Classical and quantum Hamilton formalisms for the mechanics of the Bernoulli oscillators
I - Classical framework

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Summary - In a few previous papers, we discussed the fundamentals of the so-called Bernoulli oscillators physics. The Bernoulli oscillators are classical entities whose behavior is influenced by a "hidden" degree of freedom (HDF), in its turn excited by a quantum vacuum action. Within assessed approximations and limits (uni-dimensional motion), we assumed a classical-like interpretation of quantum effects, and displayed the Newtonian motion background subtending - by our proposal - the matter wave physics. In a couple of papers, we give a formal description of the Bernoulli oscillators classical degree of freedom mechanics, by the means of Hamilton-like formalisms. These are, however, different in their conception from the standard known ones: we introduce indeed an extended form for the Hamilton function, called the Bernoulli Hamiltonian, and non-standard forms for the Hamilton equations or procedures. Due to the flexibility of our forms, both a classical framework and a quantum-like one will be shown able to provide a full description of all the cases relevant to us. In the present paper I, the classical framework is discussed. Physical interpretation matching the mentioned formal procedures is provided step-by-step.

PACS 05.90. - Other topics in statistical physics and thermodynamics

1 Introduction

In a few previous papers [1÷4], we introduced a classical-like, theoretical framework describing the behavior of the so-called Bernoulli oscillators. These ones are classical oscillators perturbed by the action of the quantum "vacuum". The vacuum is able to drive distinguished parts of the oscillators space and momentum co-ordinates, which we comprehensively call the hidden degree of freedom (HDF). This last perturbs, in turn, the oscillation center time-law x(t) (x = classical degree of freedom space-co-ordinate), and causes deviations from the oscillators classical behavior. The particle-vacuum interaction is dependent on the time and on a parametric function named $\xi(\mathrm{x})$, a sort of generalized de-Broglie wavelength describing the radius of the interaction. This last is submitted to Heisenberg's indetermination principle as to a parametric constraint effective on the HDF oscillation amplitude and frequency. In our framework, the Heisenberg principle takes a (proposed) classical-like interpretation, so that the HDF behavior itself is described as a classical parametric oscillation [2]. The oscillator classical degree of freedom (CDF) is then found submitted to a Newtonian motion equation, whose energy theorem has the form of a (generalized) Kapitza theorem (see equation (4)). This equation sets a bridge between classical and quantum mechanics. It is able indeed to give a classical-like description of tunnelling phenomena and, by the effect of non-local initial conditions of motion, determines a CDF behavior looking to us consistent with a matter-wave one. In our framework, the Schrödinger equation is indeed interpreted as the statistical appearance of a initial conditions manifold set - statistics being inherent to the distribution of $\xi(\mathrm{x})$ parameters, in turn determined by the motion initial conditions [4].

To resume, within the proposed framework, we developed various contexts.

The first one (which we call PHME) is relevant to the description of a partial, but primary effect we identified as the core of the particle-vacuum interaction: this is a locally induced mass effect, interpreted as the vacuum local reaction to the Eulerian velocity field of the particle in motion.

The second one (which we call SPND) takes into account the comprehensive interaction resulting from "space distributed" vacuum reaction points. It includes a complete interaction scheme, thus describing our proposed Newtonian dynamics of the single particle. This is represented by the equation that we consider to be the physical background below the matter-wave appearance.

A third context (here referred to as DBOE) is constituted by a framework where, basing on the SPND equation, we describe both the density and drift velocity field pertaining to a flowing ensemble of particles (always classically modelled). By the means of a double-solution assumption in the SPND equation, we advanced indeed in previous papers the hypothesis that it also includes the information about the average velocity field relevant to a particles ensemble.

The SPND and DBOE contexts must therefore be submitted to extensive comparative analysis with the orthodox quantum mechanical context (called OQMC). Amongst other, some investigation can be carried out within the frame of Hamilton formalisms. These last are indeed especially appropriate to give evidence to both physical properties and mathematical symmetries.

In this paper, we want therefore to show that the Hamilton formalisms - both classical and quantum-like - are able to describe all the physical contexts of our interest. To this end, however, we introduce and investigate the performances of some peculiar Hamilton functions which we call the "Bernoulli" Hamiltonians. They will be shown flexible enough to provide the desired complete descriptions. The tasks will be accomplished through different steps introduced in the following sections.

In the present paper I we resume the various contexts equations first, and investigate classical-like procedures. In the following paper II the quantum-like procedures will be discussed.

2 Theoretical Background

2.1 The primary HDF-effect (PHME)

The SPND equation (shown in equation (4)) can also be derived by a model where a peculiar "space distributed" vacuum reaction to the Eulerian velocity field of the particle in motion is assumed [4]. By that model, we draw a distinction between what we call "primary HDF effect" (a local one, evaluated at zero value of the parameter $\xi(\mathrm{x})$), and the effect of the distribution of the vacuum distant reaction source-points in space (the "distant effects" of the interaction).

In reference [4], equation (51), is shown indeed that the primary particle-vacuum interaction (stationary part) has the form

$$\Phi_{\mathrm{HDF}}(\mathrm{x}) = -\frac{1}{2}\delta \mathrm{m}\mathrm{v}^2(\mathrm{x}) \tag{1}$$

As anticipated, this is a (negative) mass effect, expressed by the coefficient $-\delta\mathrm{m}$. The energy theorem expression corresponding to the mentioned action is

$$\frac{1}{2}\,\mathrm{m}\mathrm{v}^2(\mathrm{x})+\Phi(\mathrm{x})+\Phi_{\mathrm{HDF}}(\mathrm{x}) = \frac{1}{2}\,\mathrm{m}_{eff}\mathrm{v}^2(\mathrm{x})+\Phi(\mathrm{x}) = \mathrm{E} \tag{2}$$

$$\mathrm{m}_{eff} = \mathrm{m} - \delta\mathrm{m} \tag{3}$$

In these equations, $\mathrm{v}(\mathrm{x})$ is the particle (CDF) Eulerian velocity field (x-componen t), $\Phi(\mathrm{x})$ is the potential energy corresponding to the classical force field imposed to the particle and m_{eff} is the effective mass resulting from the vacuum+HDF action. We will refer, in the following, to equation (2) as the primary HDF-affected energy theorem form for the single-particle dynamics, and the corresponding physical context will be recalled by the acronym PHME (primary HDF-induced mass effect).

In reference [4] we made the hypothesis that the effective mass can assume values included between the classical mass value m and zero. The same assumption subtends our present papers where m_{eff} (and its statistical counterpart m^{D}_{eff}) will however play a simple parametric role and will be processed as constants by simplicity.

2.2 The single-particle energy theorem for Bernoulli oscillators (SPND case)

The single-particle energy theorem expression accounting for the "full" (i.e., including the "distant effects" of the interaction) HDF effect has been found in reference [4] - equation (36), to have the form:

$$\frac{1}{2}\,\mathrm{m}_{eff}\mathrm{v}^2+\Phi(\mathrm{x})-\frac{\hbar^2}{2\mathrm{m}}\frac{\mathrm{v}^{2''}}{\mathrm{v}^2}+\mathrm{C}(\mathrm{v}^2)=\mathrm{E}_n \tag{4}$$

It is a relevant matter to add here the corresponding continuity equation ([1]):

$$\rho_c(\mathrm{x})\mathrm{v}(\mathrm{x})=2\nu_0 \tag{5}$$

In these equations, $''$ is the second derivative with respect to the space coordinate x, $\hbar$ is the reduced Planck's constant, E_n is the quantum mechanical energy eigenvalue for the particle in the potential $\Phi(\mathrm{x})$, and $\mathrm{C}(\mathrm{v}^2)$ is a weak function of v^2. In this paper, we will generally assume $\mathrm{C}(\mathrm{v}^2)=0$ for the sake of simplicity ([2]). In the present section, the volume-flow ν_0 is a constant and $\rho_c(\mathrm{x})$ is clearly defined as a classical-like statistical density. Major discussion of equations (4) and (5) is in the quoted reference. We will refer in the following to equation (4) as the energy theorem form holding for the "fully" HDF-affected single-particle dynamics, and will recall the corresponding equations and physical context by the means of a comprehensive acronym SPND (single particle Newtonian dynamics).

