New Wendroff's type inequalities for discontinuous functions and their applications

A. Gallo [a], A.M. Piccirillo [b,*]

[a] *Department of Mathematics and Applications "R. Caccioppoli"*
University of Naples "Federico II"
Via Claudio 21 - 80125 Napoli, Italy
angallo@unina.it

[b] *Department of Civil Engineering*
Second University of Naples
Via Roma, 29 - 81031 Aversa (CE), Italy
annamaria.piccirillo@unina2.it

Abstract

In the present paper we found new analogies of Wendroff's inequality for discontinuous functions with finite jumps on some curves and non-Lipschitz' type discontinuities. New conditions of boundedness for solutions of the impulsive nonlinear hyperbolic equations are obtained.

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1 Introduction

The strong evolution of the impulsive differential systems theory is based on the fundamental results by Krylov and Bogolyubov [10], Bainov and Simeonov [1,14], Blaquiere [2], Borysenko [3-6], Deo and Pandit [7], Kaul [9], Halanay and Wexler [8], Hu [11], Lakshmikantham [9,11-14], Leela [9,11,12], Milman and Myshkis [15], Mitropolskiy, Iovane and Borysenko [16], Myshkis [17],

* Corresponding author. Phone +39 081 7683548 - fax +39 081 7683643

Samoilenko et al. [20-22], and others. In the last 30 years most of the investigations about impulsive systems connected with ordinary impulsive differential equations gave very important results.

Some results in the impulsive systems theory are based on the application of the integral inequalities method for discontinuous functions in qualitative analysis of solutions: boundedness, stability, continuous dependence, existence, etc... Let us point out the monographs by Bainov, Borysenko, Lakshmikantham, Samoilenko, all appeared in the last 20 years.

The first generalization of Wendroff's result for discontinuous functions was obtained in [4]; later in [20] the more important results in the integral inequalities theory for discontinuous functions with several variables were described (see also [6]).

Our paper is devoted to generalization of the results [4], [6], [16] and it is based on new analogies with Wendroff's type inequality.

In Section 2 we state some theorems about solvability for nonlinear inequality for functions with two independent variables with Hölder's type discontinuities.

In Section 3 we obtain estimates for solutions of hyperbolic equations with impulse perturbation and moreover the conditions for their boundedness.

2 Main results

Let us consider some set $D^* \subset \Re^2_+$, where

$$D^* = D \backslash \Gamma, \quad B = \bigcup_j B_j, \quad \Gamma = \bigcup_j \Gamma_j, \quad \Gamma_j \stackrel{\text{def}}{=} \{(x_1 x_2) : \varphi_j(x_1 x_2) = 0, \ \ \forall j \in N\}$$

$\Gamma_j \cap \Gamma_{j+1} = \emptyset \ \ \forall j \in N$. Let us suppose that $\varphi_j(x_1 x_2) \ \forall j \in N$ are real valued continuously differentiable functions such as $\operatorname{grad} \varphi_j(x_1 x_2) > 0 \ \forall j \in N$. Let us consider the set B_j

1. $B_1 \stackrel{\text{def}}{=} \{(x_1 x_2) : x_1 \geq 0, \quad x_2 \geq 0, \quad \varphi_1(x_1 x_2) < 0\}$
2. $B_k \stackrel{\text{def}}{=} \{(x_1 x_2) : x_1 \geq 0, \quad x_2 \geq 0, \quad \varphi_{k-1}(x_1 x_2) \geq 0,$
   $\varphi_k(x_1 x_2) < 0, \ \ \forall k \geq 2, \ \ k \in N\}$

and define the set G_j:

$$G_j = \{(y, z) : (x_1 x_2) \in B_j\,, \quad 0 \leq y \leq x_1\,, \ \ 0 \leq z \leq x_2 \ \ \forall j \in N\}.$$

