l}}}\right)C_{mnp}C_{lqr}\cdot \\ \cdot G^{m\bar{k}}G^{l\bar{l}}G^{n\bar{n}}G^{p\bar{p}}\cdot \\ \cdot G^{q\bar{q}}G^{r\bar{r}}\left(\overline{D_{\bar{q}}\bar{Z}}\right)\overline{D_{\bar{r}}\bar{Z}} \end{bmatrix} \end{pmatrix}_{non-susy}$$

where the subscript "$non-susy$" means that everything inside the matrix is evaluated at the point(s) $\left(z_{non-susy}\left(p,q\right),\bar{z}_{non-susy}\left(p,q\right)\right)$ in $\mathcal{M}_{z,\bar{z}}$ defined by the differential conditions (5.3.2.1)-(5.3.2.2).

It is worth pointing out that at the critical points of V_{BH} the Kähler-covariant Hessian of V_{BH} coincides with the "flat", ordinary Hessian, defined through ordi-

nary derivatives:

$$
H_{\hat{\imath}\hat{\jmath}}^{V_{BH}}(z,\bar{z};p,q)\Big|_{\partial_r V_{BH}=0,\ \forall r\in\{1,...,n_V\}} = \begin{pmatrix} H_{ij}^{V_{BH}} & H_{i\bar{\jmath}}^{V_{BH}} \\ \\ H_{\bar{\jmath}i}^{V_{BH}} & H_{\bar{\imath}\bar{\jmath}}^{V_{BH}} \end{pmatrix}_{\partial_r V_{BH}=0,\ \forall r\in\{1,...,n_V\}} \equiv
$$
(5.3.2.10)

$$
\equiv \begin{pmatrix} D_i D_j V_{BH} & D_i \overline{D}_{\bar{\jmath}} V_{BH} \\ \\ D_j \overline{D}_{\bar{\imath}} V_{BH} & \overline{D}_{\bar{\imath}} \overline{D}_{\bar{\jmath}} V_{BH} \end{pmatrix}_{\partial_r V_{BH}=0,\ \forall r\in\{1,...,n_V\}}
$$

This is clearly due to the fact that the (regular) Special Kähler moduli space $\mathcal{M}_{z,\bar{z}}$ is linearly connected (see Eqs. (5.3.1.19) and (5.3.1.20)).

Now, by knowing the explicit expressions of the functions $Z(z,\bar{z};p,q)$, $K(z,\bar{z})$ and $C_{ijk}(z,\bar{z})$, and by solving the differential conditions (5.3.2.1)-(5.3.2.2), one should explicitly calculate the Hessian $H_{\hat{\imath}\hat{\jmath}}^{V_{BH}}\left(z_{non-susy}(p,q),\bar{z}_{non-susy}(p,q);p,q\right)$ given on the last page and study, case by case (if more than one solution exists to (5.3.2.1)-(5.3.2.2)), the definiteness of such an Hessian, i.e. the sign of its eigenvalues.

By denoting $H_{\hat{\imath}\hat{\jmath}}^{V_{BH}}\left(z_{non-susy}(p,q),\bar{z}_{non-susy}(p,q);p,q\right) \equiv H_{\hat{\imath}\hat{\jmath},non-susy}^{V_{BH}}(p,q)$, it is immediate to check that the Hessian $H_{\hat{\imath}\hat{\jmath},non-susy}^{V_{BH}}(p,q)$ is not Hermitian

$$
\left(H_{\hat{\imath}\hat{\jmath},non-susy}^{V_{BH}}(p,q)\right)^{\dagger} = \begin{pmatrix} H_{\bar{\imath}\bar{\jmath},non-susy}^{V_{BH}}(p,q) & H_{i\bar{\jmath},non-susy}^{V_{BH}}(p,q) \\ \\ H_{\bar{\jmath}i,non-susy}^{V_{BH}}(p,q) & H_{ij,non-susy}^{V_{BH}}(p,q) \end{pmatrix} \neq
$$

$$
\neq \begin{pmatrix} H_{ij,non-susy}^{V_{BH}}(p,q) & H_{i\bar{\jmath},non-susy}^{V_{BH}}(p,q) \\ \\ H_{\bar{\jmath}i,non-susy}^{V_{BH}}(p,q) & H_{\bar{\imath}\bar{\jmath},non-susy}^{V_{BH}}(p,q) \end{pmatrix} =
$$

$$
= H_{\hat{\imath}\hat{\jmath},non-susy}^{V_{BH}}(p,q).
$$
(5.3.2.11)

Such a non-Hermiticity is, in general, due to the diagonal terms of the above block-diagonal arrangement, i.e. essentially to the $n_V \times n_V$ matrix $H_{ij,non-susy}^{V_{BH}}(p,q)$, which is symmetric but, in general, not real, and therefore not Hermitian (see also [72]).

Let us now evaluate the "BH effective potential" function V_{BH} at its critical points. By using Eqs. (5.3.4), (5.3.5) and (5.3.2.4)-(5.3.2.6), one gets that the (semiclassical,

leading order) BH entropy reads

$$S_{BH} = \pi \left. V_{BH} \right|_{\partial_r V_{BH}=0, \, \forall r \in \{1,...,n_V\}} =$$

$$= \pi \left\{ |Z|^2 + G^{i\bar{i}} \left(D_i Z \right) \overline{D}_{\bar{i}} \overline{Z} \right\}_{\partial_r V_{BH}=0, \, \forall r \in \{1,...,n_V\}} = \tag{5.3.2.12}$$

$$= \pi \left\{ \begin{array}{l} |Z|^2 + \\ \\ + \frac{1}{4|Z|^2} C^{\bar{i}\bar{m}\bar{k}} \overline{C}_{\bar{i}}^{mn} \left(\overline{D}_{\overline{m}} \overline{Z} \right) \left(\overline{D}_{\bar{k}} \overline{Z} \right) \left(D_m Z \right) D_n Z \end{array} \right\}_{\partial_r V_{BH}=0, \, \forall r \in \{1,...,n_V\}} .$$

Thus, the (semiclassical, leading order) BH entropy at the *NON-(BPS-)SUSY* extreme BH "attractor(s)" is

$$S_{BH,non-susy} \equiv S_{BH} \left(z_{non-susy} \left(p,q \right), \overline{z}_{non-susy} \left(p,q \right); p,q \right) =$$

$$= \pi \left\{ |Z|^2 + \frac{1}{4|Z|^2} \left| C_{ijk} G^{j\overline{m}} G^{k\overline{k}} \left(\overline{D}_{\overline{m}} \overline{Z} \right) \overline{D}_{\bar{k}} \overline{Z} \right|^2 \right\}_{non-susy} , \tag{5.3.2.13}$$

where the subscript "*non − susy*" in the r.h.s. has the same meaning as above. $\left| C_{ijk} G^{j\overline{m}} G^{k\overline{k}} \left(\overline{D}_{\overline{m}} \overline{Z} \right) \overline{D}_{\bar{k}} \overline{Z} \right|^2$ is the square norm of the complex, Kähler gauge-invariant covariant vector $C_{ijk} G^{j\overline{m}} G^{k\overline{k}} \left(\overline{D}_{\overline{m}} \overline{Z} \right) \overline{D}_{\bar{k}} \overline{Z}$ in $\mathcal{M}_{z,\bar{z}}$. Since we assume the SKG of $\mathcal{M}_{z,\bar{z}}$ to be regular, i.e. that the metric tensor $G_{i\bar{j}}$ is strictly positive definite in all $\mathcal{M}_{z,\bar{z}}$, it holds true that

$$\left| C_{ijk} G^{j\overline{m}} G^{k\overline{k}} \left(\overline{D}_{\overline{m}} \overline{Z} \right) \overline{D}_{\bar{k}} \overline{Z} \right|^2 \equiv C^{\bar{i}\bar{m}\bar{k}} \overline{C}_{\bar{i}}^{mn} \left(\overline{D}_{\overline{m}} \overline{Z} \right) \left(\overline{D}_{\bar{k}} \overline{Z} \right) \left(D_m Z \right) D_n Z \geqslant 0, \tag{5.3.2.14}$$

vanishing iff

$$C_{ijk} G^{j\overline{m}} G^{k\overline{k}} \left(\overline{D}_{\overline{m}} \overline{Z} \right) \overline{D}_{\bar{k}} \overline{Z} = 0, \forall i \in \{1,...,n_V\} . \tag{5.3.2.15}$$

Notice that the condition (5.3.2.15) is trivially satisfied at the $\frac{1}{2}$-*BPS-SUSY preserving* extreme BH "attractor" point(s) defined by the differential conditions (5.3.1.2). However, it might happen also that, depending on the BH charge configuration $\left(p^\Lambda, q_\Lambda \right) \in \Gamma$ and on the explicit expressions of C_{ijk}, K and Z, condition (5.3.2.15) is satisfied at some particular *NON-(BPS-)SUSY* extreme BH "attractor(s)".

Thus, by recalling Eq. (5.3.1.15) one gets that the BH entropy $S_{BH,non-susy}$ at the *NON-(BPS-)SUSY* extreme BH "attractor(s)" is larger than the entropy $S_{BH,susy}$ at the $\frac{1}{2}$-*BPS-SUSY preserving* extreme BH "attractor" point(s) (having the same $|Z|^2$). In other words, by assuming

$$|Z|^2 \left(z_{susy} \left(p,q \right), \overline{z}_{susy} \left(p,q \right); p,q \right) =$$

$$= |Z|^2 \left(z_{non-susy} \left(p,q \right), \overline{z}_{non-susy} \left(p,q \right); p,q \right) \equiv |Z|^2_{cr} \left(p,q \right), \tag{5.3.2.16}$$

it holds that

$$\Delta\left(p,q\right) \equiv S_{BH,non-susy} - S_{BH,susy} =$$

$$= \frac{\pi}{4}\left[\frac{1}{|Z|^2}C^{\bar{i}\bar{m}\bar{k}}\bar{C}_{\bar{i}}{}^{mn}\left(\overline{D}_{\overline{m}}\overline{Z}\right)\left(\overline{D}_{\overline{k}}\overline{Z}\right)\left(D_m Z\right)D_n Z\right]_{non-susy} \geqslant 0.$$

(5.3.2.17)

The above expressions can be further elaborated by using the SKG constraints expressed by Eq. (7.2.1.39), yielding

$$C_{pil}\overline{C}_{\overline{p}\overline{j}\overline{m}}G^{p\overline{p}} = G_{i\overline{j}}G_{l\overline{m}} + G_{i\overline{m}}G_{l\overline{j}} - R_{i\overline{j}l\overline{m}}.$$

(5.3.2.18)

Consequently, at the *NON-(BPS-)SUSY* extreme BH "attractor(s)" it holds that

$$\left[G^{i\bar{i}}\left(D_i Z\right)\overline{D}_{\bar{i}}\overline{Z}\right]_{non-susy} =$$

$$= \left[\frac{1}{4}\frac{G^{i\bar{i}}}{|Z|^2}C_{ilm}\overline{C}_{\overline{i}\overline{n}\overline{p}}G^{l\bar{l}}G^{m\overline{m}}G^{n\overline{n}}G^{p\overline{p}}\left(\overline{D}_{\bar{l}}\overline{Z}\right)\left(\overline{D}_{\overline{m}}\overline{Z}\right)\left(D_n Z\right)D_p Z\right]_{non-susy} =$$

(5.3.2.19)

$$= \left\{\frac{1}{4|Z|^2}\left[2\left(G^{i\bar{i}}D_i Z\overline{D}_{\bar{i}}\overline{Z}\right)^2 - R_{l\overline{n}m\overline{p}}\left(\overline{D}^l\overline{Z}\right)\left(\overline{D}^m\overline{Z}\right)\left(D^n Z\right)D^p Z\right]\right\}_{non-susy}.$$

Now, by recalling that in a (commutative) Kähler manifold the completely covariant Riemann-Christoffel tensor $R_{i\overline{j}l\overline{m}}$ is given by Eq. (4.1.8) and the SKG constraints may correspondingly be rewritten as in Eqs. (4.1.9)-(4.1.11), the obtained result may be further elaborated by writing

$$\left[G^{i\bar{i}}D_i Z\overline{D}_{\bar{i}}\overline{Z}\right]_{non-susy} =$$

$$= \left\{\frac{1}{4|Z|^2}\left\{2\left(G^{i\bar{i}}D_i Z\overline{D}_{\bar{i}}\overline{Z}\right)^2 + \left[\begin{array}{c}\overline{\partial}_{\overline{n}}\partial_l\overline{\partial}_{\overline{p}}\partial_m K + \\ -G^{r\overline{s}}\left(\overline{\partial}_{\overline{s}}\partial_l\partial_m K\right)\left(\partial_r\overline{\partial}_{\overline{n}}\overline{\partial}_{\overline{p}}K\right)\end{array}\right]\left(\overline{D}^l\overline{Z}\right)\left(\overline{D}^m\overline{Z}\right)\left(D^n Z\right)D^p Z\right\}\right\}_{non-susy}.$$

(5.3.2.20)

Summarizing, in the (regular) SKG of $\mathcal{M}_{z,\overline{z}}$, the following expressions for the (semi-classical, leading order) BH entropy $S_{BH} = \pi V_{BH}$ at the *NON-(BPS-)SUSY* extreme

BH "attractor(s)" are equivalent:

$$S_{BH,non-susy} = \pi \left[|Z|^2 + G^{i\bar{i}} D_i Z \overline{D}_{\bar{i}} \overline{Z} \right]_{non-susy} =$$

$$= \pi \left\{ \begin{array}{l} |Z|^2 + \\ \\ + \frac{1}{4} \frac{G^{i\bar{i}}}{|Z|^2} C_{ijk} \overline{C}_{\bar{i}\bar{j}\bar{n}} G^{j\bar{m}} G^{k\bar{k}} G^{m\bar{j}} G^{n\bar{n}} \left(\overline{D}_{\bar{m}} \overline{Z} \right) \left(\overline{D}_{\bar{k}} \overline{Z} \right) (D_m Z) D_n Z \end{array} \right\}_{non-susy} =$$

$$= \pi \left\{ \begin{array}{l} |Z|^2 + \\ \\ + \frac{1}{4|Z|^2} \left[2 \left(G^{i\bar{i}} D_i Z \overline{D}_{\bar{i}} \overline{Z} \right)^2 - R_{l\bar{n}m\bar{p}} \left(\overline{D}^l \overline{Z} \right) \left(\overline{D}^m \overline{Z} \right) \left(D^{\bar{n}} Z \right) D^{\bar{p}} Z \right] \end{array} \right\}_{non-susy} =$$

$$\text{(5.3.2.21)}$$

$$= \pi \left\{ \begin{array}{l} |Z|^2 + \frac{1}{4|Z|^2} G^{l\bar{l}} G^{m\bar{m}} \left(\overline{D}_{\bar{l}} \overline{Z} \right) \left(\overline{D}_{\bar{m}} \overline{Z} \right) \cdot \\ \\ \cdot \left[2 (D_l Z) D_m Z - R_{l\bar{n}m\bar{p}} G^{n\bar{n}} G^{p\bar{p}} (D_n Z) D_p Z \right] \end{array} \right\}_{non-susy} =$$

$$= \pi \left\{ \begin{array}{l} |Z|^2 + \frac{1}{4|Z|^2} G^{l\bar{l}} G^{m\bar{m}} \left(\overline{D}_{\bar{l}} \overline{Z} \right) \left(\overline{D}_{\bar{m}} \overline{Z} \right) \cdot \\ \\ \cdot \left\{ \begin{array}{l} 2 (D_l Z) D_m Z + \\ \\ - \left[\overline{\partial}_{\bar{n}} \partial_l \overline{\partial}_{\bar{p}} \partial_m K - G^{r\bar{s}} \left(\overline{\partial}_{\bar{s}} \partial_l \partial_m K \right) \left(\partial_r \overline{\partial}_{\bar{n}} \overline{\partial}_{\bar{p}} K \right) \right] \cdot \\ \\ \cdot G^{n\bar{n}} G^{p\bar{p}} (D_n Z) D_p Z \end{array} \right\} \end{array} \right\}_{non-susy} .$$

In the following two chapters, we will present a self-contained analysis of our work highlighting the interplay between Special Kähler geometry, Supergravity, Black Holes and Attractor Mechanism which are commonly dubbed as "On Quantum Special Kähler Geometry" 6 and "Topics in Cubic Special Geometry" 7.

Bibliography

[1] M. K. Gaillard, B. Zumino, *"Duality Rotations for Interacting Fields"*, Nucl. Phys. **B193**, 221 (1981).

[2] P. Frè, *"Lectures on Special Kähler Geometry and Electric-Magnetic Duality Rotations"*, Lectures given at ICTP Workshop on String Theory, Gauge Theory and Quantum Gravity, Trieste, Italy, 5-7 Apr 1995, Nucl. Phys. Proc. Suppl. **45BC**, 59 (1996), `hep-th/9512043`.

[3] T. Levi-Civita, R.C. Acad. Lincei **26**, 519 (1917).

[4] B. Bertotti, *"Uniform Electromagnetic Field in the Theory of General Relativity"*, Phys. Rev. **116**, 1331 (1959).

[5] I. Robinson, Bull. Acad. Polon. **7**, 351 (1959).

[6] P. Claus, M. Derix, R. Kallosh, J. Kumar, P.K. Townsend and A. Van Proeyen, *"Black Holes and Superconformal Mechanics"*, Phys. Rev. Lett. **81**, 4553 (1998), `hep-th/9804177`.

[7] S. Bellucci, A. Galajinsky, E. Ivanov and S. Krivonos, *"$AdS_2 \times S^2$ Canonical Transformations and Superconformal Mechanics"*, Phys. Lett. **B555**, 99 (2003), `hep-th/0212204`.

[8] S. Bellucci, A. Galajinsky, E. Ivanov and S. Krivonos, *"$AdS_2 \times S^2$ AdS/CFT Equivalence Transformation"*, Phys. Rev. **D66**, 086001 (2002); Erratum-ibid. **D67**, 049901 (2003), `hep-th/0206126`.

[9] G. W. Gibbons, in *"Supersymmetry, Supergravity and Related Topics"*, eds. F. del Aguila, J. de Azcárraga and L. Ibáñez (World Scientific, Singapore 1985), p. 147.

[10] S. Ferrara, *"Bertotti-Robinson Geometry and Supersymmetry"*, talk given at the 12th Italian Conference on General Relativity and Gravitational Physics, Sept. 1996, `hep-th/9701163`.

[11] R. Kallosh, *"Supersymmetric Black Holes"*, Phys. Lett. **B282**, 80 (1992),

`hep-th/9201029`.

[12] R. Kallosh and A. Peet, *"Dilaton Black Holes near the Horizon"*, Phys. Rev. **D46**, 5223 (1992), `hep-th/9209116`.

[13] D. A. Lowe and A. Strominger, *"Exact Four-Dimensional Dyonic Black Holes and Bertotti-Robinson Space-Times in String Theory"*, Phys. Rev. Lett. **73**, 1468 (1994), `hep-th/9403186`.

[14] G. W. Gibbons and P. Townsend, *"Vacuum Interpolation in Supergravity via Super p-Branes"*, Phys. Rev. Lett. **71**, 3754 (1993), `hep-th/9307049`.

[15] S. Ferrara and J. M. Maldacena, *"Branes, central charges and U-duality invariant BPS conditions"*, Class. Quant. Grav. **15**, 749 (1998), `hep-th/9706097`.

[16] L. Frappat, P. Sorba and A. Sciarrino, *"Dictionary on Lie Superalgebras"*, `hep-th/9607161`.

[17] W. Nahm, V. Rittenberg, M. Scheunert, *"The Classification of Graded Lie Algebras"*, Phys. Lett. **B61**, 383 (1976).

[18] W. Nahm, V. Rittenberg, M. Scheunert, *"Classification of all Simple Graded Lie Algebras whose Lie Algebra is Reductive. 1"*, J. Math. Phys. **17**, 1626 (1976).

[19] W. Nahm, V. Rittenberg, M. Scheunert, *"Classification of all Simple Graded Lie Algebras whose Lie Algebra is Reductive. 2. Construction of the Exceptional Lie Algebras "*, J. Math. Phys. **17**, 1640 (1976).

[20] W. Nahm, V. Rittenberg, M. Scheunert, *"Graded Lie Algebras : Generalization of Hermitian Representations"*, J. Math. Phys. **18**, 146 (1977).

[21] F. R. Tangherlini, Nuovo Cimento **27**, 636 (1963).

[22] R. C. Myers and M. J. Perry, *"Black Holes in Higher Dimensional Space-Times"*, Ann. Phys. **172**, 304 (1986).

[23] J. D. Bekenstein, *"Black Holes and Entropy"*, Phys. Rev. **D7**, 2333 (1973).

[24] S. W. Hawking, Phys. Rev. Lett. **26**, 1344 (1971); in C. DeWitt and B. S. DeWitt, *"Black Holes"* (Les Houches 1972), Gordon and Breach, New York 1973.

[25] S. W. Hawking, Nature **248**, 30 (1974).

[26] S. W. Hawking, Comm. Math. Phys. **43**, 199 (1975).

[27] S. Ferrara, R. Kallosh and A. Strominger, *"$\mathcal{N} = 2$ Extremal Black Holes"*, Phys. Rev. **D52**, 5412 (1995), `hep-th/9508072`.

[28] A. Strominger, *" Macroscopic Entropy of $\mathcal{N} = 2$ Extremal Black Holes"*, Phys. Lett. **B383**, 39 (1996), `hep-th/9602111`.

[29] R. Kallosh and A. Linde, *"Strings, Black Holes, and Quantum Information"*, `hep-th/0602061`.

[30] S. Ferrara and R. Kallosh, *" Supersymmetry and Attractors"*, Phys. Rev. **D54**, 1514 (1996), `hep-th/9602136`.

[31] G. Moore, *"Attractors and Arithmetic"*, `hep-th/9807056`.

[32] G. Moore, *"Arithmetic and Attractors"*, `hep-th/9807087`.

[33] G. Moore, *"Les Houches Lectures on Strings and Arithmetic"*, `hep-th/0401049`.

[34] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Frè and T. Magri, *"$\mathcal{N} = 2$ Supergravity and $\mathcal{N} = 2$ Super Yang-Mills Theory on General Scalar Manifolds: Symplectic Covariance, Gaugings and the Momentum Map"*, J. Geom. Phys. **23**, 111 (1997), `hep-th/9605032`.

[35] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, and P. Frè, *"General Matter Coupled $\mathcal{N} = 2$ Supergravity"*, Nucl. Phys. **B476**, 397 (1996), `hep-th/9603004`.

[36] A. Strominger, *"Special Geometry"*, Commun. Math. Phys. **133**, 163 (1990).

[37] E. J. Flaherty, *"Hermitian and Kählerian Geometry in Relativity"*, Springer-Verlag, 1976.

[38] A. Ceresole, R. D'Auria, S. Ferrara and A. Van Proeyen, *"Duality Transformations in Supersymmetric Yang-Mills Theories coupled to Supergravity"*, Nucl. Phys. **B444**, 92 (1995) , `hep-th/9502072`.

[39] B. de Wit and A. Van Proeyen, *"Potentials and Symmetries of General Gauged $\mathcal{N} = 2$ Supergravity - Yang-Mills Models"*, Nucl. Phys. **B245**, 89 (1984).

[40] B. Craps, F. Roose, W. Troost and A. Van Proeyen, *"The Definitions of Special Geometry"*, `hep-th/9606073`.

[41] B. Craps, F. Roose, W. Troost and A. Van Proeyen, *"What is Special Kähler Geometry?"*, Nucl. Phys. **B503**, 565 (1997), `hep-th/9703082`.

[42] E. Cremmer, C. Kounnas, A. Van Proeyen, J. P. Derendinger, S. Ferrara, B. de Wit and L. Girardello, *"Vector Multiplets coupled to $\mathcal{N} = 2$ Supergravity : SuperHiggs Effect, Flat Potentials and Geometric Structure"*, Nucl. Phys. **B250**, 385 (1985).

[43] B. de Wit, P. G. Lauwers and A. Van Proeyen, *"Lagrangians of $\mathcal{N} = 2$ Supergravity - Matter Systems"*, Nucl. Phys. **B255**, 569 (1985).

[44] L. Castellani, R. D'Auria and S. Ferrara, *" Special Geometry without Special Coordinates"*, Class. Quant. Grav. **7**, 1767 (1990).

[45] S. Ferrara and A. Strominger, *"$\mathcal{N} = 2$ Spacetime Supersymmetry and Calabi-Yau Moduli Space"*, in Proceedings of College Station Workshop *"Strings 89"*, eds. E. Arnowitt et al. (World Scientific, Singapore 1989), p. 245.

[46] P. Candelas and X. C. de la Ossa, *"Moduli Space of Calabi-Yau Manifolds"*, Nucl. Phys. **B355**, 455 (1991).

[47] L. Castellani, R. D'Auria and S. Ferrara, *"Special Kähler Geometry : an Intrinsic Formulation from $\mathcal{N} = 2$ Space-Time Supersymmetry"*, Phys. Lett. **B241**, 57 (1990).

[48] A. Ceresole, R. D'Auria, S. Ferrara, W. Lerche and J. Louis, *"Picard-Fuchs Equations and Special Geometry"*, Int. J. Mod. Phys. **A8**, 79 (1993).

[49] A. Ceresole, R. D'Auria, S. Ferrara and A. Van Proeyen, *"On Electromagnetic Duality in Locally Supersymmetric $\mathcal{N} = 2$ Yang–Mills Theory"*, contributed to Joint U.S.-Polish Workshop on Physics from Planck Scale to Electro-Weak Scale (SUSY 94), Warsaw, Poland, 21-24 Sep 1994, `hep-th/9412200`.

[50] C. M. Hull and P. K. Townsend, *"Unity of Superstring Dualities"*, Nucl. Phys. **B438**, 109 (1995), `hep-th/9410167`.

[51] A. Ceresole, R. D'Auria and S. Ferrara, *"The Symplectic Structure of $\mathcal{N} = 2$ Supergravity and its Central Extension"*, talk given at ICTP Trieste Conference on Physical and Mathematical Implications of Mirror Symmetry in String Theory, Trieste, Italy, 5-9 Jun 1995, Nucl. Phys. Proc. Suppl. **46**, 67 (1996), `hep-th/9509160`.

[52] R. Kallosh, *"New Attractors"*, `hep-th/0510024`.

[53] A. Giryavets, *"New Attractors and Area Codes"*, `hep-th/0511215`.

[54] S. Ferrara, M. Bodner and A. C. Cadavid, *"Calabi-Yau Supermoduli Space, Field Strength Duality and Mirror Manifolds"*, Phys. Lett. **B247**, 25 (1990).

[55] S. Ferrara, G. W. Gibbons and R. Kallosh, *"Black Holes and Critical Points in Moduli Space"*, Nucl. Phys. **B500**, 75 (1997), `hep-th/9702103`.

[56] L. J. Dixon, V. Kaplunovsky and J. Louis, *"On Effective Field Theories describing $(2, 2)$ Vacua of the Heterotic String"*, Nucl. Phys. **B329**, 27 (1990).

[57] P. Breitenlohner, D. Maison and G. W. Gibbons, *"Four-Dimensional Black Holes from Kaluza-Klein Theories"*, Commun. Math. Phys. **120**, 295 (1988).

[58] G. W. Gibbons, *"Antigravitating Black Hole Solitons with Scalar Hair in $\mathcal{N} = 4$ Supergravity"*, Nucl. Phys. **B207**, 337 (1982).

[59] G. W. Gibbons, R. Kallosh and B. Kol, *"Moduli, Scalar Charges, and the First Law of Black Hole Thermodynamics"*, Phys. Rev. Lett. **77**, 4992 (1996),

`hep-th/9607108`.

[60] D. A. Rasheed, *"Nonlinear Electrodynamics : Zeroth and First Laws of Black Hole Mechanics"*, `hep-th/9702087`.

[61] R. Kallosh, N. Sivanandam and M. Soroush, *"The Non-BPS Black Hole Attractor Equation"*, `hep-th/0602005`.

[62] J. Scherk, *"Antigravity: a Crazy Idea?"*, Phys. Lett. **B88**, 265 (1979).

[63] J. Scherk, in *"Supergravity"*, eds. D. Freedman and P. van Nieuwenhuizen, North-Holland, Amsterdam, 1979.

[64] G. W. Gibbons and C. M. Hull, *"A Bogomol'ny Bound for General Relativity and Solitons in $\mathcal{N} = 2$ Supergravity"*, Phys. Lett. **B109**, 190 (1982).

[65] P. Frè, *"Supersymmetry and First Order Equations for Extremal States: Monopoles, Hyperinstantons, Black-Holes and p-Branes"*, invited Seminar given at Santa Margherita Conference on Contrained Dynamics and Quantum Gravity, September 1995, Nucl. Phys. Proc. Suppl. **57**, 52 (1997), `hep-th/9701054`.

[66] K. Goldstein, N. Iizuka, R. P. Jena and S. P. Trivedi, *"Non-Supersymmetric Attractors"*, `hep-th/0507096`.

[67] R. Kallosh and A. W. Peet, *"Dilaton Black Holes near the Horizon"*, Phys. Rev. **D46**, 5223 (1992), `hep-th/9209116`.

[68] R. Kallosh, T. Ortin and A. W. Peet, *"Entropy and Action of Dilaton Black Holes"*, Phys. Rev. **D47**, 5400 (1993), `hep-th/9211015`.

[69] G.W. Gibbons and R. Kallosh, *"Topology, Entropy and Witten Index of Dilaton Black Holes"*, Phys. Rev. **D51**, 2839 (1995), `hep-th/9407118`.

[70] E. Bergshoeff, R. Kallosh and T. Ortin, *"Stationary Axion/Dilaton Solutions and Supersymmetry"*, Nucl. Phys. **B478**, 156 (1996), `hep-th/9605059`.

[71] K. Goldstein, R. P. Jena, G. Mandal and S. P. Trivedi, *"A C-Function for Non-Supersymmetric Attractors"*, `hep-th/0512138`.

[72] S. Bellucci, S. Ferrara and A. Marrani, *"On some properties of the Attractor Equations"*, `hep-th/0602161`.

[73] B. de Wit and A. Van Proeyen, *"Broken Sigma Model Isometries in Very Special Geometry"*, Phys. Lett. **B293**, 94 (1992), `hep-th/9207091`.

[74] R. D'Auria, S. Ferrara and P. Frè, *"Special and Quaternionic Isometries : General Couplings in $\mathcal{N} = 2$ Supergravity and the Scalar Potential"*, Nucl. Phys. **B359**, 705 (1991).

[75] B. de Wit and A. Van Proeyen, *"Isometries of Special Manifolds"*, Proceedings of the Meeting on Quaternionic Structures in Mathematics and Physics, Trieste, September 1994; ILAS/FM-6/1996, p.109, `hep-th/9505097`.

[76] B. de Wit and A. Van Proeyen, *"Special Geometry and Symplectic Transformations"*, talk given at Spring School and Workshop on String Theory, Gauge Theory and Quantum Gravity, Trieste, Italy, 11-22 Apr 1994, Nucl. Phys. Proc. Suppl. **45BC**, 196 (1996), `hep-th/9510186`.

[77] A. Chou, R. Kallosh, J. Rahmfeld, S.-J. Rey, M. Shmakova and W. K. Wong, *"Critical Points and Phase Transitions in 5-D Compactifications of M Theory"*, Nucl. Phys. **B508**, 147 (1997), `hep-th/9704142`.

[78] P. Tripathy and S. P. Trivedi, *"Non-Supersymmetric Attractors in String Theory"*, `hep-th/0511117`.

[79] M. Alishahiha, H. Ebrahim, *"Non-Supersymmetric Attractors and Entropy Function"*, `hep-th/0601016`.

[80] J. P. Hsu, A. Maloney and A. Tomasiello, *"Black Hole Attractors and Pure Spinors"*, `hep-th/0602142`.

[81] K. P. Tod, Phys. Lett. **121B**, 241 (1983).

Chapter 6

Quantum Special Kähler Geometry

In [1] we compute the effective black hole potential V_{BH} of the most general $\mathcal{N} = 2$, $d = 4$ (*local*) special Kähler geometry with quantum perturbative corrections, consistent with axion-shift Peccei-Quinn symmetry and with cubic leading order behavior.

We determine the charge configurations supporting axion-free attractors, and explain the differences among various configurations in relations to the presence of "flat" directions of V_{BH} at its critical points and also elucidate the role of the sectional curvature at the non-supersymmetric critical points of V_{BH}, and compute the Riemann tensor (and related quantities), as well as the so-called *E*-tensor. The latter expresses the non-symmetricity of the considered quantum perturbative special Kähler geometry.

6.1 Introduction

The *Attractor Mechanism* was discovered in the mid 90's [2]- [7] in the context of dynamics of scalar fields coupled to BPS (Bogomol'ny-Prasad-Sommerfeld [7]) black holes (BHs). In recent years, a number of studies (see *e.g.* [8]-[13] for recent reviews, and lists of Refs., see also [13]) have been devoted to the investigation of the properties of *extremal BH attractors*. This renewed interest can be essentially be traced back to the (re)discovery of new classes configurations of scalar fields at the BH horizon, which do not saturate the BPS bound. When embedded into a supergravity theory, such non-BPS configurations break all supersymmetries at the BH event horizon.

The geometry of the scalar manifold determines the various classes of BPS and non-BPS attractors. The richest case study is provided by the theory in which the

Attractor Mechanism was originally discovered, i.e. by $\mathcal{N} = 2$, $d = 4$ *ungauged* supergravity coupled to n_V Abelian vector multiplets. In such a theory, the scalar fields coordinatize a Kähler manifold of (*local*) *special* type (see *e.g.* [14], [15], and [16], and Refs. therein), determined by an holomorphic prepotential function $\mathcal{F}$. In general, (*local*, as understood throughout unless otherwise noted) special Kähler (SK) geometry admits three classes of extremal BH attractors (see *e.g.* [17] for an analysis in *symmetric* SK geometry):

- $\frac{1}{2}$-BPS (preserving four supersymmetries out of the eight pertaining to asymptotical $\mathcal{N} = 2$, $d = 4$ superPoincaré algebra);

- non-BPS with non-vanising $\mathcal{N} = 2$ central charge function Z (shortly named non-BPS $Z \neq 0$);

- non-BPS with vanishing Z (shortly named non-BPS $Z = 0$).

6.1.1   Quantum Corrections to Prepotential

Dealing with the stringy origin of $\mathcal{N} = 2$, $d = 4$ supergravity, the classical prepotential F receives quantum (perturbative and non-perturbative) corrections, of polynomials or non-polynomial (usually polylogarithmic) nature, which in some cases can spoil the holomorphicity of $\mathcal{F}$ itself (see *e.g.* [18]- [31]).

A typical (and simple) example is provided by the *large volume limit* of CY_3-compactifications of Type IIA superstrings, which determines a SK geometry with *purely cubic* $\mathcal{F}$ at the classical level. Thus, the sub-leading nature of the quantum corrections constrains the most general polynomial correction to $\mathcal{F}$ to be at most of degree two in the moduli, with a priori complex coefficients. Moreover, some symmetries can further constrain the structure of such sub-leading polynomial quantum corrections to $\mathcal{F}$. As shown in [32], the only polynomial quantum perturbative correction to classical cubic $\mathcal{F}$ which is consistent with the perturbative (continuous) *axion-shift symmetry* [33] is the constant purely imaginary term ($i = 1, ..., n_V$ throughout):

$$\mathcal{F}_{class} = \frac{1}{3!}d_{ijk}z^i z^j z^k \longrightarrow \mathcal{F}_{quant-pert.} = \frac{1}{3!}d_{ijk}t^i t^j t^k + i\xi, \ \xi \in \mathbb{R}, \qquad (6.1.1.1)$$

where d_{ijk} is the real, constant, completely symmetric tensor defining the cubic geometry (which is then usually named d-SK geometry [34; 35]). All other polynomial perturbative corrections (quadratic, linear and real constant terms in the scalar fields

$z^{i'}$s) can be proved not to affect the classical d-SK geometry, also because the Kähler potential is insensitive to their presence [32].

The explicit form of the quantum corrections to $\mathcal{F}$ depends on the superstring theory under consideration, and non-trivial relations among the various corrections arise due to the (perturbative and non-perturbative) dualities relating the various superstring theories.

For instance, in a certain class of compactifications of the heterotic $E_8 \times E_8$ superstring over $K_3 \times T^2$, the whole quantum-corrected $\mathcal{F}$ reads (see *e.g.* [19], and Refs. therein)

$$
\begin{aligned}
\mathcal{F}^{het} &= stu - s \sum_{a=4}^{n_V} \left(\widetilde{t}^a\right)^2 + h_1\left(t, u, \widetilde{t}\right) + f_{non-pert.}\left(e^{-2\pi s}, t, u, \widetilde{t}\right), \\
z^1 &\equiv s, \ z^2 \equiv t, \ z^3 \equiv u, \ z^a \equiv \widetilde{t}^a.
\end{aligned}
\tag{6.1.1.2}
$$

This compactifications exhibits the peculiar feature that the axio-dilaton s belongs to a vector multiplet, and this determines the presence of (T-symmetric) quantum perturbative string-loop corrections and non-perturbative corrections, as well. The tree-level, classical term

$$
\mathcal{F}^{het}_{class} = stu - s \sum_{a=4}^{n_V} \left(\widetilde{t}^a\right)^2
\tag{6.1.1.3}
$$

is the prepotential of the so-called *generic Jordan sequence*

$$
\frac{SU(1,1)}{U(1)} \times \frac{SO(2, n_V - 1)}{SO(2) \times SO(n_V - 1)}
\tag{6.1.1.4}
$$

of homogeneous *symmetric* SK manifolds (see *e.g.* [32] and [12; 17], and Refs. therein). Notice that $\mathcal{F}^{het}_{class}$ given by Eq. (6.1.1.3) exhibits its maximal (non-compact) symmetry, i.e. $SO(1, n_V - 2)$, pertaining to its $d = 5$ uplift. Non-renormalization theorems state that all quantum perturbative string-loop corrections are encoded in the 1-loop contribution h_1, made out of a constant term, a purely cubic polynomial term and a polylogarithmic part (see *e.g.* [19] and Refs. therein). Finally, $f_{non-pert.}$ encodes the non-perturbative corrections, exponentially suppressed in the limit $s \to \infty$ (see *e.g.* [19; 22; 31; 36]).

As mentioned above, superstring dualities play a key role in relating the quantum corrected $\mathcal{F}$'s in various theories. In our framework, the Type IIA/heterotic duality allows for an identification of the relevant scalar fields (*moduli* of the geometry of the internal manifold in stringy language) such that the heterotic prepotential (6.1.1.2) becomes structurally identical to the one determined by Type IIA compactifications over Calabi-Yau threefolds (CY_3s). Within this latter framework, the $\mathcal{F}$

governing the resulting low-energy $\mathcal{N} = 2$, $d = 4$ supergravity is of purely classical origin. Indeed, there are only Kähler structure *moduli*, and the axio-dilaton s belongs to an hypermultiplet; this leads to no string-loop corrections, and all corrections to the *large volume limit* cubic prepotential come from the world-sheet sigma-model [37]. In particular, as shown in [36; 38], there are no 1-, 2- and 3-loop contributions. It is here worth pointing out that the non-perturbative, world-sheet instanton corrections (which we will disregard in the treatment below) spoil the continuous nature of the axion-shift symmetry, by making it discrete [33].

Thus, the relevant part of the prepotential $\mathcal{F}$ in Type IIA compactifications reads ($n_V = h_{1,1}$) [19; 36; 39]:

$$\mathcal{F}^{IIA} = \frac{1}{3!} \mathcal{C}_{ijk} t^i t^j t^k + \mathcal{W}_{0i} t^i - i \frac{\chi \zeta(3)}{16\pi^3}. \qquad (6.1.1.5)$$

The $\mathcal{C}_{ijk}$ are the real classical intersection numbers, determining the classical d-SK geometry in the *large volume limit*. On the other hand, the quantum perturbative contributions from 2-dimensional CFT on the world-sheet are encoded only in a linear and in a constant term:

- the *linear* term is determined by

$$\mathcal{W}_{0i} = \frac{1}{4!} c_2 \cdot J_i = \frac{1}{4!} \int_{CY_3} c_2 \wedge J_i, \qquad (6.1.1.6)$$

  which are the real expansion coefficients of the second Chern class c_2 of CY_3 with respect to the basis J_i^* of the cohomology group $H^4(CY_3, \mathbb{R})$, *dual* to the basis of the $(1,1)$-forms J_i of the cohomology $H^2(CY_3, \mathbb{R})$. The linear term $\mathcal{W}_{0i} t^i$ has been shown to be reabsorbed by a suitable symplectic transformation of the period vector; thus, in the dual heterotic picture it has just the effect of a *constant shift* in Ims ([40]; see *e.g.* also discussion in [19]).

- The constant term $-i\frac{\chi\zeta(3)}{16\pi^3} (\equiv i\xi$ in Eq. (6.1.1.1)) in (6.1.1.5) is the only relevant one, as proved in general in [32]. It is determined by the Riemann zeta-function ζ, by the Euler character[1] χ of CY_3, and it has a 4-loop origin in the non-linear sigma-model [36–38]. It is worth noticing that $\chi = 0$ for *self-mirror* CY_3's, such that all have $\xi = 0$. Furthermore, some arguments lead to argue

---

[1]For typical CY_3's

$$|\chi| \leqslant 10^3 \Leftrightarrow \frac{\chi\zeta(3)}{16\pi^3} \sim \mathcal{O}(1).$$

This motivates the statement that attractor solutions with $\xi = 0$ can be (in certain BH charge configurations) a good approximation for the solutions computed with $\xi \neq 0$ (see *e.g.* the remark after Eq. (3.42) of [19]).

that (*at least*) for some particular *self-mirror* models (such as the so-called FHSV one [41] and the octonionic magic [42]), non-perturbative, world-sheet instanton corrections vanish, as well (see e.g. discussion in Sects. 12 and 13 of [43], and Refs. therein). As a consequence, such models, up to suitable symplectic transformations of the period vector, would have their classical *cubic* prepotential unaffected by any perturbative and non-perturbative correction.

It should be here pointed out that CY_3-compactifications of Type IIB do not admit a large volume limit; moreover, in Type IIB the *Attractor Eqs.* only depend on the complex structure *moduli* (which in supergravity description are the scalars of the $\mathcal{N} = 2$ vector multiplets). The solutions to $\mathcal{N} = 2, d = 4$ *Attractor Eqs.* for the resulting SK geometries were studied (in proximity of the Landau-Ginzburg point) in [44] for the particular class of Fermat CY_3's with $n_V = 1$, and in [45] for a particular CY_3 with $n_V = 2$.

In [46], extending the BPS analysis of [19], the $\mathcal{N} = 2, d = 4$ *Attractor Eqs.* were studied in the simplest case of perturbative quantum corrected *d*-SK geometry, i.e. in the SK geometry with $n_V = 1$ scalar fields, described (in a *special coordinates*) by the holomorphic Kähler gauge-invariant prepotential[2]

$$\mathcal{F} = t^3 + i\xi, \tag{6.1.1.7}$$

which, up to overall rescaling, is nothing but $\mathcal{F}_{quant-pert.}$ of Eq. (6.1.1.1) for $n_V = 1$. Despite the (apparently) minor correction to the classical prepotential, in [46] new phenomena, absent in the classical limit $\xi = 0$, were observed:

- The *"separation"* of attractors, i.e. the existence of multiple stable solutions to the Attractor Eqs. (for a given BH charge configuration). This can be ultimately related to the existence of *basins of attractions* [47–49], specified by suitable *"area codes"* [50] (*i.e.* asymptotical boundary conditions) in the radial evolution dynamics of scalar fields in the extremal BH background.

- The *"transmutation"* of attractors, i.e. the change in the supersymmetry preserving features of stable critical points of V_{BH}, depending on the value of the quantum parameter ξ, suitably *"renormalized"* in terms of the relevant BH charges. For example, by varying such a *"renormalized"* quantum parameter, a $\frac{1}{2}$-BPS attractor becomes non-BPS (and vice versa). This can ultimately be related to the lack of an orbit structure in the space of Bh charges; this is no surprise, by noticing that the SK geometry determined by $\mathcal{F}$ given by Eq. (6.1.1.7) is generally not symmetric nor homogeneous.

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[2]In [46] ξ was named λ.

6.1.2   Critical *"Flat"* Directions of Black Hole Potential

Let us now shortly recall the fundamentals of the *Attractor Mechanism*. In the critical implementation given in [6], the *Attractor Mechanism related* extremal BH attractors to *stable* critical points[3] z_H^i of a suitably defined BH effective potential V_{BH}:

$$z_H(Q) : \left.\frac{\partial V_{BH}(z,Q)}{\partial z^i}\right|_{z=z_H(Q)} = 0 \qquad (6.1.2.1)$$

where Q denotes the $Sp(2n_V + 2, \mathbb{R})$-vector of magnetic and electric BH charges (see Eq. (6.2.0.9) below). The n_V complex Eqs. (6.1.2.1) are usually called *Attractor Equations*. Then, a critical point $z_H(Q)$ is an *attractor* in strict sense *iff* the (Hermitian) $2n_V \times 2n_V$ Hessian matrix $\mathcal{H}^{V_{BH}}$ of V_{BH} evaluated at $z_H(Q)$ is *positive definite*:

$$\left.\mathcal{H}^{V_{BH}}\right|_{z=z_H(Q)} \geqslant 0, \qquad (6.1.2.2)$$

with "$\geqslant 0$" here expressing the non-negativity of the $2n_V$ eigenvalues.

As shown in [51; 52], in $\mathcal{N} = 2$, $d = 4$ *ungauged* supergravities with homogeneous (not necessarily symmetric) SK manifolds (as well as in $\mathcal{N} > 2$-extended *ungauged* $d = 4$ supergravities, which we however do not consider here) the critical matrix $\left.\mathcal{H}^{V_{BH}}\right|_{\partial V_{BH}=0}$ has the following general signature: all strictly positive eigenvalues, up to some eventual vanishing eigenvalues (*massless Hessian modes*), which have been proved to be *"flat"* directions of V_{BH} itself.

Thus, one can claim that in all homogeneous SK geometries the critical points of V_{BH} satisfying the "non-degeneracy" condition

$$V_{BH}|_{\partial V_{BH}=0} \neq 0 \qquad (6.1.2.3)$$

are all *stable*, up to some eventual *"flat"* directions. Such directions of the SK scalar manifold $\mathcal{M}_{SK}$ coordinatize the so-called *moduli space* $\mathfrak{M} \subsetneq \mathcal{M}$ of the considered (class of) solution(s) to Eqs. (6.1.2.1). In other words, such *"flat"* directions span a subset of the scalar fields which is *not* stabilized by the Attractor Eqs. (6.1.2.1) at the BH event horizon in terms of the BH charges Q. It is worth pointing out that, somewhat surprisingly, the existence of *"flat"* directions at the critical points of V_{BH} does *not* plague the thermodynamical macroscopic description of *extremal* BHs with inconsistencies. Indeed, at the considered class of critical points, V_{BH} does *not* actually turn out to depend on the unstabilized scalars; therefore, through the relation [6]

$$S_{BH}(Q) = \pi V_{BH}|_{\partial V_{BH}=0}, \qquad (6.1.2.4)$$

---

[3]The subscript *"H"* denotes the value at the event horizon of the considered extremal BH.

the BH entropy S_{BH} can be consistently defined. Notice that the condition (6.1.2.3) implies the (classical) *Attractor Mechanism*[4] to work only for the so-called *"large"* BHS, *i.e.* for those BHs with non-vanishing classical entropy.

As known since [6], *"flat"* directions cannot arise at $\frac{1}{2}$_BPS critical points of V_{BH}. This is no more true for the remaining two classes of non-supersymmetric critical points, namley for non-BPS $Z \neq 0$ and non-BPS $Z = 0$ ones [51; 52]. Tables 2 and 3 of [52] respectively list the *moduli spaces* of non-BPS $Z \neq 0$ and non-BPS $Z = 0$ attractors for symmetric SK geometries, whose classification is known after [54] (see also [35] and [32], as well as Refs. therein). Let us mention that non-BPS $Z \neq 0$ *moduli spaces* are nothing but the symmetric *real special* scalar manifolds of the corresponding $\mathcal{N} = 2$, $d = 5$ supergravity.

6.1.3 Quantum Removal/Survival of Critical *"Flat"* Directions

It should be pointed out clearly that the issue of the *"flat"* directions of V_{BH} at its critical points, reported in Subsect. 6.1.2 hold only at the classical, Einstein supergravity level. It is conceivable that such *"flat"* directions are removed by quantum (perturbative *and/or* non-perturbative) corrections. Consequently, at the quantum (perturbative and/or non-perturbative) regime, *no moduli spaces for attractor solutions might exist at all* (and also the actual attractive nature of the critical points of V_{BH} might be destroyed). *However, this might not be the case for $\mathcal{N} = 8$*, or for some particular charge configurations in $\mathcal{N} < 8$ supergravities (see below).

By relating the issues reported in Subsects. 6.1.1 and 6.1.2, one might thus ask about the fate of classical *"flat"* directions of V_{BH} at its (non-BPS) critical points, in presence of quantum (perturbative *and/or* non-perturbative) corrections to the prepotential $\mathcal{F}$ of SK geometry.

This issue, crucial in order to understand the features of the *Attractor Mechanism* in the quantum regime (and thus its consistent embedding in the high-energy theories whose supergravity is an effective low-energy limit, i.e. superstrings and *M*-theory), was started to be investigated in [55], and it is the object of the investigation carried out in the present paper.

Let us start by recalling the simplest symmetric *d*-SK geometries, and their eventual non-BPS *"flat"* directions. For our purpose, it will suffice to consider only the

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[4]*Attractor Mechanism* can be consistently implemented at the quantum level, *at least* in some frameworks, for instance within the so-called entropy function formalism (see *e.g.* [9] and Refs. therein, see also [13]). See also [53] (and Refs. therein) for recent developments concerning *Attractor Mechanism* and higher derivatives corrections to Einstein (super)gravity theories.

so-called t^3 and st^2 models:

- The t^3 model is based on the rank-1 symmetric SK manifold

$$\frac{SU(1,1)}{U(1)},$$

(6.1.3.1)

endowed with prepotential ($z^1 \equiv t$, *Imt* < 0)

$$\mathcal{F} = t^3,$$

(6.1.3.2)

which is the classical limit $\xi \to 0$ of Eq. (6.1.1.7). As yielded by the analysis of [54], it is an isolated case in the classification of symmetric SK geometries (see also [32]). Furthermore, such a model can also be conceived as the "$s = t = u$ *degeneration*" of the so-called *stu* model [56]-[63], or equivalently as the "$s = t$ *degeneration*" of the so-called st^2 model (see below). Beside the $\frac{1}{2}$-BPS attractors, the t^3 model (whose $d = 5$ uplift is *pure* $\mathcal{N} = 2$, $d = 5$ supergravity) admits only non-BPS $Z \neq 0$ critical points of V_{BH} with no *"flat"* directions (and thus no associated *moduli space*) [17; 52].

- The st^2 model is based on the rank-2 symmetric SK manifold

$$\left(\frac{SU(1,1)}{U(1)}\right)^2,$$

(6.1.3.3)

endowed with prepotential ($z^1 \equiv s$, $z^2 \equiv t$, *Ims* < 0, *Imt* < 0)

$$\mathcal{F} = st^2.$$

(6.1.3.4)

It has one non-BPS $Z \neq 0$ *"flat"* direction, spanning the *moduli space SO* $(1,1)$ (i.e., the scalar manifold of the st^2 model in $d = 5$), but *no* non-BPS $Z = 0$ *"flat"* directions at all. Such a model is the smallest (*i.e.* the fewest-moduli) *symmetric* model exhibiting a non-BPS $Z \neq 0$ *"flat"* direction. Remarkably, the st^2 model constitutes the unique example of homogeneous *d*-SK geometry with $n_V = 2$ scalar fields [34; 35]. Furthermore, as evident from the structure of the cubic norm (see *e.g.* the discussion in [35], as well as Eq. (3.2.3) and Sect. 5 of [64]), the st^2 model is the unique $n_V = 2$ SK geometry to be uplifted to anomaly-free *pure* $(1,0)$, $d = 6$ supergravity (at least in presence of neutral matter).

As mentioned at the end of Subsect. 6.1.1, the non-homogeneous model "$t^3 + i\xi$" (with prepotential given by Eq. (6.1.1.7)) was studied in [46]. The "$t^3 + i\xi$" model can be conceived as the prototype of quantum perturbative corrected SK geometry,

because it is the $n_V = 1$ SK geometry with the most general quantum perturbative correction consistent with the (continuous, perturbative) axion-shift symmetry [32]. However, since the t^3 model has no non-BPS *"flat"* directions at all, the study performed in [46] is not relevant for the aforementioned issue of the fate of the *moduli spaces* of classical attractors in the quantum regime.

From the above analysis, the st^2 model is the simplest example of SK geometry in which the study of the fate of classical non-BPS $Z \neq 0$ moduli space can be investigated in quantum perturbative regime, i.e. considering the "$st^2 + i\xi$" model, whose prepotential in special coordinates reads

$$\mathcal{F} = st^2 + i\xi. \qquad (6.1.3.5)$$

Notice that Eq. (6.1.3.5) is the unique homogeneous $n_V = 2$ determination of Eq. (6.1.1.1). Such a study was performed in [55], within the (supergravity analogues of the) so-called *magnetic* $(D0 - D4)$, *electric* $(D2 - D6)$ and $D0 - D6$ BH charge configurations. As somewhat intuitively expected, in the magnetic and electric configurations the classical non-BPS $Z \neq 0$ moduli space $SO(1,1)$ was shown not to survive after the introduction of the quantum parameter $\xi \neq 0$. Interestingly, the investigation of [55] showed the that the quantum removal of classical *"flat"* directions occurs more often towards *repeller* directions (thus destabilizing the whole critical solution, and *destroying the attractor in strict sense*), rather than towards *attractive* directions.

Surprisingly, the study of [55] also revealed that the $D0 - D6$ configuration exhibits a qualitatively different phenomenon, i.e. that the non-BPS $Z \neq 0$ classical *"flat"* direction *survives* the considered quantum perturbative corrections effectively encoded in the "$+i\xi$" term in Eq.(6.1.3.5), despite acquiring a *non-vanishing* axionic part.