Equation (4) holds for a single particle but also takes a peculiar statistical-ensemble average form, able to describe a drifting particle ensemble behavior [4]. We have to introduce this case with a few details. This is in the next section.

2.3 The drifting Bernoulli oscillators ensemble (DBOE case)

The mentioned form is (see equation (56) in ref. [4]):

$$\frac{1}{2}\,\mathrm{m}^{D}_{eff}\mathrm{v}_{D}^2+\Phi(\mathrm{x})-\frac{\hbar^2}{2\mathrm{m}}\frac{\mathrm{v}_{D}^{2''}}{\mathrm{v}_{D}^2}\;=\mathrm{E}_n \tag{6}$$

Here m^{D}_{eff} is an equivalent statistical mass ([3]) and $\mathrm{v}_{D}(\mathrm{x})$ is the drift velocity field (x-component) of a particles ensemble. The correlation between $\mathrm{v}_{D}(\mathrm{x})$ and the density $\rho(\mathrm{x})$ is

$$\rho(\mathrm{x})\mathrm{v}_{D}(\mathrm{x})=2\nu(\mathrm{x}) \tag{7}$$

[1] The quantities $\mathrm{v}(\mathrm{x})$ and $\mathrm{v}_{D}(\mathrm{x})$ afterwards are originally defined as x-components of the respective vector fields. By conciseness, they missed their index "x" so that in some circumstances (as in equations (5), (7) and similar ones) the same symbols will just be used for their corresponding absolute values.

[2] In the contrary case, all the treatments here should be renormalized by embodying $\mathrm{C}(\mathrm{v}^2)$ into an effective kinetic term, i.e. by defining a new effective mass; due to the function weakness, a constant assumption for this effective mass still looks appropriate for a first glance as in these papers.

[3] Here again, the function $\mathrm{C}(\mathrm{v}_{D}^2)$ is taken negligible or may be thought embodied into m^{D}_{eff}.

The function $\nu(\mathrm{x})$ is the ensemble volume-flow rate. An expression for it is given in [4]. In the present context, the density $\rho(\mathrm{x})$ is, from a mathematical point of view, coincident with a quantum statistical density ([4]). Indeed, the context itself is intended for a close comparison with a standard (deep) quantum-mechanical framework (for comparison with the quasi-classical case the continuity equation associated to this context should turn out back to the form (5)).

In the following, we will rarely specify again whether an assumed density ρ is a classical or a quantum one. That will be easily understood by the context: in practice, the density is a classical one (continuity equation (5)) in the PHME, SPND cases and in the $\hbar\rightarrow 0$ limit of the DBOE case. It is instead a quantum-like one in the DBOE (deep quantum context) and OQMC (continuity equation (7) or (9) respectively) cases.

Equations (6)/(7) are conclusively the energy theorem form holding for the case of a drifting ensemble of Bernoulli oscillators. These equations, as well as the corresponding physical context, will be recalled all across this paper by the use of the quoted acronym DBOE.

2.4 The orthodox quantum mechanical (OQMC) case

This is represented by the known hydrodynamic equations [5]:

$$\frac{\nabla S^2}{2\mathrm{m}} + \Phi(\mathrm{x}) - \frac{\hbar^2}{2\mathrm{m}}\frac{\sqrt{\rho}''}{\sqrt{\rho}} = \mathrm{E}_n \tag{8}$$

$$\rho \nabla S = const \tag{9}$$

The particle wave-function corresponding to this formulation takes the form

$$\Psi_n(\mathrm{x}) \equiv \Psi(\mathrm{x}) = \pm\sqrt{\rho}\; exp(iS(\mathrm{x})/\hbar) \tag{10}$$

where the quantity $S(\mathrm{x})$ is the relevant quantum phase function.

2.5 Remarks

In order to give a unified expression to various cases at investigation in our papers, we will use an auxiliary quantity ∇S^*. Depending on the cases, ∇S^* may be set equal to $\sqrt{\mathrm{mm}_{eff}}\mathrm{v}$ or $\sqrt{\mathrm{mm}^D_{eff}}\mathrm{v}_D$ (in the SPND and DBOE cases), or just to the quantum-mechanical quantity ∇S when the orthodox quantum mechanical context OQMC is at hand instead.

[4] Yet on the interpretational side the physical sense of the density, both in the SPND and DBOE cases, is a classical one: it classically comes out from statistical ensembles of Eulerian velocity fields and is not a "quantum probabilistic" one.

In the present paper, we want to show that all the energy theorem forms (2), (4)/(5), (6)/(7) and (8)/(9) can be obtained by the means of (generalized) Hamilton formalisms. The task will be accomplished in a number of steps. We will first show that the primary form (2) can be obtained by means of a classical-like formalism. Then we will show that the expressions (4)÷(7) (with C=0 or "embodied"), and the orthodox quantum-mechanical Madelung formulation (8)/(9) as well, can - all of them - be described by an analogous, generalized formalism. These tasks are demanded to the present paper I. Quantum-like procedures will instead be introduced in paper II, and we will show that they are equally able to describe all of the relevant cases to us.

What we will call an "optimized" (in some respects, to be discussed) quantum formalism will also be conclusively introduced at the end of paper II.

2.6 Classical Hamilton formalism

The classical orthodox Hamilton equations [6] for the particle dynamics base on the definition of the canonical momentum p=mv so that we have ($\Phi_{\rm HDF}(\rm x)=0$ now):

$$\rm H(p,x) \; = Q(p,x) + \Phi(x) = \frac{1}{2m}\,p^2 + \Phi(x) = E \tag{11}$$

As is well known, the Hamiltonian H(p,x) turns out to be a motion constant over the {p,x} - domain relevant to the particle classical dynamics when the following (Hamilton) equations are satisfied:

$$\frac{\partial \rm H(p,x)}{\partial \rm p} = \frac{\rm p}{\rm m} = \frac{d\rm x}{dt} \equiv \rm v(x) \tag{12}$$

$$-\frac{\partial \rm H(p,x)}{\partial \rm x} = -\nabla\Phi(\rm x) = \frac{d\rm p}{dt} = \rm m\frac{d\rm v}{dt} \equiv \rm m\frac{d^2\rm x}{dt^2} \tag{13}$$

Now we want to set up an extended formalism able to account for our described PHME framework first. This is done in the next section.

3 The Bernoulli-Hamilton formalism

In this section, what we call the Bernoulli Hamiltonian form will be introduced. It consists of a simple mathematical extension of the classical orthodox form. As it will be discussed more in the following, the Bernoulli Hamiltonians can be operated with both classical, standard Hamilton equations or non-standard ones. These last will be introduced in the sequel. Although the application domain of the Bernoulli Hamiltonians and non-standard formalisms appears to be a very broad one, we will discuss their properties essentially by working out our physics of the Bernoulli oscillators.

