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*Analysis and numerical solution of a
non-standard non-linear integro-differential boundary value
problem*

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IL COORDINATORE
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To my Parents

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Preface

The aim of this thesis is the analytical study and the development of a numerical method to solve the following non-linear, integro-differential boundary value problem on the half line:

$$\begin{cases} \nu(y)g(y) - \int_0^{+\infty} k(x)g(x)dx [D(y)g'(y)]' = p(y) \\ g'(0) = 0, \quad \lim_{y \rightarrow +\infty} g(y) = 0 \end{cases}, \quad y \geq 0. \quad (0.0.1)$$

This problem is representative of a class of *non-standard* integral equations where the derivatives of the unknown function are multiplied by an integral term depending on the unknown itself. The problems of this class are related to some real phenomena like plasmas kinetics (see for instance [16]-[18], [27], [41]) and population dynamics (see i. e. [6]).

In particular equation (0.0.1) contains all the peculiarities of a more complicated model of kinetic equations arising from the study of dust production and dynamics in the vacuum chamber of the experimental fusion devices [41]. This equation is defined on the half line, and its *non-standard* nature makes the analytical and numerical study rather complicated.

As a matter of fact, the knowledge about the existence, uniqueness and other properties of the solution is missing and the numerical methods are still undeveloped.

Hence, the first part of this thesis concerns with the theoretical analysis of problem (0.0.1), which provides useful informations about the existence and other qualitative properties of the solution itself and represents an essential preparation for a numerical resolution of the problem.

The approach followed in this thesis consists in rewriting problem (0.0.1) in the equivalent form as follows:

$$\begin{cases} \nu(y) g(y, q) - q [D(y) g'(y, q)]' = p(y) \\ g'(0, q) = 0, \quad \lim_{y \rightarrow +\infty} g(y, q) = 0 \end{cases}, \quad y \geq 0, \quad (0.0.2)$$

$$F(q) = q - \int_0^{+\infty} k(x)g(x, q)dx = 0, \quad (0.0.3)$$

where, (0.0.2) is a classical Sturm-Liouville boundary value problem depending on a parameter q and it coincides with the problem (0.0.1) when q is a zero of the non-linear function $F(q)$ defined by (0.0.3).

Theoretical results show that for any $q > 0$ fixed, there exists a unique solution $g(y, q) \in C^4([0, +\infty))$ of problem (0.0.2), which is positive and bounded together with its derivatives up to order 4. Furthermore, we have proved other important properties, like the uniform boundedness of $g(y, q)$ and its derivatives up to order 4, with respect to the parameter q , and the uniform continuity of $g(y, q)$ with respect to q . In particular this last property allows us to get the uniform continuity of the function $F(q)$.

In order to prove the existence of a solution of (0.0.1), we show that there exists an interval $[a, b]$ where the function $F(q)$ has at least a zero q^* , and $g(y, q^*)$, solution of the differential problem (0.0.2) with q^* fixed, is a solution of the integro-differential problem (0.0.1).

It remains an open problem the uniqueness of the solution of the integro-differential problem (0.0.1).

The second part of this thesis concerns with the development of the numerical method to solve the integro-differential problem (0.0.1) and the study of the convergence of it.

At first, we rewrite problems (0.0.2)-(0.0.3) in a compact interval $[0, T]$, where the end point T is sufficiently large. Then, we discretize problem (0.0.2) by a finite difference scheme and the integral term in (0.0.3) by a truncated composite trapezoidal rule, on a uniform mesh, $\{y_i\}_{i=0}^N$, on $[0, T]$. In this way we get the discrete version of (0.0.2)-(0.0.3)

$$A(q) g(q) = p. \quad (0.0.4)$$

$$F_h(q) = q - h \sum_{i=0}^N \omega_i k(y_i) g_i(q) = 0, \quad (0.0.5)$$

where h is step size.

Under suitable assumptions the system (0.0.4) has a unique solution $g(q)$, positive and continuous with respect to q . This last property ensures the continuity of the function $F_h(q)$ with respect to q . Furthermore we have proved the convergence of F_h to F when h tends to zero, hence, for h sufficiently small, F_h has at least a zero, q_h^* , in the same interval $[a, b]$ found for function $F(q)$ in continuous case. The corresponding value $g(q_h^*) = A^{-1}(q_h^*)p$ is the numerical approximation of the solution of the integro-differential problem (0.0.1).

Starting from the interval $[a, b]$, we develop an iterative process based on bisection method, which converges to the zero of F_h .

An important part of this research work is devoted to the analysis and proof of the convergence of the numerical method for $h \rightarrow 0$ and $T \rightarrow +\infty$. It is possible by means of a non-standard technique which involves the virtual computing of a sequence $\{q^r\}_{r \in \mathbb{N}}$, obtained applying the same iteration process to the continuous problem (0.0.2)-(0.0.3).

The thesis is divided into 4 chapters.

The aim of Chapter 1 is to furnish the basic concepts used in the remaining part of the thesis. At first we give some notions about two point boundary value problems for ordinary differential equations on finite intervals, the theorem of the existence and uniqueness of the solution and some qualitative properties of the solution itself and its derivatives. All these results will play an important role in Chapter 2 where we focus on the analysis of the solution of the integro-differential problem (0.0.1). Finally, we give some results about finite difference method for boundary value problems for ordinary differential equations.

In Chapter 2 we present a theoretical analysis of the integro-differential problem (0.0.1), which provides useful informations about the existence of a solution itself and its derivatives and represents an essential preparation for a numerical approach to the problem. All the studies reported in this section on the properties of the solution of problem (0.0.1) turn to be essential for the comprehension of the problem itself as well as for its numerical analysis.

In Chapter 3 we design and analyze a numerical method to solve the integro-differential problem (0.0.1). This method consists of two steps: discretization of the differential and integral terms by using finite differences and a quadrature formula respectively, resolution of the non-linear system which comes out from this discretization. In order to prove the convergence of our method, at first, we analyze the convergence properties of the finite differences method related to the differential problem (0.0.2) with a fixed value of the parameter q . Then, we describe the algorithm that we implement, and finally we prove the convergence of our method.

In Chapter 4 some numerical experiments on problems of type (0.0.1) are described. The tests performed consists of the choice of the best interval $[0, T]$ on which to apply the numerical method , and to verify the experimental order of convergence when $h \rightarrow 0$.

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Chapter 1

Preliminary notions

1.1 Introduction

The aim of this chapter is to recall some basic notions that we need in the future chapters. To start we present an overview about two point boundary value problems for ordinary differential equations on finite intervals, the most important results about the existence and uniqueness of the solution and some properties of the solution itself and its derivatives (see i. e. [7]-[11], [14], [15]). All these results will play an important role in Chapter 2 where we focus on the analysis of the solution of the integro-differential problem (0.0.1). Finally, we provide some notions about finite difference method for boundary value problems for ordinary differential equations.

1.2 Two point boundary value problems on compact intervals

A boundary value problem in one dimension is an ordinary differential equation together with conditions involving values of the solution and/or its derivatives at two or more points. The number of conditions imposed is equal to the order of the differential equation. Usually, boundary value problems of any physical relevance have these characteristics: the conditions are imposed at two different points; the solution is of interest only between those two points. We are concerned with cases where the differential equation is linear and of second order. In contrast to initial value problems, even the most innocent looking boundary value problem may have exactly one solution, no solution, or an infinite number of solutions.

In this chapter we consider the following linear second order boundary value problem:

$$\begin{cases} \nu(y)g(y) - [D(y)g'(y)]' = p(y) \\ g'(0) = 0, \quad g(T) = 0 \end{cases}, \quad 0 \leq y \leq T. \quad (1.2.1)$$

Definition 1.1 *Boundary value problem (1.2.1) is called regular if both the end points are*

finite, and the function $D(y) \neq 0$ for all $0 \leq y \leq T$. If one and/or both end points are infinite and/or $D(y) = 0$ for at least one point $y \in [0, T]$, then the problem is said to be singular.

In this chapter we focus on regular boundary value problems. We report some results about the existence and uniqueness of the solution $g(y)$ of the problem (1.2.1) together with the analysis of other useful properties such as the sign of g and the boundedness of $g(y)$, $g'(y)$, $g''(y)$. These results will play an important role in next chapter.

1.2.1 Existence and uniqueness of the solution

The existence and uniqueness theory for the boundary value problems is more difficult than that of initial value problems. In fact, in the case of boundary value problems a slight change in the boundary conditions can lead to significant changes in the behaviour of the solutions. For example, the initial value problem:

$$\begin{cases} g'' + g = 0 \\ g(0) = c_1, \quad g'(0) = c_2 \end{cases},$$

has a unique solution $g(y) = c_1 \cos y + c_2 \sin y$ for any set of values c_1, c_2 . However, the boundary value problem

$$\begin{cases} g'' + g = 0 \\ g(0) = 0, \quad g(\pi) = \epsilon (\neq 0) \end{cases},$$

has no solution; the problem

$$\begin{cases} g'' + g = 0 \\ g(0) = 0, \quad g(\beta) = \epsilon \end{cases},$$

$0 < \beta < \pi$ has a unique solution $g(y) = \epsilon \frac{\sin y}{\sin \beta}$, while the problem

$$\begin{cases} g'' + g = 0 \\ g(0) = 0, \quad g(\pi) = 0 \end{cases},$$

has an infinite number of solutions $g(y) = c \sin y$, where c is an arbitrary constant.

The theory of existence and uniqueness of bounded solutions of linear and non linear boundary value problems over a compact interval is well developed, see i.e. [7]-[11], [14], [15]. Below we briefly report a theorem about the existence and uniqueness of the solution of problem (1.2.1).

Theorem 1.1 *Assume that:*

1. $D \in C^1([0, T])$, $\nu, p \in C([0, T])$,
2. $D(y) > 0$, $y \in [0, T]$,
3. $\nu(y) > 0$, $y \in [0, T]$,

are satisfied. Then the boundary value problem (1.2.1) has a unique solution

$g \in C^2([0, T])$.

Proof.

Let us consider the following homogeneous problem:

$$\begin{cases} \nu(y) g(y) - [D(y)g'(y)]' = 0 \\ g'(0) = 0, \quad g(T) = 0 \end{cases}, \quad 0 \leq y \leq T \quad (1.2.2)$$

It is clear that the function

$$g^*(y) = 0, \quad \forall y \in [0, T],$$

is a solution of (1.2.2). Under the assumptions made, this problem has only the trivial solution. Indeed, let $g(y)$ be a non trivial solution of (1.2.2), we have:

$$\nu(y)g(y) - [D(y)g'(y)]' = 0.$$

Multiplying by $g(y)$ the last equation and integrating from 0 to T , we get:

$$\begin{aligned} 0 &= \int_0^T g(y) [\nu(y)g(y) - [D(y)g'(y)]'] dy = \\ &= \int_0^T \nu(y)g^2(y)dy - \int_0^T g(y) [D(y)g'(y)]' dy. \end{aligned} \quad (1.2.3)$$

Assumption 3., ensures that

$$\int_0^T \nu(y)g^2(y)dy > 0.$$

Then, integrating by parts the second integral in (1.2.3) we get:

$$\begin{aligned} &\int_0^T \nu(y)g^2(y)dy - \int_0^T g(y) [D(y)g'(y)]' dy > \\ &> - [g(y) [D(y)g'(y)]]_0^T + \int_0^T g'(y, q) [D(y)g'(y, q)] dy = \\ &= \int_0^T [D(y)g'^2(y)] dy \geq 0 \end{aligned}$$

a contradiction for (1.2.3). Hence, $g(y) = 0$ is the only solution of (1.2.2).

If the problem (1.2.2) has only the trivial solution, then the inhomogeneous problem

$$\begin{cases} [D(y)g'(y)]' = \nu(y)g(y) - p(y) \\ g'(0) = 0, \quad g(T) = 0 \end{cases}, \quad 0 \leq y \leq T \quad (1.2.4)$$

has a unique solution $g(y)$, for all $p(y) \in C([0, T])$. In fact, consider the following IVPs:

$$\begin{cases} [D(y)g'_1(y)]' = \nu(y)g_1(y) - p(y) \\ g_1(0) = 0, \quad g'_1(0) = 0 \end{cases}, \quad 0 \leq y \leq T \quad (1.2.5)$$

$$\begin{cases} [D(y)g_2'(y)]' = \nu(y)g_2(y) \\ g_2(0) = 1, \quad g_2'(0) = 0 \end{cases}, \quad 0 \leq y \leq T \quad (1.2.6)$$

and let $g_1(y)$ and $g_2(y)$ be their respective solutions . Let us define a function $g(y, s)$ depending on a parameter s by:

$$g(y, s) = g_1(y) + sg_2(y), \quad y \in [0, T], \quad (1.2.7)$$

it satisfies both the inhomogeneous differential equation of (1.2.5) and the first boundary condition of (1.2.4). Therefore, the boundary value problem (1.2.4) has a unique solution if and only if there exists an only value of the parameter s , such that $g(y, s)$ satisfies the second boundary condition of (1.2.4):

$$g(T, s) = g_1(T) + sg_2(T) = 0. \quad (1.2.8)$$

It happens if and only if $g_2(T) \neq 0$. Hence

$$s = -\frac{g_1(T)}{g_2(T)}, \quad (1.2.9)$$

and the second boundary condition of (1.2.4) can be solved uniquely for s in (1.2.9). Note that $g_2(T)$ is not equal to zero, otherwise it would be a solution of the homogeneous boundary value problem, but the latter has only the trivial solution, and $g_2(0) = 1$.

Furthermore, by hypothesis 1. it follows that $g \in C^2([0, T])$. ■

1.3 Some properties of the solution and its derivatives

In this section we report some qualitative properties of the solution of the problem (1.2.1) as the sign of g and the boundedness of g, g', g'' ; these properties will be crucial in order to develop a theoretical analysis of the integro-differential problem (0.0.1). We want to specify that these results are reported in non-linear case by several authors [7]-[11]. We follow this way to get these properties for linear equations.

1.3.1 Positivity and boundedness of the solution

We have seen that, under the assumptions 1., 2., 3., the differential boundary value problem (1.2.1) has a unique solution $g(y) \in C^2([0, T])$. About the sign of g , the following theorem holds.

Theorem 1.2 *In addition to 1.-3., assume*

$$4. \quad p(y) \geq 0, \quad y \in [0, T],$$

then $\exists r_0 > 0$, such that

$$0 \leq g(y) \leq r_0, \quad y \in [0, T]. \quad (1.3.10)$$

Proof.

Hypotheses 1., 3. and 4. allow us to define the following constant:

$$r_0 := \max_{0 \leq y \leq T} \frac{p(y)}{\nu(y)}. \quad (1.3.11)$$

By the assumptions made on the functions $D(y)$, $p(y)$, $\nu(y)$, the solution $g(y)$ cannot have a minimum negative at $s \in (0, T)$, since if it did we would have:

$$D(s)g''(s) = \nu(s)g(s) - p(s) < 0,$$

because $g'(s) = 0$, and $g(s) < 0$, a contradiction.

