Comments on "Mechanics and Thermodynamics of the Bernoulli oscillators I and II"

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1 Summary

We revise here the fundamental structure of our oscillators model aimed at removing (some of) the incongruities between quantum and classical mechanics. To the purpose of clarity and completeness, we provide a number of additional comments, refinements, calculation details and corriges compared to published work. We suggest a physical interpretation of some subtle circumstances we have met with in numerical analysis. To make the reader easy with calculations, we include links to graphic and math programs ([1]).

PACS. 45.50.-j - Dynamics and kinematics of a particle and a system of particles

PACS. 03.65.Ta - Foundations of Quantum Mechanics

2 Introduction

For the interested reader, concerning the model expounded in refs. [1,2], we provide in these notes a few auxilia to those lectures. A very synthetic but more ordered exposition of the main topics with a few additional comments, theoretical refinements and details of numerical solution procedures is given here. So the basic model equations will be found in this paper again, resumed step by step in the subsequent sections. For each section, we also give new elements of solution and numerical details ([2]). For the sake of brevity, the obvious definitions of the most current quantities used in standard models are assumed to be known to the reader; however, they are easily recovered by the context or via specific references we provide to the main papers.

3 Theoretical basis

In our model, the quantum wave equation is modified in such a way to accommodate a space-variable current density probability even in the stationary case. Using a specific law (see eq. (9)), the resulting quantum density turns out consistent with the classical ensemble density corresponding to a collection of oscillators with mechanical energies included in a certain interval ΔE. This last quantity corresponds to a typical quantum incertitude. Both the new

[1] Use Adobe handtool with boxed items to connect. In case of link failure, Google-search files as indicated in the section Demonstrative Software. For now, we give connection to the pdf prints of these files. Whenever interested to the operational math files, they will be made available writing to *mastroci@unina.it*.

[2] We add an errata corrige concerning the so called Second Step Procedure produced in ref. [2].

wave equation and the classical energy theorem implied in this theoretical amalgama include, by assumption, the existence of the peculiar mass effect formally described by what we call the "mass eigenfunctions" $m_{eff}(x)$ (eq. (15)). So far, the model applies to one-dimensional (1D) closed motions. The quantum system description essentially follows the Madelung formalism; so the equivalent statistical ensemble of oscillators is regarded as a fluid-dynamic system or gas, composed by two counterrunning beams of particles. These ones are taken as forward/backward, periodic flows of particles in the quantum mechanically admitted space domain. At the same time, we look at the gas element at a position x as to a particles packet moving along the admitted (eulerian-like) velocity fields. Once a solution to the equations set for these quantum-classical motions is made available, it is intended that the (many particles) statistical ensemble state can be commuted into a single-particle dynamical state, i.e. is resolvable in classical time laws, by the help of the ergodic principle (see eqs. (62)÷(65) in [1]).

We start with a section dedicated to the wave equation as expressed by our model. It is a non-linear equation whose solution implies accounting for a number of conditions dictated by various physical conditions for completion. All of them are reported in this first section; in the subsequent sections, quantities more specifically relevant to to the classical-like part of the model are calculated. A main example of calculation (the 3rd level of the harmonic oscillator) is expounded with details. As far as the present notes may be updated (at the same http address), other practical examples might be found added in the future.

4 Wave equation

The basic wave equation is expressed as follows :

1) basic ansatz for the wave function (1D case; x ≡space coordinate; $\rho(x)$ ≡quantum density; $\varphi(x) = S(x)/\hbar \equiv$ quantum wave phase):

$$\Psi_n(x) = \sqrt{\rho(x)} exp\left[i\varphi(x)\right] = \sqrt{\rho(x)} exp\left[iS(x)/\hbar\right] \tag{1}$$

$$\frac{1}{2}\oint \rho(x)dx = 1 \tag{2}$$

Comment : the same that in standard QM. The factor 1/2 in the norm condition is because $\rho(x)$ accounts for the 2 counterrunning beams of particles accommodated in the oscillator statistical space domain. As is obvious, eq. (2) specifies a typical amplitude parameter, to be attached to the un-normalised wave function as given by the wave equation; we choose this parameter to be the maximum

value attained by the density in space, i.e. $\rho_{\max}$. Therefore, named x_n (a positive determination, we assume symmetry around x = 0) the position where this maximum is attained, $\rho(x)$ takes the form $\rho_{\max}\rho_n(x)$ ([3]), with $\rho_n(x_n) = 1$. The density $\rho(x)$ fits into both : the standard quantum interpretation, as the probability to find the particle at position x ([4]); and the classical one, as the statistical particle density in the fluid-dynamic flow composed by the assumed equivalent collection of classical oscillators.

2) wave equation ($\Phi(x)$ = imposed potential energy, with even symmetry in x assumed; $\nabla \equiv d/d_x \equiv '$):

$$\frac{\nabla S^2(x)}{2m} - \frac{\hbar^2}{2m}\frac{\sqrt{\rho(x)}''}{\sqrt{\rho(x)}} = E_n - \Phi(x) \tag{3}$$

$$Lim[\rho(x),\ |x| \to x_0^*] = 0 \tag{4}$$

$$\rho'(\pm x_n) = \nabla S'(\pm x_n) = S''(\pm x_n) = 0 \tag{5}$$

$$\rho'(0) = 0 \tag{6}$$

$$S''(0) = \nabla S'(0) = 0 \tag{7}$$

Comment : Madelung hydrodynamic-like expression; formally coincident with the standard one in orthodox QM, but actually $non - standard$ because the current density $J(x) = \rho(x)\nabla S(x)$ is a *variable* quantity in our model. The occurrence is best enlightened by the $\nabla S(x)$ definition given in eq. (44) of ref. [1]; also reported here in (9). $E_n \equiv nth$ energy eigenvalue (same value as in orthodox QM); the coordinate x_0^* is the extreme boundary (intended in its positive determination) attainable by the quantum particles in space, where $\rho(x)$ evanesces (for the harmonic oscillator HO, $x_0^* \to \infty$). Boundary conditions in x = 0 are due to the assumed even parity in x of the potential energy $\Phi(x)$, with $\Phi'(0) = 0$. Due to symmetry therefore, we formally confine our treatment to the right half part of the space domain, included between x = 0 and $x = x_0^* > 0$. See note ([7]) and its context in ref. [1]. For the same reason, we assume the $\nabla S(x)$ determination symmetrical and ≥ 0 everywhere.

3) Phase or *phase quantization* condition (for periodic closed motions; n = quantum number ≥ 1) :

$$\oint \nabla S(x)dx = 2\pi(n-1)\hbar \tag{8}$$

Comment : same as in standard QM of quasi-classical travelling modes in a cavity. In our model, generally $\nabla S(x) \neq 0$ (except for n = 1) and this is even in the stationary cases where standard QM wavefunctions are real quantities.

[3] here n stays for $normalised\,to\,the\,maximum\ value$, ρ_n is an adimensional quantity.

[4] In our previous papers, we somewhat erroneously named "*Bohr* postulate" the current interpretation of $|\Psi_n(x)|^2$ as a probability density; not for fussiness, we note that it is instead quoted in the literature as *Born*'s statistical principle.

4.1 Expression of $\nabla S(\mathbf{x})$

Solving our wave equation asks for the expression of $\nabla S(\mathrm{x})$. As anticipated, here we deviate from the orthodox QM model. We correlate the phase gradient to classical-like quantities, in such a way:

4) Constitutive equation for the phase gradient ($\mathrm{v}_{_D}(\mathrm{x}) =$ center of mass packet velocity at x; $\mathrm{m}_{eff}(\mathrm{x}) = \mathrm{n}th$ level mass eigenfunction; $c_n = const$):

$$\nabla S(\mathrm{x}) = \mathrm{m}_{eff}(\mathrm{x})\mathrm{v}_{_D}(\mathrm{x}) - \frac{c_n \mathrm{h}}{4}\rho(\mathrm{x}) \tag{9}$$

Comment : *non standard* expression; in standard QM (stationary cases), $\nabla S(\mathrm{x})$ is simply $J_0/\rho(\mathrm{x})$, with current density $J_0 = const$. Expression (9) connects the quantum wave phase to a classical-like momentum field with variable mass $\mathrm{m}_{eff}(\mathrm{x})$. Note that the corresponding expression of $\nabla S(\mathrm{x})$ in standard spinless Bohmian mechanics (or in pilot wave theories) would be written here as $\mathrm{m}\,\mathrm{v}_{_D}(\mathrm{x}) \propto \rho(\mathrm{x})^{-1}$. On one hand, equation (9) can be interpreted in a three waves interaction scheme; on the other, it is consistent with the typical expression of the statistical average of classical densities fields (see eqs. (17), (22) and (60) in [1]) corresponding to an ensemble of mechanical energies E. These ones take indeed all the values between the extreme values E_{ni}, E_{nf} of the incertitude interval. The last is centered in $\mathrm{E}_n = (\mathrm{E}_{ni} + \mathrm{E}_{nf})/2$. See eq. (22) in the sequel and (126) in [3].

4.1.1 Expressions for $\mathbf{v}_{_D}(\mathbf{x})$ and $\mathbf{m}_{eff}(\mathbf{x})$

To substantiate (9) we need to give:

5) Expression for $\mathrm{v}_{_D}(\mathrm{x})$:

$$\mathrm{v}_{_D}(\mathrm{x}) = \frac{2\nu_n(\mathrm{x})}{\rho(\mathrm{x})} \tag{10}$$

Here $\nu_n(\mathrm{x})$ $[\mathrm{s}^{-1}]$ is the one-beam (f.i. the forward beam ([5])) volume flow $\equiv$ average number of particles belonging to the statistical oscillators ensemble and flowing out of a section placed at position x per unit time).

Comment: The factor 2 is due to the definition of the density as including both forward and backward beams. Velocity fields and particles flows are taken in their absolute values all throughout our calculations.