3.1 Hamilton formalism for the Bernoulli oscillator with HDF-induced mass effect (PHME case)

In this case, we could accomplish our task by simply replacing the mass m in the orthodox formalism with the effective mass m_{eff}, and we could then exploit the classical method; but we have another interesting procedure to propose. To this end, let us consider the following function and positions:

$$\mathrm{H}_B(\mathrm{p,x}) = \frac{1}{2\mathrm{m}}\,\mathrm{p}^2 + \Phi(\mathrm{x}) + i\frac{\mathrm{p}}{\mathrm{m}}\sigma(\mathrm{x}) = \frac{1}{2\mathrm{m}}\,\mathrm{p}^2 + \Phi(\mathrm{x}) - \sqrt{\frac{\delta\mathrm{m}}{\mathrm{m}}}\mathrm{pv(x)} \equiv$$

$$\equiv \frac{1}{2\mathrm{m}}\,\mathrm{p}^2 + \Phi(\mathrm{x}) + \frac{\mathrm{P(p)}}{\rho_c(\mathrm{x})} \tag{14}$$

$$\sigma(\mathrm{x}) = i\sqrt{\mathrm{m}\delta\mathrm{m}}\,\mathrm{v(x)} = -i\frac{\mathrm{m}}{\mathrm{p}}\frac{\mathrm{P(p)}}{\rho_c(\mathrm{x})} \tag{15}$$

$$\mathrm{v(x)} = \frac{2\nu_0}{\rho_c(\mathrm{x})} \tag{16}$$

$$\mathrm{P(p)} = -\sqrt{\frac{\delta\mathrm{m}}{\mathrm{m}}}\,2\nu_0\mathrm{p} \tag{17}$$

The Hamiltonian H_B(p,x) (14) differs from the orthodox classical form by the addition of a linear term in the momentum p. This last is defined different from the orthodox one, see equation (19). The proposed form is originated by analogy with the standard expression of the Bernoulli theorem in classical hydrodynamics. By equation (15) we see that the momentum σ(x) is defined proportional to the classical momentum mv(x). By equations (14)÷(17), now, we also see that if the Eulerian velocity field v(x) is written as a function of the volume-flow $2\nu_0$ and of the corresponding classical statistical density ρ_c(x), then the additional term in the Hamiltonian expression looks as a (uni-dimensional) pressure divided by the density ρ_c(x) - i.e. is a hydrodynamic-like term. Therefore we will call the $\mathrm{H_B}$(p,x) function and similar generalized expressions the Bernoulli Hamiltonians.

Equations (14)÷(17) therefore provide us with a specific example of a Bernoul li Hamiltonian. Some other examples will be found in the following. The general Bernoulli Hamiltonian form includes complex functions p and σ, and can also be worked out by the means of (so-called here, to be expounded next) non-standard Hamilton equations. Here the simple protoype case (14) still rests on standard, classical forms for the Hamilton equations. The form (15) for the function σ is specific for the case at hand here (the PHME case).

Using the standard, classical Hamilton equations (12) and (13) in (14) we find

$$\frac{\partial \mathrm{H}_B(\mathrm{p,x})}{\partial \mathrm{p}} = \frac{\mathrm{p}}{\mathrm{m}} - \sqrt{\frac{\delta\mathrm{m}}{\mathrm{m}}}\mathrm{v(x)} = \frac{d\mathrm{x}}{d\mathrm{t}} \equiv \mathrm{v(x)} \tag{18}$$

$$\mathrm{p} = \left(1 + \sqrt{\frac{\delta \mathrm{m}}{\mathrm{m}}}\right) \mathrm{mv(x)} \tag{19}$$

$$-\frac{\partial \mathrm{H}_B(\mathrm{p,x})}{\partial \mathrm{x}} = \frac{d\mathrm{p}}{dt} \equiv -\nabla\Phi(\mathrm{x}) + \sqrt{\frac{\delta \mathrm{m}}{\mathrm{m}}}\mathrm{p}\frac{d\mathrm{v}}{d\mathrm{x}} \tag{20}$$

Then we also find

$$-\nabla\Phi(\mathrm{x}) = (\mathrm{m} - \delta \mathrm{m})\,\mathrm{v}\frac{d\mathrm{v}}{d\mathrm{x}} = \mathrm{m}_{eff}\frac{d^2\mathrm{x}}{dt^2} \tag{21}$$

$$\mathrm{H}_B(\mathrm{p,x}) \equiv \frac{1}{2}\,\mathrm{m}_{eff}\mathrm{v}^2 + \Phi(\mathrm{x}) = \ \mathrm{E} = const \tag{22}$$

These equations represent our PHME context.

By the same equations it is clear that, in our proposed formalism, we have taken into account the potential $\Phi_{\mathrm{HDF}}(\mathrm{x})$ (1) by inserting it into the expression of the quadratic momentum function $Q_B(\mathrm{p,x})$ defined as follows:

$$Q_B(\mathrm{p,x}) = \frac{\mathrm{p}^2}{2\mathrm{m}} + i\frac{\mathrm{p}}{\mathrm{m}}\sigma(\mathrm{x}) = \frac{\mathrm{p}^2}{2\mathrm{m}} + i\mathrm{p}i\sqrt{\frac{\delta \mathrm{m}}{\mathrm{m}}}\mathrm{v(x)} \equiv$$

$$\equiv \frac{\mathrm{p}^2}{2\mathrm{m}} + i\mathrm{p}\sqrt{\frac{2}{\mathrm{m}}}\sqrt{-\frac{\delta \mathrm{mv}^2}{2}} \equiv \frac{\mathrm{p}^2}{2\mathrm{m}} + i\frac{\mathrm{p}}{\mathrm{m}}\sqrt{2\mathrm{m}\Phi_{\mathrm{HDF}}(\mathrm{x})} \tag{23}$$

This equation enlightens the correlation between the momentum $\sigma(\mathrm{x})$ (15) and the potential $\Phi_{\mathrm{HDF}}(\mathrm{x})$. It is clear, however, that this correlation holds as such, in the proposed example, because the Bernoulli Hamiltonian has been operated with standard, classical Hamilton equations. This is not always the case, as will be seen in the next section.

Conclusively, we have shown by the means of equations (14)÷(22) that the energy theorem form (22) can be obtained via the standard, classical Hamilton equations applied to the Bernoulli Hamiltonian function $\mathrm{H}_B(\mathrm{p,x})$ (14).

As a last remark we note here that a simple Bernoulli Hamiltonian form

$$\mathrm{H}_B(\mathrm{p,x}) = \frac{1}{2\mathrm{m}}\,\mathrm{p}^2 + i\frac{\mathrm{p}}{\mathrm{m}}\sqrt{2\mathrm{m}\Phi(\mathrm{x})} = \mathrm{E} \tag{24}$$

with associated standard, classical Hamilton equations is able to reproduce the orthodox classical particle mechanics. If we write indeed

$$\frac{\partial \mathrm{H}_B(\mathrm{p,x})}{\partial \mathrm{p}} = \frac{\mathrm{p}}{\mathrm{m}} + \frac{i}{\mathrm{m}}\sqrt{2\mathrm{m}\Phi(\mathrm{x})} = \frac{d\mathrm{x}}{dt} \equiv \mathrm{v(x)} \tag{25}$$

$$-\frac{\partial \mathrm{H}_B(\mathrm{p,x})}{\partial \mathrm{x}} = -i\mathrm{p}\frac{\nabla\Phi(\mathrm{x})}{\sqrt{2\mathrm{m}\Phi(\mathrm{x})}} = \frac{d\mathrm{p}}{d\mathrm{t}} \tag{26}$$

then we find

$$-\nabla\Phi(\mathrm{x}) = \mathrm{m}\frac{d\mathrm{v}}{d\mathrm{t}} \tag{27}$$

$$\frac{1}{2}\,\mathrm{m}\,\mathrm{v}^2 + \Phi(\mathrm{x}) = \mathrm{E} = const \tag{28}$$

3.2 An invariance property

The previous considerations also enlighten an interesting property of the Bernoulli Hamiltonians: once the function σ is written in the form $\sqrt{\sigma^2}$, is very easy to show that the following forms

$$\mathrm{H}_B(\mathrm{p,x}) = \frac{1}{2\mathrm{m}}\,\mathrm{p}^2 + i\frac{\mathrm{p}}{\mathrm{m}}\sqrt{\sigma^2} + \Phi(\mathrm{x}) = \mathrm{E} \tag{29}$$

$$\mathrm{H}_B(\mathrm{p,x}) = \frac{1}{2\mathrm{m}}\,\mathrm{p}^2 + i\frac{\mathrm{p}}{\mathrm{m}}\sqrt{\sigma^2 + 2\mathrm{m}\Theta(\mathrm{x})} + \Phi(\mathrm{x}) - \Theta(\mathrm{x}) = \mathrm{E} \tag{30}$$

are equivalent to each other, whatever the potential $\Theta(\mathrm{x})$ is, if the Hamilton equations are assumed. This invariance property will also hold in the case of the non-standard formalism introduced in the next section, and will be used in the following.

