Multidimensional Impulse Inequalities and General Bihari–Type Inequalities for Discontinuous Functions with Delay

A. Gallo [a,*], A.M. Piccirillo [b]

[a] *Department of Mathematics and Applications "R. Caccioppoli"*
University of Naples "Federico II"
Via Claudio 21 - 80125 Napoli, Italy
angallo@unina.it

[b] *Department of Civil Engineering*
Second University of Naples
Via Roma, 29 - 81031 Aversa (CE), Italy
annamaria.piccirillo@unina2.it

Abstract

In this article we investigate some impulsive integro-functional inequalities for functions of n independent variables. The problem of reducing multidimensional integro-sum functional inequalities to one-dimensional inequalities is also considered (using conditions of Chaplygin problem solvability for impulsive integral inequalities). Some new analogies of Wendroff–type inequalities for discontinuous functions with finite jumps are obtained.

MSC: 34B15, 26D15, 26D20

Key words: Inequalities, discontinuous functions, delay, impulse integral inequalities

1 Introduction

Over the last 20 years, the theory of ordinary impulsive differential systems has undergone extensive development, and this explains the appearance of

* Corresponding author. Phone +39 081 7683548 - fax +39 081 7683643

new problems in several fields of the investigation such as: physics, biology, chemistry, electronics and many others.

The integral representations of the solutions of ordinary impulsive differential systems usually have both a continuous and a discrete part and may be written in the following form:

$$x(t) = \varphi(t) + \int_{t_0}^{t} k(t, s, x(s))ds + \sum_{t_0 < t_k < t} \psi_i(t, \tau_i) I_i(u(\tau_i - 0)) \tag{A}$$

where $\varphi(t)$ characterizes the initial perturbations, $k(t, s, x)$ describes the continuous perturbations (right–hand side of system for $t \neq \tau_i$) and I_i characterizes the impulse perturbations value of the system ($x \in R^n$), in fixed moments of time $\{\tau_i\}$, ψ_i certain functions.

The representation (A) is preserved when the impulse perturbations take place not only in fixed moments of time $\{\tau_i\} : \tau_0 < \tau_1 < \ldots \lim\limits_{i\to\infty} \tau_i(x) = \infty$, but also on some hyper–surfaces

$$\{\tau_i(x) : \tau_0(x) < \tau_1(x) < \ldots \lim_{i\to\infty} \tau_i(x) = \infty\,, \text{ uniformly for } x\}.$$

In this case, τ_i^* is a moment of time when the impulsive system solution $x(t)$ intersects with $\{\tau_i(x)\}$ (under certain conditions this occurs in one place).

If we investigate some concrete processes, which can be described by impulsive differential systems, we have to use information about the values of initial, continuous and discrete perturbations (for example, estimates of $\|\varphi\|,\ \|k\|,\ \|I_i\|$). Then, for $\|x\|$, we obtain an inequality, which can be described by

$$u(t) \leq \varphi^*(t) + \int_{t_0}^{t} k^*(t, s, u(s))\, ds + \sum_{t_0 < t_i < t} \psi_i^*(t, \tau_i) I_i^*(u(\tau_i - 0))\,, \tag{B}$$

where u, φ^*, ψ_i^*, I_i^* are nonnegative functions and a comparison impulsive equation

$$v(t) = \varphi^*(t) + \int_{t_0}^{t} k^*(t, s, v)(s))ds + \sum_{t_0 < t_i < t} \psi_i^*(t, \tau_i) I_i^*(v(\tau_i - 0)) \tag{C}$$

Equation (C) and inequality (B) have strong relations. Under certain conditions (see [36]) $u(t) \leq v_{\varphi^*}(t)$, where v_{φ^*} represents one solution of the equation (C).

In the continuous case ($I_i \equiv 0$) the integral inequalities theory is based on fundamental results of Bellman [6], Bihari [7], and their numerous generalizations [1]–[5], [22], [23], [25], [26], [30]–[32], [35], [38]–[42] and many others. For investigations of discrete inequalities ($k \equiv 0$) and its applications, see [2] and references therein.

For the generalization of the integral inequalities method for discontinuous functions and their applications for qualitative analysis impulsive systems: existence, uniqueness, boundedness, comparison, stability, etc. we refer to the results [5], [8]–[21], [27], [33], [34], [36], [37]. While for investigating periodic boundary value problems, where representation (A) is used, see [28], [29] (and references therein).

In paper [11] Wendroff type impulsive integral inequality (B) is studied, when $t = (t_1, t_2)$, $t_0 = (t_{01}, t_{02})$, $k = f(t_1, t_2) \cdot u^m(t_1, t_2)$, $m > 0$, $\tau_i = (\tau_i^1 \tau_i^2)$, later different kinds of multidimensional impulsive inequalities (B) are investigated in monograph [36], for function $u(t)$ of n-independent variables $t = (t_1, \dots, t_n)$, $k = f(t_1, \dots, t_n) u^m(t_1, \dots, t_n) \tau_i = (\tau_i^1, \dots, \tau_i^n)$ under assumptions that $\psi_i I_i = \beta_i u(t_i - 0)$, $\beta_i = \text{const.} > 0, i = 1, 2, \dots$

The results of [17], [18] ,[20], [27] generalize the investigations of [36] to the case when $\psi_i I_i = \beta_i u^m(t_i - 0)$, $m > 0$, and kernel $k = f \cdot u^m$.

The following cases have not been explored: when function $u(t)$ of two (many) independent variables satisfies inequality (B) with $k(t, s, u) = f(t) W[u]$, $I_i = \beta_i u^m(t_i - 0)$, where W is one function which includes also delay for t in $u = u(\sigma(t))$, $(\sigma(t) \leq t)$.

Our paper is devoted to investigate impulsive integral inequalities for discontinuous functions with n-independent variables ($n \geq 2$).

In Section 2 we consider an impulsive integro-functional inequality for discontinuous functions of n-independent variables.

By using the results in [11], we reduce, under some assumptions, a multidimensional inequality to a unidimensional one.

