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KAM Theory for PDEs

Relatore: Ch.mo Prof. Massimiliano Berti

Coordinatore: Ch.mo Prof. Francesco De Giovanni

Tesi di Dottorato di:
Elisa Magistrelli

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Introduction

This thesis deals with KAM theory for Hamiltonian partial differential equations. This theory concerns the following subject: since the solutions of a linear equation are periodic, quasi periodic or almost periodic (for they are superpositions of periodic motions), the problem is to investigate what happens when we add a (sufficiently) small nonlinearity.

This thesis contains two new results: an abstract KAM theorem for degenerate infinite–dimensional systems, with an application to the nonlinear wave equation, and a KAM theorem for a completely resonant nonlinear Schrödinger equation.

The KAM theory is born in the context of perturbation of integrable Hamiltonian systems with finitely many degrees of freedom in order to prove the persistence of invariant tori. The original result for analytic Hamiltonian systems was due to Kolmogorov [**Kol54**], a new proof was given by Arnold [**Arn63a**] and then Moser [**Mos62**] extended it to differentiable Hamiltonian systems.

Roughly speaking (see Theorem 1.7), Kolmogorov's theorem states that for nearly integrable Hamiltonian systems of the form

$$H = H_0(I) + \varepsilon H_P(\theta, I)$$

with $(\theta, I) \in \mathbb{T}^n \times \mathbb{R}^n$ angle–action coordinates, the most, with respect to Lebesgue measure, of the invariant tori persists under sufficiently small perturbations. This result holds for non–degenerate systems, namely for systems whose frequency–to–action map

$$I \mapsto \omega(I) = \frac{\partial H_0(I)}{\partial I}$$

is a local diffeomorphism (Kolmogorov's non–degeneracy condition), and states the persistence of those tori whose frequencies are strongly non–resonant in a diophantine sense, namely there exist constants $\alpha > 0, \tau > n-1$ such that

$$|k \cdot \omega| \geq \frac{\alpha}{|k|^\tau} \text{ for all } k \in \mathbb{Z}^n \setminus \{0\}.$$

These tori that persist are only slightly deformed and still completely filled with quasi-periodic motions, namely the dynamics on these tori is described by a finite number of incommensurable frequencies. The base of this partial foliation on the phase space into invariant tori is no longer open, but has the structure of a Cantor set.

Another typical situation is the research of periodic or quasi–periodic solutions near an elliptic equilibrium point (see Section 1.3 for the definition and Section 1.4 for detailed explanation). In this case we consider the parameter–dependent Hamiltonians

$$H = H_0 + \varepsilon H_P$$

where the linear system

$$H_0 = \sum_{j=1}^{m} \omega_j(\xi) I_j + \sum_{j=m+1}^{n} \Omega_j(\xi)(p_j^2 + q_j^2)$$

is the superposition of m uncoupled harmonic oscillators with frequencies ω_j depending on an m–dimensional parameter $\xi \in \mathbb{R}^m$. The Kolmogorov's theorem for nonlinear Hamiltonian systems can be reduced to a KAM theorem for systems of this type assuming the non–degeneracy condition on the map $\xi \mapsto \omega(\xi)$ (see [**Pös01**] and [**KP03**]).

In the particular case $m = 1$, we have perturbations of periodic orbits near the equilibrium. If $2 \leq m \leq n-1$ we focus on the m–dimensional invariant torus $\{I = \text{constant}, p = q = 0\}$ for the linear system. The persistence of this lower dimensional torus in the complete system is obtained assuming the non–degeneracy condition on the frequency map

$$\xi \in \mathbb{R}^m \mapsto \omega(\xi) \in \mathbb{R}^m$$

and the Melnikov's non–resonance condition

$$|k \cdot \omega + l \cdot \Omega| \geq \frac{\alpha}{1 + |k|^\tau}$$

for some $\alpha, \tau > 0$ and for any $(k, l) \in \mathbb{Z}^m \times \mathbb{Z}^{n-m} \setminus \{(0, 0)\}$ with $1 \leq |l| \leq 2$ (see results by Moser [**Mos67**], Eliasson [**Eli88**] and Pöschel [**Pös89**]).

A natural problem concerns the extension of these results to infinite–dimensional systems. Indeed many typical partial differential equations arising from physical problems, for example the nonlinear wave equation

$$u_{tt} - u_{xx} + V(x)u + f(u) = 0$$

can be written as an infinite–dimensional Hamiltonian system near the origin

$$H = \sum_{j \geq 1} \omega_j I_j + \varepsilon H_P$$

where, in the case of the nonlinear wave equation, ω_j^2 are the eigenvalues of the operator $A = -\frac{d^2}{dx^2} + V(x)$.

In this direction the first results are due to Kuksin [**Kuk93**] and Wayne [**Way90**] concerning perturbation of parameter–dependent linear wave and Schrödinger equations. Further results are due also to Pöschel [**Pös96b**] [**Pös96a**], Kuksin [**Kuk98**], Eliasson [**Eli88**], Bourgain [**Bou94**], Kuksin–Pöschel [**KP96**], Craig–Wayne [**CW93**] [**CW94**] for PDEs in one space dimension, while for PDEs in higher dimensions we cite results by Berti–Bolle [**BB10**], Berti–Bolle–Procesi [**BBP10**], Berti–Procesi [**BP**], Bourgain [**Bou95**] for periodic solutions, and by Eliasson–Kuksin [**EK10**], Bourgain [**Bou98**] [**Bou05a**], Berti–Bolle [**BB**], Geng–Xu–You [**GXY11a**], Procesi–Xu [**PX**] for quasi–periodic solutions.

These results prove the existence of finite–dimensional tori in infinite–dimensional systems seen as small perturbation of an unperturbed Hamiltonian

$$H_0 = \sum_{j=1}^{n} \omega_j(\xi) I_j + \sum_{j \geq n+1} \Omega_j(\xi)(p_j^2 + q_j^2)$$

with frequencies ω, Ω depending on an n–dimensional parameter $\xi \in \mathbb{R}^n$ and satisfying the non–degeneracy condition as above, namely that the frequency map

$$\xi \in \mathbb{R}^n \mapsto \omega(\xi) \in \mathbb{R}^n$$

is a local diffeomorphism, and the above Melnikov's conditions. The main difficulty with respect to the finite dimensional case is to verify infinitely many non–resonance conditions, in particular the most difficult are the second order ones, namely

$$|k \cdot \omega + \Omega_i \pm \Omega_j| \geq \frac{\alpha}{1 + |k|^\tau}$$

for some $\alpha, \tau > n - 1$ and for any $k \in \mathbb{Z}^n$, $i, j \in \mathbb{Z}$ with $i, j \geq n + 1$.

In Chapter 3 we present in detail two abstract KAM theorems for infinite dimensional systems that we shall use later for the new results. The former by Pöschel [**Pös96a**] is an improved version of the result by Kuksin [**Kuk93**]. The latter is a recent result by Berti–Biasco [**BB11**].

In order to apply these theorems to concrete nonlinear partial differential equations, one has to verify the non–degeneracy condition on the frequency map. In general this could be a hard task, in particular for systems depending on a small number of parameters (degenerate systems).

The extension of KAM Theorem to this kind of systems is an already known problem also in finite–dimensional systems, since, for example, it arises in the study of celestial mechanics. Arnold himself devoted of his most important work [**Arn63b**] to this problem, see also recent results by Herman–Jejoz [**Féj04**] and Chierchia–Pinzari [**CP**]. Since the result of Arnold, the Kolmogorov's non–degeneracy condition has been then weakened till Rüssmann [**Rüs01**] and Xu–Qiu–You [**XYQ97**]. These authors assume that the range of the frequency map in not confined on any hyperplane in the frequency space. The range may be a curve, for example, but it has to twist in all directions.

It is then natural to extend these results to infinite dimensional systems, in order to obtain a KAM theorem for systems with frequencies depending only on a few number of parameters. In Chapter 4 we prove an abstract degenerate KAM theorem for infinite–dimensional systems, see [**BBM11**]. This theorem is an extension of the result of Rüssmann to nonlinear PDEs whose linear operator depends analytically only on one parameter. The main difficulty is the bound of the maximal order of the zeros of infinitely many analytic functions, a fact which is generically impossible. We exploit the asymptotic growth of the frequencies to reduce the effective number of functions to a finite one. This idea allows to deduce a quantitative non-resonant property of the kind of the Melnikov non-resonance conditions.

This theorem is then applied to the nonlinear wave equation with Dirichlet boundary conditions

$$\begin{cases} u_{tt} - u_{xx} + V(x)u + \xi u + f(x,u) = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases}$$

where the unique real parameter is the mass ξ varying in a compact real set $\mathcal{I} \subset \mathbb{R}$, $V(x)$ is an analytic, 2π–periodic, even potential and the nonlinearity f is odd, real analytic and $f(x,0) = (\partial uf)(x,0) = 0$. Section 4.1.3 is dedicated to the study of this system proving the existence of quasi–periodic solution for a large set of masses. More precisely, we prove the following result.

THEOREM 0.1. *For every choice of indexes $\mathcal{J} := \{j_1 < j_2 < \ldots < j_N\}$, there exists $r_* > 0$ such that, for any $A = (A_1, \ldots, A_N) \in \mathbb{R}^N$ with $|A| =: r \le r_*$, there is a Cantor set $\mathcal{I}^* \subset \mathcal{I}$ with asymptotically full measure as $r \to 0$, such that, for all the masses $\xi \in \mathcal{I}^*$, the nonlinear wave equation has a quasi–periodic solution of the form*

$$u(t,x) = \sum_{h=1}^{N} A_h \cos(\widetilde{\lambda}_h t + \theta_h)\, \phi_{j_h}(x) + o(r),$$

where $o(r)$ is small in some analytic norm and $\widetilde{\lambda}_h - \lambda_{j_h} \to 0$ as $r \to 0$, being λ_{j_h} the frequencies of the linear equation.

This generalizes the results in [**Way90**] where the potential is taken as an infinite dimensional parameters, and the result in Kuksin [**Kuk93**] where the potential depends on n parameters. Regarding the nonlinearity, we only require $f(x,u) = O(u^2)$, while the result in [**Pös96b**] is valid for $f(x,u) = u^3$+higher order terms.

In the previous result, the role of the parameter ξ is to control the frequencies in order to verify non–resonance conditions.

The second result of this thesis concerns completely resonant systems as the nonlinear Schrodinger equations

$$\mathrm{i}u_t - u_{xx} + |u|^{2p}u = 0, \qquad x \in \mathbb{T}^d,$$

with $p \in \mathbb{N}$, where the frequencies of the linearized system are all integers $\omega_k = |k|^2$, hence the orbits of the linearized equation are all periodic of period 2π, and obviously the Melnikov's non–resonance conditions are not verified.

This situation has been widely studied in finite dimension. The persistence of periodic solutions near an elliptic equilibrium point for completely resonant systems has been proved by Weinstein [**Wei73**], Moser [**Mos76**] [**Mos78**] and Fadell–Rabinowitz [**FR78**] (we refer to [**Ber07**] for a detailed exposition).

The existence of periodic solutions in infinite–dimensional systems has been proved in Gentile–Mastropietro–Procesi [**GMP05**], Berti–Bolle [**BB03**] [**BB04**] [**BB06b**] [**BB08**], Gentile–Procesi [**GP06**], Baldi–Berti [**BB06a**].

The problem of proving the existence of quasi–periodic solutions is even more complicated, first because the small divisors problem is more difficult, and also because the linear system does not possess any quasi–periodic solution, hence their bifurcation is a purely nonlinear phenomenon. The main tool is the introduction of the Birkhoff normal form.

The Birkhoff normal form is proved to be completely integrable for the cubic NLS

$$\mathrm{i}u_t - u_{xx} + mu + |u|^2u + O(u^5) = 0, \quad x \in [0,\pi].$$

by Kuksin–Poschel [**KP96**], since it is a reflex of the completely integrability of $\mathrm{i}u_t - u_{xx} + mu + |u|^2u = 0$. Then Geng–Yi [**GY07**] proved the completely integrability of the normal form for the quintic NLS. For generic p Liang [**Lia08**] considered the 1–dimensional nonlinear Schrödinger equation

$$\mathrm{i}u_t - u_{xx} + |u|^{2p}u = 0, \qquad x \in \mathbb{T},$$

and proved the existence of quasi–periodic solutions with only two frequencies. The reason for this limitation is that only in this way he could obtain a normal form with constant coefficients (this is not true for general non-linearities for any number of frequencies), suitable for the application of the KAM Theorem. Recently C.Procesi-M.Procesi [**PP**] showed the construction of a reducible normal form (namely, with constant coefficients) for the Schrödinger equation with analytic non–linearities in any dimension under a finite number of conditions on the tangential sites.

Taking in mind these results, in Chapter 5 we focus on the 1–dimensional case with $p = 3$ and prove the existence of quasi–periodic solutions with any number of frequencies, namely we prove the following result.

THEOREM 0.2. *For "generic" choices of indexes* $\mathcal{J} := \{j_1, j_2, \dots, j_m\}$ *there exist* $\rho_* > 0$ *such that for any* $\rho < \rho_*$ *there exists a Cantor set* $\Pi^*_\rho \subset B_\rho(0)$ *of positive Lebesgue measure such that, for any* $\xi \in \Pi^*_\rho$, *the nonlinear Schrödinger equation admits a quasi–periodic solution of the form*

$$u(t,x) = \sum_{i=1}^{m} \sqrt{\xi_i} e^{\mathrm{i}((j_i^2 + \omega_i^*(\xi))t + \theta_i)} + o(\xi).$$

where the map $\xi \mapsto \omega^*(\xi)$ *is a lipeomorphism,* $\theta \in \mathbb{R}^m$ *are arbitrary phases and* $o(\xi)$ *is small in some analytical norm. The measure of the set* Π^*_ρ *is greater than* $c\rho^m$, *where* c *is a constant independent on* ρ.

In proving this result, we first reduce the system to normal form, imposing a finite number of choices on the indexes $\mathcal{J}$ in order to make it reducible and developing all the computations on the normal form and the needed conditions also in the case of three frequencies. Then we use the obtained normal form as the unperturbed Hamiltonian to apply the KAM Theorem as stated in [**BB11**]. We expect that this result hold for any $p \in \mathbb{N}$. We focus on the case $p = 3$ to check in details all the assumptions of the KAM Theorem.

We have cited so far results for 1–d NLS because KAM theories in higher dimensions are very difficult to obtain. Recently Eliasson–Kuksin [**EK10**] proved a KAM theorem for nonlinear Schrödinger equation using Töplitz–Lipschitz properties of the perturbative terms to control the frequencies. We refer also to results by Geng–Xu–You [**GXY11a**] and Procesi–Xu [**PX**].

CHAPTER 1

Classical background

In this chapter we recall some classical definitions and results for finite dimensional Hamiltonian systems, taking as a reference the book of Kappeler-Pöschel [**KP03**]. We first recall some definitions and properties for Hamiltonian vector fields, then we consider the case of integrable systems (in the senso of Liouville) and finally we study the behavior of systems that are small perturbations of integrable ones.

1.1. Hamiltonian formalism

Let $n \in \mathbb{N}$. Let M be a smooth (i.e. infinitely many differentiable) manifold of finite dimension $2n$, without boundary and connected.

DEFINITION 1. A *symplectic form* on M is a closed and non–degenerate 2–form α on M. The pair (M, ν) is called *symplectic manifold*.

The symplectic form α induces an isomorphism between the tangent and the cotangent bundle of M

$$\begin{aligned} S\colon TM &\longrightarrow T^*M \\ X &\longmapsto \nu \circ X = \alpha(X, \cdot). \end{aligned}$$

Let $J := S^{-1}\colon T^*M \to TM$ the inverse of S.

Consider a smooth function $H\colon M \to \mathbb{R}$. This defines a vector field

$$X_H = JdH$$

on M, that is the unique one satisfying

$$\alpha \circ X_H = dH.$$

DEFINITION 2. X_H is called the *Hamiltonian vector field* associated to the *Hamiltonian* H on the phase space M. The *Hamiltonian flow* of H is the flow defined by the vector field X_H and is indicated with X_H^t.

The Hamiltonian H is constant along the flow lines of its Hamiltonian vector field, namely by definition

$$\frac{d}{dt} H \circ X_H^t = dH(X_H) = \alpha(X_H, X_H) = 0$$

and this is also known as the conservation of energy.

DEFINITION 3. On the symplectic manifold M define the *Poisson bracket* of two smooth functions G, H as

$$\{G, H\} := \alpha(X_G, X_H).$$

The Poisson bracket is a skew form on the linear space of all the smooth functions on M. One fundamental property, that follows by the definition, is that

$$\{G, H\} = dG(X_H)$$

and so, for any smooth function G, the flow X_H^t has the property that

$$\dot{G} = \{G, H\},$$

where $\dot{G}$ denotes the derivative of G with respect to the vector field X_H, namely

$$\dot{G} = \frac{d}{dt} G \circ X_H^t\big|_{t=0} = dG(X_H).$$

The Poisson bracket satisfies the Leibniz rule

$$\{FG, H\} = F\{G, H\} + G\{F, H\}$$

and the Jacobi identity

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0.$$

DEFINITION 4. The *Lie bracket* of to vector fields X, Y is defined as

$$[X, Y] := YX - XY.$$

This is bilinear and skew–symmetric. Moreover, the Lie bracket of two Hamiltonian vector field is again Hamiltonian, namely

$$[X_G, X_H] = X_{\{G,H\}}$$

for any two Hamiltonian G, H on the symplectic manifold M.

DEFINITION 5. A smooth non–constant function G is called an *integral* of a Hamiltonian system with Hamiltonian function H if

$$\{G, H\} = 0.$$

Since $\{G, H\} = X_H G$, this means that G is constant along the flow lines of X_H. By the skew symmetry of the Poisson bracket, if G is an integral for X_H, then also H is an integral for X_G, and then the two Hamiltonians G, H are said to be *in involution*. Finally, G and H are in involution if and only if

$$[X_G, X_H] = 0$$

and we say that the two vector field X_G, X_H *commute*.

To preserve the Hamiltonian nature of vector fields, a diffeomorphism of a symplectic manifold has to preserve the underlying structure.

DEFINITION 6. A diffeomorphism Φ of the symplectic manifold M is called *symplectic* or a *symplectomorphism* if it preserves the symplectic form, that is if $\Phi^*\alpha = \alpha$.

A symplectomorphism Φ is also *canonical*, namely it preserves also the Poisson bracket:

$$\{F, G\} \circ \Phi = \{F \circ \Phi, G \circ \Phi\}$$

for any two functions F, G.

All the linear symplectic vector spaces of the same dimension are symplectically isomorphic, but this is no longer true for nonlinear ones. The following result states that this is true locally around every point of a symplectic manifold.

THEOREM 1.1 (Darboux). *A symplectic manifold* (M, α) *of dimension* $2n$ *is locally symplectomorphic to an open subset of* $(\mathbb{R}^{2n}, \alpha_0)$.

This theorem states that, given any point $p \in M$, there are a neighborhood W of p in M and a diffeomorphism $Phi \colon V \to W$ of an open set V in $\mathbb{R}^{2n}$ onto W such that $\Phi^*\alpha = \alpha_0$.

The coordinates provided by Φ are called *Darboux coordinates*.

1.2. Liouville integrable systems

Integrable systems are particular Hamiltonian systems that can be solved for any initial data by quadratures. In order to be integrable, the system has to admit sufficiently many conserved quantities in involution. It turns out that for a system with n degrees of freedom it is sufficient to have n independent integrals in involution. More precisely, we give the following definitions.

DEFINITION 7. A family of m functions $F_1, \ldots, F_m$ on M is called *independent* if their 1–forms $dF_1, \ldots, dF_m$ are linearly independent at every point in M.

DEFINITION 8. A Hamiltonian system on a symplectic manifold M of dimension $2n$ is called *integrable* if its Hamiltonian H admits n independent integrals $F_1, \ldots, F_m$ in involution, namely

(1) $\{H, F_i\} = 0$ for any $i = 1, \ldots, n$
(2) $\{F_i, F_j\} = 0$ for any $i, j = 1, \ldots, n$
(3) $dF_1 \wedge \ldots \wedge dF_m \neq 0$

everywhere on M.

EXAMPLE. In standard action–angle coordinates $(\theta, I) \in \mathbb{T}^n \times \mathbb{R}^n$ any Hamiltonian of the form $H = H(I)$ is integrable with integrals $F_i = I_i$, for $i = 1, \ldots, n$.

EXAMPLE. In standard cartesian coordinates $(q, p) \in \mathbb{R}^n \times \mathbb{R}^n$ any Hamiltonian of the form $H = H(q_1^2 + p_1^2, \ldots, q_n^2 + p_n^2)$ is integrable with integrals $F_i = q_i^2 + p_i^2$, for $i = 1, \ldots, n$.

We now give a geometric description of an integrable system. Consider an arbitrary number of smooth independent functions $F_1, \ldots, F_m$ on M and the map $F = (F_1, \ldots, F_m) \colon M \to \mathbb{R}^m$. Every non–empty leaf

$$M^c := F^{-1}(c) = \{p \in M \colon F(p) = c\}$$

is a smooth submanifold of M of codimension m. The whole manifold M is foliated into these leaves. The following result holds.

LEMMA 1.2. *Assume that the map $F = (F_1, \ldots, F_m)$ defines a foliation of M with leaves $M^c = F^{-1}(c)$. Then the following statements are equivalent:*

(1) *the functions $F_1, \ldots, F_m$ are in involution, namely $\{F_i, F_j\} = 0$ for $i, j = 1, \ldots, m$*
(2) *the Hamiltonian vector fields X_{F_i} are everywhere tangent to the leaves of F, namely $X_{F_i}(p) \in T_pM^c$ for $i = 1, \ldots, m$, $p \in M^c$.*

On a symplectic manifold of dimension $2n$ there are at most n independent functions in involution.

DEFINITION 9. If the number of independent function in involution is n then each leaf is called *Lagrangian submanifold* of M.

COROLLARY 1.3. *If $F_1, \ldots, F_n$ are independent functions in involution on M, then the map $F = (F_1, \ldots, F_m)$ defines a foliation of M into Lagrangian submanifolds $M^c = F^{-1}(c)$.*

Suppose that the Hamiltonian H admits $F_1, \ldots, F_n$ as independent integrals, hence $\{H, F_i\} = 0$. It follows that the Hamiltonian vector field X_H is tangent to the leaves M^c and then these are invariant manifolds with respect to its flow.

COROLLARY 1.4. *A Hamiltonian system is integrable if and only if it admits a foliation of its phase space into Lagrangian submanifolds.*

Assume now that the Hamiltonian H is integrable with integrals $F_1, \ldots, F_n$ in involution. Liouville showed that locally around each point one can introduce standard symplectic coordinates (q, p) such that the Hamiltonian assumes the form $H = H(p)$. Then the coordinates $p_1, \ldots, p_n$ become integrals of the Hamiltonian. There is then a global version of this result due to Arnold and Jost that leads to the following theorem.

THEOREM 1.5 (Liouville-Arnold-Jost). *Let (M, α) be a symplectic manifold of dimension $2n$ and let $F = (F_1, \dots, F_n)$ be n independent functions in involution on M. Suppose that one of he leaves of F, say $M^0 = F^{-1}(0)$ is compact and connected. Then*

(1) M^0 in an n–dimensional embedded torus
(2) there exist an open neighborhood U of M^0, an open neighborhood D of 0 in $\mathbb{R}^n$ and a diffeomorphism $\Psi\colon \mathbb{T}^n \times D \to U$ introducing action–angle variables with

$$\Psi * \alpha = \alpha_0, \qquad \Psi^* M^0 = \mathbb{T}^n \times \{0\},$$

such that the functions $F_i \circ \Psi$ are independent of the angular coordinates.

Consider now an integrable Hamiltonian $H = H(I)$ in action–angle coordinates. The equation of motion are

$$\begin{cases} \dot{\theta}_i = \omega_i(I) \\ \dot{I}_i = 0, \end{cases}$$

where $\omega_i(I) = \frac{dH(I)}{dI_i}$ for $i = 1, \dots, n$. These equations are easily integrable and their general solution is

$$\theta(t) = \theta^0 + \omega(I^0)t, \qquad I(t) = I^0.$$

Every solution is a straight line which, due to the identification of the angular coordinates θ modulo 2π, is winding around the underlying torus $\mathbb{T}_{I^0} := \mathbb{T}^n \times \left\{I^0\right\}$ with constant frequencies $\omega(I^0) = \big(\omega_1(I^0), \dots, \omega_n(I^0)\big)$. They completely determine the dynamics on this torus, which consists of parallel translations. These tori are called *Kronecker* (or *rotational*) *tori* and the associated frequencies are called the *frequencies of the invariant torus*. We give now a more general definition.

DEFINITION 10. Let X be a smooth vector field on a manifold M of arbitrary dimension. An invariant n–torus T of X is called a *Kronecker torus* (or *torus with linear flow*) if there exist a diffeomorphism $\Phi\colon \mathbb{T}^n \to T$ such that $\Phi^* X$ is a constant n–vector ω on $\mathbb{T}^n$ called the *frequency vector* of the Kronecker torus.

From a geometrical point of view, an integrable Hamiltonian system around a compact connected leaf is then completely foliated into an n–parameter family of invariant and Lagrangian tori with linear flow. From an analytical point of view, all solution curves on an invariant Kronecker torus T with frequencies ω are represented as $\Phi(\theta^0 + \omega t)$, with $\theta^0 \in \mathbb{T}^n$, hence they are quasi–periodic function of t, in the sense of the following definition.

DEFINITION 11. A continuous function $q\colon \mathbb{R} \to \mathbb{R}$ is called *quasi–periodic* with frequencies $\omega = (\omega_1, \dots, \omega_n)$ if there exists a continuous function

$$Q\colon \mathbb{T}^n \to \mathbb{R}$$

such that $q(t) = Q(\omega t)$ for all $t \in \mathbb{R}$.

The flow on a Kronecker torus is rather simple and depends on arithmetical properties of its frequency ω. There are two cases.

(1) The frequencies ω are *non–resonant* or *rationally independent*. This means that

$$\langle k, \omega \rangle \neq 0 \qquad \text{for all } 0 \neq k \in \mathbb{Z}^n.$$

Then each orbit on this torus is dense and the flow is ergodic.

(2) The frequencies are *resonant* or *rationally dependent*. This means that there exist integer relations

$$\langle k, \omega \rangle = 0 \qquad \text{for some } 0 \neq k \in \mathbb{Z}^n.$$

The prototype is $\omega = (\omega_1, \dots, \omega_m, 0, \dots, 0)$with $n - m \geq 1$ trailing zeros and $(\omega_1, \dots, \omega_m)$ non–resonant. Then the torus decomposes into an $n-m$–parameter family of identical invariant m–tori. Each orbit is dense on such a lower dimensional torus but not in the entire Kronecker torus. If there are $n-1$ independent resonant relations, then each frequency $\omega_1, \dots, \omega_n$ is an integer multiple of a fixed non–zero frequency $\overline{\omega}$ and the whole torus is filled by periodic orbits with one and the same period $2\pi/\overline{\omega}$.

In an integrable system the frequencies on the tori may or may not vary with the torus, depending on the nature of the frequency map $I \mapsto \omega(I)$. If it is *non–degenerate* in the sense that

$$\det \frac{\partial \omega}{\partial I} = \det \frac{\partial^2 H}{\partial I^2} \neq 0,$$

then the frequency map is a local diffeomorphism.

Non–resonant and resonant tori form dense subsets in the phase space. The resonant ones sit among the non–resonant ones like rational numbers among the irrational numbers.

In Section 1.4 we will understand through the KAM theory the behavior of nearly integrable Hamiltonian systems, namely of those systems which are close to integrable ones.

1.3. Birkhoff Normal Form theorem

On $\mathbb{R}^{2n}$ consider a Hamiltonian H with an equilibrium point at zero (this is always possible, eventually using Darboux coordinates).

DEFINITION 12. The equilibrium point is said to be elliptic if there exists a canonical system of coordinates (p, q) in which the Hamiltonian takes the form

$$H(p, q) = H_0(p, q) + H_P(p, q), \tag{1.1}$$

where

$$H_0(p, q) = \sum_{j=1}^{n} \omega_j \frac{p_j^2 + q_j^2}{2}, \qquad \omega_j \in \mathbb{R}$$

and H_P is a smooth function having a zero of order 3 at the origin.

In the linear approximation, since $H_P = O(\|(p, q)\|^3)$, the system consists in n independent harmonic oscillators.

THEOREM 1.6 (Theorem 1 in [**Bam08**]). *For any integer $r \geq 0$, there exists a neighborhood $\mathcal{U}(r)$ of the origin and a canonical transformation $\tau_r : \mathcal{U}(r) \to \mathbb{R}^{2n}$ which puts the system* (1.1) *in Birkhoff Normal Form up to order r, namely*

$$H^{(r)} := H \circ \tau_r = H_0 + Z^{(r)} + R^{(r)}, \tag{1.2}$$

where

(1) $Z^{(r)}$ is a polynomial of degree $r + 2$ which Poisson commutes with H_0, namely $\{H_0, Z^{(r)}\} = 0$,
(2) $R^{(r)}$ is small, namely

$$\left|R^{(r)}(z)\right| \leq C_r \|z\|^{r+3}, \qquad \forall z \in \mathcal{U}(r).$$

Moreover,

$$\|z - \tau_r(z)\| \leq C_r \|z\|^2, \qquad \forall z \in \mathcal{U}(r),$$

and the same holds also for the inverse τ_r^{-1}.

If the frequencies ω are nonresonant up to order $r + 2$, namely

$$\omega \cdot k \neq 0 \qquad \forall k \in \mathbb{Z}^n,\ 0 < |k| \leq r + 2,$$

then the function $Z^{(r)}$ depends only on the actions $I_j := \frac{p_j^2 + q_j^2}{2}$.

The idea of the proof is to construct a canonical transformation obtained as the time–1–flow of a suitable Hamiltonian function, pushing the non–normalized part of the Hamiltonian to order four, followed by a transformation pushing it to order five, and so on.

The Birkhoff theorem ensures then the existence of a canonical transformation putting the Hamiltonian system in normal form up to a remainder of a given order. The dynamics of the system in normal form depends on the resonance relations fulfilled by the frequencies.

1.4. KAM theory

Integrable systems are the exception, but many interesting Hamiltonian systems may be viewed as small perturbation of an integrable system, for example the planetary system. So the goal now it to understand what happens to a foliation of invariant tori with their quasi–periodic under small perturbation of the Hamiltonian.

So, consider a Hamiltonian in action–angle coordinates (θ, I) of the form

$$H = H_0(I) + H_\varepsilon(\theta, I)$$

where H_0 is the unperturbed integrable Hamiltonian and H_ε is a general perturbation that we assume of the form $\varepsilon H_1(\theta, I)$, so that ε measures the size of the perturbation.

We assume the unperturbed system to be non–degenerate, namely we assume that the frequency map

$$I \mapsto \omega(I) = \frac{\partial H_0(I)}{\partial I}$$

is a local diffeomorphism (this is also called *Kolmogorov's condition*, see [**Kol54**]).

The first result due to Poincaré is of negative nature and states that the resonant tori are in general destroyed by any arbitrary small perturbation and that a generic Hamiltonian system is not integrable.

