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Nonlinear stability of nonautonomous Lotka-Volterra models.

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Introduction

This thesis deals with the study of the stability for the equilibria of ordinary differential nonautonomous systems generalizing the classical Lotka-Volterra bidimensional predator-prey model.

The classical predator-prey model was introduced by A. J. Lotka in 1925 and successively by V. Volterra in 1926, to explain the increasing of some predator fishes in the Atlantic during the World War I. Indicated with x an y the preys and predators densities, respectively, the system is the following

$$\begin{cases} \dot{x} = ax - bxy, \\ \\ \dot{y} = -cy + dxy, \end{cases}$$

with a, b, c, d positive constants. The meaningful equilibrium point is $(\overline{x}, \overline{y}) = \left(\frac{c}{d}, \frac{a}{b}\right)$ and it is (simply) stable in the sense of Liapunov. For this model the existence of cycles is proved. The cycles are very important from a biological point of view for two reasons, since they assure

- the survival of both species;
- the existence of a positive constant T such that

  $$x(t+T) = x(t), \qquad y(t+T) = y(t),$$

  i.e. the solution is periodic.

However, the hypotheses at the base of this model are unrealistic since, for example, they don't take in account of limitation of time and space resources, or don't consider how the random effects in the environment influence the interaction between the two population. Hence the problem to modify the assumptions at the basis of the model arises.

During the years, a lot of variations on the Lotka-Volterra model have been studied. Some perturbations to the model consist in to introduce a dependence on time in the coefficients of the classical model; others add some functions to the equations of the Lotka-Volterra model. In these last cases the pertubed models are of the type

$$\dot{x} = (a - by)x + F_1, \qquad \dot{y} = (-c + dx)y + F_2,$$

where $F_i\,(i = 1, 2)$, are sufficiently smooth in order to guarantee that there is an unique solution existing globally in time.

Some authors ([16]-[19]) consider the functions $F_i(i = 1, 2)$ independent of time and depending on a small parameter ε, others ([13]-[15], [41], [44]) study nonautonomous perturbed models.

In most of the works, the nonautonomous perturbations introduced are of the kind

$$F_1(t) = D_1(t)(y - x), \qquad F_2(t) = D_2(t)(x - y),$$

since these terms take in to account of the “diffusion” of the populations among heterogeneous patches.

By using the Liapunov Direct Method, the nonlinear stability of the biological meaningful equilibrium $(\overline{x}, \overline{y})$ has been studied under hypotheses of positive perturbations and $\dfrac{c}{d} = \dfrac{a}{b}$.

Now we consider classes of generalized Lotka-Volterra models, with either coefficients or perturbation functions F_i, $(i = 1, 2)$, depending on time, to both equations of the classical system. These functions F_i are chosen in a way that the perturbed model admits the same equilibrium $(\overline{x}, \overline{y})$ of the classical system. We don't make any assumption on the sign of the perturbations F_i, but we only suppose that all the hypotheses assuring the global existence in time and uniqueness of the solutions, are satisfied.

The aim of this thesis is to study the influence of nonautonomous perturbation terms

on the asymptotic behaviour of the solutions around $(\overline{x}, \overline{y})$.
In particular it is shown that there are conditions on the perturbations assuring the non linear (local) asymptotic stability. When these conditions hold, then the cycles are not possible and the solutions can't be periodic even if all the perturbations considered are periodic of the same period.

The thesis is organized as follows.

In Chapter 1, a series of concepts and definitions on the dynamical systems are furnished.
In Chapter 2, the Liapunov Direct Method for autonomous and nonautonomous ordinary differential equations is introduced. In particular, differences between these two cases are focused.
In Chapter 3, the classical bidimensional predator-prey model is presented and some properties on the stability of the equilibrium point $(\overline{x}, \overline{y})$ are given. It is also considered the Lotka-Volterra model in the case of a logistic growth for the preys in absence of the predators.
In Chapter 4-5, some generalized nonautonomous Lotka-Volterra models are presented. In particular, Chapter 4 concerns perturbations to the model of the type

$$\dot{x} = f_1(t)(a - by)x + F_1(t), \qquad \dot{y} = f_2(t)(-c + dy)x + F_2(t),$$

where $f_i, F_i (i = 1, 2)$ are sufficiently smooth to assure the global existence in time and uniqueness of the solutions, and F_i $(i = 1, 2)$ depending on time and on the difference of the populations densities in a nonlinear way.
In Chapter 5, different perturbed models are introduced. They came from considering higher nonlinearities in the classical nonautonomous Lotka-Volterra model, and hence

give rise to the following model

$$\begin{cases} \dot{x} = af_1(t)x - bf_1(t)\dfrac{x^{1+p}y^{1+q}}{\overline{x}^p\overline{y}^q}, \\ \\ \dot{y} = -cf_2(t)y + df_2(t)\dfrac{x^{1+p}y^{1+q}}{\overline{x}^p\overline{y}^q}, \end{cases}$$

under the hypotheses which assure the existence and uniqueness of the solution and $(p, q) \in \mathbb{N}^2$.

In Chapter 6, a nonautonomous perturbed Lotka-Volterra model is studied in the case of a logistic growth for the preys in absence of the predators. Also in this case, the perturbations are chosen in order to guarantee that the perturbed model admits the same equilibria of the classical Lotka-Volterra model with logistic growth for the preys. Conditions on the perturbations which lead to the extinction of the species are furnished.

Our analysis is based essentially on the Liapunov Direct Method and on the use of Liapunov functions depending - together with the temporal derivative along the solutions - on the eigenvalues of the problem in a simple direct way [38].

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Chapter 1

Dynamical systems and ordinary differential equations

An *observable phenomenon* is a source of notable signals. In the study of a phenomenon there are at least three phases which can be distinguished:

1) the first consists in observing the phenomenon, selecting the meaningful parameters adapt to describe the *state* of the phenomenon and determining a numerical sequence of their values during a large interval of time (experimental data);

2) the second phase deals with the construction of a mathematical model governing the aforesaid parameters and in analyzing the model;

3) the third phase consists in a validation of the model through the comparison with the experimental data.

Let $x_i \, (i = 1, ..., n)$ be the fundamental parameters selected during the phase 1. The vector $\mathbf{x} = (x_1, ..., x_n)$ is the *vector state* of the phenomenon. In the general case, x_i are functions of time and space. Here and in the sequel we deal with vector state depending only on time. A phenomenon is governed by ordinary differential equations

(ODEs) of first order if the vector state $\mathbf{x}$ verifies equations of the type

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(t, \mathbf{x}(t)), \tag{1.1}$$

where $\mathbf{F}$ is a well known function defined in a subset of $\mathbb{R}^{n+1}$ and takes values in $\mathbb{R}^n$. If $\mathbf{F}$ doesn't depend explicitly on time, the model (1.1) is said *autonomous*, in the other case it is said *nonautonomous*. (1.1) is also called *evolution equation* since it allows to understand how the vector state evolves in time.
Given a model, one has to verify the *well posedness* according to Hadamard, i.e.

- the solution of the model exists globally in time, that is for every (finite) interval of time;
- once linked to the model the initial data (the vector state in an initial time), the solution has to be unique;
- the solution depends continuously on the initial data, i.e. sufficiently small changes in the initial data make the consequential change in the solution arbitrarily small.

1.1 Existence and uniqueness for the solutions of ordinary differential equations

Let be $\mathcal{I} =]\tau, \infty[$ and

$$\mathbf{F} : (t, \mathbf{x}) \in \mathcal{I} \times D \to \mathbf{F}(t, \mathbf{x}) \in \mathbb{R}^n,$$

with D an open subset of $\mathbb{R}^n$ containing the origin. The problem

$$\begin{cases} \dfrac{d\mathbf{x}}{dt} = \mathbf{F}(t, \mathbf{x}), \\ \\ \mathbf{x}(t_0) = \mathbf{x}_0, \end{cases} \tag{1.2}$$

where $(t_0, \mathbf{x}_0) \in \mathcal{I} \times D$, is called *Cauchy problem.* It consists in determining a function $\mathbf{x}$ of t derivable, which verifies $(1.2)_1$ together with the initial data $(1.2)_2$. Hence it is useful to recall the conditions on the function $\mathbf{F}$ guaranteeing the existence and the uniqueness of the solutions.

Let's set

$$\mathcal{R}^* = \{x_{0i} - b \leq x_i \leq x_{0i} + b,\, t_0 - a \leq t \leq t_0 + a\}, \tag{1.3}$$

with a, b positive constants such that $\mathcal{R}^* \subset \mathcal{I} \times D$.

Definition 1.1 *The function* $\mathbf{F}$ *verifies the uniform Lipschitz condition in* $\mathcal{R}^*$ *with respect to* $\mathbf{x}$ *if there exists a positive constant* k *such that*

$$|\mathbf{F}(\mathbf{x}_2, t) - \mathbf{F}(\mathbf{x}_1, t)| \leq k|\mathbf{x}_2 - \mathbf{x}_1|, \quad \forall (x_i, t) \in \mathcal{R}^* \, (i = 1, 2). \tag{1.4}$$

The conditions on $\mathbf{F}$ to guarantee the (local) existence and uniqueness of the solution of the Cauchy problem, are summarized in the following theorem [24].

Theorem 1.1 *If* $\mathbf{F}$ *is continuous in* $\mathcal{R}^*$ *and verifies the uniform Lipschitz condition, then there exists an unique function* $\mathbf{x}(t)$ *continuous and derivable in* $[t_0 - \delta, t_0 + \delta]$, *with*

$$\delta = \min\left(a, \frac{b}{M}\right), \quad M \geq \sup_{\mathcal{R}^*} |\mathbf{F}|, \tag{1.5}$$

verifying (1.2).

The continuity is only sufficient to prove the local existence but not the uniqueness.

The theorem 1.1 provides conditions for the local existence of the solution, i.e. the existence in the interval $[t_0 - \delta, t_0 + \delta]$. The existence of the solution in the whole temporal interval in which t can assume values in the definition of the function $\mathbf{F}$, is called the *global existence.*

The global existence theorems are the following.

Theorem 1.2 *If* $\mathbf{F}$ *is defined in* $S = [t_0 - a, t_0 + a] \times \mathbb{R}^n$*, with* a *a positive constant, and one of the following hypotheses is verified*

- $\mathbf{F}$ *is continuous and bounded in* S *and verifies the Lipschitz condition in every rectangular domain contained in* S*;*
- $\mathbf{F}$ *is continuous in* S *and verifies the uniform Lipschitz condition in* S*,*

then there exists a global solution of (1.2).

Theorem 1.3 *If* $\mathbf{F}$ *is defined in* $\mathbb{R}^{n+1}$ *and in every strip* $S = [t_0 - \alpha, t_0 + \alpha] \times \mathbb{R}^n$*, with* α *a positive constant, and the hypotheses of theorem 1.2 are satisfied, then (1.2) admits an unique solution defined in* $] - \infty, \infty[$.

Theorem 1.4 *If* $\mathbf{F}$ *is defined in* $[\bar{t}, \infty[\times \mathbb{R}^n$ *and in every strip* $S = [t_0 - \alpha, t_0 + \alpha] \times \mathbb{R}^n$*, with* α *positive constant, and the hypotheses of theorem 1.2 are satisfied, then (1.2) (with* $t_0 > \bar{t}$*) admits an unique solution defined in* $[t_0, \infty[$.

Theorem 1.5 *If* $\mathbf{F}$ *is defined and continuous in the strip* $S = [t_0, t_1] \times \mathbb{R}^n$ *and verifies the Lipschitz condition in every rectangle* $\mathcal{R}^* \subset S$ *and if a solution of the problem (1.2) is bounded, then it exists in* $[t_0, t_1]$.

Theorem 1.6 *If* $\mathbf{F}$ *is defined and continuous in the strip* $\mathcal{I} = [t_0, \infty[\times \mathbb{R}^n$ *and verifies the Lipschitz condition in every rectangle* $\mathcal{R}^* \subset \mathcal{I}$*, then every bounded solution of the problem (1.2) is global (in time).*

1.2 Dynamical systems on $\mathbb{R}^n$ generated by differential models

Let $\mathbf{x}(\mathbf{x}_0, t)$ with $\mathbf{x}(\mathbf{x}_0, t_0) = \mathbf{x}_0$ be a global solution to the Cauchy problem (1.2). Then $\mathbf{x}$ is a dynamical system according to the following definition [3], [20], [26]-[27], [42].

Definition 1.2 *A map*

$$\mathbf{x}:\ (\mathbf{x}_0,t)\in\mathbb{R}^n\times\mathbb{R}^+\to\mathbf{x}(\mathbf{x}_0,t)\in\mathbb{R}^n, \tag{1.6}$$

is a dynamical system on $\mathbb{R}^n$ *if verifies the following semigroup properties*

$$\begin{cases} \mathbf{x}(\mathbf{x}_0,t_0)=\mathbf{x}_0, \\ \\ \mathbf{x}(\mathbf{x}_0,t+\tau)=\mathbf{x}[\mathbf{x}(\mathbf{x}_0,\tau),t], \end{cases} \qquad \forall\mathbf{x}_0\in\mathbb{R}^n,\ \forall\, t,\ \tau\in\mathbb{R}^+. \tag{1.7}$$

If $\mathbf{x}$ is also continuous with respect to t and $\mathbf{x}_0$, then it is a C_0-semigroup.
If $\mathbf{x}$ is a dynamical system, the *motion* associated to an initial data $\mathbf{x}_0\in\mathbb{R}^n$ is a function

$$\mathbf{x}(\mathbf{x}_0,\cdot): t\in\mathbb{R}^+\to\mathbf{x}(\mathbf{x}_0,t)\in\mathbb{R}^n, \tag{1.8}$$

and it is also denoted with $\mathbf{x}(\mathbf{x}_0,t)$ or $\mathbf{x}(t)$.
If

$$\mathbf{x}(t)=\mathbf{x}_0, \qquad \forall\, t\in\mathbb{R}^+, \tag{1.9}$$

the motion is said *stationary* and $\mathbf{x}_0$ is an *equilibrium point* or *critical point*.

The dynamical system assures the forward (in time) uniqueness. This means that only one motion is associated to an assigned initial data for $t\in\mathbb{R}^+$, i.e.

$$\mathbf{x}_0=\mathbf{y}_0\ \Rightarrow\ \mathbf{x}(t)=\mathbf{y}(t), \qquad \forall\, t>0,$$

where $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are two motions.
Analogously, one can define the backward (in time) uniqueness.

The *positive graph* of the motion $\mathbf{x}$, is the set $\{t,\mathbf{x}(t)\}$ with $t\in\mathbb{R}^+$ and its projection on $\mathbb{R}^n$, that is the set $\gamma:\{\mathbf{x}(t): t\in\mathbb{R}^+\}$ is said *positive orbit* of the motion starting from $\mathbf{x}_0$. Analogously one can define the negative graph and the negative orbit for a motion.

Definition 1.3 *A motion is said to be periodic with respect to time of period* $T\in\mathbb{R}^+$*, if*

$$\mathbf{x}(t+T)=\mathbf{x}(t),\ \ \forall t\in\mathbb{R}^+. \tag{1.10}$$

1.3 Stability for dynamical systems generated by differential equations

In modelling a real world phenomenon, inevitably one can make some errors in the measurements of the initial data or in the formulation of the model. Hence, the problem to see how these errors influence the motion, arises. This is the problem of stability (with respect to the initial data).

The principal idea of the stability of a motion $\mathbf{x}(\mathbf{x}_0, \cdot)$ (the *basic* or *unperturbed* motion), is to verify if another motion $\mathbf{x}(\mathbf{x}_1, \cdot)$ (*perturbed* motion), starting from a position $\mathbf{x}_1$ sufficiently next to $\mathbf{x}_0$, for how much time will be next to the unperturbed motion. If this happens for any finite interval of time, then there is *continue dependence* with respect to variations of initial data. If this happens for $t \in [0, \infty[$, then $\mathbf{x}(\mathbf{x}_0, \cdot)$ is said to be stable.

Denoting with $B_r(\mathbf{x})$ with $r > 0$, the open ball in $\mathbb{R}^n$ centered at $\mathbf{x}$ and having radius r, i.e.

$$B_r(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n : d(\mathbf{x}, \mathbf{y}) < r\},$$

with d the usual distance in $\mathbb{R}^n$, the following definitions hold [6], [20], [22], [34].

Definition 1.4 *A motion $\mathbf{x}(\mathbf{x}_0, \cdot)$, depends continuously on the initial data iff $\forall t_0 \in \mathcal{I}$, $\forall T > 0$, $\forall \varepsilon \in]0, \chi[$, $\exists \delta(\varepsilon, T, t_0) \in]0, \varepsilon[$ such that*

$$\mathbf{x}_1 \in B_\delta(\mathbf{x}_0) \Rightarrow \mathbf{x}(\mathbf{x}_1, t) \in B_\varepsilon(\mathbf{x}(\mathbf{x}_0, t)), \quad \forall t \in [t_0, t_0 + T], \tag{1.11}$$

where $\chi = dist(\mathbf{0}, \partial D)$.

Remark 1.1 *In the study of some models, the function $\mathbf{F}$ can depend on some constitutive parameters. In estimating these parameters, one can make some errors. Hence the problem to study the stability with respect to the constitutive parameters, arises. The continuous dependence on the parameters can be traced back to the continuous* *dependence on the initial data. In fact, for example, the following Cauchy problem*

$$\begin{cases} \dot{u} = f(t, u, \lambda), \\ \\ u(t_0) = u_0, \end{cases}$$

with $f : (t, u) \in [t_0, \infty[\times\mathbb{R} \to \mathbb{R}$ *depending on the parameter* λ*, is equivalent to the following problem*

$$\begin{cases} \dot{u} = f(t, u, v), \\ \\ \dot{v} = 0, \\ \\ u(t_0) = u_0, \\ \\ v(t_0) = \lambda. \end{cases}$$

Hence the dependence on λ *is transferred to the dependence on the initial data.*

For the uniqueness and the continuous dependence on the initial data, the following theorem holds.

Theorem 1.7 *If* $\mathbf{F}$ *verifies the Lipschitz condition and if there is an unique solution of (1.2) in* $[t_0, t_1]$ *with* $t_1 < \infty$*, then it is unique and depends continuously on the initial data.*

Definition 1.5 *A motion* $\mathbf{x}(\mathbf{x}_0, \cdot)$ *is Liapunov stable (with respect to perturbations in the initial data) iff:*

$$\forall t_0 \in \mathcal{I}, \forall \varepsilon \in]0, \chi[, \exists \delta(\varepsilon, t_0) \in]0, \varepsilon[: \mathbf{x}_1 \in B_\delta(\mathbf{x}_0) \Rightarrow \mathbf{x}(\mathbf{x}_1, t) \in B_\varepsilon(\mathbf{x}(\mathbf{x}_0, t)), \forall t \geq t_0. \quad (1.12)$$

If δ doesn't depend on t_0, the stability is *uniform.*

Obviously, if $\mathbf{x}(\mathbf{x}_0, \cdot)$ is a stable motion associated to an autonomous differential equation, then the stability is uniform.

The stability is a stronger condition then that of the continuous dependence on the initial data, since it holds for $t \geq t_0$ and not for a finite temporal interval.