In Section 3, for $n = 2$ we obtain new analogies of Bihari's result for a Wendroff type inequality with different kinds of W and non-Lipschitz type discontinuities of the function $u(t, x)$ in fixed points (t_i, x_i).

Our results are based on investigations [1–42].

2 Multidimensional Inequalities. The problem of reduction

Following the investigation of papers [3], [11], [42], let us consider n-dimensional Euclidean space R^n with points $t = (t_1, \ldots, t_n)$, $t_0 = (t_{10}, \ldots, t_{n0})$ and natural ordering $t_0 \leq t \Leftrightarrow t_{i0} \leq t_i$ for $i = 1, \ldots, n$.

By using term integro-sum inequality [11], [36] (or impulse integral inequality [5], [24]) we have a one-dimensional inequality of the following type

$$\begin{aligned} &u(t) \leq \varphi(t) + \int\limits_{t_0}^{t} K(t, s, u(s))ds + \sum_{t_0 < t_i < t} \mu(t, t_i) \cdot \eta_i(u(t_i - 0)) \\ &t \geq t_0 \geq 0\,, \quad t_{i_1} < t_i \quad \forall\, i = 1, 2, \ldots, \lim_{i \to \infty} t_i = \infty, \end{aligned} \tag{2.1}$$

where $u(t)$ is a piecewise continuous function with 1-st kind discontinuity points $\{t_i\}$; we consider the most general kind of n-dimensional impulse integral inequalities with delay (integro-functional inequality) of the following type:

$$\begin{aligned} &u(t_1, \ldots, t_n) \leq \varphi(t_1, \ldots, t_n) + \\ &+ \int\limits_{c_1}^{t_1} \ldots \int\limits_{c_n}^{t_n} K[t_1, \ldots, t_n, s_1, \ldots, s_n, u(\sigma(s_1, \ldots, s_n))]ds_1 \ldots ds_n + \\ &+ \sum_{t_0 < t_i < t} \mu(t_1, \ldots, t_n, \tau_i^{(1)}, \ldots, \tau_i^{(n)}) \cdot \eta_i[u(\rho(\tau_i^{(i)} - 0, \ldots, \tau_i^{(n)} - 0))], \end{aligned}$$

$$c_1 < \tau_i^{(1)} < t_1, \ldots, c_n < \tau_i^{(n)} < t_n \tag{2.2}$$

Let us suppose that the following assumptions are fulfilled:

(H_1) $u(t_1, \ldots, t_n)$ is a real valued nonnegative discontinuous function with finite type of discontinuities in the points

$$(\tau_1^{(1)}, \ldots, \tau_1^{(n)}),\ (\tau_2^{(1)}, \ldots \tau_2^{(n)}),\ \ldots :\ u(\tau_j^{(1)} - 0, \ldots, \tau_j^{(n)} - 0) \neq$$
$$\neq u(\tau_j^{(1)} + 0, \ldots, \tau_j^{(n)} + 0), \quad j = 1, 2 \tag{2.3}$$

Remark 2.1 *If $u(t_1, \ldots, t_n)$ satisfies (2.3), we say that the function $u(t_1, \ldots, t_n)$ has a finite jump in the points $\{\tau_j\} = (\tau_j^{(1)}, \ldots, \tau_j^{(n)})$, $j = 1, 2$.*

(H_2) In a finite bounded domain $[c, T] = \{[c_1, T_1], \ldots [c_n, T_n]\}$ the set $\{\tau_j\}$ is bounded and for infinite domain $[c, \infty]$ the set of points $\{\tau_j\}$ is N, the set of natural numbers.

(H_3) The functions $\varphi(t) \geq 0$, $\mu_i(t, \tau) \geq 0$, $\eta_i(\rho(u)) \geq 0$ and also f, μ_i are continuous functions, η_i are continuous and nondecreasing at u.

(H_4) The kernel $K(t, s, y) \geq 0$ and it is nondecreasing at y with fixed t, s.

(H_5) $c < \tau_i < \tau_{i+1}$, $\forall\, i = 0, 1, 2, \ldots \lim\limits_{i \to \infty} \tau_i = \infty$

(H_6) $\sigma(s), \rho(s)$ belong to the space $\mathcal{T}$ of continuous functions $F : R^n \to R^n$ which satisfy the following conditions [3]:

i) $F(x) = F_1(x), \ldots, F_n(x)$, $F_i : R^n \to R$, $i = 1, \ldots, n$

ii) $\lim\limits_{|x| \to \infty} = \infty$, $i = 1, \ldots, n$

iii) $F(x) \leq x$.

In the following, as done in [11] (see also [36], [42]) we denote by $t = (t_1, \ldots, t_n)$, $t^* = (t_2, \ldots, t_n)$, $t = (t_1, t^*)$, $u_+ = \max\{0, u(t)\}$, $\bar{u}(t) = \sup u_+(s)$, $s_i \in [c_i, t_i]$, $\tau_i = (\tau_i^{(1)}, \ldots, \tau_i^{(n)})$, $t_i^* = (\tau_i^{(2)}, \ldots, \tau_i^{(n)})$, $\tau_i = (\tau_i^{(1)}, \tau_i^*)$.

