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**DOTTORATO DI RICERCA IN
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PERTURBATION THEORY FOR ELECTROMAGNETIC WAVE SCATTERING IN RANDOM LAYERED STRUCTURES

PASQUALE IMPERATORE

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Chapter 1

General Introduction

"Quelli che s'innamoran di pratica sanza scienza son come il nocchiere, ch'entra in navilio sanza timone o bussola, che mai ha certezza di dove si vada."
Leonardo

This chapter is aimed at providing a general discussion on the relevance of the scattering problem in random layered structures, emphasizing both the pertinent applications contest and the modeling issue. The role of the perturbation theory approach to the scattering problem is also addressed in a conceptual perspective.

1.1 Scattering Models and Application Context

The electromagnetic wave interaction with layered structures constitutes a crucial topic of current interest in theoretical and experimental research. Indeed, the analysis of layered structures poses challenging questions from the electromagnetic theoretical investigation point of view and certainly is of enormous interest in the applications perspective. Accordingly, the scattering problem from layered structures is of a paramount interest in many scientific and engineering areas. Generally speaking, several modelling and design problems, encountered, for instance, in SAR (Synthetic Aperture Radar) application, GPR (Ground Penetrating Radar) sensing, radar altimeter for planetary exploration, microstrip antennas and MMICs (Monolithic Microwave Integrated Circuits), radio-propagation in urban environment for wireless communications, through-the-wall detection technologies, optics, biomedical diagnostic of layered biological tissues, geophysical and seismic exploration, lead to the analysis of the electromagnetic wave interaction with multilayered structure, whose boundaries can exhibit some amount of roughness.

Furthermore, the evaluation of the wave propagation through layered media with rough boundaries (eventually with spatially inhomogeneous dielectric properties) is crucial in several research fields such as radar, remote sensing, wireless communication and detection technologies, geophysics and optics.

In particular, from the remote sensing applications point of view, scattering from layered media with rough interfaces has been subject of ongoing research and is becoming of increasing importance. In this field the proliferation of the proposed methods for the simulation of wave propagation in a natural stratified medium and the continuous interest in this topic are indicative of the need of appropriate modelling and interpretation of the complex physical phenomena that take place in realistic environment structures.

Properly, each region of the Earth's crust can be morphologically modelled as a suitable multi-layered structure, in which some amount of roughness is presented by every interface, especially when the remote sensing applications scenario is concerned. Stratified soil, sand cover of arid regions, forest canopies, urban buildings, snow blanket, snow cover ice, sea ice and glaciers, oil flood on sea surface, constitute typical natural scenarios, of interest for remote sensing, for which a layered representation is adequate for studying the problem of the electromagnetic (EM) interaction with radar signals. Furthermore, a layered model is usually employed in extraterrestrial scenario, when the revealing of the content under a Planet's surface illuminated by a sensor is concerned. Indeed, in order to properly model the electromagnetic interaction between the radar signal and natural layered structures, the layered media can be modelled most likely as discrete (piecewise-constant) systems, rather than continuous, with some amount of roughness presented by every interface.

On the other hand, in order to exploit efficiently the information contained in the high-resolution backscattering map produced by the nowadays remote sensing airborne or space-borne imaging radar platforms, realistic and comprehensive scattering model for natural solid Earth's cover, as well as for surface and subsurface layers of extraterrestrial scene, are required and cannot rely entirely on heuristic approach. Moreover, explicit analytical forms are advisable for the effective design of processing algorithms and simulation of *Synthetic Aperture Radar* signals.

In this regard, it is worth noting that the amount of data acquired by microwave sensors is continuously increasing; however, the prediction capability of the available electromagnetic (EM) models certainly not always turns out to be satisfactory: major reason for that resides in the intrinsic complexity of modeling the wave interaction with a broad class of inherent natural and man-made structures. In the perspective of overcoming this challenging difficulty, the developing of new and reliable electromagnetic models, possibly leading to closed form solutions, gains new stimulus, because of its crucial role in concretely achieving an accurate understanding and a reliable interpretation of the wide assortment of obtained experimental data.

Therefore, appropriate electromagnetic modeling is fundamental for the exploitation of this dataset. Therefore, the availability of accurate, sound physical and manageable models turns out still to be a strong necessity, in perspective of their application in retrieving add-valued information from the data acquired by microwave sensors.

1.2 Electromagnetic Modelling

It is well known that a comprehensive scattering theory to (exactly) solve the *Maxwell*'s equations (with generic boundary conditions) does not exist; many treatments have been developed to obtain (approximate) solutions to those equations for specific classes of problems, in particular for scattering from random media. Main problem in deriving these solutions is ascribable to the intrinsic mathematical difficulties involved in treating electromagnetic interaction with random structures, whose description can turn out to be extremely complex. It is worth recalling that electromagnetic scattering problems involving general distributions for the inhomogeneities and boundaries shapes are analytical intractable and require intensive, or prohibitive, numerical evaluations.

Thus, in the last decades, only structures with extremely idealized assumptions, i.e., with boundaries coincident to canonical coordinate surfaces, were considered: even for these highly idealized situations cumbersome analytical formulation with formidable mathematical difficulties and many restrictive assumptions have been presented as tractable solutions to the scattering problem.

Electromagnetic propagation in layered media has been considered since the beginning of the 20’th century and several approaches have been developed with application in several branches, such as remote sensing, geophysics, optics and plasma physics.

Broadly speaking, at microwave frequency the scattering from this kind of structures is essentially governed by the scattering properties of corrugated interfaces, the dielectric permittivity vertical profile, and the volume inhomogenity.

Unfortunately, the extremely complicated nature of the physical processes involved, associated with the interactions between volume inhomogeneity and rough interfaces scattering, poses strong limitations to the development of comprehensive models. As a matter of fact, no analytical solution exists that takes into account, in conjunction and in rigorous manner, electromagnetic interaction between the volume scattering and the rough boundary interfaces scattering that take place in a real stratification. For instance, several authors suggest to evaluate total scattering as the heuristic incoherent addition of surface scattering contribution and volume scattering contribution, evaluated separately. Moreover, although the evidence of subsurface scattering has been assessed, the unavailability of adequate theoretical model is tangible. In fact, in several work, the evaluation of the scattering from heterogeneous stratification is treated neglecting the roughness of the interface, whereas some other authors use some empirical extension starting from classical rough surface scattering model. In particular, the effect of stratification on the surface scattering is taken into account in some cases by introducing an empirical beam divergence factor.

Therefore, approximate models are still a necessity due to the insurmountable complexity of real scattering problems. To deal with the problem of scattering by rough multilayer, the available methods differ in the type of approximation made, in how the layered medium is characterized, and in the applicability to different frequency regimes.

When random stratified media are concerned, the possible approaches to cope with the EM scattering problem fall within three main categories.

First, the numerical approaches require a proper specification of the layer structure and rely purely on computational power. Although a plethora of numerical techniques have been developed to give possible solutions to the scattering problem, general conclusions on the general functional dependence of the scattering response on the layered structure's material parameters cannot be easily reached only on the basis of numerical simulations. In addition, they do not permit to attain a comprehensive understanding of physics of the problem , as well as do not allow capturing the physics of the involved scattering mechanisms. In addition, the numerical approach turns out to be feasible for non-fully 3D geometry or configurations in which a only a very limited number of rough interfaces is accounted for. Therefore, we underline that, even if such an approach is in principle viable in analyzing an arbitrary complex structure, in practice the associated computational load precludes the general application of the existing numerical methods to arbitrary layered structures.

On the other hand, layered structures with rough interfaces have been also treated resorting to *Radiative Transfer theory* (RT). RT approaches preclude the consideration of the coherent effects, since they neglect the absolute phase information. As a result, coherent effects, which are not properly accounted for in RT theory, could not be contemplated without employing full wave analysis, which preserves phase information.

Another approach relies on the full-wave methods. We here focus our attention on the wave theory approach, because it simultaneously considers multiple interferential interaction with layer boundaries and preserves phase information, so that it is possible to properly model the well-known backscattering enhancement phenomenon. In addition, only within the wave theory approach the phase is considered and a full application to coherent remote sensing instruments is allowed. Therefore, in the following we do not consider the RT approach.

Full-wave analytical approaches have been conducted relying on different suitable approximations, leading to different domain of applicability. Although several analytical formulation have been conducted in last decades, involving some idealized cases and suitable approximations, to deal with the electromagnetic propagation and scattering in complex random layered media, the relevant solutions are usually too complicated to be generally useful in the remote sensing scenario, even if simplified geometries are accounted for.

However, while there are many analytical techniques dealing with the surface scattering problem which apply to different scattering regimes, unfortunately, for rough interfaces layered media knowledge of the relation between radar response patterns and stratification structure is less advanced.

On the other hand, the proliferation of the proposed methods for the simulation of wave propagation and scattering in stratified media and the continuous interest in this topic are indicative of the need of appropriate modelling and interpretation of the complex physical phenomena that take place especially in layered structures.

Another approach relays on perturbation-based methods, which have been extensively applied in many areas, to attain.

1.3 Scattering and Perturbation Methods

As previously discussed, an exact analytical solution of *Maxwell* equations can be found only for a few idealized problems; subsequently, appropriate approximation methods are needed.

Within this framework, the approaches typical of the *perturbation theory* can be sometimes conveniently employed. Perturbation theory is introduced to deal with systems that can be regarded as obtained from a solvable system by the addition of a small effect (perturbation).

This approach offers a powerful and valuable theoretical technique and allows us attaining approximate solutions of the actual system by suitably adopting some exact solutions relevant to approximate version of the system: this is to say that conveniently approximate solution for perturbed systems can be attained by suitably transforming exact solution of the approximate system, which are known in closed form. The perturbative solution can capture as many features of the analyzed system as many terms of the perturbative development are accounted for.

A variety of perturbation methods has been widely adopted in several research areas, such as Acoustics, Celestial Mechanics, Quantum Mechanics, Optics, Atomic Physics, and Quantum Chemistry. More specifically, in applied electromagnetics the

Perturbation Theory formulation of *Maxwell*'s equations has been conveniently applied in several contexts.

Generally speaking, scattering theory can be regarded as a form of perturbation analysis. Its goal is to predict the perturbation experienced by an electromagnetic wave that interacts with a medium whose properties, with respect to the ones of the original unperturbed medium, are changed. The scattered field is then the difference between the actual and the unperturbed EM wave. The problem is mathematically susceptible also of a formulation in terms of perturbation of linear operators.

Some general considerations are now in order. Rigorously speaking, scattering field itself has no legitimacy from a physical standpoint; only the overall EM field has an objective legitimacy instead. As a matter of fact, the scattering concept intrinsically implies a perturbative description, i. e. it concerns purely a representation matter: This is to say that conceptually scattering itself is a perturbative concept and the overall scattering theory is a form of perturbative analysis and. Therefore, when both surface and volume scattering, respectively ascribable to different kind of inhomogeneities, are concerned, the distinction between these two kinds of phenomena in random media is somehow arbitrary and the adoption of a certain structural description for the scattering medium is only a matter of convenience.

As a practical counterpart, surface and volume scattering contributions turn out difficult to separate if experimental data are concerned. It should be noted that this possibility is essentially denied by the lacking of comprehensive mathematical models, instead of the ability to devise an appropriate data processing algorithm. This appears extremely problematic when extraction of value added information from scattering data, concerning random structures of the inherent natural scenario, is addressed.

Conversely, the coexistence of interfacial roughness and volumetric fluctuations in actual structures should be taken into account methodologically and an inclusive scattering analysis, even though approximate, should be fulfilled, in order to clear understand the distinguishing characteristics of these two different scattering mechanisms.

First of all, some considerations on electromagnetic wave scattering from a rough surface, which is a classical problem in physics and engineering, are in order. Although several perturbation strategies have been proposed (such as Small Perturbation Method (SPM), phase perturbation method , self-energy perturbation method, etc.) to cope with EM scattering, SPM remains the one widely adopted.

The pertinent scattering problem becomes analytically tractable just under suitable approximations, and the only effective approaches reside definitely in *Kirchoff Approach* and *Small Perturbation Method* (SPM). As a matter of fact, concerning a gently rough surface between two half-spaces, closed form SPM scattering solution has been used extensively in applications and constitutes a well consolidated result in the current literature. Concerning surface scattering, SPM solution to an arbitrary order can be derived by using the Rayleigh method (also referred to as Rayleigh-Rice or Rayleigh-Fano procedure), which relies on the Rayleigh hypothesis for expanding the scattered field in power series of the surface-profile function. The same solution can be alternatively obtained by means of the extended boundary conditions, which does not require this a priori assumption, but is formally more involved (note there was some controversy on the legitimacy of the Rayleigh hypothesis).

When layered structure are concerned, the scattering problem has been analytically treated, in the perturbative framework, in remote sensing and antennas and microwave engineering communities; in addition, the problem has been studied independently in the optics community, to deal with properties of optical thin films, and in the theoretical physics community. Specifically, in the analysis is limited to a specific layered configuration with one or, in, two rough interfaces. On the other hand, the relevant works in optics, concerning the scattering through stratifications, have not been taken into consideration by remote sensing and microwave communities, most likely because they are not in a direct closed form and are, at best, of difficult use in practice. Therefore, the existing approaches for the evaluation of the wave scattering through layered structures with rough interfaces are still lacking under the viewpoints of the usability in the applications and of the theoretical investigation clearness, or else of the generality of the structures geometry; and this, despite the problem is encountered in several practical applications.

Moreover, due to the analytical difficulties, commonly the problem of wave scattering by random stratifications has been investigated by separately treating surface roughness and volume scattering effects.

This is particularly true when the scattering phenomena occur in actual media, whose volume structure can be involved and eventually stratified is concerned, the scattering contributions arising from both interfacial roughness and volume inhomogeneities must be taken into account because, the electromagnetic waves significantly penetrate inhomogeneous media as roughly predicted by the values of the penetration depth. To the best of our knowledge, hitherto there is no comprehensive model that is able to rigorously take into account both these wave scattering mechanisms. Moreover, each of the involved scattering mechanisms is generally presented in isolate form in current literature.

We finally underline that, even though the fundamentals of perturbation theory is very simple, however there are not general guidelines for the analytical derivation of a perturbed solution, and very often a significant amount of tedious algebraic manipulation can be required. In this thesis, it is considered as a fundamental guideline in proposed perturbation development the use of a sound physical justification in the pertinent mathematical developments.

Conversely, when mechanically applied without physical justifications, the perturbative techniques can lead to final solutions that are unnecessarily involved and obscure.

In this regard, a final note is in order. Generally speaking, electromagnetic fields are generally regarded as unobservable: they can only be indirectly measured through their interactions with observable quantities. To emphasize the neat physical significance of the methodological approach developed in this thesis, a remarkable interpretation of the scattering solutions, obtained within the innovative theoretical construct presented in this thesis, in terms of the (observable) *Rumsey*'s reaction concept is also provided.

1.5 Scope and Contributions of the Thesis

The problem of electromagnetic wave scattering in 3-D random layered structures, is analytical treated by relying on original results of the *Boundary Perturbation Theory* (BPT) and *Volumetric-Perturbative Reciprocal Theory* (VPRT), whose structured presentation of the pertinent theoretical body of innovative results is proposed and developed in this thesis.

The systematic formulation of *Boundary Perturbation Theory* (BPT) is here introduced to deal with the analysis of a layered structure with an arbitrary number of gently rough interfaces: in this case the proposed theoretical construct is based on a suitable perturbation pertinent to the geometry of the problem and the scattering problem is treated by adopting a proper perturbation of boundary conditions. Specifically, it is demonstrated that, in the first-order approximation, BPT leads to fully polarimetric, formally symmetric and physical revealing closed form solution: the relevant innovative scattering models obtained in this perturbation framework permit to deal with bistatic scattering, from and through three-dimensional layered structures with an arbitrary number of gently rough interfaces.

Furthermore, *Volumetric-Perturbative Reciprocal Theory* (VPRT) is also formulated in this thesis. VPRT methodologically adopts a different approach, which is based on two key elements: the use of the Reciprocity Theorem and an appropriate description of the scattering structure in terms of space-variant volumetric perturbation of the dielectric constant distribution. The VPRT construct also provides meaningful reaction-based expressions for the scattering field, which are straightforward and rich in descriptive power.

It is important to emphasize that VPRT, which is methodologically conceived to consistently treat both interfacial and volumetric random inhomogeneities (so providing a unified mathematical formulation and conceptual understanding of two inherent scattering mechanisms), is also fully consistent with the results of BPT. Accordingly, within VPRT framework, both rough-interface and volume scattering are take into account methodologically.

Furthermore, within this new theoretical framework, a new look at the classical SPM solution for rough surface is also offered: even such a specific solution (whose derivation hitherto obtained via unnecessary, involved and obscure algebraic manipulations) is derived a surprisingly simple way, clarifying all the same the lacking inherent physical meaning.

Beyond a certain compactness of the pertinent closed-form solutions, the fundamental scattering interactions can be revealed, gaining a coherent explanation and a neat picture of the physical meaning of the proposed theoretical constructs. In fact, it is important to note that a deep comprehension of the physical phenomena involved in the electromagnetic wave scattering interaction with such kind of complex structures would have been a rather hopeless task before the introduction of these theories.

Finally, it is noteworthy that this theoretical body of results enables a new way to systematically construct meaningful and general expressions for the scattering field pertinent to wide class of scattering configurations, involving complex structures that can be arranged in a perturbation framework, and it is successful in that it exhibit: conceptual clearness, descriptive power and general applicability to random layered structures.

Chapter 2

Stochastic Characterization for 3-D Layered Structures

"Tutti sanno che una cosa è impossibile da realizzare, finché arriva uno sprovveduto che non lo sa e la inventa."
Albert Einstein

"Se un uomo non è disposto a correre qualche rischio per le proprie idee, o le sue idee non valgono nulla o è lui che non vale nulla."
Ezra Pound

Some features, which can be observed, of natural objects can be described in terms of randomness inherent to a certain spatial irregularity. The source of randomness is intimately related to the lack of detailed knowledge about the processes involved.

Therefore, it is necessary and instructive to introduce herein some basic mathematical notions regarding the representation theory for random processes, with particular emphasis to layered structures whose properties exhibit random spatial fluctuations. Specifically, the focus of this Chapter is on spectral representation for random properties of relevant layered structures, and the notion of wide-sense stationary process is also detailed.

2.1 Spectral Representation of the Interfacial Roughness Stochastic Description

First of all, when the description of a rough interface (see Fig. 1) by means of deterministic function $\zeta_m(\mathbf{r}_\perp)$ is concerned, the corresponding *ordinary* 2-D *Fourier Transform* pair can be defined as

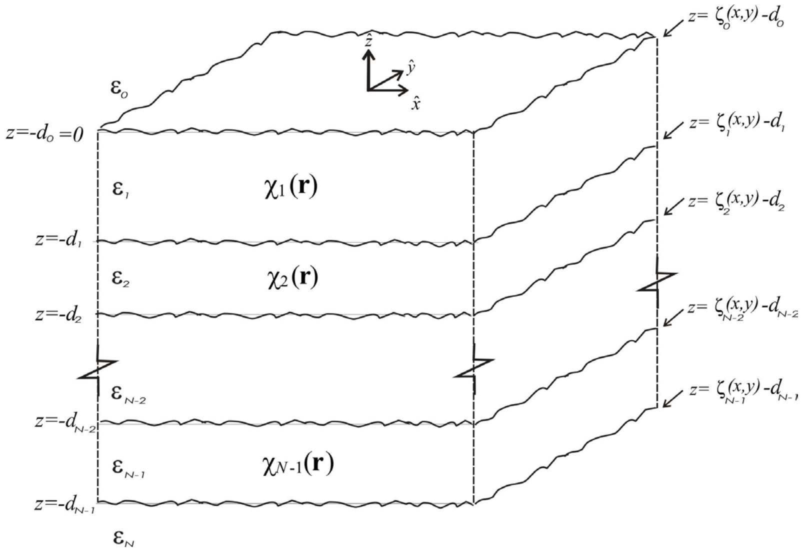

Fig. 1. Layered medium structure.

$$\tilde{\zeta}_m(\mathbf{k}_\perp) = (2\pi)^{-2} \iint d\mathbf{r}_\perp \, e^{-j\mathbf{k}_\perp \cdot \mathbf{r}_\perp} \zeta_m(\mathbf{r}_\perp), \tag{2.1}$$

$$\zeta_m(\mathbf{r}_\perp) = \iint d\mathbf{k}_\perp \, e^{j\mathbf{k}_\perp \cdot \mathbf{r}_\perp} \tilde{\zeta}_m(\mathbf{k}_\perp). \tag{2.2}$$

Let us assume now that $\zeta_m(\mathbf{r}_\perp)$, which describes the generic (*m*-th) rough interface, is a 2-D stochastic process satisfying the conditions

$$< \zeta_m(\mathbf{r}_\perp) > = 0, \tag{2.3}$$

$$< \zeta_m(\mathbf{r}_\perp + \boldsymbol{\rho}_\perp) \zeta_m(\mathbf{r}_\perp) > = B_{\zeta_m}(\boldsymbol{\rho}_\perp), \tag{2.4}$$

where the *angular bracket* denotes statistical ensemble averaging, and where $B_{\zeta_m}(\boldsymbol{\rho}_\perp)$ is the interface *autocorrelation* function, which quantifies the similarity of the spatial fluctuations with a displacement $\boldsymbol{\rho}_\perp$. Equations (2)-(3) constitute the basic assumptions defining a *wide* *sense stationary* (*WSS*) stochastic process (statistical homogeneities): the statistical properties of the process under consideration are invariant to a spatial shift.

Similarly, concerning two mutually correlated random rough interfaces ζ_m and ζ_n, we also assume that they are *jointly WSS*, i.e.

$$<\zeta_m(\mathbf{r}_\perp + \boldsymbol{\rho}_\perp)\zeta_n(\mathbf{r}_\perp)> = B_{\zeta_m\zeta_n}(\boldsymbol{\rho}_\perp)\,, \tag{2.5}$$

where $B_{\zeta_m\zeta_n}(\boldsymbol{\rho}_\perp)$ is the corresponding *cross-correlation function* of the two random processes. It can be readily derived that

$$B_{\zeta_m\zeta_n}(\boldsymbol{\rho}_\perp) = B_{\zeta_n\zeta_m}(-\boldsymbol{\rho}_\perp)\,. \tag{2.6}$$

The integral in (1) is a *Riemann* integral representation for $\zeta_m(\mathbf{r}_\perp)$, and it exists if $\zeta_m(\mathbf{r}_\perp)$ is piecewise continuous and *absolutely integrable*. On the other hand, when the spectral analysis of a stationary random process is concerned, the integral (1) does not in general exist in the framework of theory of the ordinary functions. Indeed, a WSS process describing an interface $\zeta_m(\mathbf{r}_\perp)$ of infinite lateral extension, for its proper nature, is not *absolutely integrable*, so the conditions for the existence of the Fourier Transform are not satisfied.

In order to obtain a spectral representation for a *WSS* random process, this difficulty can be circumvented by resorting to the more general *Fourier-Stieltjes* integral [1]; otherwise one can define space-truncated functions. When a finite patch of the rough interface with area A is concerned, the space-truncated version of (1) can be introduced as

$$\tilde{\zeta}_m(\mathbf{k}_\perp; A) = (2\pi)^{-2}\iint_A d\mathbf{r}_\perp\, e^{-j\mathbf{k}_\perp\cdot\mathbf{r}_\perp}\zeta_m(\mathbf{r}_\perp)\,, \tag{2.7}$$

subsequently, $\tilde{\zeta}_m(\mathbf{k}_\perp) = \lim_{A\to\infty}\tilde{\zeta}_m(\mathbf{k}_\perp; A)$ is not an ordinary function. Nevertheless, here we use again the (1)-(2), regarding them as symbolic formulas, which hold a rigorous mathematical meaning beyond the ordinary function theory (*generalized Fourier Transform*).

We underline that by virtue of the condition (3) directly follows also that $<\tilde{\zeta}_m(\mathbf{k}_\perp)>=0$.

Let us consider

$$<\zeta_m(\mathbf{r}'_\perp)\zeta_n^*(\mathbf{r}''_\perp)>=\iint d\mathbf{k}'_\perp \iint d\mathbf{k}''_\perp\, e^{j(\mathbf{k}'_\perp\cdot\mathbf{r}'_\perp-\mathbf{k}''_\perp\cdot\mathbf{r}''_\perp)} <\tilde{\zeta}_m(\mathbf{k}'_\perp)\tilde{\zeta}_n^*(\mathbf{k}''_\perp)> \tag{2.8}$$

where the asterisk denotes the complex conjugated, and where the operations of average and integration have been interchanged.

When *jointly WSS* processes ζ_m and ζ_n are concerned, accordingly to (29), the LHS of (8) must be a function of the displacement $\boldsymbol{\rho}_\perp=\mathbf{r}'_\perp-\mathbf{r}''_\perp$ only; therefore, it is required that

$$<\tilde{\zeta}_m(\mathbf{k}'_\perp)\tilde{\zeta}_n^*(\mathbf{k}''_\perp)>=W_{mn}(\mathbf{k}'_\perp)\delta(\mathbf{k}'_\perp-\mathbf{k}''_\perp)\,, \tag{2.9}$$

where δ(·) is the *Dirac* delta function, and where $W_{mn}(\boldsymbol{\kappa})$ is called the (spatial) *cross power spectral density* of two interfaces ζ_m and ζ_n, for the spatial frequencies of the roughness.

Equation (9) states that the different spectral components of the two considered interfaces must be uncorrelated.

Indeed, by using (9) into (8), we obtain

$$<\zeta_m(\mathbf{r}'_\perp)\zeta_n(\mathbf{r}''_\perp)>=\iint d\mathbf{k}''_\perp\, e^{j\cdot\mathbf{k}''_\perp\cdot(\mathbf{r}'_\perp-\mathbf{r}''_\perp)}W_{mn}(\mathbf{k}''_\perp)\,, \tag{2.10}$$

where the RHS of (10) involves an (ordinary) 2D Fourier Transform. Note also that as a direct consequence of the fact that $\zeta_n(\mathbf{r}_\perp)$ is real we have the relation $\tilde{\zeta}_n(\mathbf{k}_\perp)=\tilde{\zeta}_n^*(-\mathbf{k}_\perp)$. Therefore, setting $\boldsymbol{\rho}_\perp=\mathbf{r}'_\perp-\mathbf{r}''_\perp$ in (10), we have

$$B_{\zeta_m\zeta_n}(\boldsymbol{\rho}_\perp)=\iint d\boldsymbol{\kappa}\; e^{j\boldsymbol{\kappa}\cdot\boldsymbol{\rho}_\perp}\, W_{mn}(\boldsymbol{\kappa})\;. \tag{2.11}$$

The *cross-correlation function* $B_{\zeta_m\zeta_n}(\boldsymbol{\rho})$ of two interfaces ζ_m and ζ_n is then given by the (inverse) 2-D Fourier Transform of their (spatial) *cross power spectral density*, and Equation (11) together with its Fourier inverse

$$W_{mn}(\mathbf{\kappa}) = (2\pi)^{-2} \iint d\mathbf{\rho}_\perp \, e^{-j\mathbf{\kappa}\cdot\mathbf{\rho}_\perp} B_{\zeta_m \zeta_n}(\mathbf{\rho}), \qquad (2.12)$$

may be regarded as the (generalized) *Wiener-Khinchin* theorem. In particular, when $n = m$, (9) reduces to

$$< \tilde{\zeta}_m(\mathbf{k}'_\perp)\tilde{\zeta}^*_m(\mathbf{k}''_\perp) > = W_m(\mathbf{k}'_\perp)\delta(\mathbf{k}'_\perp - \mathbf{k}''_\perp), \qquad (2.13)$$

where $W_m(\mathbf{\kappa})$ is called the (spatial) *power spectral density* of *n*th corrugated interface ζ_m and can be expressed as the (*ordinary)* 2-D Fourier transform of *n*-corrugated interface autocorrelation function, i.e., satisfying the transform pair:

$$W_m(\mathbf{\kappa}) = (2\pi)^{-2} \iint d\mathbf{\rho}_\perp \, e^{-j\mathbf{\kappa}\cdot\mathbf{\rho}_\perp} B_{\zeta_m}(\mathbf{\rho}_\perp) \ , \qquad (2.14)$$

$$B_{\zeta_m}(\mathbf{\rho}_\perp) = \iint d\mathbf{\kappa} \, e^{j\mathbf{\kappa}\cdot\mathbf{\rho}_\perp} W_m(\mathbf{\kappa}) \ , \qquad (2.15)$$

which is the statement of the classical *Wiener-Khinchin* theorem.

We emphasize the physical meaning of $W_m(\mathbf{\kappa})d\mathbf{\kappa} = W_m(\kappa_x,\kappa_y)d\kappa_x d\kappa_y$: it represents the power of the spectral components of the *m*th rough interface having spatial wave number between κ_x and $\kappa_x + d\kappa_x$ and κ_y and $\kappa_y + d\kappa_y$, respectively, in *x* and *y* direction.

Furthermore, from (6) and (12) it follows that

$$W_{mn}(\mathbf{\kappa}) = W^*_{nm}(\mathbf{\kappa}) . \qquad (2.16)$$

This is to say that, unlike the power spectral density, the cross power spectral density is, in general, neither real nor necessarily positive.

Furthermore, it should be noted that the *Dirac*'s delta function can be defined by the integral representation

$$\delta(\mathbf{\kappa}) = (2\pi)^{-2} \iint d\mathbf{\rho}_\perp \, e^{-j\mathbf{\kappa}\cdot\mathbf{\rho}_\perp} = \lim_{A\to\infty} \delta(\mathbf{\kappa}\,; A) . \qquad (2.17)$$

By using the relation $\delta(0;A) = A/(2\pi)^2$ in (13) and (9) we have, respectively, that the (spatial) power *spectral density* of *n*-th corrugated interface can be also expressed as

$$W_m(\boldsymbol{\kappa}) = (2\pi)^2 \lim_{A\to\infty} \frac{1}{A} < \left|\widetilde{\zeta}_m(\boldsymbol{\kappa};A)\right|^2 >, \tag{2.18}$$

and the (spatial) *cross power spectral density* of two interfaces ζ_m and ζ_n is given by

$$W_{mn}(\boldsymbol{\kappa}) = (2\pi)^2 \lim_{A\to\infty} \frac{1}{A} < \widetilde{\zeta}_m(\boldsymbol{\kappa};A)\widetilde{\zeta}_n^*(\boldsymbol{\kappa};A) >. \tag{2.19}$$

It should be noted that the domain of a rough interface is physically limited by the illumination beamwidth.

Note also that the different definitions of the Fourier transform are available and used in the literature: the sign of the complex exponential function are sometimes exchanged and a multiplicative constant $(2\pi)^{-2}$ may appear in front of either integral or its square root in front of each expression (1)-(2).

Finally, we recall that the theory of random process predicts only the averages over many realizations.

3.2 Spectral Representation of 3-D Homogeneous Complex Random Function

Let us consider a three-dimensional (3-D) complex random function $\chi(\mathbf{r}_\perp, z)$, the corresponding 3-D *Fourier Transform* pair,

$$\widetilde{\chi}(\mathbf{k}_\perp, \beta_z) = (2\pi)^{-3} \iiint d\mathbf{r}_\perp dz\, e^{-j\mathbf{k}_\perp\cdot\mathbf{r}_\perp} e^{-j\beta_z z} \chi(\mathbf{r}_\perp, z), \tag{2.20}$$

$$\chi(\mathbf{r}_\perp, z) = \iiint d\mathbf{k}_\perp d\beta_z\, e^{j\mathbf{k}_\perp\cdot\mathbf{r}_\perp} e^{j\beta_z z} \widetilde{\chi}(\mathbf{k}_\perp, \beta_z), \tag{2.21}$$

and 2-D *Fourier Transform* pair with respect to transverse coordinates

$$\tilde{\chi}(\mathbf{k}_\perp, z) = (2\pi)^{-2} \iint d\mathbf{r}_\perp e^{-j\mathbf{k}_\perp \cdot \mathbf{r}_\perp} \chi(\mathbf{r}_\perp, z), \tag{2.22}$$

$$\chi(\mathbf{r}_\perp, z) = \iint d\mathbf{k}_\perp \, e^{j\mathbf{k}_\perp \cdot \mathbf{r}_\perp} \tilde{\chi}(\mathbf{k}_\perp, z). \tag{2.23}$$

can be introduced, which have to be regarded in a *generalized* sense as discussed in previous Section. Accordingly,

$$\tilde{\chi}(\mathbf{k}_\perp, \beta_z) = (2\pi)^{-1} \int dz \, e^{-j\beta_z z} \tilde{\chi}(\mathbf{k}_\perp, z). \tag{2.24}$$

Hence, also here, for the sake of simplicity, the *Fourier-Stieltjes* integral formalism is understood. Similarly as discussed in previous Section, let us assume now that $\chi(\mathbf{r}) = \chi(\mathbf{r}_\perp, z)$ is a 3-D *stochastic process* describing volumetric inhomogeneity that satisfies the conditions

$$< \chi(\mathbf{r}) > = 0, \tag{2.25}$$

$$< \chi(\mathbf{r} + \boldsymbol{\rho}) \chi^*(\mathbf{r}) > = B_\chi(\boldsymbol{\rho}), \tag{2.26}$$

where $B_\chi(\boldsymbol{\rho})$ is the *autocorrelation* function of the volumetric fluctuations.

Let us consider

$$< \chi(\mathbf{r}') \chi^*(\mathbf{r}'') > = \iiint d\mathbf{k}'_\perp d\beta'_z \iiint d\mathbf{k}''_\perp d\beta''_z \, e^{j\mathbf{k}'_\perp \cdot \mathbf{r}'_\perp} e^{-j\mathbf{k}''_\perp \cdot \mathbf{r}''_\perp} e^{j\beta'_z z'} e^{-j\beta''_z z''} < \tilde{\chi}(\mathbf{k}'_\perp, \beta'_z) \tilde{\chi}^*(\mathbf{k}''_\perp, \beta''_z) > . \tag{2.27}$$

It is then evident that, in order to have homogeneous spatial statistics, we must have:

$$< \tilde{\chi}(\mathbf{k}'_\perp, \beta'_z) \tilde{\chi}^*(\mathbf{k}''_\perp, \beta''_z) > = W_\chi(\mathbf{k}'_\perp, \beta'_z) \, \delta(\mathbf{k}'_\perp - \mathbf{k}''_\perp) \, \delta(\beta'_z - \beta''_z), \tag{2.28}$$

where $W_\chi(\boldsymbol{\kappa},\beta)$ is (spatial) *power spectral density* of the volumetric fluctuation. Therefore, by substituting (28) into (27), the corresponding expression of the *Wiener-Khinchin* theorem is obtained:

$$B_\chi(\boldsymbol{\rho}) = B_\chi(\mathbf{r}'-\mathbf{r}'') = \iiint d\mathbf{k}_\perp d\beta_z \, e^{j\mathbf{k}_\perp\cdot\boldsymbol{\rho}_\perp} e^{j\beta_z\rho_z} \, W_\chi(\mathbf{k}_\perp,\beta_z), \qquad (2.29)$$

$$W_\chi(\mathbf{k}_\perp,\beta_z) = (2\pi)^{-3}\iiint d\boldsymbol{\rho}_\perp dz \, e^{-j\mathbf{k}_\perp\cdot\boldsymbol{\rho}_\perp} e^{-j\beta_z\rho_z} \, B_\chi(\boldsymbol{\rho}_\perp,\rho_z), \qquad (2.30)$$

where we have set $\boldsymbol{\rho} = \mathbf{r}'-\mathbf{r}''$ so that, $\boldsymbol{\rho}_\perp = \mathbf{r}'_\perp - \mathbf{r}''_\perp$ and $\rho_z = z'-z''$.

In addition, a suitable 2-D *spectral representation* for the process $\chi(\mathbf{r}) = \chi(\mathbf{r}_\perp, z)$ can be also introduced. Hence, the following two conditions have to be satisfied:

$$<\tilde{\chi}(\mathbf{k}_\perp, z)> = 0\,, \qquad (2.31)$$

$$<\tilde{\chi}(\mathbf{k}'_\perp, z')\tilde{\chi}^*(\mathbf{k}''_\perp, z'')> = \tilde{B}_\chi(\mathbf{k}'_\perp, z'-z'')\,\delta(\mathbf{k}'_\perp - \mathbf{k}''_\perp). \qquad (2.32)$$

where the dependence of $\tilde{B}_\chi$ on the difference variable $\rho_z = z'-z''$ reflects the aforementioned assumption of statistical homogeneities. From (23), we get

$$<\chi(\mathbf{r}'_\perp, z')\,\chi^*(\mathbf{r}''_\perp, z'')> = \iint d\mathbf{k}'_\perp \iint d\mathbf{k}''_\perp \; e^{j\mathbf{k}'_\perp\cdot\mathbf{r}'_\perp} e^{-j\mathbf{k}''_\perp\cdot\mathbf{r}''_\perp} <\tilde{\chi}(\mathbf{k}'_\perp, z')\tilde{\chi}^*(\mathbf{k}''_\perp, z'')> \qquad (2.33)$$

By using (32) in (33), it can be obtained the following relationships:

$$B_\chi(\mathbf{r}'_\perp - \mathbf{r}''_\perp, z'-z'') = B_\chi(\boldsymbol{\rho}_\perp, \rho_z) = \iint d\mathbf{k}_\perp \; e^{j\mathbf{k}_\perp\cdot\boldsymbol{\rho}_\perp} \tilde{B}_\chi(\mathbf{k}_\perp, \rho_z)\,, \qquad (2.34)$$

$$\tilde{B}_\chi(\mathbf{k}_\perp, z'-z'') = \tilde{B}_\chi(\mathbf{k}_\perp, \rho_z) = (2\pi)^{-2}\iint d\boldsymbol{\rho}_\perp \, e^{-j\mathbf{k}_\perp\cdot\boldsymbol{\rho}_\perp} B_\chi(\boldsymbol{\rho}_\perp, \rho_z), \qquad (2.35)$$

In addition, from (24), considering that

$$\widetilde{\chi}(\mathbf{k}_\perp, z) = \int d\beta_z \, e^{j\beta_z z} \widetilde{\chi}(\mathbf{k}_\perp, \beta_z),$$

we get

$$< \widetilde{\chi}(\mathbf{k}'_\perp, z')\widetilde{\chi}^*(\mathbf{k}''_\perp, z'') > = \\ \int d\beta'_z \int d\beta''_z \, e^{j\beta'_z z'} e^{-j\beta''_z z''} < \widetilde{\chi}(\mathbf{k}'_\perp, \beta'_z)\widetilde{\chi}^*(\mathbf{k}''_\perp, \beta''_z) > . \tag{2.36}$$

By using (28) and (32), the corresponding expression of the *Wiener-Khinchin* theorem is obtained:

$$\widetilde{B}_\chi(\boldsymbol{\kappa}_\perp, \rho_z) = \int_{-\infty}^{+\infty} d\beta_z \, e^{j\beta_z \rho_z} W_\chi(\boldsymbol{\kappa}_\perp, \beta_z), \tag{2.37}$$

$$W_\chi(\boldsymbol{\kappa}_\perp, \beta_z) = (2\pi)^{-1} \int_{-\infty}^{+\infty} dz \;\; e^{-j\beta_z \rho_z} \widetilde{B}_\chi(\boldsymbol{\kappa}_\perp, \rho_z), \tag{2.38}$$

which provide the relations between W_χ and $\widetilde{B}_\chi$.

Furthermore, it should be noted that the *Dirac*'s delta function can be defined by the integral representation (see also (17)):

$$\delta(\beta) = (2\pi)^{-1} \iint dz \, e^{-j\beta z} = \lim_{\Delta\to\infty} \delta(\beta; \Delta). \tag{2.39}$$

By using the relations $\delta(0; A) = A/(2\pi)^2$ and $\delta(0; \Delta) = \Delta/(2\pi)$ in (28), we have that the *power spectral density* $W_\chi(\boldsymbol{\kappa}, \beta)$ of the volumetric fluctuation can be also expressed as

$$W_\chi(\boldsymbol{\kappa}, \beta) = (2\pi)^3 \lim_{A\to\infty} \frac{1}{A} \lim_{\Delta\to\infty} \frac{1}{\Delta} < \left|\widetilde{\chi}(\boldsymbol{\kappa}, \beta; A, \Delta)\right|^2 > . \tag{2.40}$$

In addition, taking into account (17) and $\delta(0; A) = A/(2\pi)^2$ in (32), $\widetilde{B}_\chi(\boldsymbol{\kappa}_\perp, \rho_z)$ can be also expressed as

$$\widetilde{B}_{\chi}(\boldsymbol{\kappa}_{\perp}, z' - z'') = (2\pi)^2 \lim_{A\to\infty} \frac{1}{A} < \widetilde{\chi}(\boldsymbol{\kappa}_{\perp}, z')\, \widetilde{\chi}^{*}(\boldsymbol{\kappa}_{\perp}, z'') > . \tag{2.41}$$

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Chapter 3

Electromagnetic Propagation in 3-D Arbitrary Multi-Layer Structure with Flat Boundaries

"Le convinzioni, più delle bugie, sono nemiche pericolose della verità."

Friedrich Wilhelm Nietzsche

"Quando si ricerca la verità, può darsi che il criterio migliore sia quello di cominciare col criticare le nostre credenze più care."

Karl Raimund Popper

The goal of this chapter is to establish fundamental mathematical properties of electromagnetic waves in a multilayer. Therefore, it will shown that the problem of propagation of a planar wave impinging on the flat boundary layered media is a generalization of the problem of reflection/transmission on the flat interface between two half-spaces.

Accordingly, a proper formalism for electromagnetic propagation in three-dimensional layered media is defined, and a general closed-form solution for the unperturbed vectorial field in the overall structure in terms of the *generalized reflection/transmission* coefficients is provided.

The formalism introduced here is methodologically employed the rest of this thesis. Indeed, this chapter constitutes a conceptual basis in the perspective of the analytical treatment of electromagnetic scattering in layered random structures.

3.1 Introduction and Motivation

Planar multilayer structures with flat boundaries are useful for modelling physical phenomena such as electromagnetic propagation in media that are best modelled as discrete (piecewise-constant), rather than continuous, systems. Layered structures have been the subject of intensive investigation for their application in a number of important research areas: such as the remote sensing, geophysics, ocean engineering, the design of optical instrument as well plasma physics.

There now exist a number of excellent texts in which this subject is discussed [1][2][3]: some of them resort to the transfer matrix operator, others differently adopt some definitions for generalized reflection and transmission coefficients. However, these existing formalisms are here revised in a comprehensive perspective, in order to establish a power formalism that methodologically enables the evaluation of the vectorial field general expressions directly.

In this perspective, the inherent limitations of well-known approach involving the transmission line formalism, which is usually adopted, are overcome: The adopted formalism has the advantage of illustrating clearly the meaning of the general vectorial equations, which are essential for the study of more complex 3-D cases.

3.2 Preliminary notation and definitions

In this Section, the employed notation is briefly defined.

The flat boundaries layered medium (Fig.1) is defined as a stack of parallel slabs, sandwiched in between two half-spaces, whose structure is shift invariant in the direction of x and y (infinite lateral extent in x–y directions). Each layer is assumed to be homogeneous and characterized by arbitrary and deterministic parameters: the dielectric relative permittivity ε_m, the magnetic relative permeability μ_m and the thickness $\Delta_m = d_m - d_{m-1}$. The parameters pertaining to layer m with boundaries $-d_{m-1}$ and $-d_m$ are distinguished by a subscript m. With reference to Fig.1, it has been assumed that in particular, $d_0=0$. In the following, the symbol $\perp$ denotes the projection of the corresponding vector on the plane z=0. Here $\mathbf{r} = (\mathbf{r}_\perp, z)$, so we

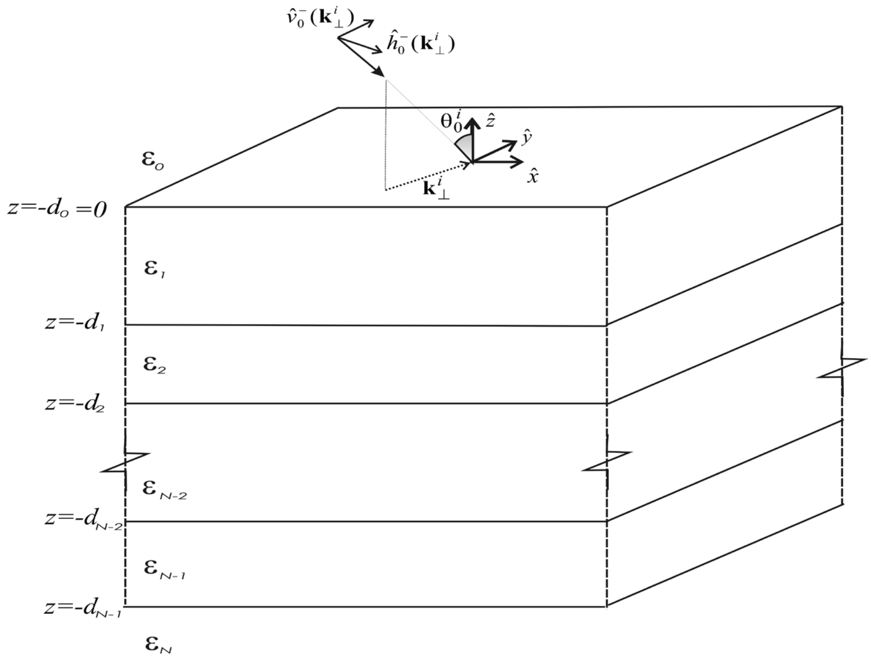

Fig. 1. Geometry for a flat boundaries layered medium.

distinguish the transverse spatial coordinates $\mathbf{r}_\perp = (x, y)$ and the longitudinal coordinate z.

3.2.1 Reflection and Transmission coefficients

This section is devoted preliminary to introduce the basic formalism used in the following of this chapter.

It is important firstly to recall that the *plane of incidence* is a plane that contains the incidence direction $\mathbf{k}_0^i$ and the normal $\hat{z}$ to flat boundary.

When the incident electric field is completely perpendicular to the plane of incidence we have TE condition; this is also known as being perpendicularly or *horizontally* polarized. The TM incident wave is linearly polarized with the electric vector lying in the plane of incidence; this is also known as parallel polarization or vertical polarized.

Accordingly, the incident field can be decomposed into TE and TM components, so that the corresponding reflected and transmission components can be related to the incident field components through the reflection/transmission coefficients. At the m-th plane interface ($z = -d_m$), which separates the m and $m+1$ homogeneous media, the components of the propagating electromagnetic wave encounter impedance mismatches, and a portion of the incident wave is reflected, while the remainder is transmitted, according to the *ordinary* (Fresnel) *Reflection* and *Transmission Coefficients*, respectively indicated with the notations $T^{p}_{m|m+1}$ and $R^{p}_{m|m+1}$, and given by:

$$R^{h}_{m|m+1} = \frac{\mu_{m+1}k_{zm} - \mu_m k_{z(m+1)}}{\mu_{m+1}k_{zm} + \mu_m k_{z(m+1)}}, \tag{3.1}$$

$$R^{v}_{m|m+1} = \frac{\varepsilon_{m+1}k_{zm} - \varepsilon_m k_{z(m+1)}}{\varepsilon_{m+1}k_{zm} + \varepsilon_m k_{z(m+1)}}, \tag{3.2}$$

$$T^{h}_{m|m+1} = \frac{2\mu_{m+1}k_{zm}}{\mu_{m+1}k_{zm} + \mu_m k_{z(m+1)}}, \tag{3.3}$$

$$T^{v}_{m|m+1} = \frac{2\varepsilon_{m+1}k_{zm}}{\varepsilon_{m+1}k_{zm} + \varepsilon_m k_{z(m+1)}}, \tag{3.4}$$

where the superscripts $p \in \{v, h\}$ denote the polarization state for the incident wave and may stand for *horizontal* (h) or *vertical* (v) polarization, $k_0 = \omega\sqrt{\mu_0\varepsilon_0} = \omega / c = 2\pi / \lambda$, $k_m = k_0\sqrt{\mu_m\varepsilon_m}$ is the wave number for the electromagnetic medium in the m-th layer, and

$$k_{zm} = \sqrt{k_m^2 - |\mathbf{k}_\perp|^2} = k_m \cos\theta_m, \tag{3.5}$$

where $\mathbf{k}_\perp = k_x\hat{x} + k_y\hat{y}$ is the two-dimensional projection of vector wave-number on the plane $z=0$.

It is important to emphasize that the coefficients for the *v* polarization can be obtained from the ones for the *h* polarization for *duality* (interchanging $\varepsilon_m \leftrightarrow \mu_m$), and vice versa.

Moreover, if we consider stratified media which are isotropic in the planes z = const, the coefficients (3)-(6) depend only on the amplitude of the vector $\mathbf{k}_\perp$, regardless of the direction.

In addition, it should be also noted that:

$$R^p_{i|j} = -R^p_{j|i} \qquad i=j\pm 1 \qquad (3.6)$$
$$T^p_{i|j} = 1 + R^p_{i|j}$$

3.2.2 Field Representation

In this section, the solution for the flat-boundaries structure is addressed. An arbitrary polarized monochromatic plane wave is considered to be incident on the layered medium at an angle θ_0^i relative to the $\hat{z}$ direction from the upper half-space, as schematically shown in Fig.1,

$$\mathbf{E}_0^i(\mathbf{r}) = [E_0^{ih}\hat{h}_0^-(\mathbf{k}_\perp^i) + E_0^{iv}\hat{v}_0^-(\mathbf{k}_\perp^i)]\, e^{j(\mathbf{k}_\perp^i\cdot\mathbf{r}_\perp - k_{z0}^i z)}, \qquad (3.7)$$

where in the field expression a time factor *exp(-jωt)* is understood, and, using a spherical frame representation, the incident vector wave direction is individuated by θ_0^i, φ_0^i:

$$k_0\hat{k}_0^i = \mathbf{k}_0^i = \mathbf{k}_\perp^i - \hat{z}k_{z0}^i = k_0(\hat{x}\sin\theta_0^i\cos\varphi_0^i + \hat{y}\sin\theta_0^i\sin\varphi_0^i - \hat{z}\cos\theta_0^i) \qquad (3.8)$$

with

$$\hat{h}_0^-(\mathbf{k}_\perp^i) = \frac{\hat{k}_0^i \times \hat{z}}{\left|\hat{k}_0^i \times \hat{z}\right|} = \sin\varphi_0^i\hat{x} - \cos\varphi_0^i\hat{y} \qquad (3.9)$$

$$\hat{v}_0^-(\mathbf{k}_\perp^i) = \hat{h}_0^-(\mathbf{k}_\perp^i)\times\hat{k}_0^i = (\hat{x}\cos\varphi_0^i + \hat{y}\sin\varphi_0^i)\cos\theta_0^i + \hat{z}\sin\theta_0^i \qquad (3.10)$$

where $\mathbf{k}_{\perp}^{i} = k_{x}^{i}\hat{x} + k_{y}^{i}\hat{y}$ is the two dimensional projection of incident wave-number on the plane *z=0*.

When the electromagnetic properties of an isotropic medium are varying only in one direction, e.g., the *z* direction, the vector wave equation can be reduced to two scalar equations that are decoupled from each other, so the electromagnetic propagation can be expressed in terms of the propagation, reflection and transmission of two *up*- and *down*-going decoupled polarized modes (*transverse electric* TE and *transverse magnetic* TM) in each piecewise constant region. Hence, a unique solution is found by matching boundary condition across the discontinuities at the interfaces.

The fact that the vector wave equation can be reduced to two scalar equations that are decoupled from each other derives directly from the mutually independence of the condition of continuity, on the generic boundary, of the TE and TM component, respectively. Therefore, in accordance with the *duality* of electromagnetic theory, we can limit ourselves to consider the TE case.

Within the *m*-th layer (see Fig.1) the field solutions can be expressed in the form:

$$\mathbf{E}_{m}^{(0)} = \mathbf{E}_{m}^{-(0)} + \mathbf{E}_{m}^{+(0)} \tag{3.11}$$

$$\mathbf{H}_{m}^{(0)} = \mathbf{H}_{m}^{-(0)} + \mathbf{H}_{m}^{+(0)} \tag{3.12}$$

where $\mathbf{E}_{m}^{-(0)}$ and $\mathbf{E}_{m}^{+(0)}$ represent the electric-field waves propagating *up*- and *down*-going directions, respectively:

$$\mathbf{E}_{m}^{\pm(0)}(\mathbf{r}) = \sum_{p=h,v} e^{j\mathbf{k}_{\perp}^{i}\cdot\mathbf{r}_{\perp}}\hat{p}_{m}^{\pm}(\mathbf{k}_{\perp}^{i})S_{m}^{\pm p(0)}(k_{\perp}^{i})e^{\pm jk_{zm}^{i}z} \tag{3.13}$$

$$\mathbf{H}_{m}^{\pm(0)}(\mathbf{r}) = \frac{1}{Z_{m}}\hat{k}_{m}^{\pm}(\mathbf{k}_{\perp}^{i}) \times \mathbf{E}_{m}^{\pm(0)} \tag{3.14}$$

where a parenthesized superscript indicates the order of perturbation: the superscript (0) refers to the *unperturbed field* associated with the flat boundaries stratification, $k_{zm}^{i} = k_{zm}(\mathbf{k}_{\perp}^{i})$, Z_m is the *intrinsic impedance* of the *m*-th layer medium, $S_{m}^{\pm p(0)}$ are respectively the zero-order complex amplitudes of the *up-going* (+) and *down-going* (–) field components in the m-th layer in p polarization, $p \in \{v,h\}$ denotes the polarization, and the wave-vectors in the m-th layer are given by

$$\mathbf{k}_m^{\pm}(\mathbf{k}_{\perp}) = \mathbf{k}_{\perp} \pm \hat{z} k_{zm}, \tag{3.15}$$

and

$$\hat{h}_m^{\pm}(\mathbf{k}_{\perp}) = \hat{k}_m^{\pm} \times \hat{z} / \left| \hat{k}_m^{\pm} \times \hat{z} \right|, \tag{3.16}$$

$$\hat{v}_m^{\pm}(\mathbf{k}_{\perp}) = \hat{h}_m^{\pm} \times \hat{k}_m^{\pm}. \tag{3.17}$$

It should be observed that

$$\hat{h}_m^{\pm}(\mathbf{k}_{\perp}) = \hat{k}_{\perp} \times \hat{z} = \hat{h}, \tag{3.18}$$

$$\hat{v}_m^{\pm}(\mathbf{k}_{\perp}) = \mp \frac{k_{zm}}{k_m} \hat{k}_{\perp} + \frac{k_{\perp}}{k_m} \hat{z} \tag{3.19}$$

is a basis for the horizontal/vertical polarization vectors.

The solution (11)-(14) is also named zero-order or *unperturbed* solution. It should be noted that, in the case of a plane-wave incident upon an infinite surface, the angular distribution of the specular component of the scattering intensity can be regarded as a delta function centered at a specular direction.

3.2.3 Boundary Conditions

The continuity of the electric and magnetic tangent components on the boundary derives directly from the Maxwell equation's boundary conditions:

$$\hat{z} \times \Delta \mathbf{E}_m^{(0)} \Big|_{z=-d_m} = 0 \tag{3.20}$$

$$\hat{z} \times \Delta \mathbf{H}_m^{(0)} \Big|_{z=-d_m} = 0 \tag{3.21}$$

where $\Delta\mathbf{E}_m^{(0)} = \mathbf{E}_{m+1}^{(0)} - \mathbf{E}_m^{(0)}, \Delta\mathbf{H}_m^{(0)} = \mathbf{H}_{m+1}^{(0)} - \mathbf{H}_m^{(0)}$

Therefore, a unique solution is found by matching boundary condition across the discontinuities at the interfaces. It should be noted that for the *phase matching* condition the projection of the vector wave $\mathbf{k}_\perp$ must be invariant in the stratification.

Here in after we focus on the *TE case* only, since the TM solution can be directly obtained straightforwardly for duality. Accordingly, without loss of generality, we consider primarily the following unitary amplitude $(E_0^{ih} = 1)$ horizontal polarized (TE) electric incident field:

$$\mathbf{E}_0^i = \hat{h}_0^-(\mathbf{k}_\perp^i) e^{j(\mathbf{k}_\perp^i \cdot \mathbf{r}_\perp - k_{z0}^i z)} \tag{3.22}$$

Noting that

$$\mathbf{H}_m^{+(0)} = \frac{1}{Z_m} \hat{k}_m^+ \times \mathbf{E}_m^{+(0)} \tag{3.23}$$

$$\mathbf{H}_m^{(-)} = \frac{1}{Z_m} \hat{k}_m^- \times \mathbf{E}_m^{-(0)} \tag{3.24}$$

we have

$$\mathbf{H}_{m+1}^{(0)} = \frac{1}{Z_{m+1}} [\hat{k}_{m+1}^- \times \mathbf{E}_{m+1}^{-(0)} + \hat{k}_{m+1}^+ \times \mathbf{E}_{m+1}^{+(0)}] \tag{3.25}$$

$$\mathbf{H}_m^{(0)} = \frac{1}{Z_m} [\hat{k}_m^- \times \mathbf{E}_m^{-(0)} + \hat{k}_m^+ \times \mathbf{E}_m^{+(0)}] \tag{3.26}$$

Utilizing Eqs. (23)-(26), Eqs. (20)-(21) can be rewritten as follows:

$$\hat{z} \times [\mathbf{E}_{m+1}^{+(0)} + \mathbf{E}_{m+1}^{-(0)}]\Big|_{z=-d_m} = \hat{z} \times [\mathbf{E}_m^{+(0)} + \mathbf{E}_m^{-(0)}]\Big|_{z=-d_m} \tag{3.27}$$

$$\begin{aligned} &[\hat{z} \times \hat{k}_{m+1}^+ \times \mathbf{E}_{m+1}^{+(0)} + \hat{z} \times \hat{k}_{m+1}^- \times \mathbf{E}_{m+1}^{-(0)}]\Big|_{z=-d_m} = \\ &= \frac{Z_{m+1}}{Z_m} [\hat{z} \times \hat{k}_m^+ \times \mathbf{E}_m^{+(0)} + \hat{z} \times \hat{k}_m^- \times \mathbf{E}_m^{-(0)}]\Big|_{z=-d_m} \end{aligned} \tag{3.28}$$

Taking into account the following relation:

$$\hat{z}\times\hat{k}_m^{\pm}\times\mathbf{E}_m^{\pm(0)}=\hat{k}_m^{\pm}(\hat{z}\cdot\mathbf{E}_m^{\pm(0)})-\mathbf{E}_m^{\pm(0)}(\hat{k}_m^{\pm}\cdot\hat{z}),\tag{3.29}$$

and pre-multiplying across two sides of eq. (28) by $\hat{z}$, we get:

$$\hat{z}\times[-\mathbf{E}_m^{+(0)}+\mathbf{E}_m^{-(0)}]\Big|_{z=-d_m}=\frac{k_{z(m+1)}\mu_m}{k_{zm}\mu_{m+1}}\hat{z}\times[-\mathbf{E}_{m+1}^{+(0)}+\mathbf{E}_{m+1}^{-(0)}]\Big|_{z=-d_m}\tag{3.30}$$

Adding and subtracting eq. (30) to and from eq. (27), respectively, we get on the plane z = $-d_m$:

$$\begin{aligned}\hat{z}\times\mathbf{E}_m^{-(0)}&=\frac{1}{T^h_{m|m+1}}[(\hat{z}\times\mathbf{E}_{m+1}^{-(0)})+R^h_{m|m+1}(\hat{z}\times\mathbf{E}_{m+1}^{+(0)})]\\ \hat{z}\times\mathbf{E}_m^{+(0)}&=\frac{1}{T^h_{m|m+1}}[R^h_{m|m+1}(\hat{z}\times\mathbf{E}_{m+1}^{-(0)})+(\hat{z}\times\mathbf{E}_{m+1}^{+(0)})]\end{aligned}\tag{3.31}$$

Therefore, Eqs. (31) has been obtained by enforcing the continuity of the tangential fields across the discontinuity of the interfaces, for TE case, on the plane z = $-d_m$.

3.3 Transfer Matrix formalism

The transition of the wave through each layer can be completely described by a matrix, called transfer matrix, with elements depending on the character of the wave and on the properties of the layer and its thickness.

In each layer up-going and down-going field components are present: their complex amplitudes can be arranged in a single vector $\mathbf{S}_m^{p(0)}$ according to the following notation:

$$\mathbf{S}_m^{p(0)}(k_\perp,d_m)=\begin{bmatrix}S_m^{+p(0)}(k_\perp)e^{-jk_{zm}d_m}\\ S_m^{-p(0)}(k_\perp)e^{+jk_{zm}d_m}\end{bmatrix}\tag{3.32}$$

This notation is convenient to discuss *up-going* end *down-going* components by means of single equation instead of using two equations as done in Eq. (31). In fact, by using Eq. (13) into Eqs. (31)

the system of recurrent equations (31) may then be resolved in the form:

$$\mathbf{S}_m^{h(0)}(k_\perp^i, d_m) = \boldsymbol{N}_{m|m+1}^h(k_\perp^i)\mathbf{S}_{m+1}^{h(0)}(k_\perp^i, d_m) \qquad (3.33)$$

where the fundamental *transfer matrix operator* $\boldsymbol{N}_{m|m+1}^p$ is given by:

$$\boldsymbol{N}_{m|m+1}^p(k_\perp) = \frac{1}{T_{m|m+1}^p}\begin{bmatrix} 1 & R_{m|m+1}^p \\ R_{m|m+1}^p & 1 \end{bmatrix} \qquad (3.34)$$

with the superscripts $p \in \{v, h\}$ denoting the polarization.

It is clear the physical meaning of this matrix that relates the amplitude of the waves propagating in the (m+1)-layer, up going and down going respectively, to the corresponding waves in the m-layer.

It should be emphasized that eqs. (33) states in a simpler form the problem originally set by eqs. (31): as matter of fact, solving Eq. (33) $\forall m$ implies dealing with the determination of unknown scalar amplitudes $S_m^{\pm p(0)}(\mathbf{k}_\perp)$ instead of working with the corresponding vector unknowns $\mathbf{E}_m^{(0)}, \mathbf{H}_m^{(0)}$. Accordingly, when the structure with flat interfaces is considered, the enforcement of the boundary conditions through the stratification (m= 0,...,N-1) can be addressed by writing down a linear system of equations with the aid of the matrix formalism (33) with m=0,..,N-1.

Therefore, as it will discussed in Section 3.6, the adoption of the *matrix re-formulation* of the boundary conditions (31), which essentially works *recursively* to match boundary conditions at each successive interface, implies that the problem in each m-th layer is reduced to the algebraic calculation of the unknown expansion coefficients vector (32). Letting

$$\boldsymbol{\Pi}_m(k_\perp) = \begin{bmatrix} e^{jk_{zm}\Delta_m} & 0 \\ 0 & e^{-jk_{zm}\Delta_m} \end{bmatrix} \qquad (3.35)$$

which accounts for the propagation in the m-th layer, with $\Delta_m = d_m - d_{m-1}$ the thickness of the m-th layer, allows writing:

$$\mathbf{S}_m^{h(0)}(k_\perp^i, d_m) = \boldsymbol{N}_{m|m+1}^h(k_\perp^i)\boldsymbol{\Pi}_{m+1}(k_\perp^i)\mathbf{S}_{m+1}^{h(0)}(k_\perp^i, d_{m+1}). \tag{3.36}$$

We emphasize that the matrixes $\boldsymbol{N}_{m|m+1}^h$ and $\boldsymbol{\Pi}_{m+1}$ are dependent on the medium proprieties and on the projection of the wave-number vector on the plane z=0. By using (34) by (35), it turns out that:

$$\boldsymbol{N}_{m-1|m}^p\boldsymbol{\Pi}_m(k_\perp) = \frac{1}{T_{m-1|m}^p}\begin{bmatrix} e^{jk_{zm}\Delta_m} & R_{m-1|m}^p e^{-jk_{zm}\Delta_m} \\ R_{m-1|m}^p e^{jk_{zm}\Delta_m} & e^{-jk_{zm}\Delta_m} \end{bmatrix}. \tag{3.37}$$

It is also useful to consider that

$$[\boldsymbol{N}_{j|j+1}^p(k_\perp)]^{-1} = \boldsymbol{N}_{j+1|j}^p(k_\perp) = \frac{1}{T_{j+1|j}^p}\begin{bmatrix} 1 & R_{j+1|j}^p \\ R_{j+1|j}^p & 1 \end{bmatrix}, \tag{3.38}$$

$$\boldsymbol{\Pi}_j^{-1}(k_\perp) = \begin{bmatrix} e^{-jk_{zj}\Delta_j} & 0 \\ 0 & e^{+jk_{zj}\Delta_j} \end{bmatrix}, \tag{3.39}$$

so that

$$[\boldsymbol{N}_{j|j+1}^p]^{-1}\boldsymbol{\Pi}_j^{-1} = \frac{1}{T_{j+1|j}^p}\begin{bmatrix} e^{-jk_{zj}\Delta_j} & R_{j+1|j}^p e^{+jk_{zj}\Delta_j} \\ R_{j+1|j}^p e^{-jk_{zj}\Delta_j} & e^{+jk_{zj}\Delta_j} \end{bmatrix}. \tag{3.40}$$

Accordingly, the condition (36) can be rewritten in the form:

$$\mathbf{S}_k^{h(1)}(k_\perp, d_k) = \boldsymbol{\Pi}_k^{-1}(k_\perp)[\boldsymbol{N}_{k-1|k}^h(k_\perp)]^{-1}\mathbf{S}_{k-1}^{h(1)}(k_\perp, d_{k-1}). \tag{3.41}$$

This matrix formalism provides an elegant technique not only for the calculation of the field in flat-boundaries stratification, but also for the evaluation of the perturbative field as clarified in the next chapters. In fact, the transfer operators include an intrinsic characterization of the layered structure, and as such they can be successfully exploited even in the derivation of the approximation of the scattered field.

3.4 Generalized reflection formalism

This section is devoted to introduce the formalism used in the following of this thesis. The *generalized reflection coefficients* $\Re^p_{m-1|m}$, for the *p*-polarization (TE or TM), at the interface between the regions (*m*-1) and *m* are defined as the ratio of the amplitudes of *upward*- and *downward*-propagating waves immediately above the interface, respectively. They can be expressed by recursive relations as:

$$\Re^p_{m-1|m} = \frac{R^p_{m-1|m} + \Re^p_{m|m+1} e^{j2k_{zm}\Delta_m}}{1 + R^p_{m-1|m}\Re^p_{m|m+1} e^{j2k_{zm}\Delta_m}}, \tag{3.42}$$

where the denominator $1 + R^p_{m|m-1}\Re^p_{m|m+1} e^{j2k_{zm}\Delta_m}$ takes into account the multiple reflection in the *m*-th layer. Likewise, at the interface between the regions (*m*+1) and *m*, $\Re^p_{m+1|m}$ is given by:

$$\Re^p_{m+1|m} = \frac{R^p_{m+1|m} + \Re^p_{m|m-1} e^{j2k_{zm}\Delta m}}{1 + R^p_{m+1|m}\Re^p_{m|m-1} e^{j2k_{zm}\Delta_m}}. \tag{3.43}$$

Furthermore, the following notations are introduced:

$$\vec{M}^p_m(k_\perp) = 1 - R^p_{m|m-1}\Re^p_{m|m+1} e^{j2k_{zm}\Delta_m} \quad , \tag{3.44}$$

$$\overleftarrow{M}^p_m(k_\perp) = 1 - \Re^p_{m|m-1} R^p_{m|m+1} e^{j2k_{zm}\Delta_m}, \tag{3.45}$$

$$\overleftrightarrow{M}^p_m(k_\perp) = 1 - \Re^p_{m|m-1}\Re^p_{m|m+1} e^{j2k_{zm}\Delta_m}, \tag{3.46}$$

It should be noted that the inverse of (44)-(46) take into account the multiple reflections in the *m*-th layer.

It can be easily shown that the following identity holds:

$$\overleftarrow{M}^p_{m-1}\overleftrightarrow{M}^p_m = \overleftrightarrow{M}^p_{m-1}\vec{M}^p_m. \tag{3.47}$$

In order to provide the proof of the identity (47), by considering the definitions (44)-(46), it can be rewritten in the form:

$$[1-\Re^p_{m-1|m-2}R^p_{m-1|m}e^{j2k_{z(m-1)}\Delta_{m-1}}][1-\Re^p_{m|m-1}\Re^p_{m|m+1}e^{j2k_{zm}\Delta_m}]=$$
$$[1-\Re^p_{m-1|m-2}\Re^p_{m-1|m}e^{j2k_{z(m-1)}\Delta_{m-1}}][1-R^p_{m|m-1}\Re^p_{m|m+1}e^{j2k_{zm}\Delta_m}]. \tag{3.48}$$

By substituting the definition

$$\Re^p_{m|m-1}=\frac{R^p_{m|m-1}+\Re^p_{m-1|m-2}e^{j2k_{z(m-1)}\Delta_{m-1}}}{1+R^p_{m|m-1}\Re^p_{m-1|m-2}e^{j2k_{z(m-1)}\Delta_{m-1}}} \tag{3.49}$$

in the left-hand-side of (48), we obtain:

$$1+\Re^p_{m-1|m-2}R^p_{m|m-1}e^{j2k_{z(m-1)}\Delta_{m-1}}-R^p_{m|m-1}\Re^p_{m|m+1}e^{j2k_{zm}\Delta_m}$$
$$-\Re^p_{m-1|m-2}\Re^p_{m|m+1}e^{j2k_{z(m-1)}\Delta_{m-1}}e^{j2k_{zm}\Delta_m}. \tag{3.50}$$

In the same way, by substituting (42) in the right-hand-side of (48), an expression identical to (50) is obtained. Therefore, the identity (47) is proved. In addition, using definitions (42)-(46) and equations (6), after suitable algebraic manipulations, it can be verified that:

$$1\pm\Re^p_{m|m+1}=[1\pm R^p_{m|m+1}][\vec{M}^p_{m+1}]^{-1}[1\pm\Re^p_{m+1|m+2}e^{j2k_{z(m+1)}\Delta_{m+1}}], \tag{3.51}$$

$$1\pm\Re^p_{m+1|m}=[1\pm R^p_{m+1|m}][\overleftarrow{M}^p_{m}]^{-1}[1\pm\Re^p_{m|m-1}e^{j2k_{zm}\Delta_m}]. \tag{3.52}$$

3.5 Generalized transmission formalism

The *generalized transmission coefficients* in *downward* direction $\Im^p_{0|m}$ can, which are defines as $\Im^p_{0|m}S_0^{-p(0)}e^{jk_{z0}d_0}=S_m^{-p(0)}e^{jk_{zm}d_{m-1}}$, can be written as:

$$\Im^{p}_{0|m}(k_{\perp}) = \exp\left[j\sum_{n=1}^{m-1} k_{zn}\Delta_n\right]\prod_{n=0}^{m-1} T^{p}_{n|n+1}\left[\prod_{n=1}^{m}\vec{M}^{p}_{n}\right]^{-1}, \tag{3.53}$$

where $p \in \{v, h\}$.

Here the notion of layered *slab* is introduced, which refers to a layered structure sandwiched between two half-spaces. The *generalized transmission coefficients* in *downward* direction for the layered slab between two half-spaces (*0*, *m*), $\Im^{p(slab)}_{0|m}$, can be then defined as

$$\Im^{p(slab)}_{0|m}(k_{\perp}) = \exp\left[j\sum_{n=1}^{m-1} k_{zn}\Delta_n\right]\prod_{n=0}^{m-1} T^{p}_{n|n+1}\left[\prod_{n=1}^{m-1}\vec{M}^{p}_{n}\right]^{-1}. \tag{3.54}$$

Note also that coefficients $\Im^{p}_{0|m}$ are distinct from the coefficients $\Im^{p(slab)}_{0|m}$, because in the evaluation of $\Im^{p}_{0|m}$ the effect of all the layers under the layer *m* is taken into account, whereas $\Im^{p(slab)}_{0|m}$ are evaluated referring to a different configuration in which the intermediate layers 1...*m* are bounded by the half-spaces 0 and *m*.

Accordingly, the *generalized transmission coefficients* in *downward* direction for a layered slab between two half-spaces (0*, N*), $\Im^{p(slab)}_{0|N}$, which are defined as $\Im^{h(slab)}_{0|N}\, S^{-h(0)}_{0} e^{jk_{z0}d_0} = S^{-h(0)}_{N} e^{jk_{zN}d_{N-1}}$, can be written:

$$\Im^{p(slab)}_{0|N}(k_{\perp}) = \exp\left[j\sum_{n=1}^{N-1} k_{zn}\Delta_n\right]\prod_{n=0}^{N-1} T^{p}_{n|n+1}\left[\prod_{n=1}^{N-1}\vec{M}^{p}_{n}\right]^{-1}. \tag{3.55}$$

It should be noted that the parenthesized superscript *slab* indicates that both the media 0 and *N* are half-space.

