Covariant Holographic Renormalization of Non-Conformal Field Theories

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November 30, 2010

Abstract

We present a Covariant Holographic Renormalization (CHR) procedure of fake supergravity (sugra) systems. Holography is a strong/weak coupling duality between gravitational and gauge theories, and fake sugra is a theory of gravity coupled to scalars in which the scalar potential can be derived from a superpotential in some specific way.

Other systematic methods of holographic renormalization of fake sugra exist, but they are all restricted to spacetimes that are asymptotically anti-de Sitter (aAdS). According to the AdS/CFT correspondence, their dual quantum field theories are conformal in the ultraviolet.

Covariant Holographic Renormalization can be applied to fake sugra systems in both aAdS and non-aAdS backgrounds. The field theories dual to non-aAdS sugra systems are non-conformal. So far, the method is limited to the renormalization of two-point functions.

The CHR procedure shows that the divergences from the bare fake sugra action can be cancelled by covariant counterterms without introducing new divergences.

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Chapter 1

Introduction and outline

◇ This thesis is divided into four parts. In the first part we discuss the required background. This comprises an overview of the literature on the subject of holographic renormalization until the present day including an exposition of all the formulas that are interesting to us, ending in a formulation of our research question. In the second part we specifically discuss our research, *i.e* the development of the Covariant Holographic Renormalization method. In the third part we apply CHR to some case studies in aAdS and non-aAdS. We end in the fourth part with a summary and outlook, followed by the appendix and references. Each part is divided into a number of chapters. Below we give a qualitative overview of the content of each chapter, while we simultaneously introduce our notation and terminology. ◇

1.1 Part I: Background

Chapter 2: Renormalization in QFT Quantum Field Theory (QFT) is very successful in describing interactions between fundamental particles. The heart of the theory is a *generating functional* that is built from the action, from which correlation functions can be found by functional differentiation. In turn, the correlation functions, or *correlators* for short, appear in formulas that predict cross-sections and decay widths. These are physically measurable predictions that have been scrutinously tested at particle colliders, with great precision and success.

A major stumbling block in the development of QFT was the obiquitous occurence of infinities. In an interacting quantum field theory, the so-called bare action leads to a generating functional that generates infinite correlators, yielding senseless physical predictions such as an infinite probablity for two particles to collide. The infinities arise because the correlators are calculated by taking integrals over the virtual momenta, which go from zero to infinity, and are therefore called *ultraviolet divergences*.

Luckily this problem was overcome by the procedure of *renormalization*. The philosophy of renormalization is that the parameters in the action that in the free theory have the interpretation of physical particle masses and coupling constants can no longer be interpreted this way in the interacting theory. This reinterpretation is justified because the parameters in the action are not directly measurable quantities.

First, the interacting theory must be *regularized* in some artificial way, for example by integrating the momenta from zero to some finite cut-off instead of to infinity. Then, in many theories the dependence on the cut-off can be absorbed into a redefinition of the masses and coupling constants. When the action is rewritten in terms of the physical masses and couplings, it reproduces the bare action, but this time with the physical parameters, plus other terms called *counterterms*. The regularized action is defined as the sum of the bare action plus the counterterms, and the renormalized action is defined as the regularized action in the limit where the cut-off goes to infinity. A field theory is called renormalizable when this limit exists. All quantum field theories of the Standard Model are renormalizable.

Counterterms do not alter the equations of motion. Their only job is to cancel the divergences from the bare action by introducing equal divergences with opposite sign. The renormalization procedure has an inherent ambiguity, since we are always free to add finite terms to the action. Each finite term can be multiplied by an arbitrary constant called a *scheme constant*. The dependence of the renormalized action (and hence the correlators) on the scheme constants is called *scheme dependence*.

Chapter 3: Holographic Renormalization To make predictions with quantum field theory, one often has to rely on perturbation theory. Unfortunately, perturbation theory only works in the regime where the couplings are small, and is unfitted for strongly coupled interactions, such as the confinement of quarks in mesons and hadrons. To describe such a phenomenon, we need a theory that is predictive at strong coupling.

At first sight, one might think that we are forced to look for a theory that is not based on perturbation, since we are precisely interested in the regime where the gauge coupling becomes too strong to be used as a perturbation parameter. But physics knows many examples of strong/weak coupling dualities. These are dualities between two equivalent theories, one of which has a strong coupling while the other has a weak one. We may then perform a pertubation in the weak coupling parameter, and afterwards map the result to the strongly decoupled description using the duality.

$$\begin{array}{ccccc}
\text{Weak theory} & \Longleftrightarrow & \text{Dual} & \Longleftrightarrow & \text{Strong theory} \\
\downarrow & & & & \downarrow \\
\text{Perturbation theory} & & & & ? \\
\downarrow & & & & \downarrow \\
\text{Result 1} & \Longleftrightarrow & \text{Dual} & \Longleftrightarrow & \text{Result 2}
\end{array}$$

Indeed, there exists a strong/weak coupling duality between quantum field theories on one side and classical supergravity theories on the other. This duality is called *holography* [25, 26], since the sugra theories live in $d+1$ dimensions while their dual quantum field theories live in only d dimensions.

According to the holographic duality, *the on-shell action of a supergravity theory in $d+1$ dimensions is the generating functional of the connected correlators of a quantum field theory in d dimensions.* In some cases we may even define a quantum field theory by its gravitational dual. The duality is quite remarkable, not only because one theory lives in a different number of spatial dimensions than the other, but also because one theory describes classical gravity while the other describes quantum gauge interactions. The duality has not yet been proven, but by now the accumulated evidence in favor of it is enormous.

When two theories are dual, all symmetries and properties in one theory have their counterparts in the dual theory. For example, the ultraviolet (UV) divergences of quantum field theory are related to infrared (IR) divergences in the supergravity theory that arise due to the infinite volume of spacetime. Both theories therefore require regularization and renormalization (which in both cases is scheme dependent). In quantum field theory, we can renormalize the bare action by adding counterterms to it. The dual counterpart of this procedure is called *holographic renormalization*: instead of adding counterterms to the bare quantum field theory action, we add them to the bare supergravity action.

Bare Sugra Action	$\Longleftrightarrow$	Dual	$\Longleftrightarrow$	Bare QFT Action
$\downarrow$				$\downarrow$
Holographic Renormalization: Add counterterms				*QFT Renormalization:* Add counterterms
$\downarrow$				$\downarrow$
Renormalized Sugra Action	$\Longleftrightarrow$	Dual	$\Longleftrightarrow$	Renormalized QFT Action

The occurence of divergences in the on-shell bare action was noted already in [12,13,19]. The first divergence that was removed by adding a counterterm was the divergence of the boundary volume in the setting of pure gravity (no scalar fields) on an Anti-de Sitter (AdS) background [15]. According to the *AdS/CFT correspondence* [16, 28], the gravitational theory in AdS has a dual quantum field theory with conformal symmetry, and is therefore called *conformal field theory* (CFT). The general structure of the divergent terms and the relation of the logarithmically divergent terms to the *conformal anomaly*, also known as *Weyl anomaly*, of the dual CFT was discussed in [14], but still in the context of pure gravity in AdS.

So far, only three systematic procedures of holographic renormalization are known for gravity coupled to scalar fields Φ: the *standard method* [4], the *Hamilton-Jacobi method* [8, 17], and *Hamiltonian Holographic Renormalization* [21]. However, these methods are restricted to supergravity theories living in a spacetime that is asymptotically anti-de Sitter. Hence, their duals are always conformal field theories. We will discuss the standard method and the Hamilton-Jacobi method in section 7, but we will not discuss Hamiltonian Holographic Renormalization in this thesis.

In order to calculate strongly coupled correlators in non-conformal quantum field theories, it is therefore necessary to extend the method of holographic renormalization to spacetimes that are not aAdS. Indeed, the AdS/CFT correspondence is believed to hold more generally, including a *non-AdS/non-CFT correspondence*. In general the holographic duality is called the *gauge/gravity correspondence*. There are many reviews on this subject, for example [1, 9–11, 20, 23, 24].

Chapter 4: Fake Supergravity In order to extend the holographic renormalization program for scalars coupled to gravity from aAdS to general spacetimes, we consider a special type of theory, called *fake supergravity*. The action couples gravity to an arbitrary number of scalars Φ^a by some sigma-model metric G_{ab}, where a and b run from one to the number of scalars n_s. By definition of a fake supergravity theory, the scalar potential $V(\Phi)$ must be derivable from a superpotential $W(\Phi)$ by some specific prescription. Fake supergravity theories can be supersymmetric, but they are not necessarily so, hence the adjective "fake". The relation between supergravity and fake supergravity was studied in [7,29]. The sigma-model metric and the superpotential together define the theory completely. The fake sugra system allows for solutions which are the gravitational duals of renormalization group flows in quantum field theory.

Fake supergravity theories arise in a variety of physical models, such as consistent truncations of type IIB supergravity, see for example section 3 in [2]. Fake sugra is well suited for our research, because the generic action is simple yet general enough to allow for solutions that are either aAdS or not, depending on the choice of superpotential. If the superpotential approaches a constant at the boundary, called a *fixed point*, then the solutions are aAdS, otherwise they are not. The existence of a fixed point simplifies the holographic renormalization procedure considerably, since it allows for a Taylor expansion of the potential and superpotential around the fixed point.

Chapter 5: Gauge Invariant Fluctuations Our work builds especially upon [6], which focussed on the renormalization of two-point functions in a general spacetime. To study two-point functions, an expansion of the action up to quadratic order in the fluctuations around the background is enough. On the background, the scalars are functions of the radial coordinate r only. When we consider small fluctuations around the background, the scalar fields become dependent on the boundary coordinates x as well. Similarly, we consider small metric fluctuations.

In [2] the degrees of freedom of the scalar and metric fluctuations are combined into combinations that are invariant under diffeomorphisms (gauge invariant), denoted by $\mathfrak{a}^a$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}^i$, and $\mathfrak{e}^i_j$. The unphysical (gauge) degrees of freedom are denoted by ϵ^i, H and h.

The generating functional of the quantum field theory is the *on-shell* supergravity action, so the physical fluctuations satisfy their equations of motion (The unphysical degrees of freedom are not dynamical so they do not obey any equations of motion.) We use the equations of motion to eliminate $\mathfrak{b}$ and $\mathfrak{c}$ in favor of $\mathfrak{a}^a$, and find $\mathfrak{d}^i$ equal to zero to the required order. We are then left with only the equations of motion for $\mathfrak{a}^a$ and $\mathfrak{e}^i_j$. A great avantage of the gauge invariant formalism is that the equation of motion for the gauge invariant scalar fluctuations $\mathfrak{a}^a$ decouples from the equation of motion for the traceless and transverse metric fluctuations $\mathfrak{e}^i_j$, so that they can be studied separately. Another advantage is that $\mathfrak{e}^i_j$ behaves like a massless scalar.

Chapter 6: Ward Identities Before we discuss the existing methods of holographic renormalization in aAdS, we first discuss the translational Ward identity, which should always hold (since the dual field theory should have translational invariance), and the conformal Ward identity that holds only in aAdS (since then the dual field theory is conformal). This allows us to define the conformal anomaly, which we need to discuss the Hamilton-Jacobi method.

Chapter 7: Holographic Renormalization in aAdS We show how the existence of a fixed point allows for a Taylor expansion around this fixed point, so we can write down general expansions for the potential $V(\Phi)$ and the superpotential $W(\Phi)$. Then we respectively discuss the standard method and the Hamilton-Jacobi method of holographic renormalization in aAdS, which rely on this Taylor expansion right from the beginning.

1.2 Part II: Developments

Chapter 8: Divergences of the Bare Action We start our research by explicitly writing down all divergences of the bare action in terms of the gauge invariant variables. There are three divergent terms, involving respectively only the fields $\mathfrak{a}^a$, $\mathfrak{e}^i_j$ and h. Each term is of quadratic order of the same field without any mixing to other fields.

Chapter 9: Non-Covariant Counterterms In [6] a *counterterm matrix* $\mathcal{U}_{ab}$ was constructed to renormalize the two-point functions of quantum field theory operators dual to the fluctuations of the gauge invariant bulk scalars $\mathfrak{a}^a$ that live in a general spacetime. Since the physical metric behaves like a massless scalar, we can immediately write down the equivalent counterterm function $\mathcal{T}$ for the traceless transverse metric fluctuations $\mathfrak{e}^i_j$. For simplicity, we will refer to $\mathcal{U}_{ab}$ and $\mathcal{T}$ collectively as "counterterm matrices". The counterterm matrices appear in the counterterm action "sandwiched" between the fluctuations, like

$$\mathfrak{a}^a \mathcal{U}_{ab} \mathfrak{a}^b, \qquad \mathfrak{e}^i_j \mathcal{T} \mathfrak{e}^j_i.$$

Clearly, the counterterms obtained this way are not covariant. As a non-covariant counterterm for the gauge field h we just write down the divergent term from the bare action with opposite sign. These three counterterms together form a counterterm action S_{cnt} that cancels all divergences from the bare action, but is not covariant.

Chapter 10: Differential Equations for Counterterm Matrices The counterterm matrices $\mathcal{U}_{ab}$ and $\mathcal{T}$ are defined in terms of the dominant solutions to the equations of motion for the scalars and metric respectively. Since these solutions satisfy differential equations, so do the counterterm matrices. These differential equations turn out to be very useful in what follows.

Chapter 11: Boundary Covariant Counterterms The goal of this thesis is to show how the required non-covariant counterterm action S_{cnt} arises consistently from a boundary covariant one S_{cov}. By "boundary covariant" we mean that the counterterm action is allowed to depend explicitly on the cut-off of the radial variable, but must be covariant with respect to the boundary coordinates such that the dual field theory, that lives on the boundary, is fully covariant. We will often abbreviate "boundary covariance" to just "covariance" when it is clear from the context what we mean.

We construct the covariant counterterms in S_{cov} such that, after an expansion up to quadratic order in the fluctuations, they produce the required non-covariant part S_{cnt}. Starting with covariant counterterms, the expansion in fluctuations automatically produces another part S_{fin}, which has to be finite by itself because the divergences from the bare action are already cancelled by S_{cnt}. After the expansion in fluctuations, we therefore have $S_{\text{cov}} = S_{\text{cnt}} + S_{\text{fin}}$. Working with covariant counterterms has the advantage that the unphysical degrees of freedom can be studied along with the physical ones, without any gauge fixing.

First, we write down the general structure of all possible covariant counterterms. Each counterterm is classified by its number of derivatives. We keep terms up to four derivatives, which after the expansion and partial integration yields terms up to quadratic order in the d'Alembertian $\Box$. This order is sufficient for four-dimensional field theories. There are seven counterterms to be determined: one at zeroth order, two at first order and three at second order in $\Box$.

The starting point we described above is the same as in the Hamilton-Jacobi method. However, in the Hamilton-Jacobi method, the next step is to expand the unknown functions of the scalars into a Taylor series around the fixed point. In non-aAdS spacetimes there is no fixed point, so this step is impossible. Instead, we expand the counterterms *around the background* up to quadratic order in the fluctuations.

We then fix the background values of the unknown functions by demanding that the renormalized action is finite. Five out of seven counterterms are fixed by the requirement that we reproduce the non-covariant counterterms proportional to $\mathfrak{a}^a \mathcal{U}_{ab} \mathfrak{a}^b$ and $\mathfrak{e}^i_j \mathcal{T} \mathfrak{e}^j_i$. The other two counterterms are determined by the requirement that the remaining part of the expanded action is finite by itself.

Chapter 12: Renormalized Action We show that we do not introduce any new divergences. When we expand the covariant counterterm S_{cov} up to quadratic order in the fluctuations, not only do we find the required non-covariant part S_{cnt} that kills the divergences from the bare action by construction, but also another part S_{fin}, which has to be finite by itself. The requirement that S_{cov} produces the required part S_{cnt} fixes S_{fin} up to first order in $\Box$, but we have the freedom to choose the counterterms such that all terms of order $\Box^2$ are finite. At zeroth and first order, we explicitly show that there are no divergences in S_{fin} under a reasonable assumption.

Chapter 13: One-Scalar Systems in aAdS We show that Covariant Holographic Renormalization reproduces the same results as the Hamilton-Jacobi method for aAdS systems with one scalar.

1.3 Part III: Case Studies

In the third part we test our method on three aAdS systems: the GPPZ flow, the Coulomb Branch Flow, and the two scalar $SU(2)\times U(1)$ flow, and finally on a non-aAdS system, the Klebanov-Strassler theory. In one-scalar systems, we distinguish between operator flows and vev flows. In an operator flow, the QFT operator $\mathcal{O}$ dual to the bulk scalar Φ has a vacuum expectation value (vev) that is purely scheme dependent, while in a vev flow it has a non-zero vev independently of the scheme. We construct the counterterm matrices $\mathcal{U}_{ab}$ and $\mathcal{T}$ in two ways. First by their explicit definition in terms of the dominant solutions to the equations of motion, and then by solving the differential equations they obey. Either way leads to the same result, but the second way is much faster.

Chapter 14: GPPZ The GPPZ flow is an operator flow. The operator dual to the bulk scalar has conformal dimension $\Delta = 3$. We find the counterterms, calculate the one-point functions and show that the Ward identites are satsified.

Chapter 15: Coulomb Branch Flow The Coulomb Branch Flow is a vev flow. The operator dual to the bulk scalar has conformal dimension $\Delta = 2$. We find the counterterms, calculate the one-point functions and show that the Ward identities are satsified.

Chapter 16: Two Scalars in aAdS The $SU(2) \times U(1)$ flow has two scalars, one with $\Delta = 2$ and one with $\Delta = 3$. We find the counterterms and show that when the $\Delta = 3$ scalar is made inert, the counterterms reduce to the counterterms for the Coulomb Branch Flow, such that the operator dual to the remaining scalar has a non-zero vev independently of the scheme.

Chapter 17: Non-aAdS: Klebanov-Strassler The Klebanov-Strassler system is non-aAdS. Because of its complexity, we are not yet able to construct the counterterms, but we are able to show a non-trivial cancellation of divergences that confirms one of the predictions of Covariant Holographic Renormalization. The cancellation of divergences takes place at first order in $\Box$ within the counterterms themselves. This is an important result, because it shows how the counterterms conspire to cancel divergences from the bare action without introducing new divergences.

1.4 Part IV: Summary, Outlook and Appendix

We summarize our work and indicate the direction of our future research. Appendix A gives an overview of our notation, appendix B shows the details of the expansion of the counterterm action up to second order in the fluctuations, and appendix C shows how to calculate one-point functions up to linear order in the fluctuations from the renormalized action.

Part I

Background

Chapter 2

Renormalization in QFT

$\diamond$ We review renormalization in Quantum Field Theory (QFT), assuming the reader is already familiar with this subject. Our working example is scalar field theory. The action leads to physical predictions through a generating functional from which correlators can be derived. We discuss the physical interpretation of the correlators and the philosophy of regularization and renormalization. In this thesis, we will always use a cut-off for the regularization and counterterms for the renormalization. This section relies heavily on section 10.2 in [22]. $\diamond$

A quantum field theory is defined by an action $S[\phi]$, which is a functional of some set of fields ϕ. Our goal is to make physical predictions with the action, so we need to subtract physical quantities from it.

2.1 Free Scalar Field Theory

Consider a free scalar field theory defined by the Klein-Gordon Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2, \tag{2.1.1}$$

where m has the interpretation of the mass of the scalar field ϕ. From the Lagrangian, we construct the *generating functional*[1]

$$Z[\mathfrak{s}] \equiv e^{-iW[\mathfrak{s}]} \equiv \int \mathcal{D}\phi \exp\left(i\int d^4x\,[\mathcal{L} + \mathfrak{s}(x)\phi(x)]\right), \tag{2.1.2}$$

where $\mathfrak{s}(x)$ is the *source* coupling to the QFT operator $\phi(x)$. The physical quantities we can subtract from the generating functional are called *n-point correlation functions*, or n-point functions for short. The n-point function is defined by taking n functional

[1]The quantity $W[\mathfrak{s}] = i\log Z[\mathfrak{s}]$ is the generating functional of *connected* correlators. In [22] the symbol $J(x)$ is used for the source, while we use $\mathfrak{s}(x)$.

differentiations of the generating functional with respect to the source. For example, the one- and two-point functions are given by

$$\langle\phi(x)\rangle = -\frac{i}{Z[\mathfrak{s}]}\frac{\delta Z[\mathfrak{s}]}{\delta\mathfrak{s}(x)}\Big|_{\mathfrak{s}=0}, \qquad \langle\phi(x)\phi(y)\rangle = (-i)^2\,\frac{1}{Z[\mathfrak{s}]}\frac{\delta^2 Z[\mathfrak{s}]}{\delta\mathfrak{s}(x)\delta\mathfrak{s}(y)}\Big|_{\mathfrak{s}=0}. \tag{2.1.3}$$

The physical interpretation of the one-point function is the vacuum expectation value (vev) of the field $\phi(x)$, while the interpretation of the two-point function is the amplitude for the propagation of a particle between spacetime coordinates x and y, and the square of this amplitude is proportional to the probability for this to happen. In general, from the n-point functions we can calculate collision cross-sections and decay widths, which of course have to be finite values.

In the free Klein-Gordon theory, the two-point function is the Feynman propagator

$$\langle\phi(x)\phi(y)\rangle = \int\frac{d^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}, \tag{2.1.4}$$

where ϵ is a small positive constant needed to define the integral. In momentum space, the propagator takes the form

$$\int d^4x e^{ip\cdot x}\,\langle\phi(x)\phi(0)\rangle = \frac{i}{p^2-m^2+i\epsilon}. \tag{2.1.5}$$

In the free theory the two-point function is finite, but, without renormalization, this is not the case in the interacting theory.

2.2 Interacting Scalar Field Theory

Consider the interacting Klein-Gordon theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4, \tag{2.2.1}$$

where λ is the coupling of the scalar field to itself. From the new Lagrangian we construct a new generating functional, and from the new generating functional we calculate the new n-point functions. The two-point function in momemtum space that we calculated in the free theory eq.(2.1.5) changes in the interacting theory to

$$\int d^4x e^{ip\cdot x}\,\langle\phi(x)\phi(0)\rangle = \frac{iF}{p^2-Fm^2+\delta_m+i\epsilon} + \text{finite}, \tag{2.2.2}$$

where F and δ_m are divergent quantities. When we consider four-point functions, we will find another divergent quantity δ_λ. Higher point functions are finite. Thus the interacting Klein-Gordon theory contains three divergent quantities: $F, \delta_m, \delta_\lambda$. The divergences appear because on the left-hand side the integral over the spacetime position x of the incoming particle comes arbitrarily close to zero, the position of the outgoing particle. In momentum space the divergence comes from taking the integral over arbitrarily high momenta. Divergences of this kind are called ultraviolet divergences.

2.3 Regularization

In order to proceed, we have to take a step back to the point just before things go wrong and diverge. We must *regularize* the theory so that we are always working with finite quantities. One way to do this is by introducing a cut-off Λ in the integration over the four-momentum,

$$\langle \phi(x)\phi(y) \rangle = \int_0^\infty d^4p \ldots \to \int_0^\Lambda d^4p \ldots \qquad (2.3.1)$$

The value of the cut-off is arbitrary, because eventually the limit $\Lambda \to \infty$ must be taken. At this point we have

$$\lim_{\Lambda \to \infty} F(\Lambda) = \infty, \qquad (2.3.2)$$

but $F(\Lambda)$ itself is finite (regularized), so it is well defined and we can safely work with it. The same argument applies to δ_m and δ_λ.

Since both F and δ_m are divergent functions of Λ, we can not identify the pole $Fm^2 - \delta_m$ in the propagator with the physical mass. This means that, in going from the free to the interaction theory, either m loses its interpretation of the physical mass (interpretation one), or the interacting Lagrangian is missing a second mass term (counterterm) that will cancel the divergence from the pole, resulting in a finite physical mass (interpretation two). Consequently, we may *renormalize* the interacting theory in two equivalent ways.

2.4 Redefinition of parameters

First, we may think of the Lagrangian as physical (renormalized), but its parameters as unphysical (bare),

$$\mathcal{L}_r = \frac{1}{2}\partial_\mu\phi_b\partial^\mu\phi_b - \frac{1}{2}m_b^2\phi_b^2 - \frac{\lambda_b}{4!}\phi_b^4. \qquad (2.4.1)$$

We already saw that in the free theory, the interpretation of the bare quantities coincides with the physical quantities, but in the interacting theory this interpretation must be modified. In other words, if we set $\lambda_b = 0$, then m_b has the interpretation of the physical mass m, otherwise m_b diverges and can not have this physical interpretation anymore. Since we replaced $\phi \to \phi_b$ and $m \to m_b$ in the Lagrangian, we must do the same in the two-point function that follows from it,

$$\int d^4x e^{ip\cdot x} \langle \phi_b(x)\phi_b(0) \rangle = \frac{iF}{p^2 - Fm_b^2 + \delta_m + i\epsilon} + \text{finite}. \qquad (2.4.2)$$

The divergences can be absorbed into the bare parameters as follows

$$\phi_r \equiv \frac{\phi_b}{\sqrt{F}}, \qquad \delta_F \equiv F - 1, \qquad \delta_m \equiv Fm_b^2 - m_r^2, \qquad \delta_\lambda \equiv F^2\lambda_b - \lambda_r. \qquad (2.4.3)$$

From now on we consider ϕ_r, m_r, λ_r as the physical quantities. The *physical* two-point function then becomes finite, since we can divide by F on both sides,

$$\int d^4x e^{ip\cdot x} \langle \phi_r(x)\phi_r(0)\rangle = \frac{i}{p^2 - m_r^2 + i\epsilon} + \text{finite}, \tag{2.4.4}$$

and thanks to the renormalized coupling constant λ_r the physical four-point function becomes finite as well, but we will not discuss that here. In terms of the physical quantities, the Lagrangian reads

$$\mathcal{L}_r = \frac{1}{2}\partial_\mu\phi_r\partial^\mu\phi_r - \frac{1}{2}m_r^2\phi_r^2 - \frac{\lambda_r}{4!}\phi_r^4 + \frac{1}{2}\delta_F\partial_\mu\phi_r\partial^\mu\phi_r - \frac{1}{2}\delta_m\phi_r^2 - \frac{\delta_\lambda}{4!}\phi_r^4. \tag{2.4.5}$$

Let us summarize what we have done. We started with a Lagrangian that is known to give the correct equation of motion for a free scalar field. From the Lagrangian we constructed a generating functional, from which we calculated n-point functions. Unfortunately, when we consider interactions, the two- and four-point functions become infinite. We then reinterpreted the three quantities in the interacting Lagrangian ϕ_b, m_b, λ_b as *unphysical*, and rewrote the same Lagrangian in terms of physical quantities ϕ_r, m_r, λ_r, that are defined by the requirement that the correlators calculated from ϕ_r give finite values in terms of the measurable parameters m_r and λ_r.

2.5 Counterterms

We may use the equivalent method of counterterms: *instead of viewing the parameters ϕ, m, λ in the Lagrangian as unphysical (bare), we may view the interacting Lagrangian itself as unphysical (bare)*,

$$\mathcal{L}_b = \frac{1}{2}\partial_\mu\phi_r\partial^\mu\phi_r - \frac{1}{2}m_r^2\phi_r^2 - \frac{\lambda_r}{4!}\phi_r^4. \tag{2.5.1}$$

As we have seen, calculating correlators from this Lagrangian yields infinite and thus unphysical results. But we can construct the physical (renormalized) Lagrangian by

$$\mathcal{L}_r = \mathcal{L}_b + \mathcal{L}_c, \tag{2.5.2}$$

where the *counterterm* Lagrangian is given by, see eq.(2.4.5):

$$\mathcal{L}_c = \frac{1}{2}\delta_F\partial_\mu\phi_r\partial^\mu\phi_r - \frac{1}{2}\delta_m\phi_r^2 - \frac{\delta_\lambda}{4!}\phi_r^4. \tag{2.5.3}$$

Thus, neither the bare Lagrangian $\mathcal{L}_b$ nor the counterterm Lagrangian $\mathcal{L}_c$ is physical by itself, but their sum is. It is $\mathcal{L}_r$ that appears in the generating functional, which then produces finite correlators.

We shall use the counterterm renormalization method in the rest of this thesis. This has the advantage that all fields and their masses and couplings are always physical.

Chapter 3

Holographic Renormalization

◇ The heart of holography is the idea that the on-shell action of a gravitational theory acts as the generating functional of connected correlators in a dual quantum field theory. To obtain finite correlators, in holographic renormalization we add counterterms to the gravitational action instead of directly to the quantum field theory action. ◇

According to the AdS/CFT correspondence [9, 16, 24], a classical theory of gravity in a $d+1$ dimensional Anti-de Sitter geometry (*the bulk*) is dual to a Conformal Field Theory (CFT) living on the d-dimensional boundary of the bulk. Similarly, there exists an non-AdS/non-CFT correspondence. In general, the duality is called *holographic duality* or *gauge/gravity duality*.

We can visualize the bulk geometry as a $(d+1)$-dimensional ball with radial coordinate r. The boundary is a d-dimensional sphere with boundary coordinates x^i $(i = 1, \ldots, d)$, which we collectively denote by x. Infinities arise since the quantum field theory lives on the boundary, which lies infinitely far away at $r \to \infty$, called the infrared (IR) limit. The IR divergences are the holographic duals of the infamous ultraviolet (UV) divergences in quantum field theory.

3.1 Holographic Generating Functional

Let us rewrite the QFT generating functional eq.(2.1.2) in Eucledian signature for a general number of dimensions d and a general number n_s of operators $\mathcal{O}_i$ coupled to sources $\mathfrak{s}_i$ $(i = 1, \ldots, n_s)$,

$$\exp\left(-W[\mathfrak{s}]_{\mathrm{QFT}}\right) \equiv \int \mathcal{D}\phi \exp\left(\int d^d x \left[-\mathcal{L}_{\mathrm{QFT}} + \mathfrak{s}_i(x)\mathcal{O}_i(x)\right]\right). \tag{3.1.1}$$

The holographic duality states that[1]

The renormalized, on-shell action of the gravitational theory $S_{ren}[\mathfrak{s}]$
is identified with
the generating functional of connected correlators $W[\mathfrak{s}]$ in the dual quantum field theory

$$\boxed{W[\mathfrak{s}]_{\text{QFT}} = S_{\text{ren}}[\mathfrak{s}]}. \tag{3.1.2}$$

The gravitational action consists of bulk terms, which are $(d+1)$-dimensional integrals, plus boundary terms, which are d-dimensional integrals. The equation of motion for a physical field is found by the requirement that the bulk terms varied with respect to that field vanish.[2] The on-shell action thus contains only integrals over the d-dimensional boundary. This agrees with the idea that the quantum field theory lives in d dimensions, while the gravitational theory lives in $d+1$ dimensions.

The goal of holographic renormalization is to obtain finite quantum field theory correlators by adding counterterms to the on-shell gravitational action instead of the quantum field theory action. The one- and two-point functions for the quantum field theory operators $\mathcal{O}_i$ can thus be calculated from the renormalized, on-shell gravitational action S_{ren} through the following formulas

$$\boxed{\langle \mathcal{O}_i(x) \rangle = -\frac{\delta S_{\text{ren}}}{\delta \mathfrak{s}_i(x)}\bigg|_{\mathfrak{s}=0}, \qquad \langle \mathcal{O}_i(x) \mathcal{O}_j(y) \rangle = \frac{\delta^2 S_{\text{ren}}}{\delta \mathfrak{s}_i(x) \delta \mathfrak{s}_j(y)}\bigg|_{\mathfrak{s}=0}.} \tag{3.1.3}$$

The *exact* one-point function is defined without setting the sources to zero,

$$\langle \mathcal{O}_i(x) \rangle_{\mathfrak{s}\neq 0} = -\frac{\delta S_{\text{ren}}}{\delta \mathfrak{s}_i(x)}_{\mathfrak{s}\neq 0}. \tag{3.1.4}$$

The exact one-point function thus carries information of higher point functions. For example, the two-point function can be obtained from the exact one-point function as follows

$$\langle \mathcal{O}_i(x) \mathcal{O}_j(y) \rangle = -\frac{\delta}{\delta \mathfrak{s}_j(y)} \langle \mathcal{O}_i(x) \rangle_{\mathfrak{s}} \bigg|_{\mathfrak{s}=0}. \tag{3.1.5}$$

Given a bare action S_{bare}, the challenge is to find the counterterm action S_{cov} such that the renormalized action $S_{\text{ren}} \equiv S_{\text{bare}} + S_{\text{cov}}$ yields finite correlators. The bare action that we will work with is the fake supergravity action, which we will now present.

[1] Since $S_{\text{ren}}[\mathfrak{s}]$ is the generating functional of *connected* correlators, we will always imply the connected correlators whenever we talk about correlators or use the brackets $\langle \dots \rangle$.

[2] Before the variation, it is ambiguous to speak of bulk terms and boundary terms, since by Stokes' theorem a boundary term can always be written as a bulk term and vice versa. However, after the variation with respect to some field is performed, the *varied* action can be unambiguously split into bulk and boundary terms.

Chapter 4

Fake Supergravity

⋄ We present the fake supergravity action, its background solutions and small fluctuations around the background. The fluctuations are decomposed into irreducible components so that the physical degrees of freedom can be isolated from the gauge degrees of freedom, which we shall do in the next section. ⋄

4.1 Bare Action

The bare gravitational action we consider is that of n_s scalars with potential $V(\Phi)$ coupled to gravity through the sigma-model metric G_{ab}, where $a, b = 1, \ldots n_s$,

$$\begin{aligned} S_{\text{bare}} &= \int dr d^d x \sqrt{\mathbf{g}} \left(-\frac{1}{4}\mathbf{R}[\mathbf{g}] + \frac{1}{2}\mathbf{g}^{\mu\nu} G_{ab}(\Phi)\partial_\mu \Phi^a \partial_\nu \Phi^b + V(\Phi) \right) + \\ &\quad + \frac{1}{2}\int d^d x \sqrt{\gamma}\mathcal{K}. \end{aligned} \tag{4.1.1}$$

Here, $\mathcal{K} \equiv \mathcal{K}^m_m$ is the trace of the second fundamental form, which we present later in eq.(8.1.3). The first integral runs over the $(d+1)$-dimensional bulk while the second integral, the Gibbons-Hawking boundary term, runs over the d-dimensional boundary. The equations of motion follow from the bulk term only. The counterterms are boundary terms, and therefore do not modify the equations of motion.

Fake supergravity systems are defined as systems for which the scalar potential $V(\Phi)$ can be derived from a function called superpotential $W(\Phi)$ as follows

$$V(\Phi) = \frac{1}{2}W^a W_a - \frac{d}{d-1}W^2, \tag{4.1.2}$$

where we have used the notation

$$W^a \equiv G^{ab} W_b, \qquad W_a \equiv \partial_a W \equiv \frac{\partial W}{\partial \Phi^a}. \tag{4.1.3}$$

The motivation for the requirement eq.(4.1.2) becomes clear when we consider the Hamilton-Jacobi method in section 7.3. The equations of motion that follow from the action are second order differential equations, so there are two independent solutions, one of which is dominant and the other one is subdominant at the boundary,

$$\Phi^a(r,x) = \hat{\Phi}^a_i(r)\Phi_{\mathfrak{s}i}(x) + \check{\Phi}^a_i(r)\Phi_{\mathfrak{r}i}(x). \tag{4.1.4}$$

where $\Phi_{\mathfrak{s}i}$ is the *source* of the *dominant solution* $\hat{\Phi}^a_i$ and $\Phi_{\mathfrak{r}i}$ the *response* of the *subdominant solution* $\check{\Phi}^a_i$. The indices a and i both run from 1 to n_s, the number of scalars.

Background solutions The fake sugra system allows for background solutions which are the gravitational duals of renormalization group flows in quantum field theory,

$$ds^2 = dr^2 + e^{2A(r)}\eta_{ij}dx^i dx^j, \qquad \phi^a(r), \qquad i,j = 1,\dots d, \tag{4.1.5}$$

where η_{ij} is the constant Minkowski metric and $A(r)$ is the *warp function*. The warp function goes to infinity when r goes to infinity, which allows us to think of A as an alternative radial coordinate. The domain wall background satisfies the equations of motion that follow from the fake supergravity action if (but not only if)[1] the warp function A and the scalar background solution ϕ^a satisfy

$$\boxed{\dot{A} = -\frac{2\mathcal{W}}{d-1}, \qquad \dot{\phi}^a = \mathcal{W}^a.} \tag{4.1.6}$$

Here $\dot{A} \equiv \partial_r A$, $\dot{\phi} \equiv \partial_r \phi$ and $\mathcal{W} \equiv \mathcal{W}(\phi)$ is the background value of the superpotential.

4.2 Fluctuations around the background

In this thesis, we are only interested in calculating one- and two-point functions. Then it is sufficient to know the solutions to the equations of motion up to second order in small fluctuations around the background. Since we want to formulate the fluctuation dynamics gauge invariantly, we expand the scalar fields in a way that is sigma-model covariant by using the exponential map [27],

$$\Phi^a(r,x) = \exp_\phi(\varphi)^a = \phi^a(r) + \varphi^a(r,x) - \frac{1}{2}\bar{\mathcal{G}}^a_{bc}(r)\varphi^b(r,x)\varphi^c(r,x) + \dots, \tag{4.2.1}$$

where $\bar{\mathcal{G}}^a_{bc}$ is the sigma-model connection evaluated on the background,

$$\bar{\mathcal{G}}^a_{bc} \equiv \frac{1}{2}\bar{G}^{ab}\left(\partial_c\bar{G}_{db} + \partial_b\bar{G}_{dc} - \partial_d\bar{G}_{bc}\right). \tag{4.2.2}$$

[1] The Euler-Lagrange equations are second order differential equations, so when we substitute the domain wall background, we obtain second order differential equations for A and ϕ. These second order equations are solved automatically when the first order equations above are satisfied, but not the other way around. The first order equations for A and ϕ are therefore sufficient but not necessary, and thus represent only a subclass of solutions.

For further details, see section (4.1) in [2].

To study metric fluctuations, we conveniently rewrite the background eq.(4.1.5) as

$$ds^2 = (n^2 + n_i n^i)dr^2 + 2n_i dr dx^i + \gamma_{ij} dx^i dx^j. \tag{4.2.3}$$

The matrix γ^{ij} is the inverse of γ_{ij}, and is used to raise hypersurface quantities,

$$\gamma_{ik}\gamma^{kj} = \delta_i^j, \qquad n^i = \gamma^{ij} n_j. \tag{4.2.4}$$

However, indices on the partial derivative are raised with the inverse Minkowski metric,

$$\partial^i \equiv \eta^{ij}\partial_j, \qquad \Box \equiv \partial^i \partial_i = \eta^{ij}\partial_i\partial_j. \tag{4.2.5}$$

The source of the boundary metric $\gamma_{\mathfrak{s}ij}(x)$ is defined by

$$\gamma_{ij}(A, x) = e^{2A}\gamma_{\mathfrak{s}ij}(x) + \dots \tag{4.2.6}$$

The metric fluctuations can be written as

$$n = 1 + \nu, \qquad n_i = \nu_i, \qquad \gamma_{ij} = e^{2A}\left(\eta_{ij} + h_{ij}\right). \tag{4.2.7}$$

All the dependence on the boundary coordinates x in Φ^a, n, n_i and γ comes through the fluctuations. We recover the background solutions eq.(4.1.5) by setting the fluctuations to zero.