Definition 1.6 *A motion is unstable if it is not stable, that is*

$$\exists t_0 \in \mathcal{I}, \exists \varepsilon \in]0, \chi[: \forall \delta \in]0, \varepsilon], \exists \mathbf{x}_1 : \mathbf{x}_1 \in B_\delta(\mathbf{x}_0), \exists t_1 > t_0 : d\left[\mathbf{x}(\mathbf{x}_1, t_1), \mathbf{x}(\mathbf{x}_0, t_1)\right] \geq \varepsilon. \quad (1.13)$$

Definition 1.7 *A motion* $\mathbf{x}(\mathbf{x}_0, \cdot)$ *is said to be an attractor (or attractive) on a set* Y *if*

$$\mathbf{x}_1 \in Y \Rightarrow \lim_{t \to \infty} d\left[\mathbf{x}(\mathbf{x}_0, t), \mathbf{x}(\mathbf{x}_1, t)\right] = 0. \quad (1.14)$$

From (1.14) it follows that

$$\forall \nu > 0,\ \exists T(\nu, t_0, \mathbf{x}_0) > 0 :\ t > t_0 + T \Rightarrow\ \mathbf{x}(\mathbf{x}_1, t) \in B_\nu(\mathbf{x}(\mathbf{x}_0, t)). \quad (1.15)$$

The biggest set Y satisfying (1.14) is called the *basin (or domain) of attraction* of $\mathbf{x}(\mathbf{x}_0, \cdot)$.

Definition 1.8 *A motion* $\mathbf{x}(\mathbf{x}_0, \cdot)$ *is asymptotically stable if it is stable and if* $\forall t_0 \in \mathcal{I}$ *there exists* $\delta_1(t_0) > 0$ *such that* $\mathbf{x}(\mathbf{x}_0, \cdot)$ *is attractive on* $B_{\delta_1}(\mathbf{x}_0)$.

In particular:

Definition 1.9 *A motion* $\mathbf{x}(\mathbf{x}_0, \cdot)$ *is exponentially stable if* $\forall t_0 \in \mathcal{I},\ \exists \delta_1(t_0) > 0$, $\exists \lambda(\delta_1) > 0$, $\exists M(\delta_1) > 0$ *such that*

$$\mathbf{x}_1 \in B_{\delta_1}(\mathbf{x}_0) \Rightarrow d(\mathbf{x}(\mathbf{x}_0, t), \mathbf{x}(\mathbf{x}_1, t)) \leq M e^{-\lambda t} d(\mathbf{x}_1, \mathbf{x}_0). \quad (1.16)$$

If $\delta_1 = \infty$, then $\mathbf{x}(\mathbf{x}_0, \cdot)$ is *asymptotically (exponentially) unconditionally (or globally) stable.*

If T doesn't depend on $\mathbf{x}_0$, then the motion is said *equiattractive.*

If T doesn't depend on $\mathbf{x}_0$ and t_0, and δ_1 doesn't depend on t_0, there is *uniform attractivity.*

If the motion is stable and attractive, it is *asymptotically stable.*
If $\mathbf{x}(\mathbf{x}_0, \cdot)$ is uniformly stable and uniformly attractive, then it is *uniformly asymptotically stable.*
Analogous definitions can be given for the stability of an equilibrium point.

Remark 1.2 *One can always refer to the case in which the critical point is the null solution. In fact, if $\overline{x}$ is a critical point, then by the substitution*

$$z = x - \overline{x},$$

$(1.2)_1$ *reduces to $\dot{z} = F(z + \overline{x})$ which admits the null solution as critical point.*

In the applications the asymptotic stability is more important then the stability. In some cases the question about the extent of asymptotic stability arises. The desirable feature is asymptotic stability *in the large.* If one cannot assure that, one may have to be content with the assurance that when the perturbations are not too large the system tends to return to the equilibrium. Hence one needs to have information about the size of the region of asymptotic stability. To determine possible restrictions to asymptotic stability one must examine nonlinearities.
In order to proceed with our topic is necessary to give some definitions.

Definition 1.10 *A subset A of $\mathbb{R}^n$ is positively invariant if*

$$\mathbf{x}_0 \in A \Rightarrow \mathbf{x}(\mathbf{x}_0, t) \in A, \ \forall t \in \mathbb{R}^+. \tag{1.17}$$

Definition 1.11 *A set $\mathcal{I}$ in the phase space, is said attractor if there is an open set $\mathcal{H} \supset \mathcal{I}$ such that*

$$\lim_{t \to \infty} d(\mathbf{x}_0, \mathcal{I}) = 0, \tag{1.18}$$

for any initial data $\mathbf{x}_0 \in \mathcal{H}$, and where d is the distance of $\mathbf{x}_0$ from $\mathcal{I}$, i.e.

$$d(t) = \inf_{\mathcal{I}} |\mathbf{x} - \mathbf{x}_0| .$$

Definition 1.12 *The biggest set* $\mathcal{H}$ *verifying (1.18) is said basin of attraction of* $\mathcal{I}$.

Definition 1.13 *A set* $\mathcal{I}$ *in the phase space is an absorbing set if it is invariant and attractor.*

Definition 1.14 *A set* $\mathcal{I}$ *is a global attractor if it is compact and its basin of attraction is the whole phase space.*

Chapter 2

The Liapunov Direct Method

Sometime solving an evolution equation which models a phenomena, is quite difficult and hence some qualitative information on the behaviour of its solutions can be useful. The Liapunov Direct Method allows to obtain information about the asymptotic behaviour of the solutions around the equilibrium point of a system, without the integration of the equations. The technique consists into introduce "auxiliary" functions, the *Liapunov functions*, and to study some properties of these functions and of their temporal derivatives along the solutions of the system. The method is said "direct" in the sense that the temporal derivatives of the Liapunov functions, calculated along the solutions of the equation, can be linked directly to the second member of the same equation. In this chapter we refer to [20], [23], [28], [29], [45].

2.1 Autonomous equations

Let's consider an autonomous system

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}). \tag{2.1}$$

These kind of equations are very important especially for two reasons:

i) they represent the motion of mechanical systems with one degree of freedom under the action of forces independent of time;

ii) they describe a model of various eco-biological phenomena.

For the sake of simplicity, from now on, we refer to (2.1) with $n = 2$, and hence we refer to the following system

$$\begin{cases} \dot{x} = f(x, y), \\ \\ \dot{y} = g(x, y), \end{cases} \tag{2.2}$$

where we suppose f and g sufficiently smooth to guarantee the global existence of the solutions. Without loss of generality, we can assume that

$$f(0, 0) = g(0, 0) = 0, \tag{2.3}$$

i.e. (2.2) admits the null solution. Applying the differential theorem to f and g in the origin, one has

$$\begin{cases} f(x, y) = f_x(0, 0)x + f_y(0, 0)y + f_1(x, y), \\ \\ g(x, y) = g_x(0, 0)x + g_y(0, 0)y + g_1(x, y), \end{cases} \tag{2.4}$$

where

$$f_1 = o(r), \qquad g_1 = o(r), \qquad r = \sqrt{x^2 + y^2}.$$

Hence (2.2) can be written as

$$\begin{cases} \dot{x} = ax + by + f_1(x, y), \\ \\ \dot{y} = cx + dy + g_1(x, y), \end{cases} \tag{2.5}$$

with

$$\begin{cases} a = f_x(0, 0), \qquad b = f_y(0, 0), \\ \\ c = g_x(0, 0), \qquad d = g_y(0, 0). \end{cases} \tag{2.6}$$

The *linearized system* associated to (2.5) is obtained disregarding the nonlinear terms, and it is given by

$$\begin{cases} \dot{x} = ax + by, \\ \\ \dot{y} = cx + dy. \end{cases} \tag{2.7}$$

2.2 Stability for linear autonomous systems

In this section we want to study the stability of the null solution of the system (2.7). The behaviour of the trajectory around the critical point will depend on the two invariants of the coefficient matrix (A_{ij}) of system (2.7)

$$A = ad - bc, \qquad I = a + d. \tag{2.8}$$

In fact, searching solutions of the system (2.7) of the type $e^{\lambda t}$, one obtains the eigenvalue equation of the matrix (A_{ij})

$$\begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = \lambda^2 - I\lambda + A = 0. \tag{2.9}$$

(2.9) admits the following solutions

$$\lambda = \frac{I \pm \sqrt{I^2 - 4A}}{2}, \tag{2.10}$$

and hence the nature of the eigenvalues depends on the sign of $\Delta = I^2 - 4A$.
We recall some definitions.

Definition 2.1 *If the eigenvalues of* (A_{ij}) *are complex with non null real part, then the critical point is said focus and, precisely, it is a stable focus if the real part is negative, unstable in the other case.*

In the case of a focus, the trajectory is a spiral converging to the critical point in the case of stability, diverging in the case of an unstable focus.

Definition 2.2 *If the eigenvalue of* (A_{ij}) *are purely imaginary, then the critical point is a center.*

A center is linearly stable but it is not an attractor like the stable focus.

Definition 2.3 *If the eigenvalues of* (A_{ij}) *are real and of different sign, the critical point is said a saddle point.*

A saddle point is unstable.

Definition 2.4 *If the eigenvalues of* (A_{ij}) *are negative and real, then the critical point is a stable node, in the case that both eigenvalues are positive and real, the critical point is an unstable node.*

Definition 2.5 *If the eigenvalues of* (A_{ij}) *are real and coincident, then in the case* $b = c = 0$*, the critical point is said a star node and it is stable if* $a = d < 0$*, unstable in the other case.*

Definition 2.6 *If the eigenvalues of* (A_{ij}) *are real and coincident, then in the case* $b^2 + c^2 > 0$*, the critical point is said a one tangent node.*

The following theorem holds.

Theorem 2.1 *The critical point of the system (2.7) is*

i) unstable, if at least one eigenvalue of (A_{ij}) *has positive real part;*

ii) stable, if all the eigenvalues are imaginary;

iii) asymptotically (exponentially) stable if all the eigenvalues have negative real part.

2.3 The Liapunov Direct Method for autonomous equations

The conditions assuring the stability of the null solution of the system (2.7) are said *conditions for the linear stability.* If a critical point is linearly stable, it doesn't mean that it is also nonlinearly stable. In fact considering, for example, the equation

$$\dot{x} = x^2,$$

it follows, very easily, that the null solution is linearly stable but nonlinearly unstable.

The problem to see when it is possible to transfer the conditions for the linear stability to the non linear system (2.5) arises.

In order to study the nonlinear stability, we introduce the Liapunov Direct Method. We refer to the autonomous equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}), \tag{2.11}$$

with $\mathbf{F}(\mathbf{0}) = \mathbf{0}$.

Definition 2.7 *A scalar function $V(\mathbf{x}) = V(x_1, ..., x_n) \in C^1(\Omega)$, with Ω an open subset of $\mathbb{R}^n$ containing the origin, is a Liapunov function, if verifies the following properties:*

i) $V(\mathbf{0}) = 0$;

ii) $V(\mathbf{x}) > 0, \quad \forall\, \mathbf{x} \in \Omega, \mathbf{x} \neq \mathbf{0}$;

iii) $\mathbf{F}(\mathbf{x}) \cdot \nabla V \leq 0, \quad \forall\, \mathbf{x} \in \Omega.$

Definition 2.8 *If a function verifies the first two properties of the definition above, it is positive definite in Ω.*

Definition 2.9 *If a function V_1 is such that $-V_1$ is positive definite, then it is negative definite.*

We observe that the temporal derivative of V, $\dot{V}$, along the solution of (2.1) is given by

$$\dot{V} = \frac{d}{dt}V(x_1, ..., x_n) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\frac{dx_i}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}F_i = \nabla V \cdot \mathbf{F},$$

and hence the third property $iii)$ assures that $\dot{V}$, calculated along the solutions of the system, is not increasing.

Definition 2.10 *A function $V \in C(\Omega)$ is positive semidefinite if*

$$V(\mathbf{0}) = 0, \qquad V(\mathbf{x}) \geq 0 \quad \forall \mathbf{x} \in \Omega \backslash \{\mathbf{0}\}.$$

Example For example the functions $V = x^2 + y^2$ and $V = (x - y)^2$ are, respectively, positive definite and positive semidefinite.

Finding a Liapunov function is not very easy. The quadratic forms or the first integral[1], are good candidates.

The following theorems for the stability of the null solution of (2.1) hold.

Theorem 2.2 *If in an open set containing the origin there exists a Liapunov function V, then the origin is stable.*

Theorem 2.3 *In the hypotheses of theorem 2.2, if $\dot{V}$ is negative definite, then the origin is asymptotically stable.*

The instability of the null solution of (2.1) is guaranteed by the following theorems.

[1] A first integral is a function $G(x) \in C^1$ such that, along the solutions of the system, $\dot{G} = 0$.

Theorem 2.4 *If in an open subset of* $\Omega \subset \mathbb{R}^n$ *containing the origin, there exists a function* $V \in C^1(\Omega)$ *such that* $V(\mathbf{0}) = 0$, $\dot{V}$ *is positive definite and* V *takes some positive values in each spherical set contained in* Ω *and containing the origin, then the origin is unstable.*

Theorem 2.5 *Let* S *be a spherical subset of* $\mathbb{R}^n$ *containing the origin and* $\Omega_1 \subset S$ *a closed set such that* $O \in \partial\Omega_1$. *If there exists a function* $V \in C^1(\Omega_1)$ *such that* V *and* $\dot{V}$ *are positive in* $\Omega_1 \backslash \partial\Omega_1$ *and* V *is null on* $\partial\Omega_1 \cap (S \backslash \partial S)$, *then the null solution of (2.1) is unstable.*

This last theorem is also said Chetaiev instability theorem and has hypotheses more general then that of theorem 2.4.

2.4 Nonlinear stability for autonomous equations

Now we return to the problem to transfer the conditions for the linear stability to the nonlinear system. Considering the system (2.5), we introduce the function

$$V = A(x^2 + y^2) + (ay - cx)^2 + (by - dx)^2. \tag{2.12}$$

It follows, very easily, that the temporal derivative of V along the solutions of (2.5) is given by

$$\frac{1}{2}\frac{dV}{dt} = IA(x^2 + y^2) + \Psi(x, y), \tag{2.13}$$

with

$$\begin{cases} \Psi(x, y) = (\alpha_1 x - \alpha_3 y) f_1 + (\alpha_2 y - \alpha_3 x) g_1, \\ \\ \alpha_1 = A + c^2 + d^2, \quad \alpha_2 = A + a^2 + b^2, \quad \alpha_3 = ac + bd. \end{cases} \tag{2.14}$$

It is proved the following result.

Theorem 2.6 *If* f_1 *and* g_1 *are* $o(r)$, *then the theorem 2.1 for the linear stability-instability, continues to hold also for the system (2.5) except in the* critical case.

Definition 2.11 *The critical case is the case in which the eigenvalues of* (A_{ij}) *are null or pure imaginary.*

In the critical case, the nonlinear part of $\mathbf{F}$ can be stabilizing or destabilizing. In this case the only way to obtain information about the stability/instability of the null solution of (2.5) is to apply the theorems of the Liapunov Direct Method.

We conclude this chapter with an important remark.

Remark 2.1 *In the autonomous case, the largest conditions assuring the stability of the null solution for the system (2.7), are the Hurwitz conditions*

$$A = ad - bc > 0, \quad I = a + d < 0. \tag{2.15}$$

2.5 The Liapunov Direct Method for nonautonomous equations

In analyzing the stability-instability problems of nonautonomous systems, *a big caution is needed* since the conditions guaranteeing the stability-instability of the null solution of the autonomous systems, in general, *do not continue to guarantee* the stability-instability also for the null solution of the nonautonomous systems. In fact, in the *nonautonomous case* not even the uniform Hurwitz conditions

$$I^* = \sup_{t \geq t_0} I < 0, \quad A_* = \inf_{t \geq t_0} A > 0, \quad t_0 \geq 0, \tag{2.16}$$

are able to guarantee the stability of the null solution. This happens, for instance, in the case [38]

$$\begin{cases} \dot{x} = \left(-1 + \dfrac{3}{2}\cos^2 t\right) x + \left(1 - \dfrac{3}{4}\sin 2t\right) y, \\ \\ \dot{y} = -\left(1 + \dfrac{3}{4}\sin 2t\right) x + \left(-1 + \dfrac{3}{2}\sin^2 t\right) y. \end{cases} \tag{2.17}$$

Although

$$I = -\frac{1}{2}, \qquad A = \frac{1}{2}, \qquad \forall t \in \mathbb{R}^+,$$

(2.17) admits the unbounded solution

$$x = -ce^{\frac{t}{2}} \cos t, \; y = ce^{\frac{t}{2}} \sin 2t, \qquad \forall c \in \mathbb{R}.$$

We want to describe the Liapunov Direct Method for the equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, t), \tag{2.18}$$

where $\mathbf{F}$ is a function defined in $D \equiv \overset{\circ}{D} \times \mathcal{I} \subset \mathbb{R}^{n+1}$ (with $\mathcal{I} =]\tau, \infty[$, $\mathbf{0} \in D$), smooth enough to guarantee the global existence of the solution and such that $\mathbf{F}(\mathbf{0}, t) \equiv \mathbf{0}$. We recall here some useful definitions. Let be

$$V : (t, \mathbf{x}) \in \mathcal{I} \times \Gamma \to V(t, \mathbf{x}) \in \mathbb{R}, \tag{2.19}$$

where Γ is an open subset of $\mathbb{R}^n$ containing the origin.

Definition 2.12 *V is positive semidefinite if there exists a closed sphere contained in Γ centered in the origin (say B_γ) such that*

$$\forall (t, \mathbf{x}) \in \mathcal{I} \times B_\gamma,\; V(t, \mathbf{x}) \geq 0, \; \textit{and}\; V(t, \mathbf{0}) = 0. \tag{2.20}$$

Definition 2.13 *V is positive definite if*

i) $V(t, \mathbf{0}) = 0, \quad \forall t \in \mathcal{I},$

ii) there exists a continuous function W

$$W : \mathbf{x} \in B_\gamma \to W(\mathbf{x}) \in \mathbf{R}^+,$$

such that

$$W(\mathbf{0}) = 0, \; W(\mathbf{x}) > 0, \; \forall x \in B_\gamma \backslash \{\mathbf{0}\},$$

and

$$V(t, \mathbf{x}) \geq W(\mathbf{x}), \quad \forall (t, \mathbf{x}) \in \mathcal{I} \times B_\gamma.$$

Analogously one can give the definitions for negative semidefiniteness and negative definiteness.

Definition 2.14 *V admits an infinitely small upper bound, if there exists a continuous function W*

$$W : \mathbf{x} \in B_\gamma \to W(\mathbf{x}) \in \mathbb{R}^+,$$

with W positive semidefinite such that

$$|V(t, \mathbf{x})| \leq W(\mathbf{x}), \qquad \forall (t, \mathbf{x}) \in \mathcal{I} \times B_\gamma.$$

The stability of the null solution of (2.18) is guaranteed by the following Liapunov stability theorems.

Theorem 2.7 *Let (2.18) holds with $\mathbf{F}$ sufficiently smooth to guarantee the global existence and uniqueness of the solution. If there exists a function $V : (t, \mathbf{x}) \in \mathcal{I} \times B_\gamma \to V(t, \mathbf{x}) \in \mathbb{R}$, where $\gamma \in]0, \chi[$ and $\chi = dist(\mathbf{0}, \partial D)$, $V \in C^1$ and*

i) V positive definite;

ii) $\dot{V}$ negative semidefinite along the solutions of (2.18),

then the null solution of (2.18) is (simply) stable.

Theorem 2.8 *In the hypotheses of the theorem before, if V has an upper bound which is infinitely small at the origin, then the null solution of (2.18) is uniformly stable.*

Theorem 2.9 *If there exists a function $V : (t, \mathbf{x}) \in \mathcal{I} \times B_\gamma \to V(t, \mathbf{x}) \in \mathbb{R}$, with $\gamma \in]0, \chi[$, $V \in C^1$ such that*

i) V is positive definite;

ii) V has an upper bound which is infinitely small at the origin;

iii) $\dot{V}$, along the solution of (2.18), is negative definite, *then the null solution of (2.18) is uniformly, asymptotically, stable.*

The instability of the null solution of (2.18) is guaranteed by the following instability Liapunov theorems.

