A stress-based approach to the solution of Saint Venant problem

Ph.D. Dissertation

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Chapter 1

Introduction

The solution to Saint-Venant problem, concerning the elastic equilibrium problem of beams, is classically addressed by the displacement approach. Still to date, more than one hundred and fifty years since the publication of the two fundamental papers by A.J.C. Barré de Saint Venant [4, 5], the solution of this classical problem is derived by adopting the celebrated semi-inverse approach pioneered by the french mathematician, and is classically taught in the form contributed by Clebsch [17].

At a first sight the displacement approach appears to be straightforward and effective, basically for two reasons: the Cauchy-Navier equations, governing the elastic equilibrium problem of solids in which the displacement field is the primary unknown, are easier to derive and by far simpler with respect to the corresponding ones, due to Beltrami and Michell [9, 40], which formulate the elastic equilibrium problem in terms of the stress field.

Moreover, a solution expressed in terms of stresses would require an integration procedure to derive the corresponding displacement field, as opposite to the straightforward differentiation procedure which allows one to derive the stress field once the displacement field has been assigned.

Only one hundred years after Saint-Venant papers Riccardo Baldacci [6], professor of Scienza delle Costruzioni (the italian acronym for Structural and Solid Mechanics) at the University of Genova, presented a stress-based formulation of the elastic equilibrium problem of beams which turn out to be particularly compact and elegant.

In the author's opinion the main reason which makes Baldacci treatment preferable to the classical one is that there is no need to figure out the displacement field and to check a posteriori its correctness according to Saint-Venant semi-inverse method.

Conversely, the stress field solution of de Saint-Venant problem is directly and consistently derived from Beltrami-Michell equation without invoking any a-priori assumption apart from the classical hypotheses which characterize Saint-Venant model of beams.

The price to pay for using the approach pioneered by Baldacci is that the displacement field solution of Saint-Venant problem is not easy to derive since it requires to integrate the strain field associated with stress field via the linear isotropic elastic law.

However, it is shown in the dissertation that this issue, not addressed by Baldacci, can be solved elegantly and in full generality by an integration procedure, basically equivalent to Cesaro's formula, which can be applied indifferently, though with a different degree of complexity, to any kind of internal force, that is axial force, torsion, biaxial bending and biaxial shear.

In order to achieve this result the original treatment of Saint-Venant problem due to Baldacci is reformulated more synthetically by an extensive use of tensor calculus.

In this respect use is made of an original derivation of Beltrami-Michell equation which can be defined of algebraic nature in the sense that the final result is obtained by means of basic operations of tensor calculus which formally replace the differential manipulations usually exploited in classical textbooks of continuum mechanics.

Basically these manipulations are of two kinds: the first one, mainly coincident with that initially formulated by Beltrami [9] and later generalized by Michell [40], hinges on the systematic use of the indicial notation and, in particular, of the $\epsilon - \delta$ identities connecting Ricci tensor to the identity one. The second approach [29] to the derivation of Beltrami-Michell equations basically reformulates the above-mentioned identities in intrinsic form by means of tensor identities which are very elegant but completely hinder the meaning and the consequentiality of the several not-trivial steps required to achieve the final result.

It is shown, on the contrary, that the approach illustrated in this dissertation makes the whole procedure transparent though formally involved.

Basically, it is based on the extensive use of Gibbs calculus to formally address gradient, divergence, curl and Laplace operator of vector or tensor fields by applying the rules of tensor calculus to the field and to a fictitious vector known as the $\boldsymbol{\nabla}$ (nabla) operator by Hamilton.

A suitable extension of classical Gibbs calculus is presented in the dissertation by introducing an original definition of vector product between vectors and tensors which is required to derive more elaborate results.

Following this approach the compatibility conditions for the infinitesimal strain field as well as Bianchi identities can be derived elegantly and in a straightforward manner. Moreover, invoking Rivlin identities [46], Beltrami-Michell equations are obtained by a constructive approach in which any step of the procedure follows naturally and consequentially from the previous one.

Accordingly, the dissertation is basically divided in two main parts. The first one includes the above mentioned issues on differential calculus while the second one is completely devoted to the tensor reformulation of Baldacci approach to Saint-Venant problem and to the presentation of some original results. The first one, as previously specified, is concerned with the explicit derivation, in tensor form, of the displacement field, separately for axial force, torsion, biaxial bending and biaxial shear. Since the reference frame is completely arbitrary no a priori use of the inertia principal directions is required for addressing bending and shear.

The second original result presented in the dissertation is represented by a frame-independent solution of the shear problem which, to the best of the author's knowledge, has not yet been proposed. For instance, the matrix field used by Baldacci to implicitly define the shear stress solution does not constitute the matrix representation of a tensor field. The same drawback holds as well for the shear center in the sense that, differently from the centroid, it is not yet available a frame-independent expression of the shear center; the same comment applies also to the shear flexibility tensor and to the shear factor tensor.

In the light of the considerations pointed out above, a solution method for the determination of the shear stress field, alternative to the treatment by Baldacci [6], is illustrated in the dissertation thus allowing to represent the stress field, as well as the displacement one, in a completely intrinsic form.

The proposed formulation of the shear problem is based on the solution of the Neumann problem associated with the shear stress field which is obtained by exploiting an intrinsic particular integral of the differential problem emerging from Baldacci's approach to Saint-Venant problem. In this way the representation obtained for the shear stress field, for the shear center and for the shear flexibility tensor presents the advantage of being independent from the particular reference frame and of being written in intrinsic, and hence more synthetic, form.

As a final contribution the numerical solution of shear and torsion problem is carried out by a novel BEM approach in which only the vertices of the cross section, assumed to be polygonal, need to be assigned. In this way the input data required for analyzing the cross section subject to any kind of internal force are identical to those traditionally employed for axial force and biaxial bending.

Chapter 2

Beltrami-Michell equation

In this chapter it will be shown that the basic set of differential relations of the stress-based formulation of linear isotropic elastostatics can be derived by a constructive approach based upon an algebraic path of reasoning. The result will be obtained by extending Gibbs symbolic calculus to tensor fields and introducing an original definition of vector product between vectors and second-order tensors. In particular, an algebraic reformulation of the compatibility condition for the linearized strain tensor, made possible by the exploitation of Rivlin's identities for tensor polynomials, allow one to derive Beltrami-Michell equation by a direct approach. The same considerations do apply as well to the Saint Venant compatibility condition and to Bianchi identity.

2.1 Background

Given a three- or two-dimensional inner product space V over the reals, one denotes by Lin the space of linear transformation (second-order tensors) on V , and by $\mathbb{L}$in the space of all tensors on Lin. Unless differently stated, elements of V and Lin will be denoted respectively by lowercase, e.g. $\mathbf{a}$, and uppercase, e.g. $\mathbf{A}$, bold symbols; furthermore fourth-order tensors, which represent the elements of Lin, will be denoted by boldblackboard uppercase symbols, e.g. $\mathbb{A}$.

Well-known composition rules involving vectors and tensors are:

$$\mathbf{Ab} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{A}^t\mathbf{c} = \mathbf{A} \cdot (\mathbf{c} \otimes \mathbf{b}) \tag{2.1}$$

where $(\cdot)^t$ stands for transpose, and:

$$\mathbf{A}(\mathbf{b} \otimes \mathbf{c}) = \mathbf{Ab} \otimes \mathbf{c} \qquad (\mathbf{b} \otimes \mathbf{c})\mathbf{A} = \mathbf{b} \otimes \mathbf{A}^t\mathbf{c} \tag{2.2}$$

see, e.g., [29].

This chapter introduces in particular the definition of vector product between vectors and second-order tensors and presents some of its properties; in addition some properties of tensor products between second-order tensors and basic results of differential calculus for tensors valued functions of tensors will be briefly reviewed.

2.1.1 Vector product of vectors and tensors

It is well-known [14] that there exists a one-to-one correspondence between vectors and skew-simmetric tensors which is expressed by:

$$\mathbf{a} \times \mathbf{b} = \mathbf{W}_\mathbf{a}\mathbf{b} \qquad \forall\, \mathbf{b} \in V. \tag{2.3}$$

Assuming that the argument $\mathbf{b}$ in the previous relationship represents the result of a linear transformation $\mathbf{T} : V \rightarrow V$ one has:

$$\mathbf{a} \times (\mathbf{Sc}) = \mathbf{W}_\mathbf{a}(\mathbf{Sc}) = (\mathbf{W}_\mathbf{a}\mathbf{T})\mathbf{c} \qquad \forall\, \mathbf{c} \in V. \tag{2.4}$$

Since the left-hand side of the previous relation is linear in $\mathbf{c}$, we can define the vector product between a vector and a tensor as the linear operator $\mathbf{a} \times \mathbf{T}$ fulfilling the property:

$$(\mathbf{a} \times \mathbf{T})\mathbf{c} = \mathbf{a} \times \mathbf{Tc} = (\mathbf{W}_\mathbf{a}\mathbf{T})\mathbf{c} \qquad \forall\, \mathbf{c} \in V. \tag{2.5}$$

and we write:

$$\mathbf{a} \times \mathbf{T} \stackrel{def}{=} \mathbf{W_a T} \tag{2.6}$$

It is interesting to notice that the i-th column of the matrix associated with $\mathbf{a} \times \mathbf{T}$ in a cartesian frame is simply the vector product between $\mathbf{a}$ and the i-th column of the matrix associated with $\mathbf{T}$. Actually, recalling the composition rule (2.2), one has:

$$\mathbf{a} \times \mathbf{T} = \mathbf{a} \times \sum_{i=1}^{3} \mathbf{T}\mathbf{e}_i \otimes \mathbf{e}_i = \sum_{i=1}^{3} (\mathbf{W_a T e}_i) \otimes \mathbf{e}_i = \sum_{i=1}^{3} (\mathbf{a} \times \mathbf{T}\mathbf{e}_i) \otimes \mathbf{e}_i \tag{2.7}$$

This is exactly the definition reported, e.g., in [39].

Using the definition (2.6) one obtains an alternative way of expressing the relation between a skew tensor and the associated axial vector:

$$\mathbf{W_a} = \mathbf{a} \times \mathbf{I}. \tag{2.8}$$

We shall also denote by *axial* the linear operator which associates with every skew tensor $\mathbf{W_a}$ the relevant axial vector $\mathbf{a}$, that is:

$$\mathrm{axial}\mathbf{W_a} = \mathbf{a} \tag{2.9}$$

in particular, it turns out to be:

$$\mathrm{axial}\mathbf{W}_\mathbf{a}^t = -\mathrm{axial}\mathbf{W_a} \tag{2.10}$$

Some additional properties stemming from the definition (2.6) are

$$\mathbf{a} \times (\mathbf{b} \otimes \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \otimes \mathbf{c} \tag{2.11}$$

an identity addresses in [2], and

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{T})\mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{T}\mathbf{c}) \tag{2.12}$$

Furthermore, using the vector identity:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \tag{2.13}$$

one obtains:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{T}) = (\mathbf{b} \otimes \mathbf{a})\mathbf{S} - (\mathbf{a} \cdot \mathbf{b})\mathbf{S} \tag{2.14}$$

and its trivial specializations:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{I}) = \mathbf{W_a W_b} = (\mathbf{b} \otimes \mathbf{a}) - (\mathbf{a} \cdot \mathbf{b})\mathbf{I} \tag{2.15}$$

$$\mathbf{a} \times (\mathbf{a} \times \mathbf{I}) = \mathbf{W_a^2} = (\mathbf{a} \otimes \mathbf{a}) - (\mathbf{a} \cdot \mathbf{a})\mathbf{I} \tag{2.16}$$

Exploiting the anticommutativity property of the vector product one gets from (2.13) the further identity:

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{I} = \mathbf{b} \otimes \mathbf{a} - \mathbf{a} \otimes \mathbf{b} \tag{2.17}$$

according to (2.8) the previous relation states that:

$$\mathbf{a} \times \mathbf{b} = -\mathrm{axial}(\mathbf{a} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{a}) \tag{2.18}$$

Starting from the interpretation of vector product between vectors and tensors provided in (2.7) a further product can be defined:

$$\mathbf{T} \times \mathbf{a} \overset{def}{=} -\mathbf{a} \times \mathbf{T} \tag{2.19}$$

this amounts to set:

$$(\mathbf{S} \times \mathbf{a})\mathbf{c} = \mathbf{Sc} \times \mathbf{a} \tag{2.20}$$

since:

$$\mathbf{Tc} \times \mathbf{a} = -\mathbf{a} \times \mathbf{Tc} = -(\mathbf{a} \times \mathbf{T})\mathbf{c} \tag{2.21}$$

It will be shown in the next sections that the previous formulas, of purely algebraic nature, are particularly useful to provide a simple derivation of differential identities.

2.1.2 Tensor product of second-order tensors

Given $\mathbf{A}, \mathbf{B} \in \mathrm{Lin}$ the tensor product $\mathbf{A} \otimes \mathbf{B}$, usually termed dyadic product, is the element of Lin such that:

$$(\mathbf{A} \otimes \mathbf{B})\mathbf{C} = (\mathbf{B} \cdot \mathbf{C})\mathbf{A} = \mathrm{tr}\,(\mathbf{B}^t\mathbf{C})\mathbf{A} \qquad \forall\, \mathbf{C} \in \mathrm{Lin} \tag{2.22}$$

where the symbol $\mathrm{tr}(\cdot)$ denotes the trace operator.

More recently Del Piero [19] has introduced an additional tensor product $\mathbf{A} \boxtimes \mathbf{B}$ between second-order tensors defined by:

$$(\mathbf{A} \boxtimes \mathbf{B})\mathbf{C} = \mathbf{A}\mathbf{C}\mathbf{B}^t \qquad \forall\, \mathbf{C} \in \mathrm{Lin} \tag{2.23}$$

which will be referred to in the sequel as square tensor product. The previous product allows one to represent the identity tensor $\mathbb{I} \in \mathbb{L}\mathrm{in}$ as:

$$\mathbb{I} = \mathbf{I} \boxtimes \mathbf{I} \tag{2.24}$$

where $\mathbf{I}$ is the identity tensor in Lin. The following composition rules can be shown to hold:

$$\begin{aligned} (\mathbf{A} \boxtimes \mathbf{B})(\mathbf{C} \boxtimes \mathbf{D}) &= (\mathbf{A}\mathbf{C}) \boxtimes (\mathbf{B}\mathbf{D}) \\ (\mathbf{A} \boxtimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) &= (\mathbf{A}\mathbf{C}\mathbf{B}^t) \otimes \mathbf{D} \\ (\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \boxtimes \mathbf{D}) &= \mathbf{A} \otimes (\mathbf{C}^t\mathbf{B}\mathbf{D}) \end{aligned} \tag{2.25}$$

for every $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathrm{Lin}$

2.1.3 Derivatives

Let $\mathbf{G} : \mathrm{Lin} \to \mathrm{Lin}$ be a tensor valued function of tensors. $\mathbf{G}$ is said to be differentiable at $\mathbf{A}$ if there exists a linear transformation $D_{\mathbf{A}}\mathbf{G}(\mathbf{A})$, called the derivative of $\mathbf{G}$ at $\mathbf{A}$, such that:

$$\mathbf{G}(\mathbf{A} + \mathbf{B}) - \mathbf{G}(\mathbf{A}) = D_{\mathbf{A}}\mathbf{G}(\mathbf{A})[\mathbf{B}] + o(\mathbf{B}) \quad \text{as}\, \mathbf{B} \to \mathbf{0}; \tag{2.26}$$

$D_{\mathbf{A}}\mathbf{G}(\mathbf{A})[\mathbf{B}]$ represents the value of the derivative $D_{\mathbf{A}}\mathbf{G}(\mathbf{A})$ on the increment $\mathbf{B}$. If two tensor valued functions $\mathbf{G}$ and $\mathbf{K}$ are differentiable at $\mathbf{A}$, their product:

$$\mathbf{P}(\mathbf{A}) = \mathbf{G}(\mathbf{A})\mathbf{K}(\mathbf{A}) \tag{2.27}$$

is also differentiable at $\mathbf{A}$ and:

$$D_{\mathbf{A}}\mathbf{P}(\mathbf{A})[\mathbf{B}] = D_{\mathbf{A}}\mathbf{G}(\mathbf{A})[\mathbf{B}]\mathbf{K}(\mathbf{A}) + \mathbf{G}(\mathbf{A})D_{\mathbf{A}}\mathbf{K}(\mathbf{A})[\mathbf{B}] \tag{2.28}$$

Applying the definition (2.28) it is an easy matter to derive the following derivatives of the powers of $\mathbf{A}$:

$$\begin{aligned}
D_{\mathbf{A}}(\mathbf{A}) &= \mathbb{I} = \mathbf{I} \boxtimes \mathbf{I} \\
D_{\mathbf{A}}(\mathbf{A}^2) &= \mathbf{A} \boxtimes \mathbf{I} + \mathbf{I} \boxtimes \mathbf{A}^t \\
D_{\mathbf{A}}(\mathbf{A}^3) &= \mathbf{A}^2 \boxtimes \mathbf{I} + \mathbf{I} \boxtimes (\mathbf{A}^2)^t + \mathbf{A} \boxtimes \mathbf{A}^t \\
D_{\mathbf{A}}(\mathbf{A}^4) &= \mathbf{A}^3 \boxtimes \mathbf{I} + \mathbf{I} \boxtimes (\mathbf{A}^3)^t + \mathbf{A}^2 \boxtimes \mathbf{A}^t + \mathbf{A} \boxtimes (\mathbf{A}^2)^t
\end{aligned} \tag{2.29}$$

which share an amazing symmetry in their expressions and are very easy to remember. We shall also need the derivatives of the three invariants of a tensor:

$$I_{\mathbf{A}} = \operatorname{tr}\mathbf{A} \qquad II_{\mathbf{A}} = \frac{1}{2}[(\operatorname{tr}\mathbf{A})^2 - (\operatorname{tr}\mathbf{A}^2)] \qquad III_{\mathbf{A}} = \det\mathbf{A}$$

they are given in turn [59] by:

$$D_{\mathbf{A}}(I_{\mathbf{A}}) = \mathbf{I} \qquad D_{\mathbf{A}}(II_{\mathbf{A}}) = I_{\mathbf{A}}\mathbf{I} - \mathbf{A}^t \qquad D_{\mathbf{A}}(III_{\mathbf{A}}) = \mathbf{A}^* \tag{2.30}$$

where the tensor $\mathbf{A}^*$, which fulfills the properties:

$$\mathbf{A}^*\mathbf{A}^t = \mathbf{A}^t\mathbf{A}^* = III_{\mathbf{A}}\mathbf{I}, \tag{2.31}$$

represents the cofactor of $\mathbf{A}$.

2.2 The pseudo-vectorial operator ∇ and symbolic differential calculus

It is well-known [13, 26, 32, 38] that Gibbs notation [24, 25] allows one to express the most common differential operators by means of the pseudo-vectorial operator $\boldsymbol{\nabla}$ defined in a Cartesian frame by the expression:

$$\boldsymbol{\nabla} = \frac{\partial}{\partial x}\mathbf{e}_1 + \frac{\partial}{\partial y}\mathbf{e}_2 + \frac{\partial}{\partial z}\mathbf{e}_3$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are unit vectors directed along the axes.

Introduced by Hamilton in [30], although with a different symbol, and usually termed *nabla* operator, definition and properties of the $\boldsymbol{\nabla}$ operator have been recently extended to orthogonal curvilinear coordinates in [43].

Mainly after Gibbs [24, 25] it has become a common practice to denote the *gradient*, *divergence*, *curl* and *laplacian* respectively by the symbols $\boldsymbol{\nabla}$, $\boldsymbol{\nabla}\cdot$, $\boldsymbol{\nabla}\times$ and $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$.

Although the properties and limitations of $\boldsymbol{\nabla}$ symbolic calculus are well established, it will be reported hereafter, mainly for completeness, a comparative presentation of Gibbs terminology and coordinate-free notation of differential operators as well as some additional properties of symbolic calculus resulting from the newly introduced definition of vector product between vectors and tensors.

Referring to [29, 28, 21] for intrinsic definition of differential operators, the symbolic counterpart for gradient, divergence and curl of (sufficiently smooth) scalar, e.g. φ, and vector, e.g. $\mathbf{a}$, fields are given by:

$$\begin{array}{ll} \varphi\boldsymbol{\nabla} \overset{def}{=} \boldsymbol{\nabla}\varphi \overset{def}{=} \mathrm{grad}\varphi & \mathbf{a}\otimes\boldsymbol{\nabla} \overset{def}{=} \mathrm{grad}\,\mathbf{a} \\ \boldsymbol{\nabla}\cdot\mathbf{a} \overset{def}{=} \mathbf{a}\cdot\boldsymbol{\nabla} \overset{def}{=} \mathrm{div}\,\mathbf{a} & \boldsymbol{\nabla}\times\mathbf{a} \overset{def}{=} \mathrm{curl}\,\mathbf{a} \end{array} \tag{2.32}$$

see also [3].

The extension of the previous definitions to the case of second-order tensor fields $\mathbf{A}$ can be made on the basis of the intrinsic expressions reported in [29] and of the definition of vector product between vectors and second-order tensors:

$$\mathbf{A}\boldsymbol{\nabla} \overset{def}{=} \mathrm{div}\mathbf{A} \qquad \boldsymbol{\nabla}\times\mathbf{A}^t \overset{def}{=} \mathbf{W}_{\boldsymbol{\nabla}}\mathbf{A}^t \overset{def}{=} \mathrm{curl}\mathbf{A} \tag{2.33}$$

accordingly, we define:

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}^t)^t \stackrel{def}{=} \mathbf{W}_{\boldsymbol{\nabla}}(\mathbf{W}_{\boldsymbol{\nabla}}\mathbf{A}^t)^t \stackrel{def}{=} (\mathbf{W}_{\boldsymbol{\nabla}} \boxtimes \mathbf{W}_{\boldsymbol{\nabla}})\mathbf{A} \stackrel{def}{=} \mathrm{curl}\,\mathrm{curl}\mathbf{A} \tag{2.34}$$

Finally, the laplacian Δ of a scalar, vector or tensor field is defined by:

$$\begin{aligned} &(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\varphi \stackrel{def}{=} \mathrm{div}\,\mathrm{grad}\,\varphi \stackrel{def}{=} \Delta\varphi \qquad (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\mathbf{a} \stackrel{def}{=} \mathrm{div}\,\mathrm{grad}\,\mathbf{a} \stackrel{def}{=} \Delta\mathbf{a} \\ &(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\mathbf{A} \stackrel{def}{=} \mathrm{div}\,\mathrm{grad}\,\mathbf{A} \stackrel{def}{=} \Delta\mathbf{A} \end{aligned} \tag{2.35}$$

It is interesting to notice that $(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\varphi$ can be formally obtained by applying Gibbs definition of divergence to the pseudo-vector $\boldsymbol{\nabla}\varphi$ and considering the scalar product of the two $\boldsymbol{\nabla}$' s involved in the operation. Analogously, one gets:

$$\Delta\mathbf{a} = \mathrm{div}\,\mathrm{grad}\mathbf{a} = (\mathbf{a} \otimes \boldsymbol{\nabla})\boldsymbol{\nabla} \tag{2.36}$$

by invoking the definition of tensor product between vectors [14, 29] on the basis of $(2.32)_2$ and $(2.33)_1$. A similar path of reasoning can be followed for $\Delta\mathbf{A}$ by defining:

$$\mathbf{A} \otimes \boldsymbol{\nabla} \stackrel{def}{=} \mathrm{grad}\,\mathbf{A} \tag{2.37}$$

as natural extension of $(2.32)_2$.

Furthermore, by applying basic rules of vector calculus to the symbolic vector $\boldsymbol{\nabla}$, the following identities can be shown to hold:

$$\begin{aligned} &\mathbf{a} \times \boldsymbol{\nabla} = -\boldsymbol{\nabla} \times \mathbf{a} \\ &\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{a}) = \boldsymbol{\nabla} \cdot (\mathbf{a} \times \boldsymbol{\nabla}) = \mathbf{a} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\nabla}) = 0 \\ &\boldsymbol{\nabla} \times (\mathbf{a} \otimes \boldsymbol{\nabla}) = (\boldsymbol{\nabla} \times \mathbf{a}) \otimes \boldsymbol{\nabla} \\ &\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \otimes \mathbf{a}) = (\boldsymbol{\nabla} \times \boldsymbol{\nabla}) \otimes \mathbf{a} = \mathbf{0} \end{aligned} \tag{2.38}$$

where $\mathbf{a}$ is an arbitrary vector field. Specifically, the first one is the anticommutativity property of the vector product while the second one follows from a well-known property of the mixed triple product; furthermore, the last two relationships are based upon (2.11), the second one being zero since the vector product of two identical vectors vanishes.

As a consequence of the previous definitions, all standard vectorial identities can be expressed symbolically in terms of $\boldsymbol{\nabla}$. However, in order to prove such identities, special care has to be paid when applying the $\boldsymbol{\nabla}$ operator to the product of two fields since this definitively amounts to formally applying either the product or the chain rule of differential calculus.

In this respect we quote the illuminating sentence reported at page 77 of the second volume collecting the scientific papers by Gibbs [25]:

Gibbs rule - *... The principle in all these cases (i.e. composition of scalar and vector fields) is that if we have one of the operators $\boldsymbol{\nabla}$, $\boldsymbol{\nabla}\cdot$, $\boldsymbol{\nabla}\times$ prefixed to a product of any kind, and we make any transformation of the expression which would be allowable if the $\boldsymbol{\nabla}$ were a vector (viz. by changes in the order of the factors, in the signs of multiplication, in the parentheses written or implied, etc.) by which changes the $\boldsymbol{\nabla}$ is brought into connection with one particular factor, the expression thus transformed will represent the part of the value of the original expression which result from the variation of that factor.*

In order to extend the previous rule to tensor fields, in which the symbolic operator $\otimes\boldsymbol{\nabla}$ basically comes into play, we postulate the following:

Extended Gibbs rule - *The composition of scalar, vector or tensor fields postfixed by the operator $\otimes\boldsymbol{\nabla}$ is carried out by complying with Gibbs rule and bringing each particular factor into direct connection with $\otimes\boldsymbol{\nabla}$ provided that the resulting expression, representing the part of the original formula which result from the variation of that factor, makes sense.*

A practical application of the previous rules, with special emphasis on the case of tensor fields, is provided in the Appendix A.

Finally it is reported a reformulation, expressed in terms of Gibbs notation, of two classical theorems on solenoidal vector fields and irrotational tensor fields. The reader is referred to [29, 21] for analytical details and the traditional proof.

Theorem on Solenoidal Vector Fields *Let* $\mathbf{a}$ *be a class* C^1 *solenoidal vector field*

$$\operatorname{div}\mathbf{a} = 0 \tag{2.39}$$

defined on a domain B *having a boundary consisting of a simple closed surface. Then, there exists a class* C^1 *vector field* $\mathbf{b}$ *on* B *such that*

$$\mathbf{a} = \operatorname{curl}\mathbf{b} \tag{2.40}$$

A formal way to express the previous result is to observe that, being:

$$\mathbf{a} = \operatorname{div}\mathbf{a} = \boldsymbol{\nabla}\cdot\mathbf{a} = 0 \tag{2.41}$$

the vector field $\mathbf{a}$ has to be *perpendicular* to $\boldsymbol{\nabla}$. Thus $\mathbf{a}$ has to be of the form:

$$\mathbf{a} = \boldsymbol{\nabla}\times\mathbf{b} = \operatorname{curl}\mathbf{b} \tag{2.42}$$

what represents the statement of the theorem.

Theorem on Irrotational Tensor Fields *Let* B *be denote a simply-connected domain and* $\mathbf{A}$ *a tensor field of class* C^N $(N \geq 1)$ *on* B *that satisfies*

$$\operatorname{curl}\mathbf{A} = \mathbf{0} \tag{2.43}$$

Then, there exists a single-valued class C^{N+1} vector field $\mathbf{a}$ on B such that

$$\mathbf{A} = \operatorname{grad}\mathbf{a} = \mathbf{a}\otimes\boldsymbol{\nabla} \tag{2.44}$$

A formal way to reformulate the theorem is to apply the condition

$$\operatorname{curl}\mathbf{A} = \mathbf{0} \tag{2.45}$$

to an arbitrary (constant) vector field $\mathbf{b}$. We thus get, by means of $(2.33)_2$

$$(\operatorname{curl}\mathbf{A})\mathbf{b} = \mathbf{0} \quad \Leftrightarrow \quad \mathbf{W}_{\boldsymbol{\nabla}}\mathbf{A}^t\mathbf{b} = \mathbf{0} \quad \Leftrightarrow \quad \boldsymbol{\nabla}\times\mathbf{A}^t\mathbf{b} = \mathbf{0} \tag{2.46}$$

The last relation implies that $\mathbf{A}^t\mathbf{b}$ has to be *parallel* to $\boldsymbol{\nabla}$, i.e.:

$$\mathbf{A}^t\mathbf{b} = \alpha\boldsymbol{\nabla} \tag{2.47}$$

for some α. Being the vector field $\mathbf{b}$ arbitrary, it has to be

$$\mathbf{A} = \mathbf{a} \otimes \boldsymbol{\nabla} \tag{2.48}$$

for some vector field $\mathbf{a}$.

The previous review and the examples reported in the Appendix A illustrate the basic rules and properties which have to adopted in the application of $\boldsymbol{\nabla}$ symbolic calculus.

Additional results stemming from the definition of vector product between vectors and rank-two tensors introduced in subsection 2.1 as well as a convenient reformulation of the symbolic operator $\mathbf{W}_{\boldsymbol{\nabla}} \boxtimes \mathbf{W}_{\boldsymbol{\nabla}}$ appearing in (2.34) will be presented in the following sections.

2.3 Compatibility

To show a first application of the results presented in the previous section we shall prove the well known compatibility theorem ensuring the existence, in a simply connected body, of a single-valued displacement field $\mathbf{u}$ associated with a given strain field through the *strain-displacement* relation:

$$\mathbf{E} = \operatorname{sym}\operatorname{grad}\mathbf{u} = \frac{1}{2}\left[\operatorname{grad}\mathbf{u} + (\operatorname{grad}\mathbf{u})^t\right] = \frac{1}{2}\left(\mathbf{u} \otimes \boldsymbol{\nabla} + \boldsymbol{\nabla} \otimes \mathbf{u}\right). \tag{2.49}$$

In particular, we shall provide two separate proofs of the theorem: the first one is more constructive but unavoidably longer than the second one; in turn this is particularly compact once the properties of Gibbs symbolic calculus are properly mastered.

Compatibility Theorem *The strain field E associated with a class C^3 displacement field* satisfies the equation of compatibility:

$$\operatorname{curl}\operatorname{curl}\mathbf{E} = \mathbf{0} \tag{2.50}$$

Conversely, given a class $C^N (N \geq 2)$ symmetric tensor field $\mathbf{E}$ on a simply-connected body B, the fulfillment of (2.50) is sufficient to ensure the existence *of a single-valued displacement field* $\mathbf{u}$ *of class* C^{N+1} *on* B *such that* $\mathbf{E}$ *and* $\mathbf{u}$ *satisfy the strain-displacement relation.*

Proof *Necessary condition (Long version)* We have to prove that a symmetric tensor field expressed in the form (2.49) fulfills the compatibility condition (2.50). To this end let us re-write the strain displacement relation in the equivalent form:

$$\mathbf{E} + \mathbf{W} = \operatorname{grad} \mathbf{u} = \mathbf{u} \otimes \boldsymbol{\nabla} \tag{2.51}$$

and take the curl of both sides:

$$\operatorname{curl} \mathbf{E} + \operatorname{curl} \mathbf{W} = \boldsymbol{\nabla} \times (\mathbf{u} \otimes \boldsymbol{\nabla})^t \tag{2.52}$$

According to $(2.38)_4$ the right-hand side of the previous relation vanishes so that:

$$\operatorname{curl} \mathbf{E} = -\operatorname{curl} \mathbf{W} \tag{2.53}$$

Thus we are led to evaluate the curl of a swek-symmetric tensor; on account of (2.33) and (2.15) it is given by:

$$\operatorname{curl} \mathbf{W} = -\mathbf{W}_{\boldsymbol{\nabla}} \mathbf{W} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\mathbf{I} - (\boldsymbol{\omega} \otimes \boldsymbol{\nabla}) = (\operatorname{div} \boldsymbol{\omega})\mathbf{I} - \operatorname{grad} \boldsymbol{\omega} \tag{2.54}$$

where $\boldsymbol{\omega}$ is the axial vector associated with $\mathbf{W}$.

Since, by definition:

$$\mathbf{W} = \frac{1}{2}(\mathbf{u} \otimes \boldsymbol{\nabla} - \boldsymbol{\nabla} \otimes \mathbf{u}) \tag{2.55}$$

we get from formula (2.18) and property $(2.38)_1$

$$\boldsymbol{\omega} = \text{axial } \mathbf{W} = \text{axial } \left[\frac{1}{2}(\mathbf{u} \otimes \boldsymbol{\nabla} - \boldsymbol{\nabla} \otimes \mathbf{u})\right] = -\frac{1}{2}\mathbf{u} \times \boldsymbol{\nabla} = \frac{1}{2}\operatorname{curl} \mathbf{u} \tag{2.56}$$

Accordingly:

$$\operatorname{div} \boldsymbol{\omega} = \frac{1}{2}\operatorname{div} (\operatorname{curl} \mathbf{u}) = \frac{1}{2}\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{u}) = 0 \tag{2.57}$$

on account of $(2.38)_2$.

The previous result, combined with (2.53) and (2.54) yields finally:

$$\operatorname{curl}\mathbf{E} = \operatorname{grad}\boldsymbol{\omega} = \boldsymbol{\omega}\otimes\boldsymbol{\nabla} \tag{2.58}$$

so that making the curl of the previous result yields finally

$$\operatorname{curl}\operatorname{curl}\mathbf{E} = \operatorname{curl}\operatorname{grad}\boldsymbol{\omega} = \boldsymbol{\nabla}\times(\omega\otimes\boldsymbol{\nabla})^t = (\boldsymbol{\nabla}\times\boldsymbol{\nabla})\otimes\boldsymbol{\omega} = \mathbf{0} \tag{2.59}$$

where property $(2.38)_4$ has been invoked.

Proof *Sufficient condition (Long version)* We have to prove that, if a class $C^N(N \geq 2)$ symmetric tensor field $\mathbf{E}$ fulfills the property:

$$\operatorname{curl}\operatorname{curl}\mathbf{E} = \mathbf{0} \tag{2.60}$$

it admits the representation formula (2.49) where $\mathbf{u}$ denotes a single-valued class C^{N+1} vector field. Setting:

$$\mathbf{A} = \operatorname{curl}\mathbf{E} \tag{2.61}$$

the compatibility condition is written equivalently:

$$\operatorname{curl}\mathbf{A} = \mathbf{0} \tag{2.62}$$

The theorem on irrotational tensor fields, see section 2.2, ensures that, in a simply-connected domain, it exists a single-valued vector field $\mathbf{a} \in C^N$ such that:

$$\operatorname{curl}\ \mathbf{E} = \mathbf{A} = \operatorname{grad}\ \mathbf{a} = \mathbf{a}\otimes\boldsymbol{\nabla} \tag{2.63}$$

Recalling (2.54) it is natural to consider the curl of the skew tensor $\mathbf{W_a}$ associated with $\mathbf{a}$:

$$\operatorname{curl}\mathbf{W_a} = (\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{I} - (\mathbf{a}\otimes\boldsymbol{\nabla}) = (\operatorname{div}\mathbf{a})\mathbf{I} - \operatorname{grad}\mathbf{a} \tag{2.64}$$

so that the sum of (2.63) and (2.64) provides:

$$\operatorname{curl}\mathbf{E} + \operatorname{curl}\mathbf{W_a} = (\operatorname{div}\mathbf{a})\mathbf{I} \tag{2.65}$$

On the other hand, we get from (2.63),

$$\mathrm{tr}\,(\mathbf{a}\otimes\boldsymbol{\nabla}) = \mathrm{tr}\,(\mathrm{curl}\,\mathbf{E}) = \mathrm{tr}\,(\mathbf{W}_{\boldsymbol{\nabla}}\mathbf{E}^t) = \mathbf{W}_{\boldsymbol{\nabla}}\cdot\mathbf{E} = 0 \tag{2.66}$$

due to the orthogonality between skew and symmetric tensors. Hence

$$\mathrm{div}\,\mathbf{a} = \mathrm{tr}\,(\mathrm{grad}\,\mathbf{a}) = \mathrm{tr}\,(\mathbf{a}\otimes\boldsymbol{\nabla}) = 0 \tag{2.67}$$

In conclusion, formula (2.65) supplies

$$\mathrm{curl}\,(\mathbf{E}+\mathbf{W}_{\mathbf{a}}) = \mathbf{0} \tag{2.68}$$

what ensures, in a simply-connected domain, that

$$\mathbf{E}+\mathbf{W}_{\mathbf{a}} = \mathrm{grad}\,\mathbf{u} \tag{2.69}$$

the symmetric part of both sides provides finally the strain-displacement relation. Finally, the relation between $\mathbf{W}_{\mathbf{a}}$ and $\mathbf{u}$ can be inferred as in the proof of the necessity by tracing back formulas from (2.51) to (2.56).

Let us now provide a shorter version of the previous proof.