Suppose $g(y)$ has a negative minimum at $y_0 = 0$, then since $g(0) < 0$, there exists $\delta > 0$, such that $g(y) < 0$ for $y \in (0, \delta)$. This implies:

$$[D(y)g'(y)]' = \nu(y)g(y) - p(y) < 0$$

for $y \in (0, \delta)$ and

$$D(y)g'(y) = \int_0^y [D(t)g'(t)]' dt = \int_0^y [\nu(t)g(t) - p(t)] dt < 0.$$

This is a contradiction because should be $g'(y) > 0$ for $y \in (0, \eta) \subseteq (0, \delta)$.

Moreover, $g(T) = 0$ hence

$$g(y) \geq 0, \quad 0 \leq y \leq T.$$

By previous assumptions, $g(y)$ cannot have a maximum at $x \in (0, n)$ such that $g(x) > r_0$, since if it did we would have:

$$D(x)g''(x) = \nu(x)g(x) - p(x) > 0$$

because $g'(x) = 0$, and $g(x) > r_0$, a contradiction. Suppose $g(0) > r_0$ is a local positive maximum then, there exists $\delta > 0$ such that $g(y) > r_0$ for $y \in (0, \delta)$. This implies

$$[D(y)g'(y)]' = \nu(y)g(y) - p(y) > 0$$

for $y \in (0, \delta)$ and

$$D(y)g'(y) = \int_0^y [D(t)g'(t)]' dt = \int_0^y [\nu(t)g(t) - p(t)] dt > 0.$$

This is a contradiction because should be $g'(y) < 0$ for $y \in (0, \eta) \subseteq (0, \delta)$. Thus, in conclusion,

$$0 \leq g(y) \leq r_0, \quad 0 \leq y \leq T. \quad (1.3.12)$$

■

Remark 1.1 *The positiveness of the solution g arises from the positiveness of the right hand side p in (1.2.1). However, if no information on the sign of p is given, we can still say that*

a unique solution $g(y)$ of the problem (1.2.1) exists, and

(1.3.10) becomes:

$$|g(y)| < r_0, \quad 0 \leq y \leq T. \quad (1.3.13)$$

1.3.2 A priori bounds for derivatives

In this section we provide a priori bounds for derivatives of the solution of problem (1.2.1). Also in this case we apply an idea stated in [7]-[11] to the linear case and the following theorem holds.

Theorem 1.3 *Assume that the hypotheses of Theorem 1.2 are satisfied. Then, there exist*

$r_1, r_2 > 0$ such that:

$$|g'(y)| < r_1, \quad y \in [0, T], \quad (1.3.14)$$

$$|g''(y)| < r_2, \quad y \in [0, T], \quad (1.3.15)$$

Proof.

Starting from the equation of (1.2.1) and under the assumptions made we get:

$$\begin{aligned} |g''(y)| &\leq \left| \frac{D'(y)}{D(y)} \right| |g'(y)| + \left| \frac{\nu(y)}{D(y)} \right| |g(y)| + \left| \frac{p(y)}{D(y)} \right| \\ &\leq |D_1| |g'(y)| + \left| \frac{\nu_{max}}{D_{min}} \right| |r_0| + \left| \frac{P_{max}}{D_{min}} \right|, \quad 0 \leq y \leq T. \end{aligned} \quad (1.3.16)$$

Where, as usual, $f_{max} = \max_{0 \leq y \leq T} f(y)$, $f_{min} = \min_{0 \leq y \leq T} f(y)$. Furthermore,

$$D_1 = \max_{0 \leq y \leq T} \left| \frac{D'(y)}{D(y)} \right| \quad (1.3.17)$$

and r_0 is defined in (1.3.11). Thanks to the inequality $|x| < x^2 + 1$, we deduce the so called *Bernstein growth condition*:

$$|g''| \leq Ag'^2 + B, \quad 0 \leq y \leq T, \quad (1.3.18)$$

where:

$$A = D_1 \quad (1.3.19)$$

$$B = D_1 + \left| \frac{\nu_{max}}{D_{min}} \right| |r_0| + \left| \frac{P_{max}}{D_{min}} \right|. \quad (1.3.20)$$

In order to obtain (1.3.14) we note that $g \in C^2[0, T]$ and since $g'(0) = 0$, there exists $\delta > 0$ such that $g'(y)$ does not change sign in $[0, \delta] \subset [0, T]$. Assume that $g'(y) \geq 0$, $y \in [0, \delta]$ then,

$$g''(y) \leq Ag'^2(y) + B, \quad y \in [0, \delta].$$

Multiplying both sides by $2Ag'(y)$, gives:

$$\frac{2Ag'(y)g''(y)}{Ag'^2(y) + B} \leq 2Ag'(y).$$

Integrating from 0, to y , with $y \in [0, \delta]$:

$$\begin{aligned} \int_0^y \frac{2Ag'(t)g''(t)}{Ag'^2(t) + B} dt &\leq \int_0^y 2Ag'(t) dt, \\ \log \left(\frac{Ag'^2(y) + B}{B} \right) &\leq 2Ar_0, \\ |g'(y)| &\leq \left[\frac{B}{A} (e^{2Ar_0} - 1) \right]^{1/2}, \quad y \in [0, \delta]. \end{aligned} \quad (1.3.21)$$

If $g'(y)$ changes sign on $[0, T]$, because of continuity of $g'(y)$ we can choose δ such that $g'(\delta) = 0$ and assume $g'(y) \leq 0$, $\forall y \in [\delta, \mu] \subset [0, T]$. Then, from (1.3.18), we have:

$$-(Ag'^2(y) + B) \leq g''(y), \quad y \in [\delta, \mu] \quad (1.3.22)$$

and in a easy way we obtain that:

$$|g'(y)| \leq \left[\frac{B}{A} (e^{2Ar_0} - 1) \right]^{1/2}, \quad y \in [\delta, \mu]. \quad (1.3.23)$$

Then, $|g'(y)| < r_1$, $y \in [0, T]$ with,

$$r_1 = \left[\frac{B}{A} (e^{2Ar_0} - 1) \right]^{1/2}. \quad (1.3.24)$$

Finally, from (1.3.10), (1.3.14), (1.3.16), (1.3.17) and hypotheses 1.-4., we get

$$|g''(y)| \leq r_2, \quad (1.3.25)$$

with,

$$r_2 := D_1 r_1 + \max_{0 \leq y \leq T} \left| \frac{\nu(y)}{D(y)} \right| r_0 + \max_{0 \leq y \leq T} \left| \frac{P(y)}{D(y)} \right| \quad (1.3.26)$$

■

Finally, the following proposition is straightforward.

Proposition 1.1 *Assume that the hypotheses of the Theorem 1.3 are satisfied. If in addition we assume that $\nu, p \in C^2([0, T])$ and $D \in C^3([0, T])$ then, the solution $g(y)$ of (1.2.1) is in $C^4([0, T])$.*

1.4 Finite Difference Method

In this section we report some notions about finite difference methods (see for instance [22], [30]-[33], [37]). Let us consider the boundary value problem (1.2.1) which we write as:

$$L\{g\} := -D(y)g'' - D'(y)g' + \nu(y)g = p(y), \quad y \in [0, T], \quad (1.4.27)$$

$$g'(0) = 0, \quad g(T) = 0. \quad (1.4.28)$$

Let us assume that the hypotheses of the Proposition 1.1 are satisfied, in this way the boundary value problem has a unique solution g in $C^4([0, T])$. Furthermore, we make the following assumption

$$\nu(y) \geq \nu^* > 0, \quad y \in [0, T]. \quad (1.4.29)$$

In order to solve numerically this problem we define a uniform mesh on $[0, T]$:

$$\begin{aligned} \Pi_h : 0 = y_0 < y_1 < y_2 < \dots < y_{N-1} < y_N = T, \\ y_i = ih, \quad i = 0, \dots, N, \quad h = \frac{T}{N}. \end{aligned} \quad (1.4.30)$$

Then we replace the derivative terms in (1.4.27) with finite difference approximations. We choose the centered difference formulas, which means we use the following expressions

$$g'(y_i) = \frac{g(y_{i+1}) - g(y_{i-1}))}{2h} - \frac{1}{6}h^2g^{(3)}(\eta_i), \quad (1.4.31)$$

$$g''(y_i) = \frac{g(y_{i+1}) - 2g(y_i) + g(y_{i-1}))}{h^2} - \frac{1}{12}h^2g^{(4)}(\bar{\eta}_i), \quad (1.4.32)$$

with $\eta_i, \bar{\eta}_i \in [y_{i-1}, y_{i+1}]$. Let us set

$$g_i \approx g(y_i), \quad i = 0, \dots, N-1, \quad (1.4.33)$$

and consider the following difference equations:

$$\begin{aligned} L_h\{g_i\} &:= \nu(y_i)g_i - D'(y_i)\frac{g_{i+1} - g_{i-1}}{2h} - D(y_i)\frac{g_{i+1} - 2g_i + g_{i-1}}{h^2} \\ &= p(y_i), \quad i = 0, \dots, N-1. \end{aligned} \quad (1.4.34)$$

The boundary conditions in (1.4.28) are replaced by:

$$g_{-1} = g_1, \quad g_N = 0. \quad (1.4.35)$$

We note that the first one comes out from the fact that $g'(0) = 0$. Equations (1.4.34) together with (1.4.35) lead us to a system of N algebraic equations:

$$Ag = p, \quad (1.4.36)$$

where $g = [g_0, \dots, g_{N-1}]^T \in \mathbb{R}^N$, $p = [p(y_0), \dots, p(y_{N-1})]^T \in \mathbb{R}^N$ and the tridiagonal matrix $A = (a_{i,j}) \in \mathbb{R}^{N \times N}$ is the coefficient matrix,

$$A = \begin{pmatrix} a_0 & b_0 & 0 & 0 & \cdots & 0 \\ c_1 & a_1 & b_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \cdots & 0 \\ 0 & 0 & \cdots & c_{N-2} & a_{N-2} & b_{N-2} \\ 0 & 0 & \cdots & \cdots & c_{N-1} & a_{N-1} \end{pmatrix} \quad (1.4.37)$$

with:

$$a_0 := \nu(y_0) + \frac{2D(y_0)}{h^2}, \quad b_0 := -\frac{2}{h^2}D(y_0); \quad (1.4.38)$$

$$c_i := -\left[\frac{D(y_i)}{h^2} - \frac{D'(y_i)}{2h} \right],$$

$$a_i := \nu(y_i) + \frac{2D(y_i)}{h^2}, \quad i = 1, \dots, N-1; \quad (1.4.39)$$

$$b_i := -\left[\frac{D'(y_i)}{2h} + \frac{D(y_i)}{h^2} \right], \quad i = 1, \dots, N-2. \quad (1.4.40)$$

To solve the difference problem in (1.4.34)-(1.4.35) we must, in fact, solve the Nth order linear system (1.4.36).

We assume that the step size h is so small that:

$$hD_1 < 2 \quad (1.4.41)$$

where we recall that $D_1 = \max_{0 \leq y \leq T} \left| \frac{D'(y)}{D(y)} \right|$. So we deduce that

$$\begin{aligned} |a_0| &> |b_0| \\ |a_i| &> |b_i| + |c_i|, \quad i = 1, \dots, N-2 \\ |a_{N-1}| &> |c_{N-1}| \end{aligned} \quad (1.4.42)$$

Of course, this furnishes a proof of the existence of a unique solution of the difference equations (1.4.36) provided (1.4.41) is satisfied.

The local truncation errors, τ_i , are defined by:

$$L_h\{g(y_i)\} = p(y_i) + \tau_i; \quad i = 0, \dots, N - 1. \quad (1.4.43)$$

Since $g(y)$ is a solution of (1.4.27) and we have assumed that g is continuous together that its derivatives up to order four, using Taylor's theorem we get

$$\begin{aligned} \tau_i &= L_h\{g(y_i)\} - L\{g(y_i)\} = \\ &= D(y_i) \left[\frac{g(y_{i+1}) - 2g(y_i) + g(y_{i-1}))}{h^2} - g''(y_i) \right] \\ &= D'(y_i) \left[\frac{g(y_{i+1}) - g(y_{i-1}))}{2h} - g'(y_i) \right] = \\ &= -\frac{D(y_i)}{12} h^2 \left[g^{(4)}(\bar{\eta}_i) + 2\frac{D'(y_i)}{D(y_i)} g^{(3)}(\eta_i) \right] \end{aligned} \quad (1.4.44)$$

Now we consider the classical theorem about the error estimate.

Theorem 1.4 *Assume that 1.-3. hold. If in addition we assume that the stepsize h satisfies*

(1.4.41), then

$$|g_i - g(y_i)| \leq \frac{h^2}{12} \frac{D_{max}}{\nu^*} [G_4 + 2D_1 G_3], \quad (1.4.45)$$

where $G_i = \max_{0 \leq y \leq T} |g^{(i)}(y)|$, $i = 3, 4$.

Proof.

Let us define

$$e_i = g_i - g(y_i), \quad i = 0, \dots, N. \quad (1.4.46)$$

Then subtracting (1.4.43) from (1.4.34) we have

$$a_i e_i = b_i e_{i-1} + c_i e_{i+1} - \tau_i. \quad (1.4.47)$$

We set: $e = \max_{0 \leq y \leq T} |e_i|$, $\tau = \max_{0 \leq i \leq N} |\tau_i|$ and we have:

$$|a_i e_i| = [|b_i| + |c_i|]e + \tau. \quad (1.4.48)$$

By (1.4.38)-(1.4.40) and (1.4.41) we get

$$e \leq \frac{\tau}{\nu^*} \quad (1.4.49)$$

and by (1.4.44) we get

$$e \leq \frac{h^2}{12} \frac{D_{max}}{\nu^*} [G_4 + 2D_1 G_3] \quad (1.4.50)$$

■

From this theorem we can see that the difference solution converges to the exact solution as $h \rightarrow 0$ and, in fact, the error is at most $O(h^2)$.

Chapter 2

The continuous problem

2.1 Introduction

The aim of this chapter is to introduce a theoretical analysis of the following non-linear non-standard integro-differential boundary value problem:

$$\begin{cases} \nu(y)g(y) - \int_0^{+\infty} k(x)g(x)dx [D(y)g'(y)]' = p(y) \\ g'(0) = 0, \quad g(+\infty) = 0 \end{cases}, \quad y \geq 0. \quad (2.1.1)$$

The peculiarity of this equation is that the coefficients of the derivatives of the unknown function depend on the unknown function itself by means of an integral over the semi-axis. This feature makes the analytical study of this problem rather complicated. Furthermore, the knowledge about the solution and its qualitative properties is poor, hence the analytical study that we present provides useful information about the solution of the problem (2.1.1) and represents an essential preparation for a numerical approach to the problem that we will see in the next chapter.

In order to analyze the integro-differential problem (2.1.1), it is useful to consider the classical Sturm-Liouville differential problem depending on a parameter q :

$$\begin{cases} \nu(y)g(y, q) - q[D(y)g'(y, q)]' = p(y) \\ g'(0, q) = 0, \quad g(+\infty, q) = 0 \end{cases}, \quad y \geq 0, \quad (2.1.2)$$

and observe that, when

$$q = \int_0^{+\infty} k(x)g(x)dx, \quad (2.1.3)$$

it coincides with problem (2.1.1). Theoretical results about two point boundary value problems on infinite intervals for ordinary differential equations are well known in literature. Several authors (see for instance [12]-[13]), have investigated about the existence of the solution and some of its qualitative properties as the positiveness and boundedness of the solution itself and of its derivatives, in non-linear case. We have fitted these results on the form of the parametric problem (2.1.2) and connected them in order to obtain a complete theory and prepare the basis for the analysis of the problem (2.1.1).

