Classical and quantum Hamilton formalisms for the mechanics of the Bernoulli oscillators
II - Quantum framework

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Summary - In a few previous papers, we discussed the fundamentals of the so-called Bernoulli oscillators physics. The Bernoulli oscillators are classical entities whose behavior is influenced by a "hidden" degree of freedom, in its turn excited by a quantum vacuum action. The investigation brought us - within assessed approximations and limits (uni-dimensional motion) - to propose a Newtonian motion background below the matter-wave behavior. In the present couple of papers, we give a formal description of the Bernoulli oscillators classical degree of freedom mechanics, by the means of both a classical (in paper I) and a quantum-like (in the present paper II) Hamilton formalisms. The quantum procedure operates on what we call the Bernoulli Hamiltonian functions. There are many different sets of parameters bringing the quantum formalism to the desired fits, so that the most interesting (at the present investigative stage) cases are discussed. We provide finally an "optimal" framework, operating on the particle-vacuum primary interaction (a source context to us, called PHME), and we show it able to take out the distant effects of the interaction. This procedure describes our contexts with improved compactness and interpretability. Physical interpretation matching the mentioned formalisms is provided step-by-step.

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1 Introduction

By our investigations [$1 \div 4$], we promoted a peculiar model of the oscillators physics which we named the Bernoulli oscillators physics. The model bases on a classical-like description of the oscillator-vacuum interaction. It originates various physical contexts, which we call PHME, SPND, DBOE (to be compared with both the corresponding purely classical and quantum-mechanical orthodox (OQMC) contexts). Details about these models and topics can be found in the quoted references and are not to be repeated here. This paper is the second one of a couple, analyzing both our proposed "Bernoulli" Hamiltonians (with correlated non-standard formalism) and our physical models when submitted to the formalism. In paper I [5], classical-like procedures have been investigated already. The present paper II is dedicated to quantum-like procedures.

2 Generalized quantum formalism

In this section, we introduce a quantization procedure applied to our "Bernoulli Hamiltonian" forms. We want to show that this non-standard formalism is able to match the equations pertaining to each of the three cases relevant to us, which we call SPND (see eqs. (47)/(48)), OQMC (see eqs. (33)/(34)) and DBOE^{II} (eqs. (41)/(42)). These contexts will also be identified in this paper by an index j, with values 0, 1 and 2 respectively for the three.

Let us take, as a starting point, the following (Bernoulli) expression for the Hamilton function [5]:

$$\mathrm{H}_B(\mathrm{p,x}) = \frac{1}{2\mathrm{m}}\,\mathrm{p}^2 + \Phi(\mathrm{x}) + i\mathrm{p}\frac{\sigma(\mathrm{x})}{\mathrm{m}} \equiv \mathrm{E} \tag{1}$$

Equation (1) has already been discussed, from a classical point of view, in paper I. According to a definition given there, we assumed in the classical procedure that the Hamiltonian (1), and the implicated momentum p, are "canonical" quantities. The term means to us that p is defined in such a way that $\mathrm{H}_B(\mathrm{p,x})$ is equal to the energy expression for the physical context which we called PHME in previous papers (see equations (14)÷(22) in [5]):

$$\mathrm{H}_B(\mathrm{p,x}) \equiv \ \frac{1}{2}\,\mathrm{m}_{eff}\mathrm{v}^2(\mathrm{x}){+}\Phi(\mathrm{x}) \tag{2}$$

The PHME context is also taken as a reference for the quantization procedures to be described in this paper. This means that when we will take a $\hbar\rightarrow 0$ limit in the procedure, the quantum momentum $\hat{\mathrm{p}}\Psi_n(\mathrm{x})$ (see (4)) will attain an appropriate "canonical" expression.

The quantity σ is dimensionally a momentum, and is generally a complex function :

$$\sigma(\mathrm{x}) = \sigma_1(\mathrm{x}) \ + i\sigma_2(\mathrm{x}) \tag{3}$$

Appropriate forms for $\sigma(\mathrm{x})$ will be found in the sequel.

In order to set up a quantization procedure ([1]) based on the form (1), we also introduce a quantum momentum operator

$$\hat{\mathrm{p}} \equiv -i\alpha\hbar\frac{d}{d\mathrm{x}} \tag{4}$$

[1] Quantization in this paper only affects the space-dependence of the wavefunctions, and not their time-dependence (assumed to be of stationary form).

Here a constant parameter α has been introduced. If α is set equal to 1, the standard quantum-mechanical momentum operator is recovered; but, in our extended formalism, we will also consider α-values different from 1. These last will be determined by imposing congruence of the operational results with the various equations to be matched at each time.

By equations (1) and (4) we now define the Hamilton operator

$$\hat{\mathrm{H}}_B(\hat{\mathrm{p}},\,\mathrm{x},\,a) \equiv \frac{1}{2\mathrm{m}}\,\hat{\mathrm{p}}^2 + \Phi(\mathrm{x}) + ia\hat{\mathrm{p}}\frac{\sigma(\mathrm{x})}{\mathrm{m}} + i\,(1-a)\,\frac{\sigma(\mathrm{x})}{\mathrm{m}}\hat{\mathrm{p}} \tag{5}$$

The constant parameter a appearing in this equation is to introduce a linear mixing of the quantities $\hat{\mathrm{p}}\sigma(\mathrm{x})$ and $\sigma(\mathrm{x})\hat{\mathrm{p}}$. This is for the sake of generality ($\hat{\mathrm{p}}$ and $\sigma(\mathrm{x})$ are non-commutating operators). In standard quantization procedures, it is generally recommended to set the parameter a equal to 1/2 by the sake of symmetry. In our framework, the values for a can, instead, be different from 1/2 and range - in principle - all over the field of real numbers (we even have $a \to \infty$ in some cases). The a values in the sequel will be determined, they too, by a sake of congruence with the appropriate contexts.