Let us consider an inequality such as:

$$u(x_1,x_2) \le \varphi(x_1,x_2) + \iint_{G_n} f(\sigma_1,\sigma_2)u^\alpha(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2 + \\ + \sum_{j=1}^{n-1} \int_{\Gamma_j \cap G_n} \beta_j(x_1,x_2)u^m(x_1,x_2)d\mu_{\varphi_j} \tag{2.1}$$

where $u(x_1,x_2)$ is a real valued non-negative discontinuous function in D^*, with finite jumps on the curves Γ_j, $\varphi(x_1,x_2)$ is a positive non-decreasing function in $\Re_+^2$, μ_{φ_j} is the Lebesgue–Stiltjes measure concentrated on the curves Γ_k. In the papers [4], [21] inequality (2.1) was investigated in the case $\alpha > 0$, $m = 1$. We generalize the idea from [6,16] in the more general case of Hölder's type nonlinearities for the discontinuous function (2.1) by using also the results [3], [4], [18], [19], [21].

Theorem 2.1 *Let us suppose that the function $u(x_1,x_2)$ satisfies inequality (2.1) with $f \ge 0$, $\beta_j \ge 0$, $\alpha = 1$, $m > 0$. Then such estimates are valid:*

$$u(x_1,x_2) \le \varphi(x_1,x_2) \prod_{j=1}^{\infty} \left(1 + \int_{\Gamma_j \cap G_{j+1}} \varphi^{m-1}(x_1,x_2)\beta_j(x_1,x_2)d\mu_{\varphi_j}\right) \cdot \\ \cdot \exp\left[\int_0^{x_1}\int_0^{x_2} f(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\right], \qquad \textit{if } 0 < m \le 1; \tag{2.2}$$

$$u(x_1,x_2) \le \varphi(x_1,x_2) \prod_{j=1}^{\infty} \left(1 + \int_{\Gamma_j \cap G_{j+1}} \varphi^{m-1}(x_1,x_2)\beta_j(x_1,x_2)d\mu_{\varphi_j}\right) \cdot \\ \cdot \exp\left[m\int_0^{x_1}\int_0^{x_2} f(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\right], \qquad \textit{if } m \ge 1. \tag{2.3}$$

Proof. Because φ is a positive and non-decreasing function, we have:

$$\frac{u(x_1,x_2)}{\varphi(x_1,x_2)} \le 1 + \iint_{G_n} \frac{f(\sigma_1,\sigma_2)}{\varphi(x_1,x_2)} u(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2 + \\ + \sum_{j=1}^{n-1} \int_{\Gamma_j \cap G_n} \beta_j(x_1,x_2)\varphi^{m-1}(x_1,x_2)\left[\frac{u(x_1,x_2)}{\varphi(x_1,x_2)}\right]^m d\mu_{\varphi_j}.$$

Denoting by $W(x_1,x_2) = \dfrac{u(x_1,x_2)}{\varphi(x_1,x_2)}$ and considering domain B_1, inequality (2.1) assumes in B_1 the following form:

$$W(x_1,x_2) \le 1 + \int_0^{x_1}\int_0^{x_2} f(\sigma_1,\sigma_2)W(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2. \tag{2.4}$$

Inequality (2.4) is the classical Wendroff's type inequality for continuous function W in domain B_1. Then it results in:

$$W(x_1,x_2) \le \exp\left[\int_0^{x_1}\int_0^{x_2} f(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\right], \qquad \forall\,(x_1,x_2)\in B_1. \tag{2.5}$$