We decompose ν_i into longitudinal and transversal parts,

$$\nu_L \equiv \partial_i \nu^i, \qquad \nu_T^i \equiv \Pi_j^i \nu^j, \tag{4.2.8}$$

where the transverse projector is defined by

$$\Pi_j^i \equiv \delta_j^i - \Box^{-1}\partial^i\partial_j. \tag{4.2.9}$$

We decompose h_{ij} into irreducible components as well,

$$\boxed{h_j^i = h^{TT}{}_j^i + \partial^i\epsilon_j + \partial_j\epsilon^i + \frac{\partial^i\partial_j}{\Box}H + \frac{\delta_j^i h}{d-1}.} \tag{4.2.10}$$

Here, $h^{TT}{}_j^i$ is traceless and transversal, and ϵ^i is transversal,

$$h^{TT}{}_i^i = 0, \qquad \partial_i h^{TT}{}_j^i = 0, \qquad \partial_i \epsilon^i = 0. \tag{4.2.11}$$

Thus we find the useful relations

$$\begin{aligned} \partial_i h_j^i &= \Box\epsilon_j + \partial_j\left(H + \frac{h}{d-1}\right) \\ \partial_i\partial^j h_j^i &= \Box\left(H + \frac{h}{d-1}\right) \\ h_k^k &= H + \frac{d}{d-1}h. \end{aligned} \tag{4.2.12}$$

The following fields are all independent and irreducible,

$$\varphi^a, \quad \nu, \quad \nu_L, \quad \nu_T^i, \quad h^{TT}{}_j^i, \quad \epsilon^i, \quad H, \quad h. \tag{4.2.13}$$

Chapter 5

Gauge Invariant Fluctuations

⋄ Due to diffeomorphism invariance (or gauge invariance), the degrees of freedom in eq.(4.2.13) are not all physical degrees of freedom. In this section we isolate the physical fluctuations from the gauge fields, and recombine the fluctuations into new fields that are invariant under diffeomorphisms. For further details, see [2]. ⋄

5.1 Diffeomorphism invariance

Physics is invariant under diffeomorphisms. Consider a diffeomorphism generated by an infinitesimal vector field ξ^μ

$$x^\mu = \exp_{x'}\left[\xi(x')\right]^\mu = x'^\mu + \xi^\mu(x') - \frac{1}{2}\Gamma[\mathbf{g}]^\mu_{\nu\rho}(x')\xi^\nu(x')\xi^\rho(x') + \mathcal{O}\big(\xi^3\big)\,, \tag{5.1.1}$$

which again uses the exponential map. Under this coordinate transformation, the scalar fields transform as $\Phi \to \Phi + \delta\Phi$, with $\delta\Phi$ given by

$$\delta\Phi = \xi^\mu\partial_\mu\Phi + \frac{1}{2}\xi^\mu\xi^\nu\nabla[\mathbf{g}]_\mu\partial_\nu\Phi + \mathcal{O}\big(\xi^3\big)\,. \tag{5.1.2}$$

Note that this expression is covariant, since $\partial_\mu\Phi = \nabla[\mathbf{g}]_\mu\Phi$. The metric transforms as

$$\delta\mathbf{g}_{\mu\nu} = \nabla[\mathbf{g}]_{(\mu}\xi_{\nu)} + \mathcal{O}\big(\xi^2\big)\,. \tag{5.1.3}$$

The metric is thus not gauge invariant, and neither are the connections $\Gamma[\mathbf{g}]^\mu_{\nu\rho}$ derived from it. The important thing is that the gravitational action is invariant, $\delta S = 0$. The situation is analogous to classical electrodynamics. The potential A_μ is not directly measurable, only the field strength tensor $F_{\mu\nu}$. Any variation δA_μ leading to the same field strength is therefore physically equivalent. The requirement $\delta F_{\mu\nu} = 0$ is automatically satisfied by $\delta A_\mu = \partial_\mu\lambda$, for any arbitrary λ. Since A_μ and $A_\mu + \partial_\mu\lambda$ are physically equivalent potentials, λ is an unphysical (gauge) degree of freedom of A_μ. The connections $\Gamma[\mathbf{g}]^\mu_{\nu\rho}$ are the gravitational analogue of the gauge potential A_μ, and ξ^μ

is the analogue of λ. Since μ runs from 1 to $d+1$, ξ^μ encodes $d+1$ unphysical degrees of freedom of the metric.

As we will show below, we can view the fields ϵ^i, H and h as unphysical. The scalars H and h carry one degree of freedom each, and the vector ϵ^i carries $d-1$ degrees of freedom (i runs from 1 to d, but one degree of freedom is killed by the constraint of transversality, $\partial_i \epsilon^i = 0$). In total, ϵ^i, H and h carry all $d+1$ degrees of freedom of ξ^μ.

Gauge invariant fields We combine the seven fields in eq.(4.2.13) into five new fields such that they become invariant under diffeomorphisms (gauge invariant) up to first order, see [2, 5]:

$$\begin{aligned}
\mathfrak{a}^a &= \varphi^a + \frac{\mathcal{W}^a}{4\mathcal{W}} h + \mathcal{O}(f^2) \\
\mathfrak{b} &= \nu + \partial_r \left(\frac{h}{4\mathcal{W}} \right) + \mathcal{O}(f^2) \\
\mathfrak{c} &= e^{-2A} \nu_L + e^{-2A} \frac{\Box h}{4\mathcal{W}} - \frac{1}{2} \partial_r H + \mathcal{O}(f^2) \\
\mathfrak{d}^i &= e^{-2A} \nu^i_T - \partial_r \epsilon^i + \mathcal{O}(f^2) \\
\mathfrak{e}^i_j &= h^{TT}{}^i_j + \mathcal{O}(f^2) \, .
\end{aligned} \tag{5.1.4}$$

Sets of fields Three sets of fields are distinguished,

$$I = \{\mathfrak{a}^a, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}^i, \mathfrak{e}^i_j\}, \qquad Y = \{\varphi^a, \nu, \nu_L, \nu^i_T, h^{TT}{}^i_j\}, \qquad X = \{\epsilon^i, H, h\}. \tag{5.1.5}$$

The fields in I are identified as the physical fields while the fields in X are gauge degrees of freedom. The inverse relations $Y = I + X$ are

$$\begin{aligned}
\varphi^a &= \mathfrak{a}^a - \frac{\mathcal{W}^a}{4\mathcal{W}} h \\
\nu &= \mathfrak{b} - \partial_r \left(\frac{h}{4\mathcal{W}} \right) \\
e^{-2A} \nu^i &= \mathfrak{d}^i + \partial_r \epsilon^i + \frac{\partial^i}{\Box} \left(\mathfrak{c} - e^{-2A} \frac{\Box h}{4\mathcal{W}} + \frac{1}{2} \partial_r H \right) \\
h^{TT}{}^i_j &= \mathfrak{e}^i_j.
\end{aligned} \tag{5.1.6}$$

Linearized equations of motion The linearized equations of motion for the gauge invariant fields that follow from the action are derived in section (4.4) of [2], and read

$$\mathfrak{b} = -\frac{\mathcal{W}_a}{\mathcal{W}} \mathfrak{a}^a, \qquad \mathfrak{c} = \frac{\mathcal{W}_a}{\mathcal{W}} \left(\delta^a_b \mathcal{D}_r - \mathcal{M}^a_b \right) \mathfrak{a}^b, \qquad \mathfrak{d}^i = 0. \tag{5.1.7}$$

$$\left[\left(\delta^a_b \mathcal{D}_r + \mathcal{M}^a_b - \frac{2d}{d-1} \delta^a_b \mathcal{W} \right) \left(\delta^b_c \mathcal{D}_r - \mathcal{M}^b_c \right) + e^{-2A} \delta^a_c \Box \right] \mathfrak{a}^c = 0. \tag{5.1.8}$$

$$\left[\left(\partial_r - \frac{2d}{d-1}\mathcal{W}\right)\partial_r + e^{-2A}\,\Box\right]\mathfrak{e}^i_j = 0. \tag{5.1.9}$$

The mass matrix is symmetric,

$$\boxed{M^a_b \equiv D_b W^a - \frac{W^a W_b}{W}.} \tag{5.1.10}$$

Covariant derivatives The *background covariant derivative* and *sigma-model covariant derivative* are respectively defined by

$$D_r F^a \equiv \frac{d}{dr}F^a + \mathcal{G}^a_{bc}W^b F^c, \qquad D_b F^a \equiv \partial_b F^a + \mathcal{G}^c_{ab}F^c. \tag{5.1.11}$$

On the background we have, see eq.(4.1.6),

$$\mathcal{D}_r F^a(\phi) = \dot{\phi}^b \partial_b F^a + \bar{\mathcal{G}}^a_{bc}\mathcal{W}^b F^c = \mathcal{W}^b\left(\partial_b F^a + \bar{\mathcal{G}}^a_{bc}F^c\right) = \mathcal{W}^b \mathcal{D}_b F^a. \tag{5.1.12}$$

Note that $\mathfrak{e}^i_j$ satisfies the equation of motion of a massless scalar, and that the equations of motion for $\mathfrak{e}^i_j$ and $\mathfrak{a}^a$ are decoupled. These are two advantages of working with gauge invariant variables.

Solutions Like in eq.(4.1.4), the fluctuations have two independent solutions, one of which is dominant and the other one is subdominant at the boundary, see eq.(2.10) in [6]

$$\begin{aligned}\mathfrak{a}^a(A,x) &= \hat{\mathfrak{a}}^a_k(A,\Box)\mathfrak{a}_{\mathfrak{s}k}(x) + \check{\mathfrak{a}}^a_k(A,\Box)\mathfrak{a}_{\mathfrak{r}k}(x)\\ \mathfrak{e}^i_j(A,x) &= \hat{\mathfrak{e}}(A,\Box)\mathfrak{e}^i_{\mathfrak{s}j}(x) + \check{\mathfrak{e}}(A,\Box)\mathfrak{e}^i_{\mathfrak{r}j}(x),\end{aligned} \tag{5.1.13}$$

where $\mathfrak{a}_{\mathfrak{s}k}$ and $\mathfrak{e}^i_{\mathfrak{s}j}$ are the sources of the dominant solutions $\hat{\mathfrak{a}}^a_k$ and $\hat{\mathfrak{e}}$, and $\mathfrak{a}_{\mathfrak{r}k}$ and $\mathfrak{e}^i_{\mathfrak{r}j}$ the responses of the subdominant solutions $\check{\mathfrak{a}}^a_k$ and $\check{\mathfrak{e}}$. Since $\mathfrak{e}^i_j$ is traceless and transversal, so are its source and response functions. Since the indices a and k both run from 1 to n_s, the number of scalars, the most general solution to the scalar equation of motion $\mathfrak{a}^a(A,x)$ is therefore an $n_s \times n_s$ matrix. The equations of motion show that we can expand the dominant solutions as

$$\begin{aligned}\hat{\mathfrak{a}}^a_i(A,\Box) &= \hat{\mathfrak{a}}^a_{0i}(A) + \hat{\mathfrak{a}}^a_{1i}(A)e^{-2A}\,\Box + \hat{\mathfrak{a}}^a_{2i}(A)e^{-4A}\,\Box^2 + \dots\\ \hat{\mathfrak{e}}(A,\Box) &= 1 + \hat{\mathfrak{e}}_1(A)e^{-2A}\,\Box + \hat{\mathfrak{e}}_2(A)e^{-4A}\,\Box^2 + \dots\end{aligned} \tag{5.1.14}$$

The leading terms are independent of $\Box$. We can make a similar expansion for the subdominant solutions,

$$\check{\mathfrak{a}}^a_i(A,\Box) = \check{\mathfrak{a}}^a_{0i}(A) + \check{\mathfrak{a}}^a_{1i}(A)e^{-2A}\,\Box + \check{\mathfrak{a}}^a_{2i}(A)e^{-4A}\,\Box^2 + \dots \tag{5.1.15}$$

Then we can write

$$\mathfrak{a}^a = \mathfrak{a}^a_0 + \mathfrak{a}^a_1 e^{-2A}\,\Box + \mathfrak{a}^a_2 e^{-4A}\,\Box^2 + \dots \tag{5.1.16}$$

with

$$\mathfrak{a}^a_0 = \hat{\mathfrak{a}}^a_{0i}\mathfrak{a}_{\mathfrak{s}i} + \check{\mathfrak{a}}^a_{0i}\mathfrak{a}_{\mathfrak{r}i}, \qquad \mathfrak{a}^a_1 = \hat{\mathfrak{a}}^a_{1i}\mathfrak{a}_{\mathfrak{s}i} + \check{\mathfrak{a}}^a_{1i}\mathfrak{a}_{\mathfrak{r}i}, \qquad \dots \tag{5.1.17}$$

Gauge fields We saw in eq.(3.1.3) that we can obtain field theory one- and two-point functions by taking functional differentiations of the on-shell, bare supergravity action with respect to the sources. The sources of the gauge invariant scalar fluctuations are given by $\mathfrak{a}_{\mathfrak{s}i}(x)$ and the source of the traceless transverse metric fluctuation is given by $\mathfrak{e}^i_{\mathfrak{s}j}(x)$, see eq.(5.1.13).

The gauge fields H, h and ϵ^i do not have equations of motion, and therefore we can not write them in the form eq.(5.1.13). To calculate correlation functions involving operators dual to the gauge fields, we must take the functional differentiation of the on-shell action with respect to the full gauge fields $H(r,x)$, $h(r,x)$ and $\epsilon^i(r,x)$, even though the gauge fields generally depend on r. The way the gauge fields depend on r therefore does not affect the correlation functions. This is as it should, because the way the gauge fields depend on r depends on the choice of gauge. The asymptotic behaviour of the gauge fields can therefore never cause divergences in the correlation functions, in contrast to the asymptotic behaviour of $\mathfrak{a}(r,x)$ and $\mathfrak{e}(r,x)$.

Zero modes In section (4.4) of [2], one starts with the equations of motion for the scalars and the normal component and the mixed components of Einsteins equation, and solves these equations algebraically for $\mathfrak{b}$, $\mathfrak{c}$ and $\mathfrak{d}^i$. However, one assumes that $\mathfrak{c} \neq 0$ in order to derive the equation of motion for $\mathfrak{b}$.

The equation of motion for $\mathfrak{b}$ in eq.(5.1.7) is not valid for zero mode solutions of the scalars $\bar{\mathfrak{a}}^a(r)$. A zero mode means that the solution depends only on r, not on $\Box$. From the scalar equation of motion eq.(5.1.8) we see that a zero mode solution is given by

$$\bar{\mathfrak{a}}^a \sim \frac{\mathcal{W}^a}{\mathcal{W}}, \tag{5.1.18}$$

which follows directly from the identity

$$\mathcal{D}_r \frac{\mathcal{W}^a}{\mathcal{W}} = \mathcal{M}^a_b \frac{\mathcal{W}^b}{\mathcal{W}}. \tag{5.1.19}$$

From this identity also follows that $\mathfrak{c} = 0$ for the zero mode solution $\bar{\mathfrak{a}}^a$, in which case the equation of motion for $\mathfrak{b}$ is altered. When we consider the on-shell action, we will use the equation of motion for $\mathfrak{b}$ given by eq.(5.1.7), so this excludes the zero modes. Finally, we notice that in general we may write

$$\frac{\mathcal{W}^a}{\mathcal{W}} = \hat{c}_i \hat{\mathfrak{a}}^a_{0i} + \check{c}_i \check{\mathfrak{a}}^a_{0i}, \tag{5.1.20}$$

where $\hat{c}_i, \check{c}_i$ are constants and the zero modes $\hat{\mathfrak{a}}^a_{0i}$, $\check{\mathfrak{a}}^a_{0i}$ are the leading terms of the expansions for $\hat{\mathfrak{a}}^a_i$, $\check{\mathfrak{a}}^a_i$ respectively, see eq.(5.1.14).

Chapter 6

Ward Identities

⋄ Dual theories share the same symmetries. Each symmetry has an associated Ward identity. Invariance under diffeomorphisms leads to the translational Ward identity, and invariance under Weyl transformations (only in aAdS) leads to the conformal Ward identity. Checking explicitly that the Ward identities are satisfied provides a good consistency check. When a classical symmetry is broken after quantization, this leads to an anomalous Ward identity. The most important goal of this section is the concept and definition of the Weyl anomaly, also known as conformal anomaly [14]. ⋄

6.1 Translational Ward identity

From invariance under diffeomorphisms, see (4.18) in [4],

$$\delta\gamma_{\mathfrak{s}}^{ij} = -\nabla^{(i}\xi^{j)}, \qquad \delta\Phi_{\mathfrak{s}i} = \xi^j\nabla_j\Phi_{\mathfrak{s}i}, \tag{6.1.1}$$

we derive the *translational Ward identity*

$$\nabla^i \langle T_{ij}\rangle = -\langle \mathcal{O}_i\rangle \nabla_j\Phi_{\mathfrak{s}i}, \tag{6.1.2}$$

where T_{ij} is the energy-momentum tensor of the field theory. Since we know $\langle T_{ij}\rangle$ only up to first order, we can keep the right-hand side to first order as well

$$\nabla^i \langle T_{ij}\rangle = -\langle \mathcal{O}_i\rangle_0 \partial_j\varphi_{\mathfrak{s}i} + \mathcal{O}\big(f^2\big)\,. \tag{6.1.3}$$

6.2 Conformal Ward identity

If the gravitational theory is aAdS, its dual field theory is conformal. Consider the Weyl transformations in a one-scalar system, see eq.(4.20) in [4]

$$\delta\gamma_{\mathfrak{s}}^{ij} = -2\sigma\gamma_{\mathfrak{s}}^{ij}, \qquad \delta\Phi_{\mathfrak{s}} = -(d-\Delta)\sigma\Phi_{\mathfrak{s}}, \tag{6.2.1}$$

where Δ is the conformal dimension of the tree-level operator dual to the scalar, related to the spacetime dimension d of the field theory and the square of the scalar mass $m^2 = V''(0)$ by

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2}. \tag{6.2.2}$$

The invariance under the Weyl transformations leads to the conformal Ward identity

$$\left\langle T_i^i \right\rangle + (d - \Delta)\Phi_{\mathbf{5}} \left\langle \mathcal{O} \right\rangle = \hat{\mathcal{A}}, \tag{6.2.3}$$

where the *rescaled conformal anomaly* is given by

$$\boxed{\hat{\mathcal{A}} \equiv -\lim_{A\to\infty} \frac{1}{\sqrt{\gamma_{\mathbf{5}}}} \frac{\delta S_{\text{cov}}}{\delta A}.} \tag{6.2.4}$$

The variation is with respect to the explicit cut-off, and therefore comes only from the counterterm action. Furthermore, we may split the counterterm action into a part that depends explicitly on the cut-off and a part that does not

$$S_{\text{cov}}(\gamma, \Phi, A) = \bar{S}(\gamma, \Phi) + \tilde{S}(\gamma, \Phi, A), \tag{6.2.5}$$

showing that only $\tilde{S}$ contributes to the anomaly,

$$\hat{\mathcal{A}} = -\lim_{A\to\infty} \frac{1}{\sqrt{\gamma_{\mathbf{5}}}} \frac{\delta \tilde{S}}{\delta A}. \tag{6.2.6}$$

We will see later that $\bar{S}$ kills the power divergences from the bare action, while $\tilde{S}$ kills the logarithmic divergences.

Since we take the limit of the cut-off going to infinity in eq.(6.2.4), the rescaled anomaly is by definition independent of the cut-off. It is useful to introduce the *un-rescaled conformal anomaly* $\mathcal{A}$ by the definition

$$\boxed{\sqrt{\gamma}\mathcal{A} \equiv \sqrt{\gamma_{\mathbf{5}}}\hat{\mathcal{A}}.} \tag{6.2.7}$$

We will refer to both $\hat{\mathcal{A}}$ and $\mathcal{A}$ as "the anomaly" when it is clear from the context which quantity we intend. Since the right-hand side is finite, the left-hand side is finite as well, although both $\sqrt{\gamma}$ and $\mathcal{A}$ depend on the cut-off. In general, $\mathcal{A}$ has a part that does not explicitly depend on the cut-off A and a part that does, see also eq.(6.2.5),

$$\mathcal{A}(\gamma, \Phi, A) = \bar{\mathcal{A}}(\gamma, \Phi) + \tilde{\mathcal{A}}(\gamma, \Phi, A), \tag{6.2.8}$$

where we shall refer to $\bar{\mathcal{A}}$ as the *linear anomaly* and to $\tilde{\mathcal{A}}$ as the *non-linear anomaly*. The linear anomaly only appears in even dimensions d and the non-linear anomaly only appears for scalars dual to operators with conformal dimension $\Delta = d/2$. The anomalous part of the Lagrangian thus reads

$$\tilde{\mathcal{L}} = -A\bar{\mathcal{A}}(\gamma, \Phi) - \int^A dA' \tilde{\mathcal{A}}(\gamma, \Phi, A'). \tag{6.2.9}$$

6.2.1 Anomaly, scheme dependence and finite terms

The anomalous part of the counterterm depends explicitly on the cut-off, and therefore always introduces scheme dependence due to the inherent ambiguity in choosing the cut-off at any arbitrary finite value. Instead of choosing the cut-off at A, we may just as well choose the cut-off at $A + C$, where C is any finite constant, called a *scheme constant*.

The general expression for the anomalous Lagrangian thus reads[1]

$$\tilde{\mathcal{L}} = -\left(A + C_1\right)\bar{\mathcal{A}}(\gamma, \Phi) - \int^{A+C_2} dA' \tilde{\mathcal{A}}(\gamma, \Phi, A'). \tag{6.2.10}$$

Making a Taylor expansion in the second term yields

$$\tilde{\mathcal{L}} = -A\bar{\mathcal{A}}(\gamma, \Phi) - \int^{A} dA' \tilde{\mathcal{A}}(\gamma, \Phi, A') - C_1\bar{\mathcal{A}}(\gamma, \Phi) - C_2\tilde{\mathcal{A}}(\gamma, \Phi, A) + \dots \tag{6.2.11}$$

The first two terms are the divergent parts of the anomalous counterterm, the terms proportional to the scheme constants give the finite contributions, and the terms on the dots go to zero at the boundary. This is consistent with the fact that $\sqrt{\gamma}\mathcal{A} \equiv \sqrt{\gamma_5}\hat{\mathcal{A}}$ is finite.

[1] Both $\bar{\mathcal{A}}$ and $\tilde{\mathcal{A}}$ stand for a collection of anomalous terms. Although it was impossible to indicate here, each anomalous term has its own independent scheme constant in general.

Chapter 7

Holographic Renormalization in aAdS

⋄ In this section we focus on one-scalar systems in aAdS. We respectively discuss the standard method and the Hamiltonian-Jacobi method of holographic renormalization in aAdS. Both methods rely on a Taylor expansion around the aAdS fixed point, and can therefore not be applied to non-aAdS backgrounds. ⋄

7.1 Fixed point

The fake supergravity system is general enough to describe both aAdS and non-aAdS spaces, depending on the choice of the superpotential $\mathcal{W}$. If we choose a potential with a *fixed point*, then the spacetime becomes aAdS. A fixed point means that $\mathcal{W}$ approaches a non-zero constant value at the boundary,

$$\lim_{r\to\infty} \mathcal{W} = -\frac{d-1}{2L}. \tag{7.1.1}$$

We have chosen to write the unknown non-zero constant value in terms of the finite constant L, because now we find from eq.(4.1.6)

$$\lim_{r\to\infty} \dot{A} = \frac{1}{L}, \qquad \lim_{r\to\infty} A = \frac{r}{L} + \text{constant}, \tag{7.1.2}$$

where the constant can be removed by a change of coordinate. Then we see that the metric given by eq.(4.1.5) becomes AdS at the boundary

$$\lim_{r\to\infty} ds^2 = \frac{dr^2}{L^2} + e^{2r/L}\eta_{ij}dx^i dx^j. \tag{7.1.3}$$

Any AdS metric can be brought into this form by a suitable coordinate transformation. An *asymptotically* AdS metric metric is defined as

$$\lim_{r\to\infty} ds^2_{\text{aAdS}} = ds^2_{\text{AdS}}. \tag{7.1.4}$$

From eq.(7.1.3), we see that the constant L represents the characteristic AdS length scale, which is conventionally set to unity, $L = 1$.

7.1.1 Expansion around the fixed point

For simplicity, we will consider one scalar systems only in the rest of this section. In aAdS, the leading behaviour of a scalar field on the background is

$$\phi = e^{-(d-\Delta)r}\phi_{\mathfrak{s}} + \dots \tag{7.1.5}$$

where the background scalar source $\phi_{\mathfrak{s}}$ is a constant. The constant Δ is given by eq.(6.2.2) and can be interpreted as the conformal scaling dimension of the operator dual to the scalar. Operators with $\Delta < d$ are called *relevant*, while operators with $\Delta = d$ are called *marginal*. For scalar fields dual to relevant operators $d > \Delta$, the fixed point is zero,

$$\lim_{r\to\infty} \phi = 0, \tag{7.1.6}$$

while for scalars dual to marginal operators $d = \Delta$ the fixed point is a non-zero constant,

$$\lim_{r\to\infty} \phi = \phi_{\mathfrak{s}}. \tag{7.1.7}$$

From now on we will consider only relevant operators. Then we have, away from the background,

$$\lim_{r\to\infty} \Phi(r, x) = 0, \tag{7.1.8}$$

which is true for any finite but otherwise arbitrary value of the source $\Phi_{\mathfrak{s}}(x)$. In aAdS we can always Taylor expand the potential $V(\Phi)$ and the superpotential $W(\Phi)$ around the fixed point $\Phi = 0$,

$$V(\Phi) = V(0) + \frac{1}{2}V''(0)\Phi^2 + \dots, \qquad W(\Phi) = W(0) + W'(0)\Phi + \frac{1}{2}W''(0)\Phi^2 + \dots \tag{7.1.9}$$

The potential $V(\Phi)$ has no term linear in Φ. The mass of the scalar m can be found in the usual way from the quadratic part of the potential,

$$m^2 = V''(0). \tag{7.1.10}$$

Furthermore, for $L = 1$ we know that $V(0)$ and $W(0)$ are given by, see eq.(7.1.1),

$$V(0) = -\frac{d(d-1)}{4}, \qquad W(0) = -\frac{d-1}{2}. \tag{7.1.11}$$

Thus the expansions of $V(\Phi)$ and $W(\Phi)$ around the fixed point read

$$\begin{aligned} V &= -\frac{d(d-1)}{4} + \frac{1}{2}m^2\Phi^2 + \frac{1}{3!}v_3\Phi^3 + \frac{1}{4!}v_4\Phi^4 + \mathcal{O}\big(\Phi^5\big) \\ W &= -\frac{d-1}{2} + w_1\Phi + \frac{1}{2}w_2\Phi^2 + \frac{1}{3!}w_3\Phi^3 + \frac{1}{4!}w_4\Phi^4 + \mathcal{O}\big(\Phi^5\big)\,. \end{aligned} \tag{7.1.12}$$

We will now determine the coefficients w_1, w_2, ... for one-scalar, fake supergravity systems, which must satisfy eq.(4.1.2),

$$V = \frac{1}{2}\left(W'\right)^2 - \frac{d}{d-1}W^2. \tag{7.1.13}$$

To lowest order in the expansion around the fixed point, this requires $w_1 = 0$. Thus, in fake supergravity neither V nor W has a linear term. At quadratic order in the expansion, eq.(7.1.13) becomes

$$m^2 = w_2^2 + dw_2. \tag{7.1.14}$$

This is a quadratic equation for w_2, so there is a sign amibiguity

$$w_2 = -\frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2}. \tag{7.1.15}$$

However, we have the requirement

$$\dot{\phi} = \mathcal{W}' = w_2\phi, \tag{7.1.16}$$

which has the solution

$$\phi = e^{w_2}\phi_{\mathfrak{s}}, \tag{7.1.17}$$

with some constant background source $\phi_{\mathfrak{s}}$. Since we require ϕ to vanish at the boundary, we require w_2 to be negative. Since the square root is always positive, we must choose the plus sign to ensure that w_2 is always negative, for any d and m,

$$w_2 = -(d - \Delta), \tag{7.1.18}$$

where we have used the definition of Δ from eq.(6.2.2). Our requirement $w_2 < 0$ is only satisfied for $\Delta < d$, and hence we consider only these cases.

From the cubic order, we find

$$w_3 = \frac{v_3}{3\Delta - 2d}, \qquad \text{for} \qquad \Delta \neq \frac{2d}{3}. \tag{7.1.19}$$

If $\Delta = 2d/3$, the relation between V and W that defines a fake sugra system can not be satisfied at the cubic order, *unless* $v_3 = 0$.[1]We conclude that fake supergravity systems in aAdS have the following superpotential

$$W = -\frac{d-1}{2} - \frac{d-\Delta}{2}\Phi^2 + \frac{v_3}{18\Delta - 12d}\Phi^3 + \mathcal{O}\left(\Phi^4\right). \tag{7.1.21}$$

[1] In general, the relation eq.(7.1.13) can not be satisfied (and hence the system is not a fake supergravity system) if there exists some integer $k > 2$ such that

$$\Delta = \frac{k-1}{k}d, \tag{7.1.20}$$

unless the coefficients of the potential v_i coincidentally have some specific values. An example is GPPZ, where $\Delta = 3$ and $d = 4$, so for $k = 4$ we indeed have $\Delta = d(k-1)/k$. However, in GPPZ the values of v_3 and v_4 are such that this is still a fake supergravity system, see eq.(28) in [17], where the GPPZ case is discussed below.

7.2 Standard method

To keep things as simple as possible, we discuss the standard method of holographic renormalization using as an example the Coulomb Branch Flow in a *fixed* background metric. This way we only obtain the scalar counterterms, but this suffices to demonstrate all features of the standard method. In general one has to include the metric in each of the steps.

Given a potential $V(\Phi)$, the standard method of holographic renormalization consists of the following steps:

1. Taylor expand the potential $V(\Phi)$ around the fixed point $\Phi = 0$. The mass term is given by $m^2 = V''(0)$ and the conformal dimension Δ by eq.(6.2.2).

2. Derive the equation of motion for Φ from the bare action using the variational principle.

3. Find the solution for $\Phi(x,r)$ recursively by substituting the ansatz into the equation of motion, each time using a higher order in the expansion. The result will be an expansion in e^{-2r} with leading terms
$$\Phi = \begin{cases} e^{-(d-\Delta)r}\Phi_{\mathfrak{s}} + e^{-\Delta r}\Phi_{\mathfrak{r}} + \dots & \Delta \neq \dfrac{d}{2} \\ e^{-dr/2}\left(r\Phi_{\mathfrak{s}} + \Phi_{\mathfrak{r}}\right) + \dots & \Delta = \dfrac{d}{2}, \end{cases} \tag{7.2.1}$$
where $\Phi_{\mathfrak{s}}(x)$ is the source and $\Phi_{\mathfrak{r}}(x)$ the response. The required order to which we need to keep the terms in the expanded potential depends on d and Δ. For example, a term like
$$\sqrt{\mathbf{g}}\Phi^n \sim e^{[d-n(d-\Delta)]r}$$
goes to zero when
$$n > \frac{d}{d-\Delta}.$$

4. Substitute the solution for Φ into the bare action $S_{\text{bare}}[\Phi]$ to obtain the on-shell bare action $S_{\text{bare}}[\Phi_{\mathfrak{s}}, r]$.

5. Integrate $S_{\text{bare}}[\Phi_{\mathfrak{s}}, r]$ over r from zero until the cut-off r to obtain $S_{\text{bare}}[\Phi_{\mathfrak{s}}, r]$.

6. Isolate the finite number of divergent terms in $S_{\text{bare}}[\Phi_{\mathfrak{s}}, r]$ as $r \to \infty$.

7. Define the *non-covariant* counterterm action $S_{\text{cnt}}[\Phi_{\mathfrak{s}}, r]$ as minus the divergent terms from the bare, on-shell action $S_{\text{bare}}[\Phi_{\mathfrak{s}}, r]$.

8. The *covariant* counterterm $S_{\text{cov}}[\Phi, r]$ is the *unique* covariant on-shell action that contains the non-covariant counterterm $S_{\text{cnt}}[\Phi_{\mathfrak{s}}, r]$ as the *only* divergent terms, plus finite terms $S_{\text{fin}}[\Phi_{\mathfrak{s}}]$. The covariant counterterm $S_{\text{cov}} = S_{\text{cnt}} + S_{\text{fin}}$ can be found by inverting the expansion $\Phi(x,r)$ in terms of $\Phi_{\mathfrak{s}}(x)$; that is by writing $\Phi_{\mathfrak{s}}(x)$ in terms of $\Phi(x,r)$, up to the required order.

Let us now discuss each step in more detail, using as an example the Coulomb Branch Flow (CBF, see section 15), given by the potential

$$V(\Phi) = -3 - 2\Phi^2 + \mathcal{O}\big(\Phi^3\big)\,. \tag{7.2.2}$$

The mass is given by

$$m^2 = V''(0) = -4, \tag{7.2.3}$$

and the conformal dimension of the operator dual to the scalar follows from eq.(6.2.2), $\Delta = 2$. The action, with $V(\Phi)$ expanded as above, yields an equation of motion for the scalar that has the solution

$$\Phi(x,r) = e^{-2r}\left[r\Phi_{\mathfrak{s}}(x) + \Phi_{\mathfrak{r}}(x)\right] + \mathcal{O}\big(e^{-4r}\big)\,, \tag{7.2.4}$$

where the source $\Phi_{\mathfrak{s}}(x)$ and the response $\Phi_{\mathfrak{r}}(x)$ are two independent, unknown functions of x. We need to keep terms in the potential including terms of order Φ^n, where

$$n = \frac{d}{d-\Delta} = 2. \tag{7.2.5}$$

Thus, cubic terms of order $\mathcal{O}(\Phi^3)$ can be neglected. To keep things as simple as possible, we keep the background metric *fixed* so we can use the familiar result that AdS is a maximally symmetric spacetime with constant negative curvature

$$\mathbf{R}[\mathbf{g}] = -d(d-1) = -12, \tag{7.2.6}$$

where we used $d = 4$. The action thus reads[2]

$$S_{\text{bare}}[\Phi] = \int dr d^4x \sqrt{\mathbf{g}}\left(\frac{1}{2}\mathbf{g}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - 2\Phi^2 + \mathcal{O}\big(\Phi^3\big)\right). \tag{7.2.9}$$

Since $\sqrt{\mathbf{g}} = e^{4r}$, $\gamma^{ij} = e^{-2r}\eta^{ij}$, and $\Phi \sim re^{-2r}$, the only contributions to the on-shell action are

$$S_{\text{bare}}[\Phi] = \int dr d^4x e^{4r}\left(\frac{1}{2}\left(\partial_r\Phi\right)^2 - 2\Phi^2 + \dots\right). \tag{7.2.10}$$

Substituting the solution for Φ yields the on-shell, bare action

$$S_{\text{bare}}[\Phi_{\mathfrak{s}}, r] = \int dr d^4x\left(-2r\Phi_{\mathfrak{s}}{}^2 + \frac{1}{2}\Phi_{\mathfrak{s}}{}^2 - 2\Phi_{\mathfrak{s}}\Phi_{\mathfrak{r}}\right). \tag{7.2.11}$$

[2]Notice that the leading (constant) in the potential $V(0) = -3$ cancelled against the Ricci scalar

$$-\frac{1}{4}\mathbf{R}[\mathbf{g}] + V(0) = 0. \tag{7.2.7}$$

This is no coincidence, since in general we have, in aAdS

$$V(0) = -\frac{d(d-1)}{4L^2}, \qquad \mathbf{R}[\mathbf{g}] = -\frac{d(d-1)}{L^2} = 4V(0). \tag{7.2.8}$$

Now we perform the integration over r from zero to the cut-off r

$$S_{\text{bare}}[\Phi_{\mathfrak{s}}, r] = \int d^4x \left(-r^2 {\Phi_{\mathfrak{s}}}^2 + \frac{1}{2} r {\Phi_{\mathfrak{s}}}^2 - 2r \Phi_{\mathfrak{s}} \Phi_{\mathfrak{r}} \right). \tag{7.2.12}$$

Notice that the on-shell bare action has only divergent terms; there are no finite contributions. We define the non-covariant counterterm as minus the divergent terms coming from the bare action

$$S_{\text{cnt}}[\Phi_{\mathfrak{s}}, r] = \int d^4x \left(r^2 {\Phi_{\mathfrak{s}}}^2 - \frac{1}{2} r {\Phi_{\mathfrak{s}}}^2 + 2r \Phi_{\mathfrak{s}} \Phi_{\mathfrak{r}} \right). \tag{7.2.13}$$

Now there is a unique covariant counterterm that contains these divergent terms plus finite terms

$$S_{\text{cov}}[\Phi, r] = \int d^4x \sqrt{\gamma} \left(1 - \frac{1}{2r} \right) \Phi^2, \tag{7.2.14}$$

where the scalars are evaluated at the cut-off, $\Phi(x, r)$. The uniqueness is actually only true up to scheme dependence. Since we could have chosen the cut-off at $r + C$ for any finite constant C, we find the general counterterm, including the scheme dependence

$$S_{\text{cov}}[\Phi, r{+}C] = \int d^4x \sqrt{\gamma} \left(1 - \frac{1}{2\,(r + C)} \right) \Phi^2 = \int d^4x \sqrt{\gamma} \left(1 - \frac{1}{2r} + \frac{C}{2r^2} + \mathcal{O}\big(r^{-3}\big) \right) \Phi^2, \tag{7.2.15}$$

where the scalars are evaluated at the shifted cut-off $\Phi(x, r + C)$. The covariant counterterm can be split into the required non-covariant counterterm plus a non-covariant finite part

$$S_{\text{cov}}[\Phi, r + C] \; = \; S_{\text{cnt}}[\Phi_{\mathfrak{s}}, r] + S_{\text{fin}}[\Phi_{\mathfrak{s}}, C], \tag{7.2.16}$$

where the finite part is given by

$$S_{\text{fin}}[\Phi_{\mathfrak{s}}, C] = \int d^4x \left({\Phi_{\mathfrak{r}}}^2 - \Phi_{\mathfrak{s}} \Phi_{\mathfrak{r}} + \frac{C}{2} {\Phi_{\mathfrak{s}}}^2 \right). \tag{7.2.17}$$

By construction, the divergences from the counterterm cancel against those coming from the bare action, but there are remaining finite terms. The renormalized, on-shell action is explicitly finite

$$S_{\text{ren}} = S_{\text{bare}} + S_{\text{cov}} = S_{\text{fin}}. \tag{7.2.18}$$

We can now calculate the exact one-point function as follows

$$\langle \mathcal{O} \rangle = -\frac{1}{\sqrt{\gamma_{\mathfrak{s}}}} \frac{\delta S_{\text{ren}}}{\delta \Phi_{\mathfrak{s}}} = -\frac{\delta S_{\text{fin}}}{\delta \Phi_{\mathfrak{s}}} = \Phi_{\mathfrak{r}} - C \Phi_{\mathfrak{s}}. \tag{7.2.19}$$

The vev is what remains after we put the source to zero, $\langle \mathcal{O} \rangle_0 = \Phi_{\mathfrak{r}}$. The vev is non-vanishing independently of the scheme. This shows that the CBF is a vev flow instead of an operator flow.

In this case, the contributions to the one-point function come exclusively from the counterterms, but this is not always the case. In general, the on-shell bare action has a finite part as well. In any case the procedure above demonstrates the importance of using covariant counterterms: the difference between the covariant counterterm S_{cov} and the non-covariant one S_{cnt} is precisely the finite part S_{fin} that contributes to the correlators. Furthermore, covariance of the counterterms ensures that the translational Ward identity is satisfied. Notice that the standard method relies on the Taylor expansion of the potential $V(\Phi)$ around the fixed point. This restricts the procedure to aAdS, because in non-aAdS there is no fixed point.

7.3 Hamilton-Jacobi method

The starting point of the Hamilton-Jacobi method is similar to Covariant Holographic Renormalization, as we will discuss later in section 11. One writes down a general set of counterterms, where each term is multiplied by some unknown function of the scalar. Then one determines these functions by requiring that the Hamiltonian constraint is satisfied. This constraint yields a *descent equation* at each order in $\Box$. When the descent equation can not be resolved completely, the unresolved remainder contributes to the linear anomaly. For further details, we refer to the original paper [17].

We recall from eq.(9.1.1) that the regularized action is defined as

$$S_{\text{reg}} = S_{\text{bare}} + S_{\text{cov}}, \tag{7.3.1}$$

where the Lagrangian of the covariant counterterm can be split as follows, see eq.(6.2.5),

$$\mathcal{L}_{\text{cov}}(\gamma, \Phi, A) = \bar{\mathcal{L}}(\gamma, \Phi) + \tilde{\mathcal{L}}(\gamma, \Phi, A). \tag{7.3.2}$$

Now we make a further split

$$\bar{\mathcal{L}}(\gamma, \Phi) = \bar{\mathcal{L}}_{[0]}(\gamma, \Phi) + \bar{\mathcal{L}}_{[2]}(\gamma, \Phi) + \ldots \tag{7.3.3}$$

In the Hamilton-Jacobi method, one assumes that the counterterms are covariant and that $\bar{S}_{[2k]}$ contains k inverse metrics. Then

$$\begin{aligned} \bar{\mathcal{L}}_{[0]} &= C_0(\Phi) \\ \bar{\mathcal{L}}_{[2]} &= \frac{1}{2} C_1(\Phi) \left(\nabla\Phi\right)^2 + C_2(\Phi) R \\ \bar{\mathcal{L}}_{[4]} &= \ldots \\ \tilde{\mathcal{L}} &= -A\bar{\mathcal{A}}(\gamma, \Phi) - \int^A dA' \tilde{\mathcal{A}}(\gamma, \Phi, A'). \end{aligned} \tag{7.3.4}$$

The functions $C_0(\Phi), C_1(\Phi), C_2(\Phi), \ldots$ are unknown functions of the scalar Φ. The Hamiltonian constraint yields a number of descent equations that determines the unknown functions and the anomaly, and therefore the complete covariant counterterms.

The unknown functions are determined by the parts of the descent equations that can be resolved, while the linear anomaly $\bar{\mathcal{A}}$ is identified with the remaining part of the descent equations that can not be resolved (the non-linear anomaly $\tilde{\mathcal{A}}$ has to be derived separately). There are remaining parts at each level of the descent equations, so there are contributions to the linear anomaly at each level,

$$\bar{\mathcal{A}} = \bar{\mathcal{A}}_{[0]} + \bar{\mathcal{A}}_{[2]} + \bar{\mathcal{A}}_{[4]} + \dots \tag{7.3.5}$$

Level 0 descent equation The level zero descent equation reads

$$V + \frac{1}{2}\left(C_0'\right)^2 - \frac{d}{d-1}C_0^2 = 0. \tag{7.3.6}$$

We remarked in eq.(4.1.2) that in fake supergravity the potential $V(\Phi)$ can be derived from a superpotential $W(\Phi)$ as follows

$$V = \frac{1}{2}\left(W'\right)^2 - \frac{d}{d-1}W^2. \tag{7.3.7}$$

Hence, the level 0 descent equation is automatically satisfied for $C_0 = -W$. Since this solves the level 0 descent equation completely, there is no remainder and hence no contribution to the linear anomaly at this level. This means $\bar{\mathcal{A}}_{[0]} = 0$ in fake supergravity systems.