We now introduce a $\hat{\mathrm{H}}_B$-eigenfunction $\Psi_n(\mathrm{x})$, solution of the equation

$$\hat{\mathrm{H}}_B(\hat{\mathrm{p}},\,\mathrm{x},\,a)\Psi_n(\mathrm{x}) = \mathrm{E}_n\Psi_n(\mathrm{x}) \tag{6}$$

This equation can be written

$$\left\{\frac{1}{2\mathrm{m}}\,\hat{\mathrm{p}}^2 + \Phi(\mathrm{x}) + ia\hat{\mathrm{p}}\frac{\sigma(\mathrm{x})}{\mathrm{m}} + i\,(1-a)\,\frac{\sigma(\mathrm{x})}{\mathrm{m}}\hat{\mathrm{p}}\right\}\Psi_n(\mathrm{x}) = \mathrm{E}_n\Psi_n(\mathrm{x}) \tag{7}$$

As anticipated, amongst our purposes here there is demonstrating that equation (7) is able to result into the quoted equations sets in the paper: (47)/(48) holding for the j=0 (SPND) case, (33)/(34) describing the j=1 (OQMC) case, and (41)/(42) for the j=2 (so called DBOE^{II}) case.

To our end, specific determinations for the quantities α, a, $\sigma(\mathrm{x})$ according to each of the cases at hand will be assumed. Note that the case $\alpha = 1$, $\sigma(\mathrm{x}) = 0$ trivially identifies the standard quantization of the purely classical Hamiltonian, resulting in the OQMC case indeed; yet we are specifically including this case into our analysis, both for the sake of comparison and because we want to show that it is also possible to obtain the standard wave-equation by a non-standard procedure with $\sigma(\mathrm{x}) \neq 0$.

We have also to remark that in equation (7) the energy E_n will take the role of an $\hat{\mathrm{H}}_B$-"eigenvalue" (in the standard mathematical sense, meaning that it can be determined by solving the equation) in the cases j=1 and j=2 only. This is a very well known property in the (OQMC) j=1 case, but here we want to stress specifically that it also holds in the (DBOE) j=2 case. In this specific respect, indeed, this last can be made mathematically equivalent to the OQMC, as discussed in [4, 5], in a region of space (called Region II) where the quantity $\mathrm{v}_{_D}^4$ turns out proportional to the quantum density.

In the j =0 case, instead, we face a situation where we cannot consider E_n as determined by the equation itself (at our present investigation stage at least). In our model by now, it has to be taken equal to the appropriate quantum-mechanical energy eigenvalue by assumption.

Another most important remark, mirroring the one given in [5] ([2]) already, is as follows. A comprehensive description (throughout all the definition space) of our DBOE case by a quantum formalism with a wavefunction of the form (8) ([3]) has proved too much involved analytically, because the coefficient ϵ defined in equation (12) turns then out into a function of x, and inextricable second-order non-linear differential equations arise. Our treatment is plainly analytical with ϵ taking constant values. Fortunately, in our physical model - as shown in previous quoted references - the wavefunction definition space can be divided into two Regions, I and II (Region I is the "internal" one, while Region II is defined as the extreme external boundary attainable by the particle). In both regions, typical (asymptotic) behaviors of ϵ in the DBOE case are constants: it is found $\epsilon \approx -1$ in Region I, and $\epsilon = 1/4$ in Region II, so that the ϵ variation vs x is actually limited to the transition zone between the two domains. For this reason, we chose in this paper to investigate the DBOE case only in the Region II (DBOE^{II}), where is affected by $\epsilon = 1/4$; in the other region, the same context with $\epsilon \approx -1$ would just be confluent into what we find for the SPND case, for which $\epsilon = -1$ indeed. Keeping this in mind, we will have conceptually to assume that a full description of the DBOE case asks for an interpolation of results across the two Regions of space.

3 General solution

Equation (7) can be mathematically investigated for different values of parameters a, α, σ we may choose. For the relevant cases to us, one finds that a number of values sets demonstrate able to bring this equation to the desired fits. Sometimes, these sets will appear rather exotic for physical interpretation or adherence to known references, so that in a final section a reelaboration bringing to more plane circumstances will be given (case C). By our investigations, however, we have found that interesting cases are obtained both in the limit $\sigma \to 0$ (case A) and $\sigma \neq 0$ (cases B and C). We will discuss with mathematical details all of them. On the interpretative level, a comparison between them will also be performed. This is also demanded to the final section in the paper - where the most important results will be conclusively discussed.

In order to solve equation (7) the following ansatzs and symbols will be used:

$$\Psi_n(\mathrm{x}) \equiv \Psi(\mathrm{x}) = \pm\sqrt{\rho}\,\exp\left[iR(\mathrm{x})/\beta\hbar\right] \qquad (8)$$

[2]The circumstances expounded in the very following have indeed been exploited already in that paper, where some details more can be found.

[3]i.e. matching the Copenhagen interpretation: the squared modulus is the density.

$$|\Psi(\mathrm{x})|^2 = \rho(\mathrm{x}) \tag{9}$$

$$R'(\mathrm{x}) \equiv \frac{dR}{d\mathrm{x}} = c\nabla S^* \equiv c\frac{dS^*}{d\mathrm{x}} \equiv cS^{*\prime}(\mathrm{x}) = 2\frac{\mathrm{cm}\nu(\mathrm{x})}{\rho(\mathrm{x})} \tag{10}$$

The variety of symbols and constants is due to our sake of synthetic analysis of all the cases together: as a matter of fact, the mathematical procedures are unified, but the physical meaning of analogous quantities is different and they must be kept distinguished. In these and following equations, the index n in principle affecting all the quantities is dropped off by simplicity. The index j has been introduced instead in the formalism, so that this last becomes able to account in a general way for each of the three cases we are interested in. The cases will be however also analyzed one-by-one in the next sections. In equation (10), c is some appropriate constant to be determined for each case, and the other quantities will be introduced step by step. In the same equation, the factor 2 is due to the density definition. The density includes indeed the two countermoving particles streams in the space domain V* available to the oscillators ([4]) and is normalized to unity according to the following equation :

$$\int_{\mathrm{V}^*} \rho(\mathrm{x})\mathrm{dx} = \frac{1}{2}\oint \rho(\mathrm{x})\mathrm{dx} = 1 \tag{11}$$

The volume-flow rate $\nu(\mathrm{x})$ may assume a functional dependence on x or a constant value $\nu_0 \neq 0$ in the different cases relevant to us. When the DBOE or the SPND cases will be at hand, we will assign to the phase function derivative $\nabla R/\beta$ the role of classical-like momenta, see equations (42) and (48). We recall that, physically, the velocity field $\mathrm{v}_{_D}$ appearing in the DBOE case is the statistical average velocity of an open group of particles in our model: it can be expressed as a function of the density $\rho(\mathrm{x})/2$ and volume flow-rate $\nu(\mathrm{x})$, in agreement with the stationary continuity equation. In the SPND case, conversely, the velocity field v(x) corresponds to $\nu = \nu_0 = const$ and represents the single-particle time-law dx/dt.