Aim and Plan of the Chapter

This unexpected fact was not completely understood in [55], and it is the starting point of the present investigation, which aims at thoroughly investigating, within the effective BH potential formalism, the d-SK geometries with the most general quantum perturbative correction consistent with continuous Peccei-Quinn axion-shift symmetry, i.e. the SK geometries with prepotential (in *special coordinates*) given by Eq. (6.1.1.1). As already found in the simple cases investigated in [46] ($n_V = 1$) and [55] ($n_V = 2$), the Attractor Eqs. (especially the non-supersymmetric ones) cannot be solved analytically for a generic BH charge configuration, because they turn out to be algebraic Eqs. of high (> 4) order. However, by explicitly computing V_{BH}

for the prepotential (6.1.1.1), we will explain the peculiarity of the $D0 - D6$ configuration as being , in presence of $\xi \neq 0$, somewhat the *"minimal"* configuration which does not support *axion-free* attractor solutions. In light of new results concerning the relation between the so-called *sectional curvature of matter charges* at the (non-supersymmetric) critical points of V_{BH} and the BH entropy S_{BH}, we will then compute the relevant tensors characterizing the quantum SK geometry (6.1.1.1), i.e. the Riemann tensor and related contractions, and the *E*-tensor.

The plan of the chapter is as follows.

In Sect. 6.2 we explicitly compute the effective BH potential V_{BH} for the most general quantum perturbatively corrected SK geometry consistent with continuous axion-shift symmetry, i.e. the one with prepotential (6.1.1.1), in general form, *i.e.* for an arbitrary number n_V of vector multiplets and for a generic configuration Q of BH charges. We then determine the *axion-free*-supporting Bh charge configurations, commenting on the role of $D0 - D6$, and (partially) explaining the findings of [55].

Sect. 6.3 is devoted to the computation of the $\mathcal{N} = 2$ central charge Z and the related *matter charges $D_i Z$* in the considered framework. Such a computations allows one to draw some general statements on the $\frac{1}{2}$-BPS solutions, connecting to the few results already known from literature [19].

In Sect. 6.4 the role of the so-called *E*-tensor in SK geometry (and in the *Attractor Mechanism* within) is recalled, and its explicit computation for the geometry (6.1.1.1) is presented. By performing the classical limit $\xi \to 0$, the *E*-tensor for a generic *d*-SK geometry is explicitly obtained. The factorizability of some functional dependences for the classical *E*-tensor is explicitly found, highlighting the possibility to uplift the theory to $d = 5$. The same does not happen when $\xi \neq 0$, thus confirming the well known fact that only *d*-SK geometry admits an uplift to $d = 5$ (see *e.g.* [65] and Refs. therein).

Then, in Sect. 6.5 a number of original results are derived, pointing out the role of the so-called *sectional curvature of matter charges* $\mathcal{R}$ in the theory of non-supersymmetric attractors. Indeed, $\mathcal{R}$ vanishes at $\frac{1}{2}$-BPS attractors, but it is proportional to the critical value of V_{BH} (and thus, through Eq. (6.1.2.4), to the BH entropy S_{BH}). In particular, in symmetric SK geometries it has the same sign of the quartic invariant $\mathcal{I}_4$ at non-BPS $Z \neq 0$ critical points, where as it is opposite to (the double of) $\mathcal{I}_4$ at non-BPS $Z = 0$ critical points, thus being strictly negative in both cases.

Since $\mathcal{R}$ is nothing but the contraction of the Riemann tensor with the matter charges vectors (*i.e.* with covariant derivatives of Z itself), interesting role of $\mathcal{R}$ at

non-supersymmetric critical points of V_{BH} elucidated in Sect. 6.5 calls for an explicit computation of the Riemann tensor itself. This is carried out in Sect. 6.6, where also the Ricci tensor and the Ricci scalar curvature are determined. We proceed by exploiting two different approaches, one merely based on Kähler geometry (Subsect. 6.6.1) and the second one (Subsect. 6.6.2) based instead on the fundamental constraints of SK geometry (see Eq. (6.4.0.5) below). We explicitly show the equivalence of these two approaches, by shortly commenting on the results of [54] and on the eventual (unlikely) Einstein nature of the SK geometries (6.1.1.1).

Finally, Sect. 6.7 makes a brief comment and outlook, and lists some of the various open issues, originated or highlighted by the present investigation, which we leave for future study.

6.2 Effective Black Hole Potential

As recalled in previous Section and as firstly found in [32], the most general holomorphic prepotential with leading *cubic* behavior consistent with (perturbative, continuous) Peccei-Quinn axion-shift symmetry [33], and which affects the Kähler potential K of SK geometry, reads

$$F(X;\xi) = \frac{1}{3!}d_{ijk}\frac{X^i X^j X^k}{X^0} + i\xi(X^0)^2,$$
(6.2.0.1)

which is nothing but Eq. (6.1.1.1) before projectivizing, and before switching to *special coordinates* and suitably fixing the Kähler gauge (see below). Let us recall once again that $i = 1, ..., n_V$ throughout (n_V denoting the number of Abelian vector multiplets coupled to the $\mathcal{N} = 2$, $d = 4$ supergravity one), and $\xi \in \mathbb{R}$.

Aim of the present Section is to compute the effective BH potential V_{BH} for the SK geometry determined by the holomorphic prepotential (6.2.0.1). Below we will present only the main formulæ, addressing the reader to Appendix A of [1] for the further details of the calculations.

A general formula determining the kinetic vector matrix $\mathcal{N}_{\Lambda\Sigma}$ reads (see *e.g.* [66]) ($\Lambda = 0, 1, ..., n_V$ throughout)

$$\mathcal{N}_{\Lambda\Sigma} = \overline{\mathcal{F}}_{\Lambda\Sigma} + 2i\frac{Im\left(\mathcal{F}_{\Lambda\Omega}\right) Im\left(\mathcal{F}_{\Sigma\Delta}\right) X^\Omega X^\Delta}{Im\left(\mathcal{F}_{\Theta\Xi}\right) X^\Theta X^\Xi}.$$
(6.2.0.2)

After projectivizing, it is convenient to switch to the so-called *special* coordinates (see *e.g.* [15] and Refs. therein), defined by ($a = 1, ..., n_V$)

$$e_i^a(z) \equiv \frac{\partial \left(\frac{X^a}{X^0}\right)}{\partial z^i} \equiv \delta_i^a,$$
(6.2.0.3)

where (x^i, $\lambda^i \in \mathbb{R}$)

$$z^i \equiv x^i - i\lambda^i \tag{6.2.0.4}$$

are the n_V complex scalar fields, and further suitably fix the Kähler gauge as

$$X^0 \equiv 1. \tag{6.2.0.5}$$

Within such a framework, one can thus write:

$$Im\left[\mathcal{F}_{\Lambda\Sigma}(z;\xi)\right] = \begin{pmatrix} \frac{1}{3}d_{ijk}Im\left(z^iz^jz^k\right) + 2\xi & -\frac{1}{2}d_{jkl}Im\left(z^kz^l\right) \\ \\ -\frac{1}{2}d_{ikl}Im\left(z^kz^l\right) & d_{ijk}Im\left(z^k\right) \end{pmatrix}, \tag{6.2.0.6}$$

and the block components of $\mathcal{N}_{\Lambda\Sigma}$ are computed in Appendix A of [1]

In order to compute the V_{BH} governing the *Attractor Mechanism* [2–6], it is worth recalling that in $\mathcal{N} = 2$, $d = 4$ *ungauged* Maxwell-Einstein supergravity the following expression holds [4; 5; 15]:

$$V_{BH} = |Z|^2 + g^{j\bar{j}}\left(D_jZ\right)\overline{D}_{\bar{j}}\overline{Z}, \tag{6.2.0.7}$$

where Z is the $\mathcal{N} = 2$ *central charge* function. On the other hand, an equivalent (and independent of the number of supercharge generators) expression of V_{BH} reads [6]

$$V_{BH} = -\frac{1}{2}Q^T\mathcal{M}\left(\mathcal{N}\right)Q. \tag{6.2.0.8}$$

Q is the $(Sp\left(2n_V + 2, \mathbb{R}\right))$-vector of magnetic and electric charges, which in the *special coordinate* basis of $\mathcal{N} = 2$ theory reads as follows:

$$Q = \begin{pmatrix} p^0 \\ p^i \\ q_0 \\ q_i \end{pmatrix}. \tag{6.2.0.9}$$

The $(2n_V + 2) \times (2n_V + 2)$ real symmetric symplectic matrix $\mathcal{M}\left(\mathcal{N}\right)$ is defined as [4; 5; 15]

$$\mathcal{M}\left(\mathcal{N}\right) = \mathcal{M}\left(Re\left(\mathcal{N}\right), Im\left(\mathcal{N}\right)\right) \equiv$$

$$\equiv \begin{pmatrix} Im\left(\mathcal{N}\right) + Re\left(\mathcal{N}\right)\left(Im\left(\mathcal{N}\right)\right)^{-1}Re\left(\mathcal{N}\right) & -Re\left(\mathcal{N}\right)\left(Im\left(\mathcal{N}\right)\right)^{-1} \\ \\ -\left(Im\left(\mathcal{N}\right)\right)^{-1}Re\left(\mathcal{N}\right) & \left(Im\left(\mathcal{N}\right)\right)^{-1} \end{pmatrix}. \tag{6.2.0.10}$$

Thus, in order to compute V_{BH} for the $\mathcal{N} = 2$, $d = 4$ specified by the (perturbative) quantum corrected holomorphic prepotential (6.2.0.1), one has to compute the inverse of matrix $Im\mathcal{N}_{\Lambda\Sigma}$.

It is also convenient to further simplify the notation, by recalling the definitions used in [65], and suitably changing them[5] (taking into account the presence of effective quantum parameter ξ):

$$d_{ij} \equiv d_{ijk}\lambda^k; \tag{6.2.0.11}$$

$$d_i \equiv d_{ijk}\lambda^j\lambda^k; \tag{6.2.0.12}$$

$$v \equiv \frac{1}{3!}d_{ijk}\lambda^i\lambda^j\lambda^k; \tag{6.2.0.13}$$

$$\widetilde{v} \equiv v + \frac{1}{4}\xi; \tag{6.2.0.14}$$

$$h_{ij} \equiv d_{ijk}x^k; \tag{6.2.0.15}$$

$$h_i \equiv d_{ijk}x^jx^k; \tag{6.2.0.16}$$

$$h \equiv d_{ijk}x^ix^jx^k, \tag{6.2.0.17}$$

thus *e.g.* yielding

$$h_{ij}\lambda^i\lambda^j = d_ix^i. \tag{6.2.0.18}$$

By further introducing *"rescaled dilatons"* [65]

$$\widehat{\lambda}^i \equiv \frac{\lambda^i}{v^{1/3}} \Rightarrow \frac{1}{3!}d_{ijk}\widehat{\lambda}^i\widehat{\lambda}^j\widehat{\lambda}^k = 1, \tag{6.2.0.19}$$

one can then define the following quantities:

$$\widehat{d}_{ij} \equiv d_{ijk}\widehat{\lambda}^k = v^{-1/3}d_{ij}; \tag{6.2.0.20}$$

$$\widehat{d}_i \equiv d_{ijk}\widehat{\lambda}^j\widehat{\lambda}^k = v^{-2/3}d_i. \tag{6.2.0.21}$$

Let us also recall Eq. (31) of [46], giving the expression of covariant metric tensor $g_{i\bar{j}}$ for the prepotential (6.2.0.1) within the assumptions (6.2.0.3)-(6.2.0.5):

$$g_{i\bar{j}} = g_{ij} = -\frac{1}{4(v - \frac{1}{2}\xi)}\left[d_{ij} - \frac{d_id_j}{4(v - \frac{1}{2}\xi)}\right] = -\frac{v^{1/3}}{4(v - \frac{1}{2}\xi)}\left[\widehat{d}_{ij} - \frac{v\widehat{d}_i\widehat{d}_j}{4(v - \frac{1}{2}\xi)}\right]. \tag{6.2.0.22}$$

---

[5]Notice that in [65] a different notation was used, i.e.:

$$\kappa_{ij} \equiv d_{ijk}\lambda^k;$$

$$\kappa_i \equiv d_{ijk}\lambda^j\lambda^k;$$

$$\kappa \equiv d_{ijk}\lambda^i\lambda^j\lambda^k = 6v;$$

$$\kappa^{ij}\kappa_{jl} \equiv \delta_l^i.$$

The corresponding inverse metric ($g^{ij}g_{jk} \equiv \delta^i_k$) is computed as

$$g^{\bar{i}\bar{j}} = g^{ij} = -4(\nu - \frac{1}{2}\xi)\left[d^{ij} - \frac{\lambda^i\lambda^j}{2(\nu + \xi)}\right] = -4(\nu - \frac{1}{2}\xi)\left[\nu^{-1/3}\widehat{d}^{ij} - \frac{\nu^{2/3}\widehat{\lambda}^i\widehat{\lambda}^j}{2(\nu + \xi)}\right]$$

$$(6.2.0.23)$$

where

$$d^{ij}d_{jk} \equiv \delta^i_k \Leftrightarrow \widehat{d}^{ij}\widehat{d}_{jk} \equiv \delta^i_k. \tag{6.2.0.24}$$

The limit $\xi \to 0$ consistently yields the analogue results for d-SKG, given by Eqs. (2.4) and (2.6) of [65]:

$$\lim_{\xi \to 0} g_{ij} = -\frac{1}{4}\nu^{-2/3}\left(\widehat{d}_{ij} - \frac{\widehat{d}_i\widehat{d}_j}{4}\right) \equiv \breve{g}_{ij}; \tag{6.2.0.25}$$

$$\lim_{\xi \to 0} g^{ij} = 2\nu^{2/3}\left(\widehat{\lambda}^i\widehat{\lambda}^j - 2\widehat{d}^{ij}\right) \equiv \breve{g}^{ij}, \quad \breve{g}^{ij}\breve{g}_{jk} \equiv \delta^i_k, \tag{6.2.0.26}$$

where $\breve{g}_{ij}$ and $\breve{g}^{ij}$ denote the covariant and contravariant *classical* ($\xi \to 0$) metric tensor.

After a lengthy but straightforward computations (see details in Appendix A of

[1]), the following explicit expression of V_{BH} is achieved:

$$V_{BH}\left(x^j,\widehat{\lambda}^j,v;Q,\xi\right) = \frac{1}{2\tilde{v}}\left(1-\left(\tfrac{3}{4}\right)^2\tfrac{\xi^2}{\tilde{v}^2}\right)^{-1}\cdot$$

$$\cdot\left[\begin{array}{l}\left[\begin{array}{l}\tilde{v}^2\left(1-\left(\tfrac{3}{4}\right)^2\tfrac{\xi^2}{\tilde{v}^2}\right)^2+\tfrac{h^2}{36}-\tfrac{1}{8}\xi\tfrac{v^{2/3}}{\tilde{v}}h\widehat{d}_ix^i+\\[2mm]+\left(\tfrac{3}{8}\right)^2\xi^2\tfrac{v^{4/3}}{\tilde{v}^2}\left(\widehat{d}_ix^i\right)^2-\tilde{v}^2\left(1-\left(\tfrac{3}{4}\right)^2\tfrac{\xi^2}{\tilde{v}^2}\right)12\mathcal{A}+\\[2mm]-\tfrac{1}{12}\left(1-\left(\tfrac{3}{4}\right)^2\tfrac{\xi^2}{\tilde{v}^2}\right)\mathcal{A}^{ij}\left(\tfrac{1}{4}h_ih_j+\tfrac{3}{8}\xi\tfrac{v^{2/3}}{\tilde{v}}h_i\widehat{d}_j+\left(\tfrac{3}{8}\right)^2\xi^2\tfrac{v^{4/3}}{\tilde{v}^2}\widehat{d}_i\widehat{d}_j\right)\end{array}\right](p^0)^2+\\[20mm]+2\left[\begin{array}{l}-\tfrac{h}{12}h_i+\tfrac{1}{16}\xi\tfrac{v^{2/3}}{\tilde{v}}h\widehat{d}_i-\left(\tfrac{3}{8}\right)^2\xi^2\tfrac{v^2}{\tilde{v}^2}(\widehat{d}_jx^j)\widehat{d}_i+\\[2mm]+\tfrac{3}{16}\xi\tfrac{v^{2/3}}{\tilde{v}}(\widehat{d}_jx^j)h_i+\tilde{v}^2\left(1-\left(\tfrac{3}{4}\right)^2\tfrac{\xi^2}{\tilde{v}^2}\right)12\mathcal{A}_i+\\[2mm]+\tfrac{1}{12}\left(1-\left(\tfrac{3}{4}\right)^2\tfrac{\xi^2}{\tilde{v}^2}\right)\mathcal{A}^{kl}\left(\tfrac{1}{2}h_{ik}h_l+\tfrac{3}{8}\xi\tfrac{v^{2/3}}{\tilde{v}}h_{ik}\widehat{d}_l\right)\end{array}\right]p^0p^i+\\[20mm]+\left[\begin{array}{l}\tfrac{1}{4}h_ih_j-\tfrac{3}{16}\xi\tfrac{v^{2/3}}{\tilde{v}}(h_i\widehat{d}_j+h_j\widehat{d}_i)+\left(\tfrac{3}{8}\right)^2\xi^2\tfrac{v^{4/3}}{\tilde{v}^2}\widehat{d}_i\widehat{d}_j+\\[2mm]-\tilde{v}^2\left(1-\left(\tfrac{3}{4}\right)^2\tfrac{\xi^2}{\tilde{v}^2}\right)12\mathcal{A}_{ij}+\\[2mm]-\tfrac{1}{12}\left(1-\left(\tfrac{3}{4}\right)^2\tfrac{\xi^2}{\tilde{v}^2}\right)\mathcal{A}^{kl}h_{ik}h_{jl}\end{array}\right]p^ip^j+\\[20mm]+\left(\tfrac{h}{3}-\tfrac{3}{4}\xi\tfrac{v^{2/3}}{\tilde{v}}\widehat{d}_ix^i\right)q_0p^0+\\[6mm]+2\left[\begin{array}{l}\tfrac{h}{6}x^i-\tfrac{3}{8}\xi\tfrac{v^{2/3}}{\tilde{v}}\widehat{d}_jx^jx^i-\tfrac{1}{24}\left(1-\left(\tfrac{3}{4}\right)^2\tfrac{\xi^2}{\tilde{v}^2}\right)\mathcal{A}^{ij}h_j+\\[2mm]-\tfrac{3}{8}\cdot\tfrac{1}{12}\left(1-\left(\tfrac{3}{4}\right)^2\tfrac{\xi^2}{\tilde{v}^2}\right)\xi\tfrac{v^{2/3}}{\tilde{v}}\mathcal{A}^{ij}\widehat{d}_j\end{array}\right]q_ip^0+\\[12mm]+2\left(-\tfrac{h_i}{2}+\tfrac{3}{8}\xi\tfrac{v^{2/3}}{\tilde{v}}\widehat{d}_i\right)q_0p^i+\\[6mm]+2\left[-\tfrac{1}{2}h_jx^i+\tfrac{3}{8}\xi\tfrac{v^{2/3}}{\tilde{v}}\widehat{d}_jx^i+\tfrac{1}{12}\left(1-\left(\tfrac{3}{4}\right)^2\tfrac{\xi^2}{\tilde{v}^2}\right)\mathcal{A}^{ik}h_{jk}\right]q_ip^j+\\[6mm]+q_0^2+\\[4mm]+2x^iq_0q_i+\\[4mm]+\left[x^ix^j-\tfrac{1}{12}\left(1-\left(\tfrac{3}{4}\right)^2\tfrac{\xi^2}{\tilde{v}^2}\right)\mathcal{A}^{ij}\right]q_iq_j\end{array}\right]$$

$$(6.2.0.27)$$

By using the results (7.2.1.34) and (7.2.1.36) of Appendix A of [1], it is easy to check that in the *classical* limit $\xi \to 0$ Eq.(6.2.0.27) yields the effective BH potential $\check{V}_{BH}\left(x^j, \widehat{\lambda}^j, v; Q, 0\right)$ for a generic d-SKG, given by Eq. (2.13) of [65], which we report here for ease of comparison:

$$
\begin{aligned}
2 \lim_{\xi \to 0} V_{BH} \;=\;\; & 2\check{V}_{BH} = \\
=\;\; & \left[v\left(1 + 4\check{g}\right) + \frac{h^2}{36v} + \frac{3}{48v}\check{g}^{ij}h_i h_j \right]\left(p^0\right)^2 + \\
& + \left[4v\check{g}_{ij} + \frac{1}{4v}\left(h_i h_j + \check{g}^{mn}h_{im}h_{nj}\right) \right] p^i p^j + \\
& + \frac{1}{v}\left[q_0^2 + 2x^i q_0 q_i + \left(x^i x^j + \frac{1}{4}\check{g}^{ij}\right) q_i q_j \right] + \\
& + 2\left[v\check{g}_i - \frac{h}{12v}h_i - \frac{1}{8v}\check{g}^{jm}h_m h_{ij} \right] p^0 p^i + \\
& - \frac{1}{3v}\left[-hp^0 q_0 + 3q_0 p^i h_i - \left(hx^i + \frac{3}{4}\check{g}^{ij}h_j \right) p^0 q_i \right] \\
& - \frac{1}{v}\left[\left(h_j x^i + \frac{1}{2}\check{g}^{im}h_{mj} \right) q_i p^j \right].
\end{aligned}
$$

$$(6.2.0.28)$$

$\check{g}_{ij}$ and $\check{g}^{ij}$ have been respectively defined in Appendix A of [1], with contractions consistently defined as

$$
\check{g}_i \;\equiv\; \check{g}_{ij}x^j = \lim_{\xi \to 0} g_{ij}x^j; \tag{6.2.0.29}
$$

$$
\check{g} \;\equiv\; \check{g}_{ij}x^i x^j = \lim_{\xi \to 0} g_{ij}x^i x^j. \tag{6.2.0.30}
$$

6.2.1 Axion-Free-Supporting Configurations

Let us now consider the terms of V_{BH} given by Eq. (6.2.0.27) which are linear in the *axions x^i's*; they read as follows:

$$
V_{BH}\big|_{linear\ in\ \{x^i\}} = \frac{1}{2\tilde{v}} \left(1 - \left(\tfrac{3}{4}\right)^2 \frac{\xi^2}{\tilde{v}^2}\right)^{-1} \cdot
$$

$$
\cdot \left[
\begin{array}{l}
2\left[
\begin{array}{l}
-\left(\tfrac{3}{8}\right)^2 \xi^2 \frac{v^{4/3}}{\tilde{v}^2}(\widehat{d}_j x^j)\widehat{d}_i + \\[2mm]
+\tilde{v}^2\left(1 - \left(\tfrac{3}{4}\right)^2 \frac{\xi^2}{\tilde{v}^2}\right) 12\mathcal{A}_i + \\[2mm]
+\tfrac{3}{8}\cdot\tfrac{1}{12}\left(1 - \left(\tfrac{3}{4}\right)^2 \frac{\xi^2}{\tilde{v}^2}\right)\xi\frac{v^{2/3}}{\tilde{v}}\mathcal{A}^{kl}h_{ik}\widehat{d}_l
\end{array}
\right] p^0 p^i + \\[10mm]
-\tfrac{3}{4}\xi\frac{v^{2/3}}{\tilde{v}}\widehat{d}_i x^i q_0 p^0 + \\[4mm]
+2\left[\tfrac{3}{8}\xi\frac{v^{2/3}}{\tilde{v}}\widehat{d}_j x^i + \tfrac{1}{12}\left(1 - \left(\tfrac{3}{4}\right)^2 \frac{\xi^2}{\tilde{v}^2}\right)\mathcal{A}^{ik}h_{jk}\right] q_i p^j + \\[6mm]
+2x^i q_0 q_i
\end{array}
\right].
$$

$$(6.2.1.1)$$

As a consequence, for a *d*-SKG corrected by $\xi \neq 0$ (with prepotential given by Eq. (6.2.0.1)), only two *axion-free-supporting* BH charge configuration exist, i.e. the *electric* $(D2 - D6)$ and *magnetic* $(D0 - D4)$ ones:

$$
electric\ :\ Q_{el} \equiv \begin{pmatrix} p^0 \\ 0 \\ 0 \\ q_i \end{pmatrix}; \tag{6.2.1.2}
$$

$$
magnetic\ :\ Q_{magn} \equiv \begin{pmatrix} 0 \\ p^i \\ q_0 \\ 0 \end{pmatrix}. \tag{6.2.1.3}
$$

For such BH charge configurations $x^i = 0 \ \forall i$ is a(n *at least)* particular solution of the axionic Attractor Eqs.:

$$\left.\frac{\partial V_{BH}}{\partial x^i}\right|_{Q=Q_{el}} = 0 \Leftarrow x^i = 0 \ \forall i; \tag{6.2.1.4}$$

$$\left.\frac{\partial V_{BH}}{\partial x^i}\right|_{Q=Q_{magn}} = 0 \Leftarrow x^i = 0 \ \forall i. \tag{6.2.1.5}$$

This fact is a major difference with respect to the classical limit $\xi \to 0$, in which the linear term in x^i's proportional to $q_0 p^0$ (see Eq. (6.2.1.1)) vanishes. Indeed, it consistently holds that

$$2 \lim_{\xi \to 0} V_{BH}|_{linear\ in\ \{x^i\}} = 2\ \breve{V}_{BH}|_{linear\ in\ \{x^i\}} = \frac{1}{v} 2 x^i q_0 q_i + 2v \breve{g}_i p^0 p^i - \frac{1}{2v} \breve{g}^{ik} h_{kj} q_i p^j. \tag{6.2.1.6}$$

This implies that also the Kaluza-Klein ($D0 - D6$) BH charge configuration

$$KK : Q_{KK} \equiv \begin{pmatrix} p^0 \\ 0 \\ q_0 \\ 0 \end{pmatrix} \tag{6.2.1.7}$$

supports *axion-free* (*at least* particular) attractor solutions [65]. Thus, besides the classical limits of Eqs. (6.2.1.4) and (6.2.1.5), i.e.:

$$\left.\frac{\partial \breve{V}_{BH}}{\partial x^i}\right|_{Q=Q_{el}} = 0 \Leftarrow x^i = 0 \ \forall i; \tag{6.2.1.8}$$

$$\left.\frac{\partial \breve{V}_{BH}}{\partial x^i}\right|_{Q=Q_{magn}} = 0 \Leftarrow x^i = 0 \ \forall i, \tag{6.2.1.9}$$

for $\xi = 0$ it also holds that

$$\left.\frac{\partial \breve{V}_{BH}}{\partial x^i}\right|_{Q=Q_{KK}} = 0 \Leftarrow x^i = 0 \ \forall i. \tag{6.2.1.10}$$

The *non-axion-free-supporting* nature of the $D0 - D6$ BH charge configuration in perturbatively quantum corrected *d*-SKG (determined by the holomorphic prepotential (6.2.0.1)) is consistent with, and sheds new light on, the results of [55].

Such a paper (developing the analysis of [46]) addressed the issue of the fate of the unique non-BPS $Z \neq 0$ *flat* direction in the $\mathcal{N} = 2$, $d = 4$ *ungauged* Maxwell-Einstein supergravity described by Eq. (6.2.0.1) with $n_V = 2$ (*i.e.* the so-called "$st^2 + i\xi$ model"). By analyzing the (supergravity analogues of the) $D0 - D4$, $D2 - D6$ and $D0 - D6$ charge configurations, the following results were obtained:

- In $D0 - D4$ and $D2 - D6$ charge configurations the classical solutions ($\xi = 0$) were found to lift at the quantum level ($\xi \neq 0$). Remarkably, it was found that the *quantum lift* occurs more often towards *repeller* directions (thus destabilizing the whole critical solution, and *destroying the attractor in strict sense*), rather than towards *attractor* directions.

- The $D0 - D6$ charge configuration yielded a somewhat surprising result: the classical solution gets modified at the *quantum* level, acquiring a *non-vanishing* axionic part. However, despite being no more purely imaginary, *such a quantum non-BPS $Z \neq 0$ solution still exhibits a flat direction*. The origin of such a deep difference among *electric/magnetic* and $D0 - D6$ configurations was unclear in [55], but it is clarified (and further generalized to an arbitrary number n_V of Abelian vector multiplets) from the results of the analysis performed above: due to the very structure of V_{BH} (see Eqs. (6.2.0.27) and (6.2.1.1)) for $\xi \neq 0$, the *electric/magnetic* still support axion-free solutions, whereas the $D0 - D6$ configuration do *not*.

On the other hand, the persistence of the flat direction also in presence of *quantum generated axions* is still not completely understood, and we left the study of such issues for future work.

6.3 Central Charge and *Matter Charges*

As given by Eq. (6.2.0.7), the effective BH potential V_{BH} enjoys a rewriting in terms of the $\mathcal{N} = 2$, $d = 4$ central charge Z and of its covariant derivatives $D_i Z$ (usually named *matter charges*), which is therefore worth computing.

In order to do this, let us recall that under the assumptions (6.2.0.3)-(6.2.0.5) the holomorphic prepotential (6.2.0.1) reduces to

$$\mathcal{F}(z;\xi) \equiv \frac{1}{3!} d_{ijk} z^i z^j z^k + i\xi. \tag{6.3.0.1}$$

Furthermore, the Kähler potential reads ($\mathcal{F}_i \equiv \partial \mathcal{F} / \partial z^i$; see *e.g.* [15; 66])

$$
\begin{aligned}
K &= -\log \left\{ i \left[2(\mathcal{F} - \overline{\mathcal{F}}) + (\bar{z}^{\bar{\imath}} - z^i)(\mathcal{F}_i + \overline{\mathcal{F}}_{\bar{\imath}}) \right] \right\} = \\
&= -\log \left[-\frac{i}{3!} d_{ijk} (z^i - \bar{z}^{\bar{\imath}})(z^j - \bar{z}^{\bar{\jmath}})(z^k - \bar{z}^{\bar{k}}) - 4\xi \right] = \\
&= -\log \left(8 \left(\nu - \frac{\xi}{2} \right) \right),
\end{aligned}
\tag{6.3.0.2}
$$

where definitions (6.2.0.4) and (6.2.0.13) were used. Eq. (7.2.1.2) thus implies

$$\exp\left(-K\right) = 8\left(v - \frac{\xi}{2}\right) \Leftrightarrow \exp\left(K/2\right) = \frac{1}{2\sqrt{2v - \xi}}, \qquad (6.3.0.3)$$

with the global condition of consistency (relevant also for previous treatment, see for instance Eq. (6.2.0.22))

$$2v - \xi > 0. \qquad (6.3.0.4)$$

Therefore, by recalling its very definition (see *e.g.* [6] and Refs. therein)

$$Z \equiv e^{K/2}(X^\Lambda q_\Lambda - F_\Lambda p^\Lambda) \equiv e^{K/2}W, \qquad (6.3.0.5)$$

where W is the holomorphic superpotential, and under the assumptions (6.2.0.3)-(6.2.0.5), the $\mathcal{N} = 2$, $d = 4$ central charge function for the holomorphic prepotential (6.2.0.1) can be computed to be:

$$Z\left(x^j, \widehat{\lambda^j}, v; Q, \xi\right) = \frac{1}{2\sqrt{2v - \xi}}W\left(x^j, \widehat{\lambda^j}, v; Q, \xi\right) =$$

$$= \frac{1}{2\sqrt{2v - \xi}}\left[\begin{array}{l} q_0 + q_i x^i - \frac{p^0}{2}v^{2/3}\widehat{d}_i x^i + \frac{p^0}{6}h - \frac{p^i}{2}h_i + v^{2/3}\frac{p^i}{2}\widehat{d}_i + \\[2mm] + iv^{1/3}\left(-q_i\widehat{\lambda^i} - \frac{p^0}{2}\widehat{d}_{ij}x^i x^j + p^0 v^{2/3} - 2\frac{\xi}{v^{1/3}}p^0 + p^i\widehat{d}_{ij}x^j\right)\end{array}\right];$$

$$(6.3.0.6)$$

$$D_i Z\left(x^j, \widehat{\lambda^j}, v; Q, \xi\right) = \frac{1}{2\sqrt{2v - \xi}}D_i W\left(x^j, \widehat{\lambda^j}, v; Q, \xi\right) =$$

$$= \frac{1}{2\sqrt{2v - \xi}}\left\{\begin{array}{l} q_i + \frac{p^0}{2}h_i - \frac{p^0}{4}v^{2/3}\widehat{d}_i + \\[1mm] -p^j h_{ij} + iv^{1/3}\left(-p^0 x^j + p^j\right)\widehat{d}_{ij} + \\[3mm] -\frac{i}{2}\frac{v^{2/3}}{(2v - \xi)}\widehat{d}_i\left[\begin{array}{l} q_0 + q_j x^j - \frac{p^0}{2}v^{2/3}\widehat{d}_j x^j + \frac{p^0}{6}h + \\[1mm] -\frac{p^j}{2}h_j + \frac{p^j}{2}v^{2/3}\widehat{d}_j + \\[2mm] + iv^{1/3}\left(\begin{array}{l}-q_j\widehat{\lambda^j} - \frac{p^0}{2}\widehat{d}_{jk}x^j x^k + \\[1mm] +p^0 v^{2/3} - 2\frac{\xi}{v^{1/3}}p^0 + p^j\widehat{d}_{jk}x^k\end{array}\right)\end{array}\right]\end{array}\right\}.$$

$$(6.3.0.7)$$

Clearly, due to the different Kähler weights of Z and W (respectively $(1, -1)$ and $(2, 0)$), the covariant differential operator acting on them has different definitions, i.e.:

$$D_i Z \equiv \partial_i Z + \frac{1}{2}\left(\partial_i K\right) Z; \qquad (6.3.0.8)$$

$$D_i W \equiv \partial_i W + \left(\partial_i K\right) W. \qquad (6.3.0.9)$$

Notice that in the limit $\xi \to 0$ Eqs. (6.3.0.6) and (6.3.0.7) exactly matches with known results for d-SK geometries, given by Eq.(4.9) and (4.10) of [65]. It is worth remarking that Eq. (6.3.0.6) yields that the holomorphic superpotential W gets modified, with respect to its classical ($\xi \to 0$) counterpart, only by a *global* shift of its imaginary part:

$$W\left(x^j, \widehat{\lambda}^j, v; Q, \xi\right) = W\left(x^j, \widehat{\lambda}^j, v; Q, 0\right) - 2\xi i p^0. \tag{6.3.0.10}$$

In particular, for *axion-free* critical solutions (supported for $\xi \neq 0$ only by the charge configurations (6.2.1.2) and (6.2.1.3)) it holds that the superpotential W (*on-shell* for axions x^i's) is purely imaginary and real, respectively:

$$W\left(x^j = 0, \widehat{\lambda}^j, v; Q_{el}, \xi\right) = iv^{1/3}\left(-q_i\widehat{\lambda}^i + p^0 v^{2/3} - 2\frac{\xi}{v^{1/3}}p^0\right) \tag{6.3.0.11}$$

$$W\left(x^j = 0, \widehat{\lambda}^j, v; Q_{magn}, \xi\right) = q_0 + v^{2/3}\frac{p^i}{2}\widehat{d}_i. \tag{6.3.0.12}$$

Concerning supersymmetric critical points of V_{BH}, the ($\frac{1}{2}$-)BPS conditions

$$D_iW = 0 \ \forall i = 1, ..., n_V \tag{6.3.0.13}$$

for *axion-free* critical solutions within the charge configurations (6.2.1.2) and (6.2.1.3) respectively read ($\forall i = 1, ..., n_V$):

$$D_iW = 0 \Leftrightarrow q_i - \frac{p^0}{4}v^{2/3}\widehat{d}_i + \frac{1}{2}\frac{v}{(2v - \xi)}\widehat{d}_i\left(-q_j\widehat{\lambda}^j + p^0 v^{2/3} - 2\frac{\xi}{v^{1/3}}p^0\right) = 0; \tag{6.3.0.14}$$

$$D_iW = 0 \Leftrightarrow p^j\widehat{d}_{ij} - \frac{1}{2}\frac{v^{1/3}}{(2v - \xi)}\widehat{d}_i\left(q_0 + \frac{p^j}{2}v^{2/3}\widehat{d}_j\right) = 0, \tag{6.3.0.15}$$

reducing to n_V (ξ-parametrized) real algebraic Eqs. in n_V real unknowns $\left\{\widehat{\lambda}^i, v\right\}$.

In [19] the *axion-free* $\frac{1}{2}$-BPS critical points of V_{BH} determined by the holomorphic prepotential (6.3.0.1) were determined by introducing the Kähler gauge-invariant sections

$$Y \equiv \overline{Z}\begin{pmatrix} L^0 \\ L^i \\ M_0 \\ M_i \end{pmatrix} = \exp(K)\,\overline{W}\begin{pmatrix} X^0 \\ X^i \\ F_0 \\ F_i \end{pmatrix} \tag{6.3.0.16}$$

and evaluating the identities of the SK geometries (see *e.g.* [15; 66] and Refs. therein*)* along the BPS conditions (6.3.0.13), thus obtaining ($\Xi \in \mathbb{R}$)

$$
\begin{cases}
Y^0 = \frac{1}{2} \left(\Xi + ip^0 \right); \\[2ex]
Y^i = ip^i \frac{\left(\Xi + ip^i \right)}{\Xi}.
\end{cases}
\tag{6.3.0.17}
$$

In the case of $\Xi \neq 0$, the ζ-dependent value of V_{BH} at its $\frac{1}{2}$-BPS *axion-free* critical points can be computed to be[6]

$$
V_{BH,BPS,axion-free} = -2 \left[\Xi + \frac{\left(p^0 \right)^2}{\Xi} \right] \left(q_0 - 2\xi\Xi + \frac{\zeta}{2}\Xi \right),
\tag{6.3.0.18}
$$

where Ξ satisfies the ζ-parametrized Eq. (see Eq. (3.34) of [19]):

$$
3p^0 q_0 + p^i q_i = 6\xi\Xi p^0,
\tag{6.3.0.19}
$$

along with the condition [19]

$$
\left(q_0 - 2\Xi \right) d_{ijk} p^i p^j p^k > 0.
\tag{6.3.0.20}
$$

On the other hand, the $\frac{1}{2}$-BPS *axion-free* solutions with $\Xi = 0$ are necessarily supported only by the *electric* configuration (6.2.1.2), and the dependence on ζ drops out: the resolution of the Attractor Eqs. in terms of the sections Y^Λ's and the determination of the critical value of V_{BH} go as for a generic d-SK geometry [58].

Concerning *non-axion-free* supersymmetric (*if any*) and non-supersymmetric (either *axion-free* or *non-axion-free*) critical points of V_{BH}, the case study becomes much more complicated.

As yielded by the analysis of $t^3 + i\xi$ model ($n_V = 1$) [46] and of $st^2 + i\xi$ model ($n_V = 2$ particular case) [55], in general the corresponding Attractor Eqs. are higher-order algebraic Eqs. which cannot be solved analytically, but only investigated numerically. Furthermore, interesting phenomena occur, such as: the *"separation"* of attractors (related to presence of *basins of attraction / area codes* in the dynamical system describing the radial evolution of the scalar fields in the BH space-time background) [46]; the *"transmutation"* of the supersymmetry-preserving properties of the attractors [46]; and the *"lifting"* (with or without removal) of the *"flat"* directions of the critical potential, which exist in the classical ($\xi = 0$) regime, at least for *symmetric* d-SK geometries [55].

---

[6]Eq. (6.3.0.18) fixes a typo in Eq. (3.35) of [19]. For the configuration $D0 - D6$ (which however, as explicitly shown above, is *not axion-free-supporting* for $\xi \neq 0$) this was noticed in [46].

Despite the lack of analytical expressions of non-supersymmetric (non-BPS $Z \neq 0$ *and/or* non-BPS $Z = 0$) critical points of V_{BH} for $\xi \neq 0$, many issues are still to be carefully investigated (we list some of them in the concluding Sect. 6.7).

The intricacy of the SK geometry described by the holomorphic prepotential (6.2.0.1) (or, equivalently by Eq. (6.3.0.1)) calls for a deeper analysis of the fundamental quantities characterizing such a geometry, and also for a deeper understanding of the conditions determining the (various classes of) critical points of V_{BH} itself. The study of these issues, needed for a deeper investigation of the dynamics of the *Attractor Mechanism* in the generally non-homogeneous geometries under consideration, will be the object of Sects. 6.4 and 6.6.

6.4   *E- Tensor*

The first quantity we want to determine is the so-called *E*-tensor. This rank-5 tensor was firstly introduced in [35] (see also the treatment of [54]), and it expresses the deviation of the considered geometry from being symmetric. Its definition reads (see *e.g.* [10] for a recent treatment, and Refs. therein):

$$\overline{E}_{\overline{m}ijkl} \equiv \frac{1}{3}\overline{D}_{\overline{m}}D_iC_{jkl}. \tag{6.4.0.1}$$

This definition can be elaborated further, by recalling the properties of the so-called *C*-tensor C_{ijk}. This is a rank-3 tensor with Kähler weights $(2, -2)$, defined as (see *e.g.* [15; 67; 68]):

$$\begin{aligned} C_{ijk} &\equiv \left\langle D_iD_jV, D_kV \right\rangle = e^K\left(\partial_i\mathcal{N}_{\Lambda\Sigma}\right)D_jX^\Lambda D_kX^\Sigma = \\ &= e^K\left(\partial_iX^\Lambda\right)\left(\partial_jX^\Sigma\right)\left(\partial_kX^\Xi\right)\partial_\Xi\partial_\Sigma F_\Lambda\left(X\right) \equiv \\ &\equiv e^KW_{ijk}, \quad \overline{\partial}_{\overline{l}}W_{ijk} = 0, \end{aligned} \tag{6.4.0.2}$$

where the second line holds only in *special coordinates*. C_{ijk} is completely symmetric and covariantly holomorphic:

$$C_{ijk} = C_{(ijk)}; \tag{6.4.0.3}$$

$$\overline{D}_{\overline{i}}C_{jkl} = 0. \tag{6.4.0.4}$$

Furthermore, it enters the fundamental constraints on the Riemann tensor $R_{i\overline{j}k\overline{l}}$ of SK geometry [7] (see *e.g.* [15; 67; 68], [15] and Refs. therein; see also *e.g.* [69] and [10]

---

[7]Notice that the third of Eqs. (7.3.0.2) correctly defines the Riemann tensor $R_{i\overline{j}k\overline{l}}$, and it is actually the opposite of the one which may be found in a large part of existing literature. Indeed, such a formulation yields negative values of the constant scalar curvature homogeneous symmetric non-compact SK manifolds, as given by the treatment of [54].

for more recent reviews):

$$R_{i\bar{j}k\bar{l}} = -g_{i\bar{j}}g_{k\bar{l}} - g_{i\bar{l}}g_{k\bar{j}} + C_{ikm}\overline{C}_{\bar{j}\bar{l}\bar{n}}g^{m\bar{n}}. \tag{6.4.0.5}$$

The Bianchi identities for $R_{i\bar{j}k\bar{l}}$ (see *e.g.* [67]) and constraints (6.4.0.5) yield the following result

$$D_{[i}C_{j]kl} = 0, \tag{6.4.0.6}$$

where (round) square brackets denote (symmetrization) anti-symmetrization with respect to enclosed indices throughout. Due to its holomorphic Kähler weight, the covariant derivative of C_{ijk} reads:

$$D_iC_{jkl} = D_{(i}C_{j)kl} = \partial_iC_{jkl} + (\partial_iK)\,C_{jkl} + \Gamma_{ij}{}^m C_{mkl} + \Gamma_{ik}{}^m C_{mjl} + \Gamma_{il}{}^m C_{mjk}, \tag{6.4.0.7}$$

where the Christoffel connection Γ is defined as

$$\Gamma_{ij}{}^m \equiv -g^{m\bar{l}}\partial_i g_{j\bar{l}}. \tag{6.4.0.8}$$

By using Eqs. (7.3.0.2)-(6.4.0.8), $\overline{E}_{\bar{m}ijkl}$ defined by (7.3.0.1) can thus be further elaborated as follows:

$$\begin{aligned}
\overline{E}_{\bar{m}ijkl} &= \frac{1}{3}\overline{D}_{\bar{m}}D_{(i}C_{jkl)} = C_{p(kl}C_{ij)n}g^{n\bar{n}}g^{p\bar{p}}\overline{C}_{\overline{npm}} - \frac{4}{3}g_{(l|\bar{m}}C_{|ijk)} = \\
&= g^{n\bar{n}}R_{(i|\bar{m}|j|\bar{n}}C_{n|kl)} + \frac{2}{3}g_{(i|\bar{m}}C_{|jkl)} = \overline{E}_{\bar{m}(ijkl)}.
\end{aligned} \tag{6.4.0.9}$$

It thus holds that $\overline{E}_{\bar{m}ijkl} = 0$ *globally* in (homogeneous) symmetric SK manifolds, defined by the covariant constancy of $R_{i\bar{j}k\bar{l}}$ itself:

$$D_m R_{i\bar{j}k\bar{l}} = 0. \tag{6.4.0.10}$$

Eq. (7.3.0.17), through the covariant holomorphicity of C_{ijk} and the constraints (6.4.0.5), yields the *global* covariant constancy of C_{ijk} itself, and thus the global vanishing of $\overline{E}_{\bar{m}ijkl}$:

$$D_iC_{jkl} = D_{(i}C_{j)kl} = 0 \Rightarrow \overline{E}_{\bar{m}ijkl} = 0, \tag{6.4.0.11}$$

which in turn, through Eq. (7.3.0.4), implies

$$C_{p(kl}C_{ij)n}g^{n\bar{n}}g^{p\bar{p}}\overline{C}_{\overline{npm}} = \frac{4}{3}g_{(l|\bar{m}}C_{|ijk)} \Leftrightarrow g^{n\bar{n}}R_{(i|\bar{m}|j|\bar{n}}C_{n|kl)} = -\frac{2}{3}g_{(i|\bar{m}}C_{|jkl)}. \tag{6.4.0.12}$$

It is worth noticing that, while (7.3.0.17) defines the symmetricity of a Kähler manifold, Eq. (6.4.0.11) (or equivalently Eq. (7.3.0.18)) is a necessary (but not necessarily sufficient) condition of symmetricity.

Recently, in [8] the *E*-tensor was used in the expression of the value of V_{BH} at its non-BPS $Z \neq 0$ critical points (see also the treatment in [71], and Refs. therein):

$$V_{BH,nBPS,Z\neq0} = \left[4\left|Z\right|^2 + \Delta \right]_{nBPS,Z\neq0}, \tag{6.4.0.13}$$

where $(Z^{\bar{i}} \equiv g^{j\bar{i}}D_jZ)$

$$\Delta \equiv -\frac{3}{4}\frac{E_{m\overline{ijkl}}Z^{\bar{i}}Z^{\bar{j}}Z^{\bar{k}}Z^{\bar{l}}\overline{Z}^m}{\left(\overline{C_{npq}}Z^{\bar{n}}Z^{\bar{p}}Z^{\bar{q}}\right)}, \tag{6.4.0.14}$$

such that (see *e.g.* [8; 10]).

$$\left[\frac{g^{i\bar{j}}\left(D_iZ\right)\overline{D_{\bar{j}}Z}}{\left|Z\right|^2}\right]_{nBPS,Z\neq0} = 3; \tag{6.4.0.15}$$

$$\Updownarrow$$

$$\Delta_{nBPS,Z\neq0} = 0 \Leftrightarrow \left(E_{m\overline{ijkl}}Z^{\bar{i}}Z^{\bar{j}}Z^{\bar{k}}Z^{\bar{l}}\overline{Z}^m\right)_{nBPS,Z\neq0} = 0, \tag{6.4.0.16}$$

where in the last step the non-degeneracy of the cubic norm $C_{ijk}\overline{Z}^i\overline{Z}^j\overline{Z}^k$ (*at least* at non-BPS $Z \neq 0$ critical points of V_{BH}) was used. Therefore, Eq. (6.4.0.11) (or equivalently Eq. (7.3.0.18)) is a sufficient (but not necessary) condition for the so-called *"rule of three"* (6.4.0.15) to hold at non-BPS $Z \neq 0$ critical points of V_{BH}.

These results (and further relations with the sectional curvature treated further below; see Eqs. (6.5.0.21)-(6.5.0.25) as well as the treatment given in [10]) call for an explicit determination of the *E*-tensor in the SK geometries described by the holomorphic prepotential (6.3.0.1), and through the limit $\xi \to 0$, in a generic *d*-SK geometry.

Thus, after a lengthy but straightforward algebra (detailed in Appendix B of [1]), the covariant derivative of the *C*-tensor can be written as follows:

$$D_iC_{jkl} = \frac{i}{2^5}\frac{1}{\left(\nu - \frac{\xi}{2}\right)^2}\left[\begin{array}{l} -\frac{\left(\nu - \frac{\xi}{2}\right)}{\nu + \xi}\nu^{2/3}\left(d_{ij}d_{kl} + d_{ik}d_{jl} + d_{il}d_{jk}\right) + \\ -2\left(\nu - \frac{\xi}{2}\right)\nu^{-1/3}\left(d_{ijn}d_{mkl} + d_{ikn}d_{mjl} + d_{iln}d_{mjk}\right)\widehat{d}^{mn} + \\ +\nu^{2/3}\left(d_id_{jkl} + d_jd_{ikl} + d_kd_{ijl} + d_ld_{ijk}\right) \end{array}\right]. \tag{6.4.0.17}$$

Notice that for $\xi \neq 0$ there is no way to make $D_iC_{jkl} = 0$ *globally*. This confirms the result of [54] that, with the exception of the sequence of the *minimal coupling* sequence $\mathbb{CP}^n$, all homogeneous symmetric (non-compact) SK are given by *d*-geometries (i.e., by the $\xi \to 0$ limit of prepotential (6.3.0.1)). Thus, the SK geometries described by the holomorphic prepotential (6.3.0.1) are not symmetric, nor

homogeneous (*at least* of the *d*-type studied and classified in [34; 35; 54], and Refs. therein).

Through definition (7.3.0.1) and Eq. (6.4.0.17), the *E*-tensor can then be explicitly computed:

$$
\overline{E}_{\overline{m}ijkl} \equiv \frac{1}{3}\overline{D}_{\overline{m}}D_iC_{jkl} = \frac{1}{3}\left[\overline{\partial}_{\overline{m}}D_iC_{jkl} - (\overline{\partial}_{\overline{m}}K)D_iC_{jkl}\right] =
$$

$$
= \frac{1}{12\cdot 2^4}\frac{1}{\left(\nu - \frac{\xi}{2}\right)^2}
\begin{bmatrix}
(2\nu - 7\xi)\frac{\nu^{4/3}}{4(\nu+\xi)^2}\left(\widehat{d}_{ij}\widehat{d}_{kl} + \widehat{d}_{ik}\widehat{d}_{jl} + \widehat{d}_{il}\widehat{d}_{jk}\right)\widehat{d}_m + \\[2mm]
-\frac{\nu^{4/3}}{2\left(\nu-\frac{\xi}{2}\right)}\left(\widehat{d}_i d_{jkl} + \widehat{d}_j d_{ikl} + \widehat{d}_k d_{ijl} + \widehat{d}_l d_{ijk}\right)\widehat{d}_m + \\[2mm]
-\frac{\left(\nu-\frac{\xi}{2}\right)}{\nu+\xi}\nu^{1/3}\left(\begin{array}{c} d_{ijm}\widehat{d}_{kl} + d_{klm}\widehat{d}_{ij} + d_{ikm}\widehat{d}_{jl} + \\ +d_{jlm}\widehat{d}_{ik} + d_{ilm}\widehat{d}_{jk} + d_{jkm}\widehat{d}_{il} \end{array}\right) + \\[2mm]
+2\nu^{1/3}\left(\widehat{d}_{im}d_{jkl} + \widehat{d}_{jm}d_{ikl} + \widehat{d}_{km}d_{ijl} + \widehat{d}_{lm}d_{ijk}\right) + \\[2mm]
-2\left(\nu - \frac{\xi}{2}\right)\left(d_{ijn}d_{pkl} + d_{ikn}d_{pjl} + d_{iln}d_{pjk}\right)\frac{\partial(\tilde{d}^{-1})^{pn}}{\partial\lambda^m}
\end{bmatrix},
$$

$$
\text{(6.4.0.18)}
$$

where it is easy to show that

$$
\frac{\partial d^{pn}}{\partial\lambda^m} = -d_{ijm}d^{ip}d^{jn} = -\nu^{-2/3}d_{ijm}\widehat{d}^{ip}\widehat{d}^{jn}. \qquad\text{(6.4.0.19)}
$$

By standard symmetrization procedures and using Eq. (6.4.0.19), Eq. (6.4.0.18) can be further elaborated as follows:

$$
\overline{E}_{\overline{m}ijkl} = -\frac{1}{3\cdot 2^7}\frac{1}{\left(\nu - \frac{\xi}{2}\right)^3}
\begin{bmatrix}
\left[4\,\widehat{d}_{(i}d_{jkl)} - 3\frac{\left(\nu-\frac{\xi}{2}\right)}{\nu+\xi}\,\widehat{d}_{(ij}\widehat{d}_{kl)}\right]\nu^{4/3}\widehat{d}_m + \\[2mm]
+12\frac{\left(\nu-\frac{\xi}{2}\right)^2}{\nu+\xi}\nu^{1/3}d_{m(ij}\widehat{d}_{kl)} - 2^4\left(\nu-\frac{\xi}{2}\right)\nu^{1/3}\,\widehat{d}_{m(i}d_{jkl)} + \\[2mm]
-12\left(\nu-\frac{\xi}{2}\right)^2\nu^{-2/3}d_{p(ij}d_{kl)n}\,d_{mrs}\widehat{d}^{rp}\widehat{d}^{sn} + \\[2mm]
+\frac{3}{2}\xi\frac{\left(\nu-\frac{\xi}{2}\right)}{(\nu+\xi)^2}\nu^{4/3}\,\widehat{d}_m\,\widehat{d}_{(ij}\widehat{d}_{kl)}
\end{bmatrix}.
$$

$$
\text{(6.4.0.20)}
$$

It is here worth remarking that the observation made above that for $\xi \neq 0$ it is not possible to make $D_i C_{jkl} = 0$ *globally* does not imply that $\overline{E}_{\overline{m}ijkl} = 0$, *and/or* $E_{\overline{mijkl}} \overline{Z}^i \overline{Z}^j \overline{Z}^k \overline{Z}^l \overline{Z}^m = 0$, *locally*, i.e. on a (set of) point(s), eventually at non-BPS $Z \neq 0$ critical points of V_{BH}. Thus, the interesting question arises (which we leave for future investigation) whether for some charge configurations (and eventually for some value(s) of ξ itself) the *"rule of three"* (6.4.0.15) still holds at non-BPS $Z \neq 0$ critical points of V_{BH} in SK geometries determined by the prepotential (6.3.0.1). Let us here recall that, as explicitly found in [72], *at least* in some homogeneous *non-symmetric d-*SK geometries, the *"rule of three"* (6.4.0.15) still holds, despite the fact that $\overline{E}_{\overline{m}ijkl}$ does *not* vanish globally.