In Section 2.2 we examine problem (2.1.2), assume that $q > 0$ is a fixed parameter and report results about existence, uniqueness, positiveness, regularity and boundedness of the solution. All these results will be crucial for the investigations carried out in Section 2.3 on the complete problem (2.1.1) or the equivalent problem (2.1.2) with (2.1.3), where we will prove the existence of a non-negative solution g which is uniformly bounded together with its derivatives. All the studies reported in this chapter about the properties of the solution of problem (2.1.1) turn to be essential for the comprehension of the problem itself as well as for its numerical analysis. Finally Section 2.4 contains some conclusions.

2.2 Analysis of the solution of the parametric differential

boundary value problem

Boundary value problems on infinite intervals frequently occur in mathematical modelling of various applied problems. As examples (see i.e. [13]), in the study of unsteady flow of a gas through a semi-infinite porous medium, discussion of electrostatic probe measurements in solid-propellant rocket exhausts, analysis of the mass transfer on a rotating disk in a non-Newtonian fluid, heat transfer in the radial flow between parallel circular disks, investigation of the temperature distribution in the problem of phase change of solids with temperature dependent thermal conductivity, as well as numerous problems arising in the study of draining flows, circular membranes, plasma physics, radially symmetric solutions of semi-linear elliptic equations, non-linear mechanics, and non-Newtonian fluid flows.

2.2.1 Existence of global solution

In this section we consider the differential boundary value problem (2.1.2) with $q > 0$ fixed and, following a standard procedure (see i. e. [12], [13]), report some results on the existence and the uniqueness of the solution $g(y, q)$ together with the analysis of other useful properties such as the sign of $g(y, q)$ and the boundedness of $g(y, q)$ and its derivatives.

The proof of the existence of the solution of problem (2.1.2) can be divided into two steps. At first we consider sufficient conditions on the functions which define the problem that yield a global solution of the equation in (2.1.2), which is continuous and bounded together with its derivatives up to order two, on $[0, +\infty)$. Arzelà-Ascoli Theorem and the results presented in the previous chapter about boundary value problems on finite intervals will play an important role in order to obtain a global solution to (2.1.2). Furthermore, it is easy to get a global solution to (2.1.2) which also satisfies the boundary condition at zero, $g'(0, q) = 0$. Next, to satisfy the boundary condition at infinity is more complicated, in fact, constant a priori bound alone is not suffice to give the existence of a solution that satisfies the second boundary condition of (2.1.2). We need further a priori information which implies that the solution tends to zero at infinity, as we will see in the following section.

From now on we will consider boundary value problems of the kind (2.1.2) with q positive and fixed and we make the following assumptions on the involved functions:

1. $D \in C^1([0, +\infty))$, $\nu, p \in C([0, +\infty))$,
2. $0 < D_{inf} \leq D(y) \leq D_{sup}$, $y \geq 0$,
3. $0 < \sup_{y \geq 0} \left| \frac{D'(y)}{D(y)} \right| < +\infty$,
4. $0 < \nu_{inf} \leq \nu(y) \leq \nu_{sup}$, $y \geq 0$,
5. $0 \leq p(y) \leq P$, $y \geq 0$,
6. $\int_0^{+\infty} p(y) dy < +\infty$
7. $\lim_{y \rightarrow +\infty} p(y) = 0$

As usual, we denote by $BC^{(r)}[0, +\infty)$ the space of functions $f(x)$ with $f^{(j)}(x)$, $j = 0, 1, \dots, r$, bounded and continuous on $[0, +\infty)$ and $\|f\|_\infty = \sup_{y \geq 0} |f(y)|$.

In order to show the boundedness of $g(y, q)$ and its first and second derivatives, we note that hypotheses 2.-5. allow us to define the following constants:

$$A := \left\| \frac{D'}{D} \right\|_\infty, \quad B := \left\| \frac{D'}{D} \right\|_\infty + \left\| \frac{\nu}{qD} \right\|_\infty \left\| \frac{p}{\nu} \right\|_\infty + \left\| \frac{p}{qD} \right\|_\infty, \quad (2.2.4)$$

and we set

$$r_0 := \left\| \frac{p}{\nu} \right\|_\infty, \quad (2.2.5)$$

$$r_1 := \left[\frac{B}{A} (e^{2Ar_0} - 1) \right]^{1/2}, \quad (2.2.6)$$

$$r_2 := \left\| \frac{D'}{D} \right\|_\infty r_1 + \left\| \frac{\nu}{qD} \right\|_\infty r_0 + \left\| \frac{p}{qD} \right\|_\infty. \quad (2.2.7)$$

Moreover, in order to make the discussion clear, we report the following classical definitions about uniform boundedness and equicontinuity of a sequence of continuous functions, and the Arzelà-Ascoli Theorem .

Definition 2.1 A sequence $\{f_n\}_{n \in \mathbb{N}}$ of continuous functions on an interval

$I = [a, b]$ is uniformly bounded if there exists a positive constant $M < +\infty$ such that

$\forall n \in \mathbb{N}$,

$$|f_n(x)| < M, \quad x \in [a, b].$$

Definition 2.2 A sequence of functions, $\{f_n\}_{n \in \mathbb{N}}$, is equicontinuous if, for every $\epsilon > 0$,

there exists $\delta_\epsilon > 0$, such that $|y - s| < \delta_\epsilon$, $|f_n(y) - f_n(s)| < \epsilon$, $\forall n \in \mathbb{N} - \{0\}$.

Theorem 2.1 (Arzelà-Ascoli Theorem) *Let I be a compact interval. If a sequence of functions $\{f_n\}_{n \in \mathbb{N}}$ in $C(I)$ is uniformly bounded and equicontinuous then it has a uniformly convergent subsequence.*

Now, following a classical procedure (see for instance [12], [13]) based on particular applications of Arzelà-Ascoli Theorem, we report the result on existence of a global solution $g \in BC^2([0, +\infty))$ of equation in (2.1.2), satisfying the boundary condition at zero. In the following theorem we need all the results about the solution of the differential boundary value problem on finite intervals, reported in Sections 1.2-1.3

Theorem 2.2 *Assume 1.-7. are satisfied. Then, for any fixed $q > 0$, the ordinary differential equation in (2.1.2) has at least one non negative solution such that*

$$g \in BC^2([0, +\infty)), \quad (2.2.8)$$

$$\lim_{y \rightarrow 0^+} g'(y, q) = 0. \quad (2.2.9)$$

Proof.

Let us consider, for all $n \in \mathbb{N} - \{0\}$, the boundary value problem:

$$\begin{cases} \nu(y) g_n(y, q) - q [D(y)g_n'(y, q)]' = p(y) \\ g_n'(0, q) = 0, \quad g_n(n, q) = 0 \end{cases}, \quad y \in [0, n]. \quad (2.2.10)$$

Under the assumptions 1.-5., theoretical results described in the previous chapter show that, for all $n \in \mathbb{N} - \{0\}$, problem (2.2.10) has a unique solution

$g_n(y, q) \in C^2([0, n])$, positive and such that

$$0 \leq g_n(y, q) \leq r_0, \quad y \in [0, n], \quad (2.2.11)$$

$$|g'_n(y, q)| < r_1, \quad y \in [0, n], \quad (2.2.12)$$

$$|g''_n(y, q)| < r_2, \quad y \in [0, n], \quad (2.2.13)$$

where the constants r_0 , r_1 and r_2 are defined in (2.2.5)-(2.2.7). Hypotheses 4., 5. and (2.2.11) enable us to define the following estimate:

$$\left| [qD(y)g'_n(y)]' \right| \leq M_0 \quad y \in [0, n], \quad \forall n \in N - \{0\} \quad (2.2.14)$$

with,

$$M_0 = |\nu_{sup}| r_0 + |P|. \quad (2.2.15)$$

Let us consider the equation

$$q [D(y)g'_n(y, q)]' = \nu(y)g_n(y, q) - p(y),$$

an integration from 0 to $y \in (0, n]$ yields

$$g'_n(y, q) = \frac{1}{qD(y)} \int_0^y [\nu(\tau)g_n(\tau, q) - p(\tau)] d\tau$$

and another integration produces

$$g_n(y, q) - g_n(0, q) = \int_0^y \frac{1}{qD(t)} \int_0^t [\nu(\tau)g_n(\tau, q) - p(\tau)] d\tau dt.$$

At the same way, for $s \in (0, n]$ we have

$$g_n(s, q) - g_n(0, q) = \int_0^s \frac{1}{qD(t)} \int_0^t [\nu(\tau)g_n(\tau, q) - p(\tau)] d\tau dt,$$

subtracting the two previous relations we get

$$\begin{aligned} g_n(y, q) - g_n(s, q) &= \int_s^y \frac{1}{qD(t)} \int_0^t [\nu(\tau)g_n(\tau, q) - p(\tau)] d\tau dt \\ |g_n(y, q) - g_n(s, q)| &\leq \frac{1}{q} \int_s^y \frac{1}{D(t)} \int_0^t |\nu(\tau)g_n(\tau, q) - p(\tau)| d\tau dt \\ &\leq \frac{M_0}{q} \int_s^y \frac{1}{D(t)} \int_0^t d\tau dt = \frac{M_0}{q} \int_s^y \frac{t}{D(t)} dt, \quad (2.2.16) \\ &\forall n \in N - \{0\}. \end{aligned}$$

It is easy to see that

$$|g'_n(y, q) - g'_n(s, q)| \leq r_2 \left| \int_s^y dt \right| = r_2 |y - s| \quad (2.2.17)$$

with r_2 given in (2.2.7).

Let us define the sequence of functions $\{z_n(y, q)\}_{n \in N}$ in this way

$$z_n(y, q) = \begin{cases} g_n(y, q), & y \in [0, n] \\ 0, & y > n \end{cases}. \quad (2.2.18)$$

It is clear that $z_n(y, q)$ is continuous with respect to y on $[0, +\infty)$ and twice continuously differentiable except possibly at $y = n$, where there is a jump. By (2.2.11) and (2.2.12) it comes out that:

$$0 \leq z_n(y, q) \leq r_0, \quad 0 \leq y \leq +\infty, \quad n \in N - \{0\} \quad (2.2.19)$$

$$|z'_n(y, q)| < r_1, \quad 0 \leq y \leq +\infty, \quad n \in N - \{0\} \quad (2.2.20)$$

and by (2.2.16) for $y, s \in [0, \infty)$ we have:

$$|z_n(y, q) - z_n(s, q)| \leq \frac{M_0}{q} \int_y^s \frac{t}{D(t)} dt, \quad \forall n \in N - \{0\}.$$

By assumptions made on $D(y)$, the integral function: $\frac{M_0}{q} \int_0^y \frac{t}{D(t)} dt$, is continuous hence, $\forall \epsilon > 0, \exists \delta > 0$, such that $|y - s| < \delta$,

$$|g_n(y, q) - g_n(s, q)| \leq \frac{M_0}{q} \int_s^y \frac{t}{D(t)} dt < \epsilon, \quad \forall n \in N - \{0\}.$$

so $z_n(y, q)$ are equicontinuous with respect to y .

The functions $z'_n(y, q)$ with $n = 2, 3, \dots$ are equicontinuous with respect to y on $[0, 1]$, because from (2.2.17)

$$|z'_n(y, q) - z'_n(s, q)| \leq r_1 |y - s|, \quad y, s \in [0, 1]. \quad (2.2.21)$$

By Arzelà-Ascoli Theorem there is a subsequence of index N_1^* of $N - \{0, 1\}$ and a continuous function

$$h_1(y, q) \quad \text{on } [0, 1]$$

such that

$$z'_{1n}(y, q) \rightarrow h_1(y, q)$$

uniformly with respect to $y \in [0, 1]$, as $n \rightarrow +\infty$ through N_1^* . We consider the subsequence of $\{z_n(y, q)\}$ of index $N_1^* \subset N - \{0, 1\}$, $\{z_{1n}(y, q)\}_{1n \in N_1^*}$, these functions are uniformly bounded and equicontinuous with respect to y on $[0, 1]$, hence by Arzelà-Ascoli Theorem there is a subsequence of index N_1 of N_1^* and a continuous function

$$k_1(y, q) \text{ on } [0, 1]$$

such that

$$z_{n,1}(y, q) \rightarrow k_1(y, q)$$

uniformly with respect to $y \in [0, 1]$, as $n, 1 \rightarrow +\infty$ through N_1 .

We note that the subsequence of $\{z'_{n,1}(y, q)\}_{n,1 \in N_1}$ is uniformly convergent to the function $h_1(y, q)$, and thanks to the uniform convergence we have:

$$h_1(y, q) = \frac{d}{dy} k_1(y, q) = k'_1(y, q). \quad (2.2.22)$$

Again we consider the sequence $\{z'_{n,2}(y, q)\}$ with $n, 2 \geq 3$ these functions are equicontinuous w.r.t y on $[0, 2]$, and the Arzelà-Ascoli Theorem implies that there is a subsequence of index N^2 of N^1 and a continuously differentiable function

$$k_2(y, q) \text{ defined on } [0, 2],$$

such that

$$z_{n,2}(y, q) \rightarrow k_2(y, q)$$

$$z'_{n,2}(y, q) \rightarrow k'_2(y, q)$$

both uniformly with respect to $y \in [0, 2]$, as $n \rightarrow +\infty$ through N^2 .

Note that

$$k_2(y, q) = k_1(y, q), \quad y \in [0, 1],$$

because $N^2 \subset N^1$.

In this way we obtain for $j = 1, 2, \dots$ a subsequence of index $N^j \subset N - \{0\}$ with $N^j \subset N^{j-1}$ and a continuously differentiable functions k_j on $[0, j]$, such that,

$$\begin{aligned} z_{n,j}(y, q) &\rightarrow k_j(y, q) \\ z'_{n,j}(y, q) &\rightarrow k'_j(y, q) \end{aligned}$$

uniformly with respect to $y \in [0, j]$, as $n \rightarrow +\infty$ through N^j . Also

$$k_j = k_{j-1} \text{ on } [0, j-1].$$

Define $g(y, q)$ and $g'(y, q)$ as follows: fix $y \in [0, +\infty)$ and consider $j \in N - \{0\}$ with $y \leq j$, then:

$$\begin{aligned} g(y, q) &= k_j(y, q) \\ g'(y, q) &= k'_j(y, q) \end{aligned}$$

g and its first derivative are well defined and $g \in C[0, +\infty)$ (because of the uniform convergence). Hence, for $n \in N^j$ and $n \geq j$ we have:

$$z_n(y, q) = - \int_y^j \frac{1}{qD(t)} \int_0^t [\nu(\tau)z_n(\tau, q) - p(\tau)] d\tau dt + z_n(j, q).$$

Thanks to the uniform convergence of $z_n(y, q)$ to $k_j(y, q)$, for $n \rightarrow +\infty$ through N^j we have:

$$k_j(y, q) = - \int_y^j \frac{1}{qD(t)} \int_0^t [\nu(\tau)k_j(\tau, q) - p(\tau)] d\tau dt + k_j(j, q).$$

Hence,

$$g(y, q) = - \int_y^j \frac{1}{qD(t)} \int_0^t [\nu(\tau)g(\tau, q) - p(\tau)] d\tau dt + g(j, q).$$

Then

$$g'(y, q) = \frac{1}{qD(y)} \int_0^y [\nu(\tau)g(\tau, q) - p(\tau)] d\tau$$

is a continuous function with respect to y on $[0, +\infty)$.