Alternative proof of the compatibility theorem

Necessary condition (Short version) The necessity of (2.50) follows by considering the curl of the strain-displacement relation (2.49). Specifically, invoking definition (2.33) one gets:

$$\begin{aligned}\mathrm{curl}\,\mathbf{E} &= \frac{1}{2}\mathrm{curl}[(\mathbf{u}\otimes\boldsymbol{\nabla}) + (\boldsymbol{\nabla}\otimes\mathbf{u})] = \\ &= \frac{1}{2}\boldsymbol{\nabla}\times(\mathbf{u}\otimes\boldsymbol{\nabla})^t + \frac{1}{2}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\otimes\mathbf{u})^t = \\ &= \frac{1}{2}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\otimes\mathbf{u}) + \frac{1}{2}\boldsymbol{\nabla}\times(\mathbf{u}\otimes\boldsymbol{\nabla})\,.\end{aligned} \tag{2.70}$$

which, on account of $(2.38)_4$ becomes:

$$\mathrm{curl}\,\mathbf{E} = \frac{1}{2}(\boldsymbol{\nabla}\times\boldsymbol{\nabla})\otimes\mathbf{u} + \frac{1}{2}(\boldsymbol{\nabla}\times\mathbf{u})\otimes\boldsymbol{\nabla} = \frac{1}{2}\mathrm{curl}\,(\mathbf{u}\otimes\boldsymbol{\nabla}) = \boldsymbol{\omega}\otimes\boldsymbol{\nabla} = \mathrm{grad}\,\boldsymbol{\omega}. \tag{2.71}$$

Thus, the curl of the previous relation yields:

$$\operatorname{curl}\operatorname{curl}\mathbf{E} = \operatorname{curl}\operatorname{grad}\boldsymbol{\omega} = \boldsymbol{\nabla} \times (\boldsymbol{\omega} \otimes \boldsymbol{\nabla})^t = \frac{1}{2}(\boldsymbol{\nabla} \times \boldsymbol{\nabla}) \otimes \boldsymbol{\omega} = \mathbf{0} \tag{2.72}$$

Sufficient condition (Short version)

The first part of the proof is similar to the one reported in the long version till formula (2.63). Thus, taking the trace of (2.63) one gets:

$$\operatorname{div}\, \mathbf{a} = \mathbf{a} \cdot \boldsymbol{\nabla} = \operatorname{tr}(\mathbf{a} \otimes \boldsymbol{\nabla}) = \operatorname{tr}(\operatorname{curl}\mathbf{E}) = \operatorname{tr}(\mathbf{W}_{\boldsymbol{\nabla}}\mathbf{E}^t) = \mathbf{W}_{\boldsymbol{\nabla}} \cdot \mathbf{E} = 0 \tag{2.73}$$

since $\mathbf{E}$ is symmetric and $\mathbf{W}_{\boldsymbol{\nabla}}$ antysimmetric.

Hence, the theorem on solenoidal tensor fields, see section 2.2, yields:

$$\mathbf{a} = \boldsymbol{\nabla} \times \mathbf{b} \tag{2.74}$$

which can be substituted in (2.63) to provide:

$$\boldsymbol{\nabla} \times \mathbf{E} = \operatorname{curl}\mathbf{E} = (\boldsymbol{\nabla} \times \mathbf{b}) \otimes \boldsymbol{\nabla} = \boldsymbol{\nabla} \times (\mathbf{b} \otimes \boldsymbol{\nabla}). \tag{2.75}$$

on account of property $(2.38)_3$. Invoking $(2.38)_4$ and being $\mathbf{E}$ symmetric one infers,

$$\mathbf{E} = \mathbf{u} \otimes \boldsymbol{\nabla} + \boldsymbol{\nabla} \otimes \mathbf{u} \tag{2.76}$$

where $\mathbf{u} = \mathbf{b}/2$

For the purposes of this treatment it is more convenient to reformulate the equation of compatibility (2.50) by means of the definition (2.34). To further emphasize the algebraic character of such an expression we report the matrix representation of the symmetric second-order tensor ($\operatorname{curl}\operatorname{curl}\mathbf{E}$).

Specifically, invoking formulas (7.10) and (4.45) reported in the appendix B, we deduce that the six compatibility conditions in a cartesian frame can be obtained in a straightforward manner by performing a row-by-column product of the symbolic matrix $[\mathbf{W}_{\boldsymbol{\nabla}} \boxtimes \mathbf{W}_{\boldsymbol{\nabla}}]$ by the column vector $[\mathbf{E}]$ to obtain:

$$[\operatorname{curl}\operatorname{curl}\mathbf{E}] = [\mathbf{0}] \Longleftrightarrow [\mathbf{W}_{\boldsymbol{\nabla}} \boxtimes \mathbf{W}_{\boldsymbol{\nabla}}]\mathbf{E} = [\mathbf{0}] \Longleftrightarrow$$

$$
\begin{bmatrix}
0 & D_{33} & D_{22} & 0 & -2D_{23} & 0 \\
D_{33} & 0 & D_{11} & 0 & 0 & -2D_{13} \\
D_{22} & D_{11} & 0 & -2D_{12} & 0 & 0 \\
0 & 0 & -D_{12} & -D_{33} & D_{31} & D_{23} \\
-D_{23} & 0 & 0 & D_{13} & -D_{11} & D_{12} \\
0 & -D_{13} & 0 & D_{23} & D_{12} & -D_{22}
\end{bmatrix}
\begin{bmatrix}
E_{11} \\ E_{22} \\ E_{33} \\ E_{12} \\ E_{23} \\ E_{31}
\end{bmatrix}
= [\mathbf{0}] \qquad (2.77)
$$

having introducing, for clarity

$$
D_{ij} = \frac{\partial^2}{\partial x_i \partial x_j} \qquad i, j = 1, 2, 3
$$

Stated equivalently, compatibility equations can be formally obtained by a trivial matrix multiplication.

2.4 Bianchi identities

To get further evidence of the usefulness of the symbolic calculus presented in subsection 2.2 it is shown how the so-called Bianchi identities [56] can be obtained by straightforward manipulations of purely algebraic nature.

The usual way of introducing Bianchi identities in classical textbooks of continuum mechanics is to observe that the six differential relations embedded in the compact notation $\operatorname{curl}\operatorname{curl}\mathbf{E} = \mathbf{0}$ are associated with three displacement components. This is the hint to realize that the six compatibility conditions have to be subjected to three independent relations.

Using Gibbs notation Bianchi identities follow immediately:

$$
\begin{aligned}
\operatorname{div}(\operatorname{curl}\operatorname{curl}\mathbf{E}) &= (\mathbf{W}_{\boldsymbol{\nabla}}\mathbf{E}\mathbf{W}^t_{\boldsymbol{\nabla}})\boldsymbol{\nabla} = (\mathbf{W}_{\boldsymbol{\nabla}}\mathbf{E})(\mathbf{W}^t_{\boldsymbol{\nabla}}\boldsymbol{\nabla}) = \\
&= -\mathbf{W}_{\boldsymbol{\nabla}}\mathbf{E}(\boldsymbol{\nabla} \times \boldsymbol{\nabla}) = \mathbf{0}
\end{aligned} \qquad (2.78)
$$

since the vector product of two $\boldsymbol{\nabla}$'s vanishes as in ordinary vector calculus, see also $(2.38)_2$ and $(2.38)_4$.

Invoking the matrix representation (2.77) of (curl curl $\mathbf{E}$) and the definition (2.33) of divergence of a tensor, the first one of the three Bianchi identities can be explicitly written as follows:

$$\frac{\partial^3 E_{22}}{\partial x_1 \partial x_3^2} + \frac{\partial^3 E_{33}}{\partial x_1 \partial x_2^2} - 2\frac{\partial^3 E_{23}}{\partial x_1 \partial x_2 \partial x_3} - \frac{\partial^3 E_{33}}{\partial x_2 \partial x_1 \partial x_2} - \frac{\partial^3 E_{12}}{\partial x_2 \partial x_3^2} - \frac{\partial^3 E_{23}}{\partial x_2 \partial x_3 \partial x_1} +$$

$$+\frac{\partial^3 E_{13}}{\partial x_2^2 \partial x_3} + \frac{\partial^3 E_{22}}{\partial x_3 \partial x_1 \partial x_3} - \frac{\partial^3 E_{12}}{\partial x_3 \partial x_2 \partial x_3} + \frac{\partial^3 E_{23}}{\partial x_3 \partial x_1 \partial x_2} - \frac{\partial^3 E_{13}}{\partial x_3 \partial x_2^2} = 0$$

while the remaining ones are obtained by cyclic permutation of the indices.

2.5 Rivlin identities

Stress-based elastostatics takes its steps from a convenient reformulation of the compatibility condition (2.50). The task is however not trivial since an adequate mastership of $\epsilon - \delta$ relationships or intrinsic tensor identities are required to convert the rather akward expression curl curl $\mathbf{E}$ in a more tractable form. For instance the following identity is reported in [28]

$$\begin{aligned} \text{curl curl } \mathbf{E} = 2 \text{ sym grad } (\text{div } \mathbf{E}) - \Delta \mathbf{E} + \\ - \text{grad grad } (\text{tr } \mathbf{E}) + [\boldsymbol{\Delta}(\text{tr } \mathbf{E}) - \text{div div } \mathbf{E}]\mathbf{I} \end{aligned}$$

For this reason, within the framework of the symbolic differential calculus illustrated in the section 2.2, it is presented a different approach which is based on the systematic use of Rivlin identities for tensor polynomials [46].

Originally derived for modeling the constitutive behaviour of isotropic materials, Rivlin identities have been recently employed for representing the class of solution of a tensor equation [50] occurring in several branches of continuum mechanics and for deriving the constitutive algorithm of isotropic elastoplastic models depending upon all the three invariants of the stress tensor [41]. For additional applications of Rivlin identities the reader is referred to [37].

To make the thesis as complete as possible, one provides a derivation of such identities, much simpler than the original one [46], based upon an idea by Itskov [31].

To be more precise a restricted class of identities is presented, namely the one containing only products of two tensors. Actually, the original derivation by Rivlin was concerned with expressions involving products of three, and even more, second-order tensors.

Although the conceptual framework exploited in our derivation is common to the three identities, they will be considered separately in order to simplify the subsequent cross-reference.

2.5.1 First Rivlin Identity

As originally proved by Itskov [31] the first Rivlin identity can be obtained by differentiating the Cayley-Hamilton identity for an arbitrary element $\mathbf{A} \in$ Lin:

$$\mathbf{A}^3 - I_{\mathbf{A}}\mathbf{A}^2 + II_{\mathbf{A}}\mathbf{A} - III_{\mathbf{A}}\mathbf{I} = \mathbf{0} \ , \tag{2.79}$$

In this respect we first notice that, taking the transpose of the previous relation and invoking (2.31), the derivative of the third invariant $III_{\mathbf{A}}$, provided by $(2.30)_3$, can be equivalently expressed as:

$$D_{\mathbf{A}}(III_{\mathbf{A}}) = (\mathbf{A}^2 - I_{\mathbf{A}}\mathbf{A} + II_{\mathbf{A}}\mathbf{I})^t \tag{2.80}$$

so that the product rule (2.28) and the formulas (2.29)- (2.1.3) yield:

$$\begin{cases} D_{\mathbf{A}}(I_{\mathbf{A}}\mathbf{A}^2) & = & \mathbf{A}^2 \otimes \mathbf{I} + I_{\mathbf{A}}(\mathbf{A} \boxtimes \mathbf{I} + \mathbf{I} \boxtimes \mathbf{A}^t) \\ D_{\mathbf{A}}(II_{\mathbf{A}}\mathbf{A}) & = & \mathbf{A} \otimes (I_{\mathbf{A}}\mathbf{I} - \mathbf{A}^t) + II_{\mathbf{A}}(\mathbf{I} \boxtimes \mathbf{I}) \\ D_{\mathbf{A}}(III_{\mathbf{A}}\mathbf{I}) & = & \mathbf{I} \otimes (\mathbf{A}^2 - I_{\mathbf{A}}\mathbf{A} + II_{\mathbf{A}}\mathbf{I})^t \end{cases} \tag{2.81}$$

Thus, the derivative of the Cayley-Hamilton identity supplies:

$$\mathbf{A}^2 \boxtimes \mathbf{I} + \mathbf{I} \boxtimes (\mathbf{A}^2)^t + \mathbf{A} \boxtimes \mathbf{A}^t - \mathbf{A}^2 \otimes \mathbf{I} - I_{\mathbf{A}}(\mathbf{A} \boxtimes \mathbf{I} + \mathbf{I} \boxtimes \mathbf{A}^t) +$$

$$+ I_{\mathbf{A}}(\mathbf{A} \otimes \mathbf{I}) - \mathbf{A} \otimes \mathbf{A}^t + II_{\mathbf{A}}(\mathbf{I} \boxtimes \mathbf{I}) - \mathbf{I} \otimes (\mathbf{A}^2)^t + I_{\mathbf{A}}(\mathbf{I} \otimes \mathbf{A}^t) - II_{\mathbf{A}}(\mathbf{I} \otimes \mathbf{I}) = \mathbf{0} \ ,$$

which is rewritten as follows:

$$\begin{aligned}\mathbf{A}\boxtimes\mathbf{A}^t-\mathbf{A}\otimes\mathbf{A}^t= \;& -[\mathbf{A}^2\boxtimes\mathbf{I}+\mathbf{I}\boxtimes(\mathbf{A}^2)^t]+[\mathbf{A}^2\otimes\mathbf{I}+\mathbf{I}\otimes(\mathbf{A}^2)^t]+\\ & +I_{\mathbf{A}}(\mathbf{A}\boxtimes\mathbf{I}+\mathbf{I}\boxtimes\mathbf{A}^t)-I_{\mathbf{A}}(\mathbf{A}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{A}^t)+\\ & -II_{\mathbf{A}}(\mathbf{I}\boxtimes\mathbf{I})+II_{\mathbf{A}}(\mathbf{I}\otimes\mathbf{I})\ .\end{aligned} \tag{2.82}$$

in order to separate the term $\mathbf{A}\boxtimes\mathbf{A}^t$, needed in the ensuing developments, and to emphasize the symmetric role played by the square and dyadic tensor products.

2.5.2 Second Rivlin Identity

Let us now differentiate Cayley-Hamilton's identity (2.79) multiplied by $\mathbf{A}$:

$$\mathbf{A}^4-I_{\mathbf{A}}\mathbf{A}^3+II_{\mathbf{A}}\mathbf{A}^2-III_{\mathbf{A}}\mathbf{A}=\mathbf{0}\ , \tag{2.83}$$

Similarly to (2.81) one now obtains:

$$\begin{cases} D_{\mathbf{A}}(-I_{\mathbf{A}}\mathbf{A}^3) & = & -\mathbf{A}^3\otimes\mathbf{I}-I_{\mathbf{A}}[\mathbf{A}^2\boxtimes\mathbf{I}+\mathbf{I}\boxtimes(\mathbf{A}^2)^t+\mathbf{A}\boxtimes\mathbf{A}^t]\\ D_{\mathbf{A}}(II_{\mathbf{A}}\mathbf{A}^2) & = & \mathbf{A}^2\otimes(I_{\mathbf{A}}\mathbf{I}-\mathbf{A}^t)+II_{\mathbf{A}}(\mathbf{A}\boxtimes\mathbf{I}+\mathbf{I}\boxtimes\mathbf{A}^t)\\ D_{\mathbf{A}}(-III_{\mathbf{A}}\mathbf{A}) & = & -\mathbf{A}\otimes(\mathbf{A}^2-I_{\mathbf{A}}\mathbf{A}+II_{\mathbf{A}}\mathbf{I})^t-III_{\mathbf{A}}(\mathbf{I}\boxtimes\mathbf{I})\ .\end{cases} \tag{2.84}$$

Substituting in the previous formulas the expression:

$$\mathbf{A}^3=I_{\mathbf{A}}\mathbf{A}^2-II_{\mathbf{A}}\mathbf{A}+III_{\mathbf{A}}\mathbf{I}$$

stemming from (2.79), one finally infers:

$$\begin{aligned}I_{\mathbf{A}}(\mathbf{A}\boxtimes\mathbf{A}^t-\mathbf{A}\otimes\mathbf{A}^t)= \;& [\mathbf{A}^2\boxtimes\mathbf{A}^t+\mathbf{A}\boxtimes(\mathbf{A}^2)^t]\\ & -[\mathbf{A}^2\otimes\mathbf{A}^t+\mathbf{A}\otimes(\mathbf{A}^2)^t]+\\ & III_{\mathbf{A}}[(\mathbf{I}\boxtimes\mathbf{I})-(\mathbf{I}\otimes\mathbf{I})]\end{aligned} \tag{2.85}$$

2.5.3 Third Rivlin identity

Differentiation of Cayley-Hamilton identity (2.79) times $\mathbf{A}^2$:

$$\mathbf{A}^5-I_{\mathbf{A}}\mathbf{A}^4+II_{\mathbf{A}}\mathbf{A}^3-III_{\mathbf{A}}\mathbf{A}^2=\mathbf{0}\ , \tag{2.86}$$

yields:

$$
\begin{aligned}
II_{\mathbf{A}}(\mathbf{A} \boxtimes \mathbf{A}^t - \mathbf{A} \otimes \mathbf{A}^t) = \quad & \mathbf{A}^2 \boxtimes (\mathbf{A}^2)^t - \mathbf{A}^2 \otimes (\mathbf{A}^2)^t + \\
& III_{\mathbf{A}}(\mathbf{A} \boxtimes \mathbf{I} + \mathbf{I} \boxtimes \mathbf{A}^t + \\
& -\mathbf{A} \otimes \mathbf{I} - \mathbf{I} \otimes \mathbf{A}^t)
\end{aligned}
\tag{2.87}
$$

which represents the third Rivlin identity.

2.6 Specialization of Rivlin identities to the case of skew-symmetric tensors

At this stage it is useful to remind that, in view of its applications to stress-bases elastostatics, our objective is to provide an alternative expression for the compatibility condition (2.50). Since this last one is formally expressed in terms of the symbolic operator $\mathbf{W}_{\boldsymbol{\nabla}} \boxtimes \mathbf{W}_{\boldsymbol{\nabla}}$ it is quite natural, by examining the formulas (2.82), (2.85) and (2.87) to apply Rivlin identities to the tensor $\mathbf{W}_{\boldsymbol{\nabla}}$.

However, in consideration of the general properties holding for an arbitrary skew-symmetric tensor $\mathbf{W}$ and the associated axial vector $\mathbf{w}$ [14]:

$$
I_{\mathbf{W}} = III_{\mathbf{W}} = 0; \qquad II_{\mathbf{W}} = \mathbf{w} \cdot \mathbf{w} \tag{2.88}
$$

the second Rivlin identity, e.g. formula (2.85), is not useful for these purposes. In particular one shall make use of the first identity, which specializes as follows:

$$
\begin{aligned}
\mathbf{W}_{\boldsymbol{\nabla}} \boxtimes \mathbf{W}_{\boldsymbol{\nabla}} \quad &= [\mathbf{W}^2_{\boldsymbol{\nabla}} \boxtimes \mathbf{I} + \mathbf{I} \boxtimes \mathbf{W}^2_{\boldsymbol{\nabla}}] - [\mathbf{W}^2_{\boldsymbol{\nabla}} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{W}^2_{\boldsymbol{\nabla}}] + \mathbf{W}_{\boldsymbol{\nabla}} \otimes \mathbf{W}_{\boldsymbol{\nabla}} + \\
&\quad + (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})[\mathbf{I} \boxtimes \mathbf{I} - \mathbf{I} \otimes \mathbf{I}],
\end{aligned}
\tag{2.89}
$$

since it is simpler to use in the derivation of Beltrami-Michell equation, an issue addressed in the next section.

2.7 Field equations of linear isotropic elasticity

The basic system of field equations of linear isotropic elastostatics consists of the strain-displacement relation (2.49):

$$\mathbf{E} = \operatorname{sym}\operatorname{grad}\mathbf{u} = \frac{1}{2}\left[(\mathbf{u}\otimes\boldsymbol{\nabla}) + (\boldsymbol{\nabla}\otimes\mathbf{u})\right] \ . \tag{2.90}$$

the stress-strain relation:

$$\mathbf{S} = \mathbb{D}\mathbf{E} \tag{2.91}$$

and the equilibrium equation:

$$\operatorname{div}\mathbf{S} + \mathbf{b} = \mathbf{S}\boldsymbol{\nabla} + \mathbf{b} = \mathbf{0} \tag{2.92}$$

where the rank-four tensor $\mathbb{D}$ represents the elastic operator, $\mathbf{S}$ the stress tensor and $\mathbf{b}$ the body force field per unit volume.

Assuming linear isotropic elasticity it turns out to be:

$$\mathbb{D} = 2G(\mathbf{I}\boxtimes\mathbf{I}) + \lambda(\mathbf{I}\otimes\mathbf{I}) \tag{2.93}$$

where G and λ denote the Lamé moduli.

Provided that the displacement field $\mathbf{u}$ is sufficiently smooth, the equations above imply quite naturally the displacement equation of equilibrium:

$$[\mathbb{D}(\mathbf{u})]\boldsymbol{\nabla} + \mathbf{b} = \mathbf{0} \tag{2.94}$$

Specifically, if the body is inhomogeneous, the equation above takes the following form in Gibbs notation:

$$[G(\mathbf{u}\otimes\boldsymbol{\nabla} + \boldsymbol{\nabla}\otimes\mathbf{u}) + \lambda(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{I}]\boldsymbol{\nabla} + \mathbf{b} = \mathbf{0} \tag{2.95}$$

Thus, recalling formula (7.1) in the appendix:

$$(\mathbf{u}\otimes\boldsymbol{\nabla} + \boldsymbol{\nabla}\otimes\mathbf{u})\boldsymbol{\nabla}G + G(\mathbf{u}\otimes\boldsymbol{\nabla} + \boldsymbol{\nabla}\otimes\mathbf{u})\boldsymbol{\nabla} + [\lambda(\mathbf{u}\cdot\boldsymbol{\nabla})]\boldsymbol{\nabla} + \mathbf{b} = \mathbf{0} \tag{2.96}$$

Applying formally the definition of tensor product to the second addend and invoking formula (7.4) in the appendix for the third addend, one has:

$$\begin{aligned}(\mathbf{u}\otimes\boldsymbol{\nabla}+\boldsymbol{\nabla}\otimes\mathbf{u})\boldsymbol{\nabla}G+G(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\mathbf{u}+G(\mathbf{u}\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}+&\\ +\boldsymbol{\nabla}\lambda(\mathbf{u}\cdot\boldsymbol{\nabla})+\lambda(\mathbf{u}\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}+\mathbf{b}=\mathbf{0}&\end{aligned}\tag{2.97}$$

Upon rearranging, the displacement equation of equilibrium is finally arrived at:

$$\begin{aligned}G\Delta\mathbf{u}+(G+\lambda)\mathrm{grad}\ \mathrm{div}\,\mathbf{u}+[\mathrm{grad}\,\mathbf{u}+(\mathrm{grad}\,\mathbf{u})^t]\mathrm{grad}\,G+&\\ +(\mathrm{div}\,\mathbf{u})\mathrm{grad}\,\lambda+\mathbf{b}=\mathbf{0}&\end{aligned}\tag{2.98}$$

an expression which reduces to Navier's equation in the case of homogeneous bodies [29].

In particular, Navier's equation is usually exploited [29, 56] to prove the biharmonicity of the displacement field associated with divergence-free and curl-free body force fields $\mathbf{b}$ together with additional properties concerning $\mathrm{div}\,\mathbf{u}, \mathrm{curl}\,\mathbf{u}, \mathrm{tr}\,\mathbf{E}$ and $\mathrm{tr}\,\mathbf{S}$.

As opposite to the straightforward and natural derivation of (2.98), the basic equation of elastostatics expressed in terms of the stress tensor, known as Beltrami-Michell equation or stress equation of compatibility, is considerably more cumbersome to derive. It is classically obtained by exploiting properties of the Ricci alternator, as in [38, 56, 36], or by using differential identities which, though elegantly expressed in tensor form, are far from being intuitive [29].

It is shown, on the contrary, that the proposed approach based on the use of Rivlin identities, is considerably more constructive since it allows one to derive Beltrami-Michell equation by purely algebraic manipulations in which each step follows quite naturally from the previous ones.

2.8 Beltrami-Michell equation

To start with let us first invert the elasticity tensor (2.93) by writing:

$$\mathbb{D}^{-1} = \frac{1+\nu}{E}(\mathbf{I} \boxtimes \mathbf{I}) - \frac{\nu}{E}(\mathbf{I} \otimes \mathbf{I}) \tag{2.99}$$

in terms of the Young modulus E and the Poisson ratio ν. Thus, the fundamental system of field equations governing the elastostatic problem of a homogeneous linear isotropic body can also be written as:

$$\begin{cases} \mathbf{S}\boldsymbol{\nabla} + \mathbf{b} = \mathbf{0} & \text{equation of equilibrium} \\ \mathbf{E} = \left[\frac{1+\upsilon}{E}\mathbb{I} - \frac{\upsilon}{E}(\mathbf{I} \otimes \mathbf{I})\right]\mathbf{S} & \text{linear isotropic constitutive law} \\ (\mathbf{W}_{\boldsymbol{\nabla}} \boxtimes \mathbf{W}_{\boldsymbol{\nabla}})\mathbf{E} = \mathbf{0} & \text{equation of compatibility} \end{cases} \tag{2.100}$$

By substituting the second relation above in the third one:

$$\frac{1+\upsilon}{E}(\mathbf{W}_{\boldsymbol{\nabla}} \boxtimes \mathbf{W}_{\boldsymbol{\nabla}})\mathbf{S} - \frac{\upsilon}{E}(\mathbf{W}_{\boldsymbol{\nabla}} \boxtimes \mathbf{W}_{\boldsymbol{\nabla}})(\mathbf{I} \otimes \mathbf{I})\mathbf{S} \ .$$

and invoking the composition rule $(2.25)_2$ for the second addend:

$$(\mathbf{W}_{\boldsymbol{\nabla}} \boxtimes \mathbf{W}_{\boldsymbol{\nabla}})(\mathbf{I} \otimes \mathbf{I}) = \mathbf{W}_{\boldsymbol{\nabla}}\mathbf{W}^t_{\boldsymbol{\nabla}} \otimes \mathbf{I} = -\mathbf{W}^2_{\boldsymbol{\nabla}} \otimes \mathbf{I} \tag{2.101}$$

the set of equations (2.100) becomes:

$$\begin{cases} \mathbf{S}\boldsymbol{\nabla} + \mathbf{b} = \mathbf{0} & \text{equation of equilibrium} \\ (\mathbf{W}_{\boldsymbol{\nabla}} \boxtimes \mathbf{W}_{\boldsymbol{\nabla}})\mathbf{S} + \dfrac{\upsilon}{1+\upsilon}(\mathbf{W}^2_{\boldsymbol{\nabla}} \otimes \mathbf{I})\mathbf{S} = \mathbf{0} & \text{stress compatibility} \end{cases} \tag{2.102}$$

It is apparent that, in order to derive a unique formula expressed in terms of the stress tensor, one needs to provide alternative expressions for the symbolic tensors $(\mathbf{W}_{\boldsymbol{\nabla}} \boxtimes \mathbf{W}_{\boldsymbol{\nabla}})$ and $\mathbf{W}^2_{\boldsymbol{\nabla}} \otimes \mathbf{I}$ which explicitly contain the term $\mathbf{S}\boldsymbol{\nabla}$ appearing in the equation of equilibrium.

In this respect we first invoke (2.16) to write:

$$\mathbf{W}^2_{\boldsymbol{\nabla}} = \boldsymbol{\nabla} \otimes \boldsymbol{\nabla} - (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\mathbf{I} \tag{2.103}$$

so that:

$$(\mathbf{W}^2_{\boldsymbol{\nabla}} \otimes \mathbf{I})\mathbf{S} = (\boldsymbol{\nabla} \otimes \boldsymbol{\nabla})\mathrm{tr}\mathbf{S} - (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})(\mathrm{tr}\mathbf{S})\mathbf{I} \tag{2.104}$$

The additional term $(\mathbf{W}_{\boldsymbol{\nabla}} \boxtimes \mathbf{W}_{\boldsymbol{\nabla}})\mathbf{S}$ which appears in (2.102) can be modified by invoking the specialization of the first Rivlin identity to the case of skew-symmetric tensors see, e.g., formula (2.89). Thus, on account of (2.103) one gets:

$$
\begin{aligned}
(\mathbf{W}_{\boldsymbol{\nabla}} \boxtimes \mathbf{W}_{\boldsymbol{\nabla}})\mathbf{S} = \quad & (\boldsymbol{\nabla} \otimes \boldsymbol{\nabla})\mathbf{S} + \mathbf{S}(\boldsymbol{\nabla} \otimes \boldsymbol{\nabla}) + \\
- \quad & 2(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\mathbf{S} - (\boldsymbol{\nabla} \otimes \boldsymbol{\nabla})\mathrm{tr}\mathbf{S} + \\
- \quad & [(\boldsymbol{\nabla} \otimes \boldsymbol{\nabla}) \cdot \mathbf{S}]\mathbf{I} + 2(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})(\mathrm{tr}\mathbf{S})\mathbf{I} + \\
+ \quad & (\mathbf{W}_{\boldsymbol{\nabla}} \cdot \mathbf{S})\mathbf{W}_{\boldsymbol{\nabla}} + (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\mathbf{S} + \\
- \quad & (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})(\mathrm{tr}\mathbf{S})\mathbf{I}
\end{aligned}
\tag{2.105}
$$

where the definition of dyadic and square tensor product between second-order tensors has been exploited see, e.g., (2.22) and (2.23).

Observe that the quantity $\mathbf{W}_{\boldsymbol{\nabla}} \cdot \mathbf{S}$ vanishes owing to the skew-symmetry of $\mathbf{W}_{\boldsymbol{\nabla}}$ and the symmetry of $\mathbf{S}$ so that, by invoking the properties (2.1) and (2.2), the previous expressions become:

$$
\begin{aligned}
(\mathbf{W}_{\boldsymbol{\nabla}} \boxtimes \mathbf{W}_{\boldsymbol{\nabla}})\mathbf{S} = \quad & \boldsymbol{\nabla} \otimes \mathbf{S}\boldsymbol{\nabla} + \mathbf{S}\boldsymbol{\nabla} \otimes \boldsymbol{\nabla} - (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\mathbf{S} + \\
- \quad & (\boldsymbol{\nabla} \otimes \boldsymbol{\nabla})\mathrm{tr}\mathbf{S} - (\boldsymbol{\nabla} \cdot \mathbf{S}\boldsymbol{\nabla})\mathbf{I} + \\
+ \quad & (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})(\mathrm{tr}\mathbf{S})\mathbf{I}
\end{aligned}
\tag{2.106}
$$

Thus, recalling (2.104), the stress compatibility condition $(2.102)_2$ assumes the form:

$$
\begin{aligned}
& \boldsymbol{\nabla} \otimes \mathbf{S}\boldsymbol{\nabla} + \mathbf{S}\boldsymbol{\nabla} \otimes \boldsymbol{\nabla} - \frac{1}{1+\nu}(\boldsymbol{\nabla} \otimes \boldsymbol{\nabla})(\mathrm{tr}\mathbf{S}) + \\
& \qquad -(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\mathbf{S} + \frac{1}{1+\nu}(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})(\mathrm{tr}\mathbf{S})\mathbf{I} - (\mathbf{S}\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\mathbf{I} = \mathbf{0}
\end{aligned}
\tag{2.107}
$$

The previous expression contains the terms $(\boldsymbol{\nabla} \otimes \mathbf{S}\boldsymbol{\nabla})$, $(\mathbf{S}\boldsymbol{\nabla} \otimes \boldsymbol{\nabla})$ and $(\boldsymbol{\nabla} \cdot \mathbf{S}\boldsymbol{\nabla})$. Since the trace of the formers coincides with the latter it is natural to evaluate the trace of the previous relation, to obtain the identity:

$$
(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})(\mathrm{tr}\mathbf{S}) = \frac{1+\nu}{1-\nu}(\mathbf{S}\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}). \tag{2.108}
$$

Substituting the previous expression in (2.107) provides:

$$\begin{aligned}\boldsymbol{\nabla}\otimes\mathbf{S}\boldsymbol{\nabla}+\mathbf{S}\boldsymbol{\nabla}\otimes\boldsymbol{\nabla}-\frac{1}{1+\nu}(\boldsymbol{\nabla}\otimes\boldsymbol{\nabla})(\mathrm{tr}\mathbf{S})+&\\ -(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\mathbf{S}+\frac{\nu}{1-\nu}(\mathbf{S}\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\mathbf{I}=\mathbf{0}&\end{aligned}\tag{2.109}$$

which, on account of the equation of equilibrium $(2.100)_1$ yields finally:

$$\Delta\mathbf{S}+\frac{1}{1+\nu}(\boldsymbol{\nabla}\otimes\boldsymbol{\nabla})(\mathrm{tr}\mathbf{S})+\mathbf{b}\otimes\boldsymbol{\nabla}+\boldsymbol{\nabla}\otimes\mathbf{b}+\frac{\nu}{1-\nu}(\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{I}=\mathbf{0}\tag{2.110}$$

which represents the classical expression of the Beltrami Michell equation.

In a cartesian reference frame, equation (2.110) writes as follows

$$\left\{\begin{array}{l}
\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\sigma_x}{\partial y^2}+\frac{\partial^2\sigma_x}{\partial z^2}+\frac{1}{1+\upsilon}\frac{\partial^2(\sigma_x+\sigma_y+\sigma_z)}{\partial x^2}=-\frac{\upsilon}{1-\upsilon}\left(\frac{\partial b_x}{\partial x}+\frac{\partial b_y}{\partial y}+\frac{\partial b_z}{\partial z}\right)-2\frac{\partial b_x}{\partial x}\\
\frac{\partial^2\sigma_y}{\partial x^2}+\frac{\partial^2\sigma_y}{\partial y^2}+\frac{\partial^2\sigma_y}{\partial z^2}+\frac{1}{1+\upsilon}\frac{\partial^2(\sigma_x+\sigma_y+\sigma_z)}{\partial y^2}=-\frac{\upsilon}{1-\upsilon}\left(\frac{\partial b_x}{\partial x}+\frac{\partial b_y}{\partial y}+\frac{\partial b_z}{\partial z}\right)-2\frac{\partial b_y}{\partial y}\\
\frac{\partial^2\sigma_z}{\partial x^2}+\frac{\partial^2\sigma_z}{\partial y^2}+\frac{\partial^2\sigma_z}{\partial z^2}+\frac{1}{1+\upsilon}\frac{\partial^2(\sigma_x+\sigma_y+\sigma_z)}{\partial z^2}=-\frac{\upsilon}{1-\upsilon}\left(\frac{\partial b_x}{\partial x}+\frac{\partial b_y}{\partial y}+\frac{\partial b_z}{\partial z}\right)-2\frac{\partial b_z}{\partial z}\\
\frac{\partial^2\tau_{xy}}{\partial x^2}+\frac{\partial^2\tau_{xy}}{\partial y^2}+\frac{\partial^2\tau_{xy}}{\partial z^2}+\frac{1}{1+\upsilon}\frac{\partial^2(\sigma_x+\sigma_y+\sigma_z)}{\partial x\partial y}=-\frac{\partial b_x}{\partial y}-\frac{\partial b_y}{\partial x}\\
\frac{\partial^2\tau_{yz}}{\partial x^2}+\frac{\partial^2\tau_{yz}}{\partial y^2}+\frac{\partial^2\tau_{yz}}{\partial z^2}+\frac{1}{1+\upsilon}\frac{\partial^2(\sigma_x+\sigma_y+\sigma_z)}{\partial y\partial z}=-\frac{\partial b_y}{\partial z}-\frac{\partial b_z}{\partial y}\\
\frac{\partial^2\tau_{zx}}{\partial x^2}+\frac{\partial^2\tau_{zx}}{\partial y^2}+\frac{\partial^2\tau_{zx}}{\partial z^2}+\frac{1}{1+\upsilon}\frac{\partial^2(\sigma_x+\sigma_y+\sigma_z)}{\partial z\partial x}=-\frac{\partial b_z}{\partial x}-\frac{\partial b_x}{\partial z}
\end{array}\right.\tag{2.111}$$

Chapter 3

Saint Venant problem

The strategies for the general direct solution of Saint-Venant model for cylinders with arbitrary cross section can be basically classified into two categories known as displacement approach and stress approach according to the selected unknown primary field and to the corresponding governing equations.

For what concerns the displacements approach [36], based on Cauchy-Navier equations, the general solution was provided in the original work by Barré De Saint-Venant [4] and by Alfred Clebsch [17] while a stress-based general solution of Saint-Venant rod theory was presented by Riccardo Baldacci [7] more than a century after the original paper by Barré De Saint-Venant.

Although the research is currently focused on Saint-Venant-like models enriched by additional complexity factors with respect to the classical problem [10, 12], some basic issues of the homogeneous isotropic model are still object of study and debate in the scientific community. These are concerned, in particular, on the solution of the shear problem and on the proper definition of shear factors [54, 27, 44, 45], giving rise to very recent contributions as well [35, 22].

As well known, regardless of the selected approach, the complete solution in terms of displacements and stress fields for rods of generic cross section can be represented by means of explicit analytic expressions only partially. Actually some terms associated with torsion and shear embody auxiliary functions which are solution of Dirichlet or Neumann problems related to the cross section domain; this is true with the exception of sections having particular geometries for which a closed-form solution exists.