6) Expression for $\nu_n(\mathrm{x})$ ($\mathrm{x}_n \equiv$ position of maximum density $\equiv$ boundary between Region I and Region II):

$$\nu_n(\mathrm{x}) = \nu_{n\mathrm{I}}(\mathrm{x})\ \mathrm{UnitStep}[\mathrm{x}_n - \mathrm{x}] + \nu_{n\mathrm{II}}(\mathrm{x})\ (1 - \mathrm{UnitStep}[\mathrm{x}_n - \mathrm{x}]) \tag{11}$$

[5] Motion is closed and the forward and backward beams are x-symmetric, they have equal flow-functions absolute values at all positions.

$$\nu_{n\mathrm{I}} = \nu_{n0} = \frac{\mathrm{E}_{nf} - \mathrm{E}_{ni}}{c_n \mathrm{h}} \tag{12}$$

$$\nu_{n\mathrm{II}}(\mathrm{x}) =$$

$$= \frac{-\tau_n c_n \mathrm{h}\ \rho(\mathrm{x})^2 + \sqrt{8\mathrm{m}\,\mu_n\,\rho(\mathrm{x})^{5/2} + \tau_n^2 c_n^2 \mathrm{h}^2 \rho(\mathrm{x})^4}}{4\mathrm{m}} \tag{13}$$

$$\mu_n = \left[\tau_n c_n + \frac{2\mathrm{m}\nu_{n0}}{\mathrm{h}\ \rho_{\max}^2}\right] \frac{\mathrm{h}\nu_{n0}}{\sqrt{\rho_{\max}}} \tag{14}$$

Comment: in the oscillator core (Region I), $\nu_{n\mathrm{I}}(\mathrm{x}) = const = \nu_{n0}$ (closed packet); outside this (Region II), $\nu_n'(\mathrm{x}) \neq 0$ (open packet); approaching boundaries, $\nu_n(\mathrm{x}) \rightarrow 0$. See eqs.(29)-(30) in [1]. The expression for $\nu_{n\mathrm{II}}(\mathrm{x})$ i.e. in Region II is from eqs. (31)−(33) in [1]([6]). These equations are brought to us by the fact that in Region II, in the spirit of a double-solution principle, the quantity $\mathrm{v}_D^2(\mathrm{x})$ evanesces asymptotically as $\sqrt{\rho(\mathrm{x})}$. See also (65)÷(68) in [4] ([7]). The μ_n expression is for the sake of continuity between expressions (12) and (13) at the separation point x_n.

7) Expression for $\mathrm{m}_{eff}(\mathrm{x})$: in [1] we have found the expression

$$\mathrm{m}_{eff}(\mathrm{x}) = \frac{\mathrm{c}_n \mathrm{h} \rho(\mathrm{x})^2}{8\nu_n(\mathrm{x})} \times$$

$$\times \left\{1 + \left[\frac{\rho(\mathrm{x}_n)^2}{\rho(\mathrm{x})^2} g(\mathrm{x}) + sign[\rho'(\mathrm{x})]\ \sigma_n\,|1 - \mathrm{c}_n|\,\frac{\rho'(\mathrm{x})^2}{\rho(\mathrm{x})^2} - 1\right] \mathrm{UnitStep}[\mathrm{x}_n - \mathrm{x}]\right\} \tag{15}$$

$$\rho(\mathrm{x}_n) = \rho_{\max} \tag{16}$$

$$< \mathrm{m}_{eff}(\mathrm{x}) >= \frac{1}{2} \oint \rho(\mathrm{x})\ \mathrm{m}_{eff}(\mathrm{x}) d\mathrm{x} = \mathrm{m} \tag{17}$$

Comment: equation (17) means that mass fluctuations must keep the mean statistical value (17) identical with the predefined particles mass m.

[6] Errata: eq. (33) in [1] lacks a factor m multiplying $\rho(\mathrm{x})^{5/2}$. The right version is eq.(13) here.

[7] In this reference, please take care that the definition of the Regions is not the same than here: this is explained in [1], end of pag. 102.

5 Energy theorem

8) The previous equations are shown in [1] to be consistent with a classical-like energy theorem of the form

$$\frac{1}{2}\,\mathrm{m}_{eff}(\mathrm{x})\ \mathrm{v}(\mathrm{x,E})^2 + \mathrm{E}_{nf} - c_n \mathrm{h}\nu_n(\mathrm{x}) - \frac{\nabla S^2(\mathrm{x})}{2\mathrm{m}_{eff}(\mathrm{x})} \ = \mathrm{E} \tag{18}$$

Comment :This equation defines the eulerian velocity fields composing the equivalent fluid-dynamic system in our model. So every field v(x,E) complies with an energy E belonging to the interval (E_{ni}, E_{nf}). We name T(E) the corresponding classical period. The inverse quantity $[\mathrm{T(E)v(x,E)}/2]^{-1}$ defines the classical density named $\rho_{_{SP}}$ as shown in eq. (17) of ref. [1]. Summing all the densities $\rho_{_{SP}}$ as in eqs. (22), (23) in [1] amounts to the total quantum density ρ(x) (to show this, use the ∇S(x) expression (9); see also next eqs. (21) and (87)).

Up to now, we have met with the undefined function g(x) and the 8 unknown coefficients $\rho_{\max}$, c_n, ν_{n0}, μ_n, τ_n , σ_n, E_{ni}, E_{nf}. All of these quantities must be determined for each specific case, when solving the equations by numerical techniques. We have indeed to choose them in such a way that all boundary conditions and physical requirements are satisfied. These last also involve the additional conditions just given in the next section. The ensemble of these conditions installs a multiple eigenvalues problem to be solved in order to find a congruent solution for our model. However, some of the coefficients are linked to each other in a simple manner: f. i. we have easy expressions for ν_{n0} (eq. (12)) and μ_n (eq. (14)). The $\mathrm{x}_n$value must also be determined; see eq. (15) in [2].

6 Statistical ensemble properties

Since - by ergodicity - the admitted energy levels density is proportional to the period T(E), we also state the following:

9) Density levels expression (or probability to find a particle with energy between E, E+dE):

$$\mathrm{P(E)} = \frac{1}{c_n \mathrm{h}}\mathrm{T(E)} \tag{19}$$

$$\frac{1}{c_n \mathrm{h}} \int_{\mathrm{E}_{ni}}^{\mathrm{E}_{nf}} \mathrm{T(E)}d\mathrm{E} = 1 \tag{20}$$

10) Mean statistical density expression (see also eqs.(17) and (22) in [1]):

$$\frac{1}{c_n \mathrm{h}} \int_{\mathrm{E}_{ni}}^{\mathrm{E}_{nf}} \mathrm{T(E)} \rho_{_{SP}} \mathrm{(x,E)} d\mathrm{E} = \frac{2}{c_n \mathrm{h}} \mathrm{Re} \left\{ \int_{\mathrm{E}_{ni}}^{\mathrm{E}_{nf}} \frac{d\mathrm{E}}{\mathrm{v(x,E)}} \right\} = \rho(\mathrm{x}) \qquad (21)$$

Comment : the mean statistical value of the densities pertaining to the velocities ensemble must be equal to the quantum density as it appears in (3) for a congruent model. It is possible to demonstrate analytically that both eqs. (20) and (21) are automatically satisfied if we use the expression (9) in (18); so we have not to list these conditions amongst those we have to check for, when we choose the coefficients values to work out numerical solutions.

11) Mean statistical energy of the oscillators ensemble :

$$< \mathrm{E} >_n = \frac{1}{c_n \mathrm{h}} \int_{\mathrm{E}_{ni}}^{\mathrm{E}_{nf}} \mathrm{T(E)E} d\mathrm{E} = \frac{\mathrm{E}_{ni} + \mathrm{E}_{nf}}{2} \qquad (22)$$

$$\frac{\mathrm{E}_{ni} + \mathrm{E}_{nf}}{2} = \mathrm{E}_n \qquad (23)$$

Comment : The average value (22) must be equal to the quantum eigenvalue E_n.This condition is not satisfied *a priori* in simple virtue of (18), so it must be carefully checked for during calculations to be insured.

7 Thermodynamic model

A last condition must be satisfied in our model for maximum congruence with the quantum one: this is clearly explained in eqs. (69), (70) in [1]. It requires equality of the thermodynamic energies calculated in both models. Thus we have

12) thermodynamic congruence :

$$< \mathrm{U}_c >_{fluct} = \mathrm{U}_{qm}\,(\mathrm{T}) \qquad (24)$$

Comment : see the mentioned eqs. in [1] for details. Here note that eq. (24), together with (23), allows us to determine the values E_{ni}, E_{nf} by a separate procedure with respect to the other coefficients. For the harmonic oscillator, this appears in eq (17) of ref. [2].

8 Conclusive comments

The mentioned $g(\mathrm{x})$ and the 6 remaining coefficients in the expounded equations frame finally have to comply with the following conditions : 2 parity conditions (6), (7); 4 integral conditions (2), (8), (17), (22); the continuity condition (14) and the additional congruence condition (12). However, with our positions, if we consider only $g(\mathrm{x})$ functions with $g'(0) = 0$, it is easy to demonstrate that condition (7) is automatically satisfied when (6) holds, so that we will drop off discussion about this in the following. On the same step, we do not list here the boundary conditions (5) because we impose them "a priori" for all the coefficients sets we may check. But unfortunately, as far as the other listed conditions are concerned, finding the appropriate coefficients to demonstrate the existence of good solutions revealed not to be quite an easy matter mathematically. So to the purpose of clarity and for future developments, we want to discuss this topic here with more details than given already in [1,2].

9 Solution procedures for the wave equation

In [1,2] we examined the typical cases of the harmonic oscillator (HO) and rectangular well (RW). We give here more and new elements for the solution procedures which can be emploied. We first reelaborate the wave equation expression as follows :

$$\frac{\nabla S^2(\mathrm{x})}{2\mathrm{m}} - \frac{\hbar^2}{2\mathrm{m}} \frac{\sqrt{\rho(\mathrm{x})}''}{\sqrt{\rho(\mathrm{x})}} = \mathrm{E}_n - \Phi(\mathrm{x}) \tag{25}$$

$$\nabla S_g(\mathrm{x}) = \frac{\mathrm{c}_n \mathrm{h}\ \rho(\mathrm{x}_n)}{4} \mathrm{UnitStep}[\mathrm{x}_n - \mathrm{x}] \times$$

$$\times \left[\frac{\rho(\mathrm{x}_n)}{\rho(\mathrm{x})} g(\mathrm{x}) - \frac{\rho(\mathrm{x})}{\rho(\mathrm{x}_n)} + sign[\rho'(\mathrm{x})]\ \sigma_n\ |1 - \mathrm{c}_n|\ \frac{\rho'(\mathrm{x})^2}{\rho(\mathrm{x}_n)\rho(\mathrm{x})} \right] \tag{26}$$

$$\oint \nabla S(\mathrm{x}) d\mathrm{x} = 2\pi(\mathrm{n} - 1)\hbar \tag{27}$$

So for practical purposes we have given an index $_g$ to specify a $\nabla S(\mathrm{x})$ expression when a certain function $g(\mathrm{x})$ is used. We recall that a precise expression of the last is unknown, but a sound reference value is known to be $g(\mathrm{x}) = 1$. Therefore a good technique is taking first an approximate solution, starting with the assumption

$$\nabla S_1(\mathrm{x}) = \frac{\mathrm{c}_n \mathrm{h}\ \rho_1(\mathrm{x}_n)}{4} \mathrm{UnitStep}[\mathrm{x}_n - \mathrm{x}] \times$$

$$\times \left[\frac{\rho_1(\mathrm{x}_n)}{\rho_1(\mathrm{x})} - \frac{\rho_1(\mathrm{x})}{\rho_1(\mathrm{x}_n)} + sign[\rho_1'(\mathrm{x})]\ \sigma_n\ |1 - \mathrm{c}_n|\ \frac{\rho_1'(\mathrm{x})^2}{\rho_1(\mathrm{x}_n)\rho_1(\mathrm{x})} \right] \tag{28}$$

This step is discussed in the very next section. We will give refinements afterwards.