Now let us consider the domain B_2 and set $G_2 = G_2^1 \cup G_2^2$, where $G_2^2 = \{(x_1,x_2) : (x_1,x_2) \in D_2 \cap G_2\}$. Then for every $(x_1,x_2) \in D_2$ we have inequalities such as:

$$\begin{aligned}
W(x_1,x_2) &\le 1 + \iint_{G_2^1} f(\sigma_1,\sigma_2)W(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2 + \\
&+ \int_{\Gamma_1\cap G_2} \beta_1(x_1,x_2)\varphi^{m-1}(x_1,x_2)W^m(x_1,x_2)d\mu_{\varphi_1} + \\
&+ \iint_{G_2^2} f(\sigma_1,\sigma_2)W(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2) \le \\
&\le 1 + \iint_{G_2^1} f(\sigma_1,\sigma_2)\exp\left[\int_0^{\sigma_1}\int_0^{\sigma_2} f(u,v)dudv\right]d\sigma_1 d\sigma_2 + \\
&+ \int_{\Gamma_1\cap G_2} \beta_1(x_1,x_2)\varphi^{m-1}(x_1,x_2)\exp\left[m\int_0^{x_1}\int_0^{x_2} f(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\right]d\mu_{\varphi_1} + \\
&+ \iint_{G_2^2} f(\sigma_1,\sigma_2)W(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2.
\end{aligned}$$

We select on the curves $\Gamma_1 \cap G_2$ (such as in [21]) some fixed points $A_i = (x_i^1, y_i^1)$, $i = 0,\dots,n-1$, with $A_i \ne A_j$, $i \ne j$ and we consider the inequality:

$$\begin{aligned}
W(x_1,x_2) \le \sum_{i=0}^{n-1}\Bigg(&1 + \int_0^{x_1}\int_0^{x_2} f(\sigma_1,\sigma_2)\exp\left[\int_0^{\sigma_1}\int_0^{\sigma_2} f(u,v)dudv\right]d\sigma_1 d\sigma_2 + \\
&+\beta_1(x_i^1,y_i^1)\varphi^{m-1}(x_i^1,y_i^1)\cdot\exp\left[m\int_0^{x_i^1}\int_0^{y_i^1} f(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\right]\Delta\mu_{\varphi_1}^i +
\end{aligned}$$

$$+\int_{x_i^1}^{x_1}\int_{y_i^1}^{y_1} f(\sigma_1,\sigma_2)W(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\Bigg) =$$

$$=\sum_{i=0}^{n-1}\Bigg(\exp\left[\int_0^{x_i^1}\int_0^{y_i^1} f(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\right]+\beta_1(x_i^1,y_i^1)\varphi^{m-1}(x_i^1,y_i^1)\cdot$$

$$\cdot\exp\left[m\int_0^{x_i^1}\int_0^{y_i^1} f(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\right]\Delta\mu_{\varphi_1}^i+\int_{x_i^1}^{x_1}\int_{y_i^1}^{y_1} f(\sigma_1,\sigma_2)W(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\Bigg).$$

So, we obtain the following relations

$$W(x_1,x_2)\leq\sum_{i=1}^{n-1}\Bigg[\exp\left[\int_0^{x_i^1}\int_0^{y_i^1} f(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\right]\cdot$$
$$\cdot\;(1+\beta_1(x_i^1,y_i^1)\varphi^{m-1}(x_i^1,y_i^1)\Delta\mu_{\varphi_1}^i)\Bigg]+$$
$$+\int_{x_i^1}^{x_1}\int_{y_i^1}^{y_1} f(\sigma_1,\sigma_2)W(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\,,\qquad \text{if } m\in]0,1]; \tag{2.6}$$

$$W(x_1,x_2)\leq\sum_{i=1}^{n-1}\Bigg[\exp\left[m\int_0^{x_i^1}\int_0^{y_i^1} f(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\right]\cdot$$
$$\cdot\;(1+\beta_1(x_i^1,y_i^1)\varphi^{m-1}(x_i^1,y_i^1)\Delta\mu_{\varphi_1}^i)\Bigg]+$$
$$+\int_{x_i^1}^{x_1}\int_{y_i^1}^{y_1} f(\sigma_1,\sigma_2)W(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\,,\qquad \text{if } m>1. \tag{2.7}$$