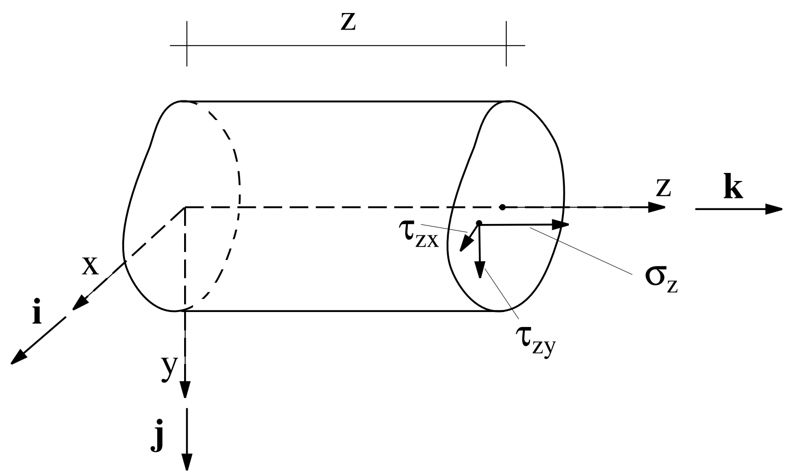
With special reference to the shear problem a further difference distinguishes the solutions available for this load case from the ones concerning axial load, biaxial bending and torsion. In these last cases, a frame-independent representation for the displacement and the stress field is still available since geometrical quantities which the solution depends upon are solely expressed by means of vector and tensor fields.

For instance, it is well known that for bending and axial load the integral quantities that characterize the dependence of the solution on the section geometry are the first area moment and the inertia tensor. These quantities in turn are defined as domain integrals extended over the section of the position vector and of its tensor product. Actually, as illustrated in section 7.3 of the appendix, the second order tensor associated with the matrix introduced by Baldacci in [6, 7] changes as function of the adopted reference frame.

3.1 Saint Venant hypothesis

As well known Saint-Venant rod theory refers to a linearly elastic isotropic cylinder of length l and cross section domain Ω.

A generic point of the cylinder is individuated by a position vector $\mathbf{p}$ whose coordinates, (x, y, z), are expressed in a Cartesian reference frame having origin in the centroid of one of the terminal bases; the coordinate z is directed along the cylinder axis whereas x and y denote the coordinates referred to arbitrary orthogonal axes in the cross section. The unit vectors associated with the coordinate axes will be denoted by $\mathbf{i}, \mathbf{j}, \mathbf{k}$, respectively.



Since some properties of the section such as the inertia tensor are defined in the cross-section, it is useful to introduce the vector

$$\mathbf{r} = \mathbf{p} - (\mathbf{p} \cdot \mathbf{k})\mathbf{k}$$

which represents the projection of the position vector in the cross plane.

$$[\mathbf{r}] = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

Let us consider that the body-forces $\mathbf{b}$ and forces per unit area $\mathbf{q}$ on the lateral surface Ω_l are equals to zero:

$$\begin{cases} \mathbf{b} = \mathbf{0} & \forall\, \mathbf{p} \in \Omega \\ \mathbf{q} = \mathbf{0} & \forall\, \mathbf{p} \in \partial\Omega_l \end{cases} \tag{3.1}$$

It is also assumed a plane stress state:

$$\sigma_x = \sigma_y = \tau_{xy} = 0 \tag{3.2}$$

3.2 Stress field in the Saint-Venant theory

Differently from the classical contributions to the Saint-Venant problem, mainly based on the displacement approach, in this work it will be referred to Baldacci's treatment. This author proposed an elegant and concise solution based on a stress approach which, as such, takes its steps from Beltrami-Michell equations [9, 40].

It is possible to derive some preliminary information on the stress field by substitution of (3.2) in (2.111). In fact, in the light of (3.2) one has

$$\begin{aligned}
&xx) \quad \frac{\partial^2(\sigma_x+\sigma_y+\sigma_z)}{\partial x^2} = \frac{\partial^2 \sigma_z}{\partial x^2} = 0 \\
&yy) \quad \frac{\partial^2(\sigma_x+\sigma_y+\sigma_z)}{\partial y^2} = \frac{\partial^2 \sigma_z}{\partial y^2} = 0 \\
&xy) \quad \frac{\partial^2(\sigma_x+\sigma_y+\sigma_z)}{\partial x \partial y} = \frac{\partial^2 \sigma_z}{\partial x \partial y} = 0 \\
&zz) \quad \frac{\partial^2 \sigma_z}{\partial x^2} + \frac{\partial^2 \sigma_z}{\partial y^2} + \frac{\partial^2 \sigma_z}{\partial z^2} + \frac{1}{1+\nu}\frac{\partial^2 \sigma_z}{\partial z^2} = \frac{2+\nu}{1+\nu}\frac{\partial^2 \sigma_z}{\partial z^2} = 0
\end{aligned} \tag{3.3}$$

By the first three equations of (3.3) one observes that σ_z is a linear function in x and y, so that the fourth one entails that σ_z is also a linear function in z:

$$\sigma_z = a_0 + \mathbf{g}\cdot\mathbf{r} + (l-z)(b_0 + \mathbf{g}_t\cdot\mathbf{r}) \tag{3.4}$$

where a_0, b_0,

$$\mathbf{g} = \begin{vmatrix} g_x \\ g_y \\ 0 \end{vmatrix} \qquad \mathbf{g}_t = \begin{vmatrix} g_{tx} \\ g_{ty} \\ 0 \end{vmatrix}$$

constitute a set of six unknowns scalars to be evaluated as function of the stress resultants applied on the end sections of the beam.

Substituting (3.2) in the equilibrium equation (2.92) provides:

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + \dfrac{\partial \tau_{xz}}{\partial z} = \dfrac{\partial \tau_{xz}}{\partial z} = & 0 \\ \dfrac{\partial \tau_{yx}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{yz}}{\partial z} = \dfrac{\partial \tau_{yz}}{\partial z} = & 0 \\ \dfrac{\partial \tau_{zx}}{\partial x} + \dfrac{\partial \tau_{zy}}{\partial y} + \dfrac{\partial \sigma_z}{\partial z} = 0 \end{cases} \tag{3.5}$$

the first two equations of (3.5) imply that τ_{zx} and τ_{zy} are independent of z, while the third one is written as

$$\mathrm{div}\, \boldsymbol{\tau} = \boldsymbol{\nabla} \cdot \boldsymbol{\tau} = -\frac{\partial \sigma_z}{\partial z} \tag{3.6}$$

where $\boldsymbol{\tau}$ is the vector parallel to xy plane whose component are:

$$\boldsymbol{\tau} = \begin{vmatrix} \tau_{zx} \\ \tau_{zy} \\ 0 \end{vmatrix}$$

On account of $(3.5)_1$ and $(3.5)_2$, the other two compatibility equations of (2.111) become

$$\begin{aligned} xz) \quad & \frac{\partial^2 \tau_{zx}}{\partial x^2} + \frac{\partial^2 \tau_{zx}}{\partial y^2} + \frac{\partial^2 \tau_{zx}}{\partial z^2} + \frac{1}{1+\nu}\frac{\partial^2 \sigma_z}{\partial z \partial x} = \frac{\partial^2 \tau_{zx}}{\partial x^2} + \frac{\partial^2 \tau_{zx}}{\partial y^2} + \frac{1}{1+\nu}\frac{\partial^2 \sigma_z}{\partial z \partial x} = 0 \\ yz) \quad & \frac{\partial^2 \tau_{zy}}{\partial x^2} + \frac{\partial^2 \tau_{zy}}{\partial y^2} + \frac{\partial^2 \tau_{zy}}{\partial z^2} + \frac{1}{1+\nu}\frac{\partial^2 \sigma_z}{\partial z \partial y} = \frac{\partial^2 \tau_{zy}}{\partial x^2} + \frac{\partial^2 \tau_{zy}}{\partial y^2} + \frac{1}{1+\nu}\frac{\partial^2 \sigma_z}{\partial z \partial y} = 0 \quad . \end{aligned} \tag{3.7}$$

In a vectorial form equations (3.7) can be equivalently expressed as

$$\mathrm{div}\,\mathrm{grad}\, \boldsymbol{\tau} = -\frac{1}{1+\nu}\mathrm{grad}\, \sigma' \ , \tag{3.8}$$

where $\sigma' = \dfrac{\partial \sigma_z}{\partial z}$.

Moreover, by deriving (3.4) with respect by z one gets the following compatibility equation:

$$\sigma' = (-b_0 + \mathbf{g_t} \cdot \mathbf{r}) \Rightarrow \mathrm{grad}\, \sigma' = -\mathbf{g}_t \tag{3.9}$$

Thus, substitution of (3.9) in (3.8) yields

$$\operatorname{div}\operatorname{grad}\boldsymbol{\tau} = \frac{1}{1+\nu}\mathbf{g}_t \ .$$

The scalar constant b_0 in (3.4) turns out to be zero if the coordinate system is barycentric. In fact, accordingly to Gauss' theorem and $(3.1)_2$ hypothesis

$$\int_{\partial A} \boldsymbol{\tau} \cdot \mathbf{n} \, ds = 0 \Rightarrow \int_{A} \operatorname{div} \boldsymbol{\tau} \, da = 0 \tag{3.10}$$

since the substitution of (3.6) and (3.9) in (3.10) yelds

$$\int_{A} (b_0 + \mathbf{g}_t \cdot \mathbf{r}) \, da = 0 \tag{3.11}$$

the scalar constant b_0 is null being also null the first moment of inertia of the section respect on the centroid:

$$\int_{A} \mathbf{r} \, da = \mathbf{0} \ .$$

Resuming, the stress field solution of the Saint Venant problem in a barycentric coordinate system is provided by a scalar component σ_z and a vector $\boldsymbol{\tau}$ fulfilling the following properties

$$\begin{cases} \sigma_z = a_0 + \mathbf{g} \cdot \mathbf{r} + (l - z)\mathbf{g}_t \cdot \mathbf{r} & \text{compatibility} \\ \operatorname{div} \boldsymbol{\tau} = \mathbf{g}_t \cdot \mathbf{r} & \text{equilibrium} \\ \operatorname{div}\operatorname{grad}\boldsymbol{\tau} = \dfrac{1}{1+\nu}\mathbf{g}_t & \text{compatibility} \end{cases} \tag{3.12}$$

and $\boldsymbol{\tau} \cdot \mathbf{n} = 0$ on the boundary of the section.

Now it will be shown how an appropriate combination of the compatibility equation with the equilibrium equation allows one to rewrite (3.12) in a more convenient form.

$$\begin{cases} \operatorname{div} \boldsymbol{\tau} = \mathbf{g}_t \cdot \mathbf{r} \\ \operatorname{div}\operatorname{grad}\boldsymbol{\tau} = \frac{1}{1+\nu}\mathbf{g}_t \end{cases} \Rightarrow \begin{cases} \operatorname{grad}\operatorname{div} \boldsymbol{\tau} = \mathbf{g}_t \\ \operatorname{div}\operatorname{grad}\boldsymbol{\tau} = \frac{1}{1+\nu}\mathbf{g}_t \end{cases} \ . \tag{3.13}$$

Subtracting the two equation in (3.13) one obtains

$$\operatorname{grad}\operatorname{div}\boldsymbol{\tau}-\operatorname{div}\operatorname{grad}\boldsymbol{\tau}=\left(1-\frac{1}{1+\nu}\right)\mathbf{g}_t=\frac{\nu}{1+\nu}\mathbf{g}_t=\bar{\nu}\mathbf{g}_t. \tag{3.14}$$

Invoking the triple vector product, see e.g.,

$$(\boldsymbol{\nabla}\cdot\boldsymbol{\tau})\boldsymbol{\nabla}-(\boldsymbol{\tau}\otimes\boldsymbol{\nabla})\boldsymbol{\nabla}=(\boldsymbol{\nabla}\cdot\boldsymbol{\tau})\boldsymbol{\nabla}-(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\boldsymbol{\tau}=\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{\tau})=\operatorname{curl}\operatorname{curl}\boldsymbol{\tau}$$

that is

$$\operatorname{grad}\operatorname{div}\boldsymbol{\tau}-\operatorname{div}\operatorname{grad}\boldsymbol{\tau}=\operatorname{curl}\operatorname{curl}\boldsymbol{\tau}\quad, \tag{3.15}$$

so that equation (3.14) can be equivalently written as

$$\operatorname{curl}\operatorname{curl}\boldsymbol{\tau}=\bar{\nu}\mathbf{g}_t \tag{3.16}$$

Due to properties $(3.5)_1$ and $(3.5)_2$ curl $\boldsymbol{\tau}$ is parallel to $\mathbf{k}$, so that one has:

$$\operatorname{curl}\boldsymbol{\tau}=\boldsymbol{\nabla}\times\boldsymbol{\tau}=(\mathbf{k}\otimes\mathbf{k})\boldsymbol{\nabla}\times\boldsymbol{\tau}=(\boldsymbol{\nabla}\times\boldsymbol{\tau}\cdot\mathbf{k})\mathbf{k}\ . \tag{3.17}$$

Accordingly, it turns out to be

$$\begin{aligned}\operatorname{curl}\operatorname{curl}\mathbf{r}\ &=\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{\tau})=\boldsymbol{\nabla}\times[(\boldsymbol{\nabla}\times\boldsymbol{\tau}\cdot\mathbf{k})\mathbf{k}]=\\&=-\mathbf{k}\times(\boldsymbol{\nabla}\times\boldsymbol{\tau}\cdot\mathbf{k})\boldsymbol{\nabla}=\mathbf{W}_{\mathbf{k}}^t\operatorname{grad}(\operatorname{curl}\boldsymbol{\tau}\cdot\mathbf{k})\end{aligned} \tag{3.18}$$

substituting the previous relation (3.18) in (3.14) one has

$$\mathbf{W}_{\mathbf{k}}^t\operatorname{grad}(\operatorname{curl}\boldsymbol{\tau}\cdot\mathbf{k})=\bar{\nu}\mathbf{g}_t\Longrightarrow\operatorname{curl}\boldsymbol{\tau}\cdot\mathbf{k}=\bar{\nu}(\mathbf{W}_{\mathbf{k}}\mathbf{g}_t\cdot\mathbf{r})+c.$$

where c is an arbitrary scalar

In conclusion, the stress vector $\boldsymbol{\tau}$ solution of the Saint Venant problem can be turned out by resolving the following linear differential problem:

$$\begin{cases}\operatorname{div}\boldsymbol{\tau}=\boldsymbol{\tau}\cdot\boldsymbol{\nabla}=\mathbf{g}_t\cdot\mathbf{r} & \text{equilibrium}\\(\operatorname{curl}\boldsymbol{\tau})_z=(\boldsymbol{\nabla}\times\boldsymbol{\tau})\cdot\mathbf{k}=\bar{\nu}\mathbf{W}_{\mathbf{k}}\mathbf{g}_t\cdot\mathbf{r}+c & \text{compatibility}\end{cases} \tag{3.19}$$

with boundary conditions

$$\boldsymbol{\tau}\cdot\mathbf{n}=0. \tag{3.20}$$

expressing the condition of stress-free lateral surface.

The linearity of the previous problem naturally prompts for a staggered solution scheme depending on the fact that $\mathbf{g}_t$ is equal to or different from zero. In order to separate the solution of the tangential stress field into shear and torsion stress, the previous system can be written as the sum of the two following systems:

$$\begin{cases} \boldsymbol{\tau}_{tor} \cdot \boldsymbol{\nabla} = 0 \\ (\boldsymbol{\nabla} \times \boldsymbol{\tau}_{tor}) \cdot \mathbf{k} = c \end{cases} \qquad \begin{cases} \boldsymbol{\tau}_{sh} \cdot \boldsymbol{\nabla} = \mathbf{g}_t \cdot \mathbf{r} \\ (\boldsymbol{\nabla} \times \boldsymbol{\tau}_{sh}) \cdot \mathbf{k} = \bar{\nu} \mathbf{W}_{\mathbf{k}} \mathbf{g}_t \cdot \mathbf{r} \end{cases} \tag{3.21}$$

The solution of the relevant differential problems have been suffixed by '*tor*' and '*sh*' to emphasize the fact that they are associated, respectively, with torque or shear. In particular it will be show in this last case that $\mathbf{g}_t$ is directly associated with the shearing force $\mathbf{t}_s$

3.3 Stress field associated with torsion

The solution of the differential problem $(3.21)_1$ is further decomposed in the form

$$\boldsymbol{\tau}_{tor} = \boldsymbol{\tau}^0_{tor} + \boldsymbol{\tau}^p_{tor} \tag{3.22}$$

where $\boldsymbol{\tau}^0_{tor}$ is the solution to the homogeneous system

$$\begin{cases} \boldsymbol{\tau}^0_{tor} \cdot \boldsymbol{\nabla} = 0 \\ (\boldsymbol{\nabla} \times \boldsymbol{\tau}^0_{tor}) \cdot \mathbf{k} = 0 \end{cases} \tag{3.23}$$

whereas $\boldsymbol{\tau}^p_{tor}$ is a particular integral

$$\begin{cases} \boldsymbol{\tau}^p_{tor} \cdot \boldsymbol{\nabla} = 0 \\ (\boldsymbol{\nabla} \times \boldsymbol{\tau}^p_{tor}) \cdot \mathbf{k} = c \end{cases} \tag{3.24}$$

To solve (3.24) the two equations are written in a more convenient form:

$$\boldsymbol{\tau}^p_{tor} \cdot \boldsymbol{\nabla} = \mathbf{W}_{\mathbf{k}} (\mathbf{W}^t_{\mathbf{k}} \boldsymbol{\tau}^p_{tor}) \cdot \boldsymbol{\nabla} = \mathbf{k} \times (\mathbf{W}^t_{\mathbf{k}} \boldsymbol{\tau}^p_{tor}) \cdot \boldsymbol{\nabla} = -\boldsymbol{\nabla} \times (W^t_{\mathbf{k}} \boldsymbol{\tau}^p_{tor}) \cdot \mathbf{k} = -\mathrm{curl}\,(\mathbf{W}^t_{\mathbf{k}} \boldsymbol{\tau}^p_{tor}) \cdot \mathbf{k}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\tau}^p_{tor} \cdot \mathbf{k} = \boldsymbol{\tau}^p_{tor} \times \mathbf{k} \cdot \boldsymbol{\nabla} = (\mathbf{W}^t_{\mathbf{k}} \boldsymbol{\tau}^p_{tor}) \cdot \boldsymbol{\nabla}$$

thus, one obtains

$$\begin{cases} \operatorname{curl}(\mathbf{W}^t_{\mathbf{k}} \boldsymbol{\tau}^p_{tor}) \cdot \mathbf{k} = 0 \\ (\mathbf{W}^t_{\mathbf{k}} \boldsymbol{\tau}^p_{tor}) \cdot \boldsymbol{\nabla} = c \ . \end{cases} \tag{3.25}$$

From the second equation in (3.25) it turns out to be

$$(\mathbf{W}^t_{\mathbf{k}} \boldsymbol{\tau}^p_{tor}) = \frac{c}{2}\mathbf{r} \qquad \Rightarrow \qquad \boldsymbol{\tau}^p_{tor} = \frac{c}{2}\mathbf{W}_{\mathbf{k}}\mathbf{r} \tag{3.26}$$

The solution of system (3.23) is a scalar harmonic function φ_{tor} so that

$$\boldsymbol{\tau}^0_{tor} = \frac{c}{2}\operatorname{grad}\varphi_{tor} = \frac{c}{2}\varphi_{tor}\boldsymbol{\nabla} \tag{3.27}$$

The solution to the homogeneous differential problem (3.23) amounts to finding a harmonic potential φ_{tor}, hence satisfying the condition $(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})_{tor} = 0$, with prescribed directional derivative along the boundary $\partial\Omega$. Being

$$\boldsymbol{\tau}^0_{tor} \cdot \mathbf{n} = -\boldsymbol{\tau}^p_{tor} \cdot \mathbf{n} \tag{3.28}$$

on account of (3.23) and (**??**), the homogeneous differential problem can be equivalently formulated in the form

$$\begin{cases} (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\varphi_{tor} = \mathbf{0} & \text{in the interior of } \Omega \\ (\varphi_{tor} \otimes \boldsymbol{\nabla})\mathbf{n} = -\frac{c}{2}\mathbf{W}_{\mathbf{k}}\mathbf{r} & \text{on the boundary } \partial\Omega \end{cases} \tag{3.29}$$

Hence on account of (3.27) and (3.26), (3.22) became

$$\boldsymbol{\tau}_{tor} = \frac{c}{2}\varphi_{tor}\boldsymbol{\nabla} + \frac{c}{2}\mathbf{W}_{\mathbf{k}}\mathbf{r} \tag{3.30}$$

in which $\boldsymbol{\tau}_{tor}$ represents the term of tangential stress associated with torsion.

3.4 Frame-independent representation of the stress field associated with pure shear

Similarly to the previous section, the solution of differential problem $(3.21)_2$ is additively decomposed in the form

$$\boldsymbol{\tau}_{sh} = \boldsymbol{\tau}^0_{sh} + \boldsymbol{\tau}^p_{sh} \tag{3.31}$$

where $\boldsymbol{\tau}^0_{sh}$ is the solution to the homogeneous system associated with $(3.21)_2$

$$\begin{cases} \boldsymbol{\tau}^0_{sh} \cdot \boldsymbol{\nabla} = 0 \\ (\boldsymbol{\nabla} \times \boldsymbol{\tau}^0_{sh}) \cdot \mathbf{k} = 0 \end{cases} \tag{3.32}$$

whereas $\boldsymbol{\tau}^p_{sh}$ is a particular integral of the non-homogeneous problem

$$\begin{cases} \boldsymbol{\tau}^p_{sh} \cdot \boldsymbol{\nabla} = \mathbf{g}_t \cdot \mathbf{r} \\ (\boldsymbol{\nabla} \times \boldsymbol{\tau}^p_{sh}) \cdot \mathbf{k} = \bar{\nu} \mathbf{W}_{\mathbf{k}} \mathbf{g}_t \cdot \mathbf{r} \end{cases} \tag{3.33}$$

System $(3.21)_2$ is supplemented by the boundary equation on $\partial\Omega$

$$\boldsymbol{\tau}_{sh} \cdot \mathbf{n} = 0. \tag{3.34}$$

expressing the condition of stress-free lateral surface.

To obtain a frame-independent expression of $\boldsymbol{\tau}^p_{sh}$ it is set

$$\boldsymbol{\tau}^p_{sh} = \mathbf{A}^p \mathbf{g}_t \tag{3.35}$$

where

$$\mathbf{A}^p = [\alpha(\mathbf{r} \otimes \mathbf{r}) + \beta(\mathbf{r} \cdot \mathbf{r})\hat{\mathbf{I}}] \tag{3.36}$$

whereas α and β are algebraic constants to be determined so as to fulfill (3.33) and $\hat{\mathbf{I}}$ is defined as

$$\hat{\mathbf{I}} = \mathbf{I} - \mathbf{k} \otimes \mathbf{k} \tag{3.37}$$

Substituting (3.35) in (3.33) and computing the divergence and the curl of the monomials appearing in the expression (3.36) of $\mathbf{A}^p$ by means of the formulas

$$\begin{aligned} &(\mathbf{r} \otimes \mathbf{r})\mathbf{g}_t \cdot \boldsymbol{\nabla} = \mathbf{g}_t \cdot (\mathbf{r} \otimes \mathbf{r})\boldsymbol{\nabla} = \mathbf{g}_t[\mathbf{r}(\mathbf{r} \cdot \boldsymbol{\nabla}) + (\mathbf{r} \otimes \boldsymbol{\nabla})\mathbf{r}] = \\ &\qquad = \mathbf{g}_t(2\mathbf{r} + \mathbf{r}) = 3\mathbf{g}_t \cdot \mathbf{r} \\ &(\mathbf{r} \cdot \mathbf{r})\mathbf{g}_t \cdot \boldsymbol{\nabla} = \mathbf{g}_t \cdot (\mathbf{r} \cdot \mathbf{r})\boldsymbol{\nabla} = 2\mathbf{g}_t \cdot \mathbf{r} \\ &\{[(\mathbf{r} \otimes \mathbf{r})\mathbf{g}_t] \times \boldsymbol{\nabla}\} \cdot \mathbf{k} = [(\mathbf{r} \cdot \mathbf{g}_t)\mathbf{r} \times \boldsymbol{\nabla}] \cdot \mathbf{k} = \mathbf{k} \times \mathbf{r} \cdot \mathbf{g}_t = \mathbf{g}_t \cdot \mathbf{r}_\perp \\ &\{[(\mathbf{r} \cdot \mathbf{r})\mathbf{g}_t] \times \boldsymbol{\nabla}\} \cdot \mathbf{k} = \mathbf{g}_t \times 2\mathbf{r} \cdot \mathbf{k} = -2\mathbf{k} \times \mathbf{r} \cdot \mathbf{g}_t = -2\mathbf{g}_t \cdot \mathbf{r}_\perp \end{aligned} \tag{3.38}$$

the following system in the unknowns coefficients α and β is arrived at

$$\begin{cases} 3\alpha\mathbf{g}_t \cdot \mathbf{r} + 2\beta\mathbf{g}_t \cdot \mathbf{r} = \mathbf{g}_t \cdot \mathbf{r} \\ \\ \alpha\mathbf{g}_t \cdot \mathbf{r}^\perp + \beta(-2\mathbf{g}_t \cdot \mathbf{r}) = \bar{\nu} \end{cases} \tag{3.39}$$

The solution of the linear system above provides finally

$$\alpha = \frac{1+\bar{\nu}}{4} \qquad \beta = \frac{1-3\bar{\nu}}{8} \tag{3.40}$$

so that the term $\mathbf{A}^p$ turns out to be defined by the following tensor expression

$$\mathbf{A}^p = \frac{1+\bar{\nu}}{4}(\mathbf{r}\otimes\mathbf{r}) + \frac{1-3\bar{\nu}}{8}(\mathbf{r}\cdot\mathbf{r})\mathbf{I}\,, \tag{3.41}$$

whose matrix representation is

$$[\mathbf{A}^p]_{xy} = \begin{bmatrix} \alpha x^2 + \beta(x^2+y^2) & \alpha xy & 0 \\ \\ \alpha xy & \alpha y^2 + \beta(x^2+y^2) & 0 \\ \\ 0 & 0 & 0 \end{bmatrix} \tag{3.42}$$

Consequently, the contribution to the tangential stress field associated with the particular integral $\boldsymbol{\tau}^p_{sh}$, solution of the differential problem (3.33)-(3.34), turns out to be

$$\boldsymbol{\tau}^p_{sh} = \frac{1+\bar{\nu}}{4}(\mathbf{r}\cdot\mathbf{g}_t)\mathbf{r} + \frac{1-3\bar{\nu}}{8}(\mathbf{r}\cdot\mathbf{r})\mathbf{g}_t \tag{3.43}$$

The solution to the homogeneous differential problem (3.32) amounts to finding a harmonic potential φ_{sh}, hence satisfying the condition $(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\varphi_{sh} = 0$, with prescribed directional derivative along the boundary $\partial\Omega$. Being

$$\boldsymbol{\tau}^0_{sh}\cdot\mathbf{n} = -\boldsymbol{\tau}^p_{sh}\cdot\mathbf{n} \tag{3.44}$$

on account of (3.34) and (3.31), the homogeneous differential problem can be equivalently formulated in the form

$$\begin{cases} (\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\varphi_{sh} = \mathbf{0} & \text{in the interior of } \Omega \\ \\ (\varphi_{sh}\otimes\boldsymbol{\nabla})\mathbf{n} = -\mathbf{A}^p\mathbf{n} & \text{on the boundary } \partial\Omega \end{cases} \tag{3.45}$$

Let us observe that due to the special form of the boundary term, the potential φ_{sh} depends on $\mathbf{g}_t$, and hence on the applied shear $\mathbf{t}$ so that this field does not have a pure geometrical nature, as it happens for the quantities φ_{tor}, $\mathbf{A}^p$ and $\mathbf{J}_G$, associated with pure torsion, bending and axial load, respectively, which depend solely on the geometry of Ω.

For this reason, to state a problem equivalent to (3.45) exploiting a field that has an exclusively geometrical nature one set

$$\varphi_{sh} = \boldsymbol{\psi} \cdot \mathbf{g}_t \tag{3.46}$$

where the harmonic nature of φ_{sh} is carried over to $\boldsymbol{\psi}$, $(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\boldsymbol{\psi} = 0$. By virtue of (3.46) the boundary condition $(3.45)_2$ can be written as

$$\begin{aligned} &[(\boldsymbol{\psi} \cdot \mathbf{g}_t)\boldsymbol{\nabla}] \cdot \mathbf{n} = -\mathbf{A}^p \mathbf{g}_t \cdot \mathbf{n} \\ &(\boldsymbol{\nabla} \otimes \boldsymbol{\psi})\mathbf{g}_t \cdot \mathbf{n} = -\mathbf{g}_t \cdot (\mathbf{A}^p)^t \mathbf{n} \\ &\mathbf{g}_t \cdot (\boldsymbol{\psi} \otimes \boldsymbol{\nabla})\mathbf{n} = -\mathbf{g}_t \cdot (\mathbf{A}^p)^t \mathbf{n} \end{aligned} \tag{3.47}$$

finally one obtains

$$\mathbf{g}_t \cdot \left[(\boldsymbol{\psi} \otimes \boldsymbol{\nabla})\mathbf{n} + (\mathbf{A}^p)^t \mathbf{n}\right] = 0 \tag{3.48}$$

Since the previous expression holds for any $\mathbf{g}_t$, the term under square brackets must be zero and the boundary condition $(3.45)_2$ can be equivalently expressed as

$$(\boldsymbol{\psi} \otimes \boldsymbol{\nabla})\mathbf{n} = -(\mathbf{A}^p)^t \mathbf{n} = -\left[\frac{1-\bar{\nu}}{4}(\mathbf{r} \otimes \mathbf{r}) + \frac{1+3\bar{\nu}}{8}(\mathbf{r} \cdot \mathbf{r})\mathbf{I}\right]\mathbf{n} \tag{3.49}$$

To sum up the harmonic vector field $\boldsymbol{\psi}$ is defined, up to a constant vector, as the solution of the following Neumann vector problem

$$\begin{cases} (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\boldsymbol{\psi} = \mathbf{0} & \text{in the interior of } \Omega \\ (\boldsymbol{\psi} \otimes \boldsymbol{\nabla})\mathbf{n} = -\mathbf{A}^p \mathbf{n} & \text{on the boundary } \partial\Omega \end{cases} \tag{3.50}$$

The tangential stress field associated with the shear force $\mathbf{t}$ is given by

$$\boldsymbol{\tau}_{sh} = (\boldsymbol{\nabla} \otimes \boldsymbol{\psi})\,\mathbf{g}_t + \mathbf{A}^p\,(\mathbf{r})\,\mathbf{g}_t \tag{3.51}$$

3.5 Tangential stress field

The solution to differential system (3.21) is obtained by summing (3.30) and (3.51)

$$\begin{aligned}\boldsymbol{\tau} &= \boldsymbol{\tau}_{tor} + \boldsymbol{\tau}_{sh} = (\boldsymbol{\nabla} \otimes \boldsymbol{\psi})\,\mathbf{g}_t + \mathbf{A}^p\,(\mathbf{r})\,\mathbf{g}_t + \frac{c}{2}\varphi_{tor}\boldsymbol{\nabla} + \frac{c}{2}\mathbf{W}_{\mathbf{k}}\mathbf{r} = \\ &= [\boldsymbol{\nabla} \otimes \boldsymbol{\psi} + \mathbf{A}^p\,(\mathbf{r})]\mathbf{g}_t + \frac{c}{2}(\varphi_{tor}\boldsymbol{\nabla} + \mathbf{W}_{\mathbf{k}}\mathbf{r})\end{aligned} \tag{3.52}$$

Notice from the previous formula that the tangential stress field $\boldsymbol{\tau}$ is defined only by purely geometrical entities

3.6 De Saint Venant stress field expressed in terms of internal forces

Combining first equation in (3.12) with (3.52) gives

$$\begin{cases} \sigma_z &= a_0 + \mathbf{g} \cdot \mathbf{r} + (l - z)\mathbf{g}_t \cdot \mathbf{r} \\ \boldsymbol{\tau} &= [\boldsymbol{\nabla} \otimes \boldsymbol{\psi} + \mathbf{A}^p\,(\mathbf{r})]\mathbf{g}_t + \frac{c}{2}(\varphi_{tor}\boldsymbol{\nabla} + \mathbf{W}_{\mathbf{k}}\mathbf{r}) \end{cases} \tag{3.53}$$

Notice from (3.53) that the stress field depends on $a_0, \mathbf{g}_t, \mathbf{g}, c$. This section illustrates relations between these constants and the internal forces in order to obtain an expression of the stress field that depends on these parameters.

Equilibrium equation to translation with respect to z direction gives

$$\begin{aligned} N\mathbf{k} &= \int_A \sigma\, da = \int_A \sigma\mathbf{k}\, da = \\ &= \left[a_0 \int_A da + \mathbf{g} \cdot \int_A \mathbf{r}\, da + (l - z)\mathbf{g}_t \cdot \int_A \mathbf{r}\, da\right] \mathbf{k} = a_0 A\mathbf{k}\end{aligned}$$

thus

$$a_0 = \frac{N}{A} \tag{3.54}$$

while, equilibrium equation to translation in the plane x, y is expressed by

$$\mathbf{t}_s = \int_A \tau\, da \tag{3.55}$$

In order to develop integral in (3.55) on the boundary it is illustrated the following property

$$(\mathbf{r}\otimes\boldsymbol{\tau})\boldsymbol{\nabla} = (\mathbf{r}\cdot\boldsymbol{\nabla})\boldsymbol{\tau} + (\boldsymbol{\tau}\cdot\boldsymbol{\nabla})\mathbf{r} = \boldsymbol{\tau} + (\boldsymbol{\tau}\cdot\boldsymbol{\nabla})\mathbf{r} \tag{3.56}$$

that can be simply verified by writing in indicial form

$$(r_i\tau_j)_{,j} = r_{i,j}\,\tau_j + r_i\tau_{j,j} = \delta_{ij}\tau_j + r_i\tau_{j,j} = \tau_i + r_i\tau_{j,j}$$

thus, in the light of (3.56), equation (3.55) become

$$\begin{aligned}
\mathbf{t}_s &= \int_A \operatorname{div}(\mathbf{r}\otimes\boldsymbol{\tau})\,da - \int_A \mathbf{r}(\operatorname{div}\boldsymbol{\tau})\,da = \\
&= \int_{\partial A}(\mathbf{r}\otimes\boldsymbol{\tau})\mathbf{n}\,ds - \int_A \mathbf{r}(\mathbf{g}_t\cdot\mathbf{r})\,da = \\
&= \int_{\partial A}(\boldsymbol{\tau}\cdot\mathbf{n})\mathbf{r}\,ds - \int_A (\mathbf{r}\otimes\mathbf{r})\mathbf{g}_t\,da
\end{aligned} \tag{3.57}$$

The first integral is equal to zero according to boundary condition $\boldsymbol{\tau}\cdot\mathbf{n} = 0$, hence from (3.57) one has

$$\mathbf{t}_s = -\mathbf{J}_G\mathbf{g}_t \Longleftrightarrow \mathbf{g}_t = -\mathbf{J}_G^{-1}\mathbf{t}_s\ . \tag{3.58}$$

Equilibrium to rotation with respect to x and y axes gives

$$\begin{aligned}
\mathbf{m}_f(z) &= \int_A \mathbf{r}\times\sigma_z\mathbf{k}\,da = \int_A \mathbf{r}\times(a_0 + \mathbf{g}\cdot\mathbf{r} + (l-z)\mathbf{g}_t\cdot\mathbf{r})\mathbf{k}\,da = \\
&= a_0\int_A \mathbf{r}\,da\times\mathbf{k} + \int_A \mathbf{r}\times(\mathbf{g}\cdot\mathbf{r})\mathbf{k}\,da + (l-z)\int_A \mathbf{r}\times(\mathbf{g}_t\cdot\mathbf{r})\mathbf{k}\,da = \\
&= -\mathbf{k}\times\left(\int_A(\mathbf{g}\cdot\mathbf{r})\mathbf{r}\,da + (l-z)\int_A(\mathbf{g}_t\cdot\mathbf{r})\mathbf{r}\,da\right) = \\
&= \mathbf{W}_{\mathbf{k}}^t\left(\int_A \mathbf{r}\otimes\mathbf{r}\,da\right)[\mathbf{g} + (l-z)\mathbf{g}_t] = \\
&= \mathbf{W}_{\mathbf{k}}^t\,(\mathbf{J}_G\mathbf{g} + (l-z)\mathbf{J}_G\mathbf{g}_t) = \mathbf{W}_{\mathbf{k}}^t\,(\mathbf{J}_G\mathbf{g} - (l-z)\mathbf{t}_s)
\end{aligned} \tag{3.59}$$

hence

$$\mathbf{m}_f(z) = \mathbf{W}_{\mathbf{k}}^t\,(\mathbf{J}_G\mathbf{g} - (l-z)\mathbf{t}_s)\ ;$$

having set

$$\mathbf{m}_f(l) = \mathbf{m}_l$$

one has

$$\mathbf{m}_l = \mathbf{W}_{\mathbf{k}}^t \mathbf{J}_G \mathbf{g} \Leftrightarrow \mathbf{g} = \mathbf{J}_G^{-1} \mathbf{W}_{\mathbf{k}} \mathbf{m}_l \ ,$$

$$\mathbf{m}_f(z) = \mathbf{m}_l + (l - z)\mathbf{W}_{\mathbf{k}} \mathbf{t}_s \ .$$

Moreover equilibrium to rotation with respect to z axis gives

$$\mathbf{m}_t = \int_A \mathbf{r} \times \boldsymbol{\tau} \, da = \left(\int_A \mathbf{W}_{\mathbf{k}} \mathbf{r} \cdot \boldsymbol{\tau} \, da \right) \mathbf{k} \tag{3.60}$$

by observing that both $\mathbf{r}$ and $\boldsymbol{\tau}$ are orthogonal with respect to $\mathbf{k}$

$$\mathbf{r} \times \boldsymbol{\tau} = (\mathbf{k} \otimes \mathbf{k})(\mathbf{r} \times \boldsymbol{\tau}) = (\mathbf{r} \times \boldsymbol{\tau} \cdot \mathbf{k})\mathbf{k} = (\mathbf{k} \times \mathbf{r} \cdot \boldsymbol{\tau})\mathbf{k} = (\mathbf{W}_{\mathbf{k}} \mathbf{r} \cdot \boldsymbol{\tau})\mathbf{k} \tag{3.61}$$

thus, by substituting (3.61) in (3.60) it results

$$\mathbf{m}_t = M_t \mathbf{k} \ . \tag{3.62}$$

Finally, the stress field that satisfy De Saint Venant hypothesis in function of the internal forces has the following expression:

$$\left\{ \begin{array}{lcl} \sigma_z & = & \dfrac{N}{A} + \mathbf{J}_G^{-1} \mathbf{W}_{\mathbf{k}} \mathbf{m}_l \cdot \mathbf{r} - (l - z)(\mathbf{J}_G^{-1} \mathbf{t}_s \cdot \mathbf{r}) \\ \boldsymbol{\tau} & = & [\boldsymbol{\nabla} \otimes \boldsymbol{\psi} + \mathbf{A}^p (\mathbf{r})] \mathbf{J}_G^{-1} + \dfrac{c}{2}(\varphi_{tor} \boldsymbol{\nabla} + \mathbf{W}_{\mathbf{k}} \mathbf{r}) \end{array} \right. \tag{3.63}$$

where $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}_{tor}$ are harmonic functions that satisfy the boundary conditions

$$\begin{gathered} (\varphi_{tor} \otimes \boldsymbol{\nabla})\mathbf{n} = -\frac{c}{2} \mathbf{W}_{\mathbf{k}} \mathbf{r} \\ (\boldsymbol{\psi} \otimes \boldsymbol{\nabla})\mathbf{n} = -\mathbf{A}^p \mathbf{n} \end{gathered} \tag{3.64}$$

3.6.1 The torsional stiffness factor

By setting

$$\mathbf{B}(\mathbf{r}) = \boldsymbol{\nabla} \otimes \boldsymbol{\psi} + \mathbf{A}^p$$

in (3.52), one can obtain an alternative expression of $\mathbf{m}_t$:

$$\begin{aligned}\mathbf{m}_t &= \left[\int_A \mathbf{W_k r}\cdot\mathbf{B}(\mathbf{r})\mathbf{g}_t\, da + \tfrac{c}{2}\int_A \mathbf{W_k r}\cdot(\operatorname{grad}\varphi_\mathbf{t} + \mathbf{W_k r})\, da\right]\mathbf{k} = \\ &= \left[\mathbf{g}_t\cdot\int_A \mathbf{B}^t(\mathbf{r})\mathbf{W_k r}\, da + \tfrac{c}{2}\int_A \mathbf{W_k r}\cdot(\operatorname{grad}\varphi_\mathbf{t} + \mathbf{W_k r})\, da\right]\mathbf{k}\ .\end{aligned} \tag{3.65}$$

In the specific case of pure torsion ($\mathbf{t}_s = 0$)

$$\mathbf{g}_t = -\mathbf{J}_G^{-1}\mathbf{t} = \mathbf{0} \tag{3.66}$$

then, by (3.65) and (3.66)

$$\begin{aligned}\mathbf{m}_t &= \left[\int_A \mathbf{W_k r}\cdot\frac{c}{2}(\operatorname{grad}\varphi_\mathbf{t} + \mathbf{W_k r})\, da\right]\mathbf{k} = \\ &= \frac{c}{2}\left[\int_A (\operatorname{grad}\varphi_t\cdot\mathbf{W_k r} + \mathbf{r}\cdot\mathbf{r})\, da\right]\mathbf{k} = \frac{c}{2}I_q\mathbf{k}\end{aligned} \tag{3.67}$$

having introduced

$$I_q = \left[\int_A (\operatorname{grad}\varphi_t\cdot\mathbf{W_k r} + \mathbf{r}\cdot\mathbf{r})\, da\right]$$

that represent the torsional stiffness factor.