Up to this section, we have introduced a formalism where the Hamilton function includes a non-standard linear term in p but the Hamilton equations still are the standard, classical ones. We will now introduce what we call in this paper a non-standard formalism. This last will rest on different definitions for the Hamilton equations.

3.3 Non-standard Bernoulli-Hamilton formalism

In order to introduce a generalized form for Hamilton equations we start considering the equation

$$\mathrm{H}(\mathrm{p,x}) = \mathrm{E} = const \qquad\qquad \{\mathrm{x}\ \epsilon\ \mathrm{V}\} \tag{31}$$

Here H(p,x) is a Hamilton function defined in a x-space domain of extension V. Taking the derivatives of the previous equation we can also write it in the form

$$\frac{\partial \mathrm{H}(\mathrm{p,x})}{\partial \mathrm{p}}d\mathrm{p} + \frac{\partial \mathrm{H}(\mathrm{p,x})}{\partial \mathrm{x}}d\mathrm{x} = 0 \tag{32}$$

We introduce now a generalized expression for the first Hamilton equation setting

$$\frac{\partial \mathrm{H(p,x)}}{\partial \mathrm{p}} = \frac{\mathrm{p}_{ns}}{\mathrm{m}} \tag{33}$$

Here p_{ns} is a non-standard momentum, i.e. it can be defined different from the orthodox quantity mv. As a consequence of equation (32), the second Hamilton equation will be written

$$\mathcal{F} \equiv -\frac{\partial \mathrm{H(p,x)}}{\partial \mathrm{x}} = \frac{\mathrm{p}_{ns}}{\mathrm{m}}\frac{d\mathrm{p}}{d\mathrm{x}} \equiv \frac{d\mathrm{p}}{d\tau} \tag{34}$$

In this equation, we introduce a "force" $\mathcal{F}$ and a time-like variable τ whose definitions are clear from the equation itself. More specifically, we have

$$\tau = \mathrm{m}\int \frac{d\mathrm{x}}{\mathrm{p}_{ns}} \tag{35}$$

The variables $\mathcal{F}$ and τ have auxiliary roles in our formalism. They can be identified with the ordinary force and time variables only in the limit when $\mathrm{p}_{ns} \rightarrow \mathrm{mv}$, i.e. when the orthodox case is recovered.

The physical interpretation we give to the formalism introduced in this section is that the Hamiltonian function H(p(x),x) represents an invariant of the co-ordinates p(x), x all over the relevant x-space domain - independently of the existence of the time-space relationship $\mathrm{x}\equiv\mathrm{x(t)}$ characterizing the standard case in classical mechanics. Once an expression for p_{ns} is assigned, indeed, we can find the expression for the "time" τ - this last has merely the role of a dependent variable here. Given a function H(p(x),x), whenever a definition for p_{ns} (or p) is assumed to characterize some relevant momentum within the physical context to be described, the first Hamilton equation will provide us with the correspondent definition of p (or p_{ns}). Within this framework, both the quantities p and p_{ns} are actually "non standard" ones. Indeed, the momentum p itself is not generally coincident with the standard classical momentum. Conclusively, the quantities p and p_{ns} define the physics of the system and, in turn, the function H(p(x),x) itself.

As a first example of use of the proposed formalism, we will show in the next section that - by the means of an appropriate momentum definition - it will be found able to provide us with the orthodox hydrodynamic formulation of the quantum mechanical wave-equation.

Before showing this, however, we want to introduce here a general case of interest. This is characterized by the following H(p(x),x) function and p_{ns} definitions:

$$\mathrm{H}_B(\mathrm{p,x}) = \frac{1}{2\mathrm{m}}\,\mathrm{p}^2 + \Phi(\mathrm{x}) + i\mathrm{p}\frac{\sigma(\mathrm{x})}{\mathrm{m}} = \mathrm{E} \tag{36}$$

$$\mathrm{p}_{ns} = \sqrt{\mathrm{mm}_{eff}\mathrm{v}^2(\mathrm{x}) - \sigma^2(\mathrm{x})} \tag{37}$$

Using for this case the first Hamilton equation (33) we find :

$$\frac{\partial \mathrm{H}_B(\mathrm{p,x})}{\partial \mathrm{p}} = \frac{\mathrm{p}}{\mathrm{m}} + i\frac{\sigma(\mathrm{x})}{\mathrm{m}} = \frac{\mathrm{p}_{ns}}{\mathrm{m}} = \frac{1}{\mathrm{m}}\sqrt{\mathrm{m}\mathrm{m}_{eff}\mathrm{v}^2(\mathrm{x}) - \sigma^2(\mathrm{x})} \tag{38}$$

so that

$$\mathrm{H}_B(\mathrm{p,x}) = \frac{1}{2\mathrm{m}}\,\mathrm{p}_{ns}^2 + \Phi(\mathrm{x}) + \frac{\sigma^2(\mathrm{x})}{2\mathrm{m}} = \frac{1}{2}\mathrm{m}_{eff}\mathrm{v}^2(\mathrm{x}) + \Phi(\mathrm{x}) = \mathrm{E} \tag{39}$$

With the assumed definition for p_{ns}, whatever the function $\sigma(\mathrm{x})$ is, the prescribed Bernoulli Hamilton function results into the primary energy theorem form (2) (PHME) relevant to us in this paper. This property will be used in the sequel. When the quoted definition (37) for p_{ns} is assumed, we will call the momentum p_{ns} itself a "central" momentum. Correspondingly, we will also call - always for the purposes of this paper - "canonical" the corresponding momentum p, Hamilton function $\mathrm{H}_B(\mathrm{p,x})$ and momentum function $\mathrm{Q}_B(\mathrm{p,x})$. With these definitions, the classical quantities themselves: p, H(p,x) and Q(p,x) appearing in equations (11)÷(13) are canonical and the momentum $\mathrm{m}\partial\mathrm{H}(\mathrm{p,x})/\partial\mathrm{p}$ $=\mathrm{mv}$ in equation (12) is central (σ is taken zero and $\mathrm{m}_{eff} = \mathrm{m}$ in the standard classical case). It is easy to show that the momentum p and Hamiltonian $\mathrm{H}_B(\mathrm{p,x})$ given in equation (14) in the previous section are also canonical and the momentum $\mathrm{m}\partial\mathrm{H}_B(\mathrm{p,x})/\partial\mathrm{p} = \mathrm{mv}(\mathrm{x})$ is central $\left(\text{with } \sigma = i\sqrt{\mathrm{m}\delta\mathrm{m}}\mathrm{v}(\mathrm{x})\right)$.

3.4 The Bernoulli Hamiltonian and the quantum mechanical wave equation

In this section, we introduce first some useful definitions of quantities - together with some discussion - with the final purpose to show that the quantum-mechanical wave-equation (hydrodynamic formulation, OQMC case) can be obtained by the means of our non-standard, classical Hamilton formalism.