Level 2 descent equation The level 2 descent equation reads:

$$\begin{aligned} 0 \;=\; & \left(-\frac{d-2}{d-1}C_0C_1 + 4C_0C_2'' - \frac{1}{2}C_0'C_1' - \frac{1}{2}\right)\left(\nabla\Phi\right)^2 + \\ & + \left(4C_0C_2' - C_0'C_1\right)\nabla^2\Phi + \left(-2\frac{d-2}{d-1}C_0C_2 + C_0'C_2' + \frac{1}{4}\right)R. \end{aligned} \tag{7.3.8}$$

We assume that the counterterms are *universal*, meaning they work for *any* aAdS system. Then, functionally independent terms must vanish separately, leading to the following three descent equations

$$\begin{aligned} 0 &= -\frac{d-2}{d-1}C_0C_1 + 4C_0C_2'' - \frac{1}{2}C_0'C_1' - \frac{1}{2} \\ 0 &= 4C_0C_2' - C_0'C_1 \\ 0 &= -2\frac{d-2}{d-1}C_0C_2 + C_0'C_2' + \frac{1}{4}. \end{aligned} \tag{7.3.9}$$

$$C_2 = c_0 + c_1\Phi + \frac{1}{2}c_2\Phi^2 + \mathcal{O}\left(\Phi^3\right). \tag{7.3.10}$$

$$\begin{aligned}
c_0 &= \frac{1}{4(d-2)}, \qquad \text{for} \qquad d \neq 2 \\
c_1 &= 0 \\
c_2 &= \frac{(d-2)(d-\Delta)}{(d-1)(\Delta - d/2 - 1)} c_0, \qquad \text{for} \qquad \Delta \neq \frac{d}{2}+1, \qquad d \neq 1 \\
C_1 &= -\frac{1}{2(\Delta - d/2 - 1)} + \mathcal{O}(\Phi)\,, \qquad \text{for} \qquad \Delta \neq \frac{d}{2}+1.
\end{aligned} \tag{7.3.11}$$

The contributions to the anomaly are those terms of the descent equations that remain unresolved

$$\begin{aligned}
\bar{\mathcal{A}}_{[2]} &= \frac{1}{2}(\nabla\Phi)^2 + \frac{d-2}{8(d-1)}\Phi^2 R, \qquad \text{for} \qquad \Delta = \frac{d}{2}+1, \\
\bar{\mathcal{A}}_{[2]} &= -\frac{1}{4}R, \qquad \text{for} \qquad d = 2 \\
\bar{\mathcal{A}}_{[4]} &= \frac{1}{16}\left(\frac{1}{3}R^2 - R^{ij}R_{ij}\right), \qquad \text{for} \qquad d = 4.
\end{aligned} \tag{7.3.12}$$

We refer to section 3.3 of [17] for a derivation of the non-linear anomaly in the Hamilton-Jacobi method. We simply present the result,

$$\tilde{\mathcal{A}} = \begin{cases} -\dfrac{1}{2}\left(\dfrac{\Phi}{A}\right)^2 & \Delta = \dfrac{d}{2} \\ 0 & \Delta \neq \dfrac{d}{2}, \end{cases} \tag{7.3.13}$$

and we notice that the non-linear anomaly is indeed finite

$$\sqrt{\gamma}\tilde{\mathcal{A}} = -\frac{1}{2}\sqrt{\gamma}\left(\frac{\Phi}{A}\right)^2 \sim 1, \qquad \text{for} \qquad \Delta = \frac{d}{2}, \tag{7.3.14}$$

which follows because

$$\Phi = e^{-dr/2}\left(r\Phi_{\mathfrak{s}} + \Phi_{\mathfrak{r}}\right), \qquad \sqrt{\gamma}\Phi^2 \sim r^2, \qquad \text{for} \qquad \Delta = \frac{d}{2}. \tag{7.3.15}$$

The anomalous part of the counterterm is thus given by, see eq.(6.2.9),

$$\tilde{\mathcal{L}} = \begin{cases} -A\bar{\mathcal{A}} - \dfrac{\Phi^2}{2A} & \Delta = \dfrac{d}{2} \\ -A\bar{\mathcal{A}} & \Delta \neq \dfrac{d}{2}. \end{cases} \tag{7.3.16}$$

Summary We can summarize the results of the Hamilton-Jacobi method as follows

$$\mathcal{L}_{\text{cov}}(\gamma, \Phi, A) = \bar{\mathcal{L}}(\gamma, \Phi) + \tilde{\mathcal{L}}(\gamma, \Phi, A), \tag{7.3.17}$$

where

$$\begin{aligned}\bar{\mathcal{L}} &= -W(\Phi) - \frac{(\nabla\Phi)^2}{4(\Delta - d/2 - 1)} + \\ &+ \left(\frac{1}{4(d-2)} - \frac{d-\Delta}{8(d-1)(\Delta - d/2 - 1)}\Phi^2\right) R + \ldots \end{aligned} \tag{7.3.18}$$

The last two terms are not present for $d = 2$ and $\Delta = d/2 + 1$. The anomalous part of the counterterm is given by eq.(7.3.16), and the linear anomaly is given by

$$\bar{\mathcal{A}} = \bar{\mathcal{A}}_{[0]} + \bar{\mathcal{A}}_{[2]} + \bar{\mathcal{A}}_{[4]} + \ldots, \tag{7.3.19}$$

where $\bar{\mathcal{A}}_{[0]} = 0$ for fake sugra systems, and

$$\begin{aligned}\bar{\mathcal{A}}_{[2]} &= \begin{cases} \frac{1}{2}(\nabla\Phi)^2 + \frac{d-2}{8(d-1)}\Phi^2 R & \Delta = \frac{d}{2} + 1, \\ -\frac{1}{4}R & d = 2 \end{cases} \\ \bar{\mathcal{A}}_{[4]} &= \frac{1}{16}\left(\frac{1}{3}R^2 - R^{ij}R_{ij}\right), \qquad \text{for} \qquad d = 4. \end{aligned} \tag{7.3.20}$$

7.3.1 Case studies: GPPZ and CBF

Example: GPPZ The GPPZ flow ($d = 4$) has the following potential

$$V(\Phi) = -3 - \frac{3}{2}\Phi^2 - \frac{1}{3}\Phi^4 + \mathcal{O}(\Phi^6). \tag{7.3.21}$$

From this we read off

$$m^2 = -3, \qquad \Delta = 3, \qquad v_3 = 0. \tag{7.3.22}$$

Then we find from eq.(7.1.21) that the GPPZ is a fake supergravity system with superpotential

$$W(\Phi) = -\frac{3}{2} - \frac{1}{2}\Phi^2 - \frac{1}{18}\Phi^4 + \mathcal{O}(\Phi^6). \tag{7.3.23}$$

The counterterm is given by $S_{\text{cov}} = \bar{S} + \tilde{S}$, where

$$\bar{\mathcal{L}} = -W(\Phi) + \frac{1}{8}R, \qquad \tilde{\mathcal{L}} = -A\bar{\mathcal{A}}, \tag{7.3.24}$$

and $\bar{\mathcal{A}} = \bar{\mathcal{A}}_{[0]} + \bar{\mathcal{A}}_{[2]} + \bar{\mathcal{A}}_{[4]}$, where

$$\bar{\mathcal{A}}_{[0]} = 0, \qquad \bar{\mathcal{A}}_{[2]} = \frac{1}{2}(\nabla\Phi)^2 + \frac{1}{12}\Phi^2 R, \qquad \bar{\mathcal{A}}_{[4]} = \frac{1}{16}\left(\frac{1}{3}R^2 - R^{ij}R_{ij}\right). \tag{7.3.25}$$

The full counterterm (excluding the finite, scheme dependent terms) thus reads

$$\begin{aligned}\mathcal{L}_{\text{cov}} &= -W(\Phi) + \frac{1}{8}R + \\ &+ A\left[-\frac{1}{2}(\nabla\Phi)^2 - \frac{1}{12}\Phi^2 R + \frac{1}{16}\left(R^{ij}R_{ij} - \frac{1}{3}R^2\right)\right]. \end{aligned} \tag{7.3.26}$$

Example: CBF The CBF flow ($d = 4$) has the following potential

$$V(\Phi) = -3 - 2\Phi^2 + \frac{4}{3\sqrt{6}}\Phi^3 + \mathcal{O}\big(\Phi^4\big)\,. \tag{7.3.27}$$

From this we read off

$$m^2 = -4, \qquad \Delta = 2, \qquad v_3 = \frac{4\sqrt{6}}{3}. \tag{7.3.28}$$

Then we find from eq.(7.1.21) that the GPPZ is a fake supergravity system with superpotential

$$W = -\frac{3}{2} - \Phi^2 - \frac{\sqrt{6}}{9}\Phi^3 + \mathcal{O}\big(\Phi^4\big)\,, \tag{7.3.29}$$

The counterterm is given by $S_{\text{cov}} = \bar{S} + \tilde{S}$, where

$$\begin{aligned} \bar{\mathcal{L}} &= -W(\Phi) + \frac{1}{4}\,(\nabla\Phi)^2 + \left(\frac{1}{8} + \frac{1}{12}\Phi^2\right) R, \\ \tilde{\mathcal{L}} &= -A\bar{\mathcal{A}} - \frac{\Phi^2}{2A}, \end{aligned} \tag{7.3.30}$$

and $\bar{\mathcal{A}} = \bar{\mathcal{A}}_{[0]} + \bar{\mathcal{A}}_{[2]} + \bar{\mathcal{A}}_{[4]}$, where

$$\bar{\mathcal{A}}_{[0]} = 0, \qquad \bar{\mathcal{A}}_{[2]} = 0, \qquad \bar{\mathcal{A}}_{[4]} = \frac{1}{16}\left(\frac{1}{3}R^2 - R^{ij}R_{ij}\right). \tag{7.3.31}$$

Putting everything together, we find

$$\mathcal{L}_{\text{cov}} \;=\; -W(\Phi) + \frac{1}{8}R + \frac{1}{16}A\left(R^{ij}R_{ij} - \frac{1}{3}R^2\right) - \frac{\Phi^2}{2A}. \tag{7.3.32}$$

Expanding the superpotential $W(\Phi)$ up to quadratic order yields

$$\mathcal{L}_{\text{cov}} \;=\; \frac{3}{2} + \frac{1}{8}R + \frac{1}{16}A\left(R^{ij}R_{ij} - \frac{1}{3}R^2\right) + \left(1 - \frac{1}{2A}\right)\Phi^2. \tag{7.3.33}$$

The last term is exactly what we found earlier in eq.(7.2.15) using the standard method and keeping the background fixed. The preceding terms are the gravitational counterterms.

Part II

Developments

Chapter 8

Divergences of the Bare Action

◇ In [6] a counterterm is presented that renormalizes the gauge invariant scalar fluctuations $\mathfrak{a}^a$. However, the counterterm is not covariant. Here we want to renormalize the full system, including the metric fluctuations $\mathfrak{e}^i_j$ and even the unphysical fluctuations H, h, ϵ, using boundary covariant counterterms. The first step towards this goal is to write down the divergences of the bare action in these gauge invariant variables. ◇

8.1 Bare Action Redefined

We find it convenient to *redefine* the fake supergravity action as

$$\begin{aligned} S_{\text{bare}} &= \int dr d^d x \sqrt{\mathbf{g}} \left(-\frac{1}{4}\mathbf{R}[\mathbf{g}] + \frac{1}{2}\mathbf{g}^{\mu\nu} G_{ab}(\Phi)\partial_\mu \Phi^a \partial_\nu \Phi^b + V(\Phi) \right) + \\ &\quad + \int d^d x \sqrt{\gamma} \left(\frac{1}{2}\mathcal{K} - W \right), \end{aligned} \tag{8.1.1}$$

where we have only added a boundary counterterm proportional to $-W(\Phi)$ to the original definition in eq.(4.1.1). This has the advantage that the expression of the bare on-shell action in terms of the gauge invariant fluctuations simplifies, as will become clear below. By Stokes' theorem, we can rewrite the Gibbons-Hawking boundary term as a bulk term,

$$\begin{aligned} S_{\text{bare}} &= \int dr d^d x n \sqrt{\gamma} \left(-\frac{1}{4}\left(R + \mathcal{K}^2 - \mathcal{K}^i_j \mathcal{K}^j_i\right) + \frac{1}{2}\mathbf{g}^{\mu\nu}\partial_\mu \Phi^a \partial_\nu \Phi_a + V(\Phi) \right) + \\ &\quad - \int d^d x \sqrt{\gamma} W(\Phi). \end{aligned} \tag{8.1.2}$$

The second fundamental form is given by

$$\mathcal{K}_{ij} = \frac{1}{2n}\left(\nabla_{(i} n_{j)} - \partial_r \gamma_{ij}\right), \tag{8.1.3}$$

where the boundary covariant derivative uses the boundary connection

$$\nabla_i n_j \equiv \partial_i n_j - \Gamma^k_{ij} n_k. \tag{8.1.4}$$

For more details on the geometry of hypersurfaces see [2]. The gravitational part of the action is now given in terms of hypersurface quantities only. We will calculate the on-shell, bare action following the steps below.

1. Vary the supergravity action with respect to Φ^a and γ_{ij},[1]
2. Split the varied action into bulk and boundary terms, and ignore the bulk terms,
3. Expand the boundary part up to second order in the fluctuations φ^a, ν, ν_L, ν^i_T, h_{ij} using eq.(4.2.1) and eq.(4.2.7),
4. Decompose h_{ij} into the irreducible components $h^{TT}{}^i_j$, ϵ^i, H and h using eq.(4.2.10),
5. Eliminate the variables φ^a, ν, ν_L, ν^i_T, $h^{TT}{}^i_j$ in favor of the gauge invariant fluctuations $\mathfrak{a}^a$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}^i$, $\mathfrak{e}^i_j$ and the gauge fields ϵ^i, H, h, using eq.(5.1.6),
6. Substitute the equations of motion for $\mathfrak{b}$, $\mathfrak{c}$ and $\mathfrak{d}^i$ from eq.(5.1.7) to render the action on-shell.

8.2 Scalar variation

Variation of S_{bare} with respect to Φ^a yields the following boundary term

$$(\delta_\Phi S_{\text{bare}})_\partial = \int d^d x \sqrt{\gamma}\, (N^\mu G_{ab} \partial_\mu \Phi^a - W_b)\, \delta\Phi^b, \tag{8.2.1}$$

where the subscript ∂ denotes the restriction to boundary terms and N^μ is the inverse of the vector N_μ normal to the boundary

$$N_\mu = (n, 0), \qquad N^\mu = \frac{1}{n}(1, -n^i). \tag{8.2.2}$$

Expanding eq.(8.2.1) up to quadratic order in the fluctuations yields[2]

$$(\delta_\Phi S_{\text{bare}})_\partial = e^{dA} \bar{G}_{ab} \int d^d x\, ((\mathcal{D}_r \varphi^a) - \nu \mathcal{W}^a - \varphi^c D_c \mathcal{W}^a)\, \delta\varphi^b. \tag{8.2.3}$$

[1] Variations with respect to n and n^i do not yield any boundary terms, and therefore do not contribute to the on-shell action.

[2] Thanks to the boundary term that we added to the definition of the bare action (the integral over $-W$), there are no zeroth order terms within the brackets. This would have created a problem, since after the substitution to gauge invariant variables eq.(5.1.6), the a zeroth order term would oblige us to use $\delta\mathfrak{a}^b$ up to second order, and the second order terms contain derivatives with respect to r, while the first order terms do not, see eq.(5.1.4).

Now we switch to the gauge invariant variables using eq.(5.1.6),

$$(\delta_\Phi S_{\text{bare}})_\partial = e^{dA}\bar{G}_{ab}\int d^dx\,\left((\mathcal{D}_r\mathfrak{a}^a) - \mathfrak{a}^c D_c\mathcal{W}^a - \mathfrak{b}\mathcal{W}^a\right)\left(\delta\mathfrak{a}^b - \frac{\mathcal{W}^b}{4\mathcal{W}}\delta h\right). \quad (8.2.4)$$

Note that the gauge field h has cancelled; we have not set this field to zero by hand. Finally, we put the action on-shell by using the equation of motion for $\mathfrak{b}$ from eq.(5.1.7) up to first order. Then we find

$$\delta_\Phi S_{\text{bare}} = e^{dA}\bar{G}_{ab}\int d^dx\,\left((\mathcal{D}_r\mathfrak{a}^a) - \mathcal{M}^a_c\mathfrak{a}^c\right)\left(\delta\mathfrak{a}^b - \frac{\mathcal{W}^b}{4\mathcal{W}}\delta h\right), \quad (8.2.5)$$

where $\mathcal{M}^a_b$ is defined in eq.(5.1.10). After we put the action on-shell, we may drop the subscript ∂ because the bulk terms vanish on-shell by definition.

8.3 Metric variation

The variation of the action with respect to the boundary metric yields the following boundary term

$$(\delta_\gamma S_{\text{bare}})_\partial = \frac{1}{8}\int d^dx\sqrt{\gamma}n^{-1}\left(\gamma^{ik}\gamma^{jm} - \gamma^{jk}\gamma^{im}\right)\left(\nabla_{(k}n_{i)} - (\partial_r\gamma_{ki}) - \frac{1}{2}W\gamma^{mj}\right)\delta\gamma_{mj}. \quad (8.3.1)$$

Expanding this up to quadratic order in the fluctuations yields

$$\begin{aligned}(\delta_\gamma S_{\text{bare}})_\partial &= \frac{e^{dA}}{8}\int d^dx\Big[\left(2e^{-2A}\partial_k\nu^k - 4\nu\mathcal{W} - 4\varphi^a\mathcal{W}_a - \left(\partial_r h^k_k\right)\right)\delta h^m_m + \\ &\quad + \left(\left(\partial_r h^i_j\right) - 2e^{-2A}\partial_j\nu^i\right)\delta h^j_i\Big]. \end{aligned} \quad (8.3.2)$$

Again, the zeroth order terms multiplying the variations have cancelled thanks to the boundary term. Now we substitute the irreducible components for h^i_j from eq.(4.2.10) and the gauge invariant variables from eq.(5.1.6),

$$\begin{aligned}(\delta_\gamma S_{\text{bare}})_\partial &= \frac{e^{dA}}{4}\int d^dx\Bigg[\frac{1}{2}\left(\partial_r\mathfrak{e}^i_j\right)\delta\mathfrak{e}^j_i - \left(2\mathcal{W}\mathfrak{b} + 2\mathcal{W}_a\mathfrak{a}^a + \partial_i\mathfrak{d}^i\right)\delta H + \\ &\quad - \left(\frac{2d}{d-1}\left(\mathcal{W}\mathfrak{b} + \mathcal{W}_a\mathfrak{a}^a\right) + \frac{\partial_i\mathfrak{d}^i}{d-1} - \mathfrak{c} + e^{-2A}\frac{\Box h}{4\mathcal{W}}\right)\delta h\Bigg]. \end{aligned} \quad (8.3.3)$$

The contribution with ϵ^i has disappeared after partial integration. We need the action on-shell, so we substitute the equations of motion for $\mathfrak{b}$ and $\mathfrak{d}^i$ from eq.(5.1.7),

$$\begin{aligned}\delta_\gamma S_{\text{bare}} &= \frac{e^{dA}}{4}\int d^dx\Bigg[\frac{1}{2}\left(\partial_r\mathfrak{e}^i_j\right)\delta\mathfrak{e}^j_i + \\ &\quad + \frac{1}{\mathcal{W}}\left(\mathcal{W}_a\left(\mathcal{D}_r\mathfrak{a}^a\right) - \mathcal{W}_a\mathcal{M}^a_b\mathfrak{a}^b - \frac{e^{-2A}}{4}\Box h\right)\delta h\Bigg]. \end{aligned} \quad (8.3.4)$$

8.4 Total variation

Adding the scalar variation eq.(8.2.5) to the metric variation eq.(8.3.4) yields

$$\delta S_{\text{bare}} = e^{dA} \int d^d x \left[\left(\mathcal{D}_r \mathfrak{a}^a - \mathcal{M}^a_b \mathfrak{a}^b \right) \delta \mathfrak{a}_a + \frac{1}{8} \left(\partial_r \mathfrak{e}^i_j \right) \delta \mathfrak{e}^j_i - \frac{e^{-2A}}{16\mathcal{W}} \left(\Box\, h \right) \delta h \right]. \tag{8.4.1}$$

Then we find the following expression for the on-shell, bare action up to quadratic order

$$\boxed{S_{\text{bare}} = \frac{e^{dA}}{2} \int d^d x \left(\mathfrak{a}^a \left(\bar{G}_{ab} \mathcal{D}_r - \mathcal{M}_{ab} \right) \mathfrak{a}^b + \frac{1}{8} \mathfrak{e}^i_j \partial_r \mathfrak{e}^j_i - \frac{e^{-2A}}{16\mathcal{W}} h \,\Box\, h \right).} \tag{8.4.2}$$

In general, each of these three terms diverges. These divergences must be cancelled by three corresponding non-covariant counterterms, which we shall now construct.

Chapter 9

Non-Covariant Counterterms

⋄ We construct non-covariant counterterms that cancel the divergences from the bare action. Since the on-shell, bare action eq.(8.4.2) only has quadratic terms, so do the non-covariant counterterms, allowing us to only calculate the two-point functions, not the vevs. We show that the two-point functions are finite and discuss scheme dependence. ⋄

9.1 Regularization and renormalization

The on-shell bare action $S_{\text{bare}}\left[\gamma_{ij}, \Phi^a\right]$ is a functional of the full boundary metric (including fluctuations) γ_{ij} and the full scalar fields (including fluctuations) Φ^a. We saw in eq.(8.4.2) that the on-shell, bare action contains divergences in the limit of $A \to \infty$. To regularize it, we assume that A has been cut-off to some finite but arbitrary constant value, for which we use the same symbol A since the distinction between A as a variable and as a fixed cut-off should be clear from the context. On the cut-off boundary, we are going to add a covariant counterterm action S_{cov} that are functionals of $\gamma_{ij}(A, x)$ and $\Phi^a(A, x)$, but may also depend *explicitly* on the cut-off A. The *regularized action* is defined as the sum of the bare action and the counterterms, evaluated at the cut-off boundary,

$$S_{\text{reg}}\left[\gamma_{ij}, \Phi^a, A\right] \equiv S_{\text{bare}}\left[\gamma_{ij}, \Phi^a\right] + S_{\text{cov}}\left[\gamma_{ij}, \Phi^a, A\right]. \tag{9.1.1}$$

The job of the counterterms is to kill the divergences of the bare action, such that the regularized action remains finite in the limit where the cut-off goes to infinity. This limit is by definition the *renormalized action*,

$$S_{\text{ren}}\left[\gamma_{\mathfrak{s}ij}(x), \gamma_{\mathfrak{r}ij}(x), \Phi_{\mathfrak{s}i}(x), \Phi_{\mathfrak{r}i}(x)\right] \equiv \lim_{A\to\infty} S_{\text{reg}}\left[\gamma_{ij}(A, x), \Phi^a(A, x), A\right], \tag{9.1.2}$$

where the sources $\gamma_{\mathfrak{s}ij}(x), \Phi_{\mathfrak{s}i}(x)$ and responses $\gamma_{\mathfrak{r}ij}(x), \Phi_{\mathfrak{r}i}(x)$ of the boundary metric and the scalars depend only on the boundary coordinates x. By construction, the renormalized action is independent of the cut-off,

$$\frac{dS_{\text{ren}}}{dA} = 0. \tag{9.1.3}$$

Since we are working to quadratic order in the fluctuations, we must also expand the counterterm action up to quadratic order in the fluctuations. Later in section 11 we will see that the expansion of the covariant counterterm action leads to two different pieces,

$$S_{\text{cov}} = S_{\text{cnt}} + S_{\text{fin}}, \tag{9.1.4}$$

where S_{cnt} kills the divergences from the bare action and S_{fin} must be finite by itself because the divergences from the bare action are already cancelled by S_{cnt}. Neither S_{cnt} nor S_{fin} is covariant by itself, but their sum is. Before we look for the covariant form of the counterterms, let us first construct the part S_{cnt} that cancels the divergences from the bare action.

9.2 Counterterms

The counterterm that renormalizes the scalar fluctuations $\mathfrak{a}^a$ is given by eq.(3.1) in [6]:

$$\frac{e^{dA}}{2}\int d^dx \mathfrak{a}^a \mathcal{U}_{ab}\mathfrak{a}^b,$$

where the $n_s \times n_s$ matrix $\mathcal{U}_{ab}$ (evaluated on the background) is given by eq.(3.3) in [6]:

$$\boxed{\mathcal{U}_{ab}(A,\phi) \equiv \mathcal{M}_{ab} - \frac{1}{2}\hat{\mathfrak{a}}^{-1}_{i(a}\mathcal{D}_r\hat{\mathfrak{a}}_{b)i}.} \tag{9.2.1}$$

We have written $\mathcal{U}_{ab}(A,\phi)$ because we assume the background value of the counterterm matrix comes from a fully covariant matrix $U_{ab}(A,\Phi)$, as we will explain later in section 11. Since the metric satisfies the equation of motion of a massless scalar, the counterterm that renormalizes the traceless transverse metric fluctuations $\mathfrak{e}^i_j$ follows directly from the scalar counterterm by setting $\mathcal{M} = 0$ and omitting the indices a, b, i (since they run from 1 to the number of scalars n_s), which allows us to simplify $\mathcal{D}_r \to \partial_r$,

$$-\frac{e^{dA}}{16}\int d^dx \mathfrak{e}^i_j \mathcal{T}\mathfrak{e}^j_i,$$

where the function $\mathcal{T}$ is the analogue of the matrix $\mathcal{U}_{ab}$[1]

$$\boxed{\mathcal{T}(A,\phi) \equiv \hat{\mathfrak{e}}^{-1}\partial_r\hat{\mathfrak{e}}.} \tag{9.2.2}$$

We have written $\mathcal{T}(A,\phi)$ because we assume the background value of the counterterm comes from a fully covariant function $T(A,\Phi)$. The matrix $\mathcal{U}_{ab}$ and the function $\mathcal{T}$ are built from the dominant solutions to the scalar equation of motion $\hat{\mathfrak{a}}^a_i(A)$ and the metric

[1]The metric counterterm has a different numerical factor with respect to the scalar counterterm, and there is a relative sign difference in our definitions of $\mathcal{U}_{ab}$ and $\mathcal{T}$.

equation of motion $\hat{\mathfrak{e}}(A)$ respectively. Since these solutions $\hat{\mathfrak{a}}_i^a$ and $\hat{\mathfrak{e}}$ have expansions in $\Box$ as given by eq.(5.1.14), we can expand $\mathcal{U}_{ab}$ and $\mathcal{T}$ in $\Box$ as well,

$$\begin{aligned}\mathcal{U}_{ab}(A,\phi,\Box) &= \mathcal{U}_{0ab}(A,\phi)+\mathcal{U}_{1ab}(A,\phi)e^{-2A}\Box+\mathcal{U}_{2ab}(A,\phi)e^{-4A}\Box^2+\mathcal{O}\big(\Box^3\big)\,,\\ \mathcal{T}(A,\phi,\Box) &= \mathcal{T}_1(A,\phi)e^{-2A}\Box+\mathcal{T}_2(A,\phi)e^{-4A}\Box^2+\mathcal{O}\big(\Box^3\big)\,.\end{aligned} \tag{9.2.3}$$

Notice that $\mathcal{T}_0=0$, because the expansion of $\hat{\mathfrak{e}}$ starts with a constant, see eq.(5.1.14).

For the h-fluctuation, we simply add the last term of eq.(8.4.2) with opposite sign,

$$\frac{e^{(d-2)A}}{32\mathcal{W}}\int d^dx h\,\Box\, h.$$

The non-covariant counterterm then reads

$$\boxed{S_{\text{cnt}}=\frac{e^{dA}}{2}\int d^dx\left(\mathfrak{a}^a\mathcal{U}_{ab}\mathfrak{a}^b-\frac{1}{8}\mathfrak{e}^i_j\mathcal{T}\mathfrak{e}^j_i+\frac{e^{-2A}}{16\mathcal{W}}h\,\Box\, h\right).} \tag{9.2.4}$$

The renormalized, on-shell action reads

$$S_{\text{ren}}=\frac{e^{dA}}{2}\int d^dx\left[\mathfrak{a}^a\left(\bar{G}_{ab}\mathcal{D}_r-\mathcal{M}_{ab}+\mathcal{U}_{ab}\right)+\frac{1}{8}\mathfrak{e}^i_j\left(\partial_r-\mathcal{T}\right)\mathfrak{e}^j_i\right]+S_{\text{fin}}. \tag{9.2.5}$$

The renormalized action eq.(9.2.5) leads to finite two-point functions for the operators $\mathcal{O}_i$ dual to the scalar sources $\mathfrak{a}_{\mathfrak{s}i}$ and for the traceless transverse part of the quantum field theory energy-momentum tensor denoted by $\mathcal{T}^i_j$, which is the dual operator of the source $\mathfrak{e}^i_{\mathfrak{s}j}$. The two-point functions follow from the linear part of the one-point functions. We can not calculate the vevs because we the terms linear in the sources $\mathfrak{a}_{\mathfrak{s}i}$ are contained in the action S_{fin}, which we do not know yet. In section 11 we take a covariant approach in which these terms appear automatically. For now, we content ourselves with the computation of two-point functions.

9.3 Finite two-point functions

The exact one-point function is defined in eq.(3.1.3),

$$\langle\mathcal{O}_i\rangle=-\frac{\delta S_{\text{ren}}}{\delta\mathfrak{a}_{\mathfrak{s}i}}=-\frac{\delta\mathfrak{a}^a}{\delta\mathfrak{a}_{\mathfrak{s}i}}\frac{\delta S_{\text{ren}}}{\delta\mathfrak{a}^a}=-\left(\hat{\mathfrak{a}}^a_i+\frac{\partial\mathfrak{a}_{\mathfrak{r}j}}{\partial\mathfrak{a}_{\mathfrak{s}i}}\check{\mathfrak{a}}^a_j\right)\frac{\delta S_{\text{ren}}}{\delta\mathfrak{a}^a}, \tag{9.3.1}$$

where we have used eq.(5.1.13) in the last step. Since the sources are not set to zero, the linear part of the exact one-point function carries information for the two-point function. Since we do not yet know S_{fin}, we can so far calculate only the linear contribution to the one-point function, denoted by the subscript 1,

$$\langle\mathcal{O}_i\rangle_1=-\left(\hat{\mathfrak{a}}^a_i+\frac{\partial\mathfrak{a}_{\mathfrak{r}j}}{\partial\mathfrak{a}_{\mathfrak{s}i}}\check{\mathfrak{a}}^a_j\right)\left(\bar{G}_{ab}\mathcal{D}_r-\mathcal{M}_{ab}+\mathcal{U}_{ab}\right)\left(\hat{\mathfrak{a}}^a_l\mathfrak{a}_{\mathfrak{s}l}+\check{\mathfrak{a}}^a_l\mathfrak{a}_{\mathfrak{r}l}\right). \tag{9.3.2}$$

The calculation is done in section 3.1 of [6] and the result is given by eq.(3.7) of that paper,

$$\langle \mathcal{O}_i \rangle_1 = Z_{ij} \mathfrak{a}_{\mathfrak{r}j} + \frac{1}{2} \tilde{Z}_{ij} \mathfrak{a}_{\mathfrak{s}j}, \tag{9.3.3}$$

where the matrices Z_{ij} and $\tilde{Z}_{ij}$ are given by eq.(3.4) in the same paper,

$$Z_{ij}(\Box) = e^{dA} \left[(\mathcal{D}_r \hat{\mathfrak{a}})_i \cdot \check{\mathfrak{a}}_j - \hat{\mathfrak{a}}_i \cdot (\mathcal{D}_r \check{\mathfrak{a}})_j \right], \qquad \tilde{Z}_{ij}(\Box) = e^{dA} \, (\mathcal{D}_r \hat{\mathfrak{a}})_{[i} \cdot \hat{\mathfrak{a}}_{j]}, \tag{9.3.4}$$

where the antisymmetrization brackets are defined without any numerical factor. It follows from the scalar equation of motion eq.(5.1.8) that these matrices are independent of r, and hence that the one-point function is finite in the limit $r \to \infty$. They do depend on $\Box$, so we can write[2]

$$Z_{ij}(\Box) = Z_{0ij} + Z_{1ij} \Box + Z_{2ij} \Box^2 + \ldots, \qquad \tilde{Z}_{ij}(\Box) = \tilde{Z}_{0ij} + \tilde{Z}_{1ij} \Box + \tilde{Z}_{2ij} \Box^2 + \ldots \tag{9.3.5}$$

The calculation of the one-point function for $\mathcal{T}^i_j$ is completely analogous, and the result reads[3]

$$\left\langle \mathcal{T}^i_j \right\rangle = \frac{1}{4} Y \mathfrak{e}^i_{\mathfrak{r}j}, \tag{9.3.6}$$

where Y is the equivalent of the matrix Z,

$$Y \equiv e^{dA} \left(\hat{\mathfrak{e}} \partial_r \check{\mathfrak{e}} - \check{\mathfrak{e}} \partial_r \hat{\mathfrak{e}} \right). \tag{9.3.7}$$

From the metric equation of motion eq.(5.1.9) one can show that Y is independent of r. Notice there is no term proportional to $\mathfrak{e}^i_{\mathfrak{s}j}$ in eq.(9.3.6). This follows because the matrix $\tilde{Z}_{ij}$ is antisymmetric, so its analogue for the metric $\tilde{Y}$ is zero.

From the exact one-point functions one can immediately obtain the two-point functions by functionally differentiation with respect to the sources once more. The results are

$$\langle \mathcal{O}_i(x) \mathcal{O}_j(y) \rangle = Z_{ik} \frac{\delta \mathfrak{a}_{\mathfrak{r}k}}{\delta \mathfrak{a}_{\mathfrak{s}j}} + \frac{1}{2} \tilde{Z}_{ij} \delta(x-y), \qquad \left\langle \mathcal{T}^i_j(x) \mathcal{T}^m_n(y) \right\rangle = \frac{1}{4} Y \frac{\delta \mathfrak{e}^i_{\mathfrak{r}j}(x)}{\delta \mathfrak{e}^n_{\mathfrak{s}m}(y)}. \tag{9.3.8}$$

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[2]Since the matrices do not depend on A, we do not include factors of e^{-2A} in the expansion.

[3]The traceless transverse part of the quantum field theory energy-momentum tensor $\mathcal{T}^i_j$ can be obtained from the full energy-momentum tensor T^i_j using the projection,

$$\mathcal{T}^i_j \equiv \Pi^{ik}_{jl} T^l_k,$$

where the traceless transverse projector is given by eq.(C.2.5).

9.4 Scheme dependence

As explained in section (3.3) of [6], the decomposition eq.(5.1.13) of $\mathfrak{a}$ and $\mathfrak{e}$ into dominant and subdominant solutions is not unique. Let us consider the scalars first. We will omit the index a for simplicity. We have the freedom to reshuffle the decomposition as follows,

$$\hat{\mathfrak{a}}_i' = \Lambda_{ij}\hat{\mathfrak{a}}_j + \lambda_{ij}\check{\mathfrak{a}}_j, \qquad \check{\mathfrak{a}}_i' = \mu_{ij}\check{\mathfrak{a}}_j, \tag{9.4.1}$$

where the non-degenerate matrices Λ_{ij}, λ_{ij} and μ_{ij} are polynomials in $\Box$. Under this transformation, the matrices Z_{ij} and $\tilde{Z}_{ij}$ transform as

$$\begin{aligned} \tilde{Z}_{ij}' &= \Lambda_{ik}\Lambda_{jl}\tilde{Z}_{kl} + \left(\Lambda_{ik}\lambda_{jl} - \Lambda_{jk}\lambda_{il}\right) Z_{kl} \\ Z_{ij}' &= \Lambda_{ik}\mu_{jl}Z_{kl}, \end{aligned} \tag{9.4.2}$$

while the source and response functions transform as

$$\mathfrak{a}_{\mathfrak{s}i}' = \mathfrak{a}_{\mathfrak{s}j}\left(\Lambda^{-1}\right)_{ji}, \qquad \mathfrak{a}_{\mathfrak{r}i}' = \left[\mathfrak{a}_{\mathfrak{r}j} - \mathfrak{a}_{\mathfrak{s}l}\left(\Lambda^{-1}\right)_{lk}\lambda_{kj}\right]\left(\mu^{-1}\right)_{ji}. \tag{9.4.3}$$

Finally, the QFT two-point function transforms as

$$\langle \mathcal{O}_i\mathcal{O}_j\rangle' = \Lambda_{ik}\Lambda_{jl}\langle \mathcal{O}_k\mathcal{O}_l\rangle - \frac{1}{2}\left(\Lambda_{ik}\lambda_{jl} + \Lambda_{jk}\lambda_{il}\right) Z_{kl}. \tag{9.4.4}$$

We see that the matrix Λ_{ij} induces a rotation of the basis of the operators, while λ_{ij} changes the contact terms. The contact terms do not influence physical scattering amplitudes, so changing them corresponds to changing the renormalization scheme. As argued in [6], it is reasonable to assume that the matrix Λ_{ij} and μ_{ij} can always be chosen such that $Z_{ij}' = \delta_{ij}$, and λ_{ij} can always be chosen such that $\tilde{Z}_{ij}' = 0$.

For the metric, the only freedom we have is

$$\hat{\mathfrak{e}}' = \Lambda\hat{\mathfrak{e}} + \lambda\check{\mathfrak{e}}, \qquad \check{\mathfrak{e}}' = \mu\check{\mathfrak{e}}, \tag{9.4.5}$$

where we omitted indices i and j for simplicity. Under this transformation, the matrix Y and the source $\mathfrak{e}_{\mathfrak{s}}$ and response functions transform as

$$Y' = \Lambda\mu Y, \qquad \mathfrak{e}_{\mathfrak{s}}' = \Lambda^{-1}\mathfrak{e}_{\mathfrak{s}}, \qquad \mathfrak{e}_{\mathfrak{r}}' = \mathfrak{e}_{\mathfrak{r}} - \Lambda^{-1}\lambda\mathfrak{e}_{\mathfrak{s}}. \tag{9.4.6}$$

Finally, the QFT two-point function transforms as

$$\left\langle \mathcal{T}^i_j\mathcal{T}^k_l\right\rangle' = \Lambda^2\left\langle \mathcal{T}^i_j\mathcal{T}^k_l\right\rangle - \Lambda\lambda Y. \tag{9.4.7}$$

Again we assume that we can always choose Λ and μ such that $Y' = 1$.