Before concluding this Section, let us notice that in the limit $\xi \to 0$ the result (6.4.0.20) yields the expression of the *E*-tensor for a generic *d*-SK geometry, i.e.:

$$\overline{E}_{\overline{m}ijkl,\xi=0} = -\frac{1}{3 \cdot 2^7} \nu^{-5/3} \left[\begin{array}{l} \left(4\, \widehat{d}_{(i} d_{jkl)} - 3\, \widehat{d}_{(ij} \widehat{d}_{kl)} \right) \widehat{d}_m + \\[2mm] +12 d_{\overline{m}(ij} \widehat{d}_{kl)} - 16\, \widehat{d}_{m(i} d_{jkl)} + \\[2mm] -12\, d_{p(ij} d_{kl)n}\, d_{mrs} \widehat{d}^{rp} \widehat{d}^{sn} \end{array} \right]. \tag{6.4.0.21}$$

It is worth noticing that Eq. (6.4.0.21) yields that the tensor

$$\widetilde{\overline{E}}_{\overline{m}ijkl,\xi=0} \equiv \nu^{5/3} \overline{E}_{\overline{m}ijkl,\xi=0} \tag{6.4.0.22}$$

is independent of ν, but it rather depends only on the *"rescaled dilatons"* $\widehat{\lambda}^i$'s (recall definitions (6.2.0.19)-(6.2.0.21)):

$$\frac{\partial \widetilde{\overline{E}}_{\overline{m}ijkl,\xi=0}}{\partial \nu} = 0. \tag{6.4.0.23}$$

By looking at Eq. (6.4.0.20), it is easy to realize that the same does not happen for $\xi \neq 0$: the non-vanishing of the quantum parameter ξ does not allow for an overall factorization of the dependence of $\overline{E}_{\overline{m}ijkl}$ on ν *and/or* other (shifted *and/or* rescaled) variables. In other words, ξ entangles the dependence of $\overline{E}_{\overline{m}ijkl}$ on ν with the dependence on $\widehat{\lambda}^i$'s, and thus the *"$\xi \neq 0$ analogue"* of $\widetilde{\overline{E}}_{\overline{m}ijkl,\xi=0}$ (defined in (6.4.0.22)) cannot be introduced. This fact is related to the impossibility to uplift the quantum perturbatively corrected SK geometry described by the prepotential (6.3.0.1) to $d = 5$ space-time dimensions. Indeed, as is well known, in general only *d*-SK geometries can be uplifted to $d = 5$ (see *e.g.* [65] and Refs. therein).

6.5 *Sectional Curvature* at Critical Points

In the present Section we reconsider the non-supersymmetric criticality conditions for the effective BH potential V_{BH} of an $\mathcal{N} = 2, d = 4$ Maxwell-Einstein supergravity coupled to a generic number n_V of Abelian vector supermultiplets. We will find that in both classes ($Z \neq 0$ and $Z = 0$) of its non-BPS critical points, the critical value of V_{BH} (and thus, through the Bekenstein-Hawking entropy-area formula, the classical BH entropy) is proportional to the local value of the so-called *sectional curvature of matter charges*.

Within the present study, this general result then motivates the explicit computation (carried out in the next Section in two different, but equivalent, approaches) of the Riemann tensor, Ricci tensor and Ricci scalar curvature for the SK geometries determined by the prepotential (6.3.0.1), as well for generic d-SK geometry, obtained as the classical limit $\xi \to 0$ of these former ones. This latter calculation extends to the inclusion of the most general axion-shift-symmetric quantum perturbative correction (see discussion in Introduction) the results on the curvature of non-compact SK manifolds, found long time ago in [54].

Along the lines of the elaborations of [10] (see also [71]), we will now determine a *"non-BPS $Z = 0$ analogue"* of the *"rule of three"* (6.4.0.15). Such a *"non-BPS $Z = 0$ analogue"* is an hitherto unaddressed issue in literature (for instance, not considered in the fairly general treatment of [59], nor in [10]). In order to derive such a result, let us contract the constraints (6.4.0.5) by $\overline{Z}^i Z^{\bar{j}} \overline{Z}^k Z^{\bar{l}}$, obtaining

$$R_{i\bar{j}k\bar{l}} \overline{Z}^i Z^{\bar{j}} \overline{Z}^k Z^{\bar{l}} = -2 \left(Z_i \overline{Z}^i \right)^2 + C_{ikm} \overline{C}_{\bar{j}\bar{l}}{}^m \overline{Z}^i Z^{\bar{j}} \overline{Z}^k Z^{\bar{l}}. \tag{6.5.0.1}$$

Therefore, by recalling the non-BPS $Z = 0$ criticality conditions for V_{BH}:

$$C_{ijk} \overline{Z}^j \overline{Z}^k = 0, \tag{6.5.0.2}$$

as well as the definition of *sectional curvature*[8] (of the *matter charges*) (see *e.g.* [73] for a recent use; notice the different definition used here, consistent with the one adopted in [10]: see Eq. (3.1.1.2.11) therein)

$$\mathcal{R}(Z) \equiv R_{i\bar{j}k\bar{l}} \overline{Z}^i Z^{\bar{j}} \overline{Z}^k Z^{\bar{l}}, \tag{6.5.0.3}$$

it follows that at (*"large"*) non-BPS $Z = 0$ critical points of V_{BH} it holds that:

$$\left(Z_i \overline{Z}^i \right)^2_{nBPS, Z=0} = \left[g^{i\bar{j}} (\partial_i Z) \overline{\partial}_{\bar{j}} \overline{Z} \right]^2_{nBPS, Z=0} = -\frac{1}{2} \mathcal{R}(Z)|_{nBPS, Z=0} > 0. \tag{6.5.0.4}$$

---

[8]Notice that in general the Riemann tensor $R_{i\bar{j}k\bar{l}}$, the Ricci tensor $R_{i\bar{j}}$, the Ricci scalar curvature R and the sectional curvature $\mathcal{R}$ itself all are real quantities.

The result (6.5.0.4) holds for *all* $\mathcal{N} = 2$, $d = 4$ *ungauged* Maxwell-Einstein supergravities, not only for the ones with symmetric scalar manifolds, and it implies that the *sectional curvature of the matter charges* $\mathcal{R}(Z)$ to be *strictly negative* at non-BPS $Z = 0$ critical points of V_{BH}.

Moreover, for *symmetric* (and actually also for homogeneous non-symmetric) SK manifolds, recalling that along the non-BPS $Z = 0$-supporting charge orbits the quartic invariant $\mathcal{I}_4$ is positive, it further holds that (see *e.g.* [17] and [10])

$$\left[g^{i\bar{j}} (\partial_i Z) \, \bar{\partial}_{\bar{j}} \overline{Z} \right]_{nBPS,Z=0} = \sqrt{-\frac{1}{2} \mathcal{R}(Z)|_{nBPS,Z=0}} = \sqrt{\mathcal{I}_4}, \qquad (6.5.0.5)$$

thus yielding the relation

$$\mathcal{R}(Z)|_{nBPS,Z=0} = -2\mathcal{I}_4 < 0. \qquad (6.5.0.6)$$

Eq. (6.5.0.6) is to be contrasted with the analogue result obtained in [10] for (*"large"*) non-BPS $Z \neq 0$ critical points of V_{BH} in *symmetric* SK geometries (see Eq. (3.1.1.2.23), as well as Eq. (3.1.1.2.20), therein):

$$\mathcal{R}(Z)|_{nBPS,Z\neq0} = -6 |Z|^4_{nBPS,Z\neq0} = \frac{3}{8}\mathcal{I}_4 < 0. \qquad (6.5.0.7)$$

Thus, *at least* in *symmetric* SK geometries, at various classes of *"large"* critical points of V_{BH} the *sectional curvature of the matter charges* $\mathcal{R}(Z)$ takes the following values:

$$\mathcal{R}(Z) = \begin{cases} \frac{1}{2} - BPS : 0; \\[2ex] nBPS, Z \neq 0 : \frac{3}{8}\mathcal{I}_4 < 0; \\[2ex] nBPS, Z = 0 : -2\mathcal{I}_4 < 0. \end{cases} \qquad (6.5.0.8)$$

Correspondingly, through the celebrated Bekenstein-Hawking entropy-area formula [74] and its implementation through the *Attractor Mechanism* [6]

$$S_{BH} = \pi \frac{A_H}{4} = \pi \, V_{BH}|_{\partial V_{BH}=0}, \qquad (6.5.0.9)$$

at (*"large"*) non-BPS critical points of V_{BH} in (*at least* symmetric) $\mathcal{N} = 2$, $d = 4$ *ungauged* Maxwell-Einstein supergravities, the value of the classical BH entropy is proportional to the local value of $\mathcal{R}(Z)$ itself:

$$\frac{S_{BH}}{\pi} = \begin{cases} nBPS, Z \neq 0 : 2\sqrt{\frac{2}{3}}\sqrt{|\mathcal{R}(Z)|}; \\[2ex] nBPS, Z = 0 : \frac{1}{\sqrt{2}}\sqrt{|\mathcal{R}(Z)|}. \end{cases} \qquad (6.5.0.10)$$

Eqs. (6.5.0.8) (and consequently Eqs. (6.5.0.10)) hold *on-shell*, *i.e.* at the various classes of critical points of V_{BH}. Actually, they can be *"unified"* into an *off-shell* (*i.e.* global) relation, involving $\mathcal{R}(Z)$ along with the true-vector (vanishing *on-shell*) $\partial_i V_{BH}$. In order to determine such a relation, let us evaluate the definition of *sectional curvature of matter charges* (6.5.0.3) along the constraints (6.4.0.5), thus obtaining:

$$\mathcal{R}(Z) = -2\left(Z_i\overline{Z}^i\right)^2 + g^{n\overline{m}}C_{ikn}\overline{C}_{\overline{j}\overline{l}\overline{m}}\overline{Z}^iZ^{\overline{j}}\overline{Z}^kZ^{\overline{l}}. \tag{6.5.0.11}$$

Now, by differentiating Eq. (6.2.0.7) and using the defining relations of SK geometry (see *e.g.* [66] and Refs. therein), one can then write [6]

$$D_iV_{BH} = \partial_iV_{BH} = 2\overline{Z}Z_i + iC_{ijk}\overline{Z}^j\overline{Z}^k \Leftrightarrow C_{ijk}\overline{Z}^j\overline{Z}^k = -i\left(\partial_iV_{BH} - 2\overline{Z}Z_i\right). \tag{6.5.0.12}$$

By using Eq. (6.5.0.12), Eq. (6.5.0.11) can thus be recast in the following way:

$$\begin{aligned}
\mathcal{R}(Z) &= -2\left(Z_i\overline{Z}^i\right)^2 + g^{i\overline{j}}\left(\partial_iV_{BH} - 2\overline{Z}Z_i\right)\left(\overline{\partial}_{\overline{j}}V_{BH} - 2Z\overline{Z}_{\overline{j}}\right) = \\
&= 2Z_i\overline{Z}^i\left(2\left|Z\right|^2 - Z_j\overline{Z}^j\right) + \\
&\quad + g^{k\overline{l}}\left[\left(\partial_kV_{BH}\right)\overline{\partial}_{\overline{l}}V_{BH} - 2Z\left(\partial_kV_{BH}\right)\overline{Z}_{\overline{l}} - 2\overline{Z}\left(\overline{\partial}_{\overline{l}}V_{BH}\right)Z_k\right]
\end{aligned} \tag{6.5.0.13}$$

Eq. (6.5.0.13) is nothing but an equivalent rewriting of the sectional curvature of matter charges in SK geometry, given by Eq. (6.5.0.11). By consistently using the criticality conditions of V_{BH} defining the various classes of (*"large"*) critical points of V_{BH} itself (i.e.: $\frac{1}{2}$-BPS - see Eq. (6.3.0.13) -, non-BPS $Z = 0$ - see Eq. (6.5.0.2) -, and non-BPS $Z \neq 0$ - see Eq. (6.5.0.16) below), the three on-shell relations (6.5.0.8) are obtained.

Aside, let us also notice that the constraints (6.4.0.5) clearly yield a constrained expression for the Ricci tensor (and for the Ricci scalar curvature) of a SK manifold, in which the partial (and complete) contractions of the *C*-tensor with its complex conjugate play a key role. i.e., Eq. (6.4.0.5) respectively imply:

$$R_{i\overline{j}} \equiv g^{k\overline{l}}R_{i\overline{l}k\overline{j}} = -\left(n_V + 1\right)g_{i\overline{j}} + g^{k\overline{l}}g^{m\overline{n}}C_{imk}\overline{C}_{\overline{j}\overline{n}\overline{l}}; \tag{6.5.0.14}$$

$$R \equiv g^{i\overline{j}}g^{k\overline{l}}R_{i\overline{l}k\overline{j}} = g^{i\overline{j}}R_{i\overline{j}} = -\left(n_V + 1\right)n_V + g^{i\overline{j}}g^{k\overline{l}}g^{m\overline{n}}C_{imk}\overline{C}_{\overline{j}\overline{n}\overline{l}}. \tag{6.5.0.15}$$

From the discussion at the end of Subsect. 6.6.1, it will be clear that the first terms on the right-hand sides of Eqs. (6.5.0.14) and (6.5.0.15) are the constribution of the *"quadratic sector"* of the SK geometry (in which $C_{ijk} = 0$, as a consequence of its

very definition (7.3.0.2); notice that the contributions of such a *"quadratic sector"* are missing in *rigid* SK geometry, see *e.g.* [16] and [70]).

A further elaboration for (*"large"*) non-BPS $Z \neq 0$ critical points of V_{BH} can be performed by plugging the non-BPS $Z \neq 0$ criticality condition of V_{BH} (see *e.g.* [10])

$$D_i \log Z = -\frac{i}{2}\frac{1}{|Z|^2}C_{ijl}\overline{Z}^j\overline{Z}^l \tag{6.5.0.16}$$

into Eq. (7.2.1.39), thus getting

$$\mathcal{R}(Z)|_{nBPS,Z\neq 0} = \left[2Z_i\overline{Z}^i\left(2|Z|^2 - Z_i\overline{Z}^i\right)\right]_{nBPS,Z\neq 0}; \tag{6.5.0.17}$$

$$\Updownarrow$$

$$\left(Z_i\overline{Z}^i\right)^2 - 2g^{i\overline{j}}Z_i\overline{Z}^i|Z|^2 + \frac{1}{2}\mathcal{R}(Z) = 0; \tag{6.5.0.18}$$

$$\Updownarrow$$

$$\left(Z_i\overline{Z}^i\right)_{\pm} = |Z|^2 \pm \sqrt{|Z|^4 - \frac{1}{2}\mathcal{R}(Z)}; \tag{6.5.0.19}$$

$$\Updownarrow$$

$$|Z|^2 = \frac{1}{4}\frac{\mathcal{R}(Z)}{Z_i\overline{Z}^i} + \frac{1}{2}Z_i\overline{Z}^i, \tag{6.5.0.20}$$

where the subscript "*nBPS, Z $\neq$ 0*" has been suppressed for simplicity's sake. Notice that, also when $\mathcal{R}(Z) > 0$ satisfying $|Z|^4 - \frac{1}{2}\mathcal{R}(Z) \geqslant 0$, only one brach of $Z_i\overline{Z}^i$ *should* be consistent with the fact that $Z_i\overline{Z}^i > 0$.

Result (6.5.0.17), holding for all $\mathcal{N} = 2$, $d = 4$ *ungauged* Maxwell-Einstein supergravities, not only for the ones with symmetric scalar manifolds, consistently reduces to Eq. (6.5.0.7) when the *"rule of three"* (6.4.0.15)holds, as is the case for symmetric SK manifolds (see discussion above).

By recalling Eqs. (6.4.0.13) and (6.4.0.14), Eq. (6.5.0.20) thus implies that

$$\mathcal{R}(Z)|_{non-BPS,Z\neq 0} = -\left[2\left(3 + \frac{\Delta}{|Z|^2}\right)\left(1 + \frac{\Delta}{|Z|^2}\right)|Z|^4\right]_{non-BPS,Z\neq 0}, \tag{6.5.0.21}$$

or, more explicitly (evaluation at *"large"* non-BPS $Z \neq 0$ critical points of V_{BH} understood)

$$\mathcal{R}(Z) = -\frac{9}{8}\frac{|Z|^4}{\left(\overline{N_3}(Z)\right)^2|Z|^4}\left[4\overline{N_3}(Z)|Z|^2 - E(Z,\overline{Z})\right]\left[\frac{4}{3}\overline{N_3}(Z)|Z|^2 - E(Z,\overline{Z})\right]. \tag{6.5.0.22}$$

where

$$\overline{N}_3\left(Z\right) \;\; \equiv \;\; \overline{C}_{\overline{ijk}}Z^{\overline{i}}Z^{\overline{j}}Z^{\overline{k}};\tag{6.5.0.23}$$

$$E\left(Z,\overline{Z}\right) \;\; \equiv \;\; E_{ij\overline{kl}\overline{m}}\overline{Z}^{i}Z^{j}Z^{\overline{k}}Z^{l}Z^{\overline{m}}.\tag{6.5.0.24}$$

Results (6.5.0.21) and (6.5.0.22) relate $\mathcal{R}\left(Z\right)$, $\overline{N}_3\left(Z\right)$ and $E\left(Z,\overline{Z}\right)$ at *"large"* non-BPS $Z \neq 0$ critical points of V_{BH} in generic $\mathcal{N} = 2$, $d = 4$ *ungauged* Maxwell-Einstein supergravities, and they consistently reduce to Eq. (6.5.0.7) (*at least*) for *symmetric* SK manifolds. They are consistent with the treatment performed in [8; 10; 71], see e.g. Eq. (3.1.1.2.17) of [71], here reported for ease of comparison (evaluation at non-BPS $Z \neq 0$ critical points of V_{BH} understood):

$$\frac{3}{4}\frac{1}{|Z|^2}\frac{E\left(Z,\overline{Z}\right)}{\overline{N}_3\left(Z\right)} - 1 = \frac{\mathcal{R}\left(Z\right)}{2\left|Z\right|^2 Z_i\overline{Z}^i} = 2\frac{\mathcal{R}\left(Z\right)}{C_{ikn}\overline{C}^i_{\overline{rs}}\overline{Z}^k\overline{Z}^n Z^{\overline{r}}Z^{\overline{s}}}.\tag{6.5.0.25}$$

Furthermore, through the definition (6.4.0.14), Eqs. (6.5.0.21) and (6.5.0.22) are implied also by Eq. (6.5.0.20). Notice that, while $\mathcal{R}\left(Z\right)$ is a real quantity, $E_{ij\overline{kl}\overline{m}}$, $E\left(Z,\overline{Z}\right)$ and Δ are generally complex. But, (at least) at non-BPS $Z \neq 0$ critical points, Δ, or equivalently the ratio $\frac{E\left(Z,\overline{Z}\right)}{\overline{N}_3\left(Z\right)}$, becomes real (consistent with Eq. (6.4.0.14); see also Eq. (276) of [8] and Eq. (5.17) of [71]).

6.6   Riemann Tensor

The new results obtained in previous Section call for an explicit computation of the Riemann tensor, Ricci tensor and Ricci scalar curvature for the SK geometries determined by the prepotential (6.3.0.1), as well for generic d-SK geometry, obtained as the classical limit $\xi \rightarrow 0$ of these former ones. We will do this in the present Section, carrying out the calculation in two different, but (proved to be) equivalent, ways.

6.6.1   First Approach

The first approach conceives SK geometry as a particular Kähler geometry, and therefore one starts with the standard formula of Riemann tensor:

$$R_{i\overline{j}k\overline{l}} \equiv g^{m\overline{n}}\left(\overline{\partial}_{\overline{l}}\overline{\partial}_{\overline{j}}\partial_m K\right)\partial_i\overline{\partial}_{\overline{n}}\partial_k K - \overline{\partial}_{\overline{l}}\partial_i\overline{\partial}_{\overline{j}}\partial_k K.\tag{6.6.1.1}$$

After a lengthy but straightforward algebra (detailed in Appendix C of [1]), the Riemann tensor of the SK geometry determined by the prepotential (6.3.0.1) is computed as

$$
R_{\bar{i}\bar{j}k\bar{l}} = R_{ijkl} = -\frac{v^{2/3}}{32\left(v - \frac{\xi}{2}\right)^2} \cdot
$$

$$
\cdot \left\{
\begin{array}{l}
-\frac{\left(v-\frac{\xi}{2}\right)}{v+\xi}\widehat{d}_{ik}\widehat{d}_{jl} + 2\widehat{d}_{ij}\widehat{d}_{kl} + 2\widehat{d}_{il}\widehat{d}_{jk} + \\[2ex]
+\frac{v^2}{4\left(v-\frac{\xi}{2}\right)^2}\widehat{d}_i\widehat{d}_j\widehat{d}_k\widehat{d}_l + \\[2ex]
-\frac{v}{2\left(v-\frac{\xi}{2}\right)}\left(\widehat{d}_{ij}\widehat{d}_k\widehat{d}_l + \widehat{d}_{jk}\widehat{d}_i\widehat{d}_l + \widehat{d}_{il}\widehat{d}_j\widehat{d}_k + \widehat{d}_{kl}\widehat{d}_i\widehat{d}_j\right) + \\[2ex]
+2\frac{\left(v-\frac{\xi}{2}\right)}{v}d_{ikn}d_{jlm}\widehat{d}^{mn}
\end{array}
\right\}. \quad (6.6.1.2)
$$

In the classical limit ($\xi \to 0$), the expression of the Riemann tensor in a generic d-SK geometry is easily obtained:

$$
R_{\bar{i}\bar{j}k\bar{l},\xi=0} = R_{ijkl,\xi=0} = -\frac{1}{32}v^{-4/3} \cdot
$$

$$
\cdot \left\{
\begin{array}{l}
-\widehat{d}_{ik}\widehat{d}_{jl} + 2\widehat{d}_{ij}\widehat{d}_{kl} + 2\widehat{d}_{il}\widehat{d}_{jk} + \frac{1}{4}\widehat{d}_i\widehat{d}_j\widehat{d}_k\widehat{d}_l + \\[2ex]
-\frac{1}{2}\left(\widehat{d}_{ij}\widehat{d}_k\widehat{d}_l + \widehat{d}_{jk}\widehat{d}_i\widehat{d}_l + \widehat{d}_{il}\widehat{d}_j\widehat{d}_k + \widehat{d}_{kl}\widehat{d}_i\widehat{d}_j\right) + 2d_{ikn}d_{jlm}\widehat{d}^{mn}
\end{array}
\right\}.
$$

$$(6.6.1.3)$$

Notice that both Eqs. (6.6.1.2) and (6.6.1.3) have all the symmetry properties suitable to the Riemann tensor.

Consequently, the Ricci tensor and Ricci curvature scalar can respectively be computed as follows (recall n_V denotes the number of Abelian vector multiplets coupled to gravity multiplet, or equivalently the complex dimension of the consid-

ered SK manifold):

$$
R_{i\bar{j}} \equiv g^{k\bar{l}} R_{i\bar{l}k\bar{j}} = -\frac{1}{16} \frac{\nu^{4/3}}{\left(\nu - \frac{\xi}{2}\right)^2}
\begin{bmatrix}
\frac{\left[n_V \nu^3 + \frac{3}{2}(n_V+2)\nu^2\xi - \frac{3}{4}\nu\xi^2 - \frac{1}{8}(4n_V+3)\xi^3\right]}{(\nu+\xi)^2\left(\nu - \frac{\xi}{2}\right)} \widehat{d}_i \widehat{d}_j + \\[2mm]
-\frac{\left[4n_V \nu^2 + 2(n_V+3)\nu\xi - (2n_V+3)\xi^2\right]}{\nu(\nu+\xi)} \widehat{d}_{ij} + \\[2mm]
-4 \frac{\left(\nu - \frac{\xi}{2}\right)^2}{\nu^2} d_{ikn} d_{jlm} \widehat{d}^{kl} \widehat{d}^{mn}
\end{bmatrix} =
$$

$$
= -\frac{1}{16} \frac{\left[n_V \nu^3 + \frac{3}{2}(n_V+2)\nu^2\xi - \frac{27}{36}\nu\xi^2 - \frac{1}{8}(4n_V+3)\xi^3\right]}{\left(\nu - \frac{\xi}{2}\right)^3 (\nu+\xi)^2} \nu^{4/3} \widehat{d}_i \widehat{d}_j +
$$

$$
+ \frac{1}{16} \frac{\left[4n_V \nu^2 + 2(n_V+3)\nu\xi - (2n_V+3)\xi^2\right]}{\left(\nu - \frac{\xi}{2}\right)^2 (\nu+\xi)} \nu^{1/3} \widehat{d}_{ij} +
$$

$$
+ \frac{1}{4} \nu^{-2/3} d_{ikn} d_{jlm} \widehat{d}^{kl} \widehat{d}^{mn}
$$

$$
= R_{ij} \tag{6.6.1.4}
$$

$$
R \equiv g^{i\bar{j}} R_{i\bar{j}} = -n_V(n_V+1) - \frac{9}{2} \frac{\left(\nu - \frac{\xi}{2}\right)\nu}{(\nu+\xi)^3}\xi + \frac{3}{2} n_V \frac{\left(\nu - \frac{\xi}{2}\right)}{\nu+\xi} - \frac{\left(\nu - \frac{\xi}{2}\right)}{\nu} d_{ikn} d_{jlm} \widehat{d}^{jk} \widehat{d}^{mn} \widehat{d}^{il}.
$$

$$
\tag{6.6.1.5}
$$

Thence, in the classical limit ($\xi \to 0$), the expression of the Ricci tensor and Ricci scalar curvature in a generic d-SK geometry is easily obtained, respectively:

$$
R_{i\bar{j},\xi=0} \equiv g^{k\bar{l}} R_{i\bar{l}k\bar{j},\xi=0} = -\frac{1}{16} \nu^{-2/3} \left(n_V \widehat{d}_i \widehat{d}_j - 4n_V \widehat{d}_{ij} - 4d_{ikn} d_{jlm} \widehat{d}^{kl} \widehat{d}^{mn} \right) = R_{ij,\xi=0};
$$

$$
\tag{6.6.1.6}
$$

$$
R_{\xi=0} \equiv g^{ij} R_{ij,\xi=0} = -n_V(n_V+1) + \frac{3}{2} n_V - d_{ikn} d_{jlm} \widehat{d}^{jk} \widehat{d}^{mn} \widehat{d}^{il} =
$$

$$
= -n_V^2 + \frac{n_V}{2} - d_{ikn} d_{jlm} \widehat{d}^{jk} \widehat{d}^{mn} \widehat{d}^{il}. \tag{6.6.1.7}
$$

Let us notice that both Eqs. (6.6.1.4) and (6.6.1.6) have the symmetry properties suitable for Ricci tensor.

As pointed out at the end of Sect. 6.4, the symmetricity conditions (7.3.0.17)-(7.3.0.18) cannot be satisfied for prepotential (6.3.0.1) with $\xi \neq 0$. As is well known,

all symmetric spaces are Einstein spaces (see *e.g.* [75], and [76] for a comprehensive list of Refs.), *i.e.* with a Ricci tensor satisfying

$$\exists \Lambda \in \mathbb{R} : R_{i\bar{j}} = \Lambda g_{i\bar{j}} \Rightarrow R = n_V \Lambda, \tag{6.6.1.8}$$

and then with a constant Ricci scalar curvature, whose sign is the one of the real constant Λ itself. However, the opposite does not generally hold true: not all Einstein spaces are symmetric. Thus, it is reasonable to ask whether the considered quantum SK geometries determined by prepotential (6.3.0.1) can be Einstein. By recalling Eq. (6.2.0.22) and using Eq. (6.6.1.4), the condition for such geometries to be Einstein can be written as follows:

$$\frac{1}{4}\frac{\nu}{\left(\nu-\frac{\xi}{2}\right)}\begin{bmatrix}\frac{\left[n_V\nu^3+\frac{3}{2}(n_V+2)\nu^2\xi-\frac{3}{4}\nu\xi^2-\frac{1}{8}(4n_V+3)\xi^3\right]}{(\nu+\xi)^2\left(\nu-\frac{\xi}{2}\right)}\widehat{d}_i\widehat{d}_j+\\ -\frac{\left[4n_V\nu^2+2(n_V+3)\nu\xi-(2n_V+3)\xi^2\right]}{\nu(\nu+\xi)}\widehat{d}_{ij}+\\ -4\frac{\left(\nu-\frac{\xi}{2}\right)^2}{\nu^2}d_{ikn}d_{jlm}\widehat{d}^{kl}\widehat{d}^{mn}\end{bmatrix}=\Lambda\left[\widehat{d}_{ij}-\frac{1}{4}\frac{\nu}{\left(\nu-\frac{1}{2}\xi\right)}\widehat{d}_i\widehat{d}_j\right],$$

$$\tag{6.6.1.9}$$

and it seems to us that such an Eq. does not admit solutions for any value of the real constants ξ and Λ.

The situation is pretty different for the classical limit ($\xi \to 0$), determining the so-called *d*-SK geometries (described by prepotential (6.3.0.1) with $\xi = 0$). For such geometries, by recalling Eq. (7.2.1.37) and using Eq. (6.6.1.6), the condition to be Einstein reads

$$\frac{1}{4}\left(n_V\widehat{d}_i\widehat{d}_j - 4n_V\widehat{d}_{ij} - 4d_{ikn}d_{jlm}\widehat{d}^{kl}\widehat{d}^{mn}\right) = \Lambda\left(\widehat{d}_{ij} - \frac{\widehat{d}_i\widehat{d}_j}{4}\right). \tag{6.6.1.10}$$

As found in [54] (see also [34]), a (proper) subset of solutions to Eq. (6.6.1.10) is given by the symmetric *d*-SK geometries, satisfying the conditions of symmetricity (7.3.0.17)-(7.3.0.18). (*At least*) in such geometries, the *d*-tensor satisfies the following relation ([17; 54; 65]; see also the treatment given in [10], and Refs. therein):

$$d_{p(kl}d_{ij)n}a^{pr}a^{ns}a^{mq}d_{rsq} = \frac{4}{3}\delta^m_{(k}d_{lij)}, \tag{6.6.1.11}$$

which is a consequence of Eq. (7.3.0.18), and in fact can be further elaborated by using the second relation (involving the Riemann tensor) in Eq. (7.3.0.18) itself. In Eq. (7.2.1.26) a^{ij} is a sort of *"rescaled"* metric tensor, defined as (recall Eq. (7.2.1.38); see *e.g.* [65] for further elucidation of $d = 5$ origin of such a quantity):

$$a^{ij} \equiv \frac{1}{4}\nu^{-2/3}\breve{g}^{ij} = \frac{1}{2}\left(\widehat{\lambda}^i\widehat{\lambda}^j - 2\widehat{d}^{ij}\right). \tag{6.6.1.12}$$

Let us also notice that, from Eq. (6.6.1.8) the constancy of the Ricci scalar curvature is necessary but not sufficient condition for Einstein, and in turn for symmetric, spaces. In other words, it holds:

$$\text{symmetric} \overset{\nLeftarrow}{\Longrightarrow} \text{Einstein} \overset{\nLeftarrow}{\Longrightarrow} \text{constant } R. \qquad (6.6.1.13)$$

Eq. (6.6.1.5) yields the condition ($\Omega \in \mathbb{R}$)

$$- n_V (n_V + 1) - \frac{9}{2} \frac{\left(v - \frac{\xi}{2}\right) v}{(v + \xi)^3} \xi + \frac{3}{2} n_V \frac{\left(v - \frac{\xi}{2}\right)}{v + \xi} - \frac{\left(v - \frac{\xi}{2}\right)}{v} d_{ikn} d_{jlm} \widehat{d}^{jk} \widehat{d}^{mn} \widehat{d}^{il} = \Omega, \qquad (6.6.1.14)$$

and it seems that it is not possible to have R constant for SK geometries determined by (6.3.0.1) with $\xi \neq 0$. On the other hand, Eq. (6.6.1.7) yields the condition

$$R_{\xi=0} = -n_V^2 + \frac{n_V}{2} - d_{ikn} d_{jlm} \widehat{d}^{jk} \widehat{d}^{mn} \widehat{d}^{il} = \Omega. \qquad (6.6.1.15)$$

As pointed out above, a (proper) set of solutions to condition (6.6.1.15) is given by the symmetric d-SK geometries. As for all Einstein spaces, for symmetric d-SK spaces it holds that

$$\Omega = \Lambda n_V. \qquad (6.6.1.16)$$

The results of [54] yields $\Lambda = -\frac{2}{3} n_V$ for the four irreducible symmetric d-SK geometries (which are nothing but the *"magic"* ones) and $\Lambda = -\frac{\left(n_V^2 + 2n_V + 3\right)}{n_V}$ for the reducible sequence $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,n_V-1)}{SO(2) \times SO(n_V-1)}$ (and $\Lambda = -(n_V + 1)$ for the *minimal coupling* $\mathbb{CP}^{n_V}$ sequence, whose prepotential is however *quadratic*).

Analogously to the comment made at the and of Sect. 6.4, it is here worth noticing that Eqs. (6.6.1.3), (6.6.1.6) and (6.6.1.7) respectively yield that the quantities

$$\widetilde{R}_{i\bar{j}k\bar{l},\xi=0} \equiv v^{4/3} R_{i\bar{j}k\bar{l},\xi=0}; \qquad (6.6.1.17)$$

$$\widetilde{R}_{i\bar{j},\xi=0} \equiv v^{2/3} R_{i\bar{j},\xi=0}; \qquad (6.6.1.18)$$

$$R_{\xi=0}; \qquad (6.6.1.19)$$

are independent of v, but they rather depend only on the *"rescaled dilatons"* $\widehat{\lambda}^{i}$'s (recall definitions (6.2.0.19)-(6.2.0.21)):

$$\frac{\partial \widetilde{R}_{i\bar{j}k\bar{l},\xi=0}}{\partial v} = 0; \qquad (6.6.1.20)$$

$$\frac{\partial \widetilde{R}_{i\bar{j},\xi=0}}{\partial v} = 0; \qquad (6.6.1.21)$$

$$\frac{\partial R_{\xi=0}}{\partial v} = 0. \qquad (6.6.1.22)$$

By looking at Eqs. (6.6.1.2), (6.6.1.4) and (6.6.1.5), it is easy to realize that the same does not happen for $\zeta \neq 0$: the non-vanishing of the quantum parameter ζ does not allow for an overall factorization of the dependence of $R_{i\bar{j}k\bar{l}}$, $R_{i\bar{j},\zeta=0}$ and R on ν *and/or* other (shifted *and/or* rescaled) variables. In other words, ζ entangles the dependence of $R_{i\bar{j}k\bar{l}}$, $R_{i\bar{j},\zeta=0}$ and R on ν with the dependence on $\widehat{\lambda}^i$'s, and thus the "$\zeta \neq 0$ *analogues*" of $\widetilde{R}_{i\bar{j}k\bar{l},\zeta=0}$ and $\widetilde{R}_{i\bar{j},\zeta=0}$ (respectively defined in (6.6.1.17) and (6.6.1.18)) cannot be introduced. As already pointed out at the end of Sect. 6.4, this fact is related to the impossibility to uplift the quantum perturbatively corrected SK geometry described by the prepotential (6.3.0.1) to $d = 5$ space-time dimensions. Indeed, as is well known, in general only d-SK geometries can be uplifted to $d = 5$ (see *e.g.* [65] and Refs. therein).

6.6.2 Second Approach

The second approach is actually the one considered in [54]: the constraints (6.4.0.5), characterizing, among others, a Kähler geometry to be *special*, are exploited in order to compute the Riemann tensor itself, yielding the same results given by Eq. (6.6.1.2) and (6.6.1.3), respectively for the prepotential (6.3.0.1) and its classical limit $\zeta \to 0$ (*d*-SK geometry). The same can explicitly be proved to hold for the Ricci tensor (6.6.1.4) and the Ricci scalar (6.6.1.5), and for their respective classical limits (6.6.1.6) and (6.6.1.7).

Thus, the approaches respectively based on (6.6.1.1) and (6.4.0.5) have been proved to be equivalent, by explicitly computing the expressions of the Riemann tensor $R_{i\bar{j}k\bar{l}}$, of Ricci tensor $R_{i\bar{j}}$ and of Ricci scalar curvature R of a SK geometry of arbitrary complex dimension n_V and determined by the holomorphic prepotential (6.3.0.1) (also considering the corresponding limit of d-SK geometry, obtained by letting the quantum parameter $\zeta \to 0$). As previously mentioned, by including in the prepotential the most general quantum perturbative correction consistent with the Peccei-Quinn axion-shift symmetry [33] (see discussion in the Introduction), the results and considerations of Sect. 6.6 are an extension of the findings of [54] to the quantum perturbative regime.

6.7 Conclusion

It is clear that the present investigation (completing, extending and generalizing the work of [46] and [55]) does not conclude the study of quantum (perturbative) SK geometries. Only some venues have been considered in the vast realm of quantum

geometries of the moduli spaces of superstring theories. Many issues still deserve a deeper understanding and call for a thorough analysis, and we leave them for further future study. Below, we list only some of the most appealing ones (to us).

1. It would be interesting to determine the extent of validity of the so-called *"rule of three"* (6.4.0.15), which is nothing but the sum rule determining the value of V_{BH} at its non-BPS $Z \neq 0$ critical points. While its *"non-BPS $Z = 0$ analogue"* (6.5.0.4) has general validity, (6.4.0.15) does not hold in general. Firstly noticed in [59], the *"rule of three"* (6.4.0.15) has been proved to hold in *symmetric* SK geometries [17], in (*at least* some of the) homogeneous *non-symmetric* d-SK geometries [72] (and in $\mathcal{N} > 2$-extended supergravities admitting non-supersymmetric attractors with non-vanishing central charge matrix [8; 77]). The most general results for d-SK geometries currently available are given in [59], but they are depending on the particular considered BH charge configurations; thus, it would be nice to see whether the *"rule of three"* (6.4.0.15) still holds in a generic BH charge configuration. On the other hand, since the condition (6.4.0.16) of validity of the *"rule of three"* does not imply symmetricity (nor homogeneity), it would be nice to see if and how the *"rule of three"* works in the quantum corrected SK geometries (6.3.0.1).

2. In the present chapter we explained the peculiarity of the $D0 - D6$ configuration in presence of the most general axion-shift-symmetric quantum perturbative parameter ξ. The $D0 - D6$ configuration turns out to be the somewhat *"minimal"* configuration which does *not* support axion-free critical points of V_{BH}. But we did not yet completely explain the results of the investigation of [55]. In other words, we did not explain why the classical non-BPS $Z \neq 0$ *"flat"* direction of V_{BH} of the st^2 model gets non-renormalized (despite acquiring a non-vanishing axion) when switching ξ on. We leave the investigation of this issue (within d-SKG geometries of arbitrary complex dimension n_V) for future study.

3. An issue concerning both d-SK geometries and their quantum corrected counterparts (6.3.0.1) is the generality of the *axion-free* solutions (*if any*) to the Attractor Eqs.. As found in [65], the axion-free-supporting BH charge configurations in d-SK geometries are the *electric* ($D2 - D6$), *magnetic* ($D0 - D4$) and $D0 - D6$ ones, whereas in the present work we obtained that for SK geometries determined by the prepotential (6.3.0.1) only *electric* and *magnetic* configurations support purely imaginary critical points of V_{BH}. It would be interesting to analyze the degree of generality of *axion-free* solutions (in a model-independent fashion, if possible) in these frameworks.

4. Concerning d-SKG geometries, the expression of the $\frac{1}{2}$-BPS attractors is known in the most explicit form possible [58], and (going beyond symmetric cases) there are various explicit (but charge-dependent) results for non-BPS $Z \neq 0$ critical points of V_{BH} (see *e.g.* [59]). On the other hand, there are currently no general results on the explicit form of non-BPS $Z = 0$ critical points of V_{BH} within the same SK geometry. Thus, it would be interesting to determine such expression and use it to elaborate the *"non-BPS Z = 0 analogue"* (6.5.0.4) (obtained in the present paper) of the *"rule of three"* (6.4.0.15).

5. Still very little is known on the explicit expression of the critical points of the quantum perturbatively corrected BH potential V_{BH} given by Eq. (6.2.0.27). The complete analysis of $\frac{1}{2}$-BPS critical points (beyond the axion-free results of [19]; see the end of Sect. 6.3) should be based on the implementation of $\frac{1}{2}$-BPS conditions (6.3.0.13) through the formula (6.3.0.7). More interestingly, the non-BPS ($Z \neq 0$ and $Z = 0$) critical points of (6.2.0.27) still need to be completely determined and studied.

6. The phenomena of *"splitting"* of attractors [46], *"transmutation"* of attractors [46], and *"lifting"* of *moduli spaces* of attractors [55], even if explicitly found by studying models with only one or two complex scalar field(s), are likely to characterize the quantum perturbatively corrected SK geometry (6.3.0.1) for an arbitrary complex dimension n_V. Thus, it would be worth studying more in depth such phenomena, eventually relating them with the presence of particular symmetry groups acting in transitive or non-transitive way on the (generally non-homogeneous) scalar manifold.

7. By extending the results obtained in [55] (at least in the *magnetic* and *electric* configurations) to the presence of more than one *"flat"* direction, and including the effects of non-perturbative corrections (see *e.g.* [19; 22; 31; 36]), one would be lead to conjecture that *only a (very) few classical attractors do remain attractors in strict sense at the quantum level*. Consequently, *at the quantum* (perturbative *and* non-perturbative) *level the set of actual extremal BH attractors should be strongly constrained and reduced*. As already noticed in the Conclusion of [55] itself, in $\mathcal{N} = 8$, $d = 4$ supergravity the (*"large"*) $\frac{1}{8}$-BPS and non-BPS BHs critical points of $V_{BH,\mathcal{N}=8}$ exhibit 40 and 42 *"flat"* directions, respectively [51; 78]. Within the possibility of $\mathcal{N} = 8$ supergravity to be a finite theory of quantum gravity (see *e.g.* [79] and [80], and Refs. therein), it would be interesting to understand whether these *"flat"* directions may be removed at all by perturbative *and/or* non-perturbative quantum effects.

Bibliography

[1] S. Bellucci, A. Marrani and R. Roychowdhury, *On Quantum Special Kähler Geometry*, Int. J. Mod. Phys. **A25**, 1891 (2010), `arXiv:0910.4249 [hep-th]`.

[2] S. Ferrara, R. Kallosh and A. Strominger, $\mathcal{N}=2$ *Extremal Black Holes*, Phys. Rev. **D52**, 5412 (1995), `hep-th/9508072`.

[3] A. Strominger, *Macroscopic Entropy of* $\mathcal{N}=2$ *Extremal Black Holes*, Phys. Lett. **B383**, 39 (1996), `hep-th/9602111`.

[4] S. Ferrara and R. Kallosh, *Supersymmetry and Attractors*, Phys. Rev. **D54**, 1514 (1996), `hep-th/9602136`.

[5] S. Ferrara and R. Kallosh, *Universality of Supersymmetric Attractors*, Phys. Rev. **D54**, 1525 (1996), `hep-th/9603090`.

[6] S. Ferrara, G. W. Gibbons and R. Kallosh, *Black Holes and Critical Points in Moduli Space*, Nucl. Phys. **B500**, 75 (1997), `hep-th/9702103`.

[7] G. W. Gibbons and C. M. Hull, *A Bogomol'ny Bound for General Relativity and Solitons in* $\mathcal{N}=2$ *Supergravity*, Phys. Lett. **B109**, 190 (1982).

[8] L. Andrianopoli, R. D'Auria, S. Ferrara and M. Trigiante, *Extremal black holes in supergravity*, Lect. Notes Phys. **737**, 661 (2008), `hep-th/0611345`.

[9] A. Sen, *Black Hole Entropy Function, Attractors and Precision Counting of Microstates*, `arXiv:0708.1270`.

[10] S. Bellucci, S. Ferrara, R. Kallosh and A. Marrani, *Extremal Black Hole and Flux Vacua Attractors*, Lect. Notes Phys. **755**, 115 (2008), `arXiv:0711.4547 [hep-th]`.

[11] S. Ferrara, K. Hayakawa and A. Marrani, *Lectures on Attractors and Black Holes*, Fortsch. Phys. **56**, 993 (2008), `arXiv:0805.2498 [hep-th]`.

[12] S. Bellucci, S. Ferrara, M. Günaydin and A. Marrani, *SAM Lectures on Extremal Black Holes in* $d=4$ *Extended Supergravity*, `arXiv:0905.3739 [hep-th]`.

[13] A. Sen, *Black hole entropy function and the attractor mechanism in higher derivative*

gravity, JHEP **0509**, 038 (2005), `hep-th/0506177`.

[14] A. Strominger, *Special Geometry*, Commun. Math. Phys. **133**, 163 (1990).

[15] A. Ceresole, R. D'Auria and S. Ferrara, *The Symplectic Structure of $\mathcal{N} = 2$ Supergravity and Its Central Extension*, Nucl. Phys. Proc. Suppl. **46** (1996), `hep-th/9509160`.

[16] D. S. Freed, *Special Kähler manifolds*, Commun. Math. Phys. **203**, 31 (1999), `hep-th/9712042`.

[17] S. Bellucci, S. Ferrara, M. Günaydin and A. Marrani, *Charge orbits of symmetric special geometries and attractors*, Int. J. Mod. Phys. **A21**, 5043 (2006), `hep-th/0606209`.

[18] G. L. Cardoso, D. Lust and T. Mohaupt, *Modular symmetries of $\mathcal{N} = 2$ black holes*, Phys. Lett. **B388**, 266 (1996), `hep-th/9608099`.

[19] K. Behrndt, G. L. Cardoso, B. de Wit, R. Kallosh, D. Lust and T. Mohaupt, *Classical and quantum $\mathcal{N} = 2$ supersymmetric black holes*, Nucl. Phys. **B488**, 236 (1997), `hep-th/9610105`.

[20] K. Behrndt, *Quantum corrections for $D = 4$ black holes and $D = 5$ strings*, Phys. Lett. **B396**, 77 (1997), `hep-th/9610232`.

[21] K. Behrndt and T. Mohaupt, *Entropy of $\mathcal{N} = 2$ black holes and their M-brane description*, Phys. Rev. **D56**, 2206 (1997), `hep-th/9611140`.

[22] K. Behrndt, G. L. Cardoso, I. Gaida, *Quantum $\mathcal{N} = 2$ supersymmetric black holes in the $S - T$ model*, Nucl. Phys. **B506**, 267 (1997), `hep-th/9704095`.

[23] J. M. Maldacena, A. Strominger and E. Witten, *Black hole entropy in M-theory*, JHEP **9712**, 002 (1997), `hep-th/9711053`.

[24] G. L. Cardoso, B. de Wit and T. Mohaupt, *Corrections to macroscopic supersymmetric black hole entropy*, Phys. Lett. **B451**, 309 (1999), `hep-th/9812082`.

[25] G. L. Cardoso, B. de Wit and T. Mohaupt, *Deviations from the area law for supersymmetric black holes,* Fortsch. Phys. **48**, 49 (2000), `hep-th/9904005`.

[26] G. L. Cardoso, B. de Wit and T. Mohaupt, *Macroscopic entropy formulae and non-holomorphic corrections for supersymmetric black holes*, Nucl. Phys. **B567**, 87 (2000), `hep-th/9906094`.

[27] G. L. Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Stationary BPS solutions in $\mathcal{N} = 2$ supergravity with R^2 interactions*, JHEP **0012**, 019 (2000), `hep-th/0009234`.

[28] G. L. Cardoso, D. Lüst and J. Perz, *Entropy maximization in the presence of higher-*

curvature interactions, JHEP **0605**, 028 (2006), `hep-th/0603211`.

[29] G. L. Cardoso, V. Grass, D. Lüst and J. Perz, *Extremal non-BPS Black Holes and Entropy Extremization*, JHEP **0609**, 078 (2006), `hep-th/0607202`.

[30] G.L. Cardoso, B. de Wit and S. Mahapatra, *Black hole entropy functions and attractor equations*, JHEP **0703**, 085 (2007), `hep-th/0612225`.

[31] G.L. Cardoso, B. de Wit and S. Mahapatra, *Subleading and non-holomorphic corrections to $\mathcal{N}=2$ BPS black hole entropy*, HEP **0902**, 006 (2009), `arXiv:0808.2627 [hep-th]`.

[32] S. Cecotti, S. Ferrara and L. Girardello, *Geometry of Type II Superstrings and the Moduli of Superconformal Field Theories*, Int. J. Mod. Phys. **A4**, 2475 (1989).

[33] R. D. Peccei and H. R. Quinn, *Constraints imposed by CP conservation in the presence of instantons*, Phys. Rev. **D16**, 1791 (1977). R. D. Peccei and H. R. Quinn, *CP conservation in the presence of instantons*, Phys. Rev. Lett. **38**, 1440 (1977). R. D. Peccei and H. R. Quinn, *Some aspects of instantons*, Nuovo Cim. **A41**, 309 (1977).

[34] B. de Wit and A. Van Proeyen, *Special geometry, cubic polynomials and homogeneous quaternionic spaces*, Commun. Math. Phys. **149**, 307 (1992), `hep-th/9112027`.

[35] B. de Wit, F. Vanderseypen and A. Van Proeyen, *Symmetry structure of special geometries*, Nucl. Phys. **B400**, 463 (1993), `hep-th/9210068`..

[36] P. Candelas, X. C. De La Ossa, P. S. Green and L. Parkes, *A Pair of Calabi-Yau Manifolds as an Exactly Soluble Superconformal Theory*, Nucl. Phys. **B359**, 21 (1991). P. Candelas, X. C. De La Ossa, P. S. Green and L. Parkes, *An Exactly Soluble Superconformal Theory from a Mirror Pair of Calabi-Yau Manifolds*, Phys. Lett. **B258**, 118 (1991).

[37] L. Alvarez-Gaume, D. Z. Freedman, *Geometrical Structure and Ultraviolet Finiteness in the Supersymmetric Sigma Model*, Commun. Math. Phys. **80**, 443 (1981).

[38] M. T. Grisaru, A. van de Ven and D. Zanon, *Four Loop Beta Function for the $\mathcal{N}=1$ and $\mathcal{N}=2$ Supersymmetric Nonlinear Sigma Model in Two Dimensions*, Phys. Lett. **B173**, 423 (1986). M. T. Grisaru, A. van de Ven and D. Zanon, *Two Dimensional Supersymmetric Sigma Models on Ricci Flat Kähler Manifolds are not Finite*, Nucl. Phys. **B277**, 388 (1986). M. T. Grisaru, A. van de Ven and D. Zanon, *Four Loop Divergences for the $\mathcal{N}=1$ Supersymmetric Nonlinear Sigma Model in Two Dimensions*, Nucl. Phys. **B277**, 409 (1986).

[39] S. Hosono, A. Klemm, S. Theisen and Shing-Tung Yau, *Mirror symmetry, mirror map and applications to Calabi-Yau hypersurfaces*, Commun. Math. Phys. **167**, 301

(1995), `hep-th/9308122`.

[40] E. Witten, *Dyons of charge e theta/2 pi*, Phys. Lett. **B86**, 283 (1979).

[41] S. Ferrara, J. A. Harvey, A. Strominger and C. Vafa, *Second quantized mirror symmetry*, Phys. Lett. **B361**, 59 (1995), `hep-th/9505162`. P. S. Aspinwall, *An $\mathcal{N} = 2$ dual pair and a phase transition*, Nucl. Phys. **B460**, 57 (1996), `hep-th/9510142`. D. R. Morrison and C. Vafa, *Compactifications of F theory on Calabi-Yau threefolds. 1*, Nucl. Phys. **B473**, 74 (1996), `hep-th/9602114`. D. R. Morrison and C. Vafa, *Compactifications of F theory on Calabi-Yau threefolds. 2*, Nucl. Phys. **B476**, 437 (1996), `hep-th/9603161`. J. A. Harvey and G. W. Moore, *Exact gravitational threshold correction in the FHSV model*, Phys. Rev. **D57**, 2329 (1998), `hep-th/9611176`. A. Klemm and M. Marino, *Counting BPS states on the Enriques Calabi-Yau*, Commun. Math. Phys. **280**, 27 (2008), `hep-th/0512227`. J. R. David, *On the dyon partition function in $\mathcal{N} = 2$ theories*, JHEP **0802**, 025 (2008), `arXiv:0711.1971`.

[42] M. Bianchi and S. Ferrara, *Enriques and Octonionic Magic Supergravity Models*, JHEP **0802**, 054 (2008), `arXiv:0712.2976 [hep-th]`.

[43] M. Günaydin, *Lectures on Spectrum Generating Symmetries and U-duality in Supergravity, Extremal Black Holes, Quantum Attractors and Harmonic Superspace*, `arXiv:0908.0374 [hep-th]`.

[44] S. Bellucci, S. Ferrara, A. Marrani and A. Yeranyan, *Mirror Fermat Calabi-Yau Threefolds and Landau-Ginzburg Black Hole Attractors*, Riv. Nuovo Cim. **029**, 1 (2006), `hep-th/0608091`.

[45] P. Kaura and A. Misra, *On the Existence of Non-Supersymmetric Black Hole Attractors for Two-Parameter Calabi-Yau's and Attractor Equations*, Fortsch. Phys. **54**, 1109 (2006), `hep-th/0607132`.

[46] S. Bellucci, S. Ferrara, A. Marrani and A. Shcherbakov, *Splitting of Attractors in 1-modulus Quantum Corrected Special Geometry*, JHEP **0802**, 088 (2008), `arXiv:0710.3559 [hep-th]`.

[47] A. Chou, R. Kallosh, J. Rahmfeld, Soo-Jong Rey, M. Shmakova and Wing Kai Wong, *Critical points and phase transitions in 5-d compactifications of M-theory*, Nucl. Phys. **B508**, 147 (1997), `hep-th/9704142`.

[48] R. Kallosh, A. D. Linde and M. Shmakova, *Supersymmetric multiple basin attractors*, JHEP **9911**, 010 (1999), `hep-th/9910021`.

[49] M. Wijnholt and S. Zhukov, *On the Uniqueness of Black Hole Attractors*, `hep-th/9912002`.

[50] G.W. Moore, *Attractors and Arithmetic*, `hep-th/9807056`. G.W. Moore, *Arithmetic and Attractors*, `hep-th/9807087`. G.W. Moore, *Les Houches Lectures on Strings and Arithmetic*, `hep-th/0401049`.

[51] S. Ferrara and A. Marrani, *$\mathcal{N}= 8$ non-BPS Attractors, Fixed Scalars and Magic Supergravities*, Nucl. Phys. **B788**, 63 (2008), `arXiV:0705.3866`.

