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THE DYNAMIC OLIGOPOLISTIC MARKET EQUILIBRIUM PROBLEM: THE VARIATIONAL FORMULATION, THE LAGRANGEAN FORMULATION, THE INVERSE PROBLEM AND COMPUTATIONAL MATTERS

TESI DI DOTTORATO DI RICERCA

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You cannot teach a man anything;
you can only help him discover it in himself.
Galileo Galilei

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Preface

Lately, the theory of variational inequalities has taken an important role in scientific research, in particular in Applied Mathematics because they are closely related with many general problems arising from Nonlinear Analysis, such as optimization, fixed point and complementarity problems. Originally, in the sixties, it represented an innovative and effective method to solve a group of partial differential equations. The responsible of the birth of this theory are, mainly, G. Fichera and G. Stampacchia (see [47, 62, 63, 100, 101]).

But in order to assist to the very first study of an optimization problem through a variational approach, we had to wait the late seventies when M.J. Smith (see [98]) and S. Dafermos (see [27]) studied the traffic network equilibrium problem in terms of a finite-dimensional variational inequality. Consequently, this gave the start to the study of the complex theory of existence, uniqueness and stability of equilibria and of computation of solutions.

In the end of nineties some authors started to investigate the optimization problems by considering also the dependence from time. Daniele, Maugeri and Oettli first investigated the traffic network equilibrium problem with feasible path flows which have to satisfy time-dependent capacity constraints and traffic demands in [37] and [38] (see also [48]). In these papers an appropriate evolutionary variational inequality was formulated and solved by means of theorems and computational procedures.

From that moment on, a lot of problems with time-dependent data were formulated in the same terms. So, the evolutionary variational formulation gave a meaningful help to the study of problems coming from economy (see [9, 29, 31, 34, 44, 45, 57, 77]), finance (see [19, 30, 36]), engineering (see [79, 82]), physics (see [85]), biology (see [52]), sociology (see [76, 81]) and other fields of applied sciences. All these problems can be handled as generalized complementarity problems and their formulation can be expressed in a unified way as projected dynamical systems (see [25, 50]).

In the same period the study of conditions under which the solution is continuous represents an important achievement because the continuity of the solution plays an important role to solve numerically the dynamic network equilibrium problems. Infact, it is possible to introduce some methods to compute dynamic equilibria by means of a discretization procedure in order to reduce the infinite-dimensional problems to the finite-dimensional problems ones (see [3, 7, 9, 16]).

In this thesis we focus our attention on a particular optimization problem coming form economy: the dynamic oligopolistic market equilibrium problem. This is the problem of finding a trade equilibrium in a supply-demand market between a finite number of spatially separated firms who produce one only commodity and ship it to some demand markets.

The heart of this thesis is just the time-dependence that allows to explore the change of behavior of equilibrium states for oligopolistic market models over a finite time interval of interest. As M.J. Beckmann and J.P. Wallace pointed out, for the first time, in [20], "the time-dependent formulation of equilibrium problems allows one to explore the dynamics of adjustment processes in which a delay on time response is operating". Of course a delay on time response always happens because the processes have not an infinite speed. Usually, such adjustment processes can be represented by means of a memory term which depends on previous equilibrium solutions according to the Volterra operator (see, for instance [10, 11]).

We start by giving a brief history of the mathematical models of such a problem.

The first author who treated the noncooperative behavior was Cournot (see [26]). He investigated the competition between only two producers of a given commodity, nowadays called the duopoly problem. Cournot precised that if both producers try, each one of his own, to increase his own profit, they will produce certain definite quantities of the commodity for the market. An equilibrium will be obtained when no one can increase his income by departing alone from his equilibrium decision, while the other retains it. Moreover Cournot proved, under suitable assumptions, the existence, the uniqueness and the stability of solution. As regards the stability of solution, Cournot showed that, if one producer, temporarily mistaken about his actual self-interest, departs from the equilibrium, he will be driven back to it through a sequence of rational reactions of each producer (called also player) to the decisions of the other producers.

Later Nash, (see [83, 84]), generalized this model by considering the behavior of n agents each acting according to his own self interest, the so called non-cooperative game. Each player $i \in N = \{1, 2, \dots, n\}$ has at his disposal a strategy x_i which he chooses from a set X_i of feasible strategies. The Nash-equilibrium is a point $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ in the common strategy set $X = X_1 \times X_2 \times \dots \times X_n$ such that no player has a rational motive to unilaterally depart from his equilibrium strategy. Rationality is defined here on the basis of individual real-valued utility functions $\{u_i : X \longrightarrow \mathbb{R},\ i = 1, 2, \dots, n,\}$ where $u_i(x) = u_i(x_1, x_2, \dots, x_n)$ represents the ith players evaluation of the collective strategy $x = (x_1, x_2, \dots, x_n)$. The rationality postulate of noncooperative behavior can be stated as follows: each player $i \in N$ chooses a strategy $x_i \in X_i$ which maximizes his utility level $u_i(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ given the decisions $(x_j)_{j\neq i}$ of the other players. Moreover, Nash proved the existence of the solution under the assumptions that the sets X_i are simplexes and the functions u_i are bilinear with respect to strategies.

Later, Rosen (see [92]) proved the existence in the more general case where the common strategy set X is a nonempty, compact and convex subset of $\mathbb{R}^n$ and the functions u_i are concave functions on X. He also established the uniqueness and the stability of such an equilibrium when the common strategy set X is described by a set of inequalities and the utility functions are such that the operator $-F_\alpha : X \longrightarrow \mathbb{R}^n$ defined by:

$$F_\alpha = \left[\alpha_1 \frac{\delta u_1}{\delta x_1}, \alpha_2 \frac{\delta u_2}{\delta x_2}, \ldots, \alpha_n \frac{\delta u_n}{\delta x_n}\right],$$

is strictly monotone on X for given $\alpha_i > 0,\ i = 1, 2, \ldots, n.$

With Karamardian (see [58]) we assisted to a relaxation of the hypotheses on the feasible set X for the existence of the solution. More precisely, he proved the existence and the uniqueness of the solution by assuming X as the nonnegative orthant of $\mathbb{R}^n$ and the utility functions such that the operator $-F : X \longrightarrow \mathbb{R}^n$ defined by:

$$F = \left[\frac{\delta u_1}{\delta x_1}, \frac{\delta u_2}{\delta x_2}, \ldots, \frac{\delta u_n}{\delta x_n}\right],$$

is continuous and strongly monotone on X (see also [59]). Such a monotonicity condition is not necessary as Gabay and Moulin (see [49]) proved. By assuming that a coercivity condition on the operator F is fulfilled, they guaranteed the existence of a Nash equilibrium even if the feasible set is no longer compact. Moreover, if the opposite of the gradient of the profit satisfies the strict monotonicity condition, as it is well known, the uniqueness of the equilibrium is guaranteed, provided that the solution exists.

A more complete and efficient study was done by Nagurney in [28, 75, 78, 80] but the problem was still faced in a static case through a finite-dimensional variational approach. In order to study the time-dependent behaviour of the model, we afford this study by considering the evolution of the market in time and, as a consequence, all the variables present in this model, such as the costs, the commodity shipments, and the excesses depend on time.

A time-dependent version of the oligopolistic market equilibrium problem was introduced by Barbagallo and Cojocaru in [9]. Moreover, the authors proved the equivalence between the dynamic Cournot-Nash principle and an evolutionary variational inequality that represents a particular parametric variational inequality where the parameter is taken to represent physical time. Via the variational formulations it is possible to give conditions under which the problem has solutions or if they are continuous with respect to time. There exists a vast literature about existence and regularity results (see [2–6, 8, 71]). The continuity allows to derive a computational procedure to numerically approximate their solutions (see [7, 16, 103]. The discretization procedure allows to reduce the infinite-dimensional problem (time-dependent) into a finite-dimensional problem where the variable time is a fixed parameter and it is possible to use the direct method for the computation of solution (see [29, 34, 70]). Moreover, in [10], a Lipschitz continuity result, which depends on the variation rate of projections onto time-dependent constraints set, is shown while in [12] some sensitivity results have been obtained each of them showing that small changes of the solution happen in correspondence with small changes of the profit function.

In [12] the authors, through the notion of quasi-relative interior of sets (see [21]), applied the infinite-dimensional duality results (or Lagrange Theory) developed in [33, 35, 39, 72] to overcome the difficulty of the voidness of the interior of the ordering cone which defines the constraints of the problem and so proved the existence of Lagrange variables which permit to describe the behaviour of the market and to highlight the presence of constraints. The equilibrium conditions established in terms of Lagrange variables do not arouse any concern because it is possible to prove that their presence is not influential in the definition of equilibrium because we can characterize such equilibrium conditions by means of an evolutionary variational formulation that does not contain the Lagrange variables. Another thing to notice is that the equilibrium conditions provided with the help of the duality theory, is equivalent to the dynamic Cournot-Nash equilibrium principle because we can prove that they are both equivalent to the same evolutionary variational inequality.

In [13, 14] the authors eliminate the serious drawback present in the theory of the oligopolistic market equilibrium problem (see [9]) coming from the unreasonable assumption that the production of a given commodity can be unbounded. This assumption can give a false solution to the problem because it is supposed that any commodity shipment from a firm to a demand market be always possible. Then, it is necessary to consider a model in which the amount of a commodity, that the producers can offer, is limited because it is reasonable to think that the resources are finite. As a consequence of this assumption, it can happen that some of the amounts of the commodity available be sold out, whereas for a part of the producers can occur an excess of production.

In [15] the authors consider capacity constraints, production and demand excesses. In order to clarify the presence of these constraints we consider some concrete economic situations: during an economic crisis period the presence of production excesses can be due to a demand decrease in demand markets and, on the other hand, the presence of demand excesses may occur when the supply can not satisfy the demand especially for fundamental goods. Moreover, since the market model presented in this paper evolves in time, the presence of both production and demand excesses is a consequence of the fact that the physical transportation of commodity between a firm and a demand market is evidently limited, therefore, there can exist some time intervals in which some of the demand markets require more commodity, though some firms produce more commodity than they can send to the demand markets.