To be defined, when we quote the OQMC operational frame we have instead kept the description adherent to the orthodox interpretation of quantum mechanics, by which we should not be allowed to interpret the quantum action derivative ∇S as a particle velocity field ([5]).

In the DBOE frame, a statistical-effective mass $\mathrm{m}^{_D}_{eff}$ is introduced (see equation (6) in paper I). This term generally displays a dependence on x. But in Region II where our calculations are effective, $\mathrm{m}^{_D}_{eff}$ has to be set equal to zero in practice (this is for congruence with quantum mechanical wavefunctions with constant phase S, the primary case of interest to us). On the other hand, even

[4] We recall that our physical context is a uni-dimensional one.

[5] But actually, many modern authors and ourselves in our models believe - following the known Bohm [6 $\div$ 12] interpretation - that Newtonian velocity fields are instead implicated into the quantum mechanical phase gradient ∇S.

in Region I, by the quoted analogy with the SPND case and just at our present investigations stage, we will take a constant value. By uniformity and simplicity of physical insight, therefore, the effective masses in both frameworks will be taken along, and processed, as constants throughout the paper.

In order to characterize the volume-flow rate $\nu(\mathrm{x})$ we introduce the following quantity

$$\epsilon = \frac{d\ln S^{*\prime}}{d\ln\rho} \equiv \frac{\nu'(\mathrm{x})\rho(\mathrm{x})}{\nu(\mathrm{x})\rho'(\mathrm{x})} - 1 \tag{12}$$

As far as the DBOE^{II} case is concerned, we will consider the specific law

$$\frac{\nu'(\mathrm{x})}{\nu(\mathrm{x})} = \frac{5}{4}\frac{\rho'(\mathrm{x})}{\rho(\mathrm{x})} \Leftrightarrow \frac{S^{*\prime\prime}(\mathrm{x})}{S^{*\prime}(\mathrm{x})} = \frac{R''(\mathrm{x})}{R'(\mathrm{x})} = \frac{1}{4}\frac{\rho'(\mathrm{x})}{\rho(\mathrm{x})} \tag{13}$$

This law ($\epsilon = 1/4$, Region II) has been found in reference [4].

The corresponding relationship for the $\nu(\mathrm{x}) = \nu_0 = const$ cases implies instead a value $\epsilon = -1$:

$$\frac{S^{*\prime\prime}(\mathrm{x})}{S^{*\prime}(\mathrm{x})} = \frac{R''(\mathrm{x})}{R'(\mathrm{x})} = -\frac{\rho'(\mathrm{x})}{\rho(\mathrm{x})} \tag{14}$$

This law characterizes both the SPND and OQMC cases (in this last case, the flow ν_0 is, orthodoxly, a quantum probability density flow).

For many of the cases to be treated in this paper, we will use the following mathematical procedure.

We use equations (4), (8)÷(10), (12) into equation (7), and the ansatz

$$\sigma_2(\mathrm{x}) = \lambda R' \tag{15}$$

After lenghty but simple calculations we obtain :

$$\sigma_1(\mathrm{x}) = \frac{\alpha\hbar}{2}\left\{\left(1-\frac{\gamma}{2}\right) + \epsilon\,(1-a\gamma)\right\}\frac{\rho'(\mathrm{x})}{\rho(\mathrm{x})} = \frac{\alpha\hbar}{2}A\frac{\rho'(\mathrm{x})}{\rho(\mathrm{x})} =$$

$$= \frac{\alpha\hbar}{2}\left(\frac{R''}{R'} + \frac{\rho'}{\rho}\right) - \frac{\beta\hbar}{2}\frac{\sigma_2}{R'}\frac{\rho'}{\rho} - \beta\hbar a\frac{\sigma_2'}{R'} \tag{16}$$

Setting $c^2 = \beta^2/(\alpha^2(1-\gamma))$ we have also

$$\frac{\nabla S^{*2}}{2\mathrm{m}} - \frac{\alpha^2\hbar^2}{2\mathrm{m}}\frac{\sqrt{\rho}''}{\sqrt{\rho}} +$$

$$+\frac{\alpha\hbar}{2\mathrm{m}}\frac{\rho'(\mathrm{x})}{\rho(\mathrm{x})}\sigma_1(\mathrm{x}) + a\frac{\alpha\hbar}{\mathrm{m}}\sigma_1'(\mathrm{x}) = \mathrm{E}_n - \Phi(\mathrm{x}) \tag{17}$$

The coefficient A (a comfortable auxiliary quantity) is just defined by the same equation (16) while the coefficient γ is given by the expression

$$\gamma = \frac{2\beta\lambda}{\alpha} \tag{18}$$

Here again, the variety of the coefficients is obliged to merge into different analysis paths. Equations (16) and (17) can now be specified for different choices of the coefficients α, a and of the function $\sigma_2(\mathrm{x})$ (i.e., of λ–values). The choices will imply different determinations of γ, A and $\sigma_1(\mathrm{x})$ consequently. Expressions for the coefficients we have used can be found in equations (24) $\div$ (30) (relevant to the case A in the next subsection), and in another subsection (relevant to the case B) moreover. Equation (17) can also be written

$$\frac{\nabla S^{*2}}{2\mathrm{m}} - \frac{\hbar^2}{2\mathrm{m}}\zeta\left(\frac{\rho'(\mathrm{x})}{\rho(\mathrm{x})}\right)' - \frac{\hbar^2}{2\mathrm{m}}\eta\frac{\rho'(\mathrm{x})^2}{\rho(\mathrm{x})^2} = \mathrm{E}_n - \Phi(\mathrm{x}) \tag{19}$$

$$\zeta = \alpha^2\left\{\frac{1}{2} - a\left(1 - \frac{\gamma}{2}\right) - a\epsilon\left(1 - a\gamma\right)\right\} = \alpha^2\left\{\frac{1}{2} - aA\right\} \tag{20}$$

$$\eta = \alpha^2\left\{-\frac{1}{2}\left(1 - \frac{\gamma}{2} + \epsilon\left(1 - a\gamma\right)\right) + \frac{1}{4}\right\} = \alpha^2\left\{\frac{1}{4} - \frac{A}{2}\right\} = \zeta^2 \tag{21}$$