In a previous paper (reference [3], equations (73) and (74)) we showed that the Bohm potential $\mathrm{U_B}(\mathrm{x})$ in the hydrodynamic formulation of the Schrödinger equation is equivalent to the indicated work expression corresponding to the "full" (thermalization constant $\kappa = 1$) action of the quantum pressure P. The relevant equations are as follows :

$$\frac{\mathrm{P}_1}{\rho} \equiv \frac{\mathrm{P}}{\rho}\,|_{\kappa=1} = -\frac{\hbar^2}{4\mathrm{m}}\left(\frac{\rho'}{\rho}\right)' \tag{40}$$

(pressure definition), and

$$\int \frac{d\mathrm{P}_1}{\rho} = -\frac{\hbar^2}{2\mathrm{m}}\frac{\sqrt{\rho}''}{\sqrt{\rho}} = \mathrm{U_B}(\mathrm{x}) \tag{41}$$

(indicated work expression). Therefore we can write again the equations (8) and (9) as follows:

$$\frac{\nabla S^2}{2\mathrm{m}} \quad + \quad \Phi(\mathrm{x}) \quad + \int \frac{d\mathrm{P}_1}{\rho} \quad = \quad \mathrm{E}_n \tag{42}$$

$$\rho \nabla S = const \tag{43}$$

These equations are coincident with the hydrodynamic formulation of the matter wave equation but they make clear our interpretation ([5]) that, for a many-particles "thermalized" ($\kappa = 1$) system, the Bohm potential is represented by the indicated work term $\int d\mathrm{P}_1/\rho$.

We have now to recall the expression of the momentum field p_{qm} in quantum mechanics:

$$\mathrm{p}_{qm} = \nabla S - i\frac{\hbar}{2}\left(\frac{\rho'}{\rho}\right) = \mathrm{p}^R_{qm} + i\mathrm{p}^I_{qm} \tag{44}$$

The quantities p^R_{qm} and p^I_{qm} are the real and imaginary parts of the complex quantity p_{qm} respectively.

Using the p_{qm} and pressure definitions in equations (44), (40) we can also write

$$\int \frac{d\mathrm{P}_1}{\rho} = \frac{\mathrm{P}_1}{\rho} - \int \mathrm{P}_1 d\frac{1}{\rho} = -\frac{\hbar^2}{4\mathrm{m}}\left(\frac{\rho'}{\rho}\right)' - \frac{\hbar^2}{8\mathrm{m}}\left(\frac{\rho'}{\rho}\right)^2 \tag{45}$$

$$\int \mathrm{P}_1 d\frac{1}{\rho} = \frac{\hbar^2}{8\mathrm{m}}\left(\frac{\rho'}{\rho}\right)^2 = \frac{{\mathrm{p}^I_{qm}}^2}{2\mathrm{m}} \tag{46}$$

This last equation allows us to interpret the quantum momentum field imaginary part p^I_{qm} as the momentum a particle would assume if (hypothetically) all the expansion work was transferred into its kinetic degree of freedom. We might call the term p^I_{qm} a "virtual" quantity, because the energy transfer mechanisms within equation (42) anyway include energy balancing with other degrees of freedom. Since the full pressure-dependent potential also includes the thrust term P/ρ, we are brought by analogy to define a similar momentum field term p^*_{qm} accounting for the virtual energy transfer from this potential to the kinetic degree of freedom. We write

$$-\frac{\mathrm{P}_1}{\rho} = \frac{{\mathrm{p}^*_{qm}}^2}{2\mathrm{m}} = \frac{\hbar^2}{4\mathrm{m}}\left(\frac{\rho'}{\rho}\right)' = -\frac{\hbar}{2\mathrm{m}}{\mathrm{p}^I_{qm}}' \tag{47}$$

Conclusively, the momenta p^I_{qm} and p^*_{qm} are representative of the motion quantities the thrust potential and the expansion work would transfer to the particle

[5] Also found in other authors: see f.i. [7 ÷ 8].

when no other mechanism is influent. The momentum p^*_{qm} is given by the expression $i\sqrt{\hbar \mathrm{p}^{I\,'}_{qm}}$. Note that this expression lends itself to be easily generalized, bringing us to consider the extended form $\sqrt{i\hbar \mathrm{p}^{'}_{qm}}$ - in which the ∇S contribution to p_{qm} has been included. The usefulness of this extension will be soon understood.

The momenta p^I_{qm} and p^*_{qm} being defined on the basis of independent (since they are virtual) effects, we can assign them a quadrature relationship, and first define a comprehensive complex momentum $\tilde{\mathrm{p}}$, as a function of $i\mathrm{p}^I_{qm}$, as follows:

$$\tilde{\mathrm{p}}(i\mathrm{p}^I_{qm}) = \mathrm{p}^*_{qm} + i\mathrm{p}^I_{qm} = i\sqrt{\hbar \mathrm{p}^{I\,'}_{qm}} + i\mathrm{p}^I_{qm} \tag{48}$$

It is interesting to go even further and we define the generalized momentum p, function of p_{qm} :

$$\mathrm{p}(\mathrm{p}_{qm}) = \sqrt{i\hbar \mathrm{p}^{'}_{qm}} + \mathrm{p}_{qm} \tag{49}$$

The momentum $\tilde{\mathrm{p}}$ can be considered as the comprehensive quantity representing the motion quantity virtually equivalent to the action of the indicated work term (45), while the momentum p is a correspondent, generalized expression importing the term ∇S into equation (48). Now we consider the following complex momentum function $Q^{qm}_B(\mathrm{p,x})$ and Bernoulli Hamiltonian:

$$Q^{qm}_B(\mathrm{p,x}) = \frac{1}{2\mathrm{m}}\,\mathrm{p}^2 + i\frac{\mathrm{p}}{\mathrm{m}}\sigma^{qm}_B = \frac{1}{2\mathrm{m}}\,\mathrm{p}^2 - i\sqrt{-i\hbar \mathrm{p}'_{qm}}\,\frac{\mathrm{p}}{\mathrm{m}} \tag{50}$$

$$\sigma^{qm}_B = -\sqrt{-i\hbar \mathrm{p}'_{qm}} \tag{51}$$

$$\mathrm{H}^{qm}_B(\mathrm{p,x}) = \frac{1}{2\mathrm{m}}\,\mathrm{p}^2 + \Phi(\mathrm{x}) - i\sqrt{-i\hbar \mathrm{p}'_{qm}}\,\frac{\mathrm{p}}{\mathrm{m}} \tag{52}$$

If we set $\mathrm{H}^{qm}_B(\mathrm{p,x}) = const = \mathrm{E}_n$ and take the expression (49) for p it is easy to show that the Madelung equations set

$$\frac{\nabla S^2}{2\mathrm{m}} \;+\; \Phi(\mathrm{x}) \;-\frac{\hbar^2}{2\mathrm{m}}\frac{\sqrt{\rho}''}{\sqrt{\rho}} = \;\mathrm{E}_n \tag{53}$$

$$\rho \nabla S = const \tag{54}$$

is recovered. Conversely, it is now clear that equations (53), (54) can be obtained starting from the proposed Hamiltonian form $\mathrm{H}^{qm}_B(\mathrm{p,x}) \equiv \mathrm{E}_n$ (52) if we assume as a first Hamilton equation the following :

$$\frac{\partial \mathrm{H}^{qm}_B(\mathrm{p,x})}{\partial \mathrm{p}} = \frac{\mathrm{p}}{\mathrm{m}} - \frac{i}{\mathrm{m}}\sqrt{-i\hbar \mathrm{p}'_{qm}} = \frac{\mathrm{p}_{qm}}{\mathrm{m}} \equiv \frac{\mathrm{p}_{ns}}{\mathrm{m}} \tag{55}$$