The relation between I_q and $\mathbf{m}_t$ can be reached by combining (3.62) and (3.67):

$$\mathbf{m}_t = M_t\mathbf{k} = \frac{c}{2}I_q\mathbf{k} \Rightarrow c = 2\frac{M_t}{I_q}\quad .$$

3.7 Closed-form analytical solutions

This section reports the analytical solution to the shear problem (3.50) in terms of tangential stresses, for some common sections for which a closed-form solution is available: the circular section, the elliptic section and the rectangular section in the case $\nu = 0$

3.7.1 Circular section

The solution to the previously stated shear problem has a significantly synthetic expression for the circular section. It can be easily verified that the field $\boldsymbol{\psi}$, solution of (3.50), associated with a section of radius R is:

$$\boldsymbol{\psi} = -(\alpha+\beta)R^2\mathbf{r} = -\frac{3+2\nu}{8(1+\nu)}R^2\mathbf{r} \tag{3.68}$$

Actually, being linear, it is also harmonic and satisfies $(3.50)_1$. Moreover, its gradient, $\boldsymbol{\psi} \otimes \boldsymbol{\nabla}$, turns out to be

$$\boldsymbol{\psi} \otimes \boldsymbol{\nabla} = -\frac{3+2\nu}{8(1+\nu)}R^2\hat{\mathbf{I}}. \tag{3.69}$$

and on the boundary one has

$$\mathbf{r} = R\mathbf{n} \tag{3.70}$$

since the origin of the reference frame is located in the centroid. Thus, upon substituting the previous relation into (3.36) one obtains

$$\mathbf{A}^p\mathbf{n} = \frac{3+2\nu}{8(1+\nu)}R^2\mathbf{r} \tag{3.71}$$

an expression which fulfills the boundary condition $(3.50)_2$ as well.

3.7.2 Elliptic section

In the case of the elliptic section it is convenient to express the solution in the reference frame of the principal axes. In such a frame, say xy, one has

$$\begin{bmatrix} \psi_x \\ \psi_y \end{bmatrix} = -\frac{1}{2(1+\nu)} \begin{bmatrix} \dfrac{2(1+\nu)R_x^2+R_y^2}{3R_x^2+R_y^2} & 0 \\ 0 & \dfrac{2(1+\nu)R_y^2+R_x^2}{3R_y^2+R_x^2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \\ + \frac{1-2\nu}{8(1+\nu)} \begin{bmatrix} \dfrac{R_y^2-R_x^2}{3R_x^2+R_y^2} & 0 \\ 0 & \dfrac{R_x^2-R_y^2}{3R_y^2+R_x^2} \end{bmatrix} \begin{bmatrix} \dfrac{x^3}{3}-xy^2 \\ \dfrac{y^3}{3}-yx^2 \end{bmatrix} \tag{3.72}$$

where R_x and R_y are the principal radii of the ellipse. It can be easily verified that the components of (3.72) are harmonic functions, as its terms x, y, $\frac{x^3}{3} - xy^2$ and $\dfrac{y^3}{3} - yx^2$ do all possess such feature and it is also recognized that (3.72) specializes to the matrix form of (3.68) whenever the axes have the same length.

3.7.3 Rectangular section ($\nu = 0$)

Provided that $\nu = 0$, the present approach yields a closed form expression for the harmonic vector potential $\boldsymbol{\psi}$, and hence the shear stress $\boldsymbol{\tau}_{sh}$ analogous to the classical solution reported in [57]. Observing that in this case the solution is similar under several respects to the one for the elliptic section

$$[\boldsymbol{\psi}] = \begin{bmatrix} \psi_x \\ \psi_y \end{bmatrix} = -\frac{1}{8} \begin{bmatrix} L_x^2 & 0 \\ 0 & L_y^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{8} \begin{bmatrix} \dfrac{x^3}{3} - xy^2 \\ \dfrac{y^3}{3} - yx^2 \end{bmatrix} \tag{3.73}$$

where x and y are parallel to the edges of the rectangle whereas L_x and L_y are their respective lengths. By virtue of (3.51) the shear stress field generated by a shear force directed along y applied at the centroid turns out to be

$$\tau_{shx} = 0, \qquad \tau_{shy} = \frac{6t_y}{L_x L_y^3} \left(\frac{L_y^2}{4} - y^2 \right) \tag{3.74}$$

Chapter 4

Displacement field, shear center and deformability tensor

This chapter illustrates the derivation of the displacement field associated with each kind of internal force, e.g. axial force, biaxial bending, torsion and biaxial shear. Moreover we provide the expression of the shear center and of deformability tensor.

4.1 Derivation of the displacement field

As detailed in the introduction aim of this section is to complete the stress-based solution of Saint-Venant problem due to Baldacci [6] by deriving the displacement field separately for axial force, biaxial bending, torsion and biaxial shear.

4.1.1 Displacement field associated with axial force

By substituting $\mathbf{g} = \mathbf{g}_t = \mathbf{0}$ in (3.4) one obtains the stress field associated with axial force.

$$\sigma_z = a_0 \tag{4.1}$$

Hence the relevant stress tensor has the following expression

$$\mathbf{S} = a_0(\mathbf{k} \otimes \mathbf{k}) \ . \tag{4.2}$$

Upon substituting the previous expression in the elastic-law $(2.100)_2$ one gets

$$\mathbf{E} = \frac{a_0(1+\nu)}{E}(\mathbf{k} \otimes \mathbf{k}) - \frac{a_0\nu}{E}\mathbf{I} \ . \tag{4.3}$$

The displacements field $\mathbf{u}$ associated with the stress field defined above can be computed by means of a direct integration procedure which is substantially equivalent to the use of Cesaro's formulas [16]. In particular, once the infinitesimal strain field $\mathbf{E}$ has been obtained, one can exploit the identity [29]

$$\boldsymbol{\omega} \otimes \boldsymbol{\nabla} = \boldsymbol{\nabla} \times \mathbf{E}^t \tag{4.4}$$

between the gradient of the axial vector $\boldsymbol{\omega}$ of the skew-symmetric part of the displacement gradient $\mathbf{W}$ and the curl of $\mathbf{E}$. Thus, by employing the definition (2.6), one can derive $\mathbf{W}$ as the cross product between $\boldsymbol{\omega}$ and $\mathbf{I}$, i.e. $\mathbf{W} = \boldsymbol{\omega} \times \mathbf{I}$. Finally, a second integration of the displacement gradient field $\mathbf{u} \otimes \boldsymbol{\nabla} = \mathbf{E} + \mathbf{W}$ provides the displacement field $\mathbf{u}$ which is looked for.

Since $\mathbf{E}$ is constant, it turns out to be

$$\operatorname{rot} \mathbf{E} = \boldsymbol{\nabla} \times \frac{a_0}{E}\left[(1+\nu)(\mathbf{k} \otimes \mathbf{k}) - \nu\mathbf{I}\right] = \mathbf{0}$$

and

$$\operatorname{grad} \boldsymbol{\omega} = \boldsymbol{\omega} \otimes \boldsymbol{\nabla} = \mathbf{0} \implies \boldsymbol{\omega} = \boldsymbol{\omega}_0$$

the final result is

$$\mathbf{W} = \boldsymbol{\omega}_0 \times \mathbf{I} \ .$$

Apart from a rigid rotation, the gradient of displacement is symmetric so that coincides with the strain tensor, i.e.

$$\operatorname{grad}\mathbf{u} = \mathbf{E} = \frac{a_0(1+\nu)}{E}(\mathbf{k}\otimes\mathbf{k}) - \frac{a_0\nu}{E}\mathbf{I}$$

Being constant integration of the previous relation provides

$$\mathbf{u} = \frac{a_0(1+\nu)}{E}(\mathbf{k}\otimes\mathbf{k})\mathbf{p} - \frac{a_0\nu}{E}\mathbf{p} = \frac{a_0}{E}\left[(1+\nu)z\mathbf{k} - \nu\mathbf{p}\right] \ .$$

Upon substituting $a_0 = N/A$ an expression explicitly related to the normal stress is obtained

$$\mathbf{u} = \frac{N}{EA}(z\mathbf{k} - \nu x\mathbf{i} - \nu y\mathbf{j}) \ .$$

4.1.2 Displacement field associated with biaxial bending

By substituting $\mathbf{g} = \mathbf{0}$ and $a_0 = 0$ in (3.4) one obtains the stress field due to biaxial bending

$$\sigma_z = \mathbf{g}\cdot\mathbf{r} \tag{4.5}$$

Hence the stress tensor and the associated strain tensor have in turn the following expressions

$$\mathbf{S} =(\mathbf{g}\cdot\mathbf{r})(\mathbf{k}\otimes\mathbf{k}) \ . \tag{4.6}$$

and

$$\mathbf{E} = \left[\frac{1+\nu}{E}\mathbb{I} - \frac{\nu}{E}(\mathbf{I}\otimes\mathbf{I})\right]\mathbf{S} = \frac{(1+\nu)}{E}(\mathbf{g}\cdot\mathbf{r})(\mathbf{k}\otimes\mathbf{k}) - \frac{\nu}{E}(\mathbf{g}\cdot\mathbf{r})\mathbf{I} \ . \tag{4.7}$$

The displacements field $\mathbf{u}$ associated with the stress field defined above can be computed by means of a direct integration procedure which is substantially equivalent to the use of Cesaro's formulas [16]. In particular, starting from (4.7) one can exploit the identity [29]

$$\boldsymbol{\omega}\otimes\boldsymbol{\nabla} = \boldsymbol{\nabla}\times\mathbf{E}^t \tag{4.8}$$

between the gradient of the axial vector $\boldsymbol{\omega}$ of the skew-symmetric part of the displacement gradient $\mathbf{W}$ and the curl of $\mathbf{E}$.

Accordingly, by employing the definition (2.8), one can derive $\mathbf{W}$ as the cross product between $\boldsymbol{\omega}$ and $\mathbf{I}$, i.e. $\mathbf{W} = \boldsymbol{\omega} \times \mathbf{I}$. Finally, a second integration of the displacement gradient field $\mathbf{u} \otimes \boldsymbol{\nabla} = \mathbf{E} + \mathbf{W}$ provides the displacement field $\mathbf{u}$ which is looked for.

Now it is necessary to compute $\operatorname{curl} \mathbf{E}$

$$\begin{aligned}
\operatorname{curl} \mathbf{E} &= \boldsymbol{\nabla} \times \frac{1}{E} \left[(1+\upsilon)(\mathbf{g} \cdot \mathbf{r})(\mathbf{k} \otimes \mathbf{k}) - \upsilon(\mathbf{g} \cdot \mathbf{r})\mathbf{I}\right]^t = \\
&= \frac{(1+\upsilon)}{E} \left[\boldsymbol{\nabla} \times (\mathbf{g} \cdot \mathbf{r})\mathbf{k}\right] \otimes \mathbf{k} - \frac{\upsilon}{E} \left[\boldsymbol{\nabla} \times (\mathbf{g} \cdot \mathbf{r})\mathbf{I}\right]
\end{aligned}$$

by observing that

$$\operatorname{curl} \left[(\mathbf{g} \cdot \mathbf{r})\mathbf{k}\right] = \boldsymbol{\nabla} \times (\mathbf{g} \cdot \mathbf{r})\mathbf{k} = \boldsymbol{\nabla}(\mathbf{g} \cdot \mathbf{r}) \times \mathbf{k} = \operatorname{grad}(\mathbf{g} \cdot \mathbf{r}) \times \mathbf{k} = \mathbf{g} \times \mathbf{k} \quad (4.9)$$

and

$$\operatorname{curl} \left[(\mathbf{g} \cdot \mathbf{r})\mathbf{I}\right] = \boldsymbol{\nabla} \times (\mathbf{g} \cdot \mathbf{r})\mathbf{I} = \boldsymbol{\nabla}(\mathbf{g} \cdot \mathbf{r}) \times \mathbf{I} = \operatorname{grad}(\mathbf{g} \cdot \mathbf{r}) \times \mathbf{I} = \mathbf{g} \times \mathbf{I} \ , \quad (4.10)$$

one can express the gradient of $\boldsymbol{\omega}$ and curl of $\mathbf{E}$ as follows

$$\operatorname{grad} \boldsymbol{\omega} = \operatorname{curl} \mathbf{E} = \frac{(1+\nu)}{E} \left[(\mathbf{g} \times \mathbf{k}) \otimes \mathbf{k}\right] - \frac{\nu}{E}(\mathbf{g} \times \mathbf{I}) \ . \quad (4.11)$$

Integration of constant tensors in (4.11) gives

$$\boldsymbol{\omega} = \frac{(1+\nu)}{E} \left[(\mathbf{g} \times \mathbf{k}) \otimes \mathbf{k}\right] \mathbf{r} - \frac{\nu}{E}(\mathbf{g} \times \mathbf{r}) + \boldsymbol{\omega}_0 = \frac{(1+\nu)}{E} z(\mathbf{g} \times \mathbf{k}) - \frac{\nu}{E}(\mathbf{g} \times \mathbf{r}) + \boldsymbol{\omega}_0$$

and then, except for a rigid rotation, the anti-symmetric part of grad $\mathbf{u}$ results

$$\mathbf{W} = \boldsymbol{\omega} \times \mathbf{I} = \frac{(1+\nu)}{E} z(\mathbf{g} \times \mathbf{k}) \times \mathbf{I} - \frac{\nu}{E}(\mathbf{g} \times \mathbf{r}) \times \mathbf{I} \ . \quad (4.12)$$

On account of the following property

$$\mathbf{A} \times (\mathbf{b} \times \mathbf{c}) = -(\mathbf{b} \times \mathbf{c}) \times \mathbf{A} = (\mathbf{b} \otimes \mathbf{c})\mathbf{A} - (\mathbf{c} \otimes \mathbf{b})\mathbf{A}$$

the term on the left side of equation (4.12) can be expressed by

$$(\mathbf{g} \times \mathbf{k}) \times \mathbf{I} = -\mathbf{I} \times (\mathbf{g} \times \mathbf{k}) = (\mathbf{k} \otimes \mathbf{g}) - (\mathbf{g} \otimes \mathbf{k})$$

$$(\mathbf{g}\times\mathbf{r})\times\mathbf{I}=(\mathbf{r}\otimes\mathbf{g})-(\mathbf{g}\otimes\mathbf{r})$$

hence, equation (4.12) becomes

$$\mathbf{W}=\frac{(1+\nu)}{E}z\left[(\mathbf{k}\otimes\mathbf{g})-(\mathbf{g}\otimes\mathbf{k})\right]-\frac{\nu}{E}\left[(\mathbf{r}\otimes\mathbf{g})-(\mathbf{g}\otimes\mathbf{r})\right]\ .$$

Finally

$$\operatorname{grad}\mathbf{u}=\mathbf{E}+\mathbf{W}=$$

$$=\frac{(1+\nu)}{E}\left\{-(\mathbf{g}\otimes z\mathbf{k})+\mathbf{k}\otimes\left[(\mathbf{r}\cdot\mathbf{g})\mathbf{k}+(\mathbf{r}\cdot\mathbf{k})\mathbf{g}\right]\right\}+\frac{\nu}{E}\left\{(\mathbf{g}\otimes\mathbf{r})-\left[(\mathbf{g}\cdot\mathbf{r})\mathbf{I}+(\mathbf{r}\otimes\mathbf{g})\right]\right\}$$

Integration of the previous formula is immediate by means of following expressions

$$z\mathbf{k}=\operatorname{grad}\frac{z^2}{2}\implies z(\mathbf{g}\otimes\mathbf{k})=\operatorname{grad}\left(\frac{z^2}{2}\mathbf{g}\right)\ ,$$

$$\left[(\mathbf{r}\cdot\mathbf{g})\mathbf{k}+(\mathbf{r}\cdot\mathbf{k})\mathbf{g}\right]=\operatorname{grad}\left[(\mathbf{r}\cdot\mathbf{g})(\mathbf{r}\cdot\mathbf{k})\right]\implies\mathbf{k}\otimes\left[(\mathbf{r}\cdot\mathbf{g})\mathbf{k}+(\mathbf{r}\cdot\mathbf{k})\mathbf{g}\right]=\operatorname{grad}\left[(\mathbf{r}\cdot\mathbf{g})(\mathbf{r}\cdot\mathbf{k})\mathbf{k}\right]\ ,$$

$$\mathbf{r}=\operatorname{grad}\frac{|\mathbf{r}|^2}{2}\implies(\mathbf{g}\otimes\mathbf{r})=\operatorname{grad}\left[\frac{(\mathbf{r}\cdot\mathbf{r})}{2}\mathbf{g}\right]\ ,$$

$$\left[(\mathbf{g}\cdot\mathbf{r})\mathbf{I}+(\mathbf{r}\otimes\mathbf{g})\right]=\operatorname{grad}\left[(\mathbf{g}\cdot\mathbf{r})\mathbf{r}\right]\ .$$

Finally, the displacement field associated with biaxial bending turns out to be

$$\mathbf{u}=\frac{(1+\nu)}{E}\left[-\frac{z^2}{2}\mathbf{g}+z(\mathbf{r}\cdot\mathbf{g})\mathbf{k}\right]+\frac{\nu}{E}\left[\frac{(\mathbf{r}\cdot\mathbf{r})}{2}\mathbf{g}-(\mathbf{g}\cdot\mathbf{r})\mathbf{r}\right]+\boldsymbol{\omega}_0\times\mathbf{r}+\mathbf{u}_0\ .$$

4.1.3 Displacement field associated with torsion

The tangential stress field $\boldsymbol{\tau}_{tor}$ is provided by (3.53) by setting $\mathbf{g}_t=\mathbf{0}$ so that the stress tensor $\mathbf{S}$ becomes

$$\mathbf{S}=(\mathbf{k}\otimes\boldsymbol{\tau})+(\boldsymbol{\tau}\otimes\mathbf{k})=\left[\mathbf{k}\otimes\frac{c}{2}\left(\operatorname{grad}\varphi_t+\mathbf{W}_{\mathbf{k}}\mathbf{r}\right)\right]+\left[\frac{c}{2}\left(\operatorname{grad}\varphi_t+\mathbf{W}_{\mathbf{k}}\mathbf{r}\right)\otimes\mathbf{k}\right]\ . \tag{4.13}$$

Substituting the previous expression in the elastic-law $(2.100)_2$ yelds

$$\mathbf{E}=\left[\frac{1}{2G}\mathbb{I}-\frac{\lambda}{2G(3\lambda+2G)}(\mathbf{I}\otimes\mathbf{I})\right]\mathbf{S} \tag{4.14}$$

Since the tensor $\mathbf{S}$ is traceless, it turns out to be

$$\mathbf{E} = \frac{1}{2G}\left[(\mathbf{k}\otimes\boldsymbol{\tau})+(\boldsymbol{\tau}\otimes\mathbf{k})\right] = \frac{c}{2G}\left\{\frac{1}{2}[\mathbf{k}\otimes(\operatorname{grad}\varphi_t+\mathbf{k}\times\mathbf{r})]+\frac{1}{2}[(\operatorname{grad}\varphi_t+\mathbf{k}\times\mathbf{r})\otimes\mathbf{k}]\right\} =$$

$$= \frac{c}{2G}\left\{\frac{1}{2}\left[(\operatorname{grad}\varphi_t\otimes\mathbf{k})+(\mathbf{k}\otimes\operatorname{grad}\varphi_t)\right]+\frac{1}{2}[(\mathbf{k}\times\mathbf{r})\otimes\mathbf{k}]+[\mathbf{k}\otimes(\mathbf{k}\times\mathbf{r})]\right\} . \tag{4.15}$$

The displacements field $\mathbf{u}$ associated with torsion field is computed by following the same steps outlined for biaxial bending; hence curl $\mathbf{E}$ is first evaluated.

$$\operatorname{curl}\mathbf{E} = \boldsymbol{\nabla}\times\frac{1}{2G}\left[(\mathbf{k}\otimes\boldsymbol{\tau})+(\boldsymbol{\tau}\otimes\mathbf{k})\right] \;\; = \frac{1}{2G}\left\{[(\boldsymbol{\nabla}\times\mathbf{k})\otimes\boldsymbol{\tau}]+[(\boldsymbol{\nabla}\times\boldsymbol{\tau})\otimes\mathbf{k}]\right\} =$$

$$= \frac{1}{2G}\left\{-\left[\mathbf{k}\times(\boldsymbol{\nabla}\otimes\boldsymbol{\tau})\right]+\left[\operatorname{curl}\boldsymbol{\tau}\otimes\mathbf{k}\right]\right\}$$

On account of $(3.24)_2$, it turns out to be $\operatorname{curl}\boldsymbol{\tau} = c\mathbf{k}$, so that the previous expression becomes

$$\operatorname{curl}\mathbf{E} = \frac{1}{2G}\left\{[c\mathbf{k}\otimes\mathbf{k}]-\left[\mathbf{k}\times(\operatorname{grad}\tau)^t\right]\right\} =$$

$$\frac{c}{2G}(\mathbf{k}\otimes\mathbf{k})-\frac{c}{2G}\left\{\frac{\mathbf{k}}{2}\times[\operatorname{grad}(\operatorname{grad}\varphi_t+\mathbf{k}\times\mathbf{r})]^t\right\} =$$

$$= \frac{c}{2G}\left\{(\mathbf{k}\otimes\mathbf{k})-\frac{1}{2}(\mathbf{k}\times\mathbf{I})\left[(\operatorname{grad}\operatorname{grad}\varphi_t)^t+(\mathbf{k}\times\mathbf{I})^t\right]\right\} =$$

$$= \frac{c}{2G}\left\{(\mathbf{k}\otimes\mathbf{k})-\frac{1}{2}\operatorname{grad}\left[(\mathbf{k}\times\mathbf{I})\operatorname{grad}\varphi_t\right]+\frac{1}{2}(\mathbf{k}\times\mathbf{I})^2\right\} .$$

where in the last algebraic manipulation proper account has been made of the fact that $\operatorname{grad}\operatorname{grad}\varphi_t$ is a symmetric tensor and $\mathbf{k}\times\mathbf{I}$ is skew-symmetric. Finally

$$\operatorname{grad}\boldsymbol{\omega} = \frac{c}{2G}\left\{(\mathbf{k}\otimes\mathbf{k})-\frac{1}{2}\operatorname{grad}\left[(\mathbf{k}\times\mathbf{I})\operatorname{grad}\varphi_t\right]+\frac{1}{2}(\mathbf{k}\times\mathbf{I})^2\right\} .$$

Integration of the previous formula provides

$$\boldsymbol{\omega} = \frac{c}{2G}\left[z\mathbf{k}-\frac{1}{2}(\mathbf{k}\times\mathbf{I})\operatorname{grad}\varphi_t+\frac{1}{2}(\mathbf{k}\times\mathbf{I})^2\mathbf{r}\right]+\boldsymbol{\omega}_0 \tag{4.16}$$

in which $\boldsymbol{\omega}_0$ is an arbitrary constant vector

Notice that projecting $\boldsymbol{\omega}$ along z axis yields

$$\omega_z = \theta_z = \boldsymbol{\omega} \cdot \mathbf{k} = \frac{c}{2G} z + \omega_{0z} \Longrightarrow \theta'_z = \frac{\partial \theta_z}{\partial z} = \frac{c}{2G} \ .$$

hence each cross-sections is characterized by a rigid rotation in the plane which varies linearly in z direction.

By virtue of (4.16), it turns out to be

$$\mathbf{W} = \theta'_z \left\{ (z\mathbf{k} \times \mathbf{I}) - \frac{1}{2} \left[(\mathbf{k} \times \operatorname{grad} \varphi_t) \times \mathbf{I} \right] + \frac{1}{2} [\mathbf{k} \times (\mathbf{k} \times \mathbf{r})] \times \mathbf{I} \right\} + \boldsymbol{\omega}_0 \times \mathbf{I}$$

and, on account of the following property

$$(\mathbf{b} \times \mathbf{c}) \times \mathbf{A} = (\mathbf{c} \otimes \mathbf{b})\mathbf{A} - (\mathbf{b} \otimes \mathbf{c})\mathbf{A}$$

the following identities

$$[(\mathbf{k} \times \operatorname{grad} \varphi_t) \times \mathbf{I}] = (\operatorname{grad} \varphi_t \otimes \mathbf{k}) - (\mathbf{k} \otimes \operatorname{grad} \varphi_t)$$

$$[\mathbf{k} \times (\mathbf{k} \times \mathbf{r})] \times \mathbf{I} = -\mathbf{I} \times [\mathbf{k} \times (\mathbf{k} \times \mathbf{r})] = [(\mathbf{k} \times \mathbf{r}) \otimes \mathbf{k}] - [\mathbf{k} \otimes (\mathbf{k} \times \mathbf{r})]$$

are inferred. Thus, except for the arbitrary rigid rotation $(\boldsymbol{\omega}_0 \times \mathbf{I})$ one obtains

$$\mathbf{W} = \theta'_z \left\{ (z\mathbf{k} \times \mathbf{I}) - \frac{1}{2} \left[(\operatorname{grad} \varphi_t \otimes \mathbf{k}) - (\mathbf{k} \otimes \operatorname{grad} \varphi_t) \right] + \frac{1}{2} [(\mathbf{k} \times \mathbf{r}) \otimes \mathbf{k}] - [\mathbf{k} \otimes (\mathbf{k} \times \mathbf{r})] \right\} \ . \tag{4.17}$$

The gradient of $\mathbf{u}$ is the sum of expressions (4.15) and (4.17)

$$\operatorname{grad} \mathbf{u} = \mathbf{E} + \mathbf{W} =$$

$$= \theta'_z \left\{ \frac{1}{2} \left[(\operatorname{grad} \varphi_t \otimes \mathbf{k}) + (\mathbf{k} \otimes \operatorname{grad} \varphi_t) \right] + \frac{1}{2} [(\mathbf{k} \times \mathbf{r}) \otimes \mathbf{k}] + [\mathbf{k} \otimes (\mathbf{k} \times \mathbf{r})] \right\} +$$

$$+ \theta'_z \left\{ (z\mathbf{k} \times \mathbf{I}) - \frac{1}{2} \left[(\operatorname{grad} \varphi_t \otimes \mathbf{k}) - (\mathbf{k} \otimes \operatorname{grad} \varphi_t) \right] + \frac{1}{2} [(\mathbf{k} \times \mathbf{r}) \otimes \mathbf{k}] - [\mathbf{k} \otimes (\mathbf{k} \times \mathbf{r})] \right\} =$$

$$= \theta'_z \left[(z\mathbf{k} \times \mathbf{I}) + (\mathbf{k} \times \mathbf{r}) \otimes \mathbf{k} + (\mathbf{k} \otimes \operatorname{grad} \varphi_t) \right] \ .$$

Furthermore, by observing that

$$\operatorname{grad} \mathbf{u} = \theta'_z \left[\operatorname{grad} (z\mathbf{k} \times \mathbf{r}) + \operatorname{grad} (\varphi_t \mathbf{k}) \right]$$

the displacement field associated with torsion is finally obtained

$$\mathbf{u} = \theta'_z (z\mathbf{k} \times \mathbf{r} + \varphi_t \mathbf{k}) + \boldsymbol{\omega}_0 \times \mathbf{r} + \mathbf{u}_0 \ .$$

4.1.4 Displacement field associated with biaxial shear

The tangential stress field $\boldsymbol{\tau}_{sh}$ is provided by (3.51) so that the stress tensor has the following expression

$$\mathbf{S} = (l - z)(\mathbf{g}_t \cdot \mathbf{r})(\mathbf{k} \otimes \mathbf{k}) + \boldsymbol{\tau}_{sh} \otimes \mathbf{k} + \mathbf{k} \otimes \boldsymbol{\tau}_{sh}. \tag{4.18}$$

We recall once more that the skew-symmetric part of the displacement gradient $\mathbf{W}$ is obtained as the cross product between $\boldsymbol{\omega}$ and $\mathbf{I}$, i.e. $\mathbf{W} = \boldsymbol{\omega} \times \mathbf{I}$, where $\boldsymbol{\omega}$ is obtained from

$$\boldsymbol{\omega} \otimes \boldsymbol{\nabla} = \boldsymbol{\nabla} \times \mathbf{E}^t \tag{4.19}$$

once the infinitesimal strain tensor $\mathbf{E}$ is derived from (4.19) via the linear elastic law. In this way, being $\mathbf{u} \otimes \boldsymbol{\nabla} = \mathbf{E} + \mathbf{W}$, a second integration provides the displacement field $\mathbf{u}$ which is looked for. To simplify the integration procedure illustrated below, it is useful to express alternatively the shear stress field (3.51) upon introducing the vector

$$\mathbf{a} = \frac{1}{8}(\mathbf{r} \cdot \mathbf{r})\mathbf{r} \tag{4.20}$$

whose gradient is the symmetric tensor

$$\mathbf{a} \otimes \boldsymbol{\nabla} = \boldsymbol{\nabla} \otimes \mathbf{a} = \frac{1}{8}[2(\mathbf{r} \otimes \mathbf{r}) + (\mathbf{r} \cdot \mathbf{r})\hat{\mathbf{I}}] \tag{4.21}$$

The vector $\mathbf{a}$ allows one to express $\mathbf{A}^p$ in (3.41) as

$$\mathbf{A}^p = (1 + \bar{\nu})(\boldsymbol{\nabla} \otimes \mathbf{a}) - \bar{\nu}\frac{\mathbf{r} \cdot \mathbf{r}}{2}\hat{\mathbf{I}} \tag{4.22}$$

so that the tangential stress field (3.51) becomes

$$\boldsymbol{\tau}_{sh} = [(\boldsymbol{\nabla} \otimes \boldsymbol{\psi}) + (1 + \bar{\nu})(\boldsymbol{\nabla} \otimes \mathbf{a}) - \bar{\nu}\frac{\mathbf{r} \cdot \mathbf{r}}{2}\hat{\mathbf{I}}]\mathbf{g}_t \tag{4.23}$$

The previous relation, upon introducing the vector $\boldsymbol{\xi}$, defined as

$$\xi = \boldsymbol{\psi} + (1 + \bar{\nu})\mathbf{a} \tag{4.24}$$

becomes

$$\boldsymbol{\tau}_{sh} = \left(\boldsymbol{\nabla} \otimes \xi - \bar{\nu} \frac{\mathbf{r} \cdot \mathbf{r}}{2} \hat{\mathbf{I}} \right) \mathbf{g}_t = (\xi \cdot \mathbf{g}_t) \boldsymbol{\nabla} - \bar{\nu} \frac{\mathbf{r} \cdot \mathbf{r}}{2} \mathbf{g}_t. \tag{4.25}$$

Applying the linear isotropic law

$$\mathbf{E} = \frac{1+\nu}{E} \mathbf{S} - \frac{\nu}{E} (\text{tr } \mathbf{S}) \mathbf{I} \tag{4.26}$$

to (4.18), the following representation is obtained for the infinitesimal strain field

$$\begin{aligned} \mathbf{E} \quad = \quad & \frac{1+\nu}{E} \left[(l-z)(\mathbf{g_t} \cdot \mathbf{a})(\mathbf{k} \otimes \mathbf{k}) + (\boldsymbol{\tau}_{sh} \otimes \mathbf{k}) + (\mathbf{k} \otimes \boldsymbol{\tau}_{sh}) \right] + \\ & - \frac{\nu}{E} \left[(l-z)(\mathbf{g_t} \cdot \mathbf{a}) \mathbf{I} \right] \end{aligned} \tag{4.27}$$

Substituting (4.25) into the previous expression and setting the relation

$$(1+\nu)\bar{\nu} = \nu \tag{4.28}$$

one finally obtains

$$\begin{aligned} \mathbf{E} \quad = \quad & \frac{1+\nu}{E} \{ (l-z)(\mathbf{g_t} \cdot \mathbf{a})(\mathbf{k} \otimes \mathbf{k}) + [(\boldsymbol{\xi} \cdot \mathbf{g}_t) \boldsymbol{\nabla}] \otimes \mathbf{k} + \mathbf{k} \otimes [(\boldsymbol{\xi} \cdot \mathbf{g}_t) \boldsymbol{\nabla}] \} + \\ & - \frac{\nu}{E} [(l-z)(\mathbf{g_t} \cdot \mathbf{a}) \mathbf{I} + \frac{\mathbf{r} \cdot \mathbf{r}}{2} \mathbf{g}_t \otimes \mathbf{k} + \mathbf{k} \otimes \frac{\mathbf{r} \cdot \mathbf{r}}{2} \mathbf{g}_t] \end{aligned} \tag{4.29}$$