9.4.1 Scheme dependence and finite terms

This shows that changing the renormalization scheme is equivalent to adding finite terms to the action. Consider the scheme transformation eq.(9.4.1) with $\Lambda_{ij} = \mu_{ij} = \delta_{ij}$,

$$\hat{\mathfrak{a}}'_i = \hat{\mathfrak{a}}_i + \lambda_{ij}\check{\mathfrak{a}}_j, \qquad \check{\mathfrak{a}}'_i = \check{\mathfrak{a}}_i. \tag{9.4.8}$$

Then the linear part of the one-point function transforms to

$$\langle \mathcal{O}_i \rangle'_1 = Z'_{ij}\mathfrak{a}'_{\mathfrak{r}j} + \frac{1}{2}\tilde{Z}'_{ij}\mathfrak{a}'_{\mathfrak{s}j}, \tag{9.4.9}$$

which reads, using eq.(9.4.2) and eq.(9.4.3),

$$\langle \mathcal{O}_i \rangle'_1 = \langle \mathcal{O}_i \rangle_1 - \lambda_{kj}Z_{ij}\mathfrak{a}_{\mathfrak{s}k} + \lambda_{jk}Z_{ik}\mathfrak{a}_{\mathfrak{s}j} - \lambda_{ik}Z_{jk}\mathfrak{a}_{\mathfrak{s}j}. \tag{9.4.10}$$

Since $(\mathfrak{a}^a)' = \mathfrak{a}^a$, the only scheme dependence in the one-point function comes from $\mathcal{U}_{ab}$,

$$\frac{1}{2}\frac{\delta}{\delta\mathfrak{a}_{\mathfrak{s}i}}e^{dA}\mathfrak{a}^a\mathcal{U}'_{ab}\mathfrak{a}^b = \frac{1}{2}\frac{\delta}{\delta\mathfrak{a}_{\mathfrak{s}i}}e^{dA}\mathfrak{a}^a\mathcal{U}_{ab}\mathfrak{a}^b - \lambda_{kj}Z_{ij}\mathfrak{a}_{\mathfrak{s}k} + \lambda_{jk}Z_{ik}\mathfrak{a}_{\mathfrak{s}j} - \lambda_{ik}Z_{jk}\mathfrak{a}_{\mathfrak{s}j}, \tag{9.4.11}$$

from which we derive that the scheme change corresponds to adding a finite term to the action,

$$e^{dA}\mathfrak{a}^a\mathcal{U}'_{ab}\mathfrak{a}^b = e^{dA}\mathfrak{a}^a\mathcal{U}_{ab}\mathfrak{a}^b - \mathfrak{a}_{\mathfrak{s}i}\lambda_{kj}Z_{ij}\mathfrak{a}_{\mathfrak{s}k}. \tag{9.4.12}$$

The last two terms in eq.(9.4.11) have cancelled each other. Similarly, if we change the scheme according to eq.(9.4.5) with $\Lambda = \mu = 1$

$$\hat{\mathfrak{e}}' = \hat{\mathfrak{e}} + \lambda\check{\mathfrak{e}}, \qquad \check{\mathfrak{e}}' = \check{\mathfrak{e}}, \tag{9.4.13}$$

then we find

$$e^{dA}\mathfrak{e}^i_j\mathcal{T}'\mathfrak{e}^i_j = e^{dA}\mathfrak{e}^i_j\mathcal{T}\mathfrak{e}^i_j + \lambda\mathfrak{e}^i_{\mathfrak{s}j}Y\mathfrak{e}^j_{\mathfrak{s}i}, \tag{9.4.14}$$

which is consistent with, see eq.(9.3.6) and eq.(9.4.6),

$$\left\langle \mathcal{T}^i_j \right\rangle' = \frac{1}{4}Y\left(\mathfrak{e}^i_{\mathfrak{r}j}\right)' = \frac{1}{4}Y\left(\mathfrak{e}^i_{\mathfrak{r}j} - \lambda\mathfrak{e}^i_{\mathfrak{s}j}\right). \tag{9.4.15}$$

Chapter 10

Differential Equations for Counterterm Matrices

⋄ We derive first order differential equations for the counterterm matrices $\mathcal{U}_{ab}$ and $\mathcal{T}$. We can use these differential equations to find the counterterm matrices $\mathcal{U}_{ab}$ and $\mathcal{T}$ by solving the differential equations, as we show later in the case studies in sections 14, 15, 16 and 17. ⋄

10.1 Counterterm Matrix

Eq.(9.2.1) shows that we can write the symmetric counterterm matrix as

$$\mathcal{U}_{ab} = \frac{1}{2}\left(\tilde{\mathcal{U}}_{ab} + \tilde{\mathcal{U}}_{ba}\right), \tag{10.1.1}$$

where we have defined the non-symmetric matrix

$$\tilde{\mathcal{U}}_{ab} \equiv \mathcal{M}_{ab} - \hat{\mathfrak{a}}^{-1}_{ia}\mathcal{D}_r\hat{\mathfrak{a}}_{ib} = -\hat{\mathfrak{a}}^{-1}_{ia}\left(\delta^c_b\mathcal{D}_r - \mathcal{M}^c_b\right)\hat{\mathfrak{a}}_{ic}. \tag{10.1.2}$$

Then we find

$$\mathcal{D}_r\tilde{\mathcal{U}}_{ab} = -\left(\mathcal{D}_r\hat{\mathfrak{a}}^{-1}_{ia}\right)\left(\delta^c_b\mathcal{D}_r - \mathcal{M}^c_b\right)\hat{\mathfrak{a}}_{ic} - \hat{\mathfrak{a}}^{-1}_{ia}\mathcal{D}_r\left[\left(\delta^c_b\mathcal{D}_r - \mathcal{M}^c_b\right)\hat{\mathfrak{a}}_{ic}\right]. \tag{10.1.3}$$

We simplify the first term using some algebra,

$$\begin{aligned}-\left(\mathcal{D}_r\hat{\mathfrak{a}}^{-1}_{ia}\right)\left(\delta^c_b\mathcal{D}_r - \mathcal{M}^c_b\right)\hat{\mathfrak{a}}_{ic} &= \left(\hat{\mathfrak{a}}^{-1}\right)^d_i\hat{\mathfrak{a}}^{-1}_{ja}\left(\mathcal{D}_r\hat{\mathfrak{a}}_{jd}\right)\left(\delta^c_b\mathcal{D}_r - \mathcal{M}^c_b\right)\hat{\mathfrak{a}}_{ic} \\ &= \tilde{\mathcal{U}}^d{}_b\left(\tilde{\mathcal{U}}_{ad} - \mathcal{M}_{ad}\right),\end{aligned} \tag{10.1.4}$$

and we simplify the second term using the scalar equation of motion eq.(5.1.8),

$$-\hat{\mathfrak{a}}^{-1}_{ia}\mathcal{D}_r\left[\left(\delta^c_b\mathcal{D}_r - \mathcal{M}^c_b\right)\hat{\mathfrak{a}}_{ic}\right] = \bar{G}_{ab}e^{-2A}\,\Box - \mathcal{M}^d_b\tilde{\mathcal{U}}_{ad} - d\dot{A}\tilde{\mathcal{U}}_{ab}. \tag{10.1.5}$$

Combining this yields

$$\left(\mathcal{D}_r + d\dot{A}\right)\tilde{\mathcal{U}}_{ab} = \bar{G}_{ab}e^{-2A}\,\Box - \mathcal{M}_{ad}\tilde{\mathcal{U}}^d{}_b - \mathcal{M}^d_b\tilde{\mathcal{U}}_{ad} + \tilde{\mathcal{U}}^d{}_b\tilde{\mathcal{U}}_{ad}. \tag{10.1.6}$$

We can write this back in terms of the symmetric matrix $\mathcal{U}_{ab}$, except for the last term,

$$\left(\mathcal{D}_r + d\dot{A}\right)\mathcal{U}_{ab} = \bar{G}_{ab}e^{-2A}\,\Box - \mathcal{M}^c_{(a}\mathcal{U}_{b)c} + \tilde{\mathcal{U}}^c{}_a\tilde{\mathcal{U}}_{bc}. \tag{10.1.7}$$

When we expand this in $\Box$ as in eq.(9.2.3), we find the following differential equations[1]

$$\begin{aligned} 0 &= \left(\mathcal{D}_r + d\dot{A}\right)\mathcal{U}_{0ab} + \mathcal{M}^c_{(a}\mathcal{U}_{0b)c} - \tilde{\mathcal{U}}^c_{0a}\tilde{\mathcal{U}}_{0bc} \\ \bar{G}_{ab} &= \left(\mathcal{D}_r + (d-2)\dot{A}\right)\mathcal{U}_{1ab} + \mathcal{M}^c_{(a}\mathcal{U}_{1b)c} - \tilde{\mathcal{U}}^c_{0(a}\tilde{\mathcal{U}}_{1b)c}. \end{aligned} \tag{10.1.8}$$

10.2 Counterterm Function

We obtain the analogue of the differential equation for $\mathcal{U}_{ab}$ eq.(10.1.7) for $\mathcal{T}$ by setting $\mathcal{M} = 0$ and substituting $\mathcal{U}_{ab} \to -\mathcal{T}$: [2]

$$\left(\frac{d}{dr} - d\dot{A}\right)\mathcal{T} = -e^{-2A}\,\Box - \mathcal{T}^2. \tag{10.2.1}$$

Again, we expand $\mathcal{T}$ as in eq.(9.2.3)

$$\mathcal{T} = \mathcal{T}_1 e^{-2A}\,\Box + \mathcal{T}_2 e^{-4A}\,\Box^2 + \mathcal{O}\left(\Box^3\right). \tag{10.2.2}$$

Then we find the analogue of eq.(10.1.8)

$$\left(\frac{d}{dr} + (d-2)\dot{A}\right)\mathcal{T}_1 = -1, \qquad \left(\frac{d}{dr} + (d-4)\dot{A}\right)\mathcal{T}_2 = -\mathcal{T}_1^2. \tag{10.2.3}$$

Since $\mathcal{T}_1(A,\phi)$ is a function of the cut-off A and the scalars ϕ^a, we can rewrite its differential equation as

$$\left(\frac{\mathcal{W}^a}{\mathcal{W}}\mathcal{D}_a - \frac{2}{d-1}\partial_A - \frac{2(d-2)}{d-1}\right)\mathcal{T}_1 = -\frac{1}{\mathcal{W}}. \tag{10.2.4}$$

This relation is only true on the background, since we have used eq.(5.1.12). Similarly, we have

$$\left(\frac{\mathcal{W}^a}{\mathcal{W}}\mathcal{D}_a - \frac{2}{d-1}\partial_A - \frac{2(d-4)}{d-1}\right)\mathcal{T}_2 = -\frac{{\mathcal{T}_1}^2}{\mathcal{W}}. \tag{10.2.5}$$

[1] There are also differential equations for $\mathcal{U}_{2ab}$, $\mathcal{U}_{3ab}$ and so on, but we will not need them.

[2] The minus sign comes from our definition $\mathcal{T} \equiv \hat{\mathfrak{e}}^{-1}\partial_r\hat{\mathfrak{e}}$.

10.3 Away from the Background

The differential equations are also valid away from the background,

$$\begin{aligned} 0 &= \left(D_r + d\dot{A}\right) U_{0ab} + M^c_{(a} U_{0b)c} - \tilde{U}^{\ c}_{0a} \tilde{U}_{0bc} \\ G_{ab} &= \left(D_r + (d-2)\dot{A}\right) U_{1ab} + M^c_{(a} U_{1b)c} - \tilde{U}^{\ c}_{0(a} \tilde{U}_{1b)c}. \end{aligned} \tag{10.3.1}$$

$$\left(\frac{d}{dr} + (d-2)\dot{A}\right) T_1 = -1, \qquad \left(\frac{d}{dr} + (d-4)\dot{A}\right) T_2 = -{T_1}^2. \tag{10.3.2}$$

Chapter 11

Boundary Covariant Counterterms

⋄ In section 9, we obtained the non-covariant counterterm action S_{cnt} that cancel the divergences from the bare action. In this section we show how a covariant counterterm action S_{cov} reproduces S_{cnt} plus another part S_{fin} that is finite by itself and has terms of zeroth and linear order in the fluctuations, from which we can calculate the vacuum energy and the vevs of the operators dual to the bulk scalars. ⋄

The start of our method is similar to the Hamilton-Jacobi method, so let us review the first steps.

11.1 Comparison to Hamilton-Jacobi method

In the Hamilton-Jacobi method, one starts with the general counterterm

$$\mathcal{L}_{\text{cov}}(\gamma, \Phi, A) = \bar{\mathcal{L}}(\gamma, \Phi) + \tilde{\mathcal{L}}(\gamma, \Phi, A), \tag{11.1.1}$$

where

$$\bar{\mathcal{L}}(\gamma, \Phi) = \bar{\mathcal{L}}_{[0]}(\gamma, \Phi) + \bar{\mathcal{L}}_{[2]}(\gamma, \Phi) + \dots \tag{11.1.2}$$

Each term in $\bar{\mathcal{L}}$ is *fully* covariant, which means that none of the terms depends explicitly on the cut-off, so we can write

$$\begin{aligned}
\bar{\mathcal{L}}_{[0]} &= C_0(\Phi) \\
\bar{\mathcal{L}}_{[2]} &= \frac{1}{2} C_1(\Phi) \left(\nabla\Phi\right)^2 + C_2(\Phi) R \\
\bar{\mathcal{L}}_{[4]} &= \dots
\end{aligned} \tag{11.1.3}$$

The functions $C_0(\Phi), C_1(\Phi), C_2(\Phi), \dots$ are unknown functions of the scalar Φ only, not of the cut-off A. The only explicit cut-off dependence is in the anomaly

$$\tilde{\mathcal{L}} = -A\bar{\mathcal{A}}(\gamma, \Phi) - \int^A dA' \tilde{\mathcal{A}}(\gamma, \Phi, A'). \tag{11.1.4}$$

Putting everything together, the counterterm reads

$$\begin{aligned}\mathcal{L}_{\rm cov} \;=\; & C_0(\Phi) + C_{1ab}(\Phi)\nabla^n\Phi^a\nabla_n\Phi^b + C_2(\Phi)R + \ldots + \\ & - A\bar{\mathcal{A}}(\gamma,\Phi) - \int^A dA'\tilde{\mathcal{A}}(\gamma,\Phi,A'). \end{aligned} \tag{11.1.5}$$

The functions $C_0, C_1, C_2, \ldots$ and the linear anomaly $\bar{\mathcal{A}}$ are simultaneously determined by the recursive descent equations, while $\tilde{\mathcal{A}}$ has to be derived separately. By definition, $\bar{\mathcal{A}}$ is the remaining part of the descent equations that can not be resolved. There are remaining parts at each level of the descent equations, so we can write

$$\bar{\mathcal{A}} = \bar{\mathcal{A}}_{[0]} + \bar{\mathcal{A}}_{[2]} + \bar{\mathcal{A}}_{[4]} + \ldots \tag{11.1.6}$$

An important ingredient in the Hamilton-Jacobi method is the expansion of the potential around the fixed point. Only then can we solve the descent equations for a generic potential. Unfortunately, the expansion around the fixed point is only possible in aAdS. Since our goal is to find a holographic renormalization procedure in general spacetimes, we can not take this step.

Instead of expanding around the fixed point, we will now *expand the general counterterm around the background*. Since we are interested in calculating two-point functions, an expansion up to quadratic order in the fluctuations is sufficient. Another important difference between our method and the Hamilton-Jacobi method, is that we will allow the functions $C_0(\Phi, A), C_1(\Phi, A), C_2(\Phi, A), \ldots$ to depend not only on the scalars Φ^a but also explicitly on the cut-off A. This has the advantage that we can treat the anomalous terms simultaneously. Finally, we do not determine the functions $C_0, C_1, C_2, \ldots$ by descent equations that follow from the Hamiltonian constraint, but by making the simple requirement that all correlators that follow from it have to be finite. This requirement can be split in two steps:

1. The non-covariant counterterm $S_{\rm cnt}$ given by eq.(9.2.4) has to be contained in the covariant one $S_{\rm cov} = S_{\rm cnt} + S_{\rm fin}$,
2. The remaining terms contained in $S_{\rm fin}$ all have to be finite.

Both requirements are necessary and sufficient to determine the covariant counterterm completely. We compare our method to the Hamilton-Jacobi method in table 11.1.

11.2 Covariant Holographic Renormalization

We want to find a gravitational theory whose dual field theory is invariant under translations. The gravitational theory must share the same symmetry, so it must be translationally invariant as well, but only on the d dimensional boundary where the field theory lives. The translational invariance is guaranteed if we work with counterterms that are *boundary covariant*. That means that the action must be covariant with respect to the boundary coordinates x, but not necessarily with respect to the radial

Hamilton-Jacobi method	Covariant Holographic Renormalization
$C_0(\Phi), C_1(\Phi), C_2(\Phi), \ldots$	$C_0(\Phi, A), C_1(\Phi, A), C_2(\Phi, A), \ldots$
Expansion around aAdS fixed point	Expansion around background
Determine counterterms by Hamiltonian constraint	Determine counterterms by requiring finite correlators

coordinate r. Thus the counterterm may explicitly depend on the cut-off r, where it is defined. Terms that depend explicitly on the cut-off always lead to scheme dependence, since we always have the freedom to redefine the arbitrary cut-off value by shifting it with any finite constant.

The most general boundary covariant counterterm reads[1]

$$\begin{aligned}\mathcal{L}_{\text{cov}} \;=\; & C_0 + C_{1ab}\nabla^i\Phi^a\nabla_i\Phi^b + C_{2ab}\nabla^2\Phi^a\nabla^2\Phi^b + C_3\nabla^i\nabla^j\Phi^a\nabla_i\nabla_j\Phi^b + \\ & + C_4 R + C_5 R^2 + C_6 R^{ij}R_{ij} + C_7 R^{klmn}R_{klmn} + \\ & + C_{8a}\nabla^i\Phi^a\nabla_i R + C_9\nabla^2 R + \mathcal{O}\big(\Box^3\big)\,. \end{aligned} \tag{11.2.1}$$

We have not included the term $\nabla^i\nabla^j R_{ij}$ because it is related to $\nabla^2 R$ by the Bianchi identity. We can leave out even more terms since we are going to expand this up to second order only. For example, to second order $R^{klmn}R_{klmn}$ is a combination of R^2 and $R^{ij}R_{ij}$, and $\nabla^i\nabla^j\Phi^a\nabla_i\nabla_j\Phi^b$ can be partially integrated to be proportional to $\nabla^2\Phi^a\nabla^2\Phi^b$. Finally, we have, using that R is linear in the fluctuations and dropping boundary terms

$$C_9\nabla^2 R = -\left(\nabla_i C_9\right)\nabla^i R = -\left(\mathcal{D}_a\mathcal{C}_9\right)\left(\nabla_i\varphi^a\right)\nabla^i R + \mathcal{O}\big(f^3\big)\,. \tag{11.2.2}$$

To second order, this term has the same structure as the term

$$C_{8a}\left(\nabla^i\Phi^a\right)\nabla_i R = \mathcal{C}_{8a}\left(\nabla^i\varphi^a\right)\nabla_i R + \mathcal{O}\big(f^3\big)\,. \tag{11.2.3}$$

We will therefore keep only the term $(\nabla^i\Phi^a)\nabla_i R$, since then we do not have to worry about integrability of the function $\mathcal{C}_9$. To second order, we therefore find the following boundary covariant counterterms

$$\boxed{\begin{aligned}\mathcal{L}_{\text{cov}} = & U_0 - \frac{1}{2}U_{1ab}\nabla^i\Phi^a\nabla_i\Phi^b + \frac{1}{2}U_{2ab}\nabla^2\Phi^a\nabla^2\Phi^b + \\ & - \frac{1}{4}T_1 R - \frac{1}{4}T_2 R^{ij}R_{ij} + BR^2 + \frac{1}{4}C_a\nabla^i\Phi^a\nabla_i R + \mathcal{O}\big(\Box^3\big)\,.\end{aligned}} \tag{11.2.4}$$

[1]We may also introduce similar counterterms but with additional insertions of the background covariant differential operator $\mathcal{D}_r$. However, we shall argue that the counterterms given here are sufficient.

For future convenience we inserted numerical factors and used the names U_0, U_{1ab}, U_{2ab}, T_1, T_2, B, C_a for the unknown functions of A and Φ^a. In the following we will use the notation

$$\mathcal{U}_{0a} \equiv \mathcal{D}_a \mathcal{U}_0, \qquad \mathcal{U}_{0ab} \equiv \mathcal{D}_a \mathcal{D}_b \mathcal{U}_0, \tag{11.2.5}$$

where $\mathcal{U}_0(A,\phi)$ is the background value of $U_0(A,\Phi)$. Similarly, we denote the background values of U_{1ab}, U_{2ab}, T_1, T_2, B and C_a by $\mathcal{U}_{1ab}$, $\mathcal{U}_{2ab}$, $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{B}$, $\mathcal{C}_a$ respectively. Expanding the counterterms up to second order in the fluctuations yields, up to $\mathcal{O}\big(\Box^3\big)$,

$$\begin{aligned}
\sqrt{\gamma}\mathcal{L}_{\mathrm{cov}} \;=\; & \frac{e^{dA}}{2}\mathfrak{a}^a\left(\mathcal{U}_{0ab} + \mathcal{U}_{1ab}e^{-2A}\,\Box + \mathcal{U}_{2ab}e^{-4A}\,\Box^2\right)\mathfrak{a}^b + \frac{e^{(d-2)A}}{32\mathcal{W}}h\,\Box\,h + \\
& -\frac{e^{dA}}{16}\mathfrak{e}^i_j\left(\mathcal{T}_1 e^{-2A}\,\Box + \mathcal{T}_2 e^{-4A}\,\Box^2\right)\mathfrak{e}^j_i + \frac{e^{dA}}{32}h\left[\frac{\mathcal{W}^a\mathcal{W}^b}{\mathcal{W}^2}\mathcal{U}_{0ab} + \right.\\
& + \left(\frac{\mathcal{W}^a\mathcal{W}^b}{\mathcal{W}^2}\mathcal{U}_{1ab} + \frac{2(d-2)}{d-1}\mathcal{T}_1 - 2\frac{\mathcal{W}^a}{\mathcal{W}}\mathcal{D}_a\mathcal{T}_1 - \frac{1}{\mathcal{W}}\right)e^{-2A}\,\Box + \\
& \left. + \left(\frac{\mathcal{W}^a\mathcal{W}^b}{\mathcal{W}^2}\mathcal{U}_{2ab} + 32\mathcal{B} - 2\frac{\mathcal{W}^a}{\mathcal{W}}\mathcal{C}_a - \frac{2d}{d-1}\mathcal{T}_2\right)e^{-4A}\,\Box^2\right]h + \\
& + \frac{e^{dA}}{4}\mathfrak{a}^a\left[-\frac{\mathcal{W}^b}{\mathcal{W}}\mathcal{U}_{0ab} + \left(\mathcal{D}_a\mathcal{T}_1 - \frac{\mathcal{W}^b}{\mathcal{W}}\mathcal{U}_{1ab}\right)e^{-2A}\,\Box + \right.\\
& \left. + \left(\mathcal{C}_a - \frac{\mathcal{W}^b}{\mathcal{W}}\mathcal{U}_{2ab}\right)e^{-4A}\,\Box^2\right]h + e^{dA}\mathcal{U}_0\left[1 + \frac{1}{2}H + \frac{d}{2(d-1)}h + \right.\\
& \left. -\frac{1}{4}\mathfrak{e}^m_k\mathfrak{e}^k_m + \frac{1}{2}\epsilon^k\,\Box\,\epsilon_k - \frac{1}{8}H^2 + \frac{d-2}{4(d-1)}\left(H + \frac{d}{2(d-1)}h\right)h\right] + \\
& + e^{dA}\mathcal{U}_{0a}\left(1 + \frac{1}{2}H + \frac{d}{2(d-1)}h\right)\left(\mathfrak{a}^a - \frac{\mathcal{W}^a}{4\mathcal{W}}h\right).
\end{aligned} \tag{11.2.6}$$

Details of this calculation can be found in appendix B. The last term is just $\sqrt{\gamma}\varphi^a\mathcal{U}_{0a}$ up to quadratic order. To obtain the required counterterm S_{cnt} given by eq.(9.2.4), we must identify the functions $\mathcal{U}_{0ab}$, $\mathcal{U}_{1ab}$, $\mathcal{U}_{2ab}$, $\mathcal{T}_1$, $\mathcal{T}_2$ with the coefficients of the expansions of the matrix $\mathcal{U}_{ab}$ and $\mathcal{T}$ respectively. The matrix $\mathcal{U}_{ab}$ is defined by eq.(9.2.1), the function $\mathcal{T}$ is defined by eq.(9.2.2), and their expansions are given by eq.(9.2.3). We anticipated this result by giving already the correct names U_0, U_{1ab}, U_{2ab}, T_1, T_2 to the functions in the counterterm. Using these identifications, the first three terms provide precisely to the required counterterm S_{cnt} given by eq.(9.2.4).[2] Since $S_{\mathrm{cov}} = S_{\mathrm{cnt}} + S_{\mathrm{fin}}$, the remaining terms form the finite part of the counterterm action S_{fin}.

We still have the freedom to choose B and C_a. These functions appear at the order $\Box^2$ in the finite part of the action S_{fin}, and they must be chosen such that the correlators are finite at order $\Box^2$. We have enough freedom to ensure that the action is

[2] We have simply added the required term proportional to $h\,\Box\,h$ on the first line and subtracted the same term on the third line.

finite at order $\Box^2$. First, we can choose C_a such that the term proportional to $\mathfrak{a}^a \Box^2 h$ in eq.(11.2.6) is finite, and then we still have the freedom to choose B such that the term proportional to $h \Box^2 h$ in eq.(11.2.6) is finite too. Then, the whole action is finite at order $\Box^2$.

We might even choose B and C_a such that the terms proportional to $\mathfrak{a}^a \Box^2 h$ and $h \Box^2 h$ are zero, which would fix the functions B and C_a unambiguously, but this requirement is too strong. We will therefore leave the functions B and C_a as unknown, and we keep in mind that they give us enough freedom to kill all divergences at order $\Box^2$.

Vacuum energy The forelast term in eq.(11.2.6) contributes $e \equiv e^{dA}\mathcal{U}_0$ to zeroth order in the fluctuations. In eq.(C.2.16) we show that the vacuum energy density is given by

$$\left\langle T^i_j \right\rangle_0 = \delta^i_j e, \tag{11.2.7}$$

where T^i_j is the energy momentum tensor of the quantum field theory. The requirement that $\mathfrak{a}^a \mathcal{U}_{ab} \mathfrak{a}^b$ yields the right counterterm fixes only $\mathcal{U}_{0ab} \equiv \mathcal{D}_a \mathcal{D}_b \mathcal{U}_0$ rather than $\mathcal{U}_0$. We have the equivalence relation

$$U_0 \sim U_0 + f(A), \tag{11.2.8}$$

where $f(A)$ is a function only of A and therefore boundary covariant. In other words, adding any function $f(A)$ to the action is boundary covariant and does not affect the matrix $\mathcal{U}_{0ab}$, but it does affect the vacuum energy,

$$e \sim e + e^{dA} f(A). \tag{11.2.9}$$

We will choose $f = -\mathcal{U}_0$, such that $e = 0$. Then we can forget about the forelast term in eq.(11.2.6).

Recombination We saw that the matrix components $\mathcal{U}_{0ab}$, $\mathcal{U}_{1ab}$, $\mathcal{U}_{2ab}$, $\mathcal{T}_1$ and $\mathcal{T}_2$ combine to form the full matrices $\mathcal{U}_{ab}$ and $\mathcal{T}$ in S_{cnt}. This seems to be happening in the finite part S_{fin} as well. Let us choose

$$\mathcal{B} = \frac{d-2}{8(d-1)} \mathcal{T}_2 + \dots, \qquad \mathcal{C}_a = \mathcal{D}_a \mathcal{T}_2 + \dots \tag{11.2.10}$$

where the terms on the dots allow for other terms that may be needed to obtain finite correlators. Now we use the differential equations for $\mathcal{T}_1$ and $\mathcal{T}_2$, given by eq.(10.2.4) and eq.(10.2.5), to eliminate $\mathcal{T}_1$ and $\mathcal{T}_2$ in favor of $\mathcal{D}_a \mathcal{T}_1$ and $\mathcal{D}_a \mathcal{T}_2$. Then the finite part of the counterterm becomes

$$\begin{aligned}\sqrt{\gamma}\mathcal{L}_{\text{fin}} &= \frac{e^{dA}}{4}\left(\mathfrak{a}^a - \frac{\mathcal{W}^a}{8\mathcal{W}}h\right)\Bigg[\mathcal{D}_a\mathcal{T}_0 - \frac{\mathcal{W}^b}{\mathcal{W}}\mathcal{U}_{0ab} + \\ &+ \left(\mathcal{D}_a\mathcal{T}_1 - \frac{\mathcal{W}^b}{\mathcal{W}}\mathcal{U}_{1ab}\right)e^{-2A}\,\Box + \\ &+ \left(\mathcal{D}_a\mathcal{T}_2 - \frac{\mathcal{W}^b}{\mathcal{W}}\mathcal{U}_{2ab} + \ldots\right)e^{-4A}\,\Box^2\Bigg]h + \\ &- \frac{e^{dA}}{16(d-1)}h\left(\partial_A\mathcal{T}_0 + e^{-2A}\partial_A\mathcal{T}_1\,\Box + e^{-4A}\partial_A\mathcal{T}_2\,\Box^2\right)h + \\ &+ e^{dA}\mathcal{U}_{0a}\left(1 + \frac{1}{2}H + \frac{d}{2(d-1)}h\right)\left(\mathfrak{a}^a - \frac{\mathcal{W}^a}{4\mathcal{W}}h\right). \qquad (11.2.11)\end{aligned}$$

We included terms with $\mathcal{T}_0 = 0$ just to show that we indeed recover the full matrices $\mathcal{U}_{ab}$ and $\mathcal{T}$,

$$\begin{aligned}\sqrt{\gamma}\mathcal{L}_{\text{fin}} &= \frac{e^{dA}}{4}\left(\mathfrak{a}^a - \frac{\mathcal{W}^a}{8\mathcal{W}}h\right)\left(\mathcal{D}_a\mathcal{T} - \frac{\mathcal{W}^b}{\mathcal{W}}\mathcal{U}_{ab} + \mathcal{O}\big(\Box^2\big)\right)h - \frac{e^{(d-2)A}}{16(d-1)}\partial_A\mathcal{T}_1 h\,\Box\, h + \\ &+ e^{dA}\mathcal{U}_{0a}\left(1 + \frac{1}{2}H + \frac{d}{2(d-1)}h\right)\left(\mathfrak{a}^a - \frac{\mathcal{W}^a}{4\mathcal{W}}h\right). \qquad (11.2.12)\end{aligned}$$

We see the full matrices $\mathcal{U}_{ab}$ and $\mathcal{T}$ appear in the first term. In the second term appears the component $\mathcal{T}_1$, but it may be that the functions $\mathcal{B}$ and $\mathcal{C}_a$ give contributions proportional to $e^{(d-4)A}\partial_A\mathcal{T}_2 h\,\Box^2\, h$ as we have indicated on the forelast line of eq.(11.2.11), and we have left this possibility implicit in the $\mathcal{O}\big(\Box^2\big)$ ambiguity that we indicated. Notice that this ambiguity affects only terms proportional to $\mathfrak{a}^a\,\Box^2\, h$ and $h\,\Box^2\, h$, and not the other terms in the action. The vector $\mathcal{U}_{0a}$ is not a component of some expansion.

Our choices for $\mathcal{B}$ and $\mathcal{C}_a$ in eq.(11.2.10) yield the following counterterms

$$(D_aT_2)\,\nabla^i\Phi^a\nabla_i R, \qquad -\frac{1}{4}T_2\left(R^{ij}R_{ij} - \frac{d-2}{2(d-1)}R^2\right).$$

The desired term proportional to $\mathcal{D}_a\mathcal{T}_2$ comes more naturally from the counterterm proportional to $T_2\nabla^2 R$, which we left out because to quadratic order it is equivalent to the counterterm proportional $(D_aT_2)\,\nabla^i\Phi^a\nabla_i R$, as we showed in eq.(11.2.2) and eq.(11.2.3). In $d = 4$ the fraction in the last term gives the correct factor of minus one third. The total, on-shell, renormalized action is given by

$$\begin{aligned}\sqrt{\gamma}\mathcal{L}_{\text{ren}} &= \frac{e^{dA}}{2}\mathfrak{a}^a\left(\bar{G}_{ab}\mathcal{D}_r - \mathcal{M}_{ab} + \mathcal{U}_{ab}\right)\mathfrak{a}^b + \frac{e^{dA}}{16}\mathfrak{e}^i_j\left(\partial_r - \mathcal{T}\right)\mathfrak{e}^j_i + \\ &+ \frac{e^{dA}}{4}h\left(\mathcal{D}_a\mathcal{T} - \frac{\mathcal{W}^b}{\mathcal{W}}\mathcal{U}_{ab} + \mathcal{O}\big(\Box^2\big)\right)\left(\mathfrak{a}^a - \frac{\mathcal{W}^a}{8\mathcal{W}}h\right) - \frac{e^{(d-2)A}}{16(d-1)}\mathcal{T}_1 h\,\Box\, h + \\ &+ e^{dA}\mathcal{U}_{0a}\left(1 + \frac{1}{2}H + \frac{d}{2(d-1)}h\right)\left(\mathfrak{a}^a - \frac{\mathcal{W}^a}{4\mathcal{W}}h\right). \qquad (11.2.13)\end{aligned}$$

Finite correlation functions In the next section we will show that the renormalized action leads to finite correlation functions. The correlation functions are obtained from the action by functional differentiation with respect to the sources $\mathfrak{a}_{\mathfrak{s}i}(x)$ and $\mathfrak{e}^i_{\mathfrak{s}j}(x)$ of the fluctuations $\mathfrak{a}^a(A,x)$ and $\mathfrak{e}^i_j(A,x)$, see eq.(5.1.13), and the full gauge fields H, h, and ϵ^i. The dependence on A of the gauge fields, which does not follow from an equation of motion but depends on the choice of gauge, is therefore irrelevant as it does not affect the physical correlators. Since the first line of eq.(11.2.13) is renormalized by contruction and the rest of the action, given explicitly by eq.(11.2.12), depends only on the fluctuations $\mathfrak{a}^a$, H and h, all we have to do is to show that the parts multiplying the source $\mathfrak{a}_{\mathfrak{s}i}$ and the gauge fields H, h go to finite values at the boundary. After functional differentiation of the finite action eq.(11.2.12) with respect to the sources $\mathfrak{a}_{\mathfrak{s}i}$ and the gauge fields H, h the following combinations appear in the one-point functions,

$$\begin{aligned}
a_R &\equiv e^{(d-2)A}\mathcal{T}_1 \\
v &\equiv -e^{dA}\mathcal{U}_{0a}\frac{\mathcal{W}^a}{\mathcal{W}} \\
x_i(\Box) &\equiv e^{dA}\frac{\partial\mathfrak{a}^a}{\partial\mathfrak{a}_{\mathfrak{s}i}}\left(\mathcal{D}_a\mathcal{T}-\frac{\mathcal{W}^b}{\mathcal{W}}\mathcal{U}_{ab}+\mathcal{O}(\Box^2)\right) \\
w(\Box) &\equiv e^{dA}\frac{\mathcal{W}^a}{\mathcal{W}}\left(\mathcal{D}_a\mathcal{T}-\frac{\mathcal{W}^b}{\mathcal{W}}\mathcal{U}_{ab}+\mathcal{O}(\Box^2)\right) \\
v_i(\Box) &\equiv -e^{dA}\mathcal{U}_{0a}\frac{\partial\mathfrak{a}^a}{\partial\mathfrak{a}_{\mathfrak{s}i}}.
\end{aligned}\tag{11.2.14}$$

In the next section we will prove that these combinations approach finite values at the boundary, under reasonable assumptions. Notice that a_R and v do not depend on $\Box$, while the other expressions are expansions up to quadratic order in $\Box$.

We can expand w and x_i in $\Box$ as follows,

$$w(\Box) = w_0 + w_1\Box + w_2\Box^2, \qquad x_i(\Box) = x_{0i} + x_{1i}\Box + x_{2i}\Box^2, \tag{11.2.15}$$

where each order in $\Box$ has to be finite by itself. We already noticed that we can always choose the counterterms proportional to B and C_a such that w_2 and x_{2i} are finite. We therefore only need to prove the finiteness up to first order in $\Box$.

11.3 Anomaly

Finite correlation functions automatically imply a finite anomaly. Nevertheless, we find it convenient to discuss both at the same time, so let us calculate the rescaled anomaly $\hat{\mathcal{A}}$ from our action, to first order in the fluctuations. The first order terms in the full renormalized action can come only from the following two terms of the covariant counterterm, given by eq.(11.2.4)

$$S_{\rm cov} = \int d^d x \sqrt{\gamma} \left(U_0(A,\Phi) - \frac{1}{4} T_1(A,\Phi) R \right) + \mathcal{O}\big(f^2\big) . \qquad (11.3.1)$$

The anomaly comes from the *explicit* dependence on A, see eq.(6.2.4),

$$\hat{\mathcal{A}} \equiv -\frac{1}{\sqrt{\gamma_5}} \frac{\delta S_{\rm cov}}{\delta A} = -e^{dA} \partial_A U_0(A,\Phi) + \frac{1}{4} e^{dA} \partial_A T_1(A,\Phi) R + \mathcal{O}\big(f^2\big) . \qquad (11.3.2)$$

Keeping only the terms up to linear order yields

$$\hat{\mathcal{A}} = -e^{dA} \partial_A \mathcal{U}_0 - e^{dA} \varphi^a \partial_A \mathcal{U}_{0a} - \frac{1}{4} e^{(d-2)A} \partial_A \mathcal{T}_1 \Box h + \mathcal{O}\big(f^2\big) . \qquad (11.3.3)$$

Let us define

$$a \equiv -e^{dA} \partial_A \mathcal{U}_0, \qquad a_R \equiv e^{(d-2)A} \partial_A \mathcal{T}_1 . \qquad (11.3.4)$$

Then we find

$$\hat{\mathcal{A}} = a - e^{dA} \varphi^a \partial_A \mathcal{U}_{0a} - \frac{1}{4} a_R \Box h + \mathcal{O}\big(f^2\big) . \qquad (11.3.5)$$

Chapter 12

Renormalized Action

⋄ We will now prove that, under some reasonable assumptions, the renormalized action eq.(11.2.13) is free of divergences. This is straightforward to zeroth order in $\Box$. To first order, we must assume that the vevs of the operators dual to the bulk scalar sources are finite. The strategy consists of three steps. First, we show that a_R and v are finite. Then we show that the next two terms in the list above are finite to zeroth order in $\Box$. Then we assume that the last term is finite to all orders in $\Box$, which we need to show that the action is also finite at first order in $\Box$. ⋄

12.1 Two finite functions

Let us start by considering a_R, which appears both in the anomaly and in the renormalized action eq.(11.2.13).

There exists an intimate relation between the homogeneous solutions to the linear differential equations for T_1 and T_2 given eq.(10.3.2), the explicit cut-off dependence in the anomalous counterterms and scheme dependence. A linear differential equation has a particular solution and a homogeneous solution, where the homogeneous solution can be multiplied by any finite constant. Looking at eq.(10.3.2), we see that T_1 and T_2 satisfy linear differential equations with known homogeneous solutions,

$$T_1 = \text{particular} + c_{T1}e^{-(d-2)A}, \qquad T_2 = \text{particular} + c_{T2}e^{-(d-4)A}. \tag{12.1.1}$$

Here, c_{T1} and c_{T2} are arbitrary finite constants with the interpretation of scheme constants. Indeed, in the action eq.(11.2.4) we have

$$\sqrt{\gamma}T_1 R, \qquad \sqrt{\gamma}T_2 R^{ij}R_{ij}. \tag{12.1.2}$$

The contributions to the action due to the homogeneous solutions are

$$c_{T1}e^{-(d-2)A}\sqrt{\gamma}R, \qquad c_{T2}e^{-(d-4)A}\sqrt{\gamma}R^{ij}R_{ij}. \tag{12.1.3}$$

These terms are finite, since $\sqrt{\gamma} \sim e^{dA}$ and $R_{ij} \sim e^{-2A}$. We see that the homogeneous solutions are related to adding finite terms to the action, which is related to scheme dependence, as we saw in subsection 9.4. We also know that the explicit dependence on the cut-off is directly related to scheme dependence, since we can always add any finite constant to the cut-off. The general solutions, including the scheme dependence, are therefore

$$T_1(A + C_{T1}, \Phi), \qquad T_2(A + C_{T2}, \Phi), \tag{12.1.4}$$

where C_{T1} and C_{T2} are scheme constants (in general not equal to c_{T1} and c_{T2}). We can always choose the cut-off A large enough with respect to the finite constants C_{T1} and C_{T2} such that we can Taylor expand around the small ratios C_{T1}/A and C_{T2}/A,

$$\begin{aligned} T_1(A + C_{T1}, \Phi) &= T_1(A, \Phi) + C_{T1}\partial_A T_1(A, \Phi) \\ T_2(A + C_{T2}, \Phi) &= T_2(A, \Phi) + C_{T2}\partial_A T_2(A, \Phi). \end{aligned} \tag{12.1.5}$$

Since we have changed the renormalization scheme, this should be equivalent to adding finite terms to the action mutliplied by arbitrary scheme constants. Indeed, comparing eq.(12.1.5) to eq.(12.1.1), we read off

$$\partial_A T_1 \sim e^{-(d-2)A}, \qquad \partial_A T_2 \sim e^{-(d-4)A}, \tag{12.1.6}$$

which shows that a_R goes to a constant at the boundary.