Then from the inequality

$$u(t) \leq \varphi(t) + \int_{t_0}^{t} K(t, s, \sigma(u(s)))ds + \sum_{t_0 < \tau_i < t} \mu(t, \tau_i) \cdot \eta_i[u(\rho(\tau_i - 0))]$$

we obtain

$$
\begin{aligned}
\bar{u}(t) &\le \varphi(t) + \int_c^t K(t,s,\sigma(\bar{u}(s)))ds + \sum_{c<\tau_i<t} \mu(t,\tau_i)\cdot \eta_i[\bar{u}(\rho(\bar{u}(\tau_i-0)))] \le \\
&\le \bar{\varphi}(t) + \int_c^t K(t,s,\sigma(\bar{u}(s_1,s^*)))ds + \\
&\quad + \sum_{c<\tau_i<t} \mu_i(t,\tau_i)\cdot \eta_i[\bar{u}(\rho(\tau_i^{(1)}-0,\tau_i^{(*)}-0))] \le \\
&\le \bar{\varphi}(t) + \int_c^t \bar{K}(t,s,\bar{u}(\sigma(s_1,s^*)))ds + \\
&\quad + \sum_{c<\tau_i<t} \bar{\mu}_i(t,\tau_i)\cdot \eta_i[\bar{u}(\rho(\tau_i^{(1)}-0,\tau_i^{(*)}-0))],
\end{aligned}
$$

where

$$
\bar{\varphi}(t) = \sup_{\theta_i\le t_i} \varphi(\theta),\ \bar{K}(t,s,\sigma(\xi)) = \sup_{\substack{\theta_i \le t_i \\ p\le \xi}} K(\theta,s,\sigma(p)),\ \bar{\mu}_i(q,\tau_i) = \sup_{w\le q} \mu_i(w,\tau_i)\,.
$$

So, denoting by

$$
\begin{aligned}
&\bar{K}(t,s,\bar{u}(\sigma(s_1,s^*))) = \int_{c_2}^{t_2}\dots\int_{c_n}^{t_n} \bar{K}(t,s,\bar{u}(\sigma(s_1,s^*)))ds_2\dots ds_n, \\
&\bar{\mu}_i(t,\tau_i^{(1)},\tau_i^*)\cdot\bar{\eta}_i[\bar{u}(\rho(\tau_i^{(1)}-0,\tau_i^{(*)}-0))] = \\
&= \sum_{\substack{c_2<\tau_i^{(2)}<t_2 \\ \dots\dots\dots\dots \\ c_n<\tau_i^{(n)}<t_n}} \bar{\mu}_i(t,\tau_i^{(1)},\tau_i^*)\cdot\eta_i[\bar{u}(\rho(\tau_i^{(1)}-0,\tau_i^{(*)}-0))],
\end{aligned}
$$

for every fixed t^* and τ_i^* the function $\bar{u}(t_1,t^*)$ satisfies the one-dimensional impulse integral inequality

$$
\begin{aligned}
&\bar{u}(t_1,t^*) \le \bar{\varphi}(t) + \int_{c_1}^{t_1} \bar{K}(t,s,\bar{u}(\sigma(s_1,t^*)))ds_1 + \\
&+ \sum_{c_1<\tau_i^{(1)}<t_1} \bar{\mu}_i(t,\tau_i^{(1)},\tau_i^*)\cdot\bar{\eta}_i[\bar{u}(\rho(\tau_i^{(1)}-0,\tau_i^{(*)}-0))]. \qquad (2.4)
\end{aligned}
$$

Let $t_* = (t_{1*},\dots,t_{n*})$ be an arbitrary point so that $t_{i*}\ge c_i$, $i=1,2,\dots,n$.

By virtue of the assumptions (H_3), (H_4), for every $t_i \in [c_i, t_{i*}]$ we obtain the following integro-sum inequality

$$\bar{u}(t_1, t^*) \leq \bar{\varphi}(t_*) + \int\limits_{c_1}^{t_1} \bar{K}(t_*, s_1, \bar{u}(\sigma(s_1, s^*)))ds_1 +$$
$$+ \sum_{c_1 < \tau_1^{(1)} < t_1} \bar{\mu}_i(t_*, \tau_i^{(1)}, \tau_i^*) \cdot \bar{\eta}_i[\bar{u}(\rho(\tau_i^{(1)} - 0, \tau_i^{(*)} - 0))]. \tag{2.5}$$

Impulse integral inequality (2.5) is a one-dimensional inequality similar to (2.1). By using the result from [34] (Theorem 3.1.1, p. 174) about the Chaplygin solvability problem for inequality (2.5) we obtain the following estimate:

$$\bar{u}(t) < \theta_{\bar{\varphi}(t_*)}(t), \tag{2.6}$$

where $\theta_{\bar{\varphi}(t_*)}(t)$ is an arbitrary solution of the equation:

$$\theta(t) = \bar{\varphi}(t_*) + \int\limits_{c_1}^{t_1} \bar{K}(t_*, s_1, \theta(\sigma))ds_1 +$$
$$+ \sum_{c_1 < \tau_i^{(1)} < t_1} \bar{\mu}_i(t_*, \tau_i) \cdot \bar{\eta}_i[\theta(\rho(\tau_i - 0))] \tag{2.7}$$

continuous in each interval $\lfloor \tau_i^{(i)}, \tau_{i+1}^{(1)} \lfloor$, $i \in N$. Finally, since the point t_* is arbitrary, we obtain

$$u(t) \leq \bar{u}(t) < \theta_{\bar{\varphi}(t)}(t) \quad \forall\, t \geq c. \tag{2.8}$$

Remark 2.2 *If in (2.2) $\sigma(t) = \rho(t) = t$, the above results are the same as the ones in [11], (§3, p.1640–1641).*

3 General Bihari theorems for Wendroff type impulse inequalities with delay

Now we consider the space R_+^2. Next statement holds:

Theorem 3.1 *Let us suppose that a nonnegative function $u(t, x)$ is defined* *in the domain* $D \subset R^2_+$*:*

$$D = \{ \bigcup_{k,j\geq 1} D_{kj}, D_{kj} = \\ = \{(t,x) : t \in [t_{k-1}, t_k[, \;\; x \in [x_{j-1}, x_j[\}, \;\; k = 1,2,\ldots, \quad j = 1,2,\ldots\},$$

continuous in D *except for* $\{t_i, x_i\}$ *points of finite jump:* $u(t_i - 0, x_i - 0) \neq u(t_i + 0, x_i + 0)$, $i = 1,2,\ldots$ *and satisfies the impulse integro-functional inequality*

$$u(t,x) \leq \varphi(t,x) + q(t,x) \int_{t_0}^{t}\int_{x_0}^{x} f(\xi,\eta) W(u(\sigma_1(\xi), \sigma_2(\eta))) d\xi d\eta + \\ + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \beta_i u^m(t_i - 0, x_i - 0), \tag{3.9}$$