The explicit computation of the curl of $\mathbf{E}$, i.e. $\boldsymbol{\nabla} \times \mathbf{E}^t$, required by the previously outlined integration procedure, needs the separate computation of the curl of the addends contained in the RHS of the previous expression

$$\begin{aligned}
\boldsymbol{\nabla}\times[(l-z)(\mathbf{g_t}\cdot\mathbf{a})(\mathbf{k}\otimes\mathbf{k})] &= (l-z)(\mathbf{g_t}\cdot\mathbf{a})\boldsymbol{\nabla}\times(\mathbf{k}\otimes\mathbf{k}) = \\
&= [(l-z)\mathbf{g_t}-(\mathbf{g_t}\cdot\mathbf{a})\mathbf{k}]\times(\mathbf{k}\otimes\mathbf{k}) = \\
&= (\mathbf{g_t}\times\mathbf{k})\otimes(l-z)\mathbf{k} \\
\boldsymbol{\nabla}\times[\mathbf{k}\otimes(\boldsymbol{\xi}\cdot\mathbf{g}_t)\boldsymbol{\nabla}] &= [\boldsymbol{\nabla}\times\mathbf{k}]\otimes(\boldsymbol{\xi}\cdot\mathbf{g}_t)\boldsymbol{\nabla} = \\
&= -(\mathbf{k}\times\boldsymbol{\nabla})\otimes(\boldsymbol{\xi}\cdot\mathbf{g}_t)\boldsymbol{\nabla} = \\
&= -\mathbf{k}\times[\boldsymbol{\nabla}\otimes(\boldsymbol{\xi}\cdot\mathbf{g}_t)\boldsymbol{\nabla}] = \\
&= -\mathbf{W_k}[(\boldsymbol{\xi}\cdot\mathbf{g}_t)\boldsymbol{\nabla}\otimes\boldsymbol{\nabla}] \\
\boldsymbol{\nabla}\times[(\boldsymbol{\xi}\cdot\mathbf{g}_t)\boldsymbol{\nabla}\otimes\mathbf{k}] &= [\boldsymbol{\nabla}\times(\boldsymbol{\xi}\cdot\mathbf{g}_t)\boldsymbol{\nabla}]\otimes\mathbf{k} = \mathbf{0} \\
\boldsymbol{\nabla}\times[(l-z)(\mathbf{g_t}\cdot\mathbf{a})\mathbf{I}] &= (l-z)(\mathbf{g_t}\cdot\mathbf{a})\boldsymbol{\nabla}\times\mathbf{I} = \\
&= [(l-z)\mathbf{g_t}+(\mathbf{g_t}\cdot\mathbf{a})\mathbf{k}]\times\mathbf{I} = \\
&= (l-z)\mathbf{W_{g_t}}+(\mathbf{g_t}\cdot\mathbf{a})\mathbf{W_k} \\
\boldsymbol{\nabla}\left(\mathbf{k}\otimes\frac{\mathbf{r}\cdot\mathbf{r}}{2}\mathbf{g}_t\right) &= -\mathbf{k}\times\left[\boldsymbol{\nabla}\frac{\mathbf{r}\cdot\mathbf{r}}{2}\otimes\mathbf{g}_t\right] = \\
&= -\mathbf{k}\times\mathbf{r}\otimes\mathbf{g}_t \\
\boldsymbol{\nabla}\left[\left(\frac{\mathbf{r}\cdot\mathbf{r}}{2}\right)\mathbf{g}_t\otimes\mathbf{k}\right] &= \left[\boldsymbol{\nabla}\left(\frac{\mathbf{r}\cdot\mathbf{r}}{2}\right)\right]\times\mathbf{g}_t\otimes\mathbf{k} = \mathbf{r}\times\mathbf{g}_t\otimes\mathbf{k}
\end{aligned} \tag{4.30}$$

The identities above can be proven by invoking the differential identities reported in the Appendix A. Using the identities (4.30) the curl of $\mathbf{E}$ becomes

$$\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{E}^t = \; & \frac{1+\nu}{E}\{(\mathbf{g_t}\times\mathbf{k})\otimes(l-z)\mathbf{k}-\mathbf{k}\times[(\boldsymbol{\xi}\cdot\mathbf{g}_t)\boldsymbol{\nabla}\otimes\boldsymbol{\nabla}]\}+ \\
& +\frac{\nu}{E}\{-\mathbf{r}\times\mathbf{g}_t\otimes\mathbf{k}+\mathbf{k}\times\mathbf{r}\otimes\mathbf{g}_t\}+ \\
& +\frac{\nu}{E}[(z-l)\mathbf{g}_t\times\mathbf{I}+(\mathbf{g}_t\cdot\mathbf{r})\mathbf{k}\times\mathbf{I}]
\end{aligned} \tag{4.31}$$

On account of identity (4.19), one knows the expression of the gradient of $\boldsymbol{\omega}$; hence, to obtain $\boldsymbol{\omega}$, it is necessary to integrate the expression above. To this end (4.31) is further developed as follows

$$
\begin{aligned}
\boldsymbol{\omega}\otimes\boldsymbol{\nabla} &= \frac{1+\nu}{E}\left\{\left(lz-\frac{z^2}{2}\right)(\mathbf{g}_t\times\mathbf{k})-\mathbf{k}\times[(\boldsymbol{\xi}\cdot\mathbf{g}_t)\boldsymbol{\nabla}]\right\}\otimes\boldsymbol{\nabla}+\\
&\quad+\frac{\nu}{E}[\mathbf{g}_t\times\mathbf{r}\otimes\mathbf{k}+\mathbf{k}\times\mathbf{r}\otimes\mathbf{g}_t+(z-l)\mathbf{g}_t\times\mathbf{I}+(\mathbf{g}_t\cdot\mathbf{r})\mathbf{k}\times\mathbf{I}]=\\
&= \frac{1+\nu}{E}\left\{\left(lz-\frac{z^2}{2}\right)(\mathbf{g}_t\times\mathbf{k})-\mathbf{k}\times[(\boldsymbol{\xi}\cdot\mathbf{g}_t)\boldsymbol{\nabla}]\right\}\otimes\boldsymbol{\nabla}+\\
&\quad+\frac{\nu}{E}\left[(z-l)\mathbf{g}_t\times\mathbf{I}+(\mathbf{g}_t\cdot\mathbf{r})\mathbf{k}\times\mathbf{r}+\frac{z^2}{2}(\mathbf{k}\times\mathbf{g}_t)\right]\otimes\boldsymbol{\nabla}
\end{aligned}
\tag{4.32}
$$

In particular, the last equality in (4.32) hinges on the identity

$$
\mathbf{g}_t\times\mathbf{r}\otimes\mathbf{k}+(z-l)\mathbf{g}_t\times\mathbf{I}=[(z-l)\mathbf{g}_t\times\mathbf{p}]\otimes\boldsymbol{\nabla}+\left[\frac{z^2}{2}(\mathbf{k}\times\mathbf{g}_t)\right]\otimes\boldsymbol{\nabla} \tag{4.33}
$$

which is inferred from the relation $\mathbf{r}=\mathbf{p}-z\mathbf{k}$.

The rightmost expression in (4.32) allows one to directly identify the expression of the axial vector $\boldsymbol{\omega}$

$$
\begin{aligned}
\boldsymbol{\omega}= &\frac{1+\nu}{E}\left\{\left(lz-\frac{z^2}{2}\right)(\mathbf{g}_t\times\mathbf{k})-\mathbf{k}\times(\boldsymbol{\xi}\cdot\mathbf{g}_t)\boldsymbol{\nabla}\right\}+\\
&+\frac{\nu}{E}\left[(z-l)\mathbf{g}_t\times\mathbf{p}+(\mathbf{g}_t\cdot\mathbf{p})\mathbf{k}\times\mathbf{p}+\frac{z^2}{2}(\mathbf{k}\times\mathbf{g}_t)\right]+\boldsymbol{\omega}_0
\end{aligned}
\tag{4.34}
$$

where $\boldsymbol{\omega}_0$ is an arbitrary constant vector field. Accordingly, the skew-symmetric component of the displacement gradient $\mathbf{W}$, whose axial vector is $\boldsymbol{\omega}$, it turns out to be:

$$
\begin{aligned}
\mathbf{W} &= \boldsymbol{\omega}\times\mathbf{I}=\frac{1+\nu}{E}\left\{\left(lz-\frac{z^2}{2}\right)(\mathbf{g}_t\times\mathbf{k})\times\mathbf{I}-[\mathbf{k}\times(\boldsymbol{\xi}\cdot\mathbf{g_t})\boldsymbol{\nabla}]\times\mathbf{I}\right\}+\\
&\quad+\frac{\nu}{E}\left\{\mathbf{k}\times\left[(\mathbf{g}_t\cdot\mathbf{p})\mathbf{p}+\frac{z^2}{2}\mathbf{g}_t\right]\times+[\mathbf{g}_t\times(z-l)\mathbf{p}]\times\mathbf{I}\right\}+\boldsymbol{\omega}_0\times\mathbf{I},
\end{aligned}
$$

The last addend $\boldsymbol{\omega}_0\times\mathbf{I}$ is the displacement gradient of an arbitrary rigid motion up to which the displacement solution is defined. For sake of brevity this term will be omitted in the following developments although it will be reported in the final expression of the displacements.

Recalling identity (2.17) one infers

$$\begin{aligned}\mathbf{W} &= \frac{1+\nu}{E}\left\{\left(lz-\frac{z^2}{2}\right)[(\mathbf{k}\otimes\mathbf{g}_t)-(\mathbf{g}_t\otimes\mathbf{k})]+[\mathbf{k}\otimes(\boldsymbol{\xi}\cdot\mathbf{g}_t)\boldsymbol{\nabla}]-[(\boldsymbol{\xi}\cdot\mathbf{g}_t)\boldsymbol{\nabla}\otimes\mathbf{k}]\right\}+\\ &+\frac{\nu}{E}\left\{\left[(\mathbf{g}_t\cdot\mathbf{p})\mathbf{p}+\frac{z^2}{2}\mathbf{g}_t\right]\otimes\mathbf{k}-\mathbf{k}\left[(\mathbf{g}_t\cdot\mathbf{p})\mathbf{p}+\frac{z^2}{2}\mathbf{g}_t\right]+\right.\\ &+\quad\left.[(z-l)\mathbf{p}\otimes\mathbf{g}_t]-[\mathbf{g}_t\otimes(z-l)\mathbf{p}]\right\}\end{aligned} \tag{4.35}$$

By definition, the sum of the expression above with that of $\mathbf{E}$ reported in (4.29) provides the displacement gradient grad $\mathbf{u}=\mathbf{u}\otimes\boldsymbol{\nabla}$. Comparing expressions (4.29) and (4.35) one recognizes the presence of two terms premultiplied either by the coefficient $\frac{1+\nu}{E}$ or by $\frac{\nu}{E}$. For this reason, for the sake of readability the following representation is exploited in the computation of the displacement gradient. In particular, the integral of grad $\mathbf{u}_{1+\nu}$ can be achieved more easily upon developing the integrand as follows

$$\begin{aligned}\text{grad }\mathbf{u}_{1+\nu} &= \{(l-z)(\mathbf{g}_t\cdot\mathbf{r})(\mathbf{k}\otimes\mathbf{k})+[(\boldsymbol{\xi}\cdot\mathbf{g}_t)\boldsymbol{\nabla}\otimes\mathbf{k}]+[\mathbf{k}\otimes(\boldsymbol{\xi}\cdot\mathbf{g}_t)\boldsymbol{\nabla}]\}+\\ &+\left\{\left(lz-\frac{z^2}{2}\right)(\mathbf{k}\otimes\mathbf{g}_t)-(\mathbf{g}_t\otimes\mathbf{k})]+[\mathbf{k}\otimes(\boldsymbol{\xi}\cdot\mathbf{g}_t)\boldsymbol{\nabla}]-[(\boldsymbol{\xi}\cdot\mathbf{g}_t)\boldsymbol{\nabla}\otimes\mathbf{k}]\right\}=\\ &= \left\{\left(lz-\frac{z^2}{2}\right)(\mathbf{g}_t\cdot\mathbf{p})\mathbf{k}+2(\boldsymbol{\xi}\cdot\mathbf{g}_t)\mathbf{k}\left(\frac{z^3}{6}-l\frac{z^2}{2}\right)\mathbf{g}_t\right\}\otimes\boldsymbol{\nabla}\end{aligned} \tag{4.36}$$

Following an analogous strategy for grad $\mathbf{u}_\nu$ one has:

$$\begin{aligned}\text{grad }\mathbf{u}_{\nu} &= (l-z)(\mathbf{g}_t\cdot\mathbf{r})\mathbf{I}-\frac{\mathbf{r}\cdot\mathbf{r}}{2}\mathbf{g}_t\otimes\mathbf{k}-\mathbf{k}\otimes\frac{\mathbf{r}\cdot\mathbf{r}}{2}\mathbf{g}_t+\\ &+\left[(\mathbf{g}_t\cdot\mathbf{p})\mathbf{p}+\frac{z^2}{2}\mathbf{g}_t\right]\otimes\mathbf{k}-\mathbf{k}\otimes\left[(\mathbf{g}_t\cdot\mathbf{p})\mathbf{p}+\frac{z^2}{2}\mathbf{g}_t\right]=\\ &+[(l-z)\mathbf{p}\otimes\mathbf{g}_t]-[\mathbf{g}_t\otimes(z-l)\mathbf{p}]=\\ &= \left\{(z-l)(\mathbf{g_t}\cdot\mathbf{p})\mathbf{p}-\left[(z-l)\frac{(\mathbf{p}\cdot\mathbf{p})}{2}-\frac{z^3}{3}\right]\mathbf{g_t}-\frac{(\mathbf{p}\cdot\mathbf{p})}{2}(\mathbf{g_t}\cdot\mathbf{p})\mathbf{k}\right\}\otimes\boldsymbol{\nabla}\end{aligned} \tag{4.37}$$

Finally, adding all displacement terms and including the integration constants

$\mathbf{u}_0$ and $\boldsymbol{\omega}_0$ as well, the required displacement field turns out to be

$$
\begin{aligned}
\mathbf{u} \; = \; & \frac{1+\nu}{E}\left\{\left(lz-\frac{z^2}{2}\right)(\mathbf{g_t}\cdot\mathbf{p})\mathbf{k}-2(\boldsymbol{\xi}\cdot\mathbf{g}_t)\mathbf{k}+\left(\frac{z^3}{6}-l\frac{z^2}{2}\right)\mathbf{g}_t\right\}+\\
& +\frac{\nu}{E}\left\{(z-l)(\mathbf{g_t}\cdot\mathbf{p})\mathbf{p}-\left[(z-l)\frac{(\mathbf{p}\cdot\mathbf{p})}{2}-\frac{z^3}{3}\right]\mathbf{g_t}-\frac{\mathbf{p}\cdot\mathbf{p}}{2}(\mathbf{g}_t\cdot\mathbf{p})\mathbf{k}\right\}+\\
& +\boldsymbol{\omega}_0\times\mathbf{p}+\mathbf{u}_0
\end{aligned}
\tag{4.38}
$$

Upon substituting $\mathbf{g}_t=-\mathbf{J}_G^{-1}\mathbf{t}$ an expression explicitly related to the shear force is obtained

$$
\begin{aligned}
\mathbf{u} \; = \; & \frac{1+\nu}{E}\left\{-\left(lz-\frac{z^2}{2}\right)(\mathbf{J}_G^{-1}\mathbf{t}\cdot\mathbf{p})\mathbf{k}+2(\boldsymbol{\xi}\cdot\mathbf{J}_G^{-1}\mathbf{t})\mathbf{k}-\left(\frac{z^3}{6}-l\frac{z^2}{2}\right)\mathbf{J}_G^{-1}\mathbf{t}\right\}+\\
& +\frac{\nu}{E}\left\{-(z-l)(\mathbf{J}_G^{-1}\mathbf{t}\cdot\mathbf{p})\mathbf{p}+\left[(z-l)\frac{(\mathbf{p}\cdot\mathbf{p})}{2}-\frac{z^3}{3}\right]\mathbf{J}_G^{-1}\mathbf{t}+\right.\\
& \left.+\,(\mathbf{J}_G^{-1}\mathbf{t}\cdot\mathbf{p})\frac{(\mathbf{p}\cdot\mathbf{p})}{2}\mathbf{k}\right\}+\boldsymbol{\omega}_0\times\mathbf{p}+\mathbf{u}_0
\end{aligned}
$$

Comparing (4.1.4) with the classic expression reported by [36], which addresses a section loaded by a shear force directed along the x axis, one finds the following relation (4.1.4)

$$
\psi_x=\frac{1}{2(1+\nu)}\left[-\chi-xy^2-\frac{1}{4}(\mathbf{r}\cdot\mathbf{r})x\right]
\tag{4.39}
$$

between the function χ introduced by Love and the component along x of the auxiliary function $\boldsymbol{\xi}$ appearing in the formula above. Actually,

$$
\xi_x=\psi_x+\frac{1}{8}(1+\bar{\nu})(\mathbf{r}\cdot\mathbf{r})x
\tag{4.40}
$$

by means of formulas (4.20) and (4.24).

4.2 Shear center

Aim of this section is to derive a frame-independent expression of the shear center on account of the previously derived representation for tangential stresses originated by pure shear (3.51).

The shear center $\boldsymbol{\rho}_C$ is defined as the point of the transverse cross plane common to all lines of action of the resultants of the shear stress fields characterized by zero torsional rotation, i.e. by

$$\theta' = \frac{M_t}{I_q} = 0. \tag{4.41}$$

According to such definition, setting $\theta' = 0$ in $(38)_2$, the tangential stress field generated by pure shear $\boldsymbol{\tau}_{sh}$, admits the following expression.

$$\boldsymbol{\tau}_{sh} = -\left[\mathbf{A}^p(\mathbf{r}) + (\boldsymbol{\nabla} \otimes \boldsymbol{\psi})\right] \mathbf{J}_G^{-1}\mathbf{t} \tag{4.42}$$

The shear center $\boldsymbol{\rho}_C$ must consequently fulfill the property

$$\boldsymbol{\rho}_C \times \mathbf{t} \cdot \mathbf{k} = \int_\Omega \mathbf{r} \times \boldsymbol{\tau}_{sh} dA \cdot \mathbf{k} \tag{4.43}$$

stating the static equivalence of the vector $\mathbf{t}$ applied at $\mathbf{r}_C$ with the field $\boldsymbol{\tau}_{sh}$.

Considering the identity

$$\mathbf{r} \times \boldsymbol{\tau}_{sh} = (\mathbf{k} \otimes \mathbf{k})(\mathbf{r} \times \boldsymbol{\tau}_{sh}) = (\mathbf{r} \times \boldsymbol{\tau}_{sh} \cdot \mathbf{k})\mathbf{k} = (\mathbf{r}^\perp \cdot \boldsymbol{\tau}_{sh})\mathbf{k} \tag{4.44}$$

one infers from (4.42) and (4.44)

$$(\mathbf{k} \times \boldsymbol{\rho}_C) \cdot \mathbf{t} = \mathbf{r}_C^\perp \cdot \mathbf{t} = -\int_\Omega \left[\mathbf{A}^p(\mathbf{r}) + (\boldsymbol{\nabla} \otimes \boldsymbol{\psi})\right]^t \mathbf{r}^\perp \cdot \mathbf{J}_G^{-1}\mathbf{t}\, dA = -\boldsymbol{\gamma} \cdot \mathbf{J}_G^{-1}\mathbf{t} \tag{4.45}$$

where it has been set

$$\boldsymbol{\gamma} = -\int_\Omega \left[\mathbf{A}^p(\mathbf{r}) + (\boldsymbol{\nabla} \otimes \boldsymbol{\psi})\right]^t \mathbf{r}^\perp\, dA \tag{4.46}$$

Notice that, recalling definition (3.41) for $\mathbf{A}^p$ and observing that $(\mathbf{r} \otimes \mathbf{r})(\mathbf{k} \times \mathbf{r} = \mathbf{0})$, one can also write

$$\boldsymbol{\gamma} = -\int_\Omega \left[(\boldsymbol{\psi} \otimes \boldsymbol{\nabla}) + \frac{1 - 3\bar{\nu}}{8}(\mathbf{r} \cdot \mathbf{r})\hat{\mathbf{I}}\right] \mathbf{r}^\perp\, dA \tag{4.47}$$

Recalling that $\mathbf{J}_G^{-1}$ is symmetric, formula (4.35) becomes

$$\mathbf{k} \times \mathbf{r}_C \cdot \mathbf{t} = -\mathbf{J}_G^{-1}\boldsymbol{\gamma} \cdot \mathbf{t} \tag{4.48}$$

and hence for the arbitrariness of $\mathbf{t}$,

$$\mathbf{k} \times \mathbf{r}_C = -\mathbf{J}_G^{-1}\boldsymbol{\gamma} \tag{4.49}$$

A vector premultiplication of both members by $\mathbf{k}$ yields

$$\mathbf{r}_C = \mathbf{k} \times (\mathbf{J}_G^{-1}\boldsymbol{\gamma})\,, \tag{4.50}$$

so that, invoking (4.46), one finally has

$$\mathbf{r}_C = \mathbf{k} \times \left\{ \mathbf{J}_G^{-1} \int_\Omega \left[(\boldsymbol{\psi} \otimes \boldsymbol{\nabla}) + \frac{1-3\bar{\nu}}{8}(\mathbf{r}\cdot\mathbf{r})\hat{\mathbf{I}} \right] (\mathbf{k} \times \mathbf{r}) dA \right\} . \tag{4.51}$$

Observe that all terms in the previous expression are frame independent quantities.

4.2.1 Boundary integral tensor expression of the shear center

We are going to illustrate the manipulations that provide a boundary integral expression of the shear center by means of Gauss theorem. Since a formula of this type is available for $\mathbf{J}_G$

$$\mathbf{J}_G = \frac{1}{4}\int_{\partial\Omega} (\mathbf{r}\otimes\mathbf{r})(\mathbf{r}\cdot\mathbf{n}) dA\,, \tag{4.52}$$

it is sufficient to develop a boundary form of the sole term given by (4.47). Invoking the product rule, the first addend in (4.47) can be written as follows

$$(\boldsymbol{\nabla} \otimes \boldsymbol{\psi})\mathbf{r}^\perp = (\boldsymbol{\psi} \otimes \mathbf{r}^\perp)\boldsymbol{\nabla} - (\mathbf{r}^\perp \cdot \boldsymbol{\nabla})\boldsymbol{\psi} = (\boldsymbol{\psi} \otimes \mathbf{r}^\perp)\boldsymbol{\nabla} \tag{4.53}$$

since $\mathbf{r}^\perp \cdot \boldsymbol{\nabla} = 0$.

Similarly, the second integrand in (4.47), containing $(\mathbf{r}\cdot\mathbf{r})\mathbf{r}^\perp$, can be also expressed as a boundary integral on account of the identity

$$[(\mathbf{r}\cdot\mathbf{r})]\mathbf{r}^\perp = [(\mathbf{r}\cdot\mathbf{r})(\mathbf{r}\otimes\mathbf{r}^\perp)]\boldsymbol{\nabla} \tag{4.54}$$

which is proved in section 7.4 of the appendix. Thus, by virtue of the divergence theorem, the integral (4.47) can be expressed as

$$\begin{aligned} \boldsymbol{\gamma} &= \int_{\partial\Omega} \left[\boldsymbol{\psi} \otimes \mathbf{r}^{\perp}\right] \mathbf{n}\, dA + \frac{1-3\bar{\nu}}{8} \int_{\partial\Omega} [(\mathbf{r}\cdot\mathbf{r})(\mathbf{r}\otimes\mathbf{r}^{\perp})]\mathbf{n}\, dA = \\ &= \int_{\partial\Omega} \left[(\mathbf{r}^{\perp}\cdot\mathbf{n})\boldsymbol{\psi}\right] \mathbf{n}\, dA + \frac{1-3\bar{\nu}}{8} \int_{\partial\Omega} (\mathbf{r}^{\perp}\cdot\mathbf{n})(\mathbf{r}\cdot\mathbf{r})\mathbf{r}\, dA\,. \end{aligned} \tag{4.55}$$

Upon substituting finally (4.55) into (4.50), one obtains the required boundary integral expression of the shear center

$$\boldsymbol{\rho}_C = \mathbf{k} \times \left\{ \mathbf{J}_G^{-1} \int_{\partial A} \left[(\boldsymbol{\psi} + \frac{1-3\bar{\nu}}{8}(\mathbf{r}\cdot\mathbf{r})\hat{\mathbf{I}} \right] (\mathbf{r}^{\perp}\cdot\mathbf{n})\, dA \right\}. \tag{4.56}$$

The previous expression turns out to be particularly useful in numerical computations

4.2.2 Further developments for polygonal sections

Introducing the following parametric representation

$$\mathbf{r}(\mu) = \mathbf{r}_l + \mu\Delta\mathbf{r}_l \tag{4.57}$$

where

$$\Delta\mathbf{r}_l = \mathbf{r}_{l+1} - \mathbf{r}_l \tag{4.58}$$

the unit vector $\mathbf{n}$ normal to the boundary can be written

$$\mathbf{n} = \frac{\Delta\mathbf{r}_l}{l_l} \times \mathbf{k} = -\frac{1}{l_l}\Delta\mathbf{r}_l^{\perp}. \tag{4.59}$$

Substituting (4.57) and (4.59) in the terms $(\mathbf{r}\cdot\mathbf{n})$ and $(\mathbf{r}\otimes\mathbf{r})$ appearing in (4.55) one has

$$\begin{aligned} &\mathbf{r}^{\perp}\cdot\mathbf{n} = (\mathbf{r}_l^{\perp} + \mu\Delta\mathbf{r}_l^{\perp})(-\frac{1}{l_l}\Delta\mathbf{r}_l^{\perp}) = -\frac{1}{l_l}(\mathbf{r}_l^{\perp}\cdot\Delta\mathbf{r}_l^{\perp}) - l_l\mu \\ &\mathbf{r}\otimes\mathbf{r} = \mathbf{r}_l\otimes\mathbf{r}_l + 2\mu(\mathbf{r}_l\otimes\Delta\mathbf{r}_l) + \mu^2(\Delta\mathbf{r}_l\otimes\Delta\mathbf{r}_l) = \mathbf{T}_1 + 2\mu\mathbf{T}_2 + \mu^2\mathbf{T}_3 \end{aligned} \tag{4.60}$$

where

$$\begin{aligned}\mathbf{T}_1 &= \mathbf{r}_l \otimes \mathbf{r}_l \\ \mathbf{T}_2 &= \mathbf{r}_l \otimes \Delta\mathbf{r}_l \\ \mathbf{T}_3 &= \Delta\mathbf{r}_l \otimes \Delta\mathbf{r}_l\end{aligned} \tag{4.61}$$

Since

$$\mu = \frac{s}{l_l} \Rightarrow d\mu = \frac{dA}{l_l}, \tag{4.62}$$

by (4.55), it turn out to be

$$\begin{aligned}
&-\sum_{l=1}^{n_v}\int_0^1 [(\mathbf{r}_l^{\perp}\cdot\Delta\mathbf{r}_l^{\perp}) + \mu l_l^2](\mathbf{T}_1 + 2\mu\mathbf{T}_2 + \mu^2\mathbf{T}_3)(\mathbf{r}_l + \mu\Delta\mathbf{r}_l)d\mu = \\
&= -\sum_{l=1}^{n_v}\Big\{(\mathbf{r}_l^{\perp}\cdot\Delta\mathbf{r}_l^{\perp})\mathbf{T}_1\mathbf{r}_l + \int_0^1 \Big[(\mathbf{r}_l^{\perp}\cdot\Delta\mathbf{r}_l^{\perp})(\mathbf{T}_1\Delta\mathbf{r}_l\mu + \mathbf{T}_2\mathbf{r}_l\mu + \\
&+\mathbf{T}_2\Delta\mathbf{r}_l\mu^2 + \mathbf{T}_3\mathbf{r}_l\mu^2 + \mathbf{T}_3\Delta\mathbf{r}_l\mu^3) + l_l^2(\mathbf{T}_1\mathbf{r}_l\mu + \mathbf{T}_1\Delta\mathbf{r}_l\mu^2 + \\
&\mathbf{T}_2\mathbf{r}_l\mu^2 + \mathbf{T}_2\Delta\mathbf{r}_l\mu^3 + \mathbf{T}_3\mathbf{r}_l\mu^3 + \mathbf{T}_3\Delta\mathbf{r}_l\mu^4)\Big]\Big\}\, d\mu = \\
&= -\sum_{l=1}^{n_v}\left[(\mathbf{r}_l^{\perp}\cdot\Delta\mathbf{r}_l^{\perp})\mathbf{T}_1\mathbf{r}_l + \int_0^1 (\mathbf{s}_1\mu + \mathbf{s}_2\mu^2 + \mathbf{s}_3\mu^3 + l_l^2\mathbf{T}_3\Delta\mathbf{r}_l\mu^4)d\mu\right]
\end{aligned} \tag{4.63}$$

being

$$\begin{aligned}
\mathbf{s}_1 &= (\mathbf{r}_l^{\perp}\cdot\Delta\mathbf{r}_l^{\perp})\mathbf{T}_1\Delta\mathbf{r}_l + (\mathbf{r}_l^{\perp}\cdot\Delta\mathbf{r}_l^{\perp})\mathbf{T}_2\mathbf{r}_l + l_l^2\mathbf{T}_1\mathbf{r}_l \\
\mathbf{s}_2 &= (\mathbf{r}_l^{\perp}\cdot\Delta\mathbf{r}_l^{\perp})\mathbf{T}_2\Delta\mathbf{r}_l + (\mathbf{r}_l^{\perp}\cdot\Delta\mathbf{r}_l^{\perp})\mathbf{T}_3\mathbf{r}_l + l_l^2\mathbf{T}_1\Delta\mathbf{r}_l + l_l^2\mathbf{T}_2\mathbf{r}_l \\
\mathbf{s}_3 &= (\mathbf{r}_l^{\perp}\cdot\Delta\mathbf{r}_l^{\perp})\mathbf{T}_3\Delta\mathbf{r}_l + l_l^2\mathbf{T}_2\Delta\mathbf{r}_l + l_l^2\mathbf{T}_3\mathbf{r}_l
\end{aligned} \tag{4.64}$$

4.3 Frame-independent form of the shear flexibility tensor

According to the energetic definition [44, 49], the shear flexibility tensor is defined as the tensor $\mathbf{D}_s$ such that the equivalence $U^{sh}_{\mathbf{D}_s} = U^{sh}$ is satisfied, where U^{sh} is the strain energy associated with shear

$$U^{sh} = \frac{1}{2G}\int_{\Omega} \boldsymbol{\tau}_{sh}\cdot\boldsymbol{\tau}_{sh} dA \tag{4.65}$$

whereas $U^{sh}_{\mathbf{D}_s}$ is the elastic energy term expressed as the $\mathbf{D}_s$-associated quadratic form operating on the shear force vectors $\mathbf{t}$

$$U^{sh}_{\mathbf{D}_s} = \frac{1}{2}\mathbf{D}_s\mathbf{t}\cdot\mathbf{t} \tag{4.66}$$

Such a definition provides

$$\mathbf{D}_s = \frac{1}{G}\mathbf{J}_G^{-1}\mathbf{\Phi}\mathbf{J}_G^{-1} \tag{4.67}$$

where

$$\mathbf{\Phi} = \int_\Omega (\boldsymbol{\psi}\otimes\nabla + \mathbf{A}^p)(\nabla\otimes\boldsymbol{\psi} + \mathbf{A}^p)dA \tag{4.68}$$

Tensor $\mathbf{\Phi}$ and the adimensional tensor of shear coefficients $\boldsymbol{\chi}$ are related, see e.g.

$$\boldsymbol{\chi} = A\mathbf{J}_G^{-1}\mathbf{\Phi}\mathbf{J}_G^{-1} \tag{4.69}$$

4.3.1 Expression of the shear flexibility tensor by means of boundary integrals

This section illustrates a procedure for transforming the domain integral (4.68) in a complete boundary form exploiting the divergence theorem. For the sake of clarity, an indicial notation is used in some of the developments reported below. Integral (4.68) turns out to be the sum of four terms

$$\mathbf{\Phi} = \int_\Omega [(\boldsymbol{\psi}\otimes\nabla)(\nabla\otimes\boldsymbol{\psi}) + (\boldsymbol{\psi}\otimes\nabla)\mathbf{A}^p + \mathbf{A}^p(\nabla\otimes\boldsymbol{\psi}) + \mathbf{A}^p\mathbf{A}^p]dA. \tag{4.70}$$

and in indicial notation reads

$$\Phi_{ij} = \int_\Omega [A^p_{ik}(\psi_{j,k} + A^p_{kj}) + \psi_{i,k}\psi_{j,k} + \psi_{i,k}A^p_{kj}]dA \tag{4.71}$$

In order to achieve a boundary integral expression of (4.71), the following identity

$$\Phi_{ij} = \int_\Omega \left[\frac{1}{2}(A^p_{ik}\psi_j + A^p_{jk}\psi_i)_{,k} - (\psi_j\rho_i + \psi_i\rho_j) + A^p_{ik}A^p_{kj}\right] dA \tag{4.72}$$

will be preliminarily derived.

Since $\boldsymbol{\psi}$ is harmonic, $\psi_{j,\,kk} = 0$, one infers from the product rule

$$\psi_{i,\,k}\psi_{j,\,k} = (\psi_i\psi_{j,\,k})_{,\,k} - \psi_i\psi_{j,\,kk} = (\psi_i\psi_{j,\,k})_{,\,k} \tag{4.73}$$

and one also has

$$\psi_{i,\,k}A^p_{kj} = (\psi_i A^p_{kj})_{,\,k} - \psi_i A^p_{kj,\,k} \tag{4.74}$$

Substituting (4.73) and (4.74), into (4.71) one obtains

$$\Phi_{ij} = \int_\Omega (A^p_{ik}\psi_{j,\,k} + A^p_{ik}A^p_{kj} - \psi_i A^p_{kj,\,k})\,dA + \int_\Omega (\psi_i\psi_{j,\,k} + \psi_i A^p_{kj})_{,\,k}\,dA \tag{4.75}$$

It is recognized that the second integral above vanishes on account of the divergence theorem; actually, invoking the symmetry of $\mathbf{A}^p$ in (3.41), such an integral can be written as follows

$$\int_\Omega \psi_i(\psi_{j,\,k} + A^p_{kj})_{,\,k}\,dA = \int_{\partial\Omega} (\psi_i\psi_{j,\,k} + \psi_i A^p_{kj})n_k\,ds = 0 \tag{4.76}$$

where the last equality holds by virtue of the boundary constraint $(3.50)_2$

$$(\psi_{j,\,k} + A^p_{jk})n_k = 0 \tag{4.77}$$

Hence (4.71) reduces to

$$\Phi_{ij} = \int_\Omega (A^p_{ik}\psi_{j,\,k} - \psi_i A^p_{kj,\,k} + A^p_{ik}A^p_{kj})\,dA \tag{4.78}$$

Upon observing that

$$A^p_{ik}\psi_{j,\,k} = (A^p_{ik}\psi_j)_{,\,k} - A^p_{ik,\,k}\psi_j \tag{4.79}$$

the integrand function of (4.78) can be further transformed as follows

$$\Phi_{ij} = \int_\Omega [(A^p_{ik}\psi_j)_{,\,k} - A^p_{ik,\,k}\psi_j + A^p_{ik}A^p_{kj} - \psi_i A^p_{kj,\,k}]\,dA \tag{4.80}$$

The previous expression can be further transformed in a form which more clearly emphasizes the symmetry of $\boldsymbol{\Phi}$. Actually, by (4.77), one has

$$\int_\Omega (A^p_{ik}\psi_j)_{,\,k}dA = \int_{\partial\Omega} (A^p_{ik}\psi_j)n_k ds = \int_{\partial\Omega} (-\psi_{i,\,k}\psi_j)n_k ds = \int_\Omega (-\psi_{i,\,k}\psi_j)_{,\,k}\,dA \tag{4.81}$$

while the harmonicity of $\boldsymbol{\psi}$, $\psi_{j,\,kk}=0$, yields

$$(-\psi_{i,\,k}\psi_j)_{,\,k}=(-\psi_i\psi_{j,\,k})_{,\,k} \tag{4.82}$$

From (4.81) and (4.82) one eventually infers

$$\int_\Omega (A^p_{ik}\psi_j)_{,k} dA = \int_\Omega (A^p_{ik}\psi_i)_{,\,k} dA = \int_\Omega \frac{1}{2}(A^p_{ik}\psi_j + A^p_{jk}\psi_i)_{,\,k}\, dA \tag{4.83}$$

so that (4.80) is equivalent to

$$\int_\Omega \left[\frac{1}{2}(A^p_{ik}\psi_j + A^p_{jk}\psi_i)_{,\,k} - (A^p_{ik\,,k}\psi_j + \psi_i A^p_{kj\,,k}) + A^p_{ik}A^p_{kj}\right] dA \tag{4.84}$$

By taking the divergence of (3.41), which provides $A^p_{ik,\,k}=\rho_i$, the second integrand term in (4.84) can be further simplified to obtain the required result (4.72). This preliminary result allows one to transform (4.72) in a boundary integral. To this end it is necessary to transform the second and third integrand terms of (4.72) into the divergence of a tensor, just as for the first term. For greater clarity we introduce the notation

$$\Phi_{ij} = \Phi^{(1)}_{ij} + \Phi^{(2)}_{ij} + \Phi^{(3)}_{ij} \tag{4.85}$$

$$\Phi^{(1)}_{ij} = \int_\Omega \frac{1}{2}(A^p_{ik}\psi_j + A^p_{jk}\psi_i)_{,\,k}\, dA \tag{4.86}$$

$$\Phi^{(2)}_{ij} = -\int_\Omega \psi_j\rho_i + \psi_i\rho_j\, dA \tag{4.87}$$

$$\Phi^{(3)}_{ij} = \int_\Omega A^p_{ik}A^p_{kj}\, dA \tag{4.88}$$

Considering the explicit expression (3.41) of $\mathbf{A}^p$ and the divergence theorem one has for $\boldsymbol{\Phi}^{(1)}$

$$\boldsymbol{\Phi}^{(1)} = \int_{\partial\Omega}\left[\frac{\alpha}{2}(\mathbf{r}\cdot\mathbf{n})(\mathbf{r}\otimes\boldsymbol{\psi} + \boldsymbol{\psi}\otimes\mathbf{r}) + \frac{\beta}{2}(\mathbf{r}\cdot\mathbf{r})(\mathbf{n}\otimes\boldsymbol{\psi} + \boldsymbol{\psi}\otimes\mathbf{n})\right] ds \tag{4.89}$$

Making use of $\mathbf{p}$, previously introduced in (4.20), the integrand of (4.88) is expressed as the divergence of a tensor

$$\psi_i r_j = \psi_i p_{j,kk}, \qquad \psi_j r_i = \psi_j p_{i,kk} \tag{4.90}$$

and on account of the property of being harmonic, which provides

$$\begin{aligned} \psi_i p_{i,kk} &= (\psi_i p_{i,k})_{,k} - (\psi_{i,k} p_i)_{,k} \\ \psi_j p_{i,kk} &= (\psi_j p_{i,k})_{,k} - (\psi_{j,k} p_i)_{,k} \end{aligned} \tag{4.91}$$

Relation (4.90) is inferred on account of the following identity

$$\psi_{j,kk} = \rho_j \tag{4.92}$$

which is proven in section 7.4.2 of the appendix.