Similarly, the *generalized transmission coefficients* in *downward* direction for the layered slab between two half-spaces (*m*+1*, N*), $\Im^{p(slab)}_{m+1|N}$, which are defined as $\Im^{h(slab)}_{m+1|N}\, S^{-h(0)}_{m+1} e^{jk_{z(m+1)}d_{m+1}} = S^{-h(0)}_{N} e^{jk_{zN}d_{N-1}}$, can be written as:

$$\Im_{m+1|N}^{p(slab)}(k_\perp) = \exp\left[j \sum_{n=m+2}^{N-1} k_{zn}\Delta_n \right] \prod_{n=m+1}^{N-1} T_{n|n+1}^{p} \left[\prod_{n=m+2}^{N-1} \vec{M}_n^p \right]^{-1} . \quad (3.56)$$

It should be also noted that

$$\Im_{m+1|N}^{p} = [\overleftrightarrow{M}_{m+1}^{p}]^{-1} \Im_{m+1|N}^{p(slab)} . \quad (3.57)$$

Moreover, we consider the *generalized transmission coefficients* in *upward* direction for the layered *slab* between two half-spaces (m, 0), $\Im_{m|0}^{p(slab)}$, which are defined as

$$\Im_{m|0}^{p(slab)}(k_\perp) = \exp\left[j \sum_{n=1}^{m-1} k_{zn}\Delta_n \right] \prod_{n=0}^{m-1} T_{n+1|n}^{p} \left[\prod_{n=1}^{m-1} \overleftarrow{M}_n^p \right]^{-1} . \quad (3.58)$$

Moreover, the *generalized transmission coefficients* in *upward* direction $\Im_{m|0}^{p}$ are then given by

$$\Im_{m|0}^{p}(k_\perp) = [\overleftrightarrow{M}_{m}^{p}(k_\perp)]^{-1} \Im_{m|0}^{p(slab)}(k_\perp) . \quad (3.59)$$

We stress that generalized reflection and transmission coefficients do not depend on the direction of $\mathbf{k}_\perp$. In the following, we shown how the employing the generalized reflection/transmission coefficient notions not only is crucial in obtaining a compact closed-form solution, but it also permit us to completely elucidate the obtained analytical expressions from a physical point of view, highlighting the role played by the *equivalent reflecting interfaces* and by *the equivalent slabs*.

Finally, it should be noted that results similar to those presented in this Section are given in[3], however the formalism is not fully equivalent to the one provided here.

3.5.1 Reciprocal Character of Generalized Coefficients

In this Section, a formal verification for the *reciprocity* of the generalized transmission coefficients for an arbitrary flat-boundaries layered structure is provided. By applying recursively the identity (47), it is quite straightforward to show that the following equality holds:

$$[\overleftarrow{M}_1^p \overleftarrow{M}_2^p \cdots \overleftarrow{M}_{m-2}^p \overleftarrow{M}_{m-1}^p] \overleftrightarrow{M}_m^p = \overrightarrow{M}_1^p [\overrightarrow{M}_2^p \cdots \overrightarrow{M}_{m-2}^p \overrightarrow{M}_{m-1}^p \overrightarrow{M}_m^p] \quad (3.60)$$

Note also that in last recursion we have taken in account that $\overleftrightarrow{M}_1^p = \overrightarrow{M}_1^p$, since the region 0 is a half-space. Furthermore, from the definition of transmission coefficients, we have:

$$\prod_{n=0}^{m-1} \frac{T_{n+1|n}^h}{T_{n|n+1}^h} = \prod_{n=0}^{m-1} \frac{\mu_n k_{z(n+1)}}{k_{zn}\mu_{n+1}} = \frac{\mu_0 k_{zm}}{\mu_m k_{z0}} \quad (3.61)$$

$$\prod_{n=0}^{m-1} \frac{T_{n+1|n}^v}{T_{n|n+1}^v} = \prod_{n=0}^{m-1} \frac{\varepsilon_n k_{z(n+1)}}{k_{zn}\varepsilon_{n+1}} = \frac{\varepsilon_0 k_{zm}}{\varepsilon_m k_{z0}} \quad (3.62)$$

In conclusion, considering (60)-(62) and the definition (53), it follows that:

$$\Im_{m|0}^p = \begin{cases} \Im_{0|m}^p \dfrac{\mu_0 k_{zm}}{\mu_m k_{z0}} & for\ p = h \\ \Im_{0|m}^p \dfrac{\varepsilon_0 k_{zm}}{\varepsilon_m k_{z0}} & for\ p = v \end{cases} . \quad (3.63)$$

Similarly, applying recursively the identity (47), we get:

$$\overleftrightarrow{M}_{m+1}^p [\overrightarrow{M}_{m+2}^p \cdots \overrightarrow{M}_{N-2}^p \overrightarrow{M}_{N-1}^p] = [\overleftarrow{M}_{m+1}^p \overleftarrow{M}_{m+2}^p \cdots \overleftarrow{M}_{N-2}^p] \overleftarrow{M}_{N-1}^p \quad (3.64)$$

where in the last recursion we have taken in account that $\overleftrightarrow{M}_{N-1}^p = \overleftarrow{M}_{N-1}^p$, since the region N is a half-space. Similarly, we have:

$$\prod_{n=m+1}^{N-1} \frac{T^h_{n+1|n}}{T^h_{n|n+1}} = \prod_{n=m+1}^{N-1} \frac{\mu_n k_{z(n+1)}}{k_{zn}\mu_{n+1}} = \frac{\mu_{m+1} k_{zN}}{\mu_N k_{z(m+1)}} \tag{3.65}$$

$$\prod_{n=m+1}^{N-1} \frac{T^v_{n+1|n}}{T^v_{n|n+1}} = \prod_{n=m+1}^{N-1} \frac{\varepsilon_n k_{z(n+1)}}{k_{zn}\varepsilon_{n+1}} = \frac{\varepsilon_{m+1} k_{zN}}{\varepsilon_N k_{z(m+1)}} \tag{3.66}$$

Therefore from (64)-(66) and the definition (56)-(57), as a counterpart of (63), the following relations is obtained:

$$\Im^p_{m+1|N} = \begin{cases} \Im^p_{N|m+1} \dfrac{\mu_N k_{z(m+1)}}{\mu_{m+1} k_{zN}} & for\ p = h \\ \Im^p_{N|m+1} \dfrac{\varepsilon_N k_{z(m+1)}}{\varepsilon_{m+1} k_{zN}} & for\ p = v \end{cases}. \tag{3.67}$$

As a result, equations (63) and (67) formally express the *reciprocity* of the generalized transmission coefficients for an arbitrary flat-boundaries layered structure [6].

3.6 Derivation of the Field Coefficients

In this section, jointly employing the *recursive matrix approach* (Sect. 3.3) and *generalized coefficients formalism* (Sect. 3.4-5), we calculate the unknown coefficients of the field in an arbitrary multilayered structure.

By multiplying the relevant matrices together, the matrix relation across the entire structure can be determined. Indeed, in order to calculate equivalent reflection and transmission response from a flat boundaries layered slab, the matrix multiplication of all propagator elements involved can be used, so that the generalized reflection/transmission coefficients can be derived from a recursive calculation trough the stack (Fig. 2).

Therefore, the equations system obtained enforcing the condition (33) (for p=h) on each flat interface $z = -d_m$ of the layered structure can be solved recursively. Therefore, using recursively (36) for every interface working from top to bottom, we get:

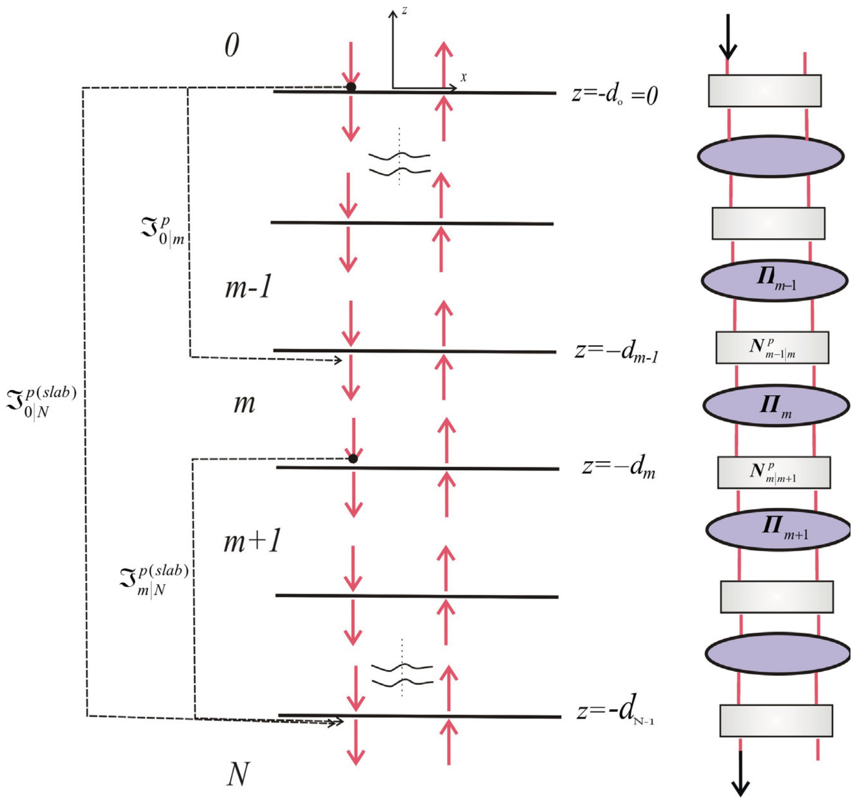

Fig. 2. Propagation in the layered structure: Scheme for propagator elements.

$$\mathbf{S}_m^{h(0)}(k_\perp, d_m) = \boldsymbol{N}^h_{m|m+1}\boldsymbol{\Pi}_{m+1}\ldots\boldsymbol{N}^h_{N-2|N-1}\boldsymbol{\Pi}_{N-1}\boldsymbol{N}^h_{N-1|N}\,\mathbf{S}_N^{h(0)}(k_\perp, d_{N-1}) \quad (3.68)$$

Similarly, from (41) solving recursively, we get:

$$\begin{aligned}\mathbf{S}_m^{p(1)}(k_\perp, d_{m-1}) = [\boldsymbol{N}^p_{m-1|m}]^{-1}\boldsymbol{\Pi}^{-1}_{m-1}[\boldsymbol{N}^p_{m-2|m-1}]^{-1}\ldots\boldsymbol{\Pi}_k^{-1}[\boldsymbol{N}^p_{k-1|k}]^{-1}\\ \ldots\boldsymbol{\Pi}_2^{-1}[\boldsymbol{N}^p_{1|2}]^{-1}\boldsymbol{\Pi}_1^{-1}[\boldsymbol{N}^p_{0|1}]^{-1}\mathbf{S}_0^{p(1)}(k_\perp, d_0)\end{aligned} \quad (3.69)$$

Assuming that an incident field from only one side of the structure, i.e., $S_N^{+p(0)} = 0$, we from (68) obtain:

$$\boldsymbol{N}^p_{N-1|N}\mathbf{S}_N^{p(0)}(k_\perp, d_{N-1}) = \begin{bmatrix} R^p_{N-1|N} \\ 1 \end{bmatrix}[T^p_{N-1|N}]^{-1}S_N^{-p(0)}e^{jk_{zN}d_{N-1}} \quad (3.70)$$

By using (37),(42) and (44), we get:

$$\boldsymbol{N}^{p}_{j-1|j}\boldsymbol{\Pi}_{j}\begin{bmatrix}\Re^{p}_{j|j+1}\\ 1\end{bmatrix}=[T^{p}_{j-1|j}e^{jk_{zj}\Delta_j}]^{-1}\vec{M}^{p}_{j}(k_{\perp})\begin{bmatrix}\Re^{p}_{j-1|j}\\ 1\end{bmatrix}. \tag{3.71}$$

Taking into account (70) and substituting recursively (71) (for *p=h*) into (68), the recursions initialized at the *N*-th interface lead us to the following recursive expression:

$$\mathbf{S}^{h(0)}_{m}(k_{\perp},d_m)=\begin{bmatrix}\Re^{h}_{m|m+1}\\ 1\end{bmatrix}[\Im^{h(slab)}_{m|N}]^{-1}S^{-h(0)}_{N}e^{jk_{zN}d_{N-1}}, \tag{3.72}$$

where the *generalized transmission coefficients* in downward direction for the layered slab, between two half-spaces (*m,N*), $\Im^{p(slab)}_{m|N}$ are defined as in (56), so that $\Im^{h(slab)}_{m|N}S^{-h(0)}_{m}e^{jk_{zm}d_m}=S^{-h(0)}_{N}e^{jk_{zN}d_{N-1}}$. Similarly, we obtain:

$$\mathbf{S}^{h(0)}_{0}(k_{\perp},d_0)=\begin{bmatrix}\Re^{h}_{0|1}\\ 1\end{bmatrix}[\Im^{h(slab)}_{0|N}]^{-1}S^{-h(0)}_{N}e^{jk_{zN}d_{N-1}}, \tag{3.73}$$

so that $\Im^{h(slab)}_{0|N}S^{-h(0)}_{0}e^{jk_{z0}d_0}=S^{-h(0)}_{N}e^{jk_{zN}d_{N-1}}$. From (72)-(73) we have:

$$S^{-h(0)}_{m}e^{jk_{zm}d_m}=[\Im^{h(slab)}_{m|N}]^{-1}\Im^{h(slab)}_{0|N}S^{-h(0)}_{0}e^{jk_{z0}d_0}. \tag{3.74}$$

Since $\Im^{h}_{0|m}e^{jk_{zm}\Delta_m}=[\Im^{h(slab)}_{m|N}]^{-1}\Im^{h(slab)}_{0|N}$, the final solution for the field expansion coefficients closed-form solution is finally obtained:

$$\mathbf{S}^{h(0)}_{m}(k^{i}_{\perp},d_m)=\begin{bmatrix}\Re^{h}_{m|m+1}(k^{i}_{\perp})\\ 1\end{bmatrix}\Im^{h}_{0|m}(k^{i}_{\perp})e^{jk^{i}_{zm}\Delta_m}S^{-h(0)}_{0}(k^{i}_{\perp}), \tag{3.75}$$

where the *generalized transmission coefficients* in downward direction are defined by (53), so that $\Im^{p}_{0|m}S^{-p(0)}_{0}e^{jk_{z0}d_0}=S^{-p(0)}_{m}e^{jk_{zm}d_{m-1}}$.

Therefore by employing a *recursive approach* we have obtained the following result: the unperturbed solution, with regard to the impinging wave (22), can be expressed in closed form within each *m-th* layer by (11)-(14), with *zero-order expansion coefficients vectors* given by (75). Indeed, starting from the solution for a horizontally polarized wave (11)-(14), (75) and applying *duality* ($\mathbf{E} \rightarrow \mathbf{H}$, $\mathbf{H} \rightarrow -\mathbf{E}$ and $\varepsilon \leftrightarrow \mu$), we obtain the solution for the vertically polarized case. Then, the coefficients $S_m^{\pm v(0)}$ can be obtained from $S_m^{\pm h(0)}$ applying duality.

3.6.1 Equivalent Layered Slab Response in upward direction

Assuming that $S_0^{-p(1)} = 0$, i.e., no downward scattered field in the upper half-space is supposed, we obtain:

$$[\boldsymbol{N}_{0|1}^{p}]^{-1}\mathbf{S}_0^{p(1)}(k_\perp, d_{k-1}) = \begin{bmatrix} 1 \\ R_{1|0}^{p} \end{bmatrix} [T_{1|0}^{p}]^{-1} S_0^{+p(1)}. \tag{3.76}$$

It is crucial to note that, from (40), (43) and (45), we obtain

$$[\boldsymbol{N}_{j|j+1}^{p}]^{-1}\boldsymbol{\Pi}_j^{-1}\begin{bmatrix} 1 \\ \Re_{j|j-1}^{p} \end{bmatrix} = [T_{j+1|j}^{p} e^{jk_{zj}\Delta_j}]^{-1} \overleftarrow{M}_j^{p} \begin{bmatrix} 1 \\ \Re_{j+1|j}^{p} \end{bmatrix}. \tag{3.77}$$

Taking into account (76) and substituting recursively (77) into (69), we have:

$$\mathbf{S}_m^{p(1)}(k_\perp, d_{m-1}) = \begin{bmatrix} 1 \\ \Re_{m|m-1}^{p} \end{bmatrix} [\Im_{m|0}^{p(slab)}]^{-1} S_0^{+p(1)}(k_\perp), \tag{3.78}$$

where $\Im_{m|0}^{p(slab)}$ are the *generalized transmission coefficients* in upward direction for the layered slab between two half-spaces (*m,*0), defined as in (58).

3.7 General closed-form Zero-th Order Solution

Therefore, for an arbitrary polarized incident wave (7), we express the unperturbed field solution, which is the total (vectorial) field in the three-dimensional multilayer flat-boundaries structure (Fig.1) in concise general close form.

By employing above notations, the unperturbed field $\mathbf{E}_m^{(0)}(\mathbf{r})$ within the generic m-th layer can be conveniently expressed in the following closed-form:

$$\begin{aligned}\mathbf{E}_m^{(0)}(\mathbf{r}) = e^{j\mathbf{k}_\perp^i \cdot \mathbf{r}_\perp} \Big[& h^i \xi_{0\to m}^{+h}(k_\perp^i, z) E_0^{ih} \\ & + \hat{k}_\perp^i \frac{k_{zm}^i}{k_0 \varepsilon_m} \xi_{0\to m}^{-v}(k_\perp^i, z) E_0^{iv} \\ & + \hat{z} \frac{k_\perp^i}{k_0 \varepsilon_m} \xi_{0\to m}^{+v}(k_\perp^i, z) E_0^{iv} \Big],\end{aligned} \quad (3.79)$$

where the orthonormal right-handed basis $\mathcal{B}_i = \{\hat{h}^i, \hat{k}_\perp^i, \hat{z}\}$ has been used and the following notation has been adopted:

$$\xi_{0\to m}^{\pm p}(k_\perp, z) = \Im_{0|m}^{p}(k_\perp) e^{jk_{zm}(-z-d_{m-1})} [1 \pm \Re_{m|m+1}^{p}(k_\perp) e^{j2k_{zm}(z+d_m)}], \quad (3.80)$$

where the symbol ± in the superscript on LHS represents a given choice linked to the symbol ± in RHS expression.

In particular, owing to rotational symmetry of the structure, without loss of generality we could assume, for instance, that the x axis is included in the incidence plane, so that the Cartesian representation of the solution within each m-region is:

$$\mathbf{E}_m^{(0)}(\mathbf{r}) = \exp j[\mathbf{k}_\perp^i \cdot \mathbf{r}_\perp + k_{zm}^i(-z - d_{m-1})] \begin{bmatrix} -\dfrac{k_{zm}^i}{k_0 \varepsilon_m} \Im_{0|\mathrm{m}}^{\mathrm{v}}(k_\perp^i) \Big[1 - \Re_{\mathrm{m|m+1}}^{\mathrm{v}}(k_\perp^i)\, \mathrm{e}^{j2k_{zm}^i(d_m+z)}\Big] E_0^{iv} \\ \Im_{0|\mathrm{m}}^{\mathrm{h}}(k_\perp^i) \Big[1 + \Re_{\mathrm{m|m+1}}^{\mathrm{h}}(k_\perp^i)\, \mathrm{e}^{j2k_{zm}^i(d_m+z)}\Big] E_0^{ih} \\ \dfrac{k_\perp^i}{k_0 \varepsilon_m} \Im_{0|\mathrm{m}}^{\mathrm{v}}(k_\perp^i) \Big[1 + \Re_{\mathrm{m|m+1}}^{\mathrm{v}}(k_\perp^i)\, \mathrm{e}^{j2k_{zm}^i(d_m+z)}\Big] E_0^{iv} \end{bmatrix}. \quad (3.81)$$

This general solution, which is given in vectorial form, can be also named unperturbed solution, i. e., the *zeroth*-order field, since it refers to an idealized structure with (piecewise) homogeneous properties flat boundaries (see also next Chapters).

These results allow us to evaluate effectively in compact closed-form the field jumps on the interfaces, as discussed in next section.

3.8 Field jumps on the interfaces

As it will be clear in next Chapters, for the calculation of first-order fields' perturbation, it is necessary to know the value at the interfaces of the unperturbed fields and of their derivatives.

Specifically, we are interested in deriving closed form expressions for field jumps on the generic *m*-th interface: $\Delta\mathbf{E}_m^{(0)} = \mathbf{E}_{m+1}^{(0)} - \mathbf{E}_m^{(0)}$, $\Delta\mathbf{H}_m^{(0)} = \mathbf{H}_{m+1}^{(0)} - \mathbf{H}_m^{(0)}$.

In the following we refer implicitly to the incident direction. With regard to the *h*-polarized case, substituting (75) in (11), (13) we have:

$$\mathbf{E}_{m+1}^{(0)}\Big|_{z=-d_m} = \hat{h}\, e^{j\mathbf{k}_\perp\cdot\mathbf{r}_\perp}\ \Im_{0|m+1}^{h}[1+\Re_{m+1|m+2}^{h} e^{j2k_{z(m+1)}\Delta_{m+1}}], \tag{3.82}$$

$$\mathbf{E}_{m}^{(0)}\Big|_{z=-d_m} = \hat{h}\, e^{j\mathbf{k}_\perp\cdot\mathbf{r}_\perp} \Im_{0|m}^{h} e^{jk_{zm}\Delta_m}[1+\Re_{m|m+1}^{h}]. \tag{3.83}$$

Taking into account (51), from (82)-(83) we obtain (93). Moreover, we get:

$$\left.\frac{\partial\Delta\mathbf{E}_m^{(0)}}{\partial z}\right|_{z=-d_m} = \hat{h}\, e^{j\mathbf{k}_\perp\cdot\mathbf{r}_\perp} j\{k_{zm}\Im_{0|m}^{h} e^{jk_{zm}\Delta_m}[1-\Re_{m|m+1}^{h}]+ \\ -k_{z(m+1)}\Im_{0|m+1}^{h}[1-\Re_{m+1|m+2}^{h} e^{j2k_{z(m+1)}\Delta_{m+1}}]\}. \tag{3.84}$$

Substituting (52) into (84), and considering (18), (95) is obtained. Similarly, substituting (75) in (12)-(14) we have:

$$\mathbf{H}_{m}^{(0)}\Big|_{z=-d_m} = -\frac{e^{j\mathbf{k}_\perp\cdot\mathbf{r}_\perp}}{Z_m} \Im_{0|m}^{h} e^{jk_{zm}\Delta_m}[\hat{v}_m^{-}(\mathbf{k}_\perp)+\hat{v}_m^{+}(\mathbf{k}_\perp)\Re_{m|m+1}^{h}] \tag{3.85}$$

$$\mathbf{H}_{m+1}^{(0)}\Big|_{z=-d_m} = -\frac{e^{j\mathbf{k}_\perp\cdot\mathbf{r}_\perp}}{Z_{m+1}}\Im_{0|m+1}^{h}[\hat{v}_{m+1}^{-}(\mathbf{k}_\perp)+\hat{v}_{m+1}^{+}(\mathbf{k}_\perp)\Re_{m+1|m+2}^{h}e^{j2k_{z(m+1)}\Delta_{m+1}}] \quad (3.86)$$

From (85)-(86) using the (19), we get:

$$\hat{k}_\perp\cdot\mathbf{H}_{m}^{(0)}\Big|_{z=-d_m} = -e^{j\mathbf{k}_\perp\cdot\mathbf{r}_\perp}\frac{k_{zm}}{k_0Z_0\mu_m}\Im_{0|m}^{h}e^{jk_{zm}\Delta_m}[1-\Re_{m|m+1}^{h}] \quad (3.87)$$

$$\hat{k}_\perp\cdot\mathbf{H}_{m+1}^{(0)}\Big|_{z=-d_m} = -e^{j\mathbf{k}_\perp\cdot\mathbf{r}_\perp}\frac{k_{z(m+1)}}{k_0Z_0\mu_{m+1}}\Im_{0|m+1}^{h}[1-\Re_{m+1|m+2}^{h}e^{j2k_{z(m+1)}\Delta_{m+1}}] \quad (3.88)$$

$$\hat{z}\cdot\mathbf{H}_{m}^{(0)}\Big|_{z=-d_m} = -e^{j\mathbf{k}_\perp\cdot\mathbf{r}_\perp}\frac{k_\perp}{k_0Z_0\mu_m}\Im_{0|m}^{h}e^{jk_{zm}\Delta_m}[1+\Re_{m|m+1}^{h}] \quad (3.89)$$

$$\hat{z}\cdot\mathbf{H}_{m+1}^{(0)}\Big|_{z=-d_m} = -e^{j\mathbf{k}_\perp\cdot\mathbf{r}_\perp}\frac{k_\perp}{k_0Z_0\mu_{m+1}}\Im_{0|m+1}^{h}[1+\Re_{m+1|m+2}^{h}e^{j2k_{z(m+1)}\Delta_{m+1}}] \quad (3.90)$$

From (89)-(90), using again (52) we obtain the final expression (94). On the other hand, from (87)-(88) it can be verified that:

$$\hat{k}_\perp\cdot\Delta\mathbf{H}_{m}^{(0)}\Big|_{z=-d_m} = 0. \quad (3.91)$$

Taking into account that $\hat{z}\times(\hat{k}_m^{\pm}\times\hat{h}_m^{\pm}(\mathbf{k}_\perp)) = -\hat{h}_m^{\pm}(\mathbf{k}_\perp)(\hat{k}_m^{\pm}\cdot\hat{z})$, from (12)-(14) we obtain:

$$\begin{aligned}\hat{z}\times\frac{\partial\Delta\mathbf{H}_m^{(0)}}{\partial z}\Bigg|_{z=-d_m} &= -\frac{jk_{z(m+1)}^2}{k_{m+1}Z_{m+1}}[\mathbf{E}_{m+1}^{-}+\mathbf{E}_{m+1}^{+}]\Big|_{z=-d_m} + \\ \frac{jk_{zm}^2}{k_mZ_m}[\mathbf{E}_m^{-}+\mathbf{E}_m^{+}]\Big|_{z=-d_m} &= \frac{-j}{k_0Z_0}\left(\frac{k_{z(m+1)}^2}{\mu_{m+1}}-\frac{k_{zm}^2}{\mu_m}\right)\mathbf{E}_{m+1}^{(0)}\Big|_{z=-d_m}\end{aligned} \quad (3.92)$$

In conclusion, from (92), (96) is obtained. As a result, by using (75) in (11)-(14) with $p=h$, we have therefore obtained:

$$\Delta\mathbf{E}_m^{(0)}\Big|_{z=-d_m} = 0 \quad (3.93)$$

$$\Delta\mathbf{H}_m^{(0)}\Big|_{z=-d_m} = -\hat{z}\,\frac{Ck_\perp^i}{k_0 Z_0 \mu_{m+1}}\left(1-\frac{\mu_{m+1}}{\mu_m}\right)[1+\Re^h_{m|m+1}(k_\perp^i)] \tag{3.94}$$

$$\hat{z}\times\frac{\partial\Delta\mathbf{E}_m^{(0)}}{\partial z}\Bigg|_{z=-d_m} = jk_{zm}^i C\,\hat{k}_\perp^i\left(1-\frac{\mu_{m+1}}{\mu_m}\right)[1-\Re^h_{m|m+1}(k_\perp^i)] \tag{3.95}$$

$$\hat{z}\times\frac{\partial\Delta\mathbf{H}_m^{(0)}}{\partial z}\Bigg|_{z=-d_m} = \frac{jC}{k_0 Z_0}(\hat{z}\times\hat{k}_\perp^i)\left(\frac{k_{z(m+1)}^{i2}}{\mu_{m+1}}-\frac{k_{zm}^{i2}}{\mu_m}\right)[1+\Re^h_{m|m+1}(k_\perp^i)] \tag{3.96}$$

with $C=\Im^h_{0|m}(k_\perp^i)e^{jk_{zm}^i\Delta_m}e^{j\mathbf{k}_\perp^i\cdot\mathbf{r}_\perp}$ and where Z_0 is the intrinsic impedance of the vacuum. Applying the *duality principle,* we have for the vertical incident polarization (p=v):

$$\Delta\mathbf{H}_m^{(0)}\Big|_{z=-d_m} = 0 \tag{3.97}$$

$$\Delta\mathbf{E}_m^{(0)}\Big|_{z=-d_m} = \hat{z}\,\frac{Ck_\perp^i}{k_0\varepsilon_{m+1}}\left(1-\frac{\varepsilon_{m+1}}{\varepsilon_m}\right)[1+\Re^v_{m|m+1}(k_\perp^i)] \tag{3.98}$$

$$\hat{z}\times\frac{\partial\Delta\mathbf{H}_m^{(0)}}{\partial z}\Bigg|_{z=-d_m} = \frac{jk_{zm}^i C}{Z_0}\hat{k}_\perp^i\left(1-\frac{\varepsilon_{m+1}}{\varepsilon_m}\right)[1-\Re^v_{m|m+1}(k_\perp^i)] \tag{3.99}$$

$$\hat{z}\times\frac{\partial\Delta\mathbf{E}_m^{(0)}}{\partial z}\Bigg|_{z=-d_m} = \frac{jC}{k_0}(\hat{k}_\perp^i\times\hat{z})\left(\frac{k_{z(m+1)}^{i2}}{\varepsilon_{m+1}}-\frac{k_{zm}^{i2}}{\varepsilon_m}\right)[1+\Re^v_{m|m+1}(k_\perp^i)] \tag{3.100}$$

with $C=\Im^v_{0|m}(k_\perp^i)e^{jk_{zm}^i\Delta_m}e^{j\mathbf{k}_\perp^i\cdot\mathbf{r}_\perp}$.

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Chapter 4

Scattering from One Rough Interface: A Unified formulation of Existing Perturbative Solutions

"Vi sono alcuni che avvertono l'urgenza di risolvere un problema e per loro questo diventa qualcosa di reale, come un elemento di disordine che debbono eliminare dal loro sistema."

Karl Raimund Popper

In this chapter we investigate analytically the connection between the existing first-order SPM solutions for the scattering from specific layered structure with one rough interface. First of all, by using effectively the concept of generalized reflection coefficients, we cast the existing models in a unified, more compact formulation, and point out the connection between the different analytical solutions.

The obtained reformulations of the available analytical solutions allow us to subsequently prove the consistency of the considered models. The obtained unified formulation also opens the way toward a general closed form solution for the problem of scattering by a layered structure with an arbitrary number of corrugated interfaces.

4.1 Introduction and Motivation

Scattering from layered media with rough interfaces has been subject of ongoing research in several branches such as remote sensing, geophysics, optics and plasma physics and is becoming of increasing importance. From the remote sensing applications point of view, multi-layer dielectric structures with rough boundaries are useful for modelling electromagnetic propagation in stratified soil [1], sand cover of arid regions [2-4], forest canopies, urban buildings, snow blanket, snow cover ice [5], sea ice [6] and glaciers, oil flood on sea surface, and other natural scenes.

To deal with the scattering problem by rough interfaces of a multilayer, the available methods differ in the type of the employed approximation, in the characterisation of the layered medium, and in the applicability to different frequency regimes. On the other hand, from the applications viewpoint (e.g., radar applications and more specifically Synthetic Aperture Radar (SAR) processing and signals simulation), it is highly desirable to deal with scattering solutions that are amenable to be analytically derived in explicit closed-form: as a matter of fact, the complex nature of the scattering phenomena cannot be completely captured by only relying on numerical scattering methods, which do not provide general information on the functional dependence between the scattered electromagnetic field and the electromagnetic and geometric parameters of the layered structure (and have obviously a much larger computational load).

However, while many analytical techniques dealing with the surface scattering problem are available and apply to different scattering regimes [7-12], knowledge of the relation between radar response patterns and stratification structure is less advanced for layered media with rough interfaces. Basically, two main approaches have been adopted to find a convenient solution to this problem: the first one is based on the wave theory, the second one relies on the radiative transfer theory (RT).

The wave theory approach simultaneously considers multiple interferential interactions with layer boundaries and preserves phase information, so that it is possible to properly model the well-known backscattering enhancement phenomenon [10]; then, a full application to coherent remote sensing instruments is allowed.

Conversely, the heuristic RT theory, derived from equation governing the propagation of energy through the scattering medium, neglects the coherent nature of the field, and therefore does not take into account coherent effects associated to parallel layers, although in some way it includes multiple scattering effects [8,10].

Therefore, in the following we do not consider the RT approach and we focus our attention on the wave theory approach.

The small perturbation method (SPM) is the oldest and the most broadly used formalism to predict the radar scattering from rough surfaces with small rms height and slope. Detailed analysis on the limit of validity of the SPM method, as well as the other analytical methods, are available [7-12].

Presently, some extensions of the SPM method to the layered media with one rough interface have been proposed. The resulting expressions, derived via different techniques, are given for different simplified geometries with a limited number of layers only [13-16]. In addition, all the considered methods are valid in the limit of first order SPM. In [15], *Fuks* analyzes a structure consisting of a flat layered medium with one rough interface on the top, and uses an equivalent current method [7]. *Sarabandi et al*. [14] consider three layers with a rough lower interface and a smooth upper interface, and use the classical perturbation series expansion. Finally, the third method, presented by *Yarovoy et al*. [13] investigates the case of four layers with a rough interface between the middle layers, and employs a Green's function approach.

These methods are very interesting, since they provide analytical expressions of the scattered power density as a function of the geometric and electromagnetic parameters describing the stratified structure. However, although it is evident that the structures considered in [14] and [15] can be somehow recognised as particular cases of the one considered in [13], it is not clear if and how the proposed SPM solutions for the different configurations are compatible. Moreover, a complete understanding of the physical meaning of the SPM existing expressions is not available.

As a matter of fact those expressions can be conveniently used to compute the field scattered by a stratified medium with one rough interface, but it is difficult from those solution to provide general information on the physical mechanisms involved in the scattering phenomenon. In addition, a full analytical comparison among those solutions is not available, and only a numerical comparison of obtained results in prescribed conditions can be obtained.

Therefore, the objective of this Chapter is to investigate analytically the connection among the existing first order SPM solutions for the scattering from a layered structure with one rough interface. To meet this goal, in this Chapter we reformulate the available solutions.

First of all, by using effectively the concept of *generalized reflection coefficients* of equivalent reflecting boundaries, we cast the solutions relevant to the existing models in a unified formalism and point out the connection between the different geometries and analytical solutions. The obtained reformulations of the correspondent analytical solutions permit us to subsequently prove, on an analytical playground, the consistency of the three methods, so that all the analyzed models can be revisited and fully compared, with the help of a unified formalism, in a common analytical framework. We also underline that the reformulation we here obtain of the Yarovoy solution is much more manageable than its original form presented in [13].

This chapter is organized as follows.

Section 4.2 reviews analytical SPM models available in literature [13, 14, 15], and focuses on the study of electromagnetic wave scattering from specific geometries of a layered medium with a single rough interface.

In Section 4.3 the existing solutions are reformulated in terms of generalized Fresnel coefficients, and the proof of the consistency of the three methods is provided.

4.2 Existing Small Perturbation Approaches

In this Section, we begin by providing an overview of the state of the art of extensions of the SPM method to the layered media with one rough interface. In previous works different simplified geometry with a limited number of layers only have been analyzed [13, 14, 15].

The resulting analytical solutions, derived with different techniques, are valid in the limit of first-order SPM.

For the sake of unitary formalism, throughout this Chapter we use the following formalism for the scattering coefficients relevant to the contribution of the *n*-th corrugated interface:

$$\sigma^0_{qp,n} = \pi k_0^4 \left|\tilde{\alpha}^{n,n+1}_{qp}(\mathbf{k}_\perp, \mathbf{k}^i_\perp)\right|^2 W_n(\mathbf{k}_\perp - \mathbf{k}^i_\perp) \qquad p, q = h, v \qquad (4.1)$$

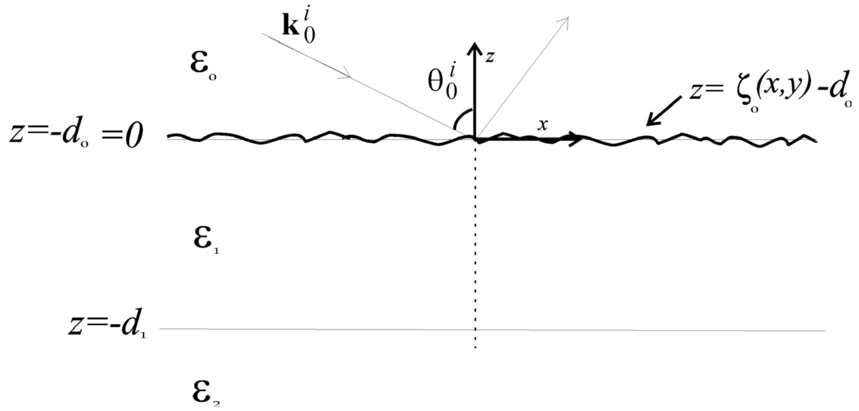

Fig.1. Geometry for the Fuks model.

where the (equivalent) coefficients $\tilde{\alpha}_{qp}^{m-1,m}$ refer to a rough interface between two layers of permittivity ε_{m-1} and ε_m, whereas $\alpha_{qp}^{m-1,m}$ are the classical SPM coefficients relative to the rough interface between two half-spaces of permittivity ε_{m-1} and ε_m, respectively. Moreover, p and q denote the incident and the scattered polarization states respectively, and may stand for h (horizontal polarisation) or v (vertical polarisation); $W_n(\boldsymbol{\kappa})$ is the spatial power spectrum of n-th corrugated interface (see Chapter 2), i.e., the Fourier transform of n-th corrugated interface autocorrelation function $B_{\zeta_n}(\boldsymbol{\rho}) = \langle \zeta_n(\mathbf{r}_\perp + \boldsymbol{\rho}_\perp)\zeta_n(\mathbf{r}_\perp)\rangle$.

Nonetheless, it should be noted that different definitions of the Fourier transform are available in the literature.

We stress that in backscattering case, in the limit of the first order SPM, backscattering cross-polarized coefficient vanishes in the plane of incidence. Full bi-static classical expression of σ_{qp}^0 for a rough surface between two different half-space media can be found in literature e.g. [10,11].

4.2.1 Fuks Model

In [15] Fuks has proposed a model to calculate scattering from a rough surface on top of a stratified medium. This model refers to the geometry of Fig.1. By using the plane wave expansion of scattered

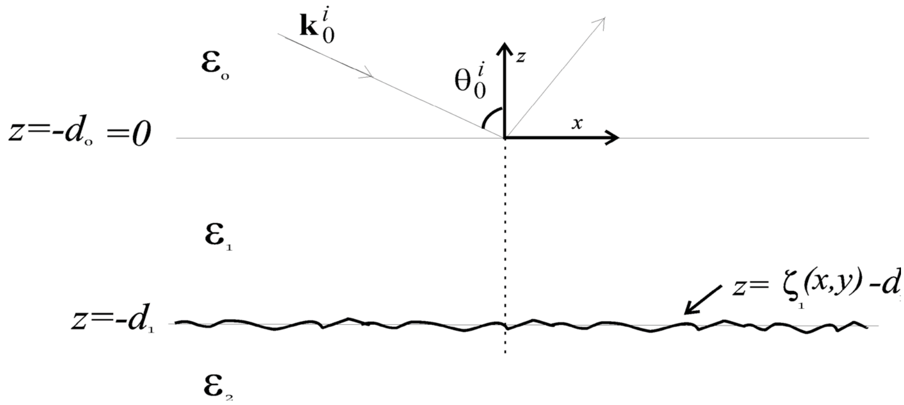

Fig.2. Geometry for the Sarabandi model.

EM fields and an equivalent current method [7], without using to the *Green*'s function method, in [15] expressions for scattering bi-static cross section were obtained. Employing the formalism consistent with the one adopted in this Chapter, the *Fuck*'s solution leads to the following expressions of the scattering coefficients:

$$\sigma^0_{qp,0} = \pi\, k_0^4 \left|\widetilde{\alpha}^{0,1}_{qp}(\mathbf{k}^s_\perp, \mathbf{k}^i_\perp)\right|^2 W_0(\mathbf{k}^s_\perp - \mathbf{k}^i_\perp) \qquad p,q = h,v \tag{4.2}$$

with

$$\widetilde{\alpha}^{0,1}_{hh} = (\varepsilon_1 - \varepsilon_0)\,(\hat{k}^i_\perp \cdot \hat{k}^s_\perp)[1+\Re^h_{0|1}(k^i_\perp)][1+\Re^h_{0|1}(k^s_\perp)] \tag{4.3}$$

$$\begin{aligned}\widetilde{\alpha}^{0,1}_{vv} = (\varepsilon_1 - \varepsilon_0)\{\,&[1+\Re^v_{0|1}(k^i_\perp)][1+\Re^v_{0|1}(k^s_\perp)]\frac{1}{\varepsilon_1}\sin\theta^i_0 \sin\theta^s_0 \\ &-[1-\Re^v_{0|1}(k^i_\perp)][1-\Re^v_{0|1}(k^s_\perp)]\cos\theta^i_0\cos\theta^s_0(\hat{k}^i_\perp\cdot\hat{k}^s_\perp)\}\end{aligned} \tag{4.4}$$

An incomplete physical interpretation is proposed in [18].

4.2.2 Sarabandi Model

The Sarabandi model [14] refers to the geometry of Fig.2, i.e., to a slightly rough interface boundary covered with a homogeneous dielectric layer. Starting from a perturbation series expansion, *Sarabandi et al.* develop a small perturbation solution to predict the first order bi-static scattering coefficients [14, eq.8-11]. The solution can be written in the form:

$$\sigma^0_{qp,1} = \pi k_0^4 \left|\tilde{\alpha}^{1,2}_{qp}(\mathbf{k}^s_\perp, \mathbf{k}^i_\perp)\right|^2 W_1(\mathbf{k}^s_\perp - \mathbf{k}^i_\perp) \qquad p, q = h, v \tag{4.5}$$

wherein

$$\left|\tilde{\alpha}^{1,2}_{hh}\right|^2 = 256\ \left|\varepsilon_2 - \varepsilon_1\right|^2 \left(\hat{k}^i_\perp \cdot \hat{k}^s_\perp\right)^2 \frac{\left|k^i_{z0}\, k^i_{z1}\right|^2}{\left|e^{jk^i_{z1}\Delta_1}(k^i_{z1} - k^i_{z2})(k^i_{z1} - k^i_{z0}) - e^{-jk^i_{z1}\Delta_1}(k^i_{z1} + k^i_{z2})(k^i_{z1} + k^i_{z0})\right|^2} \frac{\left|k^s_{z0}\, k^s_{z1}\right|^2}{\left|e^{jk^s_{z1}\Delta_1}(k^s_{z1} - k^s_{z2})(k^s_{z1} - k^s_{z0}) - e^{-jk^s_{z1}\Delta_1}(k^s_{z1} + k^s_{z2})(k^s_{z1} + k^s_{z0})\right|^2} \tag{4.6}$$

Similar expressions for other polarization combinations are provided in [14]. No physical interpretation is provided.

4.2.3 Yarovoy Model

With reference to the geometry represented schematically in Fig.3, in [13] the scattering problem for a single rough interface in a layered media is solved by means of the small perturbation method combined with the Green's function approach. It is shown that, in the *Born approximation*, the scattering from a 2-layer media can be expressed, in the first-order SPM*,* exclusively in terms of the spectral density of the roughness and the parameters referring to the flat boundary stratification.

This approach leads to some analytical expressions for backscattering coefficients [13, eqs.10-12]. It should be noted that in [13] the solution is expressed in term of σ_{pp}, whereas the more usual *radar backscattering cross section* $\sigma^0_{pp} = 4\pi\sigma_{pp}$ is here used. Therefore, the solution can be rewritten, coherently with the formalism adopted in this Chapter, as follows:

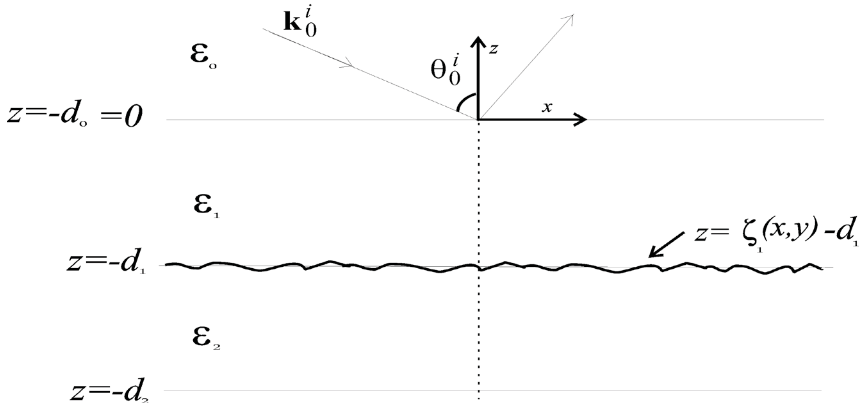

Fig.3. Geometry of the Yarovoy model

$$\sigma^0_{pp,1} = \pi\, k_0^4 \left|\tilde{\alpha}^{1,2}_{pp}(\mathbf{k}^i_\perp)\right|^2 W_1(-2\mathbf{k}^i_\perp) \qquad p = h, v \tag{4.7}$$

with

$$\begin{aligned}\left|\tilde{\alpha}^{1,2}_{hh}\right|^2 = 256\cos^4\theta_0^i \left|\varepsilon_2-\varepsilon_1\right|^2 \frac{\left|Z^h_{\Delta 1}\right|^4}{\left|1+\cos\theta_0^i Z^h_{\Delta 1}\right|^4}\left| k_{z1}(k_{z2}c_2 - jk_{z3}s_2)\right|^4 \\ \left| e^{jk_{z1}\Delta_1}(k_{z2}(k_{z1}+k_{z3})c_2 - j(k_{z1}k_{z3}+k_{z2}^2)s_2) + \right. \\ \left. + e^{-jk_{z1}\Delta_1}(k_{z2}(k_{z1}-k_{z3})c_2 - j(k_{z1}k_{z3}-k_{z2}^2)s_2)\right|^{-4}\end{aligned} \tag{4.8}$$

$$\begin{aligned}\left|\tilde{\alpha}^{1,2}_{vv}\right|^2 = 256\cos^4\theta_0^i \left|\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_1}\right|^2 \frac{\left|\dfrac{\sin^2\theta_0^i}{\varepsilon_2} + Z_0^{v^2}\right|^2}{\left|\cos\theta_0^i + Z^v_{\Delta 1}\right|^4}\left| \varepsilon_2 k_{z1}(k_{z2}\varepsilon_3 c_2 - jk_{z3}\varepsilon_2 s_2)\right|^4 \\ \left| e^{jk_{z1}\Delta_1}(\varepsilon_2 k_{z2}(\varepsilon_3 k_{z1}+\varepsilon_1 k_{z3})c_2 - j(\varepsilon_2\varepsilon_2 k_{z1}k_{z3}+\varepsilon_1\varepsilon_3 k_{z2}^2)s_2) + \right. \\ \left. e^{-jk_{z1}\Delta_1}(\varepsilon_2 k_{z2}(\varepsilon_3 k_{z1}-\varepsilon_1 k_{z3})c_2 - j(\varepsilon_2\varepsilon_2 k_{z1}k_{z3}-\varepsilon_1\varepsilon_3 k_{z2}^2)s_2)\right|^{-4}\end{aligned} \tag{4.9}$$

and where $Z_0^v = \sqrt{\varepsilon_1}\, Z^v_{\Delta_1}\big|_{\Delta_1=0}$ and

$$
\begin{aligned}
Z_{\Delta_1}^h &= \frac{k_0}{k_{z1}} \\
&\{e^{jk_{z1}\Delta_1}[k_{z2}(k_{z1}+k_{z3})c_2 - j(k_{z1}k_{z3}+k_{z2}^2)s_2]+ \\
&e^{-jk_{z1}\Delta_1}[k_{z2}(k_{z1}-k_{z3})c_2 - j(k_{z1}k_{z3}-k_{z2}^2)s_2]\}/ \\
&\{e^{jk_{z1}\Delta_1}[k_{z2}(k_{z1}+k_{z3})c_2 - j(k_{z1}k_{z3}+k_{z2}^2)s_2]- \\
&e^{-jk_{z1}\Delta_1}[k_{z2}(k_{z1}-k_{z3})c_2 - j(k_{z1}k_{z3}-k_{z2}^2)s_2]\}
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
Z_{\Delta_1}^v &= \frac{k_{z1}}{\varepsilon_1 k_0} \\
&\{e^{jk_{z1}\Delta_1}[\varepsilon_2 k_{z2}(\varepsilon_3 k_{z1}+\varepsilon_1 k_{z3})c_2 - j(\varepsilon_2\varepsilon_2 k_{z1}k_{z3}+\varepsilon_1\varepsilon_3 k_{z2}^2)s_2]- \\
&e^{-jk_{z1}\Delta_1}[\varepsilon_2 k_{z2}(\varepsilon_3 k_{z1}-\varepsilon_1 k_{z3})c_2 - j(\varepsilon_2\varepsilon_2 k_{z1}k_{z3}-\varepsilon_1\varepsilon_3 k_{z2}^2)s_2]\}/ \\
&\{e^{jk_{z1}\Delta_1}[\varepsilon_2 k_{z2}(\varepsilon_3 k_{z1}+\varepsilon_1 k_{z3})c_2 - j(\varepsilon_2\varepsilon_2 k_{z1}k_{z3}+\varepsilon_1\varepsilon_3 k_{z2}^2)s_2]+ \\
&e^{-jk_{z1}\Delta_1}[\varepsilon_2 k_{z2}(\varepsilon_3 k_{z1}-\varepsilon_1 k_{z3})c_2 - j(\varepsilon_2\varepsilon_2 k_{z1}k_{z3}-\varepsilon_1\varepsilon_3 k_{z2}^2)s_2]\}
\end{aligned}
\tag{4.11}
$$

with $c_2 = \cos(k_{z2}\Delta_2), s_2 = \sin(k_{z2}\Delta_2)$. However, the proposed solution appears very involved and difficult to manage. A physical explanation is outlined in [13], but no detailed interpretation is provided.

4.3 Connection Between Existing Functional Forms

In this Section, we discuss the relation between the three different models analyzed in Section 4.2 and establish their consistency from an analytical point of view.

The relevant question one might ask now is whether there are any possible connections between these considered models. First of all, it is important to note that the geometry analyzed by *Yarovoy* [13] is more general than the others [14, 15], because it includes the presence of a flat boundary stratification above and under the corrugated interface. On the other hand, the solution in [13] is given only in the backscattering case, whereas the other two models [14, 15] refer to a more general bi-static configuration. However, although it is evident that the structures considered in [15] and [14] are particular cases of the one considered in [13], it is not clear if and how the SPM solutions proposed by the three authors for the different configurations are compatible.

The connections between the three considered solutions are not trivial and appear at the moment difficult to establish. A way to overcome this point, in order to obtain a more transparent relation between the models, is to refer to some equivalent forms for the proposed solutions. In particular, we stress that Fuks' solution is already in the proper shape for the comparison. The final results will be surprisingly simple and recognizable. Nonetheless, building bridge across models could also be seen as a criterion of cross-validation of the corresponding functional forms.

4.3.1 Equivalent form of Sarabandi Model

In this Section, we want to considerer a more suitable expression for the model discussed in Section 4.2.2. In order to clarify the connections with the other models, we rearrange the expression of the scattering coefficient (5)-(6). Dividing nominator and denominator of (6) by

$$\left|(k_{z1}^i + k_{z2}^i)(k_{z1}^i + k_{z0}^i)(k_{z1}^s + k_{z2}^s)(k_{z1}^s + k_{z0}^s)e^{-jk_{z1}^i\Delta_1}e^{-jk_{z1}^s\Delta_1}\right|^2$$

and making use of the definitions of Chapter 3 (see (3.1)-(3.6)), we obtain the following more compact form, in terms of reflection and transmission coefficients of the boundaries:

$$\sigma_{qp,1}^0 = \pi k_0^4 \left|\tilde{\alpha}_{qp}^{1,2}(\mathbf{k}_\perp^s, \mathbf{k}_\perp^i)\right|^2 W_1(\mathbf{k}_\perp^s - \mathbf{k}_\perp^i) \qquad p, q = h, v \tag{4.12}$$

wherein

$$\begin{aligned} \left|\tilde{\alpha}_{hh}^{1,2}\right|^2 = \left|\varepsilon_2 - \varepsilon_1\right|^2 & \left|\frac{T_{0|1}^h(k_\perp^i)e^{jk_{z1}^i\Delta_1}}{1 + R_{0|1}^h(k_\perp^i)R_{1|2}^h(k_\perp^i)e^{j2k_{z1}^i\Delta_1}}\right|^2 \left|1 + R_{1|2}^h(k_\perp^i)\right|^2 \\ \left|\hat{k}_\perp^i \cdot \hat{k}_\perp^s\right|^2 & \left|\frac{T_{0|1}^h(k_\perp^s)e^{jk_{z1}^s\Delta_1}}{1 + R_{0|1}^h(k_\perp^s)R_{1|2}^h(k_\perp^s)e^{j2k_{z1}^s\Delta_1}}\right|^2 \left|1 + R_{1|2}^h(k_\perp^s)\right|^2 \end{aligned} \tag{4.13}$$

Similarly, equivalent expressions can be derived for other polarization combinations.

4.3.1 Equivalent form of the Yarovoy model

A not trivial equivalent expressions of the Yarovoy's backscattering coefficients (7)-(11) is presented in this Section.

We recognize that the use of generalized reflection coefficients, after some manipulations (see *Appendix* for the details), leads to the following less cumbersome expressions of the final solution:

$$\sigma^0_{pp,1} = \pi k_0^4 \left|\tilde{\alpha}^{1,2}_{pp}(\mathbf{k}^i_\perp)\right|^2 W_1(-2\mathbf{k}^i_\perp) \qquad p = h,v \tag{4.14}$$

wherein

$$\left|\tilde{\alpha}^{1,2}_{hh}\right|^2 = \left|\varepsilon_2 - \varepsilon_1\right|^2 \frac{\left|T^h_{0|1} e^{jk_{z1}\Delta_1}\right|^4}{\left|1 + R^h_{0|1}\Re^h_{1|2} e^{j2k_{z1}\Delta_1}\right|^4} \left|1 + \Re^h_{1|2}\right|^4 \tag{4.15}$$

$$\left|\tilde{\alpha}^{1,2}_{vv}\right|^2 = \left|\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1}\right|^2 \frac{\left|T^v_{0|1}\, e^{jk_{z1}\Delta_1}\right|^4}{\left|1 + R^v_{0|1}\Re^v_{1|2}\, e^{j2k_{z1}\Delta_1}\right|^4} \left|[1 + \Re^v_{1|2}]^2 \frac{1}{\varepsilon_2} \sin^2\theta^i_0 + [1 - \Re^v_{1|2}]^2 \cos^2\theta^i_1\right|^2 \tag{4.16}$$

where, the generalized Fresnel coefficients are given in Section 3.4, and where all the (generalized) reflection and transmission coefficients at the interfaces are evaluated at the incidence angle ($\mathbf{k}^i_\perp$).

4.3.2 Consistency of Methods

The purpose of the obtained reformulations is evident in the framework of the following considerations. The new analytical forms (12)-(13) and (14)-(16), obtained with no extra simplifying assumptions with reference to their original formulations, exhibit some important advantages with respect to the original ones. They are more suitable for practical use, and allow us to obtain compact expressions for the *first-order SPM* scattering coefficients that depend explicitly on the *(generalized) reflection coefficients* of the structure.

Furthermore, the equivalent forms allow us quite straightforwardly to find a satisfactory explanation of the relations between the correspondent solutions of the three consider models. In fact, we demonstrate that the corresponding functional forms are consistent, showing that the reformulation of the Yarovoy solution (14)-(16) reduces to the ones of others two examined models under special conditions.

To this purpose, it is easy to verify that, when the stratification above the corrugate interface disappears ($\varepsilon_0=\varepsilon_1=1$) so that the geometry of Fig.3 reduces to the one of Fig.1, we have $R^p_{0|1}=0$, $T^p_{0|1}=1$, $R^p_{1|2}=R^p_{0|2}$, $\cos\ \theta^i_1=\cos\ \theta^i_0$. In such a case, eqs. (14)-(16) are reduced to:

$$\left|\tilde{\alpha}^{0,2}_{hh}\right|^2=\left|\varepsilon_2-\varepsilon_0\right|^2\left|1+\Re^h_{0|2}\right|^4 \tag{4.17}$$

$$\left|\alpha^{0,2}_{vv}\right|^2=\left|\varepsilon_2-\varepsilon_0\right|^2\left|\frac{1}{\varepsilon_2}[1+\Re^v_{0|2}]^2\sin^2\theta^i_0+[1-\Re^v_{0|2}]^2\cos^2\theta^i_0\right|^2 \tag{4.18}$$

which are formally identical to eqs.(2)-(4) evaluated in the backscattering case ($\mathbf{k}^s_\perp=-\mathbf{k}^i_\perp$) when indexes 0,2,3 are replaced by 0,1,2.

This provides a formal proof that *Yarovoy* and *Fuks* models are consistent.

On the other hand, when the stratification under the corrugate interface disappears ($\varepsilon_2=\varepsilon_3$) so that the geometry of Fig.3 reduces to the one of Fig.2, we have $R^p_{2|3}=0$, $\Re^p_{1|2}=R^p_{1|2}$ and eq.(14)-(16) reduces to:

$$\left|\tilde{\alpha}^{1,2}_{hh}\right|^2=\left|\varepsilon_2-\varepsilon_1\right|^2\frac{\left|T^h_{0|1}e^{jk_{z1}\Delta_1}\right|^4}{\left|1+R^h_{0|1}R^h_{1|2}e^{j2k_{z1}\Delta_1}\right|^4}\left|1+R^h_{1|2}\right|^4 \tag{4.19}$$

$$\left|\tilde{\alpha}_{vv}^{1,2}\right|^2 = \left|\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1}\right|^2 \frac{\left|T_{0|1}^v \, e^{jk_{z1}\Delta_1}\right|^4}{\left|1 + R_{0|1}^v R_{1|2}^v \, e^{j2k_{z1}\Delta_1}\right|^4} \left|[1 + R_{1|2}^v]^2 \frac{1}{\varepsilon_2} \sin^2\theta_0^i + [1 - R_{1|2}^v]^2 \cos^2\theta_1^i\right|^2 \tag{4.20}$$

Equation (19) is formally identical to eq. (13) evaluated in the backscattering case ($\mathbf{k}_\perp^s = -\mathbf{k}_\perp^i$). Finally, this shows that *Yarovoy* and *Sarabandi* models are consistent. In this manner, all the analyzed models can be considered in a common framework.

It should be noted that such considerations also prove that the approaches of the different SPM models, make they use or not of the Green's functions formalism, lead after all to the same results, according to expectations.

In conclusion, the solutions of the analyzed models, which we have organized in a coherent framework with the help of a unitary formalism, have been also simplified in compact expressions and their consistency has been established analytically.

4.4 Conclusions

In this chapter three different perturbative solutions, available in literature, have been considered for the scattering from a stratified medium with one rough interface. As matter of fact, all these models, which refer to different simplified geometry, employ different perturbative procedures and different notations in the relative analytical derivation, so that the resulting solutions turn out of difficult mutual comparison. Besides, the finding of the connections between these existing functional forms is not a trivial task.

Therefore, these models have been here organized in a coherent framework with the help of a unitary formalism, and we have also simplified them in compact expressions. This has allowed us to demonstrated the equivalence of the relevant analytical procedures and establish the consistency of the respective solutions.

However, the obtained closed form solutions for the scattering from layered media with a rough interface refer to specific simplified geometries only. In [20] the Sarabandi approach was extended to the case of two rough interfaces; unfortunately, extending the proposed formulation to more layers becomes intrinsically not analytically tractable and no general closed form has been established. Other approaches have also been proposed, which are not based on the SPM, see e.g. [21], but they cannot be directly compared to the methods considered here, and their practical applicability is questionable.

Therefore, a general closed form solution for the problem of scattering by a layered structure with an arbitrary number of corrugated interfaces is not available in the literature yet, and it would be highly desirable, also in view of future advanced SAR missions (see, e.g., [22]).

The unitary formulation presented in this Chapter suggests a way to achieve this goal. In fact, the general compact closed form solution can be derived, and this is discussed in the next chapters.

4.5 Appendix: Derivation of the Equivalent form of Yarovoy Model

In this Appendix we provide the missing details of Section 4.3.1 that lead to explicit calculation of the expressions (14)-(16). We start from the analytic expression (7)-(11). Primarily, it is instructive to take into account that the following relations hold:

$$\begin{aligned} k_{z2}(k_{z1}-k_{z3}) \pm (k_{z1}k_{z3}-k_{z2}^2) &= (k_{z1} \mp k_{z2})(k_{z2} \pm k_{z3}) \\ k_{z2}(k_{z1}+k_{z3}) \pm (k_{z1}k_{z3}+k_{z2}^2) &= (k_{z1} \pm k_{z2})(k_{z2} \pm k_{z3}) \end{aligned} \tag{A.1}$$

$$\begin{aligned} \varepsilon_2 k_{z2}(\varepsilon_3 k_{z1}-\varepsilon_1 k_{z3}) \pm (\varepsilon_2\varepsilon_2 k_{z1}k_{z3}-\varepsilon_1\varepsilon_3 k_{z2}^2) &= (\varepsilon_2 k_{z1} \mp \varepsilon_1 k_{z2})(\varepsilon_3 k_{z2} \pm \varepsilon_2 k_{z3}) \\ \varepsilon_2 k_{z2}(\varepsilon_3 k_{z1}+\varepsilon_1 k_{z3}) \pm (\varepsilon_2\varepsilon_2 k_{z1}k_{z3}+\varepsilon_1\varepsilon_3 k_{z2}^2) &= (\varepsilon_2 k_{z1} \pm \varepsilon_1 k_{z2})(\varepsilon_3 k_{z2} \pm \varepsilon_2 k_{z3}) \end{aligned} \tag{A.2}$$

Furthermore, we have to consider the complex functions:

$$\begin{aligned} 2j\sin(k_{z2}\Delta_2) &= e^{jk_{z2}\Delta_2} - e^{-jk_{z2}\Delta_2} \\ 2\cos(k_{z2}\Delta_2) &= e^{jk_{z2}\Delta_2} + e^{-jk_{z2}\Delta_2} \end{aligned} \tag{A.3}$$

Substituting (A.1)-(A.2) in (7)-(11) and taking into account (A.3), eqs. (7)-(11) can be expressed as follows:

$$\left|\tilde{\alpha}_{hh}^{1,2}\right|^2 = \frac{16}{16}\left|\varepsilon_2 - \varepsilon_1\right|^2 \frac{\left|2\cos\theta_0^i Z_{\Delta 1}^h\right|^4}{\left|1+\cos\theta_0^i Z_{\Delta 1}^h\right|^4} \frac{\left|E^h + F^h\right|^4}{\left|F^h e^{-jk_{z1}\Delta_1} + E^h e^{jk_{z1}\Delta_1}\right|^4} \tag{A.4}$$

$$\left|\tilde{\alpha}_{vv}^{1,2}\right|^2 = \frac{16}{16}\left|\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1}\right|^2 \left|2\cos\ \theta_0^i\right|^4 \frac{\left|\frac{1}{\varepsilon_2}\sin^2\theta_0^i + Z_0^{v^2}\right|^2}{\left|\cos\theta_0^i + Z_{\Delta 1}^v\right|^4} \frac{\left|E^v + F^v\right|^4}{\left|E^v e^{jk_{z1}\Delta_1} + F^v e^{-jk_{z1}\Delta_1}\right|^4} \tag{A.5}$$

where

$$Z_{\Delta_1}^h = \frac{k_0}{k_{z1}}\left[\frac{F^h e^{-jk_{z1}\Delta_1} + E^h e^{jk_{z1}\Delta_1}}{F^h e^{-jk_{z1}\Delta_1} - E^h e^{jk_{z1}\Delta_1}}\right] \tag{A.6}$$

$$Z_{\Delta_1}^v = \frac{k_{z1}}{\varepsilon_1 k_0}\left[\frac{F^v e^{-jk_{z1}\Delta_1} - E^v e^{jk_{z1}\Delta_1}}{F^v e^{-jk_{z1}\Delta_1} + E^v e^{jk_{z1}\Delta_1}}\right] \tag{A.7}$$

and

$$F^h = (k_{z1}+k_{z2})(k_{z2}-k_{z3})e^{jk_{z2}\Delta_2} + (k_{z1}-k_{z2})(k_{z3}+k_{z2})e^{-jk_{z2}\Delta_2} \tag{A.8}$$

$$E^h = (k_{z1}-k_{z2})(k_{z2}-k_{z3})e^{jk_{z2}\Delta_2} + (k_{z1}+k_{z2})(k_{z2}+k_{z3})e^{-jk_{z2}\Delta_2} \tag{A.9}$$

$$F^v = (\varepsilon_2 k_{z1}+\varepsilon_1 k_{z2})(\varepsilon_3 k_{z2}-\varepsilon_2 k_{z3})e^{jk_{z2}\Delta_2} + (\varepsilon_2 k_{z1}-\varepsilon_1 k_{z2})(\varepsilon_3 k_{z2}+\varepsilon_2 k_{z3})e^{-jk_{z2}\Delta_2} \tag{A.10}$$

$$E^v = (\varepsilon_2 k_{z1}-\varepsilon_1 k_{z2})(\varepsilon_3 k_{z2}-\varepsilon_2 k_{z3})e^{jk_{z2}\Delta_2} + (\varepsilon_2 k_{z1}+\varepsilon_1 k_{z2})(\varepsilon_3 k_{z2}+\varepsilon_2 k_{z3})e^{-jk_{z2}\Delta_2} \tag{A.11}$$

Using (A.6) and (A.7), (A.4) and (A.5) can be more usefully rewritten as:

$$\left|\tilde{\alpha}_{hh}^{1,2}\right|^2 = 16\ k_0^4 \cos^4 \theta_0^i \left|\varepsilon_2 - \varepsilon_1\right|^2 \frac{\left|E^h + F^h\right|^4}{\left|(F^h e^{-jk_{z1}\Delta_1} - E^h e^{jk_{z1}\Delta_1})k_{z1} + k_{z0}(E^h e^{jk_{z1}\Delta_1} + F^h e^{-jk_{z1}\Delta_1})\right|^4} \tag{A.12}$$

$$\left|\tilde{\alpha}_{vv}^{1,2}\right|^2 = 16 \cos^4 \theta_0^i \left|\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1}\right|^2 \frac{\left|\frac{(\varepsilon_1 k_0)^2}{\varepsilon_2} \sin^2 \theta_i (E^v + F^v)^2 + \varepsilon_1 k_{z1}^2 (F^v - E^v)^2\right|^2}{\left|\varepsilon_1 k_{z0}(F^v e^{-jk_{z1}\Delta_1} + E^v e^{jk_{z1}\Delta_1}) + \varepsilon_0 k_{z1}(F^v e^{-jk_{z1}\Delta_1} - E^v e^{jk_{z1}\Delta_1})\right|^4} \tag{A.13}$$

Making use of the definition (3.1)-(3.2) and (3.3)-(3.4) in (A.12) and (A.13), after straightforward manipulation we obtain:

$$\left|\tilde{\alpha}_{hh}^{1,2}\right|^2 = \left|\varepsilon_2 - \varepsilon_1\right|^2 \left|T_{0|1}^h\ e^{jk_{z1}\Delta_1}\right|^4 \frac{\left|E^h + F^h\right|^4}{\left|F^h + R_{0|1}^h E^h e^{j2k_{z1}\Delta_1}\right|^4} \tag{A.14}$$

$$\left|\alpha_{vv}^{1,2}\right|^2 = \left|\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1}\right|^2 \left|T_{0|1}^v e^{jk_{z1}\Delta_1}\right|^4 \frac{\left|(F^v + E^v)^2 \frac{1}{\varepsilon_2} \sin^2 \theta_0^i + (F^v - E^v)^2 \cos^2 \theta_1^i\right|^2}{\left|F^v + R_{0|1}^v E^v e^{j2k_{z1}\Delta_1}\right|^4} \tag{A.15}$$

It can be recognized, see eq.(3.42), that:

$$\frac{E^p}{F^p} = [R_{1|2}^p + R_{2|3}^p e^{j2k_{z2}\Delta_2}][1 + R_{1|2}^p R_{2|3}^p e^{j2k_{z2}\Delta_2}]^{-1} = \Re_{1|2}^p \tag{A.16}$$

In conclusion, substituting (A.16) in (A.14)-(A.15) we derive the final expressions (14)-(16).

4.6 Appendix: Radar Cross Section

In this section the important concepts of *radar cross section* and *scattering coefficient* are introduced.

As active remote sensing in the microwave region of the electromagnetic spectrum is concerned, it is fundamental to describe the interaction of an electromagnetic wave with a certain object (target). As a consequence of this interaction, part of the energy carried by the incident wave is absorbed by the target itself, whereas the rest is reradiated as a new electromagnetic wave. Generally speaking, an object exposed to an electromagnetic wave disperses incident energy in all directions. This spatial distribution of energy is called scattering, and the object itself is often called a scatterer.

The *Radar Cross Section* σ, for which the abbreviation RCS has been generally recognized, provides a quantitative characterization of the electromagnetic energy intercepted and re-radiated by an object (or target), which is the energy available for detection.

Let the *Poynting* power density of the incident wave at the scattering target be S^i [W/m^2], the amount of power intercepted by the target is then related to its RCS σ, so that the *intercepted power* is $S^i\sigma$ [W]. This intercepted power is then either reradiated as the scattered power or absorbed (as heat). Assuming that it is reradiated as scattered power uniformly in all 4π [sr] of space, the *Poynting* power density of the scattered wave is given by $S^s = S^i\sigma / 4\pi r^2$ [W/m^2], where r is the distance from the scatterer (target) to the observation point.

Accordingly, as the target is considered to be in the far-field region, the radar cross section σ of an object (or target) is defined as an equivalent area intercepting that amount of power which, when scattered isotropically, produces at the radar receiver a power density which is equal to that scattered by the real object (or target):

$$\sigma = \lim_{r\to\infty} 4\pi r^2 \frac{S^s}{S^i}, \qquad \text{(B.21)}$$

which, therefore, is essentially the limit of the ratio of scattered power density at the receiver to incident power density at the target, as the distance r approaches infinity. This is to say that the radar cross section of an object is defined as the cross section of an equivalent idealized isotropic scatterer that generates the same scattered power density as the object in the observed direction.

It should be noted that the limiting process in Eq. (10) is not always an absolute requirement. However, in both measurement and analysis, the radar receiver and transmitter are usually taken to be in the far field of the target, and at that distance the scattered field decays inversely with the distance r.

The units for RSC are square meters. Although generally the larger physical size of the object the larger the inherent RSC, the RSC is not necessarily related to the physical size of a target.

A general notation for the polarization dependent *bistatic* RCS is

$$\sigma_{qp}(\hat{k}^s, \hat{k}^i) \qquad \text{(B.22)}$$

where $q \in \{v, h\}$ and $p \in \{v, h\}$ denote, respectively, the polarization of scattered field and the polarization of incident field; $\hat{k}^i$ and $\hat{k}^s$ denoting the incident and the observation directions, respectively. It is important to note that the symbol σ has been widely accepted as the designation for the radar cross section.

The RCS of a target is a function of several parameters, some of them related to the radar system (wave frequency, wave polarization, configuration, etc), other ones are related to the intrinsic (geometric and electromagnetic) properties of the target.

Accordingly, the term *bistatic cross section* refers to a configuration in which transmitter and receiver are at different locations, whereas the term *monostatic (backscattering) cross section* is used when transmitter and receiver are collocated.

The formal definition of polarization dependent radar cross section σ_{qp} is

$$\sigma_{qp} = \lim_{r\to\infty} 4\pi r^2 \frac{S_q^s}{S_p^i} = \lim_{r\to\infty} 4\pi r^2 \frac{\left|\mathbf{E}^s \cdot \hat{q}(\mathbf{k}^s)\right|^2}{\left|\mathbf{E}^i \cdot \hat{p}(\mathbf{k}^i)\right|^2}, \qquad \text{(B.23)}$$

where S_p^i is the Poynting power density of the *p*-polarized incident wave at the scattering target, S_q^s is the Poynting power density of the *q*-polarized scattered wave, $\mathbf{E}^i$ in the relevant electric field of the incident wave impinging on the target and $\mathbf{E}^s$ is the electric field of the scattered wave by the target at the observation point, and *r* is the distance from the scatterer (target) to the observation point as the target is considered to be in the far-field region; *p* is the polarization of the incident field and *q* is the polarization of the scattered field.