Here $\Delta\mu_{\varphi_1}^i$ is the variation of the measure function φ_1 on the part of curves A_iA_{i+1}. When $\max\limits_{0\leq i\leq n-1}\Delta\mu_{\varphi_1}^i\underset{n\to\infty}{\longrightarrow}0$, from inequalities (2.6) and (2.7) we have the following inequalities:

$$W(x_1,x_2)\leq\left(1+\int_{\Gamma_1\cap G_2}\beta_1(x_1,x_2)\varphi^{m-1}(x_1,x_2)d\mu_{\varphi_1}\right)\cdot$$

$$\cdot \exp\left[\int_0^{x_1}\int_0^{x_2} f(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\right], \qquad \text{if } m \in]0,1];$$

$$W(x_1,x_2) \leq \left(1 + \int\limits_{\Gamma_1\cap G_2} \beta_1(x_1,x_2)\varphi^{m-1}(x_1,x_2)d\mu_{\varphi_1}\right)\cdot$$

$$\cdot \exp\left[m\int_0^{x_1}\int_0^{x_2} f(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\right], \qquad \text{if } m \geq 1. \qquad (2.8)$$

By using (2.8) and the equality $W(x_1,x_2) = u(x_1,x_2)/\varphi(x_1,x_2)$ we obtain that estimates (2.2), (2.3) hold in B_2 for function $u(x_1,x_2)$.

Let estimates (2.2), (2.3) be true for $(x_1,x_2) \in B_k$ (we use the induction method procedure). Let us consider the domain B_{k+1}; if $(x_1,x_2) \in B_{k+1}$, we have

$$W(x_1,x_2) \leq 1 + \iint\limits_{G^1_{k+1}} f(\sigma_1,\sigma_2)W(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2) +$$
$$+ \int\limits_{\Gamma_k\cap G_{k+1}} \beta_k(x_1x_2)\varphi^{m-1}(x_1,x_2)W^m(x_1,x_2)d\mu_{\varphi_k} +$$
$$+ \iint\limits_{G^2_{k+1}} f(\sigma_1,\sigma_2)W(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2, \qquad (2.9)$$

where $G_{k+1} = G^1_{k+1} \cup G^2_{k+1}$, $G^1_{k+1} = \{(x_1,x_2) : (x_1,x_2) \in B_{k+1} \cap G_{k+1}\}$, $G^2_{k+1} = G_{k+1} \backslash G^1_{k+1}$. We select on the curves $\Gamma_k \cap G_{k+1}$ some fixed points $A^*_i = (x^k_i, y^k_i)$, $i = 0,\ldots,n-1$, with $A^*_i \neq A^*_j$, $i \neq j$. As above, we denote by $\Delta\mu^i_{\varphi_k}$ the variation of measure on the piece $A^*_k A^*_{k+1}$. Then it results in:

$$W(x_1,x_2) \leq \sum_{i=0}^{n-1}\left(1 + \int_0^{x^k_i}\int_0^{y^k_i} f(\sigma_1,\sigma_2)W(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2 + \right.$$
$$+\beta_k(x^k_i,y^k_i)\varphi^{m-1}(x^k_i,y^k_i)W(x^k_i,y^k_i)\Delta\mu^i_{\varphi_k} +$$
$$\left.\int_{x^k_i}^{x_1}\int_{y^k_i}^{y_1} f(\sigma_1,\sigma_2)W(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\right),$$

from which we obtain:

$$W(x_1,x_2) \leq \left[1 + \int_0^{x^k_i}\int_0^{y^k_i} f(\sigma_1,\sigma_2)\prod_{j=1}^{k-1}\left(1 + \int\limits_{\Gamma_j\cap G_{j+1}} \beta_j(x_1,x_2)\varphi^{m-1}(x_1,x_2)d\mu_{\varphi_j}\cdot\right.\right.$$

$$\cdot \exp\left[\int_0^{\sigma_1}\int_0^{\sigma_2} f(u,v)dudv\right] d\sigma_1 d\sigma_2\Bigg) + \beta_k(x_i^k, y_i^k)\varphi^{m-1}(x_i^k, y_i^k)\cdot$$