But in 1954 Kolmogorov observed that the majority of tori survives. More precisely, he stated the persistence of those Kronecker tori whose frequencies are not only non–resonant but *strongly non–resonant*, in the sense that there exist constants $\alpha > 0$ and $\tau > n - 1$ such that

$$|\langle k, \omega \rangle| \geq \frac{\alpha}{|k|^\tau} \qquad \text{for all } 0 \neq k \in \mathbb{Z}^n.$$

This condition is also called *diophantine* or *small divisor condition*. In order to verify the existence of these frequencies, fix α, τ and denote with Δ_α the set of all $\omega \in \mathbb{R}^n$ satisfying these infinitely many conditions. Then for any bounded set $\Omega \subset \mathbb{R}^n$ we have the following Lebesgue measure estimate

$$|\Omega \setminus \Delta_\alpha| = O(\alpha).$$

Moreover, we have that only those Kronecker tori with frequencies $\omega \in \Delta_\alpha$ with

$$\alpha \gg \sqrt{\varepsilon}$$

do survive.

To state the KAM theorem we finally consider subsets Ω_α of a bounded domain $\Omega \subset \mathbb{R}^n$ whose elements are the frequencies belonging to Δ_α and that have at least distance α to the boundary of Ω. These sets are Cantor sets and have large Lebesgue measure, $|\Omega \setminus \Omega_\alpha| = O(\alpha)$.

We can now state the main theorem of Kolmogorov, Arnold and Moser.

THEOREM 1.7 (KAM Theorem). *Suppose the Hamiltonian*

$$H = H_0 + \varepsilon H_1$$

is real analytic on the closure of $\mathbb{T}^n \times D$, where D is a bounded domain in $\mathbb{R}^n$. If the integrable Hamiltonian H_0 is non–degenerate and its frequency map is a diffeomorphism $D \to \Omega$, then there exists a constant $\delta > 0$ such that for

$$|\varepsilon| < \delta\alpha^2$$

all the Kronecker tori $(\mathbb{T}^n, \omega)$ of the unperturbed system with $\omega \in \Omega_\alpha$ persist as Lagrangian tori, being only slightly deformed. Moreover, they depend on a Lipschitz continuous way on ω and fill the phase space $\mathbb{T}^n \times D$ up to a set of measure $O(\alpha)$.

KAM theorem ensures then the persistence of invariant tori of nearly integrable Hamiltonian systems, filled by quasi–periodic solutions with frequencies satisfying strong non–resonance conditions of diophantine type.

Since its conception this theorem has been generalized and extended in several ways in order to relax some of its assumptions.

First, regarding the perturbation and the integrable Hamiltonian, it is proved that it is sufficient that they are of class C^r with $r > 2\tau + 2 > 2n$, see [**Pös80**].

The second improvements applies to the non–degeneracy condition. We have seen that in order to verify the non–resonance properties, KAM theory requires some non–degeneracy condition concerning the dependence of the frequencies on the parameters of the system (actions, potentials, masses, ...). The Kolmogorov's non–degeneracy condition is the simplest one and it is used to completely control the frequencies, so that their diophantine estimates can be preserved under perturbation, but in concrete systems it could be not verified (or it could be very difficult to check it). For example, it is never satisfied in the spatial solar system, see Arnold [**Arn63b**] and

Herman-Féjoz [**Féj04**]. This problem strongly motivated the search of weaker non–degeneracy conditions.

Degenerate KAM theory has been then widely developed since Arnold [**Arn63b**] and Pjartly [**Pja69**]. In fact, it is sufficient that the intersection of the range of the frequency map with any hyperplane has zero measure. Then, after perturbation, one can still find sufficiently many diophantine frequencies even if they are not know a priori. For example, if it happens that $\frac{\partial H_0}{\partial I}$ is a function of I_1 alone, and thus is completely degenerate, it is sufficient to require

$$\det\left(\frac{\partial^j \omega_i}{\partial I_1^j}\right)_{1\leq i,j\leq n} \neq 0,$$

as stated in the paper [**XYQ97**] by Xu-You-Qiu. We quote also other important works by Bruno [**Bru92**], Cheng-Sun [**CS94**] and Sevryuk [**Sev07**].

Then new contributions were given by Rüssmann [**Rüs90**]-[**Rüs01**] not only for Lagrangian (i.e. maximal dimensional) tori but also for lower dimensional elliptic/hyperbolic tori. For recent developments we refer to .

The main assumption in [**Rüs01**] is that the frequencies are analytic functions of the parameters and satisfy a weak non–degeneracy condition in the sense of the following definition.

Definition 13. A real analytic function $f\colon O \to \mathbb{R}^m$ defined on a domain $O \subseteq \mathbb{R}^n$ is *non–degenerate* if, for any vector of constants $(c_1, \dots, c_m) \in \mathbb{R}^m \setminus \{0\}$, the function $c_1 f_1 + \dots c_m f_m$ is not identically zero on O.

The Rüssmann weak non–degeneracy assumption on the frequency is then the following.

Definition 14. A real analytic function $(\omega, \Omega)\colon O \to \mathbb{R}^m \times \mathbb{R}^p$ defined on a domain $O \subseteq \mathbb{R}^n$ is *weakly non–degenerate* if

(1) ω is non–degenerate
(2) $l \cdot \Omega \notin \{k \cdot \omega\colon k \in \mathbb{Z}^m\}$ for all $l \in \mathbb{Z}^p$ with $0 < |l| \leq 2$.

For maximal dimensional tori this condition is equivalent to the fact that the range of the frequency map is not contained in any hyperplane.

Rüssmann's proof goes into some steps. First, he uses properties of the zero set of analytic functions to show that the *qualitative* weak non–degeneracy assumption implies a *quantitative* non–degeneracy property. Second, he shows that, notwithstanding the fact that the frequencies change during the KAM iteration process, the set of non–resonant frequencies met at each step has large measure. Third, he proves that the same is true for the final frequencies on the limiting perturbed torus constructed through the iteration. For the last two steps Rüssmann introduces the concept of "chain of frequencies".

In Chapter 4 we will see an extension of Rüssmann's result to infinite dimensional Hamiltonian system.

As seen, the classical KAM Theorem is concerned with the persistence of maximal dimensional tori with strongly non–resonant frequencies in a non–degenerate system.

In the case of resonant frequencies, we can meet lower dimensional invariant tori of dimension $m < n$. Here a typical situation is the study of the system near an elliptic equilibrium point, as we have seen in Section 1.3. This is an interesting case, since typical partial differential equation can be written in this form.

In the case of periodic orbits, Lyapunov showed that they persist, being only slightly deformed, if at the equilibrium their frequency is not in resonance with the other frequencies of the system.

For $2 \leq m \leq n-1$ first Melnikov [**Mel65**] and then Moser [**Mos67**] and Eliasson [**Eli88**] showed the existence of quasi–periodic solutions for parameter–dependent systems, namely for a nonlinear system seen as perturbation of a parameter–dependent linear system.

More precisely, consider an Hamiltonian

$$H = H(I_1, \ldots, I_n) \text{ with } I_j = \frac{1}{2}(q_j^2 + p_j^2).$$

We focus on the m–dimensional torus

$$T_{I^0} = \left\{(q,p) \colon q_j^2 + p_j^2 = 2I_j^0, \text{ for } 1 \leq j \leq n\right\}$$

with, without loss of generality, $I_1^0, \ldots I_m^0 > 0$ and $I_{m+1}^0, \ldots, I_n^0 = 0$.

For $1 \leq j \leq m$ we introduce angle–action coordinates (θ, I) on the first m modes by

$$q_j = \sqrt{2(\xi_j + I_j)}\cos\theta_j, \quad p_j = \sqrt{2(\xi_j + I_j)}\sin\theta_j$$

depending on the amplitudes $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m$, while we keep the other $m-n$ cartesian coordinates. With a series expansion of H up to the first order in I_j and the second order in q_j, p_j, we obtain the integrable Hamiltonian

$$H = H_0 + \varepsilon H_P,$$

where H_P contains the higher order terms and can be regarded as perturbation, while

$$H_0 = \sum_{j=1}^{m} \omega_j(\xi) I_j + \frac{1}{2} \sum_{j=m+1}^{n} \Omega_j(\xi)(p_j^2 + q_j^2)$$

is the superposition of uncoupled harmonic oscillators, each with frequencies $\omega_j(\xi) = \frac{\partial H}{\partial I_j}(I_1^0, \dots, I_m^0, 0, \dots, 0)$ depending on the m–dimensional parameter $\xi \in \Pi \subset \mathbb{R}^m$.

Now we want to study the behavior of this nearly integrable system on the phase space $\mathbb{T}^m \times \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}^{n-m}$.

If $\varepsilon = 0$, the system admits for each $\xi \in \Pi$ the invariant m–torus $T_0 = \mathbb{T}^m \times \{0\} \times \{0\} \times \{0\}$ and we are interested in the persistence of this torus under small perturbations of the Hamiltonian H_0, namely for $\varepsilon > 0$ small, for a large set of parameters. We point out that we have a large family of Hamiltonian systems, depending on the parameter ξ and we want to prove the persistence of one invariant torus.

In order to do this, we need the following non–degeneracy assumption.

DEFINITION 15. The parameter–dependent family of Hamiltonian H_0 is *non–degenerate* if the map

$$\xi \mapsto \omega(\xi),$$

is a local diffeomorphism on its domain, and if

$$\xi \mapsto k \cdot \omega(\xi) + l \cdot \Omega(\xi) \not\equiv 0$$

for all $(k, l) \in \mathbb{Z}^m \times \mathbb{Z}^{n-m} \setminus \{(0,0)\}$ with $1 \le |l| \le 2$.

The first condition is the usual Kolmogorov condition, while the second one is also known as *Melnikov's condition*, and is used to control the small divisors arising in the perturbation theory.

Under this assumption we have then the following result.

THEOREM 1.8. *Suppose that the Hamiltonian $H = H_0 + \varepsilon H_P$ is real analytic in a fixed neighborhood of $T_0 \times \Pi$, with $\Pi \subset \mathbb{R}^m$ closed and bounded set with positive Lebesgue measure. If H_0 is non–degenerate then, for ε sufficiently small, there exists a Cantor set $\Pi_\varepsilon \subset \Pi$ such that for each parameter $\xi \in \Pi_\varepsilon$ the perturbed system admits an elliptic invariant torus close to T_0. Moreover,* $\operatorname{meas}(\Pi \setminus \Pi_\varepsilon) \to 0$ *as $\varepsilon \to 0$.*

As we will see in Chapter 3, this will be the natural starting point in the extension to infinite–dimensional systems.

CHAPTER 2

Hamiltonian PDEs

In this chapter we recall some definitions and results for Hamiltonian partial differential equations, as a reference see [**Kuk06**].

2.1. Hilbert scales X_s

Let X be a Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and basis $\{\phi_k \colon k\in\mathbb{Z}\}$. Consider a positive sequence $\{\theta_k \colon k \in Z\}$ such that $\theta_k \to \infty$ as $k\to\infty$.

Definition 16. $\{X_s\}_s$ is an *Hilbert scale* if, for any $s\in\mathbb{R}$ (or $\mathbb{Z}$), X_s is the Hilbert space with basis $\left\{\phi_k\theta_k^{-s} \colon k\in\mathbb{Z}\right\}$. Denote with $\|\cdot\|_s, \langle\cdot,\cdot\rangle$ its norm and scalar product. Set $X_0 = X$, $X_{-\infty} := \bigcup X_s$, $X_\infty := \bigcap X_s$.

A Hilbert scale X_s satisfies the following properties:

1. X_s is compactly embedded and dense in X_r if $s > r$,
2. the spaces X_s, X_{-s} are conjugated with respect to the scalar product $\langle\cdot,\cdot\rangle$,
3. the norm $\|\cdot\|_s$ satisfies the interpolation inequality.

Example. Consider the scale of Sobolev function on the d–dimensional torus $\left\{H^s(\mathbb{T}^d;\mathbb{R}) = H^s(\mathbb{T}^d)\right\}$, where

$$H^s(\mathbb{T}^d) := \left\{ u\colon \mathbb{T}^d\to\mathbb{R} \text{ such that } u = \sum_{k\in\mathbb{Z}^d} u_k e^{\mathrm{i}k\cdot s},\ u_l = \overline{u}_{-l}\in\mathbb{C}, \right.$$
$$\left. \|u\|_s = \sum_{k\in\mathbb{Z}^d} (1+|k|)^{2s}|u_k|^2 < \infty \right\}.$$

Example. Consider the scale $\{H_0^s(0,\pi)\}$, where

$$H_0^s(0,\pi) := \left\{ u\in H^s(0,\pi), u = \sum_{k=1}^{\infty} u_j \sin kx,\ 2\pi\text{–periodic}, \right.$$
$$\left. \|u\|_s^2 = \sum |k|^{2s}|u_k|^2 < \infty \right\}.$$

Definition 17. Let $\{X_s\},\{Y_s\}$ two Hilbert scales, $L\colon X_\infty\to Y_{-\infty}$ a linear map and denote with $\|L\|_{s_1,s_2}\leq\infty$ its norm as a map from X_{s_1} to Y_{s_2}. L defines a *linear morphism of order* d of the two scales for $s\in[s_0,s_1], s_0\leq s_1$, if $\|L\|_{s,s-d}<\infty$ for every $s\in[s_0,s_1]$.

Let $\{X_s\}, \{Y_s\}$ be two scales and $O_s \subset X_s$, $s \in [a,b]$, be a system of open domains such that

$$O_{s_1} \cap O_{s_2} = O_{s_2} \qquad \text{if } a \leq s_1 \leq s_2 \leq b.$$

Let $F\colon O_a \to Y_{-\infty}$ be a map such for every $s \in [a,b]$ its restriction $F\colon O_s \to Y_{s-d}$ is an analytic (C^k–smooth) map. Then F is called an *analytic* (C^k–*smooth*) *morphism of order* d for $s \in [a,b]$.

EXAMPLE. Consider the Sobolev scale $\left\{H^s(\mathbb{T}^d)\right\}$ and a smooth function $f(u,x)$. Then the map $F\colon u(x) \mapsto f(u(x),x)$ from X_a into itself is smooth if $a > \frac{d}{2}$, so the order of F is 0 on X_a.

Let $H\colon O_d \subset X_d \to \mathbb{R}$ be a C^k–smooth function, $k \geq 1$. Consider its *gradient map* with respect to the pairing $\langle \cdot, \cdot \rangle$

$$\begin{aligned} &\nabla H\colon O_d \longrightarrow X_{-d} \\ &\langle \nabla H(u), v \rangle = dH(u)v \quad \forall u \in X_d. \end{aligned}$$

The map ∇H is C^{k-1}–smooth.

2.2. Symplectic structures

Let $\{X_s\}$ a Hilbert scale and J its anti–selfadjoint automorphism of order d for $-\infty < s < +\infty$. Define $\overline{J} := -J^{-1}$, that is an anti–selfadjoint automorphism of order $-d$.

Define the 2–form

$$\alpha = \overline{J} dx \wedge dx$$

where $\overline{J} dx \wedge dx[\xi, \eta] := \left\langle \overline{J}\xi, \eta \right\rangle$. This defines a continuous skew–symmetric bilinear form on $X_r \times X_r$ for $r \geq -\frac{d}{2}$.

DEFINITION 18. The pair (X_r, α) is called *symplectic Hilbert space*. The pair $(\{X_s\}, \alpha)$ is called *symplectic Hilbert scale*.

DEFINITION 19. Let (X_s, α), (Y, β) be two symplectic Hilbert scales, with $\alpha = \overline{J} dx \wedge dx$ and $\beta = \overline{\Gamma} dy \wedge dy$. Let $F\colon O_s \to Y_{s-d_1}$ be a C^1–smooth morphism od order d_1 on $O_s \subset X_s$, for $a \leq s \leq b$. F is a *symplectic morphism* if $F^*\beta = \alpha$. Moreover, F is a *symplectomorphism* if it is a diffeomorphism.

2.3. Hamiltonian equation

Consider a C^1–smooth function H on a domain $O_d \subset X_d$. The *Hamiltonian vector field* V_H corresponding to H is defined as

$$\alpha(V_H(x), \xi) = -dH(x)\xi \qquad \forall \xi.$$

By definition, this implies

$$V_H(x) = J\nabla H(x).$$

If H is a C^1–smooth function on $O_d \times \mathbb{R}$, then V_H is the non–autonomous vector field $V_H(x,t) = J\nabla_x H(x,t)$, where ∇_x is the gradient in x. The corresponding *Hamiltonian equations* are

$$\dot{x} = J\nabla_x H(x,t) = V_H(x,t).$$

A partial differential equation is called a *Hamiltonian partial differential equation* (in short, HPDE) if, under a suitable choice of a symplectic Hilbert scale $(\{X_s\}, \alpha)$, a domain $O_d \subset X_d$ and a Hamiltonian H, it can be written in the previous form, $\dot{x} = V_H(x,t)$.

Now we give some examples of HPDEs.

Example (Non linear Schrödinger equation, NLS). Let $X_s = H^s(\mathbb{T}^n, \mathbb{C})$, treated as a real Hilbert space with scalar product $\langle u, v\rangle = \mathrm{Re}\int u\overline{v}\,dx$. Choose $Ju(x) = \mathrm{i}u(x)$, so that its order is 0. We choose

$$H(u) = \frac{1}{2}\int_{\mathbb{T}^n}\Big(|\nabla u|^2 + V(x)|u|^2 + g(x,u,\overline{u})\Big)\,dx,$$

where V, g are smooth real functions and $u = u(t,x)$, $x \in \mathbb{T}^n$. Then

$$\nabla H(u) = -\Delta u + V(x)u + \frac{\partial}{\partial \overline{u}}g$$

and so the Hamiltonian equations are

$$\dot{u} = \mathrm{i}\Big(-\Delta u + V(x)u + \frac{\partial}{\partial\overline{u}}g(x,u,\overline{u})\Big). \tag{2.1}$$

Example (1–dimensional NLS with Dirichlet boundary conditions). Let $X_s = H_0^s([0,\pi];\mathbb{C})$, $Ju(x) = iu(x)$ and

$$H(u) = \frac{1}{2}\int_0^\pi \Big(|u_x|^2 + V(x)|u|^2 + g(x,|u|^2)\Big)\,dx,$$

where g is smooth and 2π–periodic in x. Then, setting $f = \frac{\partial}{\partial|u|^2}g$, the Hamiltonian equation is

$$\dot{u} = \mathrm{i}\Big(-u_{xx} + V(x)u + f(x,|u|^2)u\Big) \tag{2.2}$$

with Dirichlet boundary conditions $u(0) = u(\pi) = 0$.

Example (Non linear wave equation, NLW). Choose $X_s = H^s(\mathbb{T}^n) \times H^s(\mathbb{T}^n)$, $\alpha = \overline{J}d\eta \wedge d\eta$, with $\eta = (u,v)$ and $J(u,v) = \overline{J}(u,v) = (-v,u)$, and the Hamiltonian function

$$H(u,v) = \int_{\mathbb{T}^n}\Big(\frac{1}{2}v^2 + \frac{1}{2}|\nabla u|^2 - f(x,u)\Big)\,dx.$$

The corresponding Hamiltonian equation is

$$\begin{cases} \dot{u} = -v \\ \dot{v} = -\Delta u - f'_u(x,u) \end{cases} \tag{2.3}$$

or also,

$$\ddot{u} = \Delta u + f'_u(x,u), \tag{2.4}$$

with $u = u(t,x)$, $x \in \mathbb{T}^n$.

EXAMPLE (Korteweg–de Vries equation, KdV). Consider the scale $\{X_s\}$ of the Sobolev spaces $H^s(S^1)$ of zero mean–value functions. Choose $J = \frac{\partial}{\partial x}$. We choose the Hamiltonian

$$H(u) = \int_0^{2\pi} \left(\frac{1}{8}u'(x)^2 + f(u)\right) dx,$$

where $f(u)$ is some analytic function. Then the corresponding Hamiltonian equation is

$$\dot{u}(t,x) = \frac{1}{4}u''' + \frac{\partial}{\partial x} f'(u).$$

The map V_H defines an analytic morphism of order 3 of the scale $\{X_s\}$ for $s > \frac{1}{2}$.

2.4. Some results

Consider two symplectic scales $(\{X_s\},\alpha)$ and $(\{Y_s\},\beta)$ with $\alpha = \overline{J}dx \wedge dx$ and $\beta = \overline{\Gamma}dy \wedge dy$. Assume for simplicity $\operatorname{ord} J = \operatorname{ord}\Gamma = d_J \geq 0$. Let $Phi \colon Q \to O$ be a C^1–smooth symplectic map between two domains in Y_d and X_d with $d \geq 0$. If d_J then we also assume that for any $|s| \leq d$ the linearized maps $\Phi_*(y)$, $y \in Y$, define linear maps $Y_s \to X_s$ which continuously depend on y.

The following theorem states that symplectic maps transform Hamiltonian equation to Hamiltonian.

THEOREM 2.1. *Let $\Phi \colon Q \to O$ be a symplectic map as above. Consider the Hamiltonian equation*

$$\dot{x} = J\nabla_x H(x,t) = V_H(x,t)$$

and assume that the vector field V_H defines a C^1–smooth map $V_H \colon O \times \mathbb{R} \to X_{d-d_1}$ of order $d_1 \leq 2d$ and is tangent to the map Φ (i.e. for every $y \in Q$ and for every t, the vector $V_H(\Phi(y),t)$ belongs to the range of the linearized map $\Phi_*(y)$). Then Φ transforms solutions of the Hamiltonian equation $\dot{y} = \Gamma\nabla_y K(y,t)$, where $K = H \circ \Phi$, to solution of $\dot{x} = J\nabla_x H(x,t)$.

COROLLARY 2.2. *If under the assumption of Theorem 2.1,* $\{X_s\} = \{Y_s\}$, $H \circ \Phi = H$ *and* $\Phi^*\alpha = \alpha$*, then* Φ *preserves the class of solution for the equation* $\dot{x} = J\nabla_x H(x,t)$.

In order to apply Theorem 2.1 we need some regular ways to construct symplectic transformations, so we enunciate the following result.

THEOREM 2.3. *Let* $(\{X_s\},\alpha)$ *be a symplectic Hilbert scale as above and* O *be a domain in* X_d*. Let* f *be a* C^1*–smooth function on* $O \times \mathbb{R}$ *such that the map* $V_f \colon O \times \mathbb{R} \to X_d$ *is Lipschitz in* (x,t) *and* C^1*–smooth in* x*. Let* O_1 *be a subdomain of* O*. Then the flow–maps* $X_t^\tau \colon (O_1,\alpha) \to (O,\alpha)$ *are symplectomorphism. If the map* V_f *is* C^k*–smooth or analytic, then the flow–maps are* C^k*–smooth or analytic as well.*

This theorem is usually applied when the flow–maps are close to the identity. In particular we have the following result.

THEOREM 2.4. *Under the assumption of Theorem 2.3, let* H *be a* C^1*–smooth function on* O*. Then*

$$\frac{d}{d\tau} H(X_t^\tau) = \{f,H\}(X_t^\tau), \qquad x \in O_1.$$

An immediate consequences of this theorem is that for an autonomous Hamiltonian equation $\dot{x} = J\nabla f(x)$, with $\operatorname{ord} J\nabla f = 0$, a C^1–smooth function H is an integral of motion (i.e. $H(x(t))$ is time–independent for any solution $x(t)$) if and only if $\{f,H\} = 0$.

2.5. The Birkhoff Normal Form Theorem

The Birkhoff theorem 1.6 does not trivially extend to infinite dimensional system because of the problem of small divisors. In the finite dimensional case, the set of integer vectors with modulus smaller than a given γ is finite, while in the infinite dimensional case this is no more true, since the denominators accumulates to zero.

DEFINITION 20. Given a multi-index $j = (j_1,\ldots,j_r)$, let $(j_{i_1}, j_{i_2},\ldots,j_{i_r})$ be a reordering of j such that

$$|j_{i_1}| \geq |j_{i_2}| \geq \ldots \geq |j_{i_r}|.$$

Define $\mu(j) := |j_{i_3}|$ and $S(j) := \mu(j) + ||j_{i_1}| - |j_{i_2}||$.

DEFINITION 21. Let $k \geq 3$ and

$$Q(z) = \sum_{l=0}^{k} \sum_{j\in\mathbb{Z}^l} a_j z_{j_1} \ldots z_{j_l}.$$

We say that Q has *localized coefficients* if there exists $\nu \in [0, +\infty)$ such that for any $N \geq 1$ there exists $C_N > 0$ such that for any choice of the indexes $j_1, \dots, j_r$

$$|a_j| \leq C_N \frac{\mu(j)^{N+\nu}}{S(j)^N}.$$

We need now a suitable nonresonance condition.

DEFINITION 22. Fix a positive integer r. The frequency vector ω is said to fulfill the property $(r - NR)$ if there exist $\gamma > 0$ and $\tau \in \mathbb{R}$ such that for any N large enough one has

$$\left| \sum_{j \geq 1} \omega_j K_j \right| \geq \frac{\gamma}{N^\tau},$$

for any $K \in \mathbb{Z}^\infty$ with $0 \neq |K| := \sum_j |K_j| \leq r + 2$, $\sum_{j > N} |K_j| \leq 2$.

Now we are ready to tate the Birkhoff Normal Form Theorem.

THEOREM 2.5 (Theorem 4 in [**Bam08**]). *Fix $r \geq 1$. Assume that the nonlinearity H_P has localized coefficients and that the frequencies fulfill the nonresonance condition $(r - NR)$. Then there exists a finite $s_r > 0$, a neighborhood $\mathcal{U}_{s_r}$ of the origin and a canonical transformation τ defined on $\mathcal{U}_{s_r}$ which puts the system in normal form up to order $r + 3$, namely*

$$H^{(r)} := H \circ \tau = H_0 + Z^{(r)} + R^{(r)}$$

where

(1) $Z^{(r)}$ and $R^{(r)}$ have localized coefficients,
(2) $Z^{(r)}$ is a polynomial of degree $r + 2$ which Poisson commutes with J_i for all i
(3) $R^{(r)}$ has a small vector field, namely

$$\|X_{R^{(r)}}(z)\|_{s_r} \leq C\|z\|_{s_2}^{r+2} \qquad \forall z \in \mathcal{U}_{s_r},$$

(4) one has

$$\|z - \tau(z)\|_{s_r} \leq C\|z\|_{s_r}^2 \qquad \forall z \in \mathcal{U}_{s_r}$$

and the same holds for the inverse τ^{-1}.

CHAPTER 3

KAM Theory for PDEs

In Chapter 1 we have seen the classical KAM theorem for finite dimensional system, that states that the most, with respect to the Lebesgue measure, of the invariant tori of a real analytic non–degenerate integrable system persists under sufficiently small and real analytic perturbation.

Starting from the Eighties of the last century, one of the most interesting research field for partial differential equations concerns its extension to infinite–dimensional systems in order to find periodic, quasi–periodic or almost–periodic solutions. The main difficulty arises from the fact that, when the number of frequencies tends to infinity, the small divisors tends to zero very rapidly, and so also the bound of admissible perturbation. As a conclusion, a simple extension of the classical KAM Theorem does not applied to any perturbation different from zero.

Essentially up to now there is no general KAM Theorem to handle the effects of small divisors for combinations of infinitely many frequencies in systems arising from PDE's. But in such systems there are also families of finite–dimensional elliptic invariant tori filled with quasi–periodic motions. A KAM Theorem for these tori can be formulated under the Kolmogorov and Melnikov's conditions as above, but noting that in this case these conditions an infinite number of frequencies are involved.

In order to prove the persistence of finite–dimensional tori in infinite dimensional systems, the first important results are due to Kuksin [**Kuk93**] and Wayne [**Way90**]. In this chapter we present two results due to Pöschel [**Pös96a**] and Berti–Biasco [**BB11**].

3.1. Setting and assumptions

Consider a family of integrable Hamiltonians

$$N = N(x, y, z, \overline{z}; \xi) := \omega(\xi) \cdot y + \Omega(\xi) \cdot z\overline{z}$$

defined on the phase space $\mathcal{P}^{a,p} := \mathbb{T}^n_s \times \mathbb{C}^n \times \ell^{a,p} \times \ell^{a,p}$, where $\mathbb{T}^n$ is the usual n–torus $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$, and $\ell^{a,p}$ is the Hilbert space of complex–valued sequences

$$\ell^{a,p} := \left\{ z = (z_1, z_1, \ldots) \colon \|z\|_{a,p}^2 := \sum_{j\geq 1} |z_j|^2 j^{2p} e^{2aj} < +\infty \right\}$$

with $a > 0, p > \frac{1}{2}$. The normal frequencies $\omega = (\omega_1, \ldots, \omega_n)$ and the tangential frequencies $\Omega = (\Omega_{n+1}, \Omega_{n+2}, \ldots)$ depend on m parameters $\xi \in \Pi \in \mathbb{R}^m$, $m \leq n$. The set Π is a compact set with positive Lebesgue measure. The associated symplectic structure is $dx \wedge dy + \mathrm{i} dz \wedge d\overline{z}$.

For each $\xi \in \Pi$ the n–torus $T_0 := \mathbb{T}^n \times \{0\} \times \{0\} \times \{0\}$ is an invariant n–dimensional torus with frequencies $\omega(\xi)$ and with an elliptic fixed in the normal space $z\overline{z}$ with proper frequencies $\Omega(\xi)$. Hence this torus is linearly stable and we call it an elliptic rotational torus with frequencies ω.

Consider the family of Hamiltonian

$$H = N + P,$$

where P is a small analytic perturbation. In this system, the torus in general does not persist due to resonances among the modes. The aim is to prove the persistence of a large family of n–dimensional linearly stable invariant tori forming a Cantor manifold, provided the perturbation is small enough.

In order to do this, we assume the following conditions.

(A1) Parameter dependence: The map $\omega \colon \Pi \to \mathbb{R}^n$, $\xi \mapsto \omega(\xi)$, is Lipschitz continuous.

(A2) For all the integer vector $(k, l) \times \mathbb{Z}^n \times \mathbb{Z}^\infty$ with $1 \leq |l| \leq 2$,

$$|\{\xi \in \Pi \colon \omega(\xi) \cdot k + \Omega(\xi) \cdot l = 0\}| = 0$$

and

$$\Omega(\xi) \cdot l \neq 0 \qquad \text{on } \Pi.$$

(B) Asymptotic behavior: There exist $d \geq 1$ and $\delta < d - 1$ such that

$$\Omega_j(\xi) = \overline{\Omega}_j + \Omega_j^*(\xi) \in \mathbb{R}, \qquad j \geq 1,$$

where $\overline{\Omega}_j = j^d + \ldots$ and $\Omega^* \colon \Pi \to \ell_\infty^{-\delta}$ is Lipschitz continuous, where ℓ_∞^p is the space of all real sequences w with finite norm $\|w\|_p := \sup_j |w_j| j^p$.

(C) Regularity: The perturbation P is real analytic in the space coordinates and Lipschitz in the parameters. Moreover, for any $\xi \in \Pi$, the Hamiltonian vector field $X_P = (P_y, -P_x, \mathrm{i}P_{\overline{z}}, -\mathrm{i}P_z)$ defines near T_0 a map

$$X_P \colon \mathcal{P}^{a,p} \longrightarrow \mathcal{P}^{a,\overline{p}}$$

with $\overline{p} \geq p$ if $d > 1$ or $\overline{p} > p$ if $d = 1$. Moreover, we assume $p - \overline{p} \leq \delta < d - 1$.

Notations. Consider an open neighborhood of the torus $\mathbb{T}_0$

$$\mathcal{D}(s,r) := \Big\{ |\mathrm{Im}\, x| < s, |y| < r^2, \|z\|_{a,p} + \|\overline{z}\|_{a,p} < r \Big\}$$

with $0 < s, r < 1$, where $|\cdot|$ is the sup–norm of complex–vectors.