It is also worth noting that most of the models in the literature study the problems from a producer's point of view in the sense that they describe the conditions for the profit maximization of the producers. In [17, 18] a different approach has been considered: analogously to the case of the time-dependent spatial price equilibrium control problem studied in [96], here the authors draw their attention to the policy-maker's perspective who can influence the model of the problem in order to control the resource exploitations by imposing taxes or giving subsides. In these papers the main goal is to formulate an optimal regulatory tax and to give a characterization of the optimization problem as an inverse variational inequality. Inverse variational inequalities can be considered as a special case of general variational inequalities introduced in [86] and can be used to model various control problems. Only recently the strict connection between the classical variational inequalities and inverse variational inequalities has been unveiled but, despite everything, there exists an always increasing number of problems that can be described through evolutionary variational inequalities. The equivalence between the classical variational inequalities and inverse variational inequalities enables to exploit all the powerful tools of evolutionary variational inequalities, so it is possible to treat completely this problem by studying the equilibrium solution and analyze the questions about the existence, the regularity and the computation of solution.
In [54] the authors, for the first time, studied a general network economic equilibrium problem with the help of an inverse static variational inequality. Later, in [104], the power price problem is discussed in both the discrete and the evolutionary case and the optimal price is characterized as a solution of an inverse variational inequality. Recently, in [96] an optimal control perspective on the evolutionary time-dependent spatial price equilibrium problem has been afforded by underlining the equivalence with an appropriate inverse variational inequality.

Now, we conclude this introduction with a short presentation of the structure of the thesis.

In Chapter 1, we recall the theory of variational inequalities by focusing on existence results both in the finite and in the infinite cases and, moreover, we introduce the infinite-dimensional duality theory that we exploit in the next chapter to emphasize the presence of constraints and to study the behavior of the dynamic oligopolistic market equilibrium problem. In Chapter 2, we describe the dynamic oligopolistic market equilibrium problem by considering, as first, the model without excesses, then the model with production excesses and, in the end, the model with both production and demand excesses and capacity constraints. For all the models we give an equilibrium definition according to dynamic Cournot-Nash principle and we prove the equivalence with a suitable evolutionary variational inequality. Moreover, by using the infinite-dimensional duality theory, we show the equivalence between the equilibrium conditions given by Lagrange multipliers and an evolutionary variational inequality. Since both evolutionary variational inequalities are the same, we are able to say that the two equilibrium definitions are equivalent. The only difference is that the former is a more practical definition in which we underline that each firm's aim is to maximize his own profit, and the latter is more helpful to describe the behavior of the market since it highlights the presence of constraints. In Chapter 3 we give some existence, regularity and sensitivity results for equilibrium solutions. For this reason we recall some definitions of generalized continuity of functions and Kuratowski's set convergence. In Chapter 4 we face the problem from the policy-maker's point of view, namely we allow the presence of policy maker whose purpose is to control the resource exploitations by adjusting taxes or giving subsides. A complete study is made by considering the theory of inverse evolutionary variational inequality and proving the equivalence with an appropriate evolutionary variational inequality. In Chapter 5 we propose some methods to solve evolutionary variational inequalities which express dynamic oligopolistic market equilibrium problems. To this aim, we propose a discretization procedure which reduces the infinite-dimensional problem to some finite-dimensional problems. Finally, we construct an approximation of solution through linear interpolation. This is possible because the equilibrium solution is continuous. To obtain numerical results, we make use of MatLab software. In the end, we make a convergence study in L^1-sense. At last, in order to clarify the theory present in this thesis, some numerical examples are provided.

Before we go on, I would sincerely like to thank my supervisors Professor Antonino Maugeri and Doctor Annamaria Barbagallo for their constant support and encouragement.

Chapter 1

Variational inequalities and Lagrange theory

1.1 A brief introduction to variational inequalities

Recently, a lot of problems coming from Economics, Engineering, Physics, Biology and Operation Reasearch are investigated by the very powerful tool of Variational Inequalities. For these problems, searching the equilibrium solutions is equivalent to find the solution to a suitable finite or infinite dimensional variational inequality (see, for example, [32, 51, 60, 89, 90]). Moreover, they are closely related with many general problems of Nonlinear Analysis, such as fixed point, optimization and complementarity problems. This is the reason why the theory and the solution methods have made considerable advances, recently.

In the next section we present various basic concepts in optimization and variational analysis and recall their properties.

1.1.1 Preliminary concepts

Let X be a real topological vector space, let $\mathbb{K}$ be a subset of X and let X^* be the topological dual space of X.

Definition 1.1.1. A mapping $A : \mathbb{K} \to \mathbb{R} \cup \{\pm\infty\}$ is *upper semicontinuous* if for all $v \in \mathbb{K}$,

$$\limsup_{u \to v} Au \leq Av.$$

Definition 1.1.2. A mapping $A : \mathbb{K} \to \mathbb{R} \cup \{\pm\infty\}$ is *lower semicontinuous* if for all $v \in \mathbb{K}$,

$$\liminf_{u \to v} Au \geq Av.$$

Definition 1.1.3. A mapping $A : \mathbb{K} \to X^*$ is *monotone* on $\mathbb{K}$ if for all $u, v \in \mathbb{K}$,

$$\langle Au - Av, u - v \rangle \geq 0.$$

Definition 1.1.4. A mapping $A : \mathbb{K} \to X^*$ is *strictly monotone* on $\mathbb{K}$ if for all $u, v \in \mathbb{K}$,

$$\langle Au - Av, u - v \rangle > 0.$$

Definition 1.1.5. A mapping $A : \mathbb{K} \to X^*$ is *strongly monotone* on $\mathbb{K}$ if for all $u, v \in \mathbb{K}$, $\exists \nu > 0$ such that

$$\langle Au - Av, u - v \rangle \geq \nu \left\| u - v \right\|_{\mathbb{K}}^2 .$$

Definition 1.1.6. A mapping $A : \mathbb{K} \to X^*$ is *pseudomonotone* if for all $u, v \in \mathbb{K}$

$$\langle Au, u - v \rangle \geq 0 \Rightarrow \langle Av, u - v \rangle \leq 0.$$

Definition 1.1.7. A mapping $A : \mathbb{K} \to X^*$ is *pseudomonotone in the sense of Karamardian (K-pseudomonotone)* if for all $u, v \in \mathbb{K}$

$$\langle Av, u - v \rangle \geq 0 \Rightarrow \langle Au, u - v \rangle \geq 0.$$

Definition 1.1.8. A mapping $A : \mathbb{K} \to X^*$ is *strictly pseudomonotone* if for all $u, v \in \mathbb{K}$, $u \neq v$

$$\langle Av, u - v \rangle \geq 0 \Rightarrow \langle Au, u - v \rangle > 0.$$

Definition 1.1.9. A mapping $A : \mathbb{K} \to X^*$ is *pseudomonotone in the sense of Brezis (B-pseudomonotone)* if

1. For each sequence u_n weakly converging to u (in short $u_n \rightharpoonup u$) in $\mathbb{K}$ and such that $\limsup_n \langle Au_n, u_n - v \rangle \leq 0$ it results that $\liminf_n \langle Au_n, u_n - v \rangle \geq \langle Au, u - v \rangle \quad \forall v \in \mathbb{K}$.

2. For each $v \in \mathbb{K}$ the function $u \to \langle Au, u - v \rangle$ is lower bounded on the bounded subset of $\mathbb{K}$.

Let, now, $\mathbb{K}$ be a convex subset of X.

Definition 1.1.10. A mapping $A : \mathbb{K} \to X^*$ is *hemicontinuous* if for all $v \in \mathbb{K}$ the function $u \to \langle Au, v - u \rangle$ is upper semicontinuous on $\mathbb{K}$.

Definition 1.1.11. A mapping $A : \mathbb{K} \to X^*$ is *hemicontinuous along line segments,* if the function $\xi \to \langle A\xi, u - v \rangle$ is upper semicontinuous for all $u, v \in \mathbb{K}$ on the line segments $[u, v]$.

Definition 1.1.12. A mapping $A : \mathbb{K} \to X^*$ is *lower hemicontinuous along line segments,* if the function $\xi \to \langle A\xi, u - v \rangle$ is lower semicontinuous for all $u, v \in \mathbb{K}$ on the line segments $[u, v]$.

Definition 1.1.13. A mapping $A : \mathbb{K} \to X^*$ is *hemicontinuous in the sense of Fan (F-hemicontinuous)* if for all $v \in \mathbb{K}$ the function $u \to \langle Au, u - v \rangle$ is weakly lower semicontinuous on $\mathbb{K}$.

1.1.2 Finite and infinite dimensional variational inequalities

Now, we introduce finite and infinite dimensional variational inequalities and we recall some existence results.

Definition 1.1.14. Let $\mathbb{K}$ be a nonempty, convex and closed set of the m-dimensional Euclidean space $\mathbb{R}^m$ and let $A : \mathbb{K} \to \mathbb{R}^m$ be a vector-function. The *finite dimensional variational inequality* is the problem to find a vector $x^* \in \mathbb{K}$ such that

$$\langle A(x^*), x - x^* \rangle \geq 0, \qquad \forall x \in \mathbb{K}. \tag{1.1.1}$$

Here we make a list of some classical conditions showed by Stampacchia for existence of solutions to variational inequality (1.1.1).

Theorem 1.1.1. [53] *If $\mathbb{K}$ is a nonempty, convex and compact subset of $\mathbb{R}^m$ and $A : \mathbb{K} \to \mathbb{R}^m$ is a continuous operator, then the variational inequality* (1.1.1) *admits at least one solution.*

Theorem 1.1.2. [63] *If $\mathbb{K}$ is a nonempty, convex and compact subset of $\mathbb{R}^m$ and $A : \mathbb{K} \to \mathbb{R}^m$ is a continuous operator, then the the set of solutions to the variational inequality* (1.1.1) *is convex and compact.*

Theorem 1.1.3. [66] *If $A : \mathbb{K} \to \mathbb{R}^m$ is strictly monotone on $\mathbb{K}$, then the solution to variational inequality* (1.1.1) *admits a unique solution if it exists.*

Whenever the set $\mathbb{K}$ is unbounded, the existence of solutions may also be established under the coercivity condition, as shows the following results.