$$\cdot \prod_{j=1}^{k-1}\left(1+\int_{\Gamma_j\cap G_{j+1}} \beta_j(x_1,x_2)\varphi^{m-1}(x_1,x_2)d\mu_{\varphi_j}\right)\cdot$$

$$\cdot \exp\left[\int_0^{x_i^k}\int_0^{y_i^k} f(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\right]\Delta\mu^i_{\varphi_k} +$$

$$+\int_{x_i^k}^{x_1}\int_{y_i^k}^{y_1} f(\sigma_1,\sigma_2)W(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\Bigg], \qquad \text{if } m \in]0,1]$$

and

$$W(x_1,x_2) \le \sum_{i=0}^{n-1}\Bigg[1+\int_0^{x_i^k}\int_0^{y_i^k} f(\sigma_1,\sigma_2)\exp\left[\int_0^{\sigma_1}\int_0^{\sigma_2} f(u,v)dudv\right]\cdot$$

$$\cdot \prod_{j=1}^{k-1}\left(1+\int_{\Gamma_j\cap G_{j+1}} \beta_j(x_1,x_2)\varphi^{m-1}(x_1,x_2)d\mu_{\varphi_j}\right) d\sigma_1 d\sigma_2 +$$

$$+\beta_k(x_i^k, y_i^k)\varphi^{m-1}(x_i^k, y_i^k)\cdot$$

$$\cdot \prod_{j=1}^{k-1}\left(1+\int_{\Gamma_j\cap G_{j+1}} \beta_j(x_1,x_2)\varphi^{m-1}(x_1,x_2)d\mu_{\varphi_j}\right)\cdot$$

$$\cdot \exp\left[m\int_0^{x_i^k}\int_0^{y_i^k} f(\sigma_1\sigma_2)d\sigma_1 d\sigma_2\right]\Delta\mu^i_{\varphi_k} +$$

$$+\int_{x_i^k}^{x_1}\int_{y_i^k}^{y_1} f(\sigma_1,\sigma_2)W(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\Bigg], \qquad \text{if } m > 1.$$

Finally, when $\max\limits_{0\le i\le n-1} \delta\mu_{\varphi^i_k} \underset{n\to\infty}{\longrightarrow} 0$, we have these estimates:

$$W(x_1,x_2) \le \prod_{j=1}^{k}\left(1+\int_{\Gamma_j\cap G_{j+1}} \beta_j(x_1,x_2)\varphi^{m-1}(x_1,x_2)d\mu_{\varphi_j}\right)\cdot$$

$$\cdot \exp\left[\int_0^{x_1}\int_0^{x_2} f(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\right],$$

$$\text{if } m \in]0,1], \qquad \forall\,(x_1,x_2)\in B_{k+1};$$

$$W(x_1,x_2) \leq \prod_{j=1}^{k}\left(1+\int\limits_{\Gamma_j\cap G_{j+1}} \beta_j(x_1,x_2)\varphi^{m-1}(x_1,x_2)d\mu_{\varphi_j}\right)\cdot$$

$$\cdot \exp\left[m\int\limits_0^{x_1}\int\limits_0^{x_2} f(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\right],$$

$$\text{if } m \geq 1, \qquad \forall\,(x_1,x_2)\in B_{k+1}. \tag{2.10}$$

Taking into account (2.10) and equality $u(x_1,x_2) = W(x_1,x_2)\varphi(x_1,x_2)$ we obtain (2.2),(2.3).