This general result will be analyzed in the next sections where all our cases of interest will be distinctly examined. As is clearly shown in the last equation, for all the cases we (purposely) set the coefficient η equal to ζ^2, so that the equation (19) takes the final general form appropriate to us

$$\frac{\nabla S^{*2}}{2\mathrm{m}} - \frac{\hbar^2}{2\mathrm{m}}\frac{\rho^{\zeta}(\mathrm{x})''}{\rho^{\zeta}(\mathrm{x})} = \mathrm{E}_n - \Phi(\mathrm{x}) \tag{22}$$

Equation (16) now also reads

$$\rho\nabla S^* = const \times \rho^{\frac{\gamma\left(a - \frac{1}{2}\right) - A}{\gamma a - 1}} = const \times \rho^{1+\epsilon} \tag{23}$$

Equations (16)$\div$(23) are conclusively the general result we have been able to obtain by means of the quantization procedure expressed by equation (7), together with the positions (8)$\div$(15) and $\eta = \zeta^2$.

Since we affected with an index j our different contexts, we have to keep in mind that for each of them the quantity ϵ can be found all across the paper by the heuristic equation $\epsilon = -1 + 5j(j-1)/8$.

3.1 Solutions with $\sigma = 0$ (case A)

In the present context we will take

$$\lambda \rightarrow 0 \tag{24}$$

$$A \rightarrow 0 \tag{25}$$

$$aA = (j-1)\,(j-2)\,\frac{5}{16} \tag{26}$$

$$\gamma a = \frac{5}{2}j(j-1) + \frac{5}{16a}j(j-3) + \frac{5}{8a} \tag{27}$$

$$\frac{1}{2a} = j(2-j) \tag{28}$$

$$\frac{4}{5}\,(\zeta+2) - \frac{1}{2a} = j \tag{29}$$

$$\zeta = \frac{\alpha}{2} \tag{30}$$

The parameter λ has been evidenced into the momentum σ_2(x) definition so that this quantity goes to zero with λ. The quantity σ_1(x) goes to zero as well. Our equations will indeed provide the meaningful results in the limit $\lambda \rightarrow 0$. The parameter ζ is the coefficient defined in equation (20) but is simply correlated (by our position) to the parameter α in the context of the present section.

The previous equations give the (most interesting!-by now ([6])) relationships we have found, linking the different parameters encountered in our analysis, in order that the three cases of interest are described by our formalism. They have to be considered as simple ansatzs we have found to provide an unified formalism including all the relevant cases.

In order to make clear how to use the equations, we note first that in some cases they describe asymptotic behaviors : f.i., equation (28) means that in case j=0 the quantity $a \rightarrow \infty$. Consequently, equation (27) shows that γ goes to zero as $5/\left(8a^2\right)$ while A goes to zero as $5/(8a)$ (equation (26)) etc. Then our equations (24)÷(30) are just displayed in such a way that, once a value of j has been chosen, all the relevant parameters values (or behaviors) can be easily found by the reader.

[6]As remarked already, the parameters and their relationships, bringing to the desired fits within the context of this paper, are not unique nor exhausted by our choices. At present we are not able to give specifical physical meaning to all of the different quantities λ, A, a etc. Deeper analysis, if worthwhile, should be performed in order to precise a selection of parameters sets on the basis of their possible physical relevance. However at the end of paper, where a rather different but "optimal"procedure is proposed, easier interpretative circumstances will be found.

Note here - as a final remark about our positions - that the momentum $\sigma(\mathrm{x})$ turns out to be a linear combination of the two characteristic momenta $\hbar\rho'/2\rho$ and $R^{'}/\beta$ which can be identified into the $\Psi(\mathrm{x})$ expression (8) :

$$\sigma(\mathrm{x}) = \sigma_1(\mathrm{x}) + i\sigma_2(\mathrm{x}) = \alpha A \frac{\hbar}{2}\frac{\rho'(\mathrm{x})}{\rho(\mathrm{x})} + i\lambda\nabla R \tag{31}$$

This circumstance is analogous to what we have found in a previous discussion involving equation (78) in paper I of this work.

Now we discuss the three physical cases of interest to us.

3.1.1 The orthodox quantum mechanical case (OQMC, j = 1)

We first want to show here that the OQMC, or the standard quantum mechanical matter-wave equation, can be recovered using the previous formalism. Our statement here may look redundant (when we set $\sigma = 0$ and $\alpha = 1$ in equations (4)÷(5), we obviously recover the well known, standard quantization procedure of the classical Hamiltonian). However, in this section we want briefly to provide all the relevant correlations between our proposed framework and the standard quoted case. By the means of the previous equations, taking the j = 1 index case in equations (24)÷(30), we find the following values for the relevant coefficients :

$$\alpha = 2\zeta = 2\sqrt{\eta} = 2a = -\epsilon = 1 \tag{32}$$

Here we do not need to define the coefficients β and c, but we can set them equal to unity by simplicity. Now we obtain from equations (16)÷(23):

$$\frac{\nabla S^2}{2\mathrm{m}} - \frac{\hbar^2}{2\mathrm{m}}\frac{\sqrt{\rho(\mathrm{x})}''}{\sqrt{\rho(\mathrm{x})}} \;=\; \mathrm{E}_n - \Phi(\mathrm{x}) \tag{33}$$

$$\rho(\mathrm{x})\nabla S = const \equiv 2\mathrm{m}\nu_0 \tag{34}$$

To write these equations we have also identified our wavefunction with a standard quantum wavefunction so that the phase $R/\beta = S^*$ has been set equal to the quantum mechanical phase S:

$$\Psi(\mathrm{x}) = \pm\sqrt{\rho}\exp\left[iR/\left(\beta\hbar\right)\right] \equiv \pm\sqrt{\rho}\exp\left[iS/\hbar\right] \tag{35}$$

As already clear, with the values (32) and the position (35), the procedure comes to coincidence with the known standard quantization: then the OQMC context is recovered and the orthodox quantum-mechanical equations in the Madelung form are obtained.