Moreover,

$$\lim_{y \rightarrow 0^+} qD(y)g'(y, q) = \lim_{y \rightarrow 0^+} \int_0^y [\nu(\tau)g(\tau, q) - p(\tau)] d\tau = 0,$$

and thanks to the uniform convergence we have

$$0 \leq g(y, q) \leq r_0, \quad 0 \leq y < +\infty \quad (2.2.23)$$

$$|g'(y, q)| \leq r_1, \quad 0 \leq y < +\infty. \quad (2.2.24)$$

Furthermore,

$$[qD(y)g'(y, q)]' = \nu(y)g(y, q) - p(y) \quad (2.2.25)$$

and assumption 1. implies $g \in C^2[0, +\infty)$ and

$$|qD(y)g'(y, q)| < M_0 y, \quad y > 0$$

$$|[qD(y)g'(y, q)]'| < M_0, \quad y > 0$$

■

2.2.2 Study of a significative boundary value problem

The previous theorem ensures the existence of solution, $g(y, q)$, of the equation (2.1.2) satisfying the first boundary condition. In this section we analyze some properties of the solution of a particular class of boundary value problems for ODE which allows us to show that $\lim_{y \rightarrow 0} g(y, q) = 0$. By hypotheses 2. and 4. we define the following constant:

$$m := (qD_{inf}\nu_{inf})^{1/2}, \quad (2.2.26)$$

and introduce the following differential two point boundary value problem:

$$\begin{cases} [qD(y)\omega'(y, q)]' - \frac{m^2}{qD(y)}\omega(y, q) = -p(y) \\ \omega'(0, q) = 0, \quad \omega(+\infty, q) = 0 \end{cases}, \quad y \geq 0 \quad (2.2.27)$$

where the constant m , defined above, satisfies

$$\exists m > 0, \text{ s.t. } qD(y)\nu(y)g \geq m^2g, \quad y \in [0, +\infty), \quad g \in [0, r_0] \quad (2.2.28)$$

In this section we want to show some of the most important properties of the solution of this problem. At first it is easy to prove that the function

$$\begin{aligned} \omega(y, q) &= \frac{1}{2m} e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} \int_0^{+\infty} p(\tau) e^{-\frac{m}{q} \int_0^\tau \frac{ds}{D(s)}} d\tau \\ &+ \frac{1}{2m} e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} \int_0^y p(\tau) e^{\frac{m}{q} \int_0^\tau \frac{ds}{D(s)}} d\tau \\ &+ \frac{1}{2m} e^{\frac{m}{q} \int_0^y \frac{ds}{D(s)}} \int_y^{+\infty} p(\tau) e^{-\frac{m}{q} \int_0^\tau \frac{ds}{D(s)}} d\tau \end{aligned} \quad (2.2.29)$$

satisfies the equation (2.2.27).

Furthermore, $\omega(y, q)$ satisfies also the boundary conditions of the (2.2.27).

Proposition 2.1 *Assume that hypotheses 1., 2., 4.-7. hold. Then, the function $\omega(y, q)$, defined by (2.2.29), satisfies the boundary conditions of the problem (2.2.27).*

Proof.

It is easy to prove that $\omega'(0, q) = 0$.

In order to show that $\lim_{y \rightarrow +\infty} \omega(y, q) = 0$, we note

$$\lim_{y \rightarrow +\infty} e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} = 0,$$

$$\lim_{y \rightarrow +\infty} e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} \int_0^{+\infty} p(\tau) e^{-\frac{m}{q} \int_0^\tau \frac{ds}{D(s)}} d\tau = 0,$$

by Hopital's rule :

$$\lim_{y \rightarrow +\infty} e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} \int_0^y p(\tau) e^{\frac{m}{q} \int_0^\tau \frac{ds}{D(s)}} d\tau = \lim_{y \rightarrow +\infty} \frac{q}{m} p(y) D(y) = 0,$$

by Hopital's rule :

$$\lim_{y \rightarrow +\infty} e^{\frac{m}{q} \int_0^y \frac{ds}{D(s)}} \int_y^{+\infty} p(\tau) e^{-\frac{m}{q} \int_0^\tau \frac{ds}{D(s)}} d\tau = \lim_{y \rightarrow +\infty} \frac{q}{m} p(y) D(y) = 0.$$

Hence $\lim_{y \rightarrow +\infty} \omega(y, q) = 0$.

■

Moreover, hypotheses 2., 5., ensure that $\omega(y, q) \geq 0$, $y \in [0, +\infty)$.

Proposition 2.2 *Assume that 1., 2., 5. and 6. are satisfied. Then, $\omega(y, q)$ and $\omega'(y, q)$ are bounded.*

Proof. In order to prove the boundedness of $\omega(y, q)$ we note that under the assumptions 2., 5. and 6. we have:

$$\begin{aligned} \left| e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} \right| &\leq 1, \quad y \geq 0, \\ \int_0^{+\infty} p(y) dy &\leq P_{int}, \\ \int_0^y e^{\frac{m}{q} \int_0^\tau \frac{ds}{D(s)}} p(\tau) d\tau &\leq e^{\frac{m}{q} \int_0^y \frac{ds}{D(s)}} \int_0^y p(y) dy \\ \int_y^{+\infty} e^{-\frac{m}{q} \int_0^\tau \frac{ds}{D(s)}} p(\tau) d\tau &\leq e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} \int_y^{+\infty} p(y) dy \end{aligned}$$

Hence:

$$\begin{aligned} |\omega(y, q)| &\leq \frac{P_{int}}{2m} + \frac{e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} e^{\frac{m}{q} \int_0^y \frac{ds}{D(s)}}}{2m} \int_0^y p(\tau) d\tau + \\ &+ \frac{e^{\frac{m}{q} \int_0^y \frac{ds}{D(s)}} e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}}}{2m} \int_y^{+\infty} p(\tau) d\tau \leq \\ &\leq \frac{3P_{int}}{2m} = \Omega. \end{aligned} \tag{2.2.30}$$

Finally also $\omega'(y, q)$ is bounded:

$\exists \Omega_P \in R^+$, s.t. $|\omega'(y, q)| \leq \Omega_P$, for $y \in [0, +\infty)$ and for fixed q . Indeed by the assumptions 2., 5. and 6. we have:

$$|\omega'(y, q)| \leq \frac{e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}}}{2qD(y)} \int_0^{+\infty} p(t) e^{-\frac{m}{q} \int_0^t \frac{ds}{D(s)}} dt +$$

$$\begin{aligned}
& + \frac{e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}}}{2qD(y)} \int_0^y p(t) e^{\frac{m}{q} \int_0^t \frac{ds}{D(s)}} dt + \\
& + \frac{e^{\frac{m}{q} \int_0^y \frac{ds}{D(s)}}}{2qD(y)} \int_y^{+\infty} p(t) e^{-\frac{m}{q} \int_0^t \frac{ds}{D(s)}} dt \leq \\
& \leq \frac{P_{int}}{2qD_{inf}} + \frac{e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} e^{\frac{m}{q} \int_0^y \frac{ds}{D(s)}}}{2qD_{inf}} P_{int} + \\
& + \frac{e^{\frac{m}{q} \int_0^y \frac{ds}{D(s)}} e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}}}{2qD_{inf}} P_{int} = \\
& = \frac{3P_{int}}{2qD_{inf}} = \Omega_P. \tag{2.2.31}
\end{aligned}$$

■

Finally, we show how to get the solution $\omega(y, q)$ of (2.2.27) in easy way as follows.

I STEP

We multiply both sides of the equation (2.2.27) for $e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}}$ and integrate from zero to infinity:

$$\begin{aligned}
\int_0^{+\infty} e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} [qD(y)\omega'(y, q)]' dy & - \frac{m^2}{q} \int_0^{+\infty} \frac{\omega(y, q)}{D(y)} e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} dy = \\
& - \int_0^{+\infty} e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} P(y) dy \tag{2.2.32}
\end{aligned}$$

Integrating by parts the first term of the first side of the equation we obtain:

$$\begin{aligned}
\int_0^{+\infty} e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} [qD(y)\omega'(y, q)]' dy & = \left[qD(y)\omega'(y, q) e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} \right]_0^{+\infty} + \\
\int_0^{+\infty} qD(y)\omega'(y, q) e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} \frac{m}{qD(y)} dy & = m \int_0^{+\infty} \omega'(y, q) e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} dy = \\
m \left[\omega(y, q) e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} \right]_0^{+\infty} & + m \int_0^{+\infty} \omega(y, q) e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} \frac{m}{qD(y)} dy = \\
= -m\omega(0, q) + \frac{m^2}{q} \int_0^{+\infty} \frac{\omega(y, q)}{D(y)} e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} dy. & \tag{2.2.33}
\end{aligned}$$

By replacing the last result in the equation (2.2.32) we have:

$$\begin{aligned}
- m\omega(0, q) + \frac{m^2}{q} \int_0^{+\infty} \frac{\omega(y, q)}{D(y)} e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} dy & - \frac{m^2}{q} \int_0^{+\infty} \frac{\omega(y, q)}{D(y)} e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} dy = \\
- \int_0^{+\infty} P(y) e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} dy. & \tag{2.2.34}
\end{aligned}$$

Hence,

$$\omega(0, q) = \frac{1}{m} \int_0^{+\infty} P(y) e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} dy. \quad (2.2.35)$$

II STEP

We multiply both sides of the equation (2.2.27) for $e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}}$ and integrate from y to infinity:

$$\begin{aligned} & \int_y^{+\infty} e^{-\frac{m}{q} \int_0^t \frac{ds}{D(s)}} [qD(t)\omega'(t, q)]' dt - \frac{m^2}{q} \int_y^{+\infty} \frac{\omega(t, q)}{D(t)} e^{-\frac{m}{q} \int_0^t \frac{ds}{D(s)}} dt = \\ & - \int_y^{+\infty} e^{-\frac{m}{q} \int_0^t \frac{ds}{D(s)}} P(t) dt \end{aligned} \quad (2.2.36)$$

Integrating by parts the first term of the first side of the equation we obtain:

$$\begin{aligned} & \int_y^{+\infty} e^{-\frac{m}{q} \int_0^t \frac{ds}{D(s)}} [qD(t)\omega'(t, q)]' dt = \\ & = \left[qD(t)\omega'(t, q) e^{-\frac{m}{q} \int_0^t \frac{ds}{D(s)}} \right]_y^{+\infty} + \\ & + \int_y^{+\infty} qD(t)\omega'(t, q) e^{-\frac{m}{q} \int_0^t \frac{ds}{D(s)}} \frac{m}{qD(t)} dt \\ & = -qD(y)\omega'(y, q) e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} + m \int_y^{+\infty} \omega'(t, q) e^{-\frac{m}{q} \int_0^t \frac{ds}{D(s)}} dt = \\ & - qD(y) e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} + m \left[\omega(t, q) e^{-\frac{m}{q} \int_0^t \frac{ds}{D(s)}} \right]_y^{+\infty} \\ & + \frac{m^2}{q} \int_y^{+\infty} e^{-\frac{m}{q} \int_0^t \frac{ds}{D(s)}} \frac{\omega(t, q)}{D(t)} dt. \end{aligned} \quad (2.2.37)$$

By replacing the last result in the equation (2.2.36) we have:

$$\begin{aligned} & - qD(y)\omega'(y, q) e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} - m\omega(y, q) e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} + \frac{m^2}{q} \int_y^{+\infty} e^{-\frac{m}{q} \int_0^t \frac{ds}{D(s)}} \frac{\omega(t, q)}{D(t)} dt + \\ & - \frac{m^2}{q} \int_y^{+\infty} e^{-\frac{m}{q} \int_0^t \frac{ds}{D(s)}} \frac{\omega(t, q)}{D(t)} dt = - \int_y^{+\infty} P(t) e^{-\frac{m}{q} \int_0^t \frac{ds}{D(s)}} dt \\ & qD(y)\omega'(y, q) e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} + m\omega(y, q) e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} = \\ & \int_y^{+\infty} P(t) e^{-\frac{m}{q} \int_0^t \frac{ds}{D(s)}} dt \end{aligned}$$

Thus

$$qD(y)\omega'(y, q) = -m\omega(y, q) + e^{\frac{m}{q} \int_0^y \frac{ds}{D(s)}} \int_y^{+\infty} P(t) e^{-\frac{m}{q} \int_0^t \frac{ds}{D(s)}} dt \quad (2.2.38)$$

III STEP

We multiply both sides of the equation for $e^{\frac{m}{q} \int_0^y \frac{ds}{D(s)}}$ and integrate from zero to y :

$$\int_0^y e^{\frac{m}{q} \int_0^t \frac{ds}{D(s)}} [qD(t)\omega'(t, q)]' dt - \frac{m^2}{q} \int_0^y \frac{\omega(t, q)}{D(t)} e^{\frac{m}{q} \int_0^t \frac{ds}{D(s)}} dt = - \int_0^y e^{\frac{m}{q} \int_0^t \frac{ds}{D(s)}} P(t) dt \quad (2.2.39)$$

Integrating by parts the first term of the first side of the equation we obtain:

$$\begin{aligned} & \int_0^y e^{\frac{m}{q} \int_0^t \frac{ds}{D(s)}} [qD(t)\omega'(t, q)]' dt = \\ & = \left[qD(t)\omega'(t, q) e^{\frac{m}{q} \int_0^t \frac{ds}{D(s)}} \right]_0^y - \int_0^y qD(t)\omega'(t, q) e^{\frac{m}{q} \int_0^t \frac{ds}{D(s)}} \frac{m}{qD(t)} dt = \\ & = qD(y)\omega'(y, q) e^{\frac{m}{q} \int_0^y \frac{ds}{D(s)}} - m \int_0^y \omega'(t, q) e^{\frac{m}{q} \int_0^t \frac{ds}{D(s)}} dt = \\ & \quad qD(y)\omega'(y, q) e^{\frac{m}{q} \int_0^y \frac{ds}{D(s)}} - m \left[\omega(t, q) e^{\frac{m}{q} \int_0^t \frac{ds}{D(s)}} \right]_0^y + m \int_0^y \omega(t, q) e^{\frac{m}{q} \int_0^t \frac{ds}{D(s)}} \frac{m}{qD(t)} dt = \\ & = qD(y)\omega'(y, q) e^{\frac{m}{q} \int_0^y \frac{ds}{D(s)}} - m\omega(y, q) e^{\frac{m}{q} \int_0^y \frac{ds}{D(s)}} + m\omega(0, q) \\ & + \frac{m^2}{q} \int_0^y \frac{\omega(t, q)}{D(t)} e^{\frac{m}{q} \int_0^t \frac{ds}{D(s)}} dt \end{aligned} \quad (2.2.40)$$