Theorem 1.1.4. [60] *If $A : \mathbb{K} \to \mathbb{R}^m$ satisfies the coercivity condition*

$$\lim_{\|x\|_{\mathbb{R}^m} \to +\infty} \frac{\langle A(x) - A(x'), x - x' \rangle}{\|x - x'\|_{\mathbb{R}^m}} = +\infty \tag{1.1.2}$$

for $x \in \mathbb{K}$ and some $x' \in \mathbb{K}$, then the variational inequality (1.1.1) *admits a solution.*[1]

Let X be a real topological vector space, let $\mathbb{K}$ be a subset of X and let X^* be the topological dual space of X.

Definition 1.1.15. Let $\mathbb{K}$ be a nonempty, convex and closed set of X and let $A : \mathbb{K} \to X^*$ be a vector-function. The *infinite dimensional variational inequality* is the problem to find a vector $x^* \in \mathbb{K}$ such that

$$\langle A(x^*), x - x^* \rangle \geq 0, \qquad \forall x \in \mathbb{K}. \tag{1.1.3}$$

[1] The symbol $\|\cdot\|_{\mathbb{R}^m}$ denotes the norm in $\mathbb{R}^m$, for all $m > 1$.

Here we make a list of some general conditions for existence of solutions to variational inequality (1.1.3).

Let X be a reflexive Banach space and let $\mathbb{K} \subseteq X$ be a convex and closed set. Let us denote by $\|\cdot\|$ the norm in X. Let B_R be the closed ball with center in O and radius R and let us consider the closed and convex set $\mathbb{K}_R = \mathbb{K} \cap B_R$. If R is large enough, then $\mathbb{K}_R$ is nonempty. We have the following result.

Theorem 1.1.5. [101] *Let $A : \mathbb{K} \to X^*$ be a monotone and hemicontinuous along line segments function, then the variational inequality* (1.1.3) *admits a solution if and only if there exists a constant R such that at least one solution of the variational inequality*

$$x_R^* \in \mathbb{K}_R : \qquad \langle A(x_R^*), x - x_R^* \rangle \geq 0, \qquad \forall x \in \mathbb{K}_R. \tag{1.1.4}$$

satisfies the condition

$$\|x_R\| < R. \tag{1.1.5}$$

Remark 1.1.1. If the set $\mathbb{K}$ is unbounded, then the following conditions for the existence of solutions are provided:

1. let us suppose that $\exists x_0 \in \mathbb{K}$ and $R > \|x_0\|$ such that

   $$\langle A(x_0), x - x_0 \rangle < 0$$

   $\forall x \in \mathbb{K},\ \|x\| = R$, then (1.1.5) is verified.

2. let us suppose that $\exists x_0$ such that C satisfies the coercivity condition (1.1.2), then (1.1.4) holds.

3. let us suppose that C satisfies the weak coercivity requirement:

   $$\lim_{\|x\| \to +\infty} \frac{\langle A(x), x \rangle}{\|x\|} = +\infty \tag{1.1.6}$$

   $\forall x \in \mathbb{K}$, then (1.1.5) is verified.

Theorem 1.1.6. [87] *Let X be a real topological vector space and let $\mathbb{K} \subseteq X$ be a nonempty convex set. Let $A : \mathbb{K} \to X^*$ be a given function such that:*

i. *there exist $A \subseteq \mathbb{K}$ nonempty, compact and $B \subseteq \mathbb{K}$ nonempty, compact, convex such that for every $x \in \mathbb{K} \setminus A$, there exists $\hat{x} \in B$ with*

   $$\langle A(x), \hat{x} - x \rangle < 0;$$

ii. *A is pseudomonotone and hemicontinuous along line segments.*

Then, the variational inequality (1.1.3) *admits a solution.*

Theorem 1.1.7. [87] *Let X be a real topological vector space and let $\mathbb{K} \subseteq X$ be a nonempty convex set. Let $A : \mathbb{K} \to X^*$ be a given function such that:*

i. *there exist $A \subseteq \mathbb{K}$ nonempty, compact and $B \subseteq \mathbb{K}$ nonempty, compact, convex such that for every $x \in \mathbb{K} \setminus A$, there exists $\hat{x} \in B$ with*

$$\langle A(x), \hat{x} - x \rangle < 0;$$

ii. *A is hemicontinuous.*

Then, the variational inequality (1.1.3) *admits a solution.*

1.2 A brief recall to Lagrange theory

1.2.1 Motivation

In the last two sections we recall the Infinite-Dimensional duality theory, namely the problem to know conditions under which an optimization problem with cone constraints and equality constraints and its dual lagrangean problem have the same extremal points. The same problem can be seen as the problem to know the conditions under which it is possible to guarantee the existence of Lagrange multipliers associated to the constraints. This is called "strong duality problem". Well known results (see [56]) ensure such an existence by assuming that the topological interior of the ordering cone that defines the constraints is non empty. However, when the problem is infinite-dimensional, we can not apply such results since the topological interior of the ordering cones is empty.
So, it was necessary to develop a new theory that could guarantee the effectiveness of strong duality without the request that the topological interior of the ordering cone is nonempty. Such a theory has been developed in the last decade (see, for example [21, 33, 35, 39, 55, 72, 106]). To overcome this difficulty, we do not consider the topological interior, but we define an algebraic interior of sets through the notion of tangent cone and normal cone, called *quasi relative interior* (*q.r.i* in short). We can observe that in all infinite-dimensional problems that we consider, the quasi relative interior of the ordering cone is nonempty even if the topological interior is empty.
The most important assumption to ensure the strong duality is the so called Assumption S that we show throughout the chapter.
These results have gone beyond expectations because we have found that such conditions are also necessary conditions for the validity of strong duality (see [43]).

1.2.2 Lagrange theory

Let us present the infinite dimensional Lagrange duality theory which represents an important and very recent achievement (see [33, 35, 39]). At first, we remind some definitions and then we give some duality results (see [33, 39, 72]).

Let X denote a real normed space, let X^* be the topological dual of all continuous linear functionals on X and let C be a subset of X. Given an element $x \in \mathrm{Cl}(C)$, the set:

$$T_C(x) = \left\{ h \in X : \ h = \lim_{n \to \infty} \lambda_n (x_n - x), \ \lambda_n > 0, \ x_n \in C, \ \forall n \in \mathbb{N}, \ \lim_{n \to \infty} x_n = x \right\}$$

is called the tangent cone to C at x.
If C is convex, we have (see [56]):

$$T_C(x) = \mathrm{Cl}(\mathrm{Cone}(C - \{x\})),$$

where $\mathrm{Cone}(C) = \{\lambda x : \ x \in C, \ \lambda \geq 0\}$.

Following Borwein and Lewis [21], we give the following definition of quasi-relative interior for a convex set.

Definition 1.2.1. Let C be a convex subset of X. The *quasi-relative interior of* C, denoted by *qri* C, is the set of those $x \in C$ for which $T_C(x)$ is a linear subspace of X.

If we define the normal cone to C at x as the set:

$$N_C(x) = \{\xi \in X^* : \ \langle \xi, y - x \rangle \leq 0, \ \forall y \in C\},$$

the following result holds:

Proposition 1.2.1. *Let C be a convex subset of X and $x \in C$. Then $x \in \mathrm{qri}\ C$ if and only if $N_C(x)$ is a linear subspace of X^*.*

Using the notion of qri C, in [39], the following separation theorem is proved.

Theorem 1.2.1. *Let C be a convex subset of X and $x_0 \in C \setminus \mathrm{qri}\ C$. Then, there exists $\xi \in X^*$, $\xi \neq \theta_{X^*}$, such that*

$$\langle \xi, x \rangle \leq \langle \xi, x_0 \rangle, \quad \forall x \in C.$$

Vice versa, let us suppose that there exist $\xi \neq \theta_{X^*}$ and a point $x_0 \in X$ such that $\langle \xi, x \rangle \leq \langle \xi, x_0 \rangle$, $\forall x \in C$, and that $\mathrm{Cl}(T_C(x_0) - T_C(x_0)) = X$. Then, $x_0 \notin \mathrm{qri}\ C$.

Now, let us present the statement of the infinite dimensional duality theory.

Let X be a real linear topological space and S a nonempty convex subset of X; let $(Y, \|\cdot\|_Y)$ be a real normed space partially ordered by a convex cone C and let $(Z, \|\cdot\|_Z)$ be a real normed space. Let $f : S \to \mathbb{R}$ and $g : S \to Y$ be two convex functions and let $h : S \to Z$ be an affine-linear function.

Let us consider the problem

$$\min_{x \in \mathbb{K}} f(x), \tag{1.2.1}$$

where $\mathbb{K} = \{x \in S : \ g(x) \in -C, \ h(x) = \theta_Z\}$, and the dual problem

$$\max_{\substack{u \in C^* \\ v \in Z^*}} \inf_{x \in S} \{f(x) + \langle u, g(x) \rangle + \langle v, h(x) \rangle\}, \tag{1.2.2}$$

where $C^* = \{u \in Y^* : \langle u, y \rangle \geq 0 \ \forall y \in C\}$ is the dual cone of C.

We say that *Assumption S* is fulfilled at a point $x_0 \in \mathbb{K}$ if and only if it results

$$T_{\widetilde{M}}(0, \theta_Y, \theta_Z) \cap \Big( ]-\infty, 0[\times \{\theta_Y\} \times \{\theta_Z\} \Big) = \emptyset, \tag{1.2.3}$$

where $\widetilde{M} = \{(f(x) - f(x_0) + \alpha, g(x) + y, h(x)) : \ x \in S \setminus \mathbb{K}, \ \alpha \geq 0, \ y \in C\}$.

Remark 1.2.1. If $(0, \theta_Y, \theta_Z) \notin \mathrm{Cl}(\widetilde{M})$, then Assumption S holds, because $T_{\widetilde{M}}(0, \theta_Y, \theta_Z) = \emptyset$.

Remark 1.2.2. If Assumption S holds, $T_{\widetilde{M}}(0, \theta_Y, \theta_Z) \neq \emptyset$ and $(l, \theta_Y, \theta_Z) \in T_{\widetilde{M}}(0, \theta_Y, \theta_Z)$, then $l \geq 0$.

Remark 1.2.3. If Assumption S holds, then $(0, \theta_Y, \theta_Z) \notin \operatorname{qri} \widetilde{M}$.