3.1.2 The drifting Bernoulli oscillators ensemble case (DBOE^{II}, j = 2)

Here we will use the following equations, obtained from (24)÷(30) for j=2:

$$\alpha = 2\zeta = 2\sqrt{\eta} = 1 - \frac{1}{2a} = 4\epsilon = 1 \tag{36}$$

We use moreover

$$\frac{\beta}{\alpha} = c = \sqrt{\frac{\mathrm{m}^{D}_{eff}}{\mathrm{m}}} \tag{37}$$

The effective mass m^{D}_{eff} is introduced in paper I of this work, equation (6). We find

$$a \to \infty \tag{38}$$

$$\gamma a \to 5 \tag{39}$$

$$aA \to 0 \tag{40}$$

and, from (16)÷(23):

$$\frac{\nabla S^{*2}}{2\mathrm{m}} - \frac{\hbar^2}{2\mathrm{m}}\frac{\sqrt{\rho}(\mathrm{x})''}{\sqrt{\rho}(\mathrm{x})} = \frac{1}{2}\mathrm{m}^{D}_{eff}\,\mathrm{v}^2_{D}(\mathrm{x}) - \frac{\hbar^2}{2\mathrm{m}}\frac{\mathrm{v}^2_{D}(\mathrm{x})''}{\mathrm{v}^2_{D}(\mathrm{x})} = \mathrm{E}_n - \Phi(\mathrm{x}) \tag{41}$$

$$\nabla S^* = \sqrt{\frac{\mathrm{m}}{\mathrm{m}^{D}_{eff}}}\nabla R = \sqrt{\mathrm{m}\mathrm{m}^{D}_{eff}}\mathrm{v}_{D}(\mathrm{x}) = \frac{2\sqrt{\mathrm{m}\mathrm{m}^{D}_{eff}}\nu(\mathrm{x})}{\rho(\mathrm{x})} \propto \sqrt{\mathrm{m}\mathrm{m}^{D}_{eff}}\rho(\mathrm{x})^{\frac{1}{4}} \tag{42}$$

To obtain these equations, here we have also assigned, by the position expressed in the first three terms of equation (42) itself, to ∇S^* and ∇R the role of classical-like momenta proportional to the drift velocity $\mathrm{v}_{D}(\mathrm{x})$. With these equations, the DBOE^{II} context is therefore found reproduced by the procedure.

3.1.3 The single-particle Newtonian case (SPND, j = 0)

In this case we find from (24)÷(30) :

$$\frac{\alpha}{4} = \frac{\zeta}{2} = -\frac{\sqrt{\eta}}{2} = \epsilon = -1 \tag{43}$$

$$a^2 \to \frac{5}{8\gamma} \to \infty \tag{44}$$

$$aA \to a^2\gamma \to \frac{5}{8} \tag{45}$$

We also use:

$$-\frac{\beta}{\alpha} = c = \sqrt{\frac{\mathrm{m}_{eff}}{\mathrm{m}}} \tag{46}$$

Then we find, using (15)÷(23):

$$\frac{\nabla S^{*2}}{2\mathrm{m}} - \frac{\hbar^2}{2\mathrm{m}}\frac{\rho^{-2}(\mathrm{x})''}{\rho^{-2}(\mathrm{x})} = \frac{1}{2}\mathrm{m}_{eff}\,\mathrm{v}^2(\mathrm{x}) - \frac{\hbar^2}{2\mathrm{m}}\frac{\mathrm{v}^2(\mathrm{x})''}{\mathrm{v}^2(\mathrm{x})} = \mathrm{E}_n - \Phi(\mathrm{x}) \tag{47}$$

$$\nabla S^* \equiv \sqrt{\frac{\mathrm{m}}{\mathrm{m}_{eff}}}\nabla R = \sqrt{\mathrm{m}\mathrm{m}_{eff}}\mathrm{v}(\mathrm{x}) = \frac{2\sqrt{\mathrm{m}\mathrm{m}_{eff}}\nu_0}{\rho(\mathrm{x})} \tag{48}$$

Here we have identified, by position, the phase gradient ∇S^* with the classical-like momentum definition $\sqrt{\mathrm{m}\mathrm{m}_{eff}}\mathrm{v}(\mathrm{x})$. With these equations, the SPND context is recovered by the procedure.

3.1.4 Remarks and discussion

For the sake of rigor, we have to make an important comment to the cases j=0 and j=2 here discussed. We cannot look at the procedure displayed for these cases as to a quantization procedure applied to the classical-limit Hamiltonian. In the present context, indeed, the meaningful equations are found in the limit $\lambda \to 0$ and the function σ itself turns out to be zero in this limit. Expression (1) actually reduces to the purely classical Hamiltonian in the $\sigma \to 0$ limit. But, when developing the quantum procedure with the Hamiltonian (1)→ (5), we first started with $\sigma \neq 0$; we introduced the parameter a, then we took the limit of our equations for $\sigma \to 0$ and $a \to \infty$. Then the procedure provides a term $a\sigma'_1 \neq 0$ in equation (17) which would not appear if we started, in equation (1), with $\sigma = 0$. Since however the effective value of σ is zero, we are in a situation where the procedure starts with $\sigma \neq 0$ while the corresponding classical-limit Hamiltonian is a different expression, actually the purely classical one. Moreover, the σ expression (31) includes a dependence on the quantum action $\hbar$, so that the term $a\sigma'$ itself cannot be considered as a classical-like quantity.

A similar situation (the σ expression including $\hbar$) will also be found in the next section, where a quantization procedure with $\sigma \neq 0$ is proposed.

These circumstances prevent us here from stating what we would consider an interesting property of the quantization: that the $\hbar$-dependent effects of the interaction are just taken out of the PHME Hamiltonian (2) - although our starting point in the procedure is quite "close" to that. The same circumstances, however, are in agreement with the fact that a quantum mechanical context is generally considered as the one describing the "primary" physics - the classical mechanics being only considered as a peculiar limit of the quantum one for $\hbar \to 0$. But in this concern, an opposite point of view can also be taken, and in some respects this is just the one we promote in our papers: that the OQMC context can be recovered and interpreted on the basis of a classical motion background below the matter-wave behavior. One might therefore appreciate much more having available a procedure starting with a very definite classical-like Hamiltonian, and able to take out $\hbar$-dependent effects by the means of a simple quantization prescription. This is actually what we will be able to do in the section "case C". Yet we have to remark, at the same time, that in a framework where quantum effects are reduced to classical ones, the quantum procedure looses its "absolute" character, and the Planck's constant itself will turn out embodied into a comprehensive classical-like description. The PHME context itself, therefore, should in any way not be considered as the absolute source of what we call "the distant effects" of the vacuum interaction: this last simply constitutes of a comprehensive action with a few distinguished appearances which we have identified, in the theoretical context, in order to provide a detailed description.