By replacing (2.2.35) in the last equation and we have:

$$\begin{aligned} & qD(y)\omega'(y, q) e^{\frac{m}{q} \int_0^y \frac{ds}{D(s)}} - m\omega(y, q) e^{\frac{m}{q} \int_0^y \frac{ds}{D(s)}} + \\ & + \int_0^{+\infty} P(t) e^{-\frac{m}{q} \int_0^t \frac{ds}{D(s)}} + \frac{m^2}{q} \int_0^y \frac{\omega(t, q)}{D(t)} e^{\frac{m}{q} \int_0^t \frac{ds}{D(s)}} dt \end{aligned} \quad (2.2.41)$$

By replacing the last result in the equation (2.2.39) we have:

$$\begin{aligned} & qD(y)\omega'(y, q) e^{\frac{m}{q} \int_0^y \frac{ds}{D(s)}} - m\omega(y, q) e^{\frac{m}{q} \int_0^y \frac{ds}{D(s)}} + \\ & \int_0^{+\infty} P(t) e^{-\frac{m}{q} \int_0^t \frac{ds}{D(s)}} dt + \frac{m^2}{q} \int_0^y \frac{\omega(t, q)}{D(t)} e^{\frac{m}{q} \int_0^t \frac{ds}{D(s)}} dt - \frac{m^2}{q} \int_0^y \frac{\omega(t, q)}{D(t)} e^{\frac{m}{q} \int_0^t \frac{ds}{D(s)}} dt = \\ & - \int_0^y P(t) e^{\frac{m}{q} \int_0^t \frac{ds}{D(s)}} dt. \end{aligned} \quad (2.2.42)$$

Thus:

$$\begin{aligned} & qD(y)\omega'(y, q) = m\omega(y, q) - e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} \int_0^{+\infty} P(t) e^{-\frac{m}{q} \int_0^t \frac{ds}{D(s)}} dt + \\ & - e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} \int_0^y P(t) e^{\frac{m}{q} \int_0^t \frac{ds}{D(s)}} dt \end{aligned} \quad (2.2.43)$$

By comparing (2.2.43) with (2.2.38) we have:

$$\begin{aligned}
& - m\omega(y, q) + e^{\frac{m}{q} \int_0^y \frac{ds}{D(s)}} \int_y^{+\infty} P(t) e^{-\frac{m}{q} \int_0^t \frac{ds}{D(s)}} dt = \\
& = m\omega(y, q) - e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} \int_0^{+\infty} P(t) e^{-\frac{m}{q} \int_0^t \frac{ds}{D(s)}} dt - e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} \int_0^y P(t) e^{\frac{m}{q} \int_0^t \frac{ds}{D(s)}} \\
& \quad \omega(y, q) = \frac{1}{2m} e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} \int_0^{+\infty} P(t) e^{-\frac{m}{q} \int_0^t \frac{ds}{D(s)}} dt + \frac{1}{2m} e^{-\frac{m}{q} \int_0^y \frac{ds}{D(s)}} \int_0^y P(t) e^{\frac{m}{q} \int_0^t \frac{ds}{D(s)}} dt \\
& + \frac{1}{2m} e^{\frac{m}{q} \int_0^y \frac{ds}{D(s)}} \int_y^{+\infty} P(t) e^{-\frac{m}{q} \int_0^t \frac{ds}{D(s)}} dt \tag{2.2.44}
\end{aligned}$$

2.2.3 Existence and uniqueness of the solution

In Section 2.2.1 we have seen that under the assumptions 1.-7. there exists a solution, $g(y, q)$, of the equation (2.1.2) such that $g'(0, q) = 0$. Now we have to see that

$$\lim_{y \rightarrow +\infty} g(y, q) = 0.$$

In the previous section we have analyzed a particular class of boundary value problems defined by (2.2.27), with a known solution $\omega(y, q)$ given by (2.2.29) and we have seen that

- $\omega(y, q) \geq 0, y \in [0, +\infty)$,
- there exists $\Omega > 0$ such that $\omega(y, q) \leq \Omega, y \in [0, +\infty)$,
- $\lim_{y \rightarrow \infty} \omega(y, q) = 0$.

Our aim is to show that

$$g(y, q) \leq \omega(y, q), \quad \forall y \geq 0,$$

and from this it follows that

$$\lim_{y \rightarrow \infty} g(y, q) = 0. \tag{2.2.45}$$

Theorem 2.3 *Under the assumptions 1.-7. and for any $q > 0$, the boundary value problem*

(2.1.2) has a non negative solution $g \in C^2[0, +\infty)$.

Proof.

From Theorem 2.2 we have that there exists a solution $g \in C^2[0, +\infty)$ of the equation (2.1.2) such that $g'(0, q) = 0$ and $0 \leq g(y, q) \leq r_0$, with r_0 defined in (2.2.5). In order to prove (2.2.45) we define $r(y, q) = g(y, q) - \omega(y, q)$. At first we show that $r(y, q)$ cannot have a local positive maximum on $[0, +\infty)$. To see this we note that for $y > 0$

$$\begin{aligned} g''(y, q) &= -\frac{D'(y)}{D(y)}g'(y, q) + \frac{\nu(y)}{qD(y)}g(y, q) - \frac{p(y)}{qD(y)}, \\ \omega''(y, q) &= \frac{m^2}{q^2D^2(y)}\omega(y, q) - \frac{D'(y)}{D(y)}\omega'(y, q) - \frac{p(y)}{qD(y)}. \end{aligned}$$

Hence,

$$\begin{aligned} r''(y, q) &= g''(y, q) - \omega''(y, q) = \\ &= \frac{\nu(y)}{qD(y)}g(y, q) - \frac{D'(y)}{D(y)}(g'(y, q) - \omega'(y, q)) - \frac{m^2}{q^2D^2(y)}\omega(y, q) = \\ &= -\frac{D'(y)}{D(y)}(g'(y, q) - \omega'(y, q)) + \frac{\nu(y)}{qD(y)}g(y, q) - \frac{m^2}{q^2D^2(y)}\omega(y, q) \end{aligned}$$

and

$$[qD(y)r'(y, q)]' = \nu(y)g(y, q) - \frac{m^2}{qD(y)}\omega(y, q).$$

By (2.2.28) we have

$$\begin{aligned} \frac{\nu(y)qD(y)}{qD(y)}g(y, q) - \frac{m^2}{qD(y)}\omega(y, q) &\geq \frac{m^2}{qD(y)}g(y, q) - \frac{m^2}{qD(y)}\omega(y, q) = \\ &= \frac{m^2}{qD(y)}[g(y, q) - \omega(y, q)] = \frac{m^2}{qD(y)}r(y, q). \end{aligned} \quad (2.2.46)$$

Hence,

$$[qD(y)r'(y, q)]' \geq \frac{m^2}{qD(y)}r(y, q). \quad (2.2.47)$$

Suppose r has a local positive maximum at $y_0 > 0$, then $r'(y_0) = 0$ and $r''(y_0) \leq 0$. But we have

$${}_qD(y_0)r''(y_0) = [{}_qD(y_0)r'(y_0, q)]' \geq \frac{m^2}{{}_qD(y_0)}r(y_0, q) > 0,$$

a contradiction. Suppose r has a local positive maximum at $y_0 = 0$, then since

$$r(0, q) = g(0, q) - \omega(0, q) > 0,$$

there exists $\delta > 0$ such that

$$r(y, q) = g(y, q) - \omega(y, q) > 0$$

for $y \in (0, \delta)$. Hence, (2.2.47) implies $[{}_qD(y)r'(y, q)]' > 0$ for $y \in (0, \delta)$,

$${}_qD(y)r'(y, q) = \int_0^y [{}_qD(t)r'(t, q)]' dt > 0$$

a contradiction because should be $r'(y, q) < 0$ for $y \in (0, \delta)$. Thus, $r(t)$ cannot have a local positive maximum $[0, +\infty)$. Our aim is to show that

$$r(y, q) = g(y, q) - \omega(y, q) \leq 0, \quad \forall y \in [0, +\infty).$$

If $r(y, q) > 0$ for $y \in [0, +\infty)$, then there exists a $c_1 > 0$, with $r(c_1, q) > 0$. Now since $r(y, q)$ cannot have a local positive maximum on $[0, +\infty)$ it follows that for all $y_2 > y_1 \geq c_1$, $r(y_2, q) > r(y_1, q)$, otherwise $r(y, q)$ would have a local positive maximum on $[0, y_2]$. Thus $r(y, q)$ is strictly increasing for $y > c_1$. We note that both $g(y, q)$ and $\omega(y, q)$ are bounded on $[0, +\infty)$ and

$$\lim_{y \rightarrow +\infty} r(y, q) = \lim_{y \rightarrow +\infty} (g(y, q) - \omega(y, q)) = \lim_{y \rightarrow +\infty} g(y, q) = k < r_0,$$

with $0 < k \leq r_0$. Now there exists $c_2 \geq c_1$ such that $g(y, q) \geq \frac{k}{2}$ for $y \geq c_2$. For $y > 0$ we have

$$\begin{aligned} [{}_qD(y)g'(y, q)]' = \nu(y)g(y, q) - p(y) &= \frac{1}{{}_qD(y)}{}_qD(y)\nu(y)g(y, q) - (y) \\ &\geq \frac{m^2}{{}_qD(y)}g(y, q) - p(y) \end{aligned} \quad (2.2.48)$$

So for $y \geq c_2$ ($g(y, q) > \frac{k}{2}$) we have

$$[qD(y)g'(y, q)]' \geq \frac{1}{qD(y)} \left(\frac{km^2}{2} - qD(y)P(y) \right). \quad (2.2.49)$$

Assumption 7. implies that there is a constant $c_3 > c_2$ such that

$$[qD(y)g'(y, q)]' \geq \frac{1}{qD(y)} \frac{km^2}{2}, \quad y > c_3$$

Two integrations together with the fact that $g(y, q) > 0$ on $[0, +\infty)$ yields

$$\begin{aligned} \int_{c_3}^y [qD(t)g'(t, q)]' dt &\geq \frac{km^2}{2} \int_{c_3}^y \frac{1}{qD(t)} dt \\ qD(y)g'(y, q) &\geq qD(c_3)g'(c_3, q) + \frac{km^2}{2} \int_{c_3}^y \frac{1}{qD(t)} dt \\ g'(y, q) &\geq \frac{D(c_3)}{D(y)} g'(c_3, q) + \frac{km^2}{2q^2 D(y)} \int_{c_3}^y \frac{1}{D(t)} dt \end{aligned} \quad (2.2.50)$$

$$\int_{c_3}^y g'(t, q) dt \geq \int_{c_3}^y \frac{D(c_3)}{D(t)} g'(c_3, q) dt + \int_{c_3}^y \frac{km^2}{2q^2 D(t)} \int_{c_3}^t \frac{1}{D(\tau)} d\tau dt$$

Hence,

$$g(y, q) \geq D(c_3)g'(c_3, q) \int_{c_3}^y \frac{dt}{D(t)} + \frac{km^2}{2q^2} \int_{c_3}^y \frac{1}{D(t)} \int_{c_3}^t \frac{1}{D(\tau)} d\tau dt + g(c_3, q)$$

and this shows that $g(y, q)$ is unbounded on $[0, +\infty)$, because,

$$\begin{aligned} D(c_3)g'(c_3, q) \int_{c_3}^y \frac{dt}{D(t)} + \frac{km^2}{2q^2} \int_{c_3}^y \frac{1}{D(t)} \int_{c_3}^t \frac{1}{D(\tau)} d\tau dt + g(c_3, q) &\geq \\ &\geq C(y - c_3) + \frac{km^2}{4q^2 D_{sup}^2} (y - c_3)^2 \end{aligned} \quad (2.2.51)$$

where :

$$C = \begin{cases} \frac{D(c_3)}{qD_{inf}} g'(c_3, q), & \text{if } g'(c_3, q) < 0 \\ \frac{D(c_3)}{qD_{sup}} g'(c_3, q), & \text{if } g'(c_3, q) > 0 \end{cases} \quad (2.2.52)$$

a contradiction. Thus,

$$r(y, q) = g(y, q) - \omega(y, q) \leq 0, \quad \forall y \geq 0$$

and

$$0 \leq g(y, q) \leq \omega(y, q).$$

Hence, the thesis

$$\lim_{y \rightarrow +\infty} g(y, q) = 0.$$

■

Theorem 2.4 Assume that 1., 2. and 4. are satisfied. Then, for any $q > 0$ the problem

(2.1.2) has at most a solution.

Proof

In order to prove the uniqueness of the solution of the differential problem (2.1.2), it is sufficient to show that the homogeneous problem has only the trivial solution. Let $u \in C^2([0, +\infty))$ satisfy:

$$\nu(y)u(y, q) - q [D(y)u'(y, q)]' = 0, \quad y \geq 0 \quad (2.2.53)$$

$$u'(0, q) = 0, \quad u(+\infty, q) = 0 \quad (2.2.54)$$

From (2.2.53) we have

$$\int_0^{+\infty} u(y, q) [\nu(y)u(y, q) - q [D(y)u'(y, q)]'] dy = 0. \quad (2.2.55)$$

that is

$$\begin{aligned} 0 &= \int_0^{+\infty} \nu(y)u^2(y, q) dy - q \int_0^{+\infty} u(y, q) [D(y)u'(y, q)]' dy \\ &= \int_0^{+\infty} \nu(y)u^2(y, q) dy - [u(y, q)D(y)u'(y, q)]_0^{+\infty} + q \int_0^{+\infty} D(y)u'^2(y, q) dy. \\ &= \int_0^{+\infty} \nu(y)u^2(y, q) dy + q \int_0^{+\infty} D(y)u'^2(y, q) dy, \end{aligned}$$

which implies $u(y) = 0, \forall y \geq 0$, because of the positiveness of ν, q and D .

■

Remark 2.1 *The positiveness of the solution $g(y, q)$ arises from the positiveness of the right hand side p in (2.1.2). However, if no information on the sign of p is given, we can still say that a unique solution $g(y, q)$ of the problem (2.1.2) exists, and (2.2.23) becomes*

$$|g(y)| < r_0, \quad y \geq 0. \quad (2.2.56)$$

A classical result in Calculus states that if a function is lower bounded and decreasing, then it converges to a limit. However, we cannot conclude whether its derivative will decrease or not. If we want to guarantee that $g'(y, q) \rightarrow 0$ as $y \rightarrow +\infty$ we must impose some smoothness property on $g'(y, q)$. That is, we must require that g' is uniformly continuous with respect to y . We have in this way a well-known form of the Barbalat's lemma (see e.g. [28], [29]).

Lemma 2.1 (Barbalat's lemma) *Let $f(y)$ be a differentiable function with a finite limit as $y \rightarrow +\infty$. If f' is uniformly continuous, then $f'(y) \rightarrow 0$ as $y \rightarrow +\infty$.*

Corollary 2.1 *Assume 1.-7. hold, then $\lim_{y \rightarrow +\infty} g'(y) = 0$.*

Proof.