Define the set

$$\mathcal{R}_{\eta,\nu} := \Big\{ \omega \in \mathbb{R}^n \colon |\omega \cdot k| \geq \frac{\eta}{1 + |k|^\tau}, \forall k \in \mathbb{Z}^n \setminus \{0\} \Big\}.$$

Given an analytic function f defined on $\mathcal{D}(s,r) \times \Pi$, define its sup–norm as

$$|f|_{s,r} := \sup_{(x,y,z,\overline{z};\xi) \in \mathcal{D}(s,r) \times \Pi} |f(x,y,z,\overline{z};\xi)|,$$

the Lipschitz semi–norm as

$$|f|^{\mathrm{lip}}_{s,r} := \sup_{\xi,\zeta \in \Pi, \xi \neq \zeta} \frac{|f(\cdot;\xi) - f(\cdot;\zeta)|_{s,r}}{|\xi - \zeta|}$$

and, for any $\lambda \geq 0$, the Lipschitz norm

$$|\cdot|^{\lambda}_{s,r} := |\cdot|_{s,r} + \lambda |\cdot|^{\mathrm{lip}}_{s,r}.$$

Set $w = (z, \overline{z})$. Any analytic function P can be developed in a totally convergent power series

$$P(x,y,z,\overline{z};\xi) = \sum_{i,j \geq 0} P_{ij}(x;\xi) y^i z^j \overline{z}^j$$

where $P_{ij}(x) := P_{ij}(x;\xi)$ are multilinear, symmetric and bounded maps. Identify $P_{10}(x) \in \mathcal{L}(\mathbb{C}^n, \mathbb{C})$ with the vector $P_{10}(x) = \partial_{y|y=0,w=0} P \in \mathbb{C}^n$ and $P_{01}(x) \in \mathcal{L}(\ell^{a,p}, \mathbb{C})$ with the vector $P_{01}(x) = \partial_{w|y=0,w=0} P \in \ell^{a,\overline{p}}$ writing

$$P_{10}(x) y = P_{10}(x) \cdot y \ \text{ and } \ P_{01}(x) w = P_{01}(x) \cdot w.$$

Identify the form $P_{02}(x) \in \mathcal{L}(\ell^{a,p} \times \ell^{a,\overline{p}}, \mathbb{C})$ with the operator $P_{02}(x) \in \mathcal{L}(\ell^{a,p}, \ell^{a,\overline{p}})$ writing

$$P_{02}(x) w^2 = P_{02}(x) w \cdot w.$$

Define

$$P_{\leq 2} := P_{00} + P_{01} w + P_{10} y + P_{02} w \cdot w.$$

Given $W = (X, Y, U, V)$ we define the weighted phase space norm as

$$|W|_r := |X| + \frac{1}{r^2} |Y| + \frac{1}{r} \Big(\|U\|_{a,p} + \|V\|_{a,p} \Big)$$

and $|W|_{r,\mathcal{D}(s,r)} = \sup_{\mathcal{D}(s,r)} |W|_r$.

Finally fix the following notations. Given $l \in \mathbb{Z}^\infty$ define

$$|l| := \sum_{j \geq 1} |l_j|, \qquad |l|_p := \sum_{j \geq 1} j^p |l_j|, \qquad \langle l \rangle_d := \max \left(1, \left| \sum_{j \geq 1} j^d l_j \right| \right).$$

Define the space

$$\ell_\infty^{-\delta} := \left\{ \Omega = (\Omega_1, \Omega_2, \dots) \in \mathbb{R}^\infty \colon |\Omega|_{-\delta} := \sup_{j \geq 1} j^{-\delta} |\Omega_j| < \infty \right\}$$

and the Lipschitz norm

$$|\Omega|_{-\delta}^{\lambda} := \sup_{\xi \in \Pi} |\Omega(\xi)|_{-\delta} + \lambda |\Omega|_{-\delta}^{\mathrm{lip}},$$

where the Lipschitz semi–norm is defined analogously as the previous one. Finally define the set

$$\mathcal{Z} := \{(k,l) \in \mathbb{Z}^n \times \mathbb{Z}^\infty \setminus \{(0,0)\} \colon |l| \leq 2\}.$$

In the case $d = 1$, define κ as the largest positive number such that

$$\frac{\Omega_i - \Omega_j}{i - j} = 1 + O\big(j^{-\kappa}\big), \qquad \text{for } i > j$$

uniformly on Π, and assume $-\delta < \kappa$ without loss of generality.

By assumptions (A1), (B) the Lipschitz semi–norm of the frequencies satisfy

$$|\omega|^{\mathrm{lip}} + |\Omega|_{-\delta}^{\mathrm{lip}} \leq M, \qquad \left|\omega^{-1}\right|^{\mathrm{lip}} \leq L.$$

for some finite $M, L > 0$.

3.2. A KAM Theorem by Pöschel

We first enunciate the result by Pöschel in [**Pös96a**]. This is divided in two parts, an analytic and a geometric one. The first states the existence of invariant tori under the assumption that a certain set of diophantine frequencies is not empty. The second assures that this is indeed the case.

THEOREM 3.1 (Theorem A in [**Pös96a**]). *Suppose that $H = N + P$ satisfies assumptions (A1), (A2), (B), (C) with $\xi \in \Pi \subset \mathbb{R}^n$, and*

$$\varepsilon := |X_P|_{r,\mathcal{D}(s,r)} + \frac{\gamma}{M} |X_P|_{r,\mathcal{D}(s,r)}^{\mathrm{lip}} \leq \alpha\gamma,$$

where $0 < \gamma \leq 1$ is another parameter and α depends on n, τ, s. Then there exist

(1) a Cantor set $\Pi_\gamma \subset \Pi$
(2) a Lipschitz continuous family of torus embeddings $\Phi \colon \mathbb{T}^n \times \Pi_\gamma \to \mathcal{P}^{a,\overline{p}}$
(3) and a Lipschitz continuous map $\omega_* \colon \Pi_\gamma \to \mathbb{R}^n$

such that, for each $\xi \in \Pi_\gamma$, the map Φ restricted to $\mathbb{T}^n \times \{\xi\}$ is a real analytic embedding of a rotational torus with frequencies $\omega_*(\xi)$ for the Hamiltonian H in ξ.

Each embedding is real analytic on $|\mathrm{Im}\, x| < \frac{s}{2}$ *and*

$$|\Phi - \Phi_0| + \frac{\gamma}{M}|\Phi - \Phi_0|^{\mathrm{lip}} \le c\frac{\varepsilon}{\gamma}$$

$$|\omega_* - \omega| + \frac{\gamma}{M}|\omega_* - \omega|^{\mathrm{lip}} \le c\varepsilon$$

uniformly, where $\Phi_0 \colon \mathbb{T}^n \times \Pi \to \mathbb{T}^n_0$ *is the trivial embedding and* $c \le \alpha^{-1}$ *depends on the same parameters as* α.

Moreover, there exist Lipschitz maps ω^ν *and* Ω^ν *on* Π *for any* $\nu \ge 0$ *satisfying* $\omega^0 = \omega$, $\Omega^0 = \Omega$ *and*

$$|\omega^\nu - \omega| + \frac{\gamma}{M}|\omega^\nu - \omega|^{\mathrm{lip}} \le c\varepsilon$$

$$|\Omega^\nu - \Omega| + \frac{\gamma}{M}|\Omega^\nu - \Omega|^{\mathrm{lip}}_{-\delta} \le c\varepsilon$$

such that $\Pi \setminus \Pi_\gamma \subset \bigcup \mathcal{R}^\nu_{kl}(\gamma)$, *where*

$$\mathcal{R}^\nu_{kl}(\gamma) := \left\{ \xi \in \Pi \colon |\omega^\nu(\xi) \cdot k + \Omega^\nu(\xi) \cdot l| < \gamma \frac{\langle l \rangle_d}{|k|^\tau} \right\}$$

and the union is taken over all $\nu \ge 0$ *and* $(k, l) \in Z$ *such that* $|k| > K_0 2^{\nu - 1}$ *for* $\nu \ge 1$ *with a constant* $K_0 \ge 1$ *depending only on* n, τ.

The KAM Theorem is proved by a Newton–type iteration procedure, which involves an infinite sequence of coordinate change, each of which is obtained as the time–1–map of a suitable Hamiltonian vector field, in order to make the size of the perturbation smaller and smaller. In doing this, the problem of small divisors arises so, at each step of the iterative process, we have to reduce the set of admissible parameters. The following theorem ensures that the set of admissible parameters is not empty at each step, providing its measure estimate.

Theorem 3.2 (Theorem B in [**Pös96a**]). *For* $\nu \ge 0$ *let* ω^ν *and* Ω^ν *be Lipschitz maps on* Π *satisfying*

$$|\omega^\nu - \omega|, |\Omega^\nu - \Omega|_{-\delta} \le \gamma, \qquad |\omega^\nu - \omega|^{\mathrm{lip}}, |\Omega^\nu - \Omega|^{\mathrm{lip}}_{-\delta} \le \frac{1}{2L},$$

and define the sets $\mathcal{R}^\nu_{kl}(\alpha)$ *as in Theorem 3.1 choosing* τ *as*

$$\tau \ge \begin{cases} n + 1 + \dfrac{2}{d-1} & \text{for } d > 1 \\ (n+3)\dfrac{\delta - 1}{\delta} & \text{for } d = 1. \end{cases}$$

Then there exists a finite subset $X \subset Z$ *and a constant* $\widetilde{c}$ *such that*

$$\left| \bigcup_{(k,l) \notin X} \mathcal{R}^\nu_{kl}(\alpha) \right| \le \widetilde{c}\rho^{n-1}\gamma^\mu, \qquad \text{with } \mu = \begin{cases} 1 & \text{for } d > 1 \\ \dfrac{\kappa}{\kappa + 1} & \text{for } d = 1, \end{cases}$$

for all sufficiently small γ*, with* $\rho := \operatorname{diam}\Pi$*. The constant* $\widetilde{c}$ *and the index set* X *are monotone functions of the domain* Π*: they do not increase for closed subsets of* Π*. In particular, for* $\delta \leq 0$*, we have that the set* X *is contained in* $\{(k,l)\colon 0 < |k| \leq 16LM\}$*.*

By slightly sharpening the smallness condition, we have that the frequency maps of Theorem 3.1 satisfy the hypothesis of Theorem 3.2, and we can conclude that the measure of all the sets $R^{\nu}_{kl}(\gamma)$ tends to 0. Then we have the following corollary.

COROLLARY 3.3 (Corollary C in [**Pös96a**]). *If the constant* α *in Theorem 3.1 is replaced by a smaller constant* $\widetilde{\alpha} \leq \frac{\alpha}{2LM}$ *depending on the set* X*, then*

$$|\Pi \setminus \Pi_\gamma| \leq \left|\bigcup \mathcal{R}^{\nu}_{kl}(\gamma)\right| \to 0 \qquad \textit{as } \gamma \to 0.$$

In particular, if $\delta \leq 0$ *then we can take* $\widetilde{\alpha} = \frac{\alpha}{2(LM)^{\tau+1}}$*.*

In the case of the nonlinear weave equation, since γ appears with exponent $\mu < 1$, the estimate in Theorem 3.2 is not sufficient to guarantee that the set of bad frequencies is smaller than the set of all frequencies, so we need the following better estimate.

THEOREM 3.4 (Theorem D in [**Pös96a**]). *Suppose that in Theorem 3.1 the unperturbed frequencies are affine functions of the parameters. Then*

$$|\Pi \setminus \Pi_\gamma| \leq \widetilde{c}\rho^{n-1}\gamma^{\widetilde{\mu}}, \ \textit{with } \widetilde{\mu} = \begin{cases} 1 & \textit{for } d > 1 \\ \dfrac{\kappa}{\kappa + 1 - \frac{\widetilde{\pi}}{4}} & \textit{for } d = 1, \end{cases}$$

for all sufficiently small γ*, where* $\widetilde{\pi}$ *is any number in* $0 \leq \widetilde{\pi} \leq \min(\overline{p} - p, 1)$ *and* $\widetilde{c}$ *depends also on* $\widetilde{\pi}$ *and* $\overline{p} - p$*.*

3.3. A KAM Theorem by Berti–Biasco

Now we enunciate a recent result by Berti–Biasco. The main differences between this result and the previous one by Pöschel are that the KAM smallness conditions are weaker and that the final Cantor set of parameters satisfying the Melnikov non–resonance conditions for the iterative KAM process is explicitly known in terms of the final frequencies only. As a consequence, we can completely separate the question of the existence of admissible non–resonant frequencies from the iterative construction of invariant tori.

Recalling all the previous definitions and notations, we can state the result.

THEOREM 3.5 (Theorem 5.1 in [**BB11**]). *Suppose that* $H = N + P$ *satisfies assumptions (A1), (B), (C). Let* $\gamma > 0$ *be a positive parameter and*

$$\Theta := \max\left\{1, |P_{11}|_s^\lambda, |P_{03}|_s^\lambda, \sum_{2i+j=4} \left|\partial_y^i \partial_w^j P\right|_{s,r}^\lambda, r\left|\partial_y^2 \partial_w P\right|_{s,r}^\lambda\right\} \text{ with } \lambda := \frac{\gamma}{M}$$

satisfies $\Theta \leq \frac{\sqrt{\gamma}}{3r}$. *Then there exists* $\alpha = \alpha(n, \tau, s)$ *such that, if one of the following KAM–conditions holds*

(H1) $\varepsilon_1 := \max\left\{\frac{|P_{00}|_s^\lambda}{r^2\gamma^2}, \frac{|P_{01}|_s^\lambda}{r\gamma^{3/2}}, \frac{|P_{10}|_s^\lambda}{\gamma}, \frac{|P_{02}|_s^\lambda}{\gamma}\right\} \leq \alpha$,

(H2) $\varepsilon_2 := \max\left\{\frac{|P_{00}|_s^\lambda}{r^2\gamma^{5/4}}, \frac{|P_{01}|_s^\lambda}{r\gamma^{3/2}}, \frac{|P_{10}|_s^\lambda}{\gamma}, \frac{|P_{02}|_s^\lambda}{\gamma}\right\} \leq \alpha$ *and* $|P_{11}|_s^\lambda \leq \frac{\gamma^{5/4}}{r}$,

(H3) $\varepsilon_3 := \max\left\{\frac{|P_{00}|_s^\lambda}{r^2\gamma^{\mu}}, \frac{|P_{01}|_s^\lambda}{r\gamma}, \frac{|P_{10}|_s^\lambda}{\gamma}, \frac{|P_{02}|_s^\lambda}{\gamma}\right\} \leq \alpha$ *and* $|P_{11}|_s^\lambda, |P_{03}|_s^\lambda \leq \frac{\gamma}{r}$, *with* $\mu = 1$ *if* $d > 1$ *and* $0 < \mu \leq 1$ *if* $d = 1$,

then the following hold

(1) *there exist Lipschitz frequencies* $\omega_\infty\colon \Pi \to \mathbb{R}^n$, $\Omega_\infty\colon \Pi \to \ell_\infty^{-d}$ *satisfying*

$$|\omega_\infty - \omega|^\lambda, |\Omega_\infty - \Omega|_{\overline{p}-p}^\lambda \leq \alpha^{-1}\gamma\varepsilon_i \tag{3.1}$$

with $|\omega_\infty|^{\text{lip}}, |\Omega_\infty|_{-\delta}^{\text{lip}} \leq 2M$

(2) *there exists a Lipschitz family of analytic symplectic maps*

$$\Phi\colon \mathcal{D}\left(\frac{s}{4}, \frac{r}{4}\right) \times \Pi_\infty \ni (x_\infty, y_\infty, w_\infty; \xi) \longmapsto (x, y, w) \in \mathcal{D}(s, r)$$

of the form $\Phi = I + \Psi$ *with* $\Psi \in E_{s/4}$, *where* Φ_∞ *will be defined later, such that*

$$H^\infty(\cdot;\xi) := H \circ \Phi(\cdot;\xi) = \omega_\infty(\xi) y_\infty + \Omega_\infty(\xi) z_\infty \overline{z}_\infty + P^\infty$$

has $P^\infty_{\leq 2} = 0$. *Moreover,*

$$\begin{cases} |P_{11}^\infty - P_{11}|_{s/4} \leq \alpha^{-1}\varepsilon_i\Big(|P_{11}|_s + \gamma^{p_a - \frac{1}{2}}\Big) \\ |P_{03}^\infty - P_{03}|_{s/4} \leq \alpha^{-1}\varepsilon_i\Big(|P_{03}|_s + |P_{11}|_s + \gamma^{p_a - \frac{1}{2}}\Big) \end{cases}$$

(3) *the map* Ψ *satisfies*

$$|x_{00}|_{s/4}^\lambda, \quad |y_{00}|_{s/4}^\lambda \frac{\gamma^{1-p_a}}{r^2}, \quad |y_{01}|_{s/4}^\lambda \frac{\gamma^{1-p_b}}{r}, \quad |y_{10}|_{s/4}^\lambda,$$
$$|y_{02}|_{s/4}^\lambda, \quad |w_{01}|_{s/4}^\lambda, \quad |w_{00}|_{s/4}^\lambda \frac{\gamma^{1-p_b}}{r} \quad \leq \alpha^{-1}\varepsilon_i$$

if $(Hi)_{i=1,2,3}$ *holds, where*

$$p_a := \begin{cases} 2 & \text{if } (H1) \\ \frac{5}{4} & \text{if } (H2) \\ 1 & \text{if } (H3) \end{cases} \qquad \text{and} \qquad p_b := \begin{cases} \frac{3}{2} & \text{if } (H1) \text{ or } (H2) \\ 1 & \text{if } (H3) \end{cases}.$$

(4) The Cantor set $\mathbf{\Pi}_\infty$ *is explicitly*

$$\mathbf{\Pi}_\infty := \begin{cases} \Pi_\infty & \text{if } (H1) \text{ or } (H2) \text{ or } (H3) \text{ with } d>1 \\ \Pi_\infty \cap \omega^{-1}(R_{\gamma^\mu,\tau}) & \text{if } (H3) \text{ with } d=1 \end{cases}$$

where

$$\Pi_\infty := \Big\{\xi \in \Pi\colon |\omega_\infty(\xi)\cdot k + \Omega_\infty(\xi)\cdot l| \geq 2\gamma \frac{\langle l\rangle_d}{1+|k|^\tau}, \quad \forall (k,l)\in \mathbb{Z}^n\times\mathbb{Z}^\infty \setminus \{0\}, |l|\leq 2\Big\}.$$

Then, for every $\xi \in \mathbf{\Pi}_\infty$*, the map* $x_\infty \mapsto \Phi(x_\infty, 0, 0; \xi)$ *is a real analytic embedding of an elliptic, diophantine, n–dimensional torus with frequencies* $\omega_\infty(\xi)$ *for the system with Hamiltonian* H*.*

Now we make some comparison with Theorem 3.1. First, we note that the KAM condition in Theorem 3.1 is

$$\gamma^{-1}|X_P|^\lambda_{s,r} \leq \text{constant},$$

with $\lambda = \frac{\gamma}{M}$, that implies (H3), but the other two conditions (H1), (H2) are not enough to guarantee the convergence of the iterative scheme in Theorem 3.1. In the case $d = 1$ condition (H3) is strictly weaker than the KAM condition in Pöschel, since $\mu \leq 1$. This allows to prove the result of quasi–periodic solutions for the nonlinear wave equation in [**Pös96b**] without Theorem 3.4.

Moreover, as said before, the Cantor set Π_∞ depends *only* on the final frequencies $(\omega_\infty, \Omega_\infty)$. We note that a priori it can be empty, and in this case the iterative process stops after a finite number of steps and no invariant tori survives. But $\omega_\infty, \Omega_\infty$ and so Π_∞ are however well defined.

Note also that we do not claim that the final frequencies satisfy the second order Melnikov non–resonance condition as in Theorem 3.1, but we state that if the parameter ξ belongs to Π_∞ then the torus is preserved.

We now give the measure estimate for the set Π_∞.

THEOREM 3.6 (Theorem 5.2 in [**BB11**]). *Let* $\omega\colon \Phi \to \omega(\Pi)$ *be a lipeomorphism (i.e. homeomorphism which is Lipschitz in both directions) with*

$$\left|\omega^{-1}\right|^{\text{lip}} \leq L, \qquad \varepsilon_i \leq \frac{\alpha}{2LM}.$$

If

$$\Omega(\xi)\cdot l \neq 0 \quad \forall |l| = 1,2, \, \forall \xi \in \Pi \tag{3.2}$$

and

$$|\{\xi \in \Pi\colon \omega(\xi)\cdot k + \Omega(\xi)\cdot l = 0\}| = 0 \tag{3.3}$$

for any $(k,l) \in \mathbb{Z}^n\times\mathbb{Z}^\infty\setminus\{(0,0)\}, |l|\leq 2$*, then* $|\Pi\setminus\mathbf{\Pi}_\infty| \to 0$ *as* $\alpha \to 0$*.*

Moreover, if $\omega(\xi), \Omega(\xi)$ *are affine functions of* ξ *then*

$$|\Pi \setminus \mathbf{\Pi}_\infty| \leq C\rho^{n-1}\gamma^\mu \qquad \textit{with } \rho := \textit{diam}(\Pi).$$

CHAPTER 4

Degenerate KAM theory for PDEs

This chapter deals with degenerate KAM theory for lower dimensional elliptic tori of PDEs, in particular when the frequencies of the linearized system depend on *one* parameter only.

We extend to partial differential equations the results due to Rüssmann [**Rüs01**] in the context of finite dimensional systems, see Section 4.1 for the precise statements of the main theorems, and we give an application to the nonlinear wave equation, see Section 4.3.

In Chapter 1 we gave an idea of the result in [**Rüs01**] and of its proof. We recall that the main point is to deduce quantitative non–degeneracy property from the qualitative weakly non–degeneracy assumption, using property of analytic functions.

For infinite dimensional systems, the main difficulty in extending the approach of Rüssmann is met at this step, because one has to bound the maximal order of the zeros of *infinitely* many analytic functions, a fact which is generically impossible. Here we exploit the asymptotic growth of the frequencies to reduce the effective number of functions to a finite one. This idea allows to deduce a quantitative non–resonant property of the kind of the second order Melnikov non–resonance conditions, typical of infinite dimensional KAM theory, see Proposition 4.3.

Concerning the other steps, we avoid the Rüssmann construction of chains, making use of the recent formulation of the KAM theorem in Berti-Biasco [**BB11**]. As seen in Chapter 3, an advantage of this formulation is an explicit characterization of the Cantor set of parameters which satisfy the Melnikov non–resonance conditions at all the steps of the KAM iteration, in terms of the *final* frequencies only. This approach *completely* separates the question of the existence of admissible non–resonant frequencies from the iterative construction of the invariant tori. This procedure considerably simplifies the measure estimates (also for finite dimensional systems), as it allows to perform them only at the final step and not at each step of the iteration, see Section 4.2.

We apply these abstract results to nonlinear wave (NLW) equations with Dirichlet boundary conditions

$$u_{tt} - u_{xx} + V(x)u + \xi u + f(x,u) = 0$$

requiring *only* $f(x,u) = O(u^2)$. Using the mass $\xi \in \mathbb{R}$ as a parameter we prove in Theorem 4.2 the persistence of Cantor families of small amplitude elliptic invariant tori of NLW. This result generalizes the one in [**Pös96b**], valid for $f(x,u) = u^3+$ higher order terms, to arbitrary analytic nonlinearities. Actually, in [**Pös96b**] the fourth order Birkhoff normal form of NLW is non–degenerate and the action–to–frequency map is a diffeomorphism. For general nonlinearities this property could be hard to verify, if ever true. The use of degenerate KAM theory allows to avoid this computation and then it is more versatile.

Finally we recall that a KAM theorem for degenerate PDEs was already proved by Xu–You–Qiu [**XYQ96**] which extended to the infinite dimensional case the method introduced in [**XYQ97**]. The main difference is that such authors assume a quantitative (weak) non–degeneracy assumption whose verification is usually very hard. On the contrary our non–degeneracy assumption (which follows Rüssmann) is quite easy to be verified. In particular, since it is based on properties of analytic functions it is enough to verify it for one value of the parameter, a task usually not very difficult.

This chapter is organized as follows: in Section 4.1 we present the main results. In Section 4.2 we prove the measure estimates. In Section 4.3 we consider the application to the nonlinear wave equation. Finally in section 4.4 we deduce the quantitative non–resonance condition (4.13) from the qualitative non–resonance condition (ND) and the analyticity and asymptotic behavior of the linear frequencies, see assumption (A).

Notations. For $l \in \mathbb{Z}^\infty$ define the norms

$$|l| := \sum_j |l_j|, \quad |l|_\delta := \sum_j j^\delta |l_j|, \quad \langle l \rangle_d := \max\left\{1, \left|\sum_j j^d l_j\right|\right\}.$$

Given $a, b \in \mathbb{R}^M$, $M \leq +\infty$, denote the scalar product $\langle a, b\rangle := \sum_{j=1}^M a_j b_j$.

We define the set

$$\mathcal{Z}_N := \big\{(k,l) \in \mathbb{Z}^N \times \mathbb{Z}^\infty \setminus (0,0) \colon |l| \leq 2\big\} \tag{4.1}$$

and we split $\mathcal{L} := \{l \in \mathbb{Z}^\infty \colon |l| \leq 2\}$ as the union of the following four disjoint sets

$$\begin{aligned} &\mathcal{L}_0 := \{l = 0\}, \quad \mathcal{L}_1 := \{l = e_j\}, \\ &\mathcal{L}_{2+} := \{l = e_i + e_j \text{ for } i \neq j\}, \quad \mathcal{L}_{2-} := \{l = e_i - e_j \text{ for } i \neq j\}, \end{aligned} \tag{4.2}$$

where $e_i := (0, \dots 0, \underbrace{1}_{i-th}, 0, \dots)$ and $i, j \geq N+1$.

Given a map $\Omega\colon \mathcal{I} \ni \xi \mapsto \Omega(\xi) \in \mathbb{R}^\infty$ we define the norm $|\Omega|_{-\delta} := \sup_{\xi \in \mathcal{I}} \sup_j |\Omega_j| j^{-\delta}$ and the C^μ-norm, $\mu \in \mathbb{N}$, as

$$|\Omega|_{-\delta}^{C^\mu} := \sum_{\nu=0}^{\mu} \left| \frac{d^\nu}{d\xi^\nu} \Omega(\xi) \right|_{-\delta} .$$

The $|\ |^{C^\mu}$ norm of a map $\omega : \mathcal{I} \to \mathbb{R}^N$, $N < \infty$, is defined analogously.

4.1. Statement of the main results

Fix an integer $N \geq 1$ and consider the phase space

$$\mathcal{P}^{a,p} := \mathbb{T}^N \times \mathbb{R}^N \times \ell^{a,p} \times \ell^{a,p} \quad \ni (x, y, z, \overline{z})$$

for some $a > 0$, $p > 1/2$, where $\mathbb{T}^N$ is the usual N-torus and $\ell^{a,p}$ is the Hilbert space of complex valued sequences $z = (z_1, z_2, \dots)$ such that

$$\|z\|_{a,p}^2 := \sum_{j \geq 1} |z_j|^2 j^{2p} e^{2aj} < +\infty ,$$

endowed with the symplectic structure $\sum_{j=1}^N dx_j \wedge dy_j + \mathrm{i} \sum_{j \geq N+1} dz_j \wedge d\overline{z}_j$.

Consider a family of Hamiltonians

$$H := Z + P \tag{4.3}$$

depending on *one* real parameter ξ varying in a compact set $\mathcal{I} \subset \mathbb{R}$, where Z is the normal form

$$Z := \sum_{j=1}^{N} \omega_j(\xi) y_j + \sum_{j \geq N+1} \Omega_j(\xi) z_j \overline{z}_j , \tag{4.4}$$

with frequencies $\omega = (\omega_1, \dots, \omega_N) \in \mathbb{R}^N$, $\Omega = (\Omega_{N+1}, \Omega_{N+2}, \dots) \in \mathbb{R}^\infty$, real analytic in ξ, and P is a small perturbation, also real analytic in ξ.

The equations of motion of the unperturbed system Z are

$$\dot{x} = \omega(\xi), \quad \dot{y} = 0, \quad \dot{z} = \mathrm{i}\Omega(\xi) z, \quad \dot{\overline{z}} = -\mathrm{i}\Omega(\xi) \overline{z} .$$

For each $\xi \in \mathcal{I}$ the torus $\mathcal{T}_0^N = \mathbb{T}^N \times \{0\} \times \{0\} \times \{0\}$ is an invariant N–dimensional torus for Z with frequencies $\omega(\xi)$ and with an elliptic fixed point in its normal space, described by the $z\overline{z}$-coordinates, with frequencies $\Omega(\xi)$. The aim is to prove the persistence of a large family of such N–dimensional elliptic invariant tori in the complete Hamiltonian system, provided the perturbation P is sufficiently small.

To this end we shall use the abstract KAM theorem in [**BB11**]. An advantage of its formulation is an explicit characterization of the Cantor set of parameters which satisfy the Melnikov non–resonance conditions at all the steps of the KAM iteration, in terms of the *final* frequencies only, see (4.9). This approach *completely* separates the question of the existence of admissible non–resonant frequency vectors from the iterative construction of N–dimensional invariant tori.

We now state a simplified version of the KAM theorem in [**BB11**] sufficient for the applications of this paper.

4.1.1. KAM theorem. We assume:

(A) *Analyticity and Asymptotic condition*: There exist $d \geq 1$, $\delta < d-1$, $0 < \eta < 1$ fixed, and functions $\nu_j : \mathcal{I} \to \mathbb{R}$ such that

$$\Omega_j(\xi) = j^d + \nu_j(\xi) j^\delta, \qquad j \geq N+1\,,$$

where each $\nu_j(\xi)$ extends to an analytic function on the complex neighborhood of $\mathcal{I}$

$$\mathcal{I}_\eta := \bigcup_{\xi \in \mathcal{I}} \left\{\xi' \in \mathbb{C} \colon \left|\xi - \xi'\right| < \eta\right\} \subseteq \mathbb{C}.$$

Also the function $\omega \colon \mathcal{I} \to \mathbb{R}^N$ has an analytic extension on $\mathcal{I}_\eta$. Moreover there exists $\Gamma \geq 1$ such that

$$\sup_{\mathcal{I}_\eta} \sup_j |\nu_j(\xi)| \leq \Gamma\ , \quad \sup_{\mathcal{I}_\eta} |\omega(\xi)| \leq \Gamma\ .$$

Consider the complexification of $\mathcal{P}^{a,p}$ and define a complex neighborhood $\mathcal{D}_{a,p}(s,r)$ of the torus $\mathcal{T}_0^N$ by

$$\mathcal{D}_{a,p}(s,r) := \left\{|\mathrm{Im}\, x| < s, |y| < r^2, \|z\|_{a,p} + \|\overline{z}\|_{a,p} < r\right\} \tag{4.5}$$

for some $s, r > 0$, where $|\cdot|$ denotes the max–norm for complex vectors.