Before we move on to v, let us first consider

$$a \equiv -e^{dA}\partial_A \mathcal{U}_0. \tag{12.1.7}$$

In the action we have $\sqrt{\gamma}U_0(A, \Phi)$. If U_0 is independent of the explicit cut-off A then $a = 0$ identically. If $U_0(A, \Phi)$ depends on A, we can always change the scheme such that

$$\sqrt{\gamma}U_0(A + C, \Phi) = \sqrt{\gamma}U_0(A, \Phi) + C\sqrt{\gamma}\partial_A U_0(A, \Phi) + \ldots, \tag{12.1.8}$$

where the second term is finite, because it comes from changing the scheme. Thus, a is finite. Since according to eq.(11.2.9) we can always choose e to be a constant, we have

$$0 = \frac{de}{dA} = \frac{d}{dA}\left(e^{dA}\mathcal{U}_0\right) = e^{dA}\left(d\mathcal{U}_0 - \frac{d-1}{2}\frac{\mathcal{W}^a}{\mathcal{W}}\mathcal{U}_{0a} + \partial_A \mathcal{U}_0\right), \tag{12.1.9}$$

from which we find the relation,[1]

$$a = de + \frac{d-1}{2}v. \tag{12.1.10}$$

[1]This is just the conformal Ward identity up to zeroth order in the fluctuations, see eq.(6.2.3), eq.(C.2.16) and eq.(C.3.2),

$$\hat{\mathcal{A}}_0 = \langle T^k_k\rangle_0 + \phi_{\mathfrak{s}}\,\langle\mathcal{O}\rangle_0\,,$$

where $\phi_{\mathfrak{s}}$ is the background value of the scalar source

$$\phi_{\mathfrak{s}} = \frac{d-1}{2}\hat{c}.$$

Since e and a are constants, so is v. For completeness, let us show that all terms in the anomaly eq.(11.3.5) are finite. From eq.(12.1.8) we find that $\sqrt{\gamma}\partial_A U_0(A,\Phi)$ is finite, so its derivation with respect to the sources $\varphi_{\mathfrak{s}i}(x)$, which do not depend on A, is finite as well,

$$\text{finite} = e^{dA}\frac{\partial}{\partial\varphi_{\mathfrak{s}i}}\partial_A U_0 = e^{dA}\frac{\partial\varphi^a}{\partial\varphi_{\mathfrak{s}i}}\frac{\partial}{\partial\varphi^a}\partial_A U_0 = e^{dA}\frac{\partial\varphi^a}{\partial\varphi_{\mathfrak{s}i}}\partial_A U_{0a}.$$

This remains finite on the background,

$$a_i \equiv e^{dA}\frac{\partial\varphi^a}{\partial\varphi_{\mathfrak{s}i}}\partial_A \mathcal{U}_{0a} = \text{finite}. \tag{12.1.11}$$

Since the sources $\varphi_{\mathfrak{s}i}$ do not depend on A, it follows

$$e^{dA}\varphi^a\partial_A\mathcal{U}_{0a} = \text{finite}. \tag{12.1.12}$$

We have now shown that all terms in the anomaly eq.(11.3.5) are finite. Since φ^a is related to $\mathfrak{a}^a$ by eq.(5.1.6), we find

$$e^{dA}\mathfrak{a}^a\partial_A\mathcal{U}_{0a} = \text{finite}. \tag{12.1.13}$$

12.2 Zeroth order terms

We will now prove that x_{0i} and w_0 are finite. We can give an explicitly finite expression for x_{0i} by considering the identity

$$x_{0i} = -e^{dA}\frac{\partial\mathfrak{a}_0^a}{\partial\mathfrak{a}_{\mathfrak{s}i}}\mathcal{U}_{0ab}\frac{\mathcal{W}^b}{\mathcal{W}} = -e^{dA}\frac{\partial\mathfrak{a}_0^a}{\partial\mathfrak{a}_{\mathfrak{s}i}}\left(\bar{G}_{ab}\mathcal{D}_r - \mathcal{M}_{ab} + \mathcal{U}_{0ab}\right)\left(\hat{c}_i\hat{\mathfrak{a}}_{0i}^b + \check{c}_i\check{\mathfrak{a}}_{0i}^b\right), \tag{12.2.1}$$

which follows from the original definition of x_i in eq.(11.2.14) with $\mathcal{T}_0 = 0$ and the identities eq.(5.1.19) and eq.(5.1.20). Comparing this expression to eq.(9.3.2) and eq.(9.3.3), we find

$$x_{0i} = \check{c}_j Z_{0ij} + \frac{1}{2}\hat{c}_j\tilde{Z}_{0ij}. \tag{12.2.2}$$

Now consider

$$w_0 = -e^{dA}\frac{\mathcal{W}^a\mathcal{W}^b}{\mathcal{W}^2}\mathcal{U}_{0ab}. \tag{12.2.3}$$

To prove this is finite, we first note that we may write

$$x_{0i} = -e^{dA}\frac{\partial\mathfrak{a}_0^a}{\partial\mathfrak{a}_{\mathfrak{s}i}}\mathcal{U}_{0ab}\frac{\mathcal{W}^b}{\mathcal{W}} = -e^{dA}\left(\hat{\mathfrak{a}}_{0i}^a + \frac{\partial\mathfrak{a}_{\mathfrak{r}j}}{\partial\mathfrak{a}_{\mathfrak{s}i}}\check{\mathfrak{a}}_{0j}^a\right)\mathcal{U}_{0ab}\frac{\mathcal{W}^b}{\mathcal{W}} = -e^{dA}\hat{\mathfrak{a}}_{0i}^a\mathcal{U}_{0ab}\frac{\mathcal{W}^b}{\mathcal{W}}, \tag{12.2.4}$$

where the last step follows since we have seen in eq.(9.3.2) and eq.(9.3.3) that the term proportional to $\check{\mathfrak{a}}_{0j}$ does not contribute.[2] For the same reason, we may write

$$\hat{c}_i x_{0i} = -e^{dA}\left(\hat{c}_i\hat{\mathfrak{a}}_{0i}^a + \check{c}_i\check{\mathfrak{a}}_{0i}^a\right)\mathcal{U}_{0ab}\frac{\mathcal{W}^b}{\mathcal{W}} = -e^{dA}\frac{\mathcal{W}^a\mathcal{W}^b}{\mathcal{W}^2}\mathcal{U}_{0ab} = w_0. \tag{12.2.5}$$

[2] There is also a term proportional to $\check{\mathfrak{a}}_{0j}$ inside $\mathcal{W}^b/\mathcal{W}$ which does contribute, so the final result depends on $\check{c}_j$.

From eq.(12.2.2) we then find

$$w_0 = \hat{c}_i x_{0i} = \hat{c}_i \check{c}_j Z_{0ij}, \tag{12.2.6}$$

where we have used that $\tilde{Z}_{0ij}$ is antisymmetric.

Assumption To continue with the first order terms, we must first derive two equations, and to do so we must assume

$$v_i(\Box) \equiv -e^{dA} \hat{\mathfrak{a}}_i^a \mathcal{U}_{0a} = \text{finite}, \tag{12.2.7}$$

which gives an alternative definition of v_i, as we will now explain. From eq.(12.2.7) follows

$$e^{dA} \check{\mathfrak{a}}_i^a \mathcal{U}_{0a} = 0, \tag{12.2.8}$$

because the strongest term in $\check{\mathfrak{a}}_i^a$ is weaker than the weakest term in $\hat{\mathfrak{a}}_i^a$. Another way to look at this is by writing,

$$e^{dA} \mathcal{U}_{0a} \check{\mathfrak{a}}_k^a = \left(\hat{\mathfrak{a}}^{-1}\right)_{ib} \check{\mathfrak{a}}_k^b \left(e^{dA} \mathcal{U}_{0a} \hat{\mathfrak{a}}_i^a\right) = -\left(\hat{\mathfrak{a}}^{-1}\right)_{ib} \check{\mathfrak{a}}_k^b v_i. \tag{12.2.9}$$

The left-hand side is zero because v_i is finite by assumption and $\left(\hat{\mathfrak{a}}^{-1}\right)_{ib} \check{\mathfrak{a}}_k^b$ goes to zero, see also page 7 of [6]. The definition eq.(12.2.7) therefore implies the expression for v_i given in eq.(11.2.14), but not the other way around.

Since there is only one term linear in the fluctuations, it means we assume that the vevs v_{0i} of the operators $\mathcal{O}_i$ dual to the scalar sources $\mathfrak{a}_{\mathfrak{s}i}$ are all finite, see eq.(C.2.16). This is a very reasonable assumption, because at zeroth order in $\Box$ there exists only one counterterm $U_0(A, \Phi)$, which we determined by requiring that $e^{dA} \mathfrak{a}_0^a \mathcal{U}_{0ab} \mathfrak{a}_0^b$ yields the correct counterterm. Therefore, if a system has remaining divergences already at zeroth order in $\Box$, then that system can not be renormalized by covariant counterterms. However, it is very reasonable to assume that the vevs are finite in any system, because we have already shown that the other combinations that appear at zeroth order in $\Box$ in the renormalized action eq.(11.2.13) are all finite in any system,

$$\begin{aligned} \text{finite} \ = \ & e^{dA}\mathcal{U}_0, \qquad e^{dA}\partial_A \mathcal{U}_0, \qquad e^{dA} \mathfrak{a}_0^a \left(\bar{G}_{ab} \mathcal{D}_r - \mathcal{M}_{ab} + \mathcal{U}_{0ab}\right) \mathfrak{a}_0^b, \\ & e^{dA} \mathfrak{a}_0^a \mathcal{U}_{0ab} \frac{\mathcal{W}^b}{\mathcal{W}}, \qquad e^{dA} \mathfrak{a}^a \partial_A \mathcal{U}_{0a}, \qquad e^{dA} \mathcal{U}_{0a} \frac{\mathcal{W}^a}{\mathcal{W}}. \end{aligned} \tag{12.2.10}$$

Especially the last two terms are interesting are similar to v_i since they involve $\mathcal{U}_{0a}$.

12.3 Preliminaries for the first order terms

Now let us derive two equations, which we need to show that the action is finite at first order in $\Box$.

First equation To derive the first equation, we will take the derivatives with respect to r of v_i, which should go to zero because v_i is assumed to be finite. In taking the derivative, we will use, see eq.(7) from [18]

$$\mathcal{D}_r\hat{\mathfrak{a}}_i^a = -\left(d\delta_{ij} - \Delta_{ij}\right)\dot{A}\hat{\mathfrak{a}}_j^a, \tag{12.3.1}$$

In aAdS with one scalar, Δ has the interpretation of the conformal dimension of the tree-level operator $\mathcal{O}$ dual to the scalar, see eq.(6.2.2). In a physical theory, every element of the matrix Δ_{ij} is finite. Then we find

$$\begin{aligned}
0 &= \mathcal{D}_r\left(e^{dA}\hat{\mathfrak{a}}_i^a\mathcal{U}_{0a}\right)\\
&= \hat{\mathfrak{a}}_i^a\mathcal{U}_{0a}\partial_r e^{dA} + e^{dA}\mathcal{U}_{0a}\mathcal{D}_r\hat{\mathfrak{a}}_i^a + e^{dA}\hat{\mathfrak{a}}_i^a\mathcal{D}_r\mathcal{U}_{0a}\\
&= e^{dA}\left(d\dot{A}\hat{\mathfrak{a}}_i^a\mathcal{U}_{0a} + \delta_a^b\mathcal{U}_{0b}\mathcal{D}_r\hat{\mathfrak{a}}_i^a + \hat{\mathfrak{a}}_i^a\mathcal{W}^b\mathcal{U}_{0ab} + \hat{\mathfrak{a}}_i^a\dot{A}\partial_A\mathcal{U}_{0a}\right)\\
&= e^{dA}\left(d\dot{A}\hat{\mathfrak{a}}_i^a\mathcal{U}_{0a} - \hat{\mathfrak{a}}_{ka}^{-1}\hat{\mathfrak{a}}_k^b\mathcal{U}_{0b}\left(d\delta_{ij} - \Delta_{ij}\right)\dot{A}\hat{\mathfrak{a}}_j^a + \hat{\mathfrak{a}}_i^a\mathcal{W}^b\mathcal{U}_{0ab} + \hat{\mathfrak{a}}_i^a\dot{A}\partial_A\mathcal{U}_{0a}\right)\\
&= e^{dA}\left(d\dot{A}\hat{\mathfrak{a}}_i^a\mathcal{U}_{0a} - \mathcal{U}_{0b}\hat{\mathfrak{a}}_j^b\left(d\delta_{ij} - \Delta_{ij}\right)\dot{A} + \hat{\mathfrak{a}}_i^a\mathcal{W}^b\mathcal{U}_{0ab} - \hat{\mathfrak{a}}_i^a\dot{A}\partial_A\mathcal{U}_{0a}\right)\\
&= e^{dA}\left(\dot{A}\Delta_{ij}\mathcal{U}_{0b}\hat{\mathfrak{a}}_i^b + \hat{\mathfrak{a}}_i^a\mathcal{W}^b\mathcal{U}_{0ab} - \hat{\mathfrak{a}}_i^a\dot{A}\partial_A\mathcal{U}_{0a}\right).
\end{aligned} \tag{12.3.2}$$

Dividing by $\dot{A}$ yields

$$e^{dA}\left(\Delta_{ij}\mathcal{U}_{0b}\hat{\mathfrak{a}}_i^b - \frac{d-1}{2}\hat{\mathfrak{a}}_i^a\frac{\mathcal{W}^b}{\mathcal{W}}\mathcal{U}_{0ab} + \hat{\mathfrak{a}}_i^a\partial_A\mathcal{U}_{0a}\right) = 0. \tag{12.3.3}$$

We assumed in eq.(12.2.7) that the first term is finite to all orders in $\Box$, and from this assumption it follows that the last term is finite to all orders in $\Box$ as well,

$$e^{dA}\hat{\mathfrak{a}}_i^a\partial_A\mathcal{U}_{0a} = \text{finite}, \tag{12.3.4}$$

which is consistent with eq.(12.1.13). Thus we conclude that the term in the middle of eq.(12.3.3) is finite by itself,

$$e^{dA}\hat{\mathfrak{a}}_i^a\frac{\mathcal{W}^b}{\mathcal{W}}\mathcal{U}_{0ab} = \text{finite} \tag{12.3.5}$$

to all orders in $\Box$. Eq.(12.2.4) shows that this is definitely true to zeroth order in $\Box$. Then it follows that

$$e^{dA}\check{\mathfrak{a}}_i^a\frac{\mathcal{W}^b}{\mathcal{W}}\mathcal{U}_{0ab} = 0, \tag{12.3.6}$$

because the strongest term in $\check{\mathfrak{a}}_i^a$ is weaker than the weakest term in $\hat{\mathfrak{a}}_i^a$. We can combine eq.(12.3.5) and eq.(12.3.6) into

$$e^{dA}\mathfrak{a}^a\frac{\mathcal{W}^b}{\mathcal{W}}\mathcal{U}_{0ab} = \text{finite}. \tag{12.3.7}$$

To first order in $\Box$, we find

$$e^{(d-2)A}\mathfrak{a}_1^a\frac{\mathcal{W}^b}{\mathcal{W}}\mathcal{U}_{0ab} = \text{finite}, \tag{12.3.8}$$

which is the first of the two equations we will need later.

Second equation Now let us derive the second equation. In the renormalized action appears

$$e^{dA}\mathfrak{a}^a\left(\bar{G}_{ab}\mathcal{D}_r-\mathcal{M}_{ab}+\mathcal{U}_{ab}\right)\mathfrak{a}^b. \tag{12.3.9}$$

We know this is finite by construction. It must be finite order by order in $\Box$. We focus on the first order part in $\Box$,

$$\begin{aligned}1\text{st} \;=\; & e^{(d-2)A}\Big[2\mathfrak{a}_0^a\mathcal{U}_{0ab}\mathfrak{a}_1^b+\mathfrak{a}_0^a\mathcal{U}_{1ab}\mathfrak{a}_0^b+\\ & +\mathfrak{a}_1^a\left(\bar{G}_{ab}\mathcal{D}_r-\mathcal{M}_{ab}\right)\mathfrak{a}_0^b+\mathfrak{a}_0^a\left(\bar{G}_{ab}\mathcal{D}_r-\mathcal{M}_{ab}\right)\mathfrak{a}_1^b\Big],\end{aligned} \tag{12.3.10}$$

where

$$\mathfrak{a}_0^a=\hat{\mathfrak{a}}_{0i}^a\mathfrak{a}_{\mathfrak{s}i}+\check{\mathfrak{a}}_{0i}^a\mathfrak{a}_{\mathfrak{r}i},\qquad \mathfrak{a}_1^a=\hat{\mathfrak{a}}_{1i}^a\mathfrak{a}_{\mathfrak{s}i}+\check{\mathfrak{a}}_{1i}^a\mathfrak{a}_{\mathfrak{r}i}. \tag{12.3.11}$$

The first order contribution must be finite for arbitrary values of the sources $\mathfrak{a}_{\mathfrak{s}i}$ and responses $\mathfrak{a}_{\mathfrak{r}i}$, in particular for the specific values $\mathfrak{a}_{\mathfrak{s}i}\to\hat{c}_i$ and $\mathfrak{a}_{\mathfrak{r}i}\to\check{c}_i$. Then we find

$$\mathfrak{a}_0^a=\hat{\mathfrak{a}}_{0i}^a\mathfrak{a}_{\mathfrak{s}i}+\check{\mathfrak{a}}_{0i}^a\mathfrak{a}_{\mathfrak{r}i}\to\hat{c}_i\hat{\mathfrak{a}}_{0i}^a+\check{c}_i\check{\mathfrak{a}}_{0i}^a=\frac{\mathcal{W}^a}{\mathcal{W}}. \tag{12.3.12}$$

Using also the identity eq.(5.1.19), the contribution eq.(12.3.10) simplifies to

$$1\text{st}=e^{(d-2)A}\left(2\frac{\mathcal{W}^b}{\mathcal{W}}\mathcal{U}_{0ab}\mathfrak{a}_1^a+\frac{\mathcal{W}^a\mathcal{W}^b}{\mathcal{W}^2}\mathcal{U}_{1ab}+\frac{\mathcal{W}^a}{\mathcal{W}}\left(\bar{G}_{ab}\mathcal{D}_r-\mathcal{M}_{ab}\right)\mathfrak{a}_1^b\right). \tag{12.3.13}$$

We know this combination of terms is finite, and from eq.(12.3.8) it follows that the first term is finite by itself. Then it follows that the following terms are finite

$$e^{(d-2)A}\frac{\mathcal{W}^a\mathcal{W}^b}{\mathcal{W}^2}\mathcal{U}_{1ab}+e^{(d-2)A}\frac{\mathcal{W}^a}{\mathcal{W}}\left(\bar{G}_{ab}\mathcal{D}_r-\mathcal{M}_{ab}\right)\mathfrak{a}_1^b=\text{finite}. \tag{12.3.14}$$

The combination of these two terms is finite, so its derivative must vanish

$$\mathcal{D}_r\left(e^{(d-2)A}\frac{\mathcal{W}^a\mathcal{W}^b}{\mathcal{W}^2}\mathcal{U}_{1ab}\right)+\mathcal{D}_r\left(e^{(d-2)A}\frac{\mathcal{W}^a}{\mathcal{W}}\left(\bar{G}_{ab}\mathcal{D}_r-\mathcal{M}_{ab}\right)\mathfrak{a}_1^b\right)=0. \tag{12.3.15}$$

Let us calculate each term individually. Using the differential equation for $\mathcal{U}_{1ab}$ given by eq.(10.1.8) and the identity eq.(5.1.19), we find

$$\mathcal{D}_r\left(e^{(d-2)A}\frac{\mathcal{W}^a\mathcal{W}^b}{\mathcal{W}^2}\mathcal{U}_{1ab}\right)=e^{(d-2)A}\frac{\mathcal{W}^a\mathcal{W}_a}{\mathcal{W}^2}+e^{(d-2)A}\frac{\mathcal{W}^a\mathcal{W}^b}{\mathcal{W}^2}\tilde{\mathcal{U}}_{0(a}^c\tilde{\mathcal{U}}_{1b)c}. \tag{12.3.16}$$

Now we calculate the second term

$$\begin{aligned}&\mathcal{D}_r\left(e^{dA}\frac{\mathcal{W}^a}{\mathcal{W}}\left(\bar{G}_{ab}\mathcal{D}_r-\mathcal{M}_{ab}\right)\left(e^{-2A}\mathfrak{a}_1^b\right)\right)=\\ &e^{dA}\frac{\mathcal{W}^c}{\mathcal{W}}\left(d\dot{A}\delta_c^a+\mathcal{M}_c^a\right)\left(\bar{G}_{ab}\mathcal{D}_r-\mathcal{M}_{ab}\right)\left(e^{-2A}\mathfrak{a}_1^b\right)+\\ &+e^{dA}\frac{\mathcal{W}^a}{\mathcal{W}}\mathcal{D}_r\left[\left(\bar{G}_{ab}\mathcal{D}_r-\mathcal{M}_{ab}\right)\left(e^{-2A}\mathfrak{a}_1^b\right)\right].\end{aligned} \tag{12.3.17}$$

In the last term we use the scalar equation of motion eq.(5.1.8), which we can rewrite as

$$\mathcal{D}_r\left[\left(\bar{G}_{ab}\mathcal{D}_r-\mathcal{M}_{ab}\right)\mathfrak{a}^b\right]=-\left(d\dot{A}\bar{G}_{ab}+\mathcal{M}_{ab}\right)\left(\delta^b_c\mathcal{D}_r-\mathcal{M}^b_c\right)\mathfrak{a}^c-e^{-2A}\,\Box\,\mathfrak{a}_a. \tag{12.3.18}$$

The first order contribution is

$$\begin{aligned}\mathcal{D}_r\left[\left(\bar{G}_{ab}\mathcal{D}_r-\mathcal{M}_{ab}\right)e^{-2A}\mathfrak{a}^b_1\right] \;=\; &-\left(d\dot{A}\bar{G}_{ab}+\mathcal{M}_{ab}\right)\left(\delta^b_c\mathcal{D}_r-\mathcal{M}^b_c\right)\left(e^{-2A}\mathfrak{a}^c_1\right)+\\ &-e^{-2A}\frac{\mathcal{W}_a}{\mathcal{W}},\end{aligned} \tag{12.3.19}$$

where we have used eq.(12.3.12) to obtain the last term. Then we find

$$\mathcal{D}_r\left(e^{dA}\frac{\mathcal{W}^a}{\mathcal{W}}\left(\bar{G}_{ab}\mathcal{D}_r-\mathcal{M}_{ab}\right)\left(e^{-2A}\mathfrak{a}^b_1\right)\right)=-e^{(d-2)A}\frac{\mathcal{W}^a\mathcal{W}_a}{\mathcal{W}^2}. \tag{12.3.20}$$

This terms cancels exactly against the first term in eq.(12.3.16). Thus, combining eq.(12.3.15) with eq.(12.3.16) and eq.(12.3.20), we find the second equation we need

$$e^{(d-2)A}\frac{\mathcal{W}^a\mathcal{W}^b}{\mathcal{W}^2}\tilde{\mathcal{U}}^c_{0(a}\tilde{\mathcal{U}}_{1b)c}=0. \tag{12.3.21}$$

The cancellation between eq.(12.3.16) and eq.(12.3.20) indicates the cancellation of a potential divergence. For example, in section 14 we will discuss the GPPZ flow, where the term on the right-hand side of eq.(12.3.20) is a constant. Since the same term appears with opposite sign in eq.(12.3.16), we see that each individual term in eq.(12.3.14) is linearly divergent. We conclude that the first term in eq.(12.3.13) is finite by itself, while the second term has a term that may be divergent, but this divergence is always cancelled by the third term.

12.4 First order terms

Now let us continue with the first order terms. From eq.(11.2.14) we find

$$w_1=e^{(d-2)A}\frac{\mathcal{W}^a}{\mathcal{W}}\left(\mathcal{D}_a\mathcal{T}_1-\frac{\mathcal{W}^b}{\mathcal{W}}\mathcal{U}_{1ab}\right). \tag{12.4.1}$$

We use the differential equation for $\mathcal{T}_1$ given by eq.(10.2.4) to rewrite the term as

$$w_1=e^{(d-2)A}\left(\frac{2(d-2)}{d-1}\mathcal{T}_1-\frac{1}{\mathcal{W}}-\frac{\mathcal{W}^a\mathcal{W}^b}{\mathcal{W}^2}\mathcal{U}_{1ab}\right)+\frac{2}{d-1}a_R. \tag{12.4.2}$$

Since a_R is a constant by itself, the part within brackets has to be a constant by itself too. Earlier we mentioned that the three terms above are not constant by themselves. For example, we saw that the last term is linearly divergent in GPPZ. That is fine, as long as the combination of the three terms is finite. We have shown by explicit calculations that this is indeed the case in all cases we studied, including Klebanov-Strassler, and we will now give the proof. Using the differential equations for $\mathcal{T}_1$ and $\mathcal{U}_{1ab}$ given by eq.(10.2.4) and eq.(10.1.8) respectively, we find

$$-\mathcal{D}_r\left[e^{(d-2)A}\left(\frac{2(d-2)}{d-1}\mathcal{T}_1-\frac{1}{\mathcal{W}}-\frac{\mathcal{W}^a\mathcal{W}^b}{\mathcal{W}^2}\mathcal{U}_{1ab}\right)\right]=e^{(d-2)A}\frac{\mathcal{W}^a\mathcal{W}^b}{\mathcal{W}^2}\tilde{\mathcal{U}}^c_{0(a}\tilde{\mathcal{U}}_{1b)c}. \quad (12.4.3)$$

The term on the right-hand side is equal to the one in eq.(12.3.21), which goes to zero fast enough. It is interesting to see that a similar cancellation of divergences appears as before. Namely, in taking the derivative, the middle term contributes

$$\mathcal{D}_r\mathcal{W}^{-1}=-\frac{\mathcal{W}^a\mathcal{W}_a}{\mathcal{W}^2}, \quad (12.4.4)$$

which cancels an opposite contribution from the last term. There is another cancellation between the first term and the last term.

Finally, we consider x_{1i}. From eq.(11.2.14) we find

$$x_{1i}=e^{(d-2)A}\frac{\partial}{\partial\mathfrak{a}_{\mathfrak{s}i}}\left[-\mathfrak{a}^a_1\frac{\mathcal{W}^b}{\mathcal{W}}\mathcal{U}_{0ab}+\mathfrak{a}^a_0\left(\mathcal{D}_a\mathcal{T}_1-\frac{\mathcal{W}^b}{\mathcal{W}}\mathcal{U}_{1ab}\right)\right]. \quad (12.4.5)$$

Eq.(12.3.8) shows that the first term is finite by itself, so the second term must be finite by itself too

$$e^{(d-2)A}\mathfrak{a}^a_0\left(\mathcal{D}_a\mathcal{T}_1-\frac{\mathcal{W}^b}{\mathcal{W}}\mathcal{U}_{1ab}\right).$$

We can not evaluate this for arbitrary values of the sources, because we do not know T_1 away from the background. But when we replace $\mathfrak{a}_{\mathfrak{s}i}\to\hat{c}_i$ and $\mathfrak{a}_{\mathfrak{r}i}\to\check{c}_i$, we recover w_1, see eq.(12.3.12) and eq.(12.4.1). Unless this happens due to systematic cancellations independently of the system, the above term is finite too. This is a reasonable assumption, because we saw that the zeroth order term x_{0i} is finite.

12.5 Second order terms

We have mentioned several times before that we can always choose the counterterms proportional to B and C_a such that the action is finite at order $\Box^2$. We may even choose them such that the contributions propotional to $\mathfrak{a}^a\,\Box^2\,h$ and $h\,\Box^2\,h$ dissappear completely, without affecting the term proportional to $\mathfrak{a}^a\,\Box^2\,\mathfrak{a}^b$. This (overly restrictive) requirement would fix the counterterms to

$$\mathcal{C}_a=\frac{\mathcal{W}^b}{4\mathcal{W}}\mathcal{U}_{2ab}, \qquad \mathcal{B}=\frac{\mathcal{W}^a\mathcal{W}^b}{32\mathcal{W}^2}\mathcal{U}_{2ab}+\frac{d}{16(d-1)}\mathcal{T}_2. \quad (12.5.1)$$

The counterterm Lagrangian then reads

$$\boxed{\begin{aligned}\mathcal{L}_{\text{cov}} = U_0 - \frac{1}{2}U_{1ab}\nabla^n\Phi^a\nabla_n\Phi^b - \frac{1}{4}T_1 R - \frac{1}{4}T_2\left(R^{ij}R_{ij} - \frac{d}{4(d-1)}R^2\right) + \\ + \frac{1}{2}U_{2ab}\left(\nabla^2\Phi^a\nabla^2\Phi^b + \frac{W^aW^b}{16W^2}R^2 + \frac{W^a}{2W}\nabla^n\Phi^b\nabla_n R\right) + \mathcal{O}(\Box^3)\,.\end{aligned}}$$

The correlations functions that follow from these counterterms are manifestly finite at order $\Box^2$, because at this order the renormalized action contains only the terms proportional to $\mathfrak{a}^a\,\Box^2\,\mathfrak{a}^b$ and $\mathfrak{e}^i_j\,\Box^2\,\mathfrak{e}^j_i$, which are finite by construction. However, the requirement that the contributions proportional to $\mathfrak{a}^a\,\Box^2\,h$ and $h\,\Box^2\,h$ disappear completely from the action is overly restrictive, because any other choice of the counterterms proportional to B and C_a that leads to finite contributions propotional to $\mathfrak{a}^a\,\Box^2\,h$ and $h\,\Box^2\,h$ is equally acceptable. The difference between one choice of counterterms and that leads to *no* contributions propotional to $\mathfrak{a}^a\,\Box^2\,h$ and $h\,\Box^2\,h$, and another choice of counterterms that leads to *finite* contributions propotional to $\mathfrak{a}^a\,\Box^2\,h$ and $h\,\Box^2\,h$ is clearly scheme dependent, because we have seen that changin the scheme is equivalent to adding finite terms to the action.

There is therefore no contradiction between the first choice of counterterms proportional to B and C_a given by eq.(11.2.10) and the second choice by eq.(12.5.1). The difference between these choices are finite terms. All we have shown in eq.(12.5.1) is that it is possible to choose the counterterms such that there appears a pattern in the action that allows us to recombine the components $\mathcal{T}_1$, $\mathcal{T}_2$ into the full function $\mathcal{T}$ and the components $\mathcal{U}_{0ab}$, $\mathcal{U}_{1ab}$, $\mathcal{U}_{2ab}$ into the full matrix $\mathcal{U}_{ab}$. However, this choice does not guarantee that the divergences dissappear. It may be, that in order to have no divergences at order $\Box^2$, the terms on the dots eq.(12.5.1) cancel the first terms, such that no recombination of components into full matrices appears.

It is interesting to note that the choices eq.(11.2.10) and eq.(12.5.1) respectively lead to terms in the action

$$-\frac{1}{4}T_2\left(R^{ij}R_{ij} - \frac{d-2}{2(d-1)}R^2\right), \qquad -\frac{1}{4}T_2\left(R^{ij}R_{ij} - \frac{d}{4(d-1)}R^2\right),$$

which both lead to the correct term for $d = 4$,

$$-\frac{1}{4}T_2\left(R^{ij}R_{ij} - \frac{1}{3}R^2\right).$$

Again, we emphasize there is no contradiction. The ambiguity simply reflects scheme dependence. We point out that the first choice eq.(11.2.10) guarantees a recombination of components into full matrices but does not guarantee a finite action at order $\Box^2$ (but does not exclude it either), while the second choice eq.(12.5.1) guarantuees a finite action at order $\Box^2$ but excludes a recombination of components into full matrices. Since both choices lead to the correct term for $d = 4$, we can still hope that there exists a scheme in which the components recombine into the full matrices *and* thereby producing finite contributions propotional to $\mathfrak{a}^a \square^2 h$ and $h \square^2 h$. This scheme may be considered natural.

It may even be the case that the choice eq.(11.2.10), which allows for a recombination, is sufficient to produce finite contributions propotional to $\mathfrak{a}^a \square^2 h$ and $h \square^2 h$ without any additional terms on the dots. We can check this by a procedure similar to the one we just used to show the absence of divergences at first order in $\square$.

In what follows, we will use the counterterms given by eq.(12.5.2) which follows from the choice eq.(12.5.1), because these terms guarantee a finite action at order $\square^2$. We will indeed see that the other choice eq.(11.2.10) differs only by finite terms.

Finally, we remark that we expect there exists a natural scheme in which the recombination happens such that finite terms appear proportional to $\mathfrak{a}^a \square^2 h$ and $h \square^2 h$, because we see similar finite terms appear at zeroth and first order in $\square$, proportional to $\mathfrak{a}^a h$, hh and $\mathfrak{a}^a \square h$, $h \square h$ respectively.

Chapter 13

One-Scalar Systems in aAdS

⋄ We will now show that Covariant Holographic Renormalization yields the same results as the Hamilton-Jacobi method. ⋄

13.1 Preliminaries

In aAdS, we can use eq.(7.1.21)

$$W = -\frac{d-1}{2} - \frac{d-\Delta}{2}\Phi^2 + \frac{v_3}{18\Delta - 12d}\Phi^3 + \mathcal{O}\big(\Phi^4\big)\,. \tag{13.1.1}$$

This gives, see eq.(4.1.6)

$$\dot{A}(\phi) = -\frac{2\mathcal{W}}{d-1} = 1 + \frac{d-\Delta}{d-1}\phi^2 - \frac{2v_3}{(18\Delta - 12d)(d-1)}\phi^3 + \mathcal{O}\big(\phi^4\big)\,. \tag{13.1.2}$$

Now we use

$$\phi = e^{-(d-\Delta)r}\phi_{\mathfrak{s}} + \dots\,, \tag{13.1.3}$$

where $\phi_{\mathfrak{s}}$ is the background value of the scalar source, which is a constant. Using this, we find

$$\dot{A}(r) = 1 + \frac{d-\Delta}{d-1}e^{-2(d-\Delta)r}\phi_{\mathfrak{s}}^2 - \frac{2v_3}{(18\Delta - 12d)(d-1)}e^{-3(d-\Delta)r}\phi_{\mathfrak{s}}^3 + \mathcal{O}\big(e^{-4(d-\Delta)r}\big)\,. \tag{13.1.4}$$

The background value of the mass matrix becomes

$$\mathcal{M} = -(d-\Delta) + \frac{v_3}{3\Delta - 2d}e^{-(d-\Delta)r}\phi_{\mathfrak{s}} + \mathcal{O}\big(\phi^2\big)\,. \tag{13.1.5}$$

13.2 Counterterms from scalar solution

The differential equation for $\mathcal{U}_0''$ then becomes

$$\left(\frac{d}{dr} - d + 2\Delta - \mathcal{U}_0''\right)\mathcal{U}_0'' = \mathcal{O}\big(\phi^2\big)\,, \tag{13.2.1}$$

where the primes are partial derivatives with respect to the scalars ϕ, keeping A constant. The solution is given by

$$\mathcal{U}_0'' = \begin{cases} C_{U0}\phi_{\mathfrak{s}}^n e^{-(2\Delta-d)r} = C_{U0}\phi^n, \qquad n \equiv \dfrac{2\Delta - d}{d-\Delta} & \Delta \neq \dfrac{d}{2} \\ = -\dfrac{1}{A+C} & \Delta = \dfrac{d}{2}, \end{cases} \tag{13.2.2}$$

where C_{U0} is a scheme constant. Integrating twice yields

$$\mathcal{U}_0 = \frac{C_{U0}}{(n+2)(n+1)}\phi^{n+2}, \qquad \text{for} \qquad \Delta \neq \frac{d}{2}. \tag{13.2.3}$$

The following counterterm is therefore finite

$$\sqrt{\gamma}U_0 \sim \sqrt{\gamma}\Phi^{n+2} \sim e^{dr}e^{-(n+2)(d-\Delta)r} = 1, \qquad \text{for} \qquad \Delta \neq \frac{d}{2}. \tag{13.2.4}$$

The fact that the counterterm is finite is consistent with the fact that the solution for $\mathcal{U}_0''$ to the differential equation can be multiplied with any finite constant. This constant is thus scheme dependent. In a natural scheme, the vacuum energy vanishes $e = 0$ as we discussed in eq.(11.2.9). For $\Delta = d/2$ we have

$$U_0 = -\frac{\Phi^2}{2(A+C_{U0})}, \qquad \text{for} \qquad \Delta = \frac{d}{2}. \tag{13.2.5}$$

Our result is consistent with the Hamilton-Jacobi method, see eq.(7.3.16). Up to scheme dependence, our results can be summarized by

$$U_0 = -\int^A dA'\tilde{\mathcal{A}}\left(\gamma, \Phi, A'\right), \qquad \forall\Delta, \tag{13.2.6}$$

where $\tilde{\mathcal{A}}$ is given by eq.(7.3.13). Then we find

$$\tilde{\mathcal{L}} = -A\bar{\mathcal{A}}(\gamma, \Phi) + U_0(\gamma, \Phi, A), \tag{13.2.7}$$

Thus U_0 is related to the non-linear anomaly. This is consistent with the fact that

$$\hat{\tilde{\mathcal{A}}} = -e^{dA}\partial_A U_0(A, \Phi). \tag{13.2.8}$$

From eq.(7.3.15) we find

$$\sqrt{\gamma}U_0 = -\frac{\sqrt{\gamma}\Phi^2}{2(A+C_{U0})} \sim A, \qquad \text{for} \qquad \Delta = \frac{d}{2}. \tag{13.2.9}$$

The counterterm is logarithmically divergent, and hence a real counterterm. The vacuum energy automatically goes to zero when the cut-off is taken to infinity,

$$e \equiv e^{dA}\mathcal{U}_0 = \frac{\phi_{\mathfrak{s}}^2}{A+C_{U0}} \sim A^{-1}, \qquad \text{for} \qquad \Delta = \frac{d}{2}. \tag{13.2.10}$$

The differential equation for $\mathcal{U}_1$ reads:

$$\left(\frac{d}{dr} + 2\Delta - d - 2\right)\mathcal{U}_1 = 1, \tag{13.2.11}$$

with the solution given by

$$\mathcal{U}_1 = \begin{cases} \dfrac{1}{2\Delta - d - 2} + \dots & \Delta \neq \dfrac{d}{2} + 1, \quad d > 2 \\ A + C_{U1} & \Delta = \dfrac{d}{2} + 1. \end{cases} \tag{13.2.12}$$

The part depending on U_1 in our counterterm eq.(12.5.2) reads, for one scalar systems

$$-\frac{1}{2}U_1 \left(\nabla\Phi\right)^2 = \begin{cases} -\dfrac{1}{4\Delta - 2d - 4}\left(\nabla\Phi\right)^2 & \Delta \neq \dfrac{d}{2} + 1 \\ -\dfrac{A + C_{U1}}{2}\left(\nabla\Phi\right)^2 & \Delta = \dfrac{d}{2} + 1. \end{cases} \tag{13.2.13}$$

The numerical factors are exactly what we found with the Hamilton-Jacobi method, see eq.(7.3.18).

13.3 Counterterms from metric solution

The differential equation for $\mathcal{T}_1$ becomes

$$\left(\frac{d}{dr} + d - 2 + \frac{(d-\Delta)(d-2)}{d-1}\phi^2 + \mathcal{O}\left(\phi^3\right)\right)\mathcal{T}_1 = -1. \tag{13.3.1}$$

The solution reads

$$\mathcal{T}_1 = \begin{cases} -\dfrac{1}{d-2} + \dfrac{d-\Delta}{(d-1)(2\Delta - d - 2)}\phi^2 + \mathcal{O}\left(\phi^3\right) & \Delta \neq \dfrac{d}{2} + 1, \quad d > 2 \\ -\dfrac{1}{d-2} + \dfrac{d-2}{2(d-1)}\left(r + C\right)\phi^2 + \mathcal{O}\left(\phi^3\right) & \Delta = \dfrac{d}{2} + 1, \quad d > 2 \\ -(r + C_{T1}) + \mathcal{O}\left(\phi^3\right) & d = 2. \end{cases} \tag{13.3.2}$$

The part depending on T_1 in our counterterm eq.(12.5.2) reads

$$-\frac{1}{4}T_1 R = \begin{cases} \left(\dfrac{1}{4(d-2)} - \dfrac{d-\Delta}{8(d-1)(\Delta - d/2 - 1)}\Phi^2\right)R & \Delta \neq \dfrac{d}{2} + 1, \quad d > 2 \\ \left(\dfrac{1}{4(d-2)} - \dfrac{d-2}{8(d-1)}\left(r + C_{T1}\right)\Phi^2\right)R & \Delta = \dfrac{d}{2} + 1, \quad d > 2 \\ \dfrac{r + C_{T1}}{4}R & d = 2. \end{cases} \tag{13.3.3}$$

This is exactly like in eq.(7.3.18). The last line of eq.(13.2.13) and the last two lines of eq.(13.3.3) depend explicitly on the cut-off and hence give contributions to the linear anomaly

$$\bar{\mathcal{A}}_{[2]} = \begin{cases} \frac{1}{2}\left(\nabla\Phi\right)^2 + \frac{d-2}{8(d-1)}\Phi^2 R & \Delta = \frac{d}{2}+1 \\ -\frac{1}{4}R & d=2. \end{cases} \tag{13.3.4}$$

This is exactly what we found with the Hamilton-Jacobi method in eq.(7.3.20).

The differential equation for $\mathcal{T}_2$ becomes[1]

$$\left(\frac{d}{dr}+d-4\right)\mathcal{T}_2 = -\frac{1}{(d-2)^2}, \qquad \text{for} \qquad d \neq 2. \tag{13.3.5}$$

The solution reads

$$\mathcal{T}_2 = \begin{cases} -\frac{1}{(d-4)(d-2)^2} & d>4 \\ -\frac{r+C_{T2}}{4} & d=4. \end{cases} \tag{13.3.6}$$

The part depending on T_2 in our counterterm eq.(12.5.2) reads

$$-\frac{1}{4}T_2\left(R^{ij}R_{ij} - \frac{d-2}{2(d-1)}R^2\right).$$

Thus we see that

$$-\frac{1}{4}T_2\left(\ldots\right) = \begin{cases} \frac{1}{4(d-4)(d-2)^2}\left(R^{ij}R_{ij} - \frac{d-2}{2(d-1)}R^2\right) & d>4 \\ \frac{r+C_{T2}}{16}\left(R^{ij}R_{ij} - \frac{1}{3}R^2\right) & d=4. \end{cases} \tag{13.3.7}$$

The second line contributes to the linear anomaly

$$\bar{\mathcal{A}}_{[4]} = \frac{1}{16}\left(\frac{1}{3}R^2 - R^{ij}R_{ij}\right), \qquad \text{for} \qquad d=4, \tag{13.3.8}$$

which is exactly like eq.(7.3.20). We have shown that our method reproduces all results from the Hamilton-Jacobi method.

[1] For $d=2$, the counterterm proportional to T_2 goes to zero when the cut-off is taken to infinity.

13.4 Four-dimensional aAdS counterterms

For four dimensions, the counterterms are classified below.[2]

$$\begin{aligned}\mathcal{L}_{\text{cov}} &= \frac{1}{8}R + \frac{r + C_{T2}}{16}\left(R^{ij}R_{ij} - \frac{1}{3}R^2\right) + \\ &+ \begin{cases} -\dfrac{\Phi^2}{2\,(r + C_{U0})} & \Delta = 2 \\ \dfrac{C_{U0}}{12}\Phi^4 - \dfrac{r + C_{U1}}{2}\left((\nabla\Phi)^2 + \dfrac{1}{6}\Phi^2 R\right) & \Delta = 3. \end{cases}\end{aligned} \tag{13.4.1}$$

Indeed, these are the counterterms for the Coulomb Branch Flow, which is a $\Delta = 2$ vev flow as we will discuss in section 15, and for GPPZ, which is a $\Delta = 3$ operator flow as we will discuss in section 14.[3]

[2]We will see in section 14 that the conformal Ward identity requires $C_{U1} = C_{T1}$, which we have anticipated here.