From Theorem 2.2 we know that $g''(y, q)$ is bounded for any fixed value of the parameter q , hence, g' is uniformly continuous for all $y \geq 0$. From here and the Barbalat's Lemma we have that $\lim_{y \rightarrow +\infty} g'(y, q) = 0$. ■

Now we show some other useful properties of the solution of the differential boundary value problem $g(y, q)$. Let us observe that if $g(y, q)$ is a solution of (2.1.2), it satisfies

$$g''(y, q) = -\frac{D'(y)}{D(y)}g'(y, q) + \frac{\nu(y)}{qD(y)}g(y, q) - \frac{p(y)}{qD(y)}, \quad y \geq 0,$$

hence, the proof of the following theorem is straightforward.

Theorem 2.5 *Let $r \in \mathbb{N}$. In addition to 1.-7., assume $p, \nu \in BC^r[0, +\infty)$, and $D \in BC^{r+1}[0, +\infty)$. Then, for any fixed $q > 0$, the solution of (2.1.2)*

$g \in BC^{r+2}[0, +\infty)$. Moreover, for any $\bar{q} > 0$, the derivatives $g^{(j)}(y, q)$, $j = 0, \dots, r +$

2, are uniformly bounded with respect to $q \in [\bar{q}, +\infty)$.

All these properties together with the uniform continuity of $g(y, q)$ as function of q , that we are going to prove in the following section, represent the basic material to deal with the difficult task of proving the existence of the solution of the integro-problem (2.1.1).

2.3 Existence of the solution of the non-standard

integro-differential boundary value problem

In this section we focus our attention on the integro-differential problem (2.1.1), whose analysis requires all the results already described for (2.1.2). It is worthwhile to observe that whereas the results reported in the previous section for problem (2.1.2) with fixed $q > 0$ are mainly obtained by elaborations of already existing studies, the investigations we are going to start in this section represent the real innovative part of our research work.

In Section 2.2.3 it has been shown that, for any q fixed and positive there exists a unique solution $g(y, q)$ of problem (2.1.2). Thus, the following function:

$$F(q) := q - \int_0^{+\infty} k(x)g(x, q)dx, \quad q > 0, \quad (2.3.57)$$

is well defined. Let us assume that the kernel k satisfies:

8. $k \in C^2 ([0, +\infty))$,
9. $\int_0^{+\infty} k(x)dx < +\infty$,
10. $k(x) \geq 0, x \in [0, +\infty)$.

In order to prove the existence of a solution of (2.1.1), we show that there exists a solution of the equation $F(q) = 0$. In details, if F is continuous and there exist two positive values a and b such that $F(a)F(b) < 0$ then, for the Intermediate Value Theorem, equation $F(q) = 0$ has at least one solution q^* and the corresponding function $g(y, q^*)$ is the solution of the integro-differential problem (2.1.1).

Theorem 2.6 *Assume that hypotheses 1.- 10. hold. Then $\forall \bar{q} > 0, F(q)$ is uniformly continuous on $[\bar{q}, +\infty)$.*

Proof.

Note that by (2.2.23) and hypotheses 9. and 10., the improper integral $\int_0^{+\infty} k(x)g(x, q)dx$ is uniformly convergent with respect to q (see i. e. [26]). Hence, we are allowed to take the limit under the integral sign. Thus, in order to prove that the function F is uniformly continuous, we have to prove that $g(y, q)$ is uniformly continuous with respect to $q \geq \bar{q}$ and $y \geq 0$, i.e. $\forall \epsilon > 0, \exists \delta_\epsilon > 0$, such that

$$|g(y, q_1) - g(y, q_2)| < \epsilon, \quad \forall q_1, q_2 \text{ such that } |q_1 - q_2| < \delta_\epsilon, \quad \forall y \geq 0. \quad (2.3.58)$$

Let $q_1, q_2 \geq \bar{q}$ be arbitrarily fixed, then the functions $g(y, q_1)$ and $g(y, q_2)$ satisfy respectively :

$$\begin{cases} \nu(y)g(y, q_1) = q_1 [D(y)g'(y, q_1)]' + p(y) \\ g'(0, q_1) = 0, \quad g(+\infty, q_1) = 0 \end{cases}, \quad y \geq 0, \quad (2.3.59)$$

$$\begin{cases} \nu(y)g(y, q_2) = q_2 [D(y)g'(y, q_2)]' + p(y) \\ g'(0, q_2) = 0, \quad g(+\infty, q_2) = 0 \end{cases}, \quad y \geq 0. \quad (2.3.60)$$

Subtracting both hands of (2.3.59) and (2.3.60) we have that

$$e(y) = g(y, q_1) - g(y, q_2), \quad (2.3.61)$$

is a solution of

$$\begin{cases} \nu(y)e(y) = q_1 [D(y)e(y)]' + (q_1 - q_2) [D(y)g'(y, q_2)]' \\ e'(0) = 0, \quad e(+\infty) = 0 \end{cases}, \quad y \geq 0. \quad (2.3.62)$$

Thanks to (2.2.56) the following inequality holds

$$|e(y)| \leq |q_1 - q_2| \sup_{y \geq 0} \frac{|[D(y)g'(y, q_2)]'|}{\nu(y)}, \quad (2.3.63)$$

where by using hypotheses 2. and 4. and Theorem 2.5, it comes out that

$$\sup_{y \geq 0} \frac{|[D(y)g'(y, q_2)]'|}{\nu(y)} \leq M_0, \quad (2.3.64)$$

with M_0 independent of the parameters $q_1, q_2 \geq \bar{q}$. Hence, we conclude that $\forall \epsilon > 0$, $\exists \delta_\epsilon = \frac{\epsilon}{M_0} > 0$ such that (2.3.58) holds. ■

The following theorem plays an important role in order to prove the existence of the solution of the integro-differential problem (2.1.1).

Theorem 2.7 *Let $F(q)$ be the function defined in (2.3.57), assume that hypotheses 1. – 10.,*

hold and that

$$\nu \in C^2([0, +\infty)), \quad (2.3.65)$$

$$|\nu'(y)| < c, \quad \forall y \geq 0, \quad (2.3.66)$$

$$|k'(y)| < k_1, \quad \forall y \geq 0, \quad (2.3.67)$$

$$\int_0^{+\infty} \left| \left[\left(\frac{k(y)}{\nu(y)} \right)' D(y) \right]' \right| dy = C_1 < \infty, \quad (2.3.68)$$

$$\int_0^{+\infty} \frac{k(y)}{\nu(y)} p(y) dy = C_2 < \infty. \quad (2.3.69)$$

Then, there exist $a, b \in (0, +\infty)$ such that $F(a)F(b) \leq 0$.

Proof

By (2.1.2) we have

$$\begin{aligned} F(q) &= q - \int_0^{+\infty} k(x)g(x, q) dx \\ &= q \left(1 - \int_0^{+\infty} \frac{k(x)}{\nu(x)} (D(x)g'(x, q))' dx \right) - \int_0^{+\infty} \frac{k(x)}{\nu(x)} p(x) dx. \end{aligned}$$

Integrating twice by parts by (2.3.65)-(2.3.67) and Corollary 2.1 we get

$$\int_0^{+\infty} \frac{k(x)}{\nu(x)} (D(x)g'(x, q))' dx = \left[\frac{k}{\nu} \right]'(0)g(0)D(0) + \int_0^{+\infty} \left[\left(\frac{k(x)}{\nu(x)} \right)' D(x) \right]' g(x, q) dx.$$

Thus,

$$F(q) \leq q [1 + r_0(C_0 + C_1)] - C_2, \quad (2.3.70)$$

where r_0 , C_1 and C_2 are defined respectively (2.2.5), (2.3.68) and (2.3.69), and $C_0 = \left| \left[\frac{k}{\nu} \right]'(0) \right| D(0)$. By (2.3.70), $F(q) \leq 0$ for any $q \leq a$ with

$$a := \frac{C_2}{1 + r_0(C_0 + C_1)}. \quad (2.3.71)$$

Finally observe that, from (2.2.23) and 10., we have:

$$F(q) = q - \int_0^{+\infty} k(x)g(x, q) dx \geq q - r_0 \int_0^{+\infty} k(x)dx,$$

which gives $F(q) \geq 0$ for any $q \geq b$ with

$$b := r_0 \int_0^{+\infty} k(x)dx. \quad \blacksquare \tag{2.3.72}$$

From Theorem 2.7 there exist $a, b > 0$ such that $F(a)F(b) \leq 0$ and from Theorem 2.6, F is uniformly continuous in $[a, b]$. Hence, by using the Intermediate Value Theorem, we get our main result

Theorem 2.8 *Under the hypotheses of Theorem 2.7 there exists at least one solution g of the integro-differential problem (2.1.1), such that:*

$$a \leq \int_0^{+\infty} k(x)g(x)dx \leq b, \tag{2.3.73}$$

where a and b are defined in (2.3.71) and (2.3.72).

2.4 Conclusions

Observe that this theorem gives the existence but does not assure the uniqueness of the solution of (2.1.1), which remains an open problem. Moreover, since a solution of (2.1.1) is a solution of (2.1.2), it satisfies all the properties reported in Section 2. In particular, under

the hypotheses of Theorem 2.7 and from (2.2.5)-(2.2.7), we define:

$$r1_a := \left\{ \left[1 + \left(\left\| \frac{\nu}{aD} \right\|_{\infty} r_0 + \left\| \frac{p}{aD} \right\|_{\infty} \right) \left\| \frac{D'}{D} \right\|_{\infty}^{-1} \right] \left(e^{2 \left\| \frac{D'}{D} \right\|_{\infty} r_0} - 1 \right) \right\}^{1/2} \quad (2.4.74)$$

$$r2_a := \left\| \frac{D'}{D} \right\|_{\infty} r1_a + \left\| \frac{\nu}{aD} \right\|_{\infty} r_0 + \left\| \frac{p}{aD} \right\|_{\infty} \quad (2.4.75)$$

thus we have for $y \geq 0$

$$0 \leq g(y) \leq r_0 \quad (2.4.76)$$

$$|g'(y)| \leq r1_a \quad (2.4.77)$$

$$|g''(y)| \leq r2_a. \quad (2.4.78)$$

Chapter 3

The Numerical method

3.1 Introduction

In this chapter we introduce a numerical method to solve the following nonlinear integro-differential boundary value problem

$$\begin{cases} \nu(y)g(y) - \int_0^{+\infty} k(x)g(x)dx [D(y)g'(y)]' = p(y) \\ g'(0) = 0, \quad g(+\infty) = 0 \end{cases}, \quad y \geq 0, \quad (3.1.1)$$

whose theoretical analysis we developed in the previous chapter. This equation is defined on the half line, what is more, the coefficients of the first and the second derivatives of the unknown function g depend on the unknown function g itself by means of an integral over the semi-axis. These peculiarities make the numerical treatment rather complicated. In the previous chapter and in [34] we discussed the analytical study of equation (3.1.1) and proved the existence of the solution and other additional properties which are useful in the current investigation. In this chapter we focus on the numerical method to solve problem (3.1.1). In section 3.2 we describe our numerical approach which consists in two steps: discretization of the derivative and integral terms by using finite differences and a quadrature formula respectively, solution of the non linear system which comes out from this discretization. Section 3.3 is devoted to the study of the convergence of the overall method. Finally, Section 3.4 contains some conclusions.

3.2 Continuous problem and Discrete problem

In the previous chapter, in order to prove the existence of the solution of the integro-differential problem (3.1.1), we rewrote it in the following way:

$$\begin{cases} \nu(y) g(y, q) - q [D(y) g'(y, q)]' = p(y) \\ g'(0, q) = 0, \quad g(+\infty, q) = 0 \end{cases}, \quad y \geq 0, \quad (3.2.2)$$

$$F(q) = q - \int_0^{+\infty} k(x)g(x, q)dx = 0, \quad (3.2.3)$$

where $g(x, q)$ inside the sign of the integral, under suitable assumptions (see Theorem 2.3 and Theorem 2.4), is the unique solution of problem (3.2.2) for any $q > 0$ fixed. We recall that the parametric differential problem (3.2.2) coincides with the integro-differential problem (3.1.1) when q is a zero of the non-linear function defined by (3.2.3). Hence, problem (3.2.2) and the non-linear equation (3.2.3) are the equivalent version of the integro-differential problem (3.1.1), for this reason we refer to them by the name *continuous problem*.

In order to make clear the discussion we report, in a more compact form, the hypotheses that ensure the existence of a solution of problem (3.1.1)

$$h_1) D \in C^1([0, +\infty)), \quad \nu, k \in C^2([0, +\infty)), \quad p \in C([0, +\infty)),$$

$$h_2) 0 < D_{inf} \leq D(y) \leq D_{sup}, \quad |D'(y)| \leq D_1 \quad y \geq 0,$$

$$h_3) 0 < \nu_{inf} \leq \nu(y) \leq \nu_{sup}, \quad |\nu^{(i)}(y)| \leq \nu_i, \quad i = 1, 2, \quad y \geq 0,$$

$$h_4) 0 \leq p(y) \leq P, \quad y \geq 0,$$

$$h_5) \lim_{y \rightarrow +\infty} p(y) = 0,$$

$$h_6) \int_0^{+\infty} p(y)dy < +\infty,$$

$$h_7) k(y) \geq 0, \quad y \geq 0,$$

$$h_8) \int_0^{+\infty} |k^{(i)}(x)| dx < +\infty, \quad i = 0, 1, 2.$$

Moreover, when $h_2)$ holds we set

$$\bar{D} = \sup_{y \geq 0} \left| \frac{D'(y)}{D(y)} \right| < +\infty. \quad (3.2.4)$$

In the previous chapter we proved that under the assumptions $h_1)$ - $h_8)$, $\forall \bar{q} > 0$, the function

$$F(q) = q - \int_0^{+\infty} k(x)g(x, q) dx, \quad (3.2.5)$$

is uniformly continuous on $[\bar{q}, +\infty)$ (see Theorem 2.6). Furthermore, there exist $a, b \in (0, +\infty)$ such that $F(a)F(b) \leq 0$, where:

$$a := \frac{\int_0^{+\infty} \frac{k(y)p(y)}{\nu(y)} dy}{1 + \left\| \frac{p(y)}{\nu(y)} \right\|_{\infty} \left(\left| \left[\frac{k}{\nu} \right]'(0) \right| D(0) + \int_0^{+\infty} \left| \left[\left(\frac{k(y)}{\nu(y)} \right)' D(y) \right]' \right| dy \right)}, \quad (3.2.6)$$

$$b := \left\| \frac{p(y)}{\nu(y)} \right\|_{\infty} \int_0^{+\infty} k(y) dy, \quad (3.2.7)$$

as proved in Theorem 2.7.

Finally, let us denote by $BC^r[0, +\infty)$ the space of functions $f(x)$ with $f^{(j)}(x)$, $j = 0, 1, \dots, r$, bounded and continuous on $[0, +\infty)$, we report the following result which will play a very important role in this chapter.

Theorem 3.1 *In addition to $h_1)$ - $h_8)$, assume $p \in BC^2[0, +\infty)$, and $D \in BC^3[0, +\infty)$.*

Then, the solutions $g(y)$ of (3.1.1) and $g(y, q)$ of (3.2.2), for any fixed $q > 0$, are in

$BC^4[0, +\infty)$. Moreover the derivatives $g^{(j)}(y, q)$, $j = 0, \dots, 4$, are uniformly bounded

with respect to $q \in [a, b]$.