For $W = (X,Y,U,V) \in \mathbb{C}^N \times \mathbb{C}^N \times \ell^{a,p}(\mathbb{C}) \times \ell^{a,p}(\mathbb{C})$, define the weighted phase space norm

$$\|W\|_{p,r} := |X| + r^{-2}|Y| + r^{-1}\|U\|_{a,p} + r^{-1}\|V\|_{a,p}\,.$$

Finally set

$$\mathcal{E} := \mathcal{I}_\eta \times D_{a,p}(s,r)\,.$$

(R) *Regularity condition*: There exist $s > 0, r > 0$ such that, for each $\xi \in \mathcal{I}$, the Hamiltonian vector field $X_P := (\partial_y P, -\partial_x P, \mathrm{i}\partial_{\bar{z}} P, -\mathrm{i}\partial_z P)$ is a real analytic map

$$X_P \colon \mathcal{D}_{a,p}(s,r) \longrightarrow \mathcal{P}^{a,\overline{p}}, \qquad \begin{cases} \overline{p} \geq p & \text{for } d > 1 \\ \overline{p} > p & \text{for } d = 1 \end{cases}$$

with $p - \overline{p} \leq \delta < d-1$, real analytic in $\xi \in \mathcal{I}_\eta$ and

$$\|X_P\|_{r,\bar{p},\mathcal{E}} := \sup_{\mathcal{E}} \|X_P\|_{\bar{p},r} < +\infty\,.$$

KAM Theorem. [**BB11**] *Consider the Hamiltonian system $H = Z + P$ on the phase space $\mathcal{P}^{a,p}$. Assume that the frequency map of the normal form Z is analytic and satisfies condition (A). Let $9r^2 < \gamma < 1$. Suppose the perturbation P satisfies (R) and*

$$\sum_{2i+j_1+j_2=4} \sup_{\mathcal{E}} |\partial_y^i \partial_z^{j_1} \partial_{\bar{z}}^{j_2} P| \leq \frac{\sqrt{\gamma}}{3r}\,. \tag{4.6}$$

Then there is $\epsilon_* > 0$ such that, if the KAM–condition

$$\varepsilon := \gamma^{-1} \| X_P \|_{r,\bar{p},\mathcal{E}} \leq \epsilon_* \tag{4.7}$$

holds, then

1. *there exist C^∞-maps $\omega^* \colon \mathcal{I} \to \mathbb{R}^N$, $\Omega^* \colon \mathcal{I} \to \ell_\infty^{-d}$, satisfying, for any $\mu \in \mathbb{N}$,*

$$|\omega^* - \omega|^{C^\mu} \leq M(\mu)\varepsilon\gamma^{1-\mu}, \quad |\Omega^* - \Omega|_{-\delta}^{C^\mu} \leq M(\mu)\varepsilon\gamma^{1-\mu} \tag{4.8}$$

   for some constant $M(\mu) > 0$,

2. *there exists a smooth family of real analytic torus embeddings*

$$\Phi : \mathbb{T}^N \times \mathcal{I}^* \to \mathcal{P}^{a,\bar{p}}$$

   where $\mathcal{I}^*$ is the Cantor set

$$\mathcal{I}^* := \left\{ \xi \in \mathcal{I} \,:\, |\, \langle k, \omega^*(\xi)\rangle + \langle l, \Omega^*(\xi)\rangle \,| \geq \frac{2\gamma\langle l\rangle_d}{1+|k|^\tau}\,,\ \forall (k,l) \in \mathcal{Z}_N \right\}, \tag{4.9}$$

   such that, for each $\xi \in \mathcal{I}^*$, the map Φ restricted to $\mathbb{T}^N \times \{\xi\}$ is an embedding of a rotational torus with frequencies $\omega^*(\xi)$ for the Hamiltonian system H, close to the trivial embedding $\mathbb{T}^N \times \mathcal{I} \to \mathcal{T}_0^N$.

Remark. The KAM Theorem 5.1 in [**BB11**] provides also explicit estimates on the map Φ and a normal form in an open neighborhood of the perturbed torus.

Remark. The above KAM theorem follows by Theorem 5.1 in [**BB11**] and remark 5.1, valid for Hamiltonian analytic also in ξ. Actually (4.6), (4.7) and $9r^2 < \gamma < 1$ imply the assumptions (5.5) and (H3) of Theorem 5.1 of [**BB11**]. Estimate (4.8) is (5.15) in [**BB11**].

Remark. The main difference between the above KAM theorem and those in Kuksin [**Kuk93**] and Pöschel [**Pös96a**], concerns, for the assumptions, the analytic dependence of H in the parameters ξ, which is only Lipschitz in [**Kuk93**], [**Pös96a**]. For the results, the main difference is the explicit characterization of the Cantor set $\mathcal{I}^*$. Note that we do *not* only claim that the frequencies of the preserved torus satisfy the second order Melnikov non-resonance conditions, fact already proved in [**Pös96a**]. The above KAM

Theorem states that also the *converse* is true: *if* the parameter ξ belongs to $\mathcal{I}^*$, then the KAM torus with frequencies $\omega^*(\xi)$ is preserved.

The main result of the next section proves that $\mathcal{I}^*$ is non–empty, under some weak non–degeneracy assumptions.

4.1.2. The measure estimates. We first give the following definition.

DEFINITION 23. A function $f = (f_1, \ldots, f_M) \colon \mathcal{I} \to \mathbb{R}^M$ is said to be *non–degenerate* if for any vector $(c_1, \ldots, c_M) \in \mathbb{R}^M \setminus \{0\}$ the function $c_1 f_1 + \ldots + c_M f_M$ is not identically zero on $\mathcal{I}$.

We assume:

(ND) *Non–degeneracy condition*: The frequency map (ω, Ω) satisfies
 i) $(\omega, 1) \colon \mathcal{I} \to \mathbb{R}^N \times \mathbb{R}$ is non–degenerate
 ii) for any $l \in \mathbb{Z}^\infty$ with $0 < |l| \le 2$ the map $(\omega, \langle l, \Omega\rangle) \colon \mathcal{I} \to \mathbb{R}^N \times \mathbb{R}$ is non–degenerate.

REMARK. Condition *i*) implies that $\omega \colon \mathcal{I} \to \mathbb{R}^N$ is non–degenerate. Actually *i*) means that, for any $(c_1, \ldots, c_N) \in \mathbb{R}^N \setminus \{0\}$, the function $c_1\omega_1 + \ldots + c_N\omega_N$ is not identically constant on $\mathcal{I}$.

REMARK. The non–degeneracy of the first derivative of the frequency map (ω', Ω'), namely
 i') $\omega' \colon \mathcal{I} \to \mathbb{R}^N$ is non–degenerate
 ii') for any $l \in \mathbb{Z}^\infty$ with $0 < |l| \le 2$ the map $(\omega', \langle l, \Omega'\rangle) \colon \mathcal{I} \to \mathbb{R}^N \times \mathbb{R}$ is non–degenerate,

implies (ND).

THEOREM 4.1. **(Measure estimate)** *Assume that the frequency map (ω, Ω) fulfills assumptions (A) and (ND). Take*

$$M(\mu_0)\varepsilon\gamma^{1-\mu_0} \le \beta/4\,, \quad M(\mu_0+1)\varepsilon\gamma^{-\mu_0} \le 1\,, \tag{4.10}$$

where $\mu_0 \in \mathbb{N}$, $\beta > 0$ are defined in (4.13) *and $M(\mu_0)$ in* (4.8)*. Then there exist constants τ, $\gamma_* > 0$, $\mu_* \ge \mu_0$, depending on d, N, μ_0, β, η such that*

$$|\mathcal{I} \setminus \mathcal{I}^*| \le (1 + |\mathcal{I}|)\Big(\frac{\gamma}{\gamma_*}\Big)^{\frac{1}{\mu_*}}$$

for all $0 < \gamma \le \gamma_*$.

In [**Rüs01**] the constant β is called the “amount of non–degeneracy” and μ_0 the “index of non–degeneracy”.

4.1.3. Application: wave equation. The previous results apply to the nonlinear wave equation with Dirichlet boundary conditions

$$\begin{cases} u_{tt} - u_{xx} + V(x)u + \xi u + f(x,u) = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{4.11}$$

where $V(x) \geq 0$ is an analytic, 2π-periodic, even potential $V(-x) = V(x)$, the mass ξ is a real parameter on an interval $\mathcal{I} := [0, \xi_*]$, the nonlinearity $f(x,u)$ is real analytic, odd in the two variables, i.e. for all $(x,u) \in \mathbb{R}^2$,

$$f(-x,-u) = -f(x,u)\,,$$

and

$$f(x,0) = (\partial_u f)(x,0) = 0\,. \tag{4.12}$$

For every choice of the indices $\mathcal{J} := \{j_1 < j_2 < \ldots < j_N\}$ the linearized equation $u_{tt} - u_{xx} + V(x)u + \xi u = 0$ possesses the quasi–periodic solutions

$$u(t,x) = \sum_{h=1}^{N} A_h \cos(\lambda_{j_h} t + \theta_h)\, \phi_{j_h}(x)$$

where $A_h, \theta_h \in \mathbb{R}$, and ϕ_j, resp. $\lambda_j^2(\xi)$, denote the simple Dirichlet eigenvectors, resp. eigenvalues, of $-\partial_{xx} + V(x) + \xi$. For $V(x) \geq 0$ (that we can assume with no loss of generality), all the Dirichlet eigenvalues of $-\partial_{xx} + V(x)$ are strictly positive.

THEOREM 4.2. *Under the above assumptions, for every choice of indexes $\mathcal{J} := \{j_1 < j_2 < \ldots < j_N\}$, there exists $r_* > 0$ such that, for any $A = (A_1, \ldots, A_N) \in \mathbb{R}^N$ with $|A| =: r \leq r_*$, there is a Cantor set $\mathcal{I}^* \subset \mathcal{I}$ with asymptotically full measure as $r \to 0$, such that, for all the masses $\xi \in \mathcal{I}^*$, the nonlinear wave equation* (4.11) *has a quasi–periodic solution of the form*

$$u(t,x) = \sum_{h=1}^{N} A_h \cos(\widetilde{\lambda}_h t + \theta_h)\, \phi_{j_h}(x) + o(r),$$

where $o(r)$ is small in some analytic norm and $\widetilde{\lambda}_h - \lambda_{j_h} \to 0$ as $r \to 0$.

4.2. Proof of Theorem 5.1

The first step is to use the analyticity of the linear frequencies to transform the non–degeneracy assumption (ND) into a quantitative non-resonance property, extending Rüssmann's Lemma 18.2 in [**Rüs01**] to infinite dimensions.

PROPOSITION 4.3. *Let $(\omega, \Omega) : \mathcal{I} \mapsto \mathbb{R}^N \times \mathbb{R}^\infty$ satisfy assumptions (A) and (ND) on $\mathcal{I}$. Then there exist $\mu_0 \in \mathbb{N}$ and $\beta > 0$ such that*

$$\max_{0 \le \mu \le \mu_0} \left| \frac{d^\mu}{d\xi^\mu} (\langle k, \omega(\xi) \rangle + \langle l, \Omega(\xi) \rangle) \right| \ge \beta(|k| + 1) \tag{4.13}$$

for all $\xi \in \mathcal{I}$, $(k, l) \in \mathcal{Z}_N$.

Technically, this is the most difficult part of the paper and its proof is developed in Section 4.4.

As a Corollary of Proposition 4.3 and by (4.8), also the final frequencies (ω^*, Ω^*) satisfy a non–resonance property similar to (4.13).

LEMMA 4.4. *Assume $M(\mu_0)\varepsilon\gamma^{1-\mu_0} \le \beta/4$, where μ_0 and β are defined in Proposition 4.3 and $M(\mu_0)$ is the constant in* (4.8). *Then*

$$\max_{0 \le \mu \le \mu_0} \left| \frac{d^\mu}{d\xi^\mu} \langle k, \omega^*(\xi) \rangle + \langle l, \Omega^*(\xi) \rangle \right| \ge \frac{\beta}{2}(|k| + 1) \tag{4.14}$$

for all $\xi \in \mathcal{I}$ and $(k, l) \in \mathcal{Z}_N$.

PROOF. By (4.13) and (4.8) we get, for all $0 \le \mu \le \mu_0$,

$$\begin{aligned}
\left| \frac{d^\mu}{d\xi^\mu} \langle k, \omega^*(\xi) \rangle + \langle l, \Omega^*(\xi) \rangle \right| \quad &\ge \quad \left| \frac{d^\mu}{d\xi^\mu} \langle k, \omega(\xi) \rangle + \langle l, \Omega(\xi) \rangle \right| \\
&- \quad \left| \frac{d^\mu}{d\xi^\mu} \langle k, \omega^*(\xi) - \omega(\xi) \rangle + \langle l, \Omega^*(\xi) - \Omega(\xi) \rangle \right| \\
&\ge \quad \beta(|k| + 1) - 2(|k| + 1) M(\mu_0) \varepsilon \gamma^{1-\mu} \\
&\ge \quad (\beta/2)(|k| + 1)
\end{aligned}$$

since $M(\mu_0)\varepsilon\gamma^{1-\mu_0} \le \beta/4$. $\square$

We now proceed with the proof of Theorem 4.1. By (4.9) we have

$$\mathcal{I} \setminus \mathcal{I}^* \subset \bigcup_{(k,l) \in \mathcal{Z}_N} \mathcal{R}_{kl}(\gamma) \tag{4.15}$$

with resonant regions

$$\mathcal{R}_{kl}(\gamma) := \left\{ \xi \in \mathcal{I} \colon \frac{|\langle k, \omega^*(\xi) \rangle + \langle l, \Omega^*(\xi) \rangle|}{1 + |k|} < \frac{2\gamma}{1 + |k|^{\tau+1}} \langle l \rangle_d \right\}.$$

In the following we assume $0 < \gamma \le 1/8$.

LEMMA 4.5. *There is $L_* > 1$ such that*

$$\langle l \rangle_d \ge \max\{L_*, 8\Gamma|k|\} \quad \Longrightarrow \quad \mathcal{R}_{kl}(\gamma) = \emptyset\,.$$

PROOF. The asymptotic assumption (A) and (4.8) imply that

$$\frac{\langle l, \Omega^* \rangle}{\langle l \rangle_d} \to 1 \ \text{ as } \ \langle l \rangle_d \to +\infty\,, \ \text{ uniformly in } \xi \in \mathcal{I}\,.$$

So $|\langle l, \Omega^*\rangle| \geq \langle l\rangle_d/2$ for $\langle l\rangle_d \geq L_* > 1$. If $|k| \leq (1/8\Gamma)\langle l\rangle_d$ then $\mathcal{R}_{kl}(\gamma)$ is empty, because, for all $\xi \in \mathcal{I}$,

$$|\langle k, \omega^*(\xi)\rangle + \langle l, \Omega^*(\xi)\rangle| \geq \frac{\langle l\rangle_d}{2} - 2\Gamma|k| \geq 2\gamma\langle l\rangle_d \geq 2\gamma\langle l\rangle_d \frac{1+|k|}{1+|k|^{\tau+1}}$$

provided $0 < \gamma \leq 1/8$. $\square$

As a consequence, in the following we restrict the union in (4.15) to $\langle l\rangle_d < \max\{L_*, 8\Gamma|k|\}$.

LEMMA 4.6. *There exists $B := B(\mu_0, \beta, \omega, \Omega, \eta) > 0$ such that, for any $(k, l) \in \mathcal{Z}_N$ satisfying $\langle l\rangle_d < \max\{L_*, 8\Gamma|k|\}$ and for all γ with*

$$0 < \gamma < \frac{\beta}{8(\mu_0+1)\max\{L_*, 8\Gamma\}}, \tag{4.16}$$

then

$$|\mathcal{R}_{kl}(\gamma)| \leq B(1+|\mathcal{I}|)\alpha^{\frac{1}{\mu_0}} \qquad \text{where} \qquad \alpha := \frac{2\gamma}{1+|k|^{\tau+1}}\langle l\rangle_d\,. \tag{4.17}$$

PROOF. We use Theorem 17.1 in [**Rüs01**]. The C^∞–function

$$g^*_{kl}(\xi) := \frac{\langle k, \omega^*(\xi)\rangle + \langle l, \Omega^*(\xi)\rangle}{1+|k|}$$

satisfies, by (4.14),

$$\min_{\xi\in\mathcal{I}} \max_{0\leq\mu\leq\mu_0} \left|\frac{d^\mu}{d\xi^\mu} g^*_{kl}(\xi)\right| \geq \frac{\beta}{2}\,.$$

Moreover $\langle l\rangle_d < \max\{L_*, 8\Gamma|k|\}$ and (4.16) imply

$$\alpha < \max\{2L_*, 16\Gamma\}\gamma < \frac{\beta}{4(\mu_0+1)}\,.$$

Then the assumptions of Theorem 17.1 in [**Rüs01**] are satisfied and so

$$|\mathcal{R}_{kl}(\gamma)| \leq B(\mu, \beta, \eta)(1+|\mathcal{I}|)\alpha^{\frac{1}{\mu_0}} |g^*_{kl}|^{\mu_0+1}_\eta$$

where

$$|g^*_{kl}|^{\mu_0+1}_\eta := \sup_{\xi\in\mathcal{I}_\eta\cap\mathbb{R}} \max_{0\leq\nu\leq\mu_0+1} \left|\frac{d^\nu}{d\xi^\nu} g^*_{kl}(\xi)\right|.$$

By (4.10), (4.8) and $\langle l\rangle_d \leq \max\{L_*, 8\Gamma|k|\}$, we have that the norm $|g^*_{kl}|^{\mu_0+1}_\eta$ is controlled by a constant depending on ω, Ω and this implies (4.17). $\square$

Now the measure estimate proof continues as in [**Pös96a**].

LEMMA 4.7. *Assume $d > 1$, and*

$$\tau > \mu_0\left(N + \frac{2}{d-1}\right). \tag{4.18}$$

Then there is $\gamma_* := \gamma_*(N, \mu_0, \omega, \Omega, \beta, \eta, d) > 0$, such that, for any $\gamma \in (0, \gamma_*)$,

$$\left| \bigcup_{(k,l)\in\mathcal{Z}_N} \mathcal{R}_{kl}(\gamma) \right| \leq (1 + |\mathcal{I}|) \left(\frac{\gamma}{\gamma_*} \right)^{\frac{1}{\mu_0}}.$$

PROOF. By Lemma 4.5 we have

$$\left| \bigcup_{(k,l)\in\mathcal{Z}_N} \mathcal{R}_{kl}(\gamma) \right| \leq \sum_{\substack{0\leq|k|\leq\frac{L_*}{8\Gamma} \\ \langle l\rangle_d<L_*}} |\mathcal{R}_{kl}(\gamma)| + \sum_{\substack{|k|>\frac{L_*}{8\Gamma} \\ \langle l\rangle_d<8\Gamma|k|}} |\mathcal{R}_{kl}(\gamma)|. \tag{4.19}$$

We first estimate the second sum. By Lemma 4.6 and

$$\operatorname{card}\{l\colon \langle l\rangle_d < 8\Gamma|k|\} \leq (8\Gamma|k|)^{\frac{2}{d-1}}$$

we get

$$\begin{aligned}
\sum_{\substack{|k|>\frac{L_*}{8\Gamma} \\ \langle l\rangle_d<8\Gamma|k|}} |\mathcal{R}_{kl}(\gamma)| &\leq \sum_{\substack{|k|>\frac{L_*}{8\Gamma} \\ \langle l\rangle_d<8\Gamma|k|}} B(1+|\mathcal{I}|)\left(\frac{2\gamma}{|k|^{\tau+1}}\langle l\rangle_d\right)^{\frac{1}{\mu_0}} \\
&\leq C_1(1+|\mathcal{I}|)\gamma^{\frac{1}{\mu_0}} \sum_{k\in\mathbb{Z}^N\setminus\{0\}} (8\Gamma|k|)^{\frac{2}{d-1}}|k|^{-\frac{\tau}{\mu_0}} \\
&\leq C_2(1+|\mathcal{I}|)\gamma^{\frac{1}{\mu_0}}
\end{aligned}$$

by (4.18), for some constant $C_1, C_2 > 0$ depending on $N, \mu_0, \omega, \Omega, \beta, \eta, d$. Similarly the first sum in (4.19) is estimates by

$$\sum_{\substack{0\leq|k|\leq\frac{L_*}{8\Gamma} \\ \langle l\rangle_d<L_*}} |\mathcal{R}_{kl}(\gamma)| \leq C_3(1+|\mathcal{I}|)\gamma^{\frac{1}{\mu_0}}$$

with $C_3 > 0$, and so the thesis follows for some $\gamma_* > 0$ small enough. □

LEMMA 4.8. *Assume $d = 1$ and*

$$\tau > \mu_0(N+1)\Big(1 - \frac{\mu_0}{\delta}\Big). \tag{4.20}$$

Then there are positive constants γ_* and μ_* depending on $N, \mu_0, \omega, \Omega, \beta, \eta, \delta$ such that

$$\left| \bigcup_{(k,l)\in\mathcal{Z}_N} \mathcal{R}_{kl}(\gamma) \right| \leq (1 + |\mathcal{I}|) \left(\frac{\gamma}{\gamma_*} \right)^{-\frac{\delta}{\mu_0(\mu_0-\delta)}}.$$

PROOF. For $(k,l) \in \mathcal{Z}_N^+ := \mathcal{Z}_N \cap (\mathcal{L}_0 \cup \mathcal{L}_1 \cup \mathcal{L}_{2+})$, where these sets are defined in (4.2), we estimate, as in the case $d > 1$,

$$\left| \bigcup_{(k,l)\in\mathcal{Z}_N^+} \mathcal{R}_{kl}(\gamma) \right| \leq C_4(1 + |\mathcal{I}|)\gamma^{\frac{1}{\mu_0}} \tag{4.21}$$

for some $C_4 > 0$.

Let now $(k,l) \in \mathcal{Z}_N^- := \mathbb{Z}^N \times \mathcal{L}_{2-}$ and assume, without loss of generality, $i > j$, then $\langle l \rangle_d = i - j$. By the asymptotic behavior of Ω^* (see assumption (A) and (4.8)) and remembering that $\delta < 0$, there is a constant $a > 0$ such that

$$\left| \frac{\Omega_i^* - \Omega_j^*}{i-j} - 1 \right| \le \frac{a}{j^{-\delta}}, \qquad \text{for all } i > j\,. \tag{4.22}$$

Hence $\langle l, \Omega^* \rangle = \Omega_i^* - \Omega_j^* = i - j + r_{ij}$, with $|r_{ij}| \le \frac{a}{j^{-\delta}} m$ and $m := i - j$. Then we have $|\langle k, \omega^* \rangle + \langle l, \Omega^* \rangle| \ge |\langle k, \omega^* \rangle + m| - |r_{ij}|$, provided $|\langle k, \omega^* \rangle + m| \ge \left| \frac{a}{j^{-\delta}} m \right|$, from which follows that, for fixed k, l,

$$\mathcal{R}_{kl} \cap \mathcal{S}^+ \subseteq \mathcal{Q}_{kj}^m := \left\{ \xi \in \mathcal{I} \colon \frac{|\langle k, \omega^*(\xi) \rangle + m|}{1+|k|} < \frac{2\gamma}{1+|k|^{\tau+1}} m + \frac{am}{(1+|k|) j^{-\delta}} \right\}$$

where we have set for simplicity $\mathcal{R}_{kl} := \mathcal{R}_{kl}(\gamma)$, and

$$\mathcal{S}^+ := \left\{ \xi \in \mathcal{I} \colon \frac{|\langle k, \omega^*(\xi) \rangle + m|}{1+|k|} \ge \frac{am}{(1+|k|) j^{-\delta}} \right\}.$$

Calling $\mathcal{S}^-$ the complementary set of $\mathcal{S}^+$, we have

$$\mathcal{R}_{kl} = \left(\mathcal{R}_{kl} \cap \mathcal{S}^- \right) \cup \left(\mathcal{R}_{kl} \cap \mathcal{S}^+ \right) \subseteq \mathcal{Q}_{kj}^m$$

so we need to estimate $\mathcal{Q}_{kj}^m$. Notice first that $\mathcal{Q}_{kj}^m \subset \mathcal{Q}_{kj_0}^m$ if $j > j_0$, for some j_0 to be fixed later. For γ small enough the result in Lemma 4.5 applies also the set $\mathcal{Q}_{kj_0}^m$ and so we get

$$\left| \bigcup_{(k,l) \in \mathcal{Z}_N^-} \mathcal{R}_{kl} \right| \le \sum_{\substack{|k| \le \frac{L_*}{8\Gamma} \\ m < L_*}} \left(\left| \mathcal{Q}_{kj_0}^m \right| + \sum_{j < j_0} |\mathcal{R}_{kl}| \right) + \sum_{\substack{|k| > \frac{L_*}{8\Gamma} \\ m < 8\Gamma|k|}} \left(\left| \mathcal{Q}_{kj_0}^m \right| + \sum_{j < j_0} |\mathcal{R}_{kl}| \right)$$

We start with the sum over $m < 8\Gamma|k|$, that we denote with (S_2). Using Lemma 4.6 we get

$$\begin{aligned} (S_2) \quad &\le \quad C_5(1+|\mathcal{I}|) \left(\left(\frac{a}{|k| j_0^{-\delta}} \right)^{\frac{1}{\mu_0}} + \left(\frac{2\gamma}{|k|^{\tau+1}} \right)^{\frac{1}{\mu_0}} j_0 \right) \sum_{m < 8\Gamma|k|} m^{\frac{1}{\mu_0}} \\ &\le \quad C_6(1+|\mathcal{I}|) \gamma^{\frac{-\delta}{\mu_0(\mu_0-\delta)}} |k|^{1+\frac{\delta\tau}{\mu_0(\mu_0-\delta)}} \end{aligned}$$

having chosen j_0 as

$$j_0 := \left(\frac{a}{2} |k|^\tau \gamma^{-1} \right)^{\frac{1}{\mu_0 - \delta}}.$$

Summing in $|k| \ge L_*/(8\Gamma)$ and using (4.20) yields

$$\sum_{\substack{|k| \ge L_*/(8\Gamma) \\ m < 8\Gamma|k|}} \left(\left| \mathcal{Q}_{kj_0}^m \right| + \sum_{j < j_0} |\mathcal{R}_{kl}| \right) \le C_7(1+|\mathcal{I}|) \gamma^{\frac{-\delta}{\mu_0(\mu_0-\delta)}},$$

with $C_7 > 0$. The estimate of the first sum follows in a similar way. Hence we have obtained the thesis for $\gamma_* > 0$ small enough. $\square$

4.3. Proof of Theorem 5.2

We write (4.11) as an infinite dimensional Hamiltonian system introducing coordinates $q, p \in \ell^{a,p}$ by

$$u = \sum_{j\geq 1} \frac{q_j}{\sqrt{\lambda_j}}\phi_j\,, \quad v := u_t = \sum_{j\geq 1} p_j\sqrt{\lambda_j}\phi_j\,, \quad \lambda_j(\xi) := \sqrt{\mu_j + \xi}\,,$$

where μ_j and ϕ_j, are respectively the simple Dirichlet eigenvalues and eigenvectors of $-\partial_{xx} + V(x)$, normalized and orthogonal in $L^2(0,\pi)$. Note that $\mu_j > 0$ for all $j \geq 1$ because $V(x) \geq 0$. The Hamiltonian of (4.11) is

$$\begin{aligned} H_{NLW} &= \int_0^\pi \left(\frac{v^2}{2} + \frac{1}{2}(u_x^2 + V(x)u^2 + \xi u^2) + F(x,u)\right)dx \\ &= \frac{1}{2}\sum_{j\geq 1}\lambda_j(q_j^2 + p_j^2) + G(q) \end{aligned} \tag{4.23}$$

where $\partial_u F(x,u) = f(x,u)$ and

$$G(q) := \int_0^\pi F\Big(x, \sum_{j\geq 1} q_j\lambda_j^{-1/2}\phi_j\Big)\,dx\,. \tag{4.24}$$

Note that since f satisfies only (4.12) then $G(q)$ could contain cubic terms.

Now we reorder the indices in such a way that $\mathcal{J} := \{j_1 < \ldots < j_N\}$ corresponds to the first N modes. More precisely we define a reordering $k \to j_k$ from $\mathbb{N} \to \mathbb{N}$ which is bijective and increasing both from $\{1,\ldots,N\}$ onto $\mathcal{J}$ and from $\{N+1, N+2, \ldots\}$ onto $\mathbb{N}\setminus\mathcal{J}$.

Introduce complex coordinates

$$z_k := \frac{1}{\sqrt{2}}(p_{j_k} + \mathrm{i}q_{j_k})\,, \quad \bar z_k := \frac{1}{\sqrt{2}}(p_{j_k} - \mathrm{i}q_{j_k})$$

and action-angle coordinates on the first N-modes

$$z_k := \sqrt{I_k + y_k}e^{\mathrm{i}x_k}\,, \quad 1 \leq k \leq N\,,$$

with

$$I_k \in \left(\frac{r^{2\theta}}{2}, r^{2\theta}\right]\,, \quad \theta \in (0,1)\,. \tag{4.25}$$

Then the Hamiltonian (4.23) assumes the form (4.3)-(4.4) with frequencies

$$\omega(\xi) := (\lambda_{j_1}(\xi), \ldots, \lambda_{j_N}(\xi))\,, \quad \Omega(\xi) := (\lambda_{j_{N+1}}(\xi), \lambda_{j_{N+2}}(\xi), \ldots)\,.$$

The asymptotic assumption (A) holds with $d = 1$, $\delta = -1$ and $\eta = \mu_1/2$. Also the regularity assumption (R) holds with $\bar p = p + 1$, see Lemma 3.1 of [**CY00**].

By conditions (4.24), (4.12) and (4.25) the perturbation satisfies

$$\varepsilon := \gamma^{-1} \| X_P \|_{r,\overline{p},\mathcal{E}} = O(\gamma^{-1} r^{3\theta-2}) , \qquad \sum_{2i+j_1+j_2=4} \sup_{\mathcal{I}\times\mathcal{D}(s,r)} \left| \partial_y^i \partial_z^{j_1} \partial_{\overline{z}}^{j_2} P \right| = O(1) .$$

Fixed

$$\theta \in (2/3, 1) , \quad \gamma := r^{\sigma} , \quad 0 < \sigma < (3\theta - 2)/\mu_0 ,$$

then, for $r > 0$ small enough, the KAM conditions (4.6)-(4.7) are verified as well as the smallness condition (4.10). It remains to verify assumption (ND).

Lemma 4.9. *The non–degeneracy condition (ND) holds.*

Proof. It is sufficient to prove that, for any $(c_0, c_1, \ldots, c_N, c_h, c_k) \in \mathbb{R}^{N+3} \backslash \{0\}$ with $k > h > N$, the function $c_0 + c_1\lambda_{j_1} + \ldots + c_N\lambda_{j_N} + c_h\lambda_{j_h} + c_k\lambda_{j_k}$ is not identically zero on $\mathcal{I} = [0, \xi_*]$. For simplicity of notation we denote $\lambda_l := \lambda_{j_l}$.