[3]The GPPZ counterterm is given by eq.(5.61) in [4], but we note the following differences in conventions

1. We noticed that the constant term and the terms quadratic and quartic in Φ of eq.(5.61) in [4] are the first three terms in the Taylor expansion of $-W(\Phi)$. We have already included $-W(\Phi)$ in the definition of the bare action, so we have left these terms out of the counterterm.
2. In the definition of the supergravity action eq.(2.1) of [4], there is a plus sign in front of the Ricci scalar, whereas we have a minus sign in our definition. Hence, in order to compare with our model, we adapted this minus sign to our conventions in the counterterm above.
3. The two radial coordinates ϵ and r are related by $-\log\epsilon = 2r$.

The CBF counterterm is equal to eq.(5.42) in [4]. We again note that the constant term and the term quadratic in Φ of eq.(5.42) in [4] are the first terms of the expansion of $-W(\Phi)$, which we have already included in the definition of the bare action.

Part III

Case Studies

Chapter 14

GPPZ

⋄ The GPPZ flow is an operator flow. The operator dual to the bulk scalar has conformal dimension $\Delta = 3$. We find the counterterms, calculate the one-point functions and show that the Ward identites are satsified. ⋄

14.1 Conventional Construction of Counterterms

We will first construct the counterterms by solving the equations of motion for the scalars and the metric, and using the definitions eq.(9.2.1) and eq.(9.2.2). The GPPZ flow has the following superpotential, see eq.(4.1) in [6],

$$W(\Phi) = -\frac{3}{4}\left(\cosh\frac{2\Phi}{\sqrt{3}} + 1\right) = -\frac{3}{2} - \frac{1}{2}\Phi^2 - \frac{1}{18}\Phi^4 + \mathcal{O}\big(\Phi^6\big)\,. \tag{14.1.1}$$

This immediately yields, see eq.(4.1.6)

$$\dot{A} = -\frac{2}{3}\mathcal{W} = 1 + \frac{1}{3}\phi^2 + \mathcal{O}\big(\phi^4\big)\,. \tag{14.1.2}$$

The superpotential yields the following mass term, see eq.(5.1.10)

$$M = W'' - \frac{(W')^2}{W} = -1 + \mathcal{O}\big(\Phi^4\big)\,, \tag{14.1.3}$$

where the $\mathcal{O}(\Phi^2)$ terms cancelled and the primes denote partial derivatives with respect to the scalar. The background equation reads

$$\partial_r\phi = \mathcal{W}' = -\phi - \frac{2}{9}\phi^3 + \mathcal{O}\big(\phi^5\big)\,, \tag{14.1.4}$$

which has the solution

$$\phi = \phi_{\mathfrak{s}} e^{-r} + \frac{1}{9}\phi_{\mathfrak{s}}^3 e^{-3r} + \mathcal{O}\big(e^{-5r}\big)\,, \tag{14.1.5}$$

where $\phi_{\mathfrak{s}}$ is the background value of the source. On the background, the superpotential reads

$$\mathcal{W} = -\frac{3}{2} - \frac{1}{2}\phi^2 + \mathcal{O}\big(\phi^4\big) = -\frac{3}{2} - \frac{1}{2}\phi_{\mathfrak{s}}^2 e^{-2r} + \mathcal{O}\big(e^{-4r}\big)\,. \tag{14.1.6}$$

From this and eq.(14.1.2) we find

$$e^{-2A} = e^{-2r} + \frac{1}{3}\phi_{\mathfrak{s}}^2 e^{-4r}. \tag{14.1.7}$$

14.1.1 Scalars

The scalar equation of motion eq.(5.1.8) becomes

$$\big[(3+\partial_r)\,(1+\partial_r) + e^{-2r}\,\Box\big]\,\mathfrak{a} = \mathcal{O}\big(e^{-4r}\big)\,. \tag{14.1.8}$$

The solution reads, see eq.(4.12) in [6]

$$\mathfrak{a}(r,\Box) = \left(e^{-r} + \frac{1}{2}re^{-3r}\,\Box + C_0\phi_{\mathfrak{s}}^2 e^{-3r} + C_1 e^{-3r}\,\Box\right)\mathfrak{a}_{\mathfrak{s}} + e^{-3r}\mathfrak{a}_{\mathfrak{r}} + \mathcal{O}\big(e^{-5r}\big)\,, \tag{14.1.9}$$

where C_0 and C_1 are scheme constants.[1]Using $\mathfrak{a} = \hat{\mathfrak{a}}\mathfrak{a}_{\mathfrak{s}} + \check{\mathfrak{a}}\mathfrak{a}_{\mathfrak{r}}$ from eq.(5.1.13), we find

$$\begin{aligned}
\hat{\mathfrak{a}}(r,\Box) &= e^{-r} + \frac{1}{2}re^{-3r}\,\Box + C_0\phi_{\mathfrak{s}}^2 e^{-3r} + C_1\,\Box\,e^{-3r} + \mathcal{O}\big(e^{-5r}\big) \\
\check{\mathfrak{a}}(r,\Box) &= e^{-3r} + \mathcal{O}\big(e^{-5r}\big)\,.
\end{aligned} \tag{14.1.10}$$

From eq.(5.1.14) and eq.(14.1.7) follows

$$\hat{\mathfrak{a}} = \hat{\mathfrak{a}}_0 + \hat{\mathfrak{a}}_1 e^{-2r}\,\Box + \mathcal{O}\big(e^{-5r}\big)\,, \qquad \check{\mathfrak{a}} = \check{\mathfrak{a}}_0 + \mathcal{O}\big(e^{-5r}\big)\,. \tag{14.1.11}$$

Comparing eq.(14.1.10) and eq.(14.1.11), we read off

$$\hat{\mathfrak{a}}_0 = e^{-r} + C_0\phi_{\mathfrak{s}}^2 e^{-3r}, \qquad \hat{\mathfrak{a}}_1 = \left(\frac{1}{2}r + C_1\right)e^{-r}, \qquad \check{\mathfrak{a}}_0 = e^{-3r}. \tag{14.1.12}$$

The zeromode solution eq.(5.1.18) is given by (the term proportional to e^{-3r} is zero)

$$\frac{\mathcal{W}'}{\mathcal{W}} = \frac{2}{3}\phi_{\mathfrak{s}} e^{-r} + \mathcal{O}\big(e^{-5r}\big)\,. \tag{14.1.13}$$

Comparing this to the general expression given by eq.(5.1.20) and using eq.(14.1.12),

$$\frac{\mathcal{W}'}{\mathcal{W}} = \hat{c}\hat{\mathfrak{a}}_0 + \check{c}\check{\mathfrak{a}}_0 = \hat{c}e^{-r} + \left(C_0\phi_{\mathfrak{s}}^2\hat{c} + \check{c}\right)e^{-3r} + \mathcal{O}\big(e^{-5r}\big)\,, \tag{14.1.14}$$

[1] The relation between C_0 and C_1 to α_2 from [6] is

$$\alpha_2 = C_0\phi_{\mathfrak{s}}^2 + C_1\,\Box\,.$$

we read off

$$\hat{c} = \frac{2}{3}\phi_{\mathfrak{s}}, \qquad \check{c} = -\frac{2}{3}\phi_{\mathfrak{s}}^3 C_0. \tag{14.1.15}$$

From the first line of eq.(14.1.10) we construct the counterterm "matrix", see eq.(9.2.1),

$$\mathcal{U} = \mathcal{M} - \hat{\mathfrak{a}}^{-1}\partial_r\hat{\mathfrak{a}} = 2C_0\phi_{\mathfrak{s}}^2 e^{-2r} + \left(r - \frac{1}{2} + 2C_1\right) e^{-2r}\,\Box + \mathcal{O}\big(e^{-4r}\big)\,. \tag{14.1.16}$$

Using

$$\mathcal{U} = \mathcal{U}_0'' + \mathcal{U}_1 e^{-2A}\,\Box + \mathcal{O}\big(e^{-4A}\big)\,, \tag{14.1.17}$$

we read off

$$\mathcal{U}_0'' = 2C_0\phi_{\mathfrak{s}}^2 e^{-2r}, \qquad \mathcal{U}_1 = r - \frac{1}{2} + 2C_1. \tag{14.1.18}$$

To find $\mathcal{U}_0$ from $\mathcal{U}_0''$, we need to integrate twice with respect to the scalar. We therefore first need to go from $\mathcal{U}_0(r,\phi_{\mathfrak{s}})''$ to $\mathcal{U}_0(r,\phi)''$, which in this case is trivial,

$$\mathcal{U}_0(r,\phi)'' = 2C_0\phi^2. \tag{14.1.19}$$

Integrating twice yields

$$\mathcal{U}_0 = \frac{C_0}{6}\phi^4. \tag{14.1.20}$$

Any terms weaker than ϕ^4 can be neglected. In our definition of the bare action we already included $-W$, which in its expansion includes the finite term, see eq.(14.1.1),

$$\frac{1}{18}\Phi^4.$$

Below eq.(4.6) in [6] it is mentioned that the coefficient of this term respects the susy scheme, which is equivalent to the scheme choice $C_0 = 0$.

14.1.2 Metric

The metric equation of motion is given by eq.(5.1.9). Substituting $d = 4$, eq.(14.1.6) and eq.(14.1.7) yields

$$\left[\partial_r{}^2 + \left(4 + \frac{4}{3}\phi_{\mathfrak{s}}^2 e^{-2r}\right)\partial_r + e^{-2r}\,\Box + \frac{1}{3}\phi_{\mathfrak{s}}^2 e^{-4r}\,\Box\right]\mathfrak{e}^i_j = \mathcal{O}\big(e^{-6r}\big)\,. \tag{14.1.21}$$

The dominant and subdominant solutions read

$$\begin{aligned}\hat{\mathfrak{e}} \;=\; & 1 + \frac{1}{4}e^{-2r}\,\Box - \frac{1}{12}\phi_{\mathfrak{s}}^2 r e^{-4r}\,\Box + \frac{1}{16} r e^{-4r}\,\Box^2 + \qquad \check{\mathfrak{e}} = e^{-4r} + \mathcal{O}\big(e^{-6r}\big)\\ & + C_{1\hat{\mathfrak{e}}}\phi_{\mathfrak{s}}^2 e^{-4r}\,\Box + C_{2\hat{\mathfrak{e}}} e^{-4r}\,\Box^2 + \mathcal{O}\big(e^{-6r}\big)\,.\end{aligned} \tag{14.1.22}$$

From the dominant solution we calculate $\mathcal{T} \equiv \hat{\mathfrak{e}}^{-1}\partial_r\hat{\mathfrak{e}}$, see eq.(9.2.2),

$$\begin{aligned}\mathcal{T} \;&=\; -\frac{1}{2}e^{-2r}\,\Box + \left(\frac{1}{3}r - \frac{1}{12} - 4C_{1\hat{\mathfrak{e}}}\right)\phi_{\mathfrak{s}}^2 e^{-4r}\,\Box + \\ &+ \left(-\frac{1}{4}r + \frac{3}{16} - 4C_{2\hat{\mathfrak{e}}}\right)e^{-4r}\,\Box^2 + \mathcal{O}\big(e^{-6r}\big)\,. \end{aligned} \tag{14.1.23}$$

Now we use eq.(14.1.7) again to return to the variable A,

$$\begin{aligned}\mathcal{T} \;&=\; -\frac{1}{2}e^{-2A}\,\Box + \left(\frac{1}{3}A + \frac{1}{12} - 4C_{1\hat{\mathfrak{e}}}\right)\phi_{\mathfrak{s}}^2 e^{-4A}\,\Box + \\ &+ \left(-\frac{1}{4}A + \frac{3}{16} - 4C_{2\hat{\mathfrak{e}}}\right)e^{-4A}\,\Box^2 + \mathcal{O}\big(e^{-6A}\big)\,, \end{aligned} \tag{14.1.24}$$

from which we read off

$$\mathcal{T}_1 = -\frac{1}{2} + \left(\frac{1}{3}A + \frac{1}{12} - 4C_{1\hat{\mathfrak{e}}}\right)\phi_{\mathfrak{s}}^2 e^{-2A}, \qquad \mathcal{T}_2 = -\frac{1}{4}A + \frac{3}{16} - 4C_{2\hat{\mathfrak{e}}}. \tag{14.1.25}$$

It is trivial to go from $\mathcal{T}_1(A,\phi_{\mathfrak{s}})$ to $\mathcal{T}_1(A,\phi)$,

$$\mathcal{T}_1 = -\frac{1}{2} + \left(\frac{1}{3}A + \frac{1}{12} - 4C_{1\hat{\mathfrak{e}}}\right)\phi^2. \tag{14.1.26}$$

Full counterterms Away from the background, we find

$$\begin{aligned} U_0 &= \frac{C_0}{6}\Phi^4, & U_1 &= A - \frac{1}{2} + 2C_1 \\ T_1 &= -\frac{1}{2} + \left(\frac{1}{3}A + \frac{1}{12} - 4C_{1\hat{\mathfrak{e}}}\right)\Phi^2, & T_2 &= -\frac{1}{4}A + \frac{3}{16} - 4C_{2\hat{\mathfrak{e}}}. \end{aligned} \tag{14.1.27}$$

14.2 Solving differential equations

Instead of first finding the explicit dominant solutions $\hat{\mathfrak{a}}$ and $\hat{\mathfrak{e}}$, which requires solving second order differential equations, we can find the counterterms by solving the first order differential equations for $\mathcal{U}_0$, $\mathcal{U}_1$, $\mathcal{T}_1$ and $\mathcal{T}_2$, given by eq.(10.1.8) and eq.(10.2.3).

14.2.1 Scalar

For one-scalar systems, the differential equation for $\mathcal{U}_0''$ reads, see eq.(10.1.8),

$$\left(\frac{d}{dr} + d\dot{A} + 2\mathcal{M} - \mathcal{U}_0''\right)\mathcal{U}_0'' = \mathcal{O}\big(\phi^4\big)\,. \tag{14.2.1}$$

For GPPZ, this becomes, using $d = 4$, eq.(14.1.2) and eq.(14.1.3),

$$\left(\frac{d}{dr} + 2 - \mathcal{U}_0''\right)\mathcal{U}_0'' = \mathcal{O}\big(\phi^4\big)\,. \tag{14.2.2}$$

The solution reads

$$\mathcal{U}_0'' = C_{U0}\phi^2 + \mathcal{O}\big(\phi^4\big)\,, \tag{14.2.3}$$

where C_{U0} is a scheme constant. Integrating $\mathcal{U}_0''$ twice yields

$$\mathcal{U}_0 = \frac{C_{U0}}{12}\phi^4, \tag{14.2.4}$$

where all weaker terms can be neglected. For one-scalar systems, the differential equation for $\mathcal{U}_1$ reads, see eq.(10.1.8),

$$\left(\frac{d}{dr} + (d-2)\dot{A} + 2\mathcal{M}\right)\mathcal{U}_1 = 1. \tag{14.2.5}$$

For GPPZ, this becomes, using $d = 4$, eq.(14.1.2) and eq.(14.1.3),

$$\frac{d}{dr}\mathcal{U}_1 = 1. \tag{14.2.6}$$

The solution reads

$$\mathcal{U}_1 = A + C_{U1} + \mathcal{O}\big(\phi^2\big)\,, \tag{14.2.7}$$

where C_{U1} is a scheme constant. As we saw in eq.(14.1.16), we do not need the counterterm proportional to U_2.

14.2.2 Metric

The differential equation for $\mathcal{T}_1$ is given by eq.(10.2.3). Using $d = 4$ and eq.(14.1.2) yields

$$\left(\frac{d}{dr} + 2 + \frac{2}{3}\phi^2\right)\mathcal{T}_1 = -1 + \mathcal{O}\big(\phi^4\big)\,, \tag{14.2.8}$$

The solution reads

$$\mathcal{T}_1 = -\frac{1}{2} + \frac{1}{3}\left(A + C_{T1}\right)\phi^2 + \mathcal{O}\big(\phi^4\big)\,, \tag{14.2.9}$$

where C_{T1} is a scheme constant. The differential equation for $\mathcal{T}_2$ is given by eq.(10.2.3). For GPPZ, we find, using $d = 4$ and eq.(14.2.9),

$$\frac{d}{dr}\mathcal{T}_2 = -\frac{1}{4} + \mathcal{O}\big(\phi^2\big)\,. \tag{14.2.10}$$

The solution is then

$$\mathcal{T}_2 = -\frac{1}{4}\left(A + C_{T2}\right) + \mathcal{O}\big(\phi^2\big)\,, \tag{14.2.11}$$

where C_{T2} is a scheme constant. Away from the background, we find

$$\begin{aligned} U_0 &= \frac{C_{U0}}{12}\Phi^4, & U_1 &= A + C_{U1} \\ T_1 &= -\frac{1}{2} + \frac{1}{3}\left(A + C_{T1}\right)\Phi^2, & T_2 &= -\frac{1}{4}\left(A + C_{T2}\right). \end{aligned} \tag{14.2.12}$$

Comparing to eq.(14.1.27), we see that the scheme constants are related by

$$\begin{aligned} C_{U0} &= 2C_0, & C_{U1} &= -\frac{1}{2} + 2C_1, \\ C_{T1} &= \frac{1}{4} - 12C_{1\hat{\mathfrak{e}}}, & C_{T2} &= -\frac{3}{4} + 16C_{2\hat{\mathfrak{e}}}. \end{aligned} \tag{14.2.13}$$

From the general expression eq.(11.2.4) we can construct the full counterterm action for GPPZ, which is given by the second line of eq.(13.4.1).

14.2.3 One-point functions

In GPPZ, we have

$$Z = 2, \qquad \tilde{Z} = 0, \qquad Y = -4, \qquad \hat{c} = \frac{2}{3}\phi_{\mathfrak{s}}, \qquad a_R = \frac{1}{4}\phi_{\mathfrak{s}}^2, \tag{14.2.14}$$

yielding the following one-point functions, see eq.(C.4.1),

$$\partial_i \left\langle T^i_j \right\rangle = 0, \qquad \left\langle T^k_k \right\rangle = -2\phi_{\mathfrak{s}}\varphi_{\mathfrak{r}} - \phi_{\mathfrak{s}}\left(C_{U1} - C_{T1}\right)\Box\,\mathfrak{a}_{\mathfrak{s}} - \frac{1}{12}\phi_{\mathfrak{s}}^2\,\Box\,h, \tag{14.2.15}$$

$$\left\langle \mathcal{T}^i_j \right\rangle = -\mathfrak{e}^i_{\mathfrak{r}j}, \qquad \langle \mathcal{O} \rangle = 2\varphi_{\mathfrak{r}} + \frac{1}{6}\phi_{\mathfrak{s}}\left(C_{U1} - C_{T1}\right)\Box\,h, \qquad \hat{\mathcal{A}} = -\frac{1}{12}\phi_{\mathfrak{s}}^2\,\Box\,h. \tag{14.2.16}$$

We have set $C_{U0} = 0$ since in this scheme the vacuum energy vanishes, $e = 0$. The conformal Ward identity for GPPZ reads, see eq.(6.2.3),

$$\left\langle T^k_k \right\rangle + \Phi_{\mathfrak{s}}\langle \mathcal{O} \rangle = \hat{\mathcal{A}}. \tag{14.2.17}$$

Up to linear order we find, using eq.(14.1.15), eq.(C.2.7), $\Phi_{\mathfrak{s}} = \phi_{\mathfrak{s}} + \varphi_{\mathfrak{s}}$ and $\langle \mathcal{O} \rangle_0 = 0$,

$$\left\langle T^k_k \right\rangle + \phi_{\mathfrak{s}}\langle \mathcal{O} \rangle = -\phi_{\mathfrak{s}}\left(C_{U1} - C_{T1}\right)\Box\,\varphi_{\mathfrak{s}} - \frac{1}{12}\phi_{\mathfrak{s}}^2\,\Box\,h, \tag{14.2.18}$$

We see that the conformal Ward identity requires $C_{U1} = C_{T1}$, which we anticipated in the second line of eq.(13.4.1). The following one-point functions then simplify

$$\left\langle T^k_k \right\rangle = -2\phi_{\mathfrak{s}}\varphi_{\mathfrak{r}} - \frac{1}{12}\phi_{\mathfrak{s}}^2\,\Box\,h, \qquad \langle \mathcal{O} \rangle = 2\varphi_{\mathfrak{r}}. \tag{14.2.19}$$

From the first equation in eq.(14.2.15) and the fact that the GPPZ is an operator flow, *i.e.* $\langle \mathcal{O} \rangle_0 = 0$, we see that the translational Ward identity eq.(6.1.2) is satisfied up to linear order.

Chapter 15

Coulomb Branch Flow

⋄ The Coulomb Branch Flow is a vev flow. The operator dual to the bulk scalar has conformal dimension $\Delta = 2$. We find the counterterms, calculate the one-point functions and show that the Ward identities are satsified. ⋄

15.1 Conventional Construction of Counterterms

We will first construct the counterterms by solving the equations of motion for the scalars and the metric, and using the definitions eq.(9.2.1) and eq.(9.2.2). The superpotential is given by

$$W(\Phi) = -e^{-2\Phi/\sqrt{6}} - \frac{1}{2}e^{4\Phi/\sqrt{6}} = -\frac{3}{2} - \Phi^2 + \mathcal{O}\big(\Phi^3\big)\,. \tag{15.1.1}$$

This superpotential yields the mass term, see eq.(5.1.10),

$$M = W'' - \frac{(W')^2}{W} = -2 + \mathcal{O}\big(\Phi^2\big)\,. \tag{15.1.2}$$

On the background we have, see eq.(4.1.6)

$$\partial_r \phi = \mathcal{W}' = -2\phi + \mathcal{O}\big(\phi^2\big)\,, \tag{15.1.3}$$

with solution, see eq.(4.22) and eq.(4.23) in [6]

$$\phi = \phi_{\mathfrak{r}} e^{-2r}, \tag{15.1.4}$$

where the constant $\phi_{\mathfrak{r}}$ is the background value of the response function $\Phi_{\mathfrak{r}}(x)$ of the full scalar Φ. From eq.(4.1.6) we find

$$\dot{A} = 1 + \mathcal{O}\big(\phi^2\big)\,. \tag{15.1.5}$$

The warp function A is related to the radial variable by

$$e^{-2A} = e^{-2r} + \mathcal{O}\big(e^{-6r}\big)\,. \tag{15.1.6}$$

15.1.1 Scalar

The scalar equation of motion eq.(5.1.8) becomes

$$\left[(\partial_r + 2)\,(\partial_r + 2) + e^{-2r}\,\Box\right]\mathfrak{a} = \mathcal{O}\big(\phi^3\big)\,. \tag{15.1.7}$$

The solution reads

$$\hat{\mathfrak{a}} = (r + \tilde{\alpha})\,e^{-2r} + \mathcal{O}\big(e^{-3r}\big)\,, \qquad \check{\mathfrak{a}} = e^{-2r} + \mathcal{O}\big(e^{-3r}\big)\,, \tag{15.1.8}$$

where $\tilde{\alpha}$ is a scheme constant. From $\mathcal{M}$ and $\hat{\mathfrak{a}}$ we construct the counterterm "matrix", see eq.(9.2.1),

$$\mathcal{U} = \mathcal{M} - \hat{\mathfrak{a}}^{-1}\partial_r\hat{\mathfrak{a}} = \mathcal{U}_0'' + \mathcal{O}(\Box) = -\frac{1}{r} + \frac{\tilde{\alpha}}{r^2} + \mathcal{O}\big(r^{-3}\big)\,. \tag{15.1.9}$$

Integrating $\mathcal{U}_0''$ twice yields

$$\mathcal{U}_0 = \left(-\frac{1}{2r} + \frac{\tilde{\alpha}}{2r^2} + \mathcal{O}\big(r^{-3}\big)\right)\phi^2. \tag{15.1.10}$$

The zero mode solution reads, see eq.(5.1.18) and eq.(5.1.20),

$$\frac{\mathcal{W}'}{\mathcal{W}} = \frac{4}{3}\phi_{\mathfrak{r}}e^{-2r} = \hat{c}\hat{\mathfrak{a}} + \check{c}\check{\mathfrak{a}}, \tag{15.1.11}$$

from which we read off

$$\hat{c} = 0, \qquad \check{c} = \frac{4}{3}\phi_{\mathfrak{r}}. \tag{15.1.12}$$

15.1.2 Metric

The metric equation of motion eq.(5.1.9) becomes

$$\left[(\partial_r + 4)\,\partial_r + e^{-2r}\,\Box\right]\mathfrak{e}^i_j = \mathcal{O}\big(e^{-6r}\big)\,, \tag{15.1.13}$$

which has the solutions

$$\hat{\mathfrak{e}} = 1 + \frac{1}{4}e^{-2r}\,\Box + \left(\frac{1}{16}r + C_{\hat{\mathfrak{e}}}\right)e^{-4r}\,\Box^2 + \mathcal{O}\big(e^{-6r}\big)\,, \qquad \check{\mathfrak{e}} = e^{-4r} + \mathcal{O}\big(e^{-6r}\big)\,. \tag{15.1.14}$$

From the dominant solution we find, see eq.(9.2.2),

$$\mathcal{T} \equiv \hat{\mathfrak{e}}^{-1}\partial_r\hat{\mathfrak{e}} = -\frac{1}{2}e^{-2r}\,\Box + \left(-\frac{1}{4}r + \frac{3}{16} - 4C_{\hat{\mathfrak{e}}}\right)e^{-4r}\,\Box^2 + \mathcal{O}\big(e^{-6r}\big)\,. \tag{15.1.15}$$

Using eq.(15.1.6) we can rewrite this in terms of A,

$$\mathcal{T} = -\frac{1}{2}e^{-2A}\,\Box + \left(-\frac{1}{4}A + \frac{3}{16} - 4C_{\hat{\mathfrak{e}}}\right)e^{-4A}\,\Box^2 + \mathcal{O}\big(e^{-6A}\big)\,, \tag{15.1.16}$$

which allows us to read off

$$\mathcal{T}_1 = -\frac{1}{2}, \qquad \mathcal{T}_2 = -\frac{1}{4}A + \frac{3}{16} - 4C_{\hat{\mathfrak{e}}}. \tag{15.1.17}$$

Away from the background, we find

$$U_0 = \left(-\frac{1}{2A} + \frac{\tilde{\alpha}}{2A^2}\right)\Phi^2, \qquad T_1 = -\frac{1}{2}, \qquad T_2 = -\frac{1}{4}A + \frac{3}{16} - 4C_{\hat{\mathfrak{e}}}, \tag{15.1.18}$$

15.2 Solving differential equations

Alternatively, we can obtain the counterterms by solving the differential equations for $\mathcal{U}_0''$, $\mathcal{T}_1$ and $\mathcal{T}_2$.

15.2.1 Scalars

For one scalar systems, the general differential equation for $\mathcal{U}_0''$ is given by eq.(10.1.8). For CBF, it becomes, using $d = 4$, eq.(15.1.5) and eq.(15.1.2),

$$\left(\frac{d}{dr} - \mathcal{U}_0''\right)\mathcal{U}_0'' = \mathcal{O}\left(\phi^2\right). \tag{15.2.1}$$

The solution reads

$$\mathcal{U}_0'' = -\frac{1}{A} + \frac{C_{U0}}{A^2} + \mathcal{O}\left(A^{-3}\right), \tag{15.2.2}$$

where C_{U0} is a scheme constant, related to $\tilde{\alpha}$ by $C_{U0} = \tilde{\alpha}$. Integrating $\mathcal{U}_0''$ twice yields

$$\mathcal{U}_0 = \left(-\frac{1}{A} + \frac{C_{U0}}{A^2} + \mathcal{O}\left(A^{-3}\right)\right)\phi^2. \tag{15.2.3}$$

15.2.2 Metric

The differential equation for $\mathcal{T}_1$ is given by eq.(10.2.3). Substituting $d = 4$ and eq.(15.1.5) yields

$$\left(\frac{d}{dr} + 2\right)\mathcal{T}_1 = -1 + \mathcal{O}(\phi)^2. \tag{15.2.4}$$

The solution reads

$$\mathcal{T}_1 = -\frac{1}{2} + \mathcal{O}(\phi), \tag{15.2.5}$$

where the integration constant is contained in the term $\mathcal{O}(\phi)$. The differential equation for $\mathcal{T}_1$ is given by eq.(10.2.3). For CBF it becomes, using eq.(15.2.4)

$$\frac{d}{dr}\mathcal{T}_2 = -\frac{1}{4} + \mathcal{O}(\phi). \tag{15.2.6}$$

The solution reads

$$\mathcal{T}_2 = -\frac{1}{4}\left(A + C_{T2}\right) + \mathcal{O}(\phi)\,, \qquad (15.2.7)$$

where C_{T2} is a scheme constant, related to $C_{\hat{\mathfrak{e}}}$ by

$$C_{T2} = -\frac{3}{4} + 16C_{\hat{\mathfrak{e}}}. \qquad (15.2.8)$$

Away from the background, we find

$$U_0 = -\frac{\Phi^2}{2\left(A + C_{U0}\right)}, \qquad T_1 = -\frac{1}{2}, \qquad T_2 = -\frac{1}{4}\left(A + C_{T2}\right), \qquad (15.2.9)$$

which, after substitution in eq.(11.2.4), yields the first line of eq.(13.4.1).

15.2.3 One-point functions

For CBF, we have

$$Z = 1, \qquad \tilde{Z} = 0, \qquad Y = -4, \qquad \check{c} = \frac{4}{3}\phi_{\mathfrak{r}}, \qquad v_i = \phi_{\mathfrak{r}}, \qquad x_{0i} = \frac{4}{3}\phi_{\mathfrak{r}}. \quad (15.2.10)$$

The vacuum energy is independent of the scheme and vanishes automatically. Then we find to linear order, see eq.(C.4.1),

$$\begin{aligned} \left\langle \mathcal{T}^i_j \right\rangle &= -\mathfrak{e}^i_{\mathfrak{r}j}, & \partial_i \left\langle T^i_j \right\rangle &= -\phi_{\mathfrak{r}}\partial_j\varphi_{\mathfrak{s}}, \\ \left\langle T^k_k \right\rangle &= -2\phi_{\mathfrak{r}}\varphi_{\mathfrak{s}}, & \left\langle \mathcal{O} \right\rangle &= \phi_{\mathfrak{r}} + \varphi_{\mathfrak{r}}, \end{aligned} \qquad (15.2.11)$$

while the anomaly vanishes $\hat{\mathcal{A}} = 0$ to linear order. To first order in the fluctuations the conformal Ward identity reads, see eq.(6.2.3) and using $\phi_{\mathfrak{s}} = 0$,

$$\left\langle T^k_k \right\rangle + 2\varphi_{\mathfrak{s}} \left\langle \mathcal{O} \right\rangle = \hat{\mathcal{A}}. \qquad (15.2.12)$$

Using eq.(15.2.11), we see that that the conformal Ward identity is indeed satisfied. From eq.(15.2.11) also follows that the translational Ward identity is satisfied, see eq.(6.1.2). The CBF flow has a scheme independent vev, in contrast to GPPZ.

Chapter 16

Two Scalars in aAdS

⋄ The $SU(2) \times U(1)$ flow has two scalars, one with $\Delta = 2$ and one with $\Delta = 3$. We find the counterterms and show that when the $\Delta = 3$ scalar is made inert, the counterterms reduce to the counterterms for the Coulomb Branch Flow, such that the operator dual to the remaining scalar has a non-zero vev independently of the scheme. ⋄

16.1 Conventional Construction of Counterterms

We will first construct the counterterms by solving the equations of motion for the scalars and the metric, and using the definitions eq.(9.2.1) and eq.(9.2.2). The superpotential for the $SU(2) \times U(1)$ flow reads

$$W(\Phi) = -\frac{1}{2}\exp\left(-\frac{2\Phi_\beta}{\sqrt{6}}\right)\left[1+\cosh\left(2\Phi_\chi\right)\right] + \frac{1}{4}\exp\left(\frac{4\Phi_\beta}{\sqrt{6}}\right)\left[\cosh\left(2\Phi_\chi\right)-3\right], \quad (16.1.1)$$

which has the following expansion

$$\begin{aligned} W \;=\; & -\frac{3}{2} - \frac{1}{2}\Phi_\chi^2 + \sqrt{\frac{8}{3}}\Phi_\beta\Phi_\chi^2 - \Phi_\beta^2 - \frac{1}{6}\Phi_\chi^4 + \\ & + \frac{1}{3}\sqrt{\frac{8}{3}}\Phi_\chi^4\Phi_\beta + \frac{1}{3}\Phi_\beta^2\Phi_\chi^2 - \frac{1}{6}\sqrt{\frac{8}{3}}\Phi_\beta^3 - \frac{1}{45}\Phi_\chi^6 + \ldots \end{aligned} \quad (16.1.2)$$

The background equations read, see eq.(4.1.6)

$$\begin{aligned} \dot{\phi}^\beta \;&=\; \mathcal{W}^\beta = -2\phi_\beta + \sqrt{\frac{8}{3}}\phi_\chi^2 + \mathcal{O}\left(e^{-4r}\right) \\ \dot{\phi}^\chi \;&=\; \mathcal{W}^\chi = -\phi_\chi + 2\sqrt{\frac{8}{3}}\phi_\beta\phi_\chi - \frac{2}{3}\phi_\chi^3 + \mathcal{O}\left(e^{-5r}\right), \end{aligned} \quad (16.1.3)$$

with corresponding solutions

$$\begin{aligned}\phi_\beta &= \sqrt{\frac{8}{3}}\phi_1^2 r e^{-2r} + \phi_2 e^{-2r} + \mathcal{O}\big(e^{-4r}\big)\,,\\ \phi_\chi &= \phi_1 e^{-r} - \left(\frac{8}{3}\phi_1^3 r + \sqrt{\frac{8}{3}}\phi_1\phi_2 + \phi_1^3\right) e^{-3r} + \mathcal{O}\big(e^{-5r}\big)\,,\end{aligned} \tag{16.1.4}$$

where the constants ϕ_1 and ϕ_2 are combinations of the background values of the sources $\Phi_{\mathfrak{s}1}$, $\Phi_{\mathfrak{s}2}$ and responses $\Phi_{\mathfrak{r}1}$, $\Phi_{\mathfrak{r}2}$ of the full scalar solutions, see eq.(4.1.4). The mass matrix eq.(5.1.10) reads

$$\mathcal{M}_\chi^\chi = -1+\mathcal{O}\big(e^{-2r}\big)\,,\quad \mathcal{M}_\beta^\beta = -2+\mathcal{O}\big(e^{-2r}\big)\,,\quad \mathcal{M}_\chi^\beta = 2\phi_1\sqrt{\frac{8}{3}}e^{-r}+\mathcal{O}\big(e^{-3r}\big)\,. \tag{16.1.5}$$

We can switch from r to A using

$$r = A + \frac{1}{6}\phi_1^2 e^{-2A} + \mathcal{O}\big(e^{-4A}\big)\,,\qquad e^{-r} = e^{-A} - \frac{1}{6}\phi_1^2 e^{-3A} + \mathcal{O}\big(e^{-5A}\big)\,. \tag{16.1.6}$$

In terms of A the background solutions read

$$\begin{aligned}\phi_\beta &= \sqrt{\frac{8}{3}}\phi_1^2\,(A + C^*)\,e^{-2A} + \mathcal{O}\big(e^{-4A}\big)\\ \phi_\chi &= \phi_1 e^{-A} - \frac{8}{3}\phi_1^3\left(A + C^* + \frac{7}{16}\right) e^{-3A} + \mathcal{O}\big(e^{-5A}\big)\,,\end{aligned} \tag{16.1.7}$$

where we have defined

$$C^* \equiv \sqrt{\frac{3}{8}}\frac{\phi_2}{\phi_1^2}. \tag{16.1.8}$$

16.1.1 Scalars

The equations of motion for the two scalar fluctuations are, see eq.(5.1.8)

$$\begin{aligned}0 &= \left[\left(\partial_r + \mathcal{M}_\beta^\beta - \frac{2d}{d-1}\mathcal{W}\right)\left(\partial_r - \mathcal{M}_\beta^\beta\right) - \left(\mathcal{M}_\chi^\beta\right)^2 + e^{-2A}\,\Box\right]\mathfrak{a}^\beta +\\ &\quad + \left[\mathcal{M}_\chi^\beta\left(\partial_r - \mathcal{M}_\chi^\chi\right) - \left(\partial_r + \mathcal{M}_\beta^\beta - \frac{2d}{d-1}\mathcal{W}\right)\mathcal{M}_\chi^\beta\right]\mathfrak{a}^\chi\\ 0 &= \left[\mathcal{M}_\beta^\chi\left(\partial_r - \mathcal{M}_\beta^\beta\right) - \left(\partial_r + \mathcal{M}_\chi^\chi - \frac{2d}{d-1}\mathcal{W}\right)\mathcal{M}_\beta^\chi\right]\mathfrak{a}^\beta +\\ &\quad + \left[\left(\partial_r + \mathcal{M}_\chi^\chi - \frac{2d}{d-1}\mathcal{W}\right)\left(\partial_r - \mathcal{M}_\chi^\chi\right) - \left(\mathcal{M}_\chi^\beta\right)^2 + e^{-2A}\,\Box\right]\mathfrak{a}^\chi.\end{aligned} \tag{16.1.9}$$

The dominant solutions read

$$\begin{aligned}
\hat{\mathfrak{a}}^{\chi} &= a_{\chi}e^{-r} + a_{\chi}re^{-3r}\left(\frac{1}{2}\Box - 8\phi_1^2\right) + C_{\chi}e^{-3r} + \mathcal{O}\left(e^{-5r}\right) \\
\hat{\mathfrak{a}}^{\beta} &= a_{\beta}re^{-2r} - \frac{a_{\beta}}{4}(r+1)\,e^{-4r}\Box + \\
&\quad + a_{\beta}\left[\frac{4}{3}\phi_1^2 r^2 + \left(\frac{13}{3}\phi_1^2 + \frac{1}{2}\sqrt{\frac{8}{3}}\phi_2\right) r + \frac{10}{3}\phi_1^2 + \frac{1}{2}\sqrt{\frac{8}{3}}\phi_2\right] e^{-4r} + \\
&\quad + C_{\beta}\left[e^{-2r} - \frac{1}{4}e^{-4r}\Box + \frac{4}{3}\phi_1^3\left(r + \frac{9}{4} + C^*\right)e^{-4r}\right] + \\
&\quad + \frac{8}{3}\sqrt{\frac{8}{3}}a_{\chi}\phi_1^3\left(r + \frac{9}{8} + C^*\right)e^{-4r} + \mathcal{O}\left(e^{-6r}\right), \qquad (16.1.10)
\end{aligned}$$

where C_{χ} and C_{β} are scheme constants.