Theorem 2.10 *Let be* $V : (t, \mathbf{x}) \in \mathcal{I} \times \Gamma \rightarrow V(t, \mathbf{x}) \in \mathbb{R}$, *with* $\Gamma = \overset{\circ}{\Gamma} \subset D \subset \mathbb{R}^n$, $V \in C^1$. *If* $\exists t_0 \in \mathcal{I}$, $\exists \gamma > 0$, *with* $B_\gamma \subset \Gamma$ *such that*

i) $\forall \eta \in]0, \gamma[$, $\exists \mathbf{x} \in B_\eta$ *with* $V(t_o, \mathbf{x}) > 0$,

ii) V *has an upper bound which is infinitely small at the origin,*

iii) $\dot{V}$, *along the solution of (2.18), is positive definite in* $\mathcal{I} \times B_\gamma$,

then the null solution of (2.18) is unstable.

From this theorem it follows that if there exists a function V with an infinitely small upper bound and temporal derivative $\dot{V}$ along the solutions definite in sign and also if for $t \geq t_0$ (with arbitrarily large t_0) the function V has the same sign as $\dot{V}$ in a neighborhood of the origin, then the null solution is unstable.

Theorem 2.11 *Let be* $V : \mathcal{I} \times \Gamma \rightarrow \mathbb{R}$, $V \in C^1$. *Suppose that* $\exists t_0 \in \mathcal{I}$, $\exists \gamma > 0$ *with* $B_\gamma \subset \Gamma$ *such that*

i) $\forall \eta \in]0, \gamma[$ $\exists \mathbf{x} \in B_\eta$ *with* $V(t_o, \mathbf{x}) > 0$,

ii) V *bounded in* $\mathcal{I} \times B_\gamma$,

iii) $\dot{V}(t, \mathbf{x}) \geq \lambda V(t, \mathbf{x})$, *with* λ *a positive constant,*

then the null solution of (2.18) is unstable.

Now we want to state the Chetaiev theorem, which furnishes a sufficient condition for the instability of the null solution. This theorem is different from the theorems 2.10 and 2.11 since it regards an auxiliary function in a part of a particular open subset of $\mathbb{R}^n$ containing the origin.

Theorem 2.12 *Suppose that for the equation (2.18)* $\exists t_0 \in \mathcal{I}, \gamma \in]0, dist(\mathbf{0}, \partial\Gamma)[, \eta \in]0, \infty[$, *a continuous function* $V : \mathcal{I} \times \Gamma \to \mathbb{R}$, *a function* $a : [0, \gamma] \to \mathbb{R}$ *of class* k[2] *and an open subset* $\Theta \subset B_\gamma$ *such that*

i) $0 < V(t, \mathbf{x}) < \mu, \qquad \forall (t, \mathbf{x}) \in [t_0, \infty[\times\Theta,$

ii) $V(t, \mathbf{x}) = 0, \qquad \forall (t, \mathbf{x}) \in [t_0, \infty[\times(\partial\Theta \cap \mathring{B}_\gamma),$

iii) $\dot{V}(t, \mathbf{x}) \geq a(V(t, \mathbf{x})), \qquad \forall (t, \mathbf{x}) \in [t_0, \infty[\times\Theta,$

iv) $\mathbf{0} \in \partial\Theta$.

Then the null solution of (2.18) is unstable.

[2] A function $a : [0, \mu] \to \mathbb{R}$, with $\mu > 0$, is a function of class k ($a \in k$) if a is continuous, increasing and $a(0) = 0$.

Chapter 3

The Lotka-Volterra model

In 1925, during a conversation with Vito Volterra, a young zoologist, Umberto D'Ancona, pointed out that in the years following the World War I the proportion of predator fishes caught in the Upper Adriatic was up from before, whereas the proportion of prey fishes was down. This phenomenon was predicted by one of the Volterra's models. In the same year that Volterra became interested in the ecology, A. J. Lotka published a work titled *Elements of Physical Biology* [32]. In this text he discussed the same model utilized by Volterra for predator-prey interactions. The two were completely unaware of each other's work. This is the model that is known as the Lotka-Volterra model.

This model is based, as it well known, on the assumptions [35]:

i) in absence of the predators, the preys increase at a constant rate;

ii) in absence of the preys, the predators decrease at a constant rate;

iii) the rate at which preys are eaten is proportional to the product of the densities of predators and preys;

iv) the species live in a homogeneous environment.

Therefore denoting with a, b, c, d positive constants and with x and y respectively the preys and predators densities, the equations governing the model are:

$$\begin{cases} \dot{x} = ax - bxy, \\ \dot{y} = -cy + dxy. \end{cases} \tag{3.1}$$

Applying the transformation

$$u = \frac{dx}{c}, \quad v = \frac{by}{a}, \quad \tau = at, \quad \alpha = \frac{c}{a}, \tag{3.2}$$

model (3.1) becomes

$$\begin{cases} \dfrac{du}{d\tau} = u(1-v), \\ \\ \dfrac{dv}{d\tau} = \alpha v(u-1). \end{cases} \tag{3.3}$$

From (3.3), it follows that $\{u(0)>0, v(0)>0\} \Rightarrow \{u(\tau)>0, v(\tau)>0, \forall\tau>0\}$ and hence, the positive orthant is (positively) invariant. This is a property verified by all the population dynamic models

From $(3.3)_1$ and since $uv > 0$, one obtains that

$$\frac{du}{d\tau} \leq u. \tag{3.4}$$

Multiplying each member for $e^{-\tau}$, one obtains

$$\frac{d}{d\tau}(ue^{-\tau}) \leq 0 \Rightarrow u(\tau) \leq u_0 e^{\tau}. \tag{3.5}$$

Hence u is bounded in each finite interval of τ. From $(3.3)_2$, it follows that

$$\frac{dv}{d\tau} \leq \alpha v(u_0 e^{\tau} - 1), \tag{3.6}$$

and multiplying by $e^{-\alpha \int_0^\tau (u_0 e^{\xi} - 1)\, d\xi}$,

$$\frac{d}{d\tau}\left(e^{-\alpha \int_0^\tau (u_0 e^{\xi} - 1)\, d\xi} v\right) \leq 0 \Rightarrow v \leq v_0 e^{\alpha u_0 (e^{\tau} - 1) - \alpha\tau}, \tag{3.7}$$

one has that, in each finite interval of τ, v is bounded.

Because u and v can be extended in each finite interval of time, there is the global existence for the solution of (3.3).

3.1 Equilibrium points

If $v \equiv 0$, from $(3.3)_1$ one has

$$\frac{du}{d\tau} = u \Rightarrow u = u_0 e^{\tau}. \tag{3.8}$$

Hence $\{u = u_0 e^{\tau},\ v = 0\}$ is the only solution starting from $\{u_0 > 0,\ v_0 = 0\}$. Analogously $\{u = 0,\ v = v_0 e^{-\tau}\}$ is the only solution starting from $\{u_0 = 0,\ v_0 > 0\}$. We have shown that the axes u and v of the phase space are two trajectories. The first is covered in the sense of the u increasing while the second is covered in the sense of the v decreasing. The other trajectories starting from a point in the positive orthant, can not cross the axes. For this reason there can be a closed trajectory, that means a cycle. From a biological point of view, a cycle is very important because it guarantees the survival of both the species.

Model (3.3) admits $(0, 0)$ and $(1, 1)$ as equilibrium points.

Let's start to analyze the stability of $(0, 0)$.
Disregarding the nonlinear terms in (3.3), one obtains

$$\begin{cases} \dfrac{du}{d\tau} = u, \\ \\ \dfrac{dv}{d\tau} = -\alpha v. \end{cases} \tag{3.9}$$

Hence the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -\alpha \end{pmatrix}$$

has an eigenvalue with positive real part. Then $(0, 0)$ is a saddle point and it is unstable. From a biological point of view, the instability of the null solution is very important since it guarantees that the extinction of the species is not possible. This can be seen also by the fact that the axis u is covered in the sense of the u increasing.

Passing to the study of the stability of $(1,1)$, setting

$$u = x + 1, \quad v = y + 1, \tag{3.10}$$

in order to transport the equilibrium in to the origin, one obtains

$$\begin{cases} \dfrac{dx}{d\tau} = -y(1+x), \\ \\ \dfrac{dy}{d\tau} = \alpha x(1+y). \end{cases} \tag{3.11}$$

The matrix of the coefficients of the linear system associated to (3.11)

$$\begin{pmatrix} 0 & -1 \\ \alpha & 0 \end{pmatrix},$$

has two pure imaginary eigenvalues

$$\lambda = \pm \imath \sqrt{\alpha}.$$

Hence $(1,1)$ is a center and is linearly stable. It follows that, considering the linear system associated to (3.11), the motion is periodic around the equilibrium point. It is a cycle.

If we pass to study the nonlinear stability we have a critical case. If we are able to find a Liapunov function, we have nonlinear stability.

Dividing $(3.3)_2$ by $(3.3)_1$ one obtains

$$\frac{dv}{du} = \frac{\alpha v(u-1)}{u(1-v)}. \tag{3.12}$$

This is a differential equation at separable variables. Integrating (3.12), it turns out that

$$\log v - v = \alpha(u - \log u) - k, \tag{3.13}$$

with k =const. Hence, the function

$$V = \alpha(u - \log u) + v - \log v = \alpha u + v - \log(u^\alpha v), \tag{3.14}$$

is constant if calculated along the solutions of (3.3) and its value is given by k. Then, along the solutions of (3.3), $\frac{dV}{d\tau} = 0$. Now, we want to show that V is positive definite in $[-1, 1]$. Let's seek the minimum value that V assumes in the interval. The Hessian matrix calculated in $(1, 1)$ is

$$\left\| \begin{matrix} \alpha & 0 \\ 0 & 1 \end{matrix} \right\| = \alpha > 0,$$

then $\min_{[-1,1]} V = (1, 1)$. Since $V(1, 1) = \alpha + 1 > 0$, then V is positive definite. In this way we have shown that V is a Liapunov function and hence we can conclude that $(1, 1)$ is nonlinearly (simply) stable.

We have proved the following theorem.

Theorem 3.1 *The model (3.1) admits two equilibrium points* $(0, 0)$ *and* $\left(\frac{c}{d}, \frac{a}{b}\right)$*. The first is unstable while the second is nonlinearly (simply) stable.*

As an interesting sidelight, it was thought that records kept by the Hudson Bay Company for the last 200 years seemed to confirm the general oscillatory behaviour predicted by the Lotka-Volterra model. The records involved the catches of Canada lynx and its preys, the snowshoe hare. In 1973 Gilpin analyzed this data by computer and found three reverse cycles. If we did not know better, we might conclude that during these cycles the hare were eating the lynx, but the most probable explanation is an erratic trapper activity during these years.

3.2 Lotka-Volterra model with logistic growth for the preys

The hypotheses that the preys, in absence of the predators, grow in a Malthusian way, can hold only for short time, since it doesn't take in account of limitation of resources, space, ecc. In order to improve the model, the hypotheses $i)$ has been substituted with a weaker one

- the preys, in absence of the predators, grow up in a logistic way.

This means that

$$\dot{x} = (a - by - ex)x,$$

and hence the model (3.1) becomes

$$\begin{cases} \dot{x} = (a - by - ex)x, \\ \\ \dot{y} = (-c + dx)y, \end{cases} \tag{3.15}$$

where a, b, c, d, e are positive constants.

In this case there are three equilibrium points

$$(0,0), \quad \left(\frac{a}{e}, 0\right), \quad \left(\frac{c}{d}, \frac{a}{b}(1-\beta)\right), \tag{3.16}$$

where $0 < \beta = \dfrac{ec}{ad} < 1$. The meaningful equilibrium point is given by $(3.16)_3$, since $(3.16)_1$ and $(3.16)_2$ imply, respectively, the extinction of both populations and of the predators. Introducing the transformation

$$u = \frac{dx}{c}, \quad v = \frac{by}{a}, \quad \tau = at, \quad \alpha = \frac{c}{a}, \tag{3.17}$$

the model becomes

$$\begin{cases} \dfrac{du}{d\tau} = (1 - \beta u - v)u, \\ \\ \dfrac{dv}{d\tau} = \alpha v(u-1), \end{cases} \tag{3.18}$$

The equilibrium points are

$$(0,0), \quad \left(\frac{1}{\beta}, 0\right) \quad (1, 1-\beta). \tag{3.19}$$

It is proved that $(3.19)_1$ and $(3.19)_2$ are unstable, while for the meaningful equilibrium point, the following theorem holds.

Theorem 3.2 *The equilibrium* $(1, 1-\beta)$ *with* $0 < \beta < 1$, *is asymptotically stable and the attraction basin is the whole positive orthant.*

Chapter 4

Generalized Lotka-Volterra models

The assumptions at the basis of model (3.1) are unrealistic. In fact, a well established criticism can be done to $i) - iii)$ of Chapter 3 and hence to (3.1). The following remarks hold [8]:

1) the growth behaviour assumed by $i)$ is reasonable only for a limited time, since a continuous increasing of the population will exhaust its resources;

2) the density of each species does not exhibit any structure (space location, age, differences of sex or genotype,...);

3) changes in density are deterministic, ignoring the random effects in the environment that influence the interaction between x and y;

4) the effects of interactions within and between the species are instantaneous, ignoring the influence of delayed processes.

Hence the problem to modify the assumptions at the basis of the classical Lotka-Volterra model, arises.
In the literature can be found several perturbed Lotka-Volterra models of the type

$$\dot{x} = x(a - by) + F, \qquad \dot{y} = y(-c + dx) + G, \tag{4.1}$$

developed by many authors. Different types of perturbation terms F, G have been introduced in order to account of variations of the idealized hypotheses $i) - iii)$ as well as to put controls on the growth of both predators and preys (see, for instance [1]-[2], [5], [7]-[8], [11]-[19], [21], [25], [30]-[33], [36]-[41], [43]-[44] and the references therein). The influence of the perturbation terms on the stability of the positive ecological equilibrium state

$$\overline{x} = \frac{c}{d}, \quad \overline{y} = \frac{a}{b}, \tag{4.2}$$

of (3.1) or on the existence both of periodic solutions or perturbed critical points, have been studied.

In particular, in [5], [13]-[15], the perturbation terms are such that

$$[F]_{(\overline{x},\overline{y})} = [G]_{(\overline{x},\overline{y})} = 0. \tag{4.3}$$

and hence (4.1) admits the same equilibrium point of the classical model.
In [16]-[18], the perturbation terms are

$$F = -\varepsilon f_1(x, y), \qquad G = -\varepsilon f_2(x, y), \tag{4.4}$$

where $f_i\,(i = 1, 2)$, are of higher order, and $\varepsilon > 0$. These perturbations are the most general to provide damping on the predators and preys. It has been observed by Samuelson [40] that the addiction of an appropriate damping term

$$f_1(x, y) = xf(x), \quad f_2(x, y) = 0,$$

leads to oscillations of the Rayleigh Van der Pol type, i.e. leads to a stable limit cycle. The existence of a stable limit cycle provides a satisfactory explanation for animal communities in which populations oscillate in a reproducible periodic manner. It is proved that the stability of the equilibrium point of model (4.4) depends on ε. If ε takes small values, then there is linear stability. For the nonlinear stability, a Liapunov function is introduced and the Direct Method is used. Sufficient conditions for the existence of a limit cycle are determined.

In [19] the same model is analyzed but it is considered the case in which the first derivatives of the perturbation terms are zero at the critical point and the case in which the equilibrium point is fixed, i.e. the critical point doesn't depend on ε. The problem is one of perturbation in the small parameter and not of higher order and it is studied by using implicit functions techniques suggested by Loud [33]. The results indicate the type of functions f_i $(i = 1, 2)$, which yield periodic solutions of the system (4.4), but there is not estimate about the number of the solutions and it is not determined whether or not this periodic solution is unique.

In [13] time dependent perturbation terms, depending on a small parameter, have been introduced in the classic model and their influence on the stability of the equilibrium of this model has been studied by using the multiple scale method.

In [14], [15], Fergola Rionero Tenneriello, consider the same type of nonautonomous perturbation, but they do not make any assumption on the "smallness" of these perturbations. Precisely, they consider the following perturbation terms

$$F = D(t)(y - x), \qquad G = D(t)(x - y), \tag{4.5}$$

with a, b, c, d positive constants such that $\dfrac{c}{d} = \dfrac{a}{b}$ and $D : \mathbb{R}^+ \to \mathbb{R}^+$ smooth enough to guarantee the global existence and uniqueness of the solution. In this model the perturbations terms are given by the product between a very general function D depending on time and on the difference between the population densities of the preys and predators. This kind of perturbation assumes the meaning of a "diffusion" term and it was used in [1], [2] to represent the diffusion among heterogeneous patches of populations. Introduced a Liapunov energy function, depending on a coupling parameter λ between the two species

$$V(t, \lambda) = u^2 + \lambda^2 v^2,$$

with $u = x - \overline{x},\ v = y - \overline{y},\ \lambda = \text{cost.} > 0$, and setting

$$F(D, \lambda) = 2D - g, \quad \alpha(\lambda) = \frac{4}{3\sqrt{3}} \left(\frac{b}{\lambda} + d \right),$$

$$g = \frac{D - a}{\lambda} + (D + c)\lambda = \frac{m}{\lambda} + n\lambda,$$

it has been proved that

Theorem 4.1 *If*

$$\begin{aligned} &\exists \lambda \in \mathbb{R}^+ : \inf_{t \geq 0} F \geq 2\mu > 0, \\ &V^{\frac{1}{2}}(0, \lambda) < \frac{2\mu(\lambda)}{\alpha(\lambda)}, \end{aligned} \tag{4.6}$$

then, along the solutions of the system, $\dot{V}$ is negative definite and the equilibrium $(\overline{x}, \overline{y})$ (with $\overline{x} = \overline{y}$) is (locally) non linearly, asymptotically, exponentially stable.

Because in the case $D \equiv 0$, $\overline{x} = \overline{y}$ is stable, the hypotheses that D has to be a positive function of time, appears too strong. In [5] this assumption has been replaced with the weaker condition on $\int_0^t D(\tau) d\tau$

$$\exists \lambda \in \mathbb{R}^+ : \lim_{t \to \infty} \int_0^t F(\lambda, \tau) = \infty.$$

Once again a Liapunov function, depending on a coupling parameter, has been introduced and conditions assuring the stability of the equilibrium point have been determined even in the case of $D(t)$ periodic function of time.

Our aim here is to consider nonautonomous perturbations which consist in introducing a dependence on time either in the coefficients of the classical model or in functions added to the second members of the equations.

We choose the perturbations in a way that the perturbed models admit the same biological equilibrium $(\overline{x}, \overline{y})$ of (3.1). In this way we can compare all the results with those of the classical system.

We will study the following class of generalized Lotka-Volterra models [4], [9], [10]:

$$\begin{cases} \dot{x} = f_1(t)(a - by)x + D_1(t)\left(\dfrac{\overline{x}}{\overline{y}}y - x\right)^{1+p}, \\ \\ \dot{y} = f_2(t)(-c + dx)y + D_2(t)\left(\dfrac{\overline{y}}{\overline{x}}x - y\right)^{1+q}, \end{cases} \tag{4.7}$$

The hypotheses at the base of (4.7), are that:

1) f_i, D_i, $\in L^\infty(\mathbb{R}^+) \cap C^1(\mathbb{R}^+)$ $(i = 1, 2)$;

2) f_i and D_i, $(i = 1, 2)$ are such that

$$\begin{aligned} |f_i(t_1) - f_i(t_2)| &\leq L_i|t_1 - t_2|, \\ |D_i(t_1) - D_i(t_2)| &\leq K_i|t_1 - t_2|, \end{aligned}$$

with $t_1, t_2 \in \mathbb{R}^+$, and L_i, K_i $(i = 1, 2)$, positive constants;

3) a, b, c, d positive constants;

4) $(p, q) \in [\mathbb{N}^+]^2$;

5) f_i positive functions $(i = 1, 2)$ of t;

6) the preys grow up in absence of the predators, while the predators decrease in absence of the preys.