9.1 First Step approximated solution

We have worked out already in [2], for all sampled examples, the named First Step (FS) procedures. Although we are considering now the equations (25)÷(27) independently of the other ones, note that the coefficients we use for any (even approximate at First Step) solution have always to comply, rather strictly, with the overall context as previously expounded - because they also affect the remaining calculations. As in eq. (28), we have given the index $(_1)$ (i.e. $g(\mathrm{x}) = 1$) to the calculated quantities in this approximation. Since the approximate solutions $(_1)$ look rather good ones already and bring deep insight into the system physics, we give here all the details in such a way that the interested reader may soon be able to reproduce calculations by himself. We have found for instance:

9.1.1 Example : Harmonic oscillator, n=3, First Step calculation

For the third level of a harmonic oscillator with

$$\Phi(\mathrm{x}) = \frac{1}{2}\,\mathrm{m}\ 4\pi^2\nu_c^2\ \mathrm{x}^2 \tag{29}$$

$$\mathrm{E}_n = \left(\mathrm{n} - \frac{1}{2}\right) \mathrm{h}\nu_c \tag{30}$$

we have assessed the following positions and coefficient values (some of them from Table 3 in [2]) :

$$\lambda = \sqrt{\frac{\hbar}{2\pi \mathrm{m}\nu_c}} \tag{31}$$

$$\mathrm{x} = \lambda\ \xi \tag{32}$$

$$S_1'(\xi) = \lambda\, S_1'(\mathrm{x}) = \varphi_1'(\xi)\, \hbar \tag{33}$$

$$\rho_1(\xi) = \rho_1(\mathrm{x}_n)\rho_{1n}(\xi) = \rho_{1\,\max}\rho_{1n}(\xi) \tag{34}$$

$$\xi_n = 1.581139 \tag{35}$$

$$c_n = 1.190120 \tag{36}$$

$$\rho_1(\mathrm{x}_n) = \rho_{1\,\max} = \frac{0.371453}{\lambda} \tag{37}$$

$$\sigma_n\,|1 - \mathrm{c}_n| = 0.406166\,\lambda^2 \tag{38}$$

Then equations (25) and (26), (6) can be written

$$\varphi_1'(\xi)^2 - \frac{\sqrt{\rho_{1n}(\xi)}''}{\sqrt{\rho_{1n}(\xi)}} = 5 - \xi^2 \tag{39}$$

$$\varphi_1'(\xi) =$$

$$= 0.694422\left[\frac{1}{\rho_{1n}(\xi)} - \rho_{1n}(\xi) + 0.406166\ \rho_{1n}'(\xi)\frac{|\rho_{1n}'(\xi)|}{\rho_{1n}(\xi)}\right]\mathrm{UnitStep}[\xi_n - \xi] \tag{40}$$

$$\rho_{1n}'(0) = 0 \tag{41}$$

Calculations pertinent to eqs. (39)÷(41) are performed in the joint Math file ([M1]). Here we give plots for the resulting functions $\rho_{1n}(\xi)$, $\varphi_1'(\xi)$ (FS = *First Step*):

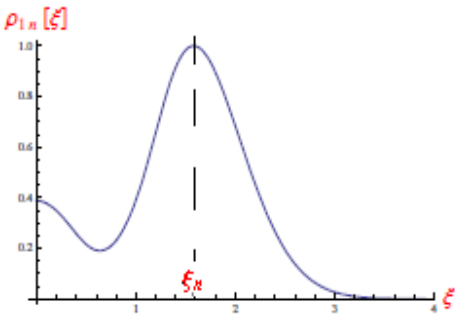


Fig.1 – FS density $\rho_{1n}(\xi)$ *for HO, n=3*

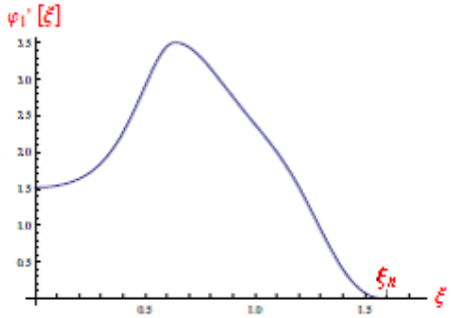


Fig.2 – FS phase gradient $\varphi_1'(\xi)$ *for HO, n=3*

A general discussion about the properties of these functions (for all cases investigated) has been given in the main reference papers [1,2]; here we want to discuss, with more details, the compliance of the present specific solution for HO n=3 with the condition (27). Indeed, as it can be checked by the math file, with the chosen coefficients the norm condition (2) is satisfied but we do not find the phase condition

$$\oint \frac{\nabla S(\mathrm{x})}{\hbar} d\mathrm{x} = \Delta\varphi_{(round\,trip)} = 4\pi \tag{42}$$

exactly verified; we find instead, by the quoted numerical program:

$$\oint \varphi_1'(\xi) d\xi = \oint \frac{\nabla S_1(\mathrm{x})}{\hbar} d\mathrm{x} = \Delta\varphi_{1(round\,trip)} = 1.97856\,(2\pi) \tag{43}$$

with a relative error $\epsilon_r \simeq -1\%$ over the requirement.

It seems to us of some importance speculating about this "small" inconvenience. This is because by working out small variations of the coefficients we emploied, we could not succed in the task of eliminating the error. The equations system behaves as a very stiff one in this respect, and decreasing ϵ_r without causing violation of any other requirement appeared to us impossible within the assessed context. As reported in [2], the same circumstances we have found for the case n=2 where the phase error is found to be $\epsilon_r \simeq -3\%$; and for the other cases examined, n=2 and n=3 of the oscillator in a Rectangular Well. However, as also noted in the main reference, when we inspect higher n levels this error rapidly decreases by increasing n; and becomes quite negligeable for (say) n $\geq$ 5, both for HO and RW. Therefore the case only matters for the first few oscillators levels ([8]). Yet for the sake of the model congruence, we must wonder about the cause of these phase defects. Are they due to some imperfect mathematical fitting of numerical data into the model somewhere, or indicate they us some conceptual refinement we should assume into the physical description for realism ? As remarked, by our investigations we could not be satisfied with various sets of values different from (35)$\div$(38), while keeping g(x) = 1. So although we are not able to exclude peremptorily the first hypothesis, we leave it apart here, and we come to the following considerations in favour of the second one.

9.1.2 Comment : a Second calculation Step is necessary

We have used, so far, a g(x) value equal to 1 but the exposed circumstances lead us to believe that a correction to this value is necessary to comply exactly with the phase quantization condition, at least for the cases quoted. Yet it turns out not possible to attain the desired result by the means of a small correction. As it

[8] For the first level with n =1 the wave phase function is zero as in orthodox QM and eq. (8) is satisfied exactly. On the same step, the n=1 wavefunctions in our model are actually identical to the standard ones.

can be seen by numerical checks, any small variation applied to the expression of $\nabla S_1(\mathrm{x})$ in (28) always seems to cause strong perturbations to appear into the solutions, and brings them soon very far from respecting the constraints. The equations appear to us as showing some kind of instability (this is however expected, because we are dealing with a stiff eigenvalues problem). Therefore a very accurate choice of the corrective function must be worked out for every single case and level. Interestingly on another hand, the same circumstance may be thought fortunate: if eq. (28) with $g(\mathrm{x}) = 1$ is not the very exact one but is very near to be, whenever by some mathematical technique we are able to find the good correction satisfying all the constraints, we can take the result for the very reliable one. This should bring noteworthy physical information to us.

So our main task in ref. [2] at this point has been first managing to find an adequate correction of our expression $\nabla S_1(\mathrm{x})$, i.e. finding a function $g(\mathrm{x})$ able to fulfill exactly the constraints set. The task has been approached by the means of the so called Second Step procedure we illustrated in that reference. It consisted in introducing a corrective function (named $\mathrm{corr(x,n)}^2$) straight into the density expression (see eqs. (1), (2) in [2]). The correction is assumed as a polynomial form of rank 7, because the new density still must be such that conditions (6), (7), (2), (8), (17), (22), (12) are satisfied. Since some of these conditions are integral ones, the mathematical work required to find the coefficients in the polynomes has been quite intricated and laborious. Here we have to warn the reader that revising calculations given in [2], we have unfortunately discovered a small numerical error propagating along the equations chain: this actually impugnes our claim of a "very good" precision having been attained in that paper (see pages 162 and 164 in [2]). Notwithstanding, all plots and comments in that work remain valid because the error stands very small anyway (as a matter of fact, we remain at the end on the same error level than in First Step): only the *numerical precision* by which the entire equations set is satisfied by our SS procedure is in question. Therefore, the interested reader will find here revisited all the matter in the very following sections. We propose and validate punctually an improved procedure and we give a physical interpretation. To do this, in the draw of this notes is enough pursuing the sampled case for HO, $\mathrm{n} = 3$. We limit ourselves to this single case here because the numerical trials necessary to find precise coefficents are exceedingly lenghty for an ordinary PC with Mathematica software. So whenever useful in the future, we may update these notes to other examples; but our real interest obviously, is just demonstrating here, by a sound although single example, the model practicability up to a great numerical finesse. Although the present machinery overtakes the previously published one with more assessed results, for the sake of clarity and completeness we give in Appendix to these notes a detailed description of the previously mentioned procedure error and its consequences.

However, since we have now a deeper insight into the question, we want to indicate here an upgraded model. This new approach looks to us based on a rather smarter technique.