Further discussion about these points will still be found in the conclusive sections. As advanced, an "optimized" formalism (case C) will be introduced, able to improve our framework performances in different respects. But the interest of the displayed quantization procedure with $\sigma = 0$ must be stressed however. Indeed, this can be resumed by the observation that the quantization of a purely classical Hamiltonian ($\sigma = 0$) generally proceeds by choosing a =1/2 and results in the ordinary quantum mechanics. If $a \to \infty$ is chosen instead, in the procedure for a Bernoulli-like Hamiltonian a limit value for $a\sigma \neq 0$ can be exploited, and we are able to result into the SPND or DBOE contexts.

3.2 Solutions with $\sigma \neq 0$ (case B)

As remarked already, accomplishing our tasks can also be done using different sets of parameters values. Here we investigate the case where $\sigma \neq 0$ (case B). We will take different paths for the j=2 case and the two other cases. The different cases are separately discussed in the next sections.

3.2.1 The orthodox quantum mechanical case (j = 1)

In case B it does not turn out possible to match the OQMC equations set (33)/(34) by the means of the ansatz (15), with $\lambda R' \neq 0$, and keeping the value for α equal to 1. This requirement is necessary to preserve the quantum momentum field definition $\{\hat{\mathrm{p}}\Psi_n(\mathrm{x})\}/\Psi_n(\mathrm{x}) \equiv \nabla S - i\hbar\rho^{'}/2\rho$ as it is meaningful in the OQMC context. However, it can be easily checked by the reader that his last can be recovered using the previous equations (4)÷(10) and the position

$$\sigma = const \times \Psi_n^{-2} = \ const \times \rho^{-1} \exp\left[-2iS/\hbar\right] \tag{49}$$

whatever the *const* value is. We are allowed to take here

$$\alpha = 2a = 1 \tag{50}$$

We do not need to define β and c, and we can set them equal to unity by simplicity. The wavefunction Ψ_n still identifies with the standard quantum one, so that the phase R/β is equal to the quantum mechanical phase S. With these positions, from equations (49)÷(50) used into (4)÷(10), we find the Madelung equations set (33)/(34) again.

The remarkable conclusion here is that the standard quantum-mechanical Madelung equations can be obtained by a "nearly orthodox" ([7]) quantization of the Bernoulli classical Hamiltonian (1), but with $\sigma \neq 0$, given by equation (49).

3.2.2 The drifting Bernoulli ensemble case (j = 2)

It is not possible to find a solution of equation (7) governed by the same ansatz (15) (with $\lambda R' \neq 0$), fitting the DBOE^{II} case equations, while keeping a value for a equal to 1/2. If we take some different values for a, instead, we can find some solutions. We will not discuss them here. If we take a different from 1/2, indeed, we are not able to set up a solution enclosing a term of the form (49). This circumstance makes here the solutions with $a \neq 1/2$ not very interesting to our main purpose of keeping, for useful comparison, the DBOE^{II} treatment as near as possible to the OQMC case. However, this last requirement will be neither satisfied in the following analyses. Then we switch on a somewhat different frame now.

A couple (i,ii) of remarkable cases is the following:

i) we can start the procedure with the following Hamiltonian:

$$\mathrm{H}_B(\mathrm{p,x}) = \frac{1}{2\mathrm{m}}\,\mathrm{p}^2 + i\mathrm{p}\frac{\sigma}{\mathrm{m}} + i\Phi_S(\mathrm{x}) + \Phi(\mathrm{x}) = \mathrm{E}_n \tag{51}$$

$$\Phi_S(\mathrm{x}) = \frac{5\hbar}{2\mathrm{m}} S^{*\prime\prime}(\mathrm{x}) \tag{52}$$

This Hamilton function and potential Φ_S have already been introduced in paper I of this work. We have shown there that a classical-like treatment of the Bernoulli function (51) - for the j=0 and j=2 cases - is more interesting, on a mathematical symmetry and physical understanding points of view, than the one based on the form (1). By operating the form (51) with the quantization procedure (4)÷(8); using σ defined by the same equation (49), $\alpha = 1$, $a = 1/2$ and the position $\nabla S^* \equiv \sqrt{\mathrm{m}/\mathrm{m}^{D}_{eff}}\nabla R = \sqrt{\mathrm{m}\mathrm{m}^{D}_{eff}}\mathrm{v}_D(\mathrm{x})$, we will find again our reference set (41)/(42).

[7] We mean here that equations (8) and (50) at least are identical to what we have in a standard quantization.

ii) whenever we start from the form (1), we can find a solution taking $\alpha = 1$, $a = 1/2$, $\nabla S^* \equiv \sqrt{\mathrm{m}/\mathrm{m}^{D}_{eff}}\nabla R = \sqrt{\mathrm{m}\mathrm{m}^{D}_{eff}}\mathrm{v}_{D}(\mathrm{x})$ and

$$\sigma = const \times \Psi_n^{-2} + \sigma_S \tag{53}$$

$$\sigma_S = 5i\Psi_n^{-2}\int \Psi_n^2 S^{*\prime\prime} dx \tag{54}$$

Here again, the *const* value is arbitrary and can be set equal to 0 by the sake of simplicity.

The two cases so far expounded look formally different; yet we note that they are strictly correlated to each other, because the expression (54) for the supplemental function σ_S is actually the one bringing to equivalence the form (7) and (51) (after quantization).

3.2.3 The single-particle Newtonian case (j = 0)

In the SPND case, solutions to equation (7) governed by the ansatz (15) (with λ different from 0) exist but they require setting up involved expressions for a and α, moreover implying some dependence on the m_{eff} parameter. In the case at hand here, we consider not necessary to keep $a = 1/2$ or $\alpha = 1$ ([8]), because the SPND context is quite different from the OQMC case. Amongst other possibilities, we give here e.g. the parameters corresponding to a solution with

$$\sigma_2(\mathrm{x}) = \lambda R' = \sqrt{\delta \mathrm{m}/\mathrm{m}}\mathrm{v}(\mathrm{x}) \tag{55}$$

δm= m$-$m$_{eff}$ is the HDF-induced mass defect in our physical theory.