Remark 4.1 *We remark that*

- $1) - 2)$ *guarantee the global (in time) existence and uniqueness of the solutions,*
- *the models (4.7) admit the same equilibrium* $(\overline{x}, \overline{y})$ *of the classical system (3.1),*

- *in the case* $D_i \equiv 0$ $(i = 1, 2)$, *the model (4.7) becomes*

$$\begin{cases} \dot{x} = f_1(t)(a - by)x, \\ \\ \dot{y} = f_2(t)(-c + dx)y, \end{cases}$$

that is a nonautonomous Lotka-Volterra model which has been analyzed by Rionero in [38]. Here the author proves the following result.

Theorem 4.2 *The equilibrium point* $(\overline{x}, \overline{y})$ *is*

i) linearly stable if, setting

$$G_1 = acf_1f_2 + \left(\frac{ad}{b}f_2\right)^2, \qquad G_2 = acf_1f_2 + \left(\frac{bc}{d}f_1\right)^2,$$

one has

$$\inf G_i > 0, \qquad \sup \frac{dG_i}{dt} \leq 0, \quad i = 1, 2; \tag{4.8}$$

ii) linearly asymptotically stable and nonlinearly (locally) asymptotically stable if $(4.8)_2$ *holds with the strict inequality;*

iii) unstable if

$$\inf \left\{ \inf \left(\frac{dG_1}{dt}\right), \inf \left(\frac{dG_2}{dt}\right) \right\} > 0.$$

Hence a first difference between the classical model and the perturbed one is put in evidence. In the autonomous case, one can only have the simple stability of the meaningful biological equilibrium point. Instead, there can be nonautonomous perturbations that, under some assumptions, lead to asymptotic stability. In these conditions, the model cannot admit periodic solutions, not even if the perturbations are all periodic with the same period.

The problem to classify the nonautonomous perturbations which, under some conditions, lead to the asymptotic stability, and hence to the absence of cycles, arises.

In order to study the stability-instability of the equilibrium $(\overline{x}, \overline{y})$, we will follow the procedure used in [38].

In [38], Rionero deals with the following nonautonomous, linear, system

$$\begin{cases} \dot{x} = a(t)x + b(t)y, \\ \dot{y} = c(t)x + d(t)y, \end{cases} \tag{4.9}$$

and the nonlinear one

$$\begin{cases} \dot{x} = a(t)x + b(t)y + f(x, y, t), \\ \dot{y} = c(t)x + d(t)y + g(x, y, t), \end{cases} \tag{4.10}$$

with f and g nonlinear functions of x and y such that

$$f(0, 0, t) \equiv g(0, 0, t) \equiv 0, \qquad \forall t \in \mathbf{R}^+, \tag{4.11}$$

and having (together with the functions a, b, c, d) the regularity guaranteeing the global existence of solutions.

In order to study the stability-instability of the null solution of such models, the following functions are introduced

$$V = \frac{1}{2}\left[A(x^2 + y^2) + (ay - cx)^2 + (by - dx)^2\right], \tag{4.12}$$

and

$$E = \frac{1}{2}[\mu_1(t)x^2 + \mu_2(t)y^2], \tag{4.13}$$

with μ_i, $(i = 1, 2)$, suitable derivable functions for $t \geq t_0$, bounded together with the derivatives $\dot{\mu}_1, \dot{\mu}_2$.

The first function V is the O.D.Es "*adaptation*" of a peculiar Liapunov function introduced by Rionero in the context of L^2-stability analysis for binary reaction-diffusion systems of P.D.Es {cfr [36]-[37] and the appendix of [38]}. This is a "peculiar" function in the sense that its temporal derivative, calculated along the solutions of the system, is linked directly to the eigenvalues of the problem. In fact, in the autonomous case, the temporal derivative of V, along the solutions of (4.9), is given by

$$\dot{V} = IA(x^2 + y^2), \tag{4.14}$$

with $A = ad - bc$ and $I = a + d$. In the nonautonomous case, one obtains

$$\dot{V} = \begin{cases} IA(x^2 + y^2) + \frac{1}{2}\dot{A}(x^2 + y^2) + \\ \\ +(ay - cx)(\dot{a}y - \dot{c}x) + (by - dx)(\dot{b}y - \dot{d}x) \end{cases} \tag{4.15}$$

and hence

$$\dot{V} = \frac{1}{2}\sum_{i=1}^{3} P_i(x, y, t), \tag{4.16}$$

with

$$\begin{cases} P_1 = (2IA + \dot{A})(x^2 + y^2), \;\; P_2 = \frac{dc^2}{dt}x^2 + \frac{da^2}{dt}y^2 - 2\frac{d(ac)}{dt}xy, \\ \\ P_3 = \frac{dd^2}{dt}x^2 + \frac{db^2}{dt}y^2 - 2\frac{d(bd)}{dt}xy. \end{cases} \tag{4.17}$$

Applying the Liapunov Direct Method, conditions assuring the stability and the instability of the null solution of (4.9) are determined. These conditions can be extended also for the system (4.10), if the following inequality holds

$$(|x| + |y|)(|f| + |g|) \leq \varepsilon_1 (x^2 + y^2)^{1+\varepsilon_2}, \tag{4.18}$$

$\varepsilon_i, (i = 1, 2)$ being positive constants.

4.1 The case $p = q = 0$

In view of (4.7) with $p = q = 0$, one obtains

$$\begin{cases} \dot{x} = af_1(t)\left(1 - \dfrac{y}{\overline{y}}\right)x + \overline{x}D_1(t)\left(\dfrac{y}{\overline{y}} - \dfrac{x}{\overline{x}}\right), \\ \\ \dot{y} = cf_2(t)\left(-1 + \dfrac{x}{\overline{x}}\right)y + \overline{y}D_2(t)\left(\dfrac{x}{\overline{x}} - \dfrac{y}{\overline{y}}\right). \end{cases} \tag{4.19}$$

Introducing the following bijective increasing transformation of time, in order to make the model dimensionless

$$\begin{cases} x = \overline{x}X, \;\; y = \overline{y}Y, \;\; \tau = a\displaystyle\int_0^t f_1(z)\,dz, \\ \\ \varphi_1(t) = \dfrac{D_1(t)}{af_1(t)}, \;\; \varphi_2(t) = \dfrac{D_2(t)}{af_1(t)}, \;\; \psi(t) = \dfrac{cf_2(t)}{af_1(t)}, \end{cases} \tag{4.20}$$

by virtue of (4.19), it turns out that the model becomes

$$\begin{cases} \dfrac{dX}{d\tau} = (1 - Y)X + \varphi_1(t)(Y - X), \\ \\ \dfrac{dY}{d\tau} = \psi(t)(-1 + X)Y + \varphi_2(t)(X - Y), \end{cases} \tag{4.21}$$

(4.21) having $(1, 1)$ as critical point when neither species is extinct. We want to study the conditions assuring the stability-instability of this equilibrium point, because the other equilibrium $(0, 0)$ is not biologically meaningful, since it means that all the species extinct.

We assume that

$$\psi, \varphi_i \in L^\infty(\mathbb{R}^+) \cap C^1(\mathbb{R}^+), \;\; (i = 1, 2), \tag{4.22}$$

with

$$\begin{cases} |\varphi_i(t_1) - \varphi_i(t_2)| \leq L_i|t_1 - t_2|, \quad (i = 1, 2), \\ \\ |\psi(t_1) - \psi(t_2)| \leq K|t_1 - t_2|, \end{cases} \tag{4.23}$$

where $t_1, t_2 \in \mathbb{R}^+$ and $L_i\,(i = 1, 2)$, K positive constants. We want that all the hypotheses at the basis of the classical Lotka-Volterra model, have to be preserved. In particular we want that

(i) the preys grow up in absence of predators;

(ii) the predators decrease in absence of preys.

Further we require that

$$\begin{cases} \varphi_1(t) < 1, \\ \qquad\qquad\qquad\qquad \forall t \in \mathbb{R}^+, \\ \psi(t) + \varphi_2(t) > 0. \end{cases} \tag{4.24}$$

We remark that $(4.22) - (4.23)$ guarantee (global) existence and uniqueness of smooth solutions of (4.21).
We want to characterize the functions ψ, φ_i $(i = 1, 2)$ guaranteeing the nonlinear stability (instability) of the biological meaningful equilibrium state $(1, 1)$, existing $\forall \psi, \varphi_i$ $(i = 1, 2)$.

4.2 Some preliminary Lemmas

In view of (4.21), by integrating the two equations, it follows that

$$\begin{cases} X = X_0 \exp \displaystyle\int_0^\tau \left\{ 1 - Y(z) + \varphi_1(z) \left[\frac{Y(z)}{X(z)} - 1 \right] \right\} dz, \\ \\ Y = Y_0 \exp \displaystyle\int_0^\tau \left\{ \psi(z)(-1 + X(z)) + \varphi_2(z) \left[\frac{X(z)}{Y(z)} - 1 \right] \right\} dz, \end{cases}$$

and hence $\{X_0 > 0, Y_0 > 0\} \Rightarrow \{X(\tau) > 0, Y(\tau) > 0,\ \forall \tau > 0\}$. This means that the positive orthant in the phase space is invariant.
Transporting the equilibrium into the origin by setting

$$X = u + 1, \qquad Y = v + 1, \tag{4.25}$$

(4.21) becomes

$$\begin{cases} \dfrac{du}{d\tau} = -\varphi_1(t)u + (\varphi_1(t) - 1)v - uv, \\ \\ \dfrac{dv}{d\tau} = (\psi(t) + \varphi_2(t))u - \varphi_2(t)v + \psi(t)uv. \end{cases} \tag{4.26}$$

In order to study the nonlinear stability/instability of $(1, 1)$ there are introduced the standard "energy", which is given by

$$E(\tau) = \frac{1}{2}(\mu_1(\tau)u^2 + \mu_2(\tau)v^2), \tag{4.27}$$

and the Rionero "energy" (4.12), which specifies as

$$V(\tau) = \frac{1}{2}\left\{A(u^2 + v^2) + [\varphi_1 v + (\psi + \varphi_2)u]^2 + [(\varphi_1 - 1)v + \varphi_2 u]^2\right\}, \tag{4.28}$$

with

$$\begin{cases} A(t) = \varphi_1\varphi_2 - (\varphi_1 - 1)(\psi + \varphi_2) = \psi(t)[1 - \varphi_1(t)] + \varphi_2(t), \\ \\ \mu_i \in L^\infty(\mathbb{R}^+) \cap C^1(\mathbb{R}^+). \end{cases} \tag{4.29}$$

Remark 4.2 *We remark that, since* $\psi \geq 0$, $\forall\tau \in \mathbb{R}^+$ *and* $(4.24)_1$ *holds, then*

$$A \geq \varphi_2,$$

and hence $\varphi_2 \geq 0$ *implies* $A \geq 0$*. Further, from (4.24), it follows that*

$$A = \varphi_1\varphi_2 - (\varphi_1 - 1)(\psi + \varphi_2) > \varphi_1\varphi_2.$$

Hence if $\varphi_1\varphi_2 \geq 0$*, then* $A > 0$*.*

Along the solutions of (4.26), it easily follows that

$$\begin{aligned}\frac{dE}{d\tau} &= \frac{1}{2}\left\{\frac{d\mu_1}{d\tau}u^2 + \frac{d\mu_2}{d\tau}v^2 + 2\mu_1 u\left[-\varphi_1 u + (\varphi_1 - 1)v - uv\right] + \right.\\ &\left. +2\mu_2 v\left[(\psi+\varphi_2)u - \varphi_2 v + \psi uv\right]\right\} = \\ &= \frac{1}{2}\left\{\left(\frac{d\mu_1}{d\tau} - 2\varphi_1\mu_1\right)u^2 + \left(\frac{d\mu_2}{d\tau} - 2\varphi_2\mu_2\right)v^2 + \right.\\ &\left. +2\left[\mu_1(\varphi_1 - 1) + \mu_2(\psi+\varphi_2)\right]uv\right\} + \Phi(\tau),\end{aligned} \tag{4.30}$$

with

$$\Phi(\tau) = (-\mu_1 u + \psi\mu_2 v)uv. \tag{4.31}$$

Moreover, setting

$$\begin{cases}\Psi(\tau) = (A_1 u - A_3 v)(-uv) + (A_2 v - A_3 u)(\psi uv), \quad I = -(\varphi_1 + \varphi_2),\\ A_1 = A + \varphi_2^2 + (\psi+\varphi_2)^2,\ A_2 = A + \varphi_1^2 + (\varphi_1 - 1)^2,\\ A_3 = -[\varphi_1(\psi+\varphi_2) + \varphi_2(\varphi_1 - 1)],\end{cases} \tag{4.32}$$

along the solutions of (4.26) it turns out that

$$\frac{dV}{d\tau} = P(\tau, u, v) + \Psi(\tau, u, v), \tag{4.33}$$

with

$$\begin{cases} P = \frac{1}{2}\sum_{i=1}^{3} P_i(\tau, u, v), \quad P_1 = \left(2IA + \frac{dA}{d\tau}\right)(u^2 + v^2),\\ P_2 = \frac{d(\psi+\varphi_2)^2}{d\tau}u^2 + \frac{d\varphi_1^2}{d\tau}v^2 + 2\frac{d[\varphi_1(\psi+\varphi_2)]}{d\tau}uv,\\ P_3 = \frac{d\varphi_2^2}{d\tau}u^2 + \frac{d(\varphi_1 - 1)^2}{d\tau}v^2 + 2\frac{d[\varphi_2(\varphi_1 - 1)]}{d\tau}uv.\end{cases} \tag{4.34}$$

Remark 4.3 *Since*

$$\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = af_1(t)\frac{d}{d\tau}, \tag{4.35}$$

and $af_1(t) > 0$, $\forall t \in \mathbb{R}^+$, *then*

$$\frac{df}{d\tau} > 0 \Leftrightarrow \frac{df}{dt} > 0, \qquad \forall f \in C^1(\mathbb{R}^+). \tag{4.36}$$

By virtue of (4.36) we can state the results for the stability-instability of the null solution of (4.26) by means of conditions on $\frac{dE}{dt}$ *and* $\frac{dV}{dt}$ *instead of* $\frac{dE}{d\tau}$ *and* $\frac{dV}{d\tau}$.

Remark 4.4 *Let be* $f : \mathbb{R}^+ \to \mathbb{R}$ *and*

$$f_* = \inf_{\mathbb{R}^+} f, \qquad f^* = \sup_{\mathbb{R}^+} f, \tag{4.37}$$

then it turns out that

(i) at any instant $\bar{t} \in \mathbb{R}^+$ *and* $\forall A(\bar{t})$, *in any disk of the phase space, centered at the origin* $O = (0, 0)$, *there exists a domain in which* $V(\bar{t}, u, v) > 0$;

(ii) if $A_* > 0$, *then there exists a positive constant* m_1 *such that*

$$A_*(u^2 + v^2) < V < m_1(u^2 + v^2), \quad \forall \tau \in \mathbb{R}^+, \tag{4.38}$$

and hence V *is bounded, has an infinitely small upper bound and is positive definite;*

(iii) the property (*i*) *holds also for the energy* E *either when* $(\mu_1)_* > 0$ *or* $(\mu_2)_* > 0$;

(iv) the property (*ii*) *holds also for the energy* E, *when* $(\mu_i)_* > 0$, $(i = 1, 2)$. *In fact one has*

$$m_2(u^2 + v^2) < E < m_3(u^2 + v^2), \tag{4.39}$$

with

$$m_2 < \frac{1}{2}\inf\left((\mu_1)_*, (\mu_2)_*\right), \quad m_3 > \frac{1}{2}\sup\left((\mu_1)^*, (\mu_2)^*\right). \tag{4.40}$$

Lemma 4.1 *The polynomial P_2 reduces to*

$$\begin{cases} P_2 = \dfrac{d(\psi+\varphi_2)^2}{d\tau}(u \pm v)^2 \quad \textit{for} \quad \varphi_1 = \pm(\psi+\varphi_2),\ \forall\tau \in \mathbb{R}^+, \\ \\ P_2 = \dfrac{d(\psi+\varphi_2)^2}{d\tau}u^2 \qquad\qquad \textit{for} \quad \varphi_1 \equiv 0,\ \forall\tau \in \mathbb{R}^+, \end{cases} \tag{4.41}$$

while P_3 reduces to

$$\begin{cases} P_3 = \dfrac{d(\varphi_1-1)^2}{d\tau}(u \pm v)^2 \quad \textit{for} \quad \varphi_1 = 1 \pm \varphi_2,\ \forall\tau \in \mathbb{R}^+, \\ \\ P_3 = \dfrac{d(\varphi_1-1)^2}{d\tau}v^2 \qquad\qquad \textit{for} \quad \varphi_2 \equiv 0,\ \forall\tau \in \mathbb{R}^+. \end{cases} \tag{4.42}$$

If no one of the functions φ_i $(i = 1, 2)$ is identically zero, the polynomials P_i $(i = 2, 3)$, cannot be negative definite.

Proof. (4.41) and (4.42) are easily implied by (4.34). Further, since

$$\begin{cases} \left[\dfrac{d(\varphi_1(\psi+\varphi_2))}{d\tau}\right]^2 - \dfrac{d\varphi_1^2}{d\tau}\dfrac{d(\psi+\varphi_2)^2}{d\tau} = \left(\dfrac{d\varphi_1}{d\tau}(\psi+\varphi_2) - \varphi_1\dfrac{d(\psi+\varphi_2)}{d\tau}\right)^2, \\ \\ \left[\dfrac{d[\varphi_2(\varphi_1-1)]}{d\tau}\right]^2 - \dfrac{d\varphi_2^2}{d\tau}\dfrac{d(\varphi_1-1)^2}{d\tau} = \left(\dfrac{d\varphi_2}{d\tau}(\varphi_1-1) - \varphi_2\dfrac{d(\varphi_1-1)}{d\tau}\right)^2, \end{cases} \tag{4.43}$$

one immediately deduces that P_i $(i = 2, 3)$, as quadratic forms of u and v, cannot be negative definite when no one of the functions φ_i $(i = 1, 2)$ is identically zero.

Remark 4.5 *We call critical case, the case in which*

$$AI \equiv 0, \qquad \forall\tau \in \mathbb{R}^+. \tag{4.44}$$

Lemma 4.2 *The quadratic polynomial $P_2 + P_3$ is*

(i) *positive semidefinite for either*

$$\begin{cases} \varphi_1(\psi+\varphi_2) + \varphi_2(\varphi_1-1) = \textit{const.},\ \dfrac{d}{d\tau}[\varphi_2^2 + (\psi+\varphi_2)^2] \geq k_1, \\ \\ \dfrac{d}{d\tau}[\varphi_1^2 + (\varphi_1-1)^2] \geq k_2,\ \forall\tau \in \mathbb{R}^+, \end{cases} \tag{4.45}$$

or

$$\frac{\varphi_1}{\psi+\varphi_2} = const., \ \frac{\varphi_1 - 1}{\varphi_2} = const., \ \frac{d(\psi+\varphi_2)^2}{d\tau} \geq k_3, \ \frac{d\varphi_2^2}{d\tau} \geq k_4, \tag{4.46}$$

or

$$\frac{\varphi_1}{\psi} = const., \ \varphi_2 = 0, \ \frac{d\psi^2}{d\tau} \geq k_5, \ \frac{d(\varphi_1 - 1)^2}{d\tau} \geq k_6, \ \forall \tau \in \mathbb{R}^+, \tag{4.47}$$

with k_i $(i = 1, ..., 6)$ *non negative constants;*

(ii) *positive definite if the constants* k_i *appearing - either in (4.45) or (4.46) or (4.47) - are positive;*

(iii) *negative semidefinite for either*

$$\begin{cases} \varphi_1(\psi+\varphi_2) + \varphi_2(\varphi_1 - 1) = const., \ \frac{d}{d\tau}[\varphi_2^2 + (\varphi_1 - 1)^2] \leq -k_1, \\ \\ \frac{d}{d\tau}[\varphi_1^2 + (\varphi_1 - 1)^2] \leq -k_2, \ \forall \tau \in \mathbb{R}^+, \end{cases} \tag{4.48}$$

or

$$\frac{\varphi_1}{\psi+\varphi_2} = const., \ \frac{\varphi_1 - 1}{\varphi_2} = const., \ \frac{d(\psi+\varphi_2)^2}{d\tau} \leq -k_3, \ \frac{d\varphi_2^2}{d\tau} \leq -k_4, \tag{4.49}$$

or

$$\frac{\varphi_1}{\psi} = const., \ \varphi_2 = 0, \ \frac{d\psi^2}{d\tau} \leq -k_5, \ \frac{d(\varphi_1 - 1)^2}{d\tau} \leq -k_6, \ \forall \tau \in \mathbb{R}^+, \tag{4.50}$$

with k_i, $(i = 1, ..., 6)$ *non negative constants;*

(iv) *negative definite if the constants* k_i *appearing - either in (4.48) or (4.49) or (4.50) - are positive;*

(v) *indefinite in the other cases.*

Proof. For the proof see [38].