Equivalently, RCS is defined as 4π times the ratio of the power per unit solid angle of the polarization *q* scattered in direction $\hat{k}^i$ to the power per unit area of a *p*-polarized plane wave incident on the scatterer from direction $\hat{k}^s$.

When the target of interest is smaller than the *footprint* of the radar system, that is, a point target, we consider the target as an isolated scatterer and from the point of view of power exchange, this target is characterized by the so-called radar cross section. Accordingly, the RCS is usually employed for discrete targets.

Conversely, for targets presenting a larger extent than the radar footprint, we need a different model to represent the target. In these situations, a target is represented as an infinite collection of statistically identical point targets. Hence, when the target of interest is significantly larger than the footprint of the radar system, it is more convenient to characterize the target independently of its extent. Indeed, since the cross section a of a patch of the extended target varies with the illuminated area and this is determined by the geometric radar parameters (pulse width, beamwidth, etc.), it is convenient to introduce a coefficient independent of these parameters. Therefore, in these situations, the target is described by the so-called polarization dependent *bistatic scattering coefficients* for the reflected intensity:

$$
\begin{aligned}
\sigma_{qp}^0 &= \lim_{A\to\infty} \frac{<\sigma_{qp}>}{A} = \lim_{r\to\infty} \lim_{A\to\infty} \frac{4\pi r^2}{A} \frac{<S_q^s>}{S_p^i} \\
&= \lim_{r\to\infty} \lim_{A\to\infty} \frac{4\pi r^2}{A} \frac{<\left|\mathbf{E}^s \cdot \hat{q}(\mathbf{k}^s)\right|^2>}{\left|\mathbf{E}^i \cdot \hat{p}(\mathbf{k}^i)\right|^2}
\end{aligned}
\qquad \text{(B.24)}
$$

where the angular brackets denotes statistical ensemble averaging, A is the illuminated area, $\mathbf{E}^s$ is the electric field of the scattered wave, resulting from the coherent addition of the scattered waves from every one of the independent targets which model the extended scatterer.

Therefore, radar return is described by σ_{qp}^0, which is the averaged radar cross section per unit area, also called the *scattering coefficient* or ''sigma-naught'' and represents the ratio of the statistically averaged scattered power density to the average incident power density over the surface of the sphere of radius r . The scattering coefficient σ_{qp}^0 is a dimensionless parameter.

It is important to note that some authors use a scattering cross section per unit *projected area* $A\cos\theta^i$ (which is the illuminated area projected onto the plane normal to the incident direction θ^i) rather than per unit *ground area A*, so that $\sigma_{qp}^0 = \gamma_{qp}\cos\theta^i$. Since, in the literature, both γ_{qp} and σ_{qp}^0 are called scattering coefficients, readers must be especially careful to determine which is being used by a particular author.

Similarly, it can be defined the polarization dependent *bistatic scattering coefficients* for the transmitted intensity:

$$\begin{aligned}\sigma_{qp}^0 &= \lim_{r\to\infty}\lim_{A\to\infty}\frac{4\pi r^2}{A}\frac{< S_q^t >}{S_p^i} \\ &= \lim_{r\to\infty}\lim_{A\to\infty}\frac{4\pi r^2}{A}\frac{< \left|\mathbf{E}^t\cdot\hat{q}(\mathbf{k}^s)\right|^2 >}{\left|\mathbf{E}^i\cdot\hat{p}(\mathbf{k}^i)\right|^2}\mathrm{Re}\left\{\sqrt{\frac{\varepsilon_1}{\varepsilon_0}}\right\}\end{aligned} \tag{B.25}$$

where ε_0 and ε_1 are, respectively, the dielectric relative permittivities of the two media separated by the pertinent surface, S_q^t is the *Poynting* power density of the q-polarized transmitted (scattered through) wave, $\mathbf{E}^t$ is the relevant electric field of the transmitted wave, resulting from the coherent addition of the scattered waves from every one of the independent targets which model the extended scatterer.

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Chapter 5

Boundary Perturbation Theory

"Io non so come mi giudica il mondo; a me sembra d'essere un bambino che giuoca sulla spiaggia del mare e si rallegra se di quando in quando trova un ciottolo più liscio degli altri o una conchiglia più bella delle altre, mentre il grande oceano della verità sta inesplorato dinanzi a lui."
Isaac Newton

This chapter is aimed primarily at providing a comprehensive analytical treatment of electromagnetic wave propagation and scattering in three-dimensional multilayered structures with rough interfaces. A general methodology is developed to analytically treat EM bistatic scattering from the class of layered structures that can be described by small changes with respect to an idealized (unperturbed) structure, whose associated problem is exactly solvable.

The emphasis is placed on the general formulation of the scattering problem in the analytic framework of the *Boundary Perturbation Theory* (BPT) whose structured presentation is proposed and developed in this chapter.

A thorough analysis of the results of this theoretical investigation (BPT), which is based on perturbation of the boundary condition, is presented methodologically emphasizing the development of the several inherent aspects.

A systematic perturbative expansion of the fields in the layered structure, based on the gently rough interfaces assumption, enables the transferring of the geometry randomness into a non-uniform boundary conditions formulation. Subsequently, the fields' expansion can be analytically evaluated by using a recursive matrix formalism approach encompassing a proper scattered field representation in each layer and a matrix reformulation of non-uniform boundary conditions. A key-point in the development resides in the appropriate exploitation of the *generalized reflection/transmission* notion, which has strong implications in order to make the mathematical treatment manageable and to effectively capture the physics of the problem.

Two relevant compact closed-form solutions, derived in the first-order limit of the perturbative development, are presented. They refer to two complementary *bi-static* configurations for the scattering, respectively, from and through layered structures with arbitrary number of rough interfaces. The employed formalism is fully-*polarimetric* and suitable for applications. In addition, it is demonstrated how the symmetrical character of the *BPT* formalism reflects the inherent conformity with the *reciprocity* theorem of the electromagnetic theory.

5.1 Introduction and Motivation

The small perturbation method (SPM) is the oldest and the most broadly used formalism to predict the radar scattering from rough surfaces with small rms height and slope. Detailed analysis on the limit of validity of the SPM methods, as well as the other analytical method, are available in literature [24][25][29]. The state of the art shows that extensions of the SPM method to the layered media with one rough interface have been proposed. The resulting expressions, derived with different techniques, are given for different simplified geometry with a limited number of layers only [1]-[7] . All the considered models are valid in the limit of first-order SPM and consider only a single rough interface.

An overview of the state of the art of extensions of the SPM method to the specific layered configurations with one rough interface, clarifying connections between the existent models [1]-[9], is provided in Chapter 4, where the considered three different perturbative solutions have been organized in a coherent framework with the help of a unitary formalism, have been also simplified in compact expressions and the proof of the consistence of the three methods has been established analytically. Methodologically, we underline that all the previously mentioned existing perturbative approaches, followed by different authors in analyzing scattering from simplified geometry, imply an inherent analytical complexity, which precludes the treatment to structures with more than one or two [9] rough interfaces. In [9] the Sarabandi approach was extended to the case of two rough interfaces; however, no general closed form has been established. This approach does not make use of the generalized reflection/transmission concepts. As matter of fact, extending this formulation to more layers intrinsically becomes analytically not tractable. In conclusion, to the best of the existing knowledge, a general closed form solution for scattering problem by an arbitrary layered structure with corrugated interface is not available in the literature yet, despite its crucial value. A few solutions are available in literature [30],[31], but they are not completely in closed form and are, at best, of difficult use in practice.

Nonetheless, it is important also to note that results obtained in Chapter 4 suggest that a key point in order to obtain a compact general closed form solution is the effectively exploiting of the generalized reflection coefficients of equivalent reflecting boundaries and the generalized transmission coefficients of equivalent slabs.

Therefore, the objective of this Chapter is to investigate analytically, in the perturbation framework, the fully polarimetric electromagnetic wave scattering from and through three-dimensional (3-D) layered structures with N-rough interfaces.

We first introduce the general perturbative expansion on which the BPT formulation is based: we perform a perturbative expansion of the fields in the rough-interfaces layered structure, assuming that deviations and slopes, with respect to the reference mean plane, exhibited by rough interfaces are small enough. In this manner, in the first-order approximation, the geometry randomness of the corrugated interfaces is translated in random current sheets imposed on unperturbed (flat) interfaces and radiating in unperturbed (flat boundaries) layered media. These uncoupled current sheets are related, in the first-order approximation, to the respective roughness and, along with the *Born* approximation, to the field components of the unperturbed solution on the respective interface of the unperturbed structure. In order to perform the evaluation of perturbative development, the scattered field is then represented as the sum of up-going and down-going waves and a systematic approach that involves the use of matrix formalism is employed.

One main factor distinguishing BPT is the formulation of non-uniform boundary conditions in matrix notation in terms of the expansion coefficients vector, the transfer matrix operators and the source vectors. This systematic matrix reformulation, which enables the formal evaluation of pertinent scattered field solutions, permits us to avoid the necessity of the cumbersome *Green functions formalism*. Consequently, we primarily consider the field scattered by a generic rough interface embedded in the layered medium, so that, by using effectively the concept of *generalized reflection/transmission coefficients*, the unknown expansion coefficients of scattered wave propagating upward in the upper half-space and downward in the lower half-space are derived via a recursive method. Subsequently, the formulation is extended straightforwardly to the *N*-rough interface case.

A relevant point is related to the reciprocal character of the developed scattering formalism. To this purpose, we furnish the general demonstration that the proposed first-order BPT solutions for the scattering *from* and *through* a layered structure with N-rough interfaces satisfy *reciprocity*.

BPT formulation leads to derive compact and easy to use close form solutions in terms of the *polarimetric* bi-static scattering coefficients of the three-dimensional layered structure, which are valid for an arbitrary layered media with an arbitrary number of gently rough interfaces and that allow us to easily deal with random surfaces parametrically including geometric and dielectric layer characteristics. In other words, the proposed model explains how the relative contribution of each corrugated boundary, on the observed scattered signal, is influenced by the layered structure. Furthermore, we discuss the formal consistency with the previous works, in the perspective of providing a unifying insight for the perturbative formulations: the demonstration of the consistency of the BPT solutions is analytically provided showing that the BPT solutions reduce to the corresponding existing ones when the stratification geometry reduces to the simplified ones considered by the other authors. In conclusion, all the existing perturbative scattering models can be regarded as particular case of the proposed general solution when the geometry configuration reduces to some simplified ones.

This chapter is organized as follows.

In section 5.2, we briefly define the problem we intend to deal with. In section 5.3, to overcome the limitations exhibited by the current models, the general formulation of the BPT is proposed. Sections 5.4 and 5.5 are devoted to the matrix reformulation of the non-uniform boundary conditions and the evaluation of the expansion coefficients, respectively. By using the new BPT formulation, general *closed-form* polarimetric solutions for the scattering from a through a layered structure with an arbitrary number of gently rough interfaces are then systematically carried out in Section 5.6. The proof of reciprocity for the perturbative solutions is addressed in Section 5.7. In Section 5.8, pertinent bistatic scattering cross sections are provided. Analytical validation of BPT models is discussed in Section 5.9. Resulting expressions are numerically computed in Section 5.10. Section 5.11 concludes the Chapter with a summary.

5.2 Statement of the problem

The general problem we intend to deal with here refers to the analytical evaluation of the electromagnetic scattering from and through layered structure with an arbitrary number of rough interfaces (Fig.1).

The parameters pertaining to layer m with boundaries $-d_{m-1}$ and $-d_m$ are distinguished by a subscript m. Each layer is assumed to be homogeneous and characterized by arbitrary and deterministic parameters: the *dielectric relative permittivity* ε_m, the *magnetic relative permeability* μ_m and the *thickness* $\Delta_m=d_m-d_{m-1}$. With reference to Fig.1, it has been assumed that in particular, $d_0=0$. In the following, the symbol $\perp$ denotes the projection of the corresponding vector on the plan $z=0$. Here $\mathbf{r}=(\mathbf{r}_\perp, z)$, so we distinguish the transverse spatial coordinates $\mathbf{r}_\perp=(x,y)$ and the longitudinal coordinate z.

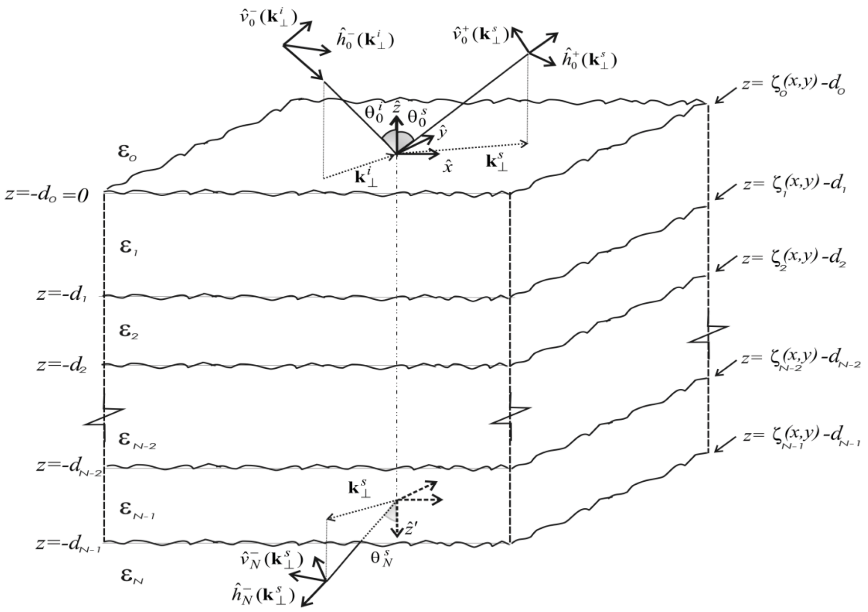


Fig. 1. Geometry for an *N*-rough boundaries layered medium.

In addition, each *m*-th rough interface is assumed to be characterized by a zero-mean two-dimensional *stochastic process* $\zeta_m = \zeta_m(\mathbf{r}_\perp)$ with normal vector $\hat{n}_m$. In addition, no constraints are imposed on the degree to which the rough interfaces are correlated.

As schematically shown in Fig.1, an arbitrary polarized monochromatic plane-wave

$$\mathbf{E}_0^i(\mathbf{r}) = [E_0^{ih}\hat{h}_0^-(\mathbf{k}_\perp^i) + E_0^{iv}\hat{v}_0^-(\mathbf{k}_\perp^i)]\, e^{j(\mathbf{k}_\perp^i\cdot\mathbf{r}_\perp - k_{z0}^i z)} \tag{5.1}$$

is considered to be incident on the layered medium at an angle θ_0^i relative to the $\hat{z}$ direction from the upper half-space, where in the field expression a time factor exp(–jωt) is understood, and where, using a spherical frame representation, the incident vector wave direction is individuated by θ_0^i, φ_0^i:

$$k_0\hat{k}_0^i = \mathbf{k}_0^i = \mathbf{k}_\perp^i - \hat{z}k_{z0}^i = k_0(\hat{x}\sin\theta_0^i\cos\varphi_0^i + \hat{y}\sin\theta_0^i\sin\varphi_0^i - \hat{z}\cos\theta_0^i)\,, \tag{5.2}$$

with

$$\hat{h}_0^-(\mathbf{k}_\perp^i) = \frac{\hat{k}_0^i \times \hat{z}}{\left|\hat{k}_0^i \times \hat{z}\right|} = \sin\varphi_0^i \hat{x} - \cos\varphi_0^i \hat{y}, \tag{5.3}$$

$$\hat{v}_0^-(\mathbf{k}_\perp^i) = \hat{h}_0^-(\mathbf{k}_\perp^i) \times \hat{k}_0^i = (\hat{x}\cos\varphi_0^i + \hat{y}\sin\varphi_0^i)\cos\theta_0^i + \hat{z}\sin\theta_0^i, \tag{5.4}$$

where $\mathbf{k}_\perp^i = k_x^i \hat{x} + k_y^i \hat{y}$ is the two-dimensional projection of incident wave-number vector on the plane z=0.

5.3 General Perturbative Formulation

In this section, a new approach for the derivation the first-order perturbative solution to the problem of the scattering from gently rough interfaces of an arbitrarily three-dimensional layered structure is presented.

In order to obtain a solution valid in each region of the structure, whose geometry is depicted in Fig.1, we have to enforce the continuity of the tangential fields:

$$[\hat{n}_m \times \Delta\mathbf{E}_m]_{z=\zeta_m(\mathbf{r}_\perp)-d_m} = 0, \tag{5.5}$$

$$[\hat{n}_m \times \Delta\mathbf{H}_m]_{z=\zeta_m(\mathbf{r}_\perp)-d_m} = 0, \tag{5.6}$$

where $\Delta\mathbf{E}_m = \mathbf{E}_{m+1} - \mathbf{E}_m$, $\Delta\mathbf{H}_m = \mathbf{H}_{m+1} - \mathbf{H}_m$, and the surface normal vector is given by:

$$\hat{n}_m = \frac{\hat{z} - \boldsymbol{\gamma}_m}{\sqrt{1+\gamma_m^2}}, \tag{5.7}$$

with the slope vector $\boldsymbol{\gamma}_m$:

$$\boldsymbol{\gamma}_m = \nabla_\perp \zeta_m = \left[\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y}\right]\zeta_m, \tag{5.8}$$

where $\nabla_{\perp}$ is the *nabla* operator in the *x*-*y* plane. In order to study the fields $\mathbf{E}_m$ and $\mathbf{H}_m$ within the generic *m*-th layer of the structure, we assume then that, for each *m*-th rough interface, the deviations and slopes of the interface, with respect to the reference mean plane $z = -d_m$, are small enough in the sense of [24][25], so that the fields can be expanded about the reference mean plane. Assume that the fields can be expanded about the reference mean plane $z = -d_m$ as:

$$\Delta\mathbf{E}_m(z) = \Delta\mathbf{E}_m\Big|_{z=-d_m} + \frac{\partial \Delta\mathbf{E}_m}{\partial z}\Bigg|_{z=-d_m}(z+d_m) + \frac{1}{2}\frac{\partial^2 \Delta\mathbf{E}_m}{\partial z^2}\Bigg|_{z=-d_m}(z+d_m)^2 + \dots, \tag{5.9}$$

$$\Delta\mathbf{H}_m(z) = \Delta\mathbf{H}_m\Big|_{z=-d_m} + \frac{\partial \Delta\mathbf{H}_m}{\partial z}\Bigg|_{z=-d_m}(z+d_m) + \frac{1}{2}\frac{\partial^2 \Delta\mathbf{H}_m}{\partial z^2}\Bigg|_{z=-d_m}(z+d_m)^2 + \dots, \tag{5.10}$$

where the dependence on $\mathbf{r}_{\perp}$ is understood. Then (9), (10) are the fields expansions in perturbative orders of the fields and their derivatives at the interfaces of the structure; next, they can be injected into the boundary conditions (5)-(6). Retaining only up to the first-order terms with respect to ζ_m and γ_m, we obtain:

$$\hat{z}\times\Delta\mathbf{E}_m\Big|_{z=-d_m} = \nabla_{\perp}\zeta_m \times \Delta\mathbf{E}_m\Big|_{z=-d_m} - \zeta_m\,\hat{z}\times\frac{\partial\Delta\mathbf{E}_m}{\partial z}\Bigg|_{z=-d_m}, \tag{5.11}$$

$$\hat{z}\times\Delta\mathbf{H}_m\Big|_{z=-d_m} = \nabla_{\perp}\zeta_m \times \Delta\mathbf{H}_m\Big|_{z=-d_m} - \zeta_m\,\hat{z}\times\frac{\partial\Delta\mathbf{H}_m}{\partial z}\Bigg|_{z=-d_m}. \tag{5.12}$$

The field solutions can then be represented formally as

$$\mathbf{E}_m(\mathbf{r}_{\perp}, z) \approx \mathbf{E}_m^{(0)} + \mathbf{E}_m^{(1)} + \mathbf{E}_m^{(2)} + \dots, \tag{5.13}$$

$$\mathbf{H}_m(\mathbf{r}_{\perp}, z) \approx \mathbf{H}_m^{(0)} + \mathbf{H}_m^{(1)} + \mathbf{H}_m^{(2)} + \dots\ . \tag{5.14}$$

where the parenthesized superscript refers to the perturbation field of order *n*: $\mathbf{E}_m^{(0)}, \mathbf{H}_m^{(0)}$ is the unperturbed solution and $\mathbf{E}_m^{(1)}, \mathbf{H}_m^{(1)}$ is correction to the first-order of ζ_m and γ_m. It should be noted that the unperturbed solution represents the field existing in flat boundaries stratification and satisfying:

$$\hat{z} \times \Delta \mathbf{E}_m^{(0)}\Big|_{z=-d_m} = 0\,, \tag{5.15}$$

$$\hat{z} \times \Delta \mathbf{H}_m^{(0)}\Big|_{z=-d_m} = 0\,. \tag{5.16}$$

The fields expansion (13)-(14) can be then injected into the boundary conditions (11)-(12), so that, retaining only up to the first-order terms, the following *non-uniform boundary conditions* can be obtained:

$$\hat{z} \times \Delta \mathbf{E}_m^{(1)}\Big|_{z=-d_m} = \nabla_\perp \zeta_m \times \Delta \mathbf{E}_m^{(0)}\Big|_{z=-d_m} - \zeta_m\, \hat{z} \times \frac{\partial \Delta \mathbf{E}_m^{(0)}}{\partial z}\Bigg|_{z=-d_m}\,, \tag{5.17}$$

$$\hat{z} \times \Delta \mathbf{H}_m^{(1)}\Big|_{z=-d_m} = \nabla_\perp \zeta_m \times \Delta \mathbf{H}_m^{(0)}\Big|_{z=-d_m} - \zeta_m\, \hat{z} \times \frac{\partial \Delta \mathbf{H}_m^{(0)}}{\partial z}\Bigg|_{z=-d_m}\,. \tag{5.18}$$

Therefore, the boundary conditions from each m-th rough interface can be transferred to the associated equivalent flat interface. In addition, the right-hand sides of Eqs. (17) and (18) can be interpreted as effective magnetic ($\mathbf{J}_{Hm}^{p(1)}$) and electric ($\mathbf{J}_{Em}^{p(1)}$) surface current densities, respectively, with p denoting the incident polarization; so that we can identify the first-order fluctuation fields as being excited by these effective surface current densities imposed on the unperturbed interfaces.

Accordingly, the geometry randomness of each corrugated interfaces is then translated in random current sheets imposed on each reference mean plane ($z = -d_m$), which radiate in an unperturbed (flat boundaries) layered medium.

Therefore, in order to derive in the overall layered media the first-order field scattered contribution by the m-th corrugated interface embedded in the stratification, we can equivalently calculate the field radiated by an effective source distribution imposed on the flat interface $z = -d_m$. Subsequently, within each generic n-th layer of the stratification, the overall first term $\mathbf{E}_n^{(1)}, \mathbf{H}_n^{(1)}$ can be obtained by the superposition of the field components radiated by each effective current distribution at $z = -d_m$.

Note also that the effective current sheets on the different reference flat interfaces are decoupled and each component is linear functional of the first-order of ζ_m and γ_m. It should be noted that, in a certain sense, our first-order perturbative approach is consistent with the classical *Born approximation*, since the derived scattered field depends on the unperturbed fields on the interfaces. As matter of fact, the Born approximation is based on the assumption that the scattering is weak, so that the scattered field is small and does not distort significantly the original field in absence of the roughness.

As a result, within the first-order approximation, the field can be than represented as the sum of an unperturbed part $\mathbf{E}_n^{(0)}, \mathbf{H}_n^{(0)}$ and a random part, so that $\mathbf{E}_n(\mathbf{r}_\perp, z) \approx \mathbf{E}_n^{(0)} + \mathbf{E}_n^{(1)}$, $\mathbf{H}_n(\mathbf{r}_\perp, z) \approx \mathbf{H}_n^{(0)} + \mathbf{H}_n^{(1)}$. The first is the primary field, which exists in absence of surface boundaries roughness (flat-boundaries stratification), detailed in Chapter 3; whereas $\mathbf{E}_n^{(1)}, \mathbf{H}_n^{(1)}$ can be interpreted as the superposition of single-scatter fields from each rough interface.

In order to perform the evaluation of perturbative development, the scattered field in each region of the layered structure is then represented as the sum of *up*- and *down-going* waves, and the first-order scattered field in each region of the layered structure can be then characterized by adopting the following field *spectral representation* in terms of the unknown coefficients $S_m^{\pm q(1)}(\mathbf{k}_\perp)$:

$$\mathbf{E}_m^{(1)} = \mathbf{E}_m^{-(1)} + \mathbf{E}_m^{+(1)}, \tag{5.19}$$

$$\mathbf{H}_m^{(1)} = \mathbf{H}_m^{-(1)} + \mathbf{H}_m^{+(1)}, \tag{5.20}$$

with

$$\mathbf{E}_m^{\pm(1)} = \sum_{q=h,v} \iint d\mathbf{k}_\perp e^{j\mathbf{k}_\perp \cdot \mathbf{r}_\perp} \hat{q}_m^{\pm}(\mathbf{k}_\perp) S_m^{\pm q(1)}(\mathbf{k}_\perp) e^{\pm jk_{zm}z}, \tag{5.21}$$

$$\mathbf{H}_m^{\pm(1)} = \sum_{q=h,v} \frac{1}{Z_m} \iint d\mathbf{k}_\perp e^{j\mathbf{k}_\perp \cdot \mathbf{r}_\perp} \hat{k}_m^{\pm} \times \hat{q}_m^{\pm}(\mathbf{k}_\perp) S_m^{\pm q(1)}(\mathbf{k}_\perp) e^{\pm jk_{zm} z}, \tag{5.22}$$

where $q \in \{v, h\}$ denotes the polarization of scattered field, Z_m is the *intrinsic impedance* of the medium m, and

$$\hat{h}_m^{\pm}(\mathbf{k}_\perp) = \hat{k}_\perp \times \hat{z} = \hat{h} \tag{5.23}$$

$$\hat{v}_m^{\pm}(\mathbf{k}_\perp) = \mp \frac{k_{zm}}{k_m} \hat{k}_\perp + \frac{k_\perp}{k_m} \hat{z} \tag{5.24}$$

is a basis for the horizontal/vertical polarization vectors.

Therefore, a solution valid in each region of the layered structure can be obtained from (19)-(22) taking into account the non-uniform boundary conditions (17)-(18).

It should be also noted that Eqs. (19)-(22) are the first-order counterpart of Eqs. (3.11)-(3.14), which have been primarily introduced to describe the zero-order fields in Chapter 3.

5.4 Matrix Reformulation of the Non-Uniform Boundary Conditions

In this section, the *non-uniform boundary conditions* (17)-(18) are reformulated, reducing the scattering problem to the formal solution of a linear system of equations; the unknowns are the scalar (complex) amplitudes, $S_m^{\pm q(1)}(\mathbf{k}_\perp)$, of the scattered fields.

Equations (17) and (18) can be rewritten by using their spectral representation:

$$\hat{z} \times \Delta\mathbf{E}_m^{(1)}\Big|_{z=-d_m} = \iint d\,\mathbf{k}_\perp\, e^{j\mathbf{k}_\perp \cdot \mathbf{r}_\perp} \tilde{\mathbf{J}}_{Hm}^{p(1)}(\mathbf{k}_\perp, \mathbf{k}_\perp^i), \tag{5.25}$$

$$\hat{z} \times \Delta\mathbf{H}_m^{(1)}\Big|_{z=-d_m} = \iint d\,\mathbf{k}_\perp\, e^{j\mathbf{k}_\perp \cdot \mathbf{r}_\perp} \tilde{\mathbf{J}}_{Em}^{p(1)}(\mathbf{k}_\perp, \mathbf{k}_\perp^i), \tag{5.26}$$

where the spectral densities $\widetilde{\mathbf{J}}_{Em}^{p(1)}$ and $\widetilde{\mathbf{J}}_{Hm}^{p(1)}$ are the two-dimensional (generalized) *Fourier transform* (2D-FT), with respect to $\mathbf{k}_\perp$, of the right-hand sides of (17) and (18), respectively, so that:

$$\widetilde{\mathbf{J}}_{Hm}^{p(1)}(\mathbf{k}_\perp,\mathbf{k}_\perp^i) = \widetilde{\zeta}_m(\mathbf{k}_\perp - \mathbf{k}_\perp^i)\left\{ j(\mathbf{k}_\perp - \mathbf{k}_\perp^i)\times \Delta\widetilde{\mathbf{E}}_m^{(0)}\Big|_{z=-d_m} - \hat{z}\times\frac{\partial \Delta\widetilde{\mathbf{E}}_m^{(0)}}{\partial z}\bigg|_{z=-d_m}\right\}, \tag{5.27}$$

$$\widetilde{\mathbf{J}}_{Em}^{p(1)}(\mathbf{k}_\perp,\mathbf{k}_\perp^i) = \widetilde{\zeta}_m(\mathbf{k}_\perp - \mathbf{k}_\perp^i)\left\{ j(\mathbf{k}_\perp - \mathbf{k}_\perp^i)\times \Delta\widetilde{\mathbf{H}}_m^{(0)}\Big|_{z=-d_m} - \hat{z}\times\frac{\partial \Delta\widetilde{\mathbf{H}}_m^{(0)}}{\partial z}\bigg|_{z=-d_m}\right\}, \tag{5.28}$$

where $\widetilde{\zeta}_m(\mathbf{k}_\perp)$ is the *spectral representation* (2D-FT) of the corrugation $\zeta_m(\mathbf{r}_\perp)$, and where $\Delta\widetilde{\mathbf{E}}_m^{(0)} = e^{-j\mathbf{k}_\perp^i\cdot\mathbf{r}_\perp}\Delta\mathbf{E}_m^{(0)}$, $\Delta\widetilde{\mathbf{H}}_m^{(0)} = e^{-j\mathbf{k}_\perp^i\cdot\mathbf{r}_\perp}\Delta\mathbf{H}_m^{(0)}$, $p\in\{v,h\}$ is associated with the incident field polarization, and where we have taken into account that the 2D-FT of $\nabla_\perp\zeta_m(\mathbf{r}_\perp)$ is $j\mathbf{k}_\perp\widetilde{\zeta}_m(\mathbf{k}_\perp)$, and that the 2D-FT of $\zeta_m(\mathbf{r}_\perp)e^{j\mathbf{k}_\perp^i\cdot\mathbf{r}_\perp}$ is $\widetilde{\zeta}_m(\mathbf{k}_\perp - \mathbf{k}_\perp^i)$.

In order to solve the scattering problem in terms of the unknown expansion coefficients $S_m^{\pm q(1)}(\mathbf{k}_\perp)$, their amplitudes are arranged in a single vector according to the notation:

$$\mathbf{S}_m^{q(1)}(\mathbf{k}_\perp, d_m) = \begin{bmatrix} S_m^{+q(1)}(\mathbf{k}_\perp)e^{-jk_{zm}d_m} \\ S_m^{-q(1)}(\mathbf{k}_\perp)e^{+jk_{zm}d_m} \end{bmatrix}. \tag{5.29}$$

In addition, we use (19)-(22) to evaluate the left-hand-side of eqs. (25), (26) and obtain (as shown in Appendix), that the *non-uniform boundary conditions* (17)-(18), for the (q=h) *horizontal* polarized scattered wave, can be reformulated by employing the following matrix notation:

$$\mathbf{S}_m^{h(1)}(\mathbf{k}_\perp, d_m) + \mathbf{\Theta}_m^p(\mathbf{k}_\perp,\mathbf{k}_\perp^i) = \boldsymbol{N}_{m|m+1}^h(k_\perp)\,\mathbf{S}_{m+1}^{h(1)}(\mathbf{k}_\perp, d_m), \tag{5.30}$$

where

$$\boldsymbol{\Theta}_m^p(\mathbf{k}_\perp,\mathbf{k}_\perp^i)=\begin{bmatrix}-\dfrac{k_0 Z_0 \mu_m}{2k_{zm}}(\hat{k}_\perp\times\hat{z})\cdot\widetilde{\mathbf{J}}_{Em}^{p(1)}+\dfrac{1}{2}\hat{k}_\perp\cdot\widetilde{\mathbf{J}}_{Hm}^{p(1)}\\ +\dfrac{k_0 Z_0 \mu_m}{2k_{zm}}(\hat{k}_\perp\times\hat{z})\cdot\widetilde{\mathbf{J}}_{Em}^{p(1)}+\dfrac{1}{2}\hat{k}_\perp\cdot\widetilde{\mathbf{J}}_{Hm}^{p(1)}\end{bmatrix}, \quad (5.31)$$

is the term associated with the *effective* source distribution, involving appropriate electric and magnetic currents imposed on the m-th unperturbed interface ($z = -d_m$), replacing the surface irregularities, and Z_0 is the intrinsic impedance of the vacuum. The fundamental *transfer matrix operator* $\boldsymbol{N}_{m|m+1}^h$ is defined by (3.34); the spectral expressions of the effective currents, $\widetilde{\mathbf{J}}_{Em}^{p(1)}$ and $\widetilde{\mathbf{J}}_{Hm}^{p(1)}$, imposed on the (flat) unperturbed boundary for an incident polarization $p \in \{v, h\}$ are given by (27)-(28).

As a matter of fact, Eq. (30) states in a simpler form the problem originally set by Eqs. (17)-(18): indeed, solving Eq. (30) $\forall m$ implies dealing with the determination of unknown scalar amplitudes $S_m^{\pm q(1)}(\mathbf{k}_\perp)$ instead of working with the corresponding vector unknowns $\mathbf{E}_m^{(1)}, \mathbf{H}_m^{(1)}$.

As a result, when a structure with rough interfaces is considered, the enforcement of the *non-uniform boundary conditions* (17)-(18) through the stratification (m=0,...,N-1) can be addressed by writing down a linear system of equations with the aid of the matrix formalism (30) with m=0,..,N-1. Therefore, the scattering problem in each m-th layer is reduced to the algebraic calculation of the unknown expansion scattering coefficients vector (29). Moreover, it should be noted that on a (k-th) flat interface Eq. (30) reduces to the *uniform boundary conditions,* thus getting:

$$\mathbf{S}_k^{h(1)}(\mathbf{k}_\perp,d_k)=\boldsymbol{N}_{k|k+1}^h(k_\perp)\,\mathbf{S}_{k+1}^{h(1)}(\mathbf{k}_\perp,d_k). \quad (5.32)$$

Note also that crossing flat boundaries the first-order expansion coefficients vectors $\mathbf{S}_m^{q(1)}$ are transformed (see (32)) in the same way as the zero-order ones $\mathbf{S}_m^{q(0)}$ (see Eq. (3.33)).

It is important to observe that resulting effective currents on different reference mean planes are decoupled each others, i.e., $\widetilde{\mathbf{J}}_{Ek}^{p(1)}$

and $\tilde{\mathbf{J}}_{Hk}^{p(1)}$ are decoupled by $\tilde{\mathbf{J}}_{Ej}^{p(1)}$ and $\tilde{\mathbf{J}}_{Hj}^{p(1)}$ $\forall\ k \neq j$. Hence, the field radiated by the effective currents system, constituted by the set of effective currents imposed on the several reference mean planes, can be obtained by superposing the fields radiated by each current system $\tilde{\mathbf{J}}_{Em}^{p(1)}, \tilde{\mathbf{J}}_{Hm}^{p(1)}$, evaluated separately.

Therefore, in the first-order limit, the general problem of the scattering from a structure with all rough interfaces can be addressed by superimposing the solutions obtained considering different configurations, evaluated separately. Each one of these configurations results to be constituted by a layered structure in which a different embedded interface (*m*-th) is rough, whereas all other interfaces ($k \neq m$) are flat.

Accordingly, we first focus our attention on the calculation of the scattering contribution from a single generic (*m*-th) rough interface embedded in an isotropic, piecewise-homogeneous and arbitrary flat-boundaries layered medium. This corresponds to solve the system of equations formed by Eq. (30) and by Eq. (32) $\forall\ k \neq m$, once the unperturbed field solution is calculated and the appropriate effective currents are evaluated.

It should be noted that, for the considered configuration, the relevant scattering coefficients $S_N^{+q(1)}(\mathbf{k}_\perp)$ and $S_0^{-q(1)}(\mathbf{k}_\perp)$ are obviously supposed to be zero. In other words, no scattered field directed inward from the infinite, for each half-space of the structure, is assumed. Consequently, by leveraging on this, it is then possible to derive recursively all the unknown expansion coefficients.

As a result, the formulation of *non-uniform boundary conditions* in matrix notation (29)-(30) enables a systematic method, which involves the effective use the matrix formalism introduced in Chapter 3, for solving the scattering problem: specifically, for the *N*-layer stratification of Fig.1, we have to find $2N$ unknown expansion coefficients, using N vectorial equations (30), i.e., $2N$ scalar equations.

The scattering problem, therefore, results to be reduced to a formal solution of a linear equation system.

Obtaining the solution at this point seems unproblematic; however, we highlight that a key-point, to resolve formally the system for a given arbitrary N, is to resort to a recursive method based on the effective use of the concept of *generalized reflection/transmission* coefficients (see also Chapter 3). Only after that this is recognized, the system of equations (30), (32) is susceptible of a straightforward closed-form solution, so that the first-order perturbation fields, which arise from each *m*-th rough interface, can be formally found anywhere in the structure (and, in particular, in the upper or the lower half-space).

In conclusion, the derivation of scattering field contribution, due to each rough interface can be then accomplished by completely avoiding the use of the cumbersome *Green functions* formalism.

5.5 Determination of expansion coefficients

In this section, we firstly focus our attention on the calculation of the scattering contribution from a generic *m*-th rough interface embedded in an isotropic, piecewise-homogeneous and arbitrary flat-boundaries layered medium. The straightforward extension to more general *N*-rough interfaces case will be addressed in next Section 5.6.

We now demonstrate how, by making use of a *recursive approach* involving the concept of generalized transmission/reflection, the system of equations (30)-(32) is susceptible of a straightforward close form solution, so that the first-order perturbation fields that arise from the *m*-th rough interface is formally found.

We also emphasize that here we are interested in the scattering from and through the stratification; therefore, the determination of the pertinent unknown expansion coefficients $S_0^{+q(1)}(\mathbf{k}_\perp)$ and $S_N^{-q(1)}(\mathbf{k}_\perp^s)$ of the scattered wave, respectively, into the upper and the lower half-space, is primary required. Full expressions for these coefficients are derived in next sections.

5.5.1 Wave Scattered upward in the upper half-space

This subsection is devoted to the formal evaluation in closed-form of the unknown scattering coefficients $S_0^{+q(1)}$ of the perturbative expansion by exploiting the matrix formulation of the *non-uniform* boundary conditions.

Accordingly, the system of equations (30)-(32) is firstly resolved in terms of the unknown expansion coefficient $S_0^{+h(1)}$ of the horizontal polarized ($q=h$) scattered wave propagating upward in the upper half-space. This is done by using a recursive approach involving the concept of *generalized transmission/reflection* coefficients.

By enforcing the condition (32) on all the interfaces for which $j \geq m+1$, the associated equations system resolved recursively with $S_N^{+q(1)} = 0$ lead to (see (3.72)):

$$\mathbf{S}_{m+1}^{q(1)}(\mathbf{k}_\perp, d_{m+1}) = \begin{bmatrix} \Re_{m+1|m+2}^{q} \\ 1 \end{bmatrix} [\Im_{m+1|N}^{q(slab)}]^{-1} S_N^{-q(1)}(\mathbf{k}_\perp) e^{jk_{zN} d_{N-1}}. \quad (5.33)$$

This is equivalent to consider the (equivalent) response, as seen from the (m+1)-th layer, of the slab constituted by the layers $m+2, m+3, .., N-1$.

On the other hand, by enforcing the condition (32) on all the j-th interfaces for which $0 \leq j < m$, the associated equations system resolved recursively with $S_0^{-q(1)} = 0$ lead to (see (3.78)):

$$\mathbf{S}_m^{q(1)}(\mathbf{k}_\perp, d_{m-1}) = \begin{bmatrix} 1 \\ \Re_{m|m-1}^{q} \end{bmatrix} [\Im_{m|0}^{q(slab)}]^{-1} S_0^{+q(1)}(\mathbf{k}_\perp), \quad (5.34)$$

where the *generalized transmission coefficients* $\Im_{m|0}^{q(slab)}$ for the layered slab in upward direction are defined as in (3.58). Equation (30) can be equivalently rewritten:

$$\boldsymbol{\Pi}_m^{-1} \mathbf{S}_m^{h(1)}(\mathbf{k}_\perp, d_{m-1}) + \boldsymbol{\Theta}_m^{p} = \boldsymbol{N}_{m|m+1}^{h}(k_\perp) \boldsymbol{\Pi}_{m+1}(k_\perp) \mathbf{S}_{m+1}^{h(1)}(\mathbf{k}_\perp, d_{m+1}). \quad (5.35)$$

where the propagation in the m-th layer is accounted for by $\boldsymbol{\Pi}_m(k_\perp)$ (see (3.35)). By substituting (33) and (34) with $q=h$ in (35), and using relation (3.36), we can formally write:

$$\begin{bmatrix} 1 \\ \Re_{m|m-1}^{h} e^{j2k_{zm}\Delta_m} \end{bmatrix} [e^{jk_{zm}\Delta_m} \Im_{m|0}^{h(slab)}]^{-1} S_0^{+h(1)}(\mathbf{k}_\perp) = \\ \begin{bmatrix} \Re_{m|m+1}^{h} \\ 1 \end{bmatrix} [\Im_{m|N}^{h(slab)}]^{-1} S_N^{-h(1)}(\mathbf{k}_\perp) e^{jk_{zN} d_{N-1}} - \boldsymbol{\Theta}_m^{p}(\mathbf{k}_\perp, \mathbf{k}_\perp^{i}). \quad (5.36)$$

In order to solve the system (36), we pre-multiply both sides by the vector $\mathbf{g} = [1 \quad -\Re^{h}_{m|m+1}(k_\perp)]$:

$$\overleftrightarrow{M}^{h}_{m}(k_\perp)\,[e^{jk_{zm}\Delta_m}\Im^{h(slab)}_{m|0}(k_\perp)]^{-1}S^{+h(1)}_{0}(\mathbf{k}_\perp) = -\mathbf{g}\cdot\mathbf{\Theta}^{p}_{m}(\mathbf{k}_\perp,\mathbf{k}^{i}_\perp), \tag{5.37}$$

where $\overleftrightarrow{M}^{p}_{m}$ is defined by (3.46). Therefore, taking into account (31), Eq. (37) can be solved in terms of the *unknown expansion coefficient* $S^{+h(1)}_{0}$ of the (q=h) horizontal polarized scattered wave propagating upward:

$$\begin{aligned} S^{+h(1)}_{0}(\mathbf{k}_\perp) &= e^{jk_{zm}\Delta_m}\Im^{h}_{m|0}(k_\perp) \\ &\{[1+\Re^{h}_{m|m+1}(k_\perp)]\frac{k_0 Z_0 \mu_m}{2k_{zm}}(\hat{k}_\perp\times\hat{z})\cdot\tilde{\mathbf{J}}^{p(1)}_{Em}(\mathbf{k}_\perp,\mathbf{k}^{i}_\perp) \\ &-[1-\Re^{h}_{m|m+1}(k_\perp)]\frac{1}{2}\hat{k}_\perp\cdot\tilde{\mathbf{J}}^{p(1)}_{Hm}(\mathbf{k}_\perp,\mathbf{k}^{i}_\perp)\}, \end{aligned} \tag{5.38}$$

where the relation (3.59) has been used.

The symmetry exhibited by the Maxwell's equations (*Duality Principle*) implies that, given a solution with $\mathbf{E},\mathbf{H},\mathbf{J}^{p}_{E},\mathbf{J}^{p}_{H}$, another solution can be obtained by the following replacements:

$$\mathbf{E}\to\mathbf{H} \quad \mathbf{H}\to-\mathbf{E} \quad \mathbf{J}^{p}_{E}\to-\mathbf{J}^{p}_{H} \quad \mathbf{J}^{p}_{H}\to\mathbf{J}^{p}_{E} \quad \mu\leftrightarrow\varepsilon.$$

We stress that interchanging $\varepsilon \leftrightarrow \mu$ the (*generalized*) reflection/ transmission coefficients corresponds to consider the dual coefficients for vertical polarized wave instead of horizontally polarized ones (changing all the superscript $h\to v$).

Looking at the dual problem, i.e. the vertical polarized (q=v) scattered wave propagating upward, from (38) we have:

$$\frac{1}{Z_0} S_0^{+v(1)}(\mathbf{k}_\perp) = -e^{jk_{zm}\Delta_m} \Im^v_{m|0}(k_\perp)$$

$$\{[1 + \Re^v_{m|m+1}(k_\perp)] \frac{k_0 \varepsilon_m}{2Z_0 k_{zm}} (\hat{k}_\perp \times \hat{z}) \cdot \widetilde{\mathbf{J}}^{p(1)}_{Hm}(\mathbf{k}_\perp, \mathbf{k}^i_\perp)) \quad (5.39)$$

$$+ [1 - \Re^v_{m|m+1}(k_\perp)] \frac{1}{2} \hat{k}_\perp \cdot \widetilde{\mathbf{J}}^{p(1)}_{Em}(\mathbf{k}_\perp, \mathbf{k}^i_\perp)\}.$$

Thus far, in order to exploit the *symmetry* of the Maxwell equations we have considered expressions that explicitly take into account the dependence from both magnetic permeability and electric permittivity. From here we focus our attention on media whose relative magnetic permeability is unitary (i.e. nonmagnetic media). This assumption is reasonable in the majority of cases of interest.

At this point, substitution of the unperturbed field jumps (see Section 3.8) in the equivalent current expression (27)-(28), and their use in (38)-(39) provide the final expressions of the upward-scattered-field expansion coefficients.

In particular, for *horizontally polarized* incident field (p=h), substituting (3.93)-(3.96) in (27)-(28), we get:

$$\widetilde{\mathbf{J}}^{h(1)}_{Hm} = 0, \quad (5.40)$$

$$\widetilde{\mathbf{J}}^{h(1)}_{Em} = j\frac{k_0}{Z_0}(\varepsilon_{m+1} - \varepsilon_m)(\hat{k}^i_\perp \times \hat{z})\widetilde{\zeta}_m(\mathbf{k}_\perp - \mathbf{k}^i_\perp)\Im^h_{0|m}(k^i_\perp)e^{jk^i_{zm}\Delta_m}[1 + \Re^h_{m|m+1}(k^i_\perp)]. \quad (5.41)$$

For *vertically polarized* incident field (p=v) substituting (3.97)-(3.100) in (27)-(28), we get:

$$\widetilde{\mathbf{J}}^{v(1)}_{Em} = \frac{j(\varepsilon_{m+1} - \varepsilon_m)}{Z_0 \varepsilon_m} k^i_{zm} \hat{k}^i_\perp \widetilde{\zeta}_m(\mathbf{k}_\perp - \mathbf{k}^i_\perp)\Im^v_{0|m}(k^i_\perp)e^{jk^i_{zm}\Delta_m}[1 - \Re^v_{m|m+1}(k^i_\perp)], \quad 5.42)$$

$$\widetilde{\mathbf{J}}^{v(1)}_{Hm} = -\frac{j(\varepsilon_{m+1} - \varepsilon_m)}{k_0 \varepsilon_{m+1} \varepsilon_m} k^i_\perp k_\perp (\hat{k}_\perp \times \hat{z}) \quad (5.43)$$

$$\widetilde{\zeta}_m(\mathbf{k}_\perp - \mathbf{k}^i_\perp)\Im^v_{0|m}(k^i_\perp)e^{jk^i_{zm}\Delta_m}[1 + \Re^v_{m|m+1}(k^i_\perp)].$$

Therefore, taking into account that

$$(\hat{k}_\perp \times \hat{z}) \cdot (\hat{k}_\perp^i \times \hat{z}) = (\hat{k}_\perp \cdot \hat{k}_\perp^i),$$
$$\hat{k}_\perp \cdot (\hat{k}_\perp^i \times \hat{z}) = -\hat{z} \cdot (\hat{k}_\perp^i \times \hat{k}_\perp), \tag{5.44}$$

and using the equivalent current density expressions (40)-(41), the expansion coefficient of q-polarized scattered wave propagating upward (38)-(39) can be rewritten as follows:

$$\begin{aligned} S_0^{+h(1)}(\mathbf{k}_\perp) = & \frac{jk_0^2}{2k_{zm}}(\varepsilon_{m+1} - \varepsilon_m)(\hat{k}_\perp \cdot \hat{k}_\perp^i)\tilde{\zeta}_m(\mathbf{k}_\perp - \mathbf{k}_\perp^i) \\ & e^{jk_{zm}\Delta_m} \Im^h_{m|0}(k_\perp)\,[1 + \Re^h_{m|m+1}(k_\perp)] \\ & e^{jk^i_{zm}\Delta_m} \Im^h_{0|m}(k_\perp^i)[1 + \Re^h_{m|m+1}(k_\perp^i)], \end{aligned} \tag{5.45}$$

$$\begin{aligned} S_0^{+v(1)}(\mathbf{k}_\perp) = & \frac{jk_0}{2}(\varepsilon_{m+1} - \varepsilon_m)\hat{z} \cdot (\hat{k}_\perp^i \times \hat{k}_\perp)\tilde{\zeta}_m(\mathbf{k}_\perp - \mathbf{k}_\perp^i) \\ & e^{jk_{zm}\Delta_m} \Im^v_{m|0}(k_\perp)[1 - \Re^v_{m|m+1}(k_\perp)] \\ & e^{jk^i_{zm}\Delta_m} \Im^h_{0|m}(k_\perp^i)[1 + \Re^h_{m|m+1}(k_\perp^i)]. \end{aligned} \tag{5.46}$$

Similarly, substituting eqs. (42)-(43) in (38)-(39), we get:

$$\begin{aligned} S_0^{+h(1)}(\mathbf{k}_\perp) = & \frac{jk_0 k^i_{zm}}{2k_{zm}\varepsilon_m}(\varepsilon_{m+1} - \varepsilon_m)\hat{z} \cdot (\hat{k}_\perp^i \times \hat{k}_\perp)\,\tilde{\zeta}_m(\mathbf{k}_\perp - \mathbf{k}_\perp^i) \\ & e^{jk_{zm}\Delta_m} \Im^h_{m|0}(k_\perp)[1 + \Re^h_{m|m+1}(k_\perp)] \\ & e^{jk^i_{zm}\Delta_m} \Im^v_{0|m}(k_\perp^i)\,[1 - \Re^v_{m|m+1}(k_\perp^i)], \end{aligned} \tag{5.47}$$

$$\begin{aligned} S_0^{+v(1)}(\mathbf{k}_\perp) = & \frac{j}{2k_{zm}\varepsilon_m}(\varepsilon_{m+1} - \varepsilon_m)\tilde{\zeta}_m(\mathbf{k}_\perp - \mathbf{k}_\perp^i) \\ & e^{jk_{zm}\Delta_m} \Im^v_{m|0}(k_\perp)\, e^{jk^i_{zm}\Delta_m} \Im^v_{0|m}(k_\perp^i) \\ & \{[1 + \Re^v_{m|m+1}(k_\perp)]\,[1 + \Re^v_{m|m+1}(k_\perp^i)]\frac{\varepsilon_m}{\varepsilon_{m+1}} k_\perp^i k_\perp \\ & - [1 - \Re^v_{m|m+1}(k_\perp)]\,[1 - \Re^v_{m|m+1}(k_\perp^i)] k^i_{zm} k_{zm} (\hat{k}_\perp \cdot \hat{k}_\perp^i)\}. \end{aligned} \tag{5.48}$$

5.5.2 Wave scattered downward in the lower half-space

This subsection is devoted to the formal evaluation in closed-form of the unknown scattering coefficients $S_N^{-q(1)}$ of the perturbative expansion by exploiting the matrix formulation of the *non uniform* boundary conditions.

Accordingly, the system of equations (30)-(32) is resolved in terms of the unknown expansion coefficient $S_N^{-q(1)}$ of the horizontal polarized (q=h) scattered wave propagating downward into the lower half-space (N-th layer). This is done by using a recursive approach involving the concept of *generalized transmission/reflection* coefficients.

Similarly as done in Section 5.5.1, by enforcing the condition (32) on all the interfaces for which $j \geq m+1$, the associated equations system resolved recursively with $S_N^{+q(1)} = 0$ leads to (see (3.72)):

$$\mathbf{S}_{m+1}^{q(1)}(\mathbf{k}_\perp, d_{m+1}) = \begin{bmatrix} \Re^q_{m+1|m+2} \\ 1 \end{bmatrix} [\Im^{q(slab)}_{m+1|N}]^{-1} S_N^{-q(1)} e^{jk_{zN}d_{N-1}}, \qquad (5.49)$$

where the *generalized transmission coefficients* $\Im^{q(slab)}_{m+1|N}(k_\perp)$ in downward direction for the layered slab between two half-spaces (m+1,N) are defined by (3.56).

On the other hand, enforcing the condition (32) on all the j-th interfaces, with $0 \leq j < m$, the associated equation system resolved recursively with $S_0^{-q(1)} = 0$ leads to (see (3.78)):

$$\mathbf{S}_{m}^{q(1)}(\mathbf{k}_\perp, d_{m-1}) = \begin{bmatrix} 1 \\ \Re^q_{m|m-1} \end{bmatrix} [\Im^{q(slab)}_{m|0}]^{-1} S_0^{+q(1)}(\mathbf{k}_\perp), \qquad (5.50)$$

where the *generalized transmission coefficients* $\Im^{q(slab)}_{m|0}(k_\perp)$ in upward direction for the layered slab between two half-spaces (m,0) are defined by (3.58). It is now useful to note that Eq.(30) can be equivalently rewritten as:

$$\boldsymbol{\Pi}_m^{-1}\mathbf{S}_m^{h(1)}(\mathbf{k}_\perp,d_{m-1})+\mathbf{\Theta}_m^p=\boldsymbol{N}_{m|m+1}^h(k_\perp)\boldsymbol{\Pi}_{m+1}(k_\perp)\mathbf{S}_{m+1}^{h(1)}(\mathbf{k}_\perp,d_{m+1}). \quad (5.51)$$

where the propagation in the m-th layer is accounted for by $\boldsymbol{\Pi}_m(k_\perp)$ (see (3.35)). By substituting (49) and (50) with $q=h$ in (51), we can formally write

$$\begin{bmatrix} 1 \\ \Re_{m|m-1}^h e^{j2k_{zm}\Delta_m}\end{bmatrix}[e^{jk_{zm}\Delta_m}\Im_{m|0}^{h(slab)}]^{-1}S_0^{+h(1)}(\mathbf{k}_\perp)+\mathbf{\Theta}_m^p(\mathbf{k}_\perp,\mathbf{k}_\perp^i)=$$
$$\begin{bmatrix}\Re_{m|m+1}^h \\ 1\end{bmatrix}\vec{M}_{m+1}^h[\Im_{m+1|N}^{h(slab)}T_{m|m+1}^h e^{jk_{z(m+1)}\Delta_{m+1}}]^{-1}S_N^{-h(1)}(\mathbf{k}_\perp)e^{jk_{zN}d_{N-1}}, \quad (5.52)$$

where the relation (3.71) has been also taken into account.

In order to solve the system (52), we pre-multiply both sides by the vector $\mathbf{g}=[-\Re_{m|m-1}^h e^{j2k_{zm}\Delta_m};\ \ 1]$, obtaining:

$$\overleftrightarrow{M}_m^h\vec{M}_{m+1}^h[\Im_{m+1|N}^{h(slab)}T_{m|m+1}^h e^{jk_{z(m+1)}\Delta_{m+1}}]^{-1}S_N^{-h(0)}(\mathbf{k}_\perp)e^{jk_{zN}d_{N-1}}=\mathbf{g}\cdot\mathbf{\Theta}_m^p(\mathbf{k}_\perp,\mathbf{k}_\perp^i). \quad (5.53)$$

By substituting Eq. (31) into (53) and applying the identity (3.47), Eq. (53) can be solved in terms of the *unknown expansion coefficient* $S_N^{-h(0)}$ of the ($q=h$) horizontal polarized scattered wave propagating into the N-th medium:

$$\begin{aligned}S_N^{-h(1)}(\mathbf{k}_\perp)e^{jk_{zN}d_{N-1}} &= e^{jk_{z(m+1)}\Delta_{m+1}} \\ &\Im_{m+1|N}^{h(slab)}(k_\perp)[\vec{M}_{m+1}^h(k_\perp)]^{-1}[\overleftarrow{M}_m^h(k_\perp)]^{-1}T_{m|m+1}^h \\ &\{[1+\Re_{m|m-1}^h(k_\perp)e^{j2k_{zm}\Delta_m}]\frac{k_0Z_0\mu_m}{2k_{zm}}(\hat{k}_\perp\times\hat{z})\cdot\tilde{\mathbf{J}}_{Em}^{p(1)} \\ &+[1-\Re_{m|m-1}^h(k_\perp)e^{j2k_{zm}\Delta_m}]\frac{1}{2}\hat{k}_\perp\cdot\tilde{\mathbf{J}}_{Hm}^{p(1)}\}.\end{aligned} \quad (5.54)$$

By exploit the definition (3.57), using the equation (3.51), Eq. (54) can be rewritten as:

$$S_N^{-h(1)}(\mathbf{k}_\perp)e^{jk_{zN}d_{N-1}} = e^{jk_{z(m+1)}\Delta_{m+1}}\Im^h_{m+1|N}(k_\perp)$$
$$\{[1+\Re^h_{m+1|m}(k_\perp)]\frac{k_0 Z_0 \mu_{m+1}}{2k_{z(m+1)}}(\hat{k}_\perp \times \hat{z})\cdot \tilde{\mathbf{J}}^{p(1)}_{Em} \tag{5.55}$$
$$+[1-\Re^h_{m+1|m}(k_\perp)]\frac{1}{2}\hat{k}_\perp \cdot \tilde{\mathbf{J}}^{p(1)}_{Hm}\}.$$

The symmetry exhibited by the Maxwell's equations (*Duality Principle*) implies that given a solution with $\mathbf{E},\mathbf{H},\mathbf{J}^p_E,\mathbf{J}^p_H$ another solution can be obtained by the following replacements:

$$\mathbf{E}\to\mathbf{H} \quad \mathbf{H}\to-\mathbf{E} \quad \mathbf{J}^p_E\to-\mathbf{J}^p_H \quad \mathbf{J}^p_H\to\mathbf{J}^p_E \quad \mu\leftrightarrow\varepsilon.$$

It is important to stress that interchanging $\varepsilon \leftrightarrow \mu$ in the (generalized) reflection/transmission coefficients corresponds to consider the dual coefficients for vertical polarized wave instead of horizontally polarized ones (changing all the superscript $h\to v$). Looking at the dual problem, i.e., the vertical polarized ($q=v$) scattered wave propagating upward, from (55) we have:

$$\frac{1}{Z_N}S_N^{-v(1)}(\mathbf{k}_\perp)e^{jk_{zN}d_{N-1}} = -e^{jk_{z(m+1)}\Delta_{m+1}}\Im^v_{m+1|N}(k_\perp)$$
$$\{[1+\Re^v_{m+1|m}(k_\perp)]\frac{k_0 \varepsilon_{m+1}}{2Z_0 k_{z(m+1)}}(\hat{k}_\perp \times \hat{z})\cdot \tilde{\mathbf{J}}^{p(1)}_{Hm} \tag{5.56}$$
$$-[1-\Re^v_{m+1|m}(k_\perp)]\frac{1}{2}\hat{k}_\perp \cdot \tilde{\mathbf{J}}^{p(1)}_{Em}\}.$$

Thus far, in order to exploit the symmetry of the Maxwell equations we have considered expressions that explicitly take into account the dependence from both magnetic permeability and electric permittivity. From here we focus our attention on media whose relative magnetic permeability is unitary (i.e. nonmagnetic media). This assumption is reasonable in the majority of cases of interest.

At this point, substituting equivalent currents expressions into Eqs. (55)-(60), the final expression of the scattered field expansion coefficients can be obtained. For horizontally polarized incident field

(p=h), substituting the effective currents expression $\widetilde{\mathbf{J}}_{Em}^{p(1)}, \widetilde{\mathbf{J}}_{Hm}^{p(1)}$ (see (40)-(41)) with p =h in (55)-(56), we get:

$$S_N^{-h(1)}(\mathbf{k}_\perp)e^{jk_{zN}d_{N-1}} = \frac{jk_0^2}{2k_{z(m+1)}}(\varepsilon_{m+1}-\varepsilon_m)(\hat{k}_\perp \cdot \hat{k}_\perp^i)\widetilde{\zeta}_m(\mathbf{k}_\perp - \mathbf{k}_\perp^i)$$
$$e^{jk_{z(m+1)}\Delta_{m+1}}\Im_{m+1|N}^h(k_\perp)[1+\Re_{m+1|m}^h(k_\perp)] \tag{5.57}$$
$$e^{jk_{zm}^i\Delta_m}\Im_{0|m}^h(k_\perp^i)[1+\Re_{m|m+1}^h(k_\perp^i)],$$

$$S_N^{-v(1)}(\mathbf{k}_\perp)e^{jk_{zN}d_{N-1}} = \frac{jk_0}{2\sqrt{\varepsilon_N}}(\varepsilon_{m+1}-\varepsilon_m)\hat{z}\cdot(\hat{k}_\perp \times \hat{k}_\perp^i)\widetilde{\zeta}_m(\mathbf{k}_\perp - \mathbf{k}_\perp^i)$$
$$e^{jk_{z(m+1)}\Delta_{m+1}}\Im_{m+1|N}^v(k_\perp)[1-\Re_{m+1|m}^v(k_\perp)] \tag{5.58}$$
$$e^{jk_{zm}^i\Delta_m}\Im_{0|m}^h(k_\perp^i)[1+\Re_{m|m+1}^h(k_\perp^i)].$$

Similarly, regarding the vertically polarized incident field (p=v), substituting the effective currents expression $\widetilde{\mathbf{J}}_{Em}^{p(1)}, \widetilde{\mathbf{J}}_{Hm}^{p(1)}$ with $p = v$ (see (42)-(43)) in the (55)-(56), we respectively get:

$$S_N^{-h(1)}(\mathbf{k}_\perp)e^{jk_{zN}d_{N-1}} = \frac{jk_0k_{zm}^i\sqrt{\varepsilon_0}}{2\varepsilon_m k_{z(m+1)}}(\varepsilon_{m+1}-\varepsilon_m)\hat{z}\cdot(\hat{k}_\perp^i \times \hat{k}_\perp)\widetilde{\zeta}_m(\mathbf{k}_\perp - \mathbf{k}_\perp^i)$$
$$e^{jk_{z(m+1)}\Delta_{m+1}}\Im_{m+1|N}^h(k_\perp)[1+\Re_{m+1|m}^h(k_\perp)] \tag{5.59}$$
$$e^{jk_{zm}^i\Delta_m}\Im_{0|m}^v(k_\perp^i)[1-\Re_{m|m+1}^v(k_\perp^i)],$$

$$S_N^{-v(1)}(\mathbf{k}_\perp)e^{jk_{zN}d_{N-1}} = j(\varepsilon_{m+1}-\varepsilon_m)\frac{\sqrt{\varepsilon_0}}{2\varepsilon_m\sqrt{\varepsilon_N}}\widetilde{\zeta}_m(\mathbf{k}_\perp - \mathbf{k}_\perp^i)$$
$$e^{jk_{z(m+1)}\Delta_{m+1}}\Im_{m+1|N}^v(k_\perp)\,\Im_{0|m}^v(k_\perp^i)e^{jk_{zm}^i\Delta_m}$$
$$\{[1+\Re_{m+1|m}^v(k_\perp)][1+\Re_{m|m+1}^v(k_\perp^i)]\frac{k_\perp^i k_\perp}{k_{z(m+1)}}+ \tag{5.60}$$
$$[1-\Re_{m+1|m}^v(k_\perp)][1-\Re_{m|m+1}^v(k_\perp^i)](\hat{k}_\perp \cdot \hat{k}_\perp^i)k_{zm}^i\}.$$

It should be noted that the identities (44) have been exploited in the derivation of (57)-(60).

5.6 BPT closed-form solutions

The aim of this section is to present the relevant BPT solutions for the scattering from and through the 3-D layered rough structure pictured schematically in Fig.1. We underline that the corresponding first-order solutions refer to two complementary *bistatic* configuration: in the first case, both the transmitter and the receiver are into the same half-space, whereas, in the second case, each one is located in a different half-space.

5.6.1 Scattering from layered structure with an arbitrary number of rough interfaces

First, we consider the case of one rough interface embedded in the layered structure. The field scattered upward in the upper half-space in the first-order limit can be written in the form (see (19)-(21)):

$$\mathbf{E}_0^{(1)}(\mathbf{r}) = \sum_{q=h,v} \iint d\,\mathbf{k}_\perp e^{j\mathbf{k}_\perp\cdot\mathbf{r}_\perp} \hat{q}_0^+(\mathbf{k}_\perp) S_0^{+q(1)}(\mathbf{k}_\perp) e^{jk_{z0}z} \,. \tag{5.61}$$

By employing the *method of stationary phase* [36], we evaluate the integral (61) in the *far field* zone, obtaining:

$$\mathbf{E}_0^{(1)}(\mathbf{r}) \cdot \hat{q}_0^+(\mathbf{k}_\perp^s) \cong -j2\pi\, k_0 \cos\theta_0^s \frac{e^{j\,k_0\,r}}{r} S_0^{+q(1)}(\mathbf{k}_\perp^s)\,, \tag{5.62}$$

with $q \in \{v,\ h\}$ is the polarization of the scattered field. Taking into account the expressions for the unknowns expansion coefficients $S_0^{+q(1)}(\mathbf{k}_\perp^s)$ given by (45)-(48), we get

$$\mathbf{E}_0^{(1)}(\mathbf{r}) \cdot \hat{q}_0^+(\mathbf{k}_\perp^s) = \pi\, k_0^2 \frac{e^{jk_0 r}}{r} \widetilde{\alpha}_{qp}^{m,m+1}(\mathbf{k}^s, \mathbf{k}^i)\, \widetilde{\zeta}_m(\mathbf{k}_\perp^s - \mathbf{k}_\perp^i)\,, \tag{5.63}$$

wherein

$$\begin{aligned}\tilde{\alpha}_{hh}^{m,m+1} &= (\varepsilon_{m+1} - \varepsilon_m)\frac{k_{z0}^s}{k_{zm}^s}(\hat{k}_\perp^s \cdot \hat{k}_\perp^i) \\ &\quad e^{jk_{zm}^s \Delta_m} \Im_{m|0}^h(k_\perp^s)[1 + \Re_{m|m+1}^h(k_\perp^s)] \\ &\quad e^{jk_{zm}^i \Delta_m} \Im_{0|m}^h(k_\perp^i)[1 + \Re_{m|m+1}^h(k_\perp^i)],\end{aligned} \tag{5.64}$$

$$\begin{aligned}\tilde{\alpha}_{vh}^{m,m+1} &= (\varepsilon_{m+1} - \varepsilon_m)\frac{k_{z0}^s}{k_0}\hat{z} \cdot (\hat{k}_\perp^i \times \hat{k}_\perp^s) \\ &\quad e^{jk_{zm}^s \Delta_m} \Im_{m|0}^v(k_\perp^s)[1 - \Re_{m|m+1}^v(k_\perp^s)] \\ &\quad e^{jk_{zm}^i \Delta_m} \Im_{0|m}^h(k_\perp^i)[1 + \Re_{m|m+1}^h(k_\perp^i)],\end{aligned} \tag{5.65}$$

$$\begin{aligned}\tilde{\alpha}_{hv}^{m,m+1} &= (\varepsilon_{m+1} - \varepsilon_m)\frac{k_{z0}^s k_{zm}^i}{k_0 k_{zm}^s \varepsilon_m}\hat{z} \cdot (\hat{k}_\perp^i \times \hat{k}_\perp^s) \\ &\quad e^{jk_{zm}^s \Delta_m} \Im_{m|0}^h(k_\perp^s)[1 + \Re_{m|m+1}^h(k_\perp^s)] \\ &\quad e^{jk_{zm}^i \Delta_m} \Im_{0|m}^v(k_\perp^i)[1 - \Re_{m|m+1}^v(k_\perp^i)],\end{aligned} \tag{5.66}$$

$$\begin{aligned}\tilde{\alpha}_{vv}^{m,m+1} &= (\varepsilon_{m+1} - \varepsilon_m)\frac{k_{z0}^s}{k_0^2 k_{zm}^s \varepsilon_m} e^{jk_{zm}^s \Delta_m} \Im_{m|0}^v(k_\perp^s) e^{jk_{zm}^i \Delta_m} \Im_{0|m}^v(k_\perp^i) \\ &\quad \{[1 + \Re_{m|m+1}^v(k_\perp^s)][1 + \Re_{m|m+1}^v(k_\perp^i)]\frac{\varepsilon_m}{\varepsilon_{m+1}} k_\perp^i k_\perp^s \\ &\quad - [1 - \Re_{m|m+1}^v(k_\perp^s)][1 - \Re_{m|m+1}^v(k_\perp^i)] k_{zm}^s k_{zm}^i (\hat{k}_\perp^s \cdot \hat{k}_\perp^i)\},\end{aligned} \tag{5.67}$$

where $k_{zm}^i = k_{zm}(k_\perp^i)$, $k_{zm}^s = k_{zm}(k_\perp^s)$; $\Im_{0|m}^p$ and $\Im_{m|0}^p$ are, respectively, the generalized transmission coefficients in *downward* direction (3.53) and the generalized transmission coefficients in *upward* direction (3.59), and $\Re_{m|m+1}^p$ are the generalized reflection coefficients defined by (3.42).

The coefficients $\tilde{\alpha}_{qp}^{m,m+1}$ are relative to the p-polarized incident wave impinging on the structure from upper half space 0 and to the q-polarized scattering contribution from structure into the upper half space, originated from the rough interface between the layers m and m+1.