The theoretical analysis developed in the previous chapter leads us in a *natural* way to the construction of the numerical method that we are going to introduce.

In order to solve numerically the integro-differential problem (3.1.1) we break off the infinite interval into a finite one $[0, T]$, with the end point T *sufficiently large* and consider problem (3.2.2) on $[0, T]$. Let us define a uniform mesh on $[0, T]$:

$$\begin{aligned} \Pi_h : 0 = y_0 < y_1 < y_2 < \dots < y_{N-1} < y_N = T, \\ y_i = ih, \quad i = 0, \dots, N, \quad h = \frac{T}{N}, \end{aligned} \quad (3.2.8)$$

and for all fixed $q > 0$, we solve problem (3.2.2) by applying the classical finite difference scheme that we described in Chapter 1. Let us set

$$g_i = g_i(q) \approx g(y_i, q), \quad i = 0, \dots, N - 1, \quad (3.2.9)$$

and approximate $g'(y_i, q)$ and $g''(y_i, q)$, for all $i = 0, \dots, N - 1$, with the centred finite differences. In this way we obtain the following difference equations:

$$\begin{aligned} L_h g_i &= \nu(y_i)g_i - qD'(y_i)\frac{g_{i+1} - g_{i-1}}{2h} - qD(y_i)\frac{g_{i+1} - 2g_i + g_{i-1}}{h^2} \\ &= p(y_i), \end{aligned} \quad (3.2.10)$$

$i = 0, \dots, N - 1$. The boundary conditions in (3.1.1) are replaced by:

$$g_{-1} = g_1, \quad g_N = 0. \quad (3.2.11)$$

While the second equation in (3.2.11) is obvious, the first one comes out from the fact that $g'(0) = 0$. Equations (3.2.10) together with (3.2.11) give rise to a system of N algebraic equations:

$$A(q)g(q) = p, \quad (3.2.12)$$

where $g(q) = [g_0(q), \dots, g_{N-1}(q)]^T \in \mathbb{R}^N$, $p = [p(y_0), \dots, p(y_{N-1})]^T \in \mathbb{R}^N$ and the

tridiagonal matrix $A(q) = (a_{i,j}(q)) \in \mathbb{R}^{N \times N}$ is the coefficient matrix,

$$A(q) = \begin{pmatrix} a_0 & b_0 & 0 & 0 & \cdots & 0 \\ c_1 & a_1 & b_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \cdots & 0 \\ 0 & 0 & \cdots & c_{N-2} & a_{N-2} & b_{N-2} \\ 0 & 0 & \cdots & \cdots & c_{N-1} & a_{N-1} \end{pmatrix} \quad (3.2.13)$$

with:

$$a_0 := \nu(y_0) + \frac{2qD(y_0)}{h^2}, \quad b_0 := -\frac{2q}{h^2}D(y_0) \quad (3.2.14)$$

$$a_i := \nu(y_i) + \frac{2qD(y_i)}{h^2},$$

$$c_i := q \left[\frac{D'(y_i)}{2h} - \frac{D(y_i)}{h^2} \right], \quad i = 1, \dots, N-1, \quad (3.2.15)$$

$$b_i := -q \left[\frac{D'(y_i)}{2h} + \frac{D(y_i)}{h^2} \right], \quad i = 1, \dots, N-2. \quad (3.2.16)$$

Since the value of q that we are looking for is a zero of the non-linear function $F(q)$ given in (3.2.5), we discretize $F(q)$ as

$$F_h(q) = q - h \sum_{i=0}^N \omega_i k(y_i) g_i(q), \quad (3.2.17)$$

where we have approximated the integral in (3.2.5) by a truncated composite trapezoidal rule ($\omega_0 = \frac{1}{2}$, $\omega_i = 1$, $i = 1, \dots, N$) and $g_i(q)$, defined in (3.2.9), comes from the solution of the algebraic system (3.2.12) for a fixed value of q . In this way we have the discrete version of $F(q)$. The algebraic system (3.2.12) and the non-linear equation

$$F_h(q) = 0 \quad (3.2.18)$$

represent the discrete version of the *continuous problem* (3.2.2)-(3.2.3), for this reason we name them *discrete problem*. In conclusion (3.2.12), (3.2.18) is the discrete problem that we are going to solve.

3.2.1 Discrete problem with q fixed

In this section we analyze the properties of the discrete problem (3.2.12), (3.2.18) with fixed $q \in [a, b]$, where a and b are defined in (3.2.6)-(3.2.7).

For convenience, we briefly collect the following classical definitions (see for instance [38], [39]).

Definition 3.1 A matrix $A \in \mathbb{R}^{n \times n}$ is called *positive* ($A \geq 0$), when $a_{ij} \geq 0$ for all its elements, and A is *strictly positive* ($A > 0$), when $a_{ij} > 0$ for all its elements.

Definition 3.2 A matrix $A \in \mathbb{R}^{n \times n}$ is *inverse positive* or *strictly inverse positive* if A^{-1} exists and $A^{-1} \geq 0$ or $A^{-1} > 0$.

Let us denote by F_n the set of $n \times n$ real matrices whose off diagonal entries are non-positive.

Definition 3.3 A non-singular matrix $A \in F_n$ is an *M-matrix* if and only if A^{-1} is non-negative.

Property 3.1 A matrix $A \in \mathbb{R}^{n \times n}$ that is strictly diagonally dominant by rows and whose entries satisfy the relations $a_{i,j} \leq 0$ for $i \neq j$ and $a_{ii} > 0$, is an M-matrix.

As we have seen, $\forall q > 0$, by applying an appropriate finite difference scheme to the differential boundary value problem (3.2.2), we obtain the algebraic system (3.2.12) whose

solution $g(q)$ is the approximation to the solution of the differential problem (3.2.2) on the grid points. Furthermore, from (3.2.14)-(3.2.16) it is clear that $A(q)$ is continuous w.r.t. q . In the following proposition we report a result well known in literature (see e.g. [21], [22]), which gives us others important informations about this matrix.

Proposition 3.1 *Assume that h_1)- h_3) hold. If in addition we assume that:*

$$h_9) \quad h\bar{D} < 2,$$

where \bar{D} is given by (3.2.4), then for all $q > 0$ and fixed, we have:

$$a_j > 0, \quad j = 0, \dots, N-1, \quad b_i < 0, \quad c_{i+1} < 0, \quad i = 0, \dots, N-2.$$

Furthermore, $A(q)$ is strictly diagonally dominant.

From the previous proposition we get that $A(q)$ is an M-matrix, hence its inverse, $A^{-1}(q)$, exists and it has positive entries. By this property we get the following result.

Proposition 3.2 *Assume that h_1)- h_4) and h_9) hold. Then, for all $q > 0$ and fixed we have*

that the system (3.2.12) has a unique solution, $g(q)$, which is non-negative.

Moreover, following the proof in [21, pp. 427-430] it comes out that, under the hypotheses of Theorem 3.1, we have the error estimate:

$$\begin{aligned} e(q, h) &= \max_{0 \leq i \leq N} |g(y_i, q) - g_i(q)| \leq \frac{q}{12} h^2 \frac{D_{sup}}{\nu_{inf}} (r_4(q) + 2\bar{D}r_3(q)) \\ &+ q \frac{D_{sup}}{\nu_{inf}} \left(\frac{1}{h^2} + \frac{\bar{D}}{2h} \right) g(y_N, q), \end{aligned} \quad (3.2.19)$$

where

$$r_i(q) = \sup_{y \geq 0} |g^{(i)}(y, q)|, \quad i = 3, 4$$

and \bar{D} is given in (3.2.4). Furthermore, since as we have already mentioned, $A(q)$ is continuous with respect to q , from (3.2.14)-(3.2.16) we get the following result.

Proposition 3.3 *Under the assumptions of Proposition 3.1, we have that $g(q)$ is continuous with respect to q .*

Hence, the following proposition is straightforward.

Proposition 3.4 *Assume the hypotheses h_1 - h_3) and h_9) hold. Then the function $F_h(q)$ defined in (3.2.17) is continuous with respect to q .*

Let us assume T sufficiently large such that

$$|g(y, q)| < Ch^4, \quad (3.2.20)$$

for any $y \geq T$. This is a reasonable request since, from the formulation of the problem (3.1.1) itself, g vanishes at infinity. Then,

$$e(q, h) \leq \bar{e}(q)h^2 \quad (3.2.21)$$

with

$$\bar{e}(q) = \frac{q}{12} \frac{D_{sup}}{\nu_{inf}} (r_4(q) + 2\bar{D}r_3(q)) + q \frac{D_{sup}}{\nu_{inf}} C \left(1 + \frac{\bar{D}}{2} h \right). \quad (3.2.22)$$

This error estimate is very important, because it ensures that, for any $q > 0$ fixed, the difference solution $\{g_i(q)\}_{i=0}^N$ converges to the exact solution $g(y, q)$ as $h \rightarrow 0$, and, in fact, the error is at most $O(h^2)$.

Now we focus on function F and its discrete version F_h and, in order to show the convergence of F_h to F , it is convenient to report the following theorem (see [5]).

Theorem 3.2 *Let a and k be fixed and let $f(x) \in C^{2k+1}[a, b]$ for all $b \geq a$. Suppose,*

further, that $\int_a^\infty f(x)dx$ exists, that

$$M = \int_a^\infty |f^{(2k+1)}(x)|dx < \infty, \quad (3.2.23)$$

and that

$$f'(a) = f'''(a) = \dots = f^{(2k-1)}(a) = 0,$$

$$f'(\infty) = f'''(\infty) = \dots = f^{(2k-1)}(\infty) = 0. \quad (3.2.24)$$

Then, for fixed $h > 0$,

$$\begin{aligned} & \left| \int_a^\infty f(x)dx - h \left[\frac{1}{2}f(a) + f(a+h) + f(a+2h) + \dots \right] \right| \leq \\ & \leq \frac{h^{2k+1}M\xi(2k+1)}{2^{2k}\pi^{2k+1}}. \end{aligned} \quad (3.2.25)$$

Here $\xi(k) = \sum_{j=1}^\infty j^{-k}$ is the Riemann zeta function.

This theorem tells us that if the integrand and all of its odd-order derivatives up to order $2k - 1$ vanish at both ends of an infinite interval, then, as $h \rightarrow 0$, the trapezoidal rule will

converge to the proper answer with the rapidity of h^{2k+1} . If all odd-order derivatives vanish, then the rapidity exceeds h^{2k+1} for all k .

The following theorem shows the convergence of F_h to F , for any $q > 0$ fixed.

Theorem 3.3 *Assume the hypotheses of Theorem 3.1 are satisfied and*

$$h_{10}) \quad k \in C^3([0, +\infty)) \text{ such that } \int_0^{+\infty} |k^{(3)}(x)| dx < +\infty,$$

$$h_{11}) \quad k'(0) = 0,$$

$$h_{12}) \quad T \text{ is large enough to have } |g(y, q)| < Ch^4, \text{ for all } y \geq T$$

then:

$$|F_h(q) - F(q)| \leq Q(q)h^2, \tag{3.2.26}$$

where

$$Q(q) = C_1 \bar{e}(q) + \frac{M(q)\xi(3)}{4\pi^3} h + C_2 h^2, \tag{3.2.27}$$

with $M(q) = \int_0^{+\infty} |k(x)g(x, q)|^{(3)} dx$, $\xi(3) = \sum_{j=1}^{+\infty} 3^{-j}$ is the Riemann zeta function,

$\bar{e}(q)$ is defined in (3.2.22), $C_1, C_2 > 0$ and F_h and F are defined in (3.2.17) and (3.2.5).

Proof.

Let us consider

$$F(q) = q - \int_0^{+\infty} k(x)g(x, q)dx \quad (3.2.28)$$

$$F_h(q) = q - h \sum_{i=0}^N \omega_i k(y_i)g_i(q), \quad (3.2.29)$$

by Theorem 3.2 and the error estimate (3.2.21), recalling the hypotheses on k , we get:

$$\begin{aligned} |F_h(q) - F(q)| &= \left| \int_0^{+\infty} k(x)g(x, q)dx - h \sum_{i=0}^N \omega_i k(y_i)g_i(q) \right| \leq \\ &\left| \int_0^{+\infty} k(x)g(x, q)dx - h \sum_{i=0}^{+\infty} \omega_i k(y_i)g(y_i, q) \right| + \\ &+ h \sum_{i=0}^N \omega_i k(y_i) |g(y_i, q) - g_i(q)| + h \sum_{i=N+1}^{+\infty} \omega_i k(y_i) |g(y_i, q)| \leq \\ &\frac{M(q)\xi(3)}{4\pi^3} h^3 + (\bar{\epsilon}(q)h^2 + Ch^4) \left(h \sum_{i=0}^{+\infty} \omega_i k(y_i) \right) \leq \\ &h^2 \left(\frac{M(q)\xi(3)h}{4\pi^3} + C_1 \bar{\epsilon}(q) + C_1 Ch^2 \right), \end{aligned} \quad (3.2.30)$$

where we used h_{12}) and $h \sum_{i=0}^{+\infty} \omega_i k(y_i) \leq C_1 < +\infty$ by virtue of Theorem 3.2 and the hypotheses on k . ■

In the previous section we have seen that $\exists a, b > 0$ such that $F(a)F(b) \leq 0$, in the discrete case it is easy to prove an analogous result.

Corollary 3.1 *Under the assumptions of Theorem 3.3 and for h sufficiently small we have:*

$$F_h(a)F_h(b) \leq 0, \quad (3.2.31)$$

where a and b are defined in (3.2.6)-(3.2.7).

Furthermore it comes out that

Theorem 3.4 *Under the assumptions of the Theorem 3.3 and for $q \in [a, b]$ we have:*

$$|F_h(q) - F(q)| \leq \bar{Q}h^2, \quad \forall q \in [a, b], \quad (3.2.32)$$

with \bar{Q} constant w.r.t. q .

Proof. For $q \in [a, b]$, in view of Theorem 3.1, we observe that $\bar{e}(q)$ and $M(q)$, appearing in (3.2.27), can be bounded by a constant which is independent of q . ■

Using the same arguments and the bound in (3.2.21) we can prove that $\forall q \in [a, b]$ also $e(q, h)$ is uniformly bounded with respect to q , so there exists a constant \bar{C} such that

$$e(q, h) \leq \bar{C}h^2, \quad q \in [a, b]. \quad (3.2.33)$$

This result will play a crucial role in next section, since we consider q not fixed.

3.3 The convergence of the overall method

As we have seen, corollary 3.1 and the continuity of $F_h(q)$ defined in (3.2.17) ensure that for h sufficiently small there exists a zero of $F_h(q)$, q_h^* in the interval $[a, b]$. Hence, it is possible to apply bisection method searching for this zero. Starting from a and b defined in (3.2.6)-(3.2.7) we get a sequence of values

$$\{q_h^r\}_{r \in \mathbb{N}} \subset (a, b)$$

which converges to q_h^* . The corresponding value

$$g(q_h^*) = A^{-1}(q_h^*)p$$

is the numerical approximation of the solution of the integro-differential problem (3.1.1).