Remark 4.6 *Apart from the critical case one immediately deduces that*

(i) if $A > 0$, $\forall \tau \in \mathbb{R}^+$, the existence of a positive constant h such that

$$P_1 \leq -h(u^2 + v^2), \tag{4.51}$$

is necessary for guaranteeing the (local) asymptotic stability;

(ii) if $A < 0$, $\forall \tau \in \mathbb{R}^+$, the existence of a positive constant h such that

$$P_1 > h(u^2 + v^2), \tag{4.52}$$

is necessary for guaranteeing the (Chetaiev) instability.

For the sake of completeness we recall here some Lemmas, proved in [38] that we will use to obtain stability/instability results.

Lemma 4.3 *Suppose that*

$$A_* > 0\,, \quad I_* > 0. \tag{4.53}$$

Then does not exist a positive constant h such that

$$P_1 \leq -h(u^2 + v^2)\,, \qquad \forall \tau \in \mathbb{R}^+, \tag{4.54}$$

and P_1 is positive semidefinite for

$$A \geq A_0 e^{-2I_* \tau}\,, \qquad A_0 = A(0), \tag{4.55}$$

and positive definite, according to

$$P_1 \geq A_* I_* (u^2 + v^2), \tag{4.56}$$

for

$$\frac{dA}{d\tau} \geq 0, \qquad \forall \tau \in \mathbb{R}^+. \tag{4.57}$$

Remark 4.7 *We observe that*

$$I_* = \inf(-(\varphi_1 + \varphi_2)) = -\sup(\varphi_1 + \varphi_2). \tag{4.58}$$

Hence (4.53) can be written as

$$A_* > 0, \qquad (\varphi_1 + \varphi_2)^* < 0. \tag{4.59}$$

Lemma 4.4 *Suppose that*

$$A_* > 0, \qquad I^* < 0. \tag{4.60}$$

Then does not exist a positive constant h such that

$$P_1 \geq h(u^2 + v^2), \tag{4.61}$$

and P_1 is negative semidefinite for

$$A \leq A_0 e^{-2I_*\tau}, \tag{4.62}$$

and negative definite, either according to

$$P_1 \leq -A_*|I_*|(u^2 + v^2), \tag{4.63}$$

for

$$\frac{dA}{d\tau} \leq 0, \tag{4.64}$$

or according to

$$P_1 \leq -2\varepsilon A_*|I_*|(u^2 + v^2), \quad 0 < \varepsilon = const. < 1, \tag{4.65}$$

for

$$A \leq A_0(1 - \varepsilon)e^{2|I_*|\tau}, \ \forall \tau > 0. \tag{4.66}$$

Lemma 4.5 *Suppose that*

$$A^* < 0, \qquad I^* < 0. \tag{4.67}$$

Then does not exist a positive constant h such that (4.54) holds. Further P_1 is positive semidefinite for

$$A \geq A_0 e^{-2I^* \tau}, \tag{4.68}$$

and positive definite according to

$$P_1 \geq A_* I_* (u^2 + v^2), \tag{4.69}$$

when (4.57) holds.

4.3 Stability criteria

Theorem 4.3 *Suppose that (4.22)-(4.24), (4.60) and (4.62) or (4.60) and (4.64) or (4.60) and (4.66) hold together with the condition (iii) or (iv) of Lemma 4.2. Then the null solution of system (4.26) is nonlinearly, asymptotically, exponentially (locally) stable.*

Proof. By virtue of the hypotheses, there exist two positive constants h_1 and h_2, such that

$$\begin{cases} P_1 \leq -h_1(u^2 + v^2), \\ \\ P_2 + P_3 \leq -h_2(u^2 + v^2)\,, \end{cases} \tag{4.70}$$

and hence, from (4.33) and (4.70), one obtains that

$$\frac{dV}{d\tau} \leq -\frac{h}{2}(u^2 + v^2) + |\Psi|, \tag{4.71}$$

with $h = \inf(h_1, h_2)(> 0)$. The boundedness of ψ, φ_i $(i = 1, 2)$, implies that

$$|\Psi| \leq M(u + v)(uv) \leq \frac{M\sqrt{2}}{2}(u^2 + v^2)^{\frac{3}{2}}, \tag{4.72}$$

with

$$M = \max(|A_3 + A_2\psi|, |A_1 + A_3\psi|). \tag{4.73}$$

Hence, starting from (4.71), one obtains:

$$\frac{dV}{d\tau} \leq -\delta_1 V + \delta_2 V^{\frac{3}{2}} = V(-\delta_1 + \delta_2 V^{\frac{1}{2}}), \tag{4.74}$$

with

$$\delta_1 = \frac{h}{2m_1}(> 0), \qquad \delta_2 = \frac{M\sqrt{2}}{2A_*^{\frac{3}{2}}}(> 0). \tag{4.75}$$

Then the assumption $V_0^{\frac{1}{2}} < \dfrac{\delta_1}{\delta_2}$ implies, by recursive argument, that:

$$V \leq V_0 e^{\delta\tau}, \qquad \delta = \delta_1 - \delta_2 V_0^{\frac{1}{2}}(> 0). \tag{4.76}$$

Moreover, by virtue of (4.38), V and $W = (u^2 + v^2)$ are equivalent and in particular (4.38) is satisfied. Now, since all the hypotheses of the Liapunov (asymptotic) stability theorem are satisfied, the null solution of (4.26) is nonlinearly, asymptotically, exponentially, locally stable.

Remark 4.8 *By virtue of Lemmas 4.2-4.5 and Theorem 4.3, apart from the critical case* $IA \equiv 0$*, the conditions (4.60) or the equivalent conditions*

$$A_* > 0, \qquad (AI)^* < 0, \tag{4.77}$$

appear to be the basic conditions to guarantee the stability of the null solution of (4.26).

Theorem 4.4 *Suppose that* (4.22) – (4.24) *hold and let (4.60) hold by virtue of*

$$(\varphi_1)_* \geq k_1, \qquad (\varphi_2)_* \geq k_2, \tag{4.78}$$

with k_i $(i = 1, 2)$ *positive constants. Then*

$$\begin{cases} \varphi_1 \geq 1 - (1 - \varphi_1)_{\tau=0} e^{2(k_1 - \varepsilon)\tau}, \\ \\ \psi + \varphi_2 \leq (\psi + \varphi_2)_{\tau=0} e^{2(k_2 - \varepsilon)\tau}, \end{cases} \tag{4.79}$$

with $0 \leq \varepsilon < \inf(k_1, k_2)$*, guarantee the (local) nonlinear asymptotic exponential stability of the null solution of* (4.26)*.*

Proof. For the proof see [38].

Remark 4.9 *Obviously (4.26) cannot admit periodic solutions when the conditions guaranteeing the asymptotic stability of the null solution hold.*

Theorem 4.5 *Suppose that* (4.22) – (4.24) *and* (4.60) *hold and let us assume that*

$$\begin{cases} (\overline{\varepsilon}_1\varphi_1)_* > 0, \ (\overline{\varepsilon}_2\varphi_2)_* > 0, \ (\varphi_1+\varphi_2)_* > 0, \\ \\ \dfrac{\psi+\varphi_2}{1-\varphi_1} < \left(\dfrac{\psi+\varphi_2}{1-\varphi_1}\right)_{\tau=0} e^{-4(1-\overline{\varepsilon}_1)F(\tau)}, \ \forall\tau>0, \\ \\ \dfrac{\psi+\varphi_2}{1-\varphi_1} > \left(\dfrac{\psi+\varphi_2}{1-\varphi_1}\right)_{\tau=0} e^{4(1-\overline{\varepsilon}_2)G(\tau)}, \ \forall\tau>0, \\ \\ F(\tau) = \displaystyle\int_0^\tau -\varphi_1(z)\,dz, \quad G(\tau) = \displaystyle\int_0^\tau -\varphi_2(z)\,dz, \end{cases} \tag{4.80}$$

with $\overline{\varepsilon}_i$ $(i=1,2)$ *constants such that*

$$(1-\overline{\varepsilon}_1)\varphi_1 + (1-\overline{\varepsilon}_2)\varphi_2 > 0, \quad \forall\tau\in\mathbb{R}^+, \tag{4.81}$$

then the zero solution of (4.26) *is nonlinearly (locally) asymptotically exponentially stable.*

Proof. Requiring

$$\begin{cases} -2\varphi_1\mu_1 + \dfrac{d\mu_1}{d\tau} < -2\overline{\varepsilon}_1\varphi_1\mu_1, \\ \\ -2\varphi_2\mu_2 + \dfrac{d\mu_2}{d\tau} < -2\overline{\varepsilon}_2\varphi_2\mu_2, \end{cases} \tag{4.82}$$

one easily obtains that

$$\begin{cases} \mu_1 < \mu_1(0)e^{-2(1-\overline{\varepsilon}_1)F(\tau)}, \\ \\ \mu_2 < \mu_2(0)e^{-2(1-\overline{\varepsilon}_2)G(\tau)}. \end{cases} \tag{4.83}$$

The hypotheses $(4.80)_4 - (4.80)_5$, are verified by

$$\mu_1 = \left(\frac{\psi + \varphi_2}{1 - \varphi_1}\right)^{\frac{1}{2}}, \ \mu_2 = \left(\frac{1 - \varphi_1}{\psi + \varphi_2}\right)^{\frac{1}{2}}. \tag{4.84}$$

With this choice, the energy $E(\tau)$ given by (4.30) has to satisfy

$$\frac{dE}{d\tau} < -\frac{1}{2}\left[2(\overline{\varepsilon}_1\varphi_1)_*\mu_1 u^2 + 2(\overline{\varepsilon}_2\varphi_2)_*\mu_2 v^2\right] + \Phi. \tag{4.85}$$

with Φ given by (4.30). Setting

$$h_2 = 2\min((\overline{\varepsilon}_1\varphi_1)_*, (\overline{\varepsilon}_2\varphi_2)_*), \tag{4.86}$$

one obtains

$$\frac{dE}{d\tau} < -h_2 E + \Phi. \tag{4.87}$$

The boundedness of ψ, φ_i $(i = 1, 2)$, implies that

$$|\Phi| \le M'(u + v)(uv) \le \frac{M'\sqrt{2}}{2}(u^2 + v^2)^{\frac{3}{2}}, \tag{4.88}$$

with

$$M' = \max(|\mu_1|, |\psi\mu_2|). \tag{4.89}$$

Hence, starting from (4.87) one obtains:

$$\frac{dE}{d\tau} \le -\delta_1' E + \delta_2' E^{\frac{3}{2}} = E(-\delta_1' + \delta_2' E^{\frac{1}{2}}), \tag{4.90}$$

with

$$\delta_1' = \frac{h_2}{m_3}(> 0), \qquad \delta_2' = \frac{M'\sqrt{2}}{2m_2^{\frac{3}{2}}}(> 0). \tag{4.91}$$

Then the assumption $E_0^{\frac{1}{2}} < \dfrac{\delta_1'}{\delta_2'}$ implies, by recursive argument, that:

$$E \le E_0 e^{\delta' \tau}, \qquad \delta' = \delta_1' - \delta_2' E_0^{\frac{1}{2}}(> 0). \tag{4.92}$$

Moreover, by virtue of (4.39), E and $W = (u^2 + v^2)$ are equivalent and in particular (4.39) $-$ (4.40) are satisfied. Now, since all the hypotheses of the Liapunov (asymptotic) stability theorem are satisfied, the null solution of (4.26) is nonlinearly, asymptotically, exponentially, locally stable.

Remark 4.10 *We observe that*

(i) $(4.80)_3$ *is necessary for the consistence of* (4.81)*, while* (4.81) *guarantees the consistence of* $(4.80)_4 - (4.80)_5$*;*

(ii) theorem 4.5 does not require necessarily

$$(\varphi_1)_* > 0, \quad (\varphi_2)_* > 0, \tag{4.93}$$

but can hold also if

$$\varphi_1\varphi_2 < 0, \quad \forall\tau > 0, \quad A_* > 0. \tag{4.94}$$

In fact, let

$$\begin{cases} \varphi_1 = -\varphi_2[\varphi(\tau) + 1], \; \varphi_2 < 0, \; \varphi > 0, \\ \\ \overline{\varepsilon}_1 = \dfrac{1}{2}, \; \overline{\varepsilon}_2 = -\dfrac{1}{2}, A_* > 0. \end{cases} \tag{4.95}$$

Then

$$(1 - \overline{\varepsilon}_1)\varphi_1 + (1 - \overline{\varepsilon}_2)\varphi_2 = -\frac{\varphi + 1}{2}\varphi_2 + \frac{3}{2}\varphi_2 > 0, \tag{4.96}$$

is verified by

$$\varphi > 2, \quad \forall\tau > 0. \tag{4.97}$$

Theorem 4.6 *Let* (4.22) – (4.24) *hold and suppose that (4.60) hold by virtue of*

$$(\varphi_1)_* > 0, \; (\varphi_2)_* > 0. \tag{4.98}$$

Assuming that

$$(1 - \varphi_1)_*(\psi + \varphi_2)^* < (\varphi_1)_*(\varphi_2)_*, \tag{4.99}$$

the zero solution of (4.26) *is nonlinearly (locally) asymptotically exponentially stable.*

Proof. Choosing

$$\mu_1 = \left(\frac{(\psi+\varphi_2)^*}{(1-\varphi_1)_*}\right)^{\frac{1}{2}},\ \mu_2 = \left(\frac{(1-\varphi_1)_*}{(\psi+\varphi_2)^*}\right)^{\frac{1}{2}}, \tag{4.100}$$

it follows that

$$\frac{dE}{d\tau} \le -\varphi_1\mu_1 u^2 - \varphi_2\mu_2 v^2 + \left[\mu_1(1-\varphi_1)_* + \mu_2(\psi+\varphi_2)^*\right] uv + \Phi. \tag{4.101}$$

In view of (4.99) – (4.100) one obtains

$$\frac{dE}{d\tau} \le -(\varphi_1)_*\mu_1 u^2 - (\varphi_2)_*\mu_2 v^2 + 2\sqrt{(1-\varphi_1)_*(\psi+\varphi_2)^*}uv + \Phi. \tag{4.102}$$

Since $\mu_1 = \mu_2^{-1}$ it turns out that

$$\begin{cases} (1-\varphi_1)_*(\psi+\varphi_2)^* < (\varphi_1)_*(\varphi_2)_* = \mu_1(\varphi_1)_*\mu_2(\varphi_2)_*, \\ \\ (1-\varphi_1)_*(\psi+\varphi_2)^* = \eta^2\mu_1(\varphi_1)_*\mu_2(\varphi_2)_*, \\ \\ 2\sqrt{(1-\varphi_1)_*(\psi+\varphi_2)^*}uv \le \eta(\mu_1(\varphi_1)_*u^2 + \mu_2(\varphi_2)_*v^2), \end{cases} \tag{4.103}$$

with $0 < \eta = const. < 1$. Then (4.102) becomes

$$\frac{dE}{d\tau} \le -(1-\eta)\left[(\varphi_1)_*\mu_1 u^2 + \mu_2(\varphi_2)_* v^2\right] + \Phi, \tag{4.104}$$

and hence

$$\frac{dE}{d\tau} \le -h_3 E + \Phi, \tag{4.105}$$

with

$$h_3 = 2(1-\eta)\min((\varphi_1)_*, (\varphi_2)_*). \tag{4.106}$$

Following the same procedure used in theorem 4.5, the thesis is hold.

Theorem 4.7 *Let* (4.22) – (4.24) *hold together with (4.60) by virtue of*

$$(\varphi_1)_* > 0, \qquad (\varphi_2)_* > 0, \tag{4.107}$$

and

$$(1-\varphi_1)_* + (\psi+\varphi_2)^* < 2\sqrt{(\varphi_1)_*(\varphi_2)_*}. \tag{4.108}$$

Then the null solution of system (4.26) *is nonlinearly (locally) asymptotically exponentially stable.*

Proof. Choosing $\mu_1 = \mu_2 = 1$, from (4.30) it follows that

$$\begin{aligned}\frac{dE}{d\tau} &= \frac{1}{2}\left[-2\varphi_1 u^2 - 2\varphi_2 v^2 + 2\left(\varphi_1 - 1 + \varphi_2 + \psi\right)uv\right] + \Phi \\ &\leq -(\varphi_1)_* u^2 - (\varphi_2)_* v^2 + ((1-\varphi_1)_* + (\psi+\varphi_2)^*)uv + \Phi.\end{aligned} \tag{4.109}$$

From (4.108), $\exists \eta = const. \in]0,1[$ such that

$$(1-\varphi_1)_* + (\psi+\varphi_2)^* = 2\eta\sqrt{(\varphi_1)_*(\varphi_2)_*}. \tag{4.110}$$

Hence

$$\begin{aligned}\frac{dE}{d\tau} &\leq -(\varphi_1)_* u^2 - (\varphi_2)_* v^2 + 2\eta\sqrt{(\varphi_1)_*(\varphi_2)_*}uv + \Phi \\ &\leq -h_4(1-\eta)(u^2+v^2) + \Phi,\end{aligned} \tag{4.111}$$

with $h_4 = \min((\varphi_1)_*, (\varphi_2)_*)$. Then following the same procedure used in theorem 4.5, the thesis is hold.

Theorem 4.8 *Let (4.22)-(4.24) hold and suppose that*

$$A_* > 0, \qquad I \equiv 0, \tag{4.112}$$

together with (*iii*) *or* (*iv*) *of Lemma 4.2. Then if*

$$\frac{dA}{d\tau} \leq 0 \qquad \forall \tau \in \mathbb{R}^+, \tag{4.113}$$

or if

$$\left(\frac{dA}{d\tau}\right)^* = -\tilde{k}, \qquad \tilde{k} = const. > 0, \tag{4.114}$$

the null solution of (4.26) is nonlinearly (locally) asymptotically, exponentially, stable.

Proof. From (4.28), by virtue of (4.112), one immediately obtains that V is positive definite. Moreover, from (4.33), on taking into account (ii) or (iv) of Lemma 4.2, it follows that

$$\frac{dV}{d\tau} \leq -h(u^2 + v^2) + |\Psi|.$$

Adopting the same procedure followed in theorem 4.3, the thesis is hold.