Finally, we emphasize that the total scattering from the N-rough interfaces layered structure can be straightforwardly obtained, in the first-order approximation, by superposition of the different contributions pertaining each rough interface:

$$\mathbf{E}_0^{(1)}(\mathbf{r}) \cdot \hat{q}_0^+(\mathbf{k}_\perp^s) = \pi k_0^2 \frac{e^{jk_0 r}}{r} \sum_{m=0}^{N-1} \widetilde{\alpha}_{qp}^{m,m+1}(\mathbf{k}^s, \mathbf{k}^i)\, \widetilde{\zeta}_m(\mathbf{k}_\perp^s - \mathbf{k}_\perp^i) \,. \quad (5.68)$$

5.6.2 Transmission Through layered structure with an arbitrary number of rough interfaces

Similarly, when one rough interface embedded in the layered structure is concerned, the field scattered into the last half-space, through the 3-D layered structure, in the first-order limit can be then written in the form:

$$\mathbf{E}_N^{(1)}(\mathbf{r}) = \sum_{q=h,v} \iint d\mathbf{k}_\perp e^{j\mathbf{k}_\perp \cdot \mathbf{r}_\perp} \hat{q}_N^-(\mathbf{k}_\perp) S_N^{-q(1)}(\mathbf{k}_\perp) e^{-jk_{zN} z} \,. \quad (5.69)$$

In order to evaluate the integral (69) in *far field* zone, we firstly consider a suitable change of variable $\mathbf{r}' = (\mathbf{r}_\perp, z')$, with $z' = -z - d_{N-1}$:

$$\mathbf{E}_N^{(1)}(\mathbf{r}') = \sum_{q=h,v} \iint d\mathbf{k}_\perp e^{j\mathbf{k}_\perp \cdot \mathbf{r}_\perp} \hat{q}_N^-(\mathbf{k}_\perp) S_N^{-q(1)}(\mathbf{k}_\perp) e^{jk_{zN} d_{N-1}} e^{jk_{zN} z'} \,, \quad (5.70)$$

then we use the method of *stationary phase* and obtain:

$$\mathbf{E}_N^{(1)}(\mathbf{r}') \cdot \hat{q}_N^-(\mathbf{k}_\perp^s) \cong -j2\pi k_N \cos\theta_N^s \frac{e^{jk_N r'}}{r'} S_N^{-q(1)}(\mathbf{k}_\perp^s) e^{jk_{zN}^s d_{N-1}} \,, \quad (5.71)$$

with $q \in \{v, h\}$. Taking into account the expressions for the unknowns expansion coefficients $S_N^{-q(1)}(\mathbf{k}_\perp^s)$ (see (57)-(60)), we get

$$\mathbf{E}_N^{(1)}(\mathbf{r}')\cdot\hat{q}_N^-(\mathbf{k}_\perp^s) = \pi\, k_0^2\,\frac{e^{jk_N r'}}{r'}\; {}_N^0\widetilde{\beta}_{qp}^{m,m+1}(\mathbf{k}^s,\mathbf{k}^i)\,\widetilde{\zeta}_m(\mathbf{k}_\perp^s-\mathbf{k}_\perp^i)\,, \tag{5.72}$$

wherein

$$\begin{aligned}{}_N^0\widetilde{\beta}_{hh}^{m,m+1} = &\left(\varepsilon_{m+1}-\varepsilon_m\right)\frac{k_{zN}^s}{k_{z(m+1)}^s}(\hat{k}_\perp^s\cdot\hat{k}_\perp^i)\\ &[1+\Re_{m+1|m}^h(k_\perp^s)]e^{jk_{z(m+1)}^s\Delta_{m+1}}\Im_{m+1|N}^h(k_\perp^s)\\ &[1+\Re_{m|m+1}^h(k_\perp^i)]e^{jk_{zm}^i\Delta_m}\Im_{0|m}^h(k_\perp^i),\end{aligned} \tag{5.73}$$

$$\begin{aligned}{}_N^0\widetilde{\beta}_{vh}^{m,m+1} = &(\varepsilon_{m+1}-\varepsilon_m)\frac{k_{zN}^s}{k_0\sqrt{\varepsilon_N}}\hat{z}\cdot(\hat{k}_\perp^i\times\hat{k}_\perp^s)\\ &[1-\Re_{m+1|m}^v(k_\perp^s)]e^{jk_{z(m+1)}^s\Delta_{m+1}}\Im_{m+1|N}^v(k_\perp^s)\\ &[1+\Re_{m|m+1}^h(k_\perp^i)]e^{jk_{zm}^i\Delta_m}\Im_{0|m}^h(k_\perp^i),\end{aligned} \tag{5.74}$$

$$\begin{aligned}{}_N^0\widetilde{\beta}_{hv}^{m,m+1} = &(\varepsilon_{m+1}-\varepsilon_m)\frac{k_{zN}^s k_{zm}^i\sqrt{\varepsilon_0}}{k_{z(m+1)}^s k_0\varepsilon_m}\hat{z}\cdot(\hat{k}_\perp^i\times\hat{k}_\perp^s)\\ &[1+\Re_{m+1|m}^h(k_\perp^s)]e^{jk_{z(m+1)}^s\Delta_{m+1}}\Im_{m+1|N}^h(k_\perp^s)\\ &[1-\Re_{m|m+1}^v(k_\perp^i)]e^{jk_{zm}^i\Delta_m}\Im_{0|m}^v(k_\perp^i),\end{aligned} \tag{5.75}$$

$$\begin{aligned}{}_N^0\widetilde{\beta}_{vv}^{m,m+1} = &(\varepsilon_{m+1}-\varepsilon_m)\frac{k_{zN}^s\sqrt{\varepsilon_0}}{k_{z(m+1)}^s k_0^2\varepsilon_m\sqrt{\varepsilon_N}}\\ &e^{jk_{z(m+1)}^s\Delta_{m+1}}\Im_{m+1|N}^v(k_\perp^s)\Im_{0|m}^v(k_\perp^i)e^{jk_{zm}^i\Delta_m}\\ &\Big\{[1+\Re_{m+1|m}^v(k_\perp^s)][1+\Re_{m|m+1}^v(k_\perp^i)]k_\perp^i k_\perp^s\\ &+[1-\Re_{m+1|m}^v(k_\perp^s)][1-\Re_{m|m+1}^v(k_\perp^i)](\hat{k}_\perp^s\cdot\hat{k}_\perp^i)k_{zm}^i k_{z(m+1)}^s\Big\}.\end{aligned} \tag{5.76}$$

where $k^i_{zm} = k_{zm}(k^i_\perp)$, $k^s_{zm} = k_{zm}(k^s_\perp)$; $\Im^p_{0|m}$ and $\Im^p_{m+1|N}$ are, respectively, the generalized transmission coefficients in *downward* direction and the generalized transmission coefficients in *downward* direction given, respectively, by (3.53) and (3.57), and $\Re^p_{m|m+1}$ are the generalized reflection coefficients (see (3.42)).

The coefficients ${}^0_N\tilde{\beta}^{m,m+1}_{qp}$ are relative to the *p*-polarized incident wave impinging on the structure from half-space 0 and to *q*-polarized scattering contribution, originated from the rough interface between the layers *m* and *m*+1, through the structure into last half-space *N*.

Finally, we emphasize that the total scattering through the *N*-rough interfaces layered structure can be straightforwardly obtained, in the first-order approximation, by superposition of the different contributions pertaining each rough interface:

$$\mathbf{E}^{(1)}_N(\mathbf{r}') \cdot \hat{q}^-_N(\mathbf{k}^s_\perp) = \pi k_0^2 \frac{e^{jk_N r'}}{r'} \sum_{m=0}^{N-1} {}^0_N\tilde{\beta}^{m,m+1}_{qp}(\mathbf{k}^s, \mathbf{k}^i)\, \tilde{\zeta}_m(\mathbf{k}^s_\perp - \mathbf{k}^i_\perp)\,. \quad (5.77)$$

As a result, the relevant final solutions (68) and (77) turn out formally identical, provided that the coefficients $\tilde{\alpha}^{m,m+1}_{qp}$ are replaced with the complementary ones ${}^0_N\tilde{\beta}^{m,m+1}_{qp}$.

Finally, it is important to better emphasize the analogy between the corresponding roles and final solutions of two scattering perturbative problems concerning the structure pictured schematically in Fig.1: the one discussed in Section 5.6.1 refers to the scattering *from*, while the other one, which is considered in this Section, concerns the scattering *through* the same structure. In this regard, we underline that, although the respective configurations are different and the associated final closed-form solutions are complementary, both the models share the same methodological background (BPT). It should be also noted that in first case, both the transmitter and the receiver are into the same half-space, whereas, in the other case, each one is located in a different half-space.

Specifically, the first-order perturbative solution (77), to the problem of the scattering into *lower* half-space *through* rough interfaces of an arbitrarily three-dimensional layered structure, can be regarded as the counterpart of solution (68), to the problem of the scattering into the *upper* half-space *from* rough interfaces of an arbitrarily three-dimensional layered structure: the corresponding final solution turn out to be formally identical provided that the coefficients $\widetilde{\alpha}_{qp}^{n,n+1}$ are replaced with the complementary ones ${}_{N}^{0}\widetilde{\beta}_{qp}^{m,m+1}$.

5.7 Reciprocal character of the BPT solutions

In this section, the emphasis is placed on the reciprocal character of the final BPT scattering solutions, which evidently constitutes a crucial point in the formal framework of the BPT.

Generally speaking, the *reciprocity principle* is a statement that expresses some form of symmetry in the laws governing a physical system. Analytically speaking, both the BPT final solutions (68) and (77), respectively, from and through the layered structure with *N*-rough interfaces can be expressed in a common formal frame exhibiting a symmetric nature:

$$\widetilde{\alpha}_{qp}^{m,m+1}(\mathbf{k}^s,\mathbf{k}^i) = \widetilde{\alpha}_{pq}^{m,m+1}(-\mathbf{k}^i,-\mathbf{k}^s), \tag{5.78}$$

$${}_{N}^{0}\widetilde{\beta}_{qp}^{m,m+1}(\mathbf{k}^s,\mathbf{k}^i) = -\,{}_{0}^{N}\widetilde{\beta}_{pq}^{m+1,m}(-\mathbf{k}^i,-\mathbf{k}^s). \tag{5.79}$$

These formal relations are not only a mere matter of aesthetic; in fact their symmetry inherently reflects the conformity with the reciprocity principle of the electromagnetic theory. We emphasize that the relations (78) and (79) imply that the wave amplitude for the scattering process $\mathbf{k}^i \to \mathbf{k}^s$ equals that of reciprocal scattering process $-\mathbf{k}^s \to -\mathbf{k}^i$.

Therefore, (78) and (79) are also *reciprocity* relationships for the scattering, respectively, from and through a layered structure with an (*m*-th) embedded rough interface.

This is to say that for the presented scattering solutions the role of the source and the receiver can be exchanged (see Fig.2), in conformity with the reciprocity principle of the electromagnetic theory.

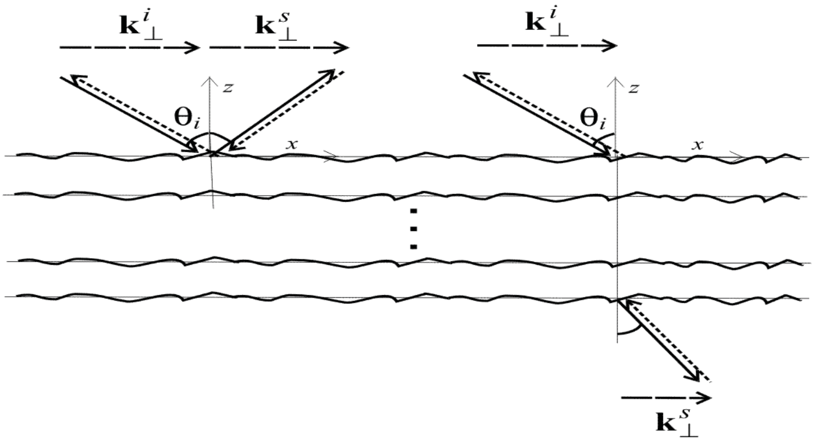


Fig. 2. Reciprocity for scattering *from* and *through* a layered structure with rough interfaces.

It should be noted that when the N-rough interfaces structure is concerned the properties (78)-(79) are satisfied as well, since the solutions in first-order limit are obtainable by superposition of the contribution of each (m-th) rough interface.

In order to provide general demonstration of these fundamental relationships, we found a more compact expression for (64)-(67) and (73)-(76), respectively. First, we introduce the following suitable notation:

$$\xi_{0\to m}^{\pm p}(k_\perp) = \Im_{0|m}^{p}(k_\perp)\, e^{jk_{zm}\Delta_m}[1 \pm \Re_{m|m+1}^{p}(k_\perp)], \tag{5.80}$$

$$\xi_{N\to m+1}^{\pm p}(k_\perp) = \Im_{N|m+1}^{p}(k_\perp)\, e^{jk_{z(m+1)}\Delta_{m+1}}[1 \pm \Re_{m+1|m}^{p}(k_\perp)]. \tag{5.81}$$

Next, when the solution for the scattering from the layered structure with an embedded rough interface is concerned, substituting relations (3.63) into (64)-(67), we obtain the alternative and more compact expressions for the relevant solution:

$$\tilde{\alpha}_{vv}^{m,m+1}(\mathbf{k}^s,\mathbf{k}^i) = \frac{\varepsilon_{m+1}-\varepsilon_m}{(k_0\varepsilon_m)^2}\left[\frac{\varepsilon_m}{\varepsilon_{m+1}} k_\perp^s \xi_{0\to m}^{+v}(k_\perp^s) k_\perp^i \xi_{0\to m}^{+v}(k_\perp^i) - (\hat{k}_\perp^s \cdot \hat{k}_\perp^i) k_{zm}^s \xi_{0\to m}^{-v}(k_\perp^s) k_{zm}^i \xi_{0\to m}^{-v}(k_\perp^i)\right], \quad (5.82)$$

$$\tilde{\alpha}_{vh}^{m,m+1}(\mathbf{k}^s,\mathbf{k}^i) = (\varepsilon_{m+1}-\varepsilon_m)\hat{z}\cdot(\hat{k}_\perp^i \times \hat{k}_\perp^s)\frac{k_{zm}^s}{k_0\varepsilon_m}\xi_{0\to m}^{-v}(k_\perp^s)\xi_{0\to m}^{+h}(k_\perp^i), \quad (5.83)$$

$$\tilde{\alpha}_{hv}^{m,m+1}(\mathbf{k}^s,\mathbf{k}^i) = (\varepsilon_{m+1}-\varepsilon_m)\hat{z}\cdot(\hat{k}_\perp^i \times \hat{k}_\perp^s)\xi_{0\to m}^{+h}(k_\perp^s)\frac{k_{zm}^i}{k_0\varepsilon_m}\xi_{0\to m}^{-v}(k_\perp^i), \quad (5.84)$$

$$\tilde{\alpha}_{hh}^{m,m+1}(\mathbf{k}^s,\mathbf{k}^i) = (\varepsilon_{m+1}-\varepsilon_m)\hat{k}_\perp^s \cdot \hat{k}_\perp^i \xi_{0\to m}^{+h}(k_\perp^s)\xi_{0\to m}^{+h}(k_\perp^i). \quad (5.85)$$

Then, by direct inspection of (82)-(85) we ultimately find Eq. (78).

On the other hand, when the solution for the scattering through the layered structure with an embedded rough interface is concerned, we proceed similarly as done previously. Substituting relations (3.67) into (73)-(76), we obtain the alternative and more compact expressions for the relevant solution:

$${}_N^0\tilde{\beta}_{vv}^{m,m+1}(\mathbf{k}_\perp^s,\mathbf{k}_\perp^i) = (\varepsilon_{m+1}-\varepsilon_m)\frac{\sqrt{\varepsilon_0\varepsilon_N}}{k_0^2\varepsilon_{m+1}\varepsilon_m} \left\{k_\perp^s k_\perp^i \xi_{N\to m+1}^{+v}(k_\perp^s)\xi_{0\to m}^{+v}(k_\perp^i) + (\hat{k}_\perp^s\cdot\hat{k}_\perp^i) k_{z(m+1)}^s k_{zm}^i \xi_{N\to m+1}^{-v}(k_\perp^s)\xi_{0\to m}^{-v}(k_\perp^i)\right\}, \quad (5.86)$$

$${}_N^0\tilde{\beta}_{vh}^{m,m+1}(\mathbf{k}^s,\mathbf{k}^i) = (\varepsilon_{m+1}-\varepsilon_m)\hat{z}\cdot(\hat{k}_\perp^s\times\hat{k}_\perp^i)\frac{\sqrt{\varepsilon_N}k_{z(m+1)}^s}{k_0\varepsilon_{m+1}}\xi_{N\to m+1}^{-v}(k_\perp^s)\xi_{0\to m}^{+h}(k_\perp^i), \quad (5.87)$$

$${}_N^0\tilde{\beta}_{hv}^{m,m+1}(\mathbf{k}^s,\mathbf{k}^i) = (\varepsilon_{m+1}-\varepsilon_m)\hat{z}\cdot(\hat{k}_\perp^i\times\hat{k}_\perp^s)\frac{\sqrt{\varepsilon_0}k_{zm}^i}{k_0\varepsilon_m}\xi_{N\to m+1}^{+h}(k_\perp^s)\xi_{0\to m}^{-v}(k_\perp^i), \quad (5.88)$$

$${}_N^0\tilde{\beta}_{hh}^{m,m+1}(\mathbf{k}^s,\mathbf{k}^i) = (\varepsilon_{m+1}-\varepsilon_m)(\hat{k}_\perp^s\cdot\hat{k}_\perp^i)\xi_{N\to m+1}^{+h}(k_\perp^s)\xi_{0\to m}^{+h}(k_\perp^i). \quad (5.89)$$

Then, by direct inspection of (86)-(89) we ultimately find Eq. (79). This is to say that BPT formalism satisfies reciprocity.

5.8 Bi-static scattering cross sections

In this section, we calculate the bi-static scattering cross sections of the layered structure arising from the *BPT* solutions, which have been derived in the first-order approximation in the previous sections. The estimate of the mean power density can be obtained by averaging over an ensemble of statistically identical interfaces.

5.8.1 Scattering Cross Section of an arbitrary layered structure with an embedded rough interface

We focus on the scattering property of a single rough interface embedded in the layered structure. According to as discussed in Section 4.6, the bi-static scattering *cross section* of a generic (n-th) rough interface embedded in the layered structure can be then defined as

$$\tilde{\sigma}^0_{qp,n} = \lim_{r\to\infty} \lim_{A\to\infty} \frac{4\pi r^2}{A} < \left| \mathbf{E}_0^{(1)}(\mathbf{r}) \cdot \hat{q}_0^+(\mathbf{k}_\perp^s) \right|^2 >, \tag{5.90}$$

where < > denotes ensemble averaging, where $q \in \{v, h\}$ and $p \in \{v, h\}$ denote, respectively, the polarization of scattered field and the polarization of incident field, and where A is the illuminated surface area.

Therefore, by substituting (63) into (90) and considering that the (spatial) *power spectral density* $W_n(\boldsymbol{\kappa})$ of nth corrugated interface is defined as in (2.18), the *scattering cross section* relative to the contribution of the nth corrugated interface, according to the formalism used in this thesis, can be expressed as

$$\tilde{\sigma}^0_{qp,n} = \pi k_0^4 \left| \tilde{\alpha}_{qp}^{n,n+1}(\mathbf{k}^s, \mathbf{k}^i) \right|^2 W_n(\mathbf{k}_\perp^s - \mathbf{k}_\perp^i), \tag{5.91}$$

with $p,\ q\ \in \{v,\ h\}$ denoting, respectively, the incident and the scattered polarization states, which may stand for *horizontal* polarization (h) or *vertical* polarization (v); $\mathbf{k}_\perp^i$ and $\mathbf{k}_\perp^s$ denote the projection on horizontal plane, respectively, of the incident and scattered vector wave-number Furthermore, we stress when the backscattering case ($\hat{k}_\perp^s \times \hat{k}_\perp^i = 0$) is concerned, cross-polarized scattering coefficients (64)-(67) evaluated in the plane of incidence vanish, in full accordance with the classical first-order SPM method for a rough surface between two different media[24][25].

5.8.2 Scattering Cross Section into last half-space of an Arbitrary Layered Structure with an Embedded Rough Interface

As counterparts of the configuration considered in the last subsection, we now refer to the complementary one in which the scattering through the structure is concerned. According to as discussed in Section 4.6, the bi-static scattering cross section into last half-space of the structure with one embedded (n-th) rough interface can be defined as

$$\tilde{\sigma}_{qp,n}^{0} = \lim_{r\to\infty} \lim_{A\to\infty} \frac{4\pi r'^2}{A} < \left|\mathbf{E}_N^{(1)}(\mathbf{r}') \cdot \hat{q}_N^-(\mathbf{k}_\perp^s)\right|^2 > \mathrm{Re}\left\{\sqrt{\frac{\varepsilon_N}{\varepsilon_0}}\right\}, \tag{5.92}$$

where < > denotes ensemble averaging, where the index $q \in \{v,\ h\}$ index $p \in \{v,\ h\}$ and denote, respectively, the polarization of scattered field and the polarization of incident field, A is the surface area, and where we have considered the *Poynthing* power density of the transmitted wave in N-th region normalized to the power density of the incident wave. Therefore, by substituting (72) into (92) and considering that the (spatial) *power spectral density* $W_n(\boldsymbol{\kappa})$ of n-th corrugated interface is defined as in (2.18), as final result, we obtain:

$$\tilde{\sigma}_{qp,n}^{0} = \pi k_0^4 \left|{}_N^0\tilde{\beta}_{qp}^{n,n+1}(\mathbf{k}^s,\mathbf{k}^i)\right|^2 W_n(\mathbf{k}_\perp^s - \mathbf{k}_\perp^i)\,\mathrm{Re}\left\{\sqrt{\frac{\varepsilon_N}{\varepsilon_0}}\right\}. \tag{5.93}$$

5.8.3 Scattering Cross Section of a Layered Structure with N-rough interfaces

We now show that the solutions, given by the expressions (91) and (93) respectively, are susceptible of a straightforward generalization to the case of arbitrary stratification with *N*-rough boundaries. Taking into account the contribution of each *n*-th corrugated interface (see (68)), the global *bi-static scattering cross section* of the *N*-rough interface layered media can be expressed as:

$$\begin{aligned}\tilde{\sigma}_{qp}^{0} &= \pi k_0^4 \sum_{n=0}^{N-1} \left|\tilde{\alpha}_{qp}^{n,n+1}(\mathbf{k}^s,\mathbf{k}^i)\right|^2 W_n(\mathbf{k}_\perp^s - \mathbf{k}_\perp^i) \\ &+ \pi k_0^4 \sum_{i\neq j} \operatorname{Re}\left\{\tilde{\alpha}_{qp}^{i,i+1}(\mathbf{k}^s,\mathbf{k}^i)\left[\tilde{\alpha}_{qp}^{j,j+1}(\mathbf{k}^s,\mathbf{k}^i)\right]^*\right\} W_{ij}(\mathbf{k}_\perp^s - \mathbf{k}_\perp^i),\end{aligned} \tag{5.94}$$

with $p,\ q \in \{v,\ h\}$, where the asterisk denotes the complex conjugated, where $\tilde{\alpha}_{qp}^{i,i+1}$ are given by (82)-(85), and where the *cross power spectral density* W_{ij}, between the interfaces *i* and *j*, for the spatial frequencies of the roughness is given by (2.19).

Likewise, the solution given by the expression (93), is susceptible of a straightforward generalization to the case of arbitrary stratification with *N*-rough boundaries. Taking into account the contribution of each *n*th corrugated interface (see (77)), the global *bi-static scattering cross section* into last half-space of the *N*-rough interface layered media can be expressed as:

$$\begin{aligned}\tilde{\sigma}_{qp}^{0} &= \pi k_0^4 \operatorname{Re}\left\{\sqrt{\frac{\varepsilon_N}{\varepsilon_0}}\right\} \sum_{n=0}^{N-1} \left|{}_N^0\tilde{\beta}_{qp}^{n,n+1}(\mathbf{k}^s,\mathbf{k}^i)\right|^2 W_n(\mathbf{k}_\perp^s - \mathbf{k}_\perp^i) + \\ &\pi k_0^4 \operatorname{Re}\left\{\sqrt{\frac{\varepsilon_N}{\varepsilon_0}}\right\} \sum_{i\neq j} \operatorname{Re}\left\{{}_N^0\tilde{\beta}_{qp}^{i,i+1}(\mathbf{k}^s,\mathbf{k}^i)\left[{}_N^0\tilde{\beta}_{qp}^{j,j+1}(\mathbf{k}^s,\mathbf{k}^i)\right]^*\right\} W_{ij}(\mathbf{k}_\perp^s - \mathbf{k}_\perp^i),\end{aligned} \tag{5.95}$$

where $p,\ q \in \{v,\ h\}$, where the asterisk denotes the complex conjugated, ${}_N^0\tilde{\beta}_{qp}^{i,i+1}$ are given by (86)-(89), and where the *cross power* *spectral density* W_{ij}, between the interfaces *i* and *j*, for the spatial frequencies of the roughness is given by (2.19).

Some final considerations are now in order.

As a matter of fact, the presented closed-form solutions permit the *full-polarimetric* evaluation of the scattering for a *bi-static* configuration, *from* or *through* the layered rough structure, once the *three-dimensional* layered structure's parameters (shape of the roughness spectra, layers thickness and complex permittivities), the incident field parameters (frequency, polarization and direction) and the observation direction are been specified.

As a result, elegant closed form solutions are established, which take into account parametrically the dependence of scattering properties on structure (geometric and electromagnetic) parameters.

Therefore, BPT formulation leads to solutions which exhibit a direct functional dependence (no integral evaluation is required) and, subsequently, permit to show that the scattered field can be parametrically evaluated considering a set of parameters: some of them refer to an unperturbed structure configuration, i.e. intrinsically the physical parameters of the smooth boundary structure, and others which are determined exclusively by (random) deviations of the corrugated boundaries from their reference position. Note also that the coefficients $\widetilde{\alpha}_{qp}^{m,m+1}$ and ${}_{N}^{0}\widetilde{\beta}_{qp}^{i,i+1}$ depend parametrically on the unperturbed structure parameters only and exhibit a direct functional dependence by the generalized transmission/reflection coefficients.

Procedurally, once the *generalized reflection/transmission coefficients* are recursively evaluated, the coefficients $\widetilde{\alpha}_{qp}^{i,i+1}$ and/or ${}_{N}^{0}\widetilde{\beta}_{qp}^{i,i+1}$ can be than directly computed, so that the scattering cross sections (94) and/or (95) for the pertinent structure with rough interfaces can be finally predicted.

Furthermore, the scattering from or through the rough layered media is sensitive to the correlation between rough profiles of different interfaces. In fact, a real layered structure will have interfaces cross-correlation somewhere between two limiting situations: perfectly correlated and uncorrelated roughness. Consequently, the degree of correlation affects the phase relation between the fields scattered by each rough interface. Obviously, when the interfaces are supposed to be uncorrelated, the second terms respectively in (94) and (95) vanish and accordingly, in the first-order approximation, the total scattering from or through the structure arises from the incoherent superposition of radiation scattered from each interface. We emphasize that the effects of the interaction between the rough interfaces can limited be treated, in the first-order approximation, only when the rough interfaces exhibit some correlation.

In addition, as it will be demonstrated in the next Chapter, the proposed global solution turns out to be completely interpretable with basic physical concepts, clearly discerning the physics of the involved scattering mechanisms.

Finally, it should be noted that the method to be applied needs only the classical gently-roughness assumption, without any further approximation (see also Chapter 8).

5.9 Analytical Validation of the Models

In this section, we give an analytical demonstration of the consistency of the proposed BPT model showing that the solution (82)-(85) agrees perfectly with the existing analytical solutions [1][2][4] when the stratification geometry reduces to those simplified ones considered by the different authors.

The aim is to reconsider the state of art in an organized mathematical framework, analytically demonstrating the formal consistency of *BPT* general scattering solution, which permits to deal with layered media with an arbitrary number of rough interfaces, with the previous existing perturbative models, whose relevant first-order solutions have been widely discussed in Chapter 4. We underline that these existing models were introduced to cope with simplified layered geometry with only one (or two) rough interface, whose derivation methods belong to the class of perturbative methods.

This also permits to set the presented general BPT model in an organized formal framework, highlighting the connections with all the previously existing simplified perturbative models. Accordingly, we discuss on the formal consistency with the previous works, in the perspective of providing both an analytical validation of the BPT general solution and a unifying insight for all the existing perturbative formulations.

First of all, it is merits to be underlined that none of the pertinent configurations of these simplified models previously considered is directly applicable, for instance, to an actual remote sensing scenario. In fact, the natural stratified media are definitely constituted by corrugated interfaces, each one exhibiting a certain amount of roughness, whereas the flatness is an idealization which does not occur in natural media. More specifically, it can occur that, for a given roughness, one might consider an operational EM wavelength for which the roughness itself can be reasonably neglected. However, in principle, there is no defensible motivation, beyond the relevant limitation of the involved analytical difficulties, for considering the effect of only one interfacial roughness, neglecting the other relevant ones. This poses not only a conceptual limitation. In fact, in the applications perspective of retrieving geo-physical parameters by scattering measurements, whether there is a dominant interfacial roughness, and, in case, which the dominant one is, should be established after the remote sensing data are analyzed and, conversely, they cannot constitute *a priori* assumptions. Conversely, the inherent distinctive features make the general BPT model suitable to be applied in remote sensing applications scenario.

Now we focus our attention on the demonstration that when the general geometry reduces to each simplified one, the consistency of the relevant solutions formally holds. First of all, we observe that when the stratification above and under a rough interface vanishes BPT solution (82)-(85) reduces to the classical SPM solution for the scattering from a rough surface between two-half-spaces. Moreover, it is important to note that emphasize that no depolarization effect is expected in backscattering case, according with the first-order classical SPM theory [24][25].

In addition, it can be straightforward verified that, when the stratification above the roughness vanishes, i.e. when the (82)-(85) are specialized for the case of Fig.1 of Chapter 4 (Fuks model), the factors $\Im^{p}_{0|m}(k_\perp)\,e^{jk_{zm}\Delta_m}$ turn out to be unitary and the general BPT solution (82)-(85) formally reduces the one discussed in Section 4.2.1.

On the other hand, to perform a direct comparison with the models discussed in Section 4.2.2 and in Section 4.2.3, we specialize the BPT solution to the simplified geometry depicted in Fig.3 of Chapter 4.

Considering that $\Re^p_{1|0} = R^p_{1|0} = -R^p_{0|1}$, and

$$\Im^p_{0|1}(k_\perp) = T^p_{0|1}(k_\perp)[1 + R^p_{0|1}(k_\perp)\Re^p_{1|2}(k_\perp)e^{j2k_{z1}\Delta_1}]^{-1},$$

from (82)-(85) we obtain:

$$\tilde{\sigma}^0_{qp,1} = \pi k_0^4 \left|\tilde{\alpha}^{1,2}_{qp}(\mathbf{k}^s_\perp, \mathbf{k}^i_\perp)\right|^2 W_1(\mathbf{k}^s_\perp - \mathbf{k}^i_\perp), \tag{5.96}$$

wherein

$$\begin{aligned}\tilde{\alpha}^{1,2}_{hh} &= (\varepsilon_2 - \varepsilon_1)\,(\hat{k}^i_\perp \cdot \hat{k}^s_\perp)\\ &\quad \frac{T^h_{0|1}(k^i_\perp)e^{jk^i_{z1}\Delta_1}}{1 + R^h_{0|1}(k^i_\perp)\Re^h_{1|2}(k^i_\perp)e^{j2k^i_{z1}\Delta_1}}[1 + \Re^h_{1|2}(k^i_\perp)]\\ &\quad \frac{T^h_{0|1}(k^s_\perp)e^{jk^s_{z1}\Delta_1}}{1 + R^h_{0|1}(k^s_\perp)\Re^h_{1|2}(k^s_\perp)e^{j2k^s_{z1}\Delta_1}}[1 + \Re^h_{1|2}(k^s_\perp)],\end{aligned} \tag{5.97}$$

$$\begin{aligned}\tilde{\alpha}^{1,2}_{vh} &= (\varepsilon_2 - \varepsilon_1)\,\frac{k^s_{z1}}{k_0\varepsilon_1}\,\hat{z}\cdot(\hat{k}^i_\perp \times \hat{k}^s_\perp)\\ &\quad \frac{T^h_{0|1}(k^i_\perp)e^{jk^i_{z1}\Delta_1}}{1 + R^h_{0|1}(k^i_\perp)\Re^h_{1|2}(k^i_\perp)e^{j2k^i_{z1}\Delta_1}}[1 + \Re^h_{1|2}(k^i_\perp)]\\ &\quad \frac{T^v_{0|1}(k^s_\perp)e^{jk^s_{z1}\Delta_1}}{1 + R^v_{0|1}(k^s_\perp)\Re^v_{1|2}(k^s_\perp)e^{j2k^s_{z1}\Delta_1}}[1 - \Re^v_{1|2}(k^s_\perp)],\end{aligned} \tag{5.98}$$

$$\tilde{\alpha}_{hv}^{1,2} = (\varepsilon_2 - \varepsilon_1)\frac{k_{z1}^i}{k_0\varepsilon_1}\hat{z}\cdot(\hat{k}_\perp^i \times \hat{k}_\perp^s)$$

$$\frac{T_{0|1}^v(k_\perp^i)e^{jk_{z1}^i\Delta_1}}{1+R_{0|1}^v(k_\perp^i)\Re_{1|2}^v(k_\perp^i)e^{j2k_{z1}^i\Delta_1}}[1-\Re_{1|2}^v(k_\perp^i)] \tag{5.99}$$

$$\frac{T_{0|1}^h(k_\perp^s)e^{jk_{z1}^s\Delta_1}}{1+R_{0|1}^h(k_\perp^s)\Re_{1|2}^h(k_\perp^s)e^{j2k_{z1}^s\Delta_1}}[1+\Re_{1|2}^h(k_\perp^s)],$$

$$\tilde{\alpha}_{vv}^{1,2} = (\varepsilon_2 - \varepsilon_1)\frac{1}{k_0^2\varepsilon_1^2}$$

$$\frac{T_{0|1}^v(k_\perp^i)e^{jk_{z1}^i\Delta_1}}{1+R_{0|1}^v(k_\perp^i)\Re_{1|2}^v(k_\perp^i)e^{j2k_{z1}^i\Delta_1}}\frac{T_{0|1}^v(k_\perp^s)e^{jk_{z1}^s\Delta_1}}{1+R_{0|1}^v(k_\perp^s)\Re_{1|2}^v(k_\perp^s)e^{j2k_{z1}^s\Delta_1}} \tag{5.100}$$

$$\{[1+\Re_{1|2}^v(k_\perp^s)][1+\Re_{1|2}^v(k_\perp^i)]\frac{\varepsilon_1}{\varepsilon_2}k_\perp^i k_\perp^s$$

$$-[1-\Re_{1|2}^v(k_\perp^s)][1-\Re_{1|2}^v(k_\perp^i)]k_{z1}^i k_{z1}^s(\hat{k}_\perp^s\cdot\hat{k}_\perp^i)\}.$$

Therefore, it is simple to verify (see Chapter 4) that BPT solution specialized to the geometry of Fig. 3 of Chapter 4, i.e. (97)-(101), when evaluated in backscattering ($\mathbf{k}_\perp^s + \mathbf{k}_\perp^i = 0$), formally reduces to the equivalent *Yarovoy* solution (see Section (4.3.2)).

Similarly, when in addition the stratification below the roughness vanishes ($R_{2|3}^p = 0$, $\Re_{1|2}^p = R_{1|2}^p$), the solution (97)-(100) formally reduces to the *Sarabandi* solution (see Section (4.3.1)).

Moreover, it can be also proved the full consistency of BPT solution (77) with one existing simplified model presented in [4], which concerns the scattering *through* a specific configuration with a rough surface on top of a stratified medium: this can be easily verified analytically by particularizing the expression of the general BPT solution (77) to the simplified case considered there.

As a result, the demonstration of the full consistency of the presented solutions has been provided analytically, showing that the BPT solutions reduce to each of the existing ones when the stratification geometry reduces to each of the corresponding simplified ones considered by the other authors.

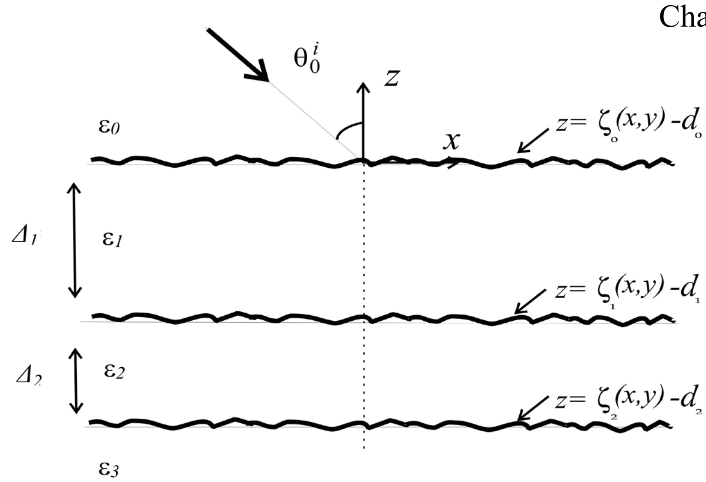


Figure 3. Three rough interfaces layered media.

Therefore, the solutions of the all existing first-order perturbative models [1], [2] and [4], which refers to a specific simplified geometry with one rough interface, can be rigorously regarded as particular cases of BPT general solution.

Analytically speaking, BPT results can be also regarded as a generalization of the classical SPM for rough surface to layered media with rough interfaces.

In conclusion, the presented analysis, which has been carried out on an analytical playground, allows us to obtain, in a unitary formal framework, a comprehensive insight into the first-order perturbation solutions formalism for scattering from stratified structure with rough interfaces, coherently highlighting the formal connections with all the previously existing simplified perturbative models.

5.10 Scattering patterns computation

In this section, we present some examples aimed at studying the bi-static scattering coefficients (94)-(95).

In order to point out the capability of the proposed BPT model, we refer to a canonical layered media with two intermediate layers and three corrugated interfaces only. This special is of interest in several applications; in addition, the evaluation of the scattering through such a structure has not been considered by other authors yet.

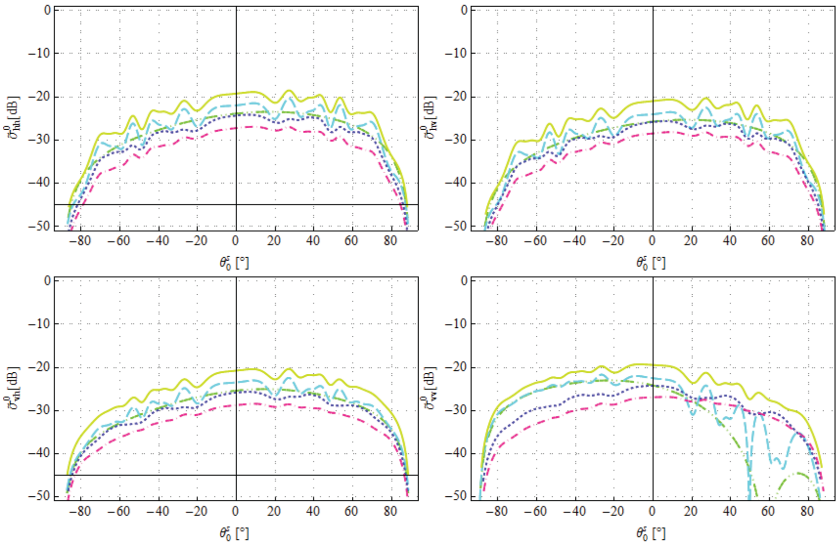


Fig.4. Bi-static scattering coefficients for three-rough-interfaces layered media: $\tilde{\sigma}^0_{qp}$ *(solid line)*, $\tilde{\sigma}^0_{qp,0}$ *(long-dashed)*, $\tilde{\sigma}^0_{qp,1}$ *(short-dashed line)*, $\tilde{\sigma}^0_{qp,2}$ *(dotted-dashed line)*, $\sigma^0_{qp,0}$ *(dot-dot-dashed line).*

In common with classical theoretical studies of the scattering of waves from random surfaces, we assume that the interfaces constitute Gaussian 2D random processes with *Gaussian correlations*, whose spectral representation is given by

$$W_n(\mathbf{\kappa}) = (\sigma_n^2 l_n^2 / 4\pi)\exp(-|\mathbf{\kappa}|^2 l_n^2 / 4) \qquad (5.101)$$

where, with regard to the n-th interface, σ_n and l_n are the surface height standard deviation and correlation length, respectively.

In the following four different cases study are considered.

5.10.1 Case Study 1

With reference to the structure depicted in Fig 3, we study the scattering patterns relevant to the bistatic scattering coefficient (94). Accordingly, we plot in Fig. 4 the bi-static scattering coefficient, for the four polarization combinations, as a function of the scattering angle θ_0^s distinguishing the three contributions $\tilde{\sigma}^0_{qp,n}$ from the correspondent considered rough interfaces (n=0,1,2). The overall scattering coefficient $\tilde{\sigma}_{qp}^{0}$ of the layered structure with three rough interfaces is also shown; we also show for comparison the contribution $\sigma_{qp,0}^{0}$ that we would have if the structure under the upper rough interface were neglected. We assume θ_0^i=45°, f=1.0 GHz and azimuthal scattering angle φ_0^s=40°. To perform a consistent comparison we refer to interfaces with the same roughness. To be more specific, we consider classical Gaussian surface height model with *Gaussian correlations* (101): we assume $k_0 l_n$=1.5, $k_0\sigma_n$=0.15 for n=0,1,2. In addition, the considered vertical profile is characterized by the following parameters: ε_0=1, ε_1=2.8+j0.001, ε_2=5.0+j0.05, ε_3=10.0+j1.0, Δ_1/λ=7.0, Δ_2/λ=5.0.

Therefore, this simple instance demonstrates the significance of taking into account the effects of the different interfaces when analyzing the response of a natural stratified structure.

5.10.2 Case Study 2

In this section, we present some numerical examples aimed at studying scattering coefficients (94). To this purpose, we consider the layered structured schematized in Fig.3, which is representative of several situations of interest, which has been parametric characterized as follows. The considered vertical profile is characterized by the following parameters: ε_0=1, ε_1=3.0+j0.0, ε_2=5.5+j0.00055, ε_3=10.5+j1.55; Δ_1/λ=1.50, Δ_2/λ=2.80.

θ_0^i	45°	$k_0\sigma_0$	0.15
Δ_1/λ	1.50	$k_0\sigma_1$	0.15
Δ_2/λ	2.80	$k_0\sigma_2$	0.15
f	1.0 GHz	$k_0 l_0$	1.5
ε_1	3.0	$k_0 l_1$	1.5
ε_2	5.5+j 0.00055	$k_0 l_2$	1.5
ε_3	10.5+j 1.55		

Table 1. A parametric characterization for the layered media of Fig.3

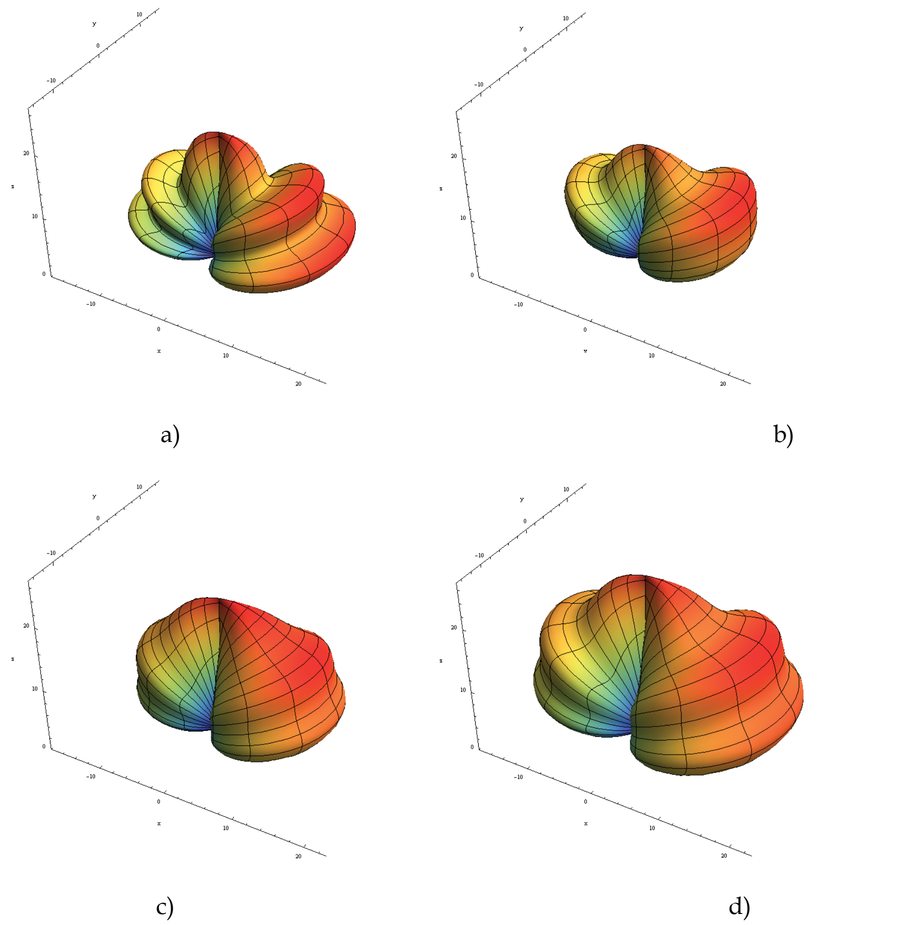


Fig.5. Bi-static scattering coefficients *hh* for a three rough interfaces layered media: ζ_0 contribution (a), ζ_1 contribution (b), ζ_2 contribution (c), total contribution (d) (note that scattering coefficients values less than -40 dB are represented by the axes origin).

In order to perform a consistent comparison, we refer to interfaces with the same roughness: we assume $k_0 l_n$=1.5, $k_0\sigma_n$=0.15 for n= 0, 1, 2. In addition, we suppose no correlation between the interfaces. Once this reference structure has been characterized (see Table 1), we study the scattering cross section of the structure as a function of the scattering direction in the upper half-space, assuming fixed the incident direction.

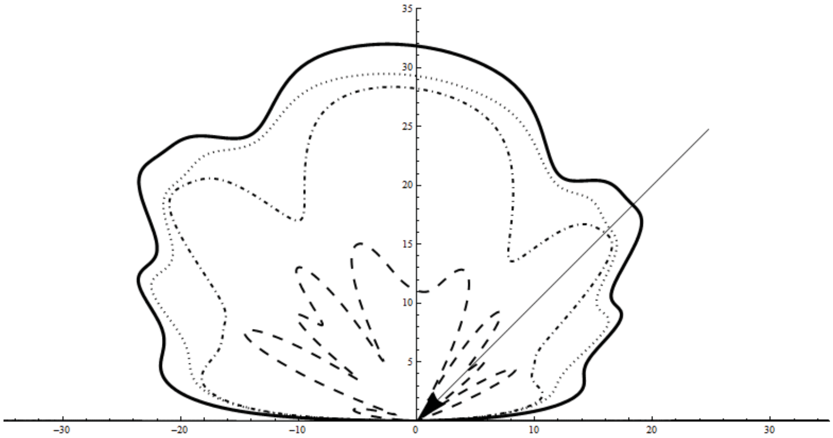
It should be noted that, also considering a limited number of layers, the number of parameters involved by the model makes difficult the jointly visualization of the multi-variables dependency. As matter of fact, once the structure has been parametrically defined and incident direction has been fixed, it is possible to visualize the scattering cross section of the structure as a function of the scattering direction in the upper half-space. Therefore, to characterize the re-irradiation pattern of the structure in three-dimensional space, scattering cross-section distributions are represented (Fig.5) as function of zenithal and azimuthal angles and are treated as three-dimensional surfaces. To save space, only the case *hh* is considered. In addition, we assume fixed the incidence angle $\Theta_0^i = 45°$ ($\hat{k}_\perp^i = \hat{x}$).

Therefore, to evaluate the effect on the global response of each rough interface, the several single contributions are shown in Fig.5a, Fig.5b, and Fig.5c, respectively. In addition, the total contribution is also pictured (Fig.5.d). It should be noted that to visualize the patterns an offset of +40dB has been considered for the radial amplitude, so that scattering coefficients less than -40dB are represented by the axes origin.

5.10.3 Case Study 3

In this section, we present some numerical examples aimed at studying scattering coefficients (94) with reference to a specific context. To this purpose, we consider the canonical layered media with three rough interfaces pictured in Fig.3, which is representative in particular of the characteristic case of *snow-covered sea ice* [40].

In addition, we consider the operational frequency $f=c/\lambda$=4.0 GHz and analyze the layered medium with three rough interfaces schematized in Fig.3, which can be parametric characterized as follows. The vertical profile, which is used for the calculation, is characterized by the following parameters: $\varepsilon_0=1.0$, $\varepsilon_1=1.51+j9.81\cdot10^{-3}$ (snow), $\varepsilon_2=4.67+j4.38\cdot10^{-2}$(sea ice), $\varepsilon_3= 63.4+j39.1$ (sea water); $\Delta_1/\lambda=2.65$, $\Delta_2/\lambda=3.80$.



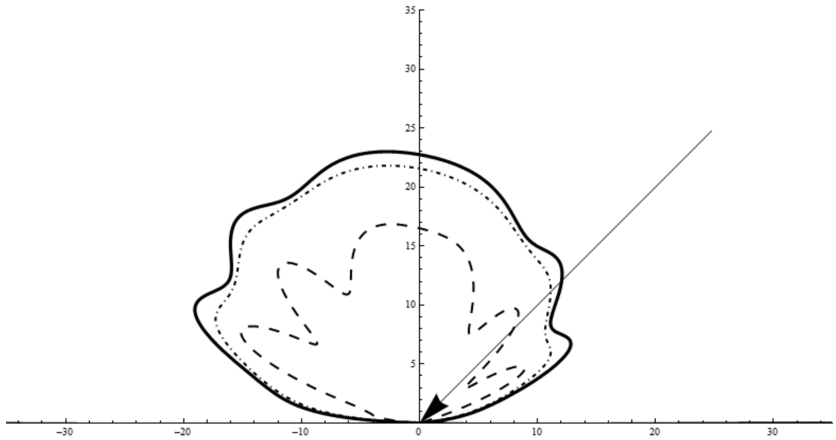


Fig. 6. Bi-static scattering coefficients *hh* for a three rough interfaces layered media: ζ_0 contribution (dashed line), ζ_1 contribution (dot dashed line), ζ_2 contribution (dotted line), and overall contribution (solid line): a) ε_2=4.67+j4.38*10^{-2}, b) ε_2=4.67+j4.38*10^{-1}

We assume that the interfaces constitute Gaussian 2D random processes with *Gaussian correlations*, where, with regard to the *n*th interface (n = 0, 1, 2), σ_n and l_n are the surface height standard deviation and correlation length, respectively. In addition, we suppose no correlation between the interfaces, so the cross products (second summation) in Eq. (94) do not contribute.

Once this reference structure has been characterized, we study the scattering cross section of the structure as a function of the scattering direction in the upper half-space, assuming fixed the incident direction.

As matter of fact, once the structure has been parametrically defined and incident direction has been fixed, it is possible to visualize the scattering cross section of the structure as a function of the scattering direction in the upper half-space. To proceed with a consistent comparison between different scattering contributions, we also conveniently assume the same value for the corresponding interfacial parameters: $k_0l_0=k_0l_1=k_0l_2=1.6$, $k_0\sigma_0=k_0\sigma_1=k_0\sigma_2=0.17$. To save space, only the case *hh* is considered. In addition, we assume fixed the incidence angle $\theta_0^i = 45°$ ($\hat{k}_\perp^i = \hat{x}$).

Therefore, to evaluate the effect on the global response of each rough interface, the several single contributions are shown in Fig.6a: each plot then refers to a different (n = 0, 1, 2) term of the first summation in Eq. (94), so representing the bistatic scattering coefficient of the same canonical layered media (Fig. 3) in which only one (with n= 0,1,2) interface is rough (while the other ones are considered flat). In Fig. 6a, the overall contribution (for which all interfaces are considered rough) is also pictured. It should be noted that to visualize the patterns an offset of +40dB has been considered for the radial amplitude, so that scattering coefficients less than -40dB are represented by the axes origin. In order to evaluate the effect of the model parameters on the scattering pattern, for instance, a different value for the losses in the sea ice permittivity ($\varepsilon_2=4.67+j4.38*10^{-1}$) is considered in Fig. 6.b: in this case it is evident that the contribution of the last interface becomes negligible.

5.10.4 Case Study 4

In this section, we present and discuss some examples aimed at studying the bi-static scattering coefficients (95).

Accordingly, we plot in Fig. 7 the bi-static scattering coefficient at 1 GHz, for the four polarization combinations, as a function of the scattering angle θ_s distinguishing the three contributions $\tilde{\sigma}_{qp,n}^0$ from the correspondent considered rough interfaces (n=0,1,2). The overall scattering coefficient $\tilde{\sigma}_{qp}^0$ of the layered structure with three rough interfaces is also shown. We assume θ_0^i=45°.

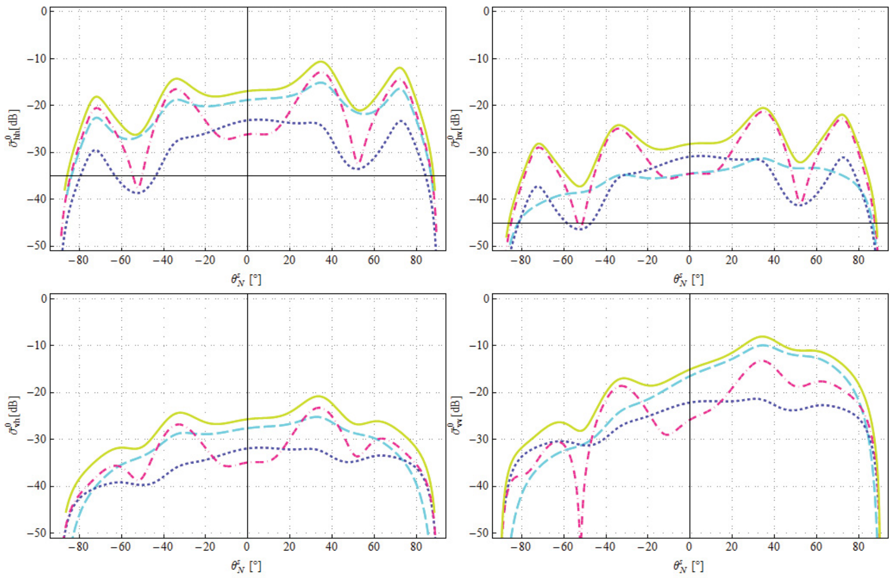


Fig.7. Bi-static scattering coefficients for three rough interfaces layered media, with ε_1=8.5+j0.02, ε_2=4.5+j0.01, $\varepsilon_0=\varepsilon_3$=1; Δ_1/λ=1.150, Δ_2/λ=2.80; $k_0l_0= k_0l_1= k_0l_2$=1.5, $k_0\sigma_0=k_0\sigma_1= k_0\sigma_2$=0.15, f=1.0 GHz, θ^i_0= 45°, φ^i_0= 0°, φ^s_N = 20°, $\tilde{\sigma}^0_{qp}$ *(solid line),* $\tilde{\sigma}^0_{qp,0}$ *(long-dashed),* $\tilde{\sigma}^0_{qp,1}$ *(short-dashed line),* $\tilde{\sigma}^0_{qp,2}$ *(dotted-dashed line).*

To perform a consistent comparison we refer to interfaces with the same roughness. To be more specific, we consider classical Gaussian surface height model with Gaussian correlations: we let k_0l_n=1.5, $k_0\sigma_n$=0.15 for n=0,1,2.

No interface cross-correlation is assumed. In addition, the considered structure (Fig. 3) is characterized by the following parameters: ε_1=8.5+j0.02, ε_2=4.5+j0.001, ε_3=1.0; Δ_1/λ=1.150, Δ_2/λ=2.80.

Note also that, by using spherical coordinates systems, the directions of the incident and scattered waves, are individuated by (θ^i_0,φ^i_0) and (θ^s_N,φ^s_N), respectively: assuming both half-spaces are vacuum, we get

$$\left|\mathbf{k}^s_\perp - \mathbf{k}^i_\perp\right|^2 = k_0^2[\sin^2\theta^i_0 + \sin^2\theta^s_N - 2\sin\theta^i_0\sin\theta^s_N\cos(\varphi^s_N - \varphi^i_0)]. \tag{5.81}$$

From Fig.7 the different effect of the several corrugated interface on the bi-static scattering coefficients can be observed. We observe that the interfaces with a greater associated dielectric contrast exhibit a more significant contribution, whereas the oscillatory behavior of the distinct contributions is more attenuated as the losses increase. The remarkable point is the importance of taking into consideration, in the evaluation of the overall bi-static scattering through the layered structure, the peculiar electromagnetic and geometric parameters of real rough interface stratifications. Therefore, this simple instance demonstrates the significance to discern the effects of the different interfaces when analyzing the response of a realistic stratified structure. We do not show further examples only to save space, since their results are qualitatively similar to the presented ones.

5.11 Conclusion

A quantitative mathematical analysis of wave propagation in three-dimensional layered rough media is fundamental in understanding intriguing scattering phenomena in such structures, especially for remote sensing applications. The problem of electromagnetic scattering in 3D layered rough structures can be analytical treated by relying on effective results of the *Boundary Perturbation Theory* (*BPT*). A structured presentation of the pertinent theoretical body of results has been provided in this Chapter.

In this context, the results of the *Boundary Perturbation Theory* (BPT), lead to compact, formally symmetric and fully polarimetric closed-form solutions that are amenable of direct and parametric numerical evaluation and, therefore, can be effectively applied to several practical situations of interest. The first-order scattering models obtained in the framework of the *BPT* allow us to polarimetrically deal with the (bi-static) scattering, from and through three-dimensional layered structures with an arbitrary number of gently rough interfaces. Analytically speaking, two relevant closed-form solutions, obtained for two different configurations, respectively, for the scattering from and through the structure, are presented in a common formal frame. As a matter of fact, beyond a certain economy and mathematical elegance in the final analytical solutions, their inherent symmetry is intimately related to the reciprocity.

BPT solutions allows us to show that the scattered field can be parametrically evaluated considering a set of parameters: some of them refer to an unperturbed structure configuration, i.e., the physical parameters of the smooth boundary structure, and the others are determined exclusively by (random) deviations of the corrugated boundaries from their reference position. To be specific, the proposed solution allows the *polarimetric* evaluation of the scattering, once the *three-dimensional* layered structure's parameters (shape of the roughness spectra, layers thickness and complex permittivities), the incident field parameters (frequency, polarization and direction) and the observation direction are been specified. Therefore, our formulation leads to a direct functional dependence (no integral evaluation is required). Procedurally, once the *generalized reflection/transmission coefficients* are recursively evaluated, the scattering coefficients of a structure with rough interfaces can be finally predicted.

It should be noted that the method to be applied only needs the classical SPM gentle-roughness assumption, without any further approximation. We underline that it can be also demonstrated that all the previous existing perturbative scattering models, introduced by other authors to deal with simplified layered structures can be all rigorously regarded as a special cases of the general BPT solutions. This analytical consistency also provides a unifying perspective on the perturbative approaches. Finally, the body of the *BPT* theoretical results can be also regarded as a generalization to the case of layered media with rough interfaces of the classical *SPM* for rough surface.

As a result, the proposed solution can be effectively applied to remote sensing of complex natural stratification as well as to the simulation of radio-wave propagation in urban environment. In addition, it is susceptible of an attractive application to the inverse problem, opening the way to new innovative techniques.

On the other hand, the main limitation of the SPM is its restricted domain of validity, as it is valid for small RMS height/wavelength ratios. In the limit of large wavelengths, however, this approximation tends to the true solution of the scattering problem. As matter of fact, SPM constitutes the reference for any approximate method in the low-frequency limit.

With regard to the validation of the proposed model, the following considerations are in order. Although full consistency of BPT solutions with existing simplified ones has been verified analytically, which is a preliminary and fundamental step for formal validation of the innovative model, proper measurement campaigns should be carried out to fully expose the proposed model to an experimental validation. With regard to that, we emphasize that a proper measurement methodology and ad-hoc measurement campaigns are needed and, for that reason, their definition and implementation are deferred to subsequent investigations. However, regarding this point, we emphasize that the simplicity of the final analytical expressions of the innovative model should suggest the appropriate experimental set-up. Nonetheless, a comparison with the results of simulations obtainable with numerical methods needs to be further investigated.

On the other hand, when the potential future developments are concerned, the following considerations are in order. First, although multiple scattering effects associated to higher-order terms of the SPM solution are generally neglected assuming gently rough interface; however, in the context of the interference phenomena that take place in stratification, the goodness of this first-order SPM approximation should need further investigations by evaluating at least the *second order* terms of the perturbative expansion.

Second, with regard to the geometric surface description, intentionally no particular restrictive assumption has been held. In fact, *fractal description* [37] can be applied as well as the classical one.

Finally, we remark the twofold advantage of the obtained compact solutions. From a theoretical viewpoint, BPT provides a general formulation for an arbitrary 3D layered geometry, in the framework of the first-order approximation, for a scattering problem whose compact solution was still lacking. On the other hand, the proposed polarimetric solutions can be effectively applied to several practical situations of interest, such as remote sensing of complex stratifications as well as to the simulation of radio-wave propagation in urban environment. For instance, the results of this work have led to transmission models that can be directly included in ray-tracing-based propagation prediction tools, in order to get a more accurate description of wave propagation in urban scenario. We also highlight that the availability of a compact closed-form solution constitutes a tremendous practical advantage to deal with the inverse problem; for instance, in order to locate and delineate subsurface structure proprieties of interest in remote sensing applications.It should be also noted that the availability of both (from and through) reciprocal scattering solutions enables the computation of the full scattering matrix to characterize the global response of a microwave or optical component, whose structure can be modeled as a rough-interfaces multilayer.

5.12 Appendix: Derivation of the non-uniform boundary conditions in matrix notation

In order to effectively use the transfer matrix approach, from (21)-(22), considering firstly only the ($q=h$) *horizontal polarization* case, we have:

$$\mathbf{E}_m^{\pm(1)} = \iint d\mathbf{k}_\perp e^{j\mathbf{k}_\perp\cdot\mathbf{r}_\perp} \hat{h}_m^{\pm}(\mathbf{k}_\perp) S_m^{\pm h(1)}(\mathbf{k}_\perp) e^{\pm jk_{zm}z}, \tag{A.1}$$

$$\mathbf{H}_m^{\pm(1)} = \iint d\mathbf{k}_\perp e^{j\mathbf{k}_\perp\cdot\mathbf{r}_\perp} \frac{1}{Z_m} \hat{k}_m^{\pm} \times \hat{h}_m^{\pm}(\mathbf{k}_\perp) S_m^{\pm h(1)}(\mathbf{k}_\perp) e^{\pm jk_{zm}z}. \tag{A.2}$$

By enforcing the non-uniform boundary condition (25), we get:

$$\begin{aligned}&\iint d\,\mathbf{k}_\perp e^{j\mathbf{k}_\perp\cdot\mathbf{r}_\perp}\, \hat{k}_\perp \mathbf{d}\cdot\mathbf{S}_{m+1}^{h(1)}(\mathbf{k}_\perp, d_m) = \\ &\iint d\,\mathbf{k}_\perp e^{j\mathbf{k}_\perp\cdot\mathbf{r}_\perp}\, \hat{k}_\perp \mathbf{d}\cdot\mathbf{S}_{m}^{h(1)}(\mathbf{k}_\perp, d_m) + \iint d\,\mathbf{k}_\perp e^{j\mathbf{k}_\perp\cdot\mathbf{r}_\perp} \widetilde{\mathbf{J}}_{Hm}^{p(1)}(\mathbf{k}_\perp, \mathbf{k}_\perp^i),\end{aligned} \tag{A.3}$$

where $\mathbf{d} = [1 \quad 1]$ and where eq. (23) has been considered. Pre-multiplying (vectorially) by $\hat{z}$ both sides of (26), we get:

$$\begin{aligned}&\iint d\,\mathbf{k}_\perp e^{j\mathbf{k}_\perp\cdot\mathbf{r}_\perp} \hat{k}_\perp \mathbf{f}\cdot\mathbf{S}_{m+1}^{h(1)}(\mathbf{k}_\perp, d_m)\frac{k_{z(m+1)}}{k_{m+1}Z_{m+1}} = \\ &\iint d\,\mathbf{k}_\perp e^{j\mathbf{k}_\perp\cdot\mathbf{r}_\perp} \hat{k}_\perp \mathbf{f}\cdot\mathbf{S}_{m}^{h(1)}(\mathbf{k}_\perp, d_m)\frac{k_{zm}}{k_m Z_m} + \iint d\,\mathbf{k}_\perp e^{j\mathbf{k}_\perp\cdot\mathbf{r}_\perp}\, (\hat{z}\times\widetilde{\mathbf{J}}_{Em}^{p(1)}(\mathbf{k}_\perp, \mathbf{k}_\perp^i)),\end{aligned} \tag{A.4}$$

with $\mathbf{f}=[-1 \quad 1]$ and where we have considerer that $\hat{z}\times(\hat{z}\times(\hat{k}_m^{\pm}\times\hat{h}_m^{+}(\mathbf{k}_\perp)))=-\hat{k}_\perp(\hat{k}_m^{\pm}\cdot\hat{z})$. Therefore, we have:

$$\hat{k}_\perp\mathbf{d}\cdot\mathbf{S}_{m+1}^{h(1)}(\mathbf{k}_\perp,d_m)-\widetilde{\mathbf{J}}_{Hm}^{p(1)}=\hat{k}_\perp\mathbf{d}\cdot\mathbf{S}_m^{h(1)}(\mathbf{k}_\perp,d_m), \tag{A.5}$$

$$\hat{k}_\perp\mathbf{f}\cdot\mathbf{S}_{m+1}^{h(1)}(\mathbf{k}_\perp,d_m)\frac{k_{z(m+1)}\mu_m}{\mu_{m+1}k_{zm}}-\frac{k_0Z_0\mu_m}{k_{zm}}(\hat{z}\times\widetilde{\mathbf{J}}_{Em}^{p(1)})=\hat{k}_\perp\mathbf{f}\cdot\mathbf{S}_m^{h(1)}(\mathbf{k}_\perp,d_m). \tag{A.6}$$

Scalar multiplying two sides of eqs. (A.5)-(A.6) by $\hat{k}_\perp$, we have:

$$\mathbf{d}\cdot\mathbf{S}_m^{h(1)}(\mathbf{k}_\perp,d_m)=\mathbf{d}\cdot\mathbf{S}_{m+1}^{h(1)}(\mathbf{k}_\perp,d_m)-\hat{k}_\perp\cdot\widetilde{\mathbf{J}}_{Hm}^{p(1)}, \tag{A.7}$$

$$\mathbf{f}\cdot\mathbf{S}_m^{h(1)}(\mathbf{k}_\perp,d_m)=\mathbf{f}\cdot\mathbf{S}_{m+1}^{h(1)}(\mathbf{k}_\perp,d_m)\frac{k_{z(m+1)}\mu_m}{\mu_{m+1}k_{zm}}-\frac{k_0Z_0\mu_m}{k_{zm}}\hat{k}_\perp\cdot(\hat{z}\times\widetilde{\mathbf{J}}_{Em}^{p(1)}). \tag{A.8}$$

Subtracting and adding (A.7) from and to (A.8), we get, respectively:

$$\begin{aligned}[1 \quad 0]\mathbf{S}_m^{h(1)}(\mathbf{k}_\perp,d_m)=&\frac{k_0Z_0\mu_m}{2k_{zm}}\hat{k}_\perp\cdot(\hat{z}\times\widetilde{\mathbf{J}}_{Em}^{p(1)})\\&-\frac{1}{2}\hat{k}_\perp\cdot\widetilde{\mathbf{J}}_{Hm}^{p(1)}+\frac{1}{T_{m|m+1}^h}\left[1 \quad R_{m|m+1}^h\right]\mathbf{S}_{m+1}^{h(1)}(\mathbf{k}_\perp,d_m),\end{aligned} \tag{A.9}$$

$$\begin{aligned}[0 \quad 1]\mathbf{S}_m^{h(1)}(\mathbf{k}_\perp,d_m)=&-\frac{k_0Z_0\mu_m}{2k_{zm}}\hat{k}_\perp\cdot(\hat{z}\times\widetilde{\mathbf{J}}_{Em}^{p(1)})\\&-\frac{1}{2}\hat{k}_\perp\cdot\widetilde{\mathbf{J}}_{Hm}^{p(1)}+\frac{1}{T_{m|m+1}^h}\left[R_{m|m+1}^h \quad 1\right]\mathbf{S}_{m+1}^{h(1)}(\mathbf{k}_\perp,d_m).\end{aligned} \tag{A.10}$$

Therefore, the boundary conditions (25)-(26), considering that $\hat{k}_\perp\cdot\hat{z}\times\mathbf{J}_{Em}^{p(1)}=\hat{k}_\perp\times\hat{z}\cdot\mathbf{J}_{Em}^{p(1)}$, can be rewritten using the matrix notation (30)-(31).

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Chapter 6

Physical Meaning of the Boundary Perturbation Theory

"I am not interested in proofs. I am only interested on how Nature works."
P.A.M. Dirac

"... è attraverso le teorie che impariamo ad osservare, cioè a porre domande che conducono a delle osservazioni e alle relative interpretazioni."
Karl Raimund Popper

The primary aim of this Chapter is to investigate on the physical meaning of first-order solutions for the field scattered by layered structures with rough interfaces, which were derived in the BPT framework presented in Chapter 5. In order to capture the intrinsic significance of the BPT closed-form scattering solutions, a mathematical description which connects the concepts of *local* scattering and *global* scattering is provided. Consequently, the functional decomposition of the BPT global scattering solution in terms of basic single-scattering local processes is rigorously established. This wave *scattering decomposition* gives insight into the BPT analytical results, so enabling a relevant physical-revealing interpretation involving ray-series representation. The performed series expansions, which can be seen as ray series, can be then accurately analyzed showing that each term has a direct physical explanation. The analysis is carried out for both from- and through-layered-structure scattering configurations.

Accordingly, in first-order limit, the way in which the character of the local scattering processes emerges is dictated by the nature of the structural filter action, which is inherently governed by the series of coherent interactions with the medium boundaries. As a result, analytical perturbative solutions turn out to be completely interpretable by simple physical concepts, so that the global scattering response can be interpreted as the superposition of single-scattering interaction mechanisms taking place locally, which are filtered by the layered structure.

As a result, scattering phenomena, which occur inside layered media and are associated with interfacial roughness, can be now completely understood on the basis of BPT; the consequent phenomenological implications on the practical applications are then noteworthy. This is to say that BPT analytical results are rigorously susceptible of a powerful physical interpretation, so that the fundamental interactions contemplated by the BPT can be revealed, gaining a coherent explanation and a neat picture of the physical meaning of the BPT theoretical construct.

The meaning of the first-order approximation is also discussed in the layered structure context. Finally, a complete explanation for the scattering enhancement phenomenon contemplated in the first-order limit is given.

6.1 Introduction and Motivation

Theoretical formulas without a clear comprehension of their intrinsic meaning are of difficult use in the context of practical applications.

Several specific perturbative solutions were found for different simplified configurations as discussed in Chapter 4 (see also [3]-[12]); however, the pertinent physical interpretations are lacking.

In previous Chapter, by analyzing the wave interaction with layered structures with rough boundaries, it has been shown that the derivation in closed-form of the *first-order perturbative solutions* for the relevant scattering problem is feasible within the general systematic BPT framework. In fact, general closed forms, involving the *generalized reflection/ transmission* formalism, has been obtained for the scattering from and through three-dimensional layered structures with an arbitrary number of gently rough interfaces has been derived in Chapter 5. Consequently, the compact expressions (5.68) and (5.77) allow us to analyze polarimetrically and parametrically the general functional dependence of the scattered electromagnetic field on the electromagnetic and geometric parameters of an arbitrary layered structure.

We emphasize that in the following we refer exclusively to the general BPT formulation, since all of the other previously existing ones [3]-[12], concerning simplified layered configurations, can be regarded as particular cases of the BPT.

On the other hand, generally speaking, even though the manageability of the analytical solutions is an essential requirement for applications, the understanding of the physical meaning can be even more crucial. Nevertheless, in analytic derivations the final results are mostly attainable in a form that can appear illegible in the physics perspective. Furthermore, modeling real situations often leads to some suitable analytical approximations whose intuitive interpretation can be lost. Conversely, when a clear physical perspective of the meaning of the obtained solutions is viable, the implications open scenarios that could not be conceived otherwise.

More in general, in the radar applications the availability of closed-form scattering solutions is even more fundamental for the comprehension and the schematic handling of the problem rather than for the actual scattering evaluation. In this perspective, the physics of the scattering mechanisms involved in the scattering from and through layered structures with rough boundaries should be better clarified.

The physics of the interaction of electromagnetic waves with complex layered structures with an arbitrary number of rough interfaces has not been completely clarified yet. On other hand, despite the fact that the general BPT solutions derived in Chapter 5 exhibit a compact and symmetric structure, the related physical meaning is not immediately obvious and a physical interpretation has not been provided yet.

Nevertheless, the relevant question one might ask now is whether, using such an analytical result, the intrinsic physical meaning of the first-order global BPT solutions can be revealed, to shed light on the contemplated scattering processes that take place *locally* inside the layered structure. Furthermore, we emphasize that in many applications, such as exploration of seismic events or GPR, time-dependent wave-trains are observed, rather than spectral intensities. Therefore, in some cases a time domain characterization of the layered structures response could result more attractive than a spectral one.

In view of the above considerations, in this Chapter considerable attention is paid to the intrinsic significance of the global scattering solution, getting more concrete insight into the physics of the problem, and a physical interpretation of the BPT solutions is carried out on an analytical playground. Our aim is then to show that detailed, physically revealing and mathematically useful information can be extracted from BPT models.

For this purpose, starting from the BPT solutions, detailed in Chapter 5, firstly we suitably expand the obtained solutions. The results we obtained in [7], [8] suggest us the usefulness to base the expansions on local descriptors, in order to analyze the meaning of the global scattering response. Once the nature of the local interaction is recognized, we demonstrate that the obtained expansions can be properly seen as a *ray series* or a *geometrical optics series*; so the basic scattering mechanisms involved can be accurately visualized showing that each term of the ray series has a direct physical explanation. Consequently, the *local/global scattering* concepts are successfully exploited, differently from [3], [9] and [10] wherein the authors resort to the *radar contrast*.

Therefore, the suitable reformulation of the scattered field expressions and the associated ray series sheds light on the relations among *global scattering* and *local scattering* phenomena in the layered structure: the expansions explain how global scattering, from and through the layered structure, arises from the (local) scattering that takes place when the waves propagating in the structure interact locally with the corrugated interfaces; whereas the multiple bounces on the flat boundaries, preceding and following the (local) single scattering occurrence, elucidate how the interference effects acting in the structure influence the global response of the structure. Consequently, in the first-order limit, the global scattering can be considered as the superposition of waves propagating in the layered structure, each one undergoing to a *local* scattering *phenomena* filtered by the layered structure; whereas the filter action arises from the interferential effects due to the coherent interaction with the boundaries. As a result, the global scattering problems, which were introduced as formal mathematics in the first-order perturbative limit (Chapter 5), turn out to be fully interpretable by simple physical concepts.