The following theorem holds (see [33]):

Theorem 1.2.2. *Under the above assumptions, if problem* (1.2.1) *is solvable and* Assumption S *is fulfilled at the extremal solution $x_0 \in \mathbb{K}$, then also problem* (1.2.2) *is solvable, the extreme values of both problems are equal and if $(x_0, \overline{u}, \overline{v}) \in \mathbb{K} \times C^* \times Z^*$ is the optimal point of problem* (1.2.2)*, it results:*

$$\langle \overline{u}, g(x_0) \rangle = 0.$$

Using Theorem 1.2.2, we are able to show the usual relationship between a saddle point of the so-called Lagrange functional

$$\mathcal{L}(x, u, v) = f(x) + \langle u, g(x) \rangle + \langle v, h(x) \rangle, \quad \forall x \in S, \ \forall u \in C^*, \ \forall v \in Z^*, \tag{1.2.4}$$

and the solution of the constraint optimization problem (1.2.1) (see [33]).

Theorem 1.2.3. *Let us assume that the assumptions of Theorem* 1.2.2 *are satisfied. Then, $x_0 \in \mathbb{K}$ is a minimal solution to problem* (1.2.1) *if and only if there exist $\overline{u} \in C^*$ and $\overline{v} \in Z^*$ such that $(x_0, \overline{u}, \overline{v})$ is a saddle point of the Lagrange functional* (1.2.4)*, namely*

$$\mathcal{L}(x_0, u, v) \leq \mathcal{L}(x_0, \overline{u}, \overline{v}) \leq \mathcal{L}(x, \overline{u}, \overline{v}), \quad \forall x \in S, \ u \in C^*, \ v \in Z^*$$

and, moreover, it results that

$$\langle \overline{u}, g(x_0) \rangle = 0. \tag{1.2.5}$$

Chapter 2

The dynamic oligopolistic market equilibrium problem

2.1 Historical development

The dynamic oligopolistic market equilibrium problem is the problem of finding a trade equilibrium in a time-dependent supply-demand market between a finite number of spatially separated firms who produce one only commodity and ship the commodity to some demand markets.
We seek to determine a non-negative commodity distribution for which the firms and the demand markets will be in a state of equilibrium as defined next.
The equilibrium principle we consider here comes from Cournot and Nash's consideration, infact it will be called the dynamic Cournot-Nash principle (see [26, 83, 84]) and regards the maximization of the firms' profits. Corunot and Nash were the first authors who described the model: Cournot investigated the problem by considering only two competitive firms, while Nash extended the concept by considering n firms.
Cournot precised that if both producers try, each one of his own, to increase his own profit, they will produce certain definite quantities of the commodity for the market and an equilibrium will be obtained when no one can increase his income by departing alone from his equilibrium decision.
Nash, moreover, stated that the equilibrium point is a strategy such that no player has a rational motive to unilaterally depart from it. Each firm chooses a strategy which maximizes his utility level given the decisions of the other firms.
As regards the stability of solution, Cournot showed that, if one producer, temporarily mistaken about his actual self-interest, departs from the equilibrium, he will be driven back to it through a sequence of rational reactions of each producer to the others decisions, called also players. As regards the existence of the solution we can refer to Nash himself, Rosen, Karamardian, Gabay and Moulin (see [49, 58, 59, 92]).

The introduction of time in all the variables present in the model was well motivated by M.J. Beckmann and J.P. Wallace who pointed out, for the first time, in [20], "the time-dependent formulation of equilibrium problems allows one to explore the dynamics of adjustment processes in which a delay on time response is operating".

In [9] a time-dependent version of the oligopolistic market equilibrium was introduced for the first time. Here the authors proved the equivalence between the dynamic Cournot-Nash principle and an evolutionary variational inequality.
Another thing to notice is that in this paper we assume that the quantity produced by each firm is equal to the commodity shipments from that firm to all the demand markets. Moreover, the quantity demanded by each demand market is equal to the commodity shipments from all the firms to that demand market. Namely, in this paper, no production or demand excess is allowed.

In this thesis we consider another different, but equivalent, definition of equilibrium that illustrates other important features of the equilibrium. Through the infinite-dimensional duality results (or Lagrange Theory) recalled in section 1.2 we prove the existence of Lagrange variables which permit to describe the behaviour of the market and to highlight the presence of constraints. Moreover, it is possible to prove that the equilibrium conditions established in terms of Lagrange variables is equivalent to the same evolutionary variational inequality coming from the Cournot-Nash principle and it does not contain the Lagrange variables. In [12] the authors, for the first time, analyzed such equilibrium conditions.

Lately, in [14, 15] we assist to an improvement of the model because here the feasible sets allow the presence of production and demand excesses. This can be consequence of problems coming from economic crisis periods, presence of fundamental goods and physical transportation.

2.2 The case with no excesses

In this section, we study the dynamic oligopolistic market equilibrium problem where all the production is shipped to the demand markets and the demand is fully satisfied.

Let us consider m firms P_i, $i = 1, \dots, m$, that produce one only commodity and n demand markets Q_j, $j = 1, \dots, n$, that are generally spatially separated. Assume that the homogeneous commodity, produced by the m firms and consumed by the n markets, is involved during a time interval $[0, T]$, $T > 0$. Let $p_i(t)$, $i = 1, \dots, m$, denote the non-negative commodity output produced by firm P_i at the time $t \in [0, T]$. Let $q_j(t)$, $j = 1, \dots, n$, denote the non-negative demand for the commodity at demand market Q_j

at the time $t \in [0,T]$. Let $x_{ij}(t)$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, denote the non-negative commodity shipment between the supply market P_i and the demand market Q_j at the time $t \in [0,T]$. In particular, let us set the vector $x_i(t) = (x_{i1}(t), \ldots, x_{in}(t))$, $i = 1, \ldots, m$, $t \in [0,T]$ as the strategy vector for the firm P_i.

Let us group the production output into a vector-function $p : [0,T] \longrightarrow \mathbb{R}^m_+$, the demand output into a vector-function $q : [0,T] \longrightarrow \mathbb{R}^n_+$ and the commodity shipments into a matrix-function $x : [0,T] \longrightarrow \mathbb{R}^{mn}_+$.

Let us assume that the following feasibility conditions hold:

$$p_i(t) = \sum_{j=1}^{n} x_{ij}(t), \quad i = 1, \ldots, m, \text{ a.e. in } [0,T], \tag{2.2.1}$$

$$q_j(t) = \sum_{i=1}^{m} x_{ij}(t), \quad j = 1, \ldots, n, \text{ a.e. in } [0,T]. \tag{2.2.2}$$

Hence, the quantity produced by each firm P_i, at the time $t \in [0,T]$, must be equal to the commodity shipments from that firm to all the demand markets, at the same time $t \in [0,T]$. Moreover, the quantity demanded by each demand market Q_j, at the time $t \in [0,T]$, must be equal to the commodity shipments from all the firms to that demand market, at the same time $t \in [0,T]$.

Furthermore, we assume that the non-negative commodity shipment between the producer P_i and the demand market Q_j has to satisfy time-dependent constraints, namely there exist two non-negative functions $\underline{x}, \overline{x} : [0,T] \longrightarrow \mathbb{R}^{mn}_+$ such that

$$0 \le \underline{x}_{ij}(t) \le x_{ij}(t) \le \overline{x}_{ij}(t), \quad \forall i = 1, \ldots, m, \ \forall j = 1, \ldots, n, \text{ a.e. in } [0,T]. \tag{2.2.3}$$

For technical reasons, let us assume that

$$x \in L^2([0,T], \mathbb{R}^{mn}_+), \quad \underline{x} \in L^2([0,T], \mathbb{R}^{mn}_+), \quad \overline{x} \in L^2([0,T], \mathbb{R}^{mn}_+).$$

As a consequence, we have

$$p \in L^2([0,T], \mathbb{R}^m_+), \quad q \in L^2([0,T], \mathbb{R}^n_+).$$

Then, the set of feasible vectors $x \in L^2([0,T], \mathbb{R}^{mn}_+)$ is

$$\mathbb{K} = \Bigg\{ x \in L^2([0,T], \mathbb{R}^{mn}_+) : \quad \underline{x}_{ij}(t) \le x_{ij}(t) \le \overline{x}_{ij}(t), \quad \forall i = 1, \ldots, m, \ \forall j = 1, \ldots, n, \text{ a.e. in } [0,T] \Bigg\}. \tag{2.2.4}$$

This set is convex, closed and bounded in the Hilbert space $L^2([0,T], \mathbb{R}^{mn}_+)$.

Furthermore, let us associate with each firm P_i a production cost f_i, $i = 1, \dots, m$, and assume that the production cost of a firm P_i may depend upon the entire production pattern, namely,

$$f_i = f_i(t, x(t)).$$

Similarly, let us associate with each demand market Q_j, a demand price for unity of the commodity d_j, $j = 1, \dots, n$, and assume that the demand price of a demand market Q_j may depend upon the entire consumption pattern, namely,

$$d_j = d_j(t, x(t)).$$

Moreover, since we allow production excesses and, consequently, the storage of commodity, we must consider the function g_i, $i = 1, \dots, m$, that denotes the storage cost of the commodity produced by the firm P_i and assume that this cost may depend upon the entire production pattern, namely,

$$g_i = g_i(t, x(t)).$$

Finally, let c_{ij}, $i = 1, \dots, m$, $j = 1, \dots, n$, denote the transaction cost, which includes the transportation cost associated with trading the commodity between firm P_i and demand market Q_j. Here we permit the transaction cost to depend upon the entire shipment pattern, namely,

$$c_{ij}(t) = c_{ij}(t, x(t)), \tag{2.2.5}$$

Hence, we have the following mappings,

$$\begin{aligned}
f &: [0,T] \times L^2([0,T], \mathbb{R}^{mn}_+) \longrightarrow L^2([0,T], \mathbb{R}^{m}_+), \\
d &: [0,T] \times L^2([0,T], \mathbb{R}^{mn}_+) \longrightarrow L^2([0,T], \mathbb{R}^{n}_+), \\
g &: [0,T] \times L^2([0,T], \mathbb{R}^{mn}_+) \longrightarrow L^2([0,T], \mathbb{R}^{m}_+), \\
c &: [0,T] \times L^2([0,T], \mathbb{R}^{mn}_+) \longrightarrow L^2([0,T], \mathbb{R}^{mn}_+).
\end{aligned}$$

The profit $v_i(t, x(t))$, $i = 1, \dots, m$, of the firm P_i at the time $t \in [0,T]$ is, then,

$$v_i(t, x(t)) = \sum_{j=1}^{n} d_j(t, x(t)) x_{ij}(t) - f_i(t, x(t)) - g_i(t, x(t)) - \sum_{j=1}^{n} c_{ij}(t, x(t)) x_{ij}(t),$$

namely, it is equal to the price that the demand markets are disposed to pay minus the production costs, the storage costs and the transportation costs.