[52] S. Ferrara and A. Marrani, *On the Moduli Space of non-BPS Attractors for $\mathcal{N}= 2$ Symmetric Manifolds*, Phys. Lett. **B652**, 111 (2007), `arXiV:0706.1667`.

[53] S. Bellucci, S. Ferrara, A. Shcherbakov and A. Yeranyan, *Black hole entropy, flat directions and higher derivatives*, `arXiv:0906.4910 [hep-th]`.

[54] E. Cremmer and A. Van Proeyen, *Classification Of Kahler Manifolds In $\mathcal{N}= 2$ Vector Multiplet Supergravity Couplings*, Class. Quant. Grav. **2**, 445 (1985).

[55] S. Bellucci, S. Ferrara, A. Marrani and A. Shcherbakov, *Quantum Lift of Non-BPS Flat Directions*, Phys. Lett. **B672**, 77 (2009), `arXiv:0811.3494 [hep-th]`.

[56] M. J. Duff, J. T. Liu and J. Rahmfeld, *Four-dimensional string/string/string triality*, Nucl. Phys. **B459**, 125 (1996), `hep-th/9508094`.

[57] K. Behrndt, R. Kallosh, J. Rahmfeld, M. Shmakova and W. K. Wong, *STU Black Holes and String Triality*, Phys. Rev. **D54**, 6293 (1996), `hep-th/9608059`.

[58] M. Shmakova, *Calabi-Yau black holes*, Phys. Rev. **D56**, 540 (1997), `hep-th/9612076`.

[59] P. K. Tripathy and S. P. Trivedi, *Non-supersymmetric attractors in string theory*, JHEP **0603**, 022 (2006), `hep-th/0511117`.

[60] K. Saraikin and C. Vafa, *Non-supersymmetric Black Holes and Topological Strings*, Class. Quant. Grav. **25**, 095007 (2008), `hep-th/0703214`.

[61] S. Nampuri, P. K. Tripathy and S. P. Trivedi, *On The Stability of Non-Supersymmetric Attractors in String Theory*, JHEP **0708**, 054 (2007), `arXiV:0705.4554`.

[62] S. Bellucci, A. Marrani, E. Orazi and A. Shcherbakov, *Attractors with Vanishing Central Charge*, Phys. Lett. **B655**, 185 (2007), `arXiV:0707.2730`.

[63] S. Bellucci, S. Ferrara, A. Marrani and A. Yeranyan, *stu Black Holes Unveiled*, Entropy Vol. **10**(4), 507 (2008), `arXiv:0807.3503 [hep-th]`.

[64] L. Andrianopoli, S. Ferrara, A. Marrani and M. Trigiante, *Non-BPS Attractors in 5d and 6d Extended Supergravity*, Nucl. Phys. **B795**, 428 (2008), `arXiv:0709.3488`.

[65] A. Ceresole, S. Ferrara and A. Marrani, *4d/5d Correspondence for the Black*

Hole Potential and its Critical Points, Class. Quant. Grav. **24**, 5651 (2007), `arXiV:0707.0964 [hep-th]`.

[66] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fré and T. Magri, $\mathcal{N}=2$ *supergravity and* $\mathcal{N}=2$ *super Yang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map,* J. Geom. Phys. **23**, 111 (1997), `hep-th/9605032`. L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara and P. Fré, *General Matter Coupled* $\mathcal{N}=2$ *Supergravity,* Nucl. Phys. **B476**, 397 (1996), `hep-th/9603004`.

[67] L. Castellani, R. D'Auria and S. Ferrara, *Special Geometry without Special Coordinates*, Class. Quant. Grav. **7**, 1767 (1990). L. Castellani, R. D'Auria and S. Ferrara, *Special Kähler Geometry: an Intrinsic Formulation from* $\mathcal{N}=2$ *Space-Time Supersymmetry*, Phys. Lett. **B241**, 57 (1990).

[68] R. D'Auria, S. Ferrara and P. Fré, *Special and Quaternionic Isometries: General Couplings in* $\mathcal{N}=2$ *Supergravity and the Scalar Potential*, Nucl. Phys. **B359**, 705 (1991).

[69] S. Bellucci, S. Ferrara and A. Marrani, *Attractors in Black*, Fortsch. Phys. **56**, 761 (2008), `ArXiv:0805.1310 [hep-th]`.

[70] Z. Lu, *A Note on Special Kähler Manifolds*, Math. Ann. **313**, 711 (1999), `math/0505577`.

[71] B. L. Cerchiai, S. Ferrara, A. Marrani, and B. Zumino, *Duality, Entropy and ADM Mass in Supergravity*, Phys. Rev. **D79**, 125010 (2009), `arXiv:0902.3973 [hep-th]`.

[72] R. D'Auria, S. Ferrara and M. Trigiante, *Critical points of the Black-Hole potential for homogeneous special geometries*, JHEP **0703**, 097 (2007), `hep-th/0701090`.

[73] M. Gomez-Reino, J. Louis and C. A. Scrucca, *No metastable de Sitter vacua in* $\mathcal{N}=2$ *supergravity with only hypermultiplets*, JHEP **0902**, 003 (2009), `arXiv:0812.0884`.

[74] J. D. Bekenstein, Phys. Rev. **D7**, 2333 (1973). S. W. Hawking, Phys. Rev. Lett. **26**, 1344 (1971); in C. DeWitt, B. S. DeWitt, *Black Holes (Les Houches 1972)* (Gordon and Breach, New York, 1973). S. W. Hawking, Nature **248**, 30 (1974). S. W. Hawking, Comm. Math. Phys. **43**, 199 (1975).

[75] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces* (Academic Press, New York, 1978).

[76] S. Ferrara and A. Marrani, *Symmetric Spaces in Supergravity*, contribution to the Proceedings of *"Symmetry in Mathematics and Physics"*, Los Angeles, CA, Jan

18-20, 2008, `arXiv:0808.3567 [hep-th]`.

[77] S. Ferrara and R. Kallosh, *On $\mathcal{N}=8$ Attractors*, Phys. Rev. **D73**, 125005 (2006), `hep-th/0603247`.

[78] L. Andrianopoli, R. D'Auria and S. Ferrara, *U invariants, black hole entropy and fixed scalars*, Phys. Lett. **B403**, 12 (1997), `hep-th/9703156`.

[79] Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson, D. A. Kosower and R. Roiban, *Three-Loop Finiteness of $\mathcal{N}=8$ Supergravity*, Phys. Rev. Lett. **98**, 161303 (2007), `hep-th/0702112`.

[80] R. Kallosh, *On UV Finiteness of the Four Loop $\mathcal{N}=8$ Supergravity*, JHEP **0909**, 116 (2009), `ArXiv:0906.3495 [hep-th]`.

[81] G. L. Cardoso, J. M. Oberreuter and J. Perz, *Entropy function for rotating extremal black holes in very special geometry*, JHEP **0705**, 025 (2007), `hep-th/0701176`.

Chapter 7

Topics in Cubic Special Geometry

In [1] we reconsider the sub-leading quantum perturbative corrections to $\mathcal{N} = 2$ cubic special Kähler geometries. Imposing the invariance under axion-shifts, all such corrections (but the imaginary constant one) can be introduced or removed through suitable, lower unitriangular symplectic transformations, dubbed Peccei-Quinn (PQ) transformations.

Since PQ transformations do not belong to the $d = 4$ U-duality group G_4, in symmetric cases they generally have a non-trivial action on the unique quartic invariant polynomial $\mathcal{I}_4$ of the charge representation $\mathbf{R}$ of G_4. This leads to interesting phenomena in relation to theory of extremal black hole attractors; i.e., the possibility to make transitions between different charge orbits of $\mathbf{R}$, with corresponding change of the supersymmetry properties of the supported attractor solutions. Furthermore, a suitable action of PQ transformations can also set $\mathcal{I}_4$ to zero, or vice versa it can generate a non-vanishing $\mathcal{I}_4$: this corresponds to transitions between "large" and "small" charge orbits, which we classify in some detail within the "special coordinates" symplectic frame.

Finally, after a brief account of the action of PQ transformations on the recently established correspondence between Cayley's hyperdeterminant and elliptic curves, we derive an equivalent, alternative expression of $\mathcal{I}_4$, with relevant application to black hole entropy.

7.1 Introduction

Special Kähler geometry (SK) characterizes the scalar manifolds of Abelian vector multiplets in $\mathcal{N} = 2$ supergravity theory in $d = 4$ space-time dimensions (see *e.g.* [2–

5], and Refs. therein). Along the years, it has played a key role in various important developments in black hole (BH) physics.

Among these, the *Attractor Mechanism* [6] shed light on the dynamics of scalar fields coupled to BPS (Bogomol'ny-Prasad-Sommerfeld) and non-BPS extremal BHs. Through the introduction of an effective BH potential V_{BH} [6], this mechanism describes the stabilization of the scalar fields in terms of the BH conserved charges in the near-horizon limit of the extremal BH background (see *e.g.* [8–12; 12], also for reviews and lists of Refs.).

Within theories with $\mathcal{N} = 2$ local supersymmetry emerging from Calabi-Yau compactifications of superstrings or M-theory, the Attractor Mechanism has played a key role in the study of connections with topological string partition functions [14] and relations with microstates counting (see for instance [10]), and also in the investigation of dynamical phenomena, such as wall crossing and split attractor flow (see *e.g.* [15], and Refs. therein).

In some seminal papers dating back to mid 90's [6], the Attractor Mechanism was discovered by Ferrara, Kallosh and Strominger in $\mathcal{N} = 2$, $d = 4$ *ungauged* supergravity coupled to n_V vector multiplets. This theory proved to be an especially relevant and rich framework for the study of the attractor dynamics of scalar flows coupled to extremal BHs.

An important arena in which many advances have been made along the years is provided by a particular yet broad class of SK geometries, i.e. the ones determined by an holomorphic prepotential function F which is purely cubic in the complex scalar fields themselves:

$$\mathcal{F}_d \equiv \frac{1}{3!} d_{ijk} z^i z^j z^k. \tag{7.1.0.1}$$

$\mathcal{F}_d$ defines the so-called d-SK geometries [16; 17]. These geometries naturally arise as the large volume limit of CY_3 compactifications of Type II(A) superstring theories, in which d_{ijk} is given by the triple intersection numbers of the CY_3 internal manifold itself (see Sec. 7.2.1 for further details, and list of Refs.).

Moreover, up to the so-called *minimal coupling* sequence (with quadratic prepotential) [18], all non-compact *symmetric* coset SK spaces $\frac{G_4}{H_4}$ are actually d-spaces, defined by a prepotential of the form [17]; G_4 is the $d = 4$ U-duality group[1], and H_4 is its maximal compact subgroup (with symmetric embedding). In symmetric SK geometries the Attractor Mechanism enjoys a noteworthy geometrical interpretation, related to the fascinating interplay among orbits of the charge irrepr. **R** of G_4

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[1] Here U-duality is referred to as the "continuous" limit (valid for large values of the charges) of the non-perturbative string theory symmetries introduced by Hull and Townsend in [19].

[20; 21], the solution of the Attractor Eqs. [21] and the related *"moduli spaces"* [22]. Through the Bekenstein-Hawking entropy (*S*) -area (*A*) formula [23]

$$\frac{S}{\pi} = \frac{A}{4} = \sqrt{|\mathcal{I}_4(\mathcal{Q})|}, \tag{7.1.0.2}$$

the entropy of the BH is given in terms of the unique invariant polynomial $\mathcal{I}_4$ of the charge irrepr. **R** of G_4, which is quartic in charges $\mathcal{Q}$. It is also worth recalling that also the recently introduced first order approach to non-BPS scalar flows [24] has been completely solved in terms of geometrical quantities (*U*-duality invariants) in [25].

It is therefore natural to ask what is the role and the effect of sub-leading corrections to the $\mathcal{N} = 2$ purely cubic prepotential (7.1.0.1). As is well known (see the recent discussion in [26], and Refs. therein), such corrections are of both quantum perturbative and non-perturbative nature, and not all of them are consistent with the Peccei-Quinn axion-shift symmetry [27], nor all of them actually affect the SK geometry of the scalar manifold itself (see *e.g.* [28]).

In this chapter, extending on some previous results in [16; 29; 30], we further develop the study of those sub-leading corrections to *d*-SK geometries (7.1.0.1) which are consistent with the axion-shift symmetry and which do not affect the geometry of the vector multiplets' scalar fields[2].

It is known [16; 29] that these sub-leading corrections can be included in (or removed from) the $\mathcal{N} = 2$ symplectic sections by acting with suitable symplectic transformations, and this provides an effective shortcut to the process of solving the Attractor Eqs. (*alias* criticality conditions for V_{BH}) in the so corrected *d*-SK geometries. As we will find in the present investigation, such symplectic transformations have a group structure (we dub them Peccei-Quinn (PQ) symplectic transformations), but they do not belong to the suitable symplectic representation of G_4 itself.

At least for symmetric *d*-SK geometries, this leads to interesting consequences in the theory of charge orbits and *"moduli spaces"* of extremal BH attractor solutions. Indeed, the PQ transformations do not affect the geometry of the scalar manifold, neither the statification of the charge irrepr. space **R** into disjoint orbits, nor the structure of the corresponding *"moduli spaces"* of attractors[3], but they can change the value and the sign of $\mathcal{I}_4$, thus possibly switching from one charge orbits to another.

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[2]For a recent discussion of the unique (constant imaginary) term which is consistent with axion-shift and affects the geometry, see *e.g.* [26].

[3]In this respect, the general analysis and findings of the present chapter explains the result obtained in Sec. 3 and App. A of [31], also providing a way to generalise them to generic BH charge configuration, and to a generic model with n_V vector multiplets.

Moreover, through the action of PQ symplectic group, also the results concerning non-perturbative

For instance, an extremal "small" BH configuration (with vanishing entropy according to formula (7.1.0.2)) within the d-SK geometry (7.1.0.1) can acquire, by introducing the quantum perturbative correction under consideration, a non-vanishing area of the event horizon, and thus a "large" nature (i.e., a non-vanishing $\mathcal{I}_4$, and thus entropy, according to (7.1.0.2)). The opposite phenomenon can occur too, i.e. that "large" extremal BH configuration can become "small" for particular choices of the supporting charge vectors.

Another possible phenomenon is that the supersymmetry preserving features of the attractor configurations of d-SK geometry (7.1.0.1) can change in presence of those sub-leading corrections accounted for by PQ transformations. This is somewhat analogous to some phenomena observed in presence of the "$+i\lambda$" correction in the prepotential in [32].

By exploiting the PQ symplectic transformation, we will also study how the effective BH potential V_{BH} gets modified in presence of the aforementioned corrections, and what is the fate of those charge configurations which support axion-free attractor solutions within the theory determined by (7.1.0.1). In general, the solutions of Attractor Eqs. for the corrected d-SK geometries can be obtained by considering the solutions in the purely cubic theory [30; 33], and by transforming the charges in such formulæ with a suitable PQ transformation.

We will also briefly comment on the action of the PQ group on the roots of certain cubic elliptic curves, which have been recently connected [34] to the Cayley's hyperdeterminant [35], i.e. to the (opposite of) $\mathcal{I}_4$ for the noteworthy *triality-symmetric* so-called *stu* supergravity model [36]. This might lead to an interpretation of the PQ transformation within the intriguing "BH/qubit correspondence" [37].

Finally, we derive an alternative expression of $\mathcal{I}_4$ for symmetric d-SK geometries, and more in general for symmetric cubic geometries (such as the ones of some $\mathcal{N} > 2$-extended, $d = 4$ supergravities). This result allows for a consistent treatment of some expressions of the BH entropy available in the literature (see *e.g.* [33]). Furthermore, its further generalisation to the case of non-symmetric geometries (in which $\mathcal{I}_4$ is not generally related to the BH entropy) explicitly shows the contribution of the so-called E-tensor [17] introducing an explicit dependence on (some of the) scalar degrees of freedom.

The plan of this chapter is as follows.

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instantonic corrections to the prepotential, obtained in Sec. 4 and App. B of [31], can be generalised to include the sub-leading quantum perturbative corrections under consideration. See treatment below for further comments.

In Sec. 7.2.1 we analyse the PQ symplectic transformations within $\mathcal{N} = 2, d = 4$ SK geometry. More specifically, in Sec. 7.2.1 we recall the general structure of subleading terms in cubic prepotential, and their consistency with axion-shift symmetry. The PQ symplectic group is introduced in Sec. 7.2.1, and its relation to the *U*-duality group clarified in Sec. 7.2.1. Moreover, Sec. 7.2.1 considers some aspects of stringy origin and topological interpretation of some generators of the PQ group.

Then, Sec. 7.2.2 applies this general formalism to relevant issues within the theory of extremal black hole attractors. Secs. 7.2.2 and 7.2.2 are devoted to the study and classification (within symmetric cubic geometries) of the PQ group on the unique invariant polynomial $\mathcal{I}_4$ of the charge representation **R** of the *U*-duality group. At the end of Sec. 7.2.2, we briefly comment on the relevance of the PQ group for the attractor values of the scalars, *i.e.* for the non-degenerate critical points of the effective BH potential V_{BH}. The transformation properties of the latter are studied in Sec. 7.2.2, with an analysis of the possible axion-free supporting charge configurations.

Sec. 7.2.3 briefly analyses the *"PQ-deformation"* of the recently established intriguing relation between Cayley's hyperdeterminant and elliptic curves.

Finally, in Sec. 7.3 an equivalent, alternative expression for $\mathcal{I}_4$ is derived, by exploiting the identities characterising symmetric cubic special geometries, with relevant consequences on the matching of known expressions of the black hole entropy. In particular, the new expression $\mathcal{I}_4$ allows one to relate its scalar-dependence in non-symmetric geometries directly to the so-called *E*-tensor.

7.2 Peccei-Quinn Symplectic Transformations

7.2.1 General Theory

Let us consider $\mathcal{N} = 2, d = 4$ *ungauged* Maxwell-Einstein supergravity, whose vector multiplets' scalar manifold is endowed with special Kähler (SK) geometry, based on an holomorphic prepotential function F, homogeneous of degree 2 in the contravariant symplectic sections X^Λ (the reader is addressed *e.g.* to [2–5] for a thorough introduction and list of Refs.).

Cubic Special Geometries and Axion-Shifts

We start and define the most general form of *cubic* prepotential as follows[4] ($d_{\Lambda\Sigma\Xi} = d_{(\Lambda\Sigma\Xi)} \in \mathbb{C}$):

$$
\begin{aligned}
F & \equiv \frac{1}{3!} d_{\Lambda\Sigma\Xi} \frac{X^\Lambda X^\Sigma X^\Xi}{X^0} = \\
& = \frac{1}{3!} \left(\mathrm{Re} d_{ijk} + i\mathrm{Im} d_{ijk} \right) \frac{X^i X^j X^k}{X^0} + \frac{1}{2} \left(\mathrm{Re} d_{0ij} + i\mathrm{Im} d_{0ij} \right) X^i X^j + \\
& \quad + \frac{1}{2} \left(\mathrm{Re} d_{00i} + i\mathrm{Im} d_{00i} \right) X^i X^0 + \frac{1}{3!} \left(\mathrm{Re} d_{000} + i\mathrm{Im} d_{000} \right) \left(X^0 \right)^2. \quad (7.2.1.1)
\end{aligned}
$$

By denoting the real and imaginary part of X^i respectively as $X^i \equiv R^i + iI^i$, the corresponding Kähler potential reads[5]

$$
\begin{aligned}
\mathcal{K} & \equiv -\ln\left[i\left(X^\Lambda \overline{F}_\Lambda - \overline{X}^\Lambda F_\Lambda \right) \right] \\
& = -\frac{4}{3} i\mathrm{Re} d_{ijk} I^i I^j I^k - \frac{2}{3} i\mathrm{Im} d_{ijk} R^i R^j R^k - 2i\mathrm{Im} d_{ijk} R^i I^j I^k \\
& \quad - 2i\mathrm{Im} d_{0ij} R^i R^j - 2i\mathrm{Im} d_{0ij} I^i I^j - 2i\mathrm{Im} d_{00i} R^i - \frac{2}{3} i\mathrm{Im} d_{000}. \quad (7.2.1.2)
\end{aligned}
$$

Thus, the invariance of $\mathcal{K}$ under Peccei-Quinn (PQ) perturbative (continuous) axion-shift symmetry [27]

$$
R^i \to R^i + \alpha^i, \ \alpha^i \in \mathbb{R} \quad (7.2.1.3)
$$

yields

$$
\mathrm{Im} d_{ijk} = \mathrm{Im} d_{0ij} = \mathrm{Im} d_{00i} = 0. \quad (7.2.1.4)
$$

The resulting axion-shift-invariant expression of $\mathcal{K}$ then simply reads

$$
\mathcal{K} = -\frac{4}{3} i\mathrm{Re} d_{ijk} I^i I^j I^k - \frac{2}{3} i\mathrm{Im} d_{000}, \quad (7.2.1.5)
$$

and the prepotential F given by (7.2.1.1) can accordingly be split as

$$
F = \mathbf{F} + \mathfrak{F}, \quad (7.2.1.6)
$$

where

$$
\mathbf{F} \equiv \frac{1}{3!} \mathrm{Re} d_{ijk} \frac{X^i X^j X^k}{X^0} + \frac{i}{3!} \mathrm{Im} d_{000} \left(X^0 \right)^2 \quad (7.2.1.7)
$$

---

[4]Greek capital and Latin lowercase indices respectively run $0, 1, ..., n_V$ and $1, ..., n_V$ throughout. The naught index pertains to the graviphoton, while n_V denotes the number of Abelian vector multiplets coupled to the supergravity one. Therefore, we work within the so-called symplectic basis of *special coordinates* (see e.g. [3; 17] and Refs. therein), which is manifestly covariant with respect to the $d = 5$ *U*-duality group G_5.

[5]For simplicity's sake, in Eqs. (7.2.1.2), (7.2.1.3) and (7.2.1.5) we give the result for $X^0 \equiv 1$, which does not imply any loss of generality for our purposes.

is the part contributing to $\mathcal{K}$ given by (7.2.1.5) and thus to the SK geometry, and

$$\mathfrak{F} \equiv \frac{1}{2}\mathrm{Re}d_{0ij}X^iX^j + \frac{1}{2}\mathrm{Re}d_{00i}X^iX^0 + \frac{1}{3!}\mathrm{Re}d_{000}\left(X^0\right)^2 \qquad (7.2.1.8)$$

is a *quadratic* form in X^Λ, which does not contribute to $\mathcal{K}$. Thus, **F** given by (7.2.1.7) is the most general cubic prepotential which is consistent with the PQ axion-shift (7.2.1.3) and which affects the geometry of the scalar manifold itself [28]. Some issues within the SK geometry based on **F** have been recently investigated in [26] (see also [29]).

On the other hand, $\mathrm{Re}d_{ijk}$ is usually denoted simply by the real symbol d_{ijk}, and the holomorphic function

$$\mathbf{F}_d \equiv \frac{1}{3!}d_{ijk}\frac{X^iX^jX^k}{X^0} \qquad (7.2.1.9)$$

is the prepotential of the so-called *d*-SK geometries[6] [17; 40]. This will be the most general framework we will be considering in the applications of Sec. 7.2.2.

For later convenience, let us compute the derivatives of $\mathfrak{F}$ with respect to the sections[7] X^Λ:

$$\mathfrak{F}_\Lambda \equiv D_\Lambda \mathfrak{F} = \frac{\partial \mathfrak{F}}{\partial X^\Lambda} = \begin{cases} \mathfrak{F}_0 = \frac{1}{2}\mathrm{Re}d_{00i}X^i + \frac{1}{3}\mathrm{Re}d_{000}X^0; \\ \\ \mathfrak{F}_i = \mathrm{Re}d_{0ij}X^j + \frac{1}{2}\mathrm{Re}d_{00i}X^0. \end{cases} \qquad (7.2.1.10)$$

It has been known (see *e.g.* [16; 29; 30]) that $\mathfrak{F}$ can be introduced (or removed) in any $\mathcal{N} = 2$ prepotential by performing suitable symplectic transformations. More specifically, through the action of particular symplectic transformations one can introduce the effect of the sub-leading quantum perturbative terms (7.2.1.8) into the explicit expression of horizon values of attractors and into the corresponding value of BH entropy [29; 30].

A major part of the present investigation is devoted to a thorough analysis of this issue in general form. In particular, we will focus on the effect of $\mathfrak{F}$ on the BH entropy in the general framework of *d*-SKG, with leading cubic prepotential given by (7.2.1.9). This will naturally lead to the study of the effect of the so-called *Peccei-Quinn transformations*, i.e. particular symplectic transformations deeply related to

---

[6]Regardless of the explicit form of d_{ijk}, the corresponding special Kähler manifold has always *at least* $n_V + 1$ global isometries, i.e. an overall scaling and PQ axion-shifts (see Eq. (7.2.1.3)), forming the group $SO(1,1) \times_s \mathbb{R}^{n_V}$, which can be considered the "minimal G_4" of *d*-SK geometries. Its relation to $d = 5$ uplift and further details can be found *e.g.* in [38] (see also Refs. therein).

[7]As shown in [39], the symplectic connection of SK geometry is *flat*.

$\mathfrak{F}$, on the duality invariants and supersymmetry properties of extremal BH attractor solutions.

The results recently obtained in Sec. 3 of [31] provide an explicit example (with $n_V = 2$ and for a particular charge configuration) of some aspects of the general treatment given here. Indeed, the prepotential given by Eq. (3.7) of [31] is nothing but a particular case[8] of the general structure (7.2.1.6)-(7.2.1.8).

The Peccei-Quinn Symplectic Group

Given an element[9]

$$S \equiv \begin{pmatrix} \mathcal{U} & \mathcal{Z} \\ \mathcal{W} & \mathcal{V} \end{pmatrix} \in GL\left(2n_V + 2, \mathbb{R}\right), \qquad (7.2.1.11)$$

it belongs to the symplectic group $Sp\left(2n_V + 2, \mathbb{R}\right) \subsetneq GL\left(2n_V + 2, \mathbb{R}\right)$ *iff*

$$S^T \Omega S = \Omega \Leftrightarrow S^{-1} = \Omega^{-1} S^T \Omega = -\Omega S^T \Omega, \qquad (7.2.1.12)$$

where Ω is the $(2n_V + 2) \times (2n_V + 2)$ symplectic metric (the subscripts denote the dimensions of the square block components):

$$\Omega \equiv \begin{pmatrix} 0_{n_V+1} & \mathbb{I}_{n_V+1} \\ -\mathbb{I}_{n_V+1} & 0_{n_V+1} \end{pmatrix}. \qquad (7.2.1.13)$$

The finite condition of symplecticity (7.2.1.12) translates on the square block components of S as follows:

$$\mathcal{U}^T \mathcal{V} - \mathcal{W}^T \mathcal{Z} = \mathbb{I}_{n_V+1}; \qquad (7.2.1.14)$$
$$\mathcal{U}^T \mathcal{W} - \mathcal{W}^T \mathcal{U} = \mathcal{Z}^T \mathcal{V} - \mathcal{V}^T \mathcal{Z} = 0_{n_V+1}. \qquad (7.2.1.15)$$

In general, the U-duality group G_4 of $\mathcal{N} = 2$, $d = 4$ supergravity is embedded into $Sp\left(2n_V + 2, \mathbb{R}\right)$ through its relevant (i.e., smallest symplectic) (ir)repr. **R** (see *e.g.* [3] and Refs. therein):

$$G_4 \overset{\mathbf{R}}{\subsetneq} Sp\left(2n_V + 2, \mathbb{R}\right). \qquad (7.2.1.16)$$

---

[8]In this respect (and referring to the equation numbering of [31]), it is worth noting that the second of Eqs. (3.8) can be directly obtained from the general expression (2.9) for d-SK geometry, because the sub-leading quantum perturbative terms appearing in Eq. (3.7) do *not* affect the Kähler potential and thus the metric.

[9]In all the following treatment, we work in the *(semi)classical* limit of large (continuous) charges, thus the field of definition of considered linear and symplectic groups is $\mathbb{R}$, and not $\mathbb{Z}$, as instead it would pertain to the *quantum* level.

The vector of the fluxes of the two-form field strengths of the Abelian vector fields and of their duals

$$\mathcal{Q} \equiv \left(p^\Lambda, q_\Lambda \right)^T = \left(p^0, p^i, q_0, q_i \right)^T, \tag{7.2.1.17}$$

as well as the vector of the holomorphic sections

$$\mathbf{V} \equiv \left(X^\Lambda, F_\Lambda \right)^T = \left(X^0, X^i, F_0, F_i \right)^T, \tag{7.2.1.18}$$

sit in **R**, and thus they are $Sp\left(2n_V + 2, \mathbb{R}\right)$-covariant, transforming under $\mathcal{S}$ as follows:

$$\mathcal{Q}' = \mathcal{S}\mathcal{Q} = \begin{pmatrix} \mathcal{U}^\Lambda_\Sigma p^\Sigma + \mathcal{Z}^{\Lambda\Sigma} q_\Sigma \\ \mathcal{W}_{\Lambda\Sigma} p^\Sigma + \mathcal{V}_\Lambda^\Sigma q_\Sigma \end{pmatrix}; \tag{7.2.1.19}$$

$$\mathbf{V}' = \mathcal{S}\mathbf{V} = \begin{pmatrix} \mathcal{U}^\Lambda_\Sigma X^\Sigma + \mathcal{Z}^{\Lambda\Sigma} F_\Sigma \\ \mathcal{W}_{\Lambda\Sigma} X^\Sigma + \mathcal{V}_\Lambda^\Sigma F_\Sigma \end{pmatrix}. \tag{7.2.1.20}$$

Now, by recalling (7.2.1.10), it is immediate to realize that $\mathfrak{F}_\Lambda$ can be generated or removed by performing a suitable symplectic finite transformation on **V**. Indeed, the identification

$$\mathfrak{F}_\Lambda \equiv F'_\Lambda - \mathcal{V}_\Lambda^\Sigma F_\Sigma = \mathcal{W}_{\Lambda\Sigma} X^\Sigma = \mathcal{W}_{\Lambda 0} X^0 + \mathcal{W}_{\Lambda i} X^i \tag{7.2.1.21}$$

defines, through Eq. (7.2.1.20), the components of the $(n_V + 1) \times (n_V + 1)$ submatrix $\mathcal{W}_{\Lambda\Sigma}$:

$$\mathcal{W}_{\Lambda\Sigma} = \begin{pmatrix} \mathcal{W}_{00} & \mathcal{W}_{0j} \\ \mathcal{W}_{i0} & \mathcal{W}_{ij} \end{pmatrix} \equiv \frac{1}{3!} \begin{pmatrix} 2\mathrm{Re}d_{000} & 3\mathrm{Re}d_{00j} \\ 3\mathrm{Re}d_{00i} & 6\mathrm{Re}d_{0ij} \end{pmatrix} \equiv \begin{pmatrix} \varrho & \mathbf{c}_j \\ \mathbf{c}_i & \Theta_{ij} \end{pmatrix} = \mathcal{W}_{(\Lambda\Sigma)}, \tag{7.2.1.22}$$

which inherits the symmetry properties from the relevant components of the $d_{\Lambda\Sigma\Xi}$ tensor. Note that we re-named the quantities for simplicity's sake ($\Theta_{ij} = \Theta_{(ij)}$).

Thus, we are going to deal with particular symplectic transformations defined as follows:

1. In order to keep the contravariant symplectic sections X^Λ (and thus the coordinates of the scalar manifold) *invariant* under the considered transformations, (7.2.1.20) imposes

$$\mathcal{Z}^{\Lambda\Sigma} \equiv 0, \, \mathcal{U}^\Lambda_\Sigma \equiv \delta^\Lambda_\Sigma. \tag{7.2.1.23}$$

2. In order to generate or remove $\mathfrak{F}_\Lambda$, as stated above one must define $\mathcal{W}_{\Lambda\Sigma}$ as in Eq. (7.2.1.22), and furthermore Eq. (7.2.1.20) yields

$$\mathcal{V}_\Lambda^\Sigma \equiv \delta_\Sigma^\Lambda. \qquad (7.2.1.24)$$

The $(n_V + 1) \times (n_V + 1)$ matrices $\mathcal{U}$, $\mathcal{Z}$, $\mathcal{V}$ and $\mathcal{W}$ defined by Eqs. (7.2.1.22), (7.2.1.24) and (7.2.1.23) do satisfy the finite symplecticity condition (7.2.1.12), and we denote the corresponding symplectic matrix as

$$\mathcal{O} \equiv \begin{pmatrix} \mathbb{I}_{n_V+1} & 0_{n_V+1} \\ \\ \mathcal{W} & \mathbb{I}_{n_V+1} \end{pmatrix}. \qquad (7.2.1.25)$$

It is easy to realize that $\mathcal{O}$ given by (7.2.1.25) belongs to the $\frac{(n_V+1)(n_V+2)}{2}$-dimensional Abelian group

$$\mathcal{PQ}(2n_V + 2, \mathbb{R}) \equiv Sp(2n_V + 2, \mathbb{R}) \cap LUT(2n_V + 2, \mathbb{R}), \qquad (7.2.1.26)$$

which we will henceforth refer to as the *Peccei Quinn symplectic group*. In (7.2.1.26) $LUT(2n_V + 2, \mathbb{R})$ is the $(n_V + 1)^2$-dimensional Abelian group of lower unitriangular
$2(n_V + 1) \times 2(n_V + 1)$ real matrices, which are unipotent (see *e.g.* [41]). Correspondingly, the Peccei-Quinn (PQ) symplectic Lie algebra $\mathfrak{pq}(2n_V + 2, \mathbb{R})$ is given by

$$\mathfrak{pq}(2n_V + 2, \mathbb{R}) \equiv \mathfrak{sp}(2n_V + 2, \mathbb{R}) \cap \mathfrak{lut}(2n_V + 2, \mathbb{R}), \qquad (7.2.1.27)$$

i.e. by the strictly lower triangular $2(n_V + 1) \times 2(n_V + 1)$ real matrices (which are nilpotent) with *symmetric* lower $(n_V + 1) \times (n_V + 1)$ block.

Matrices with structure as $\mathcal{O}$ given by (7.2.1.25), and thus belonging to the group $\mathcal{PQ}(2n_V + 2, \mathbb{R})$ defined above, appear also in other contexts. For instance, they are a particular case (with $A = \mathbb{I}_{n_V+1}$) of the quantum perturbative duality transformations in supersymmetric Yang-Mills theories coupled to supergravity (see *e.g.* [42], and Eq. (4.1) therein). In particular, Eq. (7.2.1.25) defines the structure of quantum perturbative monodromy matrices in heterotic string compactifications with classical U-duality $SL(2, \mathbb{R}) \times SO(2, n_V + 2)$ (see *e.g.* (5.4) of [42]).

Let us give here some other explicit results, useful in the subsequent treatment.

Eqs. (7.2.1.21, (7.2.1.22) and (7.2.1.24) imply

$$\mathfrak{F}_\Lambda \equiv F'_\Lambda - F_\Lambda. \qquad (7.2.1.28)$$

Thus, within the framework under consideration, it follows that

$$F_\Lambda \equiv D_\Lambda \mathbf{F} = \frac{\partial \mathbf{F}}{\partial X^\Lambda} = \begin{cases} F_0 = -\frac{1}{3!}\mathrm{Re}d_{ijk}\frac{X^i X^j X^k}{(X^0)^2} + \frac{i}{3}\mathrm{Im}d_{000}X^0; \\ \\ F_i = \frac{1}{2}\mathrm{Re}d_{ijk}\frac{X^j X^k}{X^0}; \end{cases} \tag{7.2.1.29}$$

$$F'_\Lambda \equiv D_\Lambda \mathfrak{F} + D_\Lambda \mathbf{F} = D_\Lambda F = \frac{\partial F}{\partial X^\Lambda}, \tag{7.2.1.30}$$

where Eqs. (7.2.1.6) and (7.2.1.7) were used.

Moreover, by using (7.2.1.12), the inverse of matrix $\mathcal{O}$ can be easily computed to be simply

$$\mathcal{O}^{-1} \equiv \begin{pmatrix} \mathbb{I}_{n_V+1} & 0_{n_V+1} \\ \\ -\mathcal{W} & \mathbb{I}_{n_V+1} \end{pmatrix}. \tag{7.2.1.31}$$

Thus, by recalling Eqs. (7.2.1.19), (7.2.1.20), and the expressions (7.2.1.25) and (7.2.1.31) along with Eq. (7.2.1.22), one can write down the finite transformations of $\mathcal{Q}$ and $\mathbf{V}$ under the action of a generic element of $\mathcal{PQ}(2n_V + 2, \mathbb{R})$ (the unwritten matrix components vanish throughout):

$$\mathcal{Q}' = \mathcal{O}\mathcal{Q} = \begin{pmatrix} p^0 \\ p^i \\ q_0 + \varrho p^0 + \mathbf{c}_j p^j \\ q_i + \mathbf{c}_i p^0 + \Theta_{ij}p^j \end{pmatrix} \Leftrightarrow \mathcal{Q} = \mathcal{O}^{-1}\mathcal{Q}' = \begin{pmatrix} p'^0 \\ p'^i \\ q'_0 - \varrho p'^0 - \mathbf{c}_j p'^j \\ q'_i - \mathbf{c}_i p'^0 - \Theta_{ij}p'^j \end{pmatrix}; \tag{7.2.1.32}$$

$$\mathbf{V}' = \mathcal{O}\mathbf{V} = \begin{pmatrix} X^0 \\ X^i \\ F_0 + \varrho X^0 + \mathbf{c}_j X^j \\ F_i + \mathbf{c}_i X^0 + \Theta_{ij}X^j \end{pmatrix} \Leftrightarrow \mathbf{V} = \mathcal{O}^{-1}\mathbf{V}' = \begin{pmatrix} X'^0 \\ X'^i \\ F'_0 - \varrho X'^0 - \mathbf{c}_j X'^j \\ F'_i - \mathbf{c}_i X'^0 - \Theta_{ij}X'^j \end{pmatrix}. \tag{7.2.1.33}$$

Relation with U-Duality Transformations

In order to highlight some important features of the Peccei-Quinn transformations defined above, it is here convenient to briefly recall the properties of $\mathbf{V}$ and related quantities under the action of $Sp(2n_V + 2, \mathbb{R})$ (see *e.g.* [2; 3; 43] and Refs. therein).

The holomorphic sections $\mathbf{V}$ defined in (7.2.1.18) belong to the holomorphic (chiral) ring over the Kähler-Hodge bundle defined over the vector multiplets' scalar

manifold. Under a finite symplectic transformation $\mathcal{S} \in Sp\left(2n_V + 2, \mathbb{R}\right)$ defined by
(7.2.1.11)-(7.2.1.15), $\mathbf{V}$ transform as

$$\mathbf{V}\left(z\right) \xrightarrow{\mathcal{S}} \mathcal{S}\mathbf{V}'\left(z\right) = \exp\left[-f\left(z'\right)\right] \mathcal{S}\mathbf{V}'\left(z'\right). \qquad (7.2.1.34)$$

"z" and "z'" collectively denote the scalar field parametrization (i.e., the coordinate
frame) before and after the application of $\mathcal{S}$. Thus, the action of $\mathcal{S}$ generally induces
a (generally non-linear) coordinate transformation

$$z \longrightarrow z'. \qquad (7.2.1.35)$$

Thus, the holomorphic superpotential $W \equiv \left\langle \mathcal{Q}, \mathbf{V}\left(z\right)\right\rangle \equiv \mathcal{Q}^T \Omega \mathbf{V}\left(z\right)$ transforms as
(recall (7.2.1.12))

$$W \xrightarrow{\mathcal{S}} \exp\left[-f\left(z'\right)\right] \left\langle \mathcal{Q}', \mathbf{V}'\left(z'\right)\right\rangle \equiv \exp\left[-f\left(z'\right)\right] W', \qquad (7.2.1.36)$$

i.e. with an holomorphic overall factor $\exp\left[-f\left(z'\right)\right]$. The holomorphic function
$f\left(z'\right)$ appearing in (7.2.1.34) and (7.2.1.36) is the gauge function of the Kähler trans-
formation induced by $\mathcal{S}$ on the Kähler potential $\mathcal{K}\left(z, \bar{z}\right) \equiv -\ln\left[i\left\langle \overline{\mathbf{V}}\left(\bar{z}\right), \mathbf{V}\left(z\right)\right\rangle\right]$
itself (recall Eq. (7.2.1.34)):

$$\mathcal{K}\left(z, \bar{z}\right) \xrightarrow{\mathcal{S}} -\ln\left[i\left\langle \overline{\mathbf{V}}'\left(\bar{z}'\right), \mathbf{V}'\left(z'\right)\right\rangle\right] + f\left(z'\right) + \overline{f}\left(\bar{z}'\right) \equiv \mathcal{K}'\left(z', \bar{z}'\right) + f\left(z'\right) + \overline{f}\left(\bar{z}'\right).$$
$$(7.2.1.37)$$

Eqs. (7.2.1.36) and (7.2.1.37) yield that the covariantly holomorphic sections
$\mathcal{V}\left(z, \bar{z}\right) \equiv \exp\left[\mathcal{K}\left(z, \bar{z}\right) / 2\right] \mathbf{V}\left(z\right)$, belonging to the Kähler-Hodge $U\left(1\right)$ bundle, trans-
form under $\mathcal{S}$ as follows (recall (7.2.1.34) and (7.2.1.37)):

$$\mathcal{V}\left(z, \bar{z}\right) \xrightarrow{\mathcal{S}} \exp\left[-i\mathrm{Im}\left(f\left(z'\right)\right)\right] \mathcal{S}\mathcal{V}'\left(z', \bar{z}'\right), \qquad (7.2.1.38)$$

i.e. with an overall phase (Kähler-Hodge $U\left(1\right)$ factor) $\exp\left[-i\mathrm{Im}\left(f\left(z'\right)\right)\right]$. This in
turn implies that the $\mathcal{N} = 2$ central charge $Z\left(z, \bar{z}\right) \equiv \left\langle \mathcal{Q}, \mathcal{V}\left(z, \bar{z}\right)\right\rangle$ transforms as

$$Z\left(z, \bar{z}\right) \xrightarrow{\mathcal{S}} \exp\left[-i\mathrm{Im}\left(f\left(z'\right)\right)\right] Z'\left(z', \bar{z}'\right). \qquad (7.2.1.39)$$

A general consequence of Eqs. (7.2.1.34)-(7.2.1.39) is the following.

Under a transformation $\mathcal{S} \in Sp\left(2n_V + 2, \mathbb{R}\right)$, $W\left(z\right)$ and $Z\left(z, \bar{z}\right)$ are *invariant iff*
$\mathcal{S}$ *does not induce any change in the coordinates* of the scalar manifold. By looking at the
conditions (7.2.1.14)-(7.2.1.15), it is immediate to realize that $\mathcal{O} \in \mathcal{PQ}\left(2n_V + 2, \mathbb{R}\right)$
represented by (7.2.1.25) is actually the most general element of $Sp\left(2n_V + 2, \mathbb{R}\right)$ that
does not induce any transformation of coordinates on the scalar manifold, and thus

leaves both W and Z (as well as the corresponding covariant derivatives $D_i W$ and $D_i Z$) *invariant*.

A direct consequence of this is that the effective BH potential [44]

$$V_{BH} \equiv |Z|^2 + g^{i\bar{j}} (D_i Z) \overline{D}_{\bar{j}} \overline{Z} \tag{7.2.1.40}$$

is also *invariant* under $\mathcal{PQ} (2n_V + 2, \mathbb{R})$:

$$V_{BH} (z, \bar{z}; \mathcal{Q}) \xrightarrow{\mathcal{O}} V_{BH} (z, \bar{z}; \mathcal{Q}). \tag{7.2.1.41}$$

For this reason, while $\mathcal{PQ} (2n_V + 2, \mathbb{R})$ can be efficiently used to investigate the effects of $\mathfrak{F}$ given by (7.2.1.8) on the attractor points of V_{BH} itself and on the BH entropy (through the study of the transformation properties of the quartic G_4-invariant $\mathcal{I}_4$; see Sec. 7.2.2), its use in relation to Z, $D_i Z$ and V_{BH} has some *caveats*, pointed out at the start of Sec. 7.2.2. The analysis of the latter Sec. relies on the results of [44] (see also [11] for a review, and Refs. therein) on the axion-free supporting charge configurations, and related supersymmetry properties, in d-SK geometries.

We are now going to show that

$$\mathfrak{pq} (2n, \mathbb{R}) \subsetneq \frac{\mathfrak{sp} (2n_V + 2, \mathbb{R})}{\mathfrak{g}_4}, \tag{7.2.1.42}$$

which thus implies, through exponential map:

$$\mathcal{PQ} (2n, \mathbb{R}) \subsetneq \frac{Sp (2n_V + 2, \mathbb{R})}{G_4}. \tag{7.2.1.43}$$

In other words, the PQ symplectic transformations lie in $Sp (2n_V + 2, \mathbb{R})$ outside of the $d = 4$ U-duality group G_4, whose Lie algebra is denoted by $\mathfrak{g}_4$ throughout. Thus, (7.2.1.27) and (7.2.1.26) can respectively be recast as

$$\mathfrak{pq} (2n_V + 2, \mathbb{R}) \equiv \frac{\mathfrak{sp} (2n_V + 2, \mathbb{R})}{\mathfrak{g}_4} \cap \mathfrak{lut} (2n_V + 2, \mathbb{R}) ;$$
$$\Downarrow \exp$$
$$\mathcal{PQ} (2n_V + 2, \mathbb{R}) \equiv \frac{Sp (2n_V + 2, \mathbb{R})}{G_4} \cap LUT (2n_V + 2, \mathbb{R}), \tag{7.2.1.44}$$

where "exp" denotes the exponential map.

Clearly, (7.2.1.42)-(7.2.1.44) hold whenever $\mathfrak{g}_4$ is well defined, for instance in the $\mathcal{N} = 2$ models whose vector multiplets' scalar manifold is a symmetric coset G_4 / H_4, with H_4 being the maximal compact subgroup (with symmetric embedding) of G_4

itself (see *e.g.* [17] and Refs. therein; see also [45] for a recent survey). Besides the *minimally coupled* [18] $\mathbb{CP}^n$ sequence with quadratic prepotential, these models are given by all symmetric d-SK geometries, whose prepotential is given by (7.2.1.9), with d_{ijk} satisfying the identity [46; 47]

$$d_{r(pq}d_{ij)k}d^{rkl} = \frac{4}{3}\delta^l_{(p}d_{qij)},\tag{7.2.1.45}$$

which implies that d_{ijk} and its contravariant counterpart d^{ijk} are both G_5-invariant (scalar-independent) tensors (see Sec. 7.3 for further elucidation). Moreover, for all d-SK geometries a "minimal" $G_4 \equiv SO(1,1) \times_s \mathbb{R}^{n_V}$ always exists (see Footnote 3).

Furthermore, for a symmetric d-SK geometry, the expression of the unique quartic invariant polynomial $\mathcal{I}_4(\mathcal{Q})$ of the symplectic repr. **R** of G_4 reads (in the "special coordinates" sympletic basis [20]):

$$\mathcal{I}_4(\mathcal{Q}) \equiv -\left(p^0\right)^2 q_0^2 - \left(p^i q_i\right)^2 - 2p^0 q_0 p^i q_i + 4\left[q_0 \mathcal{I}_3(p) - p^0 \mathcal{I}_3(q) + \{\mathcal{I}_3(p), \mathcal{I}_3(q)\}\right]\tag{7.2.1.46}$$

where

$$\mathcal{I}_3(p) \equiv \frac{1}{3!}d_{ijk}p^i p^j p^k; \quad \mathcal{I}_3(q) \equiv \frac{1}{3!}d^{ijk}q_i q_j q_k; \quad \{\mathcal{I}_3(p), \mathcal{I}_3(q)\} \equiv \frac{\partial \mathcal{I}_3(p)}{\partial p^i}\frac{\partial \mathcal{I}_3(q)}{\partial q_i}.\tag{7.2.1.47}$$

In d-SK geometries, the manifestly $(\mathfrak{g}_5 \oplus \mathfrak{so}(1,1))$-covariant form of the symplectic embedding of the infinitesimal transformation of the G_4 is provided by the following $2(n_V + 1) \times 2(n_V + 1)$ matrix ($i,j,k = 1,...,n_V$) [38]:

$$\mathfrak{X} \equiv \begin{pmatrix} 3\lambda & b_j & 0 & 0^j \\ c^i & \mathcal{A}^i_j + \lambda\delta^i_j & 0^i & d^{ijk}b_k \\ 0 & 0_j & -3\lambda & -c^j \\ 0_i & d_{ijk}c^k & -b_i & \mathcal{A}^j_i - \lambda\delta^j_i \end{pmatrix},\tag{7.2.1.48}$$

where $\mathcal{A}^i_j$ is the electric-magnetic representation of the $\mathfrak{g}_5$ algebra, λ is the $\mathfrak{so}(1,1)$ parameter, c^i are the parameters of the PQ axion-shift transformations $\mathfrak{l}_{+2}$, and b_i are the parameters of the additional transformations $\mathfrak{l}'_{-2}$, not implementable on the vector potentials A^0, A^i, which complete the algebra to $\mathfrak{g}_4$ (subscripts denote weights w.r.t. $\mathfrak{so}(1,1)$):

$$\mathfrak{g}_4 = (\mathfrak{g}_5)_0 \oplus (\mathfrak{so}(1,1))_0 \oplus \mathfrak{l}_{+2} \oplus \mathfrak{l}_{-2}.\tag{7.2.1.49}$$

Thus, the matrix $\mathfrak{X}$ given by (7.2.1.48) realizes the Lie algebra $\mathfrak{g}_4$ of the U-duality group G_4 in its symplectic irrepr. **R**, defining the embedding (7.2.1.16). By comparing the matrix $\mathfrak{X}$ given by (7.2.1.48) with the infinitesimal form of $\mathcal{O}$ given by

(7.2.1.25), i.e. with the strictly lower triangular matrix

$$
\mathcal{O}_{inf} = \left(\begin{array}{cc} 0_2 & 0_2 \\ \mathfrak{W}_2 & 0_2 \end{array} \right) \in \mathfrak{pq} \left(2n_V + 2, \mathbb{R} \right), \tag{7.2.1.50}
$$

one can conclude that results (7.2.1.42), and thus (7.2.1.43), hold.

Stringy Origin

It is here worth briefly commenting on the stringy origin of the components of the matrix $\mathcal{W}_{\Lambda\Sigma}$ given by (7.2.1.22). For more details, and a list of Refs., we address the reader *e.g.* to the treatment of [29; 48; 49].

In Type IIA compactifications over Calabi-Yau threefolds (CY_3), it holds that

$$
\mathcal{W}_{0i} \equiv \mathbf{c}_i = \frac{c_{2,i}}{24} \equiv \frac{c_2 \cdot J_i}{24} = \frac{1}{24} \int_{CY_3} c_2 \wedge J_i, \tag{7.2.1.51}
$$

where c_2 is the second Chern class[10] of CY_3, and $\{J_i\}_{i=1,\dots,n_V}$ is a basis of $H^2 \left(CY_3, \mathbb{R} \right)$, the second cohomology group of CY_3.

Moreover, the coefficients of $\mathbf{F}$ (as given by Eq. (7.2.1.7)) have the following stringy interpretation [29; 52–54]:

$$
\frac{1}{3!} \mathrm{Re} d_{ijk} = \mathbf{C}_{ijk}; \tag{7.2.1.52}
$$

$$
\frac{1}{3!} \mathrm{Im} d_{000} = -\frac{\zeta(3)}{(2\pi)^3} \chi, \tag{7.2.1.53}
$$

where $\mathbf{C}_{ijk}$ and χ respectively are the classical triple intersection numbers[11] and Euler character of the CY_3, and ζ is the Riemann zeta function.

Notice that the other components of $\mathcal{W}_{\Lambda\Sigma}$, i.e. $\mathcal{W}_{00} \equiv \varrho$ and $\mathcal{W}_{ij} \equiv \Theta_{ij}$, do not have an interpretation in terms of topological invariants of the internal manifold (see *e.g.* the discussion in [48]), *at least* in the compactification framework under consideration. For this reason, they are usually disregarded in the stringy literature (see *e.g.* [29], in particular the discussion of Eq. (3.48) therein; see also [30]). However, it is worth pointing out that $\mathcal{W}_{00}$ and $\mathcal{W}_{ij}$ are important for fixing the integral basis for $\mathbf{V}$ itself (see *e.g.* the discussion in [48; 55; 56].

---

[10]Note that, *e.g.* in presence of R^2-corrections, the second Chern class also contributes non-homogeneously to the BH entropy (see *e.g.* [50; 51]).

[11]Actually, quantum (perturbative and non-perturbative) effects can also affect $\mathrm{Re} d_{ijk}$, i.e. (through Eq. (7.2.1.52)) the classical triple intersection numbers (see *e.g.* [29; 48], and Refs. therein).

When setting $\varrho = \Theta_{ij} = 0$, the transformation (7.2.1.32) yields

$$
\begin{pmatrix} p^0 \\ p^i \\ q_0 \\ q_i \end{pmatrix} \xrightarrow{\mathcal{O}^{-1}} \begin{pmatrix} p'^0 \\ p'^i \\ q'_0 - \mathbf{c}_j p'^j \\ q'_i - \mathbf{c}_i p'^0 \end{pmatrix},
\tag{7.2.1.54}
$$

which is a Witten theta-shift [57] of electric charges *via* magnetic charges (in a generally axionful background).

Nevertheless, $\mathcal{W}_{00}$ and $\mathcal{W}_{ij}$ are perfectly consistent in a fully general supergravity analysis, and we will consider them non-vanishing throughout the applicative developments treated below.

In general, the term determined by $\mathrm{Re}d_{ijk}$ in the general cubic prepotential (given by Eqs. (7.2.1.6)-(7.2.1.8)) is the *leading* one for large values of the scalar fields (*moduli*), and it defines the purely cubic prepotential (7.2.1.9) of the d-SK geometry of the complex structure (or Kähler structure) deformation moduli space of the large volume limit of the internal manifold CY_3 (in Type II compactifications). All other terms in Eqs. (7.2.1.6)-(7.2.1.8) define *sub-leading* contributions, which are of quantum perturbative nature, and consistent with the continuous PQ axion-shift symmetry (7.2.1.3). All such sub-leading terms, but the purely imaginary constant determined by $i\mathrm{Im}d_{000}$ (and eventual renormalization of classical triple intersection numbers; see Footnote 6), can be taken into account by means of the group $\mathcal{PQ}(2n_V + 2, \mathbb{R})$.