Suppose, by contradiction, that there exists $(c_0, c_1, \ldots, c_N, c_h, c_k) \neq 0$ such that $c_0 + c_1\lambda_1 + \ldots + c_N\lambda_N + c_h\lambda_h + c_k\lambda_k \equiv 0$. Then, taking the first $N + 2$ derivatives, we get the system

$$\begin{cases} c_0 + c_1\lambda_1 + \ldots + c_N\lambda_N + c_h\lambda_h + c_k\lambda_k = 0 \\ c_1 \dfrac{d}{d\xi}\lambda_1 + \ldots + c_N \dfrac{d}{d\xi}\lambda_N + c_h \dfrac{d}{d\xi}\lambda_h + c_k \dfrac{d}{d\xi}\lambda_k = 0 \\ \vdots \\ c_1 \dfrac{d^{N+2}}{d\xi^{N+2}}\lambda_1 + \ldots + c_N \dfrac{d^{N+2}}{d\xi^{N+2}}\lambda_N + c_h \dfrac{d^{N+2}}{d\xi^{N+2}}\lambda_h + c_k \dfrac{d^{N+2}}{d\xi^{N+2}}\lambda_k = 0 . \end{cases}$$

Since this system admits a non–zero solution, the determinant of the associated matrix is zero. On the other hand this determinant is c_0 times the determinant of the $(N + 2) \times (N + 2)$ minor

$$D = \begin{bmatrix} \frac{d}{d\xi}\lambda_1(\xi) & \ldots & \frac{d}{d\xi}\lambda_N(\xi) & \frac{d}{d\xi}\lambda_h(\xi) & \frac{d}{d\xi}\lambda_k(\xi) \\ \frac{d^2}{d\xi^2}\lambda_1(\xi) & \ldots & \frac{d^2}{d\xi^2}\lambda_N(\xi) & \frac{d^2}{d\xi^2}\lambda_h(\xi) & \frac{d^2}{d\xi^2}\lambda_k(\xi) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{d^{N+2}}{d\xi^{N+2}}\lambda_1(\xi) & \ldots & \frac{d^{N+2}}{d\xi^{N+2}}\lambda_N(\xi) & \frac{d^{N+2}}{d\xi^{N+2}}\lambda_h(\xi) & \frac{d^{N+2}}{d\xi^{N+2}}\lambda_k(\xi) \end{bmatrix}$$

which is different from 0, as we prove below. This implies $c_0 = 0$. Moreover the unique solution $(c_1, \ldots c_N, c_h, c_k)$ of the system associated to D is zero. This is a contradiction.

In order to prove that the determinant of D is different from zero, we first observe that, by induction, for any $r \geq 1$,

$$\frac{d^r}{d\xi^r}\lambda_i(\xi) = \frac{(2r-3)!!}{2^r} \frac{(-1)^{r+1}}{(\mu^i + \xi)^{r-\frac{1}{2}}} ,$$

where, for n odd, $n!! := n(n-2)(n-4)\dots 1$ and $(-1)!! := 1$. Setting $x_i = (\mu_i+\xi)^{-1}$ and using the linearity of the determinant, we obtain

$$\det D = \prod_{r=1}^{N+2} (-1)^{r+1}\frac{(2r-3)!!}{2^r}\left(\prod_{i=1}^{N}(\mu_i+\xi)^{-\frac12}\right)(\mu_h+\xi)^{-\frac12}(\mu_k+\xi)^{-\frac12}$$
$$\cdot \det\begin{bmatrix} 1 & \dots & 1 & 1 & 1 \\ x_1 & \dots & x_N & x_h & x_k \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1^{N+1} & \dots & x_N^{N+1} & x_h^{N+1} & x_k^{N+1} \end{bmatrix}$$

The last is a Vandermonde determinant which is not zero since all the x_i are all different from each other. For a similar quantitative estimate we refer to [**Bam99**]. □

In conclusion the KAM Theorem and Theorem **??** apply proving Theorem 4.2.

4.4. Quantitative non–resonance property: Proof of Proposition 5.3

Split the set $\mathcal{L}$ as in (4.2) and discuss the four cases separately.

Case $l \in \mathcal{L}_0$**.** *There exist* $\mu_0 \in \mathbb{N}$, $\beta > 0$ *such that*

$$\max_{0\le\mu\le\mu_0}\left|\frac{d^\mu}{d\xi^\mu}\langle k,\omega(\xi)\rangle\right| \ge \beta(1+|k|)$$

for all $\xi \in \mathcal{I}$, $k \in \mathbb{Z}^N \setminus \{0\}$.

Proceed by contradiction and assume that for all $\mu_0 \in \mathbb{N}$ and for all $\beta > 0$ there exist $\xi_{\mu_0,\beta} \in \mathcal{I}$, $k_{\mu_0,\beta} \in \mathbb{Z}^N \setminus \{0\}$ such that

$$\max_{0\le\mu\le\mu_0}\left|\frac{d^\mu}{d\xi^\mu}\left\langle \frac{k_{\mu_0,\beta}}{1+|k_{\mu_0,\beta}|},\omega(\xi_{\mu_0,\beta})\right\rangle\right| < \beta.$$

In particular, for all $\lambda := \mu_0 \in \mathbb{N}$, $\beta := 1/(\lambda+1)$, there exist $\xi_\lambda \in \mathcal{I}$, $k_\lambda \in \mathbb{Z}^N \setminus \{0\}$ such that

$$\max_{0\le\mu\le\lambda}\left|\frac{d^\mu}{d\xi^\mu}\left\langle \frac{k_\lambda}{1+|k_\lambda|},\omega(\xi_\lambda)\right\rangle\right| < \frac{1}{\lambda+1},$$

namely, for all $\mu \ge 0$, for any $\lambda \ge \mu$, we have

$$\left|\frac{d^\mu}{d\xi^\mu}\left\langle \frac{k_\lambda}{1+|k_\lambda|},\omega(\xi_\lambda)\right\rangle\right| < \frac{1}{\lambda+1}. \tag{4.26}$$

By compactness there exist converging subsequences $\xi_{\lambda_h} \to \overline{\xi} \in \mathcal{I}$ and $\frac{k_{\lambda_h}}{1+|k_{\lambda_h}|} \to \overline{c} \in \mathbb{R}^N$ with $1/2 \le |\overline{c}| \le 1$ if $\lambda_h \to \infty$ as $h \to \infty$. Passing to the limit in (4.26), for any $\mu \geq 0$, we get

$$\frac{d^\mu}{d\xi^\mu}\left\langle \overline{c}, \omega(\overline{\xi}) \right\rangle = \lim_{h\to\infty} \frac{d^\mu}{d\xi^\mu}\left\langle \frac{k_{\lambda_h}}{1+|k_{\lambda_h}|}, \omega(\xi_{\lambda_h}) \right\rangle = 0\,,$$

namely the analytic function $\langle \overline{c}, \omega(\xi)\rangle$ vanishes with all its derivatives at $\overline{\xi}$. Then $\langle \overline{c}, \omega(\xi)\rangle \equiv 0$ on $\mathcal{I}$. This contradicts the assumption of non–degeneracy of ω.

Case $l \in \mathcal{L}_1$. *There exist $\mu_0 \in \mathbb{N}$, $\beta > 0$ such that*

$$\max_{0\leq\mu\leq\mu_0} \left| \frac{d^\mu}{d\xi^\mu}(\langle k, \omega(\xi)\rangle + \Omega_j(\xi)) \right| \geq \beta(1+|k|)$$

for all $\xi \in \mathcal{I}$, $k \in \mathbb{Z}^N$, $j \geq N+1$.

Arguing by contradiction as above, we assume that for all $\lambda \in \mathbb{N}$ there exist $\xi_\lambda \in \mathcal{I}$, $k_\lambda \in \mathbb{Z}^N$, $j_\lambda \geq N+1$ such that

$$\max_{0\leq\mu\leq\lambda} \left| \frac{d^\mu}{d\xi^\mu}(\langle k_\lambda, \omega(\xi_\lambda)\rangle + \Omega_{j_\lambda}(\xi_\lambda)) \right| < \frac{1}{\lambda+1}(1+|k_\lambda|). \tag{4.27}$$

The asymptotic assumption (A) implies

$$j^d \geq \Theta_1|k| + \Theta_2 \quad \Longrightarrow \quad \left| \frac{\langle k, \omega(\xi)\rangle + \Omega_j(\xi)}{1+|k|} \right| \geq \frac{1}{2}\,, \ \forall \xi \in \mathcal{I}\,,$$

with $\Theta_1 := 2\Gamma + 1$, $\Theta_2 := \max\{1, (2\Gamma)^d\}$. Then, (4.27) implies that

$$j_\lambda^d < \Theta_1|k_\lambda| + \Theta_2\,, \quad \forall \lambda \geq 1\,. \tag{4.28}$$

By compactness $\xi_{\lambda_h} \to \overline{\xi}$ as $h \to \infty$. The indexes $k_\lambda \in \mathbb{Z}^N$, $j_\lambda \geq N+1$ belong to non–compact spaces and they could converge or not. Hence we have to separate the various cases.

Case k_λ bounded. By (4.28) also the sequence j_λ is bounded. So we extract constant subsequences $k_{\lambda_h} \equiv \overline{k}$, $j_{\lambda_h} \equiv \overline{\jmath}$. Passing to the limit in (4.27), we get, for any $\mu \geq 0$,

$$\frac{d^\mu}{d\xi^\mu}\left(\left\langle \frac{\overline{k}}{1+|\overline{k}|}, \omega(\overline{\xi}) \right\rangle + \frac{\Omega_{\overline{\jmath}}(\overline{\xi})}{1+|\overline{k}|} \right) = 0\,.$$

By the analyticity of ω, Ω, the function $\left\langle \overline{k}, \omega \right\rangle(\xi) + \Omega_{\overline{\jmath}}(\xi)$ is identically zero on $\mathcal{I}$. This contradicts the non–degeneracy of (ω, Ω_j).

Case k_λ unbounded. The quantity $\frac{k_\lambda}{1+|k_\lambda|}$ converges, up to subsequence, to $\overline{c} \in \mathbb{R}^N$, with $1/2 \leq |\overline{c}| \leq 1$.

If $\{j_\lambda\}$ is bounded, there is a subsequence $\{j_{\lambda_h}\}$ that is constantly equal to $\overline{\jmath}$. Passing to the limit in (4.27), we get, for any $\mu \geq 0$,

$$\frac{d^\mu}{d\xi^\mu}\left\langle \overline{c}, \omega(\overline{\xi}) \right\rangle = \lim_{h\to\infty} \frac{d^\mu}{d\xi^\mu}\left(\left\langle \frac{k_{\lambda_h}}{1+|k_{\lambda_h}|}, \omega(\xi_{\lambda_h}) \right\rangle + \frac{j_{\lambda_h}^d + \nu_{j_{\lambda_h}}(\xi_{\lambda_h}) j_{\lambda_h}^\delta}{1+|k_{\lambda_h}|} \right) = 0.$$

By the analyticity of ω we come to a contradiction with the non–degeneracy assumption on ω.

If $\{j_\lambda\}$ is unbounded there is a divergent subsequence $j_{\lambda_h} \to \infty$. Then we consider the first derivative of the function $\langle k, \omega(\xi)\rangle + \Omega_j(\xi)$, namely, recalling assumption (A) on Ω, the function $\langle k, \omega'(\xi)\rangle + \nu_j'(\xi) j^\delta$. The analyticity assumption (A) and Cauchy estimates imply that

$$\left|\frac{d^\mu}{d\xi^\mu}\nu_j(\xi)\right| \le \frac{\Gamma}{\eta^\mu}, \quad \forall \xi \in \mathcal{I}, \ \mu \ge 0 . \tag{4.29}$$

Then, using also (4.28), there is a constant $\widetilde{\Theta}_1 > 0$ such that, for any $\mu \ge 0$,

$$\frac{d^\mu}{d\xi^\mu}\nu'_{j_{\lambda_h}} \frac{j^\delta_{\lambda_h}}{1+|k_\lambda|} \le \widetilde{\Theta}_1 \frac{j^\delta_{\lambda_h}}{j^d_{\lambda_h}} \to 0 \quad \text{ as } \ h \to \infty$$

since $\delta < d-1$. Then, passing to the limit in (4.27) yields, for any $\mu \ge 0$,

$$\frac{d^\mu}{d\xi^\mu}\left\langle \overline{c}, \omega'\big(\overline{\xi}\big)\right\rangle = 0 .$$

Hence $\langle \overline{c}, \omega'(\xi)\rangle$ and all its derivatives vanish at $\overline{\xi}$. By analyticity, $\langle \overline{c}, \omega'(\xi)\rangle$ is identically zero on $\mathcal{I}$ and then the function $\langle \overline{c}, \omega(\xi)\rangle$ is identically equal to some constant. This contradicts the non–degeneracy assumption on $(\omega, 1)$.

Case $l \in \mathcal{L}_{2+}$**.** *There exist* $\mu_0 \in \mathbb{N}$*,* $\beta > 0$ *such that*

$$\max_{0 \le \mu \le \mu_0}\left|\frac{d^\mu}{d\xi^\mu}(\langle k, \omega(\xi)\rangle + \Omega_i(\xi) + \Omega_j(\xi))\right| \ge \beta(1+|k|)$$

for all $\xi \in \mathcal{I}$*,* $k \in \mathbb{Z}^N$*,* $i, j \ge N+1$*.*

This follows by arguments similar to the case $l \in \mathcal{L}_1$.

Case $l \in \mathcal{L}_{2-}$**.** *There exist* $\mu_0 \in \mathbb{N}$*,* $\beta > 0$ *such that*

$$\max_{0 \le \mu \le \mu_0}\left|\frac{d^\mu}{d\xi^\mu}(\langle k, \omega(\xi)\rangle + \Omega_i(\xi) - \Omega_j(\xi))\right| \ge \beta(1+|k|)$$

for all $\xi \in \mathcal{I}$*,* $k \in \mathbb{Z}^N$*,* $i, j \ge N+1$*,* $i \ne j$*.*

Proceed by contradiction as above and assume that for all $\lambda \in \mathbb{N}$ there exist $\xi_\lambda \in \mathcal{I}$, $k_\lambda \in \mathbb{Z}^N$, $i_\lambda, j_\lambda \ge N+1$ such that

$$\max_{0 \le \mu \le \lambda}\left|\frac{d^\mu}{d\xi^\mu}\left(\left\langle \frac{k_\lambda}{1+|k_\lambda|}, \omega(\xi_\lambda)\right\rangle + \frac{\Omega_{i_\lambda}(\xi_\lambda)}{1+|k_\lambda|} - \frac{\Omega_{j_\lambda}(\xi_\lambda)}{1+|k_\lambda|}\right)\right| < \frac{1}{\lambda+1}.$$

In particular we have that for all $\lambda \ge \mu$

$$\left|\frac{d^\mu}{d\xi^\mu}\left(\left\langle \frac{k_\lambda}{1+|k_\lambda|}, \omega(\xi_\lambda)\right\rangle + \frac{\Omega_{i_\lambda}(\xi_\lambda)}{1+|k_\lambda|} - \frac{\Omega_{j_\lambda}(\xi_\lambda)}{1+|k_\lambda|}\right)\right| < \frac{1}{\lambda+1}. \tag{4.30}$$

The asymptotic behavior (A) of Ω implies

$$\begin{aligned}|\Omega_i(\xi) - \Omega_j(\xi)| &\ge |i^d - j^d| - |\nu_i(\xi) i^\delta| - |\nu_j(\xi) j^\delta| \\ &\ge \frac{|i-j|}{2}\Big(i^{d-1} + j^{d-1}\Big) - \Gamma\Big(i^\delta + j^\delta\Big)\end{aligned}$$

$$\geq \frac{1}{2}\Big(i^{d-1} + j^{d-1}\Big) - \Gamma\Big(i^{\delta} + j^{\delta}\Big)\,. \tag{4.31}$$

Then, remembering that $\delta < d-1$, we have that

$$\min\{i,j\}^{d-1} \geq \Theta_3|k| + \Theta_4 \implies |\langle k, \omega(\xi)\rangle + \Omega_i(\xi) - \Omega_j(\xi)| \geq \frac{1}{2}(1+|k|)$$

$\forall \xi \in \mathcal{I}$, with $\Theta_3 := 1 + 2\Gamma$ and $\Theta_4 := \max\{1, 4\Gamma^{(d-1)/(d-1-\delta)}\}$. Then (4.30) with $\mu = 0$ implies that

$$\min\{i_\lambda, j_\lambda\}^{d-1} < \Theta_3|k_\lambda| + \Theta_4\,,\ \forall \lambda \geq 1\,. \tag{4.32}$$

By compactness, $\xi_{\lambda_h} \to \overline{\xi} \in \mathcal{I}$ as $h \to \infty$. The indexes $k_\lambda, i_\lambda, j_\lambda$ can be bounded or not, and we study the various cases separately.

Case k_λ bounded. If k_λ is bounded then $k_\lambda = \overline{k}$ for infinitely many λ. Then (4.32) implies that also the sequence $\min\{i_\lambda, j_\lambda\}$ is bounded. Assuming $j_\lambda < i_\lambda$, there exists a constant subsequence $j_{\lambda_h} \equiv \overline{\jmath}$.

If also i_λ is bounded, we extract a constant subsequence $i_{\lambda_h} \equiv \overline{\imath}$. Then, passing to the limit in (4.30), we obtain, for all $\mu \geq 0$,

$$\frac{d^\mu}{d\xi^\mu}\left(\left\langle \frac{\overline{k}}{1+|\overline{k}|}, \omega(\overline{\xi})\right\rangle + \frac{\Omega_{\overline{\imath}}(\overline{\xi})}{1+|\overline{k}|} - \frac{\Omega_{\overline{\jmath}}(\overline{\xi})}{1+|\overline{k}|}\right) = 0\,.$$

By analyticity, the function $\big\langle \overline{k}, \omega(\xi)\big\rangle + \Omega_{\overline{\imath}}(\xi) - \Omega_{\overline{\jmath}}(\xi)$ is identically zero on $\mathcal{I}$, contradicting the non–degeneracy assumption on $(\omega, \langle l, \Omega\rangle)$ with $l = e_{\overline{\imath}} - e_{\overline{\jmath}}$.

If i_λ is unbounded, we extract a divergent subsequence $\{i_{\lambda_h}\}$. Since k_λ, j_λ are bounded we deduce, by the asymptotic assumption (A), that, definitively for λ large,

$$\frac{1}{1+|k_\lambda|}\Big(\langle k_\lambda, \omega(\xi_\lambda)\rangle + \Omega_{i_\lambda}(\xi_\lambda) - \Omega_{j_\lambda}(\xi_\lambda)\Big) \geq \frac{i_\lambda^d}{2(1+|k_\lambda|)}\,,$$

which tends to infinity for $\lambda \to +\infty$. This contradicts (4.30) with $\mu = 0$.

Case k_λ unbounded. If k_λ is unbounded, we extract a divergent subsequence such that $|k_{\lambda_h}| \to \infty$ as $h \to \infty$ and $\frac{k_{\lambda_h}}{1+|k_{\lambda_h}|} \to \overline{c} \in \mathbb{R}^N$ with $1/2 \leq |\overline{c}| \leq 1$.

Subcase $\max\{i_\lambda, j_\lambda\}$ *bounded.* For all $\mu \geq 0$, passing to the limit in (4.30), we have

$$\frac{d^\mu}{d\xi^\mu}\big\langle \overline{c}, \omega(\overline{\xi})\big\rangle = 0\,.$$

This contradicts the non–degeneracy of ω.

Subcase $\max\{i_\lambda, j_\lambda\}$ *unbounded,* $\min\{i_\lambda, j_\lambda\}$ *bounded.* Assume, without loss of generality, $i_\lambda > j_\lambda$. In this case

$$\sup_{\xi \in \mathcal{I}} \sup_{\lambda} |\Omega_{j_\lambda}(\xi)| =: M < +\infty\,.$$

We extract a divergent subsequence i_{λ_h} and claim that, definitively,

$$i^d_{\lambda_h} < 2\Big(1 + (1+\Gamma)|k_{\lambda_h}| + M\Big)\,. \tag{4.33}$$

Otherwise, definitively for λ large,

$$\frac{1}{1+|k_{\lambda_h}|}\Big(\,\langle k_{\lambda_h}, \omega(\xi_{\lambda_h})\rangle + \Omega_{i_{\lambda_h}}(\xi_{\lambda_h}) - \Omega_{j_{\lambda_h}}(\xi_{\lambda_h})\Big) \geq 1\,,$$

which contradicts (4.30) for $\mu = 0$.

By (4.29), (4.33), and since j_{λ_h} are bounded, there is $\widetilde{\Theta}_2 > 0$ such that, for any $\mu \geq 0$,

$$\frac{j^\delta_{\lambda_h}}{1+|k_{\lambda_h}|}\frac{d^\mu}{d\xi^\mu}\nu'_{j_{\lambda_h}}(\xi_{\lambda_h}) \leq \frac{\widetilde{\Theta}_2}{1+|k_{\lambda_h}|}\,, \qquad \frac{i^\delta_{\lambda_h}}{1+|k_{\lambda_h}|}\frac{d^\mu}{d\xi^\mu}\nu'_{j_{\lambda_h}} \leq \widetilde{\Theta}_2\frac{i^\delta_{\lambda_h}}{i^d_{\lambda_h}}$$

and both tend to zero if $h \to \infty$. Hence, passing to the limit in (4.30) (start with the first derivative), we obtain, for any $\mu \geq 0$

$$\frac{d^\mu}{d\xi^\mu}\left\langle \overline{c}, \omega'\big(\overline{\xi}\big)\right\rangle = \lim_{h\to\infty}\frac{d^\mu}{d\xi^\mu}\left\langle \frac{k_{\lambda_h}}{1+|k_{\lambda_h}|}, \omega'(\xi_{\lambda_h})\right\rangle\,. \tag{4.34}$$

By analyticity, the function $\langle \overline{c}, \omega'(\xi)\rangle$ is identically zero on $\mathcal{I}$ and consequently the function $\langle \overline{c}, \omega\rangle(\xi)$ is identically equal to some constant. This contradicts the non–degeneracy assumption on the function $(\omega, 1)$.

Subcase $\min\{i_\lambda, j_\lambda\}$ *unbounded.* Relation (4.31) implies

$$|\Omega_{i_\lambda} - \Omega_{j_\lambda}| \geq \frac{1}{4}\Big(i^{d-1}_\lambda + j^{d-1}_\lambda\Big)$$

if $i^{d-1}_\lambda + j^{d-1}_\lambda \geq 4\Gamma\big(i^\delta_\lambda + j^\delta_\lambda\big)$, that is always verified definitively since $\delta < d-1$.

We claim that

$$i^{d-1}_\lambda + j^{d-1}_\lambda < 4(\Gamma+1)|k_\lambda| + 4\,.$$

Otherwise, definitively for λ large,

$$\frac{|\langle k_\lambda, \omega(\xi_\lambda)\rangle + \Omega_{i_\lambda}(\xi_\lambda) - \Omega_{j_\lambda}(\xi_\lambda)|}{1+|k_\lambda|} \geq 1$$

which contradicts (4.30) for $\mu = 0$.

We extract diverging subsequences $i_{\lambda_h}, j_{\lambda_h}$ such that

$$i^{d-1}_{\lambda_h} \leq 4(\Gamma+1)|k_{\lambda_h}| + 4 \qquad \text{and} \qquad j^{d-1}_{\lambda_h} \leq 4(\Gamma+1)|k_{\lambda_h}| + 4\,.$$

Then, using also (4.29), there is $\widetilde{\Theta}_3 > 0$ such that, for any $\mu \geq 0$,

$$\frac{i^\delta_{\lambda_h}}{1+|k_{\lambda_h}|}\frac{d^\mu}{d\xi^\mu}\nu'_{i_{\lambda_h}} \leq \widetilde{\Theta}_3\frac{i^\delta_{\lambda_h}}{i^{d-1}_{\lambda_h}} \longrightarrow 0$$

$$\frac{j^\delta_{\lambda_h}}{1+|k_{\lambda_h}|}\frac{d^\mu}{d\xi^\mu}\nu'_{j_{\lambda_h}} \leq \widetilde{\Theta}_3\frac{j^\delta_{\lambda_h}}{j^{d-1}_{\lambda_h}} \longrightarrow 0$$

for $h \to \infty$.

We deduce as in (4.34) that all the derivatives of $\langle \overline{c}, \omega'(\overline{\xi}) \rangle$ vanish and by analyticity this contradicts the non–degeneracy assumption on $(\omega, 1)$.

CHAPTER 5

Quasi–periodic solutions for 1–d completely resonant nonlinear Schrödinger equations

The aim of this chapter is to construct quasi–periodic solutions for the nonlinear Schrödinger equation on the torus $\mathbb{T}$

$$iu_t - u_{xx} + |u|^6 u = 0, \qquad x \in \mathbb{T}. \tag{5.1}$$

This is a completely resonant system, actually it can be written as an infinite dimensional Hamiltonian dynamical system $\dot{u} = \{H, u\}$ with Hamiltonian

$$H = \int_0^{2\pi} |u_x|^2 \, dx + \frac{1}{4} \int_0^{2\pi} |u|^8 \, dx.$$

Passing to the Fourier representation

$$u(t,x) = \sum_{k \in \mathbb{Z}} u_k(t) e^{\mathrm{i}kx}$$

we have in coordinates

(5.2)

$$H = \sum_{k \in \mathbb{Z}} k^2 u_k \overline{u}_k + \frac{\pi}{2} \sum_{\substack{k_1, \dots, k_8 \in \mathbb{Z} \\ k_1 - k_2 + k_3 - k_4 + k_5 - k_6 + k_7 - k_8 = 0}} u_{k_1} \overline{u}_{k_2} u_{k_3} \overline{u}_{k_4} u_{k_5} \overline{u}_{k_6} u_{k_7} \overline{u}_{k_8}$$

where the symplectic structure in given by $\mathrm{i} \sum_k du_k \wedge d\overline{u}_k$ on the space

$$\ell^{a,p} := \left\{ u = \{u_k\}_{k \in \mathbb{Z}} \colon \|u\|_{a,p}^2 := \sum_{k \in \mathbb{Z}} |u_k|^2 e^{2a|k|} |k|^{2p} < +\infty \right\}$$

with $a > 0, p > \frac{1}{2}$.

The linearized system consists of infinitely many independent oscillators with integer frequencies k^2, and so it is completely resonant and all the solutions are periodic with period 2π.

We are now going to prove the existence of quasi–periodic solutions of equation (5.1).

In order to do so, we first perform one step of Birkhoff normal form, but a direct computation shows that this is not integrable and rather complicated. However, the study of the normal form may be simplified by an appropriate choice of the region of the phase space where we look for solutions, this is the content of Theorem 5.2.

Once we get the normal form, we use it as the unperturbed Hamiltonian to apply the KAM Theorem 3.5 of Berti–Biasco, verifying all the smallness, regularity, non–degeneracy and non–resonance assumptions.

The main result is the following. Let $\rho > 0$ and define

$$A_\rho := B_\rho(0) \cap \Big\{ \xi \in \mathbb{R}^m \colon \frac{\rho}{2} < \xi_i < \rho,\ i = 1, \dots, m \Big\}. \tag{5.3}$$

THEOREM 5.1. *For* generic *choices of indexes* $S := \{v_1, v_2, \dots, v_m\}$ *there exist* $\rho_* > 0$ *such that for any* $\rho < \rho_*$ *there exists a Cantor set* $\Pi^*_\rho \subset A_\rho$ *of positive Lebesgue measure such that, for any* $\xi \in \Pi^*_\rho$*, the nonlinear Schrödinger equation* (5.1) *admits a quasi–periodic solution of the form*

$$u(t,x) = \sum_{i=1}^{m} \sqrt{\xi_i} e^{\mathrm{i}((v_i^2 + \omega_i^*(\xi))t + \theta_i)} + o(\xi).$$

where the map $\xi \mapsto \omega^*(\xi)$ *is a lipeomorphism,* $\theta \in \mathbb{R}^m$ *are arbitrary phases and* $o(\xi)$ *is small in some analytical norm. The measure of the set* Π^*_ρ *is greater than* $c\rho^m$ *where* c *is a constant independent on* ρ.

For *generic* we mean that the indexes have to satisfy a finite number of polynomial inequalities, see Definition 24 for the precise statement.

5.1. Construction of the normal form

The following result states the existence of the normal form for our system.

THEOREM 5.2. *For all* generic *choices of the set* $S = \{v_1, \dots, v_m\} \subset \mathbb{Z}^m$ *of tangential sites there exist an open set* $O_\rho \subset A_\rho$ *and an analytic and symplectic change of variables*

$$\begin{aligned} \Phi \colon O_\rho \times \mathcal{D}_{a,p}(s,r) &\longrightarrow B_{\sqrt{\rho}}(0) \\ \xi \times (y,x,z,\overline{z}) &\longmapsto (u, \overline{u}) \end{aligned}$$

where the set $\mathcal{D}_{a,p}(s,r)$ *is defined in* (5.9)*, such that the Hamiltonian* (5.5) *becomes*

$$H = N(\xi, y, z, \overline{z}) + P(\xi, x, y, z, \overline{z})$$

with

$$N = \omega(\xi) \cdot y + \sum_{k \in S^c} \Omega_k |z_k|^2.$$

The tangential frequency ω *is a diffeomorphism and is defined in* (5.16)*, while the normal frequencies* Ω_k *are*

$$\Omega_k = k^2 + \omega_0 \cdot L(k) + \lambda_k(\xi) \qquad \forall k \in S^c,$$

where $\omega_0 = \big(v_1^2, \dots, v_m^2\big)$*, the integer vector* $L(k) \in \mathbb{Z}^m$ *and the functions* $\lambda_k(\xi) \in \mathbb{R}$ *are identically zero except for a finite number of* k*, in which case*

$|L(k)| \leq 6$ *and* $\lambda_k(\xi)$ *are homogeneous functions of degree 3 in* ξ *satisfying*

$$|\lambda_k(\xi)| \leq C\rho^3, \qquad |\nabla_\xi \lambda_k(\xi)| \leq C\rho^2, \quad \forall \xi \in O_\rho. \tag{5.4}$$

Assuming $\rho = r^{2\theta}$ *with* $\theta \in \left(\frac{1}{2}, \frac{3}{5}\right)$, *the perturbation* P *is small with respect* N, *namely*

$$|X_P|^\lambda_{s,r} \leq C r^{10\theta-2}$$

with a constant C *independent of* r.

The genericity condition on the tangential sites means that they have to satisfy a finite list of polynomial constraints , hence they can't lie in any of the varieties defined by certain polynomial equations. To be more precise, we give the following definition.

DEFINITION 24. Given a list $\mathcal{R} = \{P_1(y), \dots, P_N(y)\}$ of polynomials in the variable $y \in \mathbb{R}^b$, we say that a list of point $S = \{v_1, \dots, v_m\}$, $v_i \in \mathbb{R}$, is *generic* relative to $\mathcal{R}$ if, for any list $A = \{u_1, \dots, u_b\}$ such that $u_i \in S$ for any i and $u_i \neq u_j$ for $i \neq j$, the evaluation of the polynomials at $y_i = u_i$ is non–zero.

The rest of this section is dedicated to the proof of this theorem.

5.1.1. The normal form. Note first that the Hamiltonian has the momentum $M := \sum_{k\in\mathbb{Z}} k|u_k|^2$ and the scalar mass $L := \sum_k |u_k|^2$ as integrals of motion.

We have to perform a step of Birkhoff normal form, hence we cancel all the terms that do not Poisson commute with the quadratic part $K = \sum_{k\in\mathbb{Z}} k^2|u_k|^2$. The Hamiltonian then becomes

$$H = H_N + R^{10}, \tag{5.5}$$

with R^{10} is analytic of degree at least 10 in u and

$$H_N := \sum_{k\in\mathbb{Z}} k^2|u_k|^2 + \frac{\pi}{2} \sum^{*} u_{k_1}\overline{u}_{k_2}u_{k_3}\overline{u}_{k_4}u_{k_5}\overline{u}_{k_6}u_{k_7}\overline{u}_{k_8}, \tag{5.6}$$

where the sum $\sum^*$ is the sum restricted on the indexes $k_i \in \mathbb{Z}$ such that

$$\begin{cases} k_1 - k_2 + k_3 - k_4 + k_5 - k_6 + k_7 - k_8 = 0 \\ k_1^2 - k_2^2 + k_3^2 - k_4^2 + k_5^2 - k_6^2 + k_7^2 - k_8^2 = 0 \end{cases} \tag{5.7}$$

and this two conditions express the conservation of M and K.