Commonly, the scattering *enhancement phenomenon* is only illustrated for volume scattering or second-order scattering from rough surface [18], [19]; whereas in [9], [10], in the first-order, the backscattering enhancement from a single rough surface on the top of a layered media is pointed out. In view of that, the performed physical interpretations bring us to characterize the scattering directions, for which our first order models contemplate the enhancement scattering phenomenon, associated with the coherent effects taking place in a layered medium with rough interfaces. Consequently, *Enhancement Cones* are identified. This result is corroborated by, and could explain, the experimental results in [20].

In addition, note also that if the incident wave is a modulated pulse, each term of the expansions corresponds to an echo that will be received with a different time delay. It is then clear that the obtained expansions open the way to a time-domain analysis of the layered structures response. It should be noted that this point is of fundamental importance for instance if the considered model must be embedded in a SAR simulator, as well in a SAR data processing strategy or in a ray-tracing code for field levels prediction in urban environment.

Finally, we remark the several advantages of obtained expansions. They let us deeply understand the physics of the problem revealing the intrinsic nature of the basic scattering mechanisms involved; they elucidate the physical meaning of the first-order approximation; and they explain the enhancement phenomena contemplated in the first-order limit. What is more, the expansions are mathematically useful since they are also addressed to a direct time-domain characterization of the structure response that can be effectively applied to several situations of interest. As a result, even though our approach is primarily theoretical, the proposed analytical expansions are meaningful from the result interpretation point of view, they have interesting implications, and they open the way to new possible applications to coherent remote sensing and to radio propagation prediction in urban environment.

This Chapter is organized as follows.

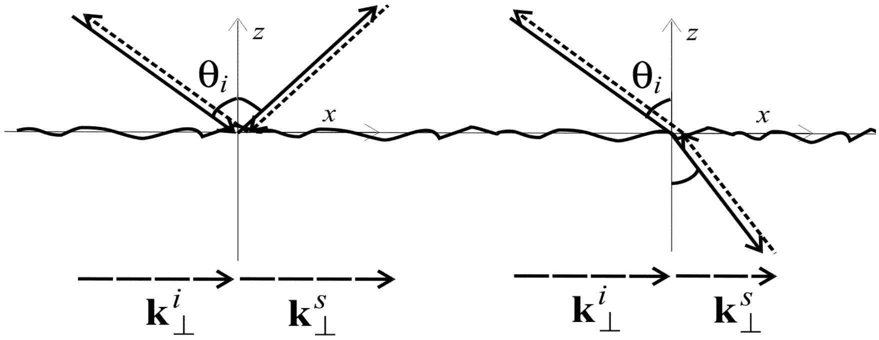

Fig.1. Reciprocity for local scattering from and through a rough interface.

The concept of physically-based local descriptors is introduced in Section 6.2. In section 6.3, a powerful physical interpretation of the pertinent analytic solutions for a specific geometry is provided. Section 6.4 is devoted to the physical interpretation of the general BPT solutions. Global and local scattering concepts are discussed in Section 6.5. The meaning of the first-order approximation is elucidated in Section 6.6. In Section 6.7, the contemplated scattering enhancement phenomenon is discussed. Section 6.8 concludes with a summary.

6.2 Local Scattering Concept

The concept of local scattering is introduced in this section, with particular emphasis to the case of local scattering from a through a rough interface.

In order deal with the scattering property of a corrugated interface of a layered structure, it is fruitful to emphasize the local response of a rough interface, i.e., the scattering properties exhibited by the roughness when the stratification surrounding the rough interface recedes to infinite.

Therefore, from the scattering mechanism point of view, it can be assumed that the wave interaction with a rough interface embedded in a layered structure can be locally assimilated to the wave interaction with a rough surface between two half-spaces. The rationale motivating this concept will appear clear in the next section, when it will be shown that the first-order BPT solutions are susceptible of a representation in terms of the local scattering properties of the corrugated interfaces.

The local scattering cross sections of the *n*-th rough interface of the structure, for the scattering from and through the roughness respectively, are then defined as:

$$\sigma^0_{qp,n} = \pi k_0^4 \left|\alpha^{n,n+1}_{qp}(\mathbf{k}^s, \mathbf{k}^i)\right|^2 W_n(\mathbf{k}^s_\perp - \mathbf{k}^i_\perp), \tag{6.1}$$

$$\sigma^{0t}_{qp,n} = \pi k_0^4 \left|\beta^{n,n+1}_{qp}(\mathbf{k}^s, \mathbf{k}^i)\right|^2 W_n(\mathbf{k}^s_\perp - \mathbf{k}^i_\perp)\,\mathrm{Re}\left\{\sqrt{\frac{\varepsilon_{m+1}}{\varepsilon_m}}\right\}, \tag{6.2}$$

where $p, q \in \{v, h\}$ and wherein, for the *hh* case, we have:

$$\alpha^{m,m+1}_{hh}(\mathbf{k}^s, \mathbf{k}^i) = (\varepsilon_{m+1} - \varepsilon_m)(\hat{k}^s_\perp \cdot \hat{k}^i_\perp)[1 + R^h_{m|m+1}(k^s_\perp)][1 + R^h_{m|m+1}(k^i_\perp)], \tag{6.3}$$

$$\beta^{m,m+1}_{hh}(\mathbf{k}^s, \mathbf{k}^i) = (\varepsilon_{m+1} - \varepsilon_m)(\hat{k}^s_\perp \cdot \hat{k}^i_\perp)[1 - R^h_{m|m+1}(k^s_\perp)][1 + R^h_{m|m+1}(k^i_\perp)], \tag{6.4}$$

$$\beta^{m+1,m}_{hh}(\mathbf{k}^s, \mathbf{k}^i) = -(\varepsilon_{m+1} - \varepsilon_m)(\hat{k}^s_\perp \cdot \hat{k}^i_\perp)[1 + R^h_{m|m+1}(k^s_\perp)][1 - R^h_{m|m+1}(k^i_\perp)], \tag{6.5}$$

$$\alpha^{m+1,m}_{hh}(\mathbf{k}^s, \mathbf{k}^i) = -(\varepsilon_{m+1} - \varepsilon_m)(\hat{k}^s_\perp \cdot \hat{k}^i_\perp)[1 - R^h_{m|m+1}(k^s_\perp)][1 - R^h_{m|m+1}(k^i_\perp)]. \tag{6.6}$$

Specifically, four distinct types of local interaction with an embedded rough interface can be distinguished: two of them identifiable as local scattering through the relevant interfacial roughness and other ones as local scattering from the roughness. We emphasize that the corresponding coefficients $\alpha^{m,m+1}_{qp}$ and $\alpha^{m+1,m}_{qp}$ refer to cases in which both the observation and incidence directions are, respectively, above and under the roughness; whereas $\beta^{m,m+1}_{qp}$ and

$\beta_{qp}^{m+1,m}$ concern the local scattering contributions that cross the roughness in opposite directions.

Therefore, the local scattering coefficients are formally identical to the classical ones relative to a rough surface between two half-spaces [18]-[22]. Consequently, the *reciprocity* for the local scattering can be expressed as follows (see Fig.1):

$$\alpha_{qp}^{m,m+1}(\mathbf{k}^s,\mathbf{k}^i) = \alpha_{pq}^{m,m+1}(-\mathbf{k}^i,-\mathbf{k}^s), \tag{6.7}$$

$$\beta_{qp}^{m,m+1}(\mathbf{k}^s,\mathbf{k}^i) = -\beta_{pq}^{m+1,m}(-\mathbf{k}^i,-\mathbf{k}^s). \tag{6.8}$$

6.3 Ray-Series Physical Interpretation for a Specific Configuration

In this Section, we first of all investigate the scattering phenomena in simplified layered structure with one rough interface and a detailed physical interpretation of the relevant formal solution is carried out, emphasizing the role of the interference phenomena that take place inside the stratification.

The relevant scattering solution (4.12)-(4.13) for bistatic configuration, which is here analyzed, is the one pertinent to the geometry depicted in Fig.2 of Chapter 4: it is important to emphasize that such a specific solution can be also directly obtained by particularizing the general BPT solution (5-82)-(5.85) with reference to a specific layered configuration (see also (5.96)-(5.97)).

To get more insight into the meaning of the first order solutions from the concerning scattering mechanism point of view, it is instructive to carry out a complete expansion of the scattering coefficients. Focusing on the *hh* case (without loss of generality), from (5.97) we have:

$$\tilde{\alpha}_{hh}^{1,2} = \alpha_{hh}^{1,2} \frac{T_{0|1}^h(k_\perp^i)e^{jk_{z1}^i\Delta_1}}{1+R_{0|1}^h(k_\perp^i)R_{1|2}^h(k_\perp^i)e^{j2k_{z1}^i\Delta_1}} \frac{k_{z0}^s}{k_{z1}^s} \frac{T_{1|0}^h(k_\perp^s)e^{jk_{z1}^s\Delta_1}}{1+R_{0|1}^h(k_\perp^s)R_{1|2}^h(k_\perp^s)e^{j2k_{z1}^s\Delta_1}}, \tag{6.9}$$

where $\alpha_{hh}^{1,2}$ is associated with the *local scattering coefficients* σ_{hh}^{0} (see Section 6.2), i.e., the classical scattering coefficients of a rough surface between two semi-infinite homogeneous media, characterized by dielectric permittivity ε_1 and ε_2, respectively [6]:

$$\alpha_{hh}^{1,2} = (\varepsilon_2 - \varepsilon_1)(\hat{k}_{\perp}^{i} \cdot \hat{k}_{\perp}^{s})[1 + R_{1|2}^{h}(k_{\perp}^{i})][1 + R_{1|2}^{h}(k_{\perp}^{s})]. \qquad (6.10)$$

It should be noted that $\tilde{\alpha}_{hh}^{1,2}$ is equal to $\alpha_{hh}^{1,2}$ when stratification above the roughness vanishes. We recognize that the factor

$$\frac{T_{0|1}^{h}(k_{\perp}^{i}) e^{jk_{z1}^{i}\Delta_1}}{1 + R_{0|1}^{h}(k_{\perp}^{i}) R_{1|2}^{h}(k_{\perp}^{i}) e^{j2k_{z1}^{i}\Delta_1}} \qquad (6.11)$$

is associated with the unperturbed local incident field on the roughness, i.e., the field that would be incident at the interface 1 if the latter were smooth. This can be considered as the wave that undergoes an equivalent coherent transmission through the layers 0-1.

It is useful to introduce the following notation:

$$\Lambda_1(k_{\perp}) = R_{1|0}^{h}(k_{\perp}) R_{1|2}^{h}(k_{\perp}) e^{j2k_{z1}\Delta_1}, \qquad (6.12)$$

and recognize that this factor corresponds to a complete round-trip in the intermediate layer with coherent reflections at the layer boundaries. We then consider the series expansion (geometric power series):

$$[1 - \Lambda_1(k_{\perp})]^{-1} = \sum_{n=0}^{\infty} [\Lambda_1(k_{\perp})]^{n}. \qquad (6.13)$$

Using (13), the expression (9) is susceptible of the following representation:

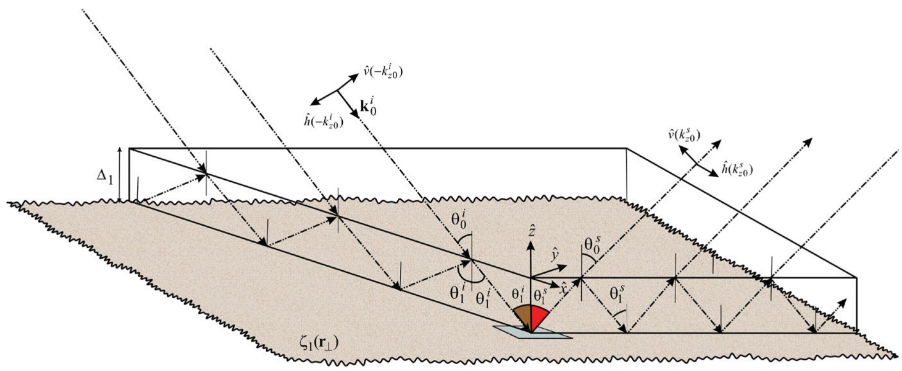

Fig.2. Physical Interpretation: bistatic configuration

$$\tilde{\alpha}_{hh}^{1,2}(\mathbf{k}_\perp^i,\mathbf{k}_\perp^s)=\alpha_{hh}^{1,2}(\mathbf{k}_\perp^i,\mathbf{k}_\perp^s)\frac{k_{z0}^s}{k_{z1}^s}T_{0|1}^h(k_\perp^i)e^{jk_{z1}^i\Delta_1}$$
$$e^{jk_{z1}^s\Delta_1}T_{1|0}^h(k_\perp^s)\sum_{m,n=0}^{\infty}[\Lambda_1(k_\perp^i)]^m[\Lambda_1(k_\perp^s)]^n. \tag{6.14}$$

This expansion, expressed as a double *absolutely summable* infinite series, is susceptible of a straightforward physical interpretation: the scattering phenomena can be expressed in terms of the (local) scattering and the series contributions take into account all the interference effects that take place in the layered structure.

Equation (14) can be thought of as a *ray series* or a *geometrical optics series*, and each term of (14) can be readily identified: each of these waves (see Fig.2) undergoes a coherent transmission through the layers 0-1 ($T_{0|1}^h(k_\perp^i)e^{jk_{z1}^i\Delta_1}$), then *m* complete round-trips ($[\Lambda_1(k_\perp^i)]^m$) in the intermediate layer with coherent reflections at the incident angle ($k_\perp^i$), then an incoherent (local) scattering from the rough interface ($\alpha_{hh}^{1,2}(\mathbf{k}_\perp^s,\mathbf{k}_\perp^i)$) in the observation plane ($\mathbf{k}_\perp^s$), subsequently *n* complete round-trips ($[\Lambda_1(k_\perp^s)]^n$) in the intermediate layer with coherent reflections at the scattering angle ($k_\perp^s$), and finally a coherent transmission through the layers 1-0 ($e^{jk_{z1}^s\Delta_1}T_{1|0}^h(k_\perp^s)$).

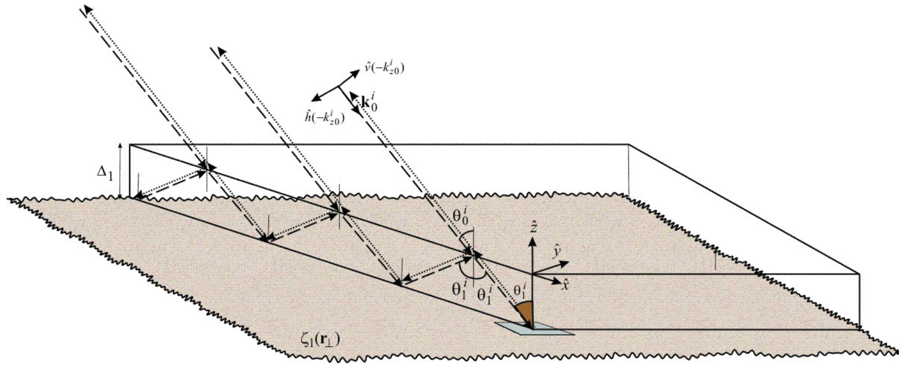

Fig.3. Physical Interpretation: mono-static configuration

When the *backscattering* configuration is concerned ($\mathbf{k}_\perp^s = -\mathbf{k}_\perp^i$), from the (14) we obtain:

$$\tilde{\alpha}_{hh}^{1,2}(\mathbf{k}_\perp^i) = \alpha_{hh}^{1,2}(\mathbf{k}_\perp^i)\frac{k_{z0}^i}{k_{z1}^i}T_{01}^h(k_\perp^i)e^{j2k_{z1}^i\Delta_1}T_{1|0}^h(k_\perp^i)\sum_{m,n=0}^{\infty}[\Lambda_1(k_\perp^i)]^{m+n}. \tag{6.15}$$

If we rearrange the individual terms in the double infinite series by collecting terms raised to the same power (such that $n+m=k$), the last equation can be rewritten as:

$$\tilde{\alpha}_{hh}^{1,2}(\mathbf{k}_\perp^i) = \alpha_{hh}^{1,2}(\mathbf{k}_\perp^i)\frac{k_{z0}^i}{k_{z1}^i}T_{0|1}^h(k_\perp^i)e^{j2k_{z1}^i\Delta_1}T_{1|0}^h(k_\perp^i)\sum_{k=0}^{\infty}(k+1)[\Lambda_1(k_\perp^i)]^k, \tag{6.16}$$

where the k+1 identified waves in (15) sum-up in phase and become indistinguishable, so that the backscattering enhancement phenomenon takes place. The k-th component in (16) corresponds to k+1 different waves. Each of these waves (see Fig.3) undergoes a coherent transmission through the upper interface ($T_{0|1}^h(k_\perp^i)e^{jk_{z1}^i\Delta_1}$), then k-r (r = 0, …, k) complete round-trips ($[\Lambda_1(k_\perp^i)]^{k-r}$) with coherent reflections at the incident angle ($k_\perp^i$) in the intermediate layer, an incoherent (local) backscattering from the rough interface, subsequently r complete round-trips ($[\Lambda_1(k_\perp^i)]^r$) with coherent reflections at the scattering angle ($k_\perp^i$) in the intermediate layer, and finally a coherent transmission through the upper interface ($T_{1|0}^h(k_\perp^i)e^{jk_{z1}^i\Delta_1}$).

As a result, these suitable expansions lead us to clearly understand the physical meaning of the pertinent analytical expressions.

Note also that if the incident wave is a modulated pulse, each term of the series corresponds to an echo that will be received with a different time delay. This point is of fundamental importance, for instance, if the considered model must be embedded in a SAR simulator.

6.4 Ray-Series Physical Interpretation for General BPT First-Order Solutions

In this section, the focus is on the intrinsic significance of the global BPT scattering solutions, getting more concrete insight into the physics of the problem of the scattering from rough interfaces of a layered media.

In this section, to get more insight into the meaning of the BPT first-order solutions (5.68) and (5.77) from the concerning scattering mechanism point of view, we consider instructive to carry out a complete expansion of the scattering coefficients.

In order to accomplish a satisfactory comprehension of how the interaction of the EM field with rough interfaces of an arbitrary layered structure takes place, a key-point is to recognize that the interaction with the structure can be expressible in terms of *local interactions* with the generic rough interface. This is to say that, in order to be able to express the solution in terms of readable basic physical phenomena, a key point is to exploit the *local scattering* concept (Section 6.2).

It should be noted that the exact *analytic decomposition* of the solution in terms of local interactions is rigorously feasible, since, in the first-order perturbative approximation, the scattering amplitude can be written as a single space integral with a kernel that depends only on the rough interface height and on its first-order derivatives at a given point [21].

As a result, in order to phenomenologically describe the scattering from and through the structure and analyze the meaning of the global scattering response, we point out the usefulness of basing the expansions on local descriptors. Specifically, four distinct types of local interaction with an embedded rough interface can be distinguished: two of them identifiable as local scattering through the roughness and the other ones as local scattering from the roughness.

Moreover, since in the limit of first-order perturbation theory the global response of a structure with all rough interfaces can be directly obtained considering the superposition of the response from each interface (see Chapter 5), we firstly focus our attention to a generic embedded rough interface. Afterwards, the general interpretation for a layered structure with an arbitrary number of rough interfaces can be addressed. Without loss of generality, since analogous considerations hold for the other polarization combinations, the analysis is conducted for the *hh* case only.

6.4.1 Scattering from an arbitrary layered media with rough interfaces

Closed form solution for the upward scattered far-field (5.63) into the upper half-space from an arbitrary layered structure with an embedded rough interface (*m*-th) has been established in Chapter 5.

In the following we focus on the relevant analytical expression (5.63) with particular reference to the *hh* case (see (5.64)).

Therefore, in order to provide a symmetrical expansion, it is possible to explicit the factor $\vec{M}_m^h(k_\perp^i)$, which is associated with the multiple round trip in the *m*-th layer and included in $\Im_{0|m}^h(k_\perp^i)$; so we can write:

$$\Im_{0|m}^h(k_\perp^i) = [\vec{M}_m^h(k_\perp^i)]^{-1} \Im_{0|m}'^h(k_\perp^i). \tag{6.17}$$

It should be noted that the $\Im_{0|m}'^p$ are distinct from the coefficients $\Im_{0|m}^{p(slab)}$, because in the evaluation of $\Im_{0|m}'^p$ the effect of all the layers under the layer *m* is taken into account, whereas $\Im_{0|m}^{p(slab)}$ are evaluated referring to a different configuration in which the intermediate layers

1...m are bounded by the half-spaces 0 and m. Moreover, noting that $\Re^{p}_{m|m+1}$ is susceptible to be written equivalently in the form [16]:

$$\Re^{p}_{m|m+1} = R^{p}_{m|m+1} + \frac{T^{p}_{m|m+1}\Re^{p}_{m+1|m+2}e^{j2k_{z(m+1)}\Delta_{m+1}}T^{p}_{m+1|m}}{1+R^{p}_{m|m+1}\Re^{p}_{m+1|m+2}e^{j2k_{z(m+1)}\Delta_{m+1}}}, \tag{6.18}$$

and applying (3.6) and (3.44), we obtain the following expansion for the sub factors in (5.64):

$$\begin{aligned} 1+\Re^{p}_{m|m+1}(k_\perp) &= T^{p}_{m|m+1}(k_\perp) \\ &\left[1+T^{p}_{m+1|m}(k_\perp)\Re^{p}_{m+1|m+2}(k_\perp)e^{j2k_{z(m+1)}\Delta_{(m+1)}}[\vec{M}^{p}_{m+1}(k_\perp)]^{-1}\right]. \end{aligned} \tag{6.19}$$

Furthermore, we introduce the following convenient notation:

$$\vec{\Lambda}^{h}_{m}(k_\perp) = R^{h}_{m|m-1}(k_\perp)\Re^{h}_{m|m+1}(k_\perp)e^{j2k_{zm}\Delta_m}, \tag{6.20}$$

$$\overleftrightarrow{\Lambda}^{h}_{m}(k_\perp) = \Re^{h}_{m|m-1}(k_\perp)\Re^{h}_{m|m+1}(k_\perp)e^{j2k_{zm}\Delta_m}, \tag{6.21}$$

and recognize that these factors correspond to a complete roundtrip in the intermediate layer with coherent reflections at the layer boundaries.

We consider then the series expansions (geometric power series) of sub-factors (3.44),(3.46):

$$[M(k_\perp)]^{-1} = \sum_{n=0}^{\infty}[\Lambda(k_\perp)]^{n}, \tag{6.22}$$

where M, Λ may stand for $\vec{M}^{h}_{m+1}, \vec{\Lambda}^{h}_{m+1}$ or $\vec{M}^{h}_{m}, \vec{\Lambda}^{h}_{m}$ or $\overleftrightarrow{M}^{h}_{m}, \overleftrightarrow{\Lambda}^{h}_{m}$, respectively. By using (22), substituting (17) and (19) in (5.64) and taking into account (3)-(6), we obtain the final expansion:

$$\tilde{\alpha}_{hh}^{m,m+1}(\mathbf{k}^s,\mathbf{k}^i) = \Im'^{h}_{0|m}(k_\perp^i)\, e^{jk_{zm}^i\Delta_m} \sum_{j_1=0}^{\infty}[\vec{\Lambda}_m^h(k_\perp^i)]^{j_1}$$

$$\left\{\alpha_{hh}^{m,m+1}(\mathbf{k}^s,\mathbf{k}^i)\frac{k_{z0}^s}{k_{zm}^s} + \right.$$

$$-T^h_{m|m+1}(k_\perp^i)\sum_{n_1=0}^{\infty}[\vec{\Lambda}_{m+1}^h(k_\perp^i)]^{n_1}\Re^h_{m+1|m+2}(k_\perp^i)e^{j2k_{z(m+1)}^i\Delta_{m+1}}\beta_{hh}^{m+1,m}(\mathbf{k}^s,\mathbf{k}^i)\frac{k_{z0}^s}{k_{zm}^s}$$

$$+\beta_{hh}^{m,m+1}(\mathbf{k}^s,\mathbf{k}^i)\frac{k_{z0}^s}{k_{z(m+1)}^s}\Re^h_{m+1|m+2}(k_\perp^s)e^{j2k_{z(m+1)}^s\Delta_{m+1}}\sum_{n_2=0}^{\infty}[\vec{\Lambda}_{m+1}^h(k_\perp^s)]^{n_2}T^h_{m+1|m}(k_\perp^s)$$

$$-T^h_{m|m+1}(k_\perp^i)\sum_{n_1=0}^{\infty}[\vec{\Lambda}_{m+1}^h(k_\perp^i)]^{n_1}\Re^h_{m+1|m+2}(k_\perp^i)e^{j2k_{z(m+1)}^i\Delta_{m+1}}\alpha_{hh}^{m+1,m}(\mathbf{k}^s,\mathbf{k}^i)\frac{k_{z0}^s}{k_{z(m+1)}^s}$$

$$\left.\Re^h_{m+1|m+2}(k_\perp^s)e^{j2k_{z(m+1)}^s\Delta_{m+1}}\sum_{n_2=0}^{\infty}[\vec{\Lambda}_{m+1}^h(k_\perp^s)]^{n_2}T^h_{m+1|m}(k_\perp^s)\right\}$$

$$\sum_{j_2=0}^{\infty}[\vec{\Lambda}_m^h(k_\perp^s)]^{j_2}e^{jk_{zm}^s\Delta_m}\Im_{m|0}^{h(slab)}(k_\perp^s). \tag{6.23}$$

Note also that, using extensively (22) the generalized transmission coefficients (3.54) and (3.58) could be as well expressed as the product of a number of summations equal to the number of layer involved. However, for the reasons substantiated before, we focus our attention on the two layers just above (m) and under (m+1) the considered roughness. In Fig.4, the remaining part of the structure is visualized condensed in two equivalent slabs constituted, respectively, by the intermediate layers m+2,...,N-1 (under the (m+1)-th layer) and 1,...,m-1(above the m-th layer).

The suitably expanded solution (23), expressed as an infinite sum of contributions, is then susceptible of a straightforward physical interpretation in terms of a ray series.

In particular, each individual term of the a*bsolutely summable* infinite series can be physically identified as a wave propagating in the structure that experiences a local single-interaction with the roughness.

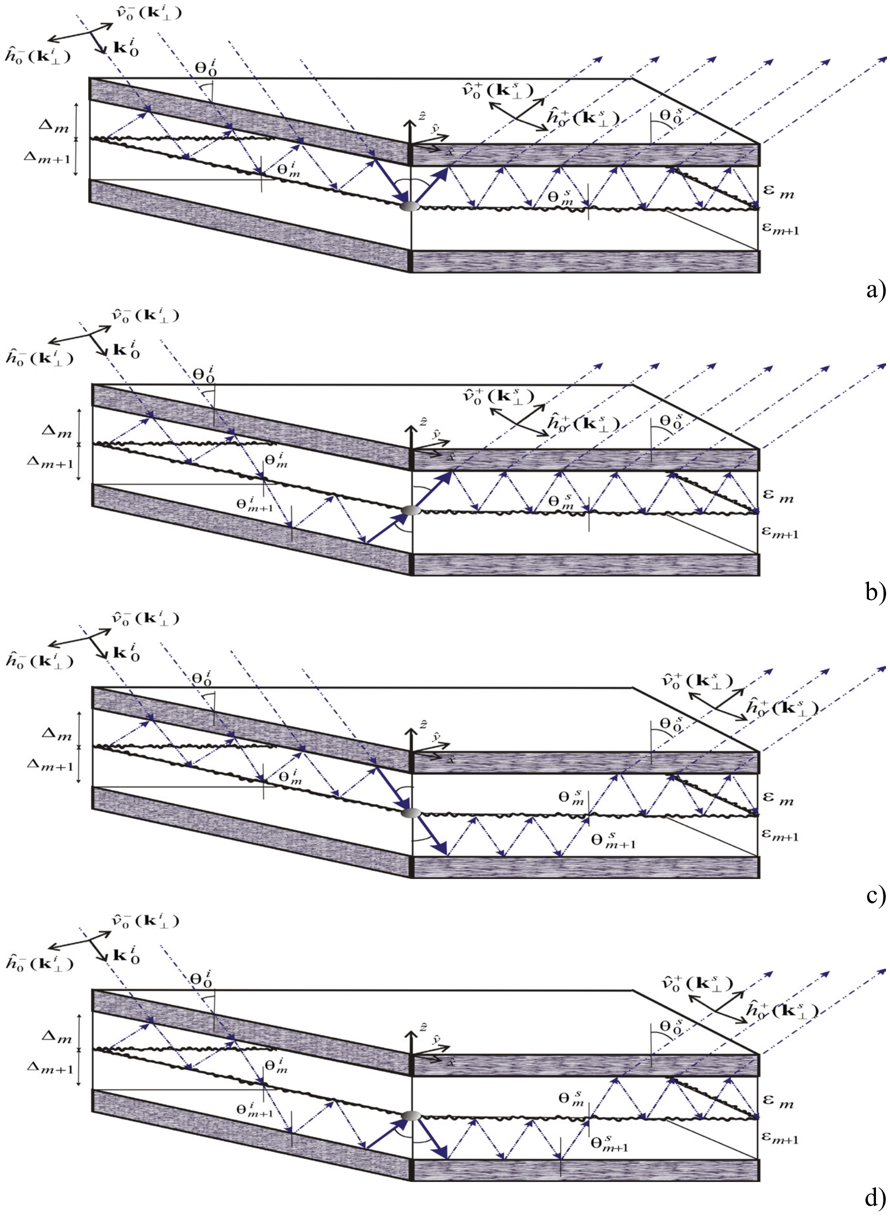

Fig.4. Physical interpretation for the scattering from an arbitrary layered structure with an embedded rough interface.

Consequently, four distinct families of rays can be recognized; each one associated to one type of local interaction, so that each term in (23) can be readily identified as follows:

a) Local upward scattered waves from rough interface: each of these waves (see Fig.4.a) undergoes a coherent transmission into m-th layer ($\Im'^{h}_{0|m}(k^i_\perp)\exp(jk^i_{zm}\Delta_m)$), through the intermediate layers 1,...,m-1, then j_1 complete round-trips ($[\vec{\Lambda}_m(k^i_\perp)]^{j_1}$) in the m-th layer with coherent reflections at the incident angle ($k^i_\perp$), then an incoherent local scattering from the rough interface ($\alpha^{m,m+1}_{hh}(\mathbf{k}^s,\mathbf{k}^i)$), upward within the observation plane ($\mathbf{k}^s_\perp$), subsequently j_2 complete round-trips ($[\vec{\Lambda}^h_m(k^s_\perp)]^{j_2}$) in the m-th layer with coherent reflections at the scattering angle ($k^s_\perp$), and finally a coherent transmission ($\exp(jk^s_{zm}\Delta_m)\Im^{h(slab)}_{m|0}(k^s_\perp)$) in the upper half-space through the intermediate layers m-1,...,1.

b) Local upward scattered waves through rough interface: each of these waves (see Fig.4.b) undergoes a coherent transmission into the m-th layer ($\Im'^{h}_{0|m}(k^i_\perp)\exp(jk^i_{zm}\Delta_m)$), through the intermediate layers 1,...,m-1, then j_1 complete round-trips ($[\vec{\Lambda}_m(k^i_\perp)]^{j_1}$) in the m-th layer with coherent reflections at the incident angle ($k^i_\perp$), subsequently a coherent transmission ($T^h_{m|m+1}(k^i_\perp)$) followed by n_1 complete round-trips in the (m+1)-th layer ($[\vec{\Lambda}^h_{m+1}(k^i_\perp)]^{n_1}$) and by further bounce on the (m+1)-th flat interface ($\Re^h_{m+1|m+2}(k^i_\perp)\exp(j2k^i_{z(m+1)}\Delta_{m+1})$) at the incident angle ($k^i_\perp$), and after that an incoherent local scattering through the rough interface ($\beta^{m+1,m}_{hh}(\mathbf{k}^s,\mathbf{k}^i)$), upward within the observation plane ($\mathbf{k}^s_\perp$), subsequently j_2 complete round-trips ($[\vec{\Lambda}^h_m(k^s_\perp)]^{j_2}$) in the m-th layer with coherent reflections at the scattering angle ($k^s_\perp$), and finally a coherent transmission ($\exp(jk^s_{zm}\Delta_m)\Im^{h(slab)}_{m|0}(k^s_\perp)$) in the upper half-space through the intermediate layers m-1,…,1.

c) Local downward scattered waves through rough interface: each of these waves (see Fig.4.c) undergoes a coherent transmission into the m-th layer ($\Im'^{h}_{0|m}(k^i_\perp)\exp(jk^i_{zm}\Delta_m)$), through the intermediate layers 1,...,m-1, then j_1 complete round-trips ($[\Lambda_m(k^i_\perp)]^{j_1}$) in the m-th layer with coherent reflections at the incident angle ($k^i_\perp$), then an incoherent local scattering through the rough interface ($\beta_{hh}^{m,m+1}(\mathbf{k}^s,\mathbf{k}^i)$)downward in the observation plane ($\mathbf{k}_\perp^s$) followed by further bounce on the (m+1)-th flat interface ($\Re_{m+1|m+2}^h(k_\perp^s)\exp(j2k_{z(m+1)}^s\Delta_{m+1})$) with subsequently n_2 complete round-trips in the (m+1)-th layer ($[\vec{\Lambda}_{m+1}^h(k_\perp^s)]^{n_2}$) at the scattering angle ($k_\perp^s$), next a coherent transmission ($T_{m+1|m}^h(k_\perp^s)$) followed by subsequently j_2 complete round-trips ($[\vec{\Lambda}_m^h(k_\perp^s)]^{j_2}$) in the m-th layer with coherent reflections at the scattering angle ($k_\perp^s$), and finally a coherent transmission ($\exp(jk_{zm}^s\Delta_m)\Im_{m|0}^{h(slab)}(k_\perp^s)$) in the upper half-space through the intermediate layers m-1,...,1.

d) Local downward scattered waves from rough interface: each of these waves (see Fig.4.d) undergoes a coherent transmission into the m-th layer ($\Im_{0|m}'^h(k_\perp^i)\exp(jk_{zm}^i\Delta_m)$), through the intermediate layers 1,...,m-1, then j_1 complete round-trips ($[\vec{\Lambda}_m(k_\perp^i)]^{j_1}$) in the m-th layer with coherent reflections at the incident angle ($k_\perp^i$), next a coherent transmission ($T_{m|m+1}^h(k_\perp^i)$) followed by n_1 complete round-trips in the (m+1)-th layer ($[\vec{\Lambda}_{m+1}^h(k_\perp^i)]^{n_1}$) and by further bounce on the (m+1)-th flat interface ($\Re_{m+1|m+2}^h(k_\perp^i)\exp(j2k_{z(m+1)}^i\Delta_{m+1})$) at the incident angle ($k_\perp^i$), and after that an incoherent local scattering from the rough interface ($\alpha_{hh}^{m+1,m}(\mathbf{k}^s,\mathbf{k}^i)$), downward in the observation plane ($\mathbf{k}_\perp^s$), followed by further bounce on the (m+1)-th flat interface ($\Re_{m+1|m+2}^h(k_\perp^s)\exp(j2k_{z(m+1)}^s\Delta_{m+1})$) with subsequently n_2 complete round-trips in the (m+1)-th layer ($[\vec{\Lambda}_{m+1}^h(k_\perp^s)]^{n_2}$) at the scattering angle ($k_\perp^s$), then a coherent transmission ($T_{m+1|m}^h(k_\perp^s)$) followed by subsequently j_2 complete round-trips ($[\vec{\Lambda}_m^h(k_\perp^s)]^{j_2}$) in the m-th layer with coherent reflections at the scattering angle ($k_\perp^s$), and finally a coherent transmission ($\exp(jk_{zm}^s\Delta_m)\Im_{m|0}^{h(slab)}(k_\perp^s)$) in the upper half-space through the intermediate layers m-1,…,1.

It should be noted that (see Chapter 3):

$$[\overleftrightarrow{M}_m^p]^{-1}\Im_{m|0}^{p(slab)} = \begin{cases} \Im_{0|m}^p \dfrac{\mu_0 k_{zm}}{\mu_m k_{z0}} & for\ p = h \\ \Im_{0|m}^p \dfrac{\varepsilon_0 k_{zm}}{\varepsilon_m k_{z0}} & for\ p = v \end{cases} \tag{6.24}$$

Furthermore, to give reason for the several factors appearing in the expansion (23) in the form k_{zn}^s / k_{zm}^s, we observe that differentiating *Snell*'s law

$$\sqrt{\varepsilon_m}\sin\theta_m^s = \sqrt{\varepsilon_n}\sin\theta_m^s, \tag{6.25}$$

we obtain:

$$\frac{k_{zn}^s}{k_{zm}^s} = \frac{d\theta_m^s}{d\theta_n^s}, \tag{6.26}$$

where the angle θ_m^s identifies, within the m-th layer, the ray scattered in observation direction. This scattered ray can be thought as the contribution in the observation direction from a spherical wave emanating from the *m*th roughness. Therefore, the factor (26) accounts for the variation of the divergence of the locally scattered rays, which cross the flat boundaries stratification from the m-th to the n-th layer.

As a result, k_{zn}^s / k_{zm}^s is interpretable as the *scattered-ray-amplitude divergence factor,* associated with the varying refractive index. Note also that when the geometry reduces to the one depicted in Fig.2 of Chapter 4, only the first term in (23) holds (see Fig.4.a), so that the interpretation is fully congruent with the one furnished in Section 6.3.

6.4.2 Scattering through an arbitrary layered media with rough interfaces.

Similarly to the analysis conducted in Section 6.4.1, the process of scattering transmitted *through* the structure can be investigated as well referring to a generic rough interface of the stratification.

Accordingly, we now focus on the relevant analytical expression (5.72) for the downward (far-field) scattered wave into the lower half-space, through an arbitrary layered structure with an embedded (*m*-th) rough interface, with particular reference to the *hh* case (see (5.73)).

Similarly to Eq. (19), the following relation can be derived:

$$\begin{gathered} 1+\Re^{p}_{m+1|m}(k_\perp)=T^{p}_{m+1|m}(k_\perp) \\ \left[1+T^{p}_{m|m+1}(k_\perp)\Re^{p}_{m|m-1}(k_\perp)e^{j2k_{zm}\Delta_m}[\overleftarrow{M}^{h}_{m}(k_\perp)]^{-1}\right]. \end{gathered} \tag{6.27}$$

Furthermore, the following additional notation is introduced:

$$\overleftarrow{\Lambda}^{h}_{m}(k_\perp)=\Re^{h}_{m|m-1}(k_\perp)R^{h}_{m|m+1}(k_\perp)e^{j2k_{zm}\Delta_m}\,. \tag{6.28}$$

Likewise, by using (22) with $\vec{M}^{h}_{m+1}, \vec{\Lambda}^{h}_{m+1}$ or $\vec{M}^{h}_{m}, \vec{\Lambda}^{h}_{m}$ or $\overleftarrow{M}^{h}_{m}, \overleftarrow{\Lambda}^{h}_{m}$ or $\overleftrightarrow{M}^{h}_{m+1}$, $\overleftrightarrow{\Lambda}^{h}_{m+1}$, respectively, in place of M, Λ; taking into account (3)-(6) and substituting (17), (19) and (27) in (5.73), the final expansion (29) is obtained.

$$
\begin{aligned}
&{}_{N}^{0}\widetilde{\beta}_{hh}^{m,m+1}(\mathbf{k}^{s},\mathbf{k}^{i}) = \Im'^{h}_{0|m}(k_{\perp}^{i})\, e^{jk_{zm}^{i}\Delta_m} \sum_{j_1=0}^{\infty}[\vec{\Lambda}_{m}^{h}(k_{\perp}^{i})]^{j_1} \\
&\left\{ \beta_{hh}^{m,m+1}(\mathbf{k}^{s},\mathbf{k}^{i})\frac{k_{zN}^{s}}{k_{z(m+1)}^{s}} + \right. \\
&- T_{m|m+1}^{h}(k_{\perp}^{i})\sum_{n_1=0}^{\infty}[\vec{\Lambda}_{m+1}^{h}(k_{\perp}^{i})]^{n_1}\Re_{m+1|m+2}^{h}(k_{\perp}^{i})e^{j2k_{z(m+1)}^{i}\Delta_{m+1}}\alpha_{hh}^{m+1,m}(\mathbf{k}^{s},\mathbf{k}^{i})\frac{k_{zN}^{s}}{k_{z(m+1)}^{s}} \\
&+ \alpha_{hh}^{m,m+1}(\mathbf{k}^{s},\mathbf{k}^{i})\frac{k_{zN}^{s}}{k_{zm}^{s}}\Re_{m|m-1}^{h}(k_{\perp}^{s})e^{j2k_{zm}^{s}\Delta_m}\sum_{n_2=0}^{\infty}[\overleftarrow{\Lambda}_{m}^{h}(k_{\perp}^{s})]^{n_2}T_{m|m+1}^{h}(k_{\perp}^{s}) \\
&- T_{m|m+1}^{h}(k_{\perp}^{i})\sum_{n_1=0}^{\infty}[\vec{\Lambda}_{m+1}^{h}(k_{\perp}^{i})]^{n_1}\Re_{m+1|m+2}^{h}(k_{\perp}^{i})e^{j2k_{z(m+1)}^{i}\Delta_{m+1}}\beta_{hh}^{m+1,m}(\mathbf{k}^{s},\mathbf{k}^{i})\frac{k_{zN}^{s}}{k_{zm}^{s}} \\
&\left. \Re_{m|m-1}^{h}(k_{\perp}^{s})e^{j2k_{zm}^{s}\Delta_m}\sum_{n_2=0}^{\infty}[\overleftarrow{\Lambda}_{m}^{h}(k_{\perp}^{s})]^{n_2}T_{m|m+1}^{h}(k_{\perp}^{s}) \right\} \\
&\sum_{j_2=0}^{\infty}[\overleftrightarrow{\Lambda}_{m+1}^{h}(k_{\perp}^{s})]^{j_2}e^{jk_{z(m+1)}^{s}\Delta_{m+1}}\Im_{m+1|N}^{h(slab)}(k_{\perp}^{s}).
\end{aligned}
\tag{6.29}
$$

Analogous considerations can be done as in the previous case; however to save space we do not repeat similar examination. The reader can verify that the expansion (29), which is the counterpart of (23) for the scattering through the structure, can be similarly interpreted in terms a series of rays identified as pictured in Fig.5.

It is important to emphasize that (see Chapter 3):

$$
[\overleftrightarrow{M}_{m+1}^{p}]^{-1}\Im_{m+1|N}^{p(slab)} = \begin{cases} \Im_{N|m+1}^{p}\dfrac{\mu_N k_{z(m+1)}}{\mu_{m+1}k_{zN}} & for\ p=h \\ \Im_{N|m+1}^{p}\dfrac{\varepsilon_N k_{z(m+1)}}{\varepsilon_{m+1}k_{zN}} & for\ p=v \end{cases} \tag{6.30}
$$

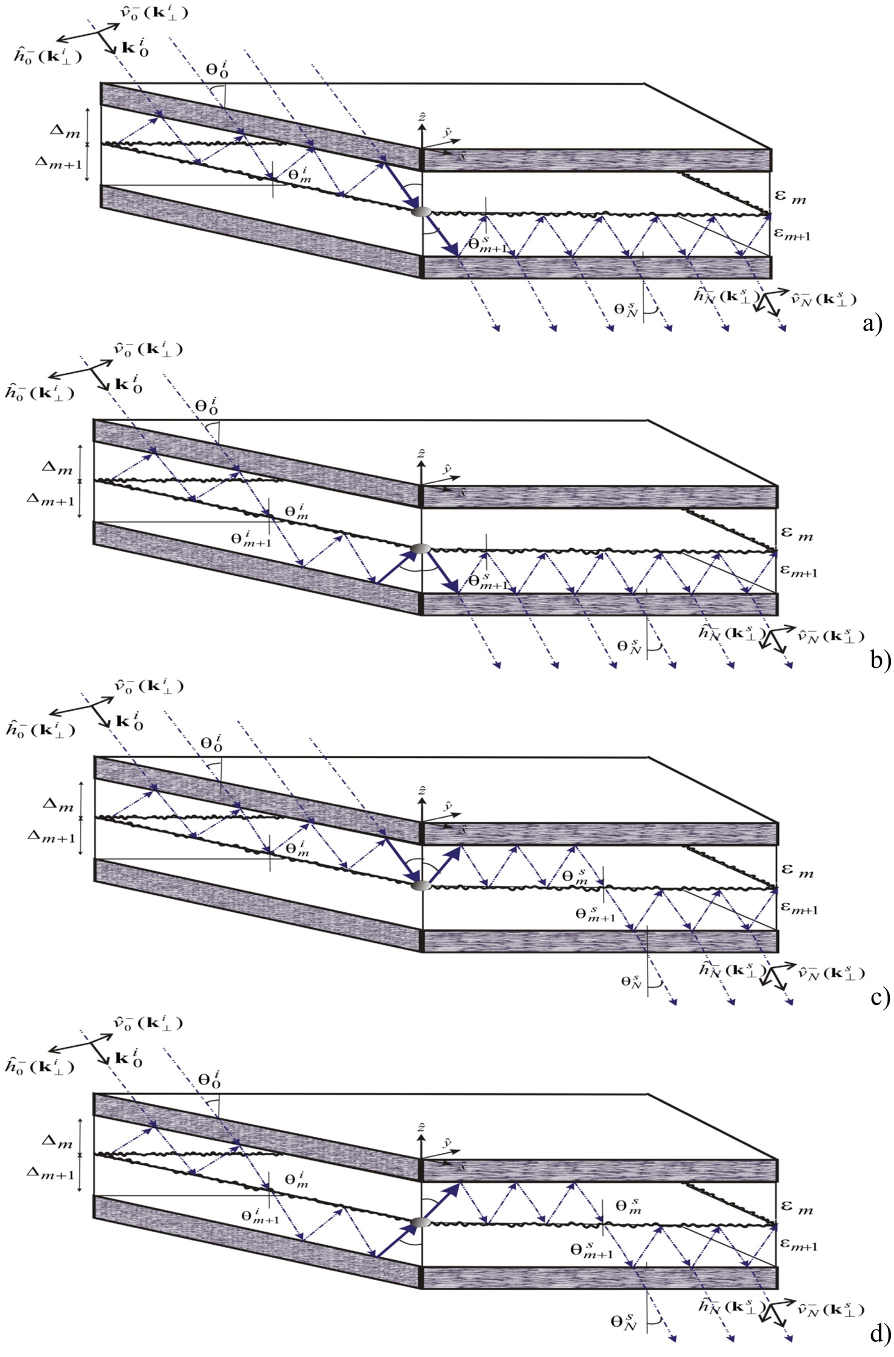

Fig.5. Physical interpretation for the scattering through an arbitrary layered structure with an embedded rough interface.

Once both the expansions, for the scattering from (23) and through (29) the structure, have been formulated and illustrated, some remarkable considerations are in order.

First of all, it should be noted that, even though the presented physical interpretations allow us directly visualizing the physics of the problem, they are not based on an intuitive approach but are carried out analytically starting from BPT solutions obtained in first-order limit. Specifically, the expansions have been carried out exploiting the *local nature* of the interaction between waves and corrugated interfaces within the layered structure; so that the global scattering response, from and through the layered structure, has been decomposed in terms of local interactions.

Although the investigation leads to expansions that can appear cumbersome, however, from the viewpoint of the comprehension of scattering mechanisms, each term of (23) and of (29) can be directly identified as a wave propagating in the structure: each of them comprises a series of coherent interactions with flat boundaries and an incoherent local single-scattering occurrence, from or through the corrugated interface. Therefore, equations (23) and (29) can be thought of as a *ray series* or a *geometrical optics series*.

Therefore, despite the expansions are attained rigorously without any further approximation with respect to the solutions proposed in Chapter 5, the resulting interpretations turn out to be extremely intuitive and surprisingly simple.

Finally, note also that when the arbitrary layered structure with all rough interfaces is concerned, since in the first-order limit the multiple scattering contributions are neglected, the relative physical interpretation can be obtained effortless by superposition of the several ray contributions obtained considering separately each rough interface.

The last but not the least factor distinguishing our approach is that, for the application point of view, the focus is often on the observed time-dependent wave-trains, rather than spectral intensities. As a matter of fact, propagation of the transmitted wave through the structure causes a superposition of the echoes scattered from the interfaces that are received by a sensor located above or under the structure. Concerning this aspect, note also that if the incident wave is a modulated pulse, each term of the ray series corresponds to an echo that will be received with a different time delay. Consequently, the obtained results also open the way toward a time-domain formulation of the problem. This aspect is of fundamental importance, for instance, if the considered models have to be embedded in a SAR simulator or in a ray tracing code to predict the characteristics of the radio channel.

6.5 Global and Local Scattering

To focus formally on the relations among *local* and *global* scattering concepts, the obtained *Wave Scattering Decomposition* (23) and (29) of the global scattering response in terms of the four types of local interactions (from and through the corrugated interface), can be expressed in a compact notation as scalar products of four-element vectors:

$$\tilde{\alpha}_{qp}^{m,m+1} = \mathbf{P}_m^{qp}(k_\perp^s, k_\perp^i) \cdot \mathbf{\Psi}_{qp}^{m,m+1}(\mathbf{k}^s, \mathbf{k}^i), \tag{6.31}$$

$${}_N^0\tilde{\beta}_{qp}^{m,m+1} = \mathbf{Q}_m^{qp}(k_\perp^s, k_\perp^i) \cdot \mathbf{\Psi}_{qp}^{m,m+1}(\mathbf{k}^s, \mathbf{k}^i), \tag{6.32}$$

wherein

$$\mathbf{\Psi}_{qp}^{m,m+1}(\mathbf{k}^s, \mathbf{k}^i) = [\alpha_{qp}^{m,m+1} \quad \beta_{qp}^{m+1,m} \quad \beta_{qp}^{m,m+1} \quad \alpha_{qp}^{m+1,m}]^T, \tag{6.33}$$

captures the local response of the *m*-th rough interface between two layer of permittivity ε_m and ε_{m+1}, and the *transfer vectors* $\mathbf{P}_m^{qp}$ and $\mathbf{Q}_m^{qp}$ are related to the coherent propagation inside the stratification. Consequently, $\tilde{\sigma}_{qp,m}^0$ depends on a combination of *local scattering proprieties* of the roughness between two homogeneous media ε_m and ε_{m+1}, ($\mathbf{\Psi}_{qp}^{m,m+1}(\mathbf{k}^s, \mathbf{k}^i)$), and of coherent propagation inside the stratification ($\mathbf{P}_m^{qp}$ and $\mathbf{Q}_m^{qp}$). This brings us to refer to $\tilde{\sigma}_{qp,m}^0$ as the (equivalent) *global scattering coefficients* of a layered structure with the *m*-th embedded corrugated boundary between two layers, respectively, of permittivity ε_m and ε_{m+1}.

As a particular situation, we mention the case in which the stratification under the roughness vanishes. This is the case of the last *(N*-1*)*-th rough interface of the layered structure, whereas $\tilde{\alpha}_{qp}^{N-1,N}$ depends on $\alpha_{qp}^{N-1,N}$ only, so that $\mathbf{P}_{N-1}^{qp}$ reduces to a scalar:

$$\frac{\tilde{\sigma}_{qp,N-1}^{0}}{\sigma_{qp,N-1}^{0}} = \left|\frac{\tilde{\alpha}_{qp}^{N-1,N}}{\alpha_{qp}^{N-1,N}}\right|^2 = \left|\mathrm{P}_{N-1}^{qp}(k_{\perp}^{s},k_{\perp}^{i})\right|^2, \qquad (6.34)$$

whereas for the *hh* case, we have:

$$P_{N-1}^{hh}(k_{\perp}^{s},k_{\perp}^{i}) = \frac{\Im_{0|N}^{h}(k_{\perp}^{s})}{T_{N-1|N}^{h}(k_{\perp}^{s})}\frac{\Im_{0|N}^{h}(k_{\perp}^{i})}{T_{N-1|N}^{h}(k_{\perp}^{i})}. \qquad (6.35)$$

As a result, the *transfer vector,* which measures the influence of the stratification on the local scattering, whatever the roughness is, can be expressed in terms of the *generalized transmission/reflection coefficients*.

It has been established that, when the observation point is located above or under the stratification, the *global* scattering by a generic rough interface embedded in the layered structure can be considered as a result of *local* scattering phenomena (from e through the embedded rough interface) *filtered* by the layered structure. The filtering action arises from the resulting interferential effects that take place in the layered structure, which are associated with the coherent interactions with the boundaries.

Moreover, when the *N*-rough interfaces structure is concerned, the global scattering (from and through layered structure) can be the thought as representative of superposition of filtered scattering phenomena that take place from and through each rough interface locally. Finally, we emphasize that the presented expansions and the introduced coefficients are not mere factorizations related to some analytical convenience, but are based on physical relevance.

6.6 Meaning of the first-order approximation

In this sub-section the physical significance of the first-order perturbative approximation in the layered structure context is clarified.

Note that the *transfer vectors* $\mathbf{P}_m^{qp}$ and $\mathbf{Q}_m^{qp}$ are affected in a global way by the unperturbed stratification properties and do not depend neither on the directions of $\mathbf{k}_\perp^i$ and $\mathbf{k}_\perp^s$, nor on the roughness. This aspect, from a different perspective, can be elucidated by means of the following further considerations. Indeed, for the phase-matching condition, the projections of the wave vectors, $\mathbf{k}_\perp^i$ and $\mathbf{k}_\perp^s$, for incident and scattered direction, respectively, must be invariant in the flat boundaries stratification, i.e., the propagation directions in multilayer flat-boundaries structure must be coplanar.

Therefore, the round trips within and the transmission through the layers are all constrained in the incidence plane or in the observation plane, individuated by the vectors $\mathbf{k}_\perp^i$ or $\mathbf{k}_\perp^s$ (and z direction), respectively.

This clarifies that the contributions contemplated by the first-order perturbative approximation are restricted within these two planes, whereas the neglected multiple scattering, associated with higher-order terms of the perturbative development, are not. In fact, it should be noted that although the considered expansions contain some infinite sums associated with multiple reflections between the interfaces, they account for a single diffuse scattering only, and only one of the surface spectral components, i.e. that specified by the so-called *momentum transfer* $\mathbf{k}_\perp^s - \mathbf{k}_\perp^i$, appears in the scattering process in the limit of the first-order perturbation method.

6.7 Scattering Enhancement Phenomenon

In this section, the focus is on one of the most interesting phenomenon, associated with coherent effects, which is perhaps a universal wave phenomenon inherent to waves of whatever physical nature; the aim of this theoretical analysis is to demonstrate that this enhancement phenomenon is contemplated by first-order BPT models. To this purpose, using the performed physical interpretations, we show that from each generic rough interface of a rough boundaries multilayer, due to the presence of the reflective action of the boundaries, coherent effects arise in the layered structure. These effects, contemplated in the *single* -scattering limit, arise in particular directions that can be clearly identified. For such a purpose, we focus our attention on waves that undergo to *local scattering through* a rough interface. Starting then from (31), which is a formal expression of the expansion (23), and evaluating it in the directions for which $|\mathbf{k}_\perp^s| = |\mathbf{k}_\perp^i|$, we get, for the *hh* case:

$$\tilde{\alpha}_{hh}^{m,m+1}(\hat{k}_\perp^s|\mathbf{k}_\perp^i|, \hat{k}_\perp^i|\mathbf{k}_\perp^i|) = \mathbf{P}_m^{hh}(k_\perp^i) \cdot \mathbf{\Psi}_{hh}^{m,m+1}(\hat{k}_\perp^s|\mathbf{k}_\perp^i|, \hat{k}_\perp^i|\mathbf{k}_\perp^i|) \tag{6.36}$$

Examination of the expansion (23) evaluated in the directions for which $|\mathbf{k}_\perp^s| = |\mathbf{k}_\perp^i|$, shows that the corresponding second and third elements of the *transfer vector* $\mathbf{P}_m^{hh}$ are identical, except for a minus sign. Both these elements are associated with the *local scattering through* ($\beta_{qp}^{m,m+1}, \beta_{qp}^{m+1,m}$) the rough interface in opposite directions (downward and upward directed).
Formally:

$$\mathbf{P}_m^{hh}(k_\perp^i) = a(k_\perp^i) b(k_\perp^i) \begin{bmatrix} \dfrac{1}{b(k_\perp^i)} & -1 & 1 & -b(k_\perp^i) \end{bmatrix} \tag{6.37}$$

where

$$a(k_\perp^i) = \Im_{0|m}^h(k_\perp^i)\, e^{j2k_{zm}^i \Delta_m} \frac{k_{z0}^i}{k_{zm}^i} \Im_{m|0}^h(k_\perp^i) \tag{6.38}$$

$$b(k_\perp^i) = T_{m|m+1}^h(k_\perp^i) \sum_{n_1=0}^{\infty} [\vec{\Lambda}_{m+1}^h(k_\perp^i)]^{n_1} \Re_{m+1|m+2}^h(k_\perp^i) e^{j2k_{z(m+1)}^i \Delta_{m+1}} \tag{6.39}$$

On the other hand, analyzing the scattering directions for which $|\mathbf{k}_\perp^s| = |\mathbf{k}_\perp^i|$, we observe that from the reciprocity (8) directly follows that:

$$-\beta_{hh}^{m+1,m}(\hat{k}_{\perp}^{s}|\mathbf{k}_{\perp}^{i}|,\hat{k}_{\perp}^{i}|\mathbf{k}_{\perp}^{i}|)=\beta_{hh}^{m,m+1}(\hat{k}_{\perp}^{s}|\mathbf{k}_{\perp}^{i}|,\hat{k}_{\perp}^{i}|\mathbf{k}_{\perp}^{i}|) \qquad (6.40)$$

Therefore, from (37) and (40) it follows that second and third terms in (36) are identical. This formal result is susceptible of an intuitive explanation in terms of coherent interaction between waves propagating through *multi-channel reciprocal paths* within the structure along scattering directions such that $|\mathbf{k}_{\perp}^{s}|=|\mathbf{k}_{\perp}^{i}|$. Preliminarily, to clarify the phenomenology we refer to the picture illustrates schematically in Fig.6a. By a solid and dashed lines, we have indicated only the propagation path of the scattering wave corresponding, respectively, to the terms $\beta_{qp}^{m,m+1}$ and $\beta_{qp}^{m+1,m}$ of (23) with the summation indexes $n_1=n_2=j_1=j_2=0$. Note that, in far field observation point, these two *reciprocal* waves interfere constructively, in spite of the randomness of the rough interface, for scattering direction such that $\mathbf{k}_{\perp}^{s}=\hat{k}_{\perp}^{s}|\mathbf{k}_{\perp}^{i}|$.

In general, the terms of (23) relative to the local scattering through the rough interface $\beta_{qp}^{m,m+1},\beta_{qp}^{m+1,m}$, for which $n_1=n_2=n$ (and $j_1+j_2=j$) constitute a family of *reciprocal local scattered ray partners*; each pair undergoes to total number (j) of round trips in the m-th layer as well as to a total number (n) of round trips in (m+1)-th layer. This mutual coherent wave partners, scattered locally through the interface in opposite direction, whatever be the random phase introduced by the roughness, sum-up in phase.

In other words, this phenomenon arises from multi-channel wave propagation of reciprocal wave partners passing through identical channels with zero phase difference. Note also that the term *reciprocal* derives from the fact that the two partners cross the roughness in opposite directions. Consequently, the waves partners, resulting from such reciprocal scattering events, have the same amplitude and phase (i.e. these waves interfere constructively) if the projection on the z=0 plane of wave vectors of the incident and scattered waves have the same modulus ($|\mathbf{k}_{\perp}^{s}|=|\mathbf{k}_{\perp}^{i}|$). As a result, the sum-up in phase of all these terms exists only for directions lying in a cone defined by the incident direction, and whose axis is parallel to z direction as in Fig.6. This brings us to refer to this family of directions as the *enhancement scattering cone.*

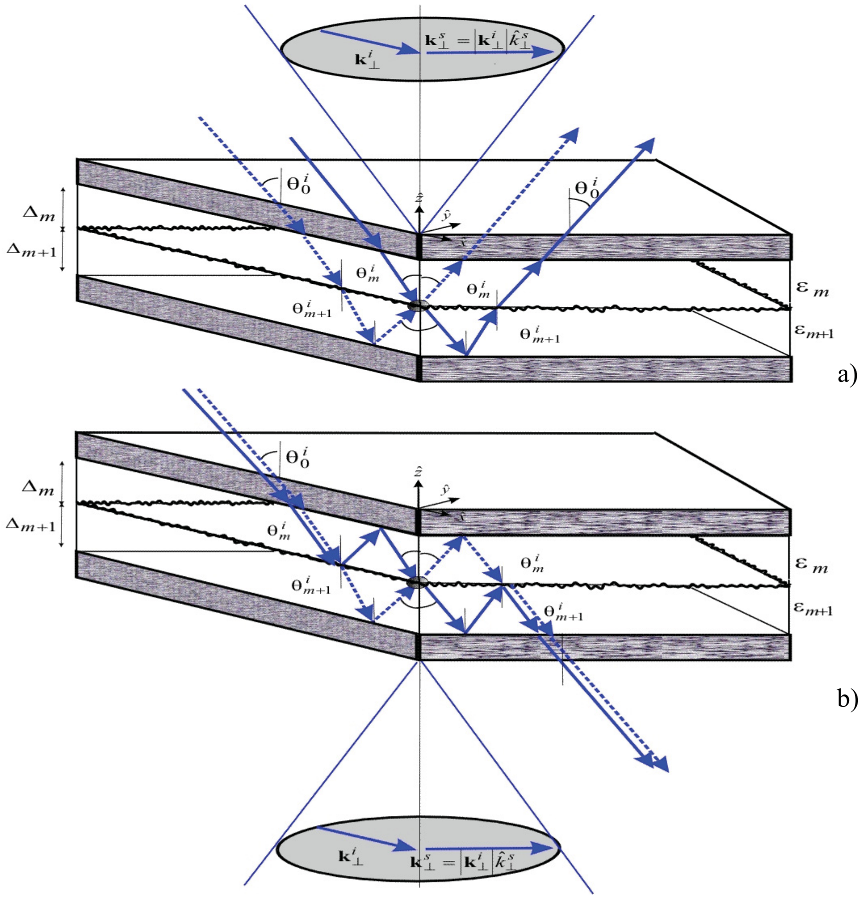

Fig.6. Physical interpretation of coherent effect in a layered structure with an embedded rough interface.

Analogous considerations hold for the case of scattering through the structure. In Fig.6.b, by dashed and solid lines we have indicated the propagation path of the scattering wave corresponding to the terms $\beta_{qp}^{m,m+1}, \beta_{qp}^{m+1,m}$ of (29) with and $n_1=n_2=0, j_1=j_2=1$ and $n_1=n_2= j_1=j_2=0$, respectively.

In conclusion, the expansions allow us to demonstrate that the well-known enhancement phenomenon is contemplated by BPT models. This analysis is accompanied by theoretical interpretations describing coherent interference between reciprocal scattered waves.

In addition, we have shown that the enhancement phenomenon manifestation appears not only in backscattering [9][10][20] and specular [20] directions, but arises also in several directions which identify an enhanced scattering cone, disappearing as the angle of scatter deviates away from these directions since the two waves partners are no longer in phase and the coherent effect weakens. However, the factor $\hat{k}_{\perp}^{s} \cdot \hat{k}_{\perp}^{i}$ (essentially related to the definition of polarization in the global coordinate system [25]) appearing in all of the terms, see (3)-(6)), is responsible for the fact that this enhancement is more evident near back- and forward-scattering directions, in agreement with the experimental data [20]. Moreover, a similar examination can be conducted for each rough interface of the structure, pointing out the corresponding Enhancement Scattering Cone.

We stress that the analysis can be obviously particularized to the backscattering configuration ($\mathbf{k}_{\perp}^{s} + \mathbf{k}_{\perp}^{i} = 0$), and analogously a ray interpretation can be used to visualize the coherent effect.

We underline that, although the two considered components sum-up in phase when the scattering directions are along the enhancement cone, however this might give rise to enhancements or reductions of the total scattered intensity, depending on how the first and fourth terms in (36) interfere with the “enhanced” second+third term. Nonetheless, to evaluate the attractiveness of this analysis a time-domain context should appear more relevant.

Finally, we emphasize that the scattering enhancement phenomenon is not accounted for by the radiative transfer theory [18]. Actually, the manifestations of these effects, that remain after ensemble averaging, could not be contemplated without employing full wave analysis which preserves phase information. Therefore, this effect should be taken into account, for instance, when data of the remote sensing of the Earth’s structures are interpreted.

6.7.1 Numerical Examples

In this section, some numerical examples aimed at better clarify the actual consequences of the coherent effects on the scattering response are presented.

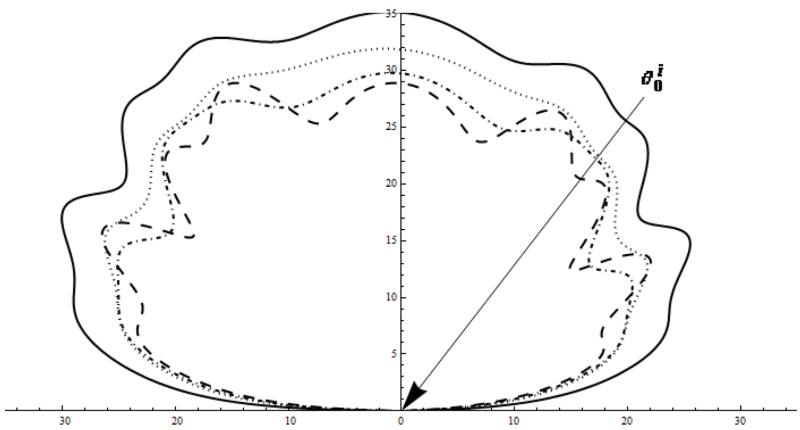

Fig. 7. Scattering coefficients *hh* for a three rough interfaces layered media with a fixed incidence angle: ζ_0 contribution (long-dashed line), ζ_1 contribution (dotted-dashed line), ζ_2 contribution (dotted line), total contribution (solid line). Note that an offset of 50 dB has been conveniently added to each scattering pattern. The incidence direction is also shown.

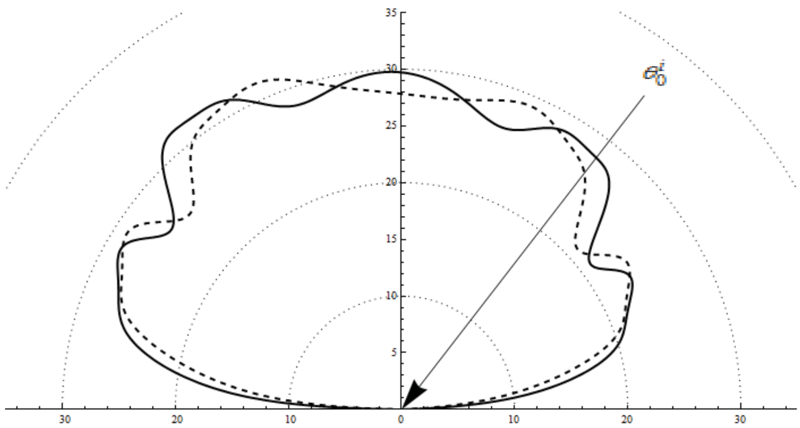

Fig. 8. Scattering contribution *hh* from the rough interfaces ζ_1 for prescribed incidence angles ($\theta_0^i = 37.8$) and for Δ_2/λ =4.30 (solid line) and 3.80 (dashed line). The associate incidence direction is also shown. Note that an offset of 50 dB has been conveniently added to each scattering pattern.

To this purpose, we refer to a canonical layered medium with three rough interfaces, which is representative of several situations of interest. The considered vertical profile is characterized by the following parameters: $\varepsilon_0 = 1$, $\varepsilon_1 = 3.0 + j0.0$, $\varepsilon_2 = 5.5 + j0.00055$, $\varepsilon_3 = 10.5 + j1.55$; $\Delta_1/\lambda = 3.20$, $\Delta_2/\lambda = 4.30$.

In addition, we model the roughness of the interfaces as Gaussian 2-D random processes with *Gaussian correlations*, characterized by the surface height standard deviation σ_n and correlation length l_n, with the subscripts n referring to the n-th interface.

In order to perform a consistent comparison, we refer to interfaces with the same roughness statistics. In addition, we suppose no correlation between the interfaces. We assume $k_0 l_n$=1.5, $k_0\sigma_n$=0.15 for n= 0, 1, 2. Once this reference structure has been characterized, we first study the scattering cross section of the structure as a function of the scattering vertical angle in the upper half-space, assuming fixed the incident direction. In the polar plot of Fig.7 the total scattered field is shown (solid line), together with the individual contributions of the different interfaces, as a function of the scattering angle θ_0^s. To save space, only the *hh* case is shown. It should be noted that to visualize the patterns an offset of 50 dB has been conveniently added to each scattering pattern. In addition, it has been assumed θ_0^i= 37.8°; we emphasize that this value of incidence angle has been calculated to have all the four local contributions of eq.(36) summing up in phase when $\theta_0^s = \theta_0^i$ for the interface ζ_1, whereas this does not happen for the other interfaces In fact, in Fig.7 the backscattering enhancement only appears in the individual return from ζ_1.

In order to better illustrate this issue, we focus our attention on the scattering contribution from the interface ζ_1: in Fig.8 we plot the scattering contribution from the interface ζ_1 with no change with respect to the previous example (solid line) and by changing the ratio Δ_2/λ to the new value 3.80 (dashed line). Again, it is clear that in the former case a backscattering enhancement is present, whereas it is not in the latter. It should be also noted that the shape of pattern responses are also affected by the coherent effects due to the unperturbed interfaces of the structure.

This simple example explains that, in agreement with the proposed analysis, a backscattering enhancement effect may appear even in the first order field, although it is not always present.

6.8 Conclusion

In this Chapter, we have investigated the physical meaning of the existing first-order perturbative solutions for the field scattered by layered structures with rough interfaces. In order to capture the physical significance of the analyzed formulations, suitable expansions of the closed form solutions are rigorously performed leveraging on local scattering descriptors. The general approach has been applied to both scattering configurations (from and through the layered structure); thus the obtained expansions render a lucid interpretation of the scattering mechanisms that take place in a layered structure, whereas the series, which can be seen as a *ray series* or a *geometrical optics series,* offer a clear physical perspective of the interferential phenomena involved. Consequently, the global scattering response can be thought as the superposition of single-scattering *local* interactions *filtered* by the layered structure, whereas the filter action arises from the interferential effects due to the coherent interactions with the boundaries. Moreover, the physical meaning of the first-order perturbative approximation has been clarified in the layered structure context.

It should be also noted that, despite the expansions are attained rigorously without any further approximation with respect to the BPT solution proposed in Chapter 5, the resulting interpretations turn out to be extremely intuitive and surprisingly simple. Therefore, the global scattering problems, which were introduced in Chapter 5 as formal mathematics in the first-order perturbative limit, turn out to be completely interpretable by simple physical concepts.

As a result, the obtained expansions also allow us to identify all the scattering directions for which the scattering enhancement phenomenon may be contemplated by our perturbative models in the first-order limit (*Enhancement Scattering Cones*).

We want to explicitly underline that the obtained expansions primarily give insight into the perturbative analytical results, so enabling a relevant physical interpretation involving ray-series representation. However, for practical calculation purposes, the more compact notation of Chapter 5 can be more conveniently used. This is certainly true if only the frequency (or, better, phasor) domain solution is of interest (this is the case, in practice, when a sinusoidal time dependence is a sufficient approximation).

However, the expansions presented in this Chapter open the way to a time domain characterization of the scattering response, since each ray of the series corresponds to an echo that will be received with a different time delay. As a result, the proposed expansions may be also useful in practice when a time-domain solution is required.

As a result, the comprehensive scattering model obtained in the BPT framework proved extremely fruitful not only in that it provides an effective tool which permits to systematically analyze the bi-static scattering patterns of 3D multilayered rough media. In fact, it is important to note that a deep comprehension of the physical phenomena involved in the electromagnetic wave scattering interaction with such kind of complex structures would have been a rather hopeless task before the introduction of the BPT.