Remark 2.1 *Theorem 2.1 generalizes the result [3, lemma1] in the case of two independent variables functions and discontinuities on some curves. When we have one-dimensional inequality for piecewise continuous functions,* $\varphi =$ *const.,* $\beta_j =$ *const. and ratio* $\sum_{j=1}^{n-1}\int\limits_{\Gamma_j\cap G_n}\beta_j(x_1,x_2)u^m(x_1,x_2)d\mu_{\varphi_j}$ *reduces itself in* $\sum_{j=1}^{n-1}\beta_j u^m(x_i-0)$*, for* $n\to 0$ *and* $\{x_i\}$ *the sequence of fixed points* $x_i \in \Re^1_+ : x_0 < x_1 < x_2 \ldots$ *and* $\lim_{i\to\infty} x_i = \infty$ *(points of discontinuities of* $u(x)$: $u(x_i-0) = \lim_{t\to x_i-0} u(x)$*,* $u(x)$ *is left continuous at* x_k*). If* $m = 1$ *the above Theorem is similar to [21, Proposition 1, p.125], see also [6, Proposition 3.1, p.60] and [16, Proposition 3.7, p.28]. For* $m \neq 0$ *in Theorem 2.1 we investigate discontinuities more general than that ones in [6, Theorem 2.1, p.6].*

The result in Theorem 2.1 is a new analogy with the Wendroff's result for discontinuous functions.

Next proposition may be proved in the same way of Theorem 2.1:

Theorem 2.2 *Let us assume that function* $u(x1,x_2)$ *satisfies inequality (2.1) with* $\alpha = m > 0$*,* $m \neq 1$ *and that the conditions of the above Theorem are valid. Then the following estimates hold:*

$$u(x_1,x_2) \leq \varphi(x_1,x_2)\prod_{j=1}^{\infty}\left(1+\int\limits_{\Gamma_j\cap G_{j+1}} \beta_j(x_1,x_2)\varphi^{m-1}(x_1,x_2)d\mu_{\varphi_j}\right)\cdot$$

$$\cdot\left[1+(1-m)\int\limits_0^{x_1}\int\limits_0^{x_2}\varphi^{m-1}(\sigma_1,\sigma_2)f(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\right]^{1/1-m},$$

$$\textit{if } 0 < m < 1; \tag{2.11}$$

$$u(x_1,x_2) \le \varphi(x_1,x_2)\prod_{j=1}^{\infty}\left(1+m\int\limits_{\Gamma_j\cap G_{j+1}}\beta_j(x_1,x_2)\varphi^{m-1}(x_1,x_2)d\mu_{\varphi_j}\right)\cdot$$

$$\cdot\left[1-(m-1)\left[\prod_{j=1}^{\infty}\left(1+m\int\limits_{\Gamma_j\cap G_{j+1}}\beta_j(x_1,x_2)\varphi^{m-1}(x_1,x_2)d\mu_{\varphi_j}\right)\right]^{m-1}\cdot\right.$$

$$\left.\int_0^{x_1}\int_0^{x_2}\varphi^{m-1}(\sigma_1,\sigma_2)f(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\right]^{1/1-m},$$

for $m > 1$, (2.12)

only if

$$\int_0^{x_1}\int_0^{x_2}\varphi^{m-1}(\sigma_1,\sigma_2)f(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2 \le \frac{1}{m}, \qquad m > 1,$$

$$\prod_{j=1}^{\infty}\left(1+m\int\limits_{\Gamma_j\cap G_{j+1}}\beta_j(x_1,x_2)\varphi^{m-1}(x_1,x_2)d\mu_{\varphi_j}\right) < \left(1+\frac{1}{m-1}\right)^{1/1-m} \tag{2.13}$$

Remark 2.2 *Theorem 2.2 generalizes the results in [3, Lemma 2, p.7], [6, Proposition 3.7, p.68], [16, Proposition 3.3, p.24; Proposition 3.7, p.28] for two independent variables discontinuous functions with Hölder type discontinuities on some curves.*

Remark 2.3 *In the next applications we use simple versions of Theorem 2.1 and Theorem 2.2. If* $\varphi(x_1,x_2) = M = const.$*, from (2.1) we have the following inequalities:*

A) $\alpha = 1, \qquad m \le 1 \Rightarrow u(x_1,x_2) \le$

$$\le M\prod_{j=1}^{\infty}\left(1+M^{m-1}\int\limits_{\Gamma_j\cap G_{j+1}}\beta_j(x_1,x_2)d\mu_{\varphi_j}\right)\cdot$$

$$\cdot\exp\left[\int_0^{x_1}\int_0^{x_2}f(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\right],$$