Non-perturbative effects (which can generally traced back to world-sheet instantons, *i.e.* to non-perturbative phenomena in the non-linear sigma model) usually exhibit exponential dependence on the moduli, and they are thus exponentially suppressed in the large volume limit (see *e.g.* [29] and [58; 59]). They break down the perturbative continuous PQ axion-shift symmetry (7.2.1.3) to its discrete form, i.e. [48]

$$
X^i \rightarrow X^i + 1.
\tag{7.2.1.55}
$$

In some stringy framework, exponential terms (*e.g.* polylogarithmic functions) can arise also from quantum perturbative corrections (see *e.g.* the discussion in [29] and [58; 59]). The effect of non-perturbative, exponential corrections to cubic prepotentials on the spectrum and the stability of extremal BH attractors has been recently addressed in [31], whose findings confirm the general belief that non-perturbative correction lift the "flat" directions (*if any*) of the perturbative theory[12]. At the level of

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[12]Actually, also quantum perturbative corrections, such as the one given by the term $i\mathrm{Im}d_{000}$ in (7.2.1.7) (with stringy origin given by (7.2.1.53)) can lift (some of the) *"flat directions"* of extremal BH attractor solutions [60].

the prepotential, this can be traced back to the fact that exponential corrections to the purely cubic holomorphic prepotential (7.2.1.9) d-SK geometries (of the kind given by Eq. (4.1) of [31]) affect the geometric properties of the scalar manifold itself.

7.2.2 Application to Black Hole Attractors, Entropy and Supersymmetry

As pointed out in Sec. 7.2.1, the Peccei-Quinn symplectic group $\mathcal{PQ}\left(2n_V + 2, \mathbb{R}\right)$ is a proper subgroup of $\frac{Sp(2n_V+2,\mathbb{R})}{G_4}$. The latter is the most general group acting linearly on the charges $\mathcal{Q}$ which can change the value and possibly the sign of the unique quartic invariant $\mathcal{I}_4\left(\mathcal{Q}\right)$ of the symplectic (ir)repr. **R** of G_4 itself.

In the following treatment, within the manifestly G_5-covariant "special coordinates" symplectic frame, we will analyse how $\mathcal{PQ}\left(2n_V + 2, \mathbb{R}\right)$ acts on $\mathcal{I}_4\left(\mathcal{Q}\right)$, on the non-degenerate critical points of the effective BH potential V_{BH} (*alias* extremal BH attractors) [6], and on their supersymmetry properties. We will work within the d-SK geometries determined by the prepotential (7.2.1.9). When they involve the contravariant tensor d^{ijk}, the results on the transformation properties of $\mathcal{I}_4$ generally hold only for d-SK geometries such that the coset G_4/H_4 is *symmetric* (see *e.g.* [17], and Refs. therein).

By suitably adapting its action, $\mathcal{PQ}\left(2n_V + 2, \mathbb{R}\right)$ reveals to be a very effective tool to investigate the effect of the quantum perturbative sub-leading corrections (7.2.1.8) to the leading d-SK prepotential (7.2.1.9), some of which have a topological interpretation (see Sec. 7.2.1).

We anticipate that, under certain conditions on the ratio between the charges $\mathcal{Q}$ and the parameters $\left(\varrho, \mathbf{c}_i, \Theta_{ij}\right)$ of the finite PQ transformation $\mathcal{O}$ (given by Eq. (7.2.1.25) and (7.2.1.22)), the action of $\mathcal{PQ}\left(2n_V + 2, \mathbb{R}\right)$ can give rise to a "transition" among the various orbits of **R** of G_4, which in turn changes the supersymmetry-preserving features of the extremal BH attractor solutions[13].

---

[13]Thus, our results should have interesting connections with the $d = 3$ timelike-reduced geodesic formalism and results of [61], whose thorough investigation we leave for further future study. For some developments in a $d = 4$ framework, see [62] (and also [8]).

Transformation of $\mathcal{I}_4$

We start and apply the finite transformation[14] $\mathcal{O}^{-1} \in \mathcal{PQ}(2n_V + 2, \mathbb{R})$ (given by (7.2.1.32)) to the G_4-invariant quartic polynomial $\mathcal{I}_4(\mathcal{Q})$ given by (7.2.1.46)-(7.2.1.47). Thus, after some algebra, the following result is achieved:

$$\mathcal{PQ}(2n_V + 2, \mathbb{R}) \ni \mathcal{O}^{-1} : \mathcal{I}_4(\mathcal{Q}) \longrightarrow \mathcal{I}_4'\left(\mathcal{O}^{-1}\mathcal{Q}'\right) = \mathcal{I}_4\left(\mathcal{O}^{-1}\mathcal{Q}'\right) = \mathcal{I}_4(\mathcal{Q}') + \mathbf{I}_4,$$
(7.2.2.1)

where the quartic quantity $\mathbf{I}_4$, describing the "PQ-deformation" of $\mathcal{I}_4(\mathcal{Q})$, is given by the following expression[15]:

$$
\begin{aligned}
\mathbf{I}_4\left(\mathcal{Q}; \varrho, \mathbf{c}_i, \Theta_{ij}\right) \equiv {} & 2\left(p^0\right)^4 \left(\frac{1}{3} d^{ijk} \mathbf{c}_i \mathbf{c}_j \mathbf{c}_k - \frac{1}{2} \varrho^2\right) \\[2mm]
& + 2\left(p^0\right)^3 \left(\varrho q_0 - \varrho \mathbf{c}_i p^i - d^{ijk} q_i \mathbf{c}_j \mathbf{c}_k + d^{ijk} \mathbf{c}_i \mathbf{c}_j \Theta_{kl} p^l\right) \\[2mm]
& + 2\left(p^0\right)^2 \left(
\begin{array}{l}
-2\left(\mathbf{c}_i p^i\right)^2 + 2q_0 \mathbf{c}_i p^i + \varrho p^i q_i - \varrho \Theta_{ij} p^i p^j - 2 d^{ijk} q_i \mathbf{c}_j \Theta_{kl} p^l \\[2mm]
+ d^{ijk} \mathbf{c}_i \Theta_{jl} \Theta_{km} p^l p^m + \frac{1}{2} d_{ijk} d^{ilm} \mathbf{c}_l \mathbf{c}_m p^j p^k + d^{ijk} q_i q_j \mathbf{c}_k
\end{array}
\right) \\[2mm]
& + 2 p^0 \left(
\begin{array}{l}
2 p^i q_i \mathbf{c}_j p^j - 2 \mathbf{c}_i \Theta_{jk} p^i p^j p^k + q_0 \Theta_{ij} p^i p^j - \frac{1}{3} \varrho d_{ijk} p^i p^j p^k \\[2mm]
+ d^{ijk} q_i q_j \Theta_{kl} p^l - d^{ijk} q_i \Theta_{jl} \Theta_{km} p^l p^m + \frac{1}{3} d^{ijk} \Theta_{il} \Theta_{jm} \Theta_{kn} p^l p^m p^n \\[2mm]
- d_{ijk} d^{ilm} p^j p^k q_l \mathbf{c}_m + d_{ijk} d^{ilm} p^j p^k \mathbf{c}_l \Theta_{ms} p^s
\end{array}
\right) \\[2mm]
& - \left(\Theta_{ij} p^i p^j\right)^2 + 2 p^i q_i \Theta_{jk} p^j p^k - \frac{2}{3} \mathbf{c}_l p^l d_{ijk} p^i p^j p^k \\[2mm]
& - 2 d_{ijk} d^{ilm} p^j p^k q_l \Theta_{ms} p^s + d_{ijk} d^{ilm} p^j p^k \Theta_{ls} \Theta_{mt} p^s p^t.
\end{aligned}
$$
(7.2.2.2)

Note that the degree-4 homogeneity of $\mathcal{I}_4$ in the charges is not spoiled, due to the linearity of the action of $\mathcal{PQ}(2n_V + 2, \mathbb{R})$ on the charges themselves.

We now analyse various particular (both "large" and "small") charge configurations, showing how the action of $\mathcal{PQ}(2n_V + 2, \mathbb{R})$ can give rise to two types of

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[14]We consider $\mathcal{O}^{-1}$ rather than $\mathcal{O}$ (a choice which is clearly immaterial at group level) because operationally (as discussed in [29]) one would like to include the effects of the sub-leading $(\varrho, \mathbf{c}_i, \Theta_{ij})$-dependent terms in the prepotential (7.2.1.6)-(7.2.1.8) on the Bekenstein-Hawking BH entropy [23] by simply performing the computations within the purely cubic prepotential (7.2.1.9) (see *e.g.* the analysis of [44]) and then by applying the transformation $\mathcal{O}^{-1}$ on $\mathcal{Q}$. Note that we will not deal here with the term $\frac{i}{3!} \mathrm{Im} d_{000} \left(X^0\right)^2$ in (7.2.1.7), which has been recently studied in [26].

[15]Throughout the subsequent treatment, we omit the priming of the $\mathcal{O}^{-1}$-transformed charges.

phenomena, both corresponding to switching among different $\mathbb{R}$-orbits:

- change of sign of $\mathcal{I}_4$:

$$\mathcal{I}_4\left(\mathcal{Q}\right) \gtrless 0 \xrightarrow{\mathcal{PQ}} \mathcal{I}_4\left(\mathcal{Q}\right) + \mathbf{I}_4\left(\mathcal{Q}; \varrho, \mathbf{c}_i, \Theta_{ij}\right) \lessgtr 0, \qquad (7.2.2.3)$$

corresponding to a switch between different "large" $\mathbb{R}$-orbits [21];

- generation of a non-vanishing $\mathcal{I}_4$:

$$\mathcal{I}_4\left(\mathcal{Q}\right) = 0 \xrightarrow{\mathcal{PQ}} \mathcal{I}_4\left(\mathcal{Q}\right) + \mathbf{I}_4\left(\mathcal{Q}; \varrho, \mathbf{c}_i, \Theta_{ij}\right) \gtrless 0, \qquad (7.2.2.4)$$

or the other way around, generation of a vanishing $\mathcal{I}_4$:

$$\mathcal{I}_4\left(\mathcal{Q}\right) \gtrless 0 \xrightarrow{\mathcal{PQ}} \mathcal{I}_4\left(\mathcal{Q}\right) + \mathbf{I}_4\left(\mathcal{Q}; \varrho, \mathbf{c}_i, \Theta_{ij}\right) = 0, \qquad (7.2.2.5)$$

both corresponding to a switch between a "large" and a "small" $\mathbb{R}$-orbit (usually named "charge orbit").

Some comments on the meaning of Eqs. (7.2.2.3)-(7.2.2.5) are in order.

- Firstly, let us recall that, through the Bekenstein-Hawking formula (7.1.0.2), "large" and "small" charge orbits respectively corresponds to $\mathcal{I}_4 \neq 0$ and $\mathcal{I}_4 = 0$; furthermore, "small" orbits split in lightlike (3-charge), critical (2-charge) and doubly-critical (1-charge) ones [20; 63–66].

Then, the general treatment of Sec. 7.2.1 implies that, in presence of $\left(\varrho, \mathbf{c}_i, \Theta_{ij}\right)$-dependent sub-leading contributions (7.2.1.8) (recall the change of notation (7.2.1.22)) to the purely cubic prepotential (7.2.1.9) of d-SK geometry, the BH entropy S becomes $\left(\varrho, \mathbf{c}_i, \Theta_{ij}\right)$-dependent:

$$\frac{S}{\pi} = \frac{A}{4} = \sqrt{\left|\mathcal{I}_4\left(\mathcal{Q}\right) + \mathbf{I}_4\left(\mathcal{Q}; \varrho, \mathbf{c}_i, \Theta_{ij}\right)\right|}, \qquad (7.2.2.6)$$

where $\mathbf{I}_4\left(\mathcal{Q}; \varrho, \mathbf{c}_i, \Theta_{ij}\right)$ is defined in (7.2.2.2). Consequently, depending on the relations between $\mathcal{I}_4\left(\mathcal{Q}\right)$ and $\mathbf{I}_4\left(\mathcal{Q}; \varrho, \mathbf{c}_i, \Theta_{ij}\right)$, the phenomena (7.2.2.3)-(7.2.2.5) can occur, and the ones related to $\mathbf{c}_i$ have, by virtue of (7.2.1.51), a clear topological interpretation within Type *II* CY_3-compactifications.

It should be remarked that the geometry of the *symmetric* coset G_4/H_4 is unaffected by the action of $Sp\left(2n_V + 2, \mathbb{R}\right)$ (which just produces a change of coordinates; see Sec. 7.2.1), and thus *a fortiori* by the action of its proper subgroup

$\mathcal{PQ}\left(2n_V + 2, \mathbb{R}\right)$. Furthermore, by virtue of the results in Sec. 7.2.1, $\mathcal{PQ}\left(2n_V + 2, \mathbb{R}\right)$ does not act on the coordinates of the scalar manifolds, and thus does not induce any Kähler gauge transformation (7.2.1.37) on $\mathcal{K}$, nor any holomorphic scaling (7.2.1.36) on W (and $D_i W$) and local phase transformation (7.2.1.39) on Z (and $D_i Z$) itself. Thus, the only effect of $\mathcal{PQ}\left(2n_V + 2, \mathbb{R}\right)$ on the BH effective potential V_{BH} and its non-degenerate critical points (*alias* extremal BH attractors) [6] is a $\left(\varrho, \mathbf{c}_i, \Theta_{ij}\right)$-dependent transformation of the charge vector $\mathcal{Q}$, as given by (7.2.1.32). This fact will allow us to analyse the axion-free-supporting nature of the BH charge configurations in presence of non-vanishing parameters ϱ, $\mathbf{c}_i$ and Θ_{ij} by relying on the results of [44] (holding for generic (7.2.1.9)). The results recently obtained in Sec. 3 of [31] are an expected confirmation of all this reasoning.

By virtue of the transition from (7.1.0.2) to (7.2.2.6) via (7.2.2.1), $Sp\left(2n_V + 2, \mathbb{R}\right)$ (and therefore its proper subgroup $\mathcal{PQ}\left(2n_V + 2, \mathbb{R}\right)$) does *not* affect the geometry of the scalar manifold, but it may affect the "magnitude" of the near-horizon space-time BH background, since its action may change the event horizon area A of the extremal BH, and thus the (semi)classical Bekenstein-Hawking BH entropy S. The phenomena described by Eqs. (7.2.2.3)-(7.2.2.5) correspond to $\left(\varrho, \mathbf{c}_i, \Theta_{ij}\right)$-dependent transformations moving from one charge orbit to another in the representation space **R** of G_4.

The geometry and the classification of BH charge orbits (and related *"moduli spaces"*[16]) is not affected by $Sp\left(2n_V + 2, \mathbb{R}\right)$ (and therefore by $\mathcal{PQ}\left(2n_V + 2, \mathbb{R}\right)$), but symplectic transformations can induce "transmutations" of the nature of the charge vector $\mathcal{Q} \longrightarrow \mathcal{Q}^{(\prime)}\left(\varrho, \mathbf{c}_i, \Theta_{ij}\right)$, and thus of its supersymmetry preserving properties. As we will see in the case study considered in Sec. 7.2.2, in the case of $\mathcal{PQ}\left(2n_V + 2, \mathbb{R}\right)$ the actual occurrence of these phenomena depends on the very relations between $\mathcal{Q}$ and the transformtaion parameters $\left(\varrho, \mathbf{c}_i, \Theta_{ij}\right)$ themselves.

Analysis of "Large" and "Small" Configurations

The above treatment will be further clarified by the various examples which we are going to treat, generalising and systematically developing some points mentioned in [29]. We will make extensive use of formulæ (7.2.1.32) and (7.2.2.1)-(7.2.2.6).

1. **"Large"** $\left(p^0, q_0\right)$ **(Kaluza-Klein) configuration**. It supports non-BPS $Z_H \neq 0$ (possibly axion-free [44]) attractors, and it is the supergravity analogue of $D0$-

---

[16]This has been recently confirmed by the analysis of the particular model of Sec. 3 of [31].

D6 configuration in Type *II*:

$$\mathcal{Q} \equiv \left(p^0, 0, q_0, 0\right)^T \Rightarrow \mathcal{I}_4\left(\mathcal{Q}\right) = -\left(p^0\right)^2 q_0^2 < 0. \tag{7.2.2.7}$$

The action of $\mathcal{PQ}\left(2n_V + 2, \mathbb{R}\right)$ reads

$$\begin{pmatrix} p^0 \\ 0 \\ q_0 \\ 0 \end{pmatrix} \xrightarrow{\mathcal{O}^{-1}} \begin{pmatrix} p^0 \\ 0 \\ q_0 - \varrho p^0 \\ -c_i p^0 \end{pmatrix}, \tag{7.2.2.8}$$

and thus it generates c_i-dependent electric charges q_i's, which in Type *II* compactifications corresponds to a stack of *D2* branes depending on the components of the second Chern class c_2 of CY_3 (recall Eq. (7.2.1.51)). The corresponding transformation of $\mathcal{I}_4$ reads

$$-\left(p^0\right)^2 q_0^2 < 0 \xrightarrow{\mathcal{O}^{-1}} \left(p^0\right)^4 \left[\frac{2}{3}d^{ijk}c_i c_j c_k - \left(\frac{q_0}{p^0} - \varrho\right)^2\right] \gtreqless 0. \tag{7.2.2.9}$$

Thus, depending on whether

$$\frac{2}{3}d^{ijk}c_i c_j c_k \gtreqless \left(\frac{q_0}{p^0} - \varrho\right)^2, \tag{7.2.2.10}$$

a "large" ($\mathcal{I}_4 > 0$:BPS or non-BPS $Z_H = 0$), a "small" ($\mathcal{I}_4 = 0$:BPS or non-BPS), or a "large" non-BPS $Z_H \neq 0$ ($\mathcal{I}_4 < 0$) BH charge configuration is generated by the action of $\mathcal{PQ}\left(2n_V + 2, \mathbb{R}\right)$. As anticipated in the above treatment, (7.2.2.10) shows that the relations among the components of $\mathcal{Q}$ and the parameters of the PQ symplectic transformation turn out to be crucial for the properties of the resulting charge configuration. The change of the axion-free-supporting nature of this configuration will be analysed in Sec. 7.2.2.

2. **"Large"** $\left(p^0, q_i\right)$ **("electric") configuration**. Depending on $\mathcal{I}_4\left(\mathcal{Q}\right) \gtreqless 0$, it supports all kind of attractors (possibly axion-free [44]). It is the supergravity analogue of *D2-D6* configuration in Type *II*:

$$\mathcal{Q} \equiv \left(p^0, 0, 0, q_i\right)^T \Rightarrow \mathcal{I}_4\left(\mathcal{Q}\right) = -\frac{2}{3}p^0 d^{ijk}q_i q_j q_k \gtreqless 0. \tag{7.2.2.11}$$

The action of $\mathcal{PQ}\left(2n_V + 2, \mathbb{R}\right)$ is

$$\begin{pmatrix} p^0 \\ 0 \\ 0 \\ q_i \end{pmatrix} \xrightarrow{\mathcal{O}^{-1}} \begin{pmatrix} p^0 \\ 0 \\ -\varrho p^0 \\ q_i - c_i p^0 \end{pmatrix}, \tag{7.2.2.12}$$

and thus it generates a ϱ-dependent electric charge q_0. The corresponding transformation of $\mathcal{I}_4$ reads

$$-\frac{2}{3}p^0 d^{ijk} q_i q_j q_k \gtrless 0$$

$$\downarrow \mathcal{O}^{-1}$$

$$-\frac{2}{3}p^0 d^{ijk} q_i q_j q_k + 2(p^0)^2 \left[(\frac{1}{3}d^{ijk} \mathbf{c}_i \mathbf{c}_j \mathbf{c}_k - \frac{1}{2}\varrho^2)(p^0)^2 - p^0 d^{ijk} q_i \mathbf{c}_j \mathbf{c}_k + d^{ijk} q_i q_j \mathbf{c}_k \right] \gtreqless 0.$$

$$(7.2.2.13)$$

Thus, depending on the sign (or on the vanishing) of the quantity on the last line of (7.2.2.13), the same comments made for configuration **1** hold in this case. The change of the axion-free-supporting nature of this configuration will be analysed in Sec. 7.2.2.

3. **"Large"** (p^i, q_0) **("magnetic") configuration**. It is the *"electric-magnetic dual"* of the "electric" configuration **2**. It is then interesting to compare the action of $\mathcal{PQ}(2n_V + 2, \mathbb{R})$ (which is asymmetric on magnetic and electric charges) on configurations **2** and **3**. Depending on $\mathcal{I}_4(\mathcal{Q}) \gtrless 0$, this configuration supports all kind of attractors (possibly axion-free [44]). It is the supergravity analogue of $D0$-$D4$ configuration in Type *II*:

$$\mathcal{Q} \equiv \left(0, p^i, q_0, 0\right)^T \Rightarrow \mathcal{I}_4(\mathcal{Q}) = \frac{2}{3}q_0 d_{ijk} p^i p^j p^k \gtrless 0. \qquad (7.2.2.14)$$

The action of $\mathcal{PQ}(2n_V + 2, \mathbb{R})$ is

$$\begin{pmatrix} 0 \\ p^i \\ q_0 \\ 0 \end{pmatrix} \xrightarrow{\mathcal{O}^{-1}} \begin{pmatrix} 0 \\ p^i \\ q_0 - \mathbf{c}_j p^j \\ -\Theta_{ij} p^j \end{pmatrix}, \qquad (7.2.2.15)$$

and thus it generates Θ_{ij}-dependent electric charges q_i's. The corresponding transformation of $\mathcal{I}_4$ reads

$$\frac{2}{3}q_0 d_{ijk} p^i p^j p^k \gtrless 0$$

$$\downarrow \mathcal{O}^{-1}$$

$$\frac{2}{3}q_0 d_{ijk} p^i p^j p^k - \left(\Theta_{ij} p^i p^j\right)^2 - \frac{2}{3}\mathbf{c}_l p^l d_{ijk} p^i p^j p^k + d_{ijk} d^{ilm} p^j p^k \Theta_{ls} \Theta_{mt} p^s p^t \gtreqless 0.$$

$$(7.2.2.16)$$

Thus, depending on the sign (or on the vanishing) of the quantity in the last line of (7.2.2.16), the same comments as made for above configurations hold.

The change of the axion-free-supporting nature of this configuration will be analysed in Sec. 7.2.2. Note that for $\Theta_{ij} = 0$, an example treated in [29] is recovered.

4. **"Small"** *lightlike* **(3-charge)** q_i **("electric") configuration**. This is the limit $p^0 = 0$ of configuration **2**. In Type *II*, it corresponds to only $D2$ branes:

$$\mathcal{Q} \equiv (0,0,0,q_i)^T \Rightarrow \mathcal{I}_4(\mathcal{Q}) = 0, \qquad (7.2.2.17)$$

such that (recall definition (7.2.1.47))

$$\mathcal{I}_3(q) \neq 0, \qquad (7.2.2.18)$$

corresponding to a "large" BH in $d = 5$, with near-horizon geometry $AdS_2 \times S^3$ (see *e.g.* [44], and Refs. therein). Since there are no magnetic charges, $\mathcal{PQ}(2n_V + 2, \mathbb{R})$ is inactive on this configuration, which is thus left unchanged:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ q_i \end{pmatrix} \xrightarrow{\mathcal{O}^{-1}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ q_i \end{pmatrix}. \qquad (7.2.2.19)$$

5. **"Small"** *lightlike* **(3-charge)** p^i **("magnetic") configuration**. This is the limit $q_0 = 0$ of configuration **3**. In Type *II*, it corresponds to only $D4$ branes:

$$\mathcal{Q} \equiv \left(0, p^i, 0, 0\right)^T \Rightarrow \mathcal{I}_4(\mathcal{Q}) = 0, \qquad (7.2.2.20)$$

such that (recall definition (7.2.1.47))

$$\mathcal{I}_3(p) \neq 0, \qquad (7.2.2.21)$$

corresponding to a "large" black string in $d = 5$, with near-horizon geometry $AdS_3 \times S^2$ (see *e.g.* [44], and Refs. therein). This configuration is the *"electric-magnetic dual"* of the "electric" configuration **4**. However, unlike what happens for configuration **4**, $\mathcal{PQ}(2n_V + 2, \mathbb{R})$ is active in this case (due to its asymmetric action on electric and magnetic charges):

$$\begin{pmatrix} 0 \\ p^i \\ 0 \\ 0 \end{pmatrix} \xrightarrow{\mathcal{O}^{-1}} \begin{pmatrix} 0 \\ p^i \\ -\mathbf{c}_j p^j \\ -\Theta_{ij} p^j \end{pmatrix}. \qquad (7.2.2.22)$$

and it generates Θ_{ij}-dependent electric charges q_i's, as well as $\mathbf{c}_i$-dependent electric charge q_0. In Type II compactifications, the latter corresponds to a stack of $D0$ branes depending on the components of the second Chern class c_2 of CY_3 (recall Eq. (7.2.1.51)). The corresponding transformation of $\mathcal{I}_4$ reads

$$0 \xrightarrow{\mathcal{O}^{-1}} - \left(\Theta_{ij}p^i p^j\right)^2 - \frac{2}{3}\mathbf{c}_l p^l d_{ijk}p^i p^j p^k + d_{ijk}d^{ilm}p^j p^k \Theta_{ls}\Theta_{mt}p^s p^t \gtrless 0. \quad (7.2.2.23)$$

Thus, according to (7.2.2.23), a "large" ($\mathcal{I}_4 > 0$:BPS or non-BPS $Z_H = 0$), a "small" ($\mathcal{I}_4 = 0$:BPS or non-BPS), or a "large" non-BPS $Z_H \neq 0$ ($\mathcal{I}_4 < 0$) BH charge configuration can be generated. In case the quantity in (7.2.2.23) does not vanish, this is an example of phenomenon (7.2.2.4). Note that for $\Theta_{ij} = 0$, an example treated in [29] is recovered.

6. **"Small"** *critical* **(2-charge)** q_i **("electric") configuration**. This is the limit $\mathcal{I}_3\left(q\right) = 0$ of configuration **4**. In Type II, it corresponds to only $D2$ branes:

$$\mathcal{Q} \equiv (0,0,0,q_i)^T \Rightarrow \mathcal{I}_4\left(\mathcal{Q}\right) = 0, \quad (7.2.2.24)$$

such that (recall definition (7.2.1.47))

$$\begin{cases} \mathcal{I}_3\left(q\right) = 0; \\ \partial\mathcal{I}_3\left(q\right)/\partial q_i \neq 0 \text{ for some } i, \end{cases} \quad (7.2.2.25)$$

corresponding to a "small" *lightlike* BH in $d = 5$. Since there are no magnetic charges, $\mathcal{PQ}\left(2n_V + 2, \mathbb{R}\right)$ is inactive on this configuration, which is thus left unchanged (see Eq. (7.2.2.19)).

7. **"Small"** *critical* **(2-charge)** p^i **("magnetic") configuration**. This is the limit $\mathcal{I}_3\left(q\right) = 0$ of configuration **5**. In Type II, it corresponds to only $D4$ branes:

$$\mathcal{Q} \equiv \left(0, p^i, 0, 0\right)^T \Rightarrow \mathcal{I}_4\left(\mathcal{Q}\right) = 0, \quad (7.2.2.26)$$

such that (recall definition (7.2.1.47))

$$\begin{cases} \mathcal{I}_3\left(p\right) = 0; \\ \partial\mathcal{I}_3\left(p\right)/\partial p^i \neq 0 \text{ for some } i, \end{cases} \quad (7.2.2.27)$$

corresponding to a "small" *lightlike* black string in $d = 5$. This configuration is the *"electric-magnetic dual"* of the "electric" configuration **6**. However, unlike what happens for configuration **6**, $\mathcal{PQ}\left(2n_V + 2, \mathbb{R}\right)$ is active in this case, due to its asymmetric action on electric and magnetic charges. As given by Eq.

(7.2.2.22), Θ_{ij}-dependent electric charges q_i's and $\mathbf{c}_i$-dependent electric charge q_0 are generated. The corresponding transformation of $\mathcal{I}_4$ reads

$$0 \xrightarrow{\mathcal{O}^{-1}} - \left(\Theta_{ij} p^i p^j \right)^2 + d_{ijk} d^{ilm} p^j p^k \Theta_{ls} \Theta_{mt} p^s p^t \gtreqless 0. \tag{7.2.2.28}$$

Thus, according to (7.2.2.28), a "large" ($\mathcal{I}_4 > 0$:BPS or non-BPS $Z_H = 0$), a "small" ($\mathcal{I}_4 = 0$:BPS or non-BPS), or a "large" non-BPS $Z_H \neq 0$ ($\mathcal{I}_4 < 0$) BH charge configuration can be generated. In case the quantity in (7.2.2.28) does not vanish, this is an example of phenomenon (7.2.2.4).

8. **"Small"** *doubly-critical* **(1-charge)** q_i **("electric") configuration**. This is the limit $\partial \mathcal{I}_3 (q) / \partial q_i = 0$ of configuration **6**. In Type *II*, it corresponds to only *D2* branes:

$$\mathcal{Q} \equiv (0,0,0,q_i)^T \Rightarrow \mathcal{I}_4 (\mathcal{Q}) = 0, \tag{7.2.2.29}$$

such that (recall definition (7.2.1.47))

$$\begin{cases} \mathcal{I}_3 (q) = 0; \\ \partial \mathcal{I}_3 (q) / \partial q_i = 0 \ \forall i; \\ q_i \neq 0 \text{ for some } i, \end{cases} \tag{7.2.2.30}$$

corresponding to a "small" *critical* BH in $d = 5$. Since there are no magnetic charges, $\mathcal{PQ} (2n_V + 2, \mathbb{R})$ is inactive on this configuration, which is thus left unchanged (see Eq. (7.2.2.19)).

9. **"Small"** *doubly-critical* **(1-charge)** p^i **("magnetic") configuration**. This is the limit $\partial \mathcal{I}_3 (p) / \partial p^i = 0$ of configuration **7**. In Type *II*, it corresponds to only *D4* branes:

$$\mathcal{Q} \equiv \left(0, p^i, 0, 0 \right)^T \Rightarrow \mathcal{I}_4 (\mathcal{Q}) = 0, \tag{7.2.2.31}$$

such that (recall definition (7.2.1.47))

$$\begin{cases} \mathcal{I}_3 (p) = 0; \\ \partial \mathcal{I}_3 (p) / \partial p^i = 0 \ \forall i; \\ p^i \neq 0 \text{ for some } i, \end{cases} \tag{7.2.2.32}$$

corresponding to a "small" *critical* black string in $d = 5$. This configuration is the *"electric-magnetic dual"* of the "electric" configuration **8**. However, unlike what happens for configuration **8**, $\mathcal{PQ} (2n_V + 2, \mathbb{R})$ is active (see Eq. (7.2.2.22)) in this case, due to its asymmetric action on electric and magnetic charges. It generates Θ_{ij}-dependent electric charges q_i's and $\mathbf{c}_i$-dependent electric charge q_0. The corresponding transformation of $\mathcal{I}_4$ reads

$$0 \xrightarrow{\mathcal{O}^{-1}} - \left(\Theta_{ij} p^i p^j \right)^2 \leqslant 0. \tag{7.2.2.33}$$

Thus, according to (7.2.2.28), a "small" ($\mathcal{I}_4 = 0$:BPS or non-BPS), or a "large" non-BPS $Z_H \neq 0$ ($\mathcal{I}_4 < 0$) BH charge configuration can be generated. In case the quantity in (7.2.2.33) is strictly negative, this is an example of phenomenon (7.2.2.4).

10. **"Small" *doubly-critical* (1-charge) p^0 ("magnetic" Kaluza-Klein) configuration**. This is the limit $q_0 = 0$ of configuration **1**. In Type II, it corresponds to only $D6$ branes:

$$\mathcal{Q} \equiv \left(p^0, 0, 0, 0 \right)^T \Rightarrow \mathcal{I}_4 \left(\mathcal{Q} \right) = 0, \qquad (7.2.2.34)$$

The action of $\mathcal{PQ} \left(2n_V + 2, \mathbb{R} \right)$ reads

$$\begin{pmatrix} p^0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{\mathcal{O}^{-1}} \begin{pmatrix} p^0 \\ 0 \\ -\varrho p^0 \\ -\mathbf{c}_i p^0 \end{pmatrix}, \qquad (7.2.2.35)$$

and thus it generates ϱ-dependent electric charge q_0 and $\mathbf{c}_i$-dependent electric charges q_i's. These latter in Type II compactifications corresponds to a stack of $D2$ branes depending on the components of the second Chern class c_2 of CY_3 (recall Eq. (7.2.1.51)). The corresponding transformation of $\mathcal{I}_4$ reads

$$0 \xrightarrow{\mathcal{O}^{-1}} \left(p^0 \right)^4 \left(\frac{2}{3} d^{ijk} \mathbf{c}_i \mathbf{c}_j \mathbf{c}_k - \varrho^2 \right) \gtrless 0. \qquad (7.2.2.36)$$

Thus, depending on whether

$$\frac{2}{3} d^{ijk} \mathbf{c}_i \mathbf{c}_j \mathbf{c}_k - \varrho^2 \gtrless 0, \qquad (7.2.2.37)$$

a "large" ($\mathcal{I}_4 > 0$:BPS or non-BPS $Z_H = 0$), a "small" ($\mathcal{I}_4 = 0$:BPS or non-BPS), or a "large" non-BPS $Z_H \neq 0$ ($\mathcal{I}_4 < 0$) BH charge configuration is generated. In case the quantity in (7.2.2.33) is non-vanishing, this is an example of phenomenon (7.2.2.4). Note that for $\varrho = 0$, an example treated in [29] is recovered, i.e.:

$$\begin{cases} 0 \xrightarrow{\mathcal{O}^{-1}} 4 \left(p^0 \right)^4 \mathcal{I}_3 \left(\mathbf{c} \right) \gtrless 0; \\ \\ \mathcal{I}_3 \left(\mathbf{c} \right) \equiv \frac{1}{3!} d^{ijk} \mathbf{c}_i \mathbf{c}_j \mathbf{c}_k. \end{cases} \qquad (7.2.2.38)$$

11. **"Small" *doubly-critical* (1-charge) q_0 ("electric" Kaluza-Klein) configuration**. This is the limit $p^0 = 0$ of configuration **1**. In Type II, it corresponds to only $D0$ branes:

$$\mathcal{Q} \equiv (0, 0, q_0, 0)^T \Rightarrow \mathcal{I}_4 \left(\mathcal{Q} \right) = 0, \qquad (7.2.2.39)$$

This configuration is the *"electric-magnetic dual"* of the "magnetic" configuration **10**. Since there are no magnetic charges, $\mathcal{PQ}(2n_V + 2, \mathbb{R})$ is inactive on this configuration:

$$
\begin{pmatrix} 0 \\ 0 \\ q_0 \\ 0 \end{pmatrix} \xrightarrow{\mathcal{O}^{-1}} \begin{pmatrix} 0 \\ 0 \\ q_0 \\ 0 \end{pmatrix}. \tag{7.2.2.40}
$$

We conclude this Sec. with a comment on the attractor values of the scalars, *i.e.* on the non-degenerate critical points of the effective BH potential V_{BH}. In presence of the sub-leading quantum perturbative corrections (7.2.1.8), the expressions of such critical points can be obtained from the ones for the uncorrected (not necessarily cubic) SK geometry, by applying a suitable transformation of $\mathcal{PQ}(2n_V + 2, \mathbb{R})$ on the charges.

This fact has been known for some time [29; 30]. In the case in which the uncorrected geometry is a d-SK geometry with prepotential (7.2.1.9), this provides a generally more efficient approach to the computation of the attractor horizon (purely charge-dependent) values of the scalars. In other words, one has to start from the general expression of the extremal BH attractors for d-SK geometries [30; 33], and then apply the suitable transformation $\mathcal{O}^{-1}$ (7.2.1.32) of $\mathcal{PQ}(2n_V + 2, \mathbb{R})$ on the charges. As an example, in this way the results recently obtained in Sec. 3 and App. A of [31] can be recovered.

Transformation of V_{BH}

As mentioned above, $\mathcal{PQ}(2n_V + 2, \mathbb{R})$, *when acting both on the charges $\mathcal{Q}$ and on the covariantly holomorphic symplectic sections $\mathcal{V}$*, leaves Z and $D_i Z$, and thus V_{BH} given by (7.2.1.40), invariant.

Actually, in order to investigate the effect of the quantum perturbative sub-leading corrections (7.2.1.8) to any $\mathcal{N} = 2$ prepotential on Z, $D_i Z$, V_{BH}, $\partial_i V_{BH}$, $D_i \partial_j V_{BH}$, $D_i \bar{\partial}_{\bar{j}} V_{BH}$ etc., one should act with $\mathcal{PQ}(2n_V + 2, \mathbb{R})$ *only on charges*. In order to show this, let us consider (without any loss of generality for our purposes) the $\mathcal{N} = 2$ central charge $Z \equiv \langle \mathcal{Q}, \mathcal{V} \rangle \equiv \mathcal{Q}^T \Omega \mathcal{V}$. By recalling that $\mathfrak{F}$ can be introduced through the action of $\mathcal{O} \in \mathcal{PQ}(2n_V + 2, \mathbb{R})$ (7.2.1.25) on the sections, the expression

of *Z for any* $\mathcal{N} = 2$ prepotential corrected with $\mathfrak{F}$ (7.2.1.8) is given by

$$
\begin{aligned}
Z' & \equiv Z\left(\mathcal{O}\mathcal{V}\left(z,\bar{z}\right);\mathcal{Q}\right) \equiv \langle\mathcal{Q},\mathcal{O}\mathcal{V}\rangle \equiv \mathcal{Q}^T\Omega\mathcal{O}\mathcal{V} \\
& = \mathcal{Q}^T\left(\mathcal{O}^T\right)^{-1}\Omega\mathcal{V} = \left\langle\mathcal{O}^{-1}\mathcal{Q},\mathcal{V}\right\rangle \equiv Z\left(\mathcal{V}\left(z,\bar{z}\right);\mathcal{O}^{-1}\mathcal{Q}\right), \quad (7.2.2.41)
\end{aligned}
$$

where in the second line the symplectic nature of $\mathcal{O}$ has been exploited. Thus, the expression of *Z for any* $\mathcal{N} = 2$ prepotential corrected with $\mathfrak{F}$ (7.2.1.8) is nothing but the expression of Z computed for the uncorrected prepotential, with the charges transformed through $\mathcal{O}$ given by (7.2.1.25). The very same holds also for W, D_iW, $D_iZ, V_{BH}, \partial_i V_{BH}, D_i\partial_j V_{BH}, D_i\bar{\partial}_{\bar{j}}V_{BH}$, and in general for all quantities depending on scalars and charges. In the case of the *locus* $\partial_i V_{BH} = 0$, this allows to easily compute the $\mathfrak{F}$-corrected attractors, once the ones for the uncorrected prepotential are known (see the discussion at the end of Sec. 7.2.2). In the case in which the uncorrected SK geometry is a cubic one, with prepotential (7.2.1.9), this reasoning provides a general alternative approach for the generalization (for all charge configurations in which the treatment of the purely cubic case is feasible [30; 33; 44]) of the computations recently performed in Sec. 3 and App. A of [31].

In light of the previous reasoning, the explicit expressions of Z, D_iZ and V_{BH} for an $\mathfrak{F}$-corrected *d*-SK geometry can be immediately obtained by applying the charge transformation $\mathcal{O}^{-1}$ (given by (7.2.1.32)) to Eqs. (4.9), (4.10) and (2.13) of [44], respectively.

Since it is crucial to our treatment, we here consider only the $\mathfrak{F}$-corrected expression of V_{BH} for *d*-SK geometries. As mentioned, the expression of V_{BH} for *d*-SK geometries (7.2.1.9) is given by Eq. (2.13) of [44], which we report here for ease of comparison:

$$
\begin{aligned}
2V_{BH}\left(z,\bar{z};\mathcal{Q}\right) = & \left[\nu\left(1+4g\right) + \frac{h^2}{36\nu} + \frac{3}{48\nu}g^{ij}h_ih_j\right]\left(p^0\right)^2 + \\
& + \left[4\nu g_{ij} + \frac{1}{4\nu}\left(h_ih_j + g^{mn}h_{im}h_{nj}\right)\right]p^ip^j + \\
& + \frac{1}{\nu}\left[q_0^2 + 2x^iq_0q_i + \left(x^ix^j + \frac{1}{4}g^{ij}\right)q_iq_j\right] + \\
& + 2\left[\nu g_i - \frac{h}{12\nu}h_i - \frac{1}{8\nu}g^{jm}h_mh_{ij}\right]p^0p^i + \\
& - \frac{1}{3\nu}\left[\begin{array}{l} -hp^0q_0 + 3q_0p^ih_i - \left(hx^i + \frac{3}{4}g^{ij}h_j\right)p^0q_i \\ +3\left(h_jx^i + \frac{1}{2}g^{im}h_{mj}\right)q_ip^j \end{array}\right]
\end{aligned} \quad (7.2.2.42)
$$

where the following notation has been introduced (see e.g. [44] for further details):

$$
\begin{cases}
z^i \equiv x^i - i\lambda^i; \\[2mm]
\nu \equiv \frac{1}{3!} d_{ijk} \lambda^i \lambda^j \lambda^k; \\[2mm]
h_{ij} \equiv d_{ijk} x^k; \; h_i \equiv d_{ijk} x^j x^k; \; h \equiv d_{ijk} x^i x^j x^k; \\[2mm]
d_{ij} \equiv d_{ijk} \lambda^k; \; d_i \equiv d_{ijk} \lambda^j \lambda^k; \; d^{ij} d_{jk} \equiv \delta^i_k; \\[2mm]
g_{ij} = -\frac{1}{4} \left(\frac{d_{ij}}{\nu} - \frac{d_i d_j}{4\nu^2} \right); \; g^{ij} = 2 \left(\lambda^i \lambda^j - 2\nu d^{ij} \right); \\[2mm]
g_i \equiv -4 g_{ij} x^j; \; g \equiv g_{ij} x^i x^j.
\end{cases}
\tag{7.2.2.43}
$$

It is worth recalling that (7.2.2.42) was recently re-obtained as the $\mathrm{Im} d_{000} = 0$ limit of the more general quantum perturbative result of [26]. Consistently with the above reasoning, straightforward computations lead to the following expression of the $\mathfrak{F}$-corrected expression of V_{BH} for d-SK geometries:

$$
V_{BH}(z, \bar{z}; \mathcal{Q}) \xrightarrow{\mathcal{O}^{-1}} V_{BH} \left(z, \bar{z}; \mathcal{O}^{-1} \mathcal{Q} \right) = V_{BH}(z, \bar{z}; \mathcal{Q}) + \mathfrak{V}_{BH} \left(z, \bar{z}; \mathcal{Q}, \varrho, \mathbf{c}_i, \Theta_{ij} \right),
\tag{7.2.2.44}
$$

where $\mathfrak{V}_{BH}$ describes the *"PQ-deformation"* of V_{BH}:

$$
2\mathfrak{V}_{BH}\left(z,\bar{z};\mathcal{Q},\varrho,\mathbf{c}_i,\Theta_{ij}\right) = \frac{1}{v}\left[
\begin{array}{c}
\varrho^2\left(p^0\right)^2 + \left(\mathbf{c}_i p^i\right)^2 - 2q_0\varrho p^0 - 2q_0\mathbf{c}_i p^i + 2\varrho p^0\mathbf{c}_i p^i \\[4pt]
+2x^i\left(\begin{array}{c} -p^0 q_0\mathbf{c}_i - q_0\Theta_{ij}p^j \\[4pt] -\varrho p^0 q_i + \varrho\left(p^0\right)^2\mathbf{c}_i + \varrho p^0\Theta_{ij}p^j \\[4pt] -\mathbf{c}_j p^j q_i + p^0\mathbf{c}_i\mathbf{c}_j p^j + \mathbf{c}_j p^j\Theta_{ik}p^k \end{array}\right) \\[30pt]
+\left(x^i x^j + \tfrac{1}{4}g^{ij}\right)\left(\begin{array}{c} -q_i\mathbf{c}_j p^0 - q_i\Theta_{jl}p^l \\[4pt] -p^0\mathbf{c}_i q_j + \left(p^0\right)^2\mathbf{c}_i\mathbf{c}_j + p^0\mathbf{c}_i\Theta_{jk}p^k \\[4pt] -\Theta_{ik}p^k q_j + p^0\Theta_{ik}p^k\mathbf{c}_j + \Theta_{ik}p^k\Theta_{jl}p^l \end{array}\right)
\end{array}
\right] +
$$

$$
-\frac{1}{3v}\left[
\begin{array}{c}
hp^0\left(\varrho p^0 + \mathbf{c}_i p^i\right) \\[6pt]
-3\left(\varrho p^0 + \mathbf{c}_j p^j\right)p^i h_i \\[6pt]
+\left(hx^i + \tfrac{3}{4}g^{ij}h_j\right)p^0\left(\mathbf{c}_i p^0 + \Theta_{ik}p^k\right) \\[6pt]
-3\left(h_j x^i + \tfrac{1}{2}g^{im}h_{mj}\right)\left(\mathbf{c}_i p^0 + \Theta_{ik}p^k\right)p^j
\end{array}
\right].
$$

$$(7.2.2.45)$$

Eqs. (7.2.2.44), (7.2.2.42) and (7.2.2.45), once specified for the particular $n_V = 2$ model treated in [31] (see Eq. (3.7) therein), allows one to easily recover Eq. (A.12) therein. Furthermore, by setting $p^0 = 0 = q_i$ (*i.e.* by considering the $D0 - D4$ configuration), Eq. (7.2.2.45) yields that the $\mathfrak{F}$-corrected V_{BH} does not depend at all on ϱ; this fact generalizes the comment below Eq. (3.1) of [31].

Let us now consider the part of V_{BH} (7.2.2.42) linear in the axions x^i. Eq. (7.2.2.42) yields

$$2V_{BH}\big|_{\text{linear in } x^i} = \frac{2}{v}x^i q_0 q_i + 2v g_i p^0 p^i - \frac{1}{2v}g^{ik}h_{kj}q_i p^j. \qquad (7.2.2.46)$$

This implies that the BH charge configurations which support the *axion-free* solution $x^i = 0 \ \forall i$ *at least* as a particular solution of the axionic Attractor Eqs. $\partial V_{BH}/\partial x^i = 0$

are the following ones [44]:

$$\begin{cases} (p^0, q_0); \\ (p^0, q_i); \\ (p^i, q_0), \end{cases}$$
(7.2.2.47)

i.e. the "large" configurations **1**, **2** and **3** treated in Sec. 7.2.2.

Through Eqs. (7.2.2.44), (7.2.2.42) and (7.2.2.45), the action of $\mathcal{PQ}(2n_V + 2, \mathbb{R})$ transforms (7.2.2.46) as follows:

$$2\left[V_{BH} + \mathfrak{V}_{BH}\right]|_{\text{linear in } x^i} = \frac{2}{v}x^i q_0 q_i + 2vg_i p^0 p^i - \frac{1}{2v}g^{ik}h_{kj}q_i p^j$$
$$+\frac{2}{v}x^i \begin{pmatrix} -p^0 q_0 \mathbf{c}_i - q_0 \Theta_{ij} p^j \\ -\varrho p^0 q_i + \varrho \left(p^0\right)^2 \mathbf{c}_i + \varrho p^0 \Theta_{ij} p^j \\ -\mathbf{c}_j p^j q_i + p^0 \mathbf{c}_i \mathbf{c}_j p^j + \mathbf{c}_j p^j \Theta_{ik} p^k \end{pmatrix}$$
$$+\frac{1}{2v}g^{im}h_{mj}\left(\mathbf{c}_i p^0 + \Theta_{ik} p^k\right) p^j.$$
(7.2.2.48)

The rather intricate expression (7.2.2.48) implies that, in presence of the sub-leading quantum perturbative corrections (7.2.1.8), the configurations (7.2.2.47) do not support axion-free solutions any more, and that in general there are no axion-free-supporting BH charge configurations at all[17], *unless some extra assumptions are made.* For instance, (7.2.2.48) yields the following axion-free-supporting conditions for the charge configurations (7.2.2.47):

$$2\left[V_{BH} + \mathfrak{V}_{BH}\right]|_{\text{linear in } x^i, (p^0, q_0)} = \frac{2}{v}x^i \mathbf{c}_i p^0 \left(-q_0 + \varrho p^0\right) = 0 \Leftrightarrow \begin{cases} \mathbf{c}_i = 0; \\ and/or \\ q_0 = \varrho p^0; \end{cases}$$
(7.2.2.49)

$$2\left[V_{BH} + \mathfrak{V}_{BH}\right]|_{\text{linear in } x^i, (p^0, q_i)} = \frac{2}{v}x^i \varrho p^0 \left(-q_i + p^0 \mathbf{c}_i\right) = 0 \Leftrightarrow \begin{cases} \varrho = 0; \\ and/or \\ q_i = p^0 \mathbf{c}_i; \end{cases}$$
(7.2.2.50)

---

[17]This result is consistent with the analysis of the particular $n_V = 2$ model in *D0-D4* configuration worked out in [31].

$$2\left[V_{BH} + \mathfrak{V}_{BH}\right]\big|_{\text{linear in } x^i, (p^i, q_0)} = \frac{2}{\nu}\left(-\delta^i_m q_0 + \delta^i_m \mathbf{c}_k p^k + \frac{1}{4}g^{il}d_{klm}p^k\right)\Theta_{ij}p^j x^m = 0$$

$$\Updownarrow$$

$$\begin{cases} \Theta_{ij} = 0; \\ \text{and/or} \\ -\delta^i_m q_0 + \delta^i_m \mathbf{c}_k p^k + \frac{1}{4}g^{il}d_{klm}p^k = 0. \end{cases} \tag{7.2.2.51}$$

It is known [22] that in symmetric d-SK geometries, the *"moduli space"* of non-BPS $Z_H \neq 0$ attractors is the scalar manifold of the $d = 5$ uplifted theory. This can be easily seen in the (p^0, q_0) configuration. Indeed, by setting $x^i = 0\ \forall i$, the effective BH potential (7.2.2.42) reads

$$2\,V_{BH}\big|_{(p^0, q_0), x^i = 0\ \forall i} = \nu\left(p^0\right)^2 + \frac{1}{\nu}q_0^2, \tag{7.2.2.52}$$

thus depending only on the Kaluza-Klein volume ν. The n_V real "rescaled" dilatons [44]

$$\widehat{\lambda}^i \equiv \nu^{-\frac{1}{3}}\lambda^i, \tag{7.2.2.53}$$

which defines the $d = 5$ scalar manifold through the cubic constraint

$$\frac{1}{3!}d_{ijk}\widehat{\lambda}^i\widehat{\lambda}^i\widehat{\lambda}^i = 1 \tag{7.2.2.54}$$

are *"flat directions"* of the critical value (7.2.2.52).

The action of $\mathcal{PQ}\left(2n_V + 2, \mathbb{R}\right)$ may make the emergence of *"moduli spaces"* of attractors less manifest but, as stated above, does not change their geometrical structure. From (7.2.2.49), in $\mathfrak{F}$-corrected d-SK geometry the Kaluza-Klein charge configuration (p^0, q_0) (with no further constraints) is axion-free-supporting for $\mathbf{c}_i = 0\ \forall i$. In such a case, Eqs. (7.2.2.49) and (7.2.2.9) respectively yield

$$2\left[V_{BH} + \mathfrak{V}_{BH}\right]\big|_{(p^0, q_0), x^i = 0\ \forall i} = \nu\left(p^0\right)^2 + \frac{1}{\nu}\left(q_0 - \varrho p^0\right)^2; \tag{7.2.2.55}$$

$$-\left(p^0\right)^2 q_0^2 \xrightarrow{\mathcal{O}^{-1}} -\left(p^0\right)^2\left(q_0 - \varrho p^0\right)^2. \tag{7.2.2.56}$$

Thus, the PQ-transformed BH charge configuration (p^0, q_0) with $\mathbf{c}_i = 0\ \forall i$ (and $q_0 \neq \varrho p^0$) still supports non-BPS $Z_H \neq 0$ (possibly axion-free) extremal BH attractors, whose *"moduli space"* is still manifest from (7.2.2.55). Note that the case $q_0 = \varrho p^0$ is troublesome, because it does not stabilize the Kaluza-Klein volume through the Attractor Mechanism.

7.2.3 Cayley's Hyperdeterminant and Elliptic Curves

Recently, in [34], an intriguing relation between elliptic curves and the Cayley's hyperdeterminant [35] was found.

More specifically, it was shown that if the cubic elliptic curve

$$y^2 = ax^3 + bx^2 + cx + d \qquad (7.2.3.1)$$

has a Mordell-Weil group containing a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then it can be transformed into the Cayley's hyperdeterminant $\mathrm{Det}(\psi)$, which is nothing but the (opposite of the) quartic scalar invariant built out of the unique rank-4 completely symmetric primitive invariant tensor of the repr. $(\mathbf{2}, \mathbf{2}, \mathbf{2})$ of $[SL\,(2, \mathbb{R})]^3$, which in turn is the U-duality group of the $\mathcal{N} = 2, d = 4$ so-called *stu* model [36]:

$$
\begin{aligned}
\mathcal{I}_{4,stu}\,(\mathcal{Q}) \;=\; & -\left(p^0\right)^2 q_0^2 - \left(p^1\right)^2 q_1^2 - \left(p^2\right)^2 q_2^2 - \left(p^3\right)^2 q_3^2 \\
& -2p^0 q_0 p^1 q_1 - 2p^0 q_0 p^2 q_2 - 2p^0 q_0 p^3 q_3 + 2p^1 q_1 p^2 q_2 + 2p^1 q_1 p^3 q_3 \\
& +2p^2 q_2 p^3 q_3 + 4q_0 p^1 p^2 p^3 - 4p^0 q_1 q_2 q_3 = -\mathrm{Det}\,(\psi)\,. \qquad (7.2.3.2)
\end{aligned}
$$

This expression can be obtained from the general one (7.2.1.46)-(7.2.1.47), by specifying the *stu* model data:

$$d_{ijk} = 6\delta_{1(i|}\delta_{2|j|}\delta_{3|k)}; \; d^{ijk} = 6\delta^{1(i|}\delta^{2|j|}\delta^{3|k)}, \qquad (7.2.3.3)$$

consistent with (7.2.1.45).