In conclusion, the implications of the obtained expansions are twofold. In fact, not only the phenomenologically successful BPT models give us deep insight into the physics of scattering problem, and as such are crucial from a speculative investigation perspective; what's more, they open the way toward new techniques for solving the inverse problem, for designing SAR processing algorithms, and for modelling the time-domain response of layered structures. These aspects will be a matter of further investigation.

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Chapter 7

Volumetric-Perturbative Reciprocal Theory

"La funzione più importante dell'osservazione e del ragionamento, come pure dell'intuizione e dell'immaginazione, è quella di aiutarci ad esaminare criticamente quelle congetture ardite che sono i mezzi con cui sondiamo l'ignoto."

Karl Raimund Popper

The aim of this Chapter is to present the *Volumetric-Perturbative Reciprocal Theory* (VPRT) formulation for the evaluation of the electromagnetic wave interaction with non trivial random stratifications; it is intrinsically reciprocal, wholly avoiding the Green's functions formalism.

The adopted structural description of media and interfaces is methodologically conceived to consistently treat both interfacial and volumetric random inhomogeneities, relying on a proper characterization of the dielectric permittivity space-variant perturbation. Accordingly, the developed comprehensive scattering approach methodologically permits to, simultaneously and rigorously, take into account both rough-interface scattering and volume scattering.

First, a new look at classical SPM for rough surface is offered within VPRT framework, so demonstrating how that the pertinent solution can be now derived in a conceptually neat way. The presented first-order VPRT formulation is then applied to the case of a layered structure with rough interfaces: the relevant polarimetric closed-form solution, formally derived for a 3-D layered geometry and a bistatic radar configuration, can be directly expressed in terms of unperturbed solutions and turns out to be fully consistent with the one obtained in the BPT theoretical framework. In order to emphasize the neat physical significance of relevant methodological approach, a remarkable interpretation of the analytical solution in terms of the *Rumsey*'s reaction concept is provided, and the concept of *multi-reaction* is introduced. Finally, the canonical case of random semi-infinite 3-D media, with both interfacial roughness and volumetric fluctuations, is addressed and the concept of *effective power spectral density* of the structure is introduced for considering the absorption in the medium. As a result, the comprehensive mathematical structure of VPRT enables a unified perturbative formulation jointly taking into account the scattering phenomena, which are formally presented within a unitary formal construct and directly interpreted in terms of the fundamental *Rumsey*'s reaction concept.

7.1 Introduction and Motivation

Generally speaking, *Perturbation Methods* have been extensively applied in many areas to obtain formal solutions to problems whose closed form was impossible, too difficult or not convenient to be obtained.

The essential idea behind *perturbation theory* applied to a physical system is the attaining of approximate solutions for such systems by suitably transforming exact solution of the approximate system, whereas the systems can be regarded as obtained from a solvable system by the addition of a small effect (perturbation). Nevertheless, this simple idea is completely obscured by the bulky classical SPM formulations, and the relative physical significance remains hidden in the available analytical derivations. In spite of its widespread recognition, the SPM solution is commonly obtained via involved procedures that require tedious manipulations even in the first-order approximation. Besides, several procedures have been applied in the derivation of the classical SPM solution. SPM solution can be derived by using Rayleigh method, which relies on the *Rayleigh hypothesis*, or else by employing the more involved *extended boundary conditions*. The *Green* functional formalism can be utilized or not. Commonly, *perturbation of the boundary conditions* is employed, even if perturbation of the dielectric constant volumetric distribution can be also considered. Indeed, the currently available procedures, as it is shown in this Chapter, not only require an unnecessary complication; in addition, they lead to obscuring the underling physics as well as the essence of the perturbative approach.

More generally, the analysis of scattering by layered and/or inhomogeneous structures with rough boundaries is of crucial importance for many applications. As a matter of fact, natural stratifications exhibit rough interfaces and volume inhomogeneities, which are both responsible for the scattering from the pertinent structure.

A brief discussion is now useful to understand the state of the art in this field. A considerable effort has been devoted to study the wave scattering by stratifications and several papers have been published: within this framework, modeling in microwave remote sensing of natural structures is of interest in [16]-[25], whereas analyzing optic thin films is the subject in [26]-[27]. The classical SPM solution for rough surface was first obtained for perfect conductive surface and then extended to dielectric surface separating two homogeneous half-spaces [19][5][6][8][9][31]. On the other hand, a closed-form solution for the volume scattering by a flat-boundaries stratification in the *Born* approximation may be easily found in literature for simplified geometry [5] including a very small number of layers; however, at best of our knowledge, a solution is not available for stratification with an arbitrary number of layers. Moreover, volumetric scattering by a perturbed random half-space was considered only limitedly to asymptotically small absorption [12] or columnar assumptions [55].

Furthermore, noticeable progress has been attained in the investigation on the extension of the classical perturbative solution for the scattering from rough surface to layered structure with an arbitrary number of rough interfaces (see Chapter 5): the systematic BPT formulation based on the perturbation of boundary conditions has been introduced to deal with the analysis of a layered structure with an arbitrary number of rough interfaces.

Specifically, the results of the *Boundary Perturbation Theory* (BPT) lead to polarimetric, formally symmetric and physical revealing first-order solutions: in this case the theoretical construct is based on a suitable perturbation in the geometry of the problem and the scattering problem is treated by adopting a proper perturbation of boundary conditions. These ensuing general solutions have been obtained in closed form for a 3-D geometry and a bi-static configuration, concerning scattering configurations from [23] and through [24] layered structures; *generalized reflection/transmission* have been adopted to get compact solutions [23]-[24]. Furthermore, these solutions enable us to express scattering amplitudes as made up of terms, each one amenable of a proper physical interpretation, that allows fully identifying the scattering mechanisms involved into the structure, as discussed in Chapter 6 (see also [25]). These solutions can be also regarded as generalization to layered media with rough interfaces of the classical SPM method originally developed for rough surfaces [5]-[6]. In addition, most of the already existing perturbative approaches [16][17][18][19], originally developed for simplified configurations in the first-order approximation, can be rigorously regarded, in a unified framework, as special case of BPT solutions [20]. Although the BPT final solution (see also [23]-[24]) is expressed in a compact form, its derivation, as presented in Chapter 5, is however very involved.

Indeed, most of the approaches for solving scattering problem, hitherto proposed by different Authors, conceive as conceptually different the nature of the scattering contributions pertaining to volume and surface inhomogeneities. Accordingly, distinct scattering formulations have been correspondingly adopted to cope with two different structural elements separately: interfacial inhomogeneity are usually described by means of suitable 2-D random processes pertinent to the geometry of the boundary, whereas volumetric inhomogeneity are modeled as 3-D random processes pertaining to the medium dielectric properties.

Although, volume and surface scattering contributions have usually been formulated separately, we quote some approaches, in which the interfacial roughness is seen as a permittivity fluctuation, that have been also presented [40]-[41]: they are, however, semi-analytical (in as much as the multi-layer Green function has to be first computed numerically) or limited to two-dimensional geometry, so that none of them providing results amenable of practical applicability.

As a result, it is worth noting that the crucial limitation resides in the different descriptions adopted for interfacial or volumetric fluctuations that preclude a unitary treatment of the two scattering contributions. Accordingly, the interaction arising between these two different scattering phenomena is not commonly taken into consideration, as no comprehensive formulation for treating their combined effects is available.

It is important to note that when both surface and volume scattering, respectively ascribable to different kind of inhomogeneities, are concerned, the distinction between these two kinds of phenomena in random media is somehow arbitrary and the adoption of a certain structural description for the scattering medium is only a matter of convenience.

Conversely, the coexistence of interfacial roughness and volumetric fluctuations in actual structures should be taken into account methodologically and an inclusive scattering analysis, even though approximate, should be fulfilled, in order to clear understand the distinguishing characteristics of these two different scattering mechanisms.

To overcome those limitation, we derive the a new formalism necessary for the theoretical description of the scattering processes, providing a mathematically consistent scheme that has the great result of the uniformity in the treatment of the two different types of scattering phenomena.

Therefore, in this Chapter, the formulation of the *Volumetric-Perturbative Reciprocal Theory* (VPRT) is developed. VPRT is based on two key elements: the use of the Reciprocity Theorem [28] and an appropriate description of the scattering structure in terms of perturbation of the dielectric constant volumetric distribution; this is a formal alternative to the perturbation of the boundary conditions, which was employed in Chapter 5.

A short discussion on the newness in use of these two key elements is in order. The description in terms of perturbation of the dielectric constant volumetric distribution is widely used in volumetric scattering problems e.g., [5], [6], [15], [31], and it leads to the well known *Born* approximation. In some cases [27] the description in terms of perturbation of the dielectric constant volumetric distribution has been also used to evaluate the scattering from a rough surface, although the connection of the obtained solution with the classical SPM solution has not been highlighted.

We are here essentially interested in presenting a general theoretical formulation whose structure maintains analytical consistency with both BPT construct and classical *Born* approximation for volume fluctuation formulations. First of all, when an inclusive scattering formulation is concerned, a conceptual key-point resides in the selection of appropriate and mutually compatible mathematical descriptions for interfacial roughness and volumetric inhomogeneities. In this perspective, we adopt a systematic description for the interfacial and volumetric inhomogeneities, in which both are regarded as an appropriate space-variant permittivity perturbation.

In addition, it is well known that Reciprocity Theorem can help solving scattering problems [28]-[30], even if its use is not so popular with respect to other approaches that may lead to equivalent results. However, at best of our knowledge, Reciprocity Theorem and volumetric distribution have never been used to compute scattering from a multi-layer. In addition, the intrinsically reciprocal formulation permits to wholly avoid the cumbersome Green functions formalism.

The VPRT construct introduced in this Chapter is based on an intrinsically reciprocal approach; we also show how our formulation methodologically will allow us to obtain, even if in the first-order limit, a rigorous and unified treatment for both volume and interface scattering.

The analytical solutions obtained in the VPRT framework are then provided directly in terms of the unperturbed solutions known in closed-form. Accordingly, the formulation here presented clearly illustrates that the first-order scattered field can be formally expressed as a proper coupling of two unperturbed solutions, thus clearly revealing the intrinsic aim of use of the perturbation theory.

First of all, since this simple idea of perturbation theory is currently completely obscured by the bulky classical SPM formulations for a gently rough surface, whose relative physical significance remains hidden in the available analytical derivations, we preliminarily demonstrate how to straightforwardly derive the classical SPM scattering solution in a surprisingly simple way. As a result, the canonical SPM can be arranged in the new methodological perspective offered by the powerful VPRT framework, in which it can be derived in a conceptually neater, more concise and clearer way, gaining a more direct comprehension in a methodological perspective.

Furthermore, the adopted methodological approach, when applied to rough multilayer, offers a certain inherent analytical convenience and can be effectively conduct to a formal solution for scattering from rough layered media, which can be carried out much more straightforwardly with respect to pertinent BPT derivation (see Chapter 5): we demonstrate that the proposed formulation, somehow surprisingly, has a reduced mathematical complexity. This can be explained by observing that the new formulation makes only use of the vector electric field, whereas the BPT formulation based on *perturbation of boundary conditions* requires the analysis of both magnetic and electric fields.

Finally, VPRT also allow us to jointly take into account the scattering phenomena from both interfacial roughness and volumetric fluctuations for random semi-infinite three dimensional (3-D) medium: the obtained perturbative solution is then given in closed-form for a bi-static configuration, providing a common analytical structure for both the scattering processes. To characterize the overall scattering response of the structure, the *effective power spectral density* (effective PSD) of the structure is also introduced. Therefore, the presented formulation helps us in gaining an analytical insight into the distinguishing character of the scattering mechanisms, neatly clarifying and quantifying the role played by the two scattering contributions in a common analytical framework.

Therefore, the VPRT framework presented in this Chapter enables a comprehensive perturbative analysis for the evaluation of the scattering from randomly inhomogeneous media. The developed unified perturbative formulation permits to treat consistently both these mentioned inhomogeneities and evaluate on an analytic playground, in the *weak* fluctuation approximation, the contributions pertinent to the two corresponding scattering mechanisms involved in a unitary theoretical framework. A proper description for the morphological features of the considered perturbed structure and formal expression for the unperturbed fields are specifically provided. The solutions here provided are all expressed in terms of the experimentally relevant quantities such as the *scattering cross* *sections*, whose expressions are directly related in terms of microscopic entities such as the structural *correlation functions*.

The Chapter is organized as follows.

In section 7.2, the innovative VPRT scattering formulation is proposed. A new look at SPM for rough surface is provided in Section 7.3. The formulation is applied to the scattering from rough interfaces of a multilayer in Section 7.4, so and the formal consistency of the solution with the one of the BPT is also analytically provided. In addition, a lucid interpretation of the perturbative solution is given in terms of the multi-reaction. In order to demonstrate how VPRT enables the joint evaluation of the scattering contributions relevant to the interfacial and volumetric inhomogeneities a canonical structure is addressed in Section 7.6. Conclusions are finally drawn in Section 7.7.

7.2 Volumetric Perturbative Formulation

In this Section we introduce the volumetric perturbative formulation: a general scattering problem is analyzed.

Let us consider a source current density $\mathbf{J}(\mathbf{r})$ radiating an electromagnetic field $\mathbf{E}(\mathbf{r})$, $\mathbf{H}(\mathbf{r})$ in an inhomogeneous medium characterized by a distribution, $\varepsilon = \varepsilon(\mathbf{r})$, of its relative dielectric permittivity . The electric field satisfies the vector *Helmholtz* equation (in the following, a factor $exp(-j\omega t)$ is understood and suppressed):

$$\nabla\times\nabla\times\mathbf{E}(\mathbf{r}) - k_0^2\varepsilon(\mathbf{r})\,\mathbf{E}(\mathbf{r}) = jk_0\eta_0\mathbf{J}(\mathbf{r}), \qquad (7.1)$$

where k_0 and η_0 are the propagation constant and the intrinsic impedance of vacuum, respectively. Let us now assume that the considered medium can be seen as an unperturbed medium with relative permittivity $\varepsilon = \varepsilon^{(0)}(\mathbf{r})$ to which a perturbation $\delta\varepsilon(\mathbf{r})$ is applied, so that $\varepsilon(\mathbf{r}) = \varepsilon^{(0)}(\mathbf{r})+\delta\varepsilon(\mathbf{r})$; let us also define the unperturbed field $\mathbf{E}^{(0)}(\mathbf{r})$ as the field radiated by $\mathbf{J}(\mathbf{r})$ in the unperturbed medium:

$$\nabla\times\nabla\times\mathbf{E}^{(0)}(\mathbf{r}) - k_0^2\varepsilon^{(0)}(\mathbf{r})\,\mathbf{E}^{(0)}(\mathbf{r}) = jk_0\eta_0\mathbf{J}(\mathbf{r}). \qquad (7.2)$$

By subtracting (2) from (1) we get

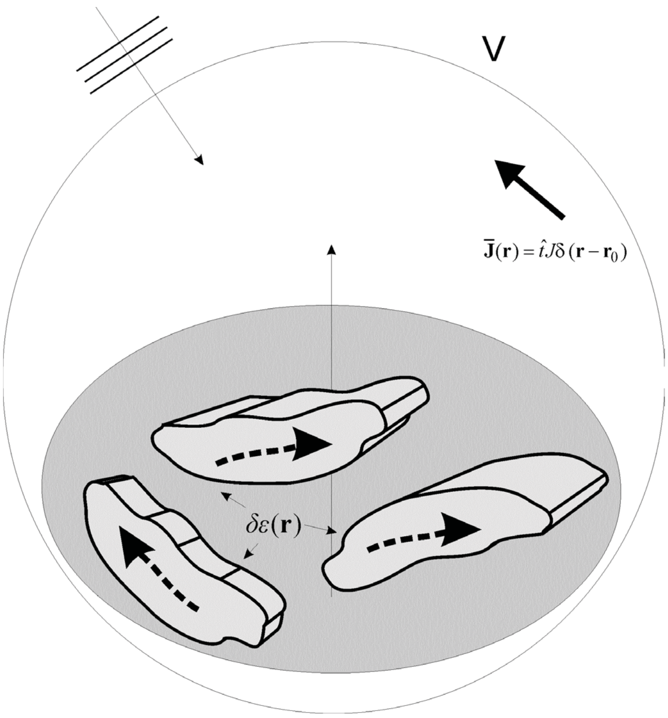

Fig. 1. Geometry of the scattering problem.

$$\nabla \times \nabla \times \mathbf{E}^{(1)}(\mathbf{r}) - k_0^2 \varepsilon^{(0)}(\mathbf{r})\,\mathbf{E}^{(1)}(\mathbf{r}) = k_0^2 \delta\varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r})\,, \qquad (7.3)$$

where $\mathbf{E}^{(1)}(\mathbf{r})= \mathbf{E}(\mathbf{r}) - \mathbf{E}^{(0)}(\mathbf{r})$ is the field perturbation. Equation (3) shows that the field perturbation can be considered as radiated by an equivalent current density $\mathbf{J}_1(\mathbf{r})$:

$$\mathbf{J}_1(\mathbf{r}) = -jk_0\eta_0^{-1}\delta\varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r}) \cong -jk_0\eta_0^{-1}\delta\varepsilon(\mathbf{r})\mathbf{E}^{(0)}(\mathbf{r})\,; \qquad (7.4)$$

in eq.(4) medium perturbations $\delta\varepsilon(\mathbf{r})$ are assumed to be small, then the field perturbation $\mathbf{E}^{(1)}(\mathbf{r})$ turns out to be small with respect to the unperturbed field $\mathbf{E}^{(0)}(\mathbf{r})$, which leaded us to replace $\mathbf{E}(\mathbf{r})$ with $\mathbf{E}^{(0)}(\mathbf{r})$ in eq.(4).

In order to compute the perturbed field in a generic point $\mathbf{r}_0$, we define an auxiliary (fictitious) source

$$\overline{\mathbf{J}}(\mathbf{r}) = \hat{t} J \delta(\mathbf{r} - \mathbf{r}_0), \tag{7.5}$$

where $\hat{t}$ is an arbitrarily oriented unit vector, $\delta(\cdot)$ is the *Dirac* delta function, and J = 1 A·m is a unitary constant introduced for dimensional consistency reasons. This (*test*) source radiates a field $\overline{\mathbf{E}}^{(0)}$ in the unperturbed medium. By applying the *Reciprocity Theorem* [28] we get:

$$\iiint_V \left\{ \mathbf{E}^{(1)}(\mathbf{r}) \cdot \overline{\mathbf{J}}(\mathbf{r}) - \overline{\mathbf{E}}^{(0)}(\mathbf{r}) \cdot \mathbf{J}_1(\mathbf{r}) \right\} d\mathbf{r} = 0, \tag{7.6}$$

where V is a volume enclosing all the sources, as pictured in Fig.1. Use of (4) and (5) in (6) leads to

$$\mathbf{E}^{(1)}(\mathbf{r}_0) \cdot \hat{t} = -j \frac{k_0}{J\eta_0} \iiint_V \overline{\mathbf{E}}^{(0)}(\mathbf{r}) \cdot \delta\varepsilon(\mathbf{r}) \mathbf{E}^{(0)}(\mathbf{r}) d\mathbf{r}. \tag{7.7}$$

Equation (7) allows us evaluating the field perturbation from knowledge of the medium perturbation and of the two unperturbed fields radiated by real and fictitious sources.

If the unperturbed medium have discontinuity planes orthogonal to the z-axis, then it is convenient to distinguish between transverse $\mathbf{E}_\perp$ and longitudinal field E_z components (i.e., the field components orthogonal and parallel to the z-axis):

$$\mathbf{E}^{(0)}(\mathbf{r}) = \mathbf{E}_\perp^{(0)}(\mathbf{r}) + \hat{z} E_z^{(0)}(\mathbf{r}) = \mathbf{E}_\perp^{(0)}(\mathbf{r}) + \hat{z} \frac{D_z^{(0)}(\mathbf{r})}{\varepsilon_v \varepsilon^{(0)}(\mathbf{r})}, \tag{7.8}$$

where ε_v is the dielectric constant of vacuum and $D_z^{(0)} = \varepsilon_v \varepsilon^{(0)} E_z^{(0)}$ is the z-component of the unperturbed electric flux density. Accordingly, eq.(7) can be rewritten as

$$\mathbf{E}_0^{(1)}(\mathbf{r}_0)\cdot\hat{t} = -j\frac{k_0}{J\eta_0}\iiint_V \Big[\overline{\mathbf{E}}_\perp^{(0)}(\mathbf{r})\cdot\delta\varepsilon(\mathbf{r})\mathbf{E}_\perp^{(0)}(\mathbf{r}) + \\ + \overline{D}_z^{(0)}(\mathbf{r})\frac{\delta\varepsilon(\mathbf{r})}{\varepsilon_\mathrm{v}^2\left[\varepsilon^{(0)}(\mathbf{r})\right]^2}D_z^{(0)}(\mathbf{r})\Big]d\mathbf{r}. \tag{7.9}$$

It is important to note that throughout this thesis with a parenthesized superscript we systematically indicate the order of perturbation.

On the other hand, where explicitly indicated, a subscript m distinguishes the pertinent m-th spatial region of the medium. Indeed, we also emphasize that a field solution assumes different expressions depending on the specific region of the structure which is concerned. Accordingly, to indicate the pertinent field expression within a specific (m-th) region, if necessary a proper subscript (m) is included in the field notation. Conversely, when the relevant subscript is omitted we indicate the overall field solution. For instance, $\mathbf{E}_0^{(1)}$ denotes the relevant expression assumed by the electric field first-order perturbation $\mathbf{E}^{(1)}$ in the 0-th region.

Finally, by noting that, for small medium perturbation,

$$\varepsilon^{-1}(\mathbf{r}) \cong \frac{1}{\varepsilon^{(0)}(\mathbf{r})} - \frac{\delta\varepsilon(\mathbf{r})}{\left[\varepsilon^{(0)}(\mathbf{r})\right]^2} \ , \tag{7.10}$$

so that

$$\delta\varepsilon^{-1}(\mathbf{r}) = -\frac{\delta\varepsilon(\mathbf{r})}{\left[\varepsilon^{(0)}(\mathbf{r})\right]^2} \ , \tag{7.11}$$

we have

$$\mathbf{E}_0^{(1)}(\mathbf{r}_0)\cdot\hat{t} = -j\frac{k_0}{J\eta_0}\iiint_V d\mathbf{r}\Big[\overline{\mathbf{E}}_\perp^{(0)}(\mathbf{r})\cdot\delta\varepsilon(\mathbf{r})\mathbf{E}_\perp^{(0)}(\mathbf{r}) + \\ - \overline{D}_z^{(0)}(\mathbf{r})\frac{\delta\varepsilon^{-1}(\mathbf{r})}{\varepsilon_\mathrm{v}^2}D_z^{(0)}(\mathbf{r})\Big]. \tag{7.12}$$

This equation, with respect to Eq. (7), has the advantage that it is expressed in terms of unperturbed field components that are all continuous across discontinuity planes orthogonal to the z-axis. Note also that the integration in Eq. (12) is effective only over the volume involving the dielectric perturbation.

7.3 A New Look at SPM for Rough Surface

In this Section, by employing the intrinsically-reciprocal VPRT formulation of Section 7.2, we preliminarily demonstrate how to derive the classical SPM scattering solution for a rough surface in a surprisingly simple way. The proposed new mathematical formulation for the relevant scattering problem then results, with respect to the classical one, neater, more concise, clearer, and as such offers a more direct comprehension in a methodological perspective. The mathematical structure presented not only represents a conceptually clean formulation for the classical SPM, but itself provides a new way of thinking about the perturbation theory applied to the scattering problem.

Some instructive and useful preliminary considerations are in order.

The essential idea behind *perturbation theory* applied to a physical system is the attaining of approximate solutions for such systems by suitably transforming exact solution of the approximate system, whereas the systems can be regarded as obtained from a solvable system by the addition of a small effect (perturbation). Nevertheless, this simple idea is completely obscured by the bulky classical SPM formulations, and the relative physical significance remains hidden in the available analytical derivations [5][6][8][9][12][44].

Therefore, the aim of this Section is not to propose a new solution to the relevant scattering problem, but to highlight how the canonical SPM for scattering by gently rough surface can be conceptually arranged in the new methodological perspective offered by the VPRT framework.

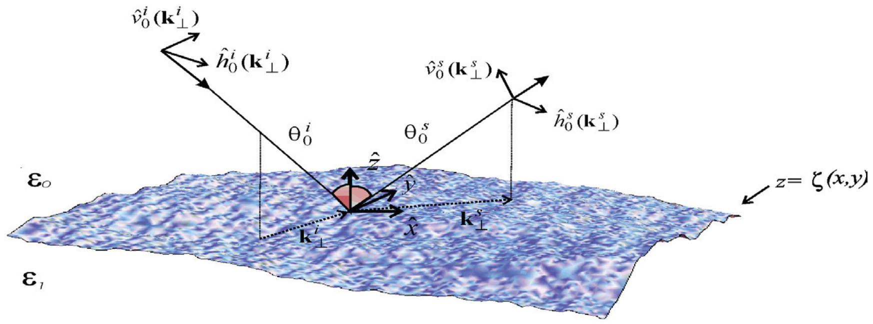

Fig. 2. Surface Scattering: Bistatic scattering configuration.

This new formulation, allows us to emphasize the crucial role of wave coupling of unperturbed solutions, which clearly exposes the intrinsic aim of the perturbation theory, thus offering a more complete comprehension in a conceptual perspective. In addition, the proposed approach permits to avoid the Green functions formalism, and is carried out referring exclusively to the vector electric field in a surprisingly simple way.

7.3.1 Dielectric Permittivity Characterization

We consider a gently rough (as in classical sense) surface between two homogeneous half-space (Fig. 2). We here assume that the magnetic relative permeability μ is uniform. Accordingly, the unperturbed permittivity distribution is

$$\varepsilon^{(0)}(\mathbf{r}) = \varepsilon^{(0)}(z) = \varepsilon_0 + (\varepsilon_1 - \varepsilon_0)\mathcal{U}(-z), \tag{7.13}$$

where $\mathcal{U}(\cdot)$ is the *Heaviside*'s unit step function, that is zero for negative argument and 1 for positive argument.The perturbed medium is now obtained by assuming that the rough surface has a roughness characterized by a *zero-mean* two-dimensional process $\zeta = \zeta(x,y) = \zeta(\mathbf{r}_\perp)$. Therefore, the perturbed permittivity distribution is

$$\varepsilon(\mathbf{r}_\perp, z) = \varepsilon_0 + (\varepsilon_1 - \varepsilon_0)\,\mathcal{U}(-z + \zeta(\mathbf{r}_\perp)), \tag{7.14}$$

and the perturbation of the dielectric permittivity is

$$\delta\varepsilon(\mathbf{r}_\perp, z) = \varepsilon(\mathbf{r}_\perp, z) - \varepsilon^{(0)}(z). \tag{7.15}$$

Note that this perturbation is non-null only in relative thin regions around the planes $z = 0$. In addition, we assume that the perturbation has a finite extent in the x, y directions, i.e., $\zeta(\mathbf{r}_\perp)$ is zero outside region of area A. Performing a series expansion of the perturbation, and assuming that roughness heights ζ are small enough, in the first-order limit we get:

$$\delta\varepsilon(\mathbf{r}_\perp, z) \cong (\varepsilon_1 - \varepsilon_0)\zeta(\mathbf{r}_\perp)\delta(z). \tag{7.16}$$

Similarly, we can write

$$\varepsilon^{-1}(\mathbf{r}_\perp, z) = \varepsilon_0^{-1} + (\varepsilon_1^{-1} - \varepsilon_0^{-1})\,\mathcal{U}(-z + \zeta(\mathbf{r}_\perp)), \tag{7.17}$$

$$\delta(\varepsilon^{-1}(\mathbf{r}_\perp, z)) \cong (\varepsilon_1^{-1} - \varepsilon_0^{-1})\,\zeta(\mathbf{r}_\perp)\delta(z). \tag{7.18}$$

In such a way, the roughness can be regarded as volume perturbations localized around the unperturbed interface and, accordingly, the roughness can be replaced by discontinuous volume inhomogeneities.

7.3.2 Unperturbed Field Evaluation

We consider an arbitrary polarized monochromatic plane wave incident from the upper half-space on the stratification at an angle θ_0^i with respect to the $\hat{z}$ direction, as schematically shown in Fig.2, whose representation (similarly as discussed in Section 3.2.2: see (3.7)-(3.10)) is given by

$$\mathbf{E}_0^i(\mathbf{r}) = [E_0^{ih}\hat{h}_0^i(\mathbf{k}_\perp^i) + E_0^{iv}\hat{v}_0^i(\mathbf{k}_\perp^i)]e^{j\mathbf{k}_\perp^i\cdot\mathbf{r}_\perp}e^{-jk_{z0}^i z}, \tag{7. 19}$$

where the incident vector wave-number direction is individuated, in a spherical coordinate frame, by θ_0^i, φ_0^i:

$$\begin{aligned} k_0 \hat{k}_0^i &= \mathbf{k}^i = \mathbf{k}_\perp^i - \hat{z} k_{z0}^i \\ &= k_0 (\hat{x} \sin\theta_0^i \cos\varphi_0^i + \hat{y} \sin\theta_0^i \sin\varphi_0^i - \hat{z} \cos\theta_0^i), \end{aligned} \tag{7.20}$$

being

$$\hat{h}_0^i(\mathbf{k}_\perp^i) = \frac{\hat{k}_0^i \times \hat{z}}{\left|\hat{k}_0^i \times \hat{z}\right|} = \hat{k}_\perp^i \times \hat{z} = \hat{h}^i, \tag{7.21}$$

$$\hat{v}_0^i(\mathbf{k}_\perp^i) = \hat{h}_0^i(\mathbf{k}_\perp^i) \times \hat{k}_0^i \ . \tag{7.22}$$

Accordingly, $\mathbf{k}_\perp^i = k_x^i \hat{x} + k_y^i \hat{y}$ is the two-dimensional projection of incident vector wave-number on the plane z=0.

By employing above notations (see also Chapter 3) the unperturbed field $\mathbf{E}^{(0)}(\mathbf{r})$, which is the one in absence of the roughness (flat interface), on the unperturbed surface can be conveniently expressed in the following closed-form:

$$\begin{aligned} \mathbf{E}_0^{(0)}(\mathbf{r}_\perp, 0^+) = e^{j\mathbf{k}_\perp^i \cdot \mathbf{r}_\perp} \Big\{ & \hat{h}^i [1 + R_{0|1}^h(k_\perp^i)] E_0^{ih} \\ & + \hat{k}_\perp^i \frac{k_{z0}^i}{k_0 \varepsilon_0} [1 - R_{0|1}^v(k_\perp^i)] E_0^{iv} + \hat{z} \frac{k_\perp^i}{k_0 \varepsilon_0} [1 + R_{0|1}^v(k_\perp^i)] E_0^{iv} \Big\} \end{aligned} \tag{7.23}$$

where the orthonormal right-handed basis $\mathcal{B}_i = \{\hat{h}^i, \hat{k}_\perp^i, \hat{z}\}$ has been used, $R_{0|1}^p$ is the usual *Fresnel* reflection coefficient (see Chapter 3), $p \in \{v,h\}$ denotes the polarization, and $k_{zm} = \sqrt{k_0^2 \mu \varepsilon_m - |\mathbf{k}_\perp|^2}$ with $m \in \{0, 1\}$.

7.3.3 Auxiliary Unperturbed Field Solution

In a similar way, if we assume that the test source is placed in the upper half-space and it is in the far zone with respect to the roughness, then the unperturbed field $\overline{\mathbf{E}}^{(0)}$ is the field present in an unperturbed medium on which a (locally) plane wave impinges, whose wave-number vector is

$$\begin{aligned} -k_0\hat{r}_s &= -\mathbf{k}_0^s = -\mathbf{k}_\perp^s - \hat{z}k_{z0}^s \\ &= k_0(-\hat{x}\sin\theta_0^s\cos\varphi_0^s - \hat{y}\sin\theta_0^s\sin\varphi_0^s - \hat{z}\cos\theta_0^s), \end{aligned} \tag{7.24}$$

and whose electric field is

$$\overline{\mathbf{E}}_0^i(\mathbf{r}) = e^{-j\mathbf{k}_\perp^s\cdot\mathbf{r}_\perp} e^{-jk_{z0}^s z}\,\overline{E}_0^i\hat{t}\,, \tag{7.25}$$

where

$$\overline{E}_0^i = j\eta_0 k_0 J \frac{e^{jk_0 r_s}}{4\pi\, r_s}\,. \tag{7.26}$$

Accordingly, the unperturbed field $\overline{\mathbf{E}}^{(0)}$ evaluated on the unperturbed surface can be written as

$$\begin{aligned} \overline{\mathbf{E}}_0^{(0)}(\mathbf{r}_\perp,0^+) = e^{-j\mathbf{k}_\perp^s\cdot\mathbf{r}_\perp}\Big\{ & \hat{h}^s[1+R_{0|1}^h(k_\perp^s)]\overline{E}_0^{ih} \\ & -\hat{k}_\perp^s\frac{k_{z0}^s}{k_0\varepsilon_0}[1-R_{0|1}^v(k_\perp^s)]\overline{E}_0^{iv} + \hat{z}\frac{k_\perp^s}{k_0\varepsilon_0}[1+R_{0|1}^v(k_\perp^s)]\overline{E}_0^{iv}\Big\}, \end{aligned} \tag{7.27}$$

where $\hat{h}^s$ and $\hat{v}^s$ are given by (21)-(22) with superscripts i replaced by superscripts s, and where

$$\overline{E}_0^{ih} = \overline{E}_0^i\,\hat{t}\cdot\hat{h}^s\,, \qquad \overline{E}_0^{iv} = \overline{E}_0^i\,\hat{t}\cdot\hat{v}^s\,. \tag{7.28}$$

7.3.4 Scattered Field Evaluation

When geometric roughness of the surface is described, as in the previous subsection, by means of an appropriate volume perturbation localized about the interfaces, then the integral in (12) become essentially a surface one. By using (16) and (18), Eq.(12) can be concisely rewritten as

$$\mathbf{E}_0^{(1)}(\mathbf{r}_0)\cdot\hat{t} = -j\frac{k_0}{J\eta_0}(\varepsilon_1-\varepsilon_0)\iint d\mathbf{r}_\perp \zeta(\mathbf{r}_\perp)\overline{\mathbf{E}}_0^{(0)}(\mathbf{r}_\perp,0^+)\cdot \mathcal{P}\ \mathbf{E}_0^{(0)}(\mathbf{r}_\perp,0^+), \quad (7.29)$$

in which 0^+ means that the longitudinal field components are evaluated immediately above the plane $z = 0$, and wherein

$$\mathcal{P} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}(\varepsilon_0/\varepsilon_1 - 1) \quad (7.30)$$

is a pseudo-horizontal projector, because it coincides with the classical horizontal one $\mathcal{P}_\perp = \mathcal{I} - \hat{z}\hat{z}$ for perfect conductivity ($\varepsilon_0/\varepsilon_1 \to 0$). In other words, the operator $\mathcal{P}$ accounts for the discontinuities of the unperturbed field across the (flat) boundary:

$$\mathbf{E}_1^{(0)}(\mathbf{r}_\perp,0^-) = \mathcal{P}\,\mathbf{E}_0^{(0)}(\mathbf{r}_\perp,0^+). \quad (7.31)$$

Note also that $\mathcal{P}$ is a symmetric operator and $\mathcal{P}^{-1}\mathbf{E}_1^{(0)}(\mathbf{r}_\perp,0^-) = \mathbf{E}_0^{(0)}(\mathbf{r}_\perp,0^+)$. Obviously, similar relationships hold also for $\overline{\mathbf{E}}^{(0)}$.

As a result, we emphasize that (29), which clearly express the perturbative solution directly in terms of unperturbed solutions, allows us to read the scattered field in terms of *wave coupling* of unperturbed solutions, whereas the roughness couples the energy of the incident wave with the scattered one at the receiver. At this point, to evaluate the scattered field $\mathbf{E}^{(1)}(\mathbf{r})$, the following representation is considered

$$\mathbf{E}_0^{(1)}(\mathbf{r}_0) = E_0^{sh}\hat{h}^s(\mathbf{k}_\perp^s) + E_0^{sv}\hat{v}^s(\mathbf{k}_\perp^s). \quad (7.32)$$

Substituting (23) and (27) in (29), evaluated for $\hat{t} = \hat{h}^s$ and for $\hat{t} = \hat{v}^s$, we directly obtain the following compact expression

$$\begin{bmatrix} E_0^{sv} \\ E_0^{sh} \end{bmatrix} = \pi k_0^2 \frac{e^{jk_0 r_0}}{r_0}\tilde{\zeta}(\mathbf{k}_\perp^s - \mathbf{k}_\perp^i)\ \boldsymbol{\alpha}(\mathbf{k}^s,\mathbf{k}^i)\begin{bmatrix} E_0^{iv} \\ E_0^{ih} \end{bmatrix}, \quad (7.33)$$

with

$$\boldsymbol{\alpha}(\mathbf{k}^s,\mathbf{k}^i)=\begin{bmatrix}\alpha_{vv}(\mathbf{k}^s,\mathbf{k}^i) & \alpha_{vh}(\mathbf{k}^s,\mathbf{k}^i)\\ \alpha_{hv}(\mathbf{k}^s,\mathbf{k}^i) & \alpha_{hh}(\mathbf{k}^s,\mathbf{k}^i)\end{bmatrix}, \tag{7.34}$$

where the superscripts *i* and *s* refer to the incident and scattered field directions, respectively;

$$\tilde{\zeta}(\mathbf{k}_\perp)=(2\pi)^{-2}\iint d\mathbf{r}_\perp\; e^{-j\mathbf{k}_\perp\cdot\mathbf{r}_\perp}\zeta(\mathbf{r}_\perp) \tag{7.35}$$

is the *Fourier* transforms (2D-FT) of the rough surfaces, and where

$$\alpha_{hh}(\mathbf{k}^s,\mathbf{k}^i)=(\varepsilon_1-\varepsilon_0)\hat{k}_\perp^s\cdot\hat{k}_\perp^i[1+R_{0|1}^h(k_\perp^s)][1+R_{0|1}^h(k_\perp^i)], \tag{7.36}$$

$$\alpha_{hv}(\mathbf{k}^s,\mathbf{k}^i)=(\varepsilon_1-\varepsilon_0)\hat{z}\cdot(\hat{k}_\perp^i\times\hat{k}_\perp^s)\frac{k_{z0}^i}{k_0\varepsilon_0}[1+R_{0|1}^h(k_\perp^s)][1-R_{0|1}^v(k_\perp^i)], \tag{7.37}$$

$$\alpha_{vh}(\mathbf{k}^s,\mathbf{k}^i)=\alpha_{hv}(-\mathbf{k}^i,-\mathbf{k}^s), \tag{7.38}$$

$$\alpha_{vv}(\mathbf{k}^s,\mathbf{k}^i)=\frac{\varepsilon_1-\varepsilon_0}{(k_0\varepsilon_0)^2}\Bigg[\frac{\varepsilon_0}{\varepsilon_1}k_\perp^s k_\perp^i[1+R_{0|1}^v(k_\perp^s)][1+R_{0|1}^v(k_\perp^i)] - \hat{k}_\perp^s\cdot\hat{k}_\perp^i k_{z0}^s k_{z0}^i[1-R_{0|1}^v(k_\perp^s)][1-R_{0|1}^v(k_\perp^i)]\Bigg] \tag{7.39}$$

Assuming that the relevant statistical properties of the process describing the interfacial roughness are invariant with respect to a spatial shift in the *x-y* plane (*wide sense stationary*), the pertinent *power spectral density* $W(\boldsymbol{\kappa})$ of the rough interface can be expressed, accordingly to (2.18), as

$$W(\boldsymbol{\kappa})=(2\pi)^2\lim_{A\to\infty}\frac{1}{A}<\left|\tilde{\zeta}(\boldsymbol{\kappa})\right|^2>, \tag{7.40}$$

where *A* is the illuminated surface area and angular brackets denote ensemble averaging. Accordingly, the *bistatic scattering cross section* for the pertinent surface can be defined as in (1.12)

$$\tilde{\sigma}_{qp}^{0} = \lim_{r\to\infty} \lim_{A\to\infty} \frac{4\pi r^2}{A} \frac{< \left| \mathbf{E}_0^{(1)}(\mathbf{r}) \cdot \hat{q}_0^{+}(\mathbf{k}_\perp^s) \right|^2 >}{\left| \mathbf{E}_0^i \cdot \hat{q}_0^{-}(\mathbf{k}_\perp^i) \right|^2}, \tag{7.41}$$

where $q \in \{v,\ h\}$ and $p \in \{v,\ h\}$ denote the scattered and incident polarizations, respectively. By substituting (33) and (40) in (41), the final expression for the bistatic scattering cross-section is obtained:

$$\sigma_{qp}^{0} = \pi\, k_0^4 \left| \alpha_{qp}(\mathbf{k}^s, \mathbf{k}^i) \right|^2 W(\mathbf{k}_\perp^s - \mathbf{k}_\perp^i) \tag{7.42}$$

This result is fully consistent with the classical SPM one obtained for the case of a homogeneous rough half-space [5]-[6].

As a result, within the VPRT framework the classical SPM scattering solution for a rough surface straightforwardly can be derived in a surprisingly simple way. The presented mathematical structure not only represents a conceptually clean formulation for the classical SPM, but itself provides a new way of thinking about the perturbation theory applied to the scattering problem, especially in view of further developments.

7.4 Scattering From Rough-Boundaries Multilayer

In this section we use the general expression reported in (12) to compute the scattering from a layered medium with rough interfaces. To this end, we have to characterize the medium, explicitly compute the medium permittivity perturbation $\delta\varepsilon$ (Section 7.4.1) and the two unperturbed fields radiated by real, $\mathbf{E}^{(0)}$, and fictitious, $\overline{\mathbf{E}}^{(0)}$, sources (Sections 7.4.2 and 7.4.3, respectively).

7.4.1 Layered Medium Characterization

For a layered medium with rough interfaces, the unperturbed medium is provided by a stack of N-1 parallel slabs, sandwiched in between two half-spaces; the entire structure is shift invariant along x and y directions (infinite lateral extent in x, y directions are assumed).

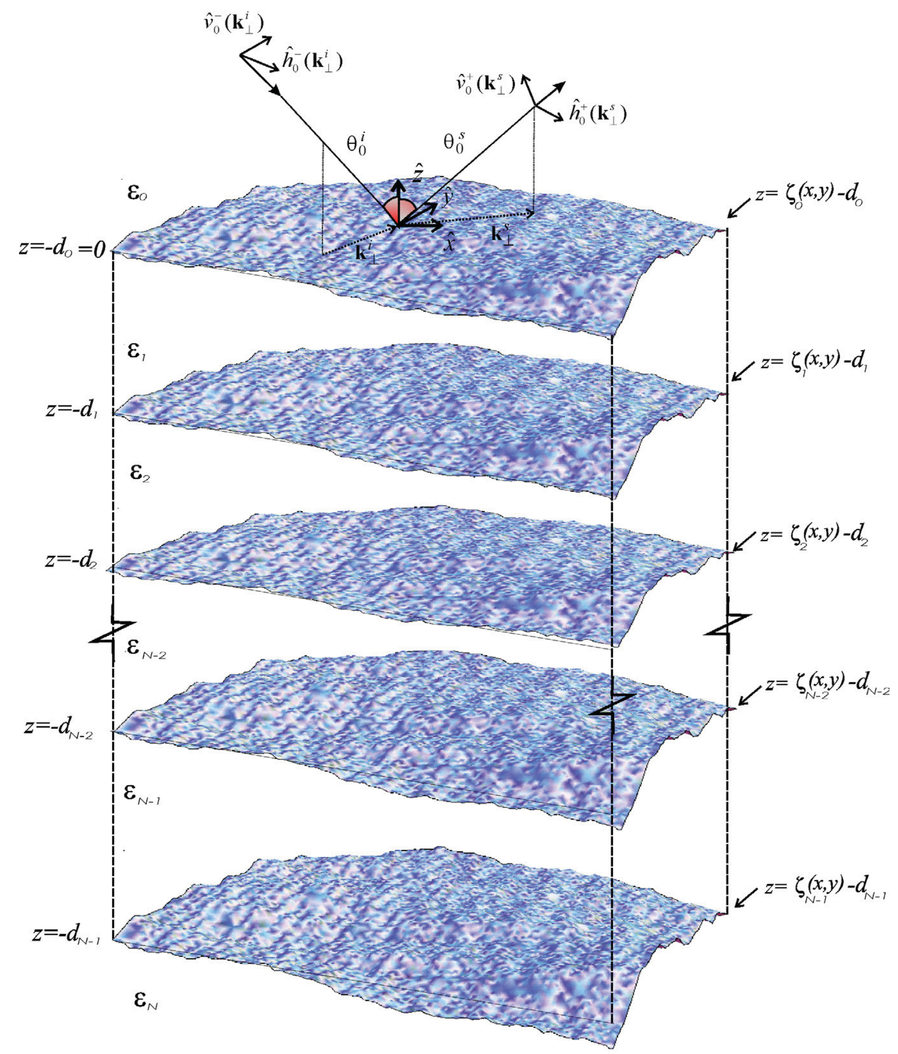

Fig. 3. Geometry of the rough-boundaries multilayer structure.

Each layer is assumed to be homogeneous and characterized by deterministic parameters: the dielectric relative permittivity ε_m, and the thickness $\Delta_m = d_m - d_{m-1}$, as depicted in see Fig.3. The parameters pertaining to m-th layer are identified by a subscript m; its boundaries are $z = -d_{m-1}$ and $z = -d_m$.

We here assume that all the layers have the same magnetic relative permeability μ (possibly, but not necessarily, equal to 1). In addition, with reference to Fig.3, we set d_0=0. Accordingly, the unperturbed permittivity distribution is

$$\varepsilon^{(0)}(\mathbf{r}) = \varepsilon^{(0)}(z) = \varepsilon_0 + \sum_{m=0}^{N-1} (\varepsilon_{m+1} - \varepsilon_m)\mathcal{U}(-z - d_m)\,, \quad (7.43)$$

where $\mathcal{U}(\cdot)$ is the *Heaviside*'s unit step function, that is zero for negative argument and 1 for positive argument.

The perturbed medium is now obtained by assuming that each interface has a roughness characterized by a *zero-mean* two-dimensional process, then for the m-th interface we have $\zeta_m = \zeta_m(x, y) = \zeta_m(\mathbf{r}_\perp)$. Therefore, the perturbed permittivity distribution is

$$\varepsilon(\mathbf{r}_\perp, z) = \varepsilon_0 + \sum_{m=0}^{N-1} (\varepsilon_{m+1} - \varepsilon_m)\,\mathcal{U}(-z - d_m + \zeta_m(\mathbf{r}_\perp))\,, \quad (7.44)$$

and the perturbation of the dielectric permittivity is given by

$$\delta\varepsilon(\mathbf{r}_\perp, z) = \varepsilon(\mathbf{r}_\perp, z) - \varepsilon^{(0)}(z)\,. \quad (7.45)$$

We assume that roughness heights ζ_m are small enough to perform a series expansion of the perturbation (45) around $\zeta_m = 0$ and truncate it to its first-order. Accordingly, by using (44) in (45) and recalling that the derivative of the *Heaviside*'s unit step function is a *Dirac*'s delta function, we get the following first-order expansion:

$$\delta\varepsilon(\mathbf{r}_\perp, z) \cong \sum_{m=0}^{N-1} (\varepsilon_{m+1} - \varepsilon_m)\zeta_m(\mathbf{r}_\perp)\delta(-z - d_m). \quad (7.46)$$

where $\delta(\cdot)$ is the *Dirac* delta function. Note that this perturbation is non-null only in thin regions around the planes $z = -d_m$. We also assume that the perturbation has a finite extent in the x–y directions, i.e., $\zeta_m(\mathbf{r}_\perp)$ is zero outside region of area A. Similarly, we can write

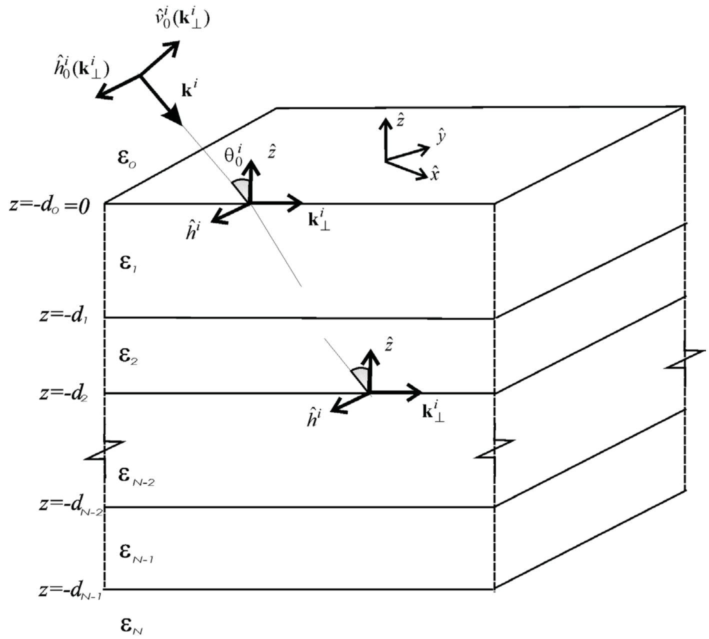

Fig. 4. Layered structure: unperturbed geometry.

$$\varepsilon^{-1}(\mathbf{r}_\perp, z) = \varepsilon_0^{-1} + \sum_{m=0}^{N-1} (\varepsilon_{m+1}^{-1} - \varepsilon_m^{-1})\, \mathcal{U}(-z - d_m + \zeta_m(\mathbf{r}_\perp)), \tag{7.47}$$

$$\delta(\varepsilon^{-1}(\mathbf{r}_\perp, z)) \cong \sum_{m=0}^{N-1} (\varepsilon_{m+1}^{-1} - \varepsilon_m^{-1})\, \zeta_m(\mathbf{r}_\perp)\delta(-z - d_m). \tag{7.48}$$

In such a way, as the interfaces description is concerned, the actual interfaces can be regarded as volume perturbations localized around the unperturbed interfaces and, accordingly, the roughness can be replaced by discontinuous volume inhomogeneities.

7.4.2 Unperturbed Field Evaluation

If we assume that the field source is placed in the upper half-space and it is in the far zone with respect to the rough interfaces, then the unperturbed field $\mathbf{E}^{(0)}(\mathbf{r})$ is the field present in the unperturbed

TABLE I

Notation	Description
N	Number of rough interfaces
Δ_m	Thickness of the *m*th layer
ζ_m	Spatial roughness of the *m*th interface
$\tilde{\zeta}_m$	Spectrum of the spatial roughness of the *m*th interface
μ	Relative permeability
ε_m	Relative permittivity of the *m*th layer
η_0	Intrinsic impedance of vacuum
k_0	Wave-number in the vacuum
k_m	Wave-number in the *m*th layer
$\mathbf{k}_\perp$	Projection on (x,y) plane of the vector wave-number
k_{zm}	*z*-component of the vector wave-number vector in the *m*th layer
θ_m	Angle in the *m*th layer
$R^p_{m-1\|m}$	*Ordinary* reflection coefficients for the *p*-polarization, at the interface between the regions *m-1* and *m*
$T^p_{m-1\|m}$	*Ordinary* transmission coefficients for the *p*-polarization in downward direction between the regions *m-1* and *m*
$\Re^p_{m-1\|m}$	*Generalized* reflection coefficients for the *p*-polarization, at the interface between the regions *m-1* and *m*
$\Im^p_{0\|m}$	*Generalized* transmission coefficients for the *p*-polarization in downward direction between the regions 0 and *m*

stratified medium (characterized by flat boundaries) on which a (locally) plane wave impinges.

We consider an arbitrary polarized monochromatic plane wave incident from the upper half-space on the stratification at an angle θ_0^i with respect to the $\hat{z}$ direction, as schematically shown in Fig.4, whose representation (similarly as discussed in Section 3.2.2: see (3.7)-(3.10)) is given by

$$\mathbf{E}_0^i(\mathbf{r}) = [E_0^{ih}\hat{h}_0^i(\mathbf{k}_\perp^i) + E_0^{iv}\hat{v}_0^i(\mathbf{k}_\perp^i)]e^{j\mathbf{k}_\perp^i\cdot\mathbf{r}_\perp}e^{-jk_{z0}^i z}, \qquad (7.49)$$

where the incident vector wave-number direction is individuated by θ_0^i, φ_0^i in a spherical coordinate frame (see also (20)-(22)).

By employing above notations (see also Chapter 3) the unperturbed field $\mathbf{E}_m^{(0)}(\mathbf{r})$ in the *m*-th layer can be conveniently expressed in the following closed-form:

$$\mathbf{E}_m^{(0)}(\mathbf{r}) = e^{j\mathbf{k}_\perp^i\cdot\mathbf{r}_\perp}\left[\hat{h}^i\xi_{0\to m}^{+h}(k_\perp^i, z)E_0^{ih} + \hat{k}_\perp^i\frac{k_{zm}^i}{k_0\varepsilon_m}\xi_{0\to m}^{-v}(k_\perp^i, z)E_0^{iv} + \hat{z}\frac{k_\perp^i}{k_0\varepsilon_m}\xi_{0\to m}^{+v}(k_\perp^i, z)E_0^{iv}\right], \quad (7.50)$$

where the orthonormal right-handed basis $\mathcal{B}_i = \{\hat{h}^i, \hat{k}_\perp^i, \hat{z}\}$ has been used (see Fig.4), and where we set:

$$\xi_{0\to m}^{\pm p}(k_\perp, z) = \Im_{0|m}^p(k_\perp)\,e^{jk_{zm}(-z-d_{m-1})}[1 \pm \Re_{m|m+1}^p(k_\perp)e^{j2k_{zm}(z+d_m)}], \quad (7.51)$$

where the symbol ± in the superscript on LHS represents a given choice linked to the symbol ± in RHS expression; the superscripts $p \in \{v, h\}$ denote the polarization, the *generalized reflection coefficient* $\Re_{m|m+1}^p(k_\perp)$ at the interface the interface between regions m and m+1 and the *generalized transmission coefficient* $\Im_{0|m}^p(k_\perp)$ can be recursively expressed, respectively, as in Section 3.4. and Section 3.5.

We stress that (generalized) reflection and transmission coefficients do not depend on the direction of $\mathbf{k}_\perp$.

7.4.3 Auxiliary Unperturbed Field Solution

Similarly, if we assume that the auxiliary (test) source is placed in the upper half-space and is located in the far zone with respect to the rough interfaces, then the unperturbed field $\overline{\mathbf{E}}^{(0)}$ is the field present in an unperturbed plane stratified medium on which a (locally) plane wave impinges; this plane wave is expressed by the electric field

$$\overline{\mathbf{E}}_0^i(\mathbf{r}) = e^{-j\mathbf{k}_\perp^s\cdot\mathbf{r}_\perp}e^{-jk_{z0}^s z}\,\overline{E}_0^i\hat{t}\,, \quad (7.52)$$

with a wave-vector (see also Fig.3)

$$\begin{aligned} -k_0\hat{r}_s &= -\mathbf{k}_0^s = -\mathbf{k}_\perp^s - \hat{z}k_{z0}^s \\ &= k_0(-\hat{x}\sin\theta_0^s\cos\varphi_0^s - \hat{y}\sin\theta_0^s\sin\varphi_0^s - \hat{z}\cos\theta_0^s), \end{aligned} \quad (7.53)$$

and amplitude

$$\overline{E}_0^i = j\eta_0 k_0 J \frac{e^{jk_0 r_s}}{4\pi r_s}. \tag{7.54}$$

Accordingly, the unperturbed field $\overline{\mathbf{E}}_m^{(0)}(\mathbf{r})$ in the m-th layer is given by:

$$\begin{aligned}\overline{\mathbf{E}}_m^{(0)}(\mathbf{r}) = e^{-j\mathbf{k}_\perp^s\cdot\mathbf{r}_\perp}\Big[\ &\hat{h}^s \xi_{0\to m}^{+h}(k_\perp^s, z)\overline{E}_0^{ih} \\ -\hat{k}_\perp^s \frac{k_{zm}^s}{k_0\varepsilon_m}\xi_{0\to m}^{-v}(k_\perp^s, z)\overline{E}_0^{iv} &+ \hat{z}\frac{k_\perp^s}{k_0\varepsilon_m}\xi_{0\to m}^{+v}(k_\perp^s, z)\overline{E}_0^{iv}\Big],\end{aligned} \tag{7.55}$$

where $\hat{h}^s$ and $\hat{v}^s$ are given by (21)-(22) with superscripts i replaced by superscripts s, and where

$$\overline{E}_0^{ih} = \overline{E}_0^i\,\hat{t}\cdot\hat{h}^s, \qquad \overline{E}_0^{iv} = \overline{E}_0^i\,\hat{t}\cdot\hat{v}^s\ . \tag{7.56}$$

7.4.4 Scattered Field Evaluation

The integral of the (12) over the volume V reduces to a multi-surface one being the geometric roughness of the interfaces described by means of an appropriate volume perturbation localized around the interfaces (Sect. 7.4.1). By substituting (46) and (48) in (12), we get

$$\begin{aligned}\mathbf{E}_0^{(1)}(\mathbf{r}_0)\cdot\hat{t} = -j\frac{k_0}{J\eta_0}\sum_{m=0}^{N-1}(\varepsilon_{m+1}-\varepsilon_m)\iint\zeta_m(\mathbf{r}_\perp)\\ \left[\overline{\mathbf{E}}_\perp^{(0)}(\mathbf{r}_\perp,-d_m)\cdot\mathbf{E}_\perp^{(0)}(\mathbf{r}_\perp,-d_m) + \frac{\overline{D}_z^{(0)}(\mathbf{r}_\perp,-d_m)D_z^{(0)}(\mathbf{r}_\perp,-d_m)}{\varepsilon_v^2\varepsilon_m\varepsilon_{m+1}}\right]d\mathbf{r}_\perp,\end{aligned} \tag{7.57}$$

or

$$\mathbf{E}_0^{(1)}(\mathbf{r}_0)\cdot\hat{t} = -j\frac{k_0}{J\eta_0}\sum_{m=0}^{N-1}(\varepsilon_{m+1}-\varepsilon_m)\iint\zeta_m(\mathbf{r}_\perp)$$
$$\left[\overline{\mathbf{E}}_\perp^{(0)}(\mathbf{r}_\perp,-d_m)\cdot\mathbf{E}_\perp^{(0)}(\mathbf{r}_\perp,-d_m)+\frac{\varepsilon_m}{\varepsilon_{m+1}}\overline{E}_z^{(0)}(\mathbf{r}_\perp,-d_m^+)E_z^{(0)}(\mathbf{r}_\perp,-d_m^+)\right]d\mathbf{r}_\perp, \tag{7.58}$$

wherein $-d_m^+$ indicates that the longitudinal field components are evaluated immediately above the plane $z = -d_m$. Equation (58) can be concisely rewritten as

$$\mathbf{E}_0^{(1)}(\mathbf{r}_0)\cdot\hat{t} = -j\frac{k_0}{J\eta_0}\sum_{m=0}^{N-1}(\varepsilon_{m+1}-\varepsilon_m)\iint d\mathbf{r}_\perp\zeta_m(\mathbf{r}_\perp)\overline{\mathbf{E}}^{(0)}(\mathbf{r}_\perp,-d_m^+)\cdot\mathcal{P}_m\mathbf{E}^{(0)}(\mathbf{r}_\perp,-d_m^+), \tag{7.59}$$

wherein

$$\mathcal{P}_m = [\hat{x}\hat{x}+\hat{y}\hat{y}+\hat{z}\hat{z}(\varepsilon_m/\varepsilon_{m+1}-1)] \tag{7.60}$$

is a pseudo-horizontal projector, because it coincides with the classical horizontal one $\mathcal{P}_\perp = \mathcal{I} - \hat{z}\hat{z}$ for perfect conductivity ($\varepsilon_m/\varepsilon_{m+1}\to 0$). In other words, the operator $\mathcal{P}_m$ accounts for the discontinuities of the unperturbed field across the m-th (flat) boundary. Note also that $\mathcal{P}_m$ is a symmetric operator:

$$\mathbf{E}_{m+1}^{(0)}(\mathbf{r}_\perp,-d_m^-) = \mathcal{P}_m\,\mathbf{E}_m^{(0)}(\mathbf{r}_\perp,-d_m^+). \tag{7.61}$$

At this point, we have all the elements to evaluate the field perturbation $\mathbf{E}^{(1)}(\mathbf{r})$, i.e., the scattered field. As a matter of fact, by substituting (50) and (55) in (59), and noting that

$$\hat{h}^s\cdot\hat{h}^i = (\hat{k}_\perp^s\times\hat{z})\cdot(\hat{k}_\perp^i\times\hat{z}) = \hat{k}_\perp^s\cdot\hat{k}_\perp^i = \cos(\varphi_0^s-\varphi_0^i), \tag{7.62}$$

$$\hat{h}^s\cdot\hat{k}_\perp^i = (\hat{k}_\perp^s\times\hat{z})\cdot\hat{k}_\perp^i = \hat{z}\cdot(\hat{k}_\perp^i\times\hat{k}_\perp^s) = \sin(\varphi_0^s-\varphi_0^i), \tag{7.63}$$

$$-\hat{k}_\perp^s\cdot\hat{h}^i = -\hat{k}_\perp^s\cdot(\hat{k}_\perp^i\times\hat{z}) = \hat{z}\cdot(\hat{k}_\perp^i\times\hat{k}_\perp^s) = \sin(\varphi_0^s-\varphi_0^i), \tag{7.64}$$

we get

$$
\begin{aligned}
\mathbf{E}_0^{(1)}(\mathbf{r}_0)\cdot\hat{t} &= -j\frac{k_0}{J\eta_0}\sum_{m=0}^{N-1}\iint d\mathbf{r}_\perp \zeta_m(\mathbf{r}_\perp)e^{-j(\mathbf{k}_\perp^s-\mathbf{k}_\perp^i)\cdot\mathbf{r}_\perp} \\
&(\varepsilon_{m+1}-\varepsilon_m)\Big[\ \hat{k}_\perp^s\cdot\hat{k}_\perp^i\ \xi_{0\to m}^{+h}(k_\perp^s)\ \xi_{0\to m}^{+h}(k_\perp^i)\overline{E}_0^{ih}E_0^{ih} \\
&+\hat{z}\cdot(\hat{k}_\perp^i\times\hat{k}_\perp^s)\frac{k_{zm}^s}{k_0\varepsilon_m}\ \xi_{0\to m}^{-v}(k_\perp^s)\ \xi_{0\to m}^{+h}(k_\perp^i)\overline{E}_0^{iv}E_0^{ih} \\
&+\hat{z}\cdot(\hat{k}_\perp^i\times\hat{k}_\perp^s)\xi_{0\to m}^{+h}(k_\perp^s)\frac{k_{zm}^i}{k_0\varepsilon_m}\ \xi_{0\to m}^{-v}(k_\perp^i)\overline{E}_0^{ih}E_0^{iv} \\
&-\hat{k}_\perp^s\cdot\hat{k}_\perp^i\ \frac{k_{zm}^s k_{zm}^i}{(k_0\varepsilon_m)^2}\ \xi_{0\to m}^{-v}(k_\perp^s)\ \xi_{0\to m}^{-v}(k_\perp^i)\overline{E}_0^{iv}E_0^{iv} \\
&+\frac{k_\perp^s k_\perp^i}{(k_0\varepsilon_m)^2}\frac{\varepsilon_m}{\varepsilon_{m+1}}\ \xi_{0\to m}^{+v}(k_\perp^s)\ \xi_{0\to m}^{+v}(k_\perp^i)\overline{E}_0^{iv}E_0^{iv}\ \Big],
\end{aligned}
\tag{7.65}
$$

where, accordingly to (51), we have set

$$\xi_{0\to m}^{\pm p}(k_\perp)=\xi_{0\to m}^{\pm p}(k_\perp,-d_m)\,. \tag{7.66}$$

It is convenient to project the scattered field onto $\hat{h}^s$ and $\hat{v}^s$ (given by (21)-(22) with superscripts *i* replaced by superscripts *s*):

$$\mathbf{E}_0^{(1)}(\mathbf{r}_0)=E_0^{sh}\hat{h}^s(\mathbf{k}_\perp^s)+E_0^{sv}\hat{v}^s(\mathbf{k}_\perp^s)\,. \tag{7.67}$$

By introducing

$$\widetilde{\zeta}_m(\mathbf{k}_\perp)=(2\pi)^{-2}\iint d\mathbf{r}_\perp\ e^{-j\mathbf{k}_\perp\cdot\mathbf{r}_\perp}\zeta_m(\mathbf{r}_\perp), \tag{7.68}$$

the *Fourier* transforms (2D-FT) of the rough interfaces ζ_m, and using (54) and (56), we have

$$E_0^{sh}(\mathbf{r}_0) = \pi k_0^2 \frac{e^{jk_0 r_0}}{r_0} \sum_{m=0}^{N-1} \widetilde{\zeta}_m (\mathbf{k}_\perp^s - \mathbf{k}_\perp^i)(\varepsilon_{m+1} - \varepsilon_m)$$
$$\Big[\ \hat{k}_\perp^s \cdot \hat{k}_\perp^i\, \xi_{0\to m}^{+h}(k_\perp^s)\, \xi_{0\to m}^{+h}(k_\perp^i) E_0^{ih}$$
$$+ \hat{z} \cdot (\hat{k}_\perp^i \times \hat{k}_\perp^s) \xi_{0\to m}^{+h}(k_\perp^s) \frac{k_{zm}^i}{k_0 \varepsilon_m} \xi_{0\to m}^{-v}(k_\perp^i) E_0^{iv}\ \Big] \tag{7.69}$$

for $\hat{t} = \hat{h}^s$, and

$$E_0^{sv}(\mathbf{r}_0) = \pi k_0^2 \frac{e^{jk_0 r_0}}{r_0} \sum_{m=0}^{N-1} \widetilde{\zeta}_m (\mathbf{k}_\perp^s - \mathbf{k}_\perp^i)(\varepsilon_{m+1} - \varepsilon_m)$$
$$\Big\{ \hat{z} \cdot (\hat{k}_\perp^i \times \hat{k}_\perp^s) \frac{k_{zm}^s}{k_0 \varepsilon_m} \xi_{0\to m}^{-v}(k_\perp^s)\, \xi_{0\to m}^{+h}(k_\perp^i) E_0^{ih}$$
$$+ \left[\frac{k_\perp^s k_\perp^i}{(k_0 \varepsilon_m)^2} \frac{\varepsilon_m}{\varepsilon_{m+1}} \xi_{0\to m}^{+v}(k_\perp^s)\, \xi_{0\to m}^{+v}(k_\perp^i) \right.$$
$$\left. - \hat{k}_\perp^s \cdot \hat{k}_\perp^i \frac{k_{zm}^s k_{zm}^i}{(k_0 \varepsilon_m)^2} \xi_{0\to m}^{-v}(k_\perp^s)\, \xi_{0\to m}^{-v}(k_\perp^i) \right] E_0^{iv} \Big\} \tag{7.70}$$

for $\hat{t} = \hat{v}^s$. Note also that in Eqs. (69)-(70) superscripts i and s refer to the incident and scattered field directions, respectively.

7.4.5 Generalized Scattering Matrix

In this section, to emphasize the polarimetric character of the obtained solution, we introduce the concept of *generalized bistatic scattering matrix* of the rough layered media. Accordingly, equations (69)-(70) can be more concisely written in matrix form as:

$$\begin{bmatrix} E_0^{sv} \\ E_0^{sh} \end{bmatrix} = \pi k_0^2 \frac{e^{jk_0 r_0}}{r_0} \sum_{m=0}^{N-1} \widetilde{\zeta}_m (\mathbf{k}_\perp^s - \mathbf{k}_\perp^i) \widetilde{\boldsymbol{\alpha}}^{m,m+1}(\mathbf{k}^s, \mathbf{k}^i) \begin{bmatrix} E_0^{iv} \\ E_0^{ih} \end{bmatrix}, \tag{7.71}$$

with *generalized bistatic scattering matrix* formally expressed by

$$\boldsymbol{\mathcal{S}}^{m|m+1}(\mathbf{k}^s, \mathbf{k}^i) = \pi k_0^2 \widetilde{\zeta}_m (\mathbf{k}_\perp^s - \mathbf{k}_\perp^i)\, \widetilde{\boldsymbol{\alpha}}^{m,m+1}(\mathbf{k}^s, \mathbf{k}^i), \tag{7.72}$$

which characterizes the polarimetric response of the generic (m-th) rough interface of the layered structure, for a plane wave incident in direction $\mathbf{k}^i$ and for a given observation direction $\mathbf{k}^s$, with

$$\tilde{\boldsymbol{\alpha}}^{m,m+1}(\mathbf{k}^s,\mathbf{k}^i)=\begin{bmatrix}\tilde{\alpha}_{vv}^{m,m+1}(\mathbf{k}^s,\mathbf{k}^i) & \tilde{\alpha}_{vh}^{m,m+1}(\mathbf{k}^s,\mathbf{k}^i)\\ \tilde{\alpha}_{hv}^{m,m+1}(\mathbf{k}^s,\mathbf{k}^i) & \tilde{\alpha}_{hh}^{m,m+1}(\mathbf{k}^s,\mathbf{k}^i)\end{bmatrix}, \tag{7.73}$$

where

$$\begin{aligned}\tilde{\alpha}_{vv}^{m,m+1}(\mathbf{k}^s,\mathbf{k}^i)=\frac{\varepsilon_{m+1}-\varepsilon_m}{(k_0\varepsilon_m)^2}\Bigg[&\frac{\varepsilon_m}{\varepsilon_{m+1}}k_\perp^s\xi_{0\to m}^{+v}(k_\perp^s)k_\perp^i\xi_{0\to m}^{+v}(k_\perp^i)\\ &-(\hat{k}_\perp^s\cdot\hat{k}_\perp^i)k_{zm}^s\xi_{0\to m}^{-v}(k_\perp^s)k_{zm}^i\xi_{0\to m}^{-v}(k_\perp^i)\Bigg]\end{aligned} \tag{7.74}$$

$$\tilde{\alpha}_{vh}^{m,m+1}(\mathbf{k}^s,\mathbf{k}^i)=(\varepsilon_{m+1}-\varepsilon_m)\hat{z}\cdot(\hat{k}_\perp^i\times\hat{k}_\perp^s)\frac{k_{zm}^s}{k_0\varepsilon_m}\xi_{0\to m}^{-v}(k_\perp^s)\xi_{0\to m}^{+h}(k_\perp^i) \tag{7.75}$$

$$\tilde{\alpha}_{hv}^{m,m+1}(\mathbf{k}^s,\mathbf{k}^i)=(\varepsilon_{m+1}-\varepsilon_m)\,\hat{z}\cdot(\hat{k}_\perp^i\times\hat{k}_\perp^s)\xi_{0\to m}^{+h}(k_\perp^s)\frac{k_{zm}^i}{k_0\varepsilon_m}\xi_{0\to m}^{-v}(k_\perp^i) \tag{7.76}$$

$$\tilde{\alpha}_{hh}^{m,m+1}(\mathbf{k}^s,\mathbf{k}^i)=(\varepsilon_{m+1}-\varepsilon_m)(\hat{k}_\perp^s\cdot\hat{k}_\perp^i)\,\xi_{0\to m}^{+h}(k_\perp^s)\,\xi_{0\to m}^{+h}(k_\perp^i) \tag{7.77}$$

where, in compliance with (51) and (66) we have

$$\xi_{0\to m}^{\pm p}(k_\perp)=\Im_{0|m}^{p}(k_\perp)\,e^{jk_{zm}\Delta_m}[1\pm\Re_{m|m+1}^{p}(k_\perp)]. \tag{7.78}$$

Eqs. (71)-(78) provide a key result of this Section. Some comments are in order to illustrate major consequences from these equations.

First of all, we emphasize that the proposed approach avoid somehow defining and using the *Green* functions, whereas our treatment directly involve the integral transform of the field. This simplifies the mathematical treatment of the problem; in addition, as it is clarified in the following, our approach leads to a meaningful physical interpretation of the perturbative solution.

Furthermore, we highlight that formally the obtained analytical solution (71)-(78), for the scattering from rough-boundaries multilayered structures, is perfectly equivalent to the one based on the perturbation of the boundary conditions approach [23]-[24]. In fact, it is easy to verify that (74)-(77) are formally coincident with (5.82)-(5.85) in Chapter 5. In particular, it is simple to verify that, when the layered medium merely reduces to a single interface between two half-space, these coefficients exactly reduce to the classical SPM ones [5][6][23].