where $q(t,x) \geq 1$, $\varphi(t,x) > 0 \;\forall\, (t,x) \in D$ *is nondecreasing with respect to* (t,x)*:* $p_1 \leq p_2$, $q_1 \leq q_2 \Rightarrow \varphi(p_1,q_1) \leq \varphi(p_2,q_2)$ *at* (p_1,q_1), $(p_2,q_2) \in D$*;* $m = const. > 0$, $\beta_i = const. \geq 0 \quad \forall\, i \in N$*; function* f *is nonnegative and satisfies such a condition*

$$f(t,x) = 0, \quad (t,x) \in D_{ij} \quad i \neq j \quad (i = 1,2,\ldots \quad j = 1,2,\ldots);$$

function $W(u)$ *belongs to the class* $\mathcal{K}$ *of functions such that*

$i_1)$ $W(\gamma\beta) = W(\gamma)W(\beta)$
$i_2)$ $W : [0,\infty[\to [0,\infty[, \;\; W(0) = 0$*;*
$i_3)$ W *is nondecreasing*
$i_4)$ $\sigma_i(s) \in \mathcal{T}, \;\; i = 1,2 \;\; (\sigma_i : R \to R, \, \sigma_i(s) \leq s, \;\; \lim_{|s|\to\infty} \sigma_i(s) = \infty)$*;*

here $(t_i, x_i) < (t_{i+1}, x_{i+1})$ *if* $t_i < t_{i+1}$, $x_i < x_{i+1}$ $\forall\, i = 1,2,\ldots$ *and* $\lim_{i\to\infty} t_i = \infty$, $\lim_{i\to\infty} x_i = \infty$*. Then for every* $(t,x) \in D$ *we obtain:*

$$u(t,x) \leq \\ \leq \varphi(t,x) q(t,x) \psi_i^{-1} \cdot \\ \cdot \left\{ \int_{t_i}^{t}\int_{x_i}^{x} \frac{f(\xi,\eta)}{\varphi(\xi,\eta)} W(\varphi(\sigma_1(\xi),\, \sigma_2(\eta)), q(\sigma_1(\xi),\, \sigma_2(\eta))) d\xi d\eta \right\} \tag{3.10}$$

with $t_i < t < t_{i+1},\ x_i < x < x_{i+1}$,

$$\int\limits_{t_i}^{t}\int\limits_{x_i}^{x}\frac{f(\xi,\eta)}{\varphi(\xi,\eta)}W(\varphi(\sigma_1(\xi),\ \sigma_2(\eta)),\ q(\sigma_1(\xi),\ \sigma_2(\eta)))d\xi d\eta \in\ Dom\,(\psi_i^{-1}) \quad (3.11)$$

$$\psi_0(v)=\int\limits_1^v\frac{d\xi}{W(\xi)},\ \ \psi_i(v)=\int\limits_{c_i}^v\frac{d\xi}{W(\xi)},\ \ i=1,2,\dots$$

$$\begin{aligned}c_i &= (1+\beta_i\varphi^{m-1}(t_i,x_i)q^m(t_i,x_i))\cdot\\ &\cdot\psi_{i-1}^{-1}\left(\int\limits_{t_{i-1}}^{t_i}\int\limits_{x_{i-1}}^{x_i}\frac{f(\xi,\eta)}{\varphi(\xi,\eta)}W(\varphi(\sigma_1(\xi),\sigma_2(\eta)),\ q(\sigma_1(\xi),\sigma_2(\eta)))d\xi d\eta\right)^A \quad (3.12)\end{aligned}$$

$$for\ \begin{cases}0<m\le 1\ \ and\ A=1 & \\ & \forall\, i=1,2,\dots\\ m\ge 1\qquad and\ A=m & \end{cases}$$

Proof. We prove the statement using the inductive method for each domain D_{ii} of continuous function $u(t,x)$. It is obvious that

$$\begin{aligned}\frac{u(t,x)}{\varphi(t,x)} &\le 1+q(t,x)\int\limits_{t_0}^{t}\int\limits_{x_0}^{x}\frac{f(\xi,\eta)}{\varphi(\xi,\eta)}W(u(\sigma_1(\xi),\sigma_2(\eta)))d\xi d\eta+\\ &+\sum_{(t_0,x_0)<(t_i,x_i)<(t,x)}\beta_i\frac{u^m(t_i-0,x_i-0)}{\varphi(t,x)}\le\\ &\le q(t,x)\left[1+\int\limits_{t_0}^{t}\int\limits_{x_0}^{x}\frac{f(\xi,\eta)}{\varphi(\xi,\eta)}W(u(\sigma_1(\xi),\sigma_2(\eta)))d\xi d\eta+\right.\\ &\left.+\sum_{(t_0,x_0)<(t_i,x_i)<(t,x)}\beta_i\varphi^{m-1}(t_i,x_i)\left(\frac{u(t_i-0,x_i-0)}{\varphi(t_i,x_i)}\right)^m\right]. \quad (3.13)\end{aligned}$$

Let us introduce

$$\begin{aligned}u^*(t,x) &= 1+\int\limits_{t_0}^{t}\int\limits_{x_0}^{x}\frac{f(\xi,\eta)}{\varphi(\xi,\eta)}W(u(\sigma_1(\xi),\sigma_2(\eta)))d\xi d\eta+\\ &+\sum_{(t_0,x_0)<(t_i,x_i)<(t,x)}\beta_i\varphi^{m-1}(t_i,x_i)\left(\frac{u(t_i-0,x_i-0)}{\varphi(t_i,x_i)}\right)^m, \quad (3.14)\\ u^*(t_0,x_0) &= 1.\end{aligned}$$