Under the aforementioned assumption on the Mordell-Weil group, the elliptic curve (7.2.3.1) can be factorised as [34]

$$y^2 = 4\,(l - kx)\,(n - mx)\,(q - px)\,, \qquad (7.2.3.4)$$

and through the positions (with u, v unknowns) [34]

$$
\begin{aligned}
y \;&=\; uv^2 - ev + g; & (7.2.3.5)\\
x \;&=\; v; & (7.2.3.6)\\
a \;&=\; -4kmp; & (7.2.3.7)\\
b \;&=\; 4kmrt + 4kpts + 4mprs; & (7.2.3.8)\\
c \;&=\; -4rts\,(kt + mr + ps)\,; & (7.2.3.9)\\
d \;&=\; 4r^2 s^2 t^2, & (7.2.3.10)
\end{aligned}
$$

finally (7.2.3.1) can be recast in the form

$$
\begin{aligned}
u^2 v^2 \;+\;& k^2 t^2 + m^2 r^2 + p^2 s^2 - 2ktuv - 2mruv - 2psuv + \\
-\;& 2kmrt - 2kpts - 2mprs + 4kmpv + 4rstu = 0, \qquad (7.2.3.11)
\end{aligned}
$$

which correspond to the vanishing of $\mathcal{I}_{4,stu}(\mathcal{Q})$ as given by (7.2.3.2), under the (non-unique) following mapping of the charge vector:

$$\mathcal{Q} \equiv \left(p^0, p^1, p^2, p^3, q_0, q_1, q_2, q_3\right)^T = (u, k, m, p, -v, t, r, s)^T. \qquad (7.2.3.12)$$

Interestingly, the two unknowns u and v corresponds to the magnetic ($D6$) and electric ($D0$) Kaluza-Klein charges in the reduction $d = 5 \to d = 4$.

Under the position (7.2.3.12), the vanishing of $\mathcal{I}_{4,stu}(\mathcal{Q})$, a necessary condition defining the "small" orbits of the $(\mathbf{2},\mathbf{2},\mathbf{2})$ of $[SL(2,\mathbb{R})]^3$ [67], can be recast in the form (7.2.3.1), with

$$
\begin{aligned}
y &= p^0 q_0^2 + q_0\left(p^1 q_1 + p^2 q_2 + p^3 q_3\right) + 2q_1 q_2 q_3; & (7.2.3.13)\\
x &= -q_0; & (7.2.3.14)\\
a &= -4p^1 p^2 p^3; & (7.2.3.15)\\
b &= 4\left(p^1 q_1 p^2 q_2 + p^1 q_1 p^3 q_3 + p^2 q_2 p^3 q_3\right); & (7.2.3.16)\\
c &= -4q_1 q_2 q_3\left(p^1 q_1 + p^2 q_2 + p^3 q_3\right); & (7.2.3.17)\\
d &= 4q_1^2 q_2^2 q_3^2, & (7.2.3.18)
\end{aligned}
$$

In light of the treatment given in Secs. 7.2.1 and 7.2.2, it is worth pointing out that the above construction admits a "$\mathcal{PQ}(8,\mathbb{R})$-deformation".

The "$\mathcal{PQ}(8,\mathbb{R})$-deformation" of the Cayley's hyperdeterminant can be obtained from the general result (7.2.2.1)-(7.2.2.2) by using the stu model data (7.2.3.3) (here $i,j = 1,2,3$):

$$
\begin{aligned}
\mathcal{I}_{4,stu}(\mathcal{Q}) + \mathbf{I}_{4,stu}\left(\mathcal{Q}; \varrho, \mathbf{c}_i, \Theta_{ij}\right) &= -\left(p^0\right)^2\left(q_0 - \varrho p^0 - \mathbf{c}_i p^i\right)^2\\
-(p^i)^2\left(q_i - \mathbf{c}_i p^0 - \Theta_{ij} p^j\right)^2 &- 2p^0 p^i\left(q_0 - \varrho p^0 - \mathbf{c}_j p^j\right)\left(q_i - \mathbf{c}_i p^0 - \Theta_{ij} p^j\right)\\
+\sum_{i=1}^{3}|\epsilon_{ijk}|\, p^j\left(q_j - \mathbf{c}_j p^0 - \Theta_{jl} p^l\right) p^k\left(q_k - \mathbf{c}_k p^0 - \Theta_{km} p^m\right) & \qquad (7.2.3.19)\\
+4\left(q_0 - \varrho p^0 - \mathbf{c}_i p^i\right) p^1 p^2 p^3 - 4p^0 \prod_{i=1}^{3}\left(q_i - \mathbf{c}_i p^0 - \Theta_{ij} p^i\right)&\\
= -\mathrm{Det}\left(\psi; \varrho, \mathbf{c}_i, \Theta_{ij}\right).&
\end{aligned}
$$

The various terms (unknowns and coefficients) of the corresponding cubic elliptic

curve (7.2.3.1) are given by the $\mathcal{PQ}\,(8,\mathbb{R})$-transformed Eqs. (7.2.3.13)-(7.2.3.18), i.e.:

$$y = p^0\left(q_0 - \varrho p^0 - \mathbf{c}_i p^i\right)^2 + \left(q_0 - \varrho p^0 - \mathbf{c}_i p^i\right) p^i \left(q_i - \mathbf{c}_i p^0 - \Theta_{ij} p^j\right)$$
$$+ 2\prod_i \left(q_i - \mathbf{c}_i p^0 - \Theta_{ij} p^i\right); \tag{7.2.3.20}$$

$$x = -\left(q_0 - \varrho p^0 - \mathbf{c}_i p^i\right); \tag{7.2.3.21}$$

$$a = -4 p^1 p^2 p^3; \tag{7.2.3.22}$$

$$b = 2\sum_i |\epsilon_{ijk}|\, p^j \left(q_j - \mathbf{c}_j p^0 - \Theta_{jl} p^l\right) p^k \left(q_k - \mathbf{c}_k p^0 - \Theta_{km} p^m\right); \tag{7.2.3.23}$$

$$c = -2\sum_k |\epsilon_{klm}|\, p^l \left(q_l - \mathbf{c}_l p^0 - \Theta_{ln} p^n\right) p^m \left(q_m - \mathbf{c}_m p^0 - \Theta_{mr} p^r\right) \prod_i \left(q_i - \mathbf{c}_i p^0 - \Theta_{ij} p^i\right); \tag{7.2.3.24}$$

$$d = 4\left[\prod_i \left(q_i - \mathbf{c}_i p^0 - \Theta_{ij} p^i\right)\right]^2. \tag{7.2.3.25}$$

Clearly, the roots of the elliptic cubic curve (7.2.3.1) (with data (7.2.3.13)-(7.2.3.18)) are not the same as the roots of (7.2.3.1) (with data (7.2.3.20)-(7.2.3.25)). In general, the action of $\mathcal{PQ}\,(8,\mathbb{R})$ amounts to a $(\varrho, \mathbf{c}_i, \Theta_{ij})$-redefinition of the vertices of the hypercube whose associate hyperdeterminant is $\mathrm{Det}(\psi)$ given by (7.2.3.2) [35].

Let us further remark that in realistic superstring compactifications leading to the *stu* model in the supergravity limit, the values of the parameters $\mathbf{c}_i = \frac{c_{2,i}}{24}$ (where c_2 is the second Chern class; see Sec. 7.2.1) can be computed to read [68; 69]:

$$\begin{array}{ll} \text{Type } IIA \text{ on } K3 \text{ fibrations} & : c_{2,1} = c_{2,3} = 24;\ c_{2,2} = 92; \\ \text{Heterotic on } T^4 \times T^2 \text{ or } K3 \times T^2 & : c_{2,2} = c_{2,3} = 0. \end{array} \tag{7.2.3.26}$$

In view of the recent progress within the fascinating BH/qubit correspondence [37], $\mathcal{PQ}\,(8,\mathbb{R})$ may well have a role on the quantum information side; we leave the study of this interesting issue for future investigation.

7.3 An Alternative Expression for $\mathcal{I}_4$

By refining and extending the analysis of [26] and considering d-SK geometries based on the purely cubic holomorphic prepotential (7.2.1.9), we will now derive an alternative expression of the quartic invariant $\mathcal{I}_4$ given by (7.2.1.46)-(7.2.1.47).

A crucial quantity in such developments is the so-called *E*-tensor. Such a rank-5 tensor was firstly introduced in [17] (see also the treatment of [47]), and it expresses

the deviation of the considered geometry from being symmetric. Its definition reads (see *e.g.* [11; 26] for a recent treatment, and Refs. therein):

$$\overline{E}_{\overline{m}ijkl} \equiv \frac{1}{3}\overline{D}_{\overline{m}}D_i C_{jkl}. \tag{7.3.0.1}$$

This definition can be elaborated further, by recalling the properties of the so-called C-tensor C_{ijk}. This is a rank-3 tensor with Kähler weights $(2, -2)$, defined as (see *e.g.* [2; 70]):

$$
\begin{aligned}
C_{ijk} &\equiv \langle D_i D_j \mathbf{V}, D_k \mathbf{V}\rangle = e^K \left(\partial_i \mathcal{N}_{\Lambda\Sigma}\right) D_j X^\Lambda D_k X^\Sigma \\
&= e^K \left(\partial_i X^\Lambda\right)\left(\partial_j X^\Sigma\right)\left(\partial_k X^\Xi\right)\partial_\Xi \partial_\Sigma F_\Lambda (X) \equiv e^K W_{ijk}, \ \ \overline{\partial}_{\bar{l}} W_{ijk} = 0,
\end{aligned}
\tag{7.3.0.2}
$$

where $\mathcal{N}_{\Lambda\Sigma}$ is the $\mathcal{N} = 2$, $d = 4$ kinetic vector matrix, and the second line holds only in "special coordinates". C_{ijk} is completely symmetric and covariantly holomorphic:

$$C_{ijk} = C_{(ijk)}; \ \ \overline{D}_{\bar{i}} C_{jkl} = 0. \tag{7.3.0.3}$$

By further steps, detailed in [26], the expression for $\overline{E}_{\overline{m}ijkl}$ defined by (7.3.0.1) can thus be further elaborated as follows:

$$C_{p(ij}C_{kl)q}g^{p\bar{r}}g^{q\bar{s}}\overline{C}_{\bar{r}\bar{s}\bar{t}} = \frac{4}{3}C_{(ijk}g_{l)\bar{t}} + \overline{E}_{\bar{t}ijkl}. \tag{7.3.0.4}$$

Formulæ (7.3.0.1) and (7.3.0.4) hold for a generic SK geometry. By considering d-SK geometries based on the purely cubic holomorphic prepotential (7.2.1.9) in the "special coordinates" symplectic basis, (7.3.0.4) can be recast as

$$\left(X^0\right)^3 e^{3K} d_{p(ij}d_{kl)q}g^{pr}g^{qs}d_{rst} = \frac{4}{3}X^0 e^K d_{(ijk}g_{l)t} + E_{tijkl}, \tag{7.3.0.5}$$

where g^{ij} and g_{ij} are defined in (7.2.2.43) (see *e.g.* [44] for further details).

Let us now introduce the "rescaled metric" [44; 71] and, for later convenience, its derivatives with respect to $\widehat{\lambda}^i$ (the unique set of scalars on which it actually depends):

$$a_{ij} \equiv 4v^{2/3}g_{ij} = \left(\frac{1}{4}\widehat{d}_i\widehat{d}_j - \widehat{d}_{ij}\right) \Leftrightarrow a^{ij} = \frac{1}{4}v^{-2/3}g^{ij} = \frac{1}{2}\widehat{\lambda}^i\widehat{\lambda}^j - \widehat{d}^{ij}; \tag{7.3.0.6}$$

$$\frac{\partial a_{ij}}{\partial\widehat{\lambda}^k} = \frac{1}{2}\left(\widehat{d}_{ik}\widehat{d}_j + \widehat{d}_{jk}\widehat{d}_i\right) - d_{ijk}; \tag{7.3.0.7}$$

$$\frac{\partial a^{ij}}{\partial\widehat{\lambda}^k} = \frac{1}{2}\left(\delta_k^i\widehat{\lambda}^j + \delta_k^j\widehat{\lambda}^i\right) + \widehat{d}^{il}\widehat{d}^{jm}d_{klm}, \tag{7.3.0.8}$$

where $\widehat{d}_{ij}$, $\widehat{d}^{ij}$ and $\widehat{d}_i$ are the "hatted" counterpart of the quantities defined in (7.2.2.43) (also recall the splitting $z^i \equiv x^i - i\lambda^i$ in the first line of (7.2.2.43), as well as (7.2.2.53) and (7.2.2.54)):

$$\widehat{d}_{ij} \equiv d_{ijk}\widehat{\lambda}^k; \ \widehat{d}_i \equiv d_{ijk}\widehat{\lambda}^j\widehat{\lambda}^k; \tag{7.3.0.9}$$

$$\widehat{d}^{ij}\widehat{d}_{jk} \equiv \delta^i_k \Rightarrow \frac{\partial \widehat{d}^{im}}{\partial \widehat{\lambda}^k} = -\widehat{d}^{ij}\widehat{d}^{ml}d_{jkl}. \tag{7.3.0.10}$$

Thus, by fixing the Kähler gauge $X^0 \equiv 1$, after some algebra one achieves the following result[18]:

$$d_{p(ij}d_{kl)q}d^{pqv} = \frac{4}{3}\delta^v_{(l}d_{ijk)} + 2^5 v^{5/3}E^v_{ijkl}, \tag{7.3.0.11}$$

where

$$d^{pqv} \equiv a^{pr}a^{qs}a^{vt}d_{rst}; \tag{7.3.0.12}$$

$$E^v_{ijkl} \equiv a^{vt}E_{tijkl}. \tag{7.3.0.13}$$

From (7.3.0.11), one can re-derive the explicit expression of E_{tijkl} given by Eq. (4.21) of[19] [26], implying that in any d-SK geometry $v^{5/3}E_{tijkl}$ depends only on the "rescaled $d = 4$ dilatons" $\widehat{\lambda}^i$.

Let us now introduce the following p^i-dependent quantities, which are scalar-independent in any d-SK geometry[20]:

$$\mathbf{d}_{ij} \equiv d_{ijk}p^k = \frac{\partial \mathcal{I}_3(p)}{\partial p^i \partial p^j}; \ \mathbf{d}^{ij}\mathbf{d}_{jk} \equiv \delta^i_k, \tag{7.3.0.14}$$

from which the following behaviors follow: $\mathbf{d}_{ij} \sim [p]^2$ and $\mathbf{d}^{ij} \sim [p]^{-2}$.

Thus, whenever $\mathbf{d}_{ij}$ has maximal rank n_V, by contracting (7.3.0.11) with $p^k p^l p^i q_v q_t \mathbf{d}^{jt}$, a little algebra leads to the result

$$-\left(p^i q_i\right)^2 + d_{ijk}d^{ilm}p^j p^k q_l q_m = \frac{1}{3}d_{ijk}p^i p^j p^k q_l q_m \mathbf{d}^{lm} + 2^5 v^{5/3}E^m_{ijkl}p^j p^k p^l q_m q_n \mathbf{d}^{in}. \tag{7.3.0.15}$$

By plugging (7.3.0.15) into the general expression of $\mathcal{I}_4$ given by (7.2.1.46)-(7.2.1.47), one obtains the following alternative expression:

$$\begin{aligned}
\mathcal{I}_4 = & -\left(p^0\right)^2 q_0^2 - 2p^0 q_0 p^i q_i + \frac{1}{3}\left(2q_0 + q_i q_j \mathbf{d}^{ij}\right) d_{klm}p^k p^l p^m \\
& -\frac{2}{3}p^0 d^{ijk}q_i q_j q_k + 2^5 v^{5/3}E^m_{ijkl}p^j p^k p^l q_m q_n \mathbf{d}^{in},
\end{aligned} \tag{7.3.0.16}$$

---

[18]Note that in d-SK geometries all geometrical quantities under consideration are real.

[19]For homogeneous non-symmetric d-SK geometries, the expression of the E-tensor was explicitly computed in [72].

[20]Attention should be paid not to confuse the scalar-independent quantities $\mathbf{d}_{ij}$ and $\mathbf{d}^{ij}$ defined by (7.3.0.14) with the λ^i-dependent quantities d_{ij} and d_{ij} defined in (7.2.2.43).

which manifestly shows the contribution of the E-tensor as a source of dependence on v and $\widehat{\lambda}^{i}$'s for non-symmetric d-SK geometries, and more in general for all d-SK geometries in which the term $E^{m}_{ijkl}p^{j}p^{k}p^{l}q_{m}q_{n}\mathbf{d}^{in}$ does not vanish. Note that (7.3.0.16) is well defined whenever $\mathbf{d}_{ij}$ (introduced in (7.3.0.14)) has maximal rank n_V.

Some comments on the alternative formula (7.3.0.16) for $\mathcal{I}_4$ are in order.

1. In *symmetric* d-SK geometries (see *e.g.* [17; 47], and Refs. therein) $E_{mijkl} = 0$, as a consequence of the covariant constancy of the Riemann tensor $R_{i\bar{j}k\bar{l}}$ itself (see *e.g.* [26] for a recent treatment):

$$D_m R_{i\bar{j}k\bar{l}} = 0. \tag{7.3.0.17}$$

This implies, through Eq. (7.3.0.4):

$$C_{p(kl}C_{ij)n}g^{n\bar{n}}g^{p\bar{p}}\overline{C}_{\overline{npm}} = \frac{4}{3}g_{(l|\overline{m}}C_{|ijk)} \Leftrightarrow g^{n\bar{n}}R_{(i|\overline{m}|j|\overline{n}}C_{n|kl)} = -\frac{2}{3}g_{(i|\overline{m}}C_{|jkl)}, \tag{7.3.0.18}$$

whose specification in the manifestly G_5-covariant "special coordinates" symplectic basis gives the identity (7.2.1.45), which is consistently the $E^{v}_{ijkl} = 0$ limit of (7.3.0.11). By recalling definition (7.3.0.12), (7.2.1.45) (holding for symmetric d-SKG, and more in general in all cases in which $E^{v}_{ijkl} = 0$ globally) implies that d^{ijk} is a constant, scalar-independent tensor:

$$\frac{\partial d^{ijk}}{\partial z^l} = 0. \tag{7.3.0.19}$$

Furthermore, the $E^{m}_{ijkl} = 0$ limit of (7.3.0.16) yields

$$\mathcal{I}_4 = -\left(p^0\right)^2 q_0^2 - 2p^0 q_0 p^i q_i + \frac{1}{3}\left(2q_0 + q_i q_j \mathbf{d}^{ij}\right)d_{klm}p^k p^l p^m - \frac{2}{3}p^0 d^{ijk}q_i q_j q_k, \tag{7.3.0.20}$$

which is a manifestly G_5-invariant, alternative simple expression of $\mathcal{I}_4$, in $\mathcal{N} = 2$ symmetric d-SK geometries, as well as in all $d = 4$ $\mathcal{N} > 2$-extended supergravity theories whose scalar manifold is characterised by a symmetric cubic geometry[21]. In particular, for $G_4 = E_{7(-25)}$ ($\mathcal{N} = 2$, $d = 4$ $J_3^{\mathbb{O}}$-based "magic" supergravity) and $G_4 = E_{7(7)}$ ($\mathcal{N} = 8$, $d = 4$ $J_3^{\mathbb{O}_s}$-based maximal supergravity), (7.3.0.20) provides an equivalent expression of the Cartan-Cremmer-Julia [73; 74] unique quartic invariant of the fundamental irrepr. **56** of the exceptional

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[21]With the exception of $\mathcal{N} = 4$ *"pure"* and of $\mathcal{N} = 5$ supergravities, these also are all $\mathcal{N} > 2$-extended theories which can be uplifted to $d = 5$ dimensions (see *e.g.* [45] for quick reference Tables, and Refs. therein).

Lie group E_7. It is also worth remarking that for symmetric d-SKG (and more in general in all cases in which $E^v_{ijkl} = 0$ globally) the expressions (7.2.1.46)-(7.2.1.47) and (7.3.0.20) actually are scalar-independent and thus purely charge-dependent, and therefore $\mathcal{I}_4$ actually is the unique quartic invariant polynomial of the relevant symplectic (ir)repr. **R** of the $d = 4$ U-duality group G_4.

2. The alternative expression (7.3.0.16) for $\mathcal{I}_4$ is necessary to consistently match some known expressions of BH entropy with the formalism of d-SK geometries. Concerning this, the $p^0 = 0$ limit of (7.3.0.20) yields

$$\mathcal{I}_4 = \frac{1}{3}\left(2q_0 + q_i q_j \mathbf{d}^{ij}\right) d_{klm} p^k p^l p^m, \tag{7.3.0.21}$$

matching Eqs. (50)-(51) of [33]. Actually, since the treatment of [33] deals with generic (not necessarily symmetric, nor homogeneous) d-SK geometries, one should actually use the full formula (7.3.0.16). Consequently, the consistency of the results (50)-(51) of [33] with the general formula (7.3.0.16) yields the following constraint on the on-shell expression of the E-tensor (*at least* for $p^0 = 0$):

$$\left. E^m_{ijkl}\right|_{\partial V_{BH}=0} p^j p^k p^l q_m q_n \mathbf{d}^{in} = 0. \tag{7.3.0.22}$$

It is known that the configuration $\left(p^i, q_0, q_i\right)$ does not support axion-free attractor solutions [44], thus (7.3.0.22) should be considered in an axionful background. However, the E-tensor is insensitive to the presence of non-vanishing axions, because it only depends on v and $\widehat{\lambda}^i$'s, as given by Eq. (4.21) of [26].

3. The observations at point **1** are no longer generally true in *non-symmetric d-SK* geometries, and in all cases in which the E-tensor does not vanish globally[22]. In this case, *$\mathcal{I}_4$ is no more an invariant of the U-duality group G_4* (whose transitive action on the scalar manifold is spoiled in the non-homogeneous case; see *e.g.* [17]). Concerning this, it is worth recalling that G_4 always contains (and for totally generic d_{ijk}'s, coincides with) the semi-direct product of PQ axion-shifts (7.2.1.3) $\mathbb{R}^{n_V}$ and an overall rescaling $SO(1, 1)$, i.e. (see *e.g.* [17]):

$$SO(1, 1) \times_s \mathbb{R}^{n_V} \subset G_4. \tag{7.3.0.23}$$

Within this framework, some analysis of the dependence on the scalar degrees of freedom can be made. First of all, one can easily verify that in d-SK geometries all relevant geometrical quantities considered above are independent of the $d = 4$ axions x^i, i.e. the real parts of the $d = 4$ complex scalars coordinatising the special Kähler vector multiplets' scalar manifolds of $\mathcal{N} = 2, d = 4$

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[22]For some elaborations on this issue, see *e.g.* the recent treatment given in [26].

supergravity. This can ultimately be traced back to the $d = 5$ origin of all d-SK geometries, which are the only SK geometries which can be uplifted to 5 space-time dimensions. Then, (7.3.0.6)-(7.3.0.12) and (7.3.0.6) yield that

$$\frac{\partial \left(v^{5/3} E^v_{ijkl} \right)}{\partial v} \;=\; 0; \tag{7.3.0.24}$$

$$
\begin{aligned}
v^{5/3}\frac{\partial E^v_{ijkl}}{\partial \widehat{\lambda}^v} \;&=\; \frac{3}{2^5} d_{p(ij}d_{kl)q}d_{rst}\left[\frac{1}{2}\left(\delta^p_v \widehat{\lambda}^r + \delta^r_v \widehat{\lambda}^p \right) + \widehat{d}^{pm}\widehat{d}^{rn}d_{vmn} \right] a^{qs}a^{vt} \\[2mm]
&=\; \frac{3}{2^6}\left[
\begin{array}{l}
d_{v(ij}d_{kl)q}d_r^{vq}\widehat{\lambda}^r + d_{p(ij}d_{kl)q}d_{svt}\widehat{\lambda}^p a^{qs}a^{vt} \\[2mm]
+2d_{p(ij}d_{kl)q}d_r^{vq}\widehat{d}^{pm}\widehat{d}^{rn}d_{vmn}
\end{array}
\right]. \tag{7.3.0.25}
\end{aligned}
$$

The result (7.3.0.24) was derived in [26]. On the other hand, (7.3.0.25) expresses the way the E-tensor depends on $\widehat{\lambda}^i$'s, encoding the non-symmetric nature of the corresponding d-SK geometry.

Bibliography

[1] S. Bellucci, A. Marrani and R. Roychowdhury, *Topics in Cubic Special Geometry*, arXiv:1011.0705 [hep-th].

[2] A. Ceresole, R. D'Auria and S. Ferrara, *The Symplectic structure of $\mathcal{N} = 2$ supergravity and its central extension*, Nucl. Phys. Proc. Suppl. **46**, 67 (1996), hep-th/9509160.

[3] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fré and T. Magri, *$\mathcal{N} = 2$ supergravity and $\mathcal{N} = 2$ superYang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map*, J. Geom. Phys. **23**, 111 (1997), hep-th/9605032.

[4] B. Craps, F. Roose, W. Troost and A. Van Proeyen, *What is Special Geometry?*, Nucl. Phys. **B503**, 565 (1997), hep-th/9703082.

[5] D. S. Freed, *Special Kähler Manifolds*, Commun. Math. Phys. **203**, 31 (1999), hep-th/9712042.

[6] S. Ferrara, R. Kallosh and A. Strominger, *$\mathcal{N} = 2$ Extremal Black Holes*, Phys. Rev. **D52**, 5412 (1995), hep-th/9508072. A. Strominger, *Macroscopic Entropy of $\mathcal{N} = 2$ Extremal Black Holes*, Phys. Lett. **B383**, 39 (1996), hep-th/9602111. S. Ferrara and R. Kallosh, *Supersymmetry and Attractors*, Phys. Rev. **D54**, 1514 (1996), hep-th/9602136. S. Ferrara and R. Kallosh, *Universality of Supersymmetric Attractors*, Phys. Rev. **D54**, 1525 (1996), hep-th/9603090.

[7] S. Ferrara, G. W. Gibbons and R. Kallosh, *Black Holes and Critical Points in Moduli Space*, Nucl. Phys. **B500**, 75 (1997), hep-th/9702103.

[8] L. Andrianopoli, R. D'Auria, S. Ferrara and M. Trigiante, *Extremal Black Holes in Supergravity*, Lect. Notes Phys. **737**, 661 (2008), hep-th/0611345.

[9] A. Sen, Black Hole Entropy Function and the Attractor Mechanism in Higher Derivative Gravity, JHEP 0509, 038 (2005), hep-th/0506177.

[10] A. Sen, *Black Hole Entropy Function, Attractors and Precision Counting of Microstates*, Gen. Rel. Grav. **40**, 2249 (2008), arXiv:0708.1270 [hep-th].

[11] S. Bellucci, S. Ferrara, R. Kallosh and A. Marrani, *Extremal Black Hole and Flux Vacua Attractors*, Lect. Notes Phys. **755**, 115 (2008), `arXiv.0711.4547 [hep-th]`.

[12] S. Ferrara, K. Hayakawa and A. Marrani, *Lectures on Attractors and Black Holes*, Fortsch. Phys. **56**, 993 (2008), `arXiv:0805.2498 [hep-th]`.

[13] S. Bellucci, S. Ferrara, M. Günaydin and A. Marrani, *SAM Lectures on Extremal Black Holes in d = 4 Extended Supergravity*, `arXiv:0905.3739 [hep-th]`.

[14] H. Ooguri, A. Strominger and C. Vafa, *Black Hole Attractors and the Topological String*, Phys. Rev. **D70**, 106007 (2004), `hep-th/0405146`.

[15] F. Denef, *Supergravity flows and D-brane stability*, JHEP **0008**, 050 (2000), `hep-th/0005049`. B. Bates and F. Denef, *Exact solutions for supersymmetric stationary black hole composites*, `arXiv:hep-th/0304094`. F. Denef and G. W. Moore, *Split States, Entropy Enigmas, Holes and Halos*, `hep-th/0702146`. F. Denef and G. W. Moore, *How Many Black Holes fit the head of a pin?*, Int. J. Mod. Phys. **D17**, 679 (2008), Gen.Rel. Grav. **39**, 1539 (2007), `arXiv:0705.2564 [hep-th]`. E. Andriyash, F. Denef, D. L. Jafferis and G. W. Moore, *Wall-crossing from supersymmetric galaxies*, `arXiv:1008.0030 [hep-th]`. E. Andriyash, F. Denef, D. L. Jafferis and G. W. Moore, *Bound state transformation walls*, `arXiv:1008.3555 [hep-th]`. S. Ferrara and A. Marrani, *Matrix Norms, BPS Bounds and Marginal Stability in $\mathcal{N} = 8$ Supergravity*, `arXiv:1009.3251 [hep-th]`. S. Ferrara, A. Marrani and E. Orazi, *Split Attractor Flow in $\mathcal{N} = 2$ Minimally Coupled Supergravity*, `arXiv:1010.2280 [hep-th]`.

[16] B. de Wit and A. Van Proeyen, *Potentials and Symmetries of General Gauged $\mathcal{N} = 2$ Supergravity: Yang-Mills Models*, Nucl. Phys. **B245**, 89 (1984). B. de Wit and A. Van Proeyen, *Special geometry and symplectic transformations*, Nucl. Phys. Proc. Suppl. **45BC**, 196 (1996), `hep-th/9510186`.

[17] B. de Wit, F. Vanderseypen and A. Van Proeyen, *Symmetry Structures of Special Geometries*, Nucl. Phys. **B400**, 463 (1993), `hep-th/9210068`.

[18] J. F. Luciani: *Coupling of O(2) Supergravity with Several Vector Multiplets*, Nucl. Phys. **B132**, 325 (1978).

[19] C. M. Hull and P. K. Townsend, *Unity of Superstring Dualities*, Nucl. Phys. **B438**, 109 (1995), `hep-th/9410167`.

[20] S. Ferrara and M. Günaydin, *Orbits of Exceptional Groups, Duality and BPS States in String Theory*, Int. J. Mod. Phys. **A13**, 2075 (1998), `hep-th/9708025`.

[21] S. Bellucci, S. Ferrara, M. Günaydin and A. Marrani, *Charge orbits of sym-*

metric special geometries and attractors, Int. J. Mod. Phys. **A21**, 5043 (2006), `hep-th/0606209`.

[22] S. Ferrara and A. Marrani, *On the Moduli Space of Non-BPS Attractors for $\mathcal{N} = 2$ Symmetric Manifolds*, Phys. Lett. **B652**, 111 (2007), `arXiv:0706.1667 [hep-th]`.

[23] S. W. Hawking, *Gravitational Radiation from Colliding Black Holes*, Phys. Rev. Lett. **26**, 1344 (1971). J. D. Bekenstein, *Black Holes and Entropy*, Phys. Rev. **D7**, 2333 (1973).

[24] A. Ceresole and G. Dall'Agata, *Flow Equations for non-BPS Extremal Black Holes*, JHEP **0703**, 110 (2007), `hep-th/0702088`.

[25] L. Andrianopoli, R. D'Auria, E. Orazi and M. Trigiante, *First Order Description of Black Holes in Moduli Space*, JHEP **0711**, 032 (2007), `arXiv:0706.0712 [hep-th]`. S. Bellucci, S. Ferrara, A. Marrani and A. Yeranyan, *stu Black Holes Unveiled*, Entropy **10 (4)**, 507 (2008), `arXiv:0807.3503 [hep-th]`. A. Ceresole, G. Dall'Agata, S. Ferrara and A. Yeranyan, *First Order Flows for $\mathcal{N} = 2$ Extremal Black Holes and Duality Invariants*, Nucl. Phys. **B824**, 239 (2010), `arXiv:0908.1110 [hep-th]`. G. Bossard, Y. Michel and B. Pioline, *Extremal Black Holes, Nilpotent Orbits and the True Fake Superpotential*, JHEP **1001**, 038 (2010), `arXiv:0908.1742 [hep-th]`. A. Ceresole, G. Dall'Agata, S. Ferrara and A. Yeranyan, *Universality of the Superpotential for d=4 Extremal Black Holes*, `arXiv:0910.2697 [hep-th]`.

[26] S. Bellucci, A. Marrani and R. Roychowdhury, *On Quantum Special Kähler Geometry*, Int. J. Mod. Phys. **A25**, 1891 (2010), `arXiv:0910.4249 [hep-th]`.

[27] R. D. Peccei and H. R. Quinn, *Constraints imposed by CP conservation in the presence of instantons*, Phys. Rev. **D16**, 1791 (1977). R. D. Peccei and H. R. Quinn, *CP conservation in the presence of instantons*, Phys. Rev. Lett. **38**, 1440 (1977). R. D. Peccei and H. R. Quinn, *Some aspects of instantons*, Nuovo Cim. **A41**, 309 (1977).

[28] S. Cecotti, S. Ferrara and L. Girardello, *Geometry of Type II Superstrings and the Moduli of Superconformal Field Theories*, Int. J. Mod. Phys. **A4**, 2475 (1989).

[29] K. Behrndt, G. Lopes Cardoso, B. de Wit, R. Kallosh, D. Lüst and T. Mohaupt, *Classical and quantum $\mathcal{N} = 2$ supersymmetric black holes*, Nucl. Phys. **B488**, 236 (1997), `hep-th/9610105`.

[30] M. Shmakova, *Calabi-Yau black holes*, Phys. Rev. **D56**, 540 (1997), `hep-th/9612076`.

[31] P. Dominic and P. K. Tripathy, *Instanton Corrected Non-Supersymmetric Attractors*,

`arXiv:1010.3373 [hep-th]`.

[32] S. Bellucci, S. Ferrara, A. Marrani and A. Shcherbakov, *Splitting of Attractors in 1-Modulus Quantum Corrected Special Geometry*, JHEP **0802**, 088 (2008), `arXiv:0710.3559 [hep-th]`. S. Bellucci, S. Ferrara, A. Marrani and A. Shcherbakov, *Quantum Lift of Non-BPS Flat Directions*, Phys. Lett. **B672**, 77 (2009), `arXiv:0811.3494 [hep-th]`.

[33] P. K. Tripathy and S. P. Trivedi, *Non-Supersymmetric Attractors in String Theory*, JHEP **0603**, 022 (2006), `hep-th/0511117`.

[34] P. Gibbs, *Elliptic Curves and Hyperdeterminants in Quantum Gravity*, `arXiv:1010.4219 [math.GM]`.

[35] A. Cayley, *On the Theory of Linear Transformations*, Camb. Math. J. **4**, 193 (1845).

[36] K. Behrndt, R. Kallosh, J. Rahmfeld, M. Shmakova and W. K. Wong, *STU Black Holes and String Triality*, Phys. Rev. **D54**, 6293 (1996), `hep-th/9608059`. M. J. Duff, J. T. Liu and J. Rahmfeld, *Four-dimensional string/string/string triality*, Nucl. Phys. **B459**, 125 (1996), `hep-th/9508094`.

[37] M. J. Duff, *String triality, black hole entropy and Cayley's hyperdeterminant*, Phys. Rev. **D76**, 025017 (2007), `hep-th/0601134`. R. Kallosh and A. Linde, *Strings, black holes, and quantum information*, Phys. Rev. **D73**, 104033 (2006). P. Levay, *Stringy black holes and the geometry of entanglement*, Phys. Rev. **D74**, 024030 (2006). M. J. Duff and S. Ferrara, *E7 and the tripartite entanglement of seven qubits*, Phys. Rev. **D76**, 025018 (2007), `arXiv:quant-ph/0609227`. P. Levay, *Strings, black holes, the tripartite entanglement of seven qubits and the Fano plane*, Phys. Rev. **D75**, 024024 (2007), `arXiv:hep-th/0610314`. M. J. Duff and S. Ferrara, *Black hole entropy and quantum information*, Lect. Notes Phys. **755**, 93 (2008), `arXiv:hepth/0612036`. P. Levay, *A Three-qubit interpretation of BPS and non-BPS STU black holes*, Phys. Rev. **D76**, 106011 (2007), `arXiv:0708.2799`. L. Borsten, D. Dahanayake, M. J. Duff, W. Rubens and H. Ebrahim, *Wrapped branes as qubits*, Phys. Rev. Lett. **100**, 251602 (2008), `arXiv:0802.0840`. L. Borsten, D. Dahanayake, M. J. Duff, H. Ebrahim and W. Rubens, *Black Holes, Qubits and Octonions*, Phys. Rept. **471**, 113 (2009), `arXiv:0809.4685 [hep-th]`. B. L. Cerchiai and B. Van Geemen, *From Qubits to E_7*, `arXiv:1003.4255 [quant-ph]`. P. Levay and S. Szalay, *The Attractor Mechanism as a Distillation Procedure*, Phys. Rev. **D82**, 026002 (2010), `arXiv:1004.2346 [hep-th]`. P. Levay, *STU Black Holes as Four Qubit Systems*, Phys. Rev. **D82**, 026003 (2010), `arXiv: 1004.3639 [hep-th]`. L. Borsten, D. Dahanayake, M. J. Duff, A. Marrani and W. Rubens, *Four-Qubit Entanglement from String Theory*, Phys. Rev.

Lett. **105**, 100507 (2010), `arXiv:1005.4915 [hep-th]`.

[38] L. Andrianopoli, R. D'Auria, S. Ferrara and M. Lledò, *Gauging of Flat Groups in Four-Dimensional Supergravity*, JHEP **0207**, 010 (2002), `hep-th/0203206`.

[39] A. Strominger, *Special Geometry*, Commun. Math. Phys. **133**, 163 (1990).

[40] B. de Wit and A. Van Proeyen, *Special Geometry, Cubic Polynomials and Homogeneous Quaternionic Spaces*, Commun. Math. Phys. **149**, 307 (1992), `hep-th/9112027`.

[41] I. N. Herstein, *"Topics in Algebra"*, John Wiley and Sons, 1975.

[42] A. Ceresole, R. D'Auria, S. Ferrara and A. Van Proeyen, *Duality Transformations in Supersymmetric Yang-Mills Theories Coupled to Supergravity*, Nucl. Phys. **B444**, 92 (1995), `hep-th/9502072`.

[43] P. Fré, *Lectures on Special Kähler Geometry and Electric-Magnetic Duality Rotations*, Nucl. Phys. Proc. Suppl. **45BC**, 59 (1996), `hep-th/9512043`.

[44] A. Ceresole, S. Ferrara and A. Marrani, *4d/5d Correspondence for the Black Hole Potential and its Critical Points*, Class. Quant. Grav. **4**, 5651 (2007), `arXiv:0707.0964 [hep-th]`.

[45] S. Ferrara and A. Marrani, *Symmetric Spaces in Supergravity*, in : *"Symmetry in Mathematics and Physics"* (D. Babbitt, D. Vyjayanthi and R. Fioresi Eds.), Contemporary Mathematics **490**, American Mathematical Society, Providence 2009, `arXiv:0808.3567 [hep-th]`.

[46] M. Günaydin, G. Sierra and P. K. Townsend, *The Geometry of $\mathcal{N} = 2$ Maxwell-Einstein Supergravity and Jordan Algebras*, Nucl. Phys. **B242**, 244 (1984).

[47] E. Cremmer and A. Van Proeyen, *Classification of Kähler Manifolds in $\mathcal{N} = 2$ Vector Multiplet Supergravity Couplings*, Class. Quant. Grav. **2**, 445 (1985).

[48] S. Hosono, A. Klemm, S. Theisen and Shing-Tung Yau, *Mirror symmetry, mirror map and applications to Calabi-Yau hypersurfaces*, Commun. Math. Phys. **167**, 301 (1995), `hep-th/9308122`. S. Hosono, A. Klemm, S. Theisen and Shing-Tung Yau, *Mirror symmetry, mirror map and applications to complete intersection Calabi-Yau spaces*, Nucl. Phys. **B433**, 501 (1995), `hep-th/9406055`.

[49] T. Mohaupt, *Black hole entropy, special geometry and strings*, Fortsch. Phys. **49**, 3 (2001), `hep-th/0007195`. T. Mohaupt, *Strings, higher curvature corrections, and black holes*, `hep-th/0512048`.

[50] J. M. Maldacena, A. Strominger and E. Witten, *Black hole entropy in M theory*, JHEP **9712**, 002 (1997), `hep-th/9711053`.

[51] C. Vafa, *Black Holes and Calabi-Yau Threefolds*, Adv. Theor. Math. Phys. 2, 207 (1998), `hep-th/9711067`.

[52] L. Alvarez-Gaume and D. Z. Freedman, *Geometrical Structure and Ultraviolet Finiteness in the Supersymmetric Sigma Model*, Commun. Math. Phys. **80**, 443 (1981). L. Alvarez-Gaumé, D. Z. Freedman and S. Mukhi, *The Background Field Method and the Ultraviolet Structure of the Supersymmetric Nonlinear Sigma Model*, Annals Phys. **134**, 85 (1981).

[53] M. T. Grisaru, A. van de Ven and D. Zanon, *Four Loop Beta Function for the $\mathcal{N}=1$ and $\mathcal{N}=2$ Supersymmetric Nonlinear Sigma Model in Two Dimensions*, Phys. Lett. **B173**, 423 (1986). M. T. Grisaru, A. van de Ven and D. Zanon, *Two Dimensional Supersymmetric Sigma Models on Ricci Flat Kähler Manifolds are not Finite*, Nucl. Phys. **B277**, 388 (1986). M. T. Grisaru, A. van de Ven and D. Zanon, *Four Loop Divergences for the $\mathcal{N}=1$ Supersymmetric Nonlinear Sigma Model in Two Dimensions*, Nucl. Phys. **B277**, 409 (1986).

[54] P. Candelas, X. C. De La Ossa, P. S. Green and L. Parkes, *A Pair of Calabi-Yau Manifolds as an Exactly Soluble Superconformal Theory*, Nucl. Phys. **B359**, 21 (1991). P. Candelas, X. C. De La Ossa, P. S. Green and L. Parkes, *An Exactly Soluble Superconformal Theory from a Mirror Pair of Calabi-Yau Manifolds*, Phys. Lett. **B258**, 118 (1991).

[55] A. Klemm and S. Theisen, *Considerations of one modulus Calabi-Yau compactifications : Picard-Fuchs equations, Kähler potentials and mirror maps*, Nucl. Phys. **B389**, 153 (1993), `hep-th/9205041`.

[56] A. Font, *Periods and Duality Symmetries in Calabi-Yau Compactifications*, Nucl. Phys. **B391**, 358 (1993), `hep-th/9203084`.

[57] E. Witten, *Dyons of Charge e theta/2 pi*, Phys. Lett. **B86**, 283 (1979).

[58] K. Behrndt and T. Mohaupt, *Entropy of $\mathcal{N}=2$ black holes and their M-brane description*, Phys. Rev. **D56**, 2206 (1997), `hep-th/9611140`. K. Behrndt, I. Gaida, D. Lüst, S. Mahapatra and T. Mohaupt, *From type IIA black holes to T dual type IIBD instantons in $\mathcal{N}=2$, $d=4$ supergravity*, Nucl. Phys. **B508**, 659 (1997), `hep-th/9706096`. K. Behrndt, G. Lopes Cardoso, B. de Wit, D. Lüst, T. Mohaupt and W. A. Sabra, *Higher order black hole solutions in $\mathcal{N}=2$ supergravity and Calabi-Yau string backgrounds*, Phys. Lett. **B429**, 289 (1998), `hep-th/9801081`.

[59] G. Lopes Cardoso, G. Curio, D. Lüst, T. Mohaupt and S.-J. Rey, *BPS spectra and nonperturbative gravitational couplings in $\mathcal{N}=2$, $\mathcal{N}=4$ supersymmetric string theories*, Nucl. Phys. **B464**, 18 (1996), `hep-th/9512129`. G. Lopes Cardoso, G. Curio, D. Lüst and T. Mohaupt, *Instanton numbers and exchange symmetries in $\mathcal{N}=2$*

dual string pairs, Phys. Lett. **B382**, 241 (1996), `hep-th/9603108`. G. Lopes Cardoso, G. Curio and D. Lüst, *Perturbative couplings and modular forms in $\mathcal{N}=2$ string models with a Wilson line*, Nucl. Phys. **B491**, 147 (1997), `hep-th/9608154`.

[60] S. Bellucci, S. Ferrara, A. Marrani and A. Shcherbakov, *Quantum Lift of Non-BPS Flat Directions*, Phys. Lett. **B672**, 77 (2009), `arXiv:0811.3494 [hep-th]`.

[61] E. Bergshoeff, W. Chemissany, A. Ploegh, M. Trigiante and T. Van Riet, *Generating Geodesic Flows and Supergravity Solutions*, Nucl. Phys. **B812**, 343 (2009), `arXiv:0806.2310 [hep-th]`. W. Chemissany, J. Rosseel, M. Trigiante and T. Van Riet, *The full integration of black hole solutions to symmetric supergravity theories*, `arXiv:0903.2777 [hep-th]`.

[62] R. Kallosh, *From BPS to non-BPS Black Holes Canonically*, `hep-th/0603003`.

[63] B. L. Cerchiai, S. Ferrara, A. Marrani and B. Zumino, *Duality, Entropy and ADM Mass in Supergravity*, Phys. Rev. **D79**, 125010 (2009), `arXiv:0902.3973 [hep-th]`.

[64] L. Andrianopoli, R. D'Auria, S. Ferrara and M. Trigiante, *Fake Superpotential for Large and Small Extremal Black Holes*, JHEP **1008**, 126 (2010), `arXiv:1002.4340 [hep-th]`.

[65] L. Borsten, D. Dahanayake, M. J. Duff, S. Ferrara, A. Marrani and W. Rubens, Class. Quant. Grav. **27**, 185003 (2010), `arXiv:1002.4223 [hep-th]`.

[66] A. Ceresole, S. Ferrara and A. Marrani, *Small $\mathcal{N}=2$ Extremal Black Holes in Special Geometry*, Phys. Lett. **B693**, 366 (2010), `arXiv:1006.2007 [hep-th]`.

[67] L. Borsten, D. Dahanayake, M. J. Duff, H. Ebrahim and W. Rubens, *Freudenthal Triple Classification of Three-Qubit Entanglement*, Phys. Rev. **A80**, 032326 (2009), `arXiv:0812.3322 [quant-ph]`.

[68] M. Huang, A. Klemm, M. Marino and A. Tavanfar, *Black Holes and Large Order Quantum Geometry*, Phys. Rev. **D79**, 066001 (2009), `arXiv:0704.2440`.

[69] B. Sahoo and A. Sen, *Higher Derivative Corrections to Non-Supersymmetric Extremal Black Holes in $\mathcal{N}=2$ Supergravity*, JHEP **0609**, 029 (2006), `hep-th/0603149`.

[70] L. Castellani, R. D'Auria and S. Ferrara, *Special Geometry without Special Coordinates*, Class. Quant. Grav. **7**, 1767 (1990). L. Castellani, R. D'Auria and S. Ferrara, *Special Kähler Geometry: an Intrinsic Formulation from $\mathcal{N}=2$ Space-Time Supersymmetry*, Phys. Lett. **B241**, 57 (1990). R. D'Auria, S. Ferrara and P. Fré, *Special and Quaternionic Isometries: General Couplings in $\mathcal{N}=2$ Supergravity and the Scalar Potential*, Nucl. Phys. **B359**, 705 (1991).

[71] S. Ferrara and M. Günaydin, *Orbits and Attractors for* $\mathcal{N} = 2$ *Maxwell-Einstein Supergravity Theories in Five Dimensions*, Nucl. Phys. **B759**, 1 (2006), `hep-th/0606108`.

[72] R. D'Auria, S. Ferrara and M. Trigiante, *Critical Points of the Black-Hole Potential for Homogeneous Special Geometries*, JHEP **0703**, 097 (2007), `hep-th/0701090`.

[73] E. Cartan, *"Œuvres Complètes"* (Editions du Centre National de la Recherche Scientifique, Paris, 1984).

[74] E. Cremmer and B. Julia, *The SO(8) Supergravity*, Nucl. Phys. **B159**, 141 (1979).

Chapter 8

The complete analytic structure of the massive gravitino propagator in four-dimensional de Sitter space

The first paper [1] of this sequel studies spinor two-point functions for spin-1/2 and spin-3/2 fields in maximally symmetric spaces such as de Sitter(dS) spacetime, by using intrinsic geometric objects. The Feynman, positive- and negative-frequency Green functions are then obtained for these cases, from which we eventually display the supercommutator and the Peierls bracket under such a setting in two-component-spinor language. In a follow-up paper [2], we complete, the explicit representation of the massive gravitino propagator in four-dimensional de Sitter space with the help of the general theory of the Heun equation. As a result of our original contribution, all weight functions which multiply the geometric invariants in the gravitino propagator are expressed through Heun functions, and the resulting plots are displayed and discussed after resorting to a suitable truncation in the series expansion of the Heun function. It turns out that there exist two ranges of values of the independent variable in which the weight functions can be divided into dominant and sub-dominant family.

8.1   Introduction

The formulation of a theory of quantum gravity requires one to thoroughly understand the particle propagation in curved spacetimes. Maximally symmetric spaces such as de Sitter and anti-de Sitter provide one with an interesting backdrop to study quantum field theory in curved spacetimes. In this background geometry,

if one needs to calculate basic quantities like scattering amplitudes, one should find out the correlation function which involves the propagators for various particles in this background. Thus, the problem of calculating the propagators has always been of much physical interest to several authors. This hunt also assumed much significance after the advent of the famous Maldacena conjecture or the AdS/CFT correspondence (for e.g. see [3],[4],[5]) which proposes a duality between a quantum gravity theory on the bulk AdS_{d+1} and a strongly coupled conformal d-dimensional gauge theory at large N on the boundary of it. Then there is the recently proposed dS/CFT correspondence [6] which might shed light on quantum gravity in de Sitter space. This conjecture, which is largely modeled on analogy with AdS/CFT [7], still lacks a clear relation to string theory which in turn hinders the explicit realization of the proposal made by Strominger. At the same time, a consistent formulation of all interactions in de Sitter space is also tempting because of the recent observational data in favor of the inflationary picture.

In field theory the Peierls bracket is a Poisson bracket which is invariant under the full infinite-dimensional invariance group of the action functional. Without invoking a definition of canonical coordinates and canonical momenta in advance, the Peierls bracket follows directly from the classical action, and is made out of the advanced and retarded Green's functions. Hence it is necessary to build the spinor parallel propagator and the spinor Green function in order to write the Peierls bracket in de Sitter and anti-de Sitter spaces for spin-1/2 and spin-3/2 particles. Here we focus our calculation on de Sitter space.

This chapter consists of two parts, the first part involves the case of ordinary spin-1/2 particles, and the second part extends the same physics to spin-3/2 fields, i.e. the gravitino. Using our pedagogical papers ([1], [2]), we first introduce the idea of the Peierls bracket in Sec.8.2, while Sec.8.3 contains an introduction to maximally symmetric bitensors. In Sec.8.4 we review a few elementary properties of the spinor parallel propagator, in Sec.8.5 we calculate the massive spinor Green functions and hence the Feynman, positive- and negative-frequency two-point functions. Then we show how to build a Peierls bracket from this for the spin-1/2 case. In Sec.8.6 we summarize all techniques developed so far in this paper and apply them to evaluate the gravitino Green functions in four-dimensional de Sitter space in two-component spinor language. Sec.8.7 is devoted to a rapid introduction to Heun differential equation while Sec.8.8 builds a Dictionary of weight functions for the gravitino propagator. In Sec.8.9 we show the qualitative behaviors of the weight functions with numerous plots and finally in 8.10 conclude this chapter with some comments.

8.2 The Peierls bracket

Since the Peierls bracket is not quite a familiar concept, a brief review about it is presented here. For more details on the subject we refer the reader to [8–10] and the references therein.

It was in the early fifties when R.E. Peierls [9] first noticed a similarity in algebraic structure between the Poisson bracket and the Peierls bracket as is called today (for theories without gauge freedom the Peierls bracket is indeed a Poisson bracket, whereas for gauge theories it becomes a Poisson under restriction to the space of observables [8; 10]), and found that this new structure could be defined directly from the action principle without performing a canonical decomposition into coordinates and momenta. His essential insight was to consider the advanced and retarded "effect of one quantity (A) on another (B)." Here, A and B are functions on the space of histories $\mathcal{H}$. The space-of-histories formulation using the DeWitt condensed-index notation often proves indeed very useful in the study of the generalized Peierls algebra and provides the opportunity to introduce in a concise way the relevant techniques. We first define the advanced and retarded effects of A and B on each other as functions on $\mathcal{H}$, from which the Peierls bracket follows. This will be straightforward by using the machinery of [10] and indeed, much of what follows is implicit in that treatment.

Once the action functional S is replaced by a new action functional $S + \epsilon A$ after the interaction with some external agent, the small disturbances $\delta\phi^i$ are ruled by an inhomogeneous differential equation (see below) which is solved after inverting a differential operator F_{ij}. On denoting by $G^{\pm jk}$ the advanced (resp. retarded) Green functions of F_{ij}, one can define

$$\delta_A^{\pm} B \equiv \epsilon B_{,i} G^{\pm ij} A_{,j}, \ D_A B \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon} \delta_A^- B, \tag{8.2.0.1}$$

and the Peierls bracket

$$(A, B) \equiv D_A B - D_B A. \tag{8.2.0.2}$$

To be more precise, following [10], recall that the undisturbed fields satisfy the equations of motion

$$0 = S_{,i} \left(\phi^j \right) \tag{8.2.0.3}$$

while the disturbed fields satisfy

$$0 = S_{\epsilon,i} \left(\phi_\epsilon^j \right) = S_{,i} \left(\phi_\epsilon^j \right) + \epsilon A_{,i} \left(\phi_\epsilon^j \right). \tag{8.2.0.4}$$

To first order, the perturbations $\delta\phi^i$ are therefore governed by the equation

$$S_{,ij} \left(\phi^k \right) \delta\phi^j = -\epsilon \, A_{,i}(\phi^k), \tag{8.2.0.5}$$

and we see that both the boundary conditions (advanced or retarded) and any gauge fixing applies only to the inversion of the operator $S_{,ij}(\phi^k)$ in the above linear equation for $\delta\phi^j$ and not to the solution of (8.2.0.3) for ϕ^i. In the case where there are no gauge symmetries, $S_{,ij}$ is invertible and has advanced and retarded Green's functions $G^{\pm jk}$ that satisfy [10]

$$S_{,ij}\, G^{\pm jk} = -\delta_i^k, \tag{8.2.0.6}$$

so that the advanced and retarded solutions to the above equations are $\delta^{\pm}\phi^j = \epsilon G^{\pm ji} A_{,i}$ where both $G^{\pm ji}$ and $A_{,i}$ depend on the unperturbed solution ϕ^i. From the definitions 8.2.0.1 and 8.2.0.2, the Peierls bracket is just

$$(A, B) = A_{,i}\widetilde{G}^{ij} B_{,j}, \tag{8.2.0.7}$$

where

$$\widetilde{G}^{ij} \equiv G^{+ij} - G^{-ij} \tag{8.2.0.8}$$

is called the *supercommutator function*, i.e. the difference of advanced and retarded Green functions.