5.1.2. Action–angle coordinates. Let us partition the set $\mathbb{Z}$ as the union of two disjoint sets

$$\mathbb{Z} = S \cup S^c$$

where $S = \{v_1, \dots, v_m\}$ and its elements are called *tangential sites*, while the elements in S^c are called *normal sites.*

The elements in S are the ones we have to choose by imposing some constraints in order to make the normal form of the system as simple as possible.

We now introduce action–angle coordinates by setting

$$\begin{cases} u_k = z_k & \text{for } k \in S^c \\ u_{v_i} = \sqrt{\xi_i + y_i}\, e^{\mathrm{i}x_i} & \text{for } i = 1, \dots, m \end{cases} \tag{5.8}$$

where $\xi_i \in A_\rho$ are positive parameters and $|y_i| < \xi_i$. This is an analytic and symplectic change of variable in the domain $A_\rho \times D_{a,p}(s,r) \subset \mathbb{R}^m \times \mathbb{T}^m_s \times \times \mathbb{C}^m \times \ell^{a,p} \times \ell^{a,p}$, with

$$\mathcal{D}_{a,p}(s,r) := \Big\{ x, y, w = (z, \overline{z}) \colon x \in \mathbb{T}^m_s, |y| \le r^2, \|w\|_{a,p} \le r \Big\} \tag{5.9}$$

where $0 < r < 1$, $s > 0$ are auxiliary parameters and $\mathbb{T}^m_s$ denotes the open neighborhood of the complex torus $\mathbb{T}^m_{\mathbb{C}} := \mathbb{C}^m / 2\pi\mathbb{Z}^m$ with $|\mathrm{Im}\, x| < s$, $x \in \mathbb{C}^m$. With this change of variables, the symplectic form becomes $dy \wedge dx + \mathrm{i} \sum_{k \in S^c} dz_k \wedge d\overline{z}_k$.

5.1.3. Constraints 1.

In order to have an integrable normal form we first impose the following constraints. For $\eta_i \in \mathbb{Z}$, $|\eta_i| \le 4$, $i = 1, \dots, m$

(i) $\eta_1 v_1 + \eta_2 v_2 + \cdots + \eta_m v_m \neq 0$ with $\sum_{i=1}^m \eta_i = 0$, $\sum_{i=1}^m |\eta_i| \le 8$,
(ii) $\sum_{i=1}^m \eta_i v_i^2 - \left(\sum_{i=1}^m \eta_i v_i\right)^2 \neq 0$ with $\sum_{i=1}^m \eta_i = 1$, $\sum_{i=1}^m |\eta_i| \le 7$.
(iii) $2\sum_{i=1}^m \eta_i v_i^2 + \left(\sum_{i=1}^m \eta_i v_i\right)^2 \neq 0$ with $\sum_{i=1}^m \eta_i = 0, -2$, $\sum_{i=1}^m |\eta_i| \le 6$

5.1.4. The Hamiltonian H_N.

With the change of variables (5.8), the normal form (5.6) becomes

$$H_N = H_0 + P^{(\ge 3)} \tag{5.10}$$

where $P^{\ge 3}$ contains the term of degree at least 3 in $y, z, \overline{z}$ and

$$\begin{aligned} H_0 = &\sum_{i=1}^{3} \omega_i(\xi) y_i + \sum_{k \in S^c} \Omega^0_k(\xi) |z_k|^2 + \sum_{h \neq k \in S^c} \Omega^1_{hk}(\xi) \overline{z}_h z_k \\ &+ \sum_{h \neq k \in S^c} \Big(\Omega^2_{hk} z_h z_k + \overline{\Omega}^2_{hk} \overline{z}_h \overline{z}_k \Big). \end{aligned} \tag{5.11}$$

Here the frequencies $\omega(\xi)$ depends only on ξ while $\Omega^0_k(\xi), \Omega^1_{hk}, \Omega^2_{hk}$ are functions also of the angles x (we will give an explicit formulation of the frequencies later and also in the case $m = 3$, see Subsection 5.3.4).

We can write the Hamiltonian H_0 in a more compact way.

Define the set

$$X_3 := \left\{ l := \sum_{j=1}^{6} \pm e_{i_j} = \sum_{i=1}^{m} l_i e_i \text{ such that } l \neq 0, -2e_i \ \forall i \right\}$$

where e_i is the standard i–th unitary vector and we have that $\sum_{i=1}^{m} |l_i| \leq 6$. Define then two sets X_3^0, X_3^{-2} as

$$X_3^0 := \left\{ l \in X_3 \text{ such that } \sum_{i=1}^{6} l_i = 0 \right\}, \tag{5.12}$$

$$X_3^{-2} := \left\{ l \in X_3 \text{ such that } \sum_{i=1}^{6} l_i = -2 \right\}. \tag{5.13}$$

See (5.28) and (5.29) for an explicit characterization of these sets in the case $m = 3$.

The Hamiltonian H_0 contains all the terms of degree at most 2 in $z, \overline{z}$ satisfying conditions (5.7).

The part of degree 0 in $z, \overline{z}$ is given when all the indexes k_i are in S. This imply that the conservation of the momentum (the linear equation in (5.7)) must hold identically, because by Constraints 1(i) all the other linear relations of the k_i are not allowed. Then, recalling (5.8), we have a contribution equal to A_4, with

$$A_r(\xi_1, \ldots, \xi_m) := \sum_{\sum_i k_i = r} \binom{r}{k_1, \ldots, k_m}^2 \prod_i \xi_i^{k_i}.$$

and so the terms of degree at most 2 are a constant part, that we ignore, and the linear term $\nabla_\xi A_4(\xi) \cdot y$.

The part of degree 1 in $z, \overline{z}$ is given when only one index is not in S. By Constrains 1(ii) these terms do not occur.

The part of degree 2 in $z, \overline{z}$ is given when only two indexes are not in S. Fix $h, k \in S^c$, then (5.7) becomes

$$\begin{cases} \sum_{j=1}^{m} l_i v_i + h - k = 0 \\ \sum_{j=1}^{m} l_i v_i^2 + h^2 - k^2 = 0 \end{cases} \qquad \text{if } l \in X_3^0 \tag{5.14}$$

and

$$\begin{cases} \sum_{j=1}^{m} l_i v_i + h + k = 0 \\ \sum_{j=1}^{m} + l_i v_i^2 + h^2 + k^2 = 0 \end{cases} \qquad \text{if } l \in X_3^{-2}. \tag{5.15}$$

It is convenient to drop another mass term. If only two equal indexes k_i are in S^c, by Constraints 1(i) we have a contribution in Ω^0_k equal to $16A_3$ and so

$$\sum_k \Omega^0_k(\xi)|z_k|^2 = \sum_k \left(k^2 + 16A_3(\xi)\right)|z_k|^2.$$

Noting that

$$\sum_k 16A_3(\xi)|z_k|^2 = 16A_3\left(\sum_k |z_k|^2 + \sum_{i=1}^m y_i\right) - 16A_3\sum_{i=1}^m y_i,$$

we have that the term in brackets is L, hence we can drop it from the Hamiltonian.

In conclusion, we have that $\Omega^0_k = k^2$ and that the frequency ω is an homogeneous polynomial in ξ of degree 3 of the form

$$\omega = \omega_0 + \nabla_\xi A_4(\xi) - 16A_3(\xi) \tag{5.16}$$

with $\omega_0 := \left(v_1^2, \dots, v_m^2\right)$.

The Hamiltonian H_0 then becomes

$$\begin{aligned} H_0 = \sum_{i=1}^m \omega_i y_i + \sum_{k\notin S} k^2|z_k|^2 + \sum_{l\in X_3^0} c(l)e^{\mathrm{i}lx} \sum_{h,k\notin S\colon (5.14)} z_h\overline{z}_k \\ + \sum_{l\in X_3^{-2}} c(l) \sum_{h,k\notin S\colon (5.15)} \left(e^{\mathrm{i}lx} z_h z_k + e^{-\mathrm{i}lx}\overline{z}_h\overline{z}_k\right) \end{aligned} \tag{5.17}$$

where $c(l)$ are some functions of the only ξ, more precisely:

$$c(l) := \begin{cases} 16\xi^{\frac{l^+ + l^-}{2}} \displaystyle\sum_{\substack{\alpha\in\mathbb{N}^m \\ |\alpha+l^+|=3}} \binom{3}{l^+ + \alpha}\binom{3}{l^- + \alpha}\xi_i^\alpha & l\in X_3^0 \\ 12\xi^{\frac{l^+ + l^-}{2}} \displaystyle\sum_{\substack{\alpha\in\mathbb{N}^m \\ |\alpha+l^+|=2}} \binom{4}{l^- + \alpha}\binom{2}{l^+ + \alpha}\xi_i^\alpha & l\in X_3^{-2} \end{cases} \tag{5.18}$$

with l^+, l^- are such that $l = l^+ - l^-$.

5.1.5. Constraints 2. Given l we consider the map $l \mapsto h(l)$ that to each $l \in X_3^0, X_3^{-2}$ associates $h(l)$ such that conditions (5.14), (5.15) hold.

Lemma 5.3. *The map $l \mapsto h(l)$ is invertible from $X_3^0 \cup X_3^{-2}$ to its image.*

Definition 25. We define the set of *special points* as the set $h(X_3^0 \cup X_3^{-2})$.

We denote with L the inverse of the map $l \mapsto h(l)$ and we extend it S^c by setting $L(h) = 0$ if h is not a special point.

PROOF. First prove that given l there exist at most two couple (h,k) such that condition (5.14) or (5.15) is satisfied. If $l \in X_3^0$ then we obtain the couple (h,k) with

$$h = \frac{\sum_{j=1}^m l_i v_i^2}{2\sum_{j=1}^m l_i v_i} - \frac{1}{2}\sum_{j=1}^m l_i v_i, \qquad k = \frac{\sum_{j=1}^m l_i v_i^2}{2\sum_{j=1}^m l_i v_i} + \frac{1}{2}\sum_{j=1}^m l_i v_i.$$

Note that if we change l with $-l$ then we obtain the couple (k,h), hence in the case $l \in X_3^0$ we can fix just one of the two components, say h.

If $l \in X_3^{-2}$ then

$$\begin{cases} h = \dfrac{-\sum_{j=1}^m l_i v_i \pm \sqrt{-\left(\sum_{j=1}^m l_i v_i\right)^2 - 2\sum_{j=1}^m l_i v_i^2}}{2} \\ k = \dfrac{-\sum_{j=1}^m l_i v_i \mp \sqrt{-\left(\sum_{j=1}^m l_i v_i\right)^2 - 2\sum_{j=1}^m l_i v_i^2}}{2}. \end{cases}$$

In order to prove that we can impose a finite number of constrain such that for each $h \in \mathbb{Z}$ there exists a unique l that satisfies condition (5.14) or (5.15) we will argue by contradiction and prove that, under some condition, it is not possible that

$$(5.19) \qquad \begin{cases} \sum l_i v_i + h \pm k = 0 \\ \sum l_i v_i^2 + h^2 \pm k^2 = 0 \end{cases}, \qquad \begin{cases} \sum \bar{l}_i v_i + h \pm \bar{k} = 0 \\ \sum \bar{l}_i v_i^2 + h^2 \pm \bar{k}^2 = 0 \end{cases}$$

with $l \neq \bar{l}$, $k \neq \bar{k}$ and the same h.

So, we assume (5.19) and we first prove that l and $\bar{l}$ have the same support. Actually, the first system defines h as a function of some tangential sites, say $h = f(v_1, \ldots .v_a)$ with $a \leq m$, and the second system defines $h = g(v_1, \ldots, v_b)$ with $b \leq m$. But $f = g$ since h has to be the same in the two systems. Then, if for example f does not depend on the variable v_j also g will not depend on that variable, and hence the two functions have to depend on the same variables, and so also l and $\bar{l}$.

For the rest of the proof we study the various cases separately.

(i) Assume (5.19) with $l, \bar{l} \in X_3^0$. Then

$$\begin{cases} h = \dfrac{\sum_{i=1}^m l_i v_i^2}{2\sum_{i=1}^m l_i v_i} - \dfrac{1}{2}\sum_{i=1}^m l_i v_i \\ h = \dfrac{\sum_{i=1}^m \bar{l}_i v_i^2}{2\sum_{i=1}^m \bar{l}_i v_i} - \dfrac{1}{2}\sum_{i=1}^m \bar{l}_i v_i \end{cases}$$

and so

$$\sum_{i=1}^m \bar{l}_i v_i \left(\sum_{i=1}^m l_i v_i^2 - \left(\sum_{i=1}^m l_i v_i \right)^2 \right) = \sum_{i=1}^m l_i v_i \left(\sum_{i=1}^m \bar{l}_i v_i^2 - \left(\sum_{i=1}^m \bar{l}_i v_i \right)^2 \right)$$

and this has to be an identity. So, comparing the coefficients of the term v_i^3 we have

$$\bar{l}_i l_i (1 - l_i) = \bar{l}_i l_i \big(1 - \bar{l}_i\big).$$

If $l_i \bar{l_i} \neq 0$ then $l = \bar{l}$ and this is impossible. The case $l_i = 0, \bar{l}_i \neq 0$ is excluded by the condition that l and $\bar{l}$ have the same support. If $l_i = \bar{l}_i = 0$ then we consider the coefficient of another term v_j (remember that l and $\bar{l}$ can't be the zero vector).

(ii) Assume (5.19) with $l \in X_3^0, \bar{l} \in X_3^{-2}$ with the following condition: impose that if $\bar{l}_i$ is -2 (resp. -1) then l_i can't be -1 (resp. -1, 1). Then

$$\begin{cases} h = \dfrac{\sum_{i=1}^m l_i v_i^2}{2\sum_{i=1}^m l_i v_i} - \dfrac{1}{2} \displaystyle\sum_{i=1}^m l_i v_i \\ 2h^2 + 2h \displaystyle\sum_{i=1}^m \bar{l}_i v_i + \sum_{i=1}^m \bar{l}_i v_i^2 + \left(\sum_{i=1}^m \bar{l}_i v_i \right)^2 = 0 \end{cases}$$

and so

$$\left(\sum_{i=1}^m l_i v_i^2 - \left(\sum_{i=1}^m l_i v_i \right)^2 \right)^2 + 2 \left(\sum_{i=1}^m l_i v_i - \left(\sum_{i=1}^m l_i v_i \right)^2 \right) \sum_{i=1}^m l_i v_i \sum_{i=1}^m \bar{l}_i v_i$$
$$+ 2 \left(\sum_{i=1}^m \bar{l}_i v_i^2 + \left(\sum_{i=1}^m \bar{l}_i v_i \right)^2 \right) \left(\sum_{i=1}^m l_i v_i \right)^2 = 0$$

and this has to be an identity. So, comparing the coefficients of the term v_i^4 we have

$$\big(l_i - l_i^2 + l_i \bar{l}_i\big)^2 + l_i^2 \bar{l}_i \big(2 + \bar{l}_i\big) = 0$$

and this can be an identity only if

$$l_i^2 \left(2\bar{l}_i + \bar{l}_i^2 \right) \leq 0$$

namely if $\bar{l}_i = -2, -1, 0$ or $l_i = 0$. Assuming $l_i \neq 0$ then also $\bar{l}_i \neq 0$ by the condition that $l, \bar{l}$ have the same support. Then if $\bar{l}_i = -1$ then l_i has to be equal to $-1, 1$, and if $\bar{l}_i = -2$ then l_i has to be equal to -1 but these cases are impossible by the previous assumption. If $l_i = \bar{l}_i = 0$ we consider the coefficient of another term, recalling that $l, \bar{l}$ can't be zero vectors.

(iii) Assume (5.19) with $l, \bar{l} \in X_3^{-2}$ with the following condition: impose that if $l_i = -1, -2$ then $\bar{l}_i$ can't be -1,-2. Then

$$\begin{cases} 2h^2 + 2h\sum_{i=1}^m l_i v_i + \sum_{i=1}^m l_i v_i^2 + \left(\sum_{i=1}^m l_i v_i\right)^2 = 0 \\ 2h^2 + 2h\sum_{i=1}^m \bar{l}_i v_i + \sum_{i=1}^m \bar{l}_i v_i^2 + \left(\sum_{i=1}^m \bar{l}_i v_i\right)^2 = 0 \end{cases}$$

and so

$$\begin{aligned} &\left[\sum_{i=1}^m (l_i - \bar{l}_i) v_i^2 + \left(\sum_{i=1}^m l_i v_i\right)^2 - \left(\sum_{i=1}^m \bar{l}_i v_i\right)^2\right] \\ &- 2\left[\sum_{i=1}^m (l_i - \bar{l}_i) v_i^2 + \left(\sum_{i=1}^m l_i v_i\right)^2 - \left(\sum_{i=1}^m \bar{l}_i v_i\right)^2\right] \sum_{i=1}^m (l_i - \bar{l}_i) v_i \sum_{i=1}^m l_i vi \\ &+ 2\left[\sum_{i=1}^m l_i v_i^2 + \left(\sum_{i=1}^m l_i v_i\right)^2\right] \left[\sum_{i=1}^m (l_i - \bar{l}_i) v_i\right]^2 = 0 \end{aligned}$$

and this has to be an identity. So, comparing the coefficients of the term v_i^4 we have

$$\left(l_i - \bar{l}_i + l_i^2 - \bar{l}_i^2 - (l_i - \bar{l}_i) l_i\right)^2 + \left(2l_i + l_i^2\right)\left(l_i - \bar{l}_i\right)^2 = 0$$

and this can be an identity only if

$$\left(2l_i + l_i^2\right)\left(l_i - \bar{l}_i\right)^2 \le 0$$

namely if $l_i = 0, -1, -2$. If $l_i = -2$ then $\bar{l}_i$ has to be equal to $-1, -2$, and if $l_i = -1$ then $\bar{l}_i$ has to be equal to $-1, -2$, but these cases are excluded by the previous assumption. If $l_i = 0$ then also $\bar{l}_i = 0$ and we consider the coefficient of another term, recalling that $l, \bar{l}$ can't be zero vectors.

□

5.1.6. Reduction to constant coefficients. Make the following symplectic change of variables

$$\begin{cases} z_h = e^{-\mathrm{i}L(h)} z'_h \\ z_k = z'_k \end{cases} \qquad \begin{cases} y = y' + \sum_{k \notin S} L(k) |z'_k|^2 \\ x = x' \end{cases} \tag{5.20}$$

where $L(k)$ is the unique l determined by the choice of k.

LEMMA 5.4. *The transformation* (5.20) *is symplectic.*

Proof. The thesis follows by a direct calculation:

$$\begin{aligned}
dy \wedge dx &= d\left(y' + \sum_{h \notin S} L(h)\left|z'_h\right|^2\right) \wedge dx' \\
&= dy' \wedge dx' + \sum_{h \notin S} L(h) d\left(\left|z'_h\right|^2\right) \wedge dx' \\
&= dy' \wedge dx' - \sum_{h \notin S} \left(L(h) dx'\right) \wedge \left(z'_h d\overline{z}'_h + \overline{z}'_h dz'_h\right), \\
\mathrm{i} dz \wedge d\overline{z} &= \mathrm{i} \sum_{h \notin S} dz_h \wedge d\overline{z}_h \\
&= \mathrm{i} \sum_{h \notin S} d\left(e^{-\mathrm{i}L(h)x'} z'_h\right) \wedge d\left(e^{\mathrm{i}L(h)x'} \overline{z}'_h\right) \\
&= \mathrm{i} \sum_{h \notin S} \left(d\left(e^{-\mathrm{i}L(h)x'}\right) z'_h + e^{-\mathrm{i}L(h)x'} dz'_h\right) \wedge \\
&\qquad\qquad \wedge \left(d\left(e^{\mathrm{i}L(h)x'}\right) \overline{z}'_h + e^{\mathrm{i}L(h)x'} d\overline{z}'_h\right) \\
&= \mathrm{i} \sum_{h \notin S} \left(-\mathrm{i}L(h) e^{-\mathrm{i}L(h)x'} dx' z'_h + e^{-\mathrm{i}L(h)x'} dz'_h\right) \wedge \\
&\qquad\qquad \wedge \left(\mathrm{i}L(h) e^{\mathrm{i}L(h)x'} dx' \overline{z}'_h + e^{\mathrm{i}L(h)x'} d\overline{z}'_h\right) \\
&= \mathrm{i} \sum_{h \notin S} dz'_h \wedge d\overline{z}'_h + \sum_{h \notin S} L(h) dx' \wedge \left(z'_h d\overline{z}'_h + \overline{z}'_h dz'_h\right),
\end{aligned}$$

hence $dy \wedge dx + \mathrm{i} dz \wedge d\overline{z} = dy' \wedge dx' + \mathrm{i} dz' \wedge d\overline{z}'$. □

Under this change of variables the Hamiltonian H_0 (5.17) becomes

$$\begin{aligned}
H_0 = \omega(\xi) \cdot y' &+ \sum_{h \notin S} \left(h^2 + \omega(\xi) \cdot L(h)\right) \left|z'_h\right|^2 \\
&+ \sum_{l \in X_3^0} c(l) \sum_{h,k \notin S\colon l \in X_3^0} z'_h \overline{z}'_k + \sum_{l \in X_3^{-2}} c(l) \sum_{h,k \notin S\colon l \in X_3^{-2}} \left(z'_h z'_k + \overline{z}'_h \overline{z}'_k\right)
\end{aligned}$$

Setting $\Omega'_h = h^2 + \omega_0 \cdot L(h)$, this is equal to k^2 if $l \in X_3^0$ and to $-k^2$ if $l \in X_3^{-2}$. Then

$$\begin{aligned}
H_0 = \omega(\xi) \cdot y' &+ \sum_{h \notin S} \Omega'_h(\xi) \left|z'_h\right|^2 \\
&+ \sum_{l \in X_3^0} c(l) \sum_{h,k \notin S\colon l \in X_3^0} z'_h \overline{z}'_k + \sum_{l \in X_3^{-2}} c(l) \sum_{h,k \notin S\colon l \in X_3^{-2}} \left(z'_h z'_k + \overline{z}'_h \overline{z}'_k\right) \\
&+ \sum_{h \notin S} (\omega - \omega_0) \cdot L(h) \left|z'_h\right|^2
\end{aligned}$$

Set $d(l) = (\omega - \omega_0) \cdot L(h)$ and denote with Q' the last two lines.

If $l \in X_3^0$ then the matrix associated to $ad(Q')$ in the basis $z'_k, \overline{z}'_k$ is

$$A^0 = \begin{pmatrix} -\Omega'_h - d(l) & -c(l) & 0 & 0 \\ -c(l) & -\Omega'_h & 0 & 0 \\ 0 & 0 & \Omega'_h + d(l) & c(l) \\ 0 & 0 & c(l) & \Omega'_h \end{pmatrix} \tag{5.21}$$

In order to study its eigenvalues, we consider the two blocks

$$A_1^0 = \begin{pmatrix} -\Omega'_h - d(l) & -c(l) \\ -c(l) & -\Omega'_h \end{pmatrix} \qquad A_2^0 = \begin{pmatrix} \Omega'_h + d(l) & c(l) \\ c(l) & \Omega'_h \end{pmatrix} \tag{5.22}$$

and note that, denoting with $\lambda(B)$ the eigenvalues of the matrix B,

$$\lambda\big(A_1^0\big) = -\lambda\big(A_2^0\big) = -\left(\Omega'_k + \lambda \begin{pmatrix} d(l) & c(l) \\ c(l) & 0. \end{pmatrix}\right)$$

If $l \in X_3^{-2}$ then the matrix associated to $ad(Q')$ in the basis $z'_k, \overline{z}'_k$ is

$$A^{-2} = \begin{pmatrix} -\Omega'_h - d(l) & c(l) & 0 & 0 \\ -c(l) & -\Omega'_h & 0 & 0 \\ 0 & 0 & \Omega'_h + d(l) & -c(l) \\ 0 & 0 & c(l) & \Omega'_h \end{pmatrix} \tag{5.23}$$

In order to study its eigenvalues, we consider the two blocks

$$A_1^{-2} = \begin{pmatrix} -\Omega'_h - d(l) & c(l) \\ -c(l) & -\Omega'_h \end{pmatrix} \qquad A_2^{-2} = \begin{pmatrix} \Omega'_h + d(l) & -c(l) \\ c(l) & \Omega'_h \end{pmatrix} \tag{5.24}$$

and note that

$$\lambda\big(A_1^{-2}\big) = -\lambda\big(A_2^{-2}\big) = -\left(\Omega'_k + \lambda \begin{pmatrix} d(l) & -c(l) \\ c(l) & 0 \end{pmatrix}\right)$$

In conclusion we obtain as eigenvalues Ω'_h plus

$$2\lambda_h = d(l) + \sqrt{d^2(l) \pm 4c^2(l)}\,, \quad 2\lambda_k = d(l) - \sqrt{d^2(l) \pm 4c^2(l)}$$

where the plus sign occurs if $l \in X_3^0$ while the minus sign occurs if $l \in X_3^{-2}$. The λ_h are the eigenvalues of the matrices:

$$A^i(l) = \begin{pmatrix} d(l) & (1+i)c(l) \\ c(l) & \Omega'_h \end{pmatrix}, \quad l \in X_3^i\,, \quad i = 1, -2. \tag{5.25}$$

The eigenvalues of these matrices are explicitly calculated in Subsection 5.3.6 in the case $m = 3$.

PROPOSITION 5.5. *There exists an open set $O_\rho \subset A_\rho$ such that for any $\xi \in O_\rho$ the two eigenvalues of each of the matrices A^0, A^{-2} are real and distinct from each other. In this set the functions $\lambda_h(\xi)$ are analytic functions and the bound* (5.4) *holds.*

Proof. For the proof see [**PP**]. One shows the existence of a region in A_ρ in where $d^2(l) - 4c^2(l) > 0$ for all $l \in X_3^{-2}$ and $d^2(l) + 4c^2(l) > 0$ for all $l \in X_3^{-2}$. The eigenvalues are analytic functions of the coefficients in the region where they are distinct. The bound follows by homogeneity and by choosing O_ρ small enough so that in its closure it is still true that all the eigenvalues are real and distinct. $\square$

Lemma 5.6. *The map $\xi \mapsto \omega(\xi)$ is a local diffeomorphism.*

Proof. Recall that ω is defined by

$$\omega = \omega_0 + \nabla_\xi A_4(\xi) - 16 A_3(\xi)$$

where

$$A_r(\xi_1, \dots, \xi_m) := \sum_{\sum_i k_i = r} \binom{r}{k_1 \dots k_m}^2 \Pi_i \xi_i^{k_i}.$$

We have to verify that the jacobian determinant of the map $\xi \mapsto \omega(\xi)$ is not identically zero. A general proof of this fact, based on algebraic (non computational) methods, is found in [**PP**], Corollary 4.7. To give a more direct proof we compute the determinant at the point $\xi_i = a$ for all $i = 1, \dots, m$. By the structure of ω the jacobian matrix has the form $\frac{\pi a^2}{2}(AI + BU)$, where I is the $m \times m$ identity matrix while U is the $m \times m$ matrix with all entries equal to one and A, B are negative integer numbers, possibly depending on m.

We compute its inverse directly. Using the fact that $U^2 = mU$, we obtain $\frac{2A}{\pi a^2}(I - \frac{B}{A+mB}U)$ which is non zero provided that $A \neq -mB$ which is trivially true since $A, B < 0$. This also gives us a bound on the Lipschitz constant of the inverse function $\xi(\omega)$ which holds true in some neighborhood of any point with all equal coordinates. Note that this proof holds for any non-linearity q.

$\square$

Finally, since the eigenvalues are all real and the eigenvalues of the same block are distinct, there exists a symplectic change of coordinates such that the system is put in a diagonal form, namely

$$H = \omega(\xi) \cdot y + \sum_{k \in S^c} \Omega_k(\xi) |z_k|^2 + P(\xi, x, y, z, \overline{z}),$$

where $\Omega_k(\xi) = k^2 + \omega_0 \cdot L(k) + \lambda_k(\xi)$ and $\lambda_k(\xi)$ are the eigenvalues of the matrices A^0 or A^{-2}.

This concludes the proof of Theorem 5.2. Now we can use the new normal form we have obtained as the unperturbed Hamiltonian in order to apply the KAM Theorem to find quasi–periodic solutions.

5.2. Application to KAM Theorem

In order to find quasi–periodic solution for the NLS 5.1 we apply the KAM Theorem 3.5, as stated by Berti–Biasco [**BB11**].

Note that we can't apply Theorem 3.6 for two reasons. Firstly, conditions (3.2) and (3.3) are not true since we are in the case of periodic boundary conditions. Secondly, the frequencies are not affine functions of the parameter ξ.

We solve the first problem using the conservation of the momentum, see Constrain 1(iii). The proof of the KAM theorem is still valid, but the Melnikov conditions need to be verified only on the subspace of functions satisfying momentum and mass conservation (see Proposition 5.8).

We verify now the KAM assumptions.

Choose $d = 2, p = \overline{p}, \delta = 0$.

The normal frequencies Ω_k are equal to $k^2 + \omega_0 \cdot L(k) + \lambda_k(\xi)$, where $L(k) \in X_3^0 \cup X_3^{-2}$, and so they have an asymptotic behavior, as stated in assumption (B). Also assumption (C) holds.

Lemma 5.7. *Choose*

$$\xi_i \in \left(\frac{r^{2\theta}}{2}, r^{2\theta}\right) \text{ with } \theta \in \left(\frac{1}{2}, \frac{3}{5}\right).$$

Then assumption (H3) holds.

Proof. We have that the following estimates hold.

$$|P_{00}|_s^\lambda < r^{10\theta}, \quad |P_{01}|_s^\lambda < r^{9\theta}, \quad |P_{11}|_s^\lambda < r^{7\theta},$$
$$|P_{10}|_s^\lambda, |P_{02}|_s^\lambda < r^{8\theta}, \quad |P_{03}|_s^\lambda < r^{5\theta}.$$

Then, if $\theta > \frac{1}{2}$, we have that

$$\max\left\{\frac{|P_{00}|_s^\lambda}{r^2\gamma}, \frac{|P_{01}|_s^\lambda}{r\gamma}, \frac{|P_{10}|_s^\lambda}{\gamma}, \frac{|P_{02}|_s^\lambda}{\gamma}\right\} \leq \frac{r^{10\theta-2}}{\gamma}.$$

Moreover, the assumption $\theta < \frac{3}{5}$ ensures that

$$|P_{11}|_s^\lambda, |P_{03}|_s^\lambda \leq r^{5\theta} \leq \frac{\gamma}{r}.$$

Finally, since the frequencies ω are homogeneous polynomial of ξ of degree 3, then $\gamma \ll r^{6\theta}$, and this is possible because $r^{10\theta-2} < r^{6\theta}$ for $\theta > \frac{1}{2}$. □

Assumption (A1) holds by Lemma 5.6.

In order to find the needed measure estimate, we can't apply Theorem 3.6 since our frequencies are not homogeneous of degree 1. We first prove the following result.