When the interfacial roughness is concerned, we emphasize that from a qualitative viewpoint, in long wavelength limit the controlling factor for the validity of our solution is not the dielectric contrast: in fact the smallness of the dielectric perturbation does not necessary requires a limitation on the dielectric contrast (whose modulus can be, and usually is, greater than 1). The relevant limitations regard the vertical extension (rms height) of the rough interface, which has to be small with respect to the wavelength of the incoming radiation. This is directly related to the role of the phase of the wave propagating inside the perturbation. In addition, regarding the roughness also a constraint on the small-slope assumption has to be considered (the gradient of the interface must be small in comparison with unit). This point is discussed in detail in Chapter 8.

Accordingly, the range of validity of the VPRT formulation applied to rough multilayer is the same as the one of the BPT formulation [23]-[24], i.e., the height deviation of the rough interfaces, about the unperturbed interface, is everywhere small compared to the wavelength of the incoming wave and the gradient of each interface is small in comparison to unity.

It should be noted that, when only first-order terms are considered, then the perturbation theory yields the *Bragg* scatter phenomenon referred to a multi-rough-interfaces scattering: in fact, the scattered field at a given angle turns out to be the linear combination of the amplitudes of the Fourier Transforms of the interfaces roughness at one specific vector wave-number. Then, the scattered power at a particular angle is directly a linear combination of energies at relevant surface scales.

We also emphasize that the scattering configuration we considered in Eq. (67) is compliant with the classical *Forward Scattering Alignment* (FSA) convention that is adopted in radar polarimetry.

Concerning the azimuthal φ-dependence, we also note that the *cos*-like scattering patterns experimentally obtained for a rough surface in [38] are taken into account in our solution. Regarding this point, we underline that our method clearly indicates how this dependence is associated with the bistatic configuration geometry in which the scattering phenomenon is observed, whereas in [38] this effect is referred to as a polarization artifact. Moreover, Eqs. (74)-(74) also shown how this behavior is also inherited by each polarization component.

Lets indicate with the superscript T the transpose. It can be easily verified that the scattering matrix whose elements are in (74)-(77) satisfies the following relationship:

$$\mathcal{S}^{m|m+1}(\mathbf{k}^s,\mathbf{k}^i) = \left[\mathcal{S}^{m|m+1}(-\mathbf{k}^i,-\mathbf{k}^s)\right]^T. \qquad (7.79)$$

This fundamental property in the radar polarimetry was first obtained with a general purpose approach in [39]: the approach we here introduced in a different way led to coefficients that satisfy that property. This can be concisely expresses as a form of the *reciprocity* principle in the electromagnetic theory. It turns out that our result is invariant for an *appropriate* exchange between the role of transmitter and receiver. As a matter of fact, the formal exchange between the projections on the z=0 plane $\mathbf{k}_\perp^i$ and $\mathbf{k}_\perp^s$ is directly related to the exchange between the incident and scattered wave-vectors $\mathbf{k}^i = \mathbf{k}_\perp^i - k_{z0}^i\hat{z}$ and $\mathbf{k}^s = \mathbf{k}_\perp^s + k_{z0}^s\hat{z}$.

Finally, some considerations on the unperturbed-waves *coupling* interpretation are in order. Taking into account that $\mathbf{E}^{(0)}(\mathbf{r}_\perp,-d_m^-) = \mathcal{P}_m\mathbf{E}^{(0)}(\mathbf{r}_\perp,-d_m^+)$, we are in the position to conveniently rewrite (59) as

$$\mathbf{E}_0^{(1)}(\mathbf{r}_0)\cdot\hat{t} = -j\frac{k_0}{J\eta_0}\sum_{m=0}^{N-1}\iiint d\mathbf{r}\,\overline{\mathbf{E}}_m^{(0)}(\mathbf{r})\cdot\delta\varepsilon_m(\mathbf{r})\,\mathbf{E}_{m+1}^{(0)}(\mathbf{r}), \qquad (7.80)$$

where, in each sub-region $\Omega_m = \{\mathbf{r}\equiv(\mathbf{r}_\perp,z): z\in(-d_m,-d_{m-1})\}$ of the layered structure, the unperturbed solution assumes the form $\mathbf{E}^{(0)}(\mathbf{r}) = \mathbf{E}_m^{(0)}(\mathbf{r})$, provided that the boundary condition on the flat interfaces are satisfied. Note that (80) is invariant when the subscripts

m and m+1, of the corresponding two unperturbed fields, are exchanged. It should be also noted that the (scalar) perturbation strength

$$\delta\varepsilon_m(\mathbf{r}) = (\varepsilon_{m+1} - \varepsilon_m)\zeta_m(\mathbf{r}_\perp)\delta(-z - d_m) \tag{7.81}$$

formally represents a perturbation operator associated with the roughness of the m-th interface. This operator reduces the integral in Eq. (80) to a multi-surface one. We highlight the crucial role played by the resulting wave coupling, which is intimately related to structural perturbation introduced in the first-order formulation: The exchange of energy is taking place as the roughness couples the energy of the incident wave with the one of the scattered field at the receiver. Consequently, for any fixed observation point the perturbation gives rise to a scattered field readable in terms of *wave coupling* of unperturbed solutions [see (80)]. In the first-order approximation, from the receiver viewpoint, the electromagnetic *coupling* between only two unperturbed waves is observed. In other words, the signal received depends on two unperturbed fields, whereas the operators $\delta\varepsilon_m$ affect the coupling between these two unperturbed solutions.

As a result, the perturbation operators (81) can be also thought as coupling coefficients, with m=0, 1, …, N-1.

7.5 Reaction-Concept-Based Interpretation of the Scattering Solution

In this section, a very useful and informative interpretation of the presented VPRT solution for rough multilayers is proposed; this is done in terms of reactions.

The concept of *reaction*, which has to be regarded as a basic physical observable, was originally introduced by *Rumsey* [35] (for a systematic exposition of this subject see also [28]). We underline that the reaction concept was applied in [35] by considering surface currents, whereas volume polarization currents case were taken into account in [36].

Let's consider two vectorial functions, $\mathbf{E}^a$ and $\mathbf{J}^b$: in an infinite-dimensional linear space for the electromagnetic fields, we can introduce the *symmetric bilinear form*

$$\left\langle \mathbf{E}^a, \mathbf{J}^b \right\rangle = \iiint d\mathbf{r}\, \mathbf{E}^a(\mathbf{r}) \cdot \mathbf{J}^b(\mathbf{r}), \tag{7.82}$$

to which is given the physical meaning of *reaction* [15] between two field quantities; the scalar on the left-hand side of eq.(82) is a measure of the reaction (or coupling) between the source field, $\mathbf{J}^b$, and the mediating field, $\mathbf{E}^a$. Note also that, the symbol $<\cdot,\cdot>$ is here adopted to mathematically represent the bilinear form (82) which is a more general concept with respect to the inner product and does not require the structure of an inner product space.

Using the notion (82), eq.(80) can be then conveniently rewritten in the form of a *multi-reaction:*

$$\mathbf{E}_0^{(1)}(\mathbf{r}_0) \cdot J\hat{t} = \sum_{m=0}^{N-1} \left\langle \overline{\mathbf{E}}_m^{(0)}(\mathbf{r}), \mathbf{J}_m^{(1)}(\mathbf{r}) \right\rangle, \tag{7.83}$$

where, to the first-order $\mathbf{J}_m^{(1)} = -jk_0\eta_0^{-1}\delta\varepsilon_m(\mathbf{r})\mathbf{E}_{m+1}^{(0)}(\mathbf{r})$ can be interpreted as the equivalent *polarization current* induced into the (localized) perturbed volume by the unperturbed field $\mathbf{E}_{m+1}^{(0)}(\mathbf{r})$. Then, the generic *m*-th term of the summation in (83) $\left\langle \overline{\mathbf{E}}_m^{(0)}, \mathbf{J}_m^{(1)} \right\rangle$ is susceptible to be physically interpreted as the unperturbed electric field $\overline{\mathbf{E}}_m^{(0)}(\mathbf{r})$, which is produced by the sampling source $\hat{t}J\delta(\mathbf{r}-\mathbf{r}_0^s)$, "measured" by the source $\mathbf{J}_m^{(1)}$. Note that if the reaction is zero, then no energy is transferred by the first-order field from the transmitter to the receiver.

The right-hand-side of Eq. (83) shows how the scattered field is intimately related to the *multi-reaction*. This remarkable interpretation, which intimately depends on the essence of perturbation approach, is straightforward and rich in descriptive power.

Note also that, since the medium is also symmetric or reciprocal, this multi-reaction is symmetric according to the reciprocity theorem. Indeed, from the symmetry of $\mathcal{P}_m$, it turns out that a form equivalent to (83) is given by:

$$\mathbf{E}_0^{(1)}(\mathbf{r}_0)\cdot J\hat{t} = \sum_{m=0}^{N-1}\left\langle \overline{\mathbf{J}}_m^{(1)}, \mathbf{E}_m^{(0)}(\mathbf{r}) \right\rangle, \tag{7.84}$$

wherein to the first-order, $\overline{\mathbf{J}}_m^{(1)} = -jk_0\eta_0^{-1}\delta\varepsilon_m(\mathbf{r})\overline{\mathbf{E}}_{m+1}^{(0)}(\mathbf{r})$ can be now interpreted, as the equivalent *polarization current* induced into the (localized) perturbed volume by the unperturbed field $\overline{\mathbf{E}}_{m+1}^{(0)}(\mathbf{r})$ produced by the test-source $\hat{t}J\delta(\mathbf{r}-\mathbf{r}_0^s)$. Accordingly, the generic m-th term of the summation in (83) can be now read as the unperturbed electric field $\mathbf{E}_m^{(0)}(\mathbf{r})$, due to the real source, "measured" by the source $\overline{\mathbf{J}}_m^{(1)}$.

This is to say that the proposed formulation is *reciprocal*.

7.6 Surface and Volume Scattering: A Joint Peturbative Formulation

A unified perturbative formulation jointly taking into account the scattering phenomena from both interfacial roughness and volumetric fluctuations for random semi-infinite three dimensional (3-D) media is presented. We first introduce a proper description for the considered spatial 3-D structure, which is described in terms of a proper perturbation of the corresponding idealized structure, afterward the formalism for the unperturbed field is provided. Finally, a general expression for the first-order scattering field is obtained by using VPRT.

7.6.1 Semi-Infinite Medium with Interfacial and Volumetric, Random Inhomogeneities

The considered structure (Fig. 5) can be regarded as obtained by a proper perturbation of the unperturbed structural properties.

The unperturbed structure (Fig. 6) is constituted of two half-spaces, separated by a flat interface, each one assumed to be homogeneous and characterized by arbitrary and deterministic dielectric permittivity ε_0 and ε_1, respectively. We hereafter assume that both the half-spaces have the same magnetic relative permeability

μ (possibly, but not necessarily, equal to 1). The unperturbed structure is then shift-invariant in the direction of x and y (infinite lateral extent in x, y directions). Accordingly, the unperturbed medium permittivity distribution is

$$\varepsilon^{(0)}(\mathbf{r}) = \varepsilon^{(0)}(z) = \varepsilon_0 + (\varepsilon_1 - \varepsilon_0)\mathcal{U}(-z), \tag{7.85}$$

where $\mathcal{U}(\cdot)$ is the *Heaviside*'s step function, which is zero for negative argument and 1 for positive argument.

In order to characterize the inherent morphology, the description of statistical fluctuations occurring in the actual spatial structures can be achieved by employing different quantities: The perturbed medium is now obtained by assuming that topography of the interfacial irregularities is characterized by a *zero-mean* two-dimensional stochastic process, $\zeta(x,y) = \zeta(\mathbf{r}_\perp)$, and the *space-variant* morphological features of the lower half-space volume are characterized by a *zero-mean* three-dimensional stochastic process, $\chi_1(\mathbf{r})$, such that the relative permittivity of the lower half-space is described by $\varepsilon_1(\mathbf{r})=\varepsilon_1+\chi_1(\mathbf{r})$. Accordingly, the perturbed medium can be seen as the truncation by the rough interface $\zeta(\mathbf{r}_\perp)$ of an infinite volume, whose permittivity fluctuations are described by a process $\chi_1(\mathbf{r})$. Therefore, the perturbed permittivity distribution is modeled by

$$\varepsilon(\mathbf{r}_\perp, z) = \varepsilon_0 + (\varepsilon_1(\mathbf{r}) - \varepsilon_0)\,\mathcal{U}(-z + \zeta(\mathbf{r}_\perp)), \tag{7.86}$$

and the perturbation of the dielectric permittivity is

$$\delta\varepsilon(\mathbf{r}_\perp, z) = \varepsilon(\mathbf{r}_\perp, z) - \varepsilon^{(0)}(z). \tag{7.87}$$

We assume that roughness heights ζ and volumetric fluctuation χ_1 are small enough to perform a series expansion of the perturbation and truncate it to its first-order:

$$\delta\varepsilon(\mathbf{r}_\perp, z) \cong (\varepsilon_1 - \varepsilon_0)\delta(z)\zeta(\mathbf{r}_\perp) + \mathcal{U}(-z)\chi_1(\mathbf{r}), \tag{7.88}$$

where $\delta(\cdot)$ is the *Dirac*'s delta function.

Similarly, we obtain

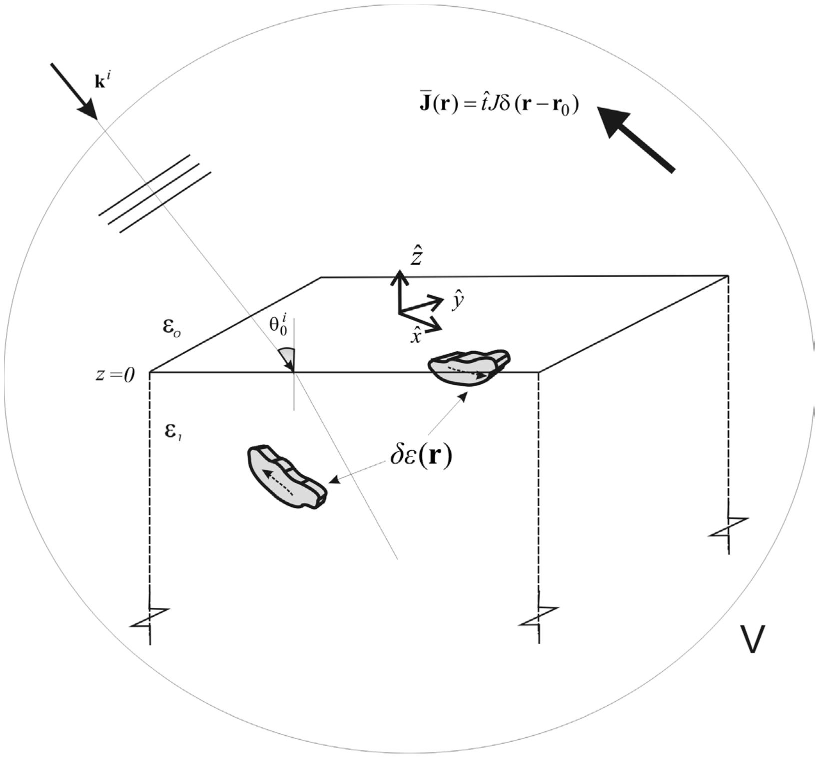

Fig. 5. Scheme for the scattering problem.

$$\delta(\varepsilon^{-1}(\mathbf{r}_\perp, z)) \cong (\varepsilon_1^{-1} - \varepsilon_0^{-1})\delta(z)\zeta(\mathbf{r}_\perp) - \varepsilon_1^{-2}\mathcal{U}(-z)\chi_1(\mathbf{r}) . \quad (7.89)$$

In addition, we can write

$$\varepsilon^{-1}(\mathbf{r}_\perp, z)\Big|_{\chi_1=0} = \varepsilon_0^{-1} + (\varepsilon_1^{-1} - \varepsilon_0^{-1})\, \mathcal{U}(-z + \zeta(\mathbf{r}_\perp)) , \quad (7.90)$$

$$\delta(\varepsilon^{-1}(\mathbf{r}_\perp, z))\Big|_{\chi_1=0} \cong (\varepsilon_1^{-1} - \varepsilon_0^{-1})\, \delta(z)\zeta(\mathbf{r}_\perp) . \quad (7.91)$$

In such a way, the roughness can be regarded as volume perturbations localized around the unperturbed interface and, accordingly, the roughness can be replaced by discontinuous volume inhomogeneities.

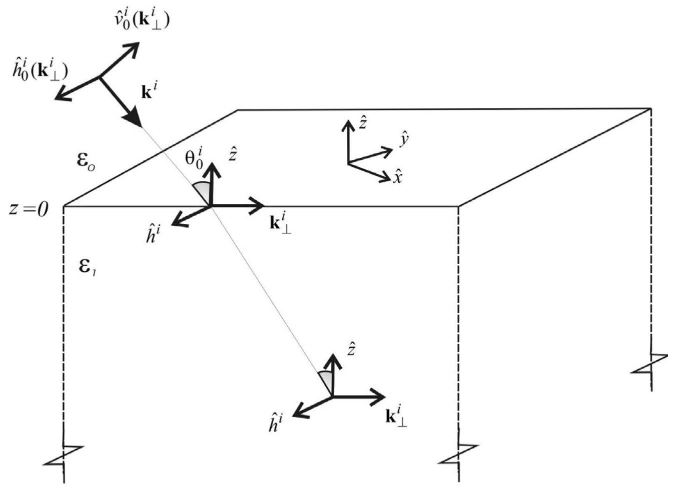

Fig. 6. Unperturbed Geometry.

We emphasize that the essential postulate of perturbation theory is that all higher terms may be neglected because $\delta\varepsilon$ is relatively small.

Anyhow, we also assume that the perturbation concerning the interface has a finite extent in the *x-y* directions, i.e., $\zeta(\mathbf{r}_\perp)$ is zero outside region of area A; and also that the perturbation $\chi_1(\mathbf{r})$ concerning the volumetric fluctuation has finite extent in the *x-y-z* directions. As matter of fact, the perturbations domain can be thought as physically limited by the illumination beam-width and by the relevant electromagnetic wave penetration-depth.

7.6.2 Unperturbed Field Evaluation

In this Section, we provide a general expression for the unperturbed vector field solution $\mathbf{E}^{(0)}(\mathbf{r})$ relevant to the pertinent unperturbed structure of Fig.6; similarly, an *auxiliary* solution $\overline{\mathbf{E}}^{(0)}$ is introduced in next subsection. If we assume that the field source is placed in the upper half-space and it is in the far zone with respect to the rough interface, then the unperturbed field $\mathbf{E}^{(0)}(\mathbf{r})$ is the field present in the unperturbed medium (characterized by flat interface and homogeneous permittivity) on which a (locally) plane wave impinges.

We consider an arbitrary polarized monochromatic plane wave incident from the upper half-space on the stratification at an angle θ_0^i with respect to the $\hat{z}$ direction, as schematically shown in Fig.6, whose representation (similarly as discussed in Section 3.2.2: see (3.7)-(3.10)) is given by

$$\mathbf{E}_0^i(\mathbf{r}) = [E_0^{ih}\hat{h}_0^i(\mathbf{k}_\perp^i) + E_0^{iv}\hat{v}_0^i(\mathbf{k}_\perp^i)]e^{j\mathbf{k}_\perp^i\cdot\mathbf{r}_\perp}e^{-jk_{z0}^i z}, \qquad (7.\ 92)$$

where the incident vector wave-number direction is individuated by θ_0^i, φ_0^i in a spherical coordinate frame. By employing above notations (see also Chapter 3), the unperturbed field $\mathbf{E}^{(0)}(\mathbf{r})$ can be conveniently expressed in the following closed-form:

$$\mathbf{E}_0^{(0)}(\mathbf{r}) = e^{j\mathbf{k}_\perp^i\cdot\mathbf{r}_\perp}e^{-jk_{z0}^i z}\Big\{\ \hat{h}^i[1+R_{0|1}^h(k_\perp^i)e^{j2k_{z0}^i z}]E_0^{ih} + \hat{k}_\perp^i\frac{k_{z0}^i}{k_0\varepsilon_0}[1-R_{0|1}^v(k_\perp^i)e^{j2k_{z0}^i z}]E_0^{iv} + \hat{z}\frac{k_\perp^i}{k_0\varepsilon_0}[1+R_{0|1}^v(k_\perp^i)e^{j2k_{z0}^i z}]E_0^{iv}\Big\} \qquad (7.93)$$

$$\mathbf{E}_1^{(0)}(\mathbf{r}) = e^{j\mathbf{k}_\perp^i\cdot\mathbf{r}_\perp}e^{-jk_{z1}^i z}\Big\{\hat{h}^i T_{0|1}^h(k_\perp^i)E_0^{ih} + \hat{k}_\perp^i\frac{k_{z1}^i}{k_0\varepsilon_1}T_{0|1}^v(k_\perp^i)E_0^{iv} + \hat{z}\frac{k_\perp^i}{k_0\varepsilon_1}T_{0|1}^v(k_\perp^i)E_0^{iv}\Big\} \qquad (7.94)$$

where the superscripts $p\in\{v,\ h\}$ denote the polarization, the subscript of the field $\in\{0, 1\}$ obviously refers to the relevant half-space, k_0 and $k_1 = k_0\sqrt{\varepsilon_1}$ are the wave-numbers in the upper and lower half-space, respectively.

In particular, the unperturbed field $\mathbf{E}^{(0)}(\mathbf{r})$ evaluated immediately above the unperturbed interface can be written as

$$\mathbf{E}_0^{(0)}(\mathbf{r}_\perp, 0^+) = e^{j\mathbf{k}_\perp^i\cdot\mathbf{r}_\perp}\Big\{\hat{h}^i[1+R_{0|1}^h(k_\perp^i)]E_0^{ih} + \hat{k}_\perp^i\frac{k_{z0}^i}{k_0\varepsilon_0}[1-R_{0|1}^v(k_\perp^i)]E_0^{iv} + \hat{z}\frac{k_\perp^i}{k_0\varepsilon_0}[1+R_{0|1}^v(k_\perp^i)]E_0^{iv}\Big\} \qquad (7.95)$$

where $R^p_{0|1}$ and $T^p_{0|1}$ are, respectively, the ordinary (Fresnel) *Reflection* and *Transmission* reflection coefficients at the interface (see Chapter 3):

$$R^h_{0|1} = \frac{k_{z0} - k_{z1}}{k_{z0} + k_{z1}}, \qquad R^v_{0|1} = \frac{\varepsilon_1 k_{z0} - \varepsilon_0 k_{z1}}{\varepsilon_1 k_{z0} + \varepsilon_0 k_{z1}}, \tag{7.96}$$

$$T^h_{0|1} = \frac{2k_{z0}}{k_{z0} + k_{z1}}, \qquad T^v_{0|1} = \frac{2\varepsilon_1 k_{z0}}{\varepsilon_1 k_{z0} + \varepsilon_0 k_{z1}}. \tag{7.97}$$

It is also important to note that $R^p_{0|1} = -R^p_{1|0}$, $T^p_{0|1} = 1 + R^p_{0|1}$, $T^p_{1|0} = 1 + R^p_{1|0}$ and

$$\frac{T^p_{0|1}}{T^p_{1|0}} = \begin{cases} \dfrac{k_{z0}}{k_{z1}} & p = h \\ \dfrac{\varepsilon_1 k_{z0}}{\varepsilon_0 k_{z1}} & p = h \end{cases} \tag{7.98}$$

7.6.3 Auxiliary Unperturbed Field Solution

Similarly, if we assume that the test source is placed in the upper half-space and is located in the far zone with respect to the structure, then the unperturbed field $\overline{\mathbf{E}}^{(0)}$ is the field present in the unperturbed medium on which a (locally) plane wave impinges; this plane wave is characterized by the electric field

$$\overline{\mathbf{E}}^i_0(\mathbf{r}) = e^{-j\mathbf{k}^s_\perp \cdot \mathbf{r}_\perp} e^{-jk^s_{z0} z} \overline{E}^i_0 \, \hat{t} \tag{7.99}$$

with a wave-vector

$$\begin{aligned} -k_0 \hat{r}_\mathbf{s} &= -\mathbf{k}^s_0 = -\mathbf{k}^s_\perp - \hat{z} k^s_{z0} \\ &= k_0(-\hat{x}\sin\theta^s_0 \cos\varphi^s_0 - \hat{y}\sin\theta^s_0 \sin\varphi^s_0 - \hat{z}\cos\theta^s_0) \end{aligned} \tag{7.100}$$

and complex amplitude

$$\overline{E}_0^i = j\eta_0 k_0 J \frac{e^{jk_0 r_s}}{4\pi r_s}. \tag{7.101}$$

Accordingly, the unperturbed field $\overline{\mathbf{E}}^{(0)}$ can be formally expressed as:

$$\begin{aligned}\overline{\mathbf{E}}_0^{(0)}(\mathbf{r}) = e^{-j\mathbf{k}_\perp^s\cdot\mathbf{r}_\perp} e^{-jk_{z0}^s z} \{ &\hat{h}^s[1 + R_{0|1}^h(k_\perp^s)e^{j2k_{z0}^s z}]\overline{E}_0^{ih} \\ &- \hat{k}_\perp^s \frac{k_{z0}^i}{k_0\varepsilon_0}[1 - R_{0|1}^v(k_\perp^s)e^{j2k_{z0}^s z}]\overline{E}_0^{iv} + \hat{z}\frac{k_\perp^s}{k_0\varepsilon_0}[1 + R_{0|1}^v(k_\perp^s)e^{j2k_{z0}^s z}]\overline{E}_0^{iv} \}\end{aligned} \tag{7.102}$$

$$\begin{aligned}\overline{\mathbf{E}}_1^{(0)}(\mathbf{r}) = e^{-j\mathbf{k}_\perp^s\cdot\mathbf{r}_\perp} e^{-jk_{z1}^s z} \{ &\hat{h}^s T_{0|1}^h(k_\perp^s)\, \overline{E}_0^{ih} \\ &- \hat{k}_\perp^s \frac{k_{z1}^s}{k_0\varepsilon_1} T_{0|1}^v(k_\perp^s)\, \overline{E}_0^{iv} + \hat{z}\frac{k_\perp^s}{k_0\varepsilon_1} T_{0|1}^v(k_\perp^s)\overline{E}_0^{iv} \}\end{aligned} \tag{7.103}$$

where $\hat{h}^s$ and $\hat{v}^s$ are given by (21)-(22) provided that superscripts i is replaced by superscripts s, and where

$$\overline{E}_0^{ih} = \overline{E}_0^i\, \hat{t}\cdot\hat{h}^s, \qquad \overline{E}_0^{iv} = \overline{E}_0^i\, \hat{t}\cdot\hat{v}^s. \tag{7.104}$$

Similarly, the unperturbed field $\overline{\mathbf{E}}^{(0)}$ evaluated immediately above the unperturbed interface can be written as

$$\begin{aligned}\overline{\mathbf{E}}_0^{(0)}(\mathbf{r}_\perp, 0^+) = e^{-j\mathbf{k}_\perp^s\cdot\mathbf{r}_\perp} \{ &\hat{h}^s[1 + R_{0|1}^h(k_\perp^s)]\overline{E}_0^{ih} \\ &- \hat{k}_\perp^s \frac{k_{z0}^s}{k_0\varepsilon_0}[1 - R_{0|1}^v(k_\perp^s)]\overline{E}_0^{iv} + \hat{z}\frac{k_\perp^s}{k_0\varepsilon_0}[1 + R_{0|1}^v(k_\perp^s)]\overline{E}_0^{iv} \}.\end{aligned} \tag{7.105}$$

7.6.4 Scattered Field Evaluation

In this section, to address the analytical evaluation of the scattered field, we here focus formally on the expression (12). By substituting (88)-(89) into (12), we get

$$\mathbf{E}_0^{(1)}(\mathbf{r}_0)\cdot\hat{t} = -j\frac{k_0}{J\eta_0}\iint_S d\mathbf{r}_\perp \overline{\mathbf{E}}_0^{(0)}(\mathbf{r}_\perp,0^+)\cdot(\varepsilon_1-\varepsilon_0)\zeta(\mathbf{r}_\perp)\mathbf{E}_1^{(0)}(\mathbf{r}_\perp,0^-)$$
$$-j\frac{k_0}{J\eta_0}\iiint_{V_1}\overline{\mathbf{E}}_1^{(0)}(\mathbf{r})\cdot\chi_1(\mathbf{r})\mathbf{E}_1^{(0)}(\mathbf{r})d\mathbf{r}, \tag{7.106}$$

in which 0^- and 0^+ mean that the fields are evaluated immediately under and above the plane $z = 0$, respectively; where V_1 denotes the half-space $z<0$, and S is the surface of the unperturbed interface z=0. It should be noted that the surface and volume integrals in (106) correspond, respectively, to the two pertinent scattering contributions (form rough interface and volume fluctuations). In particular note also that when geometric roughness of the surface is described, as in the previous subsection, by means of an appropriate volume perturbation localized about the flat interfaces, then the integral in (12) becomes essentially a surface one: the first term on the right-hand-side of (106) accounts for the contribution due to *implusive* permittivity fluctuation on the flat-boundaries, with which we equivalently describe interfacial roughness in the approximation of small roughness, and which has been treated separately because of the discontinuity of the unperturbed electric field on the flat-interfaces of the unperturbed structure.

In next Sections, we explicitly evaluate each of the integral expressions appearing in the right-hand-side of (106). To this purpose, it is convenient also to introduce a suitable spectral representation for the unperturbed solutions: let us consider the 2-D Fourier Transform of (94) and (103), respectively, with respect to the transverse coordinates

$$\mathcal{FT}_{2D}\{\mathbf{E}_1^{(0)}(\mathbf{r})\} = \widetilde{\mathbf{E}}_1^{(0)}(k_\perp^i, z|\,\mathbf{k}^i)\,\delta(\mathbf{k}_\perp - \mathbf{k}_\perp^i), \tag{7.107}$$

$$\mathcal{FT}_{2D}\{\overline{\mathbf{E}}_1^{(0)}(\mathbf{r})\} = \widetilde{\overline{\mathbf{E}}}_1^{(0)}(k_\perp^s, z|\,-\mathbf{k}^s)\,\delta(\mathbf{k}_\perp + \mathbf{k}_\perp^s), \tag{7.108}$$

wherein

$$\widetilde{\mathbf{E}}_1^{(0)}(k_\perp^i, z|\,\mathbf{k}^i) = e^{-jk_{z1}^i z}\left\{\hat{h}^i T_{0|1}^h(k_\perp^i)\,E_0^{ih} + \left[\hat{k}_\perp^i\frac{k_{z1}^i}{k_0\varepsilon_1} + \hat{z}\frac{k_\perp^i}{k_0\varepsilon_1}\right]T_{0|1}^v(k_\perp^i)\,E_0^{iv}\right\}, \tag{7.109}$$

$$\widetilde{\overline{\mathbf{E}}}_1^{(0)}(k_\perp^s, z\big| -\mathbf{k}^s) = e^{-jk_{z1}^s z}\left\{\hat{h}^s T_{0|1}^h(k_\perp^s)\,\overline{E}_0^{ih} + \left[-\hat{k}_\perp^s \frac{k_{z1}^s}{k_0\varepsilon_1} + \hat{z}\frac{k_\perp^s}{k_0\varepsilon_1}\right] T_{0|1}^v(k_\perp^s)\,\overline{E}_0^{iv}\right\}. \tag{7.110}$$

Therefore, we have

$$\mathbf{E}_1^{(0)}(\mathbf{r}) = e^{j\mathbf{k}_\perp^i\cdot\mathbf{r}_\perp} e^{-jk_{z1}^i z}\,\widetilde{\mathbf{E}}_1^{(0)}(k_\perp^i, 0\big|\,\mathbf{k}^i), \tag{7.111}$$

$$\overline{\mathbf{E}}_1^{(0)}(\mathbf{r}) = e^{-j\mathbf{k}_\perp^s\cdot\mathbf{r}_\perp} e^{-jk_{z1}^s z}\,\widetilde{\overline{\mathbf{E}}}_1^{(0)}(k_\perp^s, 0\big| -\mathbf{k}^s). \tag{7.112}$$

7.6.5 Scattering from the Interfacial Roughness

In this section, we focus on the analysis of interfacial inhomogeneities, which are responsible for surface scattering contribution from the relevant perturbated half-space. Taking into account only the first term in the RHS of (106), and proceeding quite similar as done in Section 7.3.4, from equation (106) we obtain another equivalent form, which is more convenient for our purpose,

$$\mathbf{E}_0^{(1)}(\mathbf{r}_0)\cdot\hat{t} = -j\frac{k_0}{J\eta_0}(\varepsilon_1-\varepsilon_0)\iint_S d\mathbf{r}_\perp \mathcal{P}^{-1}\overline{\mathbf{E}}_1^{(0)}(\mathbf{r}_\perp, 0^-)\cdot\zeta(\mathbf{r}_\perp)\mathbf{E}_1^{(0)}(\mathbf{r}_\perp, 0^-). \tag{7.113}$$

As a result, we emphasize that (113), which clearly express the perturbative solution directly in terms of unperturbed solutions, allows us to read the scattered field in terms of *wave coupling* of unperturbed solutions, whereas the roughness couples the energy of the incident wave with the scattered one at the receiver. By using the notations (111) and (112), we can write

$$\begin{gathered}\mathbf{E}_0^{(1)}(\mathbf{r}_0)\cdot\hat{t} = -j(2\pi)^2\frac{k_0}{J\eta_0}(\varepsilon_1-\varepsilon_0)\,\tilde{\zeta}(\mathbf{k}_\perp^s-\mathbf{k}_\perp^i)\\ \mathcal{P}^{-1}\widetilde{\overline{\mathbf{E}}}_1^{(0)}(k_\perp^s, 0\big| -\mathbf{k}^s)\cdot\widetilde{\mathbf{E}}_1^{(0)}(k_\perp^i, 0\big|\,\mathbf{k}^i),\end{gathered} \tag{7.114}$$

where $\tilde{\zeta}(\mathbf{k}_\perp)$ is the two-dimensional (generalized) *Fourier Transform* (2D-FT) of the rough surfaces (see (35)). At this point, to describe the scattered field $\mathbf{E}_0^{(1)}(\mathbf{r}_0)$, similarly as done in previous Sections, the following representation is considered

$$\mathbf{E}_0^{(1)}(\mathbf{r}_0) = E_0^{sh}\hat{h}^s(\mathbf{k}_\perp^s) + E_0^{sv}\hat{v}^s(\mathbf{k}_\perp^s). \tag{7.115}$$

Substituting (109), (110) and (30) in (115), evaluated for $\hat{t} = \hat{h}^s$ and for $\hat{t} = \hat{v}^s$, and taking into account (62)-(64), we directly obtain the following compact expression

$$\begin{bmatrix} E_0^{sv} \\ E_0^{sh} \end{bmatrix} = \frac{e^{jk_0 r_s}}{r_s}\pi k_0^2 \tilde{\zeta}(\mathbf{k}_\perp^s - \mathbf{k}_\perp^i)(\varepsilon_1 - \varepsilon_0)\boldsymbol{\gamma}'(\mathbf{k}^s, \mathbf{k}^i)\begin{bmatrix} E_0^{iv} \\ E_0^{ih} \end{bmatrix} \tag{7.116}$$

wherein

$$\boldsymbol{\gamma}'(\mathbf{k}^s, \mathbf{k}^i) = \begin{bmatrix} \gamma'_{vv}(\mathbf{k}^s, \mathbf{k}^i) & \gamma'_{vh}(\mathbf{k}^s, \mathbf{k}^i) \\ \gamma'_{hv}(\mathbf{k}^s, \mathbf{k}^i) & \gamma'_{hh}(\mathbf{k}^s, \mathbf{k}^i) \end{bmatrix}, \tag{7.117}$$

where the superscripts i and s refer to the incident and scattered field directions, respectively, and where

$$\gamma'_{vv}(\mathbf{k}^s, \mathbf{k}^i) = \frac{1}{k_0^2 \varepsilon_1^2}\left[\frac{\varepsilon_1}{\varepsilon_0} k_\perp^s k_\perp^i - \hat{k}_\perp^s \cdot \hat{k}_\perp^i k_{z1}^s k_{z1}^i\right] T_{0|1}^v(k_\perp^s) T_{0|1}^v(k_\perp^i) \tag{7.118}$$

$$\gamma'_{hh}(\mathbf{k}^s, \mathbf{k}^i) = \hat{k}_\perp^s \cdot \hat{k}_\perp^i \, T_{0|1}^h(k_\perp^s) T_{0|1}^h(k_\perp^i) \tag{7.119}$$

$$\gamma'_{hv}(\mathbf{k}^s, \mathbf{k}^i) = \hat{z} \cdot (\hat{k}_\perp^i \times \hat{k}_\perp^s) \frac{k_{z1}^i}{k_0 \varepsilon_1} T_{0|1}^h(k_\perp^s) T_{0|1}^v(k_\perp^i) \tag{7.120}$$

$$\gamma'_{vh}(\mathbf{k}^s, \mathbf{k}^i) = \gamma'_{hv}(-\mathbf{k}^i, -\mathbf{k}^s) \tag{7.121}$$

We underline that the solution furnished here is fully equivalent to the one in Section 7.3.4: this formal equivalence can be straightforwardly verified, provided that the reflection coefficients are expressed in terms of the corresponding transmission coefficients. How it will be clear in the following, the expression provided by (118)-(121) is more convenient here for our purposes. However, both these expressions essentially represent, in different forms, the classical SPM solution.

This point merits a further discussion. The available bulky procedures for the derivation of classical SPM solution require unnecessary complication and lead to obscuring the underling physics as well as the essence of the perturbation approach (see, for instance, [5][6][9]). On the contrary, the presented formulation is carried out in a surprisingly simple way, provides a conceptually clean formulation for the classical SPM, and offers, in a completely different and innovative perspective, a more complete comprehension of the resonant scattering phenomenon, emphasizing the role of wave coupling of unperturbed solutions. Moreover, VPRT approach permits also to include the volumetric scattering contribution within the same formalism, as detailed in the following.

7.6.6 Scattering from the Volumetric Fluctuations

In this section, we confine our attention to the analysis of volume inhomogeneities, which are responsible for volumetric scattering contribution from the relevant random half-space. Taking into account only the second term in the RHS of (106), equation (106) can be equivalently rewritten as

$$\mathbf{E}_0^{(1)}(\mathbf{r}_0)\cdot J\hat{t} = -j\frac{k_0}{\eta_0}\iiint_{V_1} d\mathbf{r}\;\overline{\mathbf{E}}_1^{(0)}(\mathbf{r})\cdot\chi_1(\mathbf{r})\mathbf{E}_1^{(0)}(\mathbf{r}). \tag{7.122}$$

We highlight the crucial role played by the resulting wave coupling, which is intimately related to structural perturbation introduced in the first-order formulation: $\mathbf{J}_1^{(0)}(\mathbf{r}) = -j\omega\mu\chi_1(\mathbf{r})\mathbf{E}_1^{(0)}(\mathbf{r})$ can be then interpreted as the equivalent *polarization current* induced into the (localized) perturbed half-space by the unperturbed field $\mathbf{E}_1^{(0)}$. As a result, we get

$$\mathbf{E}_0^{(1)}(\mathbf{r}_0)\cdot J\hat{t} = \iiint_{V_1} d\mathbf{r}\;\overline{\mathbf{E}}_1^{(0)}(\mathbf{r})\cdot\mathbf{J}_1^{(0)}(\mathbf{r}) = \left\langle \overline{\mathbf{E}}_1^{(0)}, \mathbf{J}_1^{(0)} \right\rangle, \tag{7.123}$$

where the symbol <·,·> is adopted to mathematically represent the *symmetric bilinear form* to which is given the physical meaning of *reaction* [15],[35]-[36].

Therefore, in the first order approximation, from the receiver viewpoint, the electromagnetic *coupling* between two unperturbed waves is observed (see Eq. (122)), and the scalar on the left-hand side of Eq.(123) is susceptible to be physically interpreted as a measure of the reaction (or coupling) between the source field, $\mathbf{J}_1^{(0)}$, and the mediating field, $\overline{\mathbf{E}}_1^{(0)}$. Therefore, we get

$$\mathbf{E}_0^{(1)}(\mathbf{r}_0)\cdot\hat{t} = -j\frac{k_0}{J\eta_0}\int_{-\infty}^{0}dz\iint_S d\mathbf{r}_\perp\,\overline{\mathbf{E}}_1^{(0)}(\mathbf{r}_\perp,z)\cdot\chi_1(\mathbf{r}_\perp,z)\mathbf{E}_1^{(0)}(\mathbf{r}_\perp,z), \tag{7.124}$$

where S is the illuminated surface. Making explicit the integration in (122) and using the expressions (109) and (110), we get

$$\begin{aligned}\mathbf{E}_0^{(1)}(\mathbf{r}_0)\cdot\hat{t} = -j\frac{k_0}{J\eta_0}\,&\widetilde{\overline{\mathbf{E}}}_1^{(0)}(k_\perp^s,0\,|\,-\mathbf{k}^s)\cdot\widetilde{\mathbf{E}}_1^{(0)}(k_\perp^i,0\,|\,\mathbf{k}^i)\\ &\int_{-\infty}^{0}dz\;e^{-j(k_{z1}^s+k_{z1}^i)z}\iint_S d\mathbf{r}_\perp e^{-j(\mathbf{k}_\perp^s-\mathbf{k}_\perp^i)\cdot\mathbf{r}_\perp}\chi_1(\mathbf{r}_\perp,z),\end{aligned} \tag{7.125}$$

By introducing the 2-D (generalized) Fourier Transform of the corresponding perturbation with respect to transverse coordinates (see also Chapter 2)

$$\widetilde{\chi}_1(\boldsymbol{\kappa},z) = (2\pi)^{-2}\iint d\mathbf{r}_\perp e^{-j\boldsymbol{\kappa}\cdot\mathbf{r}_\perp}\chi_1(\mathbf{r}_\perp,z), \tag{7.126}$$

equation (125) can be rewritten in the form

$$\mathbf{E}_0^{(1)}(\mathbf{r}_0)\cdot\hat{\mathbf{t}} = -j(2\pi)^2\frac{k_0}{J\eta_0}\widetilde{\overline{\mathbf{E}}}_1^{(0)}(k_\perp^s,0\,|\,-\mathbf{k}^s)\cdot\widetilde{\mathbf{E}}_1^{(0)}(k_\perp^i,0\,|\,\mathbf{k}^i)\,I, \tag{7.127}$$

wherein

$$I = \int_{-\infty}^{0}dz\;\widetilde{\chi}_1(\mathbf{k}_\perp^s-\mathbf{k}_\perp^i,z)e^{-j(k_{z1}^s+k_{z1}^i)z}, \tag{7.128}$$

and where $\mathbf{k}_\perp^d = \mathbf{k}_\perp^s - \mathbf{k}_\perp^i$ is the projection on the *x-y* plane of the vector $\mathbf{k}^d = \mathbf{k}^s - \mathbf{k}^i$. We emphasize that the analytical evaluation of field expression (125) essentially reduces to performing the inherent complex integral (128). It should be noted the similarity in the vectorial structure of the formal solutions (114) and (127). Proceeding as in the previous Section, we directly obtain in this case the following compact expression

$$\begin{bmatrix} E_0^{sv} \\ E_0^{sh} \end{bmatrix} = \frac{e^{jk_0 r_0}}{r_0} \pi k_0^2 I \boldsymbol{\gamma}''(\mathbf{k}^s, \mathbf{k}^i) \begin{bmatrix} E_0^{iv} \\ E_0^{ih} \end{bmatrix} \tag{7.129}$$

wherein

$$\boldsymbol{\gamma}''(\mathbf{k}^s, \mathbf{k}^i) = \begin{bmatrix} \gamma''_{vv}(\mathbf{k}^s, \mathbf{k}^i) & \gamma''_{vh}(\mathbf{k}^s, \mathbf{k}^i) \\ \gamma''_{hv}(\mathbf{k}^s, \mathbf{k}^i) & \gamma''_{hh}(\mathbf{k}^s, \mathbf{k}^i) \end{bmatrix}, \tag{7.130}$$

with

$$\gamma''_{vv}(\mathbf{k}^s, \mathbf{k}^i) = \frac{1}{k_0^2 \varepsilon_1^2} \left[k_\perp^s k_\perp^i - \hat{\mathbf{k}}_\perp^s \cdot \hat{\mathbf{k}}_\perp^i k_{z1}^s k_{z1}^i \right] T_{0|1}^v(k_\perp^s) T_{0|1}^v(k_\perp^i) \tag{7.131}$$

$$\gamma''_{hh}(\mathbf{k}^s, \mathbf{k}^i) = \hat{\mathbf{k}}_\perp^s \cdot \hat{\mathbf{k}}_\perp^i T_{0|1}^h(k_\perp^s) T_{0|1}^h(k_\perp^i) \tag{7.132}$$

$$\gamma''_{hv}(\mathbf{k}^s, \mathbf{k}^i) = \hat{\mathbf{z}} \cdot (\hat{\mathbf{k}}_\perp^i \times \hat{\mathbf{k}}_\perp^s) \frac{k_{z1}^i}{k_0 \varepsilon_1} T_{0|1}^h(k_\perp^s) T_{0|1}^v(k_\perp^i) \tag{7.133}$$

$$\gamma''_{vh}(\mathbf{k}^s, \mathbf{k}^i) = \gamma''_{hv}(-\mathbf{k}^i, -\mathbf{k}^s) \tag{7.134}$$

Note that (131)-(134) are formally identical to corresponding (118)-(121), except for a factor $\varepsilon_1/\varepsilon_0$ missing in γ''_{vv} : this reflects the role played by the dyadic operator $\boldsymbol{\mathcal{P}}$ when the scalar product of the pertinent unperturbed fields is concerned (see (114) and (127)).

It is important to note that the coefficients (131)-(134) and (118)-(121) are formally compliant with the *reciprocity principle* [39], so also (130) and (117).

7.6.7 Analysis of the Field Intensity Fluctuations

In this section, we are interested in deriving the statistical properties of the scattered far-field in terms of the structural statistical characteristics of the scattering medium. A frequently used basic assumption adopted for the description of random structures is that the pertinent structural statistics are stationary, i.e., pertinent spatial statistics are typically *homogenous*. It means that the distribution of the analyzed structure is translation-invariant in statistical sense.

In conformity with the theory of random function, the estimate of the *mean field-power-density* (PSD) is then obtained averaging over many realizations. Accordingly as discussed in Chapter 1 (see eq. (1.12)), the overall bi-static scattering cross section for the pertinent structure can be defined as

$$\tilde{\sigma}^0_{qp} = \lim_{r\to\infty} \lim_{A\to\infty} \frac{4\pi r^2}{A} \frac{< \left|\mathbf{E}_0^{(1)}(\mathbf{r}) \cdot \hat{q}_0^+(\mathbf{k}_\perp^s)\right|^2 >}{\left|\mathbf{E}_0^i \cdot \hat{q}_0^-(\mathbf{k}_\perp^i)\right|^2}, \tag{7.135}$$

where angular brackets denote ensemble averaging, $q \in \{v, h\}$ and $p \in \{v, h\}$ denote the scattered and incident polarizations, respectively, and A is the illuminated surface area. Note also that the overall scattering field from the structure, whose components are $\mathbf{E}_0^{(1)}(\mathbf{r}) \cdot \hat{q}_0^+(\mathbf{k}_\perp^s)$, arises from the superposition of the scattered fields given by (114) and (127).

Firstly, the contribution from the interfacial roughness is addressed. Assuming that suitable statistical properties (see also Chapter 2) of the process describing the interfacial roughness are invariant with respect to a spatial shift in the *x-y* plane (*wide sense stationary*), the pertinent *power spectral density* $W(\boldsymbol{\kappa})$ of the rough interface can be expressed as (40).

Accordingly, proceeding similarly as done previously, we obtain the *bi-static scattering cross-section* of the rough surface only

$$\sigma^0_{qp} = \pi k_0^4 \left|\varepsilon_1 - \varepsilon_0\right|^2 \left|\gamma'_{qp}(\mathbf{k}^s, \mathbf{k}^i)\right|^2 W(\mathbf{k}_\perp^s - \mathbf{k}_\perp^i). \tag{7.136}$$

This result corresponds to the classical SPM one formally obtained for the case of a homogeneous rough half-space. Note also that the (136) constitutes a different expression of the one carried out in Section 7.3.4.

Secondly, the contribution from the volumetric fluctuations is addressed. We assume that the volumetric permittivity fluctuations to be statistically homogeneous in vertical direction, i.e., the pertinent *vertical correlation* function is given (see also Chapter 2) by the following 2-D spectral representation for $\chi_1(\mathbf{r})$

$$B_1(\boldsymbol{\kappa}_\perp, z' - z'') = (2\pi)^2 \lim_{A\to\infty} \frac{1}{A} < \tilde{\chi}_1(\boldsymbol{\kappa}_\perp, z')\, \tilde{\chi}_1^*(\boldsymbol{\kappa}_\perp, z'') >, \qquad (7.137)$$

where the asterisk denoting the complex conjugation; the angular brackets denoting the ensemble averaging; A is the illuminated area; and where the dependence of B_1 on the difference variable $z' - z''$ reflects the aforementioned assumption of statistical homogeneities.

Accordingly, the associated (spatial) Fourier transform for the z direction of the correlation function (137) is given by

$$W_1(\boldsymbol{\kappa}_\perp, \beta_z) = (2\pi)^{-1} \int_{-\infty}^{+\infty} dz\ B_1(\boldsymbol{\kappa}_\perp, z)\ e^{-j\beta_z z}\,. \qquad (7.138)$$

where β_z is the z-directed spectral wave number. Equation (138) defines the 3-D *power spectral density* of the volumetric fluctuation and it relates its 2-D and 3-D spectral representations (as discussed in Section 1.2).

Therefore, the relevant *mean field-power-density* is

$$\lim_{A\to\infty} \frac{1}{A} < \left|\mathbf{E}_0^{(1)}(\mathbf{r}_0)\cdot \hat{t}\right|^2 > = (2\pi)^2 \left(\frac{k_0}{J\eta_0}\right)^2 \left|\tilde{\tilde{\mathbf{E}}}_1^{(0)}(k_\perp^s, 0 \,|\, -\mathbf{k}^s)\cdot \tilde{\mathbf{E}}_1^{(0)}(k_\perp^i, 0 \,|\, \mathbf{k}^i)\right|^2 \hat{W}_1, \qquad (7.139)$$

wherein

$$\hat{W}_1 = (2\pi)^2 \lim_{A\to\infty} \frac{1}{A}\langle I\, I^* \rangle$$
$$= \int_{-\infty}^{0} dz' \int_{-\infty}^{0} dz''\, B_1(\mathbf{k}_\perp^d, z'-z'')\, e^{-j(k_{z1}^s + k_{z1}^i)z'} e^{j(k_{z1}^s + k_{z1}^i)^* z''}. \tag{7.140}$$

The remaining double integral in (140) resembles the one considered in [5]: There the integration was performed in the complex plane only under asymptotically-low-absorption approximation, resorting to the residue calculus. It should be noted that, when the *asymptotically small absorption* hypothesis is not fulfilled, the result in [5] is no longer valid and the integral (140) must be evaluated explicitly. Therefore, we proceed, differently from [5], carrying out the double spatial integration, without introducing any approximation (as detailed in *Appendix*), obtaining

$$\hat{W}_1(\mathbf{k}_\perp^d + \hat{z}(\beta_z + j\alpha_z)) = \int_{-\infty}^{+\infty} d\beta_z' \,\frac{W_1(\mathbf{k}_\perp^d, \beta_z')}{\alpha_z^2 + (\beta_z - \beta_z')^2}, \tag{7.141}$$

with $\alpha_z \geq 0$, $\beta_z = \mathrm{Re}\{k_{z1}^s + k_{z1}^i\}$ and $\alpha_z = \mathrm{Im}\{k_{z1}^s + k_{z1}^i\}$. It should be noted that, for prescribed scattering ($\mathbf{k}_\perp^s$) and incident ($\mathbf{k}_\perp^i$) directions, the corresponding vector wave-numbers in the lower (unperturbed) medium, respectively, for the scattering and incident directions are given by

$$\mathbf{k}_1^s = \mathbf{k}_\perp^s + \hat{\mathbf{z}} k_{z1}^s = \mathbf{k}_\perp^s + \hat{\mathbf{z}}\, k_0 \sqrt{\varepsilon_1 - \sin^2\theta^s}\,, \tag{7.142}$$

$$\mathbf{k}_1^i = \mathbf{k}_\perp^i - \hat{\mathbf{z}} k_{z1}^i = \mathbf{k}_\perp^i - \hat{\mathbf{z}}\, k_0 \sqrt{\varepsilon_1 - \sin^2\theta^i}\,, \tag{7.143}$$

hence, we have

$$\beta_z = \mathrm{Re}\{(\mathbf{k}_1^s - \mathbf{k}_1^i)\cdot\hat{z}\}, \tag{7.144}$$

$$\alpha_z = \mathrm{Im}\{(\mathbf{k}_1^s - \mathbf{k}_1^i)\cdot\hat{z}\}, \tag{7.145}$$

$$\beta_z + j\alpha_z = (\mathbf{k}_1^s - \mathbf{k}_1^i)\cdot\hat{z}. \tag{7.146}$$

Furthermore, we emphasize that it turns out that

$$\hat{W}_1(\mathbf{k}_1^s - \mathbf{k}_1^i) = \hat{W}_1((\mathbf{k}_1^s - \mathbf{k}_1^i)^*). \tag{7.147}$$

Accordingly, the *bi-static scattering cross-section* due to the volumetric fluctuations only is

$$\sigma_{qp}^0 = \pi k_0^4 \left|\gamma''_{qp}(\mathbf{k}^s, \mathbf{k}^i)\right|^2 \hat{W}_1(\mathbf{k}_1^s - \mathbf{k}_1^i), \tag{7.148}$$

where the spectrum $\hat{W}_1$ appearing in (139), whose expression is given by (141), has to be then regarded as an *effective power spectral density*, being expressed in terms of an integral transform of the volumetric fluctuation one W_1, where formally complex spatial frequencies have to be considered. It is important to note that the prescribed spatial frequency is defined by both the incident and observation planes. Some informative considerations will be provided in the detailed examination of this formula, as discussed in next Section.

In addition, it should be emphasized that, if the first-order interfacial and volumetric perturbations are assumed statistically uncorrelated, the relevant contributions to the scattering cross-section may be superimposed. Conversely, when the assumption is not fulfilled, the scattering contributions will be coupled through cross products of the two perturbation quantities, taking into account volume-roughness spatial correlations.

7.6.8 Effective Power Spectral Density of the Structure

As discussed in previous subsection, the representation of the *mean field-power-density* appearing in (139) is then given directly in terms of the quantity $\hat{W}_1(\mathbf{k}_1^s - \mathbf{k}_1^i)$, whose expression is given by (141), and which can be evidently named the *effective power spectral density* of the structure, since it constitutes a *filtered version* of the *power spectral density* of the medium volumetric fluctuation $W_1(\boldsymbol{\kappa})$.

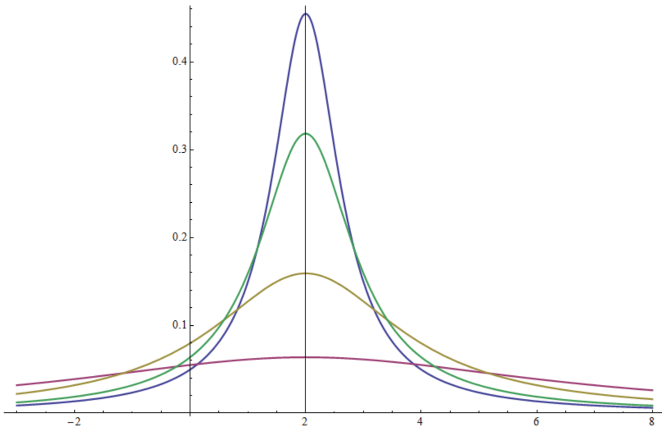

Fig. 7. Lorentzian lineshape for different value of the parameter α_z (0.7, 1.0, 2.0, 5.0).

In this regard, we highlight that the integral (141), for a given direction of scattering, clearly takes the form of a convolution:

$$\hat{W}_1(\mathbf{k}_\perp^d + \hat{z}(\beta_z + j\alpha_z)) = \frac{\pi}{\alpha_z} W_1(\mathbf{k}_\perp^d, \beta_z) \otimes L_{\alpha_z}(\beta_z)$$
$$= \frac{\pi}{\alpha_z} \int_{-\infty}^{+\infty} d\beta_z' \; W_1(\mathbf{k}_\perp^d, \beta_z') L_{\alpha_z}(\beta_z - \beta_z'), \qquad (7.149)$$

where $\otimes$ denotes the convolution and where the kernel L_{α_z} is represented by:

$$L_{\alpha_z}(\beta_z - \beta_z') = \frac{1}{\pi} \frac{\alpha_z}{\alpha_z^2 + (\beta_z - \beta_z')^2} = \frac{1}{\pi\alpha_z} \frac{1}{1 + \left(\dfrac{\beta_z - \beta_z'}{\alpha_z} \right)^2}. \qquad (7.150)$$

It should be noted that (150) is the well-know *Lorentzian* spectrum profile (Fig. 7). It is also known as *Cauchy* distribution in statistics. Note also that, in nuclear and particle physics, it is also formally representing the celebrated *Breit–Wigner* resonant formula [45][47][48]. In spectroscopy context, Lorentzian shape gives the description of the line shape of spectral lines which are subject to homogeneous broadening in which all atoms interact in the same way with the frequency range contained in the line-shape. More in general, its importance in physics is due to it being the solution to the differential equation describing forced resonance.

In mathematics, the convolution $W_1 \otimes L_{\alpha_z}$ in (149) can be also formally regarded as representative of the well-know *Poisson formula* in the half-plane resulting from the *Laplace*'s equation in a half-plane solution [46]. Figure 7 illustrates the evolution of the Lorentzian shape (150) as α_z increases: It is seen that the peak (or mode) occurs at $\beta_z' = \beta_z$ and the related amplitude is given by $(\pi\alpha_z)^{-1}$, the *full width at half maximum* (FWHM) of the Lorentzian shape is $2\alpha_z$; so that α_z plays the role of scale parameter.

As a result, the *effective* PSD of the structure (149) results from the convolution of the actual permittivity fluctuation PSD with a *Lorentzian* one, which characterizes the relevant spectrum broadening (diffraction line broadening) due to finite absorption (*spatial*) scale in the unperturbed medium. This is to say that the *effective* PSD profile, which plays a particularly important role in our investigation on the scattered power, is a precise consequence of two effects: the one concerning the morphological inhomogeneities of medium's volumetric perturbation, $\chi_1(\mathbf{r})$, and the other one intrinsically associated with absorption nature of the unperturbed lower half–space medium. It should be also noted that exponential nature of the spatial distribution of field decay in the lower unperturbed lower half-space (radiation spatial damping) determines the form of the line shape of the resonance. Moreover, we underline that the *effective* PSD of the structure measures the fluctuations of the actual PSD $W_1(\mathbf{k}_\perp^d, \beta_z')$ of the volumetric perturbation at (spatial) scale $1/\alpha_z$, providing information on the local irregularity of W_1 around the direction β_z and at scale α_z. The trend at scale α_z containing slower evolutions is essentially eliminated in $\hat{W}_1(\mathbf{k}_\perp^d + \hat{z}(\beta_z + j\alpha_z))$.

Now, we investigate the limiting case of negligible losses (*Asymptotically Small Losses*) and also show that a consistent relationship can be found between our result and the approximate one obtained in [5][12]. Under this circumstance, in which the losses in the half-space become vanishingly small, we obtain

$$\lim_{\alpha_z\to 0^+} \alpha_z \hat{W}_1(\mathbf{k}_\perp^d, \beta_z + j\alpha_z) = \pi \int_{-\infty}^{+\infty} d\beta_z' \, W_1(\mathbf{k}_\perp^d, \beta_z') \lim_{\alpha_z\to 0^+} L_{\alpha_z}(\beta_z - \beta_z') = \pi W_1(\mathbf{k}_\perp^d, \beta_z), \tag{7.151}$$

where we have observed that L_{α_z} asymptotically approaches the *Dirac*'s delta function as $\alpha_z \to 0^+$. As a matter of fact, the *Cauchy*'s definition of the Dirac's delta function is [45][46]:

$$\delta(\beta_z - \beta_z') = \lim_{\alpha_z\to 0^+} \frac{1}{\pi} \frac{\alpha_z}{\alpha_z^2 + (\beta_z - \beta_z')^2}. \tag{7.152}$$

Therefore, for $\alpha_z \to 0^+$ we have

$$\hat{W}_1(\mathbf{k}_\perp^d, \beta_z + j\alpha_z) \approx \frac{\pi}{\alpha_z} W_1(\mathbf{k}_\perp^d, \beta_z), \tag{7.153}$$

so that, when particularized for the limiting case of vanishingly small losses, our result (148) consistently reduces to the corresponding approximated one provided in [5][12]. We also stress that in such a case an infinite scattering contribution is obtained when $\alpha_z = 0$. It is important to note that in this limit case the first-order approximation could not be acceptable.

7.7 Conclusion

In this Chapter, we have proposed VPRT formulation to deal with EM scattering form rough surface and rough multilayered structures; the formulation is intrinsically reciprocal.

The comprehensive scattering formulation is based on a unified description for both interfacial and volume inhomogeneities. This formulation permits to obtain a comprehensive method to evaluate the scattering, which includes in conjunction and in rigorous manner both rough-interfaces scattering and volume scattering in complex multilayer.

We have shown that this formulation, when applied to rough surface or 3-D rough multilayer, leads to derive, in the first-order limit of the perturbative development, pertinent closed form solution in a very simple way. The obtained scattering solution is expressed in terms of unperturbed solutions, offering deep analytical insight into the physics of the problem. This clearly exposes the intrinsic aim of the perturbation theory, which relies on the assumption that the unperturbed solutions, for the problem we are dealing with, are known in closed form.

The comparison of the obtained solution with the one obtained in the theoretical framework of *perturbation of boundary conditions* (Chapter 5) reveals an intrinsic equivalence between the two different approaches, which evaluate the scattering from the same perturbed structure starting from two different kinds of description for the structure itself. In other words, the formal identity of the final solutions reflects the full consistency of the corresponding different perturbative formulations.

Indeed, a salient feature of the proposed formulation lies in its reduced mathematical complexity: In particular, the formulation here exposed can be carried out by exclusively referring to the vector electric field; conversely, the BPT formulation, based on *perturbation of boundary conditions,* requires the analysis of both magnetic and electric fields. Concerning the analytical complexity, a crucial point involves the use of polarization currents rather than equivalent surface currents (as in [23]-[24]). Although in principle both the representation in terms of surface or volume currents can be equally employed, we underline that in the analyzed problem, in which non magnetic media are concerned (i.e., whose relative permeability is unitary), the magnetic polarization currents vanish, thus we simply have to take care of the electric field distribution only. In addition, we highlight that the conducted analysis did not require to resort to the cumbersome Green functional formalism.

Therefore, we have shown that the perturbation methodology applied to rough-boundaries structure gains in generality as well in simplicity when it is considered under a different perspective of the volumetric perturbation. Therefore, the general results deduced in Chapter 5 are strengthened and, at the same time, the proposed approach offers a more complete comprehension in a conceptual perspective. We finally emphasize that when a new mathematical formulation, perhaps more general, for a physical phenomenon is conceived, the mathematical structure of the new formulation itself provides a new way of thinking about the phenomenon, especially if the results, as in our problem, are closely related to the ones obtained with a different formulation. Therefore, if applied to the case of rough-boundaries layered media, the formulation here presented is not only an (equivalent) alternative with respect to the one in Chapter 5 (see also [23]-[24]) obtained via the *perturbation of the boundary conditions* (BPT), but leads to a simpler formulation with clearer interpretation. Therefore, due to the formal full-consistency of the proposed solution with the one obtained in the theoretical framework of the boundary perturbation, we can certainly refer to the numerical examples reported in Chapter 5 (see also [23]-[25]). We are also planning to compare our results with the ones derived via numerical methods or also with the ones provided by Kirchhoff-based models; in the second case, however, the comparison can be not easy due to the different domains of validity.

On the other hand, we highlight that the mentioned validity conditions are fully consistent with the BPT theoretical result; in addition, rigorous demonstration and comparative discussion on the regime of validity are deferred to next Chapter.

Furthermore, it is worth noting that if the point source is placed in far field with respect to the illuminated volume, then the plane-wave-incidence approximation can be used, and the results of our method can be used. Otherwise, a plane wave expansion of the incident field can be performed, and the presented approach can be used for each plane wave component.

It should be noted that our method can remind other theoretical approaches [40]-[41], because in these methods the roughness interface is also seen as a permittivity fluctuation. However, for the structure mainly considered in this Chapter, the perturbation is characterized by means of zero-mean processes, so that the mean scattered (far) field is null, except that in a narrow cone around the specular direction. In addition, our formulation leads to closed form solution, whereas the approaches [40]-[41] are semi-analytical in as much as the multi-layer Green function has to be first computed numerically.

In addition, we have concerned with formulation, full vectorial treatment and solution of the scattering problem involving semi-infinite structures with fully space-variant, three-dimensional *weak* fluctuations.

Although surface and volumetric scattering were usually treated as completely disconnected phenomena, in the spirit of our approach a mutually compatible mathematical description is adopted for irregularity in the geometry of interfacial surface, which separates two different media, and the spatial fluctuation of the volumetric properties. In addition, a substantial effort is directed towards joint modeling of the corresponding two major scattering mechanisms, emphasizing the pertinent formal analogy. The perturbative analysis leads to closed form solution for bistatic configuration, including single-scattering contributions. This permits, via a unified mathematical formulation and conceptual understanding of two inherent scattering mechanisms, a detailed derivation for volume scattering contribution for a dissipative half-space, whose effect can be consequently taken into account introducing an *effective power spectral density* of the pertinent structure. In addition a practical and comprehensive scattering model, which is able to clarify relative role, common features and impact of the coexisting scattering processes is obtained. As a result, this Chapter also provides a canonical model for electromagnetic wave interactions with semi-infinite structure subject to a random perturbation: a main final purpose is to provide a unified analytic solution to be profitably used in applications, especially in the remote sensing scenario ones.

Therefore, an essential aim of this Chapter is to also furnish the conceptual and analytical treatment useful to gain a unifying perspective on the scattering phenomenon considered in its entirety. As a matter of fact, it is intellectually more satisfying to treat elementary scattering mechanisms (i.e. interfacial and volumetric scattering) consistently on the same conceptual and formal footing.

Nonetheless, the proposed solution not only turns out to be an extension to 3D semi-infinite inhomogeneous media of the classical SPM solution for rough surfaces, but it also introduces a new methodological perspective.

Furthermore, the comprehensive VPRT formulation presented in this Chapter can be used also to derive closed form solution for scattering, from and through, inhomogeneities that are possibly present in each slab of a structure with an arbitrary number of layers. Similarly, the problem of the scattering through rough boundaries multilayer can be also advantageously addressed by using the proposed approach. This opens the way to a complete analytical solution for fully space-variant dielectric multilayered structures. Indeed, the presented results provide a comprehensive basis and have important implications: This approach also furnishes a fundamental mathematical construct that can be systematically extended to layered structures. However, this case requires an suitable analysis, and therefore is deferred to subsequent publications.

Indeed, it is important to emphasize that the proposed VPRT formulation exhibits several appealing features, which are of interest for future developments. First of all, the approach that we have presented leads to a solution which is directly susceptible of a powerful interpretation in terms of the *Rumsey*'s reaction concept which allows interpreting the scattering solution in terms of *multi-reaction*. This furnishes a clear interpretation of the scattering problem and might be also useful to interpret the first-order perturbative approximation. Analogously, this approach opens the way to evaluate and interpret the higher-order terms of the perturbative development. In this regard, it is important to note that the results of this Chapter will be fundamental, since they methodologically provide a basis even for the derivation of second-order perturbative contributions, so permitting to include consistently surface-volume combined effects. This will be a matter of further publications currently in preparation. Another further investigation will be devoted to the comparison of the predictions obtained with our model with the ones provided by applying numerical methods, in order to precisely assess the pertinent limits of validity.

7.8 Appendix: Integral Analytical Evaluation

This Appendix is devoted to a full evaluation of the function $\hat{W}_1$ whose integral expression is (141). To this purpose, adopting a suitable change over to integration variables

$$\begin{cases} z_d = z' - z'' \\ z_a = (z' + z'')/2 \end{cases}, \qquad \begin{cases} z' = z_a + \dfrac{z_d}{2} \\ z'' = +z_a - \dfrac{z_d}{2} \end{cases},$$

and noting that the *Jacobian* of the pertinent transformation is unitary, we obtain

$$\hat{W}_1 = \int_{-\infty}^{0} dz_a \int_{|z_d|/2 < -z_a} dz_d \, B_1(\mathbf{k}_\perp^d, z_d) e^{2\alpha_z z_a} \; e^{-j\beta_z z_d}, \tag{A.1}$$

where

$$\beta_z = \mathrm{Re}\{k_{z1}^s + k_{z1}^i\},$$

$$\alpha_z = \mathrm{Im}\{k_{z1}^s + k_{z1}^i\}.$$

The previous integral (A.1) can be then rewritten more conveniently as

$$\hat{W}_1 = \int_{0}^{+\infty} dz_a \, e^{-2\alpha_z z_a} \int_{-\infty}^{+\infty} dz_d \; B_1(\mathbf{k}_\perp^d, z_d) \, rect\left[\frac{z_d}{2z_a}\right] e^{-j\beta_z z_d}. \tag{A.2}$$

It useful to recall the Fourier Transform pair

$$rect\left[\frac{z_d}{2z_a}\right] \;\leftrightarrow\; S(\beta_z; z_a) = \frac{\sin(2\beta_z z_a)}{\pi\beta_z}. \tag{A.3}$$

Therefore, by using (A.3) and by invoking convolution theorem, (A.2) can be rewritten as

$$\hat{W}_1 = \int_{0}^{+\infty} dz_a \, e^{-2\alpha_z z_a} \int_{-\infty}^{+\infty} d\beta_z' \; W_1(\mathbf{k}_\perp^d, \beta_z') S(\beta_z - \beta_z'; z_a), \tag{A.4}$$

where $W_1(\mathbf{\kappa}_\perp, \beta_z)$ is given by (138). To proceed further, it is convenient to observe that the integration over z_a in (A.4) can be carried out directly, exchanging the integral symbols and taking into account the following formula

$$\int_0^{+\infty} dz\, e^{-2\alpha_z z} \sin(2\beta_z z) = \frac{\beta_z}{\alpha_z^2 + \beta_z^2}. \qquad \text{(A.5)}$$

Accordingly, performing integration with respect to z_a, from (A.4) we finally obtain the formula (141).

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Chapter 8

On the Regime of Validity of Pertubative Scattering Formulations for Layered Rough Interfaces

"The engineering course influenced me very strongly:… I've learned that, in the description of nature, one has to tolerate approximations, and that even work with approximations can be interesting and can sometimes be beautiful."
P.A.M. Dirac

A theoretical analysis on the regime of validity of volumetric-perturbation-based formulations addressing electromagnetic scattering evaluation from interfacial roughness is presented: we formally establish the pertinent regime of validity, which has not been properly highlighted in previous related works. The obtained conditions, which are derived for the general case of rough multilayers, also permit to overcome the apparent theoretical incoherence between the regime of validity of volumetric-perturbative reciprocal theory (VPRT) and the one of the boundary perturbation theory (BPT). Finally, the VPRT formulation is casted within a general variational framework, enabling a wider discussion on the relevant approximation.

8.1 Introduction and Motivation

Easy physical interpretation, clear regime of validity and formal consistency with comparable analytical results are definitely matter of essential interest for theoretical constructs and analytical models.

This Chapter focus on a theoretical analysis on the regime of validity of volumetric-perturbation-based formulations addressing electromagnetic (EM) scattering evaluation from interfacial roughness: we formally establish the pertinent regime of validity, which has not been properly highlighted in previous related works [3][4][5]. The obtained conditions, which are derived for the general case of rough multilayers, also permit to overcome the apparent theoretical incoherence between the regime of validity of volumetric-perturbative reciprocal theory (VPRT) [11] and the one of the boundary perturbation theory (BPT) [11]-[15]. Finally, the VPRT formulation is casted within a proper variational framework, enabling a wider discussion on the relevant approximation.

Generally speaking, establishing the domain of validity of an analytical result is definitely of paramount interest for both practical application and theoretical investigation perspectives.

One possible approach, in order to specifically assess the conditions of validity of an (approximate) analytical method, essentially consists in a comparison between the predictions of the considered analytical solution with the “exact” results obtained by applying numerical methods. In particular, numerical methods have been widely applied to verify the validity of theoretical scattering solutions.

This point merits to be discussed more in detail.