For gauge fields, however, there exists on Φ a set of vector fields Q_α that leave the action S invariant, i.e.

$$Q_\alpha S = 0. \tag{8.2.0.9}$$

If A and B are two such gauge-invariant functionals:

$$Q_\alpha A = Q_\alpha B = 0, \tag{8.2.0.10}$$

then their Peierls bracket (A, B) is defined as follows [11; 12]:

$$(A, B) \equiv A_{,i}\widetilde{G}^{ij} B_{,j} = \int\int dx\, dy\, \frac{\delta A}{\delta\varphi^i(x)}\widetilde{G}^{ij}(x, y)\frac{\delta B}{\delta\varphi^j(y)}, \tag{8.2.0.11}$$

where the advanced and retarded Green functions used to define the supercommutator $\widetilde{G}^{ij}$ now pertain to the invertible gauge-field operator obtained from the gauge-fixing procedure. Since A and B are observables, Jacobi identity and gauge invariance hold for the Peierls bracket (for a detailed proof of these properties see, for example, [8; 10]).

8.3 Maximally symmetric bitensors

More than two decades ago Allen and co-authors used intrinsic geometric objects to calculate correlation functions in maximally symmetric spaces; their results, here exploited, were presented in a series of papers [13; 14]. In this section we would like to

review the elementary maximally symmetric bi-tensors which have been discussed previously by Allen and Jacobson [13], although more recently the calculation of spinor parallel propagator has been carried out in arbitrary dimension [15].

A maximally symmetric space is a topological manifold of dimension n, with a metric which has the maximum number of global Killing vector fields. This type of space looks exactly the same in every direction and at every point. The simplest examples are flat space and sphere, each of which has $\frac{1}{2}n(n+1)$ independent Killing fields. For S^n these generate all rotations, and for $\mathbb{R}^n$ they include both rotations and translations.

Consider a maximally symmetric space of dimension n with constant scalar curvature $n(n-1)/R^2$. For the space S^n, the radius R is real and positive, whereas for the hyperbolic space H^n, $R = il$ with l positive, and in the flat case, $\mathbb{R}^n$, $R = \infty$. Consider further two points x and x', which can be connected uniquely by a shortest geodesic. Let $\mu(x, x')$ be the proper geodesic distance along this shortest geodesic between x and x'. If $n^a(x, x')$ and $n^{a'}(x, x')$ are the tangents to the geodesic at x and x', the tangent vectors are then given in terms of the geodesic distance as follows:

$$n_a(x, x') = \nabla_a \mu(x, x') \quad \text{and} \quad n_{a'}(x, x') = \nabla_{a'} \mu(x, x'). \tag{8.3.0.1}$$

Furthermore, on denoting by $g^a_{\ b'}(x, x')$ the vector parallel propagator along the geodesic, one can then write $n^{b'} = -g^{b'}_{\ a} n^a$. Tensors that depend on two points, x and x', are bitensors [16]. They may carry unprimed or primed indices that live on the tangent space at x or x'.

These geometric objects n^a, $n^{a'}$ and $g^a_{\ b'}$ satisfy the following properties [13]:

$$\nabla_a n_b = A(g_{ab} - n_a n_b), \tag{8.3.0.2a}$$

$$\nabla_a n_{b'} = C(g_{ab'} + n_a n_{b'}), \tag{8.3.0.2b}$$

$$\nabla_a g_{bc'} = -(A + C)(g_{ab} n_{c'} + g_{ac'} n_b), \tag{8.3.0.2c}$$

where A and C are functions of the geodesic distance μ and are given by [13]

$$A = \frac{1}{R} \cot \frac{\mu}{R} \quad \text{and} \quad C = -\frac{1}{R \sin(\mu/R)}, \tag{8.3.0.3}$$

and thus they satisfy the relations

$$dA/d\mu = -C^2, \quad dC/d\mu = -AC \quad \text{and} \quad C^2 - A^2 = 1/R^2. \tag{8.3.0.4}$$

Last, our convention for covariant gamma matrices is

$$\{\Gamma^\mu, \Gamma^\nu\} = 2I g^{\mu\nu}. \tag{8.3.0.5}$$

8.4   The spinor parallel propagator

Here we will follow the conventions for two-component spinors, as well as all signature and curvature conventions, of Allen and Lutken [14], and hence we use dotted and undotted spinors instead of the primed and unprimed ones of Penrose and Rindler [17]. In our work a primed index indicates instead that it lives in the tangent space at x', while the unprimed ones live at x. The fundamental object to deal with here is the bispinor $D_A^{A'}(x, x')$ which parallel transports a two-component spinor ϕ^A at the point x, along the geodesic to the point x', yielding a new spinor $\chi^{A'}$ at x', i.e.

$$\chi^{A'} = \phi^A \, D_A^{A'}(x, x'). \tag{8.4.0.1}$$

Complex conjugate spinors are similarly transported by the complex conjugate of $D_A^{A'}(x, x')$, which is $\overline{D}_{\dot{A}}^{\dot{A}'}(x, x')$. A few elementary properties of $D_A^{A'}$ are listed below (some of them will be used for later calculations) [14]:

$$D_A^{A'}(x, x') = -D_A^{A'}(x', x), \tag{8.4.0.2a}$$

$$D_A^{A'} D_{A'}^B = \varepsilon_A^B, \tag{8.4.0.2b}$$

$$D_{AA'} D^{AA'} = 2, \tag{8.4.0.2c}$$

$$\lim_{x \to x'} D_A^{B'} = \varepsilon_A^B, \tag{8.4.0.2d}$$

$$g_a^{\ b'} = D_A^{B'} \overline{D}_{\dot{A}}^{\dot{B}'}, \tag{8.4.0.2e}$$

$$D_A^{B'} \overline{D}_{\dot{A}}^{\dot{B}'} n^{A\dot{A}} = -n^{B'\dot{B}'}, \tag{8.4.0.2f}$$

$$n_{A\dot{C}} D_{B'}^A = -n_{B'\dot{B}'} \overline{D}_{\dot{C}}^{\dot{B}'}, \tag{8.4.0.2g}$$

$$\nabla_{A\dot{A}} D_{A'}^A = \frac{3}{2}(A + C) n_{A\dot{A}} D_{A'}^A, \tag{8.4.0.2h}$$

$$D_{A'}^A \nabla_{A\dot{A}} n^{A'\dot{A}'} = -\frac{3}{2} C \overline{D}_{\dot{A}}^{\dot{A}'}, \tag{8.4.0.2i}$$

$$\nabla_{A\dot{A}} n_{\dot{B}}^A = \frac{3}{2} A \varepsilon_{\dot{A}\dot{B}}. \tag{8.4.0.2j}$$

Just to recall the previously defined notations and set up the two-component formalism, we note from 8.3.0.1 that $n_{A\dot{A}} = \nabla_{A\dot{A}} \mu$ and $n_{A'\dot{A}'} = \nabla_{A'\dot{A}'} \mu$, where $\mu(x, x')$ is the geodesic separation of x and x'. For completeness we should also find the covariant derivative of $D_A^{A'}$, which is formed out of the tangent $n_{A\dot{A}}$ to the geodesic and from $D_A^{A'}$ itself, i.e.

$$\nabla_{A\dot{A}} D_B^{B'} \equiv \alpha(\mu) n_{A\dot{A}} D_B^{B'} + \beta(\mu) n_{B\dot{A}} D_A^{B'}. \tag{8.4.0.3}$$

Here α and β are two arbitrary functions of the geodesic distance to be determined. But both of them are not independent and are related to each other because of the

fact that $D_B^{B'}$ and $n_{A\dot{A}}$, by definition, satisfy the following relations [14]:

$$n^a \nabla_a D_B^{B'} = 0, \tag{8.4.0.4}$$

$$n_{A\dot{A}} n^{B\dot{A}} = \frac{1}{2} \delta_A^B. \tag{8.4.0.5}$$

From the relations (8.4.0.4) and (8.4.0.5) it follows that $\beta(\mu) = -2\alpha(\mu)$. One then determines $\beta(\mu)$ by using the Ricci identity, i.e. the integrability condition for spinors [17], and after all dust gets settled one obtains the final form of the covariant derivative of the spinor parallel propagator as

$$\nabla_{A\dot{A}} D_B^{B'} = (A + C) \left[\frac{1}{2} n_{A\dot{A}} D_B^{B'} - n_{B\dot{A}} D_A^{B'} \right], \tag{8.4.0.6}$$

where A and C are defined as in the previous section.

8.5 The spinor Green function

First we define a four-component Dirac spinor by

$$\psi_\alpha = \begin{pmatrix} \phi_A \\ \overline{\chi}_{\dot{A}} \end{pmatrix}, \tag{8.5.0.1}$$

where ϕ_A and $\overline{\chi}_{\dot{A}}$ are a pair of two-component spinors satisfying the Dirac equation [17]

$$\nabla_{A\dot{A}} \phi^A = \frac{-m}{\sqrt{2}} \overline{\chi}_{\dot{A}}, \tag{8.5.0.2}$$

$$\nabla_A^{\dot{A}} \overline{\chi}_{\dot{A}} = \frac{m}{\sqrt{2}} \phi_A, \tag{8.5.0.3}$$

m being the mass of the spin-1/2 field. We can define two basic massive two-point functions, which are

$$P^{A\dot{B}'} = \langle \phi^A(x) \overline{\phi}^{\dot{B}'}(x') \rangle = f(\mu) D_{A'}^A n^{A'\dot{B}'}, \tag{8.5.0.4}$$

$$Q_{\dot{A}}^{\dot{B}'} = \langle \overline{\chi}_{\dot{A}}(x) \overline{\phi}^{\dot{B}'}(x') \rangle = g(\mu) \overline{D}_{\dot{A}}^{\dot{B}'}. \tag{8.5.0.5}$$

Here we temporarily assume the spacelike separation between the points x and x' such that the field operators in (8.5.0.4) and (8.5.0.5) anti-commute. On the right-hand side of (8.5.0.4) and (8.5.0.5) we have the most general maximally symmetric bispinor with the correct index structure. It is to be noted that the functions f and g appearing here in the structure, do depend only on the geodesic distance μ, and

other two-point functions like $\langle \overline{\chi}_A \chi_{B'} \rangle$ and $\langle \phi^A \chi_{B'} \rangle$ are entirely determined by f and g only. The equations of motion (8.5.0.2) and (8.5.0.3) now imply that

$$\nabla_{A\dot{A}} P^{A\dot{B}'} = \frac{-m}{\sqrt{2}} Q_{\dot{A}}^{\dot{B}'}, \tag{8.5.0.6}$$

$$\nabla_A^{\dot{A}} Q_{\dot{A}}^{\dot{B}'} = \frac{m}{\sqrt{2}} P_A^{\dot{B}'}. \tag{8.5.0.7}$$

If now we insert equations (8.5.0.4) and (8.5.0.5) into equations (8.5.0.6) and (8.5.0.7) we obtain, after a little gymnastics with two-spinor calculus, two coupled equations for the coefficients $f(\mu)$ and $g(\mu)$ as follows:

$$f' + \frac{3}{2}(A - C)f + \sqrt{2}mg = 0, \tag{8.5.0.8}$$

$$g' + \frac{3}{2}(A + C)g - \frac{m}{\sqrt{2}}f = 0, \tag{8.5.0.9}$$

where the prime stands for derivative with respect to μ. On differentiating (8.5.0.8) with respect to μ once and then using (8.3.0.4) and (8.5.0.9) successively one finds a second-order equation for f:

$$f''(\mu) + 3Af'(\mu) + \left[m^2 - \frac{9}{4}R^{-2} + \frac{3}{2}C(A - C) \right] f(\mu) = 0. \tag{8.5.0.10}$$

Now to solve for $f(\mu)$ and $g(\mu)$, one makes a change of variable

$$Z \equiv \cos^2 \left(\frac{\mu}{2R} \right) \tag{8.5.0.11}$$

to write (8.5.0.10) as [14]

$$Z(1 - Z)\frac{d^2}{dZ^2}f(Z) + 2(1 - 2Z)\frac{d}{dZ}f(Z) + \left[m^2 R^2 - \frac{9}{4} - \frac{3}{4(1 - Z)} \right] f(Z) = 0. \tag{8.5.0.12}$$

On making further a redefinition

$$w(Z) \equiv [R^2(1 - Z)]^{-1/2} f(Z), \tag{8.5.0.13}$$

one rewrites (8.5.0.12) as a hypergeometric equation in the variable w, i.e.

$$H(a, b, c; Z)w(Z) = 0, \tag{8.5.0.14}$$

where $H(a, b, c)$ is the hypergeometric operator

$$H(a, b, c; Z) = Z(1 - Z)\frac{d^2}{dZ^2} + [c - (a + b + 1)Z]\frac{d}{dZ} - ab. \tag{8.5.0.15}$$

Following our source, the factor R^2 is included in the definition (8.5.0.13) of w to ensure that the standard branch cut of the square root function lies along the timelike separations $\mu^2 > 0$. The parameters a, b, c here take the values

$$a = 2 + \sqrt{m^2 R^2}, \qquad (8.5.0.16a)$$

$$b = 2 - \sqrt{m^2 R^2}, \qquad (8.5.0.16b)$$

$$c = 2. \qquad (8.5.0.16c)$$

In the same way it can be shown that if we let $w(Z) = [R^2(Z)]^{-1/2} g(Z)$, then w satisfies a hypergeometric equation with parameters a, b and $c + 1$. Now one has to specify the boundary conditions to uniquely specify a solution to the hypergeometric equation. The correct solution to (8.5.0.14) in de Sitter space $R^2 < 0$ is obtained (following [13]) by demanding that it is only singular when $\mu = 0$, that is $Z = 1$, and not when $\mu = \pi R$, that is $Z = 0$. Two independent solutions of the hypergeometric equations [18; 19] are therefore $F(a, b; c; Z)$ and $F(a, b; c + 1; Z)$, and this yields the following solutions:

$$f_{DS} = N_{DS}(1 - Z)^{1/2} F(a, b; c; Z), \qquad (8.5.0.17)$$

$$g_{DS} = -i N_{DS} 2^{-3/2} m |R| Z^{1/2} F(a, b; c + 1; Z). \qquad (8.5.0.18)$$

The short distance behavior $\mu \to 0$ can now be used to fix the as yet undetermined constant N_{DS}. The flat-space limit as $\mu \to 0$ is

$$f \sim \frac{-i}{\sqrt{2}} \frac{1}{\pi^2} (-\mu^2)^{-3/2}. \qquad (8.5.0.19)$$

Thus, from (8.5.0.17) it follows that

$$N_{DS} = \frac{f_{DS}}{(1 - Z)^{1/2} F(a, b; c; Z)}. \qquad (8.5.0.20)$$

Furthermore, near $Z = 1$ we have

$$F(a, b; c; Z) \sim \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} (1 - Z)^{c-a-b}, \qquad (8.5.0.21)$$

and $(1 - Z) = (\mu/2R)^2$, hence one finds that

$$N_{DS} = \frac{-i}{\sqrt{2}} \frac{(-\mu^2)^{-3/2}}{\pi^2} \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(a + b - c)} (1 - Z)^{a+b-c-\frac{1}{2}} = \frac{-i}{\sqrt{2}} \frac{(-\mu^2)^{-3/2}}{\pi^2} \Gamma(a)\Gamma(b) \frac{\mu^3}{8R^3}, \qquad (8.5.0.22)$$

where we have used the fact that $\Gamma(a + b - c) = \Gamma(2) = 1$ and similarly $\Gamma(c) = 1$. On using the values of a and b and putting them together in the expression (8.5.0.22) one gets

$$N_{DS} = \frac{-i\Gamma(2 + \sqrt{m^2 R^2})\Gamma(2 - \sqrt{m^2 R^2})}{8\sqrt{2}\pi^2 |R|^3}. \qquad (8.5.0.23)$$

Furthermore, from the relations $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(1+i|mR|)\Gamma(1-i|mR|) = \frac{\pi|Rm|}{\sinh(\pi|Rm|)}$ one can rewrite the final answer for the constant N_{DS}

$$N_{DS} = \frac{-i|Rm|(1-m^2R^2)}{8\sqrt{2}\pi|R|^3\sinh\pi|Rm|}. \tag{8.5.0.24}$$

Once we determine N_{DS}, the Feynman Green function is obtained by evaluating $f_{DS}(Z)$ and $g_{DS}(Z)$ above the branch cut from $Z=1$ to ∞, i.e. by taking $f_{DS}(Z+i0)$ and $g_{DS}(Z+i0)$. This is what happens in the de Sitter case. To conclude we have the following two-point functions:

$$P^{A\dot{B}'}_{(F)} = \lim_{\epsilon\to 0^+} f_{DS}(Z+i\epsilon)D^A_{A'}n^{A'\dot{B}'}, \tag{8.5.0.25}$$

$$Q^{\dot{A}\dot{B}'}_{(F)} = \lim_{\epsilon\to 0^+} g_{DS}(Z+i\epsilon)\overline{D}^{\dot{A}\dot{B}'}, \tag{8.5.0.26}$$

where (F) stands for the Feynman Green functions.

It is now helpful to recall the discussion of various types of Green functions depending on the contours in the complex p^0-plane for the integral representation of the Green function for the simpler case of scalar fields, following [10]. From various contours the following relations among different Green's functions can be easily established:

$$G_F = G^- + G^{(-)} = G^+ - G^{(+)}, \tag{8.5.0.27a}$$

$$G^{(+)}(x,x') = -\theta(x,x')G_F(x,x') + \theta(x',x)G_F^*(x,x'), \tag{8.5.0.27b}$$

$$G^{(-)}(x,x') = \theta(x',x)G_F(x,x') - \theta(x,x')G_F^*(x,x'), \tag{8.5.0.27c}$$

$$\widetilde{G} = (G^+ - G^-) = (G^{(+)} + G^{(-)}) = -2\Big(\theta(x,x') - \theta(x',x)\Big)\mathrm{Re}G_F. \tag{8.5.0.27d}$$

With a standard notation, G^+ and G^- are the advanced and retarded functions respectively, and their difference $\widetilde{G}$ is the *supercommutator function*. G_F is the Feynman Green function and G_F^* is its complex conjugate. $G^{(+)}$ and $G^{(-)}$ are the positive- and negative-frequency parts, respectively. The $\theta(x,x')$ used above in the definition of advanced and retarded functions is the step function.

Now our approach to arrive at the Peierls bracket in the de Sitter case will be as follows: once we determine using (8.5.0.25) and (8.5.0.26) the Feynman Green function, instead of using the advanced and retarded functions, we can use (8.5.0.27b) and (8.5.0.27c) respectively to get $G^{(+)}$ and $G^{(-)}$, and then add them to get the *supercommutator function* $\widetilde{G}$. Then we use 8.2.0.11 to build the Peierls bracket $(\ ,\ )_P$ which, in terms of the spinor fields

$$\psi_\alpha = \begin{pmatrix} \phi_A \\ \overline{\chi}_{\dot{A}} \end{pmatrix}, \quad \chi_{\beta'} = \begin{pmatrix} \rho_{B'} \\ \overline{\sigma}_{\dot{B}'} \end{pmatrix} \tag{8.5.0.28}$$

reads as

$$(\psi, \chi)_P \equiv \int \int P(x, x') \sqrt{-g(x)} \sqrt{-g(x')} d^4x \, d^4x', \qquad (8.5.0.29)$$

where

$$P(x, x') \equiv -2 \Big(\theta(x, x') - \theta(x', x) \Big) \psi_\nabla (\mathrm{Re} G_F) \chi_\nabla, \qquad (8.5.0.30)$$

having set

$$\psi_\nabla (\mathrm{Re} G_F) \chi_\nabla \equiv \Big(\nabla_{A\dot{A}} \overline{\phi}^{\dot{A}} \Big) \mathrm{Re} P_{(F)}^{A\dot{B}'} \Big(\nabla_{B'\dot{B}'} \sigma^{B'} \Big) + \Big(\nabla_{A\dot{A}} \chi^A \Big) \mathrm{Re} Q_{(F)}^{\dot{A}\dot{B}'} \Big(\nabla_{B'\dot{B}'} \rho^{B'} \Big).$$
$$(8.5.0.31)$$

8.6 Massive spin-3/2 propagator

In this section we consider the propagator of the massive spin-3/2 field. Let us denote the gravitino field by $\Psi_\lambda^\alpha(x)$. In a maximally symmetric state $|s\rangle$ the propagator is

$$S_{\lambda\nu'}^{\alpha\beta'}(x, x') = \langle s | \Psi_\lambda^\alpha(x) \Psi_{\nu'}^{\beta'}(x') | s \rangle . \qquad (8.6.0.1)$$

The field equations imply that S satisfies

$$(\Gamma^{\mu\rho\lambda} D_\rho - m \, \Gamma^{\mu\lambda})^\alpha{}_\gamma S_{\lambda\nu'}{}^\gamma{}_{\beta'} = \frac{\delta(x - x')}{\sqrt{-g}} g^\mu{}_{\nu'} \delta^\alpha{}_{\beta'}. \qquad (8.6.0.2)$$

8.6.1 The ten gravitino invariants in two-component-spinor language

It is very convenient to decompose the gravitino propagator in terms of independent structures constructed out of $n_\mu, n_{\nu'}, g_{\mu\nu'}$ and $\Lambda^\alpha{}_{\beta'}$. Thus, the propagator can be written in geometric way following Anguelova et al. [20] (see also [21]):

$$\begin{aligned}
S_{\lambda\nu'}{}^\alpha{}_{\beta'} = {}& \alpha(\mu) \, g_{\lambda\nu'} \Lambda^\alpha{}_{\beta'} + \beta(\mu) \, n_\lambda n_{\nu'} \Lambda^\alpha{}_{\beta'} + \gamma(\mu) \, g_{\lambda\nu'} (n_\sigma \Gamma^\sigma \Lambda)^\alpha{}_{\beta'} \\
& + \delta(\mu) \, n_\lambda n_{\nu'} (n_\sigma \Gamma^\sigma \Lambda)^\alpha{}_{\beta'} + \varepsilon(\mu) \, n_\lambda (\Gamma_{\nu'} \Lambda)^\alpha{}_{\beta'} + \theta(\mu) \, n_{\nu'} (\Gamma_\lambda \Lambda)^\alpha{}_{\beta'} \\
& + \tau(\mu) \, n_\lambda (n_\sigma \Gamma^\sigma \Gamma_{\nu'} \Lambda)^\alpha{}_{\beta'} + \omega(\mu) \, n_{\nu'} (n_\sigma \Gamma^\sigma \Gamma_\lambda \Lambda)^\alpha{}_{\beta'} \\
& + \pi(\mu) \, (\Gamma_\lambda \Gamma_{\nu'} \Lambda)^\alpha{}_{\beta'} + \kappa(\mu) \, (n_\sigma \Gamma^\sigma \Gamma_\lambda \Gamma_{\nu'} \Lambda)^\alpha{}_{\beta'}. \qquad (8.6.1.1)
\end{aligned}$$

From here on we will be trying to re-write each of the building blocks of the invariant structure in two-spinor language and finally construct the full invariant

propagator in two-spinor form in harmony with the spin-1/2 propagator previously discussed. Following Allen and Lutken [14] we can write the gamma matrix in two-spinor language as follows (Penrose and Rindler, on page 221 of [17], do not have the $-i$ factor since their γ-matrices satisfy the anti-commutation relation $\gamma_a\gamma_b + \gamma_b\gamma_a = -2Ig_{ab}$, unlike the sign convention in our Eq. 8.3.0.5)

$$(\gamma_p)_\alpha{}^\beta = -i\sqrt{2} \begin{pmatrix} 0 & \varepsilon_{PA}\varepsilon_{\dot{P}}^{\dot{B}} \\ \varepsilon_{\dot{P}\dot{A}}\varepsilon_P^B & 0 \end{pmatrix}, \qquad (8.6.1.2)$$

where ε_{BC} is the curved epsilon symbol which raises and lowers indices within each spin-space, is skew-symmetric and encodes information on the curved spacetime metric. In the case of flat Minkowski spacetime it reduces to the well known form

$$\epsilon_{BC} = \epsilon_{BA}\epsilon_C^A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \qquad (8.6.1.3)$$

From the rules of two-spinor calculus and from the treatment of Allen and Lutken [14] we already know the following correspondences:

$$n_\alpha \longrightarrow n_{A\dot{A}}, \qquad (8.6.1.4a)$$

$$n_{\beta'} \longrightarrow n_{B'\dot{B}'}, \qquad (8.6.1.4b)$$

$$g_{\alpha\beta'} \longrightarrow D_{AB'}\overline{D}_{\dot{A}\dot{B}'}. \qquad (8.6.1.4c)$$

We also know the form of the spinor parallel propagator, which acts according to

$$\chi^{A'}(x') = D_A^{A'}(x, x')\phi^A(x), \qquad (8.6.1.5a)$$

$$\chi^A(x) = D_{B'}^A(x, x')\phi^{B'}(x'), \qquad (8.6.1.5b)$$

$$\phi^{\dot{A}}(x) = \overline{D}_{\dot{B}'}^{\dot{A}}(x, x')\chi^{\dot{B}'}(x'). \qquad (8.6.1.5c)$$

To translate the above set of equations, e.g. $\phi^\alpha = \Lambda^\alpha{}_{\beta'}\phi^{\beta'}$, into two-spinor language, both left- and right-hand sides should involve a $\begin{pmatrix} \chi \\ \phi \end{pmatrix}$ column vector, with upstairs indices at x and x' respectively. Written in matrix notation we can combine them into one reading, i.e.

$$\begin{pmatrix} \chi^A \\ \phi^{\dot{A}} \end{pmatrix} = \begin{pmatrix} 0 & D_{B'}^A \\ \overline{D}_{\dot{B}'}^{\dot{A}} & 0 \end{pmatrix} \begin{pmatrix} \chi^{\dot{B}'} \\ \phi^{B'} \end{pmatrix}. \qquad (8.6.1.6)$$

Therefore from now on we redefine $\Lambda^\rho{}_{\beta'}$ to be a (2x2) matrix, expressed in two-spinor language as $\Lambda^{R\dot{R}}_{B'\dot{B}'}$ and satisfying the correspondence rule

$$\Lambda^\rho{}_{\beta'} \longrightarrow \begin{pmatrix} 0 & D_{B'}^R \\ \overline{D}_{\dot{B}'}^{\dot{R}} & 0 \end{pmatrix}. \qquad (8.6.1.7)$$

Similarly, we go on translating each of the bits of the invariant structure into two-spinor notation. The next one is $(\Gamma^\sigma \Lambda)^\alpha{}_{\beta'}$. We note the following translation:

$$(\Gamma^\sigma \Lambda)^\alpha{}_{\beta'} = (\Gamma^\sigma)^\alpha{}_\rho \Lambda^\rho{}_{\beta'} \longrightarrow (\Gamma^{S\dot{S}})^\alpha{}_\rho \Lambda^\rho{}_{\beta'}. \tag{8.6.1.8}$$

Therefore, on using the two-spinor form 8.6.1.2 of the gamma matrix and the two-spinor version 8.6.1.7 of the spinor parallel propagator we get

$$(\Gamma^\sigma \Lambda)^\alpha{}_{\beta'} \longrightarrow -i\sqrt{2} \begin{pmatrix} 0 & \varepsilon^S_R \varepsilon^{\dot{S}\dot{A}} \\ \varepsilon^{\dot{S}}_{\dot{R}} \varepsilon^{SA} & 0 \end{pmatrix} \begin{pmatrix} 0 & D^R_{B'} \\ \overline{D}^{\dot{R}}_{\dot{B}'} & 0 \end{pmatrix} = -i\sqrt{2} \begin{pmatrix} \varepsilon^S_R \varepsilon^{\dot{S}\dot{A}} \overline{D}^{\dot{R}}_{\dot{B}'} & 0 \\ 0 & \varepsilon^{\dot{S}}_{\dot{R}} \varepsilon^{SA} D^R_{B'} \end{pmatrix}. \tag{8.6.1.9}$$

Similarly, we find

$$(n_\sigma \Gamma^\sigma \Lambda)^\alpha{}_{\beta'} \longrightarrow n_{S\dot{S}} (\Gamma^{S\dot{S}})^\alpha{}_\rho \Lambda^\rho{}_{\beta'} = -i n_{S\dot{S}} \sqrt{2} \begin{pmatrix} \varepsilon^S_R \varepsilon^{\dot{S}\dot{A}} \overline{D}^{\dot{R}}_{\dot{B}'} & 0 \\ 0 & \varepsilon^{\dot{S}}_{\dot{R}} \varepsilon^{SA} D^R_{B'} \end{pmatrix}. \tag{8.6.1.10}$$

Now we use the antisymmetry property of the epsilon symbol, i.e. $\varepsilon^{AB} = -\varepsilon^{BA}$, and the rules for raising and lowering spinor indices, i.e. $\varepsilon^{AB}\phi_B = \phi^A$, $\phi^A \varepsilon_{AB} = \phi_B$, to write $(n_\sigma \Gamma^\sigma \Lambda)^\alpha{}_{\beta'}$ in matrix form as

$$(n_\sigma \Gamma^\sigma \Lambda)^\alpha{}_{\beta'} \longrightarrow -i\sqrt{2} \begin{pmatrix} n^{\dot{A}}_R \overline{D}^{\dot{R}}_{\dot{B}'} & 0 \\ 0 & n^A_{\dot{R}} D^R_{B'} \end{pmatrix}. \tag{8.6.1.11}$$

Now let us start writing the invariants in two-spinor language. The first invariant structure (see 8.6.1.1 from now on) is

$$g_{\lambda\nu'} \Lambda^\alpha{}_{\beta'} \longrightarrow D_{LN'} \overline{D}_{\dot{L}\dot{N}'} \begin{pmatrix} 0 & D^A_{B'} \\ \overline{D}^{\dot{A}}_{\dot{B}'} & 0 \end{pmatrix}. \tag{8.6.1.12}$$

The second one is

$$n_\lambda n_{\nu'} \Lambda^\alpha{}_{\beta'} \longrightarrow n_{L\dot{L}} n_{N'\dot{N}'} \begin{pmatrix} 0 & D^A_{B'} \\ \overline{D}^{\dot{A}}_{\dot{B}'} & 0 \end{pmatrix}. \tag{8.6.1.13}$$

Then the third reads as

$$g_{\lambda\nu'} (n_\sigma \Gamma^\sigma \Lambda)^\alpha{}_{\beta'} \longrightarrow -i D_{LN'} \overline{D}_{\dot{L}\dot{N}'} \sqrt{2} \begin{pmatrix} n^{\dot{A}}_R \overline{D}^{\dot{R}}_{\dot{B}'} & 0 \\ 0 & n^A_{\dot{R}} D^R_{B'} \end{pmatrix}. \tag{8.6.1.14}$$

Next is the fourth invariant, i.e.

$$n_\lambda n_{\nu'} (n_\sigma \Gamma^\sigma \Lambda)^\alpha{}_{\beta'} \longrightarrow -i n_{L\dot{L}} n_{N'\dot{N}'} \sqrt{2} \begin{pmatrix} n^{\dot{A}}_R \overline{D}^{\dot{R}}_{\dot{B}'} & 0 \\ 0 & n^A_{\dot{R}} D^R_{B'} \end{pmatrix}. \tag{8.6.1.15}$$

The subsequent invariant structure involves $(\Gamma_{\nu'}\Lambda)^\alpha{}_{\beta'}$ and we know that

$$(\Gamma_{\nu'}\Lambda)^\alpha{}_{\beta'} \longrightarrow (\Gamma_{N'\dot{N}'})^\alpha{}_\rho \Lambda^\rho{}_{\beta'}. \tag{8.6.1.16}$$

Now from our previous discussion in this section we already have

$$(\Gamma_{N'\dot{N}'})^{\alpha'}{}_{\rho'} \longrightarrow -i\sqrt{2}\begin{pmatrix} 0 & \varepsilon_{N'R'}\varepsilon^{\dot{A}'}_{\dot{N}'} \\ \varepsilon_{\dot{N}'\dot{R}'}\varepsilon^{A'}_{N'} & 0 \end{pmatrix}. \tag{8.6.1.17}$$

The problem is that what is well defined is either $(\Gamma_\nu)^\alpha{}_\rho$ or $(\Gamma_{\nu'})^{\alpha'}{}_{\rho'}$, where everything is evaluated at the same spacetime point (either x or x'). However, here the relevant invariant consists of a mixed structure of the kind $(\Gamma_{\nu'})^\alpha{}_\rho$, and, to understand what is meant by it, we should use the parallel displacement bi-vector. Eventually, with the help of some careful thought we can write

$$(\Gamma_{\nu'})^\alpha{}_\rho \longrightarrow (\Gamma_{N'\dot{N}'})^{\alpha'}{}_{\rho'}g_{\alpha'}{}^\alpha g^{\rho'}{}_\rho = -i\sqrt{2}\begin{pmatrix} 0 & \varepsilon_{N'R'}\varepsilon^{\dot{A}'}_{\dot{N}'} \\ \varepsilon_{\dot{N}'\dot{R}'}\varepsilon^{A'}_{N'} & 0 \end{pmatrix}g_{A'\dot{A}'}{}^{A\dot{A}}g^{R'\dot{R}'}{}_{R\dot{R}}. \tag{8.6.1.18}$$

Recalling the fact that $g_{A'\dot{A}'}{}^{A\dot{A}} = D^A_{A'}\overline{D}^{\dot{A}}_{\dot{A}'}$ and $g^{R'\dot{R}'}{}_{R\dot{R}} = D^{R'}_R\overline{D}^{\dot{R}'}_{\dot{R}}$ we can write the final form of the matrix $(\Gamma_{\nu'})^\alpha{}_\rho$ as follows:

$$(\Gamma_{\nu'})^\alpha{}_\rho \longrightarrow -i\sqrt{2}\begin{pmatrix} 0 & -D_{RN'}\overline{D}^{\dot{R}'}_{\dot{R}}D^A_{A'}\overline{D}^{\dot{A}}_{\dot{N}'} \\ -D^{R'}_R\overline{D}_{\dot{R}\dot{N}'}D^A_{N'}\overline{D}^{\dot{A}}_{\dot{A}'} & 0 \end{pmatrix}. \tag{8.6.1.19}$$

Now we can build the fifth invariant quite easily as shown here,

$$n_\lambda(\Gamma_{\nu'}\Lambda)^\alpha{}_{\beta'} \longrightarrow -in_{L\dot{L}}\sqrt{2}\begin{pmatrix} 0 & -D_{RN'}\overline{D}^{\dot{R}'}_{\dot{R}}D^A_{A'}\overline{D}^{\dot{A}}_{\dot{N}'} \\ -D^{R'}_R\overline{D}_{\dot{R}\dot{N}'}D^A_{N'}\overline{D}^{\dot{A}}_{\dot{A}'} & 0 \end{pmatrix}\begin{pmatrix} 0 & D^R_{B'} \\ \overline{D}^{\dot{R}}_{\dot{B}'} & 0 \end{pmatrix}$$

$$= -in_{L\dot{L}}\sqrt{2}\begin{pmatrix} -D_{RN'}\overline{D}^{\dot{R}'}_{\dot{R}}\overline{D}^{\dot{R}}_{\dot{B}'}D^A_{A'}\overline{D}^{\dot{A}}_{\dot{N}'} & 0 \\ 0 & -D^{R'}_R D^R_{B'}\overline{D}_{\dot{R}\dot{N}'}D^A_{N'}\overline{D}^{\dot{A}}_{\dot{A}'} \end{pmatrix}. \tag{8.6.1.20}$$

The sixth invariant is constructed as follows:

$$n_{\nu'}(\Gamma_\lambda\Lambda)^\alpha{}_{\beta'} \longrightarrow n_{N'\dot{N}'}(\Gamma_\lambda)^\alpha{}_\rho\Lambda^\rho{}_{\beta'} \longrightarrow -in_{N'\dot{N}'}\sqrt{2}\begin{pmatrix} 0 & \varepsilon_{LR}\varepsilon^{\dot{A}}_{\dot{L}} \\ \varepsilon_{\dot{L}\dot{R}}\varepsilon^A_L & 0 \end{pmatrix}\begin{pmatrix} 0 & D^R_{B'} \\ \overline{D}^{\dot{R}}_{\dot{B}'} & 0 \end{pmatrix}$$

$$= -in_{N'\dot{N}'}\sqrt{2}\begin{pmatrix} \varepsilon_{LR}\varepsilon^{\dot{A}}_{\dot{L}}\overline{D}^{\dot{R}}_{\dot{B}'} & 0 \\ 0 & \varepsilon_{\dot{L}\dot{R}}\varepsilon^A_L D^R_{B'} \end{pmatrix}. \tag{8.6.1.21}$$

Now we start building the last four invariants step by step. First we express the seventh invariant $n_\lambda (n_\sigma \Gamma^\sigma \Gamma_{\nu'} \Lambda)^\alpha{}_{\beta'}$ in two-spinor language. We note that

$$
n_\lambda (n_\sigma \Gamma^\sigma \Gamma_{\nu'} \Lambda)^\alpha{}_{\beta'} = n_\lambda n_\sigma (\Gamma^\sigma)^\alpha{}_\rho (\Gamma_{\nu'})^{\rho'}{}_{\tau'} g_{\rho'}{}^\rho g^{\tau'}{}_\tau \Lambda^\tau{}_{\beta'}
$$

$$
\longrightarrow \quad -2n_{L\dot{L}} n_{S\dot{S}}
\begin{pmatrix}
0 & \varepsilon^S_R \varepsilon^{\dot{S}\dot{A}} \\
\varepsilon^{\dot{S}}_{\dot{R}} \varepsilon^{SA} & 0
\end{pmatrix}
\begin{pmatrix}
0 & \varepsilon_{N'T'} \varepsilon^{\dot{R}'}_{\dot{N}'} \\
\varepsilon_{\dot{N}'\dot{T}'} \varepsilon^{R'}_{N'} & 0
\end{pmatrix}
$$

$$
\times \quad D^R_{R'} \overline{D}^{\dot{R}}_{\dot{R}'} D_T^{T'} \overline{D}_{\dot{T}}^{\dot{T}'}
\begin{pmatrix}
0 & D^T_{B'} \\
\overline{D}^{\dot{T}}_{\dot{B}'} & 0
\end{pmatrix}.
\tag{8.6.1.22}
$$

After a little bit of algebra we arrive at the seventh invariant, i.e.

$$
n_\lambda (n_\sigma \Gamma^\sigma \Gamma_{\nu'} \Lambda)^\alpha{}_{\beta'} \longrightarrow -2n_{L\dot{L}}
\begin{pmatrix}
0 & -n^{\dot{A}}_R D^R_{N'} \overline{D}^{\dot{R}}_{\dot{R}'} D_T^{T'} \overline{D}_{\dot{T}\dot{N}'} D^T_{B'} \\
-n^A_{\dot{R}} \overline{D}^{\dot{R}}_{\dot{N}'} D^R_{R'} \overline{D}_{\dot{T}}^{\dot{T}'} D_{TN'} \overline{D}^{\dot{T}}_{\dot{B}'} & 0
\end{pmatrix}.
\tag{8.6.1.23}
$$

Now let us write the eighth invariant term as follows:

$$
n_{\nu'} (n_\sigma \Gamma^\sigma \Gamma_\lambda \Lambda)^\alpha{}_{\beta'} = n_{\nu'} n_\sigma (\Gamma^\sigma)^\alpha{}_\rho (\Gamma_\lambda)^\rho{}_\tau \Lambda^\tau{}_{\beta'}
$$

$$
\longrightarrow \quad -2n_{N'\dot{N}'} n_{S\dot{S}}
\begin{pmatrix}
0 & \varepsilon^S_R \varepsilon^{\dot{S}\dot{A}} \\
\varepsilon^{\dot{S}}_{\dot{R}} \varepsilon^{SA} & 0
\end{pmatrix}
\begin{pmatrix}
0 & \varepsilon_{LT} \varepsilon^{\dot{R}}_{\dot{L}} \\
\varepsilon_{\dot{L}\dot{T}} \varepsilon^R_L & 0
\end{pmatrix}
\begin{pmatrix}
0 & D^T_{B'} \\
\overline{D}^{\dot{T}}_{\dot{B}'} & 0
\end{pmatrix}.
\tag{8.6.1.24}
$$

The final result for the eighth invariant is

$$
n_{\nu'} (n_\sigma \Gamma^\sigma \Gamma_\lambda \Lambda)^\alpha{}_{\beta'} \longrightarrow -2n_{N'\dot{N}'}
\begin{pmatrix}
0 & n^{\dot{A}}_L \varepsilon_{\dot{L}\dot{T}} D^T_{B'} \\
n^A_{\dot{L}} \varepsilon_{LT} \overline{D}^{\dot{T}}_{\dot{B}'} & 0
\end{pmatrix}.
\tag{8.6.1.25}
$$

The ninth invariant can be translated in two-spinor form accordingly :

$$
(\Gamma_\lambda \Gamma_{\nu'} \Lambda)^\alpha{}_{\beta'} = (\Gamma_\lambda)^\alpha{}_\rho (\Gamma_{\nu'})^\rho{}_\tau \Lambda^\tau{}_{\beta'} = (\Gamma_\lambda)^\alpha{}_\rho (\Gamma_{\nu'})^{\rho'}{}_{\tau'} g_{\rho'}{}^\rho g^{\tau'}{}_\tau \Lambda^\tau{}_{\beta'}
$$

$$
\rightarrow \quad -2D^R_{R'} \overline{D}^{\dot{R}}_{\dot{R}'} D_T^{T'} \overline{D}_{\dot{T}}^{\dot{T}'}
\begin{pmatrix}
0 & \varepsilon_{LR} \varepsilon^{\dot{A}}_{\dot{L}} \\
\varepsilon_{\dot{L}\dot{R}} \varepsilon^A_L & 0
\end{pmatrix}
\begin{pmatrix}
0 & \varepsilon_{N'T'} \varepsilon^{\dot{R}'}_{\dot{N}'} \\
\varepsilon_{\dot{N}'\dot{T}'} \varepsilon^{R'}_{N'} & 0
\end{pmatrix}
\begin{pmatrix}
0 & D^R_{B'} \\
\overline{D}^{\dot{R}}_{\dot{B}'} & 0
\end{pmatrix}.
\tag{8.6.1.26}
$$

Eventually, after a few algebraic steps with matrices, we get the ninth invariant

$$
(\Gamma_\lambda \Gamma_{\nu'} \Lambda)^\alpha{}_{\beta'} \longrightarrow -2
\begin{pmatrix}
0 & \varepsilon^{\dot{A}}_{\dot{L}} D^R_{N'} \overline{D}^{\dot{R}}_{\dot{R}'} D_T^{T'} \overline{D}_{\dot{T}\dot{N}'} D_{LB'} \\
\varepsilon^A_L \overline{D}^{\dot{R}}_{\dot{N}'} D^R_{R'} \overline{D}_{\dot{T}}^{\dot{T}'} D_{TN'} \overline{D}_{\dot{L}\dot{B}'} & 0
\end{pmatrix}.
\tag{8.6.1.27}
$$

Last, but not least, the tenth invariant is built as follows:

$$(n_\sigma \Gamma^\sigma \Gamma_\lambda \Gamma_{\nu'} \Lambda)^\alpha{}_{\beta'} = n_\sigma (\Gamma^\sigma)^\alpha{}_\rho (\Gamma_\lambda)^\rho{}_\tau (\Gamma_{\nu'})_\chi{}^\tau \Lambda^\chi{}_{\beta'}$$

$$= n_\sigma (\Gamma^\sigma)^\alpha{}_\rho (\Gamma_\lambda)^\rho{}_\tau (\Gamma_{\nu'})_{\chi'}{}^{\tau'} g_{\tau'}{}^\tau g^{\chi'}{}_\chi \Lambda^\chi{}_{\beta'}$$

$$\longrightarrow -2\sqrt{2} n_{S\dot{S}} \begin{pmatrix} 0 & \varepsilon^S{}_R \varepsilon^{\dot{S}A} \\ \varepsilon^{\dot{S}}{}_{\dot{R}} \varepsilon^{SA} & 0 \end{pmatrix} \begin{pmatrix} 0 & \varepsilon_{LT} \varepsilon^{\dot{R}}_L \\ \varepsilon_{\dot{L}\dot{T}} \varepsilon^R_L & 0 \end{pmatrix} \begin{pmatrix} 0 & \varepsilon_{N'K'} \varepsilon^{\dot{T}'}_{\dot{N}'} \\ \varepsilon_{\dot{N}'\dot{K}'} \varepsilon^{T'}_{N'} & 0 \end{pmatrix}$$

$$\times \quad D^T_{T'} \overline{D}^{\dot{T}}_{\dot{T}'} D^{K'}_K \overline{D}^{\dot{K}'}_{\dot{K}} \begin{pmatrix} 0 & D^K_{B'} \\ \overline{D}^{\dot{K}}_{\dot{B}'} & 0 \end{pmatrix}. \tag{8.6.1.28}$$

At the end of the day, when all dust gets settled we obtain the final invariant in the form

$$(n_\sigma \Gamma^\sigma \Gamma_\lambda \Gamma_{\nu'} \Lambda)^\alpha{}_{\beta'} \longrightarrow -2\sqrt{2} \begin{pmatrix} n^{\dot{A}}_L D^T_{T'} \overline{D}^{\dot{K}'}_{\dot{K}} \overline{D}^{\dot{K}}_{\dot{B}'} \overline{D}_{\dot{L}\dot{N}'} D_{KN'} & 0 \\ 0 & n^A_{\dot{L}} \overline{D}^{\dot{T}}_{\dot{T}'} D^{K'}_K D^K_{B'} D_{LN'} \overline{D}_{\dot{K}\dot{N}'} \end{pmatrix}. \tag{8.6.1.29}$$

8.6.2 The weight functions multiplying the invariants

A rather tedious but straightforward calculation gives a system of 10 equations for the 10 coefficient functions $\alpha, ..., \kappa$ in (8.6.1.1) as found in (See equations (3.6)-(3.15) in [20]). It was also found there that one can easily express the algebraic solutions for $\alpha, \beta, \gamma, \delta, \varepsilon, \theta, \tau, \omega$ in terms of the (π, κ) pair in case of de Sitter space, i.e. (hereafter we set $n = 4$ in the general formulae of [20], since only in the four-dimensional case the two-component-spinor formalism can be applied)

$$\omega = \frac{2mC\kappa + ((A+C)^2 - m^2)\pi}{(m^2 + R^{-2})},$$

$$\theta = \frac{((A-C)^2 - m^2)\kappa - 2mC\pi}{(m^2 + R^{-2})},$$

$$\tau = \frac{2mC\kappa + ((A+C)^2 - m^2)\pi}{(m^2 + R^{-2})},$$

$$\varepsilon = \frac{-([(A-C)^2 + 2/R^2] + m^2)\kappa + 2mC\pi}{(m^2 + R^{-2})},$$

$$\alpha = -\tau - 4\pi,$$

$$\beta = 2\omega,$$

$$\gamma = \varepsilon - 2\kappa,$$

$$\delta = 2\varepsilon + 4(\kappa - \theta), \tag{8.6.2.1}$$

where we have used the relation $C^2 - A^2 = 1/R^2$.

Furthermore, from (8.6.2.1) we can immediately see that

$$\tau = \omega \quad \text{and} \quad \varepsilon + \theta = -2\kappa. \tag{8.6.2.2}$$

8.6.3 Peierls bracket for gravitinos

The expression 8.6.1.1 for the gravitino propagator can be written, concisely, in the form

$$S_{\lambda \nu'}^{\alpha \beta'} \longrightarrow \sum_{k=1}^{10} w_k \, _kS_{L\dot{L}N'\dot{N}'}^{A\dot{A}B'\dot{B}'}, \tag{8.6.3.1}$$

where, as k ranges from 1 through 10, $w_k = \alpha, \beta, ..., \kappa$ in (6.3), while the $_kS_{L\dot{L}N'\dot{N}'}^{A\dot{A}B'\dot{B}'}$ are the 10 spinor invariants written down in subsection 8.6.1. Two further indices are needed to characterize each w_k function, i.e. j which labels the four singular points at $z = 0, 1, a, \infty$ and the subscript F to denote the Feynman prescription to approach such singular points, i.e. from the above along the positive real axis. Thus, the definition of Peierls bracket that we propose bears analogies with Eqs.8.5.0.29 and 8.5.0.30, with ψ_∇ and χ_∇ obtained from the covariant derivative of the Rarita–Schwinger potential (see notes in 8.9), while

$$(\text{Re}G_F) \longrightarrow \text{Re}\left(\sum_{k=1}^{10} w_k^{(j_F)} \, _kS_{L\dot{L}N'\dot{N}'}^{A\dot{A}B'\dot{B}'} \right). \tag{8.6.3.2}$$

8.7 Heun's differential equation: a primer

The canonical form of the general Heun differential equation is given by ([23], [31])

$$\frac{d^2y}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a} \right) \frac{dy}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-a)} y = 0 \tag{8.7.0.1}$$

The four regular singular points of the equation are located at $z = 0, 1, a, \infty$. Here $a \in \mathbb{C}$, the location of the fourth singular point, is a parameter ($a \neq 0, 1$), and $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{C}$ are exponent-related parameters.

The solution space of the Heun differential equation is specified uniquely by the following Riemann P-symbol:

$$P \left\{ \begin{matrix} 0 & 1 & d & \infty \\ 0 & 0 & 0 & \alpha \\ 1-\gamma & 1-\delta & 1-\epsilon & \beta \end{matrix} \; ;z \right\}. \tag{8.7.0.2}$$

This does not uniquely specify the equation and its solutions, since it omits the accessory parameter $q \in \mathbb{C}$. The exponents are constrained by

$$\alpha + \beta - \gamma - \delta - \epsilon + 1 = 0. \tag{8.7.0.3}$$

This is a special case of Fuchs's relation, according to which the sum of the $2n$ characteristic exponents of any second-order Fuchsian equation on $\mathbb{CP}^1$ with n singular points must equal $n - 2$ [32].

There are $2 \times 4 = 8$ local solutions of 8.7.0.1 in all: two per singular point. If γ is not a nonpositive integer, the solution at $z = 0$ belonging to the exponent zero will be analytic. When normalized to unity at $z = 0$, it is called the local Heun function, and is denoted $Hl(a, q; \alpha, \beta, \gamma, \delta; z)$ [23]. It is the sum of a Heun series, which converges in a neighborhood of $z = 0$ [23; 33]. In general, $Hl(a, q; \alpha, \beta, \gamma, \delta; t)$ is not defined when γ is a nonpositive integer.

If $\epsilon = 0$ and $q = \alpha\beta d$, the Heun equation loses a singular point and becomes a hypergeometric equation. Similar losses occur if $\delta = 0, q = \alpha\beta$, or $\gamma = 0, q = 0$. This chapter will exclude the case when the Heun equation has fewer than four singular points. The case, in which the solution of 8.7.0.1 can be reduced to quadratures, will also be ruled out. If $\alpha\beta = 0$ and $q = 0$, the Heun equation 8.7.0.1 is said to be trivial. Triviality implies that one of the exponents at $z = \infty$ is zero (i.e., $\alpha\beta = 0$), and is implied by absence of the singular point at $z = \infty$ (i.e., $\alpha\beta = 0, \alpha + \beta = 1, q = 0$).

8.7.1   Reducing Heun to hypergeometric

The transformation to Heun ($\mathfrak{H}$) or hypergeometric ($\mathfrak{h}$) of a linear second-order Fuchsian differential equation with singular points at $z = 0, 1, d, \infty$ (resp. $z = 0, 1, \infty$), and with arbitrary exponents, is accomplished by certain linear changes of the dependent variable, called F-homotopies (see [19] and [23].) If an equation with singular points at $z = 0, 1, a, \infty$ has dependent variable u, carrying out the substitution $\tilde{u}(z) = z^{-\rho}(z - 1)^{-\sigma}(z - a)^{-\tau}u(t)$ will convert the equation to a new one, with the exponents at $z = 0, 1, d$ reduced by ρ, σ, τ respectively, and those at $z = \infty$ increased by $\rho + \sigma + \tau$. By this technique, one exponent at each finite singular point can be shifted to zero.

In fact, the Heun equation has a group of F-homotopic automorphisms isomorphic to $(\mathbb{Z}_2)^3$, since at each of $z = 0, 1, a$, the exponents $0, \zeta$ can be shifted to $-\zeta, 0$, i.e., to $0, -\zeta$. Similarly, the hypergeometric equation has a group of F-homotopic automorphisms isomorphic to $(\mathbb{Z}_2)^2$. These groups act on the 6 and 3-dimensional parameter spaces, respectively. For example, one of the latter actions is $(a, b; c) \mapsto (c - a, c - b; c)$, which is induced by an F-homotopy at $z = 1$. From this F-homotopy follows Euler's transformation [34]

$$ {}_2F_1(a, b; c; z) = (1 - z)^{c-a-b} {}_2F_1(c - a, c - b; c; z), \qquad (8.7.1.1) $$

which holds because ${}_2F_1$ is a local solution at $z = 0$, rather than at $z = 1$. If the singular points of the differential equation are arbitrarily placed, transforming it to the Heun or hypergeometric equation will require a Möbius (i.e., projective linear or homographic) transformation, which repositions the singular points to the standard locations. A unique Möbius transformation maps any three distinct points in $\mathbb{CP}^1$ to any other three; but the same is not true of four points, which is why ($\mathfrak{H}$) has the singular point a as a free parameter.

8.7.2 The cross-ratio orbit

The characterization of Heun equations that can be reduced to the hypergeometric equation will employ the cross-ratio orbit of $\{0, 1, d, \infty\}$, defined as follows. If $A, B, C, D \in \mathbb{CP}^1$ are distinct, their cross-ratio is

$$ (A, B; C, D) :\equiv \frac{(C - A)(D - B)}{(D - A)(C - B)} \in \mathbb{CP}^1 \setminus \{0, 1, \infty\}, \qquad (8.7.2.1) $$

which is invariant under Möbius transformations. Permuting A, B, C, D yields an action of the symmetric group S_4 on $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$. The cross-ratio is invariant under interchange of A, B and C, D, and also under simultaneous interchange of the two points in each pair. Thus, each orbit contains no more than $4!/4 = 6$ cross-ratios. The possible actions of S_4 on $s \in \mathbb{CP}^1 \setminus \{0, 1, \infty\}$ are generated by $s \mapsto 1 - s$ and $s \mapsto 1/s$, and the orbit of s comprises

$$ s, \quad 1 - s, \quad 1/s, \quad 1/(1 - s), \quad s/(s - 1), \quad (s - 1)/s, \qquad (8.7.2.2) $$

which may not be distinct. This is called the cross-ratio orbit of s; or, if $s = (A, B; C, D)$, the cross-ratio orbit of the unordered set $\{A, B, C, D\} \subset \mathbb{CP}^1$. Two sets of distinct points $\{A_i, B_i, C_i, D_i\}$ ($i = 1, 2$) have the same cross-ratio orbit iff they are related by a Möbius transformation.