In this formalism, we have replaced the standard first Hamilton equation, which normally defines the classical momentum mv(x), with an equation introducing instead the quantum momentum p_{qm} as the relevant non-standard momentum p_{ns}. To complete our set of Hamilton equations, we now write

$$-\frac{\partial \mathrm{H}_B^{qm}(\mathrm{p,x})}{\partial \mathrm{x}} = F = -\nabla\Phi(\mathrm{x}) + i\frac{\mathrm{p}}{\mathrm{m}}\frac{d}{d\mathrm{x}}\sqrt{-\hbar i \mathrm{p}'_{qm}} = \frac{\mathrm{p}_{qm}}{\mathrm{m}}\frac{d\mathrm{p}}{d\mathrm{x}} \equiv \frac{d\mathrm{p}}{d\tau} \tag{56}$$

$$\tau = \int \frac{\mathrm{m}d\mathrm{x}}{\mathrm{p}_{qm}} \equiv \int \frac{\mathrm{m}d\mathrm{x}}{\left[\nabla S - i\frac{\hbar}{2}\left(\frac{\rho'}{\rho}\right)\right]} \tag{57}$$

According to a previous analysis, in the second Hamilton equation we have introduced the complex, time-like variable τ which might be called "the quantum time". We want to remark here again that it has only a parametric, auxiliary role in the framework and by no means can be interpreted as the ordinary time-variable (unless we take a classical ($\hbar \to 0$) limit) in the present context.

Using equations (52) and (55)/(56) we find

$$\mathrm{H}_B^{qm}(\mathrm{p,x}) = \frac{1}{2\mathrm{m}}\,\mathrm{p}_{qm}^2 + \Phi(\mathrm{x}) - \frac{\hbar i}{2\mathrm{m}}\mathrm{p}'_{qm} = \tag{58}$$

$$-\frac{\hbar^2}{2\mathrm{m}}\frac{\sqrt{\rho}''}{\sqrt{\rho}} + \Phi(\mathrm{x}) + \frac{(\nabla S)^2}{2\mathrm{m}} - \frac{i\hbar}{2\mathrm{m}\rho}\left(\rho\nabla S\right)' = \mathrm{E}_n \tag{59}$$

so that

$$(\rho\nabla S)' = 0 \tag{60}$$

These equations representing the hydrodynamic formulation of the wave equation, we conclude that our Hamilton formalism is able to replace the standard quantization procedure. The formalism is based on the use of the complex form $Q_B^{qm}(\mathrm{p,x})$ (50) and is enlightened by the physical interpretation we have given to the variables p and p_{qm}.

Both the physical and mathematical congruence of the procedure here displayed interestingly reinforce the interpretation, expressed in equation (41) and quoted papers, of the Bohm potential as an indicated work.

The procedure we have shown in this section lends itself to some reelaboration, allowing us to display the possibility to follow an interesting "classical " path to set up the quantum-mechanical wave equation itself. This path consists in just some logical and mathematical reasoning, and will be shown able to attain the right functional dependence of the Bohm potential on the density (41) - although it is unable to provide the value of the physical constant (the Planck's constant, a quantity which can only be determined by the experiments) there involved. However, our purpose is just to show the procedure, worthy to be discussed here. We have formally to start with the equations (42), (43), here reported again (in order to stress that the starting point has a classical character, let the energy E be not yet identified with a quantum value for now). Let us call simply P the pressure and we have:

$$\frac{\nabla S^2}{2\mathrm{m}} \;+\; \Phi(\mathrm{x}) \;+ \int \frac{d\mathrm{P}}{\rho} \;=\; \mathrm{E} \tag{61}$$

$$\rho \nabla S = const \tag{62}$$

These equations have just the formal appearance of the classical flow-of-mass theorem. We do not mind here a lot about the physical meaning to be attributed to ∇S. Following some - previously introduced - considerations about the virtual momenta correlated to an indicated work potential, we can first define the non-standard momentum (let us call it p_{qm} already):

$$\mathrm{p}_{qm} = \nabla S - i\sqrt{2m\int \mathrm{P}d\frac{1}{\rho}} \tag{63}$$

to be used with a Bernoulli Hamilton formalism:

$$\mathrm{H}_B(\mathrm{p,x}) = \frac{1}{2\mathrm{m}}\,\mathrm{p}^2 + \Phi(\mathrm{x}) + i\sigma\frac{\mathrm{p}}{\mathrm{m}} = \mathrm{E} \equiv const \tag{64}$$

$$\frac{\partial \mathrm{H}_B(\mathrm{p,x})}{\partial \mathrm{p}} = \frac{\mathrm{p}_{qm}}{\mathrm{m}} \tag{65}$$

From the last two equations ([6]), we obtain

$$\mathrm{H}_B(\mathrm{p,x}) = \frac{1}{2\mathrm{m}}\,\mathrm{p}_{qm}^2 + \frac{\sigma^2}{2\mathrm{m}} + \Phi(\mathrm{x}) = \mathrm{E} \tag{66}$$

Still we need an appropriate definition for σ; by now, we are brought, by symmetry considerations, to assume that

$$\frac{\sigma^2}{2\mathrm{m}} = \hat{f}(\nabla S) + \hat{f}\left(-i\sqrt{2m\int \mathrm{P}d\frac{1}{\rho}}\right) \tag{67}$$

This equation means that the energy carried by the momentum σ is taken as the (symmetrical) addition of the same functional effects (described by an operator $\hat{f}$) we attribute to the two independent momenta we have got available within

[6] By the sake of brevity, henceforth in this paper we will always omit the second Hamilton equation and rest on the equivalent condition $\mathrm{H}_B(\mathrm{p,x})=\mathrm{E}\equiv const$ as done already in equation (64).

the context, ∇S and $-i\sqrt{2\mathrm{m}\int \mathrm{P}d\,(1/\rho)}$. Looking at equations (61) and (45), it is now reasonable to characterize the operator $\hat{f}$ as follows:

$$\hat{f}\left(-i\sqrt{2\mathrm{m}\int \mathrm{P}d\frac{1}{\rho}}\right) = \frac{\mathrm{P}}{\rho} \tag{68}$$

Equation (66) can then be written indeed

$$\frac{\nabla S^2}{2\mathrm{m}} \quad - i\frac{\nabla S}{\mathrm{m}}\sqrt{2\mathrm{m}\int \mathrm{P}d\frac{1}{\rho}} - \int \mathrm{P}d\frac{1}{\rho} \ + \frac{\mathrm{P}}{\rho} + \hat{f}(\nabla S) + \ \ \Phi(\mathrm{x}) \ = \ \mathrm{E} \tag{69}$$

so that,by comparison with equation (61), we find:

$$f\hat{(}\nabla S) - i\frac{\nabla S}{\mathrm{m}}\sqrt{2\mathrm{m}\int \mathrm{P}d\frac{1}{\rho}} = 0 \tag{70}$$

This last equation must be compared to the continuity equation (62), which can be written in the form ($S' \equiv \nabla S$):

$$S'' + \frac{\rho'}{\rho}S' = 0 \tag{71}$$

By the comparison, the easiest inference is that

$$\hat{f}(\nabla S) = s\frac{i}{\mathrm{m}}S'' \tag{72}$$

so that

$$-i\sqrt{2\mathrm{m}\int \mathrm{P}d\frac{1}{\rho}} = si\frac{\rho'}{\rho} \tag{73}$$

Here s is an unknown constant to which we have reserved the physical dimension of an action. Now the three equations (68), (70), (72) in the unknown quantities P, $\hat{f}$ can easily be solved by the following position:

$$\hat{f} \equiv s\frac{i}{\mathrm{m}}\frac{d}{d\mathrm{x}} \tag{74}$$

so that we have finally

$$\frac{\mathrm{P}}{\rho} = -\frac{s^2}{\mathrm{m}}\left(\frac{\rho'}{\rho}\right)' \tag{75}$$

$$\int \frac{d\mathrm{P}}{\rho} = -2\frac{s^2}{\mathrm{m}}\frac{\sqrt{\rho}''}{\sqrt{\rho}} \tag{76}$$

or

$$\frac{\nabla S^2}{2\mathrm{m}} \; + \; \Phi(\mathrm{x}) \; - 2\frac{s^2}{\mathrm{m}}\frac{\sqrt{\rho}''}{\sqrt{\rho}} \; = \; \mathrm{E} \equiv \mathrm{E}_n \tag{77}$$

By comparison with the wave-equation (53), then we recognize that the expounded procedure attains the right functional dependence for the Bohm potential although - as advanced - it does not allow us to know the value of the constant s. It is clear that we will determine it as $\hbar/2$ but to this end we shall have "recourse to the experimental evidence".