Then we obtain:

$$u(t,x) \leq \varphi(t,x)q(t,x)u^*(t,x),$$

$$u(\sigma_1(t),\sigma_2(x)) \leq \varphi(\sigma_1(t),\sigma_2(x))q(\sigma_1(t),\sigma_2(x))u^*(\sigma_1(t),\sigma_2(x)) \leq$$
$$\leq \varphi(\sigma_1(t),\sigma_2(x))q(\sigma_1(t),\sigma_2(x))u^*(t,x),$$

$$u(t_i-0,x_i-0) \leq \varphi(t_i,x_i)q(t_i,x_i)u^*(t_i-0,x_i-0),$$

$$W(u(\sigma_1(t),\sigma_2(x))) \leq W[\varphi(\sigma_1(t),\sigma_2(x))q(\sigma_1(t),\sigma_2(x))u^*(\sigma_1(t),\sigma_2(x))] \leq$$
$$\leq W[\varphi(\sigma_1(t),\sigma_2(x))q(\sigma_1(t),\sigma_2(x))u^*(t,x)] \leq$$
$$\leq W[\varphi(\sigma_1(t),\sigma_2(x))q(\sigma_1(t),\sigma_2(x))]W(u^*(t,x)). \qquad (3.15)$$

From (3.14), (3.15) it follows

$$u^*(t,x) \leq$$
$$\leq 1 + \int_{t_0}^{t}\int_{x_0}^{x} \frac{f(\xi,\eta)}{\varphi(\xi,\eta)} W[\varphi(\sigma_1(\xi),\sigma_2(\eta))q(\sigma_1(\xi),\sigma_2(\eta))]W(u^*(\xi,\eta))d\xi d\eta +$$
$$+ \sum_{(t_0-x_0)<(t_i,x_i)<(t,x)} \beta_i \varphi^{m-1}(t_i,x_i)q^m(t_i,x_i)u^{*m}(t_i-0,x_i-0). \qquad (3.16)$$

Let us consider the domain D_{11}; inequality (3.16) reduces itself in the next one:

$$u^*(t,x) \leq$$
$$\leq 1 + \int_{t_0}^{t}\int_{x_0}^{x} \frac{f(\xi,\eta)}{\varphi(\xi,\eta)} W[\varphi(\sigma_1(\xi),\sigma_2(\eta))q(\sigma_1(\xi),\sigma_2(\eta))]W(u^*(\xi,\eta))d\xi d\eta.$$
$$(3.17)$$

By using results [3], [34] we obtain:

$$u^*(t,x) \leq \psi_0^{-1}\left\{\int_{t_0}^{t}\int_{x_0}^{x} \frac{f(\xi,\eta)}{\varphi(\xi,\eta)} W[\varphi(\sigma_1(\xi),\sigma_2(\eta))q(\sigma_1(\xi),\sigma_2(\eta))]d\xi d\eta\right\}, \qquad (3.18)$$

where $\psi_0(v) = \int\limits_t^v \frac{d\xi}{W(\xi)}$, ψ_0^{-1} is the inverse function of ψ_0 and $t \geq t_0$, $x \geq x_0$ such that

$$\int\limits_{t_0}^{t}\int\limits_{x_0}^{x} \frac{f(\xi,\eta)}{\varphi(\xi,\eta)} W[\varphi(\sigma_1(\xi),\sigma_2(\eta))q(\sigma_1(\xi),\sigma_2(\eta))]d\xi d\eta \in \text{ Dom }(\psi_0^{-1}). \qquad (3.19)$$

Thus from (3.15), (3.17), (3.18), (3.19) it follows

$$u(t,x) \leq \varphi(t,x)q(t,x)\psi_0^{-1}\left\{\int\limits_{t_0}^{t}\int\limits_{x_0}^{x} \frac{f(\xi,\eta)}{\varphi(\xi,\eta)} W[\varphi(\sigma_1(\xi),\sigma_2(\eta))q(\sigma_1(\xi),\sigma_2(\eta))]d\xi d\eta\right\}$$
$$\forall\,(t,x) \in D_{11} \quad \text{which satisfies (3.19)} \qquad (3.20)$$

By following the scheme in [11] (see also [15], [34]) based on the inductive method, we consider domain D_{22}. Then

$$\begin{aligned}
u^*(t,x) &\leq 1 + \int\limits_{t_0}^{t_1}\int\limits_{x_0}^{x_1} \frac{f(\xi,\eta)}{\varphi(\xi,\eta)} W[\varphi(\sigma_1(\xi),\sigma_2(\eta))q(\sigma_1(\xi),\sigma_2(\eta))]W(u^*(\xi,\eta))d\xi d\eta + \\
&+ \int\limits_{t_1}^{t}\int\limits_{x_1}^{x} \frac{f(\xi,\eta)}{\varphi(\xi,\eta)} W[\varphi(\sigma_1(\xi),\sigma_2(\eta))q(\sigma_1(\xi),\sigma_2(\eta))]W(u^*(\xi,\eta))d\xi d\eta + \\
&+ \beta_1\varphi^{m-1}(t_1,x_1)q^m(t_1,x_1)u^{*m}(t_1-0,x_1-0) \leq \\
&\leq \psi_0^{-1}\left\{\int\limits_{t_0}^{t_1}\int\limits_{x_0}^{x_1} \frac{f(\xi,\eta)}{\varphi(\xi,\eta)} W[\varphi(\sigma_1(\xi),\sigma_2(\eta))q(\sigma_1(\xi),\sigma_2(\eta))]d\xi d\eta\right\} + \\
&+ \int\limits_{t_1}^{t}\int\limits_{x_1}^{x} \frac{f(\xi,\eta)}{\varphi(\xi,\eta)} W[\varphi(\sigma_1(\xi),\sigma_2(\eta))q(\sigma_1(\xi),\sigma_2(\eta))]W(u^*(\xi,\eta))d\xi d\eta + \\
&+ \beta_1\varphi^{m-1}(t_1,x_1)q^m(t_1,x_1)\cdot \\
&\cdot\left[\psi_0^{-1}\left\{\int\limits_{t_0}^{t_1}\int\limits_{x_0}^{x_1} \frac{f(\xi\eta)}{\varphi(\xi,\eta)} W[\varphi(\sigma_1(\xi),\sigma_2(\eta))q(\sigma_1(\xi),\sigma_2(\eta))]d\xi d\eta\right\}\right]^m \leq \\
&\leq c_1 + \int\limits_{t_1}^{t}\int\limits_{x_1}^{x} \frac{f(\xi,\eta)}{\varphi(\xi,\eta)} W[\varphi(\sigma_1(\xi),\sigma_2(\eta))q(\sigma_1(\xi),\sigma_2(\eta))]W(u^*(\xi,\eta))d\xi d\eta, \qquad (3.21)
\end{aligned}$$