B) $\alpha = 1, \qquad m \ge 1 \Rightarrow u(x_1,x_2) \le$

$$\le M\prod_{j=1}^{\infty}\left(1+M^{m-1}\int\limits_{\Gamma_j\cap G_{j+1}}\beta_j(x_1,x_2)d\mu_{\varphi_j}\right)\cdot$$

$$\cdot\exp\left[m\int_0^{x_1}\int_0^{x_2}f(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\right],$$

C) $\quad 0 < \alpha = m < 1 \Rightarrow u(x_1, x_2) \leq$

$$\leq M \prod_{j=1}^{\infty} \left(1 + M^{m-1} \int\limits_{\Gamma_j \cap G_{j+1}} \beta_j(x_1, x_2) d\mu_{\varphi_j} \right) \cdot$$

$$\cdot \left[1 + (1-m) M^{m-1} \int\limits_0^{x_1} \int\limits_0^{x_2} f(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 \right]^{1/1-m},$$

D) $\quad \alpha = m > 1 \Rightarrow u(x_1, x_2) \leq$

$$\leq M \prod_{j=1}^{\infty} \left(1 + m M^{m-1} \int\limits_{\Gamma_j \cap G_{j+1}} \beta_j(x_1, x_2) d\mu_{\varphi_j} \right) \cdot$$

$$\cdot \left[1 - (m-1) M^{m-1} \left[\prod_{j=1}^{\infty} \left(1 + m M^{m-1} \int\limits_{\Gamma_j \cap G_{j+1}} \beta_j(x_1, x_2) d\mu_{\varphi_j} \right) \right]^{m-1} \cdot \right.$$

$$\left. \cdot \int\limits_0^{x_1} \int\limits_0^{x_2} f(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 \right]^{-\frac{1}{m-1}},$$

only if

$$\int\limits_0^{x_1} \int\limits_0^{x_2} f(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 \leq \frac{1}{m M^{m-1}},$$

$$\prod_{j=1}^{\infty} \left(1 + m M^{m-1} \int\limits_{\Gamma_j \cap G_{j+1}} \beta_j(x_1, x_2) d\mu_{\varphi_j} \right) < \left(1 + \frac{1}{m-1} \right)^{\frac{1}{m-1}}.$$

3 Applications

By using [6], [16], [18], let us consider the hyperbolic differential equation with impulse perturbations on some curves of the type:

$$\begin{aligned}
&\frac{\partial^2 u(x_1, x_2)}{\partial x_1 \partial x_2} = F(x, u(x)), \qquad x = (x_1, x_2) \in \Gamma_i \\
&u(x_1, 0) = \psi_1(x_1) \\
&u(0, x_2) = \psi_2(x_2) \\
&\psi_1(0) = \psi_2(0) \\
&\Delta u|_{x \in \Gamma_i} = \int\limits_{\Gamma_j \cap G_n} \beta_i^*(x) u^m(x) d\mu_{\varphi_j}, \qquad m > 0,
\end{aligned} \tag{3.1}$$

where $\Delta u|_{x\in\Gamma_i}$ characterizes the values of discontinuities of solution of (3.1) when the solution of (3.1) meets curves $\Gamma_i \,:\, u(x)\cap\Gamma_i$ [6, p.70]. In (3.1) we suppose that boundary conditions $\psi_i(x)$ are bounded, i.e.

$$|\psi(x_1,x_2)| \le M = \text{const.} < \infty$$

and $F(x,u)$ satisfies the estimate:

$$|F(x,u)| \le f(x_1,x_2)|u(x_1,x_2)|^\alpha\,, \tag{3.2}$$

with $f \ge 0$, $\alpha = \text{const.} > 0$.
In the case $m = 1$, the equation of problem (3.1) was investigated in [6], [16].