Considering the further identity

$$p_{i,k} = \frac{1}{8}(2\rho_k \rho_i + \rho_h \rho_h \delta_{ik}) \tag{4.93}$$

which is proven as well in section 7.4.2 of the appendix, and relation (4.77), the following boundary integral

$$\begin{aligned} \mathbf{\Phi}^{(2)} &= \int_{\partial\Omega} \left[-\frac{1}{4}(\mathbf{r}\cdot\mathbf{n})(\mathbf{r}\otimes\boldsymbol{\psi} + \boldsymbol{\psi}\otimes\mathbf{r}) - \frac{1}{8}(\mathbf{r}\cdot\mathbf{r})(\mathbf{n}\otimes\boldsymbol{\psi} + \boldsymbol{\psi}\otimes\mathbf{n}) \right] ds \quad + \\ &+ \int_{\partial\Omega} \left[-\frac{2\alpha}{8}(\mathbf{r}\cdot\mathbf{r})(\mathbf{r}\cdot\mathbf{n})(\mathbf{r}\otimes\mathbf{r}) - \frac{\beta}{8}(\mathbf{r}\cdot\mathbf{r})^2(\mathbf{r}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{r}) \right] ds \end{aligned} \tag{4.94}$$

is derived in tensor notation.

For what concerns the third addend $\mathbf{\Phi}^{(3)}$ of (4.86), it is necessary to express in divergence form the integrand

$$\mathbf{A}^p\mathbf{A}^p = (\alpha^2 + 2\alpha\beta)(\mathbf{r}\cdot\mathbf{r})(\mathbf{r}\otimes\mathbf{r}) + \beta^2(\mathbf{r}\cdot\mathbf{r})^2\hat{\mathbf{I}} \tag{4.95}$$

This is achieved by means of the following formulas

$$\begin{aligned} (\rho_i\rho_i)\rho_h\rho_k &= \frac{1}{6}[(\rho_i\rho_i)\rho_h\rho_k\rho_l]_{,l} \\ (\rho_p\rho_p)(\rho_q\rho_q) &= \frac{1}{6}[(\rho_p\rho_p)(\rho_q\rho_q)\rho_l]_{,l} \end{aligned} \tag{4.96}$$

that are proved in sections 7.4.3 and 7.4.4 of the appendix, respectively. Owing to these formulas and to the divergence theorem one infers

$$\Phi_{ij}^{(3)} = \int_{\partial\Omega} \left\{ \frac{(\alpha^2 + 2\alpha\beta)}{6} [(\rho_i \rho_i) \rho_h \rho_k] + \frac{\beta^2}{6} [(\rho_p r_p)(\rho_q r_q) \delta_{hk}] \right\} ds \tag{4.97}$$

or, in implicit notation

$$\mathbf{\Phi}^{(3)} = \int_{\partial\Omega} \left[\frac{\alpha^2 + 2\alpha\beta}{6} (\mathbf{r} \cdot \mathbf{r})(\mathbf{r} \cdot \mathbf{n})(\mathbf{r} \otimes \mathbf{r}) + \frac{\beta^2}{6} (\mathbf{r} \cdot \mathbf{r})^2 (\mathbf{r} \cdot \mathbf{n}) \hat{\mathbf{I}} \right] ds\,. \tag{4.98}$$

Adding (4.89), (4.94), (4.98) and collecting the resulting terms the sought entirely boundary expression of the shear flexibility tensor is obtained

$$\begin{aligned}
\mathbf{\Phi} = {} & \left(\frac{\alpha}{2} - \frac{1}{4}\right) \int_{\partial\Omega} (\mathbf{r} \cdot \mathbf{n})(\mathbf{r} \otimes \boldsymbol{\psi} + \boldsymbol{\psi} \otimes \mathbf{r})\, ds && + \\
+ {} & \left(\frac{\beta}{2} - \frac{1}{8}\right) \int_{\partial\Omega} (\mathbf{r} \cdot \mathbf{r})(\mathbf{n} \otimes \boldsymbol{\psi} + \boldsymbol{\psi} \otimes \mathbf{n})\, ds && + \\
+ {} & \left(\frac{\alpha^2 + 2\alpha\beta}{6} - \frac{\alpha}{4}\right) \int_{\partial\Omega} (\mathbf{r} \cdot \mathbf{r})(\mathbf{r} \cdot \mathbf{n})(\mathbf{r} \otimes \mathbf{r})\, ds && + \\
- {} & \frac{\beta}{8} \int_{\partial\Omega} (\mathbf{r} \cdot \mathbf{r})^2 (\mathbf{r} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{r})\, ds && + \\
+ {} & \frac{\beta^2}{6} \int_{\partial\Omega} (\mathbf{r} \cdot \mathbf{r})^2 (\mathbf{r} \cdot \mathbf{n}) \hat{\mathbf{I}}\, ds
\end{aligned} \tag{4.99}$$

Substituting (3.40) in the previous formula one finally obtains

$$\begin{aligned}
\mathbf{\Phi} = {} & - \frac{1 - \bar{\nu}}{8} \int_{\partial\Omega} (\mathbf{r} \cdot \mathbf{n})(\mathbf{r} \otimes \boldsymbol{\psi} + \boldsymbol{\psi} \otimes \mathbf{r})\, ds && + \\
& - \frac{1 + 3\bar{\nu}}{16} \int_{\partial\Omega} (\mathbf{r} \cdot \mathbf{r})(\mathbf{n} \otimes \boldsymbol{\psi} + \boldsymbol{\psi} \otimes \mathbf{n})\, ds && + \\
& - \frac{2 + 3\bar{\nu} + \bar{\nu}^2}{48} \int_{\partial\Omega} (\mathbf{r} \cdot \mathbf{r})(\mathbf{r} \cdot \mathbf{n})(\mathbf{r} \otimes \mathbf{r})\, ds && + \\
& - \frac{1 - 3\bar{\nu}}{64} \int_{\partial\Omega} (\mathbf{r} \cdot \mathbf{r})^2 (\mathbf{r} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{r})\, ds && + \\
& + \frac{(3\bar{\nu} - 1)^2}{384} \int_{\partial\Omega} (\mathbf{r} \cdot \mathbf{r})^2 (\mathbf{r} \cdot \mathbf{n}) \hat{\mathbf{I}}\, ds.
\end{aligned} \tag{4.100}$$

which can be substituted in (4.69) to get, invoking also (4.52), an expression of the shear flexibility tensor expressed solely by means of boundary integrals.

4.3.2 Shear flexibility tensor for the circular section

The general expression of $\mathbf{\Phi}$ presented in (4.100) can be specified to a circular section of radius R by setting

$$\mathbf{r} = R\mathbf{n} \tag{4.101}$$

hence

$$\begin{aligned}
\mathbf{\Phi} = \quad & + \quad \frac{(3+2\nu)}{32(1+\nu)^2} R^5 \int_{\partial\Omega} (\mathbf{n} \otimes \mathbf{n})\, ds \quad + \\
& + \quad \frac{(1+4\nu)(3+2\nu)}{64(1+\nu)^2} R^5 \int_{\partial\Omega} (\mathbf{n} \otimes \mathbf{n})\, ds \quad + \\
& - \quad \frac{(2+7\nu+6\nu^2)}{48(1+\nu)^2} R^5 \int_{\partial\Omega} (\mathbf{n} \otimes \mathbf{n})\, ds \quad + \\
& + \quad \frac{2\nu-1}{32(1+\nu)} R^5 \int_{\partial\Omega} (\mathbf{n} \otimes \mathbf{n})\, ds \quad + \\
& + \quad \frac{(2\nu-1)^2}{384(1+\nu)^2} R^5 \int_{\partial\Omega} \hat{\mathbf{I}}\, ds. \quad = \\
= \quad & \quad \frac{13+32\nu+12\nu^2}{192(1+\nu)^2} R^5 \int_{\partial\Omega} (\mathbf{n} \otimes \mathbf{n})\, ds \quad + \\
& + \quad \frac{(2\nu-1)^2}{384(1+\nu)^2} R^5 \int_{\partial\Omega} \hat{\mathbf{I}}\, ds.
\end{aligned} \tag{4.102}$$

where

$$ds = Rd\theta \tag{4.103}$$

Observe that

$$\int_{\partial\Omega} \hat{\mathbf{I}}\, ds = \hat{\mathbf{I}} \int_{\partial\Omega} ds = \hat{\mathbf{I}} \int_0^{2\pi} R\, d\theta = 2R\pi\hat{\mathbf{I}} \tag{4.104}$$

Moreover, by virtue of the circular symmetry

$$\int_{\partial\Omega} (\mathbf{n} \otimes \mathbf{n})\, ds = \int_{\partial\Omega} (\mathbf{n}^{\perp} \otimes \mathbf{n}^{\perp})\, ds \tag{4.105}$$

and of the property $\mathbf{n} \otimes \mathbf{n} + \mathbf{n}^{\perp} \otimes \mathbf{n}^{\perp} = \hat{\mathbf{I}}$, the following identity holds

$$\int_{\partial\Omega} (\mathbf{n} \otimes \mathbf{n})\, ds = \frac{1}{2} \int_{\partial\Omega} (\mathbf{n} \otimes \mathbf{n} + \mathbf{n}^{\perp} \otimes \mathbf{n}^{\perp})\, ds = \frac{1}{2} \int_{\partial\Omega} \hat{\mathbf{I}}\, ds = R\pi\hat{\mathbf{I}} \tag{4.106}$$

Substituting (4.106) into (4.102) one infers

$$\boldsymbol{\Phi} = \frac{7 + 14\nu + 8\nu^2}{96(1+\nu)^2}\pi R^6 \hat{\mathbf{I}} \tag{4.107}$$

Notice that in the particular case of $\nu = 0$, one obtains

$$\begin{aligned}
\boldsymbol{\Phi} = & - \frac{1}{8}\int_{\partial\Omega}(\mathbf{r}\cdot\mathbf{n})(\mathbf{r}\otimes\boldsymbol{\psi} + \boldsymbol{\psi}\otimes\mathbf{r})\, ds \quad + \\
& - \frac{1}{16}\int_{\partial\Omega}(\mathbf{r}\cdot\mathbf{r})(\mathbf{n}\otimes\boldsymbol{\psi} + \boldsymbol{\psi}\otimes\mathbf{n})\, ds \quad + \\
& - \frac{1}{24}\int_{\partial\Omega}(\mathbf{r}\cdot\mathbf{r})(\mathbf{r}\cdot\mathbf{n})(\mathbf{r}\otimes\mathbf{r})\, ds \quad + \\
& - \frac{1}{64}\int_{\partial\Omega}(\mathbf{r}\cdot\mathbf{r})^2(\mathbf{r}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{r})\, ds \quad + \\
& - \frac{1}{384}\int_{\partial\Omega}(\mathbf{r}\cdot\mathbf{r})^2(\mathbf{r}\cdot\mathbf{n})\hat{\mathbf{I}}\, ds.
\end{aligned} \tag{4.108}$$

Thus, due to (4.67), the shear flexibility tensor of the circular section turns out to be

$$\mathbf{D}_s = \frac{1}{G}\left[\frac{7 + 14\nu + 8\nu^2}{96(1+\nu)^2}\right]\pi R^6 \mathbf{J}_G^{-2}; \tag{4.109}$$

recalling also

$$\mathbf{J}_G = \frac{\pi R^4}{4}\hat{\mathbf{I}}, \qquad A = \pi R^2 \tag{4.110}$$

one has

$$\mathbf{D}_s = \frac{1}{GA}\left[\frac{7 + 14\nu + 8\nu^2}{96(1+\nu)^2}\right]\hat{\mathbf{I}} \tag{4.111}$$

so that, on account of (4.69), the shear factor tensor becomes

$$\boldsymbol{\chi} = \frac{7 + 14\nu + 8\nu^2}{96(1+\nu)^2}\hat{\mathbf{I}} \tag{4.112}$$

The expression found is in agreement with the coefficient reported in [44]

4.4 Stiffness tensor of a beam element

The flexural behaviour of beams with shear deformability can be completely described by the stiffness tensor of the beam element whose coefficients will depend upon the shear deformability tensor.

The stiffness tensor of the beam element has been already obtained in a frame independent form [49], and it is here reported for the sake of completeness.

Let us consider a straight beam of lenght l.

Denoting by $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ the flexural rotations at the end section 1 and 2 of beam and by $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$ the corresponding transverse displacements of a the shear center, the stiffness tensor can be expressed as

$$\begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} & \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} & \mathbf{H}_{21} & \mathbf{H}_{22} \\ \mathbf{H}_{11} & \mathbf{H}_{21} & \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{H}_{12} & \mathbf{H}_{22} & \mathbf{S}_{21} & \mathbf{S}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{k} \times \mathbf{\Phi}_1 \\ \mathbf{k} \times \mathbf{\Phi}_2 \\ \mathbf{\Delta}_1 \\ \mathbf{\Delta}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{k} \times \mathbf{M}_1 \\ \mathbf{k} \times \mathbf{M}_2 \\ \mathbf{T}_1 \\ \mathbf{T}_2 \end{bmatrix}$$

where $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{T}_1$ and $\mathbf{T}_2$ are the bending moments and shearing forces at the terminal sections of the beam and the unit vector $\mathbf{k}$ points from section 1 towards section 2.

Defining the adimensional tensor

$$\mathbf{V} = 12\mathbf{J}_G \boldsymbol{\chi}/Al^2$$

and denoting by $\mathbf{I}$ the identity tensor, the elements of the stiffness tensor are given by

$$\begin{aligned} \mathbf{B}_{11} = \mathbf{B}_{22} &= \frac{1}{l}(\mathbf{I}+\mathbf{V})^{-1}(4\mathbf{I}+\mathbf{V})\mathbf{J} \\ \mathbf{B}_{12} = \mathbf{B}_{21} &= \frac{1}{l}(\mathbf{I}+\mathbf{V})^{-1}(2\mathbf{I}+\mathbf{V})\mathbf{J} \\ \mathbf{H}_{11} = \mathbf{H}_{21} &= -\frac{6}{l^2}(\mathbf{I}+\mathbf{V})^{-1}\mathbf{J} \\ \mathbf{H}_{22} = \mathbf{H}_{12} &= \frac{6}{l^2}(\mathbf{I}+\mathbf{V})^{-1}\mathbf{J} \\ \mathbf{S}_{11} = \mathbf{S}_{22} &= \frac{12}{l^3}(\mathbf{I}+\mathbf{V})^{-1}\mathbf{J} \\ \mathbf{S}_{12} = \mathbf{S}_{21} &= -\frac{12}{l^3}(\mathbf{I}+\mathbf{V})^{-1}\mathbf{J} \end{aligned}$$

The formulas above generalize the corresponding ones of the classical stiffness matrix.

Chapter 5

A BEM approach to the solution of the torsion and shear problems

Regardless of the selected approach, either based upon the displacement or the stress one, the complete solution in terms of displacements and stress fields for rods of generic cross section can be represented only partially by means of explicit analytic expressions since some terms associated with torsion and shear embody auxiliary functions that are solution of Dirichlet or Neumann problems related to the cross section domain; this is true with the exception of sections having particular geometries for which a closed-form solution exists. For all the other cases, the only way to obtain the complete solution of the problem is to adopt a numerical approach. This chapter illustrates the operative details leading to expressions which are very simple to implement in a computer code.

5.1 Weak formulation

Weak formulation of the harmonic problem for the determination of $\boldsymbol{\psi}$ (3.46), combined with the second Green identity leads to the following alternative characterization of the problem [8] :

$$-\int_{\Omega} \boldsymbol{\psi} \chi_{,ii} \, dA + \int_{\partial\Omega} \boldsymbol{\psi} \chi_{,i} n_i \, ds = -\int_{\partial\Omega} \chi \, \mathbf{A}^p \, (\mathbf{p}) \, \mathbf{n} \, ds \,. \tag{5.1}$$

in which χ is an arbitrary weight function sufficiently regular.

By adopting as weight function the so-called *fundamental solution* of Laplace equation in the plane

$$\chi * = -\frac{1}{2\pi} \ln \left(\frac{1}{||\mathbf{p} - \mathbf{p}^*||} \right) \,, \tag{5.2}$$

one obtains the classical integral identity that represent the starting point of all BEM formulations. In (5.2) $\mathbf{p}^*$ denotes the position vector of an arbitrary point of the cross section plain and is commonly called *source point*.

As stated in [8], the function χ^* is solution of the equation $\chi^*_{,ii} = \delta(\mathbf{p} - \mathbf{p}^*)$, in which δ represent the Dirac distribution; the gradient of (5.2) is given by

$$\text{grad}\chi^* = \frac{\mathbf{p} - \mathbf{p}^*}{2\pi ||\mathbf{p} - \mathbf{p}^*||^2} \,. \tag{5.3}$$

By virtue of the properties of the function χ^*, formula (5.1) becomes:

$$\begin{aligned} c(\mathbf{p}^*)\boldsymbol{\psi}(\mathbf{p}^*) - \int_{\partial\Omega} \boldsymbol{\psi}(\mathbf{p}) \frac{(\mathbf{p} - \mathbf{p}^*) \cdot \mathbf{n}}{2\pi ||\mathbf{p} - \mathbf{p}^*||^2} \, ds = \\ = -\int_{\partial\Omega} \frac{1}{2\pi} \ln \left(\frac{1}{||\mathbf{p} - \mathbf{p}^*||} \right) \mathbf{A}^p \, (\mathbf{p}) \, \mathbf{n} \, ds \,, \end{aligned} \tag{5.4}$$

where the coefficient c depends on the position of the point $\mathbf{p}^*$ with respect to the domain Ω. In particular $c(\mathbf{p}^*) = 1$ if $\mathbf{p}^* \in \Omega$, $c(\mathbf{p}^*) = 0$ if $\mathbf{p}^* \notin \Omega$ and $c(\mathbf{p}^*) = \dfrac{\theta^- - \theta^+}{2\pi}$ if $\mathbf{p}^* \in \partial\Omega$, in which θ^- and θ^+, assumed positive if counterclockwise, are the angles between the unit vector tangent to the boundary in a circulation sense respectively clockwise and counterclockwise on the boundary.

Let us now assume that the boundary $\partial\Omega$ is polygonal, i.e. it is constituted by the union of n_v sides: $\partial\Omega = \bigcup_{l=1}^{n_v} \partial\Omega_l$. In this particular case, the geometry of the section results completely defined by the vertexes' coordinates $\{\mathbf{p}_1, ..., \mathbf{p}_{n_v}\}$. Thus, by introducing the following parametric representation for the side l:

$$\mathbf{p}(\mu_l) = \frac{\mathbf{p}_l + \mathbf{p}_{l+1}}{2} + \mu_l \frac{\mathbf{p}_{l+1} - \mathbf{p}_l}{2}, \quad l \in \{1, ..., n_v\}, \mu_l \in [-1, 1], \tag{5.5}$$

the relevant arc length s is $\dfrac{ds}{d\mu} = \sqrt{\dfrac{d\mathbf{p}}{d\mu} \cdot \dfrac{d\mathbf{p}}{d\mu}} = \dfrac{l_l}{2}$, so that formula (5.4) assumes the following expression:

$$\begin{aligned} c(\mathbf{p}^*)\boldsymbol{\psi}(\mathbf{p}^*) - \sum_{l=1}^{n_v} \frac{l_l}{2} \int_{-1}^{1} \boldsymbol{\psi}(\mu_l) \frac{(\mathbf{p}(\mu_l) - \mathbf{p}^*) \cdot \mathbf{n}_l}{2\pi ||\mathbf{p}(\mu_l) - \mathbf{p}^*||^2} \, d\mu_l = \\ = -\sum_{l=1}^{n_v} \frac{l_l}{2} \int_{-1}^{1} \frac{1}{2\pi} \ln\left(\frac{1}{||\mathbf{p}(\mu_l) - \mathbf{p}^*||}\right) \mathbf{A}^p(\mu_l)\, \mathbf{n}_l \, d\mu_l \, ; \end{aligned} \tag{5.6}$$

in particular, if $\mathbf{p}^*$ belongs to one boundary's sides, excluding their extremes points, it turns out to be $c(\mathbf{p}^*) = \dfrac{1}{2}$.

One can specialize the (5.6) for the first component by obtaining

$$\begin{aligned} c(\mathbf{p}^*)\psi_1(\mathbf{p}^*) - \sum_{l=1}^{n_v} \frac{l_l}{2} \int_{-1}^{1} \psi_1(\mu_l) \frac{(\mathbf{p}(\mu_l) - \mathbf{p}^*) \cdot \mathbf{n}_l}{2\pi ||\mathbf{p}(\mu_l) - \mathbf{p}^*||^2} \, d\mu_l = \\ = -\sum_{l=1}^{n_v} \frac{l_l}{2} \int_{-1}^{1} \frac{1}{2\pi} \ln\left(\frac{1}{||\mathbf{p}(\mu_l) - \mathbf{p}^*||}\right) \\ \left[\alpha(x^2 n_1 + xyn_1) + \beta(x^2 n_1 + y^2 n_2)\right] d\mu_l \, . \end{aligned} \tag{5.7}$$

5.2 Interpolation

Integral equation (5.6) is converted in algebraic form by using for the restriction of $\boldsymbol{\psi}$ to the generic side l the same interpolation adopted in [61]:

$$\boldsymbol{\psi}(\mu_l) = \sum_{\beta=1}^{n_{Pl}} \mathbf{p}_l^{(\beta)} T^{(\beta-1)}(\mu_l) \tag{5.8}$$

i.e., in scalar form,

$$\psi_1(\mu_l) = \sum_{\beta=1}^{n_{Pl}} p_{1l}^{(\beta)} T^{(\beta-1)}(\mu_l)$$

$$\psi_2(\mu_l) = \sum_{\beta=1}^{n_{Pl}} p_{2l}^{(\beta)} T^{(\beta-1)}(\mu_l)$$

where l is the index that varies on each side of the polygonal shape, $T^{(\beta)}$ is the Chebyshev's polynomial of degree β and n_{Pl} corresponds to the number of parameters used to interpolate the unknown function $\boldsymbol{\psi}$. The first six terms of the polynomial are reported hereafter for the reader's convenience:

$$T^{(0)}(\mu) = 1 \qquad T^{(1)}(\mu) = \mu \qquad T^{(2)}(\mu) = 2\mu^2 - 1$$

$$T^{(3)}(\mu) = 4\mu^3 - 3\mu \quad T^{(4)}(\mu) = 8\mu^4 - 8\mu^2 + 1 \quad T^{(5)}(\mu) = 16\mu^5 - 20\mu^3 + 5\mu\,;$$

in general, the polynomial of order $\beta > 1$ are obtained by the following recursive expression:

$$T^{(\beta+1)}(\mu) = 2\mu T^{(\beta)}(\mu) - T^{(\beta-1)}(\mu), \tag{5.9}$$

For each side m^*, it is necessary to collocate n_{Pm^*} source points whose position $(\mathbf{p}^*)_{m^*}^{(\alpha^*)}$ is determined by (5.5) following the ordered sequence of reals $-1 < \mu_1 < \mu_2 < ... < \mu_{\alpha^*} < ... < n_{Pm^*} < 1$.

The total number of source points, that also corresponds to the number of equations to write, is expressed by N_{eq}^*, while the total number of unknowns parameters is denoted by N_{par}.

5.3 Algebraic solution system

In this section we illustrate the operative details that allow one to re-formulate (5.6), by using the adopted interpolation functions, in a system of equation $\mathbb{A}_{P^*Q}\mathbf{p}_Q = \mathbf{b}_{P^*}$, in which $P^* \in \{1, ..., N_{eq}^*\}$, $Q \in \{1, ..., N_{par}\}$ and $\mathbf{p}_Q$ is the vector containing the collection of unknown parameters p_l^β relative to all sides of the boundary Ω; elements of $\mathbb{A}_{P^*Q}$ and $\mathbf{b}_{P^*}$ will be defined afterwards.

The global indices of equations and parameters are set in relation with local indices by the following relations: $P^* \leftarrow (m^*, \alpha^*)$, $\quad Q \leftarrow (l, \beta)$; thus, the first $n_{P_{1^*}}$ equations of the solution system are those associated with the side 1, the following $n_{P_{2^*}}$ with side 2 and so on. Analogously, the columns of the $\mathbb{A}_{P^*Q}$ matrix can be merged in groups whose cardinality is defined by the number of parameters $p_l^{(\beta)}$ used for each side.

To obtain a number of equations equal to the number of unknowns it is sufficient to consider on each side a number of source points equal to the number of the parameters, so that the equivalence $N^*_{eq} = N_{par}$ is satisfied. However, fulfilling this constraint, is possible to use different choices related to particular modeling requirements.

It is convenient to compute the elements of the matrix $\mathbf{A}_{P^*Q}$ and of the vector $\mathbf{b}_{P^*}$ by iterating on the local indices m^*, α^*, l and β and, subsequently, by using correspondences between local and global indices, to lead back these terms to the global numeration. To obtain expressions that can be directly implemented it is necessary to specify the dependency of the terms in (5.6) from the local indices:

$$\begin{aligned} &\frac{1}{2}\boldsymbol{\psi}^{\alpha^*}_{m^*} - \sum_{l=1}^{n_v} \frac{l_l}{2} \int_{-1}^{1} \boldsymbol{\psi}(\mu_l)(\chi^*_{,n})^{(\alpha^*)}_{m^*}(\mu_l)\, d\mu_l = \\ &\qquad = -\sum_{l=1}^{n_v} \frac{l_l}{2} \int_{-1}^{1} (\chi^*)^{(\alpha^*)}_{m^*}(\mu_l)\, \mathbf{A}\,(\mathbf{p})\, \mathbf{n}\, d\mu_l \end{aligned} \tag{5.10}$$

For brevity, in the sequel, reference will be made to the first row of the vector equation (5.10):

$$\begin{aligned} &\frac{1}{2}\psi^{\alpha^* 1}_{m^*} - \sum_{l=1}^{n_v} \frac{l_l}{2} \int_{-1}^{1} \psi^1(\mu_l)(\chi^*_{,n})^{(\alpha^*)}_{m^*}(\mu_l)\, d\mu_l = \\ &= -\sum_{l=1}^{n_v} \frac{l_l}{2} \int_{-1}^{1} (\chi^*)^{(\alpha^*)}_{m^*}(\mu_l) \left[\alpha(x^2 n_1 + xyn_1) + \beta(x^2 n_1 + y^2 n_2)\right] d\mu_l \,, \end{aligned} \tag{5.11}$$

The term $\psi^{\alpha^* 1}_{m^*}$ is obtained from (5.8) by setting $\mu_l = \mu_{\alpha^*}$; thus, only the parameters of ψ relative to the side m^* contribute to the term $\psi_{m^*}^{\alpha^*1}$.

In order to simplify the implementation it is useful to explicit the first term in (5.10) by introducing the l index:

$$\frac{1}{2}\psi_{m^*}^{\alpha^*1} = \frac{1}{2}\sum_{l=1}^{n_v}\sum_{\beta=1}^{n_{Pl}} \delta_{lm^*} p_l^{(\beta)} T^{(\beta-1)}(\mu_{\alpha^*}) = \sum_{l=1}^{n_v}\sum_{\beta=1}^{n_{Pl}} \mathbb{A}^{(1)}_{m^*\alpha^* l\beta} p_l^{(\beta)}, \tag{5.12}$$

in which it is set $\mathbb{A}^{(1)}_{m^*\alpha^* l\beta} = \delta_{lm^*}\frac{1}{2}T^{(\beta-1)}(\mu_{\alpha^*})$. By making explicit the first integral in (5.10), one obtains:

$$\begin{aligned} &-\sum_{l=1}^{n_v}\frac{l_l}{2}\int_{-1}^{1}\sum_{\beta=1}^{n_{Pl}} p_l^{(\beta)} T^{(\beta-1)}(\mu_l)\frac{(\mathbf{p}(\mu_l)-\mathbf{p}^*)\cdot\mathbf{n}_l}{2\pi||\mathbf{p}(\mu_l)-\mathbf{p}^*||^2}\,d\mu_l = \\ &= \sum_{l=1}^{n_v}\sum_{\beta=1}^{n_{Pl}} \mathbb{A}^{(2)}_{m^*\alpha^* l\beta}\, p_l^{(\beta)} \end{aligned} \tag{5.13}$$

in which:

$$\mathbb{A}^{(2)}_{m^*\alpha^* l\beta} = -\frac{l_l}{2}\int_{-1}^{1} T^{(\beta-1)}(\mu_l)\frac{(\mathbf{p}(\mu_l)-\mathbf{p}^*)\cdot\mathbf{n}_l}{2\pi||\mathbf{p}(\mu_l)-\mathbf{p}^*||^2}\,d\mu_l \tag{5.14}$$

One observes that vector $\mathbf{p}(\mu_l) - \mathbf{p}^*$ turns out to be parallel to side l when $l = m^*$ so that, although the gradient of χ^* is singular at the source point, the scalar product at the numerator in (5.14) is equal to zero.

Finally, for the term that appears at the right-hand side of in (5.10), it turns out to be:

$$\begin{aligned} \mathbf{b}^1_{m^*\alpha^*} = -&\sum_{l=1}^{n_v}\frac{l_l}{2}\int_{-1}^{1}\frac{1}{2\pi}\ln\left(\frac{1}{||\mathbf{p}(\mu_l)-\mathbf{p}^*||}\right) \\ &\left[\alpha(x^2 n_1 + xyn_1) + \beta(x^2 n_1 + y^2 n_2)\right] d\mu_l\,, \end{aligned} \tag{5.15}$$

where the position of the source point $\mathbf{p}^*$ is defined by m^* e α^* as in the previous formulas.

5.4 Entries of the solution system matrix

We are now going to illustrate the procedure adopted for computing the integrals (5.14) and (5.18) when the integrand function is singular, what occurs when $l = m^*$ and $\mathbf{p}(\mu_{m^*}) = \mathbf{p}^*$. As a matter of fact this problem concerns only the integral (5.18), since the integrand in (5.14) is null when $l = m^*$. As it is known, this is a general problem found also in classical weak formulations [33].

We detail in the sequel the quadrature strategy adopted for integrals (5.14) and (5.18) when $l \neq m^*$; it has been obtained by suitably modifying what has been proposed in [8]. It consists of subdividing each side of $\partial\Omega$ in a number of segments, with a minimum of 4, function of the distance d_l existing between each side and the source point. The length of those segments, for which the integrand function is constant, is transformed to the one-dimensional reference element (*parent* element), defined in the interval $[-1, 1]$, by assuming for them a dimension defined by $d_l^{(\mu)} = d_l/(4l_l)$.

Thus, denoting by N_{quad} the number of subdivisions of the parent element, where $N_{quad} = \max\left\{4, \text{ int}\left(2/d_l^{(\mu)}\right) + 1\right\}$, and denoting by g the index of the generic quadrature point, W_g being the relevant weight, one assumes $W_g = \dfrac{2}{N_{quad}}$ e $\mu_g = -1 + (0.5 + g)W_g$.