where

$$c_1 = [1 + \beta_1\varphi^{m-1}(t_1,x_1)q^m(t_1,x_1)]\psi_0^{-1}\cdot$$

$$\cdot\left\{\int_{t_0}^{t_1}\int_{x_0}^{x_1}\frac{f(\xi,\eta)}{\varphi(\xi,\eta)}W[\varphi(\sigma_1(\xi),\sigma_2(\eta))q(\sigma_1(\xi),\sigma_2(\eta))]d\xi d\eta\right\},$$

$$\text{if}\quad 0<m\leq 1 \tag{3.22}$$

$$c_1=[1+\beta_1\varphi^{m-1}(t_1,x_1)q^m(t_1,x_1)]\cdot$$

$$\cdot\left[\psi_0^{-1}\left\{\int_{t_0}^{t_1}\int_{x_0}^{x_1}\frac{f(\xi,\eta)}{\varphi(\xi,\eta)}W[\varphi(\sigma_1(\xi),\sigma_2(\eta))q(\sigma_1(\xi),\sigma_2(\eta))]d\xi d\eta\right\}\right]^m$$

$$\text{if}\quad m\geq 1 \tag{3.23}$$

Let us point out that the inequality $\psi_0'(v)>0$ guarantees that ψ_0^{-1} exists and it is an increasing function. By using (3.11) with $i=1$ from (3.21) we obtain:

$$u^*(t,x)\leq\psi_1^{-1}\left\{\int_{t_1}^{t}\int_{x_1}^{x}\frac{f(\xi,\eta)}{\varphi(\xi,\eta)}W[\varphi(\sigma_1(\xi),\sigma_2(\eta))q(\sigma_1(\xi),\sigma_2(\eta))]d\xi d\eta\right\}$$

$$\text{with}\quad t_1<t<t_2\,,\qquad x_1<x<x_2;$$

so, $u(t,x)$ satisfies (3.10) for $(t,x)\in D_{22}$.

Now let us suppose that (3.10) holds for $(t,x)\in D_{kk}$; then for $(t,x)\in D_{k+1k+1}$ we have

$$u^*(t,x)\leq$$

$$\leq c_k+\int_{t_k}^{t}\int_{x_k}^{x}\frac{f(\xi,\eta)}{\varphi(\xi,\eta)}W[\varphi(\sigma_1(\xi),\sigma_2(\eta))q(\sigma_1(\xi),\sigma_2(\eta))]W(u^*(\xi,\eta))d\xi d\eta,$$

$$t_k<t<t_{k+1},\quad x_k<x<x_{k+1} \tag{3.24}$$

where c_k is defined by (3.12). Thus, on this step we have completed the proof.

Remark 3.1 *The result of Theorem 2.1 is a new Bihari analogy statement for discontinuous functions of two independent variables with non–Lipschitz type discontinuities and delay.*

Remark 3.2 A

(1) If $\beta_i=0$, $q(t,x)=1$, $W(u)=u$, $\sigma(\xi,\eta)=(\sigma_1(\xi),\sigma_2(\eta))=(\xi,\eta)$, *we*

obtain the classical result by Wendroff (see [22]).

(2) *If $\beta_i = 0$, from Theorem 2.1 Akinyele result ([3], Th. 4 as $n = 2$) follows.*

(3) *If $q(t,x) = 1$, $W(u) = u$, $\sigma(\xi,\eta) = (\xi,\eta)$, $m = 1$ we obtain Borysenko result ([11], Theorem p. 1638).*

(4) *If $q(t,x) = 1$, $W(u) = u$, $\sigma(\xi,\eta) = (\xi,\eta)$ we have: a) Borysenko–Iovane result ([16], th. 2.1); b) if $q(t,x) = 1$, $W(u) = u^m$, $m \neq 1$, $m > 0$, Theorem 2.2 in [16]; c) if $W(u) = u^m$ Theorem 2.3 in [16]; d) if $W(u) = u$ Theorem 3.1 in [16]; e) if $W(u) = u^m$, Theorem 3.2 in [16].*

(5) *If $q(x,t) = 1$, we have Gallo–Piccirillo results [18]: Theorem 2.1, if $W(u) = u$; Theorem 2.2, if $W(u) = u^m$, $n = 2$.*

B

From Theorem 2.1, for one-dimensional case $(n = 1)$, it follows:

(1) *Agarwal results ([2], Corollary 4.12 for $m = 1$, similar Theorem 4.21 for $m \neq 1$).*

(2) *Classical Bellman results [5, p. 58], Bihari [7].*

(3) *Borysenko, Gallo, Toscano results [9, 10 lemma 1, lemma 2], if in Theorem 2.1 we assume*

$$q = 1, \quad W(u) = u^m, \quad q \geq 1, \quad \sigma(t) = t\,.$$

(4) *Gallo, Piccirillo [17, Theorem 2.1], if $W(u) = u$; [17, Theorem 2.2], if $W(u) = u^m$; Borysenko [34, Theorem 3.71, p. 232], if $m = 1$; Borysenko, Samoilenko [34, Theorem 3.7.6 $(g = 0)$], if $m = 1$; Iovane [20, Theorem 2.1], if $q = 1$, $W(u) = u^m$; [20, Theorem 2.2], if $W(u) = u^m$.*