9.2 (Improved) Second Step calculation procedure

We introduce here an alternative Second Step procedure, seeming to us more appropriate than the one given in [2]. This is in the light of the following criterium: we have to improve our model with essential concern to condition (42), which we have seen (eq. (43)) to be not far at all from being satisfied. So the correction we have to impose to our phase gradient $\nabla S_1(\mathrm{x})$ should be as small as possible - taken the equation instability and all the constraints into account. While working out this second step, we will be obliged to change all the coefficients we have found in the first step; yet we want these changes to be minimal. Then we are brought to the following. We set again as in [2] the right density $\rho_n(\mathrm{x}) = \mathrm{corr}(\mathrm{x},\mathrm{n})^2\,\rho_{1n}(\mathrm{x})$ ([9]), and write equation (25) in the form

$$\frac{\nabla S_g^2(\mathrm{x})}{2\mathrm{m}} - \frac{\hbar^2}{2\mathrm{m}}\,\frac{\left[\mathrm{corr}\,(\mathrm{x},\mathrm{n})\,\sqrt{\rho_{1n}(\mathrm{x})}\right]''}{\mathrm{corr}\,(\mathrm{x},\mathrm{n})\,\sqrt{\rho_{1n}(\mathrm{x})}} = \mathrm{E}_n - \Phi(\mathrm{x}) \tag{44}$$

where from the First Step

$$\frac{\nabla S_1^2(\mathrm{x})}{2\mathrm{m}} - \frac{\hbar^2}{2\mathrm{m}}\,\frac{\sqrt{\rho_{1n}(\mathrm{x})}''}{\sqrt{\rho_{1n}(\mathrm{x})}} = \mathrm{E}_n - \Phi(\mathrm{x}) \tag{45}$$

In order to find the minimal correction corr(x,n) good to us, we first search the form of the special functions corr_0(x,n) bringing to a null correction on $\nabla S_1(\mathrm{x})$; i.e. we impose first

$$\frac{\nabla S_1^2(\mathrm{x})}{2\mathrm{m}} - \frac{\hbar^2}{2\mathrm{m}}\,\frac{\left[\mathrm{corr}_0\,(\mathrm{x},\mathrm{n})\,\sqrt{\rho_{1n}(\mathrm{x})}\right]''}{\mathrm{corr}_0\,(\mathrm{x},\mathrm{n})\,\sqrt{\rho_{1n}(\mathrm{x})}} = \mathrm{E}_n - \Phi(\mathrm{x}) \tag{46}$$

Solving this equation we easily find the (non trivial) solution

$$\mathrm{corr}_0(\mathrm{x},\mathrm{n}) = 1 + \mathrm{p}(\mathrm{n}) \int \frac{1}{\rho_{1n}(\mathrm{x})} d\mathrm{x} \tag{47}$$

where p(n) is a constant. Therefore, the interesting ansatz to us will be

$$\mathrm{corr}\,(\mathrm{x},\mathrm{n}) = 1 + \int \frac{\mathrm{p(x,n)}}{\rho_{1n}(\mathrm{x})} d\mathrm{x} \tag{48}$$

where p(x,n) shall be now taken as a variable function. With this position, equation (44) writes

$$\frac{\nabla S_g^2(\mathrm{x})}{2\mathrm{m}} - \frac{\hbar^2}{2\mathrm{m}}\,\frac{\mathrm{p'(x,n)}}{\rho_{1n}(\mathrm{x})\left[1 + \int \frac{\mathrm{p(x,n)}}{\rho_{1n}(\mathrm{x})} d\mathrm{x}\right]} - \frac{\hbar^2}{2\mathrm{m}}\,\frac{\sqrt{\rho_{1n}(\mathrm{x})}''}{\sqrt{\rho_{1n}(\mathrm{x})}} = \mathrm{E}_n - \Phi(\mathrm{x}) \tag{49}$$

[9] When comfortable, we use densities normalised to the maximum value 1 (small index n).

So finally, the good quantities $\nabla S(\mathrm{x})$, $\rho(\mathrm{x})$ must have the forms

$$\frac{\nabla S^2(\mathrm{x})}{2\mathrm{m}} \equiv \frac{\nabla S_g^2(\mathrm{x})}{2\mathrm{m}} = \frac{\nabla S_1^2(\mathrm{x})}{2\mathrm{m}} + \frac{\hbar^2}{2\mathrm{m}} \frac{\mathrm{p}'(\mathrm{x},\mathrm{n})}{\rho_{1n}(\mathrm{x}) \left[1 + \int \frac{\mathrm{p}(\mathrm{x},\mathrm{n})}{\rho_{1n}(\mathrm{x})} d\mathrm{x}\right]} \tag{50}$$

$$\rho(\mathrm{x}) \equiv \rho_g(\mathrm{x}) = \rho_{\max} \rho_{1n}(\mathrm{x}) \left[1 + \int \frac{\mathrm{p}(\mathrm{x},\mathrm{n})}{\rho_{1n}(\mathrm{x})} d\mathrm{x}\right]^2 \tag{51}$$

Here $\rho_{\max}$ is now the *new* normalisation value, to be evaluated. Before this, obviously functions p(x,n) must be found, in such a way to satisfy all the requirements. The simple technique is fitting them into a polynomial form. Here is our example.

9.2.1 Harmonic oscillator, n=3, (new) Second Step calculation

We expound the new SS calculations for the previously chosen harmonic oscillator example with n = 3. For this case, by numerical trials with $Wolfram\ Mathematica$ we have found the following function (use the normalised coordinate ξ):

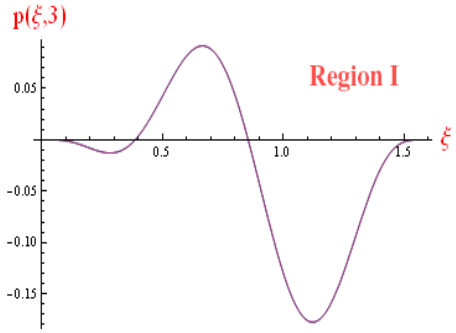
$$\mathrm{p}(\xi, 3) = 0.753724\,\xi^3 - 31.691300\,\xi^4 + 180.143974\,\xi^5 +$$

$$-412.776125\,\xi^6 + 478.745550\,\xi^7 - 298.300935\,\xi^8 +$$

$$+94.896489\,\xi^9 - 11.732129\,\xi^{10} - 0.168732\,\xi^{11} \tag{52}$$

represented in $fig.\,3$ here. The corresponding function $\mathrm{corr}(\xi, 3)$ is also plotted in $fig.\,4$, and the final density $\rho(\xi)$ and phase gradient $\varphi'(\xi)$ are in $figs.\,5$ and 6. Details of these calculations from eq. (52) to (59) further on are shown in the math file ([$M2$]). Note that the overall technique is confined to Region I

($\xi \leq \xi_n$); in Region II the obvious extension of $\mathrm{corr}(\xi, 3)$ is always the value 1, indeed $\nabla S_{g\mathrm{II}} = 0$ there and the branch II of the solution coincides in all cases with the corresponding standard quantum density (normalised to 1 in ξ_n).



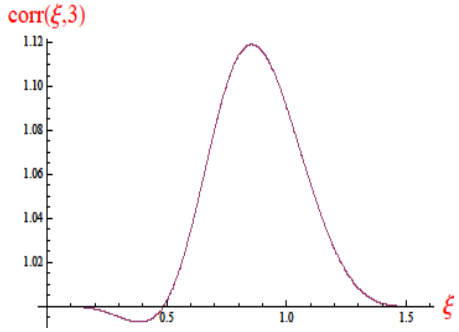
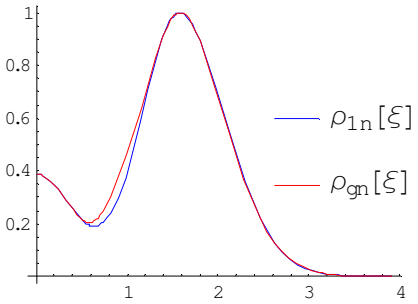


Fig.3 – Polynomial function p(ξ,n) *(HO, n=3)* *Fig.4 – Corrective function* corr(ξ,n) *(HO, n=3)*



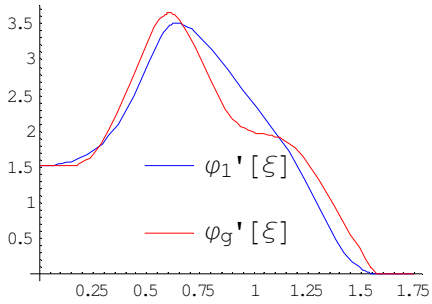


Fig.5 – 1st and 2nd Step densities *(HO, n=3)* *Fig.6 – 1st and 2nd Step phase grads* *(HO, n=3)*

As we can see from these diagrams, the essential behaviour of density and phase grad functions is not changed compared to the first step solutions, but now we have

$$\oint \varphi'_g(\xi)d\xi = \Delta\varphi_{g(round\,trip)} = 12.566368099 \equiv 4\pi(1+\epsilon) \tag{53}$$

where the relative error $\epsilon = -2.0\,10^{-7}$ is quite negligeable now. As remarked before, the density normalization parameter $\rho_{\max}$ is now changed compared to (37); to comply with the norm condition, it becomes

$$\rho_{\max} = \frac{2}{\lambda \oint \rho_{gn}(\xi)\mathrm{d}\xi} =$$

$$= \frac{1}{\lambda \mathrm{NIntegrate}[\rho_n(\xi), \{\xi, -\xi_{fin}, \xi_{fin}\}]} = \frac{0.362000}{\lambda} \tag{54}$$

The extremal points $\pm\xi_{fin}$ we have chosen to stop the numerical integration are rather arbitrary, but so great values that increasing them brings quite no difference in the result (the integrand in (54) decays fast at the HO border with a Gaussian behaviour). We took actually

$$\xi_{fin} = 3.91112 \tag{55}$$

As far as $g(\xi)$ is concerned, it changes from unity to the function in $fig.\,7$:

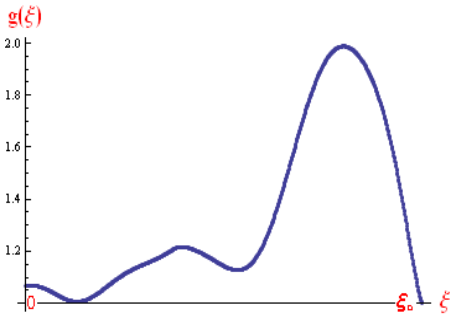


Fig.7 – Corrective function $g(\xi)$ for HO, n=3 case

Here the difference with the corresponding $fig.\ 41$ (*green curve*) in ref. [2] is in practice a doubled maximum value, and a different $g(0)$ value - this is clear from equation (26) taken in x = 0:

$$g(0) = \rho(0)\frac{4\nabla S_g(0)}{c_n h\ \rho^2_{\max}} + \frac{\rho(0)^2}{\rho^2_{\max}} = 1.0688 \tag{56}$$

Indeed we find from previous calculations

$$\rho(0) = \frac{0.140509}{\lambda} \tag{57}$$

$$\nabla S_g(0) = \frac{\varphi'_g(0)\,\hbar}{\lambda} = \frac{1.51954\,\hbar}{\lambda} \tag{58}$$

and $\rho_{\max}$ as given in eq. (54). Concerning c_n, we had to choose it (and its mate τ_n) in such a way that the subsequent constraints (17), (22) are also satisfied: to this end we assumed

$$c_n = 1.12971 \tag{59}$$

$$\tau_n = 0.117403 \tag{60}$$

We give proof of the appropriateness of these values in a subsequent math file where the named constraints are calculated. Before showing this in details, however, we calculate the functions $\nu_n(x)$ and $m_{eff}(x)$.