The first set of solutions has the basis $a_{\chi} = 1, a_{\beta} = 0$,

$$\begin{aligned}
\hat{\mathfrak{a}}_1^{\chi} &= e^{-r} + re^{-3r}\left(\frac{1}{2}\Box - 8\phi_1^2\right) + C_{1\chi}\phi_1^2 e^{-3r} + \mathcal{O}\left(e^{-5r}\right) \\
\hat{\mathfrak{a}}_1^{\beta} &= C_{1\beta}\left[\phi_1 e^{-2r} - \frac{1}{4}\phi_1 e^{-4r}\Box + \frac{4}{3}\phi_1^3\left(r + \frac{9}{4} + C^*\right)e^{-4r}\right] + \\
&\quad + \frac{8}{3}\sqrt{\frac{8}{3}}\phi_1^3\left(r + \frac{9}{8} + C^*\right)e^{-4r} + \mathcal{O}\left(e^{-6r}\right), \qquad (16.1.11)
\end{aligned}$$

and the second set of dominant solutions has the basis $a_{\chi} = 0, a_{\beta} = 1$,

$$\begin{aligned}
\hat{\mathfrak{a}}_2^{\chi} &= C_{2\chi}\phi_1 e^{-3r} + \mathcal{O}\left(e^{-5r}\right) \\
\hat{\mathfrak{a}}_2^{\beta} &= re^{-2r} - \frac{1}{4}(r+1)\,e^{-4r}\Box + \\
&\quad + \left[\frac{4}{3}\phi_1^2 r^2 + \left(\frac{13}{3}\phi_1^2 + \frac{1}{2}\sqrt{\frac{8}{3}}\phi_2\right) r + \frac{10}{3}\phi_1^2 + \frac{1}{2}\sqrt{\frac{8}{3}}\phi_2\right] e^{-4r} + \\
&\quad + C_{2\beta}\left[e^{-2r} - \frac{1}{4}e^{-4r}\Box + \frac{4}{3}\phi_1^2\left(r + \frac{9}{4} + C^*\right)e^{-4r}\right] + \mathcal{O}\left(e^{-6r}\right). \qquad (16.1.12)
\end{aligned}$$

In terms of A, the solutions read

$$\begin{aligned}
\hat{\mathfrak{a}}_1^{\chi} &= e^{-A} + \left(\frac{1}{2}A + C_{1\chi}^1\right)\Box\, e^{-3A} + \left(-8A - \frac{1}{6} + C_{1\chi}^0\right)\phi_1^2 e^{-3A} + \mathcal{O}\big(e^{-5A}\big) \\
\hat{\mathfrak{a}}_1^{\beta} &= C_{1\beta}\phi_1 e^{-2A} - \frac{1}{4}C_{1\beta}\phi_1 e^{-4A}\,\Box + \\
&\quad + \frac{8}{3}\sqrt{\frac{8}{3}}\left[\left(1 + \frac{3}{16}\sqrt{\frac{8}{3}}C_{1\beta}\right)(A + C^*) + \frac{9}{8} + \frac{3}{8}\sqrt{\frac{8}{3}}C_{1\beta}\right]\phi_1^3 e^{-4A} + \mathcal{O}\big(e^{-6A}\big) \\
\hat{\mathfrak{a}}_2^{\chi} &= C_{2\chi}\phi_1 e^{-3A} + \mathcal{O}\big(e^{-5A}\big) \\
\hat{\mathfrak{a}}_2^{\beta} &= (A + C_{2\beta})\, e^{-2A} + \\
&\quad + \frac{4}{3}\left[(A + C_{2\beta})^2 + (3 + C^* - C_{2\beta})(A + C_{2\beta}) + C^* - C_{2\beta} + \frac{21}{8}\right]\phi_1^2 e^{-4A} + \\
&\quad - \frac{1}{4}(A + C_{2\beta} + 1)\, e^{-4A}\,\Box + \mathcal{O}\big(e^{-6A}\big),
\end{aligned} \tag{16.1.13}$$

where we have defined

$$C_{1\chi} \equiv C_{1\chi}^0 + C_{1\chi}^1 \frac{\Box}{\phi_1^2}. \tag{16.1.14}$$

The solutions can be combined into a matrix as follows

$$\hat{\mathfrak{a}}_i^a = \begin{pmatrix} \hat{\mathfrak{a}}_1^{\chi} & \hat{\mathfrak{a}}_2^{\chi} \\ \hat{\mathfrak{a}}_1^{\beta} & \hat{\mathfrak{a}}_2^{\beta} \end{pmatrix}. \tag{16.1.15}$$

Counterterm Matrix From the mass matrix $\mathcal{M}_{ab}$ given by eq.(16.1.5) and the matrix of dominant solutions $\hat{\mathfrak{a}}_i^a$ whose elements are given by eq.(16.1.13) we construct the matrix $\mathcal{U}_{ab}$, defined by eq.(9.2.1),

$$\begin{aligned}
\mathcal{U}_{\chi\chi} &= \left[-\frac{32}{3}\phi_1^2\left(A - \frac{1}{2}C^* - \frac{5}{8} - \frac{3}{16}C_{1\chi}^0\right) + \left(A - \frac{1}{2} + 2C_{1\chi}^1\right)\Box\right] e^{-2A} + \mathcal{O}\left(\frac{e^{-2A}}{A}\right) \\
\mathcal{U}_{\beta\beta} &= -\frac{1}{A} + \frac{C_{2\beta}}{A^2} + \mathcal{O}\left(\frac{1}{A^3}\right) \\
\mathcal{U}_{\beta\chi} &= 2\sqrt{\frac{8}{3}}\phi_1 e^{-A} + C_{\beta\chi}\frac{e^{-A}}{A} + \mathcal{O}\left(\frac{e^{-A}}{A^2}\right),
\end{aligned} \tag{16.1.16}$$

where we have defined

$$C_{\beta\chi} \equiv C_{2\chi} + \frac{C_{1\beta}}{2}. \tag{16.1.17}$$

We read off

$$\mathcal{U}_{0\chi\chi} = -\frac{32}{3}\phi_1^2\left(A - \frac{1}{2}C^* - \frac{5}{8} - \frac{3}{16}C_{1\chi}^0\right)e^{-2A}, \qquad \mathcal{U}_{1\chi\chi} = A - \frac{1}{2} + 2C_{1\chi}^1. \tag{16.1.18}$$

From the requirements $\mathcal{U}_{0\chi\chi} = \partial_\chi^2 \mathcal{U}_0$, $\mathcal{U}_{0\beta\beta} = \partial_\beta^2 \mathcal{U}_0$ and $\mathcal{U}_{0\beta\chi} = \partial_\chi \partial_\beta \mathcal{U}_0$, we find

$$\begin{aligned}\mathcal{U}_0 \;&=\; \left(-\frac{1}{2A} + \frac{C_{2\beta}}{2A^2}\right)\phi_\beta^2 + \left(\sqrt{\frac{8}{3}} + \frac{C_{\beta\chi}}{2A}\right)\phi_\beta\phi_\chi^2 + \\ &\quad + \left(-\frac{4}{3}A + \frac{5}{9} + \frac{C^0_{1\chi}}{6} - \frac{1}{12}\sqrt{\frac{8}{3}}C_{\beta\chi}\right)\phi_\chi^4. \end{aligned} \tag{16.1.19}$$

Later we will show how to determine $\mathcal{U}_0$ systematically.

Zero modes The zero mode solutions are given by

$$\begin{aligned}\frac{\mathcal{W}^\beta}{\mathcal{W}} \;&=\; \frac{4}{3}\sqrt{\frac{8}{3}}\phi_1^2\left(A + C^* - \frac{1}{2}\right)e^{-2A} + \mathcal{O}\big(e^{-4A}\big) \\ \frac{\mathcal{W}^\chi}{\mathcal{W}} \;&=\; \frac{2}{3}\phi_1 e^{-A} - \frac{16}{3}\phi_1^3\left(A + C^* + \frac{5}{48}\right)e^{-3A} + \mathcal{O}\big(e^{-5A}\big)\,, \end{aligned} \tag{16.1.20}$$

from which we read off, see eq.(5.1.20) and eq.(16.1.13),

$$\begin{aligned}\hat{c}_1 &= \frac{2}{3}\phi_1, \qquad & \check{c}_1 &= -\frac{4}{9}\phi_1^3 - 2\sqrt{\frac{8}{3}}\phi_1\phi_2 - \frac{2}{3}\phi_1^3 C^0_{1\chi} \\ \hat{c}_2 &= \frac{4}{3}\sqrt{\frac{8}{3}}\phi_1^2, \qquad & \check{c}_2 &= -\frac{2}{3}\sqrt{\frac{8}{3}}\phi_1^2 + \frac{4}{3}\sqrt{\frac{8}{3}}\phi_1^2\left(C^* - C_{2\beta}\right). \end{aligned} \tag{16.1.21}$$

16.1.2 Metric

The results for $\hat{\mathfrak{e}}$ and $\mathcal{T}$ are exactly like in GPPZ, with $\phi_{\mathfrak{s}}$ replaced by ϕ_1.

16.2 Solving differential equations

Now we determine the counterterms using the differential equations given by eq.(10.1.8)

$$\begin{aligned} 0 \;&=\; \left(\mathcal{D}_r + d\dot{A}\right)\mathcal{U}_{0ab} + \mathcal{M}^c_{(a}\mathcal{U}_{0b)c} - \tilde{\mathcal{U}}^c_{0a}\tilde{\mathcal{U}}_{0bc} \\ \bar{G}_{ab} \;&=\; \left(\mathcal{D}_r + (d-2)\dot{A}\right)\mathcal{U}_{1ab} + \mathcal{M}^c_{(a}\mathcal{U}_{1b)c} - \tilde{\mathcal{U}}^c_{0(a}\tilde{\mathcal{U}}_{1b)c}. \end{aligned} \tag{16.2.1}$$

The two scalar system is more complicated than the one scalar system since now the differential equations are coupled, *i.e.* they mix the two scalar types β and χ together. For example, the first differential equation actually contains three differential equations, one for each function $\mathcal{U}_{0\beta\beta}, \mathcal{U}_{0\beta\chi}$, and $\mathcal{U}_{0\chi\chi}$. But since there is a sum over c, the differential equation for each function depends on at least one of the other functions as well, so, where do we start?

First, we will determine which of the functions is the strongest. Then we start with the differential equation for the strongest function (in this case $\mathcal{U}_{0\beta\beta}$, as we will see), so we can neglect the terms proportional to the weaker functions (in this case $\mathcal{U}_{0\beta\chi}$ and $\mathcal{U}_{0\chi\chi}$). The differential equation for the strongest function thus decouples from the other functions, and can easily be solved. Then we evaluate the differential equation for the next-to-strongest function (in this case $\mathcal{U}_{0\beta\chi}$). It decouples from all weaker functions (in this case only $\mathcal{U}_{0\chi\chi}$) but remains coupled to the strongest function ($\mathcal{U}_{0\beta\beta}$). We already know the solution for the strongest function, so we can simply substitute this solution, leaving us again with a decoupled differential equation that can easily be solved. This procedure can be continued down to the weakest function (in this case $\mathcal{U}_{0\chi\chi}$). Every time we have found the solution for some function, we can check for consistency if it really is of the order we expected. If everything is consistent, then we know the solutions must be correct, since they are unique.

Let us first determine the order up to which we need $\mathcal{U}_{\chi\chi}$, $\mathcal{U}_{\beta\beta}$ and $\mathcal{U}_{\beta\chi}$. In the action, they appear in the combinations

$$e^{4r}\phi_\chi\mathcal{U}_{\chi\chi}\phi_\chi, \qquad e^{4r}\phi_\beta\mathcal{U}_{\beta\chi}\phi_\chi, \qquad e^{4r}\phi_\beta\mathcal{U}_{\beta\beta}\phi_\beta. \tag{16.2.2}$$

Substituting the expressions for ϕ_χ and ϕ_β yields

$$\sim e^{2r}\mathcal{U}_{\chi\chi}, \qquad \sim re^{r}\mathcal{U}_{\beta\chi}, \qquad \sim r^2\mathcal{U}_{\beta\beta}. \tag{16.2.3}$$

We see that we need

$$\mathcal{U}_{\chi\chi} \to \mathcal{O}\left(e^{-2r}\right), \qquad \mathcal{U}_{\beta\chi} \to \mathcal{O}\left(r^{-1}e^{-r}\right), \qquad \mathcal{U}_{\beta\beta} \to \mathcal{O}\left(r^{-2}\right). \tag{16.2.4}$$

Since we need $\mathcal{U}_{\chi\chi}$ up to order e^{-2r}, we need to find $\mathcal{U}_{1\chi\chi}$ as well

$$\mathcal{U}_{\chi\chi} = \mathcal{U}_{0\chi\chi} + \mathcal{U}_{1\chi\chi}e^{-2A}\,\Box + \mathcal{O}\left(\Box^2\right). \tag{16.2.5}$$

We thus need to find $\mathcal{U}_{0\beta\beta}, \mathcal{U}_{0\beta\chi}, \mathcal{U}_{0\chi\chi}$, and $\mathcal{U}_{1\chi\chi}$. From eq.(16.2.4) we see that the strongest function is $\mathcal{U}_{0\beta\beta}$, so let us start by substituting $a = b = \beta$ in the differential equation for $\mathcal{U}_{0ab}$, given by the first line of eq.(16.2.1)

$$\frac{d}{dr}\mathcal{U}_{0\beta\beta} = \left(\mathcal{U}_{0\beta\beta}\right)^2 + \mathcal{O}\left(e^{-2r}\right). \tag{16.2.6}$$

The solution reads

$$\mathcal{U}_{0\beta\beta} = -\frac{1}{r} + \frac{C_{U0\beta\beta}}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right). \tag{16.2.7}$$

The next-to-strongest function is $\mathcal{U}_{0\beta\chi}$, whose differential equation can be found by substituting $a = \beta, b = \chi$ in eq.(16.2.1). This time we may neglect terms with $\mathcal{U}_{0\chi\chi}$ because they are weaker than $\mathcal{U}_{0\beta\chi}$. The differential equation is still coupled to $\mathcal{U}_{0\beta\beta}$, but we can substitute the solution given by eq.(16.2.7), yielding a decoupled equation for $\mathcal{U}_{0\beta\chi}$,

$$0 = \left(\frac{d}{dr} + 1\right)\mathcal{U}_{0\beta\chi} - \frac{1}{r}\left(2\sqrt{\frac{8}{3}}\phi_1 e^{-r} - \mathcal{U}_{0\beta\chi}\right) + \mathcal{O}\left(\frac{e^{-r}}{r^2}\right). \tag{16.2.8}$$

The solution reads

$$\mathcal{U}_{0\beta\chi} = 2\sqrt{\frac{8}{3}}\phi_1 e^{-r} + C_{U0\beta\chi}\frac{e^{-r}}{r} + \mathcal{O}\left(\frac{e^{-r}}{r^2}\right), \tag{16.2.9}$$

Finally, substituting $a = b = \chi$ and the solution for $\mathcal{U}_{0\beta\chi}$ in eq.(16.2.1) yields

$$\left(\frac{d}{dr} + 2\right)\mathcal{U}_{0\chi\chi} = -\frac{32}{3}\phi_1^2 e^{-2r}. \tag{16.2.10}$$

The solution reads

$$\mathcal{U}_{0\chi\chi} = -\frac{32}{3}\phi_1^2 \left(r + C_{U0\chi\chi}\right) e^{-2r}. \tag{16.2.11}$$

Substituting $a = b = \chi$ in the differential equation for $\mathcal{U}_{1ab}$ yields, up to the required order

$$\frac{d}{dr}\mathcal{U}_{1\chi\chi} = 1, \tag{16.2.12}$$

which has the solution

$$\mathcal{U}_{1\chi\chi} = r + C_{U1\chi\chi}. \tag{16.2.13}$$

Determining $\mathcal{T}$ is equivalent to the GPPZ case. Let us summarize our results

$$\begin{aligned}
\mathcal{U}_{0\beta\beta} &= -\frac{1}{r + C_{U0\beta\beta}} & \mathcal{U}_{1\chi\chi} &= r + C_{U1\chi\chi} \\
\mathcal{U}_{0\beta\chi} &= 2\sqrt{\frac{8}{3}}\phi_1 e^{-r} + C_{U0\beta\chi}\frac{e^{-r}}{r} & \mathcal{T}_1 &= -\frac{1}{2} + \frac{1}{3}\phi_1^2\left(r + C_{T1}\right)e^{-2r} \\
\mathcal{U}_{0\chi\chi} &= -\frac{32}{3}\phi_1^2\left(r + C_{U0\chi\chi}\right)e^{-2r} & \mathcal{T}_2 &= -\frac{1}{4}\left(r + C_{T2}\right).
\end{aligned} \tag{16.2.14}$$

The solution for the matrix $\mathcal{U}_{0ab}$ is consistent with our previous estimate for the strength of its components $\mathcal{U}_{0\beta\beta}$, $\mathcal{U}_{0\beta\chi}$ and $\mathcal{U}_{0\chi\chi}$. The relations between the constants are

$$\begin{aligned}
C_{U0\chi\chi} &= -\frac{1}{2}\Delta - \frac{5}{8} - \frac{3}{16}\tilde{C}^0_{1\chi}, \quad C_{U0\beta\beta} = C_{2\beta}, \quad C_{U0\beta\chi} = C_{\beta\chi} \\
C_{U1\chi\chi} &= 2\tilde{C}^1_{1\chi} - \frac{1}{2}, \quad C_{T1} = \frac{1}{4} - 12C_{1\hat{\mathfrak{e}}}, \quad C_{T2} = -\frac{3}{4} + 16C_{2\hat{\mathfrak{e}}}.
\end{aligned} \tag{16.2.15}$$

16.3 Preferred Scheme

We see from eq.(16.2.14) that all scheme constants come from the ambiguity in choosing the cut-off and can therefore be written as $r+$ constant, with the exception of $C_{U0\beta\chi}$. We may therefore argue that $C_{U0\beta\chi} = 0$ in a natural scheme.[1] Then we find

$$\mathcal{U}_0 = -\frac{\phi_\beta^2}{2\left(r + C_{U0\beta\beta}\right)} + \sqrt{\frac{8}{3}}\phi_\chi^2\phi_\beta - \frac{4}{3}\left(r + C_{U0\chi\chi}\right)\phi_\chi^4. \tag{16.3.1}$$

[1] In GPPZ, we have the finite term $C\Phi^4$. The scheme constant C can not be written as $r + C$, so it may seem unnatural. However, we can argue that the condition of zero vacuum energy is a more important condition for a natural scheme, which sets the scheme constant to $C = 1/18$. This term is already included in the counterterm $-W$, so we find $C_{U0} = 0$ in GPPZ.

We may even argue that all scheme constants at a certain order in $\Box$ have to be equal to each other when written as $r+$ constant. This means that all counterterms of the same order in $\Box$ live at the same cut-off boundary. Then we find $C_{U0\chi\chi} = C_{U0\beta\beta}$ and $C_{U1\chi\chi} = C_{T1}$. Furthermore, we may argue that the vacuum energy vanishes in a natural scheme, $e = 0$. These criteria of naturalness, namely $e = 0$, $C_{U0\beta\chi} = 0$ and $C_{U0\chi\chi} = C_{U0\beta\beta}$, lead to $C_{U0\chi\chi} = C_{U0\beta\beta} = C^*$. Then both vevs v_i defined by eq.(12.2.7) vanish automatically. Furthermore, it allows us to complete the square,

$$\mathcal{U}_0 = -\frac{4}{3}\left(A + C^*\right)\left(\phi_\chi^2 - \sqrt{\frac{3}{8}}\frac{\phi_\beta}{(A+C^*)}\right)^2 . \tag{16.3.2}$$

We see that our criteria of naturalness are sufficient to remove all scheme dependence from the correlators. We also note that we would have gotten the same result by requiring all scheme constants, at all orders in $\Box$, to be equal to C^*, implying that all counterterms live at the same cut-off boundary. The covariant counterterm then reads

$$\begin{aligned}\mathcal{L}_{\text{cov}} \;=\; & \frac{1}{8}R + (A+C^*)\Bigg[-\frac{4}{3}\left(\Phi_\chi^2 - \sqrt{\frac{3}{8}}\frac{\Phi_\beta}{(A+C^*)}\right)^2 + \\ & -\frac{1}{2}\left(\nabla\Phi_\chi\right)^2 - \frac{1}{12}\Phi_\chi^2 R + \frac{1}{16}\left(R^{ij}R_{ij} - \frac{1}{3}R^2\right)\Bigg].\end{aligned} \tag{16.3.3}$$

16.4 Spontaneous Vev

If we make the scalar χ inert by setting $\phi_1 = 0$, then $e = 0$ automatically, and the scheme constants remain unfixed. Using eq.(12.2.7), we see a spontaneous vev appear $v_2 = \phi_2$, as expected because $\Delta_\beta = 2$. All the terms proportional to Φ_χ dissappear from the counterterm, leaving exactly the counterterm for the Coulomb Branch Flow given by the first line of eq.(13.4.1).[2] It is intriguing that the mixed counterterm comes from a complete square between the individual counterterms

$$\sqrt{\frac{8}{3}}\Phi_\chi^2\Phi_\beta .$$

Furthermore, this is exactly the same term as the mixed term in the superpotential W.

[2]On the other hand, removing Φ_β from the counterterm does not give the counterterm for GPPZ.

16.5 Inversion in the two-scalar system

We will now show how to systematically obtain $\mathcal{U}_0(r,\phi_\chi,\phi_\beta)$ from $\mathcal{U}_0(r,\phi_1,\phi_2)$. We repeat eq.(16.1.4),

$$\begin{aligned} \phi_\beta &= \sqrt{\frac{8}{3}}\phi_1^2 r e^{-2r} + \phi_2 e^{-2r} + \mathcal{O}\big(e^{-4r}\big)\,, \\ \phi_\chi &= \phi_1 e^{-r} + \mathcal{O}\big(e^{-3r}\big)\,. \end{aligned} \tag{16.5.1}$$

The inverse relations are

$$\begin{aligned} \phi_1 &= e^r\phi_\chi + \mathcal{O}\big(e^{-2r}\big) \\ \phi_2 &= e^{2r}\phi_\beta - \sqrt{\frac{8}{3}} r e^{2r}\phi_\chi^2 + \mathcal{O}\big(e^{-2r}\big)\,. \end{aligned} \tag{16.5.2}$$

We repeat our result for the elements of $\mathcal{U}_{0ab}(r,\phi_1,\phi_2)$,

$$\begin{aligned} \mathcal{U}_{0\chi\chi} &= -\frac{32}{3} r\phi_1^2 e^{-2r} + \mathrm{const}(\phi_1,\phi_2)e^{-2r} \\ \mathcal{U}_{0\beta\chi} &= 2\sqrt{\frac{8}{3}}\phi_1 e^{-r} \\ \mathcal{U}_{0\beta\beta} &= -\frac{1}{r} + \frac{\mathrm{const}(\phi_1,\phi_2)}{r^2}, \end{aligned} \tag{16.5.3}$$

where "const" stands for any combination of its arguments, reflecting the scheme dependence. Using the inverse relations eq.(16.5.2), we find the desired $\mathcal{U}_{0ab}(r,\phi_\chi,\phi_\beta)$,

$$\begin{aligned} \mathcal{U}_{0\chi\chi} &= -\frac{32}{3} r\phi_\chi^2 + \mathrm{const}(r,\phi_\chi,\phi_\beta)e^{-2r} \\ \mathcal{U}_{0\beta\chi} &= 2\sqrt{\frac{8}{3}}\phi_\chi \\ \mathcal{U}_{0\beta\beta} &= -\frac{1}{r} + \frac{\mathrm{const}(r,\phi_\chi,\phi_\beta)}{r^2}, \end{aligned} \tag{16.5.4}$$

where "const" stands for any combination of its arguments such that they combine to a constant. Without any loss of generality we may write

$$\mathcal{U}_0 = f_1(r,\phi_\chi) + f_2(r,\phi_\chi,\phi_\beta) + f_3(\phi_\beta). \tag{16.5.5}$$

It is easiest to start with $\mathcal{U}_{0\beta\chi}$, because it involves only f_2

$$\mathcal{U}_{0\beta\chi} = \partial_\beta\partial_\chi f_2(r,\phi_\chi,\phi_\beta) = 2\sqrt{\frac{8}{3}}\phi_\chi. \tag{16.5.6}$$

This can easily be integrated

$$\mathcal{U}_0 = f_1(r,\phi_\chi) + \sqrt{\frac{8}{3}}\phi_\beta\phi_\chi^2 + f_3(r,\phi_\beta). \tag{16.5.7}$$

Now we find

$$\mathcal{U}_{0\beta\beta} = \partial_\beta^2 f_3(\phi_\beta) = -\frac{1}{r} + \text{const}(r, \phi_\chi, \phi_\beta) r^{-2}. \tag{16.5.8}$$

The constant may be a function of ϕ_β, and hence when we integrate we find

$$\mathcal{U}_0 = f_1(r, \phi_\chi) + \sqrt{\frac{8}{3}}\phi_\beta\phi_\chi^2 - \frac{\phi_\beta^2}{2r} + \frac{1}{r^2}\int d^2\phi_\beta \,\text{const}(r, \phi_\chi, \phi_\beta). \tag{16.5.9}$$

However, we know that the last term is a scheme dependent term, so we can write

$$\mathcal{U}_0 = f_1(r, \phi_\chi) + \sqrt{\frac{8}{3}}\phi_\beta\phi_\chi^2 - \frac{\phi_\beta^2}{2r} + \text{finite}. \tag{16.5.10}$$

From this we find

$$\mathcal{U}_{0\chi\chi} = \partial_\chi^2 f_1(r, \phi_\chi) + 2\sqrt{\frac{8}{3}}\phi_\beta = -\frac{32}{3} r\phi_\chi^2 + \text{const}(r, \phi_\chi, \phi_\beta) e^{-2r}. \tag{16.5.11}$$

Our goal is to find f_1

$$\begin{aligned}
\partial_\chi^2 f_1(r, \phi_\chi) &= -\frac{32}{3} r\phi_\chi^2 - 2\sqrt{\frac{8}{3}}\phi_\beta + \text{const}(r, \phi_\chi, \phi_\beta) e^{-2r} \\
&= -\frac{32}{3} r\phi_1^2 e^{-2r} - \frac{16}{3} r\phi_1^2 e^{-2r} - 2\sqrt{\frac{8}{3}}\phi_2 e^{-2r} + \text{const}(\phi_1, \phi_2) e^{-2r} \\
&= -16 r\phi_1^2 e^{-2r} + \text{const}(\phi_1, \phi_2) e^{-2r} \\
&= -16 r\phi_\chi^2 + \text{const}(r, \phi_\chi, \phi_\beta) e^{-2r}.
\end{aligned} \tag{16.5.12}$$

We can integrate this to find

$$f_1(r, \phi_\chi) = -\frac{4}{3} r\phi_\chi^4 + \int d^2\phi_\chi \text{const}(r, \phi_\chi, \phi_\beta) e^{-2r}. \tag{16.5.13}$$

The last term is scheme dependent so it must be finite

$$f_1(r, \phi_\chi) = -\frac{4}{3} r\phi_\chi^4 + \text{finite}. \tag{16.5.14}$$

Then we find

$$\mathcal{U}_0 = -\frac{4}{3} r\phi_\chi^4 + \sqrt{\frac{8}{3}}\phi_\beta\phi_\chi^2 - \frac{\phi_\beta^2}{2r} + \text{finite}. \tag{16.5.15}$$

Now we can go unambiguously (up to scheme dependence) from the background to the full fields,

$$U_0 = -\frac{4}{3} r\Phi_\chi^4 + \sqrt{\frac{8}{3}}\Phi_\beta\Phi_\chi^2 - \frac{\Phi_\beta^2}{2r} + \text{finite}. \tag{16.5.16}$$

Chapter 17

Non-aAdS: Klebanov-Strassler

$\diamond$ Covariant Holographic Renormalization is constructed to be applicable to both aAdS and non-aAdS spacetimes. The previous case studies were all in aAdS backgrounds. In this section we perform a first calculation in a non-aAdS background, namely the Klebanov-Strassler background. More details can be found in [2] and [3]. $\diamond$

17.1 Background

The KS theory is four-dimensional, has seven scalars fields $(x, p, y, \Phi, b, h_1, h_2)$ and the superpotential is given by, see eq.(5.2) in [6]

$$W = -\frac{1}{2}\left(e^{-2p-2x} + e^{4p}\cosh y\right) + \frac{1}{4}e^{4p-2x}\left[Q + 2P\left(bh_2 + h_1\right)\right], \tag{17.1.1}$$

The superpotential does not have a fixed point, so the metric solution will be non-aAdS. The sigma-model metric is given by eq.(5.1) in [6], but we will not need it here. We use the variable τ and in favor of r,

$$\partial_r = e^{4p}\partial_\tau. \tag{17.1.2}$$

We also introduce

$$h(\tau) \equiv \int_\tau^\infty d\theta \frac{\theta\coth\theta - 1}{\sinh^2\theta}\left[2\sinh(2\theta) - 4\theta\right]^{1/3}. \tag{17.1.3}$$

In the limit of large τ, we find

$$h(\tau) = 3\left(\tau - \frac{1}{4}\right)e^{-4\tau/3} + \mathcal{O}\left(e^{-10\tau/3}\right). \tag{17.1.4}$$

The background solutions are given by

$$\begin{aligned}
\Phi &= \Phi_0 \\
e^y &= \tanh(\tau/2) \\
b &= -\frac{\tau}{\sinh\tau} \\
h_1 &= -\frac{Q}{2P} + Pe^{\Phi_0}\coth\tau\,(\tau\coth\tau - 1) \\
h_2 &= Pe^{\Phi_0}\,(\tau\coth\tau - 1)\sinh^{-1}\tau \\
\frac{2}{3}e^{6p+2x} &= \coth\tau - \frac{\tau}{\sinh^2\tau} \\
e^{2x/3-4p} &= 3^{-2/3}2P^2e^{\Phi_0}h(\tau)\sinh^{4/3}\tau.
\end{aligned} \tag{17.1.5}$$

The warp function is given by

$$e^{-2A} \sim e^{4p}\left(e^{-2x}\sinh\tau\right)^{2/3}h(\tau) \sim \left(\tau - \frac{1}{4}\right)^{-1/3} e^{-2\tau/3} + \mathcal{O}\left(e^{-4\tau/3}\right), \tag{17.1.6}$$

where the proportionality factor sets the momentum scale. From this we find, see eq.(4.1.6),

$$\dot{A} = -\frac{3}{2}\mathcal{W} = \frac{1}{6}P^{-4/3}e^{-2\Phi_0/3}\left(\tau - \frac{1}{4}\right)^{-2/3}\left[2 + \left(\tau - \frac{1}{4}\right)^{-1}\right], \tag{17.1.7}$$

showing that $\mathcal{W} \to 0$ at the boundary so there is no aAdS solution, see subsection 7.1.

17.2 Cancellation of Divergences

In section 12, we showed that under some mild assumptions

$$e^{(d-2)A}\left(\frac{\mathcal{W}^a\mathcal{W}^b}{\mathcal{W}^2}\mathcal{U}_{1ab} + \frac{1}{\mathcal{W}} - \frac{2(d-2)}{d-1}\mathcal{T}_1\right) = \text{finite}, \tag{17.2.1}$$

even though neither of the terms is finite by itself, in general. We verified this result in all cases we considered, including the Klebanov Strassler (KS) theory. We now roughly sketch how we performed this calculation, without focussing on the details.

The vector $\mathcal{W}^a/\mathcal{W}$ is given by eq.(5.26) in [6], and the matrix $\mathcal{U}_{ab}$ is given by eq.(5.20) and eq.(5.21) in the same reference, from which we can read off $\mathcal{U}_{1ab}$ using eq.(9.2.3).[1] This allowed us to calculate the first term in eq.(17.2.1). The second term follows from eq.(17.1.7). The last term in eq.(17.2.1) follows from the differential equation for $\mathcal{T}_1$ given by eq.(10.2.3), which reads, using $d = 4$ and eq.(17.1.7),

$$-P^{4/3}e^{2\Phi_0/3}\left(\tau - \frac{1}{4}\right)^{2/3} = \left(\partial_\tau + \frac{2}{3} + \frac{1}{3\tau - 3/4}\right)\mathcal{T}_1, \tag{17.2.2}$$

[1]In [6] only the leading terms are given, but we also need the subleading terms in this calculation. This section only aims to sketch the steps of the calculation.

where the left-hand side is just $-e^{-4p}$. The solution reads

$$\begin{aligned} \mathcal{T}_1 &= -\frac{3}{2}P^{4/3}e^{2\Phi_0/3}\left[\left(\tau-\frac{1}{4}\right)^{2/3}-\frac{3}{2}\left(\tau-\frac{1}{4}\right)^{-1/3}\right] \\ &\quad + C_{T1}\left(\tau-\frac{1}{4}\right)^{-1/3}e^{-2\tau/3}, \end{aligned} \tag{17.2.3}$$

where C_{T1} is a scheme constant. We found that eq.(17.2.1) is indeed finite in KS. This is an important result because neither one of the three terms in eq.(17.2.1) is finite by itself. The counterterms proportional to U_{1ab} and T_1 apparently conspire to kill the divergences at order $\Box$ coming externally from from the bare action, while the remaining divergences within the counterterm cancel internally.

Part IV

Summary, Outlook and Appendix

Chapter 18

Summary

◇ By construction, Covariant Holographic Renormalization (CHR) can be applied to both aAdS and non-aAdS spacetimes. This is progress, because alternative holographic renormalization programs work only in aAdS. The advantage of aAdS is that there exists a fixed point around which one can make a Taylor expansion, which simplifies the procedure considerably. In non-aAdS there is no fixed point. Instead, CHR is based on an expansion in small fluctuations around the background solutions for the scalars and the metric. To calculate n-point functions, we need to keep terms to n-th order in the fluctuations. In this thesis we considered only one- and two-point functions, and therefore kept terms up to quadratic order in the fluctuations. ◇

18.1 Covariant Holographic Renormalization

Our approach was the following. First, we wrote down the divergences of the bare, on-shell action up to second order in the gauge invariant fluctuations, given by eq.(8.4.2),

$$S_{\text{bare}} = \frac{e^{dA}}{2}\int d^d x \left(\mathfrak{a}^a \left(\bar{G}_{ab}\mathcal{D}_r - \mathcal{M}_{ab}\right)\mathfrak{a}^b + \frac{1}{8}\dot{\mathfrak{e}}^i_j \mathfrak{e}^j_i - \frac{e^{-2A}}{16\mathcal{W}} h \,\square\, h\right) + \mathcal{O}\big(f^3\big)\,. \quad (18.1.1)$$

From previous work on holographic renormalization in non-aAdS, we know the explicit expression for the matrix $\mathcal{U}_{ab}$, given by eq.(9.2.1), that appears in the counterterm between the scalar fluctuations as $\mathfrak{a}^a \mathcal{U}_{ab} \mathfrak{a}^b$. Since the metric behaves as a massless scalar, we also know the equivalent expression for $\mathcal{T}$, given by eq.(9.2.2), that appears in the counterterm between the metric fluctuations as $\mathfrak{e}^i_j \mathcal{T} \mathfrak{e}^j_i$. It was therefore straightforward to write down the required counterterms, given by eq.(9.2.4),

$$S_{\text{cnt}} = \frac{e^{dA}}{2}\int d^d x \left(\mathfrak{a}^a \mathcal{U}_{ab} \mathfrak{a}^b - \frac{1}{8}\mathfrak{e}^i_j \mathcal{T} \mathfrak{e}^j_i + \frac{e^{-2A}}{16\mathcal{W}} h \,\square\, h\right) + \mathcal{O}\big(f^3\big)\,. \quad (18.1.2)$$

We showed how this non-covariant required piece arises from a boundary covariant counterterm. We therefore wrote down the most simple and general structure of the covariant counterterm action which is known to reproduce the correct results in aAdS,

$$S_{\rm cov} = \int d^d x \sqrt{\gamma} \mathcal{L}_{\rm cov}, \tag{18.1.3}$$

where $\mathcal{L}_{\rm cov}$ is given by eq.(11.2.4),

$$\begin{aligned} \mathcal{L}_{\rm cov} \;=\; & U_0 - \frac{1}{2} U_{1ab} \nabla^n \Phi^a \nabla_n \Phi^b + \frac{1}{2} U_{2ab} \nabla^2 \Phi^a \nabla^2 \Phi^b + \\ & - \frac{1}{4} T_1 R - \frac{1}{4} T_2 R^{ij} R_{ij} + B R^2 + C_a \nabla^i \Phi^a \nabla_i R + \mathcal{O}\big(\Box^3\big) \,. \end{aligned} \tag{18.1.4}$$

This way we reduced the problem to finding a set of unknown counterterm functions U_0, U_{1ab}, U_{2ab}, T_1, T_2, B, C_a. Boundary covariance allows each of these functions to depend both on the scalars Φ^a and the explicit cut-off A.

Since we focussed our attention on calculating one- and two-point functions, we expanded the action (bare action plus counterterm) in small fluctuations around the background, keeping terms up to quadratic order in the (gauge invariant) fluctuations. This reduced the problem to finding only the background solutions $\mathcal{U}_0$, $\mathcal{U}_{1ab}$, $\mathcal{U}_{2ab}$, $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{B}$, $\mathcal{C}_a$, which are functions of the background scalar modes ϕ^a and the explicit cut-off A. In turn, $\phi^a(r)$ and $A(r)$ are functions of the cut-off A only, so the background solutions $\mathcal{U}_0(A)$, $\mathcal{U}_{1ab}(A)$, ... are also functions of A only.

After the expansion, the following terms appeared in the counterterm

$$\begin{aligned} & \mathfrak{a}^a \left[\mathcal{U}_{0ab} + \mathcal{U}_{1ab}(\phi, A) e^{-2A} \, \Box + \mathcal{U}_{2ab}(\phi, A) e^{-4A} \, \Box^2 + \mathcal{O}\big(\Box^3\big) \right] \mathfrak{a}^b, \\ & \mathfrak{e}^i_j \left[\mathcal{T}_1 e^{-2A} \, \Box + \mathcal{T}_2 e^{-4A} \, \Box^2 + \mathcal{O}\big(\Box^3\big) \right] \mathfrak{e}^j_i. \end{aligned} \tag{18.1.5}$$

The terms within brackets are precisely the expansions in $\Box$ of the counterterm matrices $\mathcal{U}_{ab}$ and $\mathcal{T}$ respectively, given by eq.(9.2.3). Thus, the requirement that we find the terms $\mathfrak{a}^a \mathcal{U}_{ab} \mathfrak{a}^b$ and $\mathfrak{e}^i_j \mathcal{T} \mathfrak{e}^j_i$ in the counterterm fixed the functions $\mathcal{U}_0$, $\mathcal{U}_{1ab}$, $\mathcal{U}_{2ab}$, $\mathcal{T}_1$, $\mathcal{T}_2$. Thus we have shown how the required non-covariant piece $S_{\rm cnt}$ arises from a boundary covariant counterterm $S_{\rm cov}$.

The next task was to show that our theory does not introduce any new divergences. We therefore wrote

$$S_{\rm cov} = S_{\rm cnt} + S_{\rm fin}, \tag{18.1.6}$$

and showed in section 12 that all terms in $S_{\rm fin}$ are finite under some reasonable assumptions.

We ended up with an action that

1. is boundary covariant (satisfies the translational Ward identity),
2. includes the zeroth and first order terms (vacuum energy e and vevs),
3. includes the unphysical fields ϵ^i, H, h (no gauge has been fixed),
4. does not introduce any new divergences,
5. reproduces the correct counterterms $\mathfrak{a}^a \mathcal{U}_{ab} \mathfrak{a}^b$ and $\mathfrak{e}^i_j \mathcal{T} \mathfrak{e}^j_i$,
6. reproduces known results in aAdS and can be applied to non-aAdS as well,
7. is the generating functional of connected correlation functions for both conformal and non-conformal QFT's.

From the renormalized action that is expanded up to second order in the fluctuations eq.(11.2.13) we calculated renormalized one-point functions given by eq.(C.4.1) by functional differentiation of the renormalized action with respect to the sources. We kept the one-point functions up to linear order in the sources, so we can immediately obtain the two-point functions from them by functionally differentiating a second time. A finite renormalized action guarantees finite one- and two-point functions.

Chapter 19

Outlook

◇ We conclude by indicating the direction of our future research. ◇

19.1 Leaving the background

The counterterm matrices $\mathcal{U}_{ab}$ and $\mathcal{T}$ are determined *on the background.* We have shown that the counterterms lead to finite correlation functions, but in order to explicitly calculate the correlators we need to know the counterterm functions away from the background. For example, the vevs v_{0i} are given by, see eq.(12.2.7),

$$v_{0i} = -e^{dA}\mathcal{U}_{0a}\hat{\mathfrak{a}}^a_{0i}, \tag{19.1.1}$$

where $\mathcal{U}_{0a} \equiv \mathcal{D}_a\mathcal{U}_0(A,\phi)$. We know the matrix $\mathcal{U}_{0ab}(A,\phi)$ as a function of A, but we do not know which part of the dependence on A is implicit through the background values of the scalars, and which part is explicit. This prevents us from integrating the matrix $\mathcal{U}_{0ab}$ to obtain $\mathcal{U}_{0a}$. Hence, we must find $U_{0ab}(A,\Phi)$ from $\mathcal{U}_{0ab}(A,\phi)$. We have shown how to do this systematically for a two-scalar system in aAdS in subsection 16.5, but we do not know if we can apply the same procedure to non-aAdS. We are very interested in developping a systematic method to leave the background and explicitly determine the full counterterms in non-aAdS.

19.2 Recombination

Another interesting question that we would like to answer in future research comes from the following observation. To zeroth order in $\Box$, we have, see eq.(C.4.1),

$$\langle\mathcal{O}_i\rangle = v_{0i} + Z_{0ij}\mathfrak{a}_{\mathfrak{r}j} + \frac{1}{2}\tilde{Z}_{0ij}\mathfrak{a}_{\mathfrak{s}j} - \frac{1}{4}x_{i0}h + \mathcal{O}(\Box)\,. \tag{19.2.1}$$

Using the expression for x_{i0} from eq.(12.2.2) and the expressions for $\varphi_{\mathfrak{s}j}$ and $\varphi_{\mathfrak{r}j}$ from eq.(C.2.7), we find

$$\langle\mathcal{O}_i\rangle = v_{0i} + Z_{0ij}\varphi_{\mathfrak{r}j} + \frac{1}{2}\tilde{Z}_{0ij}\varphi_{\mathfrak{s}j} + \mathcal{O}(\Box)\,. \tag{19.2.2}$$

Possibly this recombination of $\mathfrak{a}_{\mathfrak{s}i}$, $\mathfrak{a}_{\mathfrak{r}i}$ and h into $\varphi_{\mathfrak{s}i}$ and $\varphi_{\mathfrak{r}i}$ happens at higher order in $\Box$ as well,

$$\langle \mathcal{O}_i \rangle \overset{?}{=} v_{0i} + Z_{ij}\varphi_{\mathfrak{r}j} + \frac{1}{2}\tilde{Z}_{ij}\varphi_{\mathfrak{s}j}. \qquad (19.2.3)$$

We would like to check this at linear order in $\Box$. At quadratic order in $\Box$, we can always choose the counterterms such that this happens (which then automatically leads to finite correlation functions). This choice may be consistent with the recombination of the components $\mathcal{U}_{0ab}$, $\mathcal{U}_{1ab}$, and $\mathcal{U}_{2ab}$ into the full matrix $\mathcal{U}_{ab}$, and of $\mathcal{T}_1$ and $\mathcal{T}_2$ into the full function $\mathcal{T}$, which happens when we choose the counterterms according to eq.(11.2.10). If this is true, then we consider the choice eq.(11.2.10) to be a more natural scheme than the choice eq.(12.5.1). The latter choice leads to vanishing contributions proportional to $\mathfrak{a}^a \Box^2 h$ and $h \Box^2 h$, such that $\langle \mathcal{O}_i \rangle$ has no part proportional to $\Box^2 h$ and no recombination is possible with $Z_{2ij} \Box^2 \mathfrak{a}_{\mathfrak{r}j}$ and $\tilde{Z}_{2ij} \Box^2 \mathfrak{a}_{\mathfrak{s}j}$.