By using Theorem 2.1, Theorem 2.2 and estimates A)–D) we obtain the following statement:

Theorem 3.1 *Let us suppose that for problem (3.1) the assumptions of Section 2 about curves Γ_i, domains B_k, G_k and functions φ_k are valid. Moreover let F satisfies inequality (3.2).*
I. Then the following estimates take place:

A') $\alpha = 1, \qquad m \le 1 \Rightarrow |u(x_1,x_2)| \le$

$$M\prod_{j=1}^{\infty}\left(1+M^{m-1}\int\limits_{\Gamma_j\cap G_{j+1}}|\beta_j^*(x_1,x_2)|d\mu_{\varphi_j}\right)\cdot \cdot\exp\left[\int_0^{x_1}\int_0^{x_2} f(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\right],$$

B') $\alpha = 1, \qquad m \ge 1 \Rightarrow |u(x_1,x_2)| \le$

$$\le M\prod_{j=1}^{\infty}\left(1+M^{m-1}\int\limits_{\Gamma_j\cap G_{j+1}}|\beta_j^*(x_1,x_2)|d\mu_{\varphi_j}\right)\cdot \cdot\exp\left[m\int_0^{x_1}\int_0^{x_2} f(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\right],$$

C') $0 < \alpha = m < 1 \Rightarrow |u(x_1,x_2)| \le$

$$\le M\prod_{j=1}^{\infty}\left(1+M^{m-1}\int\limits_{\Gamma_j\cap G_{j+1}}|\beta_j^*(x_1,x_2)|d\mu_{\varphi_j}\right)\cdot \cdot\left[1+(1-m)M^{m-1}\int_0^{x_1}\int_0^{x_2} f(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\right]^{1/1-m},$$

D') $\alpha = m > 1 \Rightarrow |u(x_1,x_2)| \leq$

$$\leq M \prod_{j=1}^{\infty} \left(1 + mM^{m-1} \int\limits_{\Gamma_j \cap G_{j+1}} |\beta_j^*(x_1,x_2)| d\mu_{\varphi_j}\right) \cdot$$

$$\cdot \left[1 - (m-1)M^{m-1} \left[\prod_{j=1}^{\infty} \left(1 + mM^{m-1} \int\limits_{\Gamma_j \cap G_{j+1}} |\beta_j^*(x_1,x_2)| d\mu_{\varphi_j}\right)\right]^{m-1} \cdot\right.$$

$$\left. \cdot \int\limits_0^{x_1} \int\limits_0^{x_2} f(\sigma_1,\sigma_2) d\sigma_1 d\sigma_2 \right]^{-\frac{1}{m-1}},$$

only if

$$\int\limits_0^{x_1} \int\limits_0^{x_2} f(\sigma_1,\sigma_2) d\sigma_1 d\sigma_2 \leq \frac{1}{mM^{m-1}}, \tag{3.3}$$

$$\prod_{j=1}^{\infty} \left(1 + mM^{m-1} \int\limits_{\Gamma_j \cap G_{j+1}} |\beta_j^*(x_1,x_2)| d\mu_{\varphi_j}\right) < \left(1 + \frac{1}{m-1}\right)^{\frac{1}{m-1}}. \tag{3.4}$$

II. All solutions $u(x_1,x_2)$ of (3.1) are bounded in the cases A')–C') only if the values $\prod_{j=1}^{\infty} \left(1 + \int\limits_{\Gamma_j \cap G_{j+1}} |\beta_j^*| d\mu_{\varphi_j}\right)$, $\int\limits_0^{x_1} \int\limits_0^{x_2} f(\sigma_1,\sigma_2) d\sigma_1 d\sigma_2$ *are bounded. Referring to the case D'), (3.2), (3.3) guarantee conditions of boundedness for all solutions of (3.1).*

Remark 3.1 *Theorem 3.1 generalizes the results [6, Theorem 3.4, p.31] and coincides with [6], [16] only if $m = 1$.*

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