9.3 Calculation of the flow and mass eigenfunction (HO, n=3)

To calculate $\mathrm{m}_{eff}(\mathrm{x})$ also in Reg. II, we first need to work out eq. (13) for the flow function. This is done in the math file [M3] where the following values are used (eqs. (12) here and (17) in [2]) :

$$\nu_{n0} = \nu_{30} = \frac{\mathrm{E}_{nf} - \mathrm{E}_{ni}}{c_n \mathrm{h}} = \frac{\nu_c}{c_n} = 0.885181\,\nu_c \tag{61}$$

Moreover, by eqs. (13) and (14) we have ([10]):

$$\mu_n = \mu_3 = 0.278696\,\mathrm{h}^{5/4}\nu_c^{3/4}\mathrm{m}^{-1/4} \tag{62}$$

So with the coefficients (59)÷(62), here the results are:

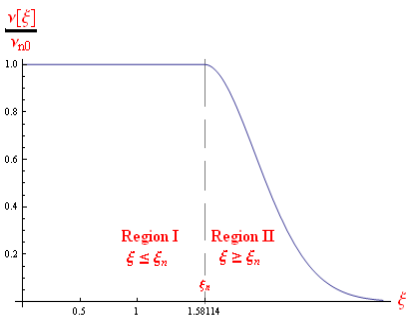


Fig.8 – Flow function for HO, n=3 case

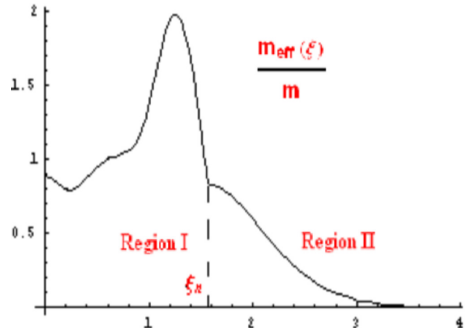


Fig. 9 – Effective mass $m_{eff}(\xi)/m$ (HO, n=3)

We can now check in the quoted math file that eq. (17) is also satisfied. The numerical calculation provides indeed

$$\lambda\ \mathrm{NIntegrate}[\mathrm{m}_{eff}(\xi)\rho(\xi), \{\xi, -\xi_{fin}, \xi_{fin}\}] = 0.9999977 \tag{63}$$

therefore unity with negligeable error.

Before continuing calculations, we have a number of remarks to do. On a physical point of view, the behavioural change of step of the quantities in *figs.* 8, 9 when crossing the boundary at ξ_n from Region I to Region II is due to the particles being reflected all along the Region II extension, starting with ξ_n up

[10] Warning: it may not be clear that the values for μ_n reported by Tables in [2] are always given in units of $\mathrm{h}\nu_{n0}/\sqrt{\rho_{\max}}$. F.i. for HO n = 3, Table 3 reports $\mu_n = 0.330593$; to comply dimensionally with eqs.(33) in [1] and (13) here, we have to take in [2] $\mu_n \to 0.330593\ \mathrm{h}\ \nu_{n0}/\sqrt{\rho_{\max}} = 0.183663\,\mathrm{h}^{5/4}\nu_c^{3/4}\mathrm{m}^{-1/4}$. This value can be compared to the new value we give in eq.(62) here, i.e. $0.278696\,\mathrm{h}^{5/4}\nu_c^{3/4}\mathrm{m}^{-1/4}$. The difference is due to the different Second Step procedure we are using in these notes compared to [2].

to the packet exhaustion in $\xi \rightarrow \infty$. The very steep descent of the effective mass in the approach of Region II is linked to the conjoint trend of density and phase gradient in (15), the $\nabla S(\mathrm{x})$ being influenced by the fast decrease of particles velocities when the turning points Region II is approached. Note that our functions are very similar, but not just identical to the ones we have published in [2] for the same case: this is not just (looking at the numerical precision attained) because of the mentioned error in that calculation (indeed staying into percents), but because we have used here, at last, an improved procedure compared to that reference. So we consider the present calculation more reliable than the previous one. However, it is clear that the real interest of $g(\mathrm{x})$ to us stays not in its detailed form (subject to variations in dependence of the procedure we may use to determine it), but rather in its physical interpretation. This last would likely be fully apparent if an analytical expression for it, f.i. as a function of such a fundamental quantity as $\rho(\mathrm{x})$, could be found. It is tantamount to say that assessing a general expression for $g(\mathrm{x})$ would bring us to discover the *exact* general equations for the mass eigenfunctions (15) and jointly for $\nabla S(\mathrm{x})$. Yet after many investigation trials, we could not succed in the specific task of discovering an analytical form; but we came to the conclusion that our circumstances seem to admit an interesting physical interpretation. We anticipate here that $g(\mathrm{x})$ may be interpreted as a form factor implied into the vacuum reaction. For now therefore, we give the following considerations. Remaining calculations to complete our example are in next sections.

10 Physical interpretation

We are led to pay more attention to the structure of equation (15). We want to go deep into the physical meaning. We can write that equation as follows (we confine now to Region I):

$$W_A(\mathrm{x}) = \frac{\mathrm{c}_n \mathrm{h}\nu_{n0}}{4} sign[\rho'(\mathrm{x})]\ \sigma_n\ |1 - \mathrm{c}_n|\ \frac{\rho'(\mathrm{x})^2}{\rho(\mathrm{x})^2} + \frac{1}{2}\mathrm{m}_{eff}(\mathrm{x}_n)\mathrm{v}_{_D}(\mathrm{x})^2 g(\mathrm{x}) =$$

$$= \frac{1}{2}\mathrm{m}_{eff}(\mathrm{x})\mathrm{v}_{_D}(\mathrm{x})^2 = \mathrm{K}_{cm} \tag{64}$$

As is clear from the very form of this expression, we have named $W_A(\mathrm{x})$ the work of the equivalent active force producing the net kinetic potential K_{cm} pertaining to the packet center. Then we can help with a simple interpretation in our model. The quantum vacuum being a sort of deformable medium perturbated by the particles flow, it expresses a reaction force whose work function $W_{VR}(\mathrm{x})$ is responsible for the mass effect. Then we write:

$$W_{VR}(\mathrm{x}) + W_A(\mathrm{x}) = \frac{1}{2}\mathrm{m}\mathrm{v}_{_D}(\mathrm{x})^2 \tag{65}$$

This is a typical equation in classical mechanics for a constrained system: f.i. for a rolling disk on the inclined plane, $W_A(\mathrm{x})$ would be the work pertinent to the weight component along the plane ($\mathrm{m}\ g\ \sin\alpha$ ([11]), α = plane inclination), and $W_{VR}(\mathrm{x})$ the one pertinent to the friction on the contact point.

Then it is appropriate to write here

$$W_{VR}(\mathrm{x}) = -\frac{1}{2}\left[\mathrm{m}_{eff}(\mathrm{x}) - \mathrm{m}\right]\mathrm{v}_{D}(\mathrm{x})^2 \tag{66}$$

just as in the case of the rolling disk ([12]). Note that both the expressions of $W_{VR}(\mathrm{x})$ and $W_A(\mathrm{x})$ are conditioned by the applied constraint. This last can be thought as a sort of physical filter for the external action with respect to the Second Law: it dictates which fraction of the active force (i.e. $\sin\alpha\times\mathrm{mg}$ in the disk example) is really effective for the thrust, and moreover opposes a reaction to it. The expression of the force sliding component pushing the disk turns out to be a formula constituted by the external force field mg imprinted with the constraint form (the factor $\sin\alpha$). Now we understand why our functions $W_A(\mathrm{x})$, $W_{VR}(\mathrm{x})$ are a sort of "made up" appearance of the active force field [5]. This last, we can trace back indeed in our case to the expression ($p \equiv$ pressure)

$$W_{MF}(\mathrm{x}) = \int p(\mathrm{x}) d\frac{1}{\rho} = \kappa\ \frac{\hbar^2}{8\mathrm{m}}\frac{\rho'(\mathrm{x})^2}{\rho(\mathrm{x})^2} + \kappa\frac{1}{2}\mathrm{m}\mathrm{v}_{D}(\mathrm{x})^2 \tag{67}$$

Such is actually the work function associated to a typical chemical potential we find in our model, sounding

$$\int \frac{dp}{\rho(\mathrm{x})} = -\ \kappa\frac{\hbar^2}{2\mathrm{m}}\ \frac{\sqrt{\rho(\mathrm{x})}''}{\sqrt{\rho(\mathrm{x})}} + \kappa\beta\frac{1}{2}\mathrm{m}\mathrm{v}_{D}(\mathrm{x})^2 \tag{68}$$

(with constant β) ([13]). So it can be now better appreciated that the function $W_A(\mathrm{x})$ is a perturbated expression of $W_{MF}(\mathrm{x})$: it is the equivalent of the disk sliding component work in our case. Eq. (64) taken into account, $W_{VR}(\mathrm{x})$ looks like a perturbated expression, too. We can even note the analogy of the presence of an alternate $sign[\rho'(\mathrm{x})]$ in those expressions, with the very similar sign alternance in $\mathrm{m\,g}\ \sin\alpha(\mathrm{x})$ if the disk moves through an undulate track. Inspecting $W_A(\mathrm{x})$ with this in mind, the function $g(\mathrm{x})$ also appears to us as a constitutive ingredient of the vacuum reaction in our model.

[11] This is sometimes called "the sliding component" of the active force.

[12] For the rigid homogeneous disk in purely rolling regime, one has $\mathrm{m}_{eff}(\mathrm{x}) = 3/2\mathrm{m}$. Would we assume the rolling body to be f.i. as a rugby ball with elliptic section, we would find a variable effective mass, with $\mathrm{m}'_{eff}(\mathrm{x}) \neq 0$.

[13] For the origin of eq. (68), see f.i. eq. (74) in ref. [7]; to that expression, we can always add a term $\kappa\beta\frac{1}{2}\mathrm{m}\mathrm{v}_{D}(\mathrm{x})^2$ because this still makes the property expressed in eqs. (83)÷(85) of ref.[7] fulfilled. Moreover, reasoning with added or subtracted kinetic energy fractions to various equations members all throughout the rest of the theory in previous papers brings no prejudice to conclusions. When calculating the pressure p from eq. (68), remember that $\mathrm{v}_{D}(\mathrm{x})$ is proportional to $\rho(\mathrm{x})^{-1}$in the Region I of space we are interested in here. The quantity κ is defined in the context of eq. (41) in ref.[7].

Note also that eq. (64) holds in the space Region I where $\nu(\mathrm{x})$ is a constant. In Region II instead, we found no match for $W_A(\mathrm{x})$ with the expression in the first term of (64). It is not difficult to figure out mathematical extensions of this last in such a way to transform it into the good expression of $W_A(\mathrm{x})$ in Region II; but we do need at all working out such an exercise, because $\nabla S(\mathrm{x}) = 0$ there and by eq. (9) we have available the exact expression for our quantities: i.e. we have

$$W_A(\mathrm{x}) = \frac{1}{2}\mathrm{m}_{eff}(\mathrm{x})\mathrm{v}_{_D}(\mathrm{x})^2 = \frac{\mathrm{c}_n \mathrm{h}\nu(\mathrm{x})}{4} \qquad \{\text{Region II}\} \qquad (69)$$

$$W_{VR}(\mathrm{x}) = \frac{2\mathrm{m}\nu(\mathrm{x})^2}{\rho(\mathrm{x})^2} - \frac{\mathrm{c}_n \mathrm{h}\nu(\mathrm{x})}{4} \qquad \{\text{Region II}\} \qquad (70)$$

The different behaviour in the two regions is, again, a circumstance much alike to the disk case: at a certain point, the dynamical friction may leave place to the static one - thus setting a boundary between two regions of space characterised by different motion regimes. So in the two regions of space, we find the vacuum response differently constituted ([14]).