In conclusion, integrals (5.14) and (5.18) become:

$$\mathbb{A}^{(2)}_{m^*\alpha^* l\beta} = -\frac{l_l}{2}\sum_{g=1}^{N_{quad}} T^{(\beta-1)}(\mu_{lg})\frac{(x(\mu_{lg}) - x^*)\, n_{lx} + (y(\mu_{lg}) - y^*)\, n_{ly}}{2\pi||\mathbf{p}(\mu_{lg}) - \mathbf{p}^*||^2} W_g \tag{5.16}$$

$$b^1_{m^*\alpha^*} = -\sum\nolimits_{l=1}^{n_v}\frac{l_l}{2}\sum\nolimits_{g=1}^{N_{quad}}\frac{1}{2\pi}\ln\left(\frac{1}{||\mathbf{p}(\mu_{lg}) - \mathbf{p}^*||}\right)$$
$$\left[\alpha(x(\mu_{lg})^2 n_1 + x(\mu_{lg})y(\mu_{lg})n_1) + \beta(x(\mu_{lg})^2 n_1 + y(\mu_{lg})^2 n_2)\right] n_1 W_g \tag{5.17}$$

Conversely, the analytical calculus of integral (5.18) leads to

$$b^1_{m^*\alpha^*} = -\sum\nolimits_{l=1}^{n_v}\frac{l_l}{2}\int_{-1}^{1}\frac{1}{2\pi}\ln\left(\frac{1}{||\mathbf{p}(\mu_l) - \mathbf{p}^*||}\right) \tag{5.18}$$
$$\left[\alpha(x(\mu_{lg})^2 n_1 + x(\mu_{lg})y(\mu_{lg})n_1) + \beta(x(\mu_{lg})^2 n_1 + y(\mu_{lg})^2 n_2)\right] n_1 d\mu_l\,,$$

To implement the analytical and numerical calculus of integrals (5.18) one needs to explicit the terms $\mathbf{p}(\mu_l)$ e $\mathbf{p}^*$ by adopting the parametric representation previously introduced:

$$\begin{aligned}
\mathbf{p}(\mu_l) &= \mathbf{p}_l^{(med)} + \frac{l_l}{2}\mu_l\left(\frac{\mathbf{p}_{l+1}-\mathbf{p}_l}{||\mathbf{p}_{l+1}-\mathbf{p}_l||}\right) = \mathbf{p}_l^{(med)} + \frac{l_l}{2}\mu_l\mathbf{e}_{tl} \\
\mathbf{p}^*(\mu_l) &= \mathbf{p}_l^{*(med)} + \frac{l_{m^*}}{2}\mu_{\alpha^*}\left(\frac{\mathbf{p}_{l+1}-\mathbf{p}_l}{||\mathbf{p}_{l+1}-\mathbf{p}_l||}\right) = \mathbf{p}_{mm^*} + \frac{l_{m^*}}{2}\mu_{m^*}\mathbf{e}_{tm^*}
\end{aligned} \tag{5.19}$$

where $\mathbf{e}_t$ is the unit vector tangent to the segment. When $l = m^*$, one gets:

$$\mathbf{p}(\mu_l) - \mathbf{p}^* = \frac{l_{m^*}}{2}\left(\mu - \mu_{\alpha^*}\right)\mathbf{e}_{tm^*} \Rightarrow ||\mathbf{p}(\mu_l) - \mathbf{p}^*|| = \frac{l_{m^*}}{2}|\mu - \mu_{\alpha^*}|$$

To explicit the expression of $\left[\alpha(x^2 n_1 + xyn_1) + \beta(x^2 n_1 + y^2 n_2)\right]$ it is necessary to write the first formula of (5.19) in components:

$$\begin{aligned}
x(\mu_l) &= x_{ml} + \frac{1}{2}\mu_l\left(x_{l+1} - x_l\right) \\
y(\mu_l) &= y_{ml} + \frac{1}{2}\mu_l\left(y_{l+1} - y_l\right)
\end{aligned} \tag{5.20}$$

By squaring both sides of (5.20), one obtains

$$\begin{aligned}
x^2(\mu_l) &= x_{ml}^2 + x_{ml}\left(x_{l+1} - x_l\right)\mu_l + \frac{1}{4}\left(x_{l+1} - x_l\right)^2\mu_l^2 \\
y^2(\mu_l) &= y_{ml}^2 + y_{ml}\left(y_{l+1} - y_l\right)\mu_l + \frac{1}{4}\left(y_{l+1} - y_l\right)^2\mu_l^2
\end{aligned} \tag{5.21}$$

It is also necessary to write the expression of the product of (5.20)

$$\begin{aligned}
&x(\mu_l)y(\mu_l) = x_{ml}y_{ml} + \frac{1}{2}y_{ml}\left(x_{l+1} - x_l\right)\mu_l + \frac{1}{2}x_{ml}\left(y_{l+1} - y_l\right)\mu_l + \\
&+\frac{1}{4}\left(x_{l+1} - x_l\right)\left(y_{l+1} - y_l\right)\mu_l^2
\end{aligned} \tag{5.22}$$

and explicit the following expression

$$\begin{aligned}
&\mathbf{A}^P \cdot \mathbf{n} = [\alpha(\mathbf{p} \otimes \mathbf{p}) + \beta(\mathbf{p} \cdot \mathbf{p})\hat{\mathbf{I}}]\mathbf{n} = \\
&\qquad = \alpha(\mathbf{p} \cdot \mathbf{n})\mathbf{p} + \beta(\mathbf{p} \cdot \mathbf{p})\mathbf{n} = \begin{bmatrix} \alpha(\mathbf{p} \cdot \mathbf{n})x + \beta(\mathbf{p} \cdot \mathbf{p})n_1 \\ \\ \alpha(\mathbf{p} \cdot \mathbf{n})y + \beta(\mathbf{p} \cdot \mathbf{p})n_2 \end{bmatrix}
\end{aligned} \tag{5.23}$$

By developing the first component of the vectorial expressions (5.23) it turns out to be

$$\alpha(x^2 n_1 + xyn_2) + \beta(x^2 + y^2)n_1 = x^2(\alpha+\beta)n_1 + \beta y^2 n_1 + \alpha xyn_2 \tag{5.24}$$

Finally, grouping constant, linear and quadratics terms one gets

$$\begin{aligned}
&x_{ml}^2(\alpha+\beta)n_1 + \beta y_{ml}^2 n_1 + \alpha x_{ml} y_{ml} n_2 + \\
&+\{x_{ml}(x_{l+1}-x_l)(\alpha+\beta)n_1 + \beta y_{ml}(y_{l+1}-y_l)n_1 + \\
&+\frac{1}{2}\alpha n_2[y_{ml}(x_{l+1}-x_l) + x_{ml}(y_{l+1}-y_l)]\}\mu_l + \\
&+\frac{1}{4}[(x_{l+1}-x_l)^2(\alpha+\beta)n_1 + \beta(y_{l+1}-y_l)^2 n_1 + \\
&+\alpha n_2(x_{l+1}-x_l)(y_{l+1}-y_l)]\mu_l^2 = L_l + M_l\mu_l + N_l\mu_l^2
\end{aligned} \tag{5.25}$$

in which:

$$\begin{aligned}
&L_1 = x_{ml}^2(\alpha+\beta)n_1 + \beta y_{ml}^2 n_1 + \alpha x_{ml} y_{ml} n_2 \\
&M_1 = +\{x_{ml}(x_{l+1}-x_l)(\alpha+\beta)n_1 + \beta y_{ml}(y_{l+1}-y_l)n_1 + \\
&+\frac{1}{2}\alpha n_2[y_{ml}(x_{l+1}-x_l) + x_{ml}(y_{l+1}-y_l)]\} \\
&N_1 = +\frac{1}{4}[(x_{l+1}-x_l)^2(\alpha+\beta)n_1 + \beta(y_{l+1}-y_l)^2 n_1 + \\
&+\alpha n_2(x_{l+1}-x_l)(y_{l+1}-y_l)]
\end{aligned}$$

Proceeding analogously one can explicit the second component of (5.23):

$$
\begin{aligned}
&\left\{\left[y_{ml}+\frac{1}{2}\mu_l\left(y_{l+1}-y_l\right)\right]^2-\bar{\nu}\left[x_{ml}+\frac{1}{2}\mu_l\left(x_{l+1}-x_l\right)\right]^2\right\}n_1=\\
&=\left[y_{ml}^2+y_{ml}\left(y_{l+1}-y_l\right)\mu_l+\frac{1}{4}\left(y_{l+1}-y_l\right)^2\mu_l^2-\bar{\nu}x_{ml}^2+\right.\\
&\left.-\bar{\nu}x_{ml}\left(x_{l+1}-x_l\right)\mu_l-\frac{1}{4}\bar{\nu}\left(x_{l+1}-x_l\right)^2\mu_l^2\right]n_1=\\
&=\left\{\left(y_{ml}^2-\bar{\nu}x_{ml}^2\right)+\left[x_{ml}\left(x_{l+1}-x_l\right)-\bar{\nu}y_{ml}\left(y_{l+1}-y_l\right)\right]\mu_l\right.\\
&\left.+\frac{1}{4}\left[\left(y_{l+1}-y_l\right)^2-\bar{\nu}\left(x_{l+1}-x_l\right)^2\right]\mu_l^2\right\}n_1=\\
&=L_2+M_2\mu_l+N_2\mu_l^2
\end{aligned}
\tag{5.26}
$$

in which it is set:

$$
\begin{aligned}
L_2&=\left(y_{ml}^2-\bar{\nu}x_{ml}^2\right)n_2\\
M_2&=\left[x_{ml}\left(x_{l+1}-x_l\right)-\bar{\nu}y_{ml}\left(y_{l+1}-y_l\right)\right]n_2\\
N_2&=\frac{1}{4}\left[\left(y_{l+1}-y_l\right)^2-\bar{\nu}\left(x_{l+1}-x_l\right)^2\right]n_2
\end{aligned}
$$

Finally, expression (5.18) becomes:

$$
\begin{bmatrix}\psi^1_{m^*\alpha^*}\\ \\ \psi^2_{m^*\alpha^*}\end{bmatrix}=\begin{bmatrix}\displaystyle\sum_{l=1}^{n_v}\frac{l_{m^*}}{4\pi}\int_{-1}^{1}\ln\left(\frac{2/l_{m^*}}{|\mu-\mu_{\alpha^*}|}\right)(L_1+M_1\mu_l+N_1\mu_l^2)d\mu_l\\ \displaystyle\sum_{l=1}^{n_v}\frac{l_{m^*}}{4\pi}\int_{-1}^{1}\ln\left(\frac{2/l_{m^*}}{|\mu-\mu_{\alpha^*}|}\right)(L_2+M_2\mu_l+N_2\mu_l^2)d\mu_l\end{bmatrix}
\tag{5.27}
$$

The integral on the right-hand side of the first row of (5.27) can be solved analytically as follows

$$\frac{l_{m^*}}{4\pi}\int_{-1}^{1}\Big[-L_1\ln\left(\frac{l_{m^*}}{2}\right)-M_1\ln\left(\frac{l_{m^*}}{2}\right)\mu_l-N_1\ln\left(\frac{l_{m^*}}{2}\right)\mu_l^2+$$
$$+L_1\ln\left(\frac{1}{|\mu_l-\mu_{\alpha^*}|}\right)+M_1\ln\left(\frac{1}{|\mu_l-\mu_{\alpha^*}|}\right)\mu_l+$$
$$+N_1\ln\left(\frac{1}{|\mu_l-\mu_{\alpha^*}|}\right)\mu_l^2\Big]\,d\mu_l=\frac{l_{m^*}}{4\pi}\Big[-\ln\left(\frac{l_{m^*}}{2}\right)(2L_1+\frac{2}{3}N_1)+$$
$$+L_1H^{(1)}_{\alpha^*}+M_1H^{(2)}_{\alpha^*}+N_1H^{(3)}_{\alpha^*}\Big]$$

where

$$\begin{aligned}
H^{(1)}_{\alpha^*}&=\int_{-1}^{1}\ln\frac{1}{|\mu_l-\mu_{\alpha^*}|}\,d\mu_l\\
H^{(2)}_{\alpha^*}&=\int_{-1}^{1}\mu_l\ln\frac{1}{|\mu-\mu_{\alpha^*}|}\,d\mu_l\\
H^{(3)}_{\alpha^*}&=\int_{-1}^{1}\mu_l^2\ln\frac{1}{|\mu-\mu_{\alpha^*}|}\,d\mu_l
\end{aligned}\tag{5.28}$$

These integrals can be calculated analytically:

$$\begin{aligned}
H^{(1)}_{\alpha^*}&=2-\ln\left(1-\mu^2_{\alpha^*}\right)+\mu_{\alpha^*}\ln\left(\frac{1-\mu_{\alpha^*}}{1+\mu_{\alpha^*}}\right)\\
H^{(2)}_{\alpha^*}&=\mu_{\alpha^*}+\frac{1}{2}\ln\left(\frac{1+\mu_{\alpha^*}}{1-\mu_{\alpha^*}}\right)+\frac{1}{2}\mu^2_{\alpha^*}\ln\left(\frac{1-\mu_{\alpha^*}}{1+\mu_{\alpha^*}}\right)\\
H^{(3)}_{\alpha^*}&=\frac{1}{3}\left[\ln(1-\mu_{\alpha^*})-\ln(1+\mu_{\alpha^*})\right]\mu^3_{\alpha^*}+\frac{2}{3}\mu^2_{\alpha^*}+\frac{2}{9}+\\
&\quad-\frac{1}{3}\ln(1-\mu_{\alpha^*})-\frac{1}{3}\ln(1+\mu_{\alpha^*})
\end{aligned}\tag{5.29}$$

by obtaining expressions that are well-defined for $-1<\mu_{\alpha^*}<1$.

5.5 Calculus of ψ for points located at the interior of the domain

The function $\boldsymbol{\psi}$ at a generic point $\bar{\mathbf{p}}$ internal to the domain can be evaluated by specializing formula (5.6) at these points:

$$\begin{aligned}\boldsymbol{\psi}(\bar{\mathbf{p}}) = \sum_{l=1}^{n_v} \frac{l_l}{2} \int_{-1}^{1} \boldsymbol{\psi}(\mathbf{p}) \frac{(\mathbf{p}(\mu_l) - \bar{\mathbf{p}}) \cdot \mathbf{n}_l}{2\pi ||\mathbf{p}(\mu_l) - \bar{\mathbf{p}}||^2} \, d\mu_l \, + \\ + \sum_{l=1}^{n_v} \frac{l_l}{2} \int_{-1}^{1} \frac{1}{2\pi} \ln \left(\frac{1}{||\mathbf{p}(\mu_l) - \bar{\mathbf{p}}||} \right) \mathbf{A}(\mathbf{p}) \mathbf{n} \, d\mu_l \, ,\end{aligned} \tag{5.30}$$

in which, now, the values of the function $\boldsymbol{\psi}$ at the boundary is known from the solution of the algebraic system referred above. For brevity we detail only the algebraic manipulations pertaining to the first row of the first equation of the vector equation (5.30)

$$\begin{aligned}\psi^1(\bar{\mathbf{p}}) = \sum_{l=1}^{n_v} \frac{l_l}{2} \int_{-1}^{1} \psi^1(\mathbf{p}) \frac{(\mathbf{p}(\mu_l) - \bar{\mathbf{p}}) \cdot \mathbf{n}_l}{2\pi ||\mathbf{p}(\mu_l) - \bar{\mathbf{p}}||^2} \, d\mu_l \, + \\ + \sum_{l=1}^{n_v} \frac{l_l}{2} \int_{-1}^{1} \frac{1}{2\pi} \ln \left(\frac{1}{||\mathbf{p}(\mu_l) - \bar{\mathbf{p}}||} \right) \left[\alpha(x^2 n_1 + xy n_1) + \beta(x^2 n_1 + y^2 n_2) \right] d\mu_l\end{aligned} \tag{5.31}$$

By substituting the first equation of formula (5.19) in the expression $||\mathbf{p}(\mu_l) - \bar{\mathbf{p}}||^2$ appearing at the denominator of the left-hand side of (5.31) one gets:

$$\begin{aligned}||\mathbf{p}(\mu_l) - \bar{\mathbf{p}}||^2 = \left\| \mathbf{r}_l^{(med)} + \frac{1}{2} \mu_l \left(\mathbf{r}_{l+1} - \mathbf{r}_l \right) - \bar{\mathbf{r}} \right\|^2 = \\ = \left\| \mathbf{r}_l^{(med)} - \bar{\mathbf{r}} + \frac{1}{2} \mu_l \left(\mathbf{r}_{l+1} - \mathbf{r}_l \right) \right\|^2 = \left\| \mathbf{a}_l + \frac{1}{2} \mu_l \mathbf{b}_l \right\|^2\end{aligned} \tag{5.32}$$

where $\quad \mathbf{a}_l = \mathbf{r}_l^{(med)} - \bar{\mathbf{r}} \quad$ and $\quad \mathbf{b}_l = (\mathbf{r}_{l+1} - \mathbf{r}_l)$

Thus

$$\begin{aligned}||\mathbf{p}(\mu_l) - \bar{\mathbf{p}}||^2 = \mathbf{a}_l \cdot \mathbf{a}_l + \mathbf{b}_l \cdot \mathbf{b}_l \mu_l + \tfrac{1}{4} \mathbf{b}_l \cdot \mathbf{b}_l \mu_l^2 = \\ = L_l + M_l \mu_l + N_l \mu_l^2\end{aligned} \tag{5.33}$$

in which

$$\bar{A}_l = \mathbf{a}_l \cdot \mathbf{a}_l = \left\| \mathbf{r}_l^{(med)} - \bar{\mathbf{r}} \right\|^2$$

$$\bar{B}_l = \mathbf{a}_l \cdot \mathbf{b}_l = (\mathbf{r}_{l+1} - \mathbf{r}_l) \left(\mathbf{r}_l^{(med)} - \bar{\mathbf{r}} \right)$$

$$\bar{C}_l = \frac{1}{4} \mathbf{b}_l \cdot \mathbf{b}_l = \| \mathbf{r}_{l+1} - \mathbf{r}_l \|^2$$

One can now substitute the first equation of (5.19) in the expression $(\mathbf{p}(\mu_l) - \bar{\mathbf{p}}) \cdot \mathbf{n}_l$ that appears at the numerator of the first integrand of (5.31):

$$\begin{aligned} &\left[\mathbf{r}_l^{(med)} + \tfrac{1}{2}\mu_l \left(\mathbf{r}_{l+1} - \mathbf{r}_l \right) - \bar{\mathbf{r}} \right] \cdot \mathbf{n}_l = \\ &\quad = \left(\mathbf{r}_l^{(med)} - \bar{\mathbf{r}} \right) \cdot \mathbf{n}_l + \tfrac{1}{2}\mu_l \left(\mathbf{r}_{l+1} - \mathbf{r}_l \right) \cdot \mathbf{n}_l \end{aligned} \tag{5.34}$$

Since $(\mathbf{r}_{l+1} - \mathbf{r}_l)$ is orthogonal to $\mathbf{n}_l$, the second term on the right-hand side of the expression above vanishes and the final result is

$$(\mathbf{p}(\mu_l) - \bar{\mathbf{p}}) \cdot \mathbf{n}_l = \left(\mathbf{r}_l^{(med)} - \bar{\mathbf{r}} \right) \cdot \mathbf{n}_l = \bar{D}_l \tag{5.35}$$

Further, by exploiting properties of the logarithm, it is possible to re-write the expression appearing in the second sum of (5.31) to obtain:

$$\begin{aligned} &\int_{-1}^{1} \ln \left(\| \mathbf{r}\left(\mu_l \right) - \bar{\mathbf{r}} \| \right)^{-1} \left(L_l + M_l \mu_l + N_l \mu_l^2 \right) d\mu_l = \\ &\qquad = -\frac{1}{2} \int_{-1}^{1} \ln \left(\| \mathbf{r}\left(\mu_l \right) - \bar{\mathbf{r}} \|^2 \right) \left(L_l + M_l \mu_l + N_l \mu_l^2 \right) d\mu_l \end{aligned}$$

Thus, formula (5.31) can be implemented by means of the following expressions:

$$\psi^1(\bar{\mathbf{p}}) = \sum_{l=1}^{n_v} \frac{l_l}{4\pi} \sum_{\beta=1}^{n_{Pl}} \bar{\Psi}_l^{(\beta)} p_{l1}^{(\beta)} - \sum_{l=1}^{n_v} \frac{l_l}{8\pi} \left(\bar{Q}_l^{(1)} + \bar{Q}_l^{(2)} + \bar{Q}_l^{(3)} \right), \tag{5.36}$$

where

$$
\begin{aligned}
\bar{\Psi}_l^{(\beta)} &= \int_{-1}^{1} T^{(\beta-1)}(\mu_l) \frac{\bar{D}_l}{\left[\bar{A}_l + \bar{B}_l \mu_l + \bar{C}_l \mu_l^2\right]} \, d\mu_l \,, \\
\bar{Q}_l^{(1)} &= L_1 \int_{-1}^{1} \ln\left[\bar{A}_l + \bar{B}_l \mu_l + \bar{C}_l \mu_l^2\right] \, d\mu \,, \\
\bar{Q}_l^{(2)} &= M_1 \int_{-1}^{1} \mu_l \ln\left[\bar{A}_l + \bar{B}_l \mu_l + \bar{C}_l \mu_l^2\right] \, d\mu_l \,, \\
\bar{Q}_l^{(3)} &= N_1 \int_{-1}^{1} \mu_l^2 \ln\left[\bar{A}_l + \bar{B}_l \mu_l + \bar{C}_l \mu_l^2\right] \, d\mu_l
\end{aligned}
\tag{5.37}
$$

Integrals (5.37), also amenable to analytical computation are evaluated numerically as in (5.16) and (5.17).

5.6 Calculus of derivatives of ψ for points located at the interior of the domain

To obtain the values of $\boldsymbol{\tau}$ for points internal to $\boldsymbol{\Omega}$ one needs to calculate the gradient of $\boldsymbol{\psi}$ at these points. By deriving (5.36) with respect to $\bar{\mathbf{p}}$ one obtains:

$$\begin{aligned} \frac{\partial \psi^1}{\partial \bar{\mathbf{p}}}(\bar{\mathbf{p}}) = \sum_{l=1}^{n_v} \frac{l_l}{4\pi} \sum_{\beta=1}^{n_{Pl}} \frac{\partial \bar{\Psi}_l^{(\beta)}}{\partial \bar{\mathbf{p}}} p_{l1}^{(\beta)} + \\ - \sum_{l=1}^{n_v} \frac{l_l}{8\pi} \left(\frac{\partial \bar{Q}_l^{(1)}}{\partial \bar{\mathbf{p}}} + \frac{\partial \bar{Q}_l^{(2)}}{\partial \bar{\mathbf{p}}} + \frac{\partial \bar{Q}_l^{(3)}}{\partial \bar{\mathbf{p}}} \right) \end{aligned} \tag{5.38}$$

Having set $\rho(\mu_l, \bar{\mathbf{p}}) = \bar{A}_l + \bar{B}_l \mu_l + \bar{C}_l \mu_l^2$, derivatives in (5.38) become:

$$\frac{\partial \bar{\Psi}_l}{\partial \bar{\mathbf{p}}}^{(\beta)} = \int_{-1}^{1} T^{(\beta-1)}(\mu_l) \frac{1}{\rho^2} \left(-\rho \mathbf{n} - \frac{\partial \rho}{\partial \bar{\mathbf{p}}} \bar{D}_l \right) d\mu_l \tag{5.39}$$

$$\begin{aligned} \frac{\partial \bar{Q}_l}{\partial \bar{\mathbf{p}}}^{(1)} &= \int_{-1}^{1} \frac{L_1}{\rho} \frac{\partial \rho}{\partial \bar{\mathbf{p}}} \, d\mu_l \\ \frac{\partial \bar{Q}_l}{\partial \bar{\mathbf{p}}}^{(2)} &= \int_{-1}^{1} \frac{M_1 \mu_l}{\rho} \frac{\partial \rho}{\partial \bar{\mathbf{p}}} \, d\mu_l \\ \frac{\partial \bar{Q}_l}{\partial \bar{\mathbf{p}}}^{(3)} &= \int_{-1}^{1} \frac{N_1 \mu_l^2}{\rho} \frac{\partial \rho}{\partial \bar{\mathbf{p}}} \, d\mu_l \end{aligned} \tag{5.40}$$

$$\frac{\partial \rho}{\partial \bar{\mathbf{p}}} = -2 \left(\mathbf{p}_l^{(med)} - \bar{\mathbf{p}} \right) - \left(\mathbf{p}_{l+1} - \mathbf{p}_l \right) \mu_l \, . \tag{5.41}$$

Chapter 6

Numerical results

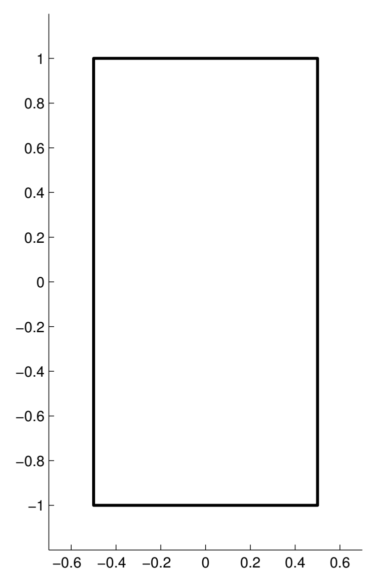
To investigate the computational efficiency as well as the accuracy of the proposed numerical solution, the expressions of stress field, of shear center and of shear deformability tensor reported in the previous chapters have been implemented in a Matlab code. In this respect we recall that the particular BEM method implemented here allows one to obtain the results by assigning only the vertices of the section, similarly to flexure.

The results of the BEM solution for circular and rectangular domain have been first compared with the available closed-form solutions. However, these last ones don't take into account the Poisson's ratio dependency. Hence, a series of numerical tests on some sections have been performed in order to make a comparison with results from literature.

6.1 Rectangular cross section

Let us consider a rectangular section assigned by means of the following vertices:

$l:$	1	2	3	4
$x:$	-0.5	0.5	0.5	-0.5
$y:$	-1	-1	1	1



The resulting shear factor χ_{11}^{NUM} is compared with the corresponding analytical value $\chi_{11}^{an.}$ [23] obtained in the case $\nu = 0$. Specifically tab.1 reports the error $e = |\chi_{11}^{NUM} - \chi_{11}^{an.}|/\chi_{11}^{an.}$ as function of the number of polynomials assumed to interpolate each side of the polygonal.

#	χ_{11}^{NUM}	e	$\chi_{11}^{an.}$ [23]
3	1.2185	1.5393	1.2000
4	1.2001	0.0110	1.2000
5	1.1994	0.0468	1.2000
6	1.1986	0.1179	1.2000
7	1.1995	0.0448	1.2000
8	1.2003	0.0273	1.2000
9	1.1997	0.0260	1.2000
10	1.1978	0.1807	1.2000

Tab.1: Error of χ_{11} ($\nu = 0$).

It is apparent from tab.1 that just 4 polynomials for each side are sufficient to obtain an error below 1% when $\nu = 0$.

To evaluate the accuracy of the proposed method for $\nu > 0$, numerical values χ_{11}^{NUM} and χ_{22}^{NUM} are compared with those reported in [42].

Notice that in the paper [42] 200 quadratic elements are considered in the FEM discretization and 100 linear elements for BEM solutions in order to achieve an error below 1% whereas the same precision is here obtained with one element for each side and 8 interpolating polynomials.

ν	χ_{11}		χ_{22}	
	Proposed method	Petrolo [42]	Proposed method	Petrolo [42]
0	1.200	1.200	1.198	1.200
0.3	1.284	1.275	1.198	1.201
0.5	1.368	1.356	1.198	1.201

Tab.1: Shear factors χ_{11} and χ_{22} (8 interpolating polynomials for each side).

We also report in fig.1 the contour plot of the shear stress intensity $\boldsymbol{\tau} = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{1/2}$ due to a unit shear in the y direction.

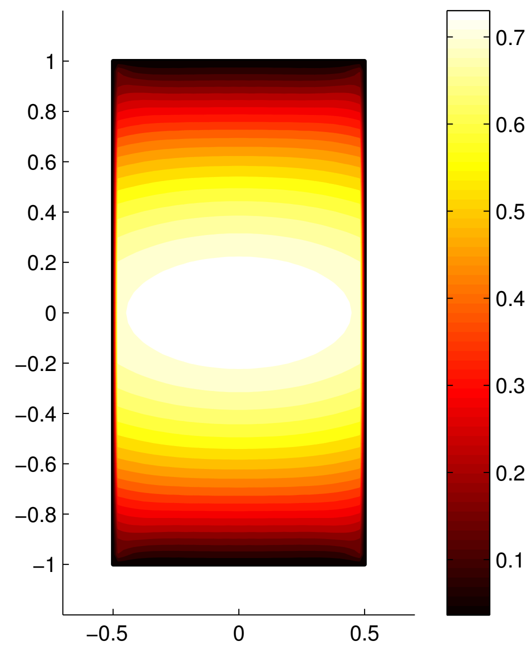


Fig. 1: Contour plot of shear stress intensity due to unit shear in y direction ($\nu = 0.3$).

6.2 Circular cross section

The circular section has been approximated by a regular polygon with 25, 50, 100 and 150 sides.

For each approximation of the boundary the shear factor χ_{11}^{NUM} is compared with the corresponding analytical value $\chi_{11}^{an.}$ [18]. In particular it is reported in tab.3 the error $e = |\chi_{11}^{NUM} - \chi_{11}^{an.}|/\chi_{11}^{an.}$ as function of the number of polynomials assumed for each side of the polygon.

25 elements

# of polynomials	χ_{11}^{NUM}	e	$\chi_{11}^{an.}$ [18]
3	1.1601	0.5657	1.1667
4	1.1601	0.5657	1.1667
5	1.1600	0.5743	1.1667
6	1.1601	0.5657	1.1667

50 elements

# of polynomials	χ_{11}^{NUM}	e	$\chi_{11}^{an.}$ [18]
3	1.1650	0.1457	1.1667
4	1.1650	0.1457	1.1667
5	1.1649	0.1543	1.1667
6	1.1650	0.1457	1.1667

100 elements

# of polynomials	χ_{11}^{NUM}	e	$\chi_{11}^{an.}$ [18]
3	1.1662	0.0429	1.1667
4	1.1662	0.0429	1.1667

150 elements

# of polynomials	χ_{11}^{NUM}	e	$\chi_{11}^{an.}$ [18]
3	1.1665	0.0171	1.1667

Tab.3: comparison between χ_{11}^{NUM} and the corresponding analytical value $\chi_{11}^{an.}$ [18].

The tables above show that a finer discretization of the boundary produces an error decrease higher then that achieved by increasing the number of polynomials for each side of the polygon.

6.3 L-Shaped cross section

Let us now consider the L-Shaped section addressed in [53] and defined by the following coordinates of the vertices

$l:$	1	2	3	4	5	6
$x:$	0	10.5	10.5	1	1	0
$y:$	0	0	1	1	15.5	15.5

Tables 4 and 5 contain the comparison between the results obtained with the proposed method and those reported in the papers by Sapountzakis [53] and Schramm [54].

ν	χ_{11}			χ_{22}		
	Proposed method	Sap.[53]	Schramm[54]	Proposed method	Sap.[53]	Schramm[54]
0	3.049	3.063	3.058	1.919	1.899	1.898
0.2	3.060	3.065	-	1.912	1.900	-
0.3	3.065	3.067	3.062	1.909	1.900	1.899
0.4	3.069	3.069	-	1.907	1.900	-

Tab.4: Shear factor χ_{11} and χ_{22} (8 interpolating polynomials for each side).

ν	x_c			y_c		
	Proposed method	Sap.[53]	Schramm[54]	Proposed method	Sap.[53]	Schramm[54]
0	-1.990	-1.998	-1.998	-4.373	-4.425	-4.424
0.2	-1.988	-1.998	-	-4.365	-4.426	-
0.3	-1.987	-1.998	-1.998	-4.362	-4.426	-4.424
0.4	-1.986	-1.998	-	-4.360	-4.426	-

Tab.5: Shear center coordinates (8 interpolating polynomials for each side).

Notice that 8 interpolating polynomials for each side are sufficient to reduce the error below 1% whereas 300 constant elements are necessary with the method illustrated in [53] to obtain the same precision.

6.4 Trapezoidal cross section

We now address the trapezoidal section defined in [42] by the following coordinates of the vertices:

$l:$	1	2	3	4
$x:$	0	5	2	0
$y:$	0	0	3	3

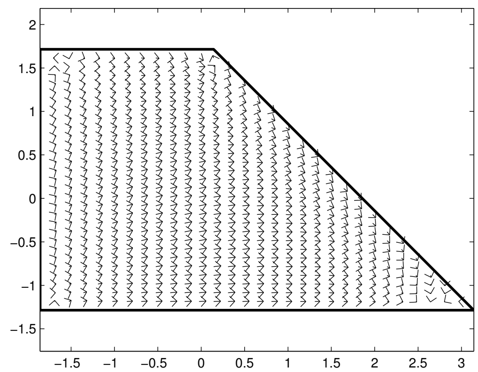


Fig.4: Stress field due to a unit shear in x direction ($\nu = 0.3$).

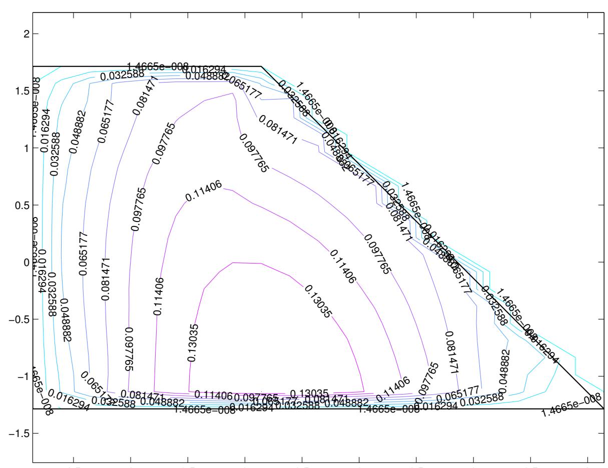


Fig.5: Contour plot of shear stress intensity due to a unit shear in x direction ($\nu = 0.3$).

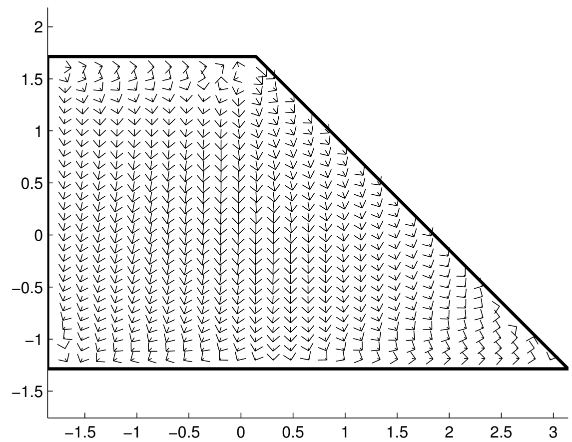


Fig.6: Stress field due to unit shear in y direction ($\nu = 0.3$).

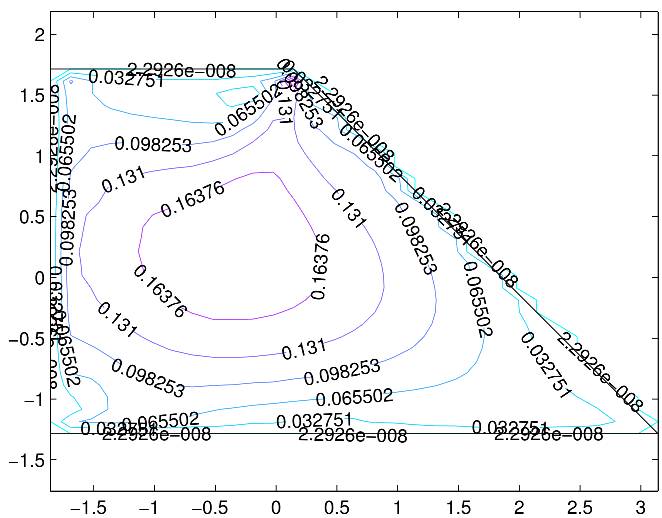


Fig.7: Contour plot of shear stress intensity due to a unit shear in y direction ($\nu = 0.3$).

	Proposed method	Petrolo[42]
x_c	1.637	1.637
y_c	1.412	1.389

Tab.6: Shear center coordinates.

ν	χ_{11} Proposed method	Petrolo[42]	χ_{22} Proposed method	Petrolo[42]
0	1.327	1.347	1.091	1.184
0.3	1.351	1.377	1.089	1.186
0.5	1.379	1.410	1.092	1.187

Tab.7: Shear factors χ_{11} and χ_{22} (4 interpolating polynomials for each side).

6.5 C-Shaped cross section

The C-shaped section considered in this example is derived from [42] and defined by the following coordinates of the vertices:

l :	1	2	3	4	5	6	7	8
x :	-5.5	0	0	-5.5	-5.5	-1	-1	-5.5
y :	-5.5	-5.5	5.5	5.5	4.5	4.5	-4.5	-4.5

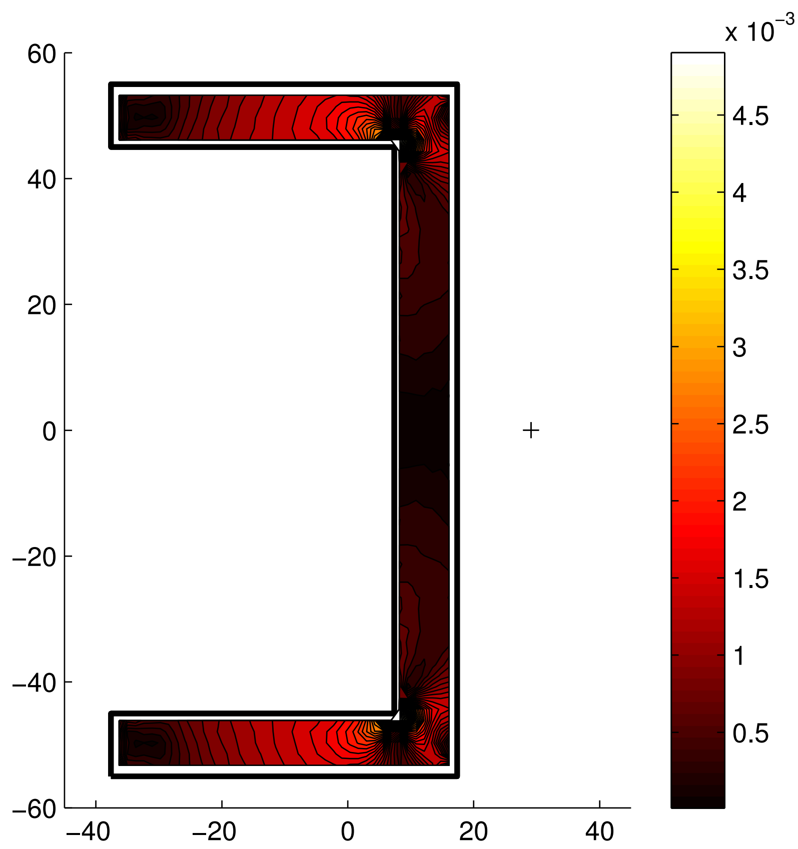


Fig.8: Contour plot of shear stress intensity due to a unit shear in x direction ($\nu = 0.3$).

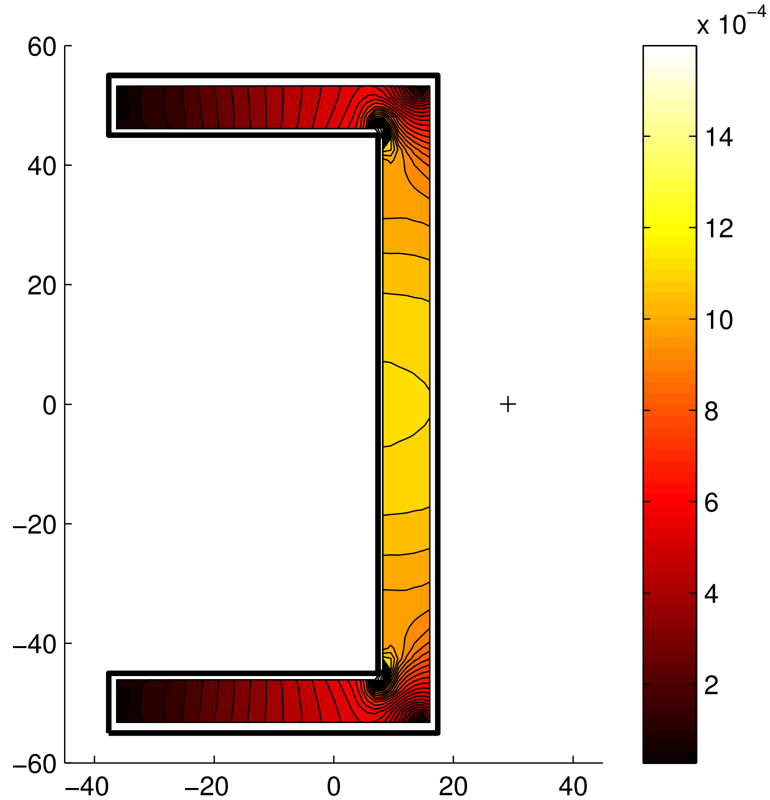


Fig.9: Contour plot of shear stress intensity due to a unit shear in y direction ($\nu = 0.3$).

ν	χ_{11} Proposed method	Petrolo[42]	χ_{22} Proposed method	Petrolo[42]
0	2.633	2.862	2.336	2.270
0.3	2.736	2.874	2.334	2.270
0.5	2.785	2.883	2.332	2.270

Tab.8: Shear factors χ_{11} e χ_{22} (4 interpolating polynomials for each side).

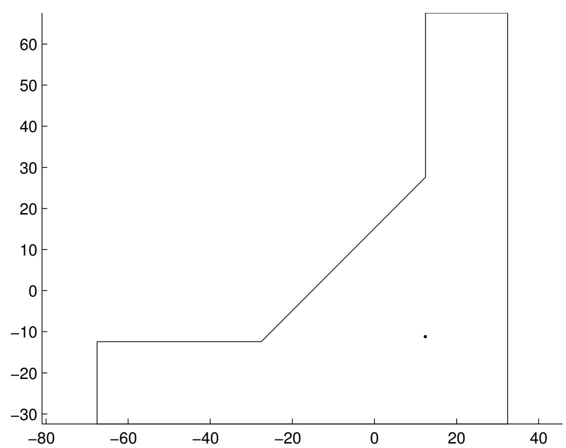
	Proposed method	Petrolo[42]
x_c	29.25	30.38
y_c	0	0

Tab.9: Shear center coordinates (4 interpolating polynomials for each side).