By using the results by Pachpatte [31] and Akinyle [3], we consider the class $\mathcal{F}'$ of functions f such that

a) $f(x)$ is nonnegative, continuous and non decreasing as $x \geq 0$;

b) $\forall\, x \geq 1, \forall\, y \geq 0 \quad x^{-1} f(y) \leq f(x^{-1} y)$

and we obtain the following statement:

Theorem 3.2 *Let us suppose that function $u(t,x)$ in domain D satisfies* *inequality (2.9) and the conditions of Theorem 3.1 (with the exception of conditions about function $W(u)$. Then, if $W \in \mathcal{F}'$, we obtain for arbitrary $t_0 \le t \le t^*$, $x_0 \le x \le x^*$ the following estimate:*

$$u(t,x) \le \varphi(t,x)q(t,x)\bar{\psi}_i^{-1} \cdot \\ \cdot \left\{ \int_{t_i}^{t}\int_{x_i}^{x} f(\xi,\eta)q(\sigma_1(\xi),\sigma_2(\eta))\frac{\varphi(\sigma_1(\xi),\sigma_2(\eta))}{\varphi(\xi,\eta)}d\xi d\eta \right\}, \qquad (3.25)$$

where $t_i < t < t_{i+1}$, $x_i < x < x_{i+1}$, $\bar{\psi}_0(v) = \int_1^v \frac{d\xi}{W(\xi)}$, $\bar{\psi}_i(v) = \int_{c_1}^v \frac{d\xi}{W(\xi)}$, $i = 1, 2, \ldots$,

$$c_i = (1 + \beta_i \varphi^{m-1}(t_i x_i) q^m(t_i, x_i)) \cdot \\ \cdot \left[\bar{\psi}_{i-1}^{-1} \left(\int_{t_{i-1}}^{t_i}\int_{x_{i-1}}^{x_i} f(\xi,\eta)W(q(\sigma_1(\xi),\sigma_2(\eta)))\, d\xi d\eta \right) \right]^A$$

$$\textit{with} \quad \begin{cases} A = 1 & \textit{if } 0 < m \le 1 \\ A = m & \textit{if } m \ge 1 \end{cases} \qquad \forall\, i = 1, 2, \ldots \qquad (3.26)$$

and t^*, x^* *are chosen such that*

$$(t^*, x^*) = \\ = \sup\left\{ (t,x) \,:\, \int_{t_j}^{t}\int_{x_j}^{x} f(\xi,\eta)q(\sigma_1(\xi),\sigma_2(\eta))\frac{\varphi(\sigma_1(\xi),\sigma_2(\eta))}{\varphi(\xi,\eta)}d\xi d\eta \in \; Dom(\psi_j^{-1}) \right\}, \\ j = 0, 1, 2, \ldots \qquad (3.27)$$

Proof. It is obvious that

$$W(u(\sigma_1(t),\sigma_2(x))) \le W[\varphi(\sigma_1(t),\sigma_2(x))q(\sigma_1(t),\sigma_2(x))u^*(t,x)) \le \\ \le \frac{\varphi(\sigma_1(t),\sigma_2(x))q(\sigma_1(t),\sigma_2(x))}{\varphi(\sigma_1(t),\sigma_2(x))q(\sigma_1(t),\sigma_2(x))} W[\varphi(\sigma_1(t),\sigma_2(x))q(\sigma_1(t),\sigma_2(x))u^*(t,x)] \le \\ \le \varphi(\sigma_1(t),\sigma_2(x))q(\sigma_1(t),\sigma_2(x))W[u^*(t,x)] \qquad (3.28)$$

where $u^*(t,x)$ is defined by (3.14). Then

$$u^*(t,x) \le 1 + \int_{t_0}^{t}\int_{x_0}^{x} f(\xi,\eta)q(\sigma_1(\xi),\sigma_2(\eta))\frac{\varphi(\sigma_1(\xi),\sigma_2(\eta))}{\varphi(\xi,\eta)} \cdot$$
$$\cdot W(u^*(\xi,\eta))d\xi d\eta +$$
$$+ \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \beta_i \varphi^{m-1}(t_i,x_i)q^m(t_i,x_i)u^{*m}(t_i-0,x_i-0). \qquad (3.29)$$

Later for inequality (3.27) we use the same inductive method procedure as for inequality (3.16) in order to conclude the proof of Theorem 2.2.

Remark 3.3 *Theorem 3.2 (similar to Theorem 3.1) is a new Bihari analogy result [7] for discontinuous functions.*

Remark 3.4 *By using the results of Section 2 we can prolong the results of Theorems 3.1, 3.2 to the n-dimensional case of impulsive integral inequalities.*

Remark 3.5 *If $\beta_i = 0$, Theorem 3.2 is similar to result by Akinyle [3, Theorem 5, if $n = 2$]; for $m = 1$, the estimate (3.25) is more precise than (7.10), the one in [36, Theorem 3.7.2].*

References

[1] R.P. Agarwal, On an integral inequality in n-independent variables, *J. Math. Anal. Appl.* **85** (1982), 192-196.

[2] R.P. Agarwal, *Difference Equations and Inequalities: Theory, Methods and Applications*, New York, 1992.

[3] O. Akinyele, On Gronwall–Bellman–Bihari type integral inequalities in several variables with retardation, *J. Math. Anal. Appl.* **104** (1) (1984), 1-26.

[4] N.V. Azbelev, Z.B. Tsalyuk, On Chaplygin's problem, *Ukrain. Mat. Zh.* **10** (1) (1958), 3-12.

[5] D. Bainov, P.S. Simeonov, *Integral Inequalities and Applications*, Kluwer Acad. Publ., Dordrecht, 1992.

[6] R. Bellman, K.L. Cooke, *Differential–Difference equations*, Academic Press, New York, 1963.

[7] I. Bihari, A generalization of lemma of Bellman and its application to uniqueness problems of differential equations, *Acta Math. Acad. Scient. Hung.* **7** (1) (1956), 71-94.

[8] D.S. Borysenko, About the integral inequality for piecewise continuous functions, in *Proc. X Int. Kravchuk Conf.*, Kyiv, 2004, p. 323.