All of this means to us that the particles packet travels along an undulate path provided by the reacting vacuum; this causes at last the mass effect. Now turning to our calculations, since having set $g(\mathrm{x}) = 1$ we have not found the condition (42) exactly satisfied, this must have happened because we missed some detail in expressing the vacuum reactive potential $W_{VR}(\mathrm{x})$ (or equivalently, $W_A(\mathrm{x})$) by the First Step position. Then better modelling is required, first of all searching for a more appropriate form of $g(\mathrm{x})$([15]) as remarked. Note that $1/\rho(\mathrm{x})$ is the specific volume of the particles ensemble at the abscissa x as in standard thermodynamics: i.e. just the constraint which the pressure is able to force while f.i. expanding the gas. Due to the imperfect result we get as in (43) with $g(\mathrm{x}) = 1$, we understand that more physical sense might be achieved assuming that the real constraint to be forced is not really the volume $1/\rho(\mathrm{x})$, but a different quantity - although it clearly should behave rather similarly in space. We will identify its physical character next.

Although eqs. (45) and (28) do not offer us the exact solution for the density $\rho(\mathrm{x})$, we have seen that the good solution is very near to $\rho_1(\mathrm{x})$, so they are a solid

[14] It is obviously questionable whether in our problem the vacuum response may be interpreted as dissipative or conservative or a mixed one; but at present, we see no reason why dissipative forces should be called in cause, and we search for totally conservative potentials to be identified into the vacuum reaction.

[15] This is not the only possibility: to comply with the general discussion, we could also perturbate f.i. the first term expression in (64) inserting some function $\sigma(\mathrm{x})$ in place of the constant σ. But it turns out that a very great perturbation is needed to comply. However, working out this case would shed no more light on this matter physically than our following determinations will do.

basis for our calculations to start with. We can help with some interpretation, writing explicitly eq. (64) with $g(\mathrm{x}) = 1$:

$$\frac{\mathrm{c}_n \mathrm{h}\nu_{n0}}{4} sign[\rho_1'(\mathrm{x})]\ \sigma_n\ |1 - \mathrm{c}_n|\ \frac{\rho_1'(\mathrm{x})^2}{\rho_1(\mathrm{x})^2} =$$

$$= \frac{1}{2}\mathrm{m}_{eff1}(\mathrm{x})\mathrm{v}_{D1}(\mathrm{x})^2 - \frac{1}{2}\mathrm{m}_{eff1}(\mathrm{x}_n)\mathrm{v}_{D1}(\mathrm{x})^2 \tag{71}$$

In agreement with previous statements, this sounds to us much like a sort of theorem where the first member is a work creating a typical kinetic energy difference; but since we know this relation *not* to be really the *exact* one, we can think that it might come from some (exact) precursor, with some approximation - although not so drastic - applied. So we focus attention on the first term, and consider that the only physical variable in our models, which strongly resembles the inverse of density but is actually different from it, is the particles packet space extension $\Delta\mathrm{x}(\mathrm{x})$. We have met with this quantity in ref. [6] already, where a classical-like definition rather than the quantum one is used and some differences are marked in comparison with the quantum incertitude parameter. In our views, a simple definition of the particles packet centered at position x includes the collection of particles around x (say in positions $\mathrm{x}' \in [\mathrm{x} - \Delta\mathrm{x}(\mathrm{x})/2,\ \mathrm{x} + \Delta\mathrm{x}(\mathrm{x})/2]$) whose motion quantities $\mathrm{m}_{eff}(\mathrm{x}')\mathrm{v}(\mathrm{x}')$ sum to an average $\mathrm{m}_{eff}(\mathrm{x})\mathrm{v}_{D}(\mathrm{x})$. We are also led to precise here, for congruence, that in such a frame the density $\rho(\mathrm{x})$ should not be viewed as simply "the number of particles observable between x and x+dx" but more practically as the number of particles belonging to the x-centered packet: in a sense, we increase the "observational slit width" in x, from dx to Δx. Yet for now, we have not worked out such a kind of complicated calculations able to assess rigorous estimates of $\Delta\mathrm{x}(\mathrm{x})$. Nevertheless, we are here at a point we can deal with on a rather conceptual stake. We introduce $\Delta\mathrm{x}(\mathrm{x})$ in place of $1/\rho_1(\mathrm{x})$, on a more proper account of the reactive constraint imposed by the vacuum to our system. We can give the interpretation that the real quantity subtending the role of $\rho_1(\mathrm{x})$ in our model is nothing else but $\Delta\mathrm{x}(\mathrm{x})^{-1}$ (although this statement is limited to Region I, in Region II more cautious views should be taken). So we still wait for some rigour in this matter, but in order to express our thinking let us just make, by hypothesis, the simple point

$$\Delta\mathrm{x}(\mathrm{x}) = \Delta\mathrm{x}_{\min}\left[\frac{\rho_{1\,\max}}{\rho_1(\mathrm{x})}\right]^{\mathrm{UnitStep}[\mathrm{x}_n - \mathrm{x}]} \tag{72}$$

Here $\rho_{1\,\max} = \rho_1(\mathrm{x}_n)$ is the maximum $\rho_1(\mathrm{x})$ value; by a physical sense, we might give an estimate of $\Delta\mathrm{x}_{\min}$ as some fraction of x_n (the characteristic space extension of the potential well). However, by continuity in $\mathrm{x}_n$we also find

$$\Delta\mathrm{x}_{\min} = \Delta\mathrm{x}(\mathrm{x}_n) \tag{73}$$

Taking into account these positions, we are brought to dress (45) and (28) with autonomous physical role and write them in terms of $\Delta \mathrm{x}(\mathrm{x})$:

$$\frac{\nabla S_1^2(\mathrm{x})}{2\mathrm{m}} - \frac{\hbar^2}{2\mathrm{m}} \frac{\Delta \mathrm{x}(\mathrm{x})^{-1/2}\,''}{\Delta \mathrm{x}(\mathrm{x})^{-1/2}} = [\mathrm{E}_n - \Phi(\mathrm{x})\,]\,\mathrm{UnitStep}[\mathrm{x}_n - \mathrm{x}] \tag{74}$$

$$\nabla S_1(\mathrm{x}) =$$

$$= \frac{\mathrm{c}_n \mathrm{h}\ \rho_{1\,\max}\ \Delta \mathrm{x}_{\min}}{4\Delta \mathrm{x}(\mathrm{x})} \left[\frac{\Delta \mathrm{x}(\mathrm{x})^2}{\Delta \mathrm{x}_{\min}^2} - 1 - sign[\Delta \mathrm{x}'(\mathrm{x})]\sigma_n\,|1 - \mathrm{c}_n| \frac{\Delta \mathrm{x}'(\mathrm{x})^2}{\Delta \mathrm{x}(\mathrm{x})^2} \right] \tag{75}$$

On this basis, it is also better understood that our solutions for $\nabla S_1(\mathrm{x})$ were actually not obliged to respect the condition (8): equations (45), (28) are indeed only similar, but not coincident with the true wave equation (3). Using the numerical result for $\rho_1(\mathrm{x})$ already shown in $fig.$ 5, we easily might come now to the $\Delta \mathrm{x}(\mathrm{x})$ behaviour as suggested by eq. (72) - i.e. the reciprocal of $\rho_1(\mathrm{x})$. However, further reasoning given in the very next lines will bring us to a slightly different plot (this last shown in $fig.\,10$, calculation is in appendix to the file [M3]).

To resume indeed, we can give physical sense to the approximate density equation (45) interpreting it as an appearance of a more basic equation for the packet extension $\Delta \mathrm{x}$, whose expression we might represent by (74) and (75). Due to our somewhat simple arguments, however, equation (75) might finally not reveal the exact one for $\Delta \mathrm{x}$, neither. But its physical sense can be appreciated considering its strong relationship with the (64): to be simple, with a small step forward more, we might dress now this last with the new form

$$W_A(\mathrm{x}) \rightarrow -sign[\Delta \mathrm{x}'(\mathrm{x})]\ \sigma_n\,|1 - \mathrm{c}_n| \frac{\Delta \mathrm{x}'(\mathrm{x})^2}{\Delta \mathrm{x}(\mathrm{x})^2} + \frac{1}{2}\mathrm{m}\mathrm{v}_{_D}(\mathrm{x})^2 =$$

$$= \frac{1}{2}\mathrm{m}_{eff}(\mathrm{x})\mathrm{v}_{_D}(\mathrm{x})^2 \tag{76}$$

Eq. (76) means conceptually to us that the real constraint to be forced is not the volume $1/\rho(\mathrm{x})$ but the packet dimension $\Delta \mathrm{x}(\mathrm{x})$, the first term at left in (76) expressing the correlated work function. This last is responsible for the evolution of the kinetic terms difference in the rest, i.e. in practice, for the mass effect. So reelaborating eq. (76) we are led to conjecture that the "good" equation for $\Delta \mathrm{x}(\mathrm{x})$ might actually be

$$\frac{\Delta \mathrm{x}(\mathrm{x})}{\Delta \mathrm{x}(\mathrm{x}_n)} =$$

$$= \mathrm{Exp}[-\frac{\gamma}{\mathrm{h}} \int_{\mathrm{x}_n}^{\mathrm{x}} sign[\mathrm{m}_{eff}(\mathrm{x}) - \mathrm{m}] \sqrt{\mathrm{m}\,|\mathrm{m}_{eff}(\mathrm{x}) - \mathrm{m}|}\mathrm{v}_{_D}(\mathrm{x}) d\mathrm{x}] \tag{77}$$

with constant γ ([16]). The argument in Exp being an action/h $\equiv$ $(Eddington - way)\, entropy$, it can be appreciated indeed that (77) is of the same form we find in classical thermodynamics where the (volume dependent part) entropy is S_T = R lnV.

Hopefully, starting with this point, further investigation might lead to the developement of a classical-like model for the quantum vacuum; but we leave this matter to future work.

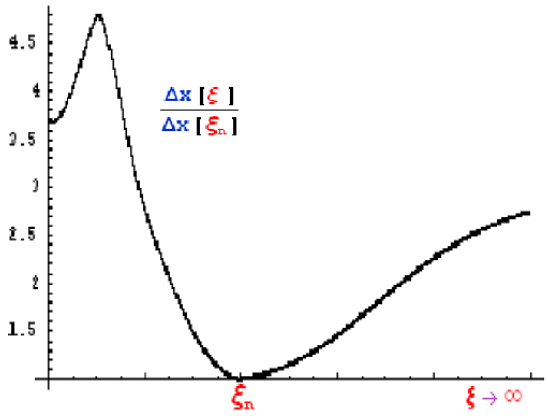


Fig. 10 – *Packet extension from (conjectured) eq.*(78). *Arbitrary units. We exploit* $m \simeq m_{eff}(0)$.