PROPOSITION 5.8. *The three Melnikov's conditions*

$$|\{\xi \in O_\rho \colon \omega(\xi)\cdot\nu + \Omega(\xi)\cdot l = 0\}| = 0$$

hold for all $(\nu, l) \in \mathbb{Z}^m \times \mathbb{Z}^\infty \setminus \{(0,0)\}, |l| \le 2$ *compatible with the conservation of* M*, namely with*

$$\sum v_i \nu_i + \sum_{k \in S^c} l_k u_k = 0,$$

where $u_k = k + L(k)\cdot v$ *for all* $k \in S^c$*. (recall that* $\sum_i L_i(k) = 0, -2$*). More precisely,*

$$\left| \bigcup_{\nu,l}^{*} \left\{ \xi \colon \left| \omega(\xi)\cdot\nu + \Omega(\xi)\cdot l < \alpha \frac{\gamma}{|\nu|^\tau} \right| \right\} \right| \le \alpha c \gamma^{\frac{1}{3}} \rho^m. \tag{5.26}$$

The union $\bigcup^*$ *denotes the union restricted to* ν, l *compatible with the conservation of* M *and* L.

PROOF. We verify that none of the (analytic) resonance functions $\omega(\xi)\cdot\nu + \Omega(\xi)\cdot l$ is identically zero. We start by computing $\omega(\xi)\cdot\nu + \Omega(\xi)\cdot l$ at the point $\xi = 0$, obtaining the integer $\omega_0\cdot\nu + \Omega'(\xi)\cdot l$ (we recall that $\Omega'_h = h^2 + L(h)\cdot\omega_0$). If this is non–zero then the function cannot be identically zero, and one has that

$$|\omega(\xi)\cdot\nu + \Omega(\xi)\cdot l| > \frac{1}{2},$$

for all $\xi \in O_\rho$ unless $|\nu| > C\rho^{-3}$. Otherwise we are left with an algebraic expression which is homogeneous of degree three in ξ.

Since ω is a diffeomorphism, we have that there exists a constant $L > 0$ such that

$$\left|J^{-1}\omega\right| \le L\left(r^{2\theta}\right)^{-2} \qquad \text{for } \xi \in O_\theta.$$

Set $\lambda(\xi) := \lambda_k(\xi) \pm \lambda_h(\xi)$. We have proved that we can choose a constant $M > 0$ (eventually big) such that

$$|\partial_\xi \lambda| < M\left(r^{2\theta}\right)^2.$$

Since the map $\xi \mapsto \omega(\xi)$ is a diffeomorphism, we invert it and consider the map $\omega \mapsto \xi(\omega)$. Assume $\omega = \frac{\nu}{|\nu|}t + w$, with w orthogonal to ν. Then

$$|\partial_\omega \lambda(\xi(\omega))| = |\partial_\xi \lambda \partial_\omega \xi| \le \left|J^{-1}\nabla_\xi \lambda\right| \le LM =: \widetilde{c}.$$

Then

$$|\partial_\omega \omega\cdot\nu + \partial_\omega \lambda(\xi(\omega))| > \nu - \widetilde{c}$$

that is greater that $\frac{|\nu|}{2}$ if $|\nu| > 2\widetilde{c}$. So in this case the resonance function cannot be zero and we have the quantitative bound

$$\left|\left\{\omega\colon |\omega\cdot\nu+\Omega(\xi(\omega))\cdot l|<\alpha\frac{\gamma}{|\nu|^\tau}\right\}\right|\le\rho^m\frac{\alpha\gamma}{|\nu|^\tau}$$

and so

$$\left|\left\{\xi\colon |\omega(\xi)\cdot\nu+\Omega(\xi)\cdot l|<\alpha\frac{\gamma}{|\nu|^\tau}\right\}\right|\le\rho^m\frac{\alpha\gamma^{\frac{1}{3}}}{|\nu|^\tau}.$$

Now we consider the case $\omega_0\cdot\nu+\Omega'(\xi)\cdot l=0$. If $|\nu|>2\widetilde{c}$ then we argue like in the previous case. Since there are only a finite number of possible functions $\lambda(\xi)$ we are left with a finite number of cases, depending on m. We wish to verify this cases numerically so we must eliminate the dependence on m. Let us leave this delicate point for the moment and conclude the proof supposing that it is true.

We notice that for the finite number of non-trivial homogeneous functions with $|\nu|<\widetilde{c}$ and $\omega_0\cdot\nu+\Omega'(\xi)\cdot l=0$, we have the bound

$$\left|\left\{\xi\colon |\omega(\xi)\cdot\nu+\Omega(\xi)\cdot l|<\alpha\frac{\gamma}{|\nu|^\tau}\right\}\right|\le\rho^m\frac{\alpha\gamma^{\frac{1}{3}}}{|\nu|^\tau}.$$

trivially provided we choose α small enough (this smallness condition does not depend on ρ but only at most on m). Indeed in this case there exists at least one direction along which the derivative doesn't vanish, and we obtain the needed estimate.

Finally, we have to take the sum over $\nu, p=h^2\pm k^2$. We can restrict to the case $|p|<\omega|\nu|$. Then we have

$$\begin{aligned}&\left|\bigcup_{\nu\in\mathbb{Z}^m,\,p\le\omega|\nu|}\left\{\xi\colon |\omega(\xi)\cdot\nu+p+\lambda(\xi)|<\alpha\frac{\gamma}{|\nu|^\tau}\right\}\right|\\
&\qquad\le\sum_{\nu\in\mathbb{Z}^m,\,p\le\omega|\nu|}\left|\left\{\xi\colon |\omega(\xi)\cdot\nu+p+\lambda(\xi)|<\alpha\frac{\gamma}{|\nu|^\tau}\right\}\right|\\
&\qquad\le\sum_{\nu\in\mathbb{Z}^m,\,p\le\omega|\nu|}\frac{\alpha\gamma^{\frac{1}{3}}}{|\nu|^\tau}\rho^m\le\sum_{\nu\in\mathbb{Z}^m}\le\frac{\alpha\gamma^{\frac{1}{3}}}{|\nu|^\tau}|\omega|^2|\nu|^2\rho^m\\
&\qquad=\alpha\gamma^{\frac{1}{3}}|\omega|^2\rho^m\sum_{\nu\in\mathbb{Z}^m}|\nu|^{2-\tau}\le c\alpha\gamma^{\frac{1}{3}}\rho^m\end{aligned}$$

if $\tau>m+3$, for some constant $c>0$. This concludes the measure estimates for the initial frequencies ω,Ω.

We now conclude the proof of the second Melnikov's condition.

Choose an index $i=1,\dots,m$ and set all the ξ_j with $j\neq i$ to zero. If our function is not identically zero under this restriction (namely as function of the only ξ_i) then it cannot be identically zero as function of all the variables.

By definition, the coefficients $c(l)$ are a finite sum of monomials which contain at least two different $\sqrt{\xi_j}$ so in our restriction $c(l)=0$ and the eigenvalues λ_h are either $d(l)$ (resp. $d(\bar{l})$ for λ_k) or zero. Then

$$(\omega-\omega_0)\cdot\nu+\lambda_h-\lambda_k=(\omega(\xi_i,0)-\omega_0)\cdot(\nu+a\,l+b\,\bar{l})\,,\quad a,b=0,\pm1.$$

A direct computation shows that

$$\omega(\xi_i,0)_i=v_i^2-12\xi_i^3\,,\quad \omega(\xi_i,0)_j=v_j^2,$$

hence $((\omega(\xi_i,0)-\omega_0)\cdot\mu)_i=-12\xi_i^3\mu_i$, which in turn implies that the function is non–zero if $\nu_i+a\,l_i+b\,\bar{l}_i$ is so. Then if $a\,l_i+b\,\bar{l}_i=0$ then also $\nu_i=0$. Since $d(l)$ can have at most 6 non–zero components, this reduces us to the case where $|\nu|\le 12$ and moreover the support of ν (i.e. its non zero components) are the same as those of l or $\bar{l}$. Without loss of generality we can suppose that this are the first (twelve) components and we can set $\xi_j=0$ whenever both l_j and $\bar{l}_j$ are zero. This finally leaves us with a finite computation (the number of functions is large but independent of m). We verify this Melnikov's conditions numerically by remarking that the condition that $\omega(\xi)\cdot\nu+\Omega(\xi)\cdot l$ is identically zero is equivalent to the condition

$$det(M)=0\,,\quad M:=\langle\omega-\omega_0,\nu\rangle\times I_{4\times4}+M_k\times I_{2\times2}-I_{2\times2}\times M_h$$

where M_h is $\pm A^i(l)$ (see formula 5.25) if $l=L(h)\in X_3^i$ $(i=0,-2)$. □

Using the previous proposition, we have the following measure estimate on the final frequencies ω_*,Ω_*. Define

$$\Pi^*_\rho:=\left\{\xi\colon|\omega_*(\xi)\cdot\nu\pm\Omega^*_k(\xi)\pm\Omega^*_h(\xi)|<\alpha\frac{\gamma}{2|\nu|^\tau}\right\}$$

for any $\nu\in\mathbb{Z}^m,h,k\in\mathbb{Z}$ with $v\cdot\nu\pm k\pm h=0$

THEOREM 5.9. *Let $\omega\colon O_\rho\to\omega(O_\rho)$ be a lipeomorphism with*

$$\left|\omega^{-1}\right|^{\text{lip}}\le L,\quad \varepsilon\le\frac{\widetilde{c}}{2LM},\tag{5.27}$$

for some constant $L,M>0$. Then $\left|O_\rho\setminus\Pi^*_\rho\right|\le c\gamma\rho^m$.

PROOF. Since (5.27) we can deduce a similar property also for the final frequencies ω_*, namely

$$\left|\omega_*^{-1}\right|^{\text{lip}}\le 2L.$$

Moreover, by (3.1), we have

$$|\omega_*-\omega|^\lambda,|\Omega_*-\Omega|^\lambda_{\bar{p}-p}\le\alpha^{-1}\gamma\varepsilon.$$

Then we have

$$\begin{aligned}&|\omega_*(\xi)\cdot\nu+\Omega^*_k(\xi)-\Omega^*_h(\xi)|\\&\qquad\ge|\omega(\xi)\cdot\nu+\Omega_k(\xi)-\Omega_h(\xi)|-|\omega_*(\xi)-\omega(\xi)||\nu|-2|\Omega^*_k-\Omega_k|\end{aligned}$$

$$\geq \frac{c\alpha\gamma^{\frac{1}{3}}}{|\nu|^{\tau}} - (|\nu| + 2)\alpha^{-1}\gamma\varepsilon \geq \frac{c\alpha\gamma^{\frac{1}{3}}}{2|\nu|^{\tau}}$$

if $|\nu| < \left(\frac{c\alpha^2\gamma^{-\frac{2}{3}}}{\varepsilon}\right)^{\frac{1}{\tau+1}}$.

On the contrary, if $|\nu| < \left(\frac{c\alpha^2\gamma^{-\frac{2}{3}}}{\varepsilon}\right)^{\frac{1}{\tau+1}}$, then we can always choose $\varepsilon \leq \frac{c\alpha^2\gamma^{-\frac{2}{3}}}{(4LM)^{1+\tau}}$, and so we have $|\nu| > 4LM$ and we can argue as in the previous proposition, obtaining the needed measure estimates on the set of the final frequencies. $\square$

5.3. Example: case $m = 3$

5.3.1. Constraints 1. We impose a finite number of constraints on the set of the normal sites $S = \{v_1, v_2, v_3\}$. The first ones are

$$\eta_1 v_1 + \eta_2 v_2 + \eta_3 v_3 \neq 0$$

with $\eta_i \in \mathbb{Z}$, $|\eta_i| \leq 4$, $i = 1, 2, 3$, and $\sum_{i=1}^{3} \eta_i = 0$, $\sum_{i=1}^{3} |\eta_i| \leq 8$. Note that this is Constraints 1(i) in 5.1.3 and that this covers also (ii) and (iii).

5.3.2. The sets X_3^0, X_3^{-2}. Under Constraints 1, the sets X_3^0, X_3^{-2} are the following.

$$X_3^0 = \{(2,-2,0), (2,-1,-1), (1,-3,2), (3,-3,0) \text{ and all their permutations}\}, \tag{5.28}$$

$$X_3^{-2} = \{(1,-2,-1), (2,-2,-2), (2,-1,-3) \text{ and all their permutations}\}. \tag{5.29}$$

Then, for $l \in X_3^0$ conditions (5.14) become the following

(Case 1)

$$\begin{cases} 2v_i - 2v_j + h - k = 0 \\ 2v_i^2 - 2v_j^2 + h^2 - k^2 = 0 \end{cases}$$

(Case 2)

$$\begin{cases} 2v_i - v_j - v_l + h - k = 0 \\ 2v_i^2 - v_j^2 - v_l^2 + h^2 - k^2 = 0 \end{cases}$$

(Case 3)

$$\begin{cases} v_i - 3v_j + 2v_l + h - k = 0 \\ v_i^2 - 3v_j^2 + 2v_l^2 + h^2 - k^2 = 0 \end{cases}$$

(Case 4)

$$\begin{cases} 3v_i - 3v_j + h - k = 0 \\ 3v_i^2 - 3v_j^2 + h^2 - k^2 = 0 \end{cases}$$

while, for $l \in X_3^{-2}$ conditions (5.15) become the following

(Case 5)

$$\begin{cases} v_i - 2v_j - v_l + h + k = 0 \\ v_i^2 - 2v_j^2 - v_l^2 + h^2 + k^2 = 0 \end{cases}$$

(Case 6)

$$\begin{cases} 2v_i - 2v_j - 2v_l + h + k = 0 \\ 2v_i^2 - 2v_j^2 - 2v_l^2 + h^2 + k^2 = 0 \end{cases}$$

(Case 7)

$$\begin{cases} 2v_i - v_j - 3v_l + h + k = 0 \\ 2v_i^2 - v_j^2 - 3v_l^2 + h^2 + k^2 = 0 \end{cases}$$

5.3.3. Constraints 2. To ensure that the function $L(h)\colon h \mapsto l$ is bijective, we impose the following constraints:

$$\begin{aligned} &18v_i^2v_j - 15v_i^3 + 27v_i^2v_l - 7v_iv_j^2 - 22v_iv_jv_l - 16v_iv_l^2 + 4v_j^2v_l \\ &\qquad + 7v_jv_l^2 + v_j^3 + 3v_l^3 \neq 0 \end{aligned}$$

$$\begin{aligned} &32v_iv_jv_l^2 - 94v_iv_j^2v_l + 62v_i^2v_jv_l - 19v_j^2v_l^2 + 44v_j^3v_l + 48v_i^3v_j - 12v_i^3v_l \\ &\qquad - 103v_i^2v_j^2 - 13v_i^2v_l^2 + 100v_iv_j^3 - 9v_i^4 - 36v_j^4 + 2v_jv_l^3 - 2v_iv_l^3 \neq 0 \end{aligned}$$

$$\begin{aligned} &75v_i^4 + 36v_l^4 + 134v_i^2v_jv_l - 40v_iv_j^2v_l - 94v_iv_l^2v_j + 293v_i^2v_l^2 - 2v_j^3v_i - 164v_iv_l^3 \\ &\qquad + 23v_i^2v_j^2 - 60v_i^3v_j + 2v_j^3v_l + 17v_j^2v_l^2 + 20v_jv_l^3 - 240v_i^3v_l \neq 0 \end{aligned}$$

$$\begin{aligned} &3v_i^4 - v_iv_j^3 - 15v_iv_l^3 - v_iv_jv_l^2 + 5v_iv_j^2v_l + 3v_l^4 + v_j^3v_l - 4v_j^2v_l^2 \\ &\qquad + 3v_jv_l^3 - 14v_i^3v_l + 2v_i^3v_j - v_i^2v_j^2 + 23v_i^2v_l^2 - 4v_i^2v_jv_l \neq 0 \end{aligned}$$

$$9v_i - 2v_j - 7v_l \neq 0$$

$$7v_i - 6v_j - v_l \neq 0$$

$$\begin{aligned} &v_j^4 + 8v_j^3v_l + 15v_j^2v_l^2 + 8v_jv_l^3 + v_l^4 - 66v_i^3v_j - 66v_i^3v_l + 45v_i^2v_j^2 + 45v_i^2v_l^2 \\ &\qquad - 12v_iv_j^3 - 12v_l^3v_i + 33v_i^4 - 54v_iv_jv_l^2 + 108v_i^2v_jv_l - 54v_iv_j^2v_l \neq 0 \end{aligned}$$

$$\begin{aligned} &11v_l^2v_iv_j + 11v_jv_lv_i^2 - 22v_j^2v_lv_i - 9v_j^4 + 24v_j^3v_l - 25v_j^2v_l^2 + 13v_jv_l^3 \\ &\qquad - 3v_l^4 - v_lv_i^3 + v_i^3v_j - 7v_i^2v_j^2 + 12v_iv_j^3 - 4v_l^2v_i^2 - v_l^3v_i \neq 0 \end{aligned}$$

$$\begin{aligned} &3v_l^2v_iv_j - 9v_jv_lv_i^2 + 6v_j^2v_lv_i - 18v_j^4 + 78v_j^3v_l - 120v_j^2v_l^2 + 79v_jv_l^3 \\ &\qquad - 19v_l^4 + v_lv_i^3 - v_i^3v_j + 6v_i^2v_j^2 - 6v_iv_j^3 + 3v_l^2v_i^2 - 3v_l^3v_i \neq 0 \end{aligned}$$

$$\begin{aligned} &3v_j^4 + 2v_j^3v_l - v_j^2v_l^2 - v_jv_l^3 - 15v_i^3v_j - 14v_iv_j^3 + 23v_i^2v_j^2 + 3v_i^3v_l \\ &\qquad - 4v_i^2v_l^2 + v_l^3v_i + 3v_i^4 - v_i^2v_jv_l + 5v_iv_jv_l^2 - 4v_iv_j^2v_l \neq 0 \end{aligned}$$

$$\begin{aligned} &4v_j^2v_iv_l + 16v_i^2v_jv_l - 20v_l^2v_iv_j + 36v_j^4 - 150v_j^3v_l + 223v_j^2v_l^2 - 142v_jv_l^3 \\ &\qquad + 33v_l^4 + 6v_j^3v_i + 10v_l^3v_i - 11v_i^2v_j^2 - 5v_i^2v_l^2 - 2v_i^3v_l + 2v_i^3v_j \neq 0 \end{aligned}$$

5.3.4. The frequencies. The explicit expression of the frequencies of the Hamiltonian (5.11) is

$$\omega_i(\xi) = v_i^2 + \frac{\pi}{2}\left[-12\xi_i^3 - 72 \sum_{\substack{j\neq i\\ j=1,\dots,3}} \xi_i\xi_j^2 - 96 \sum_{\substack{j\neq i\\ j=1,\dots,3}} \xi_i^2\xi_j \right.$$
$$\left. -144 \sum_{\substack{j\neq l\neq i, j\neq i\\ j,l=1,\dots,3}} \xi_i\xi_j\xi_l \right] \qquad \text{for } i = 1,\dots,3$$

$$\Omega_k^0(\xi) = k^2$$

$$\Omega_{hk}^1(\xi) = \frac{\pi}{2}\left[48 \sum_{\substack{i\neq j=1,\dots,3\\ \text{Case 1}}} \xi_i\xi_j^2 e^{i(2x_i-2x_j)} + 72 \sum_{\substack{i\neq j\neq l=1,\dots,3\\ \text{Case 1}}} \xi_i\xi_j\xi_l e^{i(2x_i-2x_j)} \right.$$
$$+ 48 \sum_{\substack{i\neq j\neq l=1,\dots,3\\ \text{Case 2}}} \xi_i^2 \sqrt{\xi_j\xi_l} e^{i(2x_i-x_j-x_l)}$$
$$+ 144 \sum_{\substack{i\neq j\neq l=1,\dots,3\\ \text{Case 2}}} \xi_i\xi_j \sqrt{\xi_j\xi_l} e^{i(2x_i-x_j-x_l)}$$
$$+ 48 \sum_{\substack{i\neq j\neq l=1,\dots,3\\ \text{Case 3}}} \xi_i\xi_j \sqrt{\xi_j\xi_l}\xi_i\xi_j e^{i(x_i-3x_j+2x_l)}$$
$$\left. +16 \sum_{\substack{i\neq j=1,\dots,3\\ \text{Case 4}}} \xi_i\xi_j \sqrt{\xi_i\xi_j} e^{i(3x_i-3x_j)} \right]$$

$$\Omega_{hk}^2(\xi) = \frac{\pi}{2}\left[96 \sum_{\substack{i\neq j\neq l=1,\dots,3\\ \text{Case 5}}} \xi_j^2 \sqrt{\xi_i\xi_l} e^{i(x_i-2x_j-x_l)} \right.$$
$$+ 144 \sum_{\substack{i\neq j\neq l=1,\dots,3\\ \text{Case 5}}} \xi_i\xi_j \sqrt{\xi_j\xi_l} e^{i(x_i-2x_j-x_l)}$$
$$+ 144 \sum_{\substack{i\neq j\neq l=1,\dots,3\\ \text{Case 5}}} \xi_j\xi_l \sqrt{\xi_i\xi_l} e^{i(xi_i-2\xi_j-\xi_l)}$$
$$+ 72 \sum_{\substack{i\neq j\neq l=1,\dots,3\\ \text{Case 6}}} \xi_i\xi_j\xi_l e^{i(2x_i-2x_j-2x_l)}$$
$$\left. +48 \sum_{\substack{i\neq j\neq l=1,\dots,3\\ \text{Case 7}}} \xi_i\xi_l \sqrt{\xi_j\xi_l} e^{i(2x_i-x_j-3x_l)} \right]$$

5.3.5. The values of $d(l), c(l), \Omega'_k$. We also calculate the values of $d(l), c(l), \Omega'_k$ in the several cases:

(Case 1)

$$\begin{aligned}
\Omega'_k &= k^2 = \Omega'_h \\
d(l) &= \frac{\pi}{2}\left[-24\xi_i^3 + 24\xi_j^3 + 192\xi_j^2\xi_l + 144\xi_j\xi_l^2 + 48\xi_i\xi_j^2\right. \\
&\qquad \left. -144\xi_i\xi_l^2 - 48\xi_i^2\xi_j - 192\xi_i^2\xi_l\right] \\
c(l) &= \frac{\pi}{2}\left[48\xi_i\xi_j^2 + 72\xi_i\xi_j\xi_l\right]
\end{aligned}$$

(Case 2)

$$\begin{aligned}
\Omega'_k &= k^2 = \Omega'_h \\
d(l) &= \frac{\pi}{2}\left[-24\xi_i^3 + 12\xi_j^3 + 12\xi_l^3 + 168\xi_j^2\xi_l + 168\xi_j\xi_l^2 - 48\xi_i\xi_j^2\right. \\
&\qquad \left. -48\xi_i\xi_l^2 - 120\xi_i^2\xi_j - 120\xi_i^2\xi_l\right] \\
c(l) &= \frac{\pi}{2}\left[96\xi_i^2\sqrt{\xi_j\xi_l} + 144\xi_i\xi_j\sqrt{\xi_j\xi_l}\right]
\end{aligned}$$

(Case 3)

$$\begin{aligned}
\Omega'_k &= k^2 = \Omega'_h \\
d(l) &= \frac{\pi}{2}\left[-12\xi_i^3 + 36\xi_j^3 - 24\xi_l^3 + 144\xi_j^2\xi_l + 24\xi_j\xi_l^2 + 216\xi_i\xi_j^2\right. \\
&\qquad \left. -264\xi_i\xi_l^2 + 120\xi_i^2\xi_j - 240\xi_i^2\xi_l\right] \\
c(l) &= \frac{\pi}{2}\left[48\xi_i\xi_j\sqrt{\xi_j\xi_l}\right]
\end{aligned}$$

(Case 4)

$$\begin{aligned}
\Omega'_k &= k^2 = \Omega'_h \\
d(l) &= \frac{\pi}{2}\left[-36\xi_i^3 + 36\xi_j^3 + 288\xi_j^2\xi_l + 216\xi_j\xi_l^2 + 72\xi_i\xi_j^2\right. \\
&\qquad \left. -216\xi_i\xi_l^2 - 72\xi_i^2\xi_j - 288\xi_i^2\xi_l\right] \\
c(l) &= \frac{\pi}{2}\left[16\xi_i\xi_j\sqrt{\xi_i\xi_j}\right]
\end{aligned}$$

(Case 5)

$$\begin{aligned}
\Omega'_k &= k^2 = -\Omega'_h \\
d(l) &= \frac{\pi}{2}\left[-12\xi_i^3 + 24\xi_j^3 + 12\xi_l^3 + 576\xi_i\xi_j\xi_l + 264\xi_j^2\xi_l + 24\xi_j\xi_l^2\right. \\
&\qquad \left. +120\xi_i\xi_j^2 + 24\xi_i\xi_l^2 + 48\xi_i^2\xi_j - 24\xi_i^2\xi_l\right] \\
c(l) &= \frac{\pi}{2}\left[96\xi_j^2\sqrt{\xi_i\xi_l} + 144\xi_i\xi_j\sqrt{\xi_j\xi_l} + 144\xi_j\xi_l\sqrt{\xi_i\xi_l}\right]
\end{aligned}$$

(Case 6)

$$\begin{aligned}
\Omega'_k &= k^2 = -\Omega'_h \\
d(l) &= \frac{\pi}{2}\left[-24\xi_i^3 + 24\xi_j^3 + 24\xi_l^3 + 576\xi_i\xi_j\xi_l + 336\xi_j^2\xi_l + 336\xi_j\xi_l^2\right. \\
&\qquad\qquad \left. +48\xi_i\xi_j^2 + 48\xi_i\xi_l^2 - 48\xi_i^2\xi_j - 48\xi_i^2\xi_l\right] \\
c(l) &= \frac{\pi}{2}\left[144\xi_i\xi_j\xi_l\right]
\end{aligned}$$

(Case 7)

$$\begin{aligned}
\Omega'_k &= k^2 = -\Omega'_h \\
d(l) &= \frac{\pi}{2}\left[-24\xi_i^3 + 12\xi_j^3 + 36\xi_l^3 + 576\xi_i\xi_j\xi_l + 312\xi_j^2\xi_l + 360\xi_j\xi_l^2\right. \\
&\qquad\qquad \left. -48\xi_i\xi_j^2 + 144\xi_i\xi_l^2 - 120\xi_i^2\xi_j + 24\xi_i^2\xi_l\right] \\
c(l) &= \frac{\pi}{2}\left[48\xi_i\xi_l\sqrt{\xi_j\xi_l}\right]
\end{aligned}$$

5.3.6. The eigenvalues of the matrices A^0, A^{-2}. We compute now the values of the eigenvalues of the matrices A^0, A^{-2} in the several cases:

(Case 1)

$$\begin{aligned}
\lambda_k = k^2 &- 12\xi_i^3 + 24\xi_i\xi_j^2 + 12\xi_j^3 + 96\xi_j^2\xi_l + 72\xi_j\xi_l^2 - 72\xi_i\xi_l^2 - 24\xi_i^2\xi_j \\
&- 96\xi_i^2\xi_l \pm 12\sqrt{\begin{aligned}&-140\xi_i^2\xi_j^2\xi_l^2 + 16\xi_i^2\xi_j^4 - 48\xi_i^3\xi_j^2\xi_l + 12\xi_i^3\xi_j\xi_l^2 + 32\xi_i\xi_j^4\xi_l \\ &+12\xi_i\xi_j^3\xi_l^2 - 96\xi_j^2\xi_l^3\xi_i - 72\xi_j\xi_l^4\xi_i - 96\xi_j\xi_l^3\xi_i^2 + 32\xi_i^4\xi_j\xi_l - 10\xi_i^3\xi_j^3 \\ &+76\xi_i^4\xi_l^2 + 4\xi_i^5\xi_j + 16\xi_i^5\xi_l + 4\xi_i\xi_j^5 + 16\xi_j^5\xi_l + 76\xi_j^4\xi_l^2 + 96\xi_j^3\xi_l^3 \\ &+36\xi_j^2\xi_l^4 + 36\xi_i^2\xi_l^4 + 96\xi_i^3\xi_l^3 + \xi_i^6 + \xi_j^6\end{aligned}}
\end{aligned}$$

(Case 2)

$$\begin{aligned}
\lambda_k = k^2 &+ 6\xi_j^3 - 12\xi_i^3 - 24\xi_i\xi_j^2 + 6\xi_l^3 + 84\xi_j^2\xi_l + 84\xi_j\xi_l^2 - 24\xi_i\xi_l^2 - 60\xi_i^2\xi_j \\
&- 60\xi_i^2\xi_l \pm 6\sqrt{\begin{aligned}&456\xi_i^4\xi_j\xi_l + 792\xi_i^3\xi_j^2\xi_l + 276\xi_i^2\xi_j^3\xi_l + \xi_j^6 + 4\xi_i^6 + \xi_l^6 \\ &-120\xi_j^3\xi_i\xi_l^2 + 24\xi_i^3\xi_j\xi_l^2 - 120\xi_i\xi_j^2\xi_l^3 - 112\xi_i\xi_j^4\xi_l - 528\xi_i^2\xi_j^2\xi_l^2 \\ &-300\xi_l^3\xi_i^2\xi_j - 112\xi_j\xi_l^4\xi_i + 76\xi_j^3\xi_i^3 - 8\xi_j^5\xi_i + 394\xi_j^3\xi_l^3 + 28\xi_j^5\xi_l \\ &+224\xi_j^4\xi_l^2 - 4\xi_j^4\xi_i^2 + 116\xi_i^4\xi_j^2 + 76\xi_i^3\xi_l^3 + 116\xi_i^4\xi_l^2 + 40\xi_i^5\xi_j \\ &+40\xi_i^5\xi_l + 224\xi_l^4\xi_j^2 + 28\xi_l^5\xi_j - 8\xi_l^5\xi_i - 4\xi_l^4\xi_i^2\end{aligned}}
\end{aligned}$$

(Case 3)

$$\begin{aligned}
\lambda_k = k^2 &- 6\xi_i^3 + 18\xi_j^3 + 72\xi_j^2\xi_l + 12\xi_j\xi_l^2 + 108\xi_i\xi_j^2 - 12\xi_l^3 + 60\xi_i^2\xi_j \\
&- 120\xi_i^2\xi_l - 132\xi_i\xi_l^2 \pm 6\sqrt{\begin{aligned}&-400\xi_i^4\xi_j\xi_l - 744\xi_i^3\xi_j^2\xi_l + 184\xi_i^2\xi_j^3\xi_l + 9\xi_j^6 \\ &+\xi_i^6 + 4\xi_l^6 - 60\xi_j^3\xi_i\xi_l^2 - 444\xi_i^3\xi_j\xi_l^2 - 600\xi_i\xi_j^2\xi_l^3 + 432\xi_i\xi_j^4\xi_l\end{aligned}}
\end{aligned}$$

$$\overline{-1232\xi_i^2\xi_j^2\xi_l^2 - 120\xi_l^3\xi_i^2\xi_j - 88\xi_j\xi_l^4\xi_i + 354\xi_j^3\xi_i^3 + 108\xi_j^5\xi_i + 36\xi_j^3\xi_l^3}$$
$$\overline{+72\xi_j^5\xi_l + 156\xi_j^4\xi_l^2 + 384\xi_j^4\xi_i^2 + 64\xi_i^4\xi_j^2 + 884\xi_i^3\xi_l^3 + 444\xi_i^4\xi_l^2}$$
$$\overline{-20\xi_i^5\xi_j + 40\xi_i^5\xi_l - 44\xi_l^4\xi_j^2 - 8\xi_l^5\xi_j + 88\xi_l^5\xi_i + 564\xi_l^4\xi_i^2}$$