[9] D.S. Borysenko, A. Gallo, R. Toscano, Integral inequalities Gronwall–Bellman type for discontinuous functions, *Visnik Kyiv University* **1** (2005), 63-67.

[10] D.S. Borysenko, A. Gallo, R. Toscano, Integral inequalities Gronwall–Bellman type for discontinuous functions and estimates of solutions impulsive systems, in *Proc. DE@CAS*, Brest, 2005, pp. 5-9.

[11] S.D. Borysenko, Integro–sum inequalities for functions of many independent variables, *Differential equations* **25** (9) (1989), 1638-1641.

[12] S.D. Borysenko, G. Iovane, P. Giordano, Investigations of the properties motion for essential nonlinear systems perturbed by impulses on some hypersurfaces, *Nonl. Anal.* **62** (2005), 345-363.

[13] S.D. Borysenko, M. Ciarletta, G. Iovane, Integro-sum inequalities and motion stability of systems with impulse perturbations, *Nonl. Anal.* **62** (2005), 417-428.

[14] S.D. Borysenko, G. Iovane, P. Giordano, About some hyperbolic impulsive equations and estimate solutions, in *Proc. DE@CAS*, Brest, 2005, pp. 9-14.

[15] S.D. Borysenko, G. Iovane, *Integro-Sum Inequalities and Qualitative Analysis Dynamical Systems with Perturbations*, Tipografia-Legatoria "ELDA", Università di Salerno, Salerno, 2006.

[16] S.D. Borysenko, G. Iovane, About some integral inequalities of Wendroff type for discontinuous functions, *Nonl. An.* **66** (10) (2007), 2190-2203.

[17] A. Gallo, A.M. Piccirillo, About new analogies Gronwall–Bellman–Bihari type inequalities for discontinuous functions and estimates solutions impulsive differential systems, *Nonl. An.* **67** (5) (2007), 1550–1559.

[18] A. Gallo, A.M. Piccirillo, New Wendroff type inequalities for discontinuous functions and its applications, *Nonl. Studies* (2007), *in print.*

[19] G. Iovane, S.D. Borysenko, Boundedness, stability, practical stability of motion impulsive systems, in *Proc. DE@CAS*, Brest, 2005, pp. 15-21.

[20] G. Iovane, Some new integral inequalities of Bellman–Bihari type with delay for discontinuous functions, *Nonl. Anal.* **66** (2007), 498-508.

[21] S.C. Hu, V. Lakshmikantham, S. Leela, Impulsive differential systems and the pulse phenomena, *J. Math. Anal. Appl.* **137** (2) (1989), 605-612.

[22] V. Lakshmikantham, S. Leela, *Differential and Integral Inequalities, Theory and Applications*, Academic Press, New York, 1969.

[23] V. Lakshmikantham, S. Leela, M. Mohan Rao Rama, Integral and integro-differential inequalities, *Appl. Anal.* **24** (3)(1987), 157-164.

[24] V. Lakshmikantham, D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equation, *Ser. Modern Appl. Math.* **6** (1989).

[25] A.A. Martyhyuk, V. Lakshmikantham, S. Leela, Stability of Motion: the Method of Integral Inequalities, *Naukova Dumka*, Kyiv, 1989.

[26] Yu.A. Mitropolskiy, S. Leela, A.A. Martyhyuk, About some directions of investigations of V. Lakshmikantham in theory of differential equations and applications, *Differ. Uravn.* **22** (4) (1986), 550-572.

[27] Yu.A. Mitropolskiy, G. Iovane, S.D. Borysenko, About a generalization of Bellman–Bihari type inequalities for discontinuous functions and their applications, *Nonl. Anal.* **66** (10) (2007), 2140-2165.

[28] J.J. Nieto, Basic theory for nonresonance impulsive periodic problems of first order, *J. Math. Anal. Appl.* **205** (1997), 423-433.

[29] J.J. Nieto, Impulsive resonance periodic problems of first order, *Appl. Math. Letters* **15** (2002), 489- 493.

[30] B.G. Pachpatte, On some integral inequalities similar to Bellman–Bihari inequality, *J. Math. Anal. Appl.* **49** (1975), 794-802.

[31] B.G. Pachpatte, On some new integral and integro-differential inequalities in two independent variables and their applications, *J. Differential Equations* **33** (1979), 249-272.

[32] B.G. Pachpatte, On some new integro-differential inequalities of the Wendroff type, *J. Math. Anal. Appl.* **73** (1980), 491-500.

[33] A.M. Piccirillo, Estimates of the solutions hyperbolic equations with impulse perturbations, in *Proc. XI Int. Kravchuk Conf.*, Kyiv, 2006, pp. 77-80.

[34] A.M. Piccirillo, I. Verigina, About some generalization Bihari result for integro-sum inequalities, in *Proc. XI Int. Kravchuk Conf.*, Kyiv, 2006, p. 187.

[35] L.F. Rakhmatullina, On some application of stability conditions of Chaplygin problems for the problems of boundedness and solution stability of differential equations, *Izv. Vuzov* **2** (1959), 198-201.

[36] A.M. Samoilenko, S.D. Borysenko, C. Cattani, G. Matarazzo, V. Visinsky, Differential Models: Stability, Inequalities and Estimates, *Naukova Dumka*, Kyiv, 2001.

[37] A.M. Samoilenko, N. Perestyuk, Differential Equations with Impulse Effect, *Visha Shkola*, Kyiv, 1987, pp. 423-432.

[38] E. Thahdapani, R.D. Agarwal, On some new inequalities in n-independent variables, *J. Math. Anal. Appl.* **86** (1982), 542-561.

[39] Z.B. Tsalyuk, Multidimensional integral inequalities, *Differents Uravneiya* **10** (1983), 1828–1839.

[40] C.G. Yeh, Wendroff inequality in n-independent variables, *J. Math. Anal. Appl.* **78** (1980), 78-87.

[41] W. Walter, *Differential and Integral Inequalities*, Springer–Verlag, New York, 1970.