Moreover, we recall that in the Hilbert space $L^2([0,T], \mathbb{R}^k)$, we define the canonical bilinear form on $L^2([0,T], \mathbb{R}^k)^* \times L^2([0,T], \mathbb{R}^k)$, by

$$\langle\langle \phi, \mathrm{w} \rangle\rangle := \int_0^T \langle \phi(t), \mathrm{w(t)} \rangle \, dt,$$

where $\phi \in (L^2([0,T], \mathbb{R}^k))^* = L^2([0,T], \mathbb{R}^k)$, $\mathrm{w} \in \mathrm{L}^2([0,\mathrm{T}], \mathbb{R}^\mathrm{k})$ and

$$\langle \phi(t), \mathrm{w(t)} \rangle = \sum_{l=1}^{k} \phi_l(t) \mathrm{w_l(t)}.$$

Let us denote by $\nabla_D v = \left(\frac{\partial v_i}{\partial x_{ij}} \right)_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$ and $x_i = \{x_{ij}\}_{j=1,\dots,n}$, $i = 1, \dots, m$.

Let us assume the following assumptions:

(i) $v_i(t, x(t))$ is continuously differentiable for each $i = 1, \dots, m$, a.e. in $[0, T]$,

(ii) $\nabla_D v$ is a Carathéodory function such that

$$\exists h \in L^2([0,T]) : \ \|\nabla_D v(t, x(t))\|_{mn} \leq h(t) \|x(t)\|_{mn}, \quad \text{a.e. in } [0,T], \tag{2.2.6}$$

(iii) $v_i(t, x(t))$ is pseudoconcave with respect to the variables x_i, $i = 1, \dots, m$, a.e. in $[0, T]$.

For the reader's convenience, we recall that a function v, continuously differentiable, is called *pseudoconcave* with respect to x_i, $i = 1, \dots, m$, a.e. in $[0,T]$ (see [67]), if the following condition holds, a.e. in $[0,T]$:

$$\left\langle \frac{\partial v}{\partial x_i}(t, x_1, \dots, x_i, \dots, x_m), x_i - y_i \right\rangle \geq 0$$
$$\Rightarrow v_i(t, x_1, \dots, x_i, \dots, x_m) \geq v_i(t, x_1, \dots, y_i, \dots, x_m).$$

2.2.1 The variational formulation

Now let us consider the dynamic oligopolistic market, in which the m firms supply the commodity in a noncooperative fashion, each one trying to maximize its own profit function considered the optimal distribution pattern for the other firms, at the time $t \in [0,T]$. We seek to determine a non-negative commodity distribution matrix-function x for which the m firms and the n demand markets will be in a state of equilibrium as defined below.

Definition 2.2.1. $x^* \in \mathbb{K}$ is a dynamic oligopolistic market equilibrium if and only if for each $i = 1, \dots, m$ and a.e. in $[0,T]$ we have

$$v_i(t, x^*(t)) \geq v_i(t, x_i(t), \hat{x}_i^*(t)), \tag{2.2.7}$$

where

$$\hat{x}_i^*(t) = (x_1^*(t), \dots, x_{i-1}^*(t), x_{i+1}^*(t), \dots, x_m^*(t)).$$

It is possible to prove (see [9]) that under the assumptions (i), (ii), (iii) on v_i, Definition 2.2.1 is equivalent to an evolutionary variational inequality, namely:

Theorem 2.2.1. *Let us suppose that assumptions* (i), (ii), (iii) *are satisfied. Then,* $x^* \in \mathbb{K}$ *is a dynamic oligopolistic market equilibrium in presence of excesses according to Definition* 2.2.1 *if and only if it satisfies the evolutionary variational inequality*

$$\int_0^T -\sum_{i=1}^m \sum_{j=1}^n \frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - x^*_{ij}(t))dt \geq 0 \qquad \forall x \in \mathbb{K}. \tag{2.2.8}$$

Proof. For first, let us prove that the evolutionary variational inequality (2.2.8)

$$\int_0^T \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - x^*_{ij}(t))dt = \langle\langle -\nabla_D v(x^*), x - x^*\rangle\rangle$$
$$= \int_0^T \langle -\nabla_D v(t, x^*(t)), x(t) - x^*(t)\rangle \geq 0 \quad \forall x \in \mathbb{K},$$

is equivalent to the following point-to-point formulation:

$$\langle -\nabla_D v(t, x^*(t)), x(t) - x^*(t)\rangle = \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - x^*_{ij}(t)) \geq 0$$
$$\forall x(t) \in \mathbb{K}(t), \quad \text{a.e. in } [0, T], \tag{2.2.9}$$

where

$$\mathbb{K}(t) = \left\{ x(t) \in \mathbb{R}^{mn} : \quad \underline{x}_{ij}(t) \leq x_{ij}(t) \leq \overline{x}_{ij}(t), \;\; \forall i = 1, \dots, m, \; \forall j = 1, \dots, n \right\}.$$

In fact, let us suppose by absurdum that (2.2.9) does not hold,
namely $\exists \bar{x}(t) \in \mathbb{K}$, $\exists I \subseteq [0, T]$ with $m(I) > 0$ such that

$$\langle -\nabla_D v(t, x^*(t)), \bar{x}(t) - x^*(t)\rangle < 0 \quad \text{a.e. in } I.$$

Let us choose, now,

$$x(t) = \begin{cases} x^*(t) & \text{in } [0, \mathrm{T}] \setminus I, \\ \bar{x}(t) & \text{in } I \end{cases}$$

So, by integrating over $[0, T]$, we get:

$$\begin{aligned} \langle\langle -\nabla_D v(t, x^*(t)), x - x^*\rangle\rangle \quad &= \int_{[0,T]\setminus I} \langle -\nabla_D v(t, x^*(t)), x(t) - x^*(t)\rangle \, dt \\ &+ \int_I \langle -\nabla_D v(t, x^*(t)), \bar{x}(t) - x^*(t)\rangle \, dt < 0. \end{aligned}$$

This is an absurdum.
The vice versa is immediate.

So the equivalence between the evolutionary variational inequalities (2.2.8) and (2.2.9) is proved.

Let us prove, now, the equivalence between the Cournot-Nash principle and the evolutionary variational inequality (2.2.8).
Let us suppose that $x^* \in \mathbb{K}$ is an equilibrium point according to definition 2.2.1, namely:

$$v_i(t, x^*(t)) \geq v_i(t, x(t), \hat{x}^*(t)) \qquad \forall x(t) \in \mathbb{K}(t), \quad \text{a.e. in } [0, T], \quad \forall i = 1, \ldots, m. \tag{2.2.10}$$

For well know theorems of Optimization, we have that the necessary and sufficient condition to get (2.2.10) is that $\forall i = 1, \ldots, m,\ \forall x(t) \in \mathbb{K}(t),$ a.e. in $[0, T]$

$$\langle -\nabla_D v_i(t, x^*(t)), x_i(t) - x_i^*(t) \rangle = \sum_{j=1}^{n} -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} (x_{ij}(t) - x_{ij}^*(t)) \geq 0. \tag{2.2.11}$$

By assumption ∇v_i is a Carathéodory function such that

$$\exists h \in L^2([0, T]) : \|\nabla_D v_i(t, x(t))\|_{mn} \leq h(t) \|x(t)\|_{mn} \text{ a.e. in } [0, T],$$

and, moreover x and $x^* \in L^2([0, T], \mathbb{R}^{mn})$,
so we have that

$$t \longrightarrow \langle -\nabla_D v_i(t, x^*(t)), x_i(t) - x_i^*(t) \rangle \in L^2([0, T], \mathbb{R}).$$

So, we get

$$\langle\langle -\nabla_D v_i(t, x^*(t)), x_i - x_i^* \rangle\rangle \geq 0 \qquad \forall x \in \mathbb{K},$$

from who, by summing for each firm P_i, for $i = 1, \ldots, m$, we obtain

$$\sum_{i=1}^{m} \langle\langle -\nabla_D v_i(t, x^*(t)), x - x^* \rangle\rangle = \langle\langle -\nabla_D v(t, x^*(t)), x_i - x_i^* \rangle\rangle \geq 0 \qquad \forall x \in \mathbb{K}.$$

Vice versa, let us suppose that $x^*(t)$ is a solution to the evolutionary variational inequality (2.2.8), but not an equilibrium solution according to Cournot-Nash principle, namely:
$\exists I \subseteq [0, T]$ with $m(I) > 0$, $\exists \bar{i} \in \{1, \ldots, m\}$ and $\exists \widetilde{x}_{\bar{i}}$ such that

$$v_{\bar{i}}(t, x^*(t)) < v_{\bar{i}}(t, \widetilde{x}_i(t), \hat{x}^*(t)) \quad \text{in } I$$

Since the profit function $v_{\bar{i}}(t, x(t))$ is pseudoconcave with respect to $x_{\bar{i}}$, we get:

$$\langle -\nabla_D v_{\bar{i}}(t, x^*(t)), x_{\bar{i}}^*(t) - \widetilde{x}_{\bar{i}}(t) \rangle < 0 \quad \text{in } I. \tag{2.2.12}$$

If we choose $x \in \mathbb{K}$ such that

$$x_i(t) = \begin{cases} x_i^*(t) & \text{in } [0,\text{T}] \setminus I, \forall \text{i} = 1, \dots, \text{m}, \\ x_i^*(t) & \text{in } I, \text{if i} \neq \bar{\text{i}}, \\ \widetilde{x}_i & \text{in } I, \text{if i} = \bar{\text{i}} \end{cases},$$

then

$$\int_0^T \langle -\nabla_D v(t, x^*(t)), x(t) - x^*(t) \rangle \, dt = \int_I \langle -\nabla_D v_{\bar{i}}(t, x^*(t)), \widetilde{x}_{\bar{i}}(t) - x^*_{\bar{i}}(t) \rangle \, dt < 0,$$

so we get the contradiction. □

2.2.2 The Lagrangean formulation

In this section we prove that, under the assumptions (i), (ii), (iii) on the profit function v, Definition 2.2.1 is equivalent to the equilibrium conditions defined through Lagrange variables which are very useful in order to analyze the constraints of $\mathbb{K}$:

Definition 2.2.2. $x^* \in \mathbb{K}$ is a dynamic oligopolistic market equilibrium if and only if for each $i = 1, \dots, m$ and a.e. in $[0, T]$ there exist $\alpha^*_{ij} \in L^2([0,T])$, $\beta^*_{ij} \in L^2([0,T])$ such that

$$-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} + \beta^*_{ij}(t) = \alpha^*_{ij}(t), \tag{2.2.13}$$

$$\alpha^*_{ij}(t)(\underline{x}_{ij}(t) - x^*_{ij}(t)) = 0, \qquad \alpha^*_{ij}(t) \geq 0, \tag{2.2.14}$$

$$\beta^*_{ij}(t)(x^*_{ij}(t) - \overline{x}_{ij}(t)) = 0, \qquad \beta^*_{ij}(t) \geq 0. \tag{2.2.15}$$

Conditions (2.2.13)-(2.2.15) give the optimal distribution pattern for the firm P_i. The terms $\alpha^*_{ij}(t)$, $\beta^*_{ij}(t)$ are the Lagrange multipliers associated to the constraints $x^*_{ij}(t) \geq \underline{x}_{ij}(t)$, $x^*_{ij}(t) \leq \overline{x}_{ij}(t)$, respectively. They, as it is well known, have a topical importance on the understanding and the management of the market. In fact, at a fixed time $t \in [0, T]$, we have:

(a) if $\alpha^*_{ij}(t) > 0$ then, by using (2.2.14), we obtain $x^*_{ij}(t) = \underline{x}_{ij}(t)$, namely the commodity shipment between the firm P_i and the demand market Q_j is minimum;

(b) if $x^*_{ij}(t) > \underline{x}_{ij}(t)$ then, taking into account (2.2.14), $\alpha^*_{ij}(t) = 0$ and, making use of (2.2.13), it results $\beta^*_{ij}(t) = \dfrac{\partial v_i(t, x^*(t))}{\partial x_{ij}}$, namely $\beta^*_{ij}(t)$ is equal to the marginal utility function;

(c) if $\beta^*_{ij}(t) > 0$ then, by using (2.2.15), we obtain $x^*_{ij}(t) = \overline{x}_{ij}(t)$, namely the commodity shipment between the firm P_i and the demand market Q_j is maximum;

(d) if $x^*_{ij}(t) < \overline{x}_{ij}(t)$ then, making use of (2.2.15), $\beta^*_{ij}(t) = 0$ and, taking into account (2.2.13), we get $\alpha^*_{ij}(t) = -\dfrac{\partial v_i(t, x^*(t))}{\partial x_{ij}}$, namely $\alpha^*_{ij}(t)$ is equal to the opposite of the marginal utility function.

It is worth to underline that in Definition 2.2.2, even if in (2.2.13) - (2.2.15) the unknown Lagrange variables α^*_{ij}, β^*_{ij} appear, they do not influence the equilibrium definition because the following equivalent condition in terms of evolutionary variational inequality holds (see [12]):

Theorem 2.2.2. *$x^* \in \mathbb{K}$ is a dynamic oligopolistic market equilibrium according to Definition* 2.2.2 *if and only if it satisfies the evolutionary variational inequality:*

$$\int_0^T -\sum_{i=1}^m \sum_{j=1}^n \frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} (x_{ij}(t) - x^*_{ij}(t)) dt \geq 0 \qquad \forall x \in \mathbb{K}. \tag{2.2.16}$$

Definitions 2.3.1 and 2.2.2 describe the equilibrium by means of dynamic Cournot-Nash principle and Lagrange multipliers, respectively, whereas the variational formulation gives a powerful tool for the study of the existence, the regularity and the calculus of equilibrium solutions.
Before proving this theorem, we need a preliminary lemma.

Lemma 2.2.1. *Let $x^* \in \mathbb{K}$ be a solution to the variational inequality* (2.2.8) *and let us set,*

$$\begin{aligned}
E^-_{ij} &= \left\{t \in [0,T] : x^*_{ij}(t) = \underline{x}_{ij}(t)\right\}, \quad \forall i = 1, \dots, m, \ \forall j = 1, \dots, n, \\
E^0_{ij} &= \left\{t \in [0,T] : \underline{x}_{ij}(t) < x^*_{ij}(t) < \overline{x}_{ij}(t)\right\}, \quad \forall i = 1, \dots, m, \ \forall j = 1, \dots, n, \\
E^+_{ij} &= \left\{t \in [0,T] : x^*_{ij}(t) = \overline{x}_{ij}(t)\right\}, \quad \forall i = 1, \dots, m, \ \forall j = 1, \dots, n.
\end{aligned}$$

Then, we have

$$\frac{\partial v_i(t, \underline{x}(t))}{\partial x_{ij}} \leq 0, \quad a.e. \ in \ E^-_{ij},$$

$$\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} = 0, \quad a.e. \ in \ E^0_{ij},$$

$$\frac{\partial v_i(t, \overline{x}(t))}{\partial x_{ij}} \geq 0, \quad a.e. \ in \ E^+_{ij}.$$

Proof. Le us observe that we have

$$\begin{aligned}
&\int_0^T -\sum_{i=1}^m \sum_{j=1}^n \frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - x^*_{ij}(t))dt = \\
&\int_{E^-_{ij}} -\sum_{i=1}^m \sum_{j=1}^n \frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - \underline{x}_{ij}(t))dt \\
&+ \int_{E^0_{ij}} -\sum_{i=1}^m \sum_{j=1}^n \frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - x^*_{ij}(t))dt \\
&+ \int_{E^+_{ij}} -\sum_{i=1}^m \sum_{j=1}^n \frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - \overline{x}_{ij}(t))dt \geq 0.
\end{aligned}$$

$\forall x \in \mathbb{K}$.
If we choose $x \in \mathbb{K}$ such that $x_{lr}(t) = x^*_{lr}(t)$ for $l \neq i$ and $r \neq j$, we have $\forall \underline{x}_{ij}(t) \leq x_{ij}(t) \leq \overline{x}_{ij}(t), \quad \forall i = 1, \ldots, m, \ \forall j = 1, \ldots, n,$

$$\begin{aligned}
&\int_0^T -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - x^*_{ij}(t))dt = \\
&\int_{E^-_{ij}} -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - \underline{x}_{ij}(t))dt \\
&+ \int_{E^0_{ij}} -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - x^*_{ij}(t))dt \\
&+ \int_{E^+_{ij}} -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - \overline{x}_{ij}(t))dt \geq 0.
\end{aligned} \tag{2.2.17}$$

If we choose $x \in \mathbb{K}$ such that

$$x_{ij}(t) \begin{cases} > \underline{x}_{ij}(t) & \text{in } E^-_{ij}, \\ = x^*_{ij}(t) & \text{in } E^0_{ij}, \\ = x^*_{ij}(t) & \text{in } E^+_{ij} \end{cases},$$

then, (2.2.17) becomes

$$\int_0^T -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - x^*_{ij}(t))dt = \int_{E^-_{ij}} -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - \underline{x}_{ij}(t))dt \geq 0. \tag{2.2.18}$$

Since $x_{ij}(t) > \underline{x}_{ij}(t)$, we get that $\dfrac{\partial v_i(t, x^*(t))}{\partial x_{ij}} \leq 0$. In fact if there exists a subset F of E^-_{ij} with $m(F) > 0$ such that $\dfrac{\partial v_i(t, x^*(t))}{\partial x_{ij}} > 0$ in F, choosing

$$x_{ij}(t) \begin{cases} = \underline{x}_{ij}(t) & \text{in } E^-_{ij} \setminus F, \\ > \underline{x}_{ij}(t) & \text{in } F \end{cases},$$

we get

$$\int_0^T -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - x^*_{ij}(t))dt = \int_F -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - \underline{x}_{ij}(t))dt < 0$$

in contradiction with (2.2.18).
Hence,

$$\frac{\partial v_i(t, \underline{x}(t))}{\partial x_{ij}} \leq 0, \quad \text{a.e. in } E^-_{ij}.$$

If we choose $x \in \mathbb{K}$ such that

$$x_{ij}(t) \begin{cases} = x^*_{ij}(t)) & \text{in } E^-_{ij}, \\ = x^*_{ij}(t) & \text{in } E^0_{ij}, \\ < \overline{x}_{ij}(t) & \text{in } E^+_{ij} \end{cases},$$

then, (2.2.17) becomes

$$\int_0^T -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - x^*_{ij}(t))dt = \int_{E^+_{ij}} -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - \overline{x}_{ij}(t))dt \geq 0. \tag{2.2.19}$$

Since $x_{ij}(t) < \overline{x}_{ij}(t)$, we get that $\dfrac{\partial v_i(t, x^*(t))}{\partial x_{ij}} \geq 0$. In fact if there exists a subset F of E^-_{ij} with $m(F) > 0$ such that $\dfrac{\partial v_i(t, x^*(t))}{\partial x_{ij}} < 0$ in F, choosing

$$x_{ij}(t) \begin{cases} = \overline{x}_{ij}(t) & \text{in } E^-_{ij} \setminus F, \\ < \overline{x}_{ij}(t) & \text{in } F \end{cases},$$

we get

$$\int_0^T -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - x^*_{ij}(t))dt = \int_F -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - \overline{x}_{ij}(t))dt < 0$$

in contradiction with (2.2.19).
Hence,

$$\frac{\partial v_i(t, \underline{x}(t))}{\partial x_{ij}} \geq 0, \quad \text{a.e. in } E^+_{ij}.$$

In E^0_{ij}, by using the same technique as in the previous cases, it can be easily proved that $\dfrac{\partial v_i(t, \underline{x}(t))}{\partial x_{ij}}$ cannot be either negative or positive on any set with positive measure. Hence,

$$\frac{\partial v_i(t, \underline{x}(t))}{\partial x_{ij}} = 0, \quad \text{a.e. in } E^0_{ij}.$$

□

Let us prove, now, Theorem 2.2.2.