Specifically speaking, numerical methods can provide specific answers to the considered scattering problem only for some prescribed conditions pertaining to a precise parametrical setting of the structure under analysis [6][7][8]. Indeed, by conducting several, usually time-consuming, numerical simulations is then possible to comparatively achieve, in some case (e. g. surface scattering), certain relevant conditions of validity, which turn out expressed in terms of some inequalities [2]. This inequalities cannot obviously exhibit a general validity: however, they can be of some help, inasmuch as they provide just an indication when the considered analytical solution have to be applied in specific context.

Unfortunately, when more complex structures are concerned, the same, as when the parametric dependency exhibited by the solution is relatively simple, e. g. surface scattering problem, cannot be equally achieved: obtaining such relations can turn out a neither easy nor practicable attempt. In fact, the parametric dependency exhibited by the analytical solution involving more complex problem, like scattering in rough multilayer [11]-[17], necessary leads, in order to achieve comprehensive conclusion, to a prohibitive number of possible cases to be analyzed (with associated time requirements dramatically increasing with variable space dimension) due to the intricate combination of the numerous inherent parameters involved (thickness, complex permittivities, roughness parameters, etc.). In particular, an interesting study of validity involving only two layered surfaces has been recently conducted performing numerical simulations, which employs the method of moments (MoM)[9]: the analysis was limited to one-dimensional interfaces (2-D scattering problem), fixed incidence angle and polarization, and prescribed dielectric properties: also in this case, which involves however several parameters, the numerical approach leads only to partial answers, since the verification of the relevant analytical solution for all the possible parameters combinations (multidimensional analysis) is hard to be achieved.

As a result, this kind of approach is essentially addressed to check the validity of the pertinent solution for all possible cases, in order to achieve a sort of recipe employable as a precise quantitative validity criterion. However, it is important to note that, more in general, even if the overall parameters multidimensional region was sufficiently explored, the achievable conclusion could be expressed, at the most, with an over-complicated representation, whose practical usefulness and significance remains, however, questionable. Furthermore, for the benefit of practical applications, another interesting point concerns how the range of validity can be somehow extended if relaxed error criteria are adopted, so tolerating a certain prescribed prediction error.

These preliminary considerations highlight the difficultness of the matter concerning the precise determination of the validity domain of analytical models, particularly when scattering by rough multilayers is addressed.

We now turn our attention on the pertinent theoretical scattering models, which allow us a more deep comprehension of the scattering phenomenon and pertinent direct understanding of the functional dependence of the structural scattering properties. When rough interfaces, eventually layered, are concerned, the scattering problem can be treated resorting to different (full-wave) analytical approaches.

Hitherto, two main analytical approaches are practicable: the first one relying on the *Kirchhoff* approach (and its extension), the second one is based on the perturbation theory. In the first case, the analytic derivation, which is directly based on physical concepts [1]-[2], was obtained at most for two layered rough interfaces only [10]. In the second case, closed-form solutions are obtained by rigorously applying the perturbative formulation of *Maxwell* equations to a certain class of three-dimensional (3D) structures [1]-[5], [12]-[16], whose pertinent polarimetric expression, only subsequently, turns out susceptible of a clear physical interpretation [17].

In particular, in this Chapter the focus is on analytical models referable to the perturbation theory. Within this framework, analytical solutions for scattering from and through gently rough interface [1]-[5] and rough multilayers [12]-[17] are available. Basically, two conceptually and structurally different kind of formulations, stemming from different descriptions of the inherent structural perturbation, have been adopted to analytical deal with single interface or layered interfaces scattering problem.

In a first case, the formulation relies on a suitable perturbation pertinent to the geometry of the structure, and accordingly the scattering problem is treated by adopting a proper perturbation of boundary conditions: classical *Small Perturbation Method* (SPM) [1]-[2] and *Boundary Perturbation Theory* (BPT) [14]-[16] have been developed to cope with, respectively, single surface and rough multilayers scattering. In this case, the clear validity conditions, which visibly arise from the pertinent analytical developments, regard both small (compared with the wavelength) *rms* height and small slope assumptions.

In the second case, the formulation considers a perturbation pertinent to the dielectric properties of the structure, and accordingly the scattering problem is treated by adopting suitable volumetric current distributions: various relevant volumetric-perturbative based formulations are available for a single rough interface [3]-[5],[11], and *Volumetric-Perturbative Reciprocal Theory* (VPRT) was formulated [12] for rough multilayer.

As validity condition regarding volumetric perturbation based formulations, it generally (explicitly) assumed only that the surface height variation is small compared with the wavelength of the incident wave (see for instance [3]). Perhaps, further limitations involving somehow the roughness shape (e. g. small slope assumption), which do not directly arise from the relevant developments, are not explicitly required.

Accordingly, the connection between the corresponding validity conditions of volumetric perturbation and boundary perturbation - based formulations for rough surface scattering remains obscure. Moreover, when rough multilayer scattering is concerned the same applies; however, we emphasize that the final results of this Chapter was anticipated in previous Chapter, without providing demonstration and pertinent discussion.

On the other hand, it is noteworthy that the two mentioned different kinds of perturbative formulation (volumetric perturbation and boundary perturbation -based), when applied to single rough interface (or also rough multilayer), lead to the same corresponding final solution. Although it is surprising that two completely different theoretical constructs, based on the adoption of different structural descriptions, and involving no trivial derivations (as long as they are correct), when applied to rough interface (or rough multilayer), lead to perfectly consistent final (first-order) solutions, it is reasonable to expect that this also reflects an essential consistence of the corresponding validity conditions. Nonetheless, a conceptual discrepancy arises due to the different validity conditions of the corresponding aforementioned perturbative formulations: cover this gap is of high interest.

The question now naturally arises, whether the evident inconsistency between the corresponding validity conditions of the two different perturbative formulations is essential or apparent. Due to the essentially different nature of the intrinsic theoretical constructs on which the two distinct kinds of perturbative development are based, this connection, however, cannot be directly achieved and a detailed investigation on the relevant regime of validity is required, in order to have purely intuitional considerations supported by a rigorous discussion.

In this regard, we believe that, before to proceed to onerous numerical simulations, firstly the regime of validity of the volumetric perturbation based models, for both single rough surface [3]-[5] and rough multilayer (Chapter 7, see also [12]), needs be better investigated and clarified from a formal point of view, for gaining additional insight into the conceptual coherence with respect to similar models (i. e. SPM and BPT).

Therefore, the aim of this Chapter is to overcome this conceptual discrepancy establishing and discussing proper formal validity conditions for volumetric-perturbation-based scattering models applied to both single rough interface and rough multilayer. Specifically, this investigation also enables the explanation of the apparent theoretical incoherence of the validity condition of the two different perturbative formulation (BPT and VPRT) inherent to scattering from rough multilayer.

This Chapters consists of three main parts. Sections 8.2 and 8.3 briefly present the pertinent theoretical background. In Sect. 8.4 we establish and discuss the proper regime of validity for VPRT. Finally, a the VPRT perturbative formulation is casted in a general variational framework (Section 8.5). Conclusions are drawn in Section 8.6.

8.2 Boundary Perturbation Theory

In this Section we briefly summarize the results of the Boundary Perturbation Theory *(*BPT).

8.2.1 Formulation

In order to cope with the scattering problem in layered structure with an arbitrary number of rough interfaces (see the scheme of Fig.1), closed-form (first-order) solutions have been derived in the analytic framework of BPT (Chapter 5). As schematically shown in fig.1, each layer is assumed to be homogeneous and characterized by arbitrary and deterministic parameters: the dielectric relative permittivity ε_m, the magnetic relative permeability μ_m and the thickness $\Delta_m = d_m - d_{m-1}$.

In addition, each m-th rough interface is assumed to be characterized by a zero-mean two-dimensional *stochastic process* $\zeta_m = \zeta_m(\mathbf{r}_\perp)$ with normal unit vector $\hat{n}_m$.

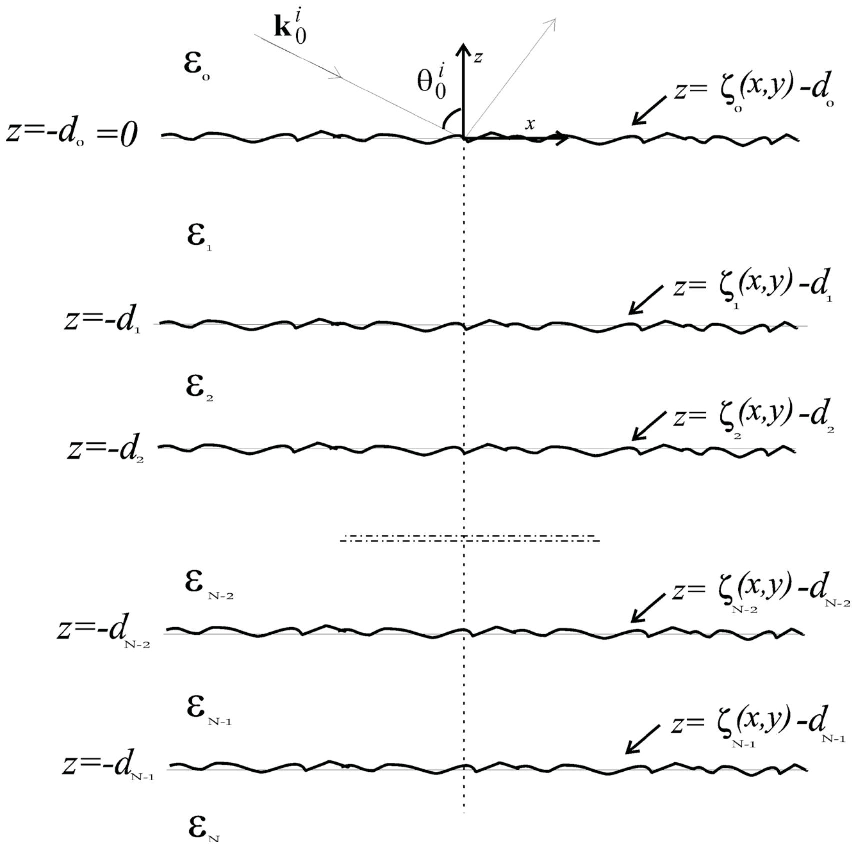

Fig. 1. Geometry of the scattering problem.

The relevant general formulation relies on a suitable perturbation pertinent to the *geometry* of the structure, and accordingly the scattering problem is treated by adopting a proper perturbation of boundary conditions. It involves a systematic perturbative expansion of the fields in the layered structure and enables the transferring of the geometry randomness into a non-uniform boundary conditions formulation (Chapter 5, [14]):

$$\hat{z} \times \Delta \mathbf{E}_m^{(1)}\Big|_{z=-d_m} = \nabla_{\perp} \zeta_m \times \Delta \mathbf{E}_m^{(0)}\Big|_{z=-d_m} - \zeta_m \, \hat{z} \times \frac{\partial \Delta \mathbf{E}_m^{(0)}}{\partial z}\Bigg|_{z=-d_m} \tag{8.1}$$

$$\hat{z} \times \Delta\mathbf{H}_m^{(1)}\Big|_{z=-d_m} = \nabla_\perp \zeta_m \times \Delta\mathbf{H}_m^{(0)}\Big|_{z=-d_m} - \zeta_m\, \hat{z} \times \frac{\partial \Delta\mathbf{H}_m^{(0)}}{\partial z}\Bigg|_{z=-d_m} \tag{8.2}$$

where $\mathbf{E}_m^{(0)}, \mathbf{H}_m^{(0)}$ and $\mathbf{E}_m^{(1)}, \mathbf{H}_m^{(1)}$ are, respectively, the unperturbed and the first-order solution in the m-th layer; $\Delta\mathbf{E}_m = \mathbf{E}_{m+1} - \mathbf{E}_m$. Subsequently, the fields' spectral expansion can be analytically evaluated by using a recursive matrix formalism approach encompassing a proper scattered field representation in each layer and a matrix reformulation of non-uniform boundary conditions, as discussed in Chapter 5.

8.2.2 Pertinent Closed Form Solution

Relevant closed-form solution have been derived for three-dimensional (3-D) layered structures with arbitrary number of rough interfaces and bistatic configuration. The field scattered into the upper half-space, form 3-D layered structure, in the first-order limit of the perturbative development can be then written in the form:

$$\begin{bmatrix} E_0^{sv} \\ E_0^{sh} \end{bmatrix} = \frac{e^{jk_0 r_0}}{r_0} \sum_{m=0}^{N-1} \boldsymbol{\mathcal{S}}^{m|m+1}(\mathbf{k}^s, \mathbf{k}^i) \begin{bmatrix} E_0^{iv} \\ E_0^{ih} \end{bmatrix}, \tag{8.3}$$

where the *generalized bistatic scattering matrix*

$$\boldsymbol{\mathcal{S}}^{m|m+1}(\mathbf{k}^s, \mathbf{k}^i) = \pi\, k_0^2 \tilde{\zeta}_m(\mathbf{k}_\perp^s - \mathbf{k}_\perp^i)\, \tilde{\boldsymbol{\alpha}}^{m,m+1}(\mathbf{k}^s, \mathbf{k}^i), \tag{8.4}$$

characterizes the polarimetric response of the generic (mth) rough interface of the layered structure, to a plane wave in the direction $\mathbf{k}^i$, in a given observation direction $\mathbf{k}^s$. $\tilde{\zeta}_m(\mathbf{k}_\perp)$ is the *spectral representation* (2D-FT) of the corrugation $\zeta_m(\mathbf{r}_\perp)$. The fully polarimetric closed-form expression for

$$\tilde{\boldsymbol{\alpha}}^{m,m+1}(\mathbf{k}^s, \mathbf{k}^i) = \begin{bmatrix} \tilde{\alpha}_{vv}^{m,m+1}(\mathbf{k}^s, \mathbf{k}^i) & \tilde{\alpha}_{vh}^{m,m+1}(\mathbf{k}^s, \mathbf{k}^i) \\ \tilde{\alpha}_{hv}^{m,m+1}(\mathbf{k}^s, \mathbf{k}^i) & \tilde{\alpha}_{hh}^{m,m+1}(\mathbf{k}^s, \mathbf{k}^i) \end{bmatrix}, \tag{8.5}$$

is provided in Chapter 5 (see also [14]-[16]). In addition, the solution is formally symmetric [15] and physically revealing [17]. Finally, we emphasize that classical SPM solution for rough surface can be rigorously regarded as a special cases of the BPT general solutions for rough interfaces of layered media [14] [16].

8.2.3 Regime of Validity

BPT validity requires that the height deviation of the rough interfaces, about the unperturbed interface, is everywhere small compared to the wavelength of the incident wave and the gradient of the surface is small in comparison to unity. Formally, this is to say:

$$\left|k_m \zeta_m\right| << 1 \ , \qquad \left|k_{m+1} \zeta_m\right| << 1 , \qquad \left|\nabla_\perp \zeta_m\right| << 1 . \quad (8.6)$$

for m=0, 1, …, N-1.

8.3 Volumetric-Perturbative Reciprocal Theory

In this Section we briefly summarize the results the volumetric perturbative reciprocal theory (VPRT) formulated in Chapter 7.

8.3.1 Formulation

The VPRT formulation considers a perturbation pertinent to the dielectric properties of the structure whose scheme is depicted in Fig.1:

$$\varepsilon(\mathbf{r}_\perp, z) = \varepsilon_0 + \sum_{m=0}^{N-1} (\varepsilon_{m+1} - \varepsilon_m)\, \mathcal{U}(-z - d_m + \zeta_m(\mathbf{r}_\perp)) \quad (8.7)$$

which can seen as an unperturbed medium with relative permittivity

$$\varepsilon^{(0)}(\mathbf{r}) = \varepsilon^{(0)}(z) = \varepsilon_0 + \sum_{m=0}^{N-1} (\varepsilon_{m+1} - \varepsilon_m) \mathcal{U}(-z - d_m) , \quad (8.8)$$

to which a perturbation $\delta\varepsilon(\mathbf{r})$ is applied, so that $\varepsilon(\mathbf{r}) = \varepsilon^{(0)}(z) + \delta\varepsilon(\mathbf{r})$. $\mathcal{U}(\cdot)$ is the *Heaviside*'s unit step function. Accordingly, the scattering problem is then treated by adopting suitable volumetric current distributions. As demonstrated in [11][12], the scattering field $\mathbf{E}^s$, at position $\mathbf{r}_0$ in upper half-space, can be written as

$$\mathbf{E}^s(\mathbf{r}_0)\cdot\hat{t} = -j\frac{k_0}{J\eta_0}\iiint_V \overline{\mathbf{E}}^{(0)}(\mathbf{r})\cdot\delta\varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r})d\mathbf{r}\cdot \tag{8.9}$$

where $\overline{\mathbf{E}}^{(0)}$ is the (unperturbed) field in the unperturbed medium radiated in the unperturbed medium by an auxiliary (fictitious) $\overline{\mathbf{J}}(\mathbf{r}) = \hat{t}J\delta(\mathbf{r}-\mathbf{r}_0)$ source located at $\mathbf{r}_0$, $\delta(\cdot)$ is the *Dirac*'s delta function; k_0 and η_0 are the propagation constant and the intrinsic impedance of vacuum, respectively. *V* is a volume enclosing all the sources: note also that the integral is effectively performed over the whole volume of the perturbation.

We highlight that the quantity $\mathbf{J}^{ind}(\mathbf{r}) = -jk_0\eta_0^{-1}\delta\varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r})$ can be interpreted as an (equivalent) polarization current density, which is induced into the perturbation volume by the total field $\mathbf{E}$. The scattering integral (10), therefore, can be interpreted as the reaction (e. g. *multi-reaction,* when a rough multilayer is concerned [12]) between the equivalent current density $\mathbf{J}^{ind}(\mathbf{r})$ and the auxiliary unperturbed field $\overline{\mathbf{E}}^{(0)}$. In the regime of small $\delta\varepsilon(\mathbf{r})$, the field $\mathbf{E}(\mathbf{r})$ in the integrand (see Eq.(10)) can be estimated by the corresponding unperturbed field $\mathbf{E}^{(0)}(\mathbf{r})$ (i.e the field radiated by the actual source in the unperturbed medium), so that the (first-order perturbation) scattered field $\mathbf{E}^{(1)}(\mathbf{r})$ turns out to be

$$\mathbf{E}^{(1)}(\mathbf{r}_0)\cdot\hat{t} = -j\frac{k_0}{J\eta_0}\iiint_V \overline{\mathbf{E}}^{(0)}(\mathbf{r})\cdot\delta\varepsilon^{(1)}(\mathbf{r})\mathbf{E}^{(0)}(\mathbf{r})d\mathbf{r}\cdot \tag{8.10}$$

with

$$\delta\varepsilon^{(1)}(\mathbf{r}) = \sum_{m=0}^{N-1}\left.\frac{\partial\delta\varepsilon(\mathbf{r})}{\partial\zeta_m}\right|_{\zeta_m=0}\zeta_m$$

Equation (10) allows us evaluating of the first-order scattered field the from knowledge of the medium perturbation $\delta\varepsilon^{(1)}$ and the two (unperturbed) field expressions, $\mathbf{E}^{(0)}(\mathbf{r})$ and $\overline{\mathbf{E}}^{(0)}$, respectively, radiated by real and fictitious sources in the unperturbed medium.

How to obtain a close form solution from (9)-(10) is discussed in Chapter 7.

8.3.1 Pertinent Closed Form Solution

It is worth noting that VPRT, when applied to the structure of Fig.1 (in far field with respect to the sources and the observation point), formally leads to the same solution (3)-(5).

This is to say that first-order scattering field expressions derived by using VPRT and BPT are essentially identical. Note also that by applying VPRT to a rough surface, the obtained solution is formally identical to the classical SPM one [11].

8.3.2 Regime of Validity

The formal validity conditions are carried out in Section 8.4.

8.4 Investigation on the VPRT Regime of Validity

In this Section we investigate on the formal validity conditions required for the volumetric perturbation based formulations, primary focusing on the more general VPRT formulation: the discussion is applied to the general case of a rough multilayer, so including the case of rough surface as a special case. The overall fields, **E** and **H**, into the structure are governed by the *Maxwell* equations

$$\begin{cases} \nabla\times\mathbf{E}(\mathbf{r}) = -j\omega\mu_v\,\mathbf{H}(\mathbf{r}) \\ \nabla\times\mathbf{H}(\mathbf{r}) = j\omega\varepsilon_v\,\varepsilon(\mathbf{r})\,\mathbf{E}(\mathbf{r}) + \mathbf{J}(\mathbf{r}) \\ \nabla\cdot(\varepsilon(\mathbf{r})\,\mathbf{E}(\mathbf{r})) = \rho(\mathbf{r})/\varepsilon_v \\ \nabla\cdot\mathbf{H}(\mathbf{r}) = 0 \end{cases},$$

where μ_v and ε_v are the permeability and the permittivity of the vacuum, respectively. Considering the structural description

$\varepsilon(\mathbf{r}) = \varepsilon^{(0)}(z) + \delta\varepsilon^{(1)}(\mathbf{r})$ and the perturbative field expansions $\mathbf{E} \cong \mathbf{E}^{(0)} + \mathbf{E}^{(1)}$, $\mathbf{H} \cong \mathbf{H}^{(0)} + \mathbf{H}^{(1)}$, we can decompose the problem in two equation systems:

$$\begin{cases} \nabla \times \mathbf{E}^{(0)}(\mathbf{r}) = -j\omega\mu_v \, \mathbf{H}^{(0)}(\mathbf{r}) \\ \nabla \times \mathbf{H}^{(0)}(\mathbf{r}) = j\omega\varepsilon_v \, \varepsilon^{(0)}(z) \, \mathbf{E}^{(0)}(\mathbf{r}) + \mathbf{J}(\mathbf{r}) \\ \nabla \cdot (\varepsilon^{(0)}(z) \, \mathbf{E}^{(0)}(\mathbf{r})) = \rho(\mathbf{r}) / \varepsilon_v \\ \nabla \cdot \mathbf{H}^{(0)}(\mathbf{r}) = 0 \end{cases},$$

$$\begin{cases} \nabla \times \mathbf{E}^{(1)}(\mathbf{r}) = -j\omega\mu_v \, \mathbf{H}^{(1)}(\mathbf{r}) \\ \nabla \times \mathbf{H}^{(1)}(\mathbf{r}) = j\omega\varepsilon_v \, \varepsilon^{(0)}(z) \mathbf{E}^{(1)}(\mathbf{r}) + \mathbf{J}^{(1)}(\mathbf{r}) \\ \nabla \cdot (\varepsilon^{(0)}(z) \mathbf{E}^{(1)}(\mathbf{r})) = \rho^{(1)}(\mathbf{r}) / \varepsilon_v \\ \nabla \cdot \mathbf{H}^{(1)}(\mathbf{r}) = 0 \end{cases}$$

where $\mathbf{J}^{(1)} = j\omega\varepsilon_v \, \delta\varepsilon(\mathbf{r})\mathbf{E}^{(0)}(\mathbf{r})$ and $\rho^{(1)} = -\varepsilon_v \nabla \cdot (\delta\varepsilon(\mathbf{r})\mathbf{E}^{(0)}(\mathbf{r}))$ are the first-order equivalent current and charge distributions, respectively. It should be noted that from the charge conservation it follows $\nabla \cdot \mathbf{J}^{(1)}(\mathbf{r}) = -j\omega\rho^{(1)}(\mathbf{r})$. First-order field $\mathbf{E}^{(1)}$ can be then regarded as produced by both these (first-order) sources $\mathbf{J}^{(1)}, \rho^{(1)}$.

8.4.1 Limitation on the shape of the perturbation

In this subsection, we clarify the origin of the required limitations involving the interfacial slope.

Since the unperturbed field is univocally determined for prescribed actual source and structural parameters of the unperturbed medium, in order to have a small scattered field, with respect to the unperturbed one, we assume sufficiently small both $\mathbf{J}^{(1)}$ and $\rho^{(1)}$. This is to say that

$$\iiint_V \nabla \cdot (\delta\varepsilon(\mathbf{r}) \, \mathbf{E}^{(0)}(\mathbf{r})) d\mathbf{r} \tag{8.11}$$

is also small.

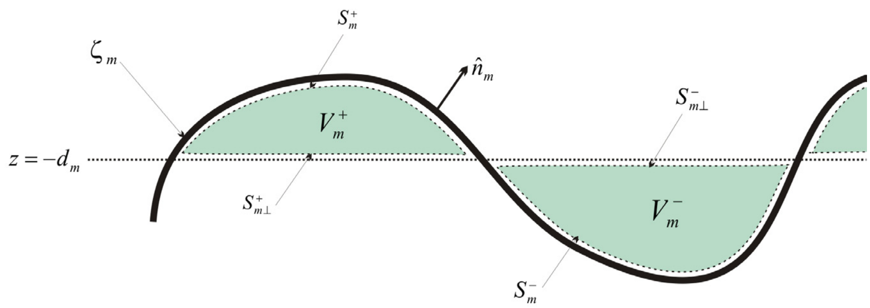

Fig. 2. Decomposition scheme for the perturbation volume relevant to the *m*th rough interface.

In order to deeply understand the necessary restrictions for the validity of the pertinent volumetric perturbative formulation, expression (11) needs to be examined accurately.

To this purpose, it is instructive to introduce a proper partitioning of the structural perturbation: we consider the following scheme. In Fig.2 we schematically represent the perturbation of the *m*-th generic (flat) interface. Here, the perturbation pertaining to the dielectric properties, denoted by $\delta\varepsilon(\mathbf{r})=\varepsilon(\mathbf{r})-\varepsilon^{(0)}(z)$ is firstly introduced without approximation. We distinguish the perturbed volumes above (V_m^+) and under (V_m^-) the *m*-th unperturbed interface; so that the pertinent entire perturbation volume is considered as the sum of two disjoint sets $V_m = V_m^+ \cup V_m^-$ (Fig. 2). Accordingly, we distinguish the positive part (ζ_m^+) and negative part (ζ_m^-) of the relevant surface ζ_m describing the generic *m*-th interfacial roughness:

$$\zeta_m^+(\mathbf{r}_\perp) = \frac{\zeta_m(\mathbf{r}_\perp)+|\zeta_m(\mathbf{r}_\perp)|}{2} = \max\{\zeta_m(\mathbf{r}_\perp),0\} \tag{8.12}$$

$$\zeta_m^-(\mathbf{r}_\perp) = \frac{\zeta_m(\mathbf{r}_\perp)-|\zeta_m(\mathbf{r}_\perp)|}{2} = \min\{\zeta_m(\mathbf{r}_\perp),0\} \tag{8.13}$$

As depicted in Fig.2, the partition of the support of the perturbation $\delta\varepsilon(\mathbf{r})$ is constituted by $V_m^+ = \{(\mathbf{r}_\perp z): 0 \le z \le \zeta_m^+(\mathbf{r}_\perp)\}$ and $V_m^- = \{(\mathbf{r}_\perp z): \zeta_m^-(\mathbf{r}_\perp) \le z \le 0\}$.

Accordingly, we get

$$\iiint_V \nabla\cdot(\delta\varepsilon(\mathbf{r})\,\mathbf{E}^{(0)}(\mathbf{r}))d\mathbf{r} = \sum_{m=0}^{N}\left[\iiint_{V_m^+}\nabla\cdot(\delta\varepsilon(\mathbf{r})\,\mathbf{E}_m^{(0)}(\mathbf{r}))d\mathbf{r} + \iiint_{V_m^-}\nabla\cdot(\delta\varepsilon(\mathbf{r})\,\mathbf{E}_{m+1}^{(0)}(\mathbf{r}))d\mathbf{r}\right] \tag{8.14}$$

and applying divergence theorem, we obtain

$$\iiint_{V_m^+}\nabla\cdot(\delta\varepsilon(\mathbf{r})\,\mathbf{E}_m^{(0)}(\mathbf{r}))d\mathbf{r} = \delta\varepsilon_m^+\iint_{S_m^+}\mathbf{E}_m^{(0)}(\mathbf{r})\cdot\hat{n}_m ds + \delta\varepsilon_m^+\iint_{S_{m\perp}^+}\mathbf{E}_m^{(0)}(\mathbf{r})\cdot\hat{n}_m ds \tag{8.15}$$

where the closed surfaces delimiting V_m^+ and V_m^- have been partitioned as, respectively, $S_m^+ \cup S_{m\perp}^+$ and $S_m^- \cup S_{m\perp}^-$. As depicted in Fig. 2, we have $S_{m\perp}^+ = \{\mathbf{r}_\perp : \zeta_m^+(\mathbf{r}_\perp) > 0\}$ and $S_{m\perp}^- = \{\mathbf{r}_\perp : \zeta_m^-(\mathbf{r}_\perp) < 0\}$. In addition, in (15) we have employed the fact that inside the perturbation volume V_m^+ we have $\delta\varepsilon(\mathbf{r}) = \delta\varepsilon_m^+ = \varepsilon_{m+1} - \varepsilon_m$ and, similarly, we have $\delta\varepsilon(\mathbf{r}) = \delta\varepsilon_m^- = \varepsilon_m - \varepsilon_{m+1}$ inside the perturbation volume V_m^-. An expression similar to (15) can be also obtained with regard to volume V_m^-. Recalling that the normal unit vector $\hat{n}_m$ to the *m*-th interface ζ_m is given by

$$\hat{n}_m = \frac{\hat{z} - \nabla_\perp\zeta_m}{\sqrt{1 + |\nabla_\perp\zeta_m|^2}} \tag{8.16}$$

and taking into account that $\hat{n}_m = -\hat{z}$ on $S_{m\perp}^+$, we get

$$\iiint_{V_m^+}\nabla\cdot(\delta\varepsilon(\mathbf{r})\,\mathbf{E}_m^{(0)}(\mathbf{r}))d\mathbf{r} = \delta\varepsilon_m^+\iint_{S_{m\perp}^+}\left[\mathbf{E}_m^{(0)}(\mathbf{r}_\perp, -d_m + \zeta_m^+) - \mathbf{E}_m^{(0)}(\mathbf{r}_\perp, -d_m^+)\right]\cdot\hat{z}\,d\mathbf{r}_\perp - \delta\varepsilon_m^+\iint_{S_{m\perp}^+}\nabla_\perp\zeta_m^+(\mathbf{r})\cdot\mathbf{E}_m^{(0)}(\mathbf{r}_\perp, -d_m + \zeta_m^+)d\mathbf{r}_\perp \tag{8.17}$$

It should be noted that the element of surface area on $S_m^{\pm}$ is $ds = \sqrt{1+|\nabla_{\perp}\zeta_m|^2}\,d\mathbf{r}_{\perp}$. Similarly, we can write

$$\iiint_{V_m^-} \nabla\cdot(\delta\varepsilon(\mathbf{r})\,\mathbf{E}_{m+1}^{(0)}(\mathbf{r}))d\mathbf{r} = -\delta\varepsilon_m^- \iint_{S_{m\perp}^-} \left[\mathbf{E}_{m+1}^{(0)}(\mathbf{r}_{\perp},-d_m+\zeta_m^-) - \mathbf{E}_{m+1}^{(0)}(\mathbf{r}_{\perp},-d_m^-)\right]\cdot\hat{z}\,d\mathbf{r}_{\perp}$$
$$+\delta\varepsilon_m^- \iint_{S_{m\perp}^-} \nabla_{\perp}\zeta_m^-(\mathbf{r})\cdot\mathbf{E}_{m+1}^{(0)}(\mathbf{r}_{\perp},-d_m+\zeta_m^-)d\mathbf{r}_{\perp} \tag{8.18}$$

By inspection of (17) and (18), it is clear that two different constraints on the shape of the perturbation are required, in order to satisfy the condition on (11), one related to the first term of the RHS of (17) and (18), and the other related to the second term. With regard to the first term, if each *m*-th interface ζ_m satisfies the condition

$$|k_m\zeta_m| << 1\ , \qquad |k_{m+1}\zeta_m| << 1\,, \tag{8.19}$$

i.e, if we assume small rough interface heights, the variability along the *z*-direction of the unperturbed field $\mathbf{E}^{(0)}(\mathbf{r})$ inside the perturbation is negligible, so that $\mathbf{E}_m^{(0)}(\mathbf{r}_{\perp},-d_m+\zeta_m^+) \cong \mathbf{E}_m^{(0)}(\mathbf{r}_{\perp},-d_m^+))$ and (17) can be rewritten as

$$\iiint_{V_m^+} \nabla\cdot(\delta\varepsilon(\mathbf{r})\,\mathbf{E}_m^{(0)}(\mathbf{r}))d\mathbf{r} \cong -\delta\varepsilon_m^+ \iint_{S_{m\perp}^+} \nabla_{\perp}\zeta_m^+(\mathbf{r})\cdot\mathbf{E}_m^{(0)}(\mathbf{r}_{\perp},-d_m^+)d\mathbf{r}_{\perp} \tag{8.20}$$

Similarly, by exploiting the small height assumption we can assume $\mathbf{E}_{m+1}^{(0)}(\mathbf{r}_{\perp},-d_m+\zeta_m^-) \cong \mathbf{E}_{m+1}^{(0)}(\mathbf{r}_{\perp},-d_m^-))$, so that (18) can be rewritten as

$$\iiint_{V_m^-} \nabla\cdot(\delta\varepsilon(\mathbf{r})\,\mathbf{E}_{m+1}^{(0)}(\mathbf{r}))d\mathbf{r} \cong \delta\varepsilon_m^- \iint_{S_{m\perp}^-} \nabla_{\perp}\zeta_m^-(\mathbf{r})\cdot\mathbf{E}_{m+1}^{(0)}(\mathbf{r}_{\perp},-d_m^-)d\mathbf{r}_{\perp} \tag{8.21}$$

The second constraint is now in order. Combining both (20) and (21), we obtain

$$\iiint_{V_m} \nabla\cdot(\delta\varepsilon(\mathbf{r})\,\mathbf{E}^{(0)}(\mathbf{r}))d\mathbf{r} \cong -\iint\left[\delta\varepsilon_m^{+}\nabla_{\perp}\zeta_m^{+}(\mathbf{r}) - \delta\varepsilon_m^{-}\nabla_{\perp}\zeta_m^{-}(\mathbf{r})\right]\cdot\mathbf{E}_{\perp}^{(0)}(\mathbf{r}_{\perp},-d_m)d\mathbf{r}_{\perp}$$
$$= -(\varepsilon_{m+1}-\varepsilon_m)\iint\nabla_{\perp}\zeta_m(\mathbf{r})\cdot\mathbf{E}_{\perp}^{(0)}(\mathbf{r}_{\perp},-d_m)d\mathbf{r}_{\perp} \qquad (8.22)$$

Therefore, for the Cauchy-Schwarz inequality, we get

$$\left|\iint\nabla_{\perp}\zeta_m(\mathbf{r})\cdot\mathbf{E}_{\perp}^{(0)}(\mathbf{r}_{\perp},-d_m)d\mathbf{r}_{\perp}\right| \le \left\|\nabla_{\perp}\zeta_m\right\|\left\|\mathbf{E}_{\perp}^{(0)}(\mathbf{r}_{\perp},-d_m)\right\|. \qquad (8.23)$$

Finally, in order to have the RHS of (23) negligible, we need $\left\|\nabla_{\perp}\zeta_m(\mathbf{r})\right\| << 1$. Accordingly, we conclude that considering $\nabla\cdot(\delta\varepsilon\,\mathbf{E}^{(0)})$ sufficiently small requires, in addition to (19), that the following condition

$$\left|\nabla_{\perp}\zeta_m\right| << 1. \qquad (8.24)$$

for m=0, 1, …, N-1, must be also fulfilled.

Therefore, although at first glance one might erroneously claim that condition (19) suffices for the validity of the volumetric-perturbative development (see also [3]-[5]), a more detailed investigation finds that, in addition to the condition (19), the further condition (24), which involves the slopes of the interfaces, is required, so restricting the class of rough interfaces to which the method applies.

As a result, the pertinent validity conditions we have pointed out are given by (19) and (24) for each (m-th) rough interface. Finally, we underline that the case of a single rough interface can be regarded as a special case and accordingly is included in our discussion.

8.4.2 The Approximation of the Internal Electric Field

In this subsection, we investigate the role of the above assumptions, emphasizing the involved internal field approximation.

Once the conditions (19) and (24) have been established, we can proceed further by using reasoning quite similar to that used in preceding subsection, again distinguishing the contributions pertinent to volumes above (V_m^+) and under (V_m^-) the inherent discontinuity plane of the unperturbed structure (see Fig.2). Thus, we decompose the integral in (9) in the form

$$\iiint_V \overline{\mathbf{E}}^{(0)}(\mathbf{r})\cdot\delta\varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r})d\mathbf{r} = \sum_{m=0}^{N}\delta\varepsilon_m^+\iiint_{V_m^+}\overline{\mathbf{E}}^{(0)}(\mathbf{r})\cdot\mathbf{E}(\mathbf{r})d\mathbf{r} + \sum_{m=0}^{N}\delta\varepsilon_m^-\iiint_{V_m^-}\overline{\mathbf{E}}^{(0)}(\mathbf{r})\cdot\mathbf{E}(\mathbf{r})d\mathbf{r}. \tag{8.25}$$

Therefore, (25) can be rewritten as

$$\begin{aligned}&\iiint_V \overline{\mathbf{E}}^{(0)}(\mathbf{r})\cdot\delta\varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r})d\mathbf{r} = \\ &\sum_{m=0}^{N}\iint d\mathbf{r}_\perp\left[\delta\varepsilon_m^+\int_0^{\zeta_m^+}dz\,\overline{\mathbf{E}}^{(0)}(\mathbf{r})\cdot\mathbf{E}\ (\mathbf{r}) + \delta\varepsilon_m^-\int_{\zeta_m^-}^{0}dz\overline{\mathbf{E}}^{(0)}(\mathbf{r})\cdot\mathbf{E}\ (\mathbf{r})\right]\end{aligned} \tag{8.26}$$

We stress that so far the considered scattering field expression (26) is rigorous. Hereinafter, proper approximation can be taken into account.

Since the total electric field $\mathbf{E}(\mathbf{r})$ inside the perturbation also includes the (unknown) scattered field, generally an estimation of this field is required in the perturbation developments. In the framework of small perturbation, it is generally assumed that the scattering field is small with respect to the unperturbed field, so that the field enters the perturbation without significant distortion. The classical (first-order) Born approximation estimates the overall field inside the perturbation with the incident field only. In the following, however, a slightly different approximation is taken into account.

In fact, it is intuitive that, instead of assuming $\mathbf{E}(\mathbf{r}_\perp, z) \cong \mathbf{E}^{(0)}(\mathbf{r}_\perp, z)$, as a direct application of the Born approximation would require, a better estimation of the actual field $\mathbf{E}(\mathbf{r})$ inside the perturbation volume V_m^+ can be obtained by considering the unperturbed field $\mathbf{E}^{(0)}$ just beneath the unperturbed interface: $\mathbf{E}(\mathbf{r}_\perp, z) \cong \mathbf{E}_{m+1}^{(0)}(\mathbf{r}_\perp, -d_m^-)$, This is sometimes referred to as “internal field approximation” [18]. Hence, we can write

$$\delta\varepsilon_m^+ \iint d\mathbf{r}_\perp \int_0^{\zeta_m^+} dz\, \overline{\mathbf{E}}^{(0)}(\mathbf{r}_\perp, z)\cdot \mathbf{E}\ (\mathbf{r}_\perp, z)$$
$$\cong \delta\varepsilon_m^+ \iint d\mathbf{r}_\perp \int_0^{\zeta_m^+} dz\, \overline{\mathbf{E}}_m^{(0)}(\mathbf{r}_\perp, -d_m^+)\cdot \mathbf{E}_{m+1}^{(0)}(\mathbf{r}_\perp, -d_m^-). \tag{8.27}$$

Note also that, in compliance with (19), the approximation $\overline{\mathbf{E}}^{(0)}(\mathbf{r}) \cong \overline{\mathbf{E}}_m^{(0)}(\mathbf{r}_\perp, -d_m^+)$ is also assumed inside the perturbation volume V_m^+. Likewise, a better estimation of the actual field $\mathbf{E}(\mathbf{r})$ inside the perturbation volume V_m^- can be obtained by considering the unperturbed field $\mathbf{E}^{(0)}$ just above the unperturbed interface: $\mathbf{E}(\mathbf{r}_\perp, z) \cong \mathbf{E}_m^{(0)}(\mathbf{r}_\perp, -d_m^+)$. Hence, we can write

$$\delta\varepsilon_m^- \iint d\mathbf{r}_\perp \int_{\zeta_m^-}^{0} dz\, \overline{\mathbf{E}}^{(0)}(\mathbf{r}_\perp, z)\cdot \mathbf{E}\ (\mathbf{r}_\perp, z)$$
$$\cong \delta\varepsilon_m^- \iint d\mathbf{r}_\perp \int_{\zeta_m^-}^{0} dz\, \overline{\mathbf{E}}_{m+1}^{(0)}(\mathbf{r}_\perp, -d_m^-)\cdot \mathbf{E}_m^{(0)}(\mathbf{r}_\perp, -d_m^+). \tag{8.28}$$

Note also that, similarly, inside the perturbation volume V_m^- the approximation $\overline{\mathbf{E}}^{(0)}(\mathbf{r}) \cong \overline{\mathbf{E}}_{m+1}^{(0)}(\mathbf{r}_\perp, -d_m^-)$ is assumed, in compliance with (19). As a result, substituting (27) and (28) into equation (26), and taking into account that $\zeta_m^+ + \zeta_m^- = \zeta_m$ and $\mathbf{E}_{m+1}^{(0)}(\mathbf{r}_\perp, -d_m^-) = [\hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}(\varepsilon_m/\varepsilon_{m+1} - 1)]\mathbf{E}_m^{(0)}(\mathbf{r}_\perp, -d_m^+)$, we get

$$\mathbf{E}^{(1)}(\mathbf{r}_0)\cdot\hat{t} = -j\frac{k_0}{J\eta_0}\iiint_V \overline{\mathbf{E}}^{(0)}(\mathbf{r})\cdot\delta\varepsilon(\mathbf{r})\mathbf{E}^{(0)}(\mathbf{r})d\mathbf{r} =$$
$$\sum_{m=0}^{N}(\varepsilon_{m+1} - \varepsilon_m)\iint d\mathbf{r}_\perp \zeta_m(\mathbf{r}_\perp)\overline{\mathbf{E}}_m^{(0)}(\mathbf{r}_\perp, -d_m^+)\cdot \mathbf{E}_{m+1}^{(0)}(\mathbf{r}_\perp, -d_m^-) \tag{8.29}$$

which is essentially the same conclusion we directly obtained in Chapter 7 via a proper perturbative expression for $\delta\varepsilon(\mathbf{r})$.

Finally, we have also demonstrated that the development in Chapter 7, which does not explicitly involve the internal field approximation, is also susceptible of an interesting interpretation in terms of internal field, which is consistent with the small height assumption.

8.4.3 Discussion

The conclusion we obtain in previous subsections is that the assumptions defining the regime of validity of volumetric perturbation based methods, and specifically VPRT, is defined by both the inequalities (19) and (24) for m=0, 1, …, N-1. Indeed, the scattering can be seen as caused by the effective induced polarization current, which is proportional to the actual field inside the perturbation. The small *rms* height assumption assures that the field enters the structure without significant distortion. Accordingly, in the first order approximation, this actual field can in turn be approximated, inside the perturbation, with the relevant (internal) unperturbed field. In this case, the role of the phase of the unperturbed wave as it propagates inside the perturbation volume is crucial.

Accordingly, in long wave limit, the controlling factor for the validity of the perturbative solution is not the dielectric contrast pertinent to the generic (mth) interface, $|\varepsilon_{m+1}-\varepsilon_m|$: indeed, the smallness of the perturbation $\delta\varepsilon(\mathbf{r})$ does not necessary require a limitation on the dielectric contrast $|\delta\varepsilon_m^+| = |\varepsilon_{m+1}-\varepsilon_m| < 1$. On the contrary, the validity condition (19) can be met even when $|\varepsilon_{m+1}-\varepsilon_m| > 1$, as usually is. Therefore, the dielectric contrast pertinent to the perturbation $\delta\varepsilon(\mathbf{r})$ can assume the prescribed value correspondent to the considered unperturbed layered structure ($\varepsilon^{(0)}(z)$) over limited spatial regions comprising the actual interfacial roughness, whose vertical extension is perhaps assumed small with respect to the wavelength of the incoming radiation.

Furthermore, we have demonstrated that the general condition (19) is insufficient to guarantee the validity of the whichever volumetric (small) perturbation based analytic development addressed to rough interfaces.

In this regard, condition (19) must be complemented with (24). This implication, regarding the volumetric perturbation based approach addressed to rough interfaces, is pointed out here for the first time (at the best of our knowledge) and plays an important role for both theoretical relevance and practical applicability perspectives.

In addition, this result, which has been established from a mathematical point instead of resorting to purely intuitional considerations, bridges the conceptual gap between the correspondent regime of validity of boundary-perturbation (e.g. classical SPM) and volumetric perturbation approaches [3]-[5],[11] applied to a rough interface.

The discussion has been conducted referring to the general case of scattering from rough multilayers. As a result, we also provide insight into the formal connection between the corresponding regimes of validity of the BPT and VPRT formulations: accordingly, these two theoretical construct addressed to rough multilayers scattering can be now regarded in a conceptually coherent frame.

8.5 Variational Formulation

In this Section, we now carefully examine the perturbation problem in the framework of variational formalism. We first derive a general variational solution for the scattering problem, then we show how the first-order VPRT solution can be arranged in the general variational framework, so enabling a wider discussion on the pertinent approximation involved.

This also demonstrates how VPRT can be reformulated as an equivalent variational principle in the square-integrable vector functions space.

The primary reason of a variational formulation of the scattering problem is that it permits an interesting physical interpretation, providing another picture for the description of the considered phenomena. In addition, this variational formulation has mathematical significance, since it enables the evaluation of the formal upper-bound to the involved prediction error.

As it has been widely demonstrated, the *Rumsey*'s reaction concept [18]-[20], and its generalized form (*multi-reaction*) (see [12]), can be effectively employed for the description of the scattering phenomena from generally layered structures. This naturally suggest us to construct the variational formulation on this concept.

Let us consider the formal linear operators describing the problem associated, respectively, with the perturbed and unperturbed medium

$$\boldsymbol{M} = (jk_0\eta_0)^{-1}[\nabla\times\nabla\times -k_0^2\varepsilon(\mathbf{r})], \tag{8.30}$$

$$\boldsymbol{M}^{(0)} = (jk_0\eta_0)^{-1}[\nabla\times\nabla\times -k_0^2\varepsilon^0(z)], \tag{8.31}$$

so that we have

$$\boldsymbol{M}\mathbf{E} = \mathbf{J}, \tag{8.32}$$

where $\mathbf{E}$ is the (unknown) vectorial function and $\mathbf{J}$ is the given (known) source function. We also introduce the associated *auxiliary* problem, for supplementing the original problem described by (33):

$$\overline{\boldsymbol{M}}\,\overline{\mathbf{E}} = \overline{\mathbf{J}}, \tag{8.33}$$

where $\overline{\mathbf{E}}$ is another (unknown) vectorial function and $\overline{\mathbf{J}}$ is another (known) suitable source function. Note that, concerning the notation adopted in this Chapter, an over-bar is used for the symbols which refer to the auxiliary problem. For the sake of clarity, we emphasize that the auxiliary source (fictitious) is located at the observation point of the bistatic scattering configuration.

In the following we exclusively refer to the real-type inner product (bilinear form):

$$<\mathbf{A},\mathbf{B}>=\iiint_V \mathbf{A}(\mathbf{r})\cdot\mathbf{B}(\mathbf{r})d\mathbf{r}, \tag{8.34}$$

which is physically related to the *reaction* between two field quantities [18]-[20],[24]. By using this formalism, the reciprocity theorem can be expressed in the form

$$<\overline{\mathbf{E}},\mathbf{J}>=<\overline{\mathbf{E}},\boldsymbol{M}\,\mathbf{E}>=<\overline{\boldsymbol{M}}\,\overline{\mathbf{E}},\mathbf{E}>=<\overline{\mathbf{J}},\mathbf{E}>, \tag{8.35}$$

which states that the reaction of the field of the auxiliary system (34) on the actual source $\mathbf{J}$ of the original system (33) is equal to that of the field of the original system (33) on the auxiliary source $\overline{\mathbf{J}}$. The statement provided by (35) needs further comments, inasmuch as the involved operator $\boldsymbol{M}$ is considered symmetric. Indeed, it is reasonable to assume that the equivalent source $\mathbf{J}^{ind}$, ascribable to the polarization induced into the perturbation, has a finite support: we can make the volume V (including the structure and the sources) infinitely large, so that it results limited by the closed surface S to infinity. Then using the fact that the fields satisfy the radiation condition and degenerate to (locally) plane waves, the relevant surface integral on S vanish. This implies that the linear operator $\boldsymbol{M}$ defined by (33) is a symmetric operator when the perturbation has a finite support [21].

We now consider the variation of the linear operator $\boldsymbol{M}$,

$$\delta\boldsymbol{M} = \boldsymbol{M} - \boldsymbol{M}^{(0)} = -(jk_0\eta_0)^{-1}k_0^2\delta\varepsilon(\mathbf{r}) = j\omega\varepsilon_0\delta\varepsilon(\mathbf{r}), \quad (8.36)$$

and the trial field $\mathbf{E} = \mathbf{E}^{(0)} + \delta\mathbf{E}$. Note that here $\mathbf{E}^{(0)}$ denotes the relevant unperturbed solution referable to whichever point source $\mathbf{J}$, which is not necessarily placed in far-field with respect to the illuminated structure. We emphasize that the actual and the unperturbed fields can be regarded as produced by the actual source $\mathbf{J}$, respectively, in the actual and unperturbed medium:

$$\boldsymbol{M}\,\mathbf{E} = \boldsymbol{M}^{(0)}\mathbf{E}^{(0)} = \mathbf{J}\,. \quad (8.37)$$

Likewise for the auxiliary source $\overline{\mathbf{J}}$, we have

$$\overline{\boldsymbol{M}}\,\overline{\mathbf{E}} = \overline{\boldsymbol{M}}^{(0)}\overline{\mathbf{E}}^{(0)} = \overline{\mathbf{J}}\,, \quad (8.38)$$

Let us consider the variation of the reaction $R = <\overline{\mathbf{J}},\mathbf{E}>$,

$$\delta R = <\overline{\mathbf{J}},\mathbf{E}> - <\overline{\mathbf{J}},\mathbf{E}^{(0)}> = <\overline{\boldsymbol{M}}^{(0)}\overline{\mathbf{E}}^{(0)},\delta\mathbf{E}> \quad (8.39)$$

By using the symmetry of the operator

$$<\overline{\boldsymbol{M}}^{(0)}\overline{\mathbf{E}}^{(0)},\delta\mathbf{E}> = <\overline{\mathbf{E}}^{(0)},\boldsymbol{M}^{(0)}\delta\mathbf{E}> \quad (8.40)$$

and taking into account that expanding the difference

$$\delta\boldsymbol{M}\,\mathbf{E} = \boldsymbol{M}\,\mathbf{E} - \boldsymbol{M}^{(0)}\mathbf{E} = \boldsymbol{M}^{(0)}\,\mathbf{E}^{(0)} - \boldsymbol{M}^{(0)}\mathbf{E} = -\boldsymbol{M}^{(0)}\delta\mathbf{E}\,, \quad (8.41)$$

Eq. (39) can be also written as

$$\delta R = -<\overline{\mathbf{E}}^{(0)}, \delta\mathcal{M}\,\mathbf{E}> = -<\overline{\mathbf{E}}^{(0)}, \delta\mathcal{M}\,\mathbf{E}^{(0)}> - <\overline{\mathbf{E}}^{(0)}, \delta\mathcal{M}\,\delta\mathbf{E}> \quad (8.42)$$

Taking into account the symmetric nature of the operator $\delta\mathcal{M}$, we finally obtain:

$$\delta R = -<\overline{\mathbf{E}}^{(0)}, \delta\mathcal{M}\,\mathbf{E}^{(0)}> - <\delta\mathcal{M}\,\overline{\mathbf{E}}^{(0)}, \delta\mathbf{E}> \quad (8.43)$$

It is important to note that the first term and second term in (43) involve, respectively, first-order and second-order effect of $\delta\mathcal{M}$.

8.5.1 Analysis of the first term

Now we analyze the first term of (43),

$$\delta R' = -<\overline{\mathbf{E}}^{(0)}, \delta\mathcal{M}\,\mathbf{E}^{(0)}> = -j\omega\varepsilon_0 <\overline{\mathbf{E}}^{(0)}, \delta\varepsilon(\mathbf{r})\mathbf{E}^{(0)}> \quad (8.44)$$

Physically speaking, Eq. (44) can be interpreted as the reaction between the unperturbed field $\overline{\mathbf{E}}^{(0)}$ and the induced current inside the perturbation volume by the unperturbed field $\mathbf{E}^{(0)}$. It is straightforward to show that

$$\left|<\overline{\mathbf{E}}^{(0)}, \delta\mathcal{M}\,\mathbf{E}^{(0)}>\right| \le \left\|\overline{\mathbf{E}}^{(0)}\right\| \left\|\delta\mathcal{M}\,\mathbf{E}^{(0)}\right\| \le \left\|\overline{\mathbf{E}}^{(0)}\right\| \left\|\delta\mathcal{M}\right\| \left\|\mathbf{E}^{(0)}\right\| \quad (8.45)$$

where $\|\mathbf{A}\| = \sqrt{<\mathbf{A}^*, \mathbf{A}>}$ is the norm of $\mathbf{A}$ in the corresponding Hilbert space, with the asterisk denoting complex conjugation. If we assume that

$$\|\delta\mathcal{M}\| = \sup_{\mathbf{A}\neq 0} \frac{\|\delta\mathcal{M}\,\mathbf{A}\|}{\|\mathbf{A}\|} \le \psi \quad (8.46)$$

with ψ <<1, then an upper bound for the reaction (45) is:

$$\left|<\overline{\mathbf{E}}^{(0)}, \delta\mathcal{M}\,\mathbf{E}^{(0)}>\right| \le \psi \left\|\overline{\mathbf{E}}^{(0)}\right\| \left\|\mathbf{E}^{(0)}\right\| \quad (8.47)$$

8.5.2 Connection with the first-order VPRT solution

In this Subsection we investigate the connection with the general formulation (44) with the first-order VPRT solution for scattering from rough multilayer (10): we demonstrate that the latter it is included in (44).

It is then instructive to evaluate the *functional derivative* of $\delta R'$ with respect to the *m*-th interfacial roughness. Assuming ζ_m as small parameters:

$$\delta R' = \sum_{m=0}^{N-1} \frac{\partial \delta R'}{\partial \zeta_m} \zeta_m = -< \overline{\mathbf{E}}^{(0)}, \sum_{m=0}^{N-1} \frac{\partial \delta \boldsymbol{M}}{\partial \zeta_m} \zeta_m \; \mathbf{E}^{(0)} > \tag{8.48}$$

In order to further manipulate the expression (48) it is useful to compute the derivatives of $\delta\varepsilon(\mathbf{r})$:

$$\delta\varepsilon(\mathbf{r}) = \sum_{m=0}^{N-1} \delta\varepsilon_m(\mathbf{r}), \tag{8.49}$$

$$\delta\varepsilon_m(\mathbf{r}) = (\varepsilon_{m+1} - \varepsilon_m)[\mathcal{U}(-z - d_m + \zeta_m) - \mathcal{U}(-z - d_m)], \tag{8.50}$$

$$\frac{\partial \delta\varepsilon(\mathbf{r})}{\partial z} = \sum_{m=0}^{N-1} (\varepsilon_{m+1} - \varepsilon_m)[-\delta(-z - d_m + \zeta_m) + \delta(-z - d_m)], \tag{8.51}$$

$$\frac{\partial \delta\varepsilon(\mathbf{r})}{\partial \zeta_m} = (\varepsilon_{m+1} - \varepsilon_m)\delta(-z - d_m + \zeta_m). \tag{8.52}$$

By comparing (51) and (52) it is easy to recognize that

$$\frac{\partial \delta\varepsilon_m(\mathbf{r})}{\partial z} = -\frac{\partial \delta\varepsilon(\mathbf{r})}{\partial \zeta_m} + \left.\frac{\partial \delta\varepsilon(\mathbf{r})}{\partial \zeta_m}\right|_{\zeta_m = 0}, \tag{8.53}$$

$$\sum_{m=0}^{N-1} \frac{\partial \delta\varepsilon_m(\mathbf{r})}{\partial z} \zeta_m = -\sum_{m=0}^{N-1} \frac{\partial \delta\varepsilon(\mathbf{r})}{\partial \zeta_m} \zeta_m + \sum_{m=0}^{N-1} \left.\frac{\partial \delta\varepsilon(\mathbf{r})}{\partial \zeta_m}\right|_{\zeta_m = 0} \zeta_m. \tag{8.54}$$

Note also that

$$\sum_{m=0}^{N-1} \left. \frac{\partial \delta\varepsilon(\mathbf{r})}{\partial \zeta_m} \right|_{\zeta_m=0} = \sum_{m=0}^{N-1} (\varepsilon_{m+1} - \varepsilon_m)\delta(-z - d_m) = -\frac{d\varepsilon^{(0)}(z)}{dz}. \qquad (8.55)$$

Therefore, by using (54), Eq. (48) assumes the form

$$\begin{aligned} \delta R' = \sum_{m=0}^{N-1} \frac{\partial \delta R'}{\partial \zeta_m} \zeta_m &= -j\omega\varepsilon_0 < \overline{\mathbf{E}}^{(0)}, \sum_{m=0}^{N-1} \left. \frac{\partial \delta\varepsilon(\mathbf{r})}{\partial \zeta_m} \right|_{\zeta_m=0} \zeta_m \, \mathbf{E}^{(0)} > \\ &+ j\omega\varepsilon_0 < \overline{\mathbf{E}}^{(0)}, \sum_{m=0}^{N-1} \frac{\partial \delta\varepsilon_m(\mathbf{r})}{\partial z} \zeta_m \, \mathbf{E}^{(0)} > \end{aligned} \qquad (8.56)$$

that can be also rewritten as

$$\begin{aligned} \delta R' = \frac{\partial \delta R'}{\partial \zeta_m} \zeta_m &= -j\omega\varepsilon_0 < \overline{\mathbf{E}}^{(0)}, \sum_{m=0}^{N-1} \left. \frac{\partial \delta\varepsilon(\mathbf{r})}{\partial \zeta_m} \right|_{\zeta_m=0} \zeta_m \, \mathbf{E}^{(0)} > \\ &- j\omega\varepsilon_0 < \overline{\mathbf{E}}^{(0)}, \sum_{m=0}^{N-1} \delta\varepsilon_m(\mathbf{r})\zeta_m \frac{\partial}{\partial z} \mathbf{E}^{(0)} > \\ &+ j\omega\varepsilon_0 < \overline{\mathbf{E}}^{(0)}, \frac{\partial}{\partial z} \left(\sum_{m=0}^{N-1} \delta\varepsilon_m(\mathbf{r})\zeta_m \mathbf{E}^{(0)} \right) >, \end{aligned} \qquad (8.57)$$

$$\begin{aligned} \delta R' = \sum_{m=0}^{N-1} \frac{\partial \delta R'}{\partial \zeta_m} \zeta_m &= -j\omega\varepsilon_0 < \overline{\mathbf{E}}^{(0)}, \sum_{m=0}^{N-1} \left. \frac{\partial \delta\varepsilon(\mathbf{r})}{\partial \zeta_m} \right|_{\zeta_m=0} \zeta_m \, \mathbf{E}^{(0)} > \\ &- j\omega\varepsilon_0 \sum_{m=0}^{N-1} < \overline{\mathbf{E}}^{(0)}, \delta\varepsilon_m(\mathbf{r}) \zeta_m \frac{\partial}{\partial z} \mathbf{E}^{(0)} > \\ &- j\omega\varepsilon_0 \sum_{m=0}^{N-1} < \zeta_m \frac{\partial}{\partial z} \overline{\mathbf{E}}^{(0)}, \delta\varepsilon_m(\mathbf{r}) \mathbf{E}^{(0)} > . \end{aligned} \qquad (8.58)$$

We promptly recognize that the first term in (58) formally coincides with the first-order VPRT solution (10). The second and third terms on RHS of (59) are clearly related to the variability along the z-direction of the unperturbed fields $\overline{\mathbf{E}}^{(0)}$ and $\mathbf{E}^{(0)}$ inside the perturbation volume. It is then clear that if, in compliance with (19), this variability is negligible, then (58) reduces to the first term only, i.e. to the VPRT first-order solution, in agreement with the discussion in Section 8.4.1.

8.5.3 Analysis of the second term

The second term in (43), considering that

$$\begin{aligned} &< \delta \boldsymbol{M}\, \overline{\mathbf{E}}^{(0)}, \delta \mathbf{E} > = < \overline{\boldsymbol{M}}\, \overline{\mathbf{E}}^{(0)}, \delta \mathbf{E} > - < \overline{\boldsymbol{M}}^{(0)}\, \overline{\mathbf{E}}^{(0)}, \delta \mathbf{E} > \\ &= - < \overline{\boldsymbol{M}}\, \delta \overline{\mathbf{E}}, \delta \mathbf{E} > + < \overline{\boldsymbol{M}}\, \overline{\mathbf{E}}\ , \delta \mathbf{E} > - < \overline{\boldsymbol{M}}^{(0)}\, \overline{\mathbf{E}}^{(0)}, \delta \mathbf{E} > \end{aligned} \tag{8.59}$$

can be rewritten as

$$\delta R'' = - < \delta \boldsymbol{M}\, \overline{\mathbf{E}}^{(0)}, \delta \mathbf{E} > = - < \overline{\boldsymbol{M}} \delta \overline{\mathbf{E}}, \delta \mathbf{E} > \tag{8.60}$$

Physically speaking, Eq. (60) can be interpreted as the reaction between the field variation $\delta \mathbf{E}$ inside the perturbation volume and the induced current inside the perturbation volume by the field variation $\delta \overline{\mathbf{E}}$ relevant to the auxiliary source. In addition, we have

$$\left| < \boldsymbol{M}\, \delta \overline{\mathbf{E}}, \delta \mathbf{E} > \right| \leq \left\| \boldsymbol{M}\, \delta \overline{\mathbf{E}} \right\| \left\| \delta \mathbf{E} \right\|. \tag{8.61}$$

The assumption that it exists a $\tau > 0$ such that

$$\left\| \boldsymbol{M}\, \delta \mathbf{E} \right\| \geq \tau \left\| \delta \mathbf{E} \right\|, \tag{8.62}$$

which is usually called a stability estimate, is consistent with the postulate that the field enters the perturbation without significant distortion:

$$\frac{\left\| \delta \mathbf{E} \right\|}{\left\| \mathbf{E}^{(0)} \right\|} << 1. \tag{8.63}$$

In fact, from (46) we get

$$\left\| \boldsymbol{M}\, \delta \overline{\mathbf{E}} \right\| = \left\| \delta \boldsymbol{M}\, \overline{\mathbf{E}}^{(0)} \right\| \leq \left\| \delta \boldsymbol{M}\ \right\| \left\| \overline{\mathbf{E}}^{(0)} \right\| \leq \psi \left\| \overline{\mathbf{E}}^{(0)} \right\|, \tag{8.64}$$

which combined with (62) gives

$$\|\delta\mathbf{E}\| \le \tau^{-1}\|\boldsymbol{M}\,\delta\mathbf{E}\| \le \tau^{-1}\psi\,\|\mathbf{E}^{(0)}\|, \qquad (8.65)$$

in accordance with (63) being $\psi/\tau << 1$.
By using (64) and (65), from (61) we obtain:

$$\left|<\boldsymbol{M}\,\delta\overline{\mathbf{E}}, \delta\mathbf{E}>\right| \le \psi\,\|\overline{\mathbf{E}}^{(0)}\|\,\|\mathbf{E}^{(0)}\|\,\frac{\psi}{\tau} \qquad (8.66)$$

The (66) merits some comments. It should be noted that as ψ/τ increases, multiple scattering phenomena included in (66) becomes increasingly important with respect to the single-scattering phenomena (see also (47)). It is important to note that the relevance of these multiple scattering phenomena can be connected to both the upper bound (ψ) for the structural perturbation $\delta\boldsymbol{M}$ and the lower bound (τ) for the structure operator $\boldsymbol{M}$. Finally, it is important to note that variational analysis maintains general validity for a generic structure whose irregularities can be described as fully space-variant perturbation.

8.6 Conclusions

This Chapter has been aimed primarily at resolving the apparent formal discrepancy between the corresponding regimes of validity of two different kinds of perturbative formulations: boundary perturbation and volumetric perturbation -based, respectively. The analysis leads to the formal demonstration of the necessity of the additional small slope assumption for the validity volumetric perturbation based formulation (and, in particular, for the VPRT), which has been not highlighted in previous works.

This finding will be important for the successful application of recently developed VPRT to canonical structures with fully space-variant perturbation.

We also emphasize that the required limitation relevant to the interfacial slope is more difficult to be directly deduced by adopting the formulations in [3]-[5]; differently our VPRT formulation enables pointing out this implication easily.

Our investigation has been addressed to the general case of layered rough interface, so that, the two relevant theoretical construct (BPT and VPT) for rough multilayer scattering can be now regarded in a conceptually coherent frame. The case of a single rough interface can be accordingly regarded as a special case of our discussion.

Finally, we have shown how the first-order VPRT solution can be arranged in the variational formalism framework, so enabling a more formal and comprehensive perspective on the discussion regarding the pertinent approximation involved. Specifically, we emphasize that, although first-order perturbative approximation constitutes a reference solution that tends to the true solution in the limit of large wavelength, second-order (and higher-order) terms of the perturbative development might turn out of particular interest, and further efforts in pertinent analytical derivation are recommended especially in the context of scattering in layered structure. Indeed, in the layer structures context, the energy directly scattered by a rough interface can be coupled into guided-wave modes and consequently could give rise to second-order processes contributing to the overall scattered field in relevant measure. It is therefore clear that the availability of second-order solutions for layered rough media is highly desirable and it is for that reason subject of current investigation. This analytical result, if available, will also enable the effective comparison of the perturbation solution prediction with the results obtainable with numerical methods, so permitting a more appropriate assessment of the precise conditions of validity for the relevant perturbation model.

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Chapter 9

Conclusion and Future Developments

"Research is what I'm doing when I don't know what I'm doing."
Wernher von Braun

"... essi godono di un ben grandissimo, e posson persuadersi d'intendere e di sapere tutte le cose, alla barba di quelli che conoscendo di non saper quel ch'e' non sanno, ed in conseguenza vedendosi non saper né anco una ben minimissima particella dello scibile, s'ammazzano con le vigilie, con le contemplazioni, e si macerano intorno a esperienze ed osservazioni."
Galileo Galilei

The problem of electromagnetic wave scattering in 3-D random layered structures, has been analytical treated by relying on original results of the *Boundary Perturbation Theory* (BPT) and *Volumetric-Perturbative Reciprocal Theory* (VPRT), whose structured presentation of the pertinent theoretical body of innovative results is proposed and developed in this thesis.

Preliminarily, the available formalisms for the evaluation of the electromagnetic field in flat-boundaries multilayer has been re-examined, providing a comprehensive and organized new perspective: this step is crucial for the derivation of pertinent (full-vectorial) general solutions in closed form, so providing a methodological basis for the perturbation analysis.

The systematic formulation of *Boundary Perturbation Theory* (BPT) has been introduced to deal with the analysis of a layered structure with an arbitrary number of gently rough interfaces: in this case the theoretical construct is based on a suitable perturbation pertinent to the geometry of the problem and the scattering problem is treated by adopting a proper perturbation of boundary conditions.

Specifically, in the first-order approximation, BPT leads to fully polarimetric, formally symmetric and physical revealing closed form solution: the relevant innovative scattering models obtained in this perturbation framework permit to deal with bistatic scattering, from and through three-dimensional layered structures with an arbitrary number of gently rough interfaces.

On the other hand, the formulation of *Volumetric-Perturbative Reciprocal Theory* (VPRT) methodologically adopts a different approach, which is based on two key elements: the use of the Reciprocity Theorem and an appropriate description of the scattering structure in terms of space-variant volumetric perturbation of the dielectric constant distribution. The VPRT construct also provides meaningful reaction-based expressions for the scattering field.

It is important to emphasize that VPRT, which is methodologically conceived to consistently treat both interfacial and volumetric random inhomogeneities (so providing a unified mathematical formulation and conceptual understanding of two inherent scattering mechanisms), is also fully consistent with the results of BPT. Accordingly, within VPRT framework, both rough-interface and volume scattering are take into account methodologically.

Furthermore, a new look at the classical SPM solution for rough surface is also offered: in this new theoretical framework even such a specific solution (whose derivation hitherto obtained via unnecessary, involved and obscure algebraic manipulations) can be now derived a surprisingly simple way, clarifying all the same the lacking inherent physical meaning.

This Thesis exploits several mathematical tools and specific key-concepts (as reaction, scattering enhancement, generalized reflection/ transmission notions, etc.) and further introduces new perspectives and concepts (local and global scattering, multi-reaction, etc.).

Beyond a certain compactness of the pertinent closed-form solutions, the fundamental scattering interactions can be revealed, gaining a coherent explanation and a neat picture of the physical meaning of the proposed theoretical constructs. In fact, it is important to note that a deep comprehension of the physical phenomena involved in the electromagnetic wave scattering interaction with such kind of complex structures would have been a rather hopeless task before the introduction of these theories.

Therefore, this theoretical body of results enables a new way to systematically construct meaningful and general expressions for the scattering field, and it is successful in that it exhibit: conceptual clearness, descriptive power and general applicability to random layered structures. In this regard, it is noteworthy that the proposed theoretical constructs methodologically provide the new way to analytically approach and study a wide class of scattering problems, involving complex structures (e.g. Semi-Infinite Media with Interfacial and Volumetric Random Inhomogeneities, Rough Multilayers, Randomly-Inhomogeneous Layers, etc.) that can be arranged in a perturbation framework. It should be noted that VPRT also provide the methodological basis to deal with scattering, from and through, three-dimensional layered structures with an arbitrary number of gently rough interfaces and inhomogeneous layers.

Finally, current investigations (not included in this thesis) indicates that the general VPRT formulation can be directly extended, so including up to second order effects, in order to address even intriguing second-order scattering effects taking place in random layered structures. This is to say, that further developments of VPRT are viable and promise to open remarkable possibilities.

Publications

Part of the work presented in this thesis is essentially based on the following publications in refereed journals, chapter books and international and national conference:

Publications in refereed journals

1. **P. Imperatore**, A. Iodice, D. Riccio, "VOLUMETRIC-PERTURBATIVE RECIPROCAL FORMULATION FOR SCATTERING FROM ROUGH MULTILAYERS", *IEEE Transaction on Antennas and Propagation*. (in print)

2. **P. Imperatore**, A. Iodice, D. Riccio, "PHYSICAL MEANING OF PERTURBATIVE SOLUTIONS FOR SCATTERING FROM AND THROUGH MULTILAYERED STRUCTURE WITH ROUGH INTERFACES", *IEEE Transaction on Antennas and Propagation*, vol.58, no.8, pp.2710-2724, Aug. 2010.

3. **P. Imperatore**, A. Iodice, D. Riccio, "REMOTE SENSING OF LAYERED MEDIA: PERTURBATIVE SCATTERING MODELS", *Italian Journal of Remote Sensing,* February 2010, n. 42 (1), pp.129-141.

4. **P. Imperatore**, A. Iodice, D. Riccio, "TRANSMISSION THROUGH LAYERED MEDIA WITH ROUGH BOUNDARIES: FIRST-ORDER PERTURBATIVE SOLUTION", *IEEE Transaction on Antennas and Propagation*, vol.57, no.5, pp.1481-1494, May 2009.

5. **P. Imperatore**, A. Iodice, D. Riccio, "ELECTROMAGNETIC WAVE SCATTERING FROM LAYERED STRUCTURES WITH AN ARBITRARY NUMBER OF ROUGH INTERFACES", *IEEE Transactions on Geoscience and Remote Sensing,* vol.47, no.4, pp.1056-1072, April 2009.

6. G. Franceschetti, **P. Imperatore**, A. Iodice, D. Riccio, and G. Ruello, "SCATTERING FROM LAYERED STRUCTURES WITH ONE ROUGH INTERFACE: A UNIFIED FORMULATION OF PERTURBATIVE SOLUTIONS", *IEEE Transactions on Geoscience and Remote Sensing,* vol.46, no.6, pp.1634-1643, June 2008.

Contributions in international books

7. **P. Imperatore**, A. Iodice, D. Riccio, "ELECTROMAGNETIC MODELS FOR REMOTE SENSING OF LAYERED ROUGH MEDIA", in *Geoscience and Remote Sensing, New achievements*, (Edited by P. Imperatore & D. Riccio) INTECH Publisher, February 2010, pp. 177-202, ISBN 978-953-7619-97-8.

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