Theorem 4.9 *Let (4.22)-(4.24) hold together with*

$$A \equiv 0. \tag{4.115}$$

Then, if (iii) *or* (iv) *of Lemma 4.2 hold, the null solution of (4.26) is nonlinearly (locally) asymptotically, exponentially, stable.*

Proof. V is positive definite. Moreover if $A \equiv 0$, then $P_1 = 0$ and hence

$$\frac{dV}{d\tau} = \frac{1}{2}(P_2 + P_3) + \Psi.$$

From (iii) or (iv) of Lemma 4.2 it follows that

$$\frac{dV}{d\tau} \leq -\delta_1 V + \delta_2 V^{\frac{3}{2}}.$$

Following the same procedure used in theorem 4.3, the thesis is hold.

4.4 Instability criteria

Instability criteria can be obtained, of course, by means either of the Liapunov function (4.28) or the function (4.27). We here recall the instability theorems obtained in [38] by the function (4.28) {cfr. Theorems 4.10-4.13} and concentrated ourselves on the instability theorems obtained by using the function (4.27).

Theorem 4.10 *Suppose that (4.22)-(4.24), (4.53), (4.57) hold together with* (i) *of Lemma 4.2. Then the null solution of (4.26) is unstable.*

Theorem 4.11 *Suppose that (4.22)-(4.24), (4.67), (4.57) and* (*i*) *of Lemma 4.2 hold. Then the null solution of (4.26) is unstable.*

Theorem 4.12 *Suppose that*

$$I \equiv 0, \tag{4.116}$$

and (*i*) *of Lemma 4.2 hold. Then if*

$$\left(\frac{dA}{d\tau}\right)_* \geq \tilde{k} = const > 0, \tag{4.117}$$

the null solution of (4.26) is unstable.

Theorem 4.13 *Suppose that (4.22)-(4.24) hold. If*

$$A \equiv 0, \tag{4.118}$$

and (*ii*) *of Lemma 4.2 hold, then the null solution of (4.26) is (Chetaiev) unstable.*

Theorem 4.14 *Suppose that (4.53) hold by virtue of*

$$\varphi_1^* \leq -h_1, \qquad \varphi_2^* \leq -h_2, \tag{4.119}$$

with h_i $(i = 1, 2)$ *positive constants. Then*

$$\begin{cases} \psi + \varphi_2 > (\psi + \varphi_2)_{\tau=0} e^{-2(h_1-\varepsilon)\tau}, \\ \\ \varphi_1 < 1 - (1 - \varphi_1)_{\tau=0} e^{-2(h_2-\varepsilon)\tau}, \end{cases} \tag{4.120}$$

with $0 < \varepsilon < \inf(h_1, h_2)$ *guarantee the instability of the null solution of (4.26).*

Proof. Choosing

$$\mu_1 = \psi + \varphi_2, \qquad \mu_2 = 1 - \varphi_1, \tag{4.121}$$

it follows that

$$E = \frac{1}{2}\left[(\psi + \varphi_2)u^2 + (1 - \varphi_1)v^2\right], \tag{4.122}$$

is positive definite and

$$\frac{dE}{d\tau}=\frac{1}{2}\left[\left(\frac{d(\psi+\varphi_2)}{d\tau}-2\varphi_1(\psi+\varphi_2)\right)u^2+\left(\frac{d(1-\varphi_1)}{d\tau}-2\varphi_2(1-\varphi_1)\right)v^2\right], \tag{4.123}$$

where we have disregarded the contribution of Φ. Then $(4.24)_2$ and (4.120) guarantee that

$$\begin{cases} \dfrac{d(\psi+\varphi_2)}{d\tau} - 2\varphi_1(\psi+\varphi_2) > 2\varepsilon(\psi+\varphi_2)_* > 0, \\ \\ \dfrac{d(1-\varphi_1)}{d\tau} - 2\varphi_2(1-\varphi_1) > 2\varepsilon(1-\varphi_1)_* > 0. \end{cases} \tag{4.124}$$

Hence all the hypotheses of the instability Liapunov theorem are verified.

Theorem 4.15 *Suppose that (4.22)-(4.24) and (4.67) hold by virtue of*

$$(\varphi_1)_* \geq h_1, \qquad |\varphi_2|^* \leq h_2, \tag{4.125}$$

with h_i $(i = 1, 2)$ *positive constants such that* $h_1 > h_2$. *Then*

$$\begin{cases} \psi+\varphi_2 > (\psi+\varphi_2)_0 e^{2(\varepsilon+h_2)\tau}, \\ \\ \varphi_1 > 1 + (\varphi_1 - 1)_0 e^{-2(\varepsilon-h_1)\tau}, \end{cases} \tag{4.126}$$

imply the instability of the null solution of (4.26).

Proof. Since (4.67) hold, we have to require

$$\varphi_1\varphi_2 < (\varphi_1 - 1)(\psi+\varphi_2) + A^*, \quad \forall\tau \in \mathbb{R}^+. \tag{4.127}$$

In fact, if $h_1 > h_2$ it follows that

$$\varphi_1 + \varphi_2 \geq h_1 - |\varphi_2|^* \geq h_1 - h_2 > 0,$$

and hence $I^* < 0$.

Choosing

$$\mu_1 = \varphi_1 - 1, \quad \mu_2 = \psi + \varphi_2, \tag{4.128}$$

one has

$$E = \frac{1}{2}\left[(\varphi_1 - 1)u^2 + (\psi + \varphi_2)v^2\right], \tag{4.129}$$

and

$$\begin{aligned} \frac{dE}{d\tau} &= \frac{1}{2}\left\{\left[\frac{d\varphi_1}{d\tau} - 2\varphi_1(\varphi_1 - 1)\right]u^2 + \left[\frac{d(\psi+\varphi_2)}{d\tau} - 2\varphi_2(\psi+\varphi_2)\right]v^2 + \right. \\ &\left. +2\left[(\varphi_1 - 1)^2 + (\psi + \varphi_2)^2\right]uv\right\}, \end{aligned} \tag{4.130}$$

where we have disregarded the contribution of Φ. Hence

$$\frac{dE}{d\tau} \geq \frac{1}{2}\left\{\left[\frac{d(\varphi_1 - 1)}{d\tau} - 2\varphi_1(\varphi_1 - 1)\right]u^2 + \left[\frac{d(\psi+\varphi_2)}{d\tau} - 2\varphi_2(\psi+\varphi_2)\right]v^2\right\}, \tag{4.131}$$

and the conditions (4.126) guarantee that

$$\begin{cases} \dfrac{d(\varphi_1 - 1)}{d\tau} - 2\varphi_1(\varphi_1 - 1) > -2\varepsilon(\varphi_1 - 1)^* > 0, \\ \\ \dfrac{d(\psi + \varphi_2)}{d\tau} - 2\varphi_2(\psi + \varphi_2) > 2\varepsilon(\psi + \varphi_2)_* > 0. \end{cases} \tag{4.132}$$

Hence E satisfies all the hypotheses of the Chetaiev instability theorem.

4.5 The case $p, q \neq 0$

Now we want to analyze the following model

$$\begin{cases} \dot{x} = f_1(t)(a - by)x + D_1(t)\left(\dfrac{\overline{x}}{\overline{y}}y - x\right)^{1+p}, \\ \\ \dot{y} = f_2(t)(-c + dx)y + D_2(t)\left(\dfrac{\overline{y}}{\overline{x}}x - y\right)^{1+q}, \end{cases} \tag{4.133}$$

with $p, q \neq 0$, under the hypotheses $1) - 6)$.

For the study of the stability of the equilibrium, we will use a different approach with respect to that used for the case $p = q = 0$. Precisely, we will study at the first the linear stability-instability and we will reconduct all the results obtained for the nonlinear system, since we proved that the nonlinear terms verify the inequality (4.18),

Applying the transformation

$$\begin{cases} x = \overline{x}X, \ y = \overline{y}Y, \ \tau = a\displaystyle\int_0^t f_1(z)\,dz, \\ \\ \varphi_1(t) = \dfrac{D_1(t)\bar{x}^p}{af_1(t)}, \ \varphi_2(t) = \dfrac{D_2(t)\bar{y}^q}{af_1(t)}, \ \psi(t) = \dfrac{cf_2(t)}{af_1(t)}, \end{cases} \tag{4.134}$$

model (4.133) becomes

$$\begin{cases} \dfrac{dX}{d\tau} = X - XY + \varphi_1(Y - X)^{1+p}, \\ \\ \dfrac{dY}{d\tau} = -\psi Y + \psi XY + \varphi_2(X - Y)^{1+q}. \end{cases} \tag{4.135}$$

Remark 4.11 *From (4.135) one has*

$$\begin{cases} X(\tau) = X_0 \exp\displaystyle\int_0^\tau \left[1 - Y(z) + \frac{\varphi_1(z)}{X(z)}(Y(z) - X(z))^{1+p}\right] dz, \\ \\ Y(\tau) = Y_0 \exp\displaystyle\int_0^\tau \left[-\psi(z) + \psi(z)X(z) + \frac{\varphi_2(z)}{Y(z)}(X(z) - Y(z))^{1+q}\right] dz, \end{cases} \tag{4.136}$$

and hence $\{X_0 > 0, Y_0 > 0\} \Rightarrow \{X(\tau) > 0, Y(\tau) > 0\} \forall \tau > 0$.

Model (4.135) admits $(1, 1)$ as equilibrium point.
Setting

$$X = u + 1, \qquad Y = v + 1, \tag{4.137}$$

the model becomes

$$\begin{cases} \dfrac{du}{d\tau} = -v - uv + \varphi_1(t)(v - u)^{1+p}, \\ \\ \dfrac{dv}{d\tau} = \psi(t)u + \psi(t)uv + \varphi_2(t)(u - v)^{1+q}, \end{cases} \tag{4.138}$$

For the study of the stability of the null solution of system (4.138), the Liapunov function [19] is given by

$$V = \frac{1}{2}\left[\psi(1+\psi)u^2 + (1+\psi)v^2\right] = \frac{1}{2}(1+\psi)\left[\psi^2u^2 + v^2\right]. \tag{4.139}$$

The temporal derivative of V, along the solution of (4.138), is

$$\frac{dV}{d\tau} = \frac{1}{2}\frac{d\psi}{d\tau}\left[(1+2\psi)u^2 + v^2\right] + \overline{F}, \tag{4.140}$$

with

$$\begin{cases} \overline{F} = (1+\psi)(\psi u F_1 + F_2), \\ \\ F_1 = -uv + \varphi_1(v-u)^{1+p}, \quad F_2 = \psi uv + \varphi_2(u-v)^{1+q}. \end{cases} \tag{4.141}$$

Remark 4.12 *Since $\psi_* > 0$, then there exists a positive constant m_1 such that*

$$\psi_*(u^2+v^2) < V < m_1(u^2+v^2), \quad \forall\tau\in\mathbb{R}^+, \tag{4.142}$$

and hence V is bounded, has an infinitely small upper bound and is positive definite.

4.6 Linear stability-instability theorems

Disregarding the contribution of nonlinear terms, model (4.138) can be written as

$$\begin{cases} \dfrac{du}{d\tau} = -v, \\ \\ \dfrac{dv}{d\tau} = \psi u. \end{cases} \tag{4.143}$$

Easily the following theorems hold.

Theorem 4.16 *The null solution of system (4.143) is stable if*

$$\frac{d\psi}{d\tau} \leq 0, \tag{4.144}$$

and it is asymptotically stable if

$$\left(\frac{d\psi}{d\tau}\right)^* = -\tilde{k}, \quad \tilde{k} = \text{ positive constant.} \tag{4.145}$$

Theorem 4.17 *If*

$$\left(\frac{d\psi}{d\tau}\right)^* \geq \tilde{k}, \quad \tilde{k} = \text{ positive constant,} \tag{4.146}$$

then the null solution of (4.143) is unstable.

Remark 4.13 *We recall that*

i) theorems 4.16-4.17 continue to hold also when the coefficients of (4.143) are periodic functions of τ,

ii) when the conditions guaranteeing the asymptotic stability of the null solution of (4.143) hold, then (4.143) cannot admit periodic solutions not even when the coefficients are periodic functions of τ of the same period.

4.7 Nonlinear stability-instability theorems

Theorem 4.18 *F_1 and F_2 in (4.141) verify*

$$(|u| + |v|)(|F_1| + |F_2|) \leq \varepsilon_1 (u^2 + v^2)^{1+\varepsilon_2}, \tag{4.147}$$

with ε_i $(i = 1, 2)$, positive constants. Hence all the results for the linear stability-instability, hold also for system (4.138).

Proof. Since

$$|u| + |v| \leq \sqrt{2}(u^2 + v^2)^{\frac{1}{2}}, \tag{4.148}$$

we have that (4.147) is satisfied when

$$|F_1| + |F_2| \leq M(u^2 + v^2)^{\frac{1}{2}+\varepsilon_2}, \tag{4.149}$$

with M positive constant. But, since the boundedness of φ_i, $(i = 1, 2)$ and ψ,

$$\begin{aligned} |F_1| + |F_2| &= |-uv + \varphi_1(v-u)^{1+p} + \psi uv + \varphi_2(u-v)^{1+q}| \leq \\ &\leq a_1|uv| + a_2|(v-u)^{1+p}| + a_3|(u-v)^{1+q}|, \end{aligned}$$

$a_i\,(i=1,2,3)$, positive constants. Setting

$$z = \max\{p, q\}, \tag{4.150}$$

one obtains

$$|F_1| + |F_2| \leq b_1|uv| + b_2(|u| + |v|)^{1+z}, \quad b_i = \text{const} > 0\,(i = 1, 2). \tag{4.151}$$

Since (4.148) and

$$|uv| \leq \frac{(u^2 + v^2)}{2}, \tag{4.152}$$

hold, then

$$|F_1| + |F_1| \leq c_1(u^2 + v^2) + c_2(u^2 + v^2)^{\frac{1+z}{2}} \quad c_i = \text{const} > 0\,(i = 1, 2). \tag{4.153}$$

We have to distinguish two case:

i) $u^2 + v^2 \leq 1$,

ii) $u^2 + v^2 \geq 1$.

In the first case one can choose

$$\varepsilon_2 = \min\left\{\frac{1}{2}, \frac{z}{2}\right\},$$

in the second case it is sufficient choose

$$\varepsilon_2 = \max\left\{\frac{1}{2}, \frac{z}{2}\right\},$$

and hence, for example, one can choose $\varepsilon_2 = \dfrac{1+z}{2}$.

Chapter 5

Generalized Lotka-Volterra models with high nonlinearities

Now we consider another kind of nonautonomous perturbations to the classical Lotka-Volterra model. In particular, we want to determine the most general conditions assuring the stability-instability of the equilibrium point $(\overline{x}, \overline{y})$ for the following model

$$\begin{cases} \dot{x} = af_1(t)x - bf_1(t)\dfrac{x^{1+p}y^{1+q}}{\overline{x}^p\overline{y}^q}, \\ \\ \dot{y} = -cf_2(t)y + df_2(t)\dfrac{x^{1+p}y^{1+q}}{\overline{x}^p\overline{y}^q}. \end{cases} \tag{5.1}$$

with $p, q \neq 0$, under the hypotheses $1) - 6)$ of Chapter 4.
Setting

$$x = \overline{x}X, \ \ y = \overline{y}Y, \ \ \tau = a\int_0^t f_1(z)\,dz, \ \ \psi(t) = \frac{cf_2(t)}{af_1(t)}, \tag{5.2}$$

(5.1) becomes

$$\begin{cases} \dfrac{dX}{d\tau} = X - X^{1+p}Y^{1+q}, \\ \\ \dfrac{dY}{d\tau} = -\psi Y + \psi X^{1+p}Y^{1+q}. \end{cases} \tag{5.3}$$

Remark 5.1 *We remark that*

i) $(1,1)$ *is a critical point of (5.3),* $\forall (p,q) \in (\mathbb{N}^+)^2$*;*

ii) in view of (5.3) it follows that

$$\begin{cases} X = X_0 \exp \displaystyle\int_0^\tau \left[1 - X^p(z)Y^{1+q}(z)\right] dz, \\ \\ Y = Y_0 \exp \displaystyle\int_0^\tau \left[-\psi(z) + \psi(z)X^{1+p}(z)Y^q(z)\right] dz, \end{cases}$$

and hence $\{X_0 > 0, Y_0 > 0\} \Rightarrow \{X(\tau) > 0, Y(\tau) > 0, \forall \tau > 0\}$.

Setting

$$X = u + 1, \qquad Y = v + 1, \tag{5.4}$$

(5.3) can be written as

$$\begin{cases} \dfrac{du}{d\tau} = u + F, \\ \\ \dfrac{dv}{d\tau} = -\psi v - \psi F, \end{cases} \tag{5.5}$$

with

$$F = 1 - (1+u)^{1+p}(1+v)^{1+q}. \tag{5.6}$$

Denoting with ξ the nonlinear part of F, it follows that

$$F(u,v) = -(1+p)u - (1+q)v + \xi. \tag{5.7}$$

Hence (5.5) becomes

$$\begin{cases} \dfrac{du}{d\tau} = -pu - (1+q)v + \xi, \\ \\ \dfrac{dv}{d\tau} = (1+p)\psi u + q\psi v - \psi\xi, \end{cases} \tag{5.8}$$

In order to study the stability/instability of the null solution of system (5.8), we will consider the the standard "energy"

$$E = \frac{1}{2}\left[\mu_1 u^2 + \mu_2 v^2\right], \tag{5.9}$$

with

$$\mu_i \in C^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+), \tag{5.10}$$

and the Liapunov function [38]

$$V = \frac{1}{2}\left\{A(u^2+v^2) + [pv + (1+p)\psi u]^2 + [(1+q)v + q\psi u]^2\right\}, \tag{5.11}$$

where

$$A = -pq\psi + (1+p)\psi(1+q) = (1+p+q)\psi, \quad I = -p + q\psi. \tag{5.12}$$

The temporal derivative of E, along the solutions of (5.8), is

$$\begin{aligned} \frac{dE}{d\tau} &= \frac{1}{2}\left[\left(\frac{d\mu_1}{d\tau} - 2p\mu_1\right)u^2 + \left(\frac{d\mu_2}{d\tau} + 2q\psi\mu_2\right)v^2\right] + \\ & +[\psi\mu_2(1+p) - \mu_1(1+q)]uv + \Phi, \end{aligned} \tag{5.13}$$

where

$$\Phi = (\mu_1 u - \mu_2 \psi v)\xi. \tag{5.14}$$

Moreover, setting

$$\begin{cases} \overline{F} = (\alpha_1 u - \alpha_3 v)\xi + (\alpha_2 v - \alpha_3 u)(-\psi\xi), \\ \alpha_1 = A + (1+p)^2\psi^2 + q^2\psi^2, \\ \alpha_2 = A + p^2 + (1+q)^2, \\ \alpha_3 = -[p(1+p) + q(1+q)]\psi, \end{cases} \tag{5.15}$$

along the solutions of (5.8) it turns out that

$$\frac{dV}{d\tau} = P(\tau, u, v) + \overline{F}, \tag{5.16}$$

with

$$
\begin{cases}
P = \dfrac{1}{2}\displaystyle\sum_{i=1}^{3} P_i(\tau, u, v), \quad P_1 = \left(2IA + \dfrac{dA}{d\tau}\right)(u^2+v^2), \\[2ex]
P_2 = (1+p)^2 \dfrac{d}{d\tau}\psi^2 \, u^2 + 2p(1+p)\dfrac{d\psi}{d\tau} uv, \\[2ex]
P_3 = q^2 \dfrac{d}{d\tau}\psi^2 \, u^2 + 2q(1+q)\dfrac{d\psi}{d\tau} uv.
\end{cases} \tag{5.17}
$$

Remark 5.2 *It turns out that*

i) since $A_* = (1+p+q)\psi_* > 0$ *and since the boundedness of* ψ*, then there exists a positive constant* m_1 *such that*

$$
A_*(u^2+v^2) < V < m_1(u^2+v^2), \quad \forall \tau \in \mathbb{R}^+, \tag{5.18}
$$

and hence V *is bounded, has an infinitely small upper bound and is positive definite;*

ii) property i) holds also for E *when* $\mu_i > 0$, $(i = 1, 2)$. *In fact one has*

$$
m_2(u^2+v^2) < E < m_3(u^2+v^2), \tag{5.19}
$$

with

$$
m_2 < \frac{1}{2}\inf((\mu_1)_*, (\mu_2)_*), \quad m_3 > \frac{1}{2}\sup((\mu_1)^*, (\mu_2)^*); \tag{5.20}
$$

iii) when either $\mu_1 > 0$ *or* $\mu_2 > 0$*, at any instant* $\overline{\tau} \in \mathbb{R}^+$ *and in any disk of the phase space, centered at the origin* $O = (0,0)$*, there exists a domain in which* $E(\overline{t}, u, v) > 0$*;*

iv) the critical case is the case in which

$$
\psi = const. = \frac{p}{q}, \tag{5.21}
$$

and hence $AI \equiv 0$, $\forall \tau \in \mathbb{R}^+$.