Here we only stress the final interpretation that the First Step calculations appear to us now as we had set them up by mimicking a more basic statement, of the form (76) or (77); indeed when we approximate Δx(x) by $1/\rho_1$(x) in (76) we are brought to (71) and (28), so that the approximate solutions ρ_1(x) and ∇S_1(x) can be worked out from (45) as we did in previous references. However, keeping ourselves independent of all such interpretations, we have now available the good solutions (50) and (51) with (52) etc. as in figs. 6 and 7, so we go further in calculations herein.

11 Velocity fields and periods

We have calculated so far the (HO, n = 3) mass eigenfunction m_{eff}(x) as shown in *fig.* 8 and we are going now to expound the next calculation steps fitted to the example completion.

Helping with eq. (18) we find :

$$v(x,E) \;= \sqrt{2\frac{E \,-\, E_{nf} + c_n h\nu_n(x)}{m_{eff}(x)} + \frac{\nabla S^2(x)}{m_{eff}(x)^2}} \qquad (78)$$

This is the general expression of the velocity fields. Let us now work out the HO n=3 example.

[16] The calculations shown in *fig.* 10 use $m_{eff}(0)$ in place of m, just for a sake of symmetry in x= 0; such kind of details may deserve attention in future model developments.

11.1 Harmonic oscillator, n=3; velocities and periods

By eq. (17) in ref. [2] we have

$$\mathrm{E}_{nf} = \mathrm{E}_{ni} + \mathrm{h}\nu_c = \mathrm{n}\,\mathrm{h}\nu_c \tag{79}$$

so we find, for n = 3 ($\tilde{\mathrm{v}} \equiv adimens.\,velocity$):

$$\tilde{\mathrm{v}}(\xi, \mathrm{E}) = \frac{\mathrm{m}\mathrm{v}(\xi, \mathrm{E})\lambda}{\hbar} \quad =$$

$$= \sqrt{\frac{2\mathrm{m}}{\mathrm{m}_{eff}(\xi, 3)}}\sqrt{\frac{\mathrm{E}}{\mathrm{h}\nu_c} - 3 + f(\xi)} \tag{80}$$

where

$$f(\xi) = \frac{\nu_n(\xi)}{\nu_{n0}} + \frac{\mathrm{m}\varphi_g'(\xi, 3)^2}{2\mathrm{m}_{eff}(\xi, 3)} \tag{81}$$

Now using the functions already shown in plots 6, 8 and 9, for all the energy values E in the interval ($2\mathrm{h}\nu_c$,$3\mathrm{h}\nu_c$) we can calculate the eulerian fields (80). A few of them we sample f.i. in $fig.\,11$ (this plot is taken from the math file [$M4$]) :

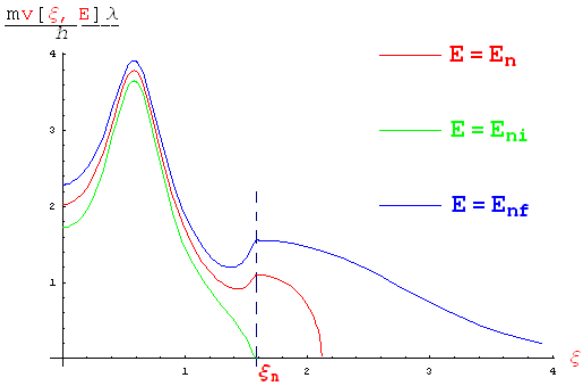


Fig. 11 – Eulerian velocity fields for sampled values of E, HO n=3

Comments on the behaviour of these functions can be found already in the quoted references. We add here only a few remarks more.

The particles turning points (i.e. the points where the velocity becomes zero) come out with monotonically increasing abscissas as a function of the energy. Particles start with turning back at ξ_n with energy E_{ni}, but those with greater energy proceed even much further: the situation configures here a tunnel-like effect, through the peculiar, classical interpretation brought by our model. Although clearly no escape of the particle is possible in this context

([17]), we see easily by eq. (78) that particles with the greatest energy value E_{nf} exhaust their velocity only in $\xi \rightarrow \infty$. On one hand, this is congruent with the standard quantum model predicting a (Gaussian-like) density extending to infinity in space for the harmonic oscillator; on the other, we face here the subtle boundary between the ordinary classical time-law with constant deceleration near the turning point (here shown by all particles with $\mathrm{E} < \mathrm{E}_{nf}$), and the extreme case $\mathrm{E}=\mathrm{E}_{nf}$: the case leads indeed to a time-law at infinity of the form

$$Lim_{\{\xi\rightarrow\infty\}}\mathrm{v}(\xi,\mathrm{E}_{nf}) \approx \frac{4\nu_n(\xi)}{\rho(\xi)} \approx \rho(\xi)^{1/4} \approx const\,\xi\, Exp[-\frac{\xi^2}{4}] \qquad (82)$$

This rather singular behaviour brought by the model worths some discussion here. A particle keeping steadily the energy E_{nf} would eventually take an infinite time to reach the oscillator boundary at $\xi \rightarrow \infty$; so the time law attached to the stationary value E_{nf} looks a bit anomalous *per se* - but we think this does not inficiates the theory, not more than quantum mechanics may not stand the linked particle at infinity in the harmonic oscillator. Indeed the Lebesgue measure of the point E_{nf} relative to the fluctuation energy interval dimension $\mathrm{E}_{nf}-\mathrm{E}_{ni}$ is just zero, so that the effective probability to find a particle with that specific energy is physically zero. For the same reason, the periods integrals (20) and (22) are unaffected by the period (singular) value in E_{nf}, being the singularity integrable: the values of the integrals are the same, would these intervals be taken closed or open in the extreme E_{nf}. On another ground, we have to remark that the model final purpose is depicting with physical coherence just the single particles motions. These last come out from the ergodic principle acting on the microcanonic system, in such a way to impose a variable energy time law to cross its level scheme - this certainly impedes a particle to persist on a steady E_{nf} value; yet keeps statistics unaltered. The simple view on this matter might be, at last, excluding the extreme E_{nf} from the energy interval: this is of no greater prejudice to the model than for standard quantum mechanics the exclusion of the point at infinity from the harmonic oscillator description would be.

Using expression (80) we can calculate now numerically the corresponding periods and densities as follows:

$$\mathrm{T}(\mathrm{E}) = \frac{1}{2\pi\nu_c}\oint \frac{d\xi}{\tilde{\mathrm{v}}(\xi,\mathrm{E})} \qquad (83)$$

$$\rho_{_{SP}}(\xi,\mathrm{E}) = \frac{2}{\mathrm{T}(\mathrm{E})\mathrm{v}(\xi,\mathrm{E})} = \frac{\lambda\sqrt{2\mathrm{m}\ \mathrm{m}_{eff}(\xi,3)}}{\hbar\mathrm{T}(\mathrm{E})\sqrt{\frac{\mathrm{E}}{\mathrm{h}\nu_c}-3+f(\xi)}} \qquad (84)$$

[17] We have also studied with details the standard case of tunnel effect through a rectangular barrier, ref. [8]. Article in preparation.

Here is f.i. the period T(E) (normalized to $1/\nu_c$) for n = 3, plotted as a function of the quantity E/E_{ni}. As advertised, the function goes asymptotically to ∞ when $E \to E_{nf}$, but is all integrable in the interval (E_{ni},E_{nf}).

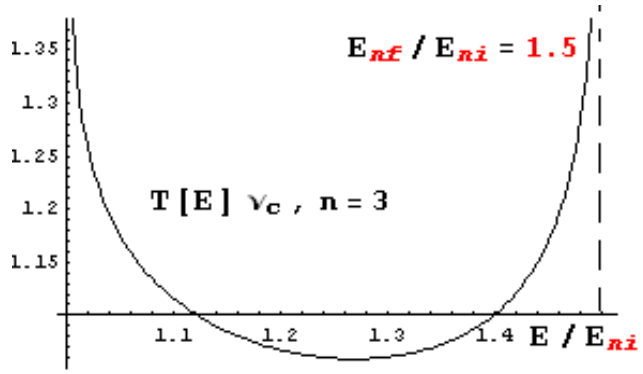


Fig.12 – Normalised periods $T(E)\nu_c$ for HO, n=3

12 Statistical averages

With the previous expressions and eq. (2), checking that eq. (20) is also satisfied can be easily done by analytical calculations in the plane (ξ,E):

$$\frac{1}{c_n \mathrm{h}} \int_{\mathrm{E}_{ni}}^{\mathrm{E}_{nf}} \mathrm{T(E)} d\mathrm{E} = \frac{1}{c_n \mathrm{h}} \frac{1}{2\pi\nu_c} \oint \int_{\mathrm{E}_{ni}}^{\mathrm{E}_{nf}} \frac{1}{\tilde{\mathrm{v}}(\xi, \mathrm{E})} d\mathrm{E} d\xi =$$

$$= \frac{1}{2} \oint \left[\frac{1}{\mathrm{h} c_n} \int_{\mathrm{E}_{ni}}^{\mathrm{E}_{nf}} \mathrm{T(E)} \rho_{_{SP}}(\xi, \mathrm{E}) d\mathrm{E} \right] \lambda \, d\xi = 1 \qquad (85)$$

Indeed the integrand in the last term of (85) is nothing else than the density $\rho(\xi)$ (see the general eq. (21); this is normalised to 1 in the space domain, as shown in eqs. (2) and (54). In the next section, we expound details for just the case n = 3.

Similarly, we have to demonstrate that eq. (22)

$$\frac{1}{c_n \mathrm{h}} \int_{\mathrm{E}_{ni}}^{\mathrm{E}_{nf}} \mathrm{T(E)E} d\mathrm{E} = \mathrm{E}_n \qquad (86)$$

is also satisfied. This cannot be done analytically but requires numerical integration case by case. Next we give the result for the case at hand here.

12.1 Statistical averages for HO, n=3

Developing the integrand in (85), we have:

$$\frac{1}{c_n \mathrm{h}} \int_{\mathrm{E}_{ni}}^{\mathrm{E}_{nf}} \mathrm{T(E)} \rho_{_{SP}}(\xi, \mathrm{E}) d\mathrm{E} \;\; |_{HO,n=3} \quad =$$

$$= \frac{\lambda\nu_c}{c_n\hbar}\sqrt{2\mathrm{m}\ \mathrm{m}_{eff}(\xi,3)}\,\mathrm{Re}\left\{\int_{\mathrm{E}_{ni}/\mathrm{h}\nu_c}^{\mathrm{E}_{nf}/\mathrm{h}\nu_c}\frac{1}{\sqrt{\frac{\mathrm{E}}{\mathrm{h}\nu_c}-3+f(\xi)}}d\frac{\mathrm{E}}{\mathrm{h}\nu_c}\right\} =$$

$$= \frac{2\mathrm{m}\lambda\nu_{30}}{\hbar}\left[\sqrt{2\frac{\mathrm{m}_{eff}(\xi,3)}{\mathrm{m}}\frac{\nu_3(\xi)}{\nu_{30}}+\varphi_g'(\xi,3)^2}-\varphi_g'(\xi,3)\right]=\rho(\xi) \qquad (87)$$

In its turn, the last occurrence is easily insured recalling that $\varphi_g'(\xi,\mathrm{n}) \equiv \lambda\nabla S/\hbar$ can also be calculated by (9) using (10). Equations (21) and (87) state that the sum of the classical densities corresponding to the given velocity fields amounts exactly to the final quantum mechanical density $\rho(\mathrm{x})$.