Conclusively, the positions (63), (64) and (67) + a "requirement of continuity" (71) are able to bring us to the Madelung equations, circumventing the ordinary quantization postulates.

3.5 Classical Hamilton formalism for the Bernoulli oscillator Newtonian dynamics (SPND and DBOE cases)

In order to obtain equations (4), (5) and (6), (7) - i.e. in order to describe the SPND and DBOE cases, we can set up a formalism analogous to that expounded in the previous section. Once established the basic equations we will switch on a slightly different version, lending itself to a more practical, physical interpretation.

We define the following (ad hoc) non-standard momentum p_n, momentum function $Q_B(\mathrm{p,x})$ and Hamilton function $\mathrm{H}_B(\mathrm{p,x})$:

$$\mathrm{p}_n = \nabla S^* - i\zeta\hbar\left(\frac{\rho'}{\rho}\right) \tag{78}$$

$$Q_B(\mathrm{p,x}) = \frac{1}{2\mathrm{m}}\,\mathrm{p}^2 + i\frac{\mathrm{p}}{\mathrm{m}}\sigma_B \tag{79}$$

$$\sigma_B(x) = -\sqrt{-i\hbar\left[-\frac{2\zeta}{\epsilon}S^{*\prime\prime} - i\zeta\hbar\left(\frac{\rho'}{\rho}\right)'\right]} \tag{80}$$

In these equations, ζ and ϵ have to be assigned according to the case at hand and ϵ is defined as

$$\epsilon = \frac{\nu'(\mathrm{x})\rho(\mathrm{x})}{\nu(\mathrm{x})\rho'(\mathrm{x})} - 1 \tag{81}$$

We also write

$$\mathrm{H}_B(\mathrm{p,x}) = \frac{1}{2\mathrm{m}}\,\mathrm{p}^2{+}\Phi(\mathrm{x}) - i\sqrt{-i\hbar\left[-\frac{2\zeta}{\epsilon}S^{*\prime\prime} - i\zeta\hbar\left(\frac{\rho'}{\rho}\right)'\right]}\,\frac{\mathrm{p}}{\mathrm{m}} = \mathrm{E}_n \tag{82}$$

As first Hamilton equation we will take

$$\frac{\partial \mathrm{H}_B(\mathrm{p,x})}{\partial \mathrm{p}} = \frac{\mathrm{p}}{\mathrm{m}} - \frac{i}{\mathrm{m}}\sqrt{-i\hbar\left[-\frac{2\zeta}{\epsilon}S^{*\prime\prime} - i\zeta\hbar\left(\frac{\rho'}{\rho}\right)'\right]} = \frac{\mathrm{p}_n}{\mathrm{m}} \tag{83}$$

Using the previous equations, we obtain by simple calculations

$$\frac{1}{2\mathrm{m}}\nabla S^{*2} - \frac{\hbar^2}{2\mathrm{m}}\frac{\rho^{\zeta}(\mathrm{x})''}{\rho^{\zeta}(\mathrm{x})} = \mathrm{E}_n - \Phi(\mathrm{x}) \tag{84}$$

$$\epsilon = \frac{d\ln S^{*'}}{d\ln\rho} \tag{85}$$

From these equations, it is now easily shown that the mentioned equations set (4), (5), SPND case) can be recovered first.

This is provided by a choice of coefficients $\zeta = -2$, $\epsilon = -1$; and by setting

$$\nabla S^* = \sqrt{\mathrm{m}\,\mathrm{m}_{eff}}\,\mathrm{v}(\mathrm{x}) \tag{86}$$

It is now interesting to note that the same general equations (78)÷(85) are also able to provide the Madelung quantum mechanical equations set (OQMC) (8) and (9) when the choice for the coefficients is $\zeta = 1/2$, $\epsilon = -1$, and ∇S^* is set equal to ∇S. In this case, indeed, the momentum p_n will be found coincident with p_{qm} (44) and the momentum σ_B equal to the corresponding momentum σ_B^{qm} (51). The framework expounded in the previous section is then completely recovered.

Concerning the DBOE case, we have to premise a brief discussion to show the formalism effectiveness. This is as follows.

On a general point of view, both the quantities ζ and ϵ introduced in our previous equations should in principle have been considered functions of x, just to account for the DBOE physical occurrences. But in practice, our formalism can plainly be worked out with constant values, in all cases. Indeed in our physical model - as shown in [4] - the space is divided into two Regions, I and II (Region I is the "internal" one, while Region II is defined as the extreme external boundary attainable by the particle). In a brute but practical approximation for the present papers, we might say that in both regions typical asymptotic behaviors of ϵ in the DBOE case are constants: we find $\epsilon \approx -1$ in Region I, and $\epsilon = 1/4$ in Region II ([7]). Now the ϵ variation vs x is actually limited to the transition zone between the two domains. Therefore, the DBOE case is characterized by the values $\zeta = 1/2$, $\epsilon = 1/4$ in Region II (then in this region, where we find $v_D^2 \propto \sqrt{\rho}$ and we also take $m_{eff}^{D} = 0$, it looks just confluent into the OQMC wave-equation (77)); and $\zeta = -2$, $\epsilon \approx -1$ in Region I (in this region it will be confluent into the SPND case). From a physical point of view, this means that the statistical ensemble of trajectories has a small spread in Region I, and a more important one in Region II.

In the present couple of papers, we chose therefore to investigate the DBOE case only in the Region II (DBOE^{II}), where is affected by $\epsilon = 1/4$; in the other region, the same context with $\epsilon \approx -1$ would just be confluent into what we find for the SPND case, for which $\epsilon = -1$ indeed. Keeping this in mind, we will have conceptually to assume that a full description of the DBOE case asks for an interpolation of results across the two Regions of space ([8]).

Equations (78)÷(85) with $\zeta = 1/2$, $\epsilon = 1/4$ will therefore provide us with the DBOE equations (6)÷(7), although only in their quoted, specific determinations for Region II.

Within the expounded circumstances, it has however still to be noted that both in the SPND and DBOE^{II} cases the momenta p and σ_B given in equations (83) and (80) respectively are not of a plane physical interpretation. The only plane case in the formalism is the quoted OQMC case where the quantity $-2\zeta/\epsilon$ is equal to 1, and p (49) and σ_B (51) are easily correlated to p_n (which last identifies in this case with p_{qm}).

As it has also been advanced at the beginning of the section, we want therefore to renormalize our formalism according to the following considerations.