5.1 Linear stability-instability results

Disregarding the contribution of nonlinear terms, model (5.8) becomes

$$\begin{cases} \dfrac{du}{d\tau} = -pu - (1+q)v, \\ \\ \dfrac{dv}{d\tau} = (1+p)\psi u + q\psi v. \end{cases} \tag{5.22}$$

The following theorems hold.

Theorem 5.1 *Suppose that*

$$0 < \psi = const. < \frac{p}{q}, \tag{5.23}$$

then the null solution of (5.22) is asymptotically (locally) stable.

Proof. When ψ is constant, then system (5.22) is autonomous. In the case (5.23), $A = \text{const.} > 0$, $I = \text{const.} < 0$ and $P_2 + P_3 \equiv 0$. Hence

$$\frac{dV}{d\tau} = AI(u^2 + v^2),$$

is negative definite.

Theorem 5.2 *In the critical case, i.e.*

$$\psi = \frac{p}{q}, \tag{5.24}$$

the null solution of (5.22) is simply stable.

Proof. The proof follows very easily since in the case (5.24) V is constant along the solutions of (5.22).

Theorem 5.3 *If*

$$\psi^* < \frac{p}{q}, \quad \frac{d\psi}{d\tau} < 0, \tag{5.25}$$

together with

$$[p(1+p)+q(1+q)]^2\left(\frac{d\psi^*}{d\tau}\right)^2 < 4\left\{AI+\frac{1}{2}\frac{dA}{d\tau}+\left[(1+p)^2+q^2\right]\psi\frac{d\psi}{d\tau}\right\}_*\cdot$$
$$\cdot\left\{AI+\frac{1}{2}\frac{dA}{d\tau}\right\}_*, \tag{5.26}$$

then the null solution of (5.22) is asymptotically stable.

Proof. If $(5.25)_1$ hold, then $I^* < 0$. $(5.25)_2$ and (5.26) assure that $\frac{dV}{d\tau}$ is negative definite and hence all the hypotheses of the Liapunov stability theorem are satisfied.

Theorem 5.4 *If*

$$\psi \le \psi_0 e^{-2h\tau}, \quad h = const. = \max\{p, 2q\psi^*\} > 0, \tag{5.27}$$

then the null solution of (5.22) is simply stable. If

$$\psi \le \psi_0 e^{-2(h+\varepsilon)\tau}, \quad \varepsilon > 0, \tag{5.28}$$

then the null solution of (5.22) is asymptotically stable.

Proof. Choosing

$$\mu_1 = \psi^2, \quad \mu_2 = \psi\frac{1+q}{1+p}, \tag{5.29}$$

it follows that E is positive definite and has an infinitely small upper bound. Moreover

$$\begin{aligned}\frac{dE}{d\tau} &= \frac{1}{2}\left[\left(\frac{d\mu_1}{d\tau}-2p\mu_1\right)u^2+\left(\frac{d\mu_2}{d\tau}+2q\psi\mu_2\right)v^2\right] = \\ &= \frac{1}{2}\left[2\psi\left(\frac{d\psi}{d\tau}-p\psi\right)u^2+\frac{1+q}{1+p}\left(\frac{d\psi}{d\tau}+2q\psi^2\right)v^2\right],\end{aligned} \tag{5.30}$$

since $\psi\mu_2(1+p)-\mu_1(1+q)=0$. (5.27) guarantees that

$$\begin{cases}\dfrac{d\psi}{d\tau}-p\psi \le 0, \\[2ex] \dfrac{d\psi}{d\tau}+2q\psi^2 \le 0,\end{cases}$$

and hence $\frac{dE}{d\tau} \leq 0$. In the case (5.28), one has

$$\begin{cases} \frac{d\psi}{d\tau} - p\psi \leq -2\varepsilon(\psi)^* < 0, \\ \\ \frac{d\psi}{d\tau} + 2q\psi^2 \leq -4\varepsilon(\psi)^* < 0, \end{cases}$$

and then the temporal derivative of E is negative definite and there is a positive constant m such that

$$\frac{dE}{d\tau} \leq -mE \Rightarrow E \leq E(0)e^{-m\tau}. \tag{5.31}$$

For the instability, the following theorems hold.

Theorem 5.5 *If*

$$\psi = const. > \frac{p}{q}, \tag{5.32}$$

then the null solution of (5.22) is unstable.

Proof. If (5.32) holds, then $I = \text{const.} > 0$ and

$$\frac{dV}{d\tau} = AI(u^2 + v^2),$$

is positive definite.

Theorem 5.6 *If (5.26) holds together with*

$$\psi_* > \frac{p}{q}, \quad \frac{d\psi}{d\tau} \geq 0, \tag{5.33}$$

then the null solution of (5.22) is unstable.

Proof. (5.26) and (5.33) assure that $\frac{dV}{d\tau}$ is positive definite.

Theorem 5.7 *Suppose that*

$$\begin{cases} 1 + q \leq pq \\ \\ \psi \leq \psi_0 e^{-2(p^2+\varepsilon)\tau}, \quad \varepsilon = const. > 0, \end{cases} \tag{5.34}$$

then the null solution of (5.22) is (Chetaiev) unstable.

Proof. Choosing

$$\mu_1 = -\psi, \quad \mu_2 = \frac{1+q}{1+p}, \tag{5.35}$$

it follows that E assumes positive values in any disk centered in the origin and

$$\begin{aligned} \frac{dE}{d\tau} &= \frac{1}{2}\left[\left(\frac{d\mu_1}{d\tau} - 2p\mu_1\right)u^2 + 2q\psi\mu_2 v^2 - 4(1+p)\mu_1\mu_2 uv\right] \\ &\geq \frac{1}{2}\left[\left(\frac{d\mu_1}{d\tau} - 2p\mu_1\right)u^2 + 2q\psi\mu_2 v^2 - 4(1+p)|\mu_1|\mu_2|uv|\right]. \end{aligned} \tag{5.36}$$

But, from $(5.34)_1$, $(1+p)|\mu_1|\mu_2 = \psi(1+q) \leq pq\psi$. Then it follows that

$$2(1+p)|\mu_1|\mu_2|uv| \leq 2\sqrt{pq\psi}\sqrt{(1+p)|\mu_1|\mu_2}|uv|,$$

and hence, applying the Schwartz inequality

$$2\sqrt{pq\psi}\sqrt{(1+p)|\mu_1|\mu_2}|uv| \leq (p(1+p)|\mu_1|u^2 + q\psi\mu_2 v^2).$$

Hence

$$\frac{dE}{d\tau} \geq \frac{1}{2}\left(\frac{d\mu_1}{d\tau} + 2p^2\mu_1\right)u^2 = -\frac{1}{2}\left(\frac{d\psi}{d\tau} + 2p^2\psi\right)u^2.$$

If $(5.34)_2$ holds, then

$$\frac{d\psi}{d\tau} + 2p^2\psi < -2\varepsilon\psi_* < 0,$$

i.e. the temporal derivative of E is positive definite. All the hypotheses of the Chetaiev instability theorem are verified.

5.2 Nonlinear stability-instability results

All the results obtained for the linear system, continue to hold also for the nonlinear one, if the nonlinear terms verify the following inequality

$$(|u| + |v|)(|\xi| + |-\psi\xi|) \leq \varepsilon_1(u^2 + v^2)^{1+\varepsilon_2}, \quad \varepsilon_i = \text{const.} > 0. \tag{5.37}$$

Theorem 5.8 *ξ verifies (5.37).*

Proof. Since ψ is bounded

$$(|u|+|v|)(|\xi|+|-\psi\xi|) = |\xi|(|u|+|v|)(1+\psi) \leq k|\xi|(|u|+|v|),$$

with k positive constant. Since

$$|u|+|v| \leq \sqrt{2}(u^2+v^2)^{\frac{1}{2}},$$

we have that (5.37) is satisfied when

$$|\xi| \leq M(u^2+v^2)^{\frac{1}{2}+\varepsilon_2}, \tag{5.38}$$

with M positive constant. We begin to prove (5.38), at first, in two particular cases.

1) Case $p=1, q=0$.
   In this case

$$F = 1-(1+u)^2(1+v) = -2u-v-u(u+uv+2v), \tag{5.39}$$

   and hence the system (5.5) becomes

$$\begin{cases} \dfrac{du}{d\tau} = -u-v-u(u+uv+2v), \\ \\ \dfrac{dv}{d\tau} = 2\psi u+\psi u(u+uv+2v). \end{cases} \tag{5.40}$$

   Hence, it turns out that

$$\begin{aligned} |\xi| &= |u^2+u^2v+2uv)| \leq u^2+u^2|v|+2|uv| \leq \\ &\leq (|u|+|v|^2)+(|u|+|v|)^2(|u|+|v|)+(u^2+v^2) \leq \\ &\leq 2(u^2+v^2)+2(u^2+v^2)\sqrt{2}(u^2+v^2)^{\frac{1}{2}}+(u^2+v^2) \leq \\ &\leq c_1(u^2+v^2)+c_2(u^2+v^2)^{\frac{3}{2}}, \quad c_i = \text{const.} > 0. \end{aligned}$$

If $u^2 + v^2 < 1$, then

$$|\xi| \leq a_1(u^2 + v^2), \quad a_1 = \text{const.} > 0,$$

then (5.38) holds with $\varepsilon_2 = \dfrac{1}{2}$.
If $u^2 + v^2 \geq 1$, then

$$|\xi| \leq a_2(u^2 + v^2)^{\frac{3}{2}}, \quad a_2 = \text{const.} > 0,$$

and one can choose $\varepsilon_2 = 1$.

2) Case $p = 0, q = 1$.

In this case

$$F = 1 - (1 + u)(1 + v)^2 = u - 2v - u(u + v^2 + 2v), \tag{5.41}$$

and hence (5.5) becomes

$$\begin{cases} \dfrac{du}{d\tau} = -2v - u(u + v^2 + 2v), \\ \\ \dfrac{dv}{d\tau} = \psi u + \psi v + \psi u(u + v^2 + 2v). \end{cases} \tag{5.42}$$

The nonlinear term is

$$|\xi| = |u^2 + uv^2 + 2uv|,$$

and following the same procedure used in the case 1), (5.38) is hold.

Now, we consider the general case $p \geq 1, q \geq 1$.

$$F = 1 - (1+u)^{1+p}(1+v)^{1+q} = 1 - \left[\sum_{h=0}^{1+p} \binom{1+p}{h} u^h \cdot \sum_{k=0}^{1+q} \binom{1+q}{k} v^k \right] =$$

$$= 1 - \left[1 + \sum_{h=1}^{1+p} \binom{1+p}{h} u^h \right] \left[1 + \sum_{k=1}^{1+q} \binom{1+q}{k} v^k \right] =$$

$$= - \left[\sum_{h=1}^{1+p} \binom{1+p}{h} u^h + \sum_{k=1}^{1+q} \binom{1+q}{k} v^k + \sum_{h=1}^{1+p} \sum_{k=1}^{1+q} \binom{1+p}{h} \binom{1+q}{k} u^h v^k \right] =$$

$$= - \left[(1+p)u + (1+q)v + \sum_{h=2}^{1+p} \binom{1+p}{h} u^h + \sum_{k=2}^{1+q} \binom{1+q}{k} v^k + \right.$$

$$\left. + \sum_{h=1}^{1+p} \sum_{k=1}^{1+q} \binom{1+p}{h} \binom{1+q}{k} u^h v^k \right],$$

and hence

$$F = -(1+p)u - (1+q)v + \xi, \tag{5.43}$$

with

$$\xi = - \left[\sum_{h=2}^{1+p} \binom{1+p}{h} u^h + \sum_{k=2}^{1+q} \binom{1+q}{k} v^k + \sum_{h=1}^{1+p} \sum_{k=1}^{1+q} \binom{1+p}{h} \binom{1+q}{k} u^h v^k \right]. \tag{5.44}$$

But

$$|\xi| \leq \sum_{h=2}^{1+p} \binom{1+p}{h} |u|^h + \sum_{k=2}^{1+q} \binom{1+q}{k} |v|^k + \sum_{i=1}^{1+p} \sum_{k=1}^{1+q} \binom{1+p}{h} \binom{1+q}{k} |u|^h |v|^k \leq$$

$$\leq \sum_{h=2}^{1+p} C^1_{h,p}(|u|+|v|)^h + \sum_{k=2}^{1+q} C^2_{k,q}(|u|+|v|)^k +$$

$$+ \sum_{i=1}^{1+p} \sum_{k=1}^{1+q} C^1_{h,p} C^2_{k,q}(|u|+|v|)^h (|u|+|v|)^k.$$

Setting

$$z = \max(p, q), \tag{5.45}$$

one obtains

$$|\xi| \leq \sum_{i=2}^{1+z} C^1_{z,i}(|u|+|v|)^i + \sum_{i=2}^{1+z} C^2_{z,i}(|u|+|v|)^i + \left[\sum_{i=1}^{1+z} C^1_{z,i} C^2_{z,i}(|u|+|v|)^i \right]^2 \leq$$

$$\leq C_1 \sum_{i=2}^{1+z} (|u|+|v|)^i + C_2 \left[\sum_{i=1}^{1+z} (|u|+|v|)^i \right]^2 .$$

If $|u|+|v| < 1$, then

$$|\xi| \leq C_1(1+z-2)(|u|+|v|)^2 + C_2 \left[(1+z-1)^2 (|u|+|v|)^2 \right] \leq C_4(u^2+v^2),$$

then it is sufficient to choose $\varepsilon_2 = \dfrac{1}{2}$.

In the case $|u|+|v| \geq 1$, then

$$|\xi| \leq C_1(1+z-2)(|u|+|v|)^{1+z} + C_2 \left[(1+z-1)^2 (|u|+|v|)^{2(1+z)} \right] \leq C_4(u^2+v^2)^{1+z},$$

then (5.38) holds with $\varepsilon_2 = \dfrac{1}{2} + z$.

Remark 5.3 *When the conditions assuring the asymptotic stability of the null solution of system (5.8) hold, then cannot exist cycles, neither if all the perturbations are periodic of the same period. Hence, in these cases, system (4.56) can not admit periodic solutions.*

Chapter 6

Generalized Lotka-Volterra models with a logistic growth for the preys

Now we want to see how the nonautonomous perturbations influence the asymptotic behaviour of the solutions of the model (3.15) around its equilibrium points. We consider the following model

$$\begin{cases} \dot{x} = f_1(t)(a - by - ex)x, \\ \\ \dot{y} = f_2(t)(-c + dx)y, \end{cases} \tag{6.1}$$

under the hypotheses

i) $f_i : \mathbb{R}^+ \to \mathbb{R}^+,\ (i = 1, 2)$, verifying

$$|f_i(t_1) - f_i(t_2)| \leq K_i |t_1 - t_2|, \quad i = 1, 2, \tag{6.2}$$

with K_i positive constants and $t_i \in \mathbb{R}^+,\ (i = 1, 2)$;

ii) $f_i \in L^\infty(\mathbb{R}^+) \cap C^1(\mathbb{R}^+)$,

iii) a, b, c, d, e are positive constants.

Remark 6.1 *We remark that*

- *i) and ii) guarantee the global (in time) existence and uniqueness of the solution;*
- *model (6.1) admits the same equilibrium points of the model (3.15)*

$$(0,0), \quad \left(\frac{a}{e}, 0\right), \quad \left(\frac{c}{d}, \frac{a}{b}(1-\beta)\right) = (\overline{x}, \overline{y}), \tag{6.3}$$

with $0 < \beta = \dfrac{e}{a}\dfrac{c}{d} < 1$*, in order to guarantee that* $(6.3)_3$ *is biologically meaningful.*

Introducing the following transformation

$$x = \overline{x}X, \quad y = \overline{y}Y, \quad \tau = a\int_0^\tau f_1(z)\,dz, \quad \psi(t) = \frac{c}{a}\frac{f_2(t)}{f_1(t)} > 0, \tag{6.4}$$

model (6.1) becomes

$$\begin{cases} \dfrac{dX}{d\tau} = [1-(1-\beta)Y - \beta X]\,X, \\ \\ \dfrac{dY}{d\tau} = \psi(-1+X)Y, \end{cases} \tag{6.5}$$

while the equilibrium points become

$$(0,0), \quad \left(\frac{1}{\beta}, 0\right), \quad (1,1). \tag{6.6}$$

6.1 Linear stability-instability results of the critical points

The meaningful equilibrium point is $(1,1)$. Setting

$$X = u + 1, \qquad Y = v + 1, \tag{6.7}$$

model (6.5) becomes

$$\begin{cases} \dfrac{du}{d\tau} = -\beta u - (1-\beta)v - \beta u^2 - (1-\beta)uv, \\ \\ \dfrac{dv}{d\tau} = \psi u + \psi uv. \end{cases} \tag{6.8}$$

For this model, the Liapunov function ([38]) is

$$V = \frac{1}{2}\left[A(u^2+v^2) + (\beta v + \psi u)^2 + (1-\beta)^2 v^2\right], \tag{6.9}$$

with $A = \psi(1-\beta) > 0,\ I = -\beta < 0.$

The temporal derivative of V, along the solutions of (6.8), is given by

$$\frac{dV}{d\tau} = \frac{1}{2}\left[\left(2AI + \frac{dA}{d\tau}\right)(u^2+v^2) + 2\beta\frac{d\psi}{d\tau}uv + 2\psi\frac{d\psi}{d\tau}u^2\right] + \overline{F} =$$

$$= \frac{1}{2}\left[\left(-2\beta(1-\beta)\psi + (1-\beta)\frac{d\psi}{d\tau} + \frac{d\psi^2}{d\tau}\right)u^2 + \left(-2\beta(1-\beta)\psi + (1-\beta)\frac{d\psi}{d\tau}\right)v^2 + \right. \tag{6.10}$$

$$\left. + 2\beta\frac{d\psi}{d\tau}uv\right] + \overline{F},$$

where

$$\begin{cases} \overline{F} = (A_1 u - A_3 v)(-\beta u^2 - (1-\beta)uv) + (A_2 v - A_3 u)(\psi uv), \\ \\ A_1 = A + \psi^2, \quad A_2 = A + \beta^2 + (1-\beta)^2, \quad A_3 = -\beta\psi. \end{cases} \tag{6.11}$$

Disregarding the contribution of nonlinear terms in (6.8), the following theorem holds.

Theorem 6.1 *If*

$$\begin{cases} \left(-2\beta(1-\beta)\psi + (1-\beta)\dfrac{d\psi}{d\tau} + \dfrac{d\psi^2}{d\tau},\ -2\beta(1-\beta)\psi + (1-\beta)\dfrac{d\psi}{d\tau}\right)^* < -h, \\ \\ \beta^2\left(\dfrac{d\psi}{d\tau}\right)^2 < \left(-2\beta(1-\beta)\psi + (1-\beta)\dfrac{d\psi}{d\tau} + \dfrac{d\psi^2}{d\tau}\right)\cdot\left(-2\beta(1-\beta)\psi + (1-\beta)\dfrac{d\psi}{d\tau}\right), \end{cases} \tag{6.12}$$

with h a positive constant, then the null solution of (6.8) is linearly asymptotically stable.