(Case 4)

$$\lambda_k = k^2 - 18\xi_i^3 + 144\xi_j^2\xi_l + 108\xi_j\xi_l^2 + 36\xi_i\xi_j^2 + 18\xi_j^3 - 36\xi_i^2\xi_j$$
$$- 144\xi_i^2\xi_l - 108\xi_i\xi_l^2 \pm 2\sqrt{2592\xi_i^4\xi_j\xi_l - 3888\xi_i^3\xi_j^2\xi_l - 3888\xi_i^2\xi_j^3\xi_l + 81\xi_j^6}$$
$$\overline{+81\xi_i^6 + 972\xi_j^3\xi_i\xi_l^2 + 972\xi_i^3\xi_j\xi_l^2 + 2592\xi_i\xi_j^4\xi_l - 14256\xi_i^2\xi_j^2\xi_l^2}$$
$$\overline{-7776\xi_l^3\xi_i^2\xi_j - 5832\xi_j\xi_l^4\xi_i - 746\xi_j^3\xi_i^3 + 324\xi_j^5\xi_i + 7776\xi_j^3\xi_l^3}$$
$$\overline{+1296\xi_j^5\xi_l + 6156\xi_j^4\xi_l^2 + 7776\xi_i^3\xi_l^3 + 6156\xi_i^4\xi_l^2}$$
$$\overline{+324\xi_i^5\xi_j + 1296\xi_i^5\xi_l + 2916\xi_l^4\xi_j^2 + 2916\xi_l^4\xi_i^2}$$

(Case 5)

$$\lambda_k = -k^2 + 6\xi_l^3 + 132\xi_j^2\xi_l + 12\xi_j\xi_l^2 + 60\xi_i\xi_j^2 + 24\xi_i^2\xi_j - 12\xi_i^2\xi_l + 12\xi_j^3$$
$$+ 12\xi_i\xi_l^2 + 288\xi_i\xi_j\xi_l - 6\xi_i^3 \pm 6\sqrt{-768\xi_j^3\sqrt{\xi_i\xi_l}\xi_i\sqrt{\xi_j\xi_l} + 1392\xi_j^3\xi_i\xi_l^2}$$
$$\overline{+552\xi_i^2\xi_j^3\xi_l - 276\xi_j^2\xi_l^3\xi_i - 1152\xi_i\xi_j^2\sqrt{\xi_j\xi_l}\xi_l\sqrt{\xi_i\xi_l} + 376\xi_j^4\xi_i\xi_l}$$
$$\overline{+192\xi_l^3\xi_i^2\xi_j + 104\xi_l^4\xi_i\xi_j + 2272\xi_j^2\xi_l^2\xi_i^2 + 300\xi_j^2\xi_l\xi_i^3 - 180\xi_j\xi_l^2\xi_i^3}$$
$$\overline{-112\xi_i^4\xi_j\xi_l + 48\xi_l^4\xi_j^2 + 4\xi_l^5\xi_j + 92\xi_l^3\xi_j^3 + 4\xi_l^5\xi_i - 10\xi_l^3\xi_i^3 + 492\xi_j^4\xi_l^2}$$
$$\overline{+88\xi_j^5\xi_l + 116\xi_i^2\xi_j^4 + 76\xi_i^3\xi_j^3 + 40\xi_i\xi_j^5 - 4\xi_i^4\xi_j^2}$$
$$\overline{-8\xi_i^5\xi_j + 4\xi_i^5\xi_l + \xi_l^6 + 4\xi_j^6 + \xi_i^6}$$

(Case 6)

$$\lambda_k = -k^2 - 12\xi_i^3 + 12\xi_j^3 + 12\xi_l^3 + 168\xi_j^2\xi_l + 168\xi_j\xi_l^2 + 24\xi_i\xi_j^2 + 288\xi_i\xi_j\xi_l$$
$$- 24\xi_i^2\xi_j - 24\xi_i^2\xi_l + 24\xi_i\xi_l^2 \pm 12\sqrt{732\xi_j^3\xi_i\xi_l^2 + 36\xi_i^2\xi_j^3\xi_l + 732\xi_j^2\xi_l^3\xi_i}$$
$$\overline{+104\xi_j^4\xi_i\xi_l + 36\xi_l^3\xi_i^2\xi_j + 104\xi_l^4\xi_i\xi_j + 328\xi_j^2\xi_l^2\xi_i^2 - 132\xi_j^2\xi_l\xi_i^3}$$
$$\overline{-132\xi_j\xi_l^2\xi_i^3 - 40\xi_i^4\xi_j\xi_l + 224\xi_l^4\xi_j^2 + 28\xi_l^5\xi_j + 394\xi_l^3\xi_j^3 + 4\xi_l^5\xi_i}$$
$$\overline{-10\xi_l^3\xi_i^3 + 224\xi_j^4\xi_l^2 + 28\xi_j^5\xi_l - 10\xi_i^3\xi_j^3 + 4\xi_i\xi_j^5 + 4\xi_i^5\xi_j}$$
$$\overline{+4\xi_i^5\xi_l + \xi_l^6 + \xi_j^6 + \xi_i^6}$$

(Case 7)

$$\lambda_k = -k^2 - 12\xi_i^3 + 6\xi_j^3 + 18\xi_l^3 + 156\xi_j^2\xi_l + 180\xi_j\xi_l^2 - 24\xi_i\xi_j^2 + 288\xi_i\xi_j\xi_l$$
$$- 60\xi_i^2\xi_j + 12\xi_i^2\xi_l + 72\xi_i\xi_l^2 \pm 6\sqrt{2280\xi_j^3\xi_i\xi_l^2 - 900\xi_i^2\xi_j^3\xi_l + 3480\xi_j^2\xi_l^3\xi_i}$$

$$\overline{-112\xi_j^4\xi_i\xi_l + 1148\xi_l^3\xi_i^2\xi_j + 1008\xi_l^4\xi_i\xi_j + 1712\xi_j^2\xi_l^2\xi_i^2 - 1080\xi_j^2\xi_l\xi_i^3}$$
$$\overline{-168\xi_j\xi_l^2\xi_i^3 - 232\xi_i^4\xi_j\xi_l + 1056\xi_l^4\xi_j^2 + 180\xi_l^5\xi_j + 1566\xi_l^3\xi_j^3 + 72\xi_l^5\xi_i}$$
$$\overline{+36\xi_l^3\xi_i^3 + 736\xi_j^4\xi_l^2 + 52\xi_j^5\xi_l - 4\xi_i^2\xi_j^4 + 76\xi_i^3\xi_j^3 - 8\xi_i\xi_j^5 + 116\xi_i^4\xi_j^2}$$
$$\overline{+40\xi_i^5\xi_j - 8\xi_i^5\xi_l + 9\xi_l^6 + \xi_j^6 + 4\xi_i^6 - 44\xi_i^4\xi_l^2 + 156\xi_l^4\xi_i^2}$$

5.3.7. Second Melnikov's conditions. We have to prove that the second Melnikov condition holds, namely that

$$\langle \omega, \nu \rangle + \lambda_k - \lambda_h \neq 0 \tag{5.30}$$

for all $\nu \in \mathbb{Z}^3$, where λ_k, λ_h are the eigenvalues of the matrices A^0, A^{-2} corresponding to $k = (h_1, k_1), h = (h_2, k_2)$ with $l \in X_3^0, X_3^{-2}$. Each of these matrices has two eigenvalues $\lambda_i = a_i \pm \sqrt{b_i}$, hence with the difference $\lambda_k - \lambda_h$ we denote the four possible differences of the two couples. We can only consider the condition

$$\langle \omega - \omega_0, \nu \rangle + \widetilde{\lambda}_k - \widetilde{\lambda}_h \tag{5.31}$$

where $\widetilde{\lambda}_i := \lambda_i - i^2$, since the rest is the constant part. Call M_i the 2×2–matrix with eigenvalues $\widetilde{\lambda}_i$.

Recall the tensor product between matrices: given A, B 2×2–matrices,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \tag{5.32}$$

then the product is 4×4–matrix $A \times B$ with elements

$$A \times B = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix} \tag{5.33}$$

Important fact: called a_1, a_2 the eigenvalues of A and b_1, b_2 the eigenvalues of B, then the eigenvalues of the matrix $A \times I - I \times B$, where I is the identity matrix, are their four differences $a_1 - b_1, a_1 - b_2, a_2 - b_1, a_2 - b_2$.

Call M_i the 2×2–matrix that has $\widetilde{\lambda}_i$ as eigenvalues. Then the 4×4 matrix

$$M := \langle \omega - \omega_0, \nu \rangle \times I_{4\times4} + M_k \times I_{2\times2} - I_{2\times2} \times M_h$$

has (5.31) as eigenvalues. In order to prove that they are not identically zero (and so the second Melnikov condition holds) it is sufficient to prove that the determinant of M is not identically zero. Call D this determinant, that is a 12^{th}–order polynomial. We divide the various case of k (from l):

(Cases 1-2) Case (h_1, k_1) such that $2v_i - 2v_j + h - k = 0$, (h_2, k_2) such that $2v_i - v_j - v_l + h - k = 0$. In this case D is not identically zero for all the choice of the integer vector $\nu \in \mathbb{Z}^3$. Set $\nu = (A, B, C)$. I calculate the derivative of the determinant. For example, $\frac{\partial^{12}}{\partial \xi_i^{12}} D = 0$ if $A^2(A-2)(A+2) = 0$. Then I can choose $(A, B, C) = (0, 0, 0)$ or $(0, 1, -1)$ such that $\frac{\partial^{12}}{\partial \xi_i^{12}} D = 0$, $\frac{\partial^{12}}{\partial \xi_i^{10} \partial \xi_j^2} D = 0$, $\frac{\partial^{12}}{\partial \xi_i^{10} \partial \xi_j \partial \xi_l} D = 0$ but $\frac{\partial^{12}}{\partial \xi_i^{9} \partial \xi_j^2 \partial \xi_l} D \neq 0$. For $A = -2, 2$ and any value of B, C we have that $\frac{\partial^{12}}{\partial \xi_i^{12}} D = 0$, but $\frac{\partial^{12}}{\partial \xi_i^{10} \partial \xi_j \partial \xi_l} D \neq 0$.

(Cases 1-3) Case (h_1, k_1) such that $2v_i - 2v_j + h - k = 0$, (h_2, k_2) such that $v_i - 3v_j + 2v_l + h - k = 0$. In this case D is not identically zero for all the choice of the integer vector $\nu \in \mathbb{Z}^3$. Set $\nu = (A, B, C)$. I calculate the derivative of the determinant. For example, $\frac{\partial^{12}}{\partial \xi_l^{12}} D = 0$ if $C^2(C-2) = 0$. Then I can choose $(A, B, C) = (0, 0, 0)$, $(-2, 2, 0)$, $(-1, -1, 2)$ or $(1, -2, 3)$ such that $\frac{\partial^{12}}{\partial \xi_l^{12}} D = 0$, $\frac{\partial^{12}}{\partial \xi_j^{2} \partial \xi_l^{10}} D = 0$ and $\frac{\partial^{12}}{\partial \xi_i \partial \xi_j^{8} \partial \xi_l^{3}} D = 0$ but $\frac{\partial^{12}}{\partial \xi_i^{2} \partial \xi_j^{3} \partial \xi_l^{7}} D \neq 0$.

(Cases 1-4) Case (h_1, k_1) such that $2v_i - 2v_j + h - k = 0$, (h_2, k_2) such that $3v_i - 3v_j + h - k = 0$.

In this case D is not identically zero for all the choice of the integer vector $\nu \in \mathbb{Z}^3$. Set $\nu = (A, B, C)$. I calculate the derivative of the determinant. For example, $\frac{\partial^{12}}{\partial \xi_l^{12}} D = 0$ if $F^4 = 0$. Then I can choose $(A, B, C) = (0, 0, 0)$, $(1, -1, 0)$, $(3, -3, 0)$ or $(-2, 2, 0)$ such that $\frac{\partial^{12}}{\partial \xi_l^{12}} D = 0$, $\frac{\partial^{12}}{\partial \xi_i^{4} \partial \xi_l^{8}} D = 0$, $\frac{\partial^{12}}{\partial \xi_i^{4} \partial \xi_j \partial \xi_l^{7}} D = 0$ but $\frac{\partial^{12}}{\partial \xi_i^{2} \partial \xi_j^{10}} D \neq 0$.

(Cases 2-3) Case (h_1, k_1) such that $2v_i - v_j - v_l + h - k = 0$, (h_2, k_2) such that $v_i - 3v_j + 2v_l + h - k = 0$.

In this case D is not identically zero for all the choice of the integer vector $\nu \in \mathbb{Z}^3$. Set $\nu = (A, B, C)$. I calculate the derivative of the determinant. For example, $\frac{\partial^{12}}{\partial \xi_i^{12}} D = 0$ if

$$A(A-1)(A+2)(A+1) = 0.$$

Then for $(A, B) = (0, 0)$, $(1, -3)$, $(-2, 1)$ or $(-1, -2)$ and any value of C we have $\frac{\partial^{12}}{\partial \xi_i^{12}} D = 0$, $\frac{\partial^{12}}{\partial \xi_i^{11} \partial \xi_j} D = 0$ but $\frac{\partial^{12}}{\partial \xi_i^{2} \partial \xi_j^{9} \partial \xi_l} D \neq 0$.

(Cases 2-4) Case (h_1, k_1) such that $2v_i - v_j - v_l + h - k = 0$, (h_2, k_2) such that $3v_i - 3v_j + h - k = 0$.

In this case D is not identically zero for all the choice of the integer vector $\nu \in \mathbb{Z}^3$. Set $\nu = (A, B, C)$. I calculate the derivative of the determinant. For example, $\frac{\partial^{12}}{\partial \xi_i^{12}} D = 0$ if

$$A(A-1)(A-3)(A+2) = 0$$

. Then for $(A, B) = (0, 0)$, $(1, -2)$ or $(3, -3)$ and any value of C we have $\frac{\partial^{12}}{\partial \xi_i^{12}} D = 0$, $\frac{\partial^{12}}{\partial \xi_i^{11} \partial \xi_j} D = 0$ but $\frac{\partial^{12}}{\partial \xi_i^{5} \partial \xi_j^{7}} D \neq 0$.

(Cases 3-4) Case (h_1,k_1) such that $v_i-3v_j+2v_l+h-k=0$, (h_2,k_2) such that $3v_i-3v_j+h-k=0$.

In this case D is not identically zero for all the choice of the integer vector $\nu\in\mathbb{Z}^3$. Set $\nu=(A,B,C)$. I calculate the derivative of the determinant. For example, $\frac{\partial^{12}}{\partial\xi_l^{12}}D=0$ if $C^2(C+2)^2=0$. Then I can choose $(A,B,C)=(0,0,0)$, $(3,-3,0)$, $(2,0,-2)$ or $(1,3,-2)$ such that $\frac{\partial^{12}}{\partial\xi_l^{12}}D=0$, $\frac{\partial^{12}}{\partial\xi_j^2\partial\xi_l^{10}}D=0$, $\frac{\partial^{12}}{\partial\xi_i\partial\xi_j^5\partial\xi_l^6}D=0$ but $\frac{\partial^{12}}{\partial\xi_i^4\partial\xi_j^4\partial\xi_l^4}D\neq0$.

(Cases 1-5) Case (h_1,k_1) such that $2v_i-2v_j+h-k=0$, (h_2,k_2) such that $v_i-2v_j-v_l+h+k=0$.

In this case D is not identically zero for all the choice of the integer vector $\nu\in\mathbb{Z}^3$. Set $\nu=(A,B,C)$. I calculate the derivative of the determinant. For example, $\frac{\partial^{12}}{\partial\xi_l^{12}}D=0$ if $C^2(C+1)^2=0$. I can choose $(A,B,C)=(0,0,0)$, $(-2,0,0)$, $(-2,2,0)$, $(0,2,0)$, $(1,1,-1)$, $(-1,1,-1)$, $(-1,3,-1)$ or $(1,3,-1)$ such that $\frac{\partial^{12}}{\partial\xi_l^{12}}D=0$, $\frac{\partial^{12}}{\partial\xi_l^{10}\partial\xi_j^2}D=0$, $\frac{\partial^{12}}{\partial\xi_l^{10}\partial\xi_i^2}D=0$ but $\frac{\partial^{12}}{\partial\xi_i\partial\xi_j^3\partial\xi_l^8}D\neq0$.

(Cases 1-6) Case (h_1,k_1) such that $2v_i-2v_j+h-k=0$, (h_2,k_2) such that $2v_i-2v_j-2v_l+h+k=0$.

In this case D is not identically zero for all the choice of the integer vector $\nu\in\mathbb{Z}^3$. Set $\nu=(A,B,C)$. I calculate the derivative of the determinant. For example, $\frac{\partial^{12}}{\partial\xi_l^{12}}D=0$ if $C^2(C+2)^2=0$. Then I can choose $(A,B,C)=(0,0,0)$, $(2,0,0)$, $(-2,0,0)$, $(-2,2,0)$, $(2,2,0)$, $(2,0,-2)$, $(-2,0,-2)$, $(0,0,-2)$,$(0,-2,-2)$, $(2,-2,-2)$ or $(-2,-2,-2)$ such that $\frac{\partial^{12}}{\partial\xi_l^{12}}D=0$, $\frac{\partial^{12}}{\partial\xi_l^{10}\partial\xi_j^2}D=0$, $\frac{\partial^{12}}{\partial\xi_i^{12}}D=0$ but $\frac{\partial^{12}}{\partial\xi_i^6\partial\xi_j^4\partial\xi_l^2}D\neq0$.

(Cases 1-7) Case (h_1,k_1) such that $2v_i-2v_j+h-k=0$, (h_2,k_2) such that $2v_i-v_j-3v_l+h+k=0$.

In this case D is not identically zero for all the choice of the integer vector $\nu\in\mathbb{Z}^3$. Set $\nu=(A,B,C)$. I calculate the derivative of the determinant. For example, $\frac{\partial^{12}}{\partial\xi_l^{12}}D=0$ if $C^2(C+3)^2=0$. Then I can choose $(A,B,C)=(0,0,0)$, $(-2,2,0)$, $(0,1,-3)$, $(2,-1,-3)$ such that $\frac{\partial^{12}}{\partial\xi_l^{12}}D=0$, $\frac{\partial^{12}}{\partial\xi_j^2\partial\xi_l^{10}}D=0$, $\frac{\partial^{12}}{\partial\xi_i\partial\xi_j^5\partial\xi_l^6}D=0$ but $\frac{\partial^{12}}{\partial\xi_i^6\partial\xi_j^4\partial\xi_l^2}D\neq0$.

(Cases 2-5) Case (h_1,k_1) such that $2v_i-v_j-v_l+h-k=0$, (h_2,k_2) such that $v_i-2v_j-v_l+h+k=0$.

In this case D is not identically zero for all the choice of the integer vector $\nu\in\mathbb{Z}^3$. Set $\nu=(A,B,C)$. I calculate the derivative of the determinant. For example, $\frac{\partial^{12}}{\partial\xi_l^{12}}D=0$ if

$$C^2(C-1)(C+1)=0.$$

Then I can choose $(A,B,C) = (0,0,0)$, $(-1,1,0)$, $(-2,1,1)$ or $(1,-1,-1)$ such that $\frac{\partial^{12}}{\partial\xi_l^{12}}D = 0$, $\frac{\partial^{12}}{\partial\xi_i^3\partial\xi_l^9}D = 0$, $\frac{\partial^{12}}{\partial\xi_i^2\partial\xi_j\partial\xi_l^9}D = 0$ but $\frac{\partial^{12}}{\partial\xi_i\partial\xi_j^3\partial\xi_l^8}D \neq 0$.

(Cases 2-6) Case (h_1,k_1) such that $2v_i - v_j - v_l + h - k = 0$, (h_2,k_2) such that $2v_i - 2v_j - 2v_l + h + k = 0$.

In this case D is not identically zero for all the choice of the integer vector $\nu \in \mathbb{Z}^3$. Set $\nu = (A,B,C)$. I calculate the derivative of the determinant. For example, $\frac{\partial^{12}}{\partial\xi_i^{12}}D = 0$ if

$$A^2(A-2)(A+2) = 0.$$

Then I can choose $(A,B,C) = (0,0,0)$, $(0,0,-1)$, $(0,-1,-1)$ or $(0,-1,0)$, $(2,-2,-2)$ or $(-2,1,1)$ such that $\frac{\partial^{12}}{\partial\xi_i^{12}}D = 0$, $\frac{\partial^{12}}{\partial\xi_i^8\partial\xi_j^4}D = 0$, $\frac{\partial^{12}}{\partial\xi_i^{10}\partial\xi_l^2}D = 0$ but $\frac{\partial^{12}}{\partial\xi_i^8\partial\xi_j\partial\xi_l^3}D \neq 0$.

(Cases 2-7) Case (h_1,k_1) such that $2v_i - v_j - v_l + h - k = 0$, (h_2,k_2) such that $2v_i - v_j - 3v_l + h + k = 0$.

In this case D is not identically zero for all the choice of the integer vector $\nu \in \mathbb{Z}^3$. Set $\nu = (A,B,C)$. I calculate the derivative of the determinant. For example, $\frac{\partial^{12}}{\partial\xi_i^{12}}D = 0$ if

$$A^2(A-2)(A+2) = 0.$$

Then I can choose $(A,B,C) = (2,-1,-3)$, value that cancels the first factor, or $(A,B,C) = (0,0,0)$, value that cancels the second factor, or $(A,B,C) = (0,0,-2)$, value that cancels the third factor, or $(A,B,C) = (-2,1,1)$, value that cancels the fourth factor, such that $\frac{\partial^{12}}{\partial\xi_i^{12}}D = 0$, $\frac{\partial^{12}}{\partial\xi_i^{11}\partial\xi_j}D = 0$, $\frac{\partial^{12}}{\partial\xi_i^{10}\partial\xi_j^2} = 0$ but $\frac{\partial^{12}}{\partial\xi_i^{10}\partial\xi_j\partial\xi_l} \neq 0$.

(Cases 3-5) Case (h_1,k_1) such that $v_i - 3v_j + 2v_l + h - k = 0$, (h_2,k_2) such that $v_i - 2v_j - v_l + h + k = 0$.

In this case D is not identically zero for all the choice of the integer vector $\nu \in \mathbb{Z}^3$. Set $\nu = (A,B,C)$. I calculate the derivative of the determinant. For example, $\frac{\partial^{12}}{\partial\xi_l^{12}}D = 0$ if

$$C(C+3)(C+2)(C+1) = 0$$

Then I can choose $(A,B,C) = (0,1,-3)$, $(1,-2,-1)$, $(-1,3,-2)$ or $(0,0,0)$ such that $\frac{\partial^{12}}{\partial\xi_l^{12}}D = 0$, $\frac{\partial^{12}}{\partial\xi_i\partial\xi_l^{11}}D = 0$, $\frac{\partial^{12}}{\partial\xi_i^2\partial\xi_l^{10}}D = 0$ but $\frac{\partial^{12}}{\partial\xi_i\partial\xi_j^2\partial\xi_l^9}D \neq 0$.

(Cases 3-6) Case (h_1,k_1) such that $v_i - 3v_j + 2v_l + h - k = 0$, (h_2,k_2) such that $2v_i - 2v_j - 2v_l + h + k = 0$.

In this case D is not identically zero for all the choice of the integer vector $\nu \in \mathbb{Z}^3$. Set $\nu = (A,B,C)$. I calculate the derivative of the determinant. For example, $\frac{\partial^{12}}{\partial \xi_i^{12}} D = 0$ if

$$A(A-1)(A-2)(A+1) = 0.$$

Then I can choose $(A, B, C) = (2, -2, -2)$, $(0, 0, 0)$, $(1, 1, -4)$ or $(-1, 3, -2)$ such that $\frac{\partial^{12}}{\partial \xi_i^{12}} D = 0$, $\frac{\partial^{12}}{\partial \xi_i^{11} \partial \xi_j} D = 0$, $\frac{\partial^{12}}{\partial \xi_i^{10} \partial \xi_j^2} D = 0$ but $\frac{\partial^{12}}{\partial \xi_i^{8} \partial \xi_j^2 \partial \xi_l^2} D \neq 0$.

(Cases 3-7) Case (h_1, k_1) such that $v_i - 3v_j + 2v_l + h - k = 0$, (h_2, k_2) such that $2v_i - v_j - 3v_l + h + k = 0$.

In this case D is not identically zero for all the choice of the integer vector $\nu \in \mathbb{Z}^3$. Set $\nu = (A, B, C)$. I calculate the derivative of the determinant. For example, $\frac{\partial^{12}}{\partial \xi_i^{12}} D = 0$ if

$$A(A-1)(A-2)(A+1) = 0.$$

Then I can choose $(A, B, C) = (2, -1, -3)$, $(0, 0, 0)$, $(1, 2, -5)$ or $(A, B, C) = (-1, 3, -2)$such that $\frac{\partial^{12}}{\partial \xi_i^{12}} D = 0$, $\frac{\partial^{12}}{\partial \xi_i^{11} \partial \xi_j} D = 0$ and $\frac{\partial^{12}}{\partial \xi_i^{10} \partial \xi_j^2} D = 0$ but $\frac{\partial^{12}}{\partial \xi_i^{8} \partial \xi_j^3 \partial \xi_l} D \neq 0$.

(Cases 4-5) Case (h_1, k_1) such that $3v_i - 3v_j + h - k = 0$, (h_2, k_2) such that $v_i - 2v_j - v_l + h + k = 0$.

In this case D is not identically zero for all the choice of the integer vector $\nu \in \mathbb{Z}^3$. Set $\nu = (A, B, C)$. I calculate the derivative of the determinant. For example, $\frac{\partial^{12}}{\partial \xi_l^{12}} D = 0$ if $\mathbb{C}^2(C+1)^2 = 0$. I can choose $(A, B, C) = (1, -2, -1)$, $(-2, 1, -1)$, $(0, 0, 0)$ or $(-3, 3, 0)$ such that $\frac{\partial^{12}}{\partial \xi_l^{12}} D = 0$, $\frac{\partial^{12}}{\partial \xi_i^{2} \partial \xi_l^{10}} D = 0$, $\frac{\partial^{12}}{\partial \xi_i^{3} \partial \xi_l^{9}} D = 0$ but $\frac{\partial^{12}}{\partial \xi_i^{2} \partial \xi_j^2 \partial \xi_l^8} D \neq 0$.

(Cases 4-6) Case (h_1, k_1) such that $3v_i - 3v_j + h - k = 0$, (h_2, k_2) such that $2v_i - 2v_j - 2v_l + h + k = 0$.

In this case D is not identically zero for all the choice of the integer vector $\nu \in \mathbb{Z}^3$. Set $\nu = (A, B, C)$. I calculate the derivative of the determinant. For example, $\frac{\partial^{12}}{\partial \xi_i^{12}} D = 0$ if

$$A(A-2)(A+3)(A+1) = 0$$

Then I can choose $(A, B, C) = (-1, 1, -2)$, $(-3, 3, 0)$, $(2, -2, -2)$ or $= (0, 0, 0)$ such that $\frac{\partial^{12}}{\partial \xi_i^{12}} D = 0$, $\frac{\partial^{12}}{\partial \xi_i^{11} \partial \xi_j} D = 0$, $\frac{\partial^{12}}{\partial \xi_i^{10} \partial \xi_j^2} D = 0$ but $\frac{\partial^{12}}{\partial \xi_i^{8} \partial \xi_j^2 \partial \xi_l^2} \neq D0$.

(Cases 4-7) Case (h_1, k_1) such that $3v_i - 3v_j + h - k = 0$, (h_2, k_2) such that $2v_i - v_j - 3v_l + h + k = 0$.

In this case D is not identically zero for all the choice of the integer vector $\nu \in \mathbb{Z}^3$. Set $\nu = (A, B, C)$. I calculate the derivative of the determinant. For example, $\frac{\partial^{12}}{\partial \xi_i^{12}} D = 0$ if

$$A(A-2)(A+3)(A+1) = 0$$

Then I can choose $(A,B,C) = (-1,2,-3)$, $(-3,3,0)$, $(2,-1,-3)$ or $(0,0,0)$, value that cancels the fourth factor, such that $\frac{\partial^{12}}{\partial \xi_i^{12}} D = 0$, $\frac{\partial^{12}}{\partial \xi_i^{11} \partial \xi_j} D = 0$, $\frac{\partial^{12}}{\partial \xi_i^{10} \partial \xi_j^2} = 0$ but $\frac{\partial^{12}}{\partial \xi_i^8 \partial \xi_j^3 \partial \xi_l} \neq 0$.

(Cases 5-6) Case (h_1,k_1) such that $v_i - 2v_j - v_l + h + k = 0$, (h_2,k_2) such that $2v_i - 2v_j - 2v_l + h + k = 0$.

In this case D is not identically zero for all the choice of the integer vector $\nu \in \mathbb{Z}^3$. Set $\nu = (A,B,C)$. I calculate the derivative of the determinant. For example, $\frac{\partial^{12}}{\partial \xi_l^{12}} D = 0$ if

$$C(C-1)(C+2)(C+1) = 0$$

Then I can choose $(A,B,C) = (2,-2,-2)$, $(0,0,0)$, $(1,0,-1)$ or $(-1,2,1)$ such that $\frac{\partial^{12}}{\partial \xi_l^{12}} D = 0$, $\frac{\partial^{12}}{\partial \xi_i \partial \xi_l^{11}} D = 0$, $\frac{\partial^{12}}{\partial \xi_i^2 \partial \xi_l^{10}} = 0$ but $\frac{\partial^{12}}{\partial \xi_i \partial \xi_j^2 \partial \xi_l^9} \neq 0$.

(Cases 5-7) Case (h_1,k_1) such that $v_i - 2v_j - v_l + h + k = 0$, (h_2,k_2) such that $2v_i - v_j - 3v_l + h + k = 0$.

In this case D is not identically zero for all the choice of the integer vector $\nu \in \mathbb{Z}^3$. Set $\nu = (A,B,C)$. I calculate the derivative of the determinant. For example, $\frac{\partial^{12}}{\partial \xi_l^{12}} D = 0$ if

$$C(C-1)(C+3)(C+2) = 0$$

Then I can choose $(A,B,C) = (2,-1,-3)$, $(-1,2,1)$, $(1,1,-2)$ or $(0,0,0)$ such that $\frac{\partial^{12}}{\partial \xi_l^{12}} D = 0$, $\frac{\partial^{12}}{\partial \xi_i \partial \xi_l^{11}} D = 0$, $\frac{\partial^{12}}{\partial \xi_i^2 \partial \xi_l^{10}} = 0$ but $\frac{\partial^{12}}{\partial \xi_i \partial \xi_j^2 \partial \xi_l^9} \neq 0$.

(Cases 6-7) Case (h_1,k_1) such that $2v_i - 2v_j - 2v_l + h + k = 0$, (h_2,k_2) such that $2v_i - v_j - 3v_l + h + k = 0$.

In this case D is not identically zero for all the choice of the integer vector $\nu \in \mathbb{Z}^3$. Set $\nu = (A,B,C)$. I calculate the derivative of the determinant. For example, $\frac{\partial^{12}}{\partial \xi_i^{12}} D = 0$ if

$$C(C-2)(C+3)(C+1) = 0.$$

Then I can choose $(A,B,C) = (2,-1,-3)$ or $(-2,2,2)$ such that $\frac{\partial^{12}}{\partial \xi_i^{12}} D = 0$, $\frac{\partial^{12}}{\partial \xi_i^{11} \partial \xi_l} D = 0$, $\frac{\partial^{12}}{\partial \xi_i^{10} \partial \xi_l^2} = 0$ but $\frac{\partial^{12}}{\partial \xi_i^6 \partial \xi_j^4 \partial \xi_l^2} \neq 0$

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