6.6 A section of arbitrary polygonal shape

In the paper by Petrolo and Casciaro [42] it has also been considered the section defined by the following coordinates of the vertices:

$l:$	1	2	3	4	5	6	7
$x:$	0	100	100	80	80	40	0
$y:$	0	0	100	100	60	20	20



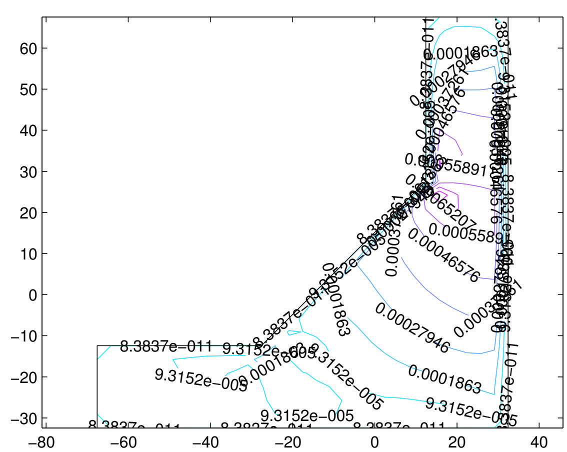


Fig.10: Contour plot of shear stress intensity due to a unit shear in y direction ($\nu = 0.3$).

	Proposed method	Petrolo[42]
x_c	79.95	79.46
y_c	21.25	20.53

Tab.10: Shear center coordinates (4 interpolating polynomials for each side).

ν	χ_{11}		χ_{22}	
	Proposed method	Petrolo[42]	Proposed method	Petrolo[42]
0.3	1.531	1.522	2.140	2.124

Tab.11: Shear factors χ_{11} e χ_{22} (4 interpolating polynomials for each side).

Chapter 7

Appendix

7.1 Examples of application of the Gibbs rule

Aim of this subsection is to present a systematic derivation of several differential identities in order to provide examples of application of the Gibbs rule. The objective is not to prove new identities but to illustrate a methodology which, on one side, simplifies the derivation of differential identities by means of the definition of vector product between vectors and tensors introduced in (2.6) and, on the other, allows one to express the final result in a manner consistent with the proposed formalism.

For greater clarity each formula is reported in intrinsic notation followed by the relevant proof in Gibbs notation. We shall make reference to scalar fields φ and ψ, vector fields $\mathbf{a}$ and $\mathbf{b}$ and a tensor field $\mathbf{A}$; each field is assumed to be smooth enough to make well defined the relevant differential operation in which it is involved.

- $\text{grad}(\varphi\psi) = (\text{grad}\ \varphi)\psi + \varphi\ \text{grad}\ \psi$

$$\boldsymbol{\nabla}(\varphi\psi) = (\boldsymbol{\nabla}\varphi)\psi + \varphi(\boldsymbol{\nabla}\psi) \tag{7.1}$$

  see, e.g., the definition $(2.32)_1$.

- $\operatorname{grad}(\varphi \mathbf{a}) = \mathbf{a} \otimes \operatorname{grad} \varphi + \varphi \operatorname{grad} \mathbf{a}$

$$(\varphi \mathbf{a}) \otimes \boldsymbol{\nabla} = \mathbf{a} \otimes (\varphi \boldsymbol{\nabla}) + \varphi(\mathbf{a} \otimes \boldsymbol{\nabla})$$

  see, e.g., the definitions $(2.32)_1$ and $(2.32)_2$

- $\operatorname{grad}(\mathbf{a} \cdot \mathbf{b}) = (\operatorname{grad} \mathbf{a})^t \mathbf{b} + (\operatorname{grad} \mathbf{b})^t \mathbf{a}$

$$\boldsymbol{\nabla}(\mathbf{a} \cdot \mathbf{b}) = \boldsymbol{\nabla}(\mathbf{a} \cdot \mathbf{b}) + \boldsymbol{\nabla}(\mathbf{b} \cdot \mathbf{a}) = (\boldsymbol{\nabla} \otimes \mathbf{a})\mathbf{b} + (\boldsymbol{\nabla} \otimes \mathbf{b})\mathbf{a} \tag{7.2}$$

  see, e.g., the definitions $(2.32)_1$ and $(2.32)_2$

- $\operatorname{grad}(\mathbf{a} \times \mathbf{b}) = (\operatorname{grad} \mathbf{a}) \times \mathbf{b} + \mathbf{a} \times (\operatorname{grad} \mathbf{b}) = \mathbf{a} \times (\operatorname{grad} \mathbf{b}) - \mathbf{b} \times (\operatorname{grad} \mathbf{a})$

$$(\mathbf{a} \times \mathbf{b}) \otimes \boldsymbol{\nabla} = (\mathbf{a} \otimes \boldsymbol{\nabla}) \times \mathbf{b} + \mathbf{a} \times (\mathbf{b} \otimes \boldsymbol{\nabla})$$

  see, e.g., the definitions $(2.19)_1$ and $(2.32)_2$

Remark 1 Notice that the extended Gibbs rule has been applied to the first addend on the right-hand side in implicit form in the sense that bringing the factor $\mathbf{a}$ into direct connection with $\otimes\boldsymbol{\nabla}$ would lead to an expression $\times\mathbf{b}(\mathbf{a}\otimes\boldsymbol{\nabla})$ deprived of any significance

- $\operatorname{grad}(\varphi \mathbf{A}) = \mathbf{A} \otimes (\operatorname{grad} \varphi) + \varphi \operatorname{grad} \mathbf{A}$

$$(\varphi \mathbf{A}) \otimes \boldsymbol{\nabla} = \mathbf{A} \otimes (\varphi \boldsymbol{\nabla}) + \varphi(\mathbf{A} \otimes \boldsymbol{\nabla})$$

  see, e.g., the definitions $(2.32)_1$, $(2.32)_2$ and (2.37)

- $\operatorname{div} (\varphi \mathbf{a}) = (\operatorname{grad} \varphi) \cdot \mathbf{a} + \varphi \operatorname{div} \mathbf{a}$

$$(\varphi \mathbf{a}) \cdot \boldsymbol{\nabla} = \mathbf{a} \cdot (\varphi \boldsymbol{\nabla}) + \varphi(\mathbf{a} \cdot \boldsymbol{\nabla})$$

  see, e.g., the definitions $(2.32)_1$ and $(2.32)_3$

- div $(\mathbf{a} \times \mathbf{b}) = $ curl $\mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot$ curl $\mathbf{b}$

  $$\boldsymbol{\nabla} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{a} \cdot (\mathbf{b} \times \boldsymbol{\nabla}) + \mathbf{b} \cdot (\boldsymbol{\nabla} \times \mathbf{a}) = -\mathbf{a} \cdot (\boldsymbol{\nabla} \times \mathbf{b}) + \mathbf{b} \cdot (\boldsymbol{\nabla} \times \mathbf{a})$$

  where the rules of the mixed triple product have been used and the definitions $(2.32)_3$ and $(2.32)_4$ have been invoked.

- div $(\mathbf{a} \otimes \mathbf{b}) = $ (grad $\mathbf{a})\mathbf{b} + $ (div $\mathbf{b})\mathbf{a}$

  $$(\mathbf{a} \otimes \mathbf{b})\boldsymbol{\nabla} = (\mathbf{a} \otimes \boldsymbol{\nabla})\mathbf{b} + \mathbf{a}(\mathbf{b} \cdot \boldsymbol{\nabla})$$

  see, e.g., the definitions $(2.32)_2$ and $(2.32)_3$

- div $(\varphi \mathbf{A}) = \mathbf{A}$grad $\varphi + \varphi$div $\mathbf{A}$

  $$(\varphi \mathbf{A})\boldsymbol{\nabla} = \mathbf{A}(\varphi \boldsymbol{\nabla}) + \varphi(\mathbf{A}\nabla)$$

  see, e.g., the definitions $(2.32)_1$, $(2.32)_3$ and $(2.33)_1$

- div $(\mathbf{A}\mathbf{b}) = \mathbf{A}^t \cdot$ grad $\mathbf{b} + \mathbf{b} \cdot$ div $\mathbf{A}^t$

  $$\mathbf{A}\mathbf{b} \cdot \boldsymbol{\nabla} = \mathbf{A} \cdot (\boldsymbol{\nabla} \otimes \mathbf{b}) + \mathbf{b} \cdot \mathbf{A}^t \boldsymbol{\nabla} \tag{7.3}$$

  see, e.g., the definition $(2.32)_3$. Notice that the first addend on the right-hand side follows from (2.1) and the second addend from the definition of transpose.

  Recalling that the product between two tensors is equal to the scalar product between their transpose, it does follow that:

  $$\mathbf{A}\mathbf{b} \cdot \boldsymbol{\nabla} = \mathbf{A}^t \cdot (\mathbf{b} \otimes \boldsymbol{\nabla}) + \mathbf{b} \cdot \mathbf{A}^t \boldsymbol{\nabla}$$

Remark 2 It would have been more immediate to express the action of $\boldsymbol{\nabla}$ upon $\mathbf{b}$ by writing $\mathbf{A}(\mathbf{b} \cdot \boldsymbol{\nabla})$ in place of the less intuitive $\mathbf{A} \cdot (\boldsymbol{\nabla} \otimes \mathbf{b})$. However, this would have led to an incorrect result since its order of tensoriality would have been incompatible with the term on the left-hand side.

- curl $(\varphi\mathbf{a})$ = grad $\varphi \times \mathbf{a} + \varphi$ curl $\mathbf{a}$

$$\boldsymbol{\nabla} \times (\varphi\mathbf{a}) = \boldsymbol{\nabla}\varphi \times \mathbf{a} + \varphi(\boldsymbol{\nabla} \times \mathbf{a})$$

  see, e.g., the definitions $(2.32)_1$ and $(2.32)_4$

- curl $(\mathbf{a} \times \mathbf{b})$ = (div $\mathbf{b})\mathbf{a}$ − (grad $\mathbf{b})\mathbf{a}$ − (div $\mathbf{a})\mathbf{b}$ + (grad $\mathbf{a})\mathbf{b}$

$$\begin{aligned}\boldsymbol{\nabla} \times (\mathbf{a} \times \mathbf{b}) \quad &= (\boldsymbol{\nabla} \cdot \mathbf{b})\mathbf{a} - (\boldsymbol{\nabla} \cdot \mathbf{a})\mathbf{b} + (\boldsymbol{\nabla} \cdot \mathbf{b})\mathbf{a} - (\boldsymbol{\nabla} \cdot \mathbf{a})\mathbf{b} = \\ &= (\boldsymbol{\nabla} \cdot \mathbf{b})\mathbf{a} - (\mathbf{b} \otimes \boldsymbol{\nabla})\mathbf{a} + (\mathbf{a} \otimes \boldsymbol{\nabla})\mathbf{b} - (\boldsymbol{\nabla} \cdot \mathbf{a})\mathbf{b}\end{aligned}$$

  see, e.g., the identity (2.13) and the definition $(2.32)_3$

Remark 3 It is worth noting that the symbol $(\mathbf{b} \otimes \boldsymbol{\nabla})\mathbf{a}$ employed in the previous relation has the same meaning as the more usual $(\boldsymbol{\nabla} \cdot \mathbf{a})\mathbf{b}$; actually, it turns out to be:

$$(\mathbf{b} \cdot \boldsymbol{\nabla})\mathbf{a} \stackrel{def}{=} b_1\frac{\partial \mathbf{a}}{\partial x_1} + b_2\frac{\partial \mathbf{a}}{\partial x_2} + b_3\frac{\partial \mathbf{a}}{\partial x_3}$$

although it represents, as a matter of fact, (grad $\mathbf{b})\mathbf{a}$.

- curl $(\mathbf{a} \otimes \mathbf{b})$ = (curl $\mathbf{b}) \otimes \mathbf{a} - \mathbf{b} \times$ (grad $\mathbf{a})^t$

$$\begin{aligned}\boldsymbol{\nabla} \times (\mathbf{a} \otimes \mathbf{b})^t = \boldsymbol{\nabla} \times (\mathbf{b} \otimes \mathbf{a}) \quad &= (\boldsymbol{\nabla} \times \mathbf{b}) \otimes \mathbf{a} + (\boldsymbol{\nabla} \otimes \mathbf{a}) \times \mathbf{b} = \\ &= (\boldsymbol{\nabla} \times \mathbf{b}) \otimes \mathbf{a} - \mathbf{b} \times (\boldsymbol{\nabla} \otimes \mathbf{a})\end{aligned}$$

  see, e.g., the identity (2.11) and the definition $(2.33)_2$

- curl $(\varphi\mathbf{A})$ = grad $\varphi \times \mathbf{A}^t + \varphi$ curl $\mathbf{A}$

$$\boldsymbol{\nabla} \times (\varphi\mathbf{A})^t = \boldsymbol{\nabla} \times (\varphi\mathbf{A})^t = \boldsymbol{\nabla}\varphi \times \mathbf{A}^t + \varphi(\boldsymbol{\nabla} \times \mathbf{A}^t)$$

  see, e.g., the definitions $(2.32)_1$ and $(2.33)_2$

- curl $(\mathbf{A}\mathbf{b})$ = (curl $\mathbf{A}^t)\mathbf{b}$ + 2axial ($\mathbf{A}$grad $\mathbf{b}$)

$$\boldsymbol{\nabla} \times (\mathbf{A}\mathbf{b}) = (\boldsymbol{\nabla} \times \mathbf{A})\mathbf{b} + \boldsymbol{\nabla} \times (\mathbf{A}\mathbf{b})$$

see, e.g., the definitions $(2.32)_4$ and $(2.33)_2$.

The second expression on the right-hand side and that on the left-hand one, though formally identical, are actually different since in the second one $\mathbf{A}$ is assumed to be constant, differently from what happens on the left-hand side. By employing the property (2.18) the second addend on the right-hand side can also be written as:

$$\begin{aligned} \boldsymbol{\nabla} \times (\mathbf{A}\mathbf{b}) &= -\text{axial}\,[\boldsymbol{\nabla} \otimes (\mathbf{A}\mathbf{b}) - (\mathbf{A}\mathbf{b}) \otimes \boldsymbol{\nabla}] = \\ &= -\text{axial}\,[(\boldsymbol{\nabla} \otimes \mathbf{b})\mathbf{A}^t - \mathbf{A}(\mathbf{b} \otimes \boldsymbol{\nabla})] = 2\text{axial}\,[\mathbf{A}(\mathbf{b} \otimes \boldsymbol{\nabla})] \end{aligned}$$

where the last identity follows (2.10), the linearity of the axial operator and the property $(\mathbf{CD})^t = \mathbf{D}^t\mathbf{C}^t$.

Clearly, differential identities involving more than two fields can be proved recursively on the basis of the previous results. For instance, invoking (7.1) and (7.2) one has:

$$\text{grad}\,[\varphi(\mathbf{a} \cdot \mathbf{b})] = (\text{grad}\,\varphi)(\mathbf{a} \cdot \mathbf{b}) + \varphi\text{grad}\,(\mathbf{a} \cdot \mathbf{b}) \tag{7.4}$$

We further prove, on the basis of our formalism, well-known relations involving combination of the differential operators grad , div and curl .

- $\text{curl}\,(\text{grad}\,\varphi) = \mathbf{0}$

  since, according to $(2.32)_1$ and $(2.32)_4$, $\text{curl}\,(\text{grad}\,\varphi) = \boldsymbol{\nabla} \times (\boldsymbol{\nabla}\varphi)$ and the vector $\boldsymbol{\nabla}\varphi$ can be intended as φ times the vector $\boldsymbol{\nabla}$.

- $\text{div}\,(\text{curl}\,\mathbf{a}) = \mathbf{0}$

  a property which follows from $(2.38)_2$ since $\text{div}\,(\text{curl}\,\mathbf{a}) = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{a})$.

- $\text{curl}\,(\text{curl}\,\mathbf{a}) = \text{grad}\,(\text{div}\,\mathbf{a}) - \Delta\mathbf{a}$

  $$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{a}) = (\boldsymbol{\nabla} \cdot \mathbf{a})\boldsymbol{\nabla} - (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\mathbf{a} = (\boldsymbol{\nabla} \cdot \mathbf{a})\boldsymbol{\nabla} - \Delta\mathbf{a}$$

  see, e.g., the identity (2.13)

- curl (grad $\mathbf{a}$) $= \mathbf{0}$

$$\boldsymbol{\nabla} \times (\mathbf{a} \otimes \boldsymbol{\nabla})^t = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \otimes \mathbf{a})$$

see, e.g., the identity $(2.38)_4$

- curl (grad $\mathbf{a})^t =$ grad (curl $\mathbf{a}$)

$$\boldsymbol{\nabla} \times [(\mathbf{a} \otimes \boldsymbol{\nabla})^t]^t = \boldsymbol{\nabla} \times (\mathbf{a} \otimes \boldsymbol{\nabla}) = (\boldsymbol{\nabla} \times \mathbf{a}) \otimes \boldsymbol{\nabla}$$

see, e.g., the identity $(2.38)_3$

- div (curl $\mathbf{A}$) $=$ curl (div $\mathbf{A}^t$)

$$(\boldsymbol{\nabla} \times \mathbf{A}^t)\boldsymbol{\nabla} = \mathbf{W}_{\boldsymbol{\nabla}}\mathbf{A}^t\boldsymbol{\nabla} = \boldsymbol{\nabla} \times (\mathbf{A}^t\boldsymbol{\nabla})$$

see, e.g., the definitions (2.6), $(2.32)_3$ and $(2.32)_4$.

- div (curl $\mathbf{A})^t = \mathbf{0}$

$$(\boldsymbol{\nabla} \times \mathbf{A}^t)^t\boldsymbol{\nabla} = (\mathbf{W}_{\boldsymbol{\nabla}}\mathbf{A}^t)^t\boldsymbol{\nabla} = \mathbf{A}\mathbf{W}^t_{\boldsymbol{\nabla}}\boldsymbol{\nabla} = -\mathbf{A}(\boldsymbol{\nabla} \times \boldsymbol{\nabla}) = \mathbf{0}$$

see, e.g., the definitions (2.6), $(2.32)_3$ and $(2.32)_4$.

- [curl (curl $\mathbf{A}$)$]^t =$ curl (curl $\mathbf{A})^t$

$$(\mathbf{W}_{\boldsymbol{\nabla}}\mathbf{A}\mathbf{W}_{\boldsymbol{\nabla}})^t = \mathbf{W}^t_{\boldsymbol{\nabla}}\mathbf{A}^t\mathbf{W}^t_{\boldsymbol{\nabla}} = \mathbf{W}_{\boldsymbol{\nabla}}\mathbf{A}^t\mathbf{W}_{\boldsymbol{\nabla}}$$

see, e.g., the definition (2.34).

- curl $\mathbf{W}_{\mathbf{a}} =$ (div $\mathbf{a})\mathbf{I} -$ grad $\mathbf{a}$

$$\boldsymbol{\nabla} \times \mathbf{W}^t_{\mathbf{a}} = -\boldsymbol{\nabla} \times \mathbf{W}_{\mathbf{a}} = -\mathbf{W}_{\boldsymbol{\nabla}}\mathbf{W}_{\mathbf{a}}$$

see, e.g., the identity (2.15) and the definition $(2.33)_2$.

- curl $\mathbf{a} = -$div $\mathbf{W_a}$

$$\boldsymbol{\nabla} \times \mathbf{a} = -\mathbf{a} \times \boldsymbol{\nabla} = -\mathbf{W_a}\boldsymbol{\nabla} - \boldsymbol{\nabla} \times \mathbf{W_a} = -\mathbf{W}_{\boldsymbol{\nabla}}\mathbf{W_a} = -(\mathbf{a} \otimes \boldsymbol{\nabla}) + (\mathbf{a} \cdot \boldsymbol{\nabla})\mathbf{I} \tag{7.5}$$

see, e.g., the definitions $(2.32)_4$ and $(2.33)_1$.

- grad $[\varphi$ div $\mathbf{a}]$ = (grad φ) div $\mathbf{a}$ + φ grad (div $\mathbf{a}$) = (grad φ)div $\mathbf{a}$ + φ div (grad $\mathbf{a})^t]$

$$\begin{aligned} \boldsymbol{\nabla}[\varphi(\boldsymbol{\nabla} \cdot \mathbf{a})] &= (\boldsymbol{\nabla}\varphi)(\boldsymbol{\nabla} \cdot \mathbf{a}) + \varphi(\boldsymbol{\nabla} \cdot \mathbf{a})\boldsymbol{\nabla} \\ &= (\boldsymbol{\nabla}\varphi)(\boldsymbol{\nabla} \cdot \mathbf{a}) + \varphi(\boldsymbol{\nabla} \otimes \mathbf{a})\boldsymbol{\nabla} \end{aligned} \tag{7.6}$$

see, e.g., the identity (7.1).

- div [(grad $\mathbf{a}$)$\mathbf{a}$] = (grad $\mathbf{a}$) $\cdot$ (grad $\mathbf{a})^t$ + $\mathbf{a} \cdot$ div (grad$\mathbf{a})^t$ = (grad $\mathbf{a}$) $\cdot$ (grad $\mathbf{a})^t + \mathbf{a} \cdot$ grad (div $\mathbf{a}$)

$$\begin{aligned} \boldsymbol{\nabla} \cdot [(\mathbf{a} \otimes \boldsymbol{\nabla})\mathbf{a}] &= (\boldsymbol{\nabla} \otimes \mathbf{a}) \cdot (\mathbf{a} \otimes \boldsymbol{\nabla}) + \mathbf{a} \cdot [(\boldsymbol{\nabla} \otimes \mathbf{a})\boldsymbol{\nabla}] \\ &= (\mathbf{a} \otimes \boldsymbol{\nabla}) \cdot (\boldsymbol{\nabla} \otimes \mathbf{a}) + \mathbf{a} \cdot [(\boldsymbol{\nabla} \cdot \mathbf{a})] \end{aligned} \tag{7.7}$$

see, e.g., the identity (7.3).

Additional differential identities and relations fulfilled by the differential operators grad , div and curl can be proved in a similar way.

7.2 Matrix representation of the fourth-order tensor $\mathbf{W_a} \boxtimes \mathbf{W_a}$

Let us denote by $\mathbf{W_a}$ the skew-symmetric rank-two tensor associated with an arbitrary vector $\mathbf{a} = \{a_1, a_2, a_3\}^t$. Recalling that the matrix $[\mathbf{W_a}]$ has cartesian components:

$$\mathbf{W}_{\mathbf{a}} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}. \tag{7.8}$$

aim of this section is to provide the matrix representation of the fourth-order tensor $\mathbf{W}_{\mathbf{a}} \boxtimes \mathbf{W}_{\mathbf{a}}$.

To this end we first recall that the cartesian components of the dyadic and of the square tensor products between rank-two tensors $\mathbf{A}$, $\mathbf{B} \in \mathrm{Lin}$ are given by:

$$(\mathbf{A} \otimes \mathbf{B})_{ijkl} = \mathbf{A}_{ij}\mathbf{B}_{kl} \qquad (\mathbf{A} \boxtimes \mathbf{B})_{ijkl} = \mathbf{A}_{ik}\mathbf{B}_{jl} \qquad \forall\, \mathbf{A}, \mathbf{B} \in \mathrm{Lin}\,. \tag{7.9}$$

Consistently with the tensor-to-matrix mapping commonly employed in computational mechanics [63], rank-two symmetric tensors $\mathbf{T}$ can be expressed in vector form as follows:

$$\mathbf{T} = [T_{11}, T_{22}, T_{33}, T_{12}, T_{23}, T_{31}]. \tag{7.10}$$

Accordingly, fourth-order tensors of the kind $\mathbf{A} \boxtimes \mathbf{A}$, which map second-order symmetric tensors into second-order symmetric tensors, can be represented by a 6x6 matrix [41]:

$$[\mathbf{A} \boxtimes \mathbf{A}] = \begin{bmatrix} \mathbf{A}_1 & 2\mathbf{A}_2 \\ \mathbf{A}_2 & \mathbf{A}_3 \end{bmatrix} \tag{7.11}$$

in witch

$$\mathbf{A}_1 = \begin{bmatrix} A_{11}^2 & A_{12}^2 & A_{13}^2 \\ A_{21}^2 & A_{22}^2 & A_{23}^2 \\ A_{31}^2 & A_{32}^2 & A_{33}^2 \end{bmatrix}, \qquad \mathbf{A}_2 = \begin{bmatrix} A_{11}A_{12} & A_{12}A_{13} & A_{11}A_{13} \\ A_{21}A_{22} & A_{22}A_{23} & A_{21}A_{23} \\ A_{31}A_{32} & A_{32}A_{33} & A_{31}A_{33} \end{bmatrix}$$

and

$$\mathbf{A}_3 = \begin{bmatrix} A_{11}A_{22} + A_{12}A_{21} & A_{12}A_{23} + A_{13}A_{22} & A_{13}A_{21} + A_{11}A_{23} \\ A_{21}A_{32} + A_{22}A_{31} & A_{22}A_{33} + A_{23}A_{32} & A_{23}A_{31} + A_{21}A_{33} \\ A_{31}A_{12} + A_{32}A_{11} & A_{32}A_{13} + A_{33}A_{12} & A_{33}A_{11} + A_{31}A_{13} \end{bmatrix}$$

Applying the previous formula to (7.8) yields:

$$[\mathbf{W_a} \boxtimes \mathbf{W_a}] = \begin{bmatrix} 0 & a_3^2 & a_2^2 & 0 & -2a_2a_3 & 0 \\ a_3^2 & 0 & a_1^2 & 0 & 0 & -2a_1a_3 \\ a_2^2 & a_1^2 & 0 & -2a_1a_2 & 0 & 0 \\ 0 & 0 & -a_1a_2 & -a_3^2 & a_3a_1 & a_2a_3 \\ -a_2a_3 & 0 & 0 & a_1a_3 & -a_1^2 & a_1a_2 \\ 0 & -a_1a_3 & 0 & a_2a_3 & a_1a_2 & -a_2^2 \end{bmatrix} \tag{7.12}$$

which is required to express compatibility equations in a matrix format, see e.g. formula (2.77).

7.3 Baldacci's frame-dependent representation of the shear stress field

Starting from Beltrami-Michell equations (2.110), Baldacci [6] proved that the non-zero component of the stress tensor, $\sigma_z = T_{33}$ and $\boldsymbol{\tau} = [T_{13}, T_{23}, 0]$, solution of the Saint-Venant problem admitted the following representation

$$\begin{cases} \sigma_z = (l - z)(-\mathbf{J}_G^{-1}\mathbf{t} \cdot \mathbf{p}) \\ \boldsymbol{\tau} = -\left\{[A_B\,(\mathbf{p})] + [(\nabla \otimes \boldsymbol{\psi})]\right\}[\mathbf{J}_G^{-1}][\mathbf{t}] \end{cases} \tag{7.13}$$

The tensor $[\mathbf{A}_B]$ appearing in (7.13) is defined as follows

$$[\mathbf{A}_B] = \begin{bmatrix} x^2 - \bar{\nu} y^2 & 0 & 0 \\ 0 & y^2 - \bar{\nu} x^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7.14}$$

where

$$\bar{\nu} = \frac{1}{1+\nu} \tag{7.15}$$

The inertia tensor has the following expression

$$\mathbf{J}_G = \int_\Omega \mathbf{r} \otimes \mathbf{r} \, dA$$

Comparing the expression of $\mathbf{J}_G$ with the field $\mathbf{A}_B$, it is important to emphasize the substantial difference existing between the formers, expressed solely in terms of operations having a frame independent nature, and the latter.

It can be proved that $\mathbf{A}_B$, whose matrix representation was originally introduced by Baldacci as defined in (7.14), does not have a truly frame independent nature.

To prove our statement let us recall the usual frame transformation law for second-order tensors. Denoting by $(\mathbf{i}, \mathbf{j})$ and $(\mathbf{i}', \mathbf{j}')$ two couples of orthogonal unit vectors and by xy and $x'y'$, respectively, the relevant coordinates, we define as $\mathbf{Q}$ an orthogonal basis-change tensor; hence $\mathbf{i}' = \mathbf{Q}\mathbf{i}$ and $\mathbf{j}' = \mathbf{Q}\mathbf{j}$. Accordingly, the components of the same tensor in two reference frames are mutually related by the expression

$$[\mathbf{J}_G]_{x'y'} = [\mathbf{Q}]^t_{xy} [\mathbf{J}_G]_{xy} [\mathbf{Q}]_{xy}$$

where $[\mathbf{Q}]_{xy}$ is the matrix containing the components of $\mathbf{Q}$ in the reference frame xy. Notice also that in the reference frames xy and $x'y'$ the same tensor

$\mathbf{J}_G$ is represented by formally similar matrixes:

$$[\mathbf{J}_G]_{xy} = \begin{bmatrix} \int_\Omega x^2 dA & \int_\Omega xy dA & 0 \\ \int_\Omega xy dA & \int_\Omega y^2 dA & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [\mathbf{J}_G]_{x'y'} = \begin{bmatrix} \int_\Omega x'^2 dA & \int_\Omega x'y' dA & 0 \\ \int_\Omega x'y' dA & \int_\Omega y'^2 dA & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

To summarize, hence, the matrixes $[\mathbf{J}_G]_{xy}$ and $[\mathbf{J}_G]_{x'y'}$ are formally similar and represent the *same* tensor.

Conversely, it can be recognized that tensor $\mathbf{A}_B$, represented in the frame xy by the matrix

$$[\mathbf{A}_B]_{xy} = \begin{bmatrix} x^2 - \nu y^2 & 0 & 0 \\ 0 & y^2 - \nu x^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

generally differs from the tensor $\mathbf{A}'_B$ which is represented in the second frame $x'y'$ by the matrix

$$[\mathbf{A}_B]_{x'y'} = \begin{bmatrix} x'^2 - \nu y'^2 & 0 & 0 \\ 0 & y'^2 - \nu x'^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

To recognize this it suffices to ascertain that the matrix representing $\mathbf{A}_B$ in the frame $x'y'$ does not generally coincide with the matrix which represents $\mathbf{A}'_B$ in such frame. Actually, considering for the base-change the rotation around the z axis of an angle α whose orthogonal matrix is

$$[\mathbf{Q}]_{xy} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

and assuming for simplicity $\nu = 0$, it is immediately recognized that the matrix representing $\mathbf{A}_B$ in the frame $x'y'$ is

$$[\mathbf{A}_B]_{x'y'} = [\mathbf{Q}]^t_{xy}[\mathbf{A}_B]_{xy}[\mathbf{Q}]_{xy} =$$

$$= \begin{bmatrix} x^2 \cos^2\alpha + y^2 \sin^2\alpha & -x^2 \sin\alpha \cos\alpha + y^2 \sin\alpha \cos\alpha & 0 \\ -x^2 \sin\alpha \cos\alpha + y^2 \sin\alpha \cos\alpha & x^2 \sin^2\alpha + y^2 \cos^2\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, being $[\mathbf{A}_B]_{x'y'}$ non-diagonal, it generally differs from $[\mathbf{A}'_B]_{x'y'}$, so that, as a general rule, $\mathbf{A}_B$ and $\mathbf{A}'_B$ are different tensors.

7.4 Further identities

We further illustrate the proofs of vector and tensor identities that appear in 4.1.4 by adopting indicial notation.

7.4.1 Proof of identity (7.16)

The identity

$$[(\mathbf{r}\cdot\mathbf{r})]\mathbf{r}^{\perp} = [(\mathbf{r}\cdot\mathbf{r})(\mathbf{r}\otimes\mathbf{r}^{\perp})]\boldsymbol{\nabla} \tag{7.16}$$

that in indicial notation reads

$$\rho_k\rho_k r_i^{\perp} = (\rho_k\rho_k\rho_i\rho_j^{\perp})_{,j} \tag{7.17}$$

is demonstrated in a straightforward way by differentiating the right-hand side of (7.17)

$$(\rho_k\rho_k\rho_i\rho_j^{\perp})_{,j} = (\rho_k\rho_k)_{,j}\rho_i\rho_j^{\perp} + \rho_k\rho_k\rho_{i,j}r_j^{\perp} + \rho_k\rho_k\rho_i r_{j,j}^{\perp} \tag{7.18}$$

Observing that

$$(\rho_k\rho_k)_{,j} = 2\rho_{k,j}\rho_k = 2\delta_{kj}\rho_k = 2\rho_j \tag{7.19}$$

one recognizes that the first term of the sum in (7.18) is zero by virtue of the orthogonality of $\mathbf{r}$ and $\mathbf{r}^{\perp}$. Moreover, also the third term on the right side of

(7.18) is zero since $\rho^{\perp}_{j,j} = 0$. One finally obtains

$$(\rho_k\rho_k\rho_i\rho_j^{\perp}),_j = \rho_k\rho_k\rho_{i,\,j}\rho_j^{\perp} = \rho_k\rho_k\delta_{ij}\rho_j^{\perp} = \rho_k\rho_k\rho_i^{\perp} \tag{7.20}$$

7.4.2 Proof of identity (4.92) and (4.92)

The developments providing the identities exploited to develop the counterparts of (4.87) and (4.88) in divergence form are reported below. Upon setting $p_j = \frac{1}{8}(\rho_h\rho_h\rho_i)$, one has

$$p_{j,\,kk} = \rho_j \tag{7.21}$$

This relation is computed below

$$\begin{aligned}(\rho_h\rho_h\rho_i),_{kk} &= [(\rho_h\rho_h),_k\rho_i + (\rho_h\rho_h)\rho_{i,\,k}],_k = [2\rho_k\rho_i + (\rho_h\rho_h)\delta_{ik}],_k = \\ &= 2\rho_{k,\,k}\rho_i + 2\rho_{i,\,k}\rho_k + (\rho_h\rho_h),_k\delta_{ik} + (\rho_h\rho_h)\delta_{ik,\,k} = \\ &= 4\rho_i + 2\rho_i + 2\rho_k\delta_{ik} = 8\rho_i\end{aligned}$$

Moreover one has

$$p_{i,\,k} = \frac{1}{8}(2\rho_k\rho_i + \rho_h\rho_h\delta_{ik}).$$

7.4.3 Proof of identity (4.96)

The identity

$$(\rho_i\rho_i)\rho_h\rho_k = \frac{1}{6}[(\rho_i\rho_i)\rho_h\rho_k\rho_l],_l \tag{7.22}$$

is demonstrated in a similar way. The derivative of the terms in square brackets on the RHS of (7.22) is

$$\begin{aligned}[(\rho_i\rho_i)\rho_h\rho_k\rho_l],_l = \;& (\rho_i\rho_i),_l\rho_h\rho_k\rho_l + (\rho_i\rho_i)\rho_{h,\,l}\rho_k\rho_l + \\ & +(\rho_i\rho_i)\rho_h\rho_{k,\,l}\rho_l + (\rho_i\rho_i)\rho_h\rho_k\rho_{l,\,l}\end{aligned} \tag{7.23}$$

and since

$$\rho_{i,\,l}\rho_i + \rho_{i,\,l} = \delta_{il}\rho_i + \rho_i\delta_{il} = 2\rho_l \tag{7.24}$$

then (7.23) achieves the following expression

$$2(\rho_l\rho_l)\rho_h\rho_k + (\rho_i\rho_i)\rho_k\rho_h + (\rho_i\rho_i)\rho_h\rho_k + 2(\rho_i\rho_i)\rho_h\rho_k = 6(\rho_i\rho_i)\rho_h\rho_k\,. \tag{7.25}$$

7.4.4 Proof of identity (4.97)

On account of (7.24) one has

$$\begin{aligned} [(\rho_i\rho_i)(\rho_j\rho_j)\rho_l]_{,l} &= 2(\rho_i\rho_i)_{,l}(\rho_j\rho_j) + (\rho_i\rho_i)(\rho_j\rho_j)\rho_{l,l} = \\ &= 2(\rho_i\rho_i)_{,l}(\rho_j\rho_j) + 2(\rho_i\rho_i)(\rho_j\rho_j) \end{aligned} \tag{7.26}$$

and hence the result

$$(\rho_p\rho_p)(\rho_q\rho_q)\delta_{hk} = \frac{1}{6}[(\rho_p\rho_p)(\rho_q\rho_q)\rho_l\delta_{hk}]_{,l}\,. \tag{7.27}$$

7.5 Nomenclature

A	cross-sectional area
G	centroid of the cross-section domain
$\mathbf{J}$	tensor of inertia
l	cylinder length
$\mathbf{M}, \mathbf{M}_t$	bending couple and torque
$\mathbf{T}, N$	shear force and normal force
x, y, z	Cartesian coordinates
i, j, k	unit vectors
σ, ϵ	longitudinal stress and elongation
E	Young's modulus
G	tangential elasticity modulus
ν	Poisson's ratio
$\mathbf{n}$	unit outward normal to the cylinder boundary
$\mathbf{S}, \mathbf{E}$	stress and strain tensors
$\mathbf{r}, \mathbf{u}$	position and displacement vectors
$\mathbf{p}, \mathbf{u}$	position and displacement vectors restricted to the cross-section plane
$\mathbf{D}_s$	shear deformability tensor

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