Parameter values are given by the following equations (still referred to the formalism deployed in equations (15)÷(23), but remind a $\lambda-$value different from zero now holds):

$$A = \frac{1}{2}(1 - 4\frac{\zeta^2}{\alpha^2}) \tag{56}$$

$$\frac{\zeta}{2} = -\frac{\sqrt{\eta}}{2} = -1 \tag{57}$$

$$a = \frac{1}{8} - \sqrt{\frac{1}{64} + \frac{5+\gamma}{8\gamma}} \tag{58}$$

[8] In the following section, however, we will see that an ansatz different from equation (8) will allow us to treat this case by the means of the standard vakues $a = 1/2$ and $\alpha = 1$.

$$\alpha^2 = \frac{16}{1+\gamma-2a\gamma} \tag{59}$$

Unfortunately, we have here a 5th order equation for γ:

$$(5-7\gamma+2\gamma^2)^2 + \frac{\mathrm{m}_{eff}}{\mathrm{m}}(-50+140\gamma-\frac{301}{2}\gamma^2+\frac{67}{2}\gamma^3+18\gamma^4+9\gamma^5)+$$

$$+\left(\frac{\mathrm{m}_{eff}}{\mathrm{m}}\right)^2(25-70\gamma+\frac{163}{2}\gamma^2-\frac{11}{2}\gamma^3-\frac{327}{16}\gamma^4-9\gamma^5)=0 \tag{60}$$

Conclusively equations (47)/(48) can be recovered by the means of the previous positions, from which we find also

$$\sigma_1(\mathrm{x}) = \frac{\alpha\hbar}{16}\left[-3\gamma-\sqrt{9\gamma^2+40\gamma}\right]\frac{\rho'(\mathrm{x})}{\rho(\mathrm{x})} \tag{61}$$

As remarked already, the previous expressions look quite involved, and make physical interpretation of the different parameters very hard (although a plane case is provided by $\mathrm{m}_{eff}=0$, implying $\gamma = 5/2$ and a =1/2). Therefore, we are brought to ask if assuming as a start Hamiltonian the one in (51), or searching for some other function σ would not bring us to easier circumstances, just alike we have found when operating the DBOE case. The answer to the question is affirmative but, for easier mathematical handling, we found that the best is giving up equation (8) and assuming a generalized form for our wavefunctions Ψ_n. The subject is discussed in the next section, where a final "optimized" formalism is introduced.

3.2.4 Remarks and discussion

A final remark concerning the present subsection, where $\sigma \neq 0$ cases have been considered, has however already been advanced: the σ expressions we have found are not "purely classical" ones because they include ħ-dependent terms. The potential $\Phi_S(\mathrm{x})$ itself (52) we have introduced in the start Hamiltonian (51) - is a quantum-like term. We can ask therefore whether it is possible to find a formalism based on quite classical-like expressions, both for σ and H_B(p,x). This point will again be met with in the sequel.

4 Optimized formalism (case C)

By the previous analyses, we have got results which we can reelaborate in such a way to set up, in this section, a compact "a priori" procedure able to describe the SPND and DBOE^{II} cases. Some of the definitions having to be renormalized, we will introduce the quantization procedure since the beginning again.

Before this, however, we have to discuss briefly the interpretative role we have given - up to now - to Ψ_n(x), because some changes will apply in the following. The wavefunction defined by equation (8) has a formal role very similar to the one drawn by the Copenhagen interpretation: the $\Psi_n(\mathrm{x})$ squared-modulus is, all across the previous sections, always equal to the statistical probability to find a particle in some small region of space around the co-ordinate x. Yet the physical sense of this statistics, in our SPND and DBOE frameworks, is a classical one: in our papers ([1]÷[4]) the basic interpretation is indeed that the densities are just what classically comes out from statistical ensembles of Eulerian velocity fields and not "quantum probabilistic" densities.

Now however we will see that procedures in this section will be based on a different definition of the "wavefunction".

We start with the following classical-like, Bernoulli Hamiltonian

$$\mathrm{H}_B(\mathrm{p,x}) = \frac{1}{2\mathrm{m}}\,\mathrm{p}^2 + \Phi(\mathrm{x}) + \frac{i}{\mathrm{m}}[\mathrm{p},\, \sigma(\mathrm{x})] = \mathrm{E} \tag{62}$$

Whenever p is chosen as a classical quantity, the commutator $[\mathrm{p},\, \sigma(\mathrm{x})]$ is 0 (then we might operate the function with the same techniques expounded in [5]). What is however relevant to us here is "quantizing" this Hamiltonian: this we do by means of the (orthodox, $\alpha = 1$) ansatz

$$p \to \hat{p} \equiv -i\hbar \frac{d}{d\mathrm{x}} \tag{63}$$

So the Hamilton operator and the energy theorem write

$$\hat{\mathrm{H}}_B(\hat{\mathrm{p}},\, \mathrm{x},\, a) \equiv \frac{1}{2\mathrm{m}}\,\hat{\mathrm{p}}^2 + \Phi(\mathrm{x}) + \frac{i}{\mathrm{m}}\,[\hat{\mathrm{p}}\sigma(\mathrm{x}) - \sigma(\mathrm{x})\hat{\mathrm{p}}] \tag{64}$$

$$\hat{\mathrm{H}}_B(\hat{\mathrm{p}},\, \mathrm{x},\, a)\Psi_n = \mathrm{E}_n\Psi_n \tag{65}$$

We assume now the following (non-standard) form for the solution $\Psi_n(\mathrm{x})$:

$$\Psi_n(\mathrm{x}) \equiv \Psi(\mathrm{x}) \equiv \pm\rho^{\zeta}\, \exp\left(\frac{i}{\hbar}S^*\right) \tag{66}$$

In this equation, the phase function $S^*/\hbar$ is defined as follows:

$$\nabla S^* = \sqrt{\mathrm{m}\mathrm{m}^{(\zeta)}_{eff}}\,\mathrm{v}_\zeta(\mathrm{x}) \tag{67}$$

Here we have introduced the quantities $\mathrm{m}^{(\zeta)}_{eff}$ and $\mathrm{v}_\zeta(\mathrm{x})$ which clearly stand for the sets: $\{\mathrm{m}_{eff}$ and $\mathrm{v}(\mathrm{x})\}$ when $\zeta = -2$, SPND case, and $\{\mathrm{m}^{D}_{eff}$ and $\mathrm{v}_{D}(\mathrm{x})\}$ when $\zeta = 1/2$, DBOE^{II} case ([9]). We define moreover the function $\sigma(\mathrm{x})$ by the very simple position

$$\sigma = i\frac{5}{2}\nabla S^* \tag{68}$$

[9] We remind for physical insight that the velocity field $\mathrm{v}_\zeta(\mathrm{x})$ interpretates as the drift velocity of a gaseous particle system in the DBOE case, and is a single particle velocity instead in the SPND context.