Proof. Let us assume that $x^* \in \mathbb{K}$ is an equilibrium solution according to Definition 2.2.2. Then, taking into account that $\alpha^*_{ij}(t)(\underline{x}_{ij}(t) - x^*_{ij}(t)) = 0$ and $\beta^*_{ij}(t)(x^*_{ij}(t) - \overline{x}_{ij}(t)) = 0$, a.e. in $[0,T]$, we have for every $x \in \mathbb{K}$, a.e. in $[0,T]$,

$$\begin{aligned} -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - x^*_{ij}(t)) &= -\beta^*_{ij}(t)(x_{ij}(t) - x^*_{ij}(t)) + \alpha^*_{ij}(t)(x_{ij}(t) - x^*_{ij}(t)) \\ &= -\beta^*_{ij}(t)(x_{ij}(t) - \overline{x}_{ij}(t)) + \alpha^*_{ij}(t)(x_{ij}(t) - \underline{x}_{ij}(t)) \geq 0, \end{aligned}$$

and, as a consequence, by summing over $i = 1, \ldots, m$ and $j = 1, \ldots, n$, integrating on $[0,T]$, it results, for each $x \in \mathbb{K}$,

$$\int_0^T \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - x^*_{ij}(t))dt \geq 0.$$

Hence, we obtain (2.2.8).

Vice versa, let $x^* \in \mathbb{K}$ be a solution to (2.2.8) and let us apply the infinite dimensional duality theory. First of all, let us prove that the Assumption S is fulfilled.

Let us set, for $x \in L^2([0,T], \mathbb{R}^{mn})$,

$$\Psi(x) = \int_0^T \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - x^*_{ij}(t))dt$$

and

$$\begin{aligned} \widetilde{M} = \Big\{ & (\Psi(x) + \alpha, -x + \underline{x} + y, x - \overline{x} + u,) : i = 1, \ldots, m, \ j = 1, \ldots, n, \ \alpha \geq 0, \\ & x \in L^2([0,T], \mathbb{R}^{mn}_+) \setminus \mathbb{K}, \quad y, u \in L^2([0,T], \mathbb{R}^{mn}_+) \Big\}, \end{aligned}$$

we must show that
if $(l, \theta_{L^2([0,T],\mathbb{R}^{mn}_+)}, \theta_{L^2([0,T],\mathbb{R}^{mn}_+)})$ belongs to $T_{\widetilde{M}}(0, \theta_{L^2([0,T],\mathbb{R}^{mn}_+)}, \theta_{L^2([0,T],\mathbb{R}^{mn}_+)})$, namely

$$\begin{aligned}
&\lim_{n\to+\infty} \lambda_n(\Psi(x^n)+\alpha_n) = l,\\
&\lim_{n\to+\infty} \lambda_n(-x^n+\underline{x}+y^n) = \theta_{L^2([0,T],\mathbb{R}^{mn}_+)},\\
&\lim_{n\to+\infty} \lambda_n(x^n-\overline{x}+u^n) = \theta_{L^2([0,T],\mathbb{R}^{mn}_+)},
\end{aligned}$$

with $\lambda_n \geq 0$, $x^n \in L^2([0,T], \mathbb{R}^{mn}_+) \setminus \mathbb{K}$, $\alpha_n \geq 0$, $y^n, u^n \in L^2([0,T], \mathbb{R}^{mn}_+)$, $\forall n \in \mathbb{N}$, and

$$\begin{aligned}
&\lim_{n\to+\infty} (\Psi(x^n)+\alpha_n) = 0,\\
&\lim_{n\to+\infty} (-x^n+\underline{x}+y^n) = \theta_{L^2([0,T],\mathbb{R}^{mn}_+)},\\
&\lim_{n\to+\infty} (x^n+\overline{x}+u^n) = \theta_{L^2([0,T],\mathbb{R}^{mn}_+)},
\end{aligned}$$

then, l is non-negative.

As a consequence, we have

$$\begin{aligned}
l &= \lim_{n\to+\infty} \lambda_n(\Psi(x^n)+\alpha_n)\\
&= \lim_{n\to+\infty} \lambda_n \left(\int_0^T \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial v_i(t,x^*(t))}{\partial x_{ij}} (x^n_{ij}(t)-x^*_{ij}(t))dt + \alpha_n \right)\\
&\geq \lim_{n\to+\infty} \lambda_n \Bigg(\int_{E^-_{ij}} \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial v_i(t,x^*(t))}{\partial x_{ij}} (x^n_{ij}(t)-\underline{x}_{ij}(t))dt\\
&\quad + \int_{E^0_{ij}} \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial v_i(t,x^*(t))}{\partial x_{ij}} (x^n_{ij}(t)-x^*_{ij}(t))dt\\
&\quad + \int_{E^+_{ij}} \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial v_i(t,x^*(t))}{\partial x_{ij}} (x^n_{ij}(t)-\overline{x}_{ij}(t))dt \Bigg).
\end{aligned}$$

We can observe that

$$\lim_{n\to+\infty} \lambda_n \int_{E^0_{ij}} \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial v_i(t,x^*(t))}{\partial x_{ij}} (x^n_{ij}(t)-x^*_{ij}(t))dt = 0 \tag{2.2.20}$$

being $-\dfrac{\partial v_i(t,x^*(t))}{\partial x_{ij}} = 0$, a.e. in E^0_{ij}, $\forall i = 1,\ldots,m$, $\forall j = 1,\ldots,n$.

We will prove that

$$\lim_{n\to+\infty} \lambda_n \int_{E^-_{ij}} \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial v_i(t,\underline{x}(t)))}{\partial x_{ij}} (x^n_{ij}(t)-\underline{x}_{ij}(t))dt \geq 0 \tag{2.2.21}$$

and

$$\lim_{n\to+\infty} \lambda_n \int_{E_{ij}^+} \sum_{i=1}^{m}\sum_{j=1}^{n} -\frac{\partial v_i(t,\overline{x}(t))}{\partial x_{ij}}(x_{ij}^n(t)-\overline{x}_{ij}(t))dt \geq 0. \tag{2.2.22}$$

It results

$$\begin{aligned}
&\lim_{n\to+\infty} \lambda_n \int_{E_{ij}^-} \sum_{i=1}^{m}\sum_{j=1}^{n} -\frac{\partial v_i(t,\underline{x}(t))}{\partial x_{ij}}(x_{ij}^n(t)-\underline{x}_{ij}(t))dt \\
&= \lim_{n\to+\infty} \lambda_n \int_{E_{ij}^-} \sum_{i=1}^{m}\sum_{j=1}^{n} \left(-\frac{\partial v_i(t,\underline{x}(t))}{\partial x_{ij}}(x_{ij}^n(t)-\underline{x}_{ij}(t)-y_{ij}^n(t)) - \frac{\partial v_i(t,\underline{x}(t))}{\partial x_{ij}}(y_{ij}^n(t))\right) dt
\end{aligned}$$

By virtue of the previous remarks and Lemma 2.2.1, for the conditions of belonging to the tangent cone, we get the inequality (2.2.21) and, with analogous considerations, we get the inequality (2.2.22).
Therefore, thanks to (2.2.20), (2.2.21), (2.2.22) we have that

$$\begin{aligned}
l &= \lim_{n\to+\infty} \lambda_n(\Psi(x^n)+\alpha_n) \\
&= \lim_{n\to+\infty} \lambda_n \left(\int_0^T \sum_{i=1}^{m}\sum_{j=1}^{n} -\frac{\partial v_i(t,x^*(t))}{\partial x_{ij}}(x_{ij}^n(t)-x_{ij}^*(t))dt+\alpha_n\right)
\end{aligned}$$

is non-negative.

Taking into account Theorems 1.2.2 and 1.2.3, if we consider the Lagrange function

$$\begin{aligned}
\mathcal{L}(x,\alpha,\beta) &= \Psi(x) \\
&\quad +\sum_{i=1}^{m}\sum_{j=1}^{n}\int_0^T \alpha_{ij}(t)(\underline{x}_{ij}(t)-x_{ij}(t))dt + \sum_{i=1}^{m}\sum_{j=1}^{n}\int_0^T \beta_{ij}(t)(x_{ij}(t)-\overline{x}_{ij}(t))dt,
\end{aligned}$$

$\forall x \in L^2([0,T],\mathbb{R}_+^{mn}),\ \alpha,\beta \in L^2([0,T],\mathbb{R}_+^{mn})$, we have that there exist $\alpha^*,\beta^* \in L^2([0,T],\mathbb{R}_+^{mn})$ such that

$$\mathcal{L}(x^*,\alpha,\beta) \leq \mathcal{L}(x^*,\alpha^*,\beta^*) \leq \mathcal{L}(x,\alpha^*,\beta^*) \tag{2.2.23}$$

$\forall x \in L^2([0,T],\mathbb{R}_+^{mn}),\ \alpha,\beta \in L^2([0,T],\mathbb{R}_+^{mn})$, and

$$\langle\langle \alpha^*, \underline{x}-x^*\rangle\rangle = \int_0^T \sum_{i=1}^{m}\sum_{j=1}^{n} \alpha_{ij}^*(t)(\underline{x}_{ij}(t)-x_{ij}^*(t))dt = 0$$

$$\langle\langle \beta^*, x^*-\overline{x}\rangle\rangle = \int_0^T \sum_{i=1}^{m}\sum_{j=1}^{n} \beta_{ij}^*(t)(x_{ij}^*(t)-\overline{x}_{ij}(t))dt = 0.$$

Hence,

$$\alpha^*_{ij}(t)(\underline{x}_{ij}(t) - x^*_{ij}(t)) = 0, \quad \forall i = 1, \dots, m, \quad \forall j = 1, \dots, n, \tag{2.2.24}$$

$$\beta^*_{ij}(t)(x^*_{ij}(t) - \overline{x}_{ij}(t)) = 0, \quad \forall i = 1, \dots, m, \quad \forall j = 1, \dots, n. \tag{2.2.25}$$

Then, for conditions (2.2.24), (2.2.25), $\mathcal{L}(x^*, \alpha^*, \beta^*) = 0$, and by virtue of the right-hand side of (2.2.23) and the equalities (2.2.24), (2.2.25), we get

$$\begin{aligned}
\mathcal{L}(x, \alpha^*, \beta^*) &= \sum_{i=1}^m \sum_{j=1}^n \int_0^T -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - x^*_{ij}(t))dt \\
&\quad - \sum_{i=1}^m \sum_{j=1}^n \int_0^T \alpha^*_{ij}(t)(x_{ij}(t) - x^*_{ij}(t))dt \\
&\quad + \sum_{i=1}^m \sum_{j=1}^n \int_0^T \beta^*_{ij}(t)(x_{ij}(t) - x^*_{ij}(t))dt \\
&\geq \mathcal{L}(x^*, \alpha^*, \beta^*) = 0, \qquad \forall x \in L^2([0, T], \mathbb{R}^{mn}_+).
\end{aligned}$$

Then, $\mathcal{L}(x, \alpha^*, \beta^*)$ has a minimal point in x^*.