It might seem natural that $\mathfrak{a}_{\mathfrak{s}i}$, $\mathfrak{a}_{\mathfrak{r}i}$ recombine with h into $\varphi_{\mathfrak{s}i}$ and $\varphi_{\mathfrak{r}i}$ because we are working with covariant counterterms, in which $\mathfrak{a}^a$ always comes from φ^a. However, when we put the action on-shell, we use the equations of motions for $\mathfrak{b}$ and $\mathfrak{c}$ to eliminate them in favor of $\mathfrak{a}^a$, see eq.(5.1.7), such that $\mathfrak{a}^a$ stems from $\mathfrak{b}$ and $\mathfrak{c}$ instead of from φ^a. We therefore can not expect that in the correlators, which come from the on-shell action, $\mathfrak{a}_{\mathfrak{s}i}$, $\mathfrak{a}_{\mathfrak{r}i}$ always recombine with h into $\varphi_{\mathfrak{s}i}$ and $\varphi_{\mathfrak{r}i}$. Still, looking at eq.(C.4.1), this does seem to happen.

19.3 Translational Ward identity

The translational Ward identity should always be satisfied, because we always require our theories to have translational invariance. Eq.(6.1.2) gives the translational Ward identity,

$$\nabla_i \left\langle T^i_j \right\rangle = - \langle \mathcal{O}_i \rangle \nabla_j \Phi_{\mathfrak{s}i}. \qquad (19.3.1)$$

Up to first order in the fluctuations, this reads[1]

$$\partial_i \left\langle T^i_j \right\rangle = -v_{0i}\partial_j \varphi_{\mathfrak{s}i} + \mathcal{O}\big(f^2\big)\,, \qquad (19.3.2)$$

where we have used $\langle \mathcal{O}_i \rangle_0 = v_{0i}$ from eq.(C.4.1). On the other hand, we find from eq.(C.4.1)

$$\partial_i \left\langle T^i_j \right\rangle = e^{dA}\mathcal{U}_{0a}\partial_j \varphi^a + \mathcal{O}\big(f^2\big)\,. \qquad (19.3.3)$$

Using eq.(5.1.6), we find

$$e^{dA}\mathcal{U}_{0a}\varphi^a = e^{dA}\mathcal{U}_{0a}\hat{\mathfrak{a}}^a_{0i}\left(\mathfrak{a}_{\mathfrak{s}i} - \frac{1}{4}\hat{c}_i h\right) + \mathcal{O}(\Box) = -v_{0i}\varphi_{\mathfrak{s}i} + \mathcal{O}(\Box)\,, \qquad (19.3.4)$$

[1] We have used

$$\nabla_i \left\langle T^i_j \right\rangle = \partial_i \left\langle T^i_j \right\rangle - \Gamma^k_{ij} \left\langle T^i_k \right\rangle_0 + \Gamma^i_{ki} \left\langle T^k_j \right\rangle_0 = \partial_i \left\langle T^i_j \right\rangle - \Gamma^i_{ij} e + \Gamma^i_{ji} e = \partial_i \left\langle T^i_j \right\rangle.$$

where the contributions proportional to $\check{c}_i$ and $\mathfrak{a}_{\mathfrak{r}i}$ can be neglected because v_{0i} is finite by assumption, see eq.(12.2.7). Substituting this into eq.(19.3.3) yields

$$\partial_i \left\langle T^i_j \right\rangle = -v_{0i}\partial_j\varphi_{\mathfrak{s}i} + \mathcal{O}(\Box) + \mathcal{O}\big(f^2\big)\,. \tag{19.3.5}$$

Comparing this result to eq.(19.3.2), we see that the translational Ward identity is satisfied if the contributions of order $\Box$ vanish. We would like to understand better how to interpret this result.

19.4 Higher n-point functions

This thesis focussed on two-point functions, but Covariant Holographic Renormalization can be generalized to higher n-point functions. If we want the counterterms for three-point functions, we have to keep terms up to cubic order in the fluctuations. We then repeat the steps, starting by identifying the divergences of the bare action up to cubic order, followed by an expansion of the counterterm action up to cubic order using

$$U_0 = \mathcal{U}_0 + \varphi^a\mathcal{U}_{0a} + \frac{1}{2}\varphi^a\varphi^b\mathcal{U}_{0ab} + \frac{1}{3!}\varphi^a\varphi^b\varphi^c\mathcal{D}_c\mathcal{U}_{0ab} + \mathcal{O}\big(f^4\big)\,, \tag{19.4.1}$$

and similarly for the other terms in eq.(11.2.4). As explained below eq.(11.2.1), for the two-point functions we can neglect counterterms proportional to

$$\nabla^i\nabla^j\Phi^a\nabla_i\nabla_j\Phi^b, \qquad R^{klmn}R_{klmn}, \qquad \nabla^2 R,$$

but we expect these counterterms to be necessary for higher n-point functions. It may happen that we can fix the counterterm matrices proportional to B and C_a more naturally when we consider higher n-point functions.

To switch terms to gauge invariant variables, we have to know the relations eq.(5.1.4) up to quadratic order in the fluctuations. These relations are much more complicated, so the level of difficulty increases rapidly. Nevertheless, the conclusion of this thesis is that one- and two-point functions of both conformal and non-conformal field theories can be holographically renormalized by covariant counterterms, which strengthens our hope that this continues to be the case for higher n-point functions.

Acknowledgements Many thanks to my supervisor Dr. W. Mück, my co-supervisors Dr. F. Lizzi and Dr. L. Cappiello, and the university of Naples "Federico II".

Appendix A

Notation

A.1 Scalars

Symbol	Meaning
$\Phi(r,x)$	Scalar
$\phi(r)$	Scalar, background
$\varphi(r,x)$	Scalar, fluctuation
$\Phi_{\mathfrak{s}}(x)$	Scalar source
$\phi_{\mathfrak{s}}$	Scalar source, background
$\varphi_{\mathfrak{s}}(x)$	Scalar source, fluctuation
$\Phi_{\mathfrak{r}}(x)$	Scalar response
$\phi_{\mathfrak{r}}$	Scalar response, background
$\varphi_{\mathfrak{r}}(x)$	Scalar response, fluctuation

A.2 Metric

Symbol	Meaning
$G(\Phi)$	Sigma model metric
$\bar{G}(\phi)$	Sigma model metric, background
$\mathcal{G}(\Phi)$	Sigma model connection
$\bar{\mathcal{G}}(\phi)$	Sigma model connection, background
$\mathbf{g}_{\mu\nu}$	Bulk metric
γ_{ij}	Boundary metric
$e^{2A}\eta_{ij}$	Boundary metric, background
$e^{2A}h_{ij}$	Boundary metric, fluctuation
$\mathbf{R}[\mathbf{g}]_{\mu\nu}$	Ricci tensor, bulk
R_{ij}	Ricci tensor, boundary
$\Gamma[\mathbf{g}]^{\rho}_{\mu\nu}$	Bulk connection
Γ^{k}_{lm}	Boundary connection
$\nabla[\mathbf{g}]_{\mu}$	Bulk nabla
∇_{m}	Boundary nabla

Appendix B

Counterterm expansion

We now show how we expanded the general counterterm eq.(11.2.4) up to second order in the gauge invariant fluctuations, leading to eq.(11.2.6).

B.1 Expansion of Individual Counterterms

The counterterm action is given by

$$S_{\rm cov} = \int d^d x \sqrt{\gamma} \mathcal{L}_{\rm cov}, \tag{B.1.1}$$

where the Lagrangian is given by eq.(11.2.4), which we repeat here for simplicity,

$$\begin{aligned}\mathcal{L}_{\rm cov} \; &= \; U_0 - \frac{1}{2} U_{1ab} \nabla^i \Phi^a \nabla_i \Phi^b + \frac{1}{2} U_{2ab} \nabla^2 \Phi^a \nabla^2 \Phi^b + \\ & - \frac{1}{4} T_1 R - \frac{1}{4} T_2 R^{ij} R_{ij} + B R^2 + C_a \nabla^i \Phi^a \nabla_i R + \ldots \end{aligned} \tag{B.1.2}$$

Up to second order and up to boundary terms, we find

$$
\begin{aligned}
\sqrt{\gamma}U_0 \;&=\; e^{dA}\mathcal{U}_0\Big(1+\frac{1}{2}H+\frac{d}{2(d-1)}h-\frac{1}{4}\mathfrak{e}^i_j\mathfrak{e}^j_i+\frac{1}{2}\epsilon^i\,\Box\,\epsilon_i+\\
&\qquad -\frac{1}{8}H^2+\frac{d-2}{4(d-1)}Hh+\frac{d(d-2)}{8(d-1)^2}h^2\Big)+\\
&\qquad +e^{dA}\mathcal{U}_{0a}\Bigg(\mathfrak{a}^a+\frac{1}{2}H\mathfrak{a}^a+\frac{d}{2(d-1)}h\mathfrak{a}^a+\\
&\qquad -\frac{\mathcal{W}^a}{4\mathcal{W}}\left[h+\frac{1}{2}Hh+\frac{d}{2(d-1)}h^2\right]\Bigg)+\\
&\qquad +\frac{e^{dA}}{2}\mathcal{U}_{0ab}\left(\mathfrak{a}^a\mathfrak{a}^b-\frac{\mathcal{W}^a}{2\mathcal{W}}\mathfrak{a}^b h+\frac{\mathcal{W}^a\mathcal{W}^b}{16\mathcal{W}^2}h^2\right)\\
-\frac{1}{2}\sqrt{\gamma}U_{1ab}\nabla^i\Phi^a\nabla_i\Phi^b \;&=\; \frac{1}{2}e^{(d-2)A}\mathcal{U}_{1ab}\left(\mathfrak{a}^a\,\Box\,\mathfrak{a}^b-\frac{\mathcal{W}^a}{2\mathcal{W}}\mathfrak{a}^b\,\Box\,h+\frac{\mathcal{W}^a\mathcal{W}^b}{16\mathcal{W}^2}h\,\Box\,h\right)\\
\frac{1}{2}\sqrt{\gamma}U_{2ab}\nabla^2\Phi^a\nabla^2\Phi^b \;&=\; \frac{1}{2}e^{(d-4)A}\mathcal{U}_{2ab}\left(\mathfrak{a}^a\,\Box^2\,\mathfrak{a}^b-\frac{\mathcal{W}^a}{2\mathcal{W}}\mathfrak{a}^b\,\Box^2\,h+\frac{\mathcal{W}^a\mathcal{W}^b}{16\mathcal{W}^2}h\,\Box^2\,h\right)\\
-\frac{1}{4}\sqrt{\gamma}T_1R \;&=\; -\frac{e^{(d-2)A}}{16}\mathcal{T}_1\left(\mathfrak{e}^i_j\,\Box\,\mathfrak{e}^j_i-\frac{d-2}{d-1}h\,\Box\,h\right)+\\
&\qquad +\frac{1}{4}e^{(d-2)A}\mathcal{D}_a\mathcal{T}_1\left(\mathfrak{a}^a\,\Box\,h-\frac{\mathcal{W}^a}{4\mathcal{W}}h\,\Box\,h\right)\\
-\frac{1}{4}\sqrt{\gamma}T_2R^{ij}R_{ij} \;&=\; -\frac{e^{(d-4)A}}{16}\mathcal{T}_2\left(\mathfrak{e}^i_j\,\Box^2\,\mathfrak{e}^j_i+\frac{d}{d-1}h\,\Box^2\,h\right)\\
\sqrt{\gamma}BR^2 \;&=\; e^{(d-4)A}\mathcal{B}h\,\Box^2\,h\\
\sqrt{\gamma}C_a\nabla^i\Phi^a\nabla_iR \;&=\; e^{(d-4)A}\mathcal{C}_a\left(\mathfrak{a}^a\,\Box^2\,h-\frac{\mathcal{W}^a}{4\mathcal{W}}h\,\Box^2\,h\right). \qquad \text{(B.1.3)}
\end{aligned}
$$

B.2 General Expansions

We made use of the following expressions below.

Metric

$$\gamma_{ij}=e^{2A}\left(\eta_{ij}+h_{ij}\right). \qquad \text{(B.2.1)}$$

Connection

$$2\Gamma^k_{ij}=\partial_{(i}h^k_{j)}-\partial^k h_{ij}-h^k_p\left(\partial_{(i}h^p_{j)}-\partial^p h_{ij}\right)+\mathcal{O}\big(h^3\big). \qquad \text{(B.2.2)}$$

Riemann tensor

$$
\begin{aligned}
2R^k{}_{mln} = \ & \partial_l\partial_{(m}h^k_{n)} - \partial_l\partial^k h_{mn} - \partial_l\left[h^k_p\left(\partial_{(m}h^p_{n)} - \partial^p h_{mn}\right)\right] + \\
& - \partial_n\partial_{(m}h^k_{l)} + \partial_n\partial^k h_{ml} + \partial_n\left[h^k_p\left(\partial_{(m}h^p_{l)} - \partial^p h_{ml}\right)\right] + \\
& + \frac{1}{2}\left(\partial_{(l}h^k_{p)} - \partial^k h_{lp}\right)\left(\partial_{(m}h^p_{n)} - \partial^p h_{mn}\right) + \\
& - \frac{1}{2}\left(\partial_{(n}h^k_{p)} - \partial^k h_{np}\right)\left(\partial_{(m}h^p_{l)} - \partial^p h_{ml}\right) + \mathcal{O}\big(h^3\big)\,. \qquad \text{(B.2.3)}
\end{aligned}
$$

Ricci tensor

$$
\begin{aligned}
2R_{mn} = \ & \partial_k\partial_{(m}h^k_{n)} - \Box\, h_{mn} - \partial_n\partial_{(m}h^k_{k)} + \partial_n\partial^k h_{mk} + \\
& + \partial_n\left[h^k_p\left(\partial_{(m}h^p_{k)} - \partial^p h_{mk}\right)\right] - \partial_k\left[h^k_p\left(\partial_{(m}h^p_{n)} - \partial^p h_{mn}\right)\right] + \\
& + \frac{1}{2}\partial_p h^k_k\left(\partial_{(m}h^p_{n)} - \partial^p h_{mn}\right) + \\
& - \frac{1}{2}\left(\partial_{(n}h^k_{p)} - \partial^k h_{np}\right)\left(\partial_{(m}h^p_{k)} - \partial^p h_{mk}\right) + \mathcal{O}\big(h^3\big)\,. \qquad \text{(B.2.4)}
\end{aligned}
$$

Ricci scalar

$$
\begin{aligned}
e^{2A}R = \ & \partial_i\partial^j h^i_j - \Box\, h^k_k + \frac{1}{4}h^i_j\,\Box\, h^j_i + \frac{1}{2}\partial^j h^i_j\partial_k h^k_i + \frac{1}{4}h^k_k\,\Box\, h^m_m + \mathcal{O}\big(h^3\big) \\
= \ & -\Box\, h + \frac{1}{4}\mathfrak{e}^i_j\,\Box\,\mathfrak{e}^j_i + \frac{1}{2}H\,\Box\, h + \frac{d+2}{4(d-1)}h\,\Box\, h + \mathcal{O}\big(h^3\big)\,. \qquad \text{(B.2.5)}
\end{aligned}
$$

Determinant To compute the determinant $\gamma \equiv \det\gamma_{ij}$, we used

$$
\gamma \equiv \det\gamma_{ij} = e^{2dA}\det\left(\eta_{ij} + h_{ij}\right), \qquad \text{(B.2.6)}
$$

where

$$
\det\left(\eta_{ij} + h_{ij}\right) = \sum_{m=0}^{\infty}\frac{1}{m!}\left(-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\,\mathrm{tr}\,\left(h^n_{ij}\right)\right)^m. \qquad \text{(B.2.7)}
$$

Up to second order, this reads

$$
\det\left(\eta_{ij} + h_{ij}\right) = 1 + \mathrm{tr}\ h_{ij} + \frac{1}{2}\left[\left(\mathrm{tr}\ h_{ij}\right)^2 - \mathrm{tr}\ \left(h^2_{ij}\right)\right] + \mathcal{O}\big(h^3_{ij}\big)\,. \qquad \text{(B.2.8)}
$$

Then we find

$$
\gamma = e^{2dA}\left[1 + h^k_k + \frac{1}{2}\left(\left(h^k_k\right)^2 - h^m_k h^k_m\right)\right] + \mathcal{O}\big(h^3\big)\,. \qquad \text{(B.2.9)}
$$

Substituting eq.(4.2.10) and eq.(4.2.12), we find

$$
\gamma = e^{2dA}\left(1 + H + \frac{d}{d-1}h - \frac{1}{2}\mathfrak{e}^m_k\mathfrak{e}^k_m + \epsilon^k\,\Box\,\epsilon_k + Hh + \frac{d}{2(d-1)}h^2\right) + \mathcal{O}\big(f^3\big)\,. \qquad \text{(B.2.10)}
$$

To calculate the square root, we use

$$\sqrt{\gamma} = e^{dA}\sqrt{1+\varepsilon} = e^{dA}\left(1+\frac{1}{2}\varepsilon-\frac{1}{8}\varepsilon^2+\mathcal{O}\big(\varepsilon^3\big)\right), \tag{B.2.11}$$

where we defined

$$\varepsilon \equiv H + \frac{d}{d-1}h - \frac{1}{2}\mathfrak{e}_k^m\mathfrak{e}_m^k + \epsilon^k\,\Box\,\epsilon_k + Hh + \frac{d}{2(d-1)}h^2. \tag{B.2.12}$$

Substituting eq.(B.2.12) in eq.(B.2.11) yields

$$\begin{aligned}\sqrt{\gamma} \;=\; & e^{dA}\Bigg(1+\frac{1}{2}H+\frac{d}{2(d-1)}h-\frac{1}{4}\mathfrak{e}_k^m\mathfrak{e}_m^k+\frac{1}{2}\epsilon^k\,\Box\,\epsilon_k+ \\ & -\frac{1}{8}H^2+\frac{d-2}{4(d-1)}Hh+\frac{d(d-2)}{8(d-1)^2}h^2\Bigg)+\mathcal{O}\big(f^3\big)\,. \end{aligned} \tag{B.2.13}$$

Appendix C

One-point functions

Here we show how to obtain one-point functions up to linear order in the sources from the renormalized action.

C.1 Renormalized Action

The renormalized action is given by eq.(11.2.13),

$$\begin{aligned}\sqrt{\gamma}\mathcal{L}_{\rm ren} \;=\; & \frac{e^{dA}}{2}\mathfrak{a}^a\left(\bar{G}_{ab}\mathcal{D}_r - \mathcal{M}_{ab} + \mathcal{U}_{ab}\right)\mathfrak{a}^b + \frac{e^{dA}}{16}\mathfrak{e}^i_j\left(\partial_r - \mathcal{T}\right)\mathfrak{e}^j_i + \\ & + \frac{e^{dA}}{4}h\left(\mathcal{D}_a\mathcal{T} - \frac{\mathcal{W}^b}{\mathcal{W}}\mathcal{U}_{ab} + \mathcal{O}(\Box^2)\right)\left(\mathfrak{a}^a - \frac{\mathcal{W}^a}{8\mathcal{W}}h\right) + \\ & - \left(1 + \frac{1}{2}H + \frac{d}{2(d-1)}h\right)\left(v_i\mathfrak{a}_{\mathfrak{s}i} - \frac{v}{4}h\right) + \\ & - \frac{e^{(d-2)A}}{16(d-1)}\mathcal{T}_1 h\,\Box\, h + e\Bigg[1 + \frac{1}{2}H + \frac{d}{2(d-1)}h + \\ & - \frac{1}{4}\mathfrak{e}^m_k\mathfrak{e}^k_m + \frac{1}{2}\epsilon^k\,\Box\,\epsilon_k - \frac{1}{8}H^2 + \frac{d-2}{4(d-1)}\left(H + \frac{d}{2(d-1)}h\right)h\Bigg]. \end{aligned} \tag{C.1.1}$$

We used the definitions,

$$e \equiv e^{dA}\mathcal{U}_0, \qquad v_i \equiv -e^{dA}\mathcal{U}_{0a}\hat{\mathfrak{a}}^a_i, \qquad v \equiv -e^{dA}\mathcal{U}_{0a}\frac{\mathcal{W}^a}{4\mathcal{W}}. \tag{C.1.2}$$

In the main text we set $e = 0$, but we keep it around here to show its interpretation. The QFT one-point functions can be obtained from eq.(C.1.1) as follows

$$\langle T_{ij}\rangle = -\frac{2}{\sqrt{\gamma_{\mathfrak{s}}}}\frac{\delta S_{\rm ren}}{\delta\gamma^{ij}_{\mathfrak{s}}}, \qquad \langle\mathcal{O}_i\rangle = -\frac{1}{\sqrt{\gamma_{\mathfrak{s}}}}\frac{\delta S_{\rm ren}}{\delta\varphi_{\mathfrak{s}i}}. \tag{C.1.3}$$

This means that the variation of the action reads

$$\delta S_{\rm ren} = -\int d^dx\sqrt{\gamma_{\mathfrak{s}}}\left[\frac{1}{2}\langle T_{ij}\rangle\,\delta\gamma^{ij}_{\mathfrak{s}} + \langle\mathcal{O}_i\rangle\,\delta\varphi_{\mathfrak{s}i}\right]. \tag{C.1.4}$$

C.2 Linear Expansion

Let us expand eq.(C.1.4) first to linear and then to quadratic order in the fluctuations. From eq.(B.2.13) we find

$$\sqrt{\gamma_{\mathfrak{s}}} = 1 + \frac{1}{2}H + \frac{d}{2(d-1)}h + \mathcal{O}\big(f^2\big)\,. \tag{C.2.1}$$

From eq.(4.2.10) we find

$$\gamma^i_{\mathfrak{s}j} = \mathfrak{e}^i_{\mathfrak{s}j} + \partial^i\epsilon_j + \partial_j\epsilon^i + \frac{\partial^i\partial_j}{\Box}H + \frac{\delta^i_j}{d-1}h. \tag{C.2.2}$$

We decompose T^i_j similar to h^i_j in eq.(4.2.10),

$$T^i_j = \mathcal{T}^i_j + \partial^i\mathcal{T}_j + \partial_j\mathcal{T}^i + \frac{\partial^i\partial_j}{\Box}\mathcal{T}_H + \frac{\delta^i_j}{d-1}\mathcal{T}_h. \tag{C.2.3}$$

Here, $\mathcal{T}^i_j$ is traceless and transversal and $\mathcal{T}^i$ is transversal

$$\mathcal{T}^i_j \equiv \Pi^{ik}_{jl}T^l_k, \qquad \partial_i\mathcal{T}^i \equiv 0, \tag{C.2.4}$$

where the traceless transverse projector Π^{ik}_{jl} is defined by

$$\Pi^{ik}_{jl} \equiv \frac{1}{2}\left(\Pi^{ik}\Pi_{jl} + \Pi^i_l\Pi^k_j\right) - \frac{1}{d-1}\Pi^i_j\Pi^k_l, \tag{C.2.5}$$

and the transverse projector Π^i_j is defined by eq.(4.2.9). We also decompose $\varphi_{\mathfrak{s}i}$ into $\mathfrak{a}_{\mathfrak{s}i}$ and h. From eq.(5.1.6) and eq.(5.1.20) we find

$$\varphi^a = \mathfrak{a}^a - \frac{\mathcal{W}^a}{4\mathcal{W}}h = \hat{\mathfrak{a}}^a_{0i}\left(\mathfrak{a}_{\mathfrak{s}i} - \frac{1}{4}\hat{c}_i h\right) + \check{\mathfrak{a}}^a_{0i}\left(\mathfrak{a}_{\mathfrak{r}i} - \frac{1}{4}\check{c}_i h\right) + \mathcal{O}(\Box)\,, \tag{C.2.6}$$

from which we read off

$$\varphi_{\mathfrak{s}i} = \mathfrak{a}_{\mathfrak{s}i} - \frac{1}{4}\hat{c}_i h, \qquad \varphi_{\mathfrak{r}i} = \mathfrak{a}_{\mathfrak{r}i} - \frac{1}{4}\check{c}_i h. \tag{C.2.7}$$

Substituting the decompositions eq.(C.2.3) and eq.(C.2.7) in eq.(C.1.4) yields

$$\begin{aligned}\delta S^{(1)}_{\text{ren}} \;=\; \int d^dx \Bigg[& \frac{1}{2}\left\langle\mathcal{T}^j_i\right\rangle_0\delta\mathfrak{e}^i_{\mathfrak{s}j} - \left\langle\Box\,\mathcal{T}_i\right\rangle_0\delta\epsilon^i + \frac{1}{2}\left(\left\langle\mathcal{T}_H\right\rangle_0 + \frac{1}{d-1}\left\langle\mathcal{T}_h\right\rangle_0\right)\delta H + \\ & + \left(\frac{1}{2(d-1)}\left\langle\mathcal{T}_H\right\rangle_0 + \frac{d}{2(d-1)^2}\left\langle\mathcal{T}_h\right\rangle_0 + \frac{\hat{c}_i}{4}\left\langle\mathcal{O}_i\right\rangle_0\right)\delta h + \\ & - \left\langle\mathcal{O}_i\right\rangle_0\delta\mathfrak{a}_{\mathfrak{s}i}\Bigg].\end{aligned} \tag{C.2.8}$$

From this we find

$$\begin{aligned}
\frac{\delta S^{(1)}_{\text{ren}}}{\delta \epsilon^i} &= -\left\langle \Box\, \mathcal{T}_i \right\rangle_0 \\
\frac{\delta S^{(1)}_{\text{ren}}}{\delta \mathfrak{e}^i_{\mathfrak{s}j}} &= \frac{1}{2}\left\langle \mathcal{T}_i^j \right\rangle_0 \\
\frac{\delta S^{(1)}_{\text{ren}}}{\delta h} &= \frac{1}{2(d-1)}\left\langle \mathcal{T}_H \right\rangle_0 + \frac{d}{2(d-1)^2}\left\langle \mathcal{T}_h \right\rangle_0 + \frac{\hat{c}_i}{4}\left\langle \mathcal{O}_i \right\rangle_0 \\
\frac{\delta S^{(1)}_{\text{ren}}}{\delta H} &= \frac{1}{2}\left\langle \mathcal{T}_H \right\rangle_0 + \frac{1}{2(d-1)}\left\langle \mathcal{T}_h \right\rangle_0 \\
\frac{\delta S^{(1)}_{\text{ren}}}{\delta \mathfrak{a}_{\mathfrak{s}i}} &= -\left\langle \mathcal{O}_i \right\rangle_0 .
\end{aligned} \tag{C.2.9}$$

We can invert these relations to

$$\begin{aligned}
\left\langle \mathcal{T}_h \right\rangle_0 &= 2(d-1)\frac{\delta S^{(1)}_{\text{ren}}}{\delta h} - 2\frac{\delta S^{(1)}_{\text{ren}}}{\delta H} + \frac{d-1}{2}\hat{c}_i\frac{\delta S^{(1)}_{\text{ren}}}{\delta \mathfrak{a}_{\mathfrak{s}i}} \\
\left\langle \mathcal{T}_H \right\rangle_0 &= -2\frac{\delta S^{(1)}_{\text{ren}}}{\delta h} + \frac{2d}{d-1}\frac{\delta S^{(1)}_{\text{ren}}}{\delta H} - \frac{\hat{c}_i}{2}\frac{\delta S^{(1)}_{\text{ren}}}{\delta \mathfrak{a}_{\mathfrak{s}i}} \\
\left\langle \Box\, \mathcal{T}^i \right\rangle_0 &= -\frac{\delta S^{(1)}_{\text{ren}}}{\delta \epsilon^i} \\
\left\langle \mathcal{T}_i^j \right\rangle_0 &= 2\frac{\delta S^{(1)}_{\text{ren}}}{\delta \mathfrak{e}^i_{\mathfrak{s}j}} \\
\left\langle \mathcal{O}_i \right\rangle_0 &= -\frac{\delta S^{(1)}_{\text{ren}}}{\delta \mathfrak{a}_{\mathfrak{s}i}} .
\end{aligned} \tag{C.2.10}$$

From eq.(C.2.3) and eq.(C.2.10) we find

$$\left\langle T^k_k \right\rangle_0 = \left\langle \mathcal{T}_H \right\rangle_0 + \frac{d}{d-1}\left\langle \mathcal{T}_h \right\rangle_0 = 2(d-1)\frac{\delta S^{(1)}_{\text{ren}}}{\delta h} + \frac{d-1}{2}\hat{c}_i\frac{\delta S^{(1)}_{\text{ren}}}{\delta \mathfrak{a}_{\mathfrak{s}i}} . \tag{C.2.11}$$

The linear part of the on-shell action eq.(C.1.1) reads

$$S^{(1)}_{\text{ren}} = \int d^d x \left[\frac{1}{2}eH + \left(\frac{d}{2(d-1)}e + \frac{v}{4}\right)h - v_{0i}\mathfrak{a}_{\mathfrak{s}i}\right] . \tag{C.2.12}$$

Notice that we were allowed to replace v_i by v_{0i}, because terms with $\Box$ acting on $\mathfrak{a}_{\mathfrak{s}i}$ are total derivatives, which drop out since we integrate them over the boundary. From eq.(C.2.12) we find

$$\frac{\delta S^{(1)}_{\text{ren}}}{\delta h} = \frac{d}{2(d-1)}e + \frac{1}{4}v, \qquad \frac{\delta S^{(1)}_{\text{ren}}}{\delta H} = \frac{1}{2}e, \qquad \frac{\delta S^{(1)}_{\text{ren}}}{\delta \mathfrak{a}_{\mathfrak{s}i}} = -v_{0i} . \tag{C.2.13}$$

This gives

$$\left\langle \mathcal{T}_i^j \right\rangle_0 = \left\langle \Box\, \mathcal{T}^i \right\rangle_0 = \left\langle \mathcal{T}_H \right\rangle_0 = 0, \qquad \left\langle \mathcal{T}_h \right\rangle_0 = (d-1)e, \qquad \left\langle \mathcal{O}_i \right\rangle_0 = v_{0i}. \tag{C.2.14}$$

From eq.(C.2.11) we then find

$$\left\langle T_k^k \right\rangle_0 = de. \tag{C.2.15}$$

We can summarize the two equations above as follows

$$\left\langle T_j^i \right\rangle_0 = \delta_j^i e, \qquad \left\langle \mathcal{O}_i \right\rangle_0 = v_{0i}. \tag{C.2.16}$$

This shows that e has the interpretation of the vacuum energy density and v_{0i} has the interpretation of the vev of the QFT operator $\mathcal{O}_i$.

C.3 Quadratic Expansion

Using the results from the expansion up to first order, we can continue to expand eq.(C.1.4) up to second order,

$$\begin{aligned}
\delta S_{\text{ren}} \;=\; \int d^d x \Bigg\{ & \frac{1}{2}\left(\left\langle \mathcal{T}_i^j \right\rangle - e\mathfrak{e}_{\mathfrak{s}i}^j\right)\delta\mathfrak{e}_{\mathfrak{s}j}^i + \Box\left(e\epsilon_k - \left\langle \mathcal{T}_k \right\rangle\right)\delta\epsilon^k + \\
& + \left[\frac{1}{2}\left\langle \mathcal{T}_H \right\rangle + \frac{1}{2(d-1)}\left\langle \mathcal{T}_h \right\rangle + \frac{e}{4}\left(\frac{d-2}{d-1}h - H\right)\right]\delta H + \\
& + \Bigg[\frac{1}{2(d-1)}\left\langle \mathcal{T}_H \right\rangle + \frac{d}{2(d-1)^2}\left\langle \mathcal{T}_h \right\rangle + \frac{1}{4}\hat{c}_i\left\langle \mathcal{O}_i \right\rangle + \\
& + \frac{1}{4}\left(\frac{d-2}{d-1}e + \frac{1}{2}v\right)\left(H + \frac{d}{d-1}h\right)\Bigg]\delta h + \\
& - \left(\left\langle \mathcal{O}_i \right\rangle + \frac{1}{2}v_{0i}H + \frac{d}{2(d-1)}v_{0i}h\right)\delta\mathfrak{a}_{\mathfrak{s}i}\Bigg\}.
\end{aligned} \tag{C.3.1}$$

We have used the relation

$$v \equiv -e^{dA}\mathcal{U}_{0a}\frac{\mathcal{W}^a}{\mathcal{W}} = -e^{dA}\mathcal{U}_{0a}\hat{c}_i\hat{\mathfrak{a}}_{0i}^a = \hat{c}_i v_{0i}, \tag{C.3.2}$$

where the contribution proportional to $\check{c}_i$ can be neglected because v_{0i} is finite by assumption, see eq.(12.2.7).

From eq.(C.3.1) we read off

$$\begin{aligned}
\frac{\delta S_{\mathrm{ren}}}{\delta \mathfrak{e}^{j}_{\mathfrak{s}i}} &= \frac{1}{2}\left(\left\langle \mathcal{T}^{i}_{j}\right\rangle - e\mathfrak{e}^{i}_{\mathfrak{s}j}\right) \\
\frac{\delta S_{\mathrm{ren}}}{\delta \epsilon^{k}} &= e\,\Box\,\epsilon_k - \Box\left\langle \mathcal{T}_k\right\rangle \\
\frac{\delta S_{\mathrm{ren}}}{\delta H} &= \frac{1}{2}\left\langle \mathcal{T}_H\right\rangle + \frac{1}{2(d-1)}\left\langle \mathcal{T}_h\right\rangle + \frac{e}{4}\left(\frac{d-2}{d-1}h - H\right) \\
\frac{\delta S_{\mathrm{ren}}}{\delta h} &= \frac{1}{2(d-1)}\left\langle \mathcal{T}_H\right\rangle + \frac{d}{2(d-1)^2}\left\langle \mathcal{T}_h\right\rangle + \frac{1}{4}\hat{c}_i\left\langle \mathcal{O}_i\right\rangle + \\
&\quad + \frac{1}{4}\left(\frac{d-2}{d-1}e + \frac{1}{2}v\right)\left(H + \frac{d}{d-1}h\right) \\
\frac{\delta S_{\mathrm{ren}}}{\delta \mathfrak{a}_{\mathfrak{s}i}} &= -\left\langle \mathcal{O}_i\right\rangle - \frac{1}{2}v_{0i}\left(H + \frac{d}{d-1}h\right).
\end{aligned} \tag{C.3.3}$$

We can invert the relations to

$$\begin{aligned}
\left\langle \mathcal{T}^{i}_{j}\right\rangle &= 2\frac{\delta S_{\mathrm{ren}}}{\delta \mathfrak{e}^{j}_{\mathfrak{s}i}} + e\mathfrak{e}^{i}_{\mathfrak{s}j} \\
\Box\left\langle \mathcal{T}_k\right\rangle &= e\,\Box\,\epsilon_k - \frac{\delta S_{\mathrm{ren}}}{\delta \epsilon^{k}} \\
\left\langle \mathcal{T}_H\right\rangle &= \frac{2d}{d-1}\frac{\delta S_{\mathrm{ren}}}{\delta H} - 2\frac{\delta S_{\mathrm{ren}}}{\delta h} + \frac{1}{2}\hat{c}_i\left\langle \mathcal{O}_i\right\rangle - \frac{d}{2(d-1)}e\left(\frac{d-2}{d-1}h - H\right) + \\
&\quad + \frac{1}{2}\left(\frac{d-2}{d-1}e + \frac{1}{2}v\right)\left(H + \frac{d}{d-1}h\right) \\
\left\langle \mathcal{T}_h\right\rangle &= 2(d-1)\frac{\delta S_{\mathrm{ren}}}{\delta h} - 2\frac{\delta S_{\mathrm{ren}}}{\delta H} - \frac{d-1}{2}\hat{c}_i\left\langle \mathcal{O}_i\right\rangle + \\
&\quad - \frac{d-1}{2}\left(e + \frac{1}{2}v\right)H - \left(\frac{1}{d-1}e + \frac{d}{4}v\right)h \\
\left\langle \mathcal{O}_i\right\rangle &= -\frac{\delta S_{\mathrm{ren}}}{\delta \mathfrak{a}_{\mathfrak{s}i}} - \frac{1}{2}v_{0i}\left(H + \frac{d}{d-1}h\right).
\end{aligned} \tag{C.3.4}$$

For the Ward identities, the following combinations of $\langle \mathcal{T}_H\rangle$ and $\langle \mathcal{T}_h\rangle$ are interesting,

$$\left\langle T^{k}_{k}\right\rangle = \left\langle \mathcal{T}_H\right\rangle + \frac{d}{d-1}\left\langle \mathcal{T}_h\right\rangle, \qquad \partial_i\left\langle T^{i}_{j}\right\rangle = \Box\,\mathcal{T}_j + \partial_j\left(\left\langle \mathcal{T}_H\right\rangle + \frac{1}{d-1}\left\langle \mathcal{T}_h\right\rangle\right). \tag{C.3.5}$$

Substituting eq.(C.3.4) in eq.(C.3.5) yields

$$\begin{aligned}
\partial_i\left\langle T^{i}_{j}\right\rangle &= -\frac{\delta S_{\mathrm{ren}}}{\delta \epsilon^{i}} + 2\partial_j\frac{\delta S_{\mathrm{ren}}}{\delta H} + e\left(\Box\,\epsilon_i - \frac{d-2}{2(d-1)}\partial_j h + \frac{1}{2}\partial_j H\right) \\
\left\langle T^{k}_{k}\right\rangle &= 2(d-1)\frac{\delta S_{\mathrm{ren}}}{\delta h} - \frac{d-1}{2}\hat{c}_i\left\langle \mathcal{O}_i\right\rangle + \\
&\quad - \frac{d-1}{2}\left(\frac{d-2}{d-1}e + \frac{1}{2}v\right)\left(H + \frac{d}{d-1}h\right).
\end{aligned} \tag{C.3.6}$$

Let us summarize our results with $e = 0$,

$$
\begin{aligned}
\Box \langle \mathcal{T}_i \rangle &= -\frac{\delta S_{\text{ren}}}{\delta \epsilon^i} \\
\left\langle \Pi^{ik}_{jm} T^m_k \right\rangle &= 2\frac{\delta S_{\text{ren}}}{\delta \mathfrak{e}^j_{\mathfrak{s}i}} \\
\partial_i \left\langle T^i_j \right\rangle &= -\frac{\delta S_{\text{ren}}}{\delta \epsilon^i} + 2\partial_j \frac{\delta S_{\text{ren}}}{\delta H} \\
\left\langle T^k_k \right\rangle + \frac{d-1}{2}\hat{c}_i \langle \mathcal{O}_i \rangle &= 2(d-1)\frac{\delta S_{\text{ren}}}{\delta h} - \frac{d-1}{4} v \left(H + \frac{d}{d-1} h \right) \\
\langle \mathcal{O}_i \rangle &= -\frac{\delta S_{\text{ren}}}{\delta \mathfrak{a}_{\mathfrak{s}i}} - \frac{1}{2} v_{0i} \left(H + \frac{d}{d-1} h \right).
\end{aligned}
\tag{C.3.7}
$$

C.4 One-point Functions

Using the renormalized action given by eq.(11.2.13) we find the following one-point functions up to linear order

$$
\begin{aligned}
\Box \langle \mathcal{T}_k \rangle &= 0 \\
\left\langle \Pi^{ik}_{jm} T^m_k \right\rangle &= \frac{1}{4} Y \mathfrak{e}^i_{\mathfrak{r}j} \\
\partial_i \left\langle T^i_j \right\rangle &= e^{dA} \mathcal{U}_{0a} \partial_j \varphi^a \\
\left\langle T^k_k \right\rangle + \frac{d-1}{2}\hat{c}_i \langle \mathcal{O}_i \rangle &= \frac{d-1}{2} v + \frac{d-1}{2} e^{dA} \left(\mathcal{D}_a \mathcal{T} - \frac{\mathcal{W}^b}{\mathcal{W}} \mathcal{U}_{ab} + \mathcal{O}\big(\Box^2\big) \right) \varphi^a + \\
&\quad + d e^{dA} \mathcal{U}_{0a} \varphi^a - \frac{a_R}{4} \Box h \\
\langle \mathcal{O}_i \rangle &= v_{0i} + Z_{ij} \mathfrak{a}_{\mathfrak{r}j} + \frac{1}{2} \tilde{Z}_{ij} \mathfrak{a}_{\mathfrak{s}j} - \frac{1}{4} x_i h \\
\hat{\mathcal{A}} &= a - e^{dA} \varphi^a \partial_A \mathcal{U}_{0a} - \frac{a_R}{4} \Box h.
\end{aligned}
\tag{C.4.1}
$$

The matrices Z_{ij} and $\tilde{Z}_{ij}$ are given by eq.(9.3.4), the operator $x_i(\Box)$ is defined by eq.(11.2.14), and the functions a and a_R are defined by eq.(11.3.4).

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