With the previous assumptions, it is easy to show, by simple calculations, that equation (65) reduces to the following equations:

$$\frac{1}{2}\mathrm{m}_{eff}^{(\zeta)}\mathrm{v}_{\zeta}^{2}(\mathrm{x}) - \frac{\hbar^2}{2\mathrm{m}}\frac{\mathrm{v}_{\zeta}^{2}(\mathrm{x})''}{\mathrm{v}_{\zeta}^{2}(\mathrm{x})} \;=\; \mathrm{E}_n - \Phi(\mathrm{x}) \tag{69}$$

$$\rho \mathrm{v}_{\zeta}(\mathrm{x}) = const \times \rho^{\frac{\zeta}{2}+1} \tag{70}$$

This last represents the continuity equation or the mass-flow law corresponding to the two contexts at hand.

As anticipated, this procedure seems to us the most interesting among the ones we have analyzed in this paper - with concern to both physical and mathematical compactness and interpretability. We give a conclusive discussion of all our results, including those in reference [5], in the next section.

5 Discussion of the results

As remarked, the definition in equation (8) is congruent with the "orthodox Copenhagen interpretation" ($|\Psi_n(\mathrm{x})|^2 \equiv \rho(\mathrm{x})$), while equation (66) is now a different definition. Using equation (70) in the $\Psi(\mathrm{x})$ expression (66) we find for the modulus of $\Psi(\mathrm{x})$ the expression

$$|\Psi(\mathrm{x})| = \rho^{\zeta} \propto \mathrm{v}_{\zeta}(\mathrm{x})^2 \tag{71}$$

This equation identifies with the Copenhagen interpretation only in the DBOE^{II} case - although just on a mathematical level, due to our expounded classical-like interpretation. By the sake of an unified interpretation of the two cases at hand here, we see by equation (71) that the present formalism brings us to a different statement: the $\Psi(\mathrm{x})$ modulus defines, in both cases, the kinetic energy field $\mathrm{v}_{\zeta}(\mathrm{x})^2$. It is interesting that, since however in the DBOE^{II} case $\mathrm{v}_{D}(\mathrm{x})^4 \propto \rho$, the Copenhagen information postulate on the density is not completely lost if we assume this alternative definition. In any case, our "new" wavefunction always encloses the density information via its absolute value ρ^{ζ}.

This circumstance calls for the following remark. We have seen in previous sections that the Bernoulli oscillators physics can actually be described by the quantum procedures, applied to a Bernoulli Hamiltonian, while resting on equation (8) which is indeed consistent with the Copenhagen interpretation. But the mathematical path has revealed quite a stiff one, and the coefficients we have met with along the procedure are of difficult physical interpretation (see f. i. equations (43)÷(46) and (55)÷(61). Basing the procedure on the Hamiltonian form (62) on the ansatz (66), instead, is a choice providing to the formalism simplicity and compactness.

As an instance of it, we want to give evidence here to the fact that the ("alternative") quantum momentum field takes the expression

$$\frac{\hat{\mathrm{p}}\Psi(\mathrm{x})}{\Psi(\mathrm{x})} \equiv -i\hbar\frac{\Psi'(\mathrm{x})}{\Psi(\mathrm{x})} = \sqrt{\mathrm{m}\mathrm{m}_{eff}^{(\zeta)}}\mathrm{v}_{\zeta}(\mathrm{x}) - i\hbar\zeta\frac{\rho'}{\rho} \equiv \mathrm{p}_n \tag{72}$$

This one is just the expression we have found in paper I - equation (78) - by the classical-like procedure already.

The formalism here proposed also shows plane adherence to a property we have invoked before: the momentum (68) being defined as a classical-like quantity, the starting Hamiltonian (62) is a "canonical", purely classical one; by quantization, we have therefore been able to take $\hbar$-dependent effects out of the original PHME context.

Note finally that the procedure also provides us with an interesting correlation between the momentum (68) and the potential $i\Phi_S$ we have met with already in reference [5]. For physical insight, we remark that the coefficient 5/2 can be interpreted in terms of the quantities ζ and ϵ, because it can be written

$$\frac{5}{2} = 1 + \frac{2\zeta}{\epsilon} \tag{73}$$

in both the SPND and DBOE cases. As an interesting corollary, we see that if the OQMC parameters set is instead considered ($\zeta = 1/2$, $\epsilon = -1$), the coefficient becomes zero, the Hamiltonian (62) reduces to the standard classical one, and the quantization procedure goes back again to the standard quantum-mechanical case.

This final setting in the procedure also seems able to make easier the extension when the assumption of constant effective masses $\mathrm{m}_{eff}^{(\zeta)}$ and other coefficients, done in these papers, will be dropped off (what will reinforce our physical models). In this context, indeed, we will be brought to consider variable functions $\zeta(x)$ and $\epsilon(x)$ to express the continuity equation: but the coefficient (73) just candidates to be kept invariant within this (next) generalized description.

6 Conclusion

We provided, in this paper, different quantum-like procedures able to fit the physical contexts at hand. The procedures are certainly non-orthodox in a number of details: they apply to peculiar ("Bernoulli") Hamiltonian functions, and use non-standard positions (for the quantities α, a etc.) and interpretations for the wavefunction formalism. In a number of cases, we discussed the properties and the consequences of these positions. In a few of them, we found rather involved expressions for the relevant parameters. Then to have a more plane procedure we turned, at the end of the paper, to a different expression of the Hamiltonian function and to a different, non-orthodox definition of the basic "wavefunction". In this way we believe having provided a rather compact and easy reference framework, both from the mathematical and physical points of view. We displayed the advantages of our formalism, and enlightened some properties more of our proposed physics of the Bernoulli oscillators.

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