Proof. Since $(\psi)_* > 0$, then $A_* > 0$. Since ψ is bounded, then there is a positive constant m such that

$$A_*(u^2 + v^2) < V < m(u^2 + v^2), \tag{6.13}$$

and hence V is positive definite and has an upper bound which is infinitesimal at the origin.
If (6.12) holds, then $\dfrac{dV}{d\tau}$ is negative definite and the thesis is hold.

Remark 6.2 *If* $\dfrac{d\psi}{d\tau} \leq 0$, *then* $(6.12)_1$ *is satisfied. An example in which all the hypotheses of the theorem 6.1 are verified is given by*

$$\psi = \alpha e^{-2\gamma\tau},$$

where α *is a positive constant and* γ *is chosen in the following way*

$$\begin{cases} \gamma = const. > 0 & \text{if } 0 < \beta \leq \dfrac{1}{2}, \\ \\ 0 < \gamma = const. < \dfrac{\beta(1-\beta)}{2\beta - 1} & \text{if } \dfrac{1}{2} < \beta < 1. \end{cases}$$

For the (linear) instability the following theorem holds.

Theorem 6.2 *If* $(6.12)_2$ *holds together with*

$$\left(-2\beta(1-\beta)\psi + (1-\beta)\frac{d\psi}{d\tau} + \frac{d\psi^2}{d\tau}, -2\beta(1-\beta)\psi + (1-\beta)\frac{d\psi}{d\tau}\right)_* \geq k, \tag{6.14}$$

with k *a positive constant, then the null solution of (6.8), disregarding the contribution of nonlinear terms, is unstable.*

Proof. Under the hypotheses (6.14), and since $\psi_* > 0$, V and its derivative along the solution of system, are positive definite.

Now we want to study the linear stability of $(0, 0)$.

Disregarding the contribution of nonlinear terms in (6.5), the model becomes

$$\begin{cases} \dfrac{dX}{d\tau} = X, \\ \\ \dfrac{dY}{d\tau} = -\psi Y. \end{cases} \tag{6.15}$$

The Liapunov function [38] is given by

$$V = \frac{1}{2}\left[A(X^2+Y^2)+Y^2+\psi^2X^2\right] \tag{6.16}$$

where $A = -\psi$, $I = 1-\psi$, i.e.

$$V = \frac{1}{2}(\psi-1)(\psi X^2 - Y^2), \tag{6.17}$$

while the standard "energy" is

$$E = \frac{1}{2}\left(\mu_1 X^2 + \mu_2 Y^2\right), \tag{6.18}$$

where $\mu_i : \mathbb{R}^+ \to \mathbb{R} \in L^\infty \cap C^1$.

Remark 6.3 *We remark that*

i) V assumes positive values in each ball centered in the origin,

ii) property i) holds also for E when μ_1 or μ_2 are positive,

iii) if μ_i, $(i = 1, 2)$, are positive then E is positive definite and the following inequality holds

$$m_1(X^2+Y^2) < E < m_2(X^2+Y^2),$$

and hence E is bounded and has an upper bound which is infinitely small at the origin.

The temporal derivative of V along the solution of (6.15) is given by

$$\begin{aligned}\frac{dV}{d\tau} &= \frac{1}{2}\left[\left(2AI + \frac{dA}{d\tau}\right)(X^2 + Y^2) + 2\psi\frac{d\psi}{d\tau}X^2\right] = \\ &= \frac{1}{2}\left[\left(2AI + \frac{dA}{d\tau} + \frac{d\psi^2}{d\tau}\right)X^2 + \left(2AI + \frac{dA}{d\tau}\right)Y^2\right] = \\ &= \frac{1}{2}\left[\left(-2\psi(1-\psi) - \frac{d\psi}{d\tau} + \frac{d\psi^2}{d\tau}\right)X^2 - \left(2\psi(1-\psi) + \frac{d\psi}{d\tau}\right)Y^2\right].\end{aligned} \tag{6.19}$$

The temporal derivative of E along the solution of (6.15) is

$$\begin{aligned}\frac{dE}{d\tau} &= \frac{1}{2}\left[\frac{d\mu_1}{d\tau}X^2 + \frac{d\mu_2}{d\tau}Y^2 + 2\mu_1 X^2 + 2\mu_2 Y(-\psi Y)\right] = \\ &= \frac{1}{2}\left[\left(\frac{d\mu_1}{d\tau} + 2\mu_1\right)X^2 + \left(\frac{d\mu_2}{d\tau} - 2\mu_2\psi\right)Y^2\right].\end{aligned} \tag{6.20}$$

Theorem 6.3 *If*

$$\psi \leq \psi(0)e^{-2\tau}, \tag{6.21}$$

then the null solution of (6.15) is (simply) stable. If

$$\psi \leq \psi(0)e^{-2(1+\varepsilon)\tau}, \tag{6.22}$$

with ε a positive constant, then the null solution of (6.15) is asymptotically stable.

Proof. Choosing

$$\mu_1 = \psi, \qquad \mu_2 = 1, \tag{6.23}$$

it follows that E is given by

$$E = \frac{1}{2}\left(\psi X^2 + Y^2\right), \tag{6.24}$$

and is positive definite and has an upper bound infinitely small at the origin. The temporal derivative of E along (6.15) is given by

$$\frac{dE}{d\tau} = \frac{1}{2}\left[\left(\frac{d\psi}{d\tau} + 2\psi\right)X^2 - 2\psi Y^2\right]. \tag{6.25}$$

(6.21) guarantees that

$$\frac{d\psi}{d\tau} + 2\psi \leq 0, \tag{6.26}$$

and hence $\dfrac{dE}{d\tau}$ is negative semidefinite, while in the case (6.22) one has

$$\frac{d\psi}{d\tau} + 2\psi \leq -2\varepsilon\psi, \tag{6.27}$$

and hence there is a positive constant m such that

$$\frac{dE}{d\tau} \leq -mE \Leftrightarrow E(\tau) \leq E_0 e^{-m\tau}. \tag{6.28}$$

Remark 6.4 *In the hypotheses of theorem 6.3, all the species extinct.*

For the linear instability, the following theorem holds.

Theorem 6.4 *If*

$$\begin{cases} \psi > \left(1, \psi(0)e^{2\left(\frac{\varepsilon - h_2}{h_1}\right)\tau}\right), \quad h_1 = (2\psi - 1)^*, \quad h_2 = (\psi - 1)_*, \\ \\ \dfrac{d\psi}{d\tau} \leq 0, \quad 0 < \varepsilon < h_2, \end{cases} \tag{6.29}$$

then the null solution of (6.15) is (Chetaiev) unstable.

Proof. Since $A = -\psi < 0$, then V assumes positive values in each ball centered at the origin.

From the hypotheses, it follows that

$$\begin{cases} \dfrac{dV}{d\tau} \geq \dfrac{1}{2}\left[(2\psi - 1)\dfrac{d\psi}{d\tau} - 2\psi(1 - \psi)\right] X^2, \\ \\ (2\psi - 1)\dfrac{d\psi}{d\tau} - 2\psi(1 - \psi) > 2\varepsilon\psi, \end{cases} \tag{6.30}$$

and hence $\dfrac{dV}{d\tau}$ is positive definite.

We analyze now the linear stability of $\left(\frac{1}{\beta}, 0\right)$.

Setting

$$u = X - \frac{1}{\beta}, \quad v = Y, \tag{6.31}$$

system (6.5) becomes

$$\begin{cases} \dfrac{du}{d\tau} = -u - \left(\dfrac{1-\beta}{\beta}\right) v - \beta u^2 - (1-\beta)uv, \\ \\ \dfrac{dv}{d\tau} = \psi \left(\dfrac{1-\beta}{\beta}\right) v + \psi uv. \end{cases} \tag{6.32}$$

In order to study the stability-instability of the null solution of model (6.32), we introduce the standard "energy" and the Liapunov function [38]

$$V = \frac{1}{2}\left[A(u^2+v^2) + v^2 + \left(\frac{1-\beta}{\beta}\right)^2 (v+\psi u)^2\right], \tag{6.33}$$

with $A = -\psi\left(\frac{1-\beta}{\beta}\right) < 0,\ I = -1 - \psi\left(\frac{\beta-1}{\beta}\right)$.
The temporal derivative of V, calculated along the solutions of (6.32), is given by

$$\begin{aligned} \frac{dV}{d\tau} =& \frac{1}{2}\left[\left(2AI + \frac{dA}{d\tau}\right)(u^2+v^2) + \frac{d\psi^2}{d\tau}\left(\frac{1-\beta}{\beta}\right)^2 u^2 + 2\frac{d\psi}{d\tau}\left(\frac{1-\beta}{\beta}\right)^2 uv\right] + \Psi = \\ &= \frac{1}{2}\left(\frac{1-\beta}{\beta}\right)\left\{\left[2\psi\left(1-\psi\left(\frac{1-\beta}{\beta}\right)\right) - \frac{d\psi}{d\tau} + \left(\frac{1-\beta}{\beta}\right)\frac{d\psi^2}{d\tau}\right]u^2 + \right. \\ &\left. + \left[2\psi\left(1-\psi\left(\frac{1-\beta}{\beta}\right)\right) - \frac{d\psi}{d\tau}\right]v^2 + 2\left(\frac{1-\beta}{\beta}\right)\frac{d\psi}{d\tau}uv\right\} + \Psi, \end{aligned} \tag{6.34}$$

where

$$\begin{cases} \Psi = (A_1 u - A_3 v)(-\beta u^2 - (1-\beta)uv) + (A_2 v - A_3 u)\psi uv, \\ A_1 = A^2 + \psi^2 + \left(\dfrac{1-\beta}{\beta}\right)^2, \ A_2 = A^2 + 1 + \left(\dfrac{1-\beta}{\beta}\right)^2, \\ A_3 = -\psi \left(\dfrac{1-\beta}{\beta}\right)^2. \end{cases} \tag{6.35}$$

The temporal derivative of E is given by

$$\frac{dE}{d\tau} = \frac{1}{2}\left[\left(\frac{d\mu_1}{d\tau} - 2\mu_1\right)u^2 + \left(\frac{d\mu_2}{d\tau} + 2\mu_2\psi\left(\frac{1-\beta}{\beta}\right)\right)v^2 - 2\mu_1\left(\frac{1-\beta}{\beta}\right)uv\right] + \Phi, \tag{6.36}$$

with

$$\Phi = u[\mu_1 u(-\beta u - (1-\beta)v) + \mu_2 \psi v^2]. \tag{6.37}$$

Theorem 6.5 *If μ_i are positive bounded functions $(i = 1, 2)$, and*

$$\begin{cases} \left(\dfrac{d\mu_1}{d\tau} - 2\mu_1, \dfrac{d\mu_2}{d\tau} + 2\mu_2\psi\left(\dfrac{1-\beta}{\beta}\right)\right)^* < -h, \\ \left(\dfrac{1-\beta}{\beta}\right)^2 \mu_1^2 < \left(\dfrac{d\mu_1}{d\tau} - 2\mu_1\right)\left(\dfrac{d\mu_2}{d\tau} + 2\mu_2\psi\left(\dfrac{1-\beta}{\beta}\right)\right), \end{cases} \tag{6.38}$$

with h a positive constant, then there is linear asymptotic stability for the null solution of (6.32).

Proof. In these hypotheses E is positive definite, admits an upper bound infinitely small at the origin, and $\dfrac{dE}{d\tau}$ is negative definite.

Theorem 6.6 *If μ_i are positive bounded functions $(i = 1, 2)$, such that*

$$\begin{cases} \mu_1 \le \mu_1(0) e^{-\left|\frac{1-3\beta}{\beta}\right|\tau}, \\ \mu_2 \le \mu_2(0) e^{-k\tau} - \dfrac{h}{k}, \end{cases} \tag{6.39}$$

with $h = \dfrac{\beta-1}{\beta}\mu_1(0) = const. < 0,\ k = 2\left(\dfrac{1-\beta}{\beta}\right)\psi^* = const. > 0$*, then the null solution of (6.32) is linearly simply stable, while if*

$$\begin{cases} \mu_1 \le \mu_1(0)e^{-\left(\left|\frac{1-3\beta}{\beta}\right|+\varepsilon\right)\tau}, \\ \\ \mu_2 \le \mu_2(0)e^{-(k+\varepsilon)\tau} - \dfrac{h}{k+\varepsilon}, \end{cases} \tag{6.40}$$

where $\varepsilon > 0$*, then there is linear asymptotic stability.*

Proof. If μ_i are positive definite, then E is positive definite and admits an upper bound infinitely small at the origin. The temporal derivative of E, along the solutions of the linear system linked to (6.32), is given by

$$\frac{dE}{d\tau} = \frac{1}{2}\left[\left(\frac{d\mu_1}{d\tau} - 2\mu_1\right)u^2 + \left(\frac{d\mu_2}{d\tau} + 2\mu_2\psi\left(\frac{1-\beta}{\beta}\right)\right)v^2 - 2\mu_1\left(\frac{1-\beta}{\beta}\right)uv\right] \le$$

$$\le \frac{1}{2}\left[\left(\frac{d\mu_1}{d\tau} - 2\mu_1\right)u^2 + \left(\frac{d\mu_2}{d\tau} + 2\mu_2\psi\left(\frac{1-\beta}{\beta}\right)\right)v^2 + 2\mu_1\left(\frac{1-\beta}{\beta}\right)|uv|\right] \le$$

$$\le \frac{1}{2}\left[\left(\frac{d\mu_1}{d\tau} + \mu_1\left(\frac{1-3\beta}{\beta}\right)\right)u^2 + \left(\frac{d\mu_2}{d\tau} + 2\mu_2\psi\left(\frac{1-\beta}{\beta}\right) + \mu_1\left(\frac{1-\beta}{\beta}\right)\right)v^2\right].$$

In the hypotheses (6.39), it follows that

$$\begin{cases} \dfrac{d\mu_1}{d\tau} + \mu_1\left(\dfrac{1-3\beta}{\beta}\right) \le 0, \\ \\ \dfrac{d\mu_2}{d\tau} + 2\mu_2\psi\left(\dfrac{1-\beta}{\beta}\right) + \mu_1\left(\dfrac{1-\beta}{\beta}\right) \le 0 \end{cases}$$

and hence $\dfrac{dE}{d\tau} \le 0$, while in the case (6.40) the temporal derivative of E is negative definite, since

$$\begin{cases} \dfrac{d\mu_1}{d\tau} + \mu_1\left(\dfrac{1-3\beta}{\beta}\right) \le -\varepsilon\mu_1, \\ \\ \dfrac{d\mu_2}{d\tau} + 2\mu_2\psi\left(\dfrac{1-\beta}{\beta}\right) + \mu_1\left(\dfrac{1-\beta}{\beta}\right) \le -\varepsilon\mu_2. \end{cases}$$

Remark 6.5 *In the hypotheses of theorems 6.5 or 6.6, the predators extinct.*

For the instability, the following result holds.

Theorem 6.7 *If*

$$\begin{cases} \left[2\psi\left(1-\psi\left(\frac{1-\beta}{\beta}\right)\right)-\frac{d\psi}{d\tau}+\left(\frac{1-\beta}{\beta}\right)\frac{d\psi^2}{d\tau},\ 2\psi\left(1-\psi\left(\frac{1-\beta}{\beta}\right)\right)-\frac{d\psi}{d\tau}\right]_* > h, \\ \left(\frac{1-\beta}{\beta}\right)^2\left(\frac{d\psi}{d\tau}\right)^2 < \left[2\psi\left(1-\psi\left(\frac{1-\beta}{\beta}\right)\right)-\frac{d\psi}{d\tau}+\left(\frac{1-\beta}{\beta}\right)\frac{d\psi^2}{d\tau}\right]\cdot \\ \cdot\left[2\psi\left(1-\psi\left(\frac{1-\beta}{\beta}\right)\right)-\frac{d\psi}{d\tau}\right], \end{cases} \tag{6.41}$$

with h a positive constant, then the null solution of (6.32) is linearly (Chetaiev) unstable.

Proof. Since $A = -\psi\left(\frac{1-\beta}{\beta}\right) < 0$, then V assumes positive values in each ball centered at the origin. The conditions (6.41) assure that the temporal derivative of V, calculated along the solutions of the linear system linked to (6.32), is positive definite.

Theorem 6.8 *If*

$$\psi \le \psi(0)e^{-\left(\left|\frac{1-2\beta}{\beta}\right|+\varepsilon\right)\tau}, \tag{6.42}$$

with $\varepsilon =$ const. > 0, then the null solution of (6.32) is linearly (Chetaiev) unstable.

Proof. Choosing

$$\mu_1 = -\psi, \qquad \mu_2 = \overline{\mu}_2 = \text{const.} > 1,$$

then E assumes positive values in each ball centered at the origin. The temporal derivative of E, along the solutions of the linear system linked to (6.32), is given by

$$\begin{aligned}\frac{dE}{d\tau} &= \frac{1}{2}\left[\left(-\frac{d\psi}{d\tau}+2\psi\right)u^2+2\overline{\mu}_2\psi\left(\frac{1-\beta}{\beta}\right)v^2+2\psi\left(\frac{1-\beta}{\beta}\right)uv\right] \geq \\ &\geq \frac{1}{2}\left[\left(-\frac{d\psi}{d\tau}+2\psi\right)u^2+2\overline{\mu}_2\psi\left(\frac{1-\beta}{\beta}\right)v^2-2\psi\left(\frac{1-\beta}{\beta}\right)|uv|\right] \geq \\ &\geq \frac{1}{2}\left[\left(-\frac{d\psi}{d\tau}+2\psi\right)u^2+2\overline{\mu}_2\psi\left(\frac{1-\beta}{\beta}\right)v^2-2\psi\left(\frac{1-\beta}{\beta}\right)(u^2+v^2)\right] = \\ &= \frac{1}{2}\left[\left(-\frac{d\psi}{d\tau}+2\psi\left(\frac{2\beta-1}{\beta}\right)\right)u^2+2\psi\left(\frac{1-\beta}{\beta}\right)(\overline{\mu}_2-1)v^2\right] \geq \\ &\geq \frac{1}{2}\left[\left(-\frac{d\psi}{d\tau}+2\psi\left(\frac{2\beta-1}{\beta}\right)\right)u^2\right].\end{aligned}$$

(6.42) assures that

$$\frac{d\psi}{d\tau}+2\psi\left(\frac{1-2\beta}{\beta}\right)\leq -2\varepsilon\psi,$$

and hence $\dfrac{dE}{d\tau}$ is positive definite.

6.2 Nonlinear stability-instability results

Theorem 6.9 *All the results guaranteeing the linear stability-instability of* $(1,1)$, $(0,0)$ *and* $\left(\frac{1}{\beta},0\right)$ *guarantee also the local nonlinear stability-instability since, respectively, the following inequalities hold*

$$(|u|+|v|)(|-\beta u^2-(1-\beta)uv|+|\psi uv|)\leq \varepsilon_1(u^2+v^2)^{1+\varepsilon_2}; \tag{6.43}$$

$$(|X|+|Y|)(|-(1-\beta)XY-\beta X^2|+|\psi XY|)\leq \varepsilon_1(X^2+Y^2)^{1+\varepsilon_2}, \tag{6.44}$$

$$(|u|+|v|)(|-\beta u^2-(1-\beta)v|+|\psi uv|)\leq \varepsilon_1(u^2+v^2)^{1+\varepsilon_2}. \tag{6.45}$$

with $\varepsilon_i\ (i=1,2)$*, positive constants.*

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