As a last step in the present context, we have to demonstrate that our previous calculations and chosen coefficients also insure that equations (22) and (23) are satisfied. The values for E_{ni}, E_{nf} assumed in eq. (79) clearly satisfy (23) and (24); finding them has been quite an easy matter for the HO case (see eqs. (16), (17) in [2]). As (22) is concerned, instead, it looks like we have to perform rather uneasy numerical calculations. Since the periods $\mathrm{T(E)}$ diverge when $\mathrm{E}\rightarrow\mathrm{E}_{nf}$, checking eq. (22) with a direct calculation technique turned out not practicable with a standard PC because the machine calculation time is very long even when not so high a precision is prescribed. Fortunately, this difficulty can be overcome analytically using the period definition (83) and inversion of the integration procedures in the plane (E,ξ). We find indeed

$$\frac{1}{c_n\mathrm{h}}\int_{\mathrm{E}_{ni}}^{\mathrm{E}_{nf}}\mathrm{T(E)E}d\mathrm{E}=\frac{1}{c_n\mathrm{h}}\int_{\mathrm{E}_{ni}}^{\mathrm{E}_{nf}}\frac{1}{2\pi\nu_c}\oint\frac{d\xi}{\tilde{\mathrm{v}}(\xi,\mathrm{E})}\mathrm{E}d\mathrm{E}=$$

$$= \frac{\mathrm{h}\nu_c}{2\pi c_n}\oint\sqrt{\frac{\mathrm{m}_{eff}(\xi,3)}{2\mathrm{m}}}\,\mathrm{Re}\left(\int_{\mathrm{E}_{ni}/\mathrm{h}\nu_c}^{\mathrm{E}_{nf}/\mathrm{h}\nu_c}\frac{1}{\sqrt{\mathrm{y}-3+f(\xi)}}\mathrm{y}d\mathrm{y}\right)d\xi =$$

$$= \frac{2\mathrm{h}\nu_c}{3\pi\mathrm{c}_n}\oint\sqrt{\frac{\mathrm{m}_{eff}(\xi,3)}{2\mathrm{m}}}\left\{\left(\frac{9}{2}-f(\xi)\right)f(\xi)^{\frac{1}{2}}-(4-f(\xi))\,\mathrm{Re}\,(f(\xi)-1)^{\frac{1}{2}}\right\}d\xi \qquad (88)$$

So it is easy to check now numerically, using the functions shown in *figs.* 6, 8 and 9 to work out the integral in (88), that this last expression amounts indeed to

$$2.50096\,\mathrm{h}\nu_c=(1+\eta)\frac{5}{2}\mathrm{h}\nu_c\equiv\mathrm{E}_3 \qquad (89)$$

i.e. is just the HO energy eigenvalue for $\mathrm{n}=3$ as required (with a very small, quite acceptable error $\eta = 0.00038$).

Details of the numerical integration are shown in the math file [M5].

13 Conclusion

By the present comments, we have given a review of previously published results; yet adding a number of remarks and improved calculation procedures. This is to the purpose of demonstrating fine solution of our equations and to make the reader easy with calculations. Due to laboriousness of these last, we limited here to the case for the harmonic oscillator with n = 3 - this is however central to demonstrative purposes. Although the numerical finesse for other cases already expounded in [1, 2] finally revealed not just to be at the same level than the present work, conceptual results and even graphics for all quantities (except for those specifical to $g(\mathrm{x})$) drawn therein remain valid due to the revealed numerical errors remaining very small anyway. Concerning the precision we are able to fulfill our constraints now with the new Second Step procedure, a resumptive glimpse can be given f.i. to eqs. (53), (89), (63). Moreover, eq. (6) is satisfied with $\rho'_{gn}(0) = 2.7\,10^{-6}$ etc. The mathematical context looks able now to support even greater precision, would it be worthwhile working out coefficients still more.

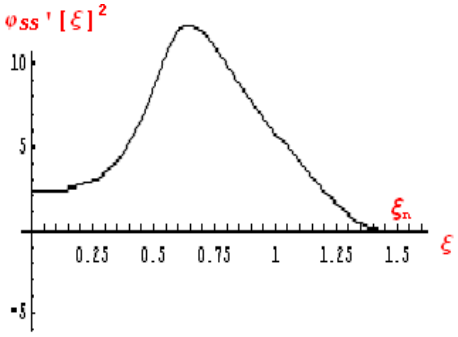
Enlightened with (65)-(66), eq. (64) looks like a sort of theorem stating that the net result of the pressure acting on the particles ensemble, the vacuum response accounted for, is the work sustaining the kinetic energy with variable mass. Conceptually, this is much analogous to the sliding component of the gravitational active force accelerating the classical disk, with an effective mass accounting for the inner degree of freedom (the rotational inertia). In our case, the active force is the particles gas pressure; the inner degree of freedom is the packet extension $\Delta\mathrm{x}$ and vacuum works upon it. Our very detailed calculations have shown us that this model, even when we use the approximated value $g(\mathrm{x}) = 1$, is quite fair almost everywhere when checked over all the oscillators level spectra; however, with the exception for the cases of very low quantum numbers (n = 2, n =3): here (very small, of the order of % 's) numerical inconveniences are encountered when trying to fulfill the wavefunction periodicity condition eq. (8) with the First Step. In order to overcome this - although minimal - difficulty, we have produced in these notes a numerical technique to show exact resolution by a (new) Second Step procedure. On a conceptual stake, we advanced the hypothesis that a more senseful account of the physical circumstances stands on the view that the real constraint to be considered for the work $W_A(\mathrm{x})$ is not properly the specific volume $1/\rho$ but the quantity $\Delta\mathrm{x}$, i.e. the spatial particles packet extension (behaviourally very similar in our context). Although here we remain on a conjectural plane, we suggest that (76), (77) might be the basic equations to rely on in order to advance in physical understanding. Conclusively, the ensemble of the theoretical elements here mentioned seem to us able to throw out a glimpse on possible modelling of the quantum vacuum action as pertinent to the present model, in the future.

14 Appendix (Errata)

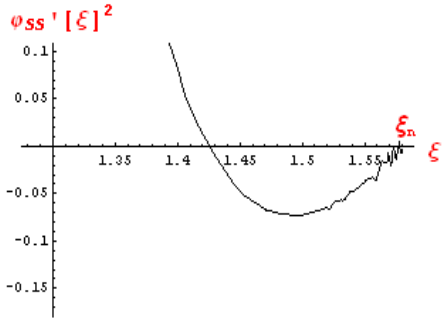
Density and phase resulting from a Second Step calculation (SS) should satisfy the wave equation with the required precision. In case HO n=3 we have (cf. (39)):

$$\varphi'_{SS}(\xi)^2 \;=\; 5 - \;\xi^2 + \frac{\sqrt{\rho_{SS}(\xi)}''}{\sqrt{\rho_{SS}(\xi)}} \tag{90}$$

Now checking again our calculations made in [2] with Table 3 parameters, the equation seems satisfied and we find the following plot ($fig.12$) for $\varphi'_{SS}(\xi)^2$:



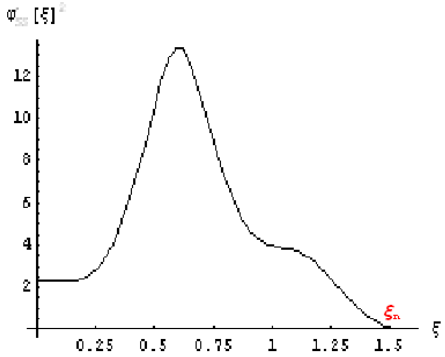
$Fig.12 - \varphi'_{SS}(\xi)^2$ *evaluated by Table 3 in* [2]



$Fig.13 - Same\ as\ fig.12, expanded\ scale\ near\ \xi_n$

On the scale of this diagram, no error is visible so we retained the data published in the Table. But examining an expanded scale near the boundary ξ_n, we have discovered a thin region of space where a small negative value is assumed before annulment ($fig.13$). This cannot be admitted because $\varphi'_{SS}(\xi)^2$ is definite positive, so the calculations we declared are affected by this error. However, it is a very small one and does not affect the form of any function plotted in [2] (except g(x)) nor the physical discussion; yet the "very good precision" claim we made of our SS calculations in [2] is obviously impugned. For comparison, we look now to the corresponding plots $figs.14$ and 15 obtained by the new SS calculation given in the present notes (eq. (50)): no such error is detectable. These plots are obtained in the quoted math file

[$M2$].



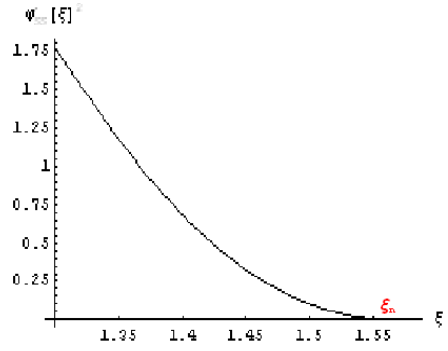


*Fig.*14 $-\varphi'^{2}_{SS}(\xi)$ *calculated by the new SS procedure.* *Fig.*15 $-$ *Same as fig.*14, *expanded scale near* ξ_n

The calculations precision can be now appreciated by the effect of eqs. (53), (89), (63) as evaluated in previous sections. So we invite the reader interested to numerical precision to rely on the procedures and results expounded in the present work.

15 References

[1] MASTROCINQUE G., Ann. de la Fond. L. de Broglie 36, 91 (2011)
[2] MASTROCINQUE G., Ann. de la Fond. L. de Broglie 36, 159 (2011)
[3] MASTROCINQUE G., Nuovo Cimento B 105, 1315 (1990)
[4] MASTROCINQUE G., Ann. de la Fond. L. de Broglie 28, 119 (2003)
[5] MASTROCINQUE G., Comm. to XCIX Congr. SIF Sez. VI, Trieste, 23-27 Sept. (2013)
[6] MASTROCINQUE G., Ann. de la Fond. L. de Broglie 27, 661 (2002)

[7] MASTROCINQUE G., Ann. de la Fond. L. de Broglie 28, 1 (2003)
[8] MASTROCINQUE G., Comm. to XCVI Congr. SIF Sez. VI, Bologna, 20-24 Sept. (2010)

16 Demonstrative software

Mathematics is worked out by Wolfram software. Printed (not operational)

versions of the files are on the web at the following addresses :
$[M1]$: FEDOA HO3 FS wave eq solve
$[M2]$: FEDOA HO3 SS wave eq solve
$[M3]$: FEDOA HO3 flow and mass functions
$[M4]$: FEDOA HO3 velocity fields
$[M5]$: FEDOA HO3 energy value
E.g. Google search: FEDOA HO FS wave eq solve. Operational programs will be made available writing to : *mastroci@unina.it*.