8.7.3   Reminder of some of the properties of Heun's function

Our aim will be to find an integral representation of the Heun function as a Frobenius' solution of the Heun equation, given in another form as follows [23]:

$$z(z-1)(z-a)y''(z) + \{\gamma(z-1)(z-a) + \delta z(z-a) + \epsilon z(z-1)\}\, y'(z)$$
$$+(\alpha\beta z - q)y(z) = 0, \tag{8.7.3.1}$$

The Frobenius' solution, noted $Hl(a,q;\alpha,\beta,\gamma,\delta;z)$ is the entire solution defined for the exponent zero at the point $z = 0$. It admits the power series expansion

$$Hl(a,q;\alpha,\beta,\gamma,\delta;z) \equiv \sum_{n=0}^{\infty} c_n z^n, \tag{8.7.3.2}$$

with $|z| < 1$ and $c_0 = 1$, $c_1 = \frac{q}{\gamma a}$ and $\gamma \neq 0, -1, -2, \dots$

The recursion relation is as follows:

$$a(n+2)(n+1+\gamma)c_{n+2}$$
$$= \left[q + (n+1)(\alpha + \beta - \delta + (\gamma + \delta - 1)a) + (n+1)^2(a+1)\right]c_{n+1}$$
$$-(n+\alpha)(n+\beta)c_n = 0 \quad n \geq 0. \tag{8.7.3.3}$$

The function $Hl(a,q;\alpha,\beta,\gamma,\delta;z)$ is normalised with the relation

$$Hl(a,q;\alpha,\beta,\gamma,\delta;0) = 1. \tag{8.7.3.4}$$

It admits the following important particular cases ([23], p9, formula(1.3.9)):

$$Hl(1,\alpha\beta;\alpha,\beta,\gamma,\delta;z) = {}_2F_1(\alpha,\beta,\gamma;z) \quad \forall\delta \in \mathbb{C}$$
$$Hl(0,0;\alpha,\beta,\gamma,\delta;z) = {}_2F_1(\alpha,\beta,\alpha+\beta-\delta+1;z) \quad \forall\gamma \in \mathbb{C}$$
$$Hl(a,a\alpha\beta;\alpha,\beta,\gamma,\alpha+\beta-\gamma+1;z) = {}_2F_1(\alpha,\beta,\gamma;z),$$
$$\tag{8.7.3.5}$$

where ${}_2F_1(\alpha,\beta,\gamma;z)$ is the usual notation for the Gauss hypergeometric function.

8.7.4   Application of Heun's equation to our problem

Finally we come to the punch line, why do we need these all and how does the Heun equation indeed find an application to our problem? The answer to this goes along the following line: On using (8.6.2.2) the differential equations for κ and π,

the equations (3.14) and (3.15) of [20], acquire the form

$$- (A + C)\theta + \kappa' + \frac{1}{2}(A - C)\kappa + m\pi = 0,$$

$$(C - A)\omega + \pi' + \frac{1}{2}(A + C)\pi + m\kappa = 0, \qquad (8.7.4.1)$$

where θ and ω are given in (8.6.2.1). Clearly one can solve algebraically the second equation for κ. By differentiating the result one obtains also κ' in terms of π, π' and π'', and substitution of these in the first equation yields a second order ODE for $\pi(\mu)$. Now let us look at the system (8.7.4.1) in case of de Sitter spacetime. On inserting A and C from (8.8.0.1) below and passing to the globally defined variable $z = \cos^2 \frac{\mu}{2R}$ (see Sec. III), we obtain the following differential equation for π:

$$\left[P_2 \frac{d^2}{dz^2} + P_1 \frac{d}{dz} + P_0 \right] \pi = 0, \qquad (8.7.4.2)$$

where P_2 in (8.7.4.2) is a quartic polynomial in z, i.e.

$$P_2 = 4 \left[m^2 R^2 + 1 \right] z^4 - 4(2m^2 R^2 + 3)z^3 + 4(m^2 R^2 + 2)z^2. \qquad (8.7.4.3)$$

Similarly, P_1 in (8.7.4.2) is a cubic polynomial in z,

$$P_1 = 16 \left[m^2 R^2 + 1 \right] z^3 - 12 \left[2m^2 R^2 + 5 \right] z^2 + 8 \left(m^2 R^2 + 2 \right) z. \qquad (8.7.4.4)$$

Last, P_0 in (8.7.4.2) is a quadratic polynomial in z, i.e.

$$P_0 = \left(4m^4 - 19m^2 + 32m^2 R^2 + 9 \right) z^2 - \left(4m^4 - 14m^2 + 32m^2 R^2 + 21 \right) z - 3m^2 R^2 - 6. \qquad (8.7.4.5)$$

On making the substitution $\pi(z) = \sqrt{z}\, \tilde{\pi}(z)$, (8.7.4.2) becomes an equation of the type

$$z(z - 1)(z - a)y''(z) + \left\{ (b + c + 1)z^2 - [b + c + 1 + a(d + e) - e] z + ad \right\} y'(z)$$
$$+ (bc\, z - q)y(z) = 0. \qquad (8.7.4.6)$$

Written in canonical form it reads as follows:

$$\frac{d^2 y}{dz^2} + \left(\frac{d}{z} + \frac{e}{z - 1} + \frac{(b + c + 1) - (d + e)}{z - a} \right) \frac{dy}{dz} + \frac{bc\, z - q}{z(z - 1)(z - a)} y = 0, \quad (8.7.4.7)$$

where the parameters in (8.7.4.7) take the values

$$a = \frac{(m^2 R^2 + 2)}{(m^2 R^2 + 1)},$$
$$b = 2 + imR,$$
$$c = 2 - imR,$$
$$d = e = 3,$$
$$q = -\frac{(m^4 R^4 + 7m^2 R^2 + 10)}{(m^2 R^2 + 1)}. \qquad (8.7.4.8)$$

The equation (8.7.4.6) is known as Heun's differential equation [22; 23]. Its solutions, here denoted by $\mathrm{Hl}(a,q;b,c,d,e;z)$, have in general four singular points as we said before, i.e. $z_0 = 0,1,a,\infty$. Near each singularity the function behaves as a combination of two terms that are powers of $(z - z_0)$ with the following exponents: $\{0, 1-d\}$ for $z_0 = 0$, $\{0, 1-e\}$ for $z_0 = 1$, $\{0, d+e-b-c\}$ for $z_0 = a$, and $\{b, c\}$ (that is, z^{-b} or z^{-c}) for $z \to \infty$.

We now insert into the second of Eq.8.7.4.1 the first of Eq.8.6.2.1, finding eventually

$$\kappa = f^{-1}\left\{\left[(A-C)((A+C)^2 - m^2) - \frac{1}{2}(A+C)(m^2 + R^{-2})\right]\pi - (m^2 + R^{-2})\pi'\right\},$$
$$(8.7.4.9)$$

where

$$f \equiv m(m^2 + R^{-2} + 2C(C-A)), \qquad (8.7.4.10)$$

and π and π' are meant to be expressed through the Heun function $\mathrm{Hl}(a,q;b,c,d,e;z)$. Eventually, we will show in the next section that all weight functions can be therefore expressed through such Heun function. The material covered in the present section and in the previous two is not new, and most of it is appropriate only for a physics-oriented choice of four-dimensional de Sitter space.

8.8 A Dictionary of weight functions for the massive gravitino propagator

Here we will explicitly list all the weight functions as functions of $z = \cos^2\frac{\mu}{2R}$, in order to analyze their qualitative behavior as a function of z and de Sitter radius R in the next section. Let us recall a few definitions in de Sitter space, where A and C are functions of the geodesic distance μ and are given by [13]

$$A = \frac{1}{R}\cot\frac{\mu}{R} \quad \text{and} \quad C = -\frac{1}{R\sin(\mu/R)}, \qquad (8.8.0.1)$$

Since all other weight functions $\alpha, \beta, \gamma, \delta, \varepsilon, \theta, \tau, \omega$ can be written in terms of the (π, κ) pair, and in the last section we have seen κ can also be expressed in a form like (8.7.4.9), it is evident that all the weight functions including κ, i.e. $\alpha, \beta, \gamma, \delta, \varepsilon, \theta, \tau, \omega, \kappa$ can be expressed in terms of $\pi(\mu)$ and $\pi'(\mu)$ only.

We can also express π as a function of z and R only as $\pi = \pi(z) = \pi(\mu = \pm 2R\cos^{-1}\sqrt{z})$. Similarly, by using a few of the familiar trigonometric identities, one

can transform $\pi'(\mu)$ as

$$\pi'(\mu) = \mp\frac{1}{R}\sqrt{z(1-z)}\pi'(z).\tag{8.8.0.2}$$

One can also write down the expressions of $(A+C)$ and $(A-C)$ in terms of z and R only as follows:

$$A + C = -\frac{1}{R}\sqrt{\frac{1-z}{z}},$$
$$A - C = \frac{1}{R}\sqrt{\frac{z}{1-z}}.\tag{8.8.0.3}$$

Another function appearing quite frequently in our evaluation of all the weight functions is f, which can be also expressed as a function of z and R only as follows:

$$f = m(m^2 + R^{-2} + R^{-2}(1-z)^{-1}).\tag{8.8.0.4}$$

Now we start by listing all the weight functions in terms of $\pi(z)$ and $\pi'(z)$, bearing in mind that

$$\tilde{\pi}(z) = \mathrm{Hl}(a,q;b,c,d,e;z),\tag{8.8.0.5}$$

$$\pi(z) = \sqrt{z}\;\mathrm{Hl}(a,q;b,c,d,e;z),\tag{8.8.0.6}$$

where $\mathrm{Hl}(a,q;b,c,d,e;z)$ is the Heun function with arguments as defined before. One has therefore the lengthy formulae for all other weight functions as follows:

$$\alpha(z) = -\frac{2mC(m^2+R^{-2})}{f(z)}\cdot\left[(A-C)((A+C)^2-m^2)-\frac{1}{2}(A+C)(m^2+R^{-2})\right]\pi(z)$$
$$\mp\frac{2mC(m^2+R^{-2})^2}{f(z)}\frac{\sqrt{z(1-z)}}{R}\pi'(z) - \left((m^2+R^{-2})[(A+C)^2-m^2]-4\right)\pi(z).$$

$$\tag{8.8.0.7}$$

$$\beta(z) = \frac{4mC}{(m^2+R^{-2})f(z)}\cdot\left[(A-C)((A+C)^2-m^2)-\frac{1}{2}(A+C)(m^2+R^{-2})\right]\pi(z)$$
$$\pm\frac{4mC}{f(z)}\frac{\sqrt{z(1-z)}}{R}\pi'(z) + 2(m^2+R^{-2})^{-1}\left[(A+C)^2-m^2\right]\pi(z).$$

$$\tag{8.8.0.8}$$

$$\gamma(z) = -\frac{[(A-C)^2-m^2]}{(m^2+R^{-2})f(z)}\cdot\left[(A-C)((A+C)^2-m^2)-\frac{1}{2}(A+C)(m^2+R^{-2})\right]\pi(z)$$
$$\pm\left[(A-C)^2-m^2\right]\frac{\sqrt{z(1-z)}}{Rf(z)}\pi'(z) - 2mC(m^2+R^{-2})^{-1}\pi(z).$$

$$\tag{8.8.0.9}$$

$$\delta(z) = -\frac{6\left[(A-C)^2-m^2\right]}{(m^2+R^{-2})f(z)} \cdot \left[(A-C)((A+C)^2-m^2) - \frac{1}{2}(A+C)(m^2+R^{-2})\right]\pi(z)$$

$$\mp 6\left[(A-C)^2-m^2\right]\frac{\sqrt{z(1-z)}}{Rf(z)}\pi'(z) + 12mC(m^2+R^{-2})^{-1}\pi(z).$$

$$(8.8.0.10)$$

$$\varepsilon(z) = -\frac{\left[(A-C)^2+\frac{2}{R^2}+m^2\right]}{(m^2+R^{-2})f(z)} \cdot \left[(A-C)((A+C)^2-m^2) - \frac{1}{2}(A+C)(m^2+R^{-2})\right]\pi(z)$$

$$\mp \left[(A-C)^2+\frac{2}{R^2}+m^2\right]\frac{\sqrt{z(1-z)}}{Rf(z)}\pi'(z) + 2mC(m^2+R^{-2})^{-1}\pi(z).$$

$$(8.8.0.11)$$

$$\theta(z) = \frac{\left[(A-C)^2-m^2\right]}{(m^2+R^{-2})f(z)} \cdot \left[(A-C)((A+C)^2-m^2) - \frac{1}{2}(A+C)(m^2+R^{-2})\right]\pi(z)$$

$$\pm \left[(A-C)^2-m^2\right]\frac{\sqrt{z(1-z)}}{Rf(z)}\pi'(z) - 2mC(m^2+R^{-2})^{-1}\pi(z).$$

$$(8.8.0.12)$$

$$\tau(z) = \frac{2mC}{(m^2+R^{-2})f(z)} \cdot \left[(A-C)((A+C)^2-m^2) - \frac{1}{2}(A+C)(m^2+R^{-2})\right]\pi(z)$$

$$\pm\frac{2mC}{f(z)}\frac{\sqrt{z(1-z)}}{R}\pi'(z) + (m^2+R^{-2})^{-1}\left[(A+C)^2-m^2\right]\pi(z) = \omega(z).$$

$$(8.8.0.13)$$

$$\kappa(z) = \left[(A-C)((A+C)^2-m^2) - \frac{1}{2}(A+C)(m^2+R^{-2})\right]\frac{\pi(z)}{f(z)}$$

$$\pm(m^2+R^{-2})\frac{\sqrt{z(1-z)}}{Rf(z)}\pi'(z). \qquad\qquad (8.8.0.14)$$

These exhaust all the weight functions multiplying the invariant structure present in the gravitino propagator, written explicitly in terms of a Heun function and its derivative.

8.9 Qualitative behaviors of the weight functions

Now using the series expansion (8.7.3.2) defined before one can numerically study the behavior of each weight function, by taking the first 10 terms of the infinite series
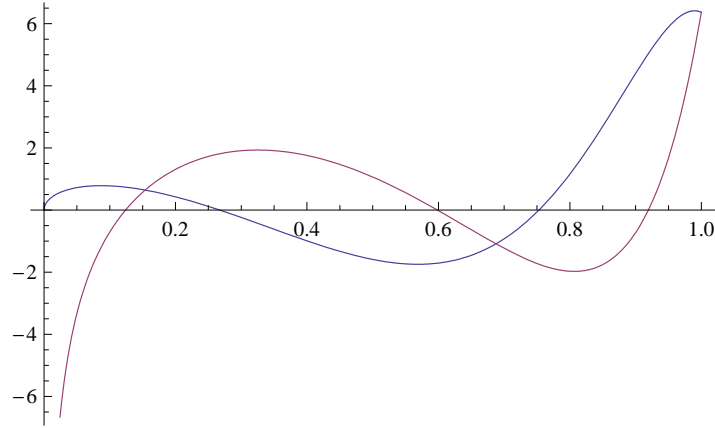
Figure 8.1: Two-dimensional plot of the weight function $\alpha(z)$. The curve has two branches, depending on whether one takes the $+$ or $-$ sign in (A1). One branch of α cuts the horizontal z-axis at the points $z = 0.25, 0.75$, whereas the other branch of α cuts the horizontal axis at the points $z = 0.12, 0.6, 0.92$. Both branches approach the vertical axis, the first one cuts it near the value 0.5, while the other has a vertical asymptote at $z = 0.02$. The two branches intersect each other at $z = 0.15, 0.69, 1$; at these points the function $\alpha(z)$ becomes single-valued.

8.7.3.2. Indeed, dealing with an infinite number of terms is impossible, and one has therefore to resort to approximations, by truncating such a series. On taking less than 10 terms, we have found minor departures from the pattern outlined below in figures 1 to 9, whereas on taking 15 terms, the pattern in such figures is essentially confirmed.

We draw for example all these weight functions in a two-dimensional plot vs z, in the range (0,1). The plots, which also include $\tilde{\pi}(z)$, are as follows.

As one can see, for values of $z < 0.1$, the main contribution to the gravitino propagator results from the weight functions $\alpha(z), \beta(z), \tau(z) = \omega(z)$, whereas the other weight functions are sub-dominating. By contrast, when $z \in]0.8, 1[$, the dominating contribution to the gravitino propagator results from the weight functions $\gamma(z), \delta(z), \varepsilon(z), \theta(z), \pi(z)$, while the others remain sub-dominating.

Rarita–Schwinger potentials

The gravitinos of supergravity are described by spinor-valued one-forms ψ_μ^A, where μ is the Greek index used to denote the one-form nature. Bearing in mind that the soldering form is obtained by contracting the tetrad $e_a^{\hat{c}}$ with the Infeld-van der
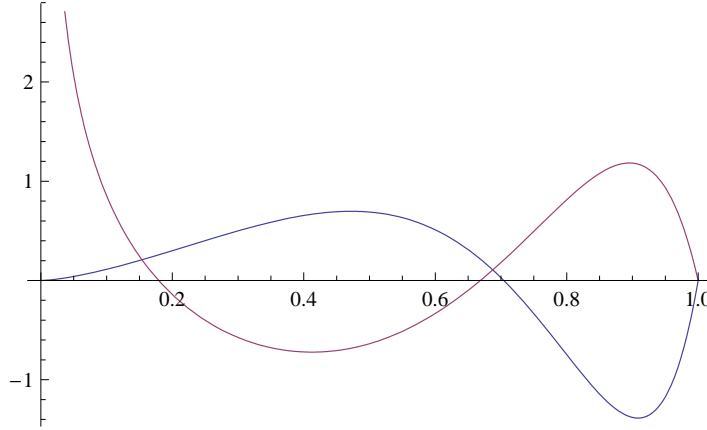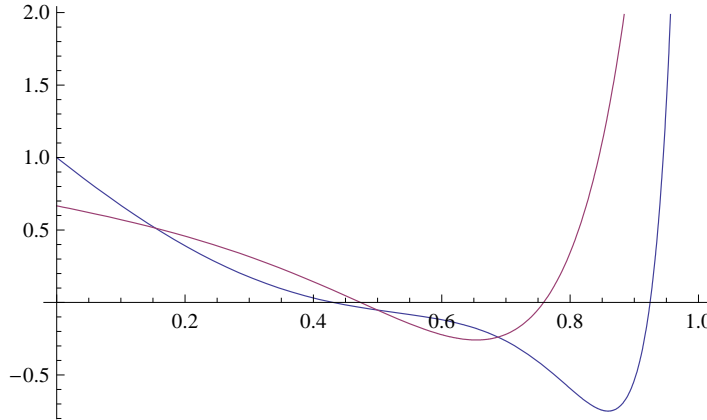
Figure 8.2: Two-dimensional plot of the weight function $\beta(z)$. The curve has two branches, depending on whether one takes the $+$ or $-$ sign in (A2). One branch of β cuts the horizontal z-axis at the points $z = 0, 0.7, 1$, whereas the other branch of β cuts the horizontal axis at the points $z = 0.18, 0.67, 1$. The first branch never cuts the vertical axis, while the other has a vertical asymptote at $z = 0.05$. The two branches intersect each other at $z = 0.15, 0.69, 1$; at these points the function $\beta(z)$ becomes single-valued.

Figure 8.3: Two-dimensional plot of the weight function $\gamma(z)$. The curve has two branches, depending on whether one takes the $+$ or $-$ sign in (A3). The first branch of γ cuts the horizontal z-axis at the points $z = 0.43, 0.92$ and the vertical axis at 1, and then it has a vertical asymptote at $z = 0.95$. The second branch of γ cuts the horizontal axis at the points $z = 0.47, 0.76$ and the vertical axis at 0.67, and then it has a vertical asymptote at $z = 0.87$. The two branches intersect each other at $z = 0.15, 0.5, 0.69$; at these points the function $\gamma(z)$ becomes single-valued. The first branch of γ has an absolute minimum, of negative sign, at $z = 0.85$.
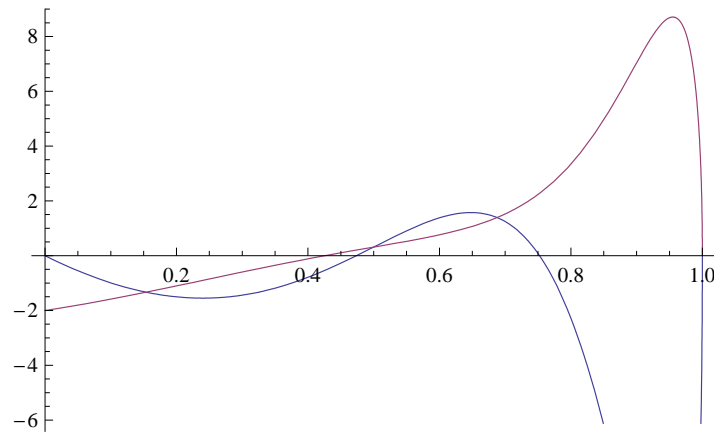
Figure 8.4: Two-dimensional plot of the weight function $\delta(z)$. The first branch cuts the horizontal axis at $z = 0, 0.47, 0.75$, and the second branch cuts the horizontal axis at $z = 0.45, 1$. While the first branch never cuts the vertical axis, the second one cuts it at -2. The two branches intersect each other at $z = 0.15, 0.5, 0.69$, where δ becomes single-valued. The first branch has a vertical asymptote at $z = 0.85$, whereas the second one does have the same at $z = 1$.
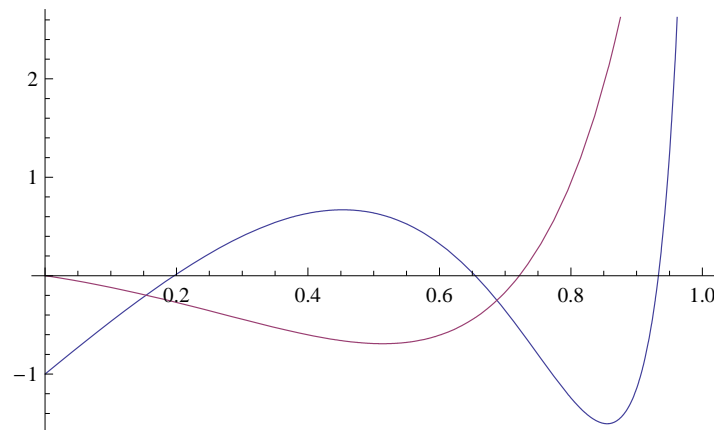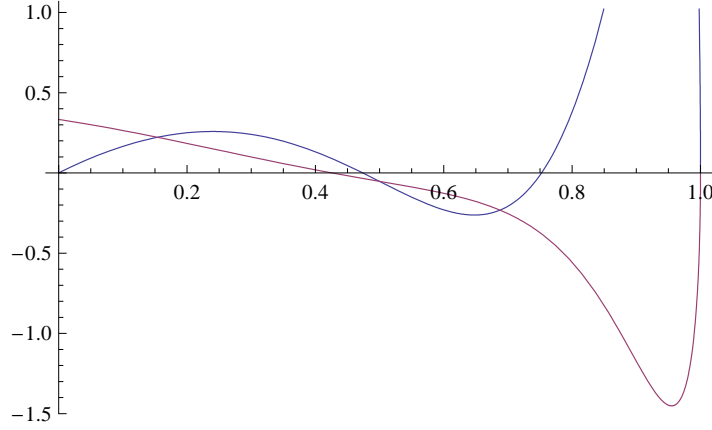

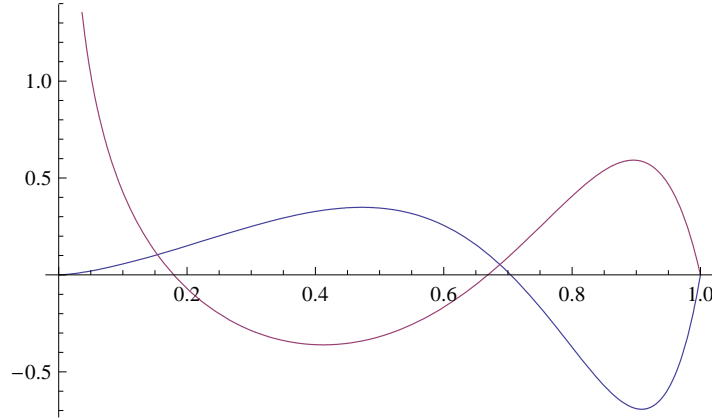Figure 8.5: Two-dimensional plot of the weight function $\varepsilon(z)$. The first branch cuts the horizontal axis at $z = 0.2, 0.65, 0.93$ and cuts the vertical axis at -1. The curve has an absolute minimum, of negative sign, at $z = 0.85$, and then reaches a vertical asymptote at $z = 0.97$. The second branch cuts the z-axis at $z = 0, 0.72$. The two branches intersect each other at $z = 0.15, 0.69$ and at these points ε is a single-valued function. The first branch has a vertical asymptote at $z = 0.95$ whereas the second does the same for z in between 0.85 and 0.9.

Figure 8.6: Two-dimensional plot of the weight function $\theta(z)$. The first branch cuts the horizontal axis at $z = 0, 0.47, 0.75$ and the second cuts the horizontal axis at $z = 0.42, 1$. The first one never cuts the vertical axis, whereas the second one does it at the functional value 0.32. The second branch has a more pronounced absolute minimum, of negative sign, at $z = 0.95$. The two branches intersect each other at $z = 0.15, 0.5, 0.69$, where θ is single-valued. The first branch has a vertical asymptote for z in between 0.85 and 0.9, whereas the second one does the same at $z = 1$.

Figure 8.7: Two-dimensional plot of the weight function $\tau(z) = \omega(z)$. The first branch cuts the horizontal z-axis at the points $z = 0, 0.7, 1$, and it never touches the vertical axis, whereas the second branch cuts the z-axis at $z = 0.18, 0.67, 1$, and it reaches a vertical asymptote at $z = 0.05$. The two branches intersect each other at $z = 0.15, 0.69, 1$, where $\tau(z)$ becomes single-valued.
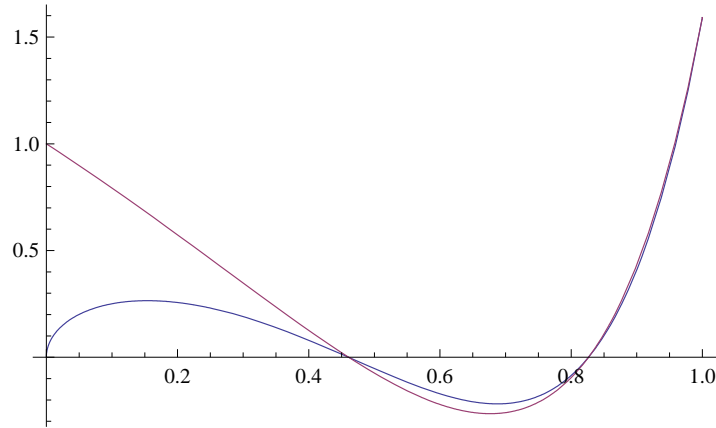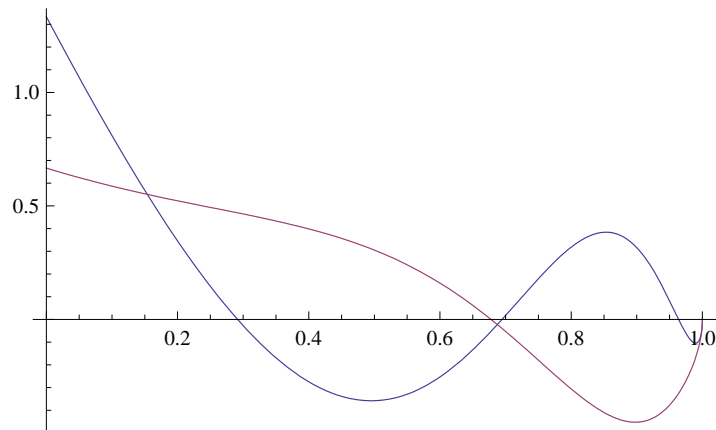
Figure 8.8: Two-dimensional plot of the weight functions $\pi(z)$ and $\tilde{\pi}(z)$. The $\pi(z)$ curve passes through the origin and cuts the horizontal axis at $z = 0.45, 0.83$. The $\tilde{\pi}$ curve never passes through the origin, it cuts the vertical axis at the functional value 1 and it cuts the z-axis at $z = 0.45, 0.83$, where it also intersects the $\pi(z)$ curve. Beyond the point $z = 0.8$ the π and $\tilde{\pi}$ curves become virtually indistinguishable. At $z = 1$ they both have a vertical asymptote.

Figure 8.9: Two-dimensional plot of the weight function $\kappa(z)$. The first branch cuts the z-axis at $z = 0.29, 0.7, 0.97$, while the second one cuts the z-axis at $z = 0.68, 1$. The first branch intersects the vertical axis at the functional value 1.35, whereas the second one does the same at the functional value 0.67. The two branches intersect each other at $z = 0.15, 0.69$, where κ becomes single-valued.

Waerden symbols $\tau_{\hat{c}}^{\ B\dot{B}}$ according to

$$e_a^{\ B\dot{B}} = e_a^{\ \hat{c}}\ \tau_{\hat{c}}^{\ B\dot{B}}, \tag{8.9.0.1}$$

one can write the spatial components of the gravitino in the form

$$\psi_{A\,i} = \Gamma^{\dot{C}}_{\ AB}\ e^{B}_{\ \dot{C}i}, \tag{8.9.0.2}$$

where Γ, the Rarita–Schwinger potential, can be obtained from a spinor field α according to [25]

$$\Gamma^A_{\ B\dot{B}} = \nabla_{B\dot{B}}\ \alpha^A. \tag{8.9.0.3}$$

It obeys the equations [26] (Λ being the cosmological constant, and Φ being the trace-free part of Ricci)

$$\varepsilon^{BC}\nabla_{\dot{A}(A}\ \Gamma^{\dot{A}}_{\ B)C} = -3\Lambda\alpha_A, \tag{8.9.0.4}$$

$$\nabla^{B(\dot{B}}\ \Gamma^{\dot{A})}_{\ BC} = \overline{\Phi}^{\dot{A}\dot{B}L}_{\qquad C}\ \alpha_L, \tag{8.9.0.5}$$

and the gauge-transformation law

$$\widehat{\Gamma}^{\dot{A}}_{\ BC} = \Gamma^{\dot{A}}_{\ BC} + \nabla^{\dot{A}}_{\ B}\ \nu_C. \tag{8.9.0.6}$$

In the Peierls bracket proposed in 8.6.3, the role of ψ_∇ and χ_∇ in 8.5.0.30 will be played by covariant derivatives of such spinor-valued one-forms, or, in purely two-component-spinor language, by spinor covariant derivatives of Rarita–Schwinger potential occurring in 8.9.0.2–8.9.0.6. A part of the existing literature on supergravity prefers instead to omit spinor indices, writing simply ψ_μ for gravitinos. With this notation, one can say that, to the functional derivative $A_{,i}$ in the definition 8.2.0.11 there corresponds the covariant derivative [27]

$$D_\nu\psi_\rho(x) = \partial_\nu\psi_\rho(x) - \Gamma^\sigma_{\ \nu\rho}\ \psi_\sigma(x) + \frac{1}{2}\omega_{\nu ab}\sigma^{ab}\psi_\rho(x), \tag{8.9.0.7}$$

where $\Gamma^\sigma_{\ \nu\rho}$ are the Christoffel symbols, $\omega_{\nu ab}$ is the spin-connection, and σ_{ab} is proportional to the commutator of "flat" γ-matrices, i.e.

$$\sigma_{ab} = \frac{1}{4}[\gamma_a,\gamma_b]. \tag{8.9.0.8}$$

8.10   Concluding remarks

Our papers [1; 2] have been devoted to geometric constructions of current interest in theoretical physics. Its original contributions, of structural nature, are as follows:

(i) A two-component-spinor analysis of geometric invariants contributing to the gravitino propagator in four-dimensional de Sitter spacetime.

(ii) A Peierls bracket for massive spin-1/2 and spin-3/2 fields in de Sitter spacetime has been proposed, by relying upon the same tools as in item (i) above.

Our use of positive- and negative-frequency Green functions to re-express the Peierls bracket is also of some interest, by virtue of the more direct link with the Feynman Green function. It now remains to be seen whether our brackets can be exploited to study quantum field theories in de Sitter spacetime from a modern perspective.

We were able to obtain the complete analytic structure of massive gravitino propagators in de Sitter space. In 8.9 we have plotted all weight functions $\alpha, \beta, \gamma, \delta, \epsilon, \theta, \tau = \omega, \pi, \kappa$ which appear in the gravitino propagator (jointly with $\tilde{\pi}$) as a function of z in a two-dimensional plot where $z = \cos^2(\mu/2R)$, μ being the geodesic distance between the points x and x', and R is the de Sitter radius. Although the series 8.7.3.2 has been truncated, it remains true that 8.9 is the first attempt to display a supersymmetric propagator in de Sitter via Heun functions. As is well-known, further interest, from the point of view of mathematical methods, arises from the possibility to expand Heun functions in terms of hypergeometric functions [29]. As we said before, direct implications of our findings on the current understanding of the propagation of gravitinos in de Sitter space are as follows: there exist two ranges of values of z in which the weight functions can be divided into dominating and sub-dominating family. In other words, when z is smaller than 0.1, the weight functions $\alpha, \beta, \tau = \omega$ are dominating while the others are sub-dominating. By contrast, when z is very close to 1, the weight functions $\gamma, \delta, \varepsilon, \theta, \pi$ take much larger values.

The plot range is between 0 and 1 for z, which is indeed the only admissible region, since the squared cos function lies always between 0 and 1. Note that the plot of $\tilde{\pi}$ is basically nothing but the plot of the Heun function with properly defined coefficients, and the plot of π is $\sqrt{z}$ times the Heun function.

The flat-space limit is instead a considerable simplification, since the functions A and C in 8.8.0.1 are then found to reduce to $A = \frac{1}{\mu}, C = -\frac{1}{\mu}$, and the formulae for the weight functions are therefore considerably simplified.

It also remains to be seen whether the familiarity acquired with Heun functions will prove useful in studying gravitino propagators in other backgrounds relevant for modern high energy physics.

Bibliography

[1] Esposito, G., Roychowdhury, R., arXiv:0902.2098 [hep-th], *Int. J. Geom. Meth. Mod. Phys.* **6** (2009) 1069.

[2] Esposito, G., Roychowdhury, R., arXiv:0907.3634 [hep-th], *Gen. Rel. Grav.* **42** 1221.

[3] J. M. Maldacena, *Adv. Theor. Math. Phys.* **2** (1998) 231 (*Preprint* hep-th/9711200).

[4] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Phys. Lett. B* **428** (1998) 105 (*Preprint* hep-th/9802109).

[5] E. Witten, *Adv. Theor. Math. Phys.* **2** (1998) 253 (*Preprint* hep-th/9802150).

[6] A. Strominger, JHEP **10** (2001) 034 (*Preprint* hep-th/0106113).

[7] M. Spradlin and A. Volovich, *Phys. Rev. D* **65** (2002) 104037 (*Preprint* hep-th/0112223).

[8] G. Bimonte, G. Esposito, G. Marmo and C. Stornaiolo, *Int. J. Mod. Phys. A* **18** (2003) 2033 (*Preprint* hep-th/0301113).

[9] R. E. Peierls, *Proc. R. Soc. Lond. A* **214** (1952) 143.

[10] B. S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon & Breach, New York, 1965).

[11] B. S. DeWitt, *The Global Approach to Quantum Field Theory*, *International Series of Monographs on Physics* **114** (Clarendon Press, Oxford, 2003); B. S. DeWitt in: *50 Years of Yang–Mills theory*, ed. G. 't Hooft (World Scientific, Singapore, 2005).

[12] G. Esposito and C. Stornaiolo, *Int. J. Geom. Meth. Mod. Phys.* **4** (2007) 349 (*Preprint* hep-th/0607114).

[13] B. Allen and T. Jacobson, *Commun. Math. Phys.* **103** (1986) 669.

[14] B. Allen and C. A. Lutken, *Commun. Math. Phys.* **106** (1986) 201.

[15] W. Mück, *J. Phys. A* **33** (2000) 3021 (*Preprint* hep-th/9912059).

[16] J. L. Synge, *Relativity: The General Theory* (North–Holland, Amsterdam, 1960).

[17] R. Penrose and W. Rindler, *Spinors and Space-Time. I* (Cambridge University Press, Cambridge, 1984).

[18] M. Abramowitz, I. A. Stegun: Handbook of mathematical functions (Dover, New York, 1964).

[19] A. Erdelyi: Higher transcendental functions (Krieger, Malabar, 1981).

[20] L. Anguelova and P. Langfelder, JHEP **03** (2003) 057.

[21] A. Basu, L. I. Uruchurtu, *Class. Quantum Grav.* **23** (2006) 6059.

[22] *Handbook of exact solutions for ordinary differential equations* (CRC Press, Boca Raton, 1995).

[23] A. Ronveaux ed., *Heun's Differential Equations* (OUP, Oxford, 1995).

[24] Y. Gaspar, *Preprint* math-ph/0305050.

[25] P. C. Aichelburg and H. K. Urbantke, *Gen. Rel. Grav.* **13** (1981) 817.

[26] G. Esposito, *Preprint* gr-qc/9911051.

[27] Freedman, van Nieuwenhuizen and Ferrara, *Phys. Rev. D* **13** (1976) 3214.

[28] Witten, E.: hep-th/0106109

[29] Sokhoyan, Melikdzanian, Ishkhanyan, et.al. arXiv:0909.1286 [math-ph]

[30] Allen, B.: Nucl. Phys. B **292**, 813 (1987)

[31] Kamke, E.: Differentialgleichungen, Lösungsmethoden und Lösungen. Vol. 1. Chelsea, New York (1974)

[32] Poole, E.G.C.: Linear Differential Equations. OUP, Oxford (1936)

[33] Snow, C. Hypergeometric and Legendre Functions with Applications to Integral Equations of Potential Theory, 2nd Edition, no. 19 in Applied Mathematics Series, National Bureau of Standards, Washington DC (1952)

[34] Andrews, G.E., Askey, R., Roy, R.: Special Functions, Vol. 71 of Encyclopedia of Mathematics and Its Applications. Cambridge University Press, Cambridge (1999)

Chapter 9

General summary and outlook

In this thesis, we have analyzed the Attractor Mechanism in various contexts, each with specific relevant features.

i) The $\mathcal{N} = 2$, $d = 4$ dilatonic extremal Reissner-Nördstrom BH arising from heterotic string theories.

ii) The $\mathcal{N} = 2$, $d = 4$, n_V-fold Maxwell-Einstein Supergravity theory (Maxwell Einstein Supergravity Theory), i.e. the $\mathcal{N} = 2$, $d = 4$ SUGRA coupled to n_V Abelian vector supermultiplets; the moduli space of such a theory has a special Kähler-Hodge geometry with additional symplectic structure.

We have discussed [1; 2] various aspects of the cubic special Kähler Geometry and treated at length the issues related to quantum perturbative corrections to the $\mathcal{N} = 2$ prepotential function. We were able to show that all such sub-leading correction terms can be either introduced or removed by means of Peccei-Quinn Symplectic transformations, leading to several interesting phenomena e.g.transitions between large and small BH charge orbits. For non-supersymmetric attractors we invoked the concept of sectional curvature to write down the Bekenstein-Hawking entropy using Attractor Mechanism. We also mention the intriguing fact of BH/Qubit correspondence, a concept developed at the interplay between Black Hole physics and Quantum Information theory.

In the final part of this thesis, we illustrated [3; 4] the complete analytic structure of the massive gravitino propagator in four-dimensional de Sitter space using the two-component spinor formalism à la Penrose. We also gave a prescription on how to divide the weight functions into dominant and sub-dominant families depending on the ranges of values taken by the ratio of geodesic distance and the de Sitter radius.

An outline for future work is tentatively the following.

In [5] the "Entropy function formalism" was developed and the complete set of higher derivative terms was considered in relation to the near-Horizon geometry $AdS_2 \times S^{d-2}$ of a d-dim. extremal BH, by applying the general formalism elaborated by Wald *et al.* in [6]-[9].

Sen's elaboration of the Wald *et al.*'s higher-order derivative Riemannian formalism is based on a set of working assumptions, which are, respectively

- asymptotic flatness,

- spherical symmetry,

- Abelian gauge fields.

Moreover, Sen's results rely on the assumption that the Lagrangian density can be expressed only in terms of gauge-invariant field strengths, and does not explicitly involve the related gauge fields. Such a condition is clearly violated in presence of Chern-Simons terms. If such terms can't be removed by switching to dual field variables, the results obtained in [5] still hold true if the additional Chern-Simons terms do not affect, for some reasons, the Eqs. of motion and the entropy of the particular BH solution under consideration.

A progressive study, by relaxing some of the working hypotheses might lead to further generalizations, which should hopefully allow one to gain new interesting insights into the Attractor Mechanism dynamics in supersymmetric and non-supersymmetric frameworks.

Below we give a list (far from being exhaustive) of just of some of the possible directions that appear to be a natural extension of the results briefly reported in this thesis, mostly being related to the removal of some hypotheses made in our treatment.

I) *Removal of the hypothesis of asymptotic flatness of metric backgrounds.*

Asymptotically non-flat (maximal) SUGRAs, in general corresponding to (maximal) gauged SUGRAs, do deserve a completely different treatment w.r.t. their asymptotically flat, ungauged counterparts.

For instance in 4 we treated the ungauged $\mathcal{N} = 2$, $d = 4$ Maxwell Einstein Supergravity Theory, i.e. the asymptotically flat, non-maximal $\mathcal{N} = 2$, $d = 4$ SUGRA coupled to n_V Abelian vector supermultiplets, and possibly to n_H hypermultiplets, too. In such a case we have seen that the scalar fields coming from the hypermultiplets are not fixed at the Event Horizon, because the central charge of the local SUSY algebra does not depend on the asymptotical configuration of these fields. Thus,

such scalars are completely decoupled from the dynamical behavior of the system, and they do remain moduli of the theory also in the asymptotical "near-Horizon" radial evolution of the considered extremal (spherically symmetric) BH.

Instead, in the non-maximal gauged SUGRAs *all* the scalars coming from the field contents of the extra matter multiplets coupled to the SUGRA one should be taken into account, including the ones related to the hypermultiplets, which now cannot be decoupled from the dynamics of the system.

Asymptotically AdS backgrounds have been quite extensively considered in the literature, also in their relation with string theories. In [10] some advances were made in the study of the Attractor Mechanism in such backgrounds, also in the de Sitter (dS) case. The obtained results are quite general, because they do not rely on SUSY, but nevertheless some other aspects of the Attractor Mechanism in asymptotically (A)dS background still wait for a detailed examination.

In particular, the moduli space dynamics related to the radial evolution of the scalars of the hypermultiplets coupled to asymptotically non-flat, non-maximal (sph. symmetric) $\mathcal{N}$-extended, d-dim. SUGRAs (e.g. to the spherically symmetric, asymptotically AdS, gauged $\mathcal{N} = 2$, $d = 4$ Maxwell Einstein Supergravity Theory) have not been considered yet, but a detailed study appears to be an interesting direction of development to be pursued.

II) *Removal of the hypotheses of spherically symmetry and /or staticity.*

All the extremal BH solutions considered in our treatment, and in most of the literature, have spherical symmetry. That is why we always considered only the evolution flow in the moduli space (see [11] for attractor flows in st^2 model for example) which was related to radial dynamics of the relevant set of scalar fields.

The study of non-spherically symmetric singular metric solutions in the context of SUGRAs should naturally lead to the "merging" of the radial and angular dynamics, and consequently to a deeper understanding of the Attractor Mechanism, possibly involving both of them.

Also the removal of the hypothesis of staticity (i.e. time-independence) of the considered solutions should shed some new light on interesting aspects. Some spinning (for example, Kerr-Newman-like) BH metrics could be considered, and their possible interpolating solitonic nature could be investigated, together with the possibility to obtain higher-dimensional spinning extensions of such backgrounds. Recently found Kerr/CFT correspondence [12] might turn out to be a good playground to test the Attractor properties.

III) *Attractor Mechanism in higher dimensions and Black Rings.*

Reasonably, the removal of the basic hypotheses about the structure of the BH metrics should possibly determine a modification of the Attractor Mechanism itself, as recent works seem to point out.

Indeed, a deeper, recently gained understanding of the BPS equations in SUGRA ([13]-[15]) has led to new examples of general solutions, such as 5-dim. Black Rings ([16] -[26]). Also multi-centered BHs in four dimensions have been considered ([27]-[30]), and they share some of the features of their 5-dim. ringy counterparts[1].

For these new classes of singular metrics the entropy turns out to be function not only of the conserved (quantized) charges related to a certain number of (Abelian) gauge symmetries exhibited by the low-energy effective SUGRA theory, but it also depends on the values of the *dipole charges*. These are non-conserved quantities, which may be defined by flux integrals on particular surfaces linked with the ring.

Thus, it could be reasonably conjectured that the near-Horizon, "attracted" configurations of the moduli should in this case *also* depend on the dipole charges.

Rather intriguingly, they actually turn out to be *exclusively* dependent on the dipole charges ([35]-[37]).

Consequently, for Black Rings the Attractor Mechanism cannot be related to some kind of Extremum Principle involving the central charge, because such a quantity depends on the conserved charges, and not on the dipole charges. In [35] Larsen and Kraus formulated a new Extremum Principle for the Attractor Mechanism in 5-dim. Black Ring solutions, in which a certain function of the dipole charges plays a key role.

A general analysis revealed the existence of two general classes of solutions, whose internal, near-Horizon dynamics is governed by the universal Attractor Mechanism, realized differently in terms of Extremum Principles for different functions of different charges.

---

[1]An interesting line of research on Black Ring solutions in 5-d. SUGRA has recently been pursued by Strominger *et al.* .

By using M-theory, in [25] Cyrier, Guica, Mateos and Strominger exploited the microscopic interpretation of the entropy of a recently discovered new Black Ring solution in 5-d. SUGRA.

Moreover, as previously mentioned, in [31] Gaiotto, Strominger and Yin proposed a simple relation between $Z_{BH}^{d=4}$ and $Z_{BH}^{d=5}$ based on the demonstration that the M-theory lift of a 4-d. CY Type IIA BH is a 5-d. BH spinning at the center of a Taub-NUT-flux geometry. Such a result on M-theory liftings was then further generalized to the case of 4-d. multi-BH geometries, which in [32] were shown to correspond to 5-d. Black Rings in a Taub-NUT-flux geometry (see also [33] and [34] for related further developments).

The discriminating, key point is the vanishing or not of certain components of the field strengths (the so-called *dipole field strengths*). The framework corresponding to non-vanishing *dipole field strengths* represents a new arena to generalize the possible realizations of the Attractor Mechanism.

As a final note, we want to make our final comments in view of the recent flurry of activities in the field of AdS/CMT (For review on this topic see [38–41]). In recent papers, [42; 43], the authors have studied extremal and non-extremal black holes and black branes in dilatonic gravity. The study is quite interesting in the sense that these extremal black hole/brane configurations exhibit the attractor mechanism regardless of supersymmetry; their near horizon geometry is universal and independent of the asymptotic values of the moduli. In such a setting different kinds of attractors correspond to different kinds of universal behavior. In the context of AdS/CFT characterizing different kinds of attractors tells us about the different kinds of IR behavior which can arise in the dual CFT which is at zero temperature but can be deformed by the addition of a chemical potential (or charge). This will be clearly of great interest in order to further develop the AdS/CFT dictionary.

Then again, as the popular saying indicates, *the road to success is always under construction,* and perhaps this is also valid for the road to AdS/CFT and thus the road to quantum gravity.

As Karl Popper remarked, "Our knowledge can only be finite, while our ignorance must necessarily be infinite".

Bibliography

[1] S. Bellucci, A. Marrani and R. Roychowdhury, *On Quantum Special Kähler Geometry*, Int. J. Mod. Phys. **A25**, 1891 (2010), `arXiv:0910.4249 [hep-th]`.

[2] S. Bellucci, A. Marrani and R. Roychowdhury, *Topics in Cubic Special Geometry*, `arXiv:1011.0705 [hep-th]`.

[3] G. Esposito, R. Roychowdhury, *Spinor Two-Point Functions and Peierls Bracket in de-Sitter Space*, Int. J. Geom. Meth. Mod. Phys. **6** 1069, (2009) `arXiv:0902.2098 [hep-th]`.

[4] G. Esposito, R. Roychowdhury, *On the Complete Analytic Structure of the Massive Gravitino Propagator in Four-Dimensional de-Sitter Space*, Gen. Rel. Grav. **42** 1221, `arXiv:0907.3634 [hep-th]`.

[5] A. Sen, *"Black Hole Entropy Function and the Attractor Mechanism in Higher Derivative Gravity"*, JHEP **0509**, 038 (2005), `hep-th/0506177`.

[6] R. M. Wald, *"Black hole entropy in the Noether charge"*, Phys. Rev. **D48**, 3427 (1993), `gr-qc/9307038`.

[7] T. Jacobson, G. Kang and R. C. Myers, *"On Black Hole Entropy"*, Phys. Rev. **D49**, 6587 (1994), `gr-qc/9312023`.

[8] V.Iyer and R. M. Wald, *"Some properties of Noether charge and a proposal for dynamical black hole entropy"*, Phys. Rev. **D50**, 846 (1994), `gr-qc/9403028`.

[9] T. Jacobson, G. Kang and R. C. Myers, *"Black hole entropy in higher curvature gravity"*, `gr-qc/9502009`.

[10] K. Goldstein, N. Iizuka, R. P. Jena and S. P. Trivedi, *"Non-Supersymmetric Attractors"*, Phys. Rev. **D72**, 124021 (2005), `hep-th/0507096`.

[11] R. Roychowdhury, *Attractor Flows in st^2 Black Holes*, `arXiv:0910.2609 [hep-th]`.

[12] M. Guica, T. Hartman, W. Song and A. Strominger, *The Kerr/CFT Correspondence*, `arXiv:0809.4266 [hep-th]`.

[13] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis and H. S. Reall, *"All su- persymmetric solutions of minimal supergravity in five dimensions"*, Class. Quant. Grav. **20**, 4587 (2003), `hep-th/0209114`.

[14] J. B. Gutowski and H. S. Reall, *"General supersymmetric AdS$_5$ black holes"*, JHEP **0404**, 048 (2004), `hep-th/0401129`.

[15] J. B. Gutowski, *"Uniqueness of five-dimensional supersymmetric black holes"*, JHEP **0408**, 049 (2004), `hep-th/0404079`.

[16] H. Elvang, R. Emparan, D. Mateos and H. S. Reall, *"A supersymmetric black ring"*, Phys. Rev. Lett. **93**, 211302 (2004), `hep-th/0407065`.

[17] I. Bena and N. P. Warner, *"One ring to rule them all...and in the darkness bind them?"*, `hep-th/0408106`.

[18] H. Elvang, R. Emparan, D. Mateos and H. S. Reall, *"Supersymmetric black rings and three-charge supertubes"*, Phys. Rev. **D71**, 024033 (2005), `hep-th/0408120`.

[19] J. P. Gauntlett and J. B. Gutowski, *"Concentric Black Rings"*, Phys.Rev. **D71**, 025013 (2005), `hep-th/0408010`.

[20] J. P. Gauntlett and J. B. Gutowski, *"General Concentric Black Rings"*, Phys. Rev. **D71**, 045002 (2005), `hep-th/0408122`.

[21] I. Bena, C. W. Wang and N. P. Warner, *"Black rings with varying charge density"*, `hep-th/0411072`.

[22] I. Bena and P. Kraus, *"Three charge supertubes and black hole hair"*, Phys. Rev. **D70**, 046003 (2004), `hep-th/0402144`.

[23] I. Bena, *"Splitting hairs of the three charge black hole"*, Phys. Rev. **D70**, 105018 (2004), `hep-th/0404073`.

[24] I. Bena and P. Kraus, *"Microscopic description of black rings in AdS/CFT"*, JHEP **0412**, 070 (2004), `hep-th/0408186`.

[25] M. Cyrier, M. Guica, D. Mateos and A. Strominger, *"Microscopic Entropy of the Black Ring"*, Phys. Rev. Lett. **94**, 191601 (2005), `hep-th/0411187`.

[26] I. Bena, C.-W. Wang and N. P. Warner, *"Sliding Rings and Spinning Holes"*, `hep-th/0512157`.

[27] F. Denef, *"Supergravity flows and D-brane stability"*, JHEP **0008**, 050 (2000), `hep-th/0005049`.

[28] F. Denef, *"On the correspondence between D-branes and stationary supergrav- ity solutions of Type II Calabi-Yau compactifications"*, talk presented at the Workshop on Strings, Duality and Geometry, CRM Montreal, March 2000,

`hep-th/0010222`.

[29] F. Denef, *"(Dis)assembling special Lagrangians"*, `hep-th/0107152`.

[30] B. Bates and F. Denef, *"Exact solutions for supersymmetric stationary black hole composites"*, `hep-th/0304094`.

[31] D. Gaiotto, A. Strominger and X. Yin, *"New Connections between 4D and 5D Black Holes"*, `hep-th/0503217`.

[32] D. Gaiotto, A. Strominger and X. Yin, *"5D Black Rings and 4D Black Holes"*, `hep-th/0504126`.

[33] I. Bena, P. Kraus and N. P. Warner, *"Black Rings in Taub-NUT"*, `hep-th/0504142`.

[34] K. Behrndt, G. L. Cardoso and S. Mahapatra, *"Exploring the relation between 4D and 5D BPS solutions"*, `hep-th/0506251`.

[35] P. Kraus and F. Larsen, *"Attractors and Black Rings"*, Phys. Rev. **D72**, 024010 (2005), `hep-th/0503219`.

[36] F. Larsen, *"Entropy of Thermally Excited Black Rings"*, `hep-th/0505152`.

[37] I. Bena and P. Kraus, *"R^2 Corrections to Black Ring Entropy"*, `hep-th/0506015`.

[38] S. Sachdev, *"Condensed matter and AdS/CFT"*, `arXiv:1002.2947`.

[39] S.A. Hartnoll, *"Quantum Critical Dynamics from Black Holes"*, `arXiv:0909.3553`.

[40] S.A. Hartnoll, *"Lectures on holographic methods for condensed matter physics"*, *Class. Quant. Grav.* **26** (2009) 224002, `arXiv:0903.3246`.

[41] J. McGreevy, *"Holographic duality with a view toward many-body physics"*, `arXiv:0909.0518`.

[42] K. Goldstein, S. Kachru, S. Prakash and S. Trivedi, *"Holography of charged dilaton Black Holes"*, `arXiv:0911:3586`.

[43] K. Goldstein, N. Iizuka, S. Kachru, S. Prakash, S. Trivedi and A. Westphal, *"Holography of dyonic dilaton Black Branes"*, `arXiv:1007:2490`.