Let $x^1_{ij} = x^*_{ij} + \varepsilon_{ij}$ and $x^2_{ij} = x^*_{ij} - \varepsilon_{ij}$. We observe that

$$\mathcal{L}(x^1, \alpha^*, \beta^*) = \sum_{i=1}^m \sum_{j=1}^n \int_0^T \left(-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} - \alpha^*_{ij}(t) + \beta^*_{ij}(t) \right) \varepsilon_{ij}(t)dt \geq 0$$
$$\forall \varepsilon \in L^2([0, T], \mathbb{R}^{mn}_+),$$

and

$$\mathcal{L}(x^2, \alpha^*, \beta^*) = -\left\{ \sum_{i=1}^m \sum_{j=1}^n \int_0^T \left(-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} - \alpha^*_{ij}(t) + \beta^*_{ij}(t) \right) \varepsilon_{ij}(t)dt \right\} \geq 0$$
$$\forall \varepsilon \in L^2([0, T], \mathbb{R}^{mn}_+),$$

from which we get

$$\sum_{i=1}^m \sum_{j=1}^n \int_0^T \left(-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} - \alpha^*_{ij}(t) + \beta^*_{ij}(t) \right) \varepsilon_{ij}(t)dt = 0 \quad \forall \epsilon \in L^2([0, T], \mathbb{R}^{mn}_+)$$

Since ε is arbitrary, we get,

$$\begin{aligned}
&-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} - \alpha^*_{ij}(t) + \beta^*_{ij}(t) = 0, \\
&\alpha^*_{ij}(\underline{x}_{ij}(t) - x^*_{ij}(t)) = 0, \quad \alpha^*_{ij}(t) \geq 0, \\
&\beta^*_{ij}(t)(x^*_{ij}(t) - \overline{x}_{ij}(t)) = 0, \quad \beta^*_{ij}(t) \geq 0,
\end{aligned}$$

$\forall i = 1, \dots, m, \ j = 1, \dots, n$, a.e. in $[0, T]$.

□

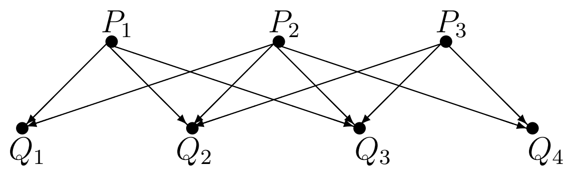

Figure 2.1: Network structure of the numerical dynamic spatial oligopoly problem.

Taking into account Theorems 2.2.1 and 2.2.2, the equivalence between Definitions 2.2.1 and 2.2.2 is proved.

2.2.3 A numerical example

Let us consider a example of the dynamical oligopolistic market equilibrium problem consisting of three firms and four demand markets, as in Figure 2.1. Let $\underline{x}, \overline{x} \in C([0,1], \mathbb{R}^{3\times 4})$ be the capacity constraints such that, in $[0,1]$,

$$\underline{x}(t) = \begin{pmatrix} 0 & \frac{1}{12}t & \frac{1}{50}t & \frac{1}{100}t \\ \frac{1}{100}t & 0 & 0 & \frac{1}{20}t \\ 0 & \frac{1}{24}t & \frac{1}{2}t & 0 \end{pmatrix}, \quad \overline{x}(t) = \begin{pmatrix} 2t & 4t & 6t & t \\ 4t & 3t & 5t & t \\ t & t & t & 3t \end{pmatrix}.$$

Hence, the feasible set is

$$\mathbb{K} = \Big\{ \quad x \in L^2([0,1], \mathbb{R}^{3\times 4}) : \\ \underline{x}_{ij}(t) \le x_{ij}(t) \le \overline{x}_{ij}(t), \quad i = 1,2,3, \ j = 1,2,3,4, \text{ a.e. in } [0,1] \Big\}.$$

Let $v \in C^1(L^2([0,1], \mathbb{R}^{3\times 4}), \mathbb{R}^3)$ be the profit function defined by

$$\begin{aligned} v_1(t, x(t)) &= -4x_{11}^2(t) - 4x_{12}^2(t) - 6x_{13}^2(t) - 6x_{14}^2(t) - 4x_{11}(t)x_{12}(t) - 6x_{13}(t)x_{14}(t) \\ &\quad +3tx_{11}(t) + 4tx_{12}(t) + tx_{13}(t) + tx_{14}(t), \\ v_2(t, x(t)) &= -5x_{21}^2(t) - 2x_{22}^2(t) - 2x_{23}^2(t) - 2x_{24}^2(t) - 2x_{21}(t)x_{22}(t) - 2x_{23}(t)x_{24}(t) \\ &\quad +2tx_{21}(t) + 2tx_{22}(t) + 3tx_{23}(t) + 2tx_{24}(t), \\ v_3(t, x(t)) &= -10x_{31}^2(t) - 4x_{32}^2(t) - 4x_{33}^2(t) - 5x_{34}^2(t) - 2x_{11}(t)x_{32}(t) - 2x_{33}(t)x_{34}(t) \\ &\quad -2x_{12}(t)x_{31}(t) + tx_{31}(t) + 2tx_{32}(t) + 10tx_{33}(t) + 3tx_{34}(t). \end{aligned}$$

Then, the operator $\nabla_D v \in C(L^2([0,1], \mathbb{R}^{3\times 4}), \mathbb{R}^{3\times 4})$ is given by

$$\nabla_D v = \begin{pmatrix} -8x_{11} - 4x_{12} + 3t & -8x_{12} - 4x_{11} + 4t & -12x_{13} - 6x_{14} + t & -12x_{14} - 6x_{13} + t \\ -10x_{21} - 2x_{22} + 2t & -4x_{22} - 2x_{21} + 2t & -4x_{23} - 2x_{24} + 3t & -4x_{24} - 2x_{23} + 2t \\ -20x_{31} - 2x_{12} + t & -8x_{32} - 2x_{11} + 2t & -8x_{33} - 2x_{34} + 10t & -10x_{34} - 2x_{33} + 3t \end{pmatrix}.$$

Now, we verify that $-\nabla_D v$ is a strongly monotone operator, in fact:

$$\begin{aligned}
&\langle -\nabla_D v(x(t)) + \nabla_D v(y(t)), x(t) - y(t)\rangle \\
&\quad = \big\{8[x_{11}(t) - y_{11}(t)] + 4[x_{12}(t) - y_{12}(t)]\big\}[x_{11}(t) - y_{11}(t)] + \big\{8[x_{12}(t) - y_{12}(t)] \\
&\qquad +4[x_{11}(t) - y_{11}(t)]\big\}[x_{12}(t) - y_{12}(t)] + \big\{12[x_{13}(t) - y_{13}(t)] + 6[x_{14}(t) - y_{14}(t)]\big\} \\
&\qquad [x_{13}(t) - y_{13}(t)] + \big\{12[x_{14}(t) - y_{14}(t)] + 6[x_{13}(t) - y_{13}(t)]\big\}[x_{14}(t) - y_{14}(t)] \\
&\qquad +\big\{10[x_{21}(t) - y_{21}(t)] + 2[x_{22}(t) - y_{22}(t)]\big\}[x_{21}(t) - y_{21}(t)] + \big\{4[x_{22}(t) - y_{22}(t)] \\
&\qquad +2[x_{21}(t) - y_{21}(t)]\big\}[x_{22}(t) - y_{22}(t)] + \big\{4[x_{23}(t) - y_{23}(t)] + 2[x_{24}(t) - y_{24}(t)]\big\} \\
&\qquad [x_{23}(t) - y_{23}(t)] + \big\{4[x_{24}(t) - y_{24}(t)] + 2[x_{23}(t) - y_{23}(t)]\big\}[x_{24}(t) - y_{24}(t)] \\
&\qquad +\big\{20[x_{31}(t) - y_{31}(t)] + 2[x_{12}(t) - y_{12}(t)]\big\}[x_{31}(t) - y_{31}(t)] + \big\{8[x_{32}(t) - y_{32}(t)] \\
&\qquad +2[x_{11}(t) - y_{11}(t)]\big\}[x_{32}(t) - y_{32}(t)] + \big\{8[x_{33}(t) - y_{33}(t)] + 2[x_{34}(t) - y_{34}(t)]\big\} \\
&\qquad [x_{33}(t) - y_{33}(t)] + \big\{10[x_{34}(t) - y_{34}(t)] + 2[x_{33}(t) - y_{33}(t)]\big\}[x_{34}(t) - y_{34}(t)] \\
&\quad \geq 2\|x(t) - y(t)\|^2_{3\times 4}.
\end{aligned}$$

Moreover, it results that $-\nabla_D v$ is Lipschitz continuous, in fact:

$$\begin{aligned}
&\| - \nabla_D v(x) + \nabla_D v(y)\|^2_{3\times 4} \\
&\quad = \big\{8[x_{11} - y_{11}] + 4[x_{12} - y_{12}]\big\}^2 + \big\{8[x_{12} - y_{12}] + 4[x_{11} - y_{11}]\big\}^2 \\
&\qquad +\big\{12[x_{13} - y_{13}] + 6[x_{14} - y_{14}]\big\}^2 + \big\{12[x_{14} - y_{14}] + 6[x_{13} - y_{13}]\big\}^2 \\
&\qquad +\big\{10[x_{21} - y_{21}] + 2[x_{22} - y_{22}]\big\}^2 + \big\{4[x_{22} - y_{22}] + 2[x_{21} - y_{21}]\big\}^2 \\
&\qquad +\big\{4[x_{23} - y_{23}] + 2[x_{24} - y_{24}]\big\}^2 + \big\{4[x_{24} - y_{24}] + 2[x