The numerical method (3.2.12), (3.2.18) is based on the following iteration process:

$$\begin{aligned}
q_h^0 &= \frac{a+b}{2}, \quad r = 0 \\
&\textit{repeat} \\
&\textit{compute } g^{r+1} \textit{ from } A(q_h^r)g^{r+1} = p \\
&\textit{compute } F_h(q_h^r) = q_h^r - h \sum_{i=0}^N \omega_i k(y_i) g_i^{r+1} \\
&\textit{if } F_h(a)F_h(q_h^r) < 0 \\
&\quad b = q_h^r \\
&\textit{else} \\
&\quad a = q_h^r \\
&\textit{endif} \\
q_h^{r+1} &= \frac{a+b}{2} \\
r &= r + 1 \\
&\textit{until convergence}
\end{aligned} \tag{3.3.34}$$

In order to prove the convergence of the method (3.2.12), (3.2.18), we consider an iteration process equivalent to algorithm (3.3.34) but applied to the continuous problem (3.2.2) and $F(q) = 0$. In fact, we do not perform this iterative process and therefore we name it *ghost*. Consider, then, the *ghost sequence*

$$\{q^r\}_{r \in \mathbb{N}} \subset (a, b)$$

obtained starting from the values a and b defined in (3.2.6)-(3.2.7), such that

$$q^r \rightarrow q^*,$$

as $r \rightarrow +\infty$, where q^* is a zero of $F(q)$. The solution $g(y, q^*)$ virtually obtained solving (3.2.2) with $q = q^*$ is of course a solution of (3.1.1). In this section we want to show the convergence of $g_n(q_h^*)$ to $g(y_n, q^*)$ when $h \rightarrow 0$.

In order to prove the convergence of the method we need the following results.

Theorem 3.5 *Under the assumptions of Theorem 3.4 we have:*

$$\lim_{h \rightarrow 0} q_h^r = q^r, \text{ for any fixed } r = 1, 2, \dots \quad (3.3.35)$$

Proof.

Thanks to Theorem 3.4 and Corollary 3.1, for h sufficiently small, say $h = h_1$, and T sufficiently large, we have that

$$\text{sign}(F_h(a)) = \text{sign}(F(a))$$

and

$$\text{sign}(F_h(b)) = \text{sign}(F(b)),$$

for all $h < h_1$, in this way

$$q^1 = \frac{a+b}{2} = q_h^1.$$

Hence, for $r = 1$, the statement is true. Then, for any fixed r , there exists $\bar{h} = \min\{h_1, \dots, h_r\}$, such that $\forall h < \bar{h}$,

$$\text{sign}(F_h(q^j)) = \text{sign}(F(q^j)),$$

$j = 1, \dots, r$, therefore,

$$q^{r+1} = q_h^{r+1}.$$

■

Theorem 3.6 *Let q_h^* and q^* be respectively the limits of the sequences $\{q_h^r\}_{r \in \mathbb{N}}$ and $\{q^r\}_{r \in \mathbb{N}}$.*

Then,

$$\lim_{h \rightarrow 0} |q_h^* - q^*| = 0. \quad (3.3.36)$$

Proof.

- From the convergence of the *ghost* iteration process

$$\forall \epsilon > 0, \exists r_1 > 0 : \forall r \geq r_1 \quad |q^r - q^*| < \frac{\epsilon}{3}$$

- from the convergence of the bisection method,

$$\forall \epsilon > 0, \exists r_0 > 0 : \forall r \geq r_0 \quad |q_h^r - q_h^*| < \frac{\epsilon}{3}, \quad (3.3.37)$$

where r_0 does not depend on h ,

- from Theorem 3.5, for $r_2 = \max\{r_0, r_1\}$

$$\forall \epsilon > 0, \exists h_0 : \forall h < h_0 \quad |q_h^{r_2} - q^{r_2}| < \frac{\epsilon}{3}. \quad (3.3.38)$$

Since

$$|q_h^* - q^*| \leq |q_h^* - q_h^{r_2}| + |q_h^{r_2} - q^{r_2}| + |q^{r_2} - q^*|,$$

then,

$$\forall \epsilon > 0, \exists h_0 : \forall h < h_0, |q_h^* - q^*| \leq \epsilon$$

■

Now we are ready to prove the main result of our research work that is the convergence of method (3.2.12), (3.2.18).

Theorem 3.7 Consider method (3.2.12), (3.2.18). Under the assumptions of Theorem 3.4

we have that there exists a constant $\bar{C} > 0$ independent of q such that for all sufficiently

small h

$$\max_{0 \leq n \leq N} |g(y_n, q^*) - g_n(q_h^*)| \leq \bar{C}h^2 + \Phi(h), \quad (3.3.39)$$

where $\lim_{h \rightarrow 0} \Phi(h) = 0$, with $y_n \in [0, Nh]$ and $Nh \rightarrow +\infty$

Proof. For the global error we have:

$$\begin{aligned} & \max_{0 \leq n \leq N} |g(y_n, q^*) - g_n(q_h^*)| \leq \\ & \max_{0 \leq n \leq N} |g(y_n, q_h^*) - g_n(q_h^*)| + \max_{0 \leq n \leq N} |g(y_n, q^*) - g(y_n, q_h^*)| \leq \\ & \bar{C}h^2 + \max_{0 \leq n \leq N} |g(y_n, q^*) - g(y_n, q_h^*)|. \end{aligned} \quad (3.3.40)$$

where $\bar{C}h^2$ comes from (3.2.33) and it is independent of q . The convergence of the overall method is proved since g is a continuous function of q , thanks to the previous theorem. ■

3.4 Conclusions

We have proposed a numerical method for the solution of the non-standard non-linear integro-differential boundary value problem (3.1.1), which is based on a finite difference scheme of order 2 and bisection method. We have proved the convergence of the overall method and the term $\bar{C}h^2$ in (3.3.39) allows us to hope in an order 2 of convergence. However, the presence of $\Phi(h) = \max_{0 \leq n \leq N} |g(y_n, q^*) - g(y_n, q_h^*)|$ in the bound of the global error prevents us to predict the theoretical order of convergence of the method. Nevertheless, the order 2 will be confirmed in the numerical experiments.

Chapter 4

Numerical Experiments

4.1 Numerical Experiments

The aim of this chapter is to present a collection of some of the most significant numerical experiments we have performed.

In the previous chapter and in [35] we showed that in order to solve numerically the integro-differential problem (3.1.1), we break off the infinite interval into a finite one $[0, T]$ with T sufficiently large. Hence, one of the aims of these tests is a check on the choice of T such that (3.2.20) occurs. Our second aim is to verify the convergence when $h \rightarrow 0$. For this reason we report the classical definitions of the number of the correct digits (cd) (see for instance [36])

$$cd_h = -\log_{10} err, \quad (4.1.1)$$

with,

$$err = \frac{\|\bar{g} - g(q_h^*)\|_\infty}{\|\bar{g}\|_\infty} \quad (4.1.2)$$

where, for $i = 0, \dots, N$, $\bar{g}_i = g(y_i)$ are the components of the solution of problem (3.1.1) on the grid points and $g(q_h^*)$ is the numerical approximation of the solution of the integro-differential problem (3.1.1) on the grid points, obtained applying the numerical method (3.2.12), (3.2.18), whose algorithm is described in (3.3.34). Moreover, we report the definition of the experimental order of convergence

$$Ord = \frac{cd_{h/2} - cd_h}{\log_{10} 2}. \quad (4.1.3)$$

Experiment 1. We consider the integro-differential problem

$$\begin{cases} \frac{10^3(y+1)}{y+2} g(y) - \int_0^{+\infty} e^{-x^2} g(x) dx [(1 + e^{-y})g'(y)]' = p(y) \\ g'(0) = 0, \quad \lim_{y \rightarrow +\infty} g(y) = 0 \end{cases}, \quad y \geq 0, \quad (4.1.4)$$

where the known term $p(y)$ is chosen such that $g(y) = e^{-y^2}$. Furthermore, the involved functions satisfy the hypotheses of the Theorem 3.7, which ensures the convergence of our method and the interval $[a, b]$ is obtained using (3.2.6), (3.2.7).

T	2	4	6	15	25
cd	1.7370	6.9484	10.6582	10.6582	10.6582

Table 4.1: cd values for increasing T in problem (4.1.4)

Test 1. In Table 4.1 we report the number of cd , computed with $h = 1.0e-04$, for increasing values of the truncation abscissa T . It is clear that $T = 6$ can be chosen as the end point with respect to this fixed value of h , because increasing T does not improve the accuracy of the solution. The fact that such a small value of T is large enough to solve problem (4.1.4) sufficient with accuracy can be explained by observing that the solution g has an exponential decay. We note that this value of T satisfies (3.2.20) with $h = 1.0e - 04$.

Test 2. The aim of this test is to show the experimental order of convergence of the method. The result of the implementation is reported in the following table, where h is the step size used. Table 4.2 shows the cd values for different values of h and $T = 6$ fixed, for problem (4.1.4). In this case we observe the convergence of our method with experimental order 2.

h	cd	Ord
0.1	4.63	-
0.05	5.24	2.03
0.0250	5.85	2.01
0.0125	6.45	2.00
0.00625	7.05	2.00
0.003125	7.66	2.00
0.0015625	8.26	2.00
0.00078125	8.86	2.00
0.000390625	9.46	2.00
0.0001953125	10.06	2.00

Table 4.2: cd values for problem (4.1.4), when $T = 6$ and h varies

Experiment 2. We next consider the integro-differential problem

$$\begin{cases} (25 + e^{-y}) g(y) - \int_0^{+\infty} \left(\frac{x}{1+x^4}\right)^2 g(x) dx [(1 + e^{-y})g'(y)]' = p(y) \\ g'(0) = 0, \quad \lim_{y \rightarrow +\infty} g(y) = 0 \end{cases}, \quad y \geq 0, (4.1.5)$$

the known term $p(y)$ is chosen in order to get $g(y) = \frac{10}{(1+y^2)^2}$. Moreover, the functions which define the problem satisfy the hypotheses which ensure the convergence of our method and the interval $[a, b]$ is obtained using (3.2.6), (3.2.7).

Test 1. In the following table we report the number of cd , computed with $h = 1.0e - 03$, for increasing values of the truncation abscissa T . This table shows that the value of T large

T	10	30	50	70	90
cd	4.0275	5.9093	6.7961	6.83311	6.83311

Table 4.3: cd values for increasing T in problem (4.1.5)

enough is $T = 70$, because increasing T does not improve the accuracy of the solution.

Test 2. This test is devoted to show the experimental order of convergence of our method. For this test we consider $T = 1100$ so that (3.2.20) is satisfied with respect to $h = 1.0e - 03$. We recall that h is the step size used.

h	cd	Ord
0.1	2.82	-
0.05	3.43	2.02
0.0250	4.03	2.01
0.0125	4.63	2.00
0.00625	5.24	2.00
0.003125	5.84	2.00
0.0015625	6.44	2.00
0.00078125	7.04	2.00
0.000390625	7.64	2.00
0.0001953125	8.25	2.00

Table 4.4: cd values for problem (4.1.5), when $T = 1100$ and h varies

This table shows that our method converges with experimental order 2.

Experiment 3. In this experiment we consider the following problem

$$\begin{cases} \frac{y+1}{y+2}g(y) - \int_0^{+\infty} e^{-x^2}g(x)dx [(1 + e^{-y})g'(y)]' = e^{-y^2} \\ g'(0) = 0, \quad \lim_{y \rightarrow +\infty} g(y) = 0 \end{cases}, \quad y \geq 0, \quad (4.1.6)$$

where the involved functions are chosen in order to satisfy the hypotheses of the convergence theorem. The interval $[a, b]$ is obtained using (3.2.6), (3.2.7). The solution g is unknown, for this reason we are not able to compute the cd values and then we report the values of $errq_h = |q_h^* - q_{\frac{h}{2}}^*|$, for $h = 1.0e - 04$.

T	1.5	3	5	8	15
$errq_h$	1.1465 e-04	8.4023 e-06	1.0604 e-07	1.1559 e-08	1.1486 e-08

Table 4.5: $errq_h$ values for increasing T in problem (4.1.6)

This table shows that the best value of T , with respect to $h = 1.0e - 04$ is $T = 8$.

In Figure 4.1 the numerical solution of problem (4.1.6) is plotted. From this plot we want to confirm some of the qualitative properties that we have theoretically proved such as the boundedness and the non-negativeness of g .

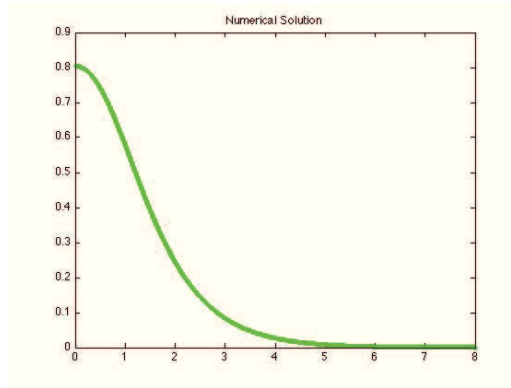


Figure 4.1: plot of the numerical solution of problem (4.1.6) with $T = 8$ and $h = 1.0e - 04$

Finally we recall that each bisection iterate involves an evaluation of the function F_h , which requires the solution of the algebraic system (3.2.12), whose dimension depends on N , the number of the grid points, and an application of the truncated trapezoidal rule. Hence, in the last experiment we compare the performance of our method, in terms of number of evaluations of function F_h , with respect to other iterative procedures, like Newton, Picard, Chord and Secant.

Experiment 4. This experiment is designed to compare the efficiency of our method, based on bisection iteration, with respect to other iterative procedures. We refer to problem (4.1.4) and the aim of this test is to show how many evaluations of function F_h , fe , are required in order to get $cd = 5$.

Methods	Bisection	Chord	Secant	Picard	Newton
fe	42	7	8	7	6

Table 4.6: Comparison with other iterative process to get $cd = 5$ for problem (4.1.4)

The method based on Newton iterations performs $cd = 5$ with less evaluations of function F_h hence it could appear more efficient. However we observe that each Newton iterate requires the computation of the first derivative of the function F_h , which is, in general, very expensive from a computational point of view.

4.2 Conclusions

In this thesis we develop a depth theoretical analysis about a particular class of the non-linear integro-differential boundary value problems on the half-line. Then we design and analyze a numerical method to solve these problems. We prove the convergence of our method and the experimental order of convergence seems to be 2.

We are still working to define a check on the choice of the end point T such that (3.2.20) is satisfied. This criterion is based on the construction of a function which is an upper bound of the solution g of the integro-differential problem (3.1.1). The study of this problem will be subject of future investigations.

About the efficiency it is important to underline that each bisection iterate requires the solution of the algebraic system (3.2.12), whose dimension depends on N , the number of the grid points. For this reason we have also performed experiments with other iterative procedures, like Secant, Chord, Picard and Newton iteration which, as expected, are more efficient from a computational point of view. On the other hand, we are still investigating on the theoretical convergence of these methods. As a matter of fact, it is known that the convergence of Newton and Picard iterations is not uniform and therefore the r_0 in (3.3.37) depends on h . In conclusion two open problems remain to be addressed: the order of the convergence of method (3.2.12), (3.2.18) and the study of convergence of methods based on different iterative processes.

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