Using a property introduced in equations (29) and (30) we can write again equation (82) as follows :

$$\mathrm{H}_B^{qm}(\mathrm{p,x}) = \frac{1}{2\mathrm{m}}\,\mathrm{p}^2 - i\sqrt{-i\hbar \mathrm{p}_n'}\,\frac{\mathrm{p}}{\mathrm{m}} + \frac{i\hbar}{2\mathrm{m}}\left(\frac{2\zeta}{\epsilon}+1\right) S^{*\prime\prime} + \Phi(\mathrm{x}) = \mathrm{E}_n \qquad (87)$$

This Hamiltonian form is equivalent to that expressed in equation (82). It displays the new function σ_B^* equal to

$$\sigma_B^* = -\sqrt{-i\hbar\left[S^{*\prime\prime} - i\hbar\zeta\left(\frac{\rho'}{\rho}\right)'\right]} = -\sqrt{-i\hbar \mathrm{p}_n'} \qquad (88)$$

and an extra potential energy term $i\hbar\,[2\zeta/\epsilon + 1]\,S^{*\prime\prime}/2\mathrm{m}$ - which we will call $i\Phi_S$

[7] In Region I the volume flow $\nu(\mathrm{x})$ appearing in equation (63) of reference [4] has a small derivative, while in Region II it is shown to have a dependence on the 5/4 power of the density.

[8] It is clear that in more complex calculations functions $\zeta(x)$ and $\epsilon(x)$ could be drawn, able to bring the procedure to the desired task. Indeed, if we assume negligeable derivatives for $\zeta(x)$ and $\epsilon(x)$, the DBOE context can practically be recovered by the same equations (78)÷(85) replacing ζ with $\zeta(x)$ $(-2 \leq \zeta(x) \leq 1/2)$, taking $-1 \leq \epsilon(x) \leq 1/4$, and setting $\nabla S^* = \sqrt{\mathrm{m}\,\mathrm{m}_{eff}^{D}}\,\mathrm{v}_D(\mathrm{x})$.

- into the x-dependent part of the Hamiltonian. By the first Hamilton equation, the momentum p will now be found equal to

$$\mathrm{p} = \mathrm{p}_n + i\sqrt{-i\hbar \mathrm{p}'_n} \tag{89}$$

The renormalized version here introduced of the formalism is equivalent to the previous one as regards the final results, but looks to us more interesting because the momenta p (89) and σ^*_B (88) can be interpreted physically in a way analogous to what has been done for the OQMC correspondents in equations (46), (47). We recall indeed, from reference [4], that the HDF potential has an expression where the term $-\hbar^2 \mathrm{v}^{2''}/2\mathrm{mv}^2$ is relevant. If we first consider here the SPND case with $\zeta = -2$, this term can be written :

$$\Phi_{\mathrm{HDF}}(\mathrm{x},\xi(\mathrm{x})) = -\frac{\hbar^2}{2\mathrm{m}}\frac{\mathrm{v}^{2''}}{\mathrm{v}^2} = -\frac{\hbar^2}{2\mathrm{m}}\frac{\rho^{-2''}}{\rho^{-2}} = -\int \mathrm{k}_0(\rho,1)\mathrm{dg}(\rho,1) =$$

$$= -\mathrm{k}_0(\rho,1)\mathrm{g}(\rho,1) + \int \mathrm{g}(\rho,1)\mathrm{dk}_0(\rho,1) \tag{90}$$

$$-\frac{\hbar^2}{2\mathrm{m}}\frac{\rho^{-2''}}{\rho^{-2}} = -\frac{\hbar^2}{2\mathrm{m}}\left\{-2\left(\frac{\rho'}{\rho}\right)' + 4\left(\frac{\rho'}{\rho}\right)^2\right\} \equiv \frac{\hbar}{2\mathrm{m}}\mathrm{p}_n^{I\ \prime} - \frac{\mathrm{p}_n^{I\ 2}}{2\mathrm{m}} \tag{91}$$

$$\frac{\mathrm{p}_n^{I\ 2}}{2\mathrm{m}} = -\int \mathrm{g}(\rho,1)\mathrm{dk}_0(\rho,1) \tag{92}$$

$$-\frac{\hbar}{2\mathrm{m}}\mathrm{p}_n^{I\ \prime} = \mathrm{k}_0(\rho,1)\mathrm{g}(\rho,1) \tag{93}$$

These equations show - in a way similar (but dual, due to a sign difference) to what expressed by equations (46) and (47) - that the momenta $i\mathrm{p}_n^I$ and $\sqrt{-\hbar \mathrm{p}_n^{I\ \prime}}$ have physical meaning as constitutive of the kinetic energy virtual equivalents of the work and thrust potentials $\int \mathrm{gdk}_0$ and $\mathrm{k}_0\mathrm{g}$ associated to the variables k_0 and g (for the definition of these quantities, see references [1 ÷ 4]: they are the HDF elastic function $\mathrm{k}_0 \propto \rho^4$ and its correspondent pressure g. Consider also that in the present context a classical-like density $\rho(\mathrm{x}) = 2\nu_0/\mathrm{v}(\mathrm{x})$ is assumed).

On the other hand, since the DBOE case is practically confluent into the SPND (Region I) and OQMC (Region II), a plane physical interpretation of the momentum fields there involved is still insured.

As a conclusion, we note that the quantity $2\zeta/\epsilon = 4$ in both regions for both cases. Then we are able to state finally that both the DBOE and SPND contexts can be described by the equations set

$$\mathrm{H}_B^{qm}(\mathrm{p},\mathrm{x}) = \frac{1}{2\mathrm{m}}\mathrm{p}^2 - i\sqrt{-i\hbar \mathrm{p}'_n}\frac{\mathrm{p}}{\mathrm{m}} + \frac{i5\hbar}{2\mathrm{m}}S^{*\prime\prime} + \Phi(\mathrm{x}) = \mathrm{E}_n \tag{94}$$

$$\mathrm{p}_n = \mathrm{m}\frac{\partial \mathrm{H}_B(\mathrm{p,x})}{\partial \mathrm{p}} = S^{*\prime} - i\hbar\zeta\left(\frac{\rho'}{\rho}\right) \tag{95}$$

with appropriate ζ values for both contexts and relevant Regions. Here, in all the cases of interest to us, the momenta p, p_n and $\sigma_B^* = -\sqrt{-i\hbar \mathrm{p}_n'}$can be regarded as quite simple physical representatives of the HDF potential. This renormalized formalism looks to us more appropriate for the sake of direct physical insight. Conversely, we have to remark that now giving a simple interpretation to the potential $\Phi_S = 5\hbar S^{*\prime\prime}/(2m)$ might be considered the challenging task (demanded to further investigation; but see also Part II of this work).

4 Conclusion

In this paper, the physical and mathematical contexts corresponding to the acronym cases PHME, SPND, DBOE and OQMC have been fitted into Hamilton-like formalisms. The Hamilton functions we introduced are named Bernoulli forms and include a linear term in their expressions. By a sake of congruence, we made the requirement that the appropriate coefficient (see equation (94)) is proportional to the square-root of p_n', where p_n is the momentum defined in the first Hamilton equation (95). The requirement caused the potential $i\Phi_S$ to pop out of the quadratic function Q_B (79).

In our opinion, this circumstance is likely due to the fact that the physical information brought by the potential $i\Phi_S$ is quite different and independent of the one expressed by p and p_n characterizing our formalism. It must be directly correlated to some microscopic effect with a very definite phenomenological base.

Further re-normalization of the momenta p and p_n definitions, able to bring back the potential $i\Phi_S$ into the quadratic form Q_B - and saving congruence, is however conceptually possible. But it is easy to show that, to pursue this task, we should be able to solve a Riccati-like equation in the complex plane. If even we could, the resulting momenta expressions would certainly be found too much involved to be usefully interpreted within such a framework as we have introduced.

The potential $i\Phi_S$, however, is zero in the OQMC case and different from zero in the two other cases, SPND and DBOE. These two last are just the contexts we want to promote, within our investigations, as the classical equivalents (or competitors) of the OQMC - the orthodox quantum-mechanical case. Then whenever the physical interpretation we give to the quadratic Bernoulli forms Q_B as the energy associated to the momentum p is accepted, investigating the physical meaning of the excerpted term $i\Phi_S$ or correlated quantities should display a potential for deeper understanding and comparison amongst the various cases.

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Last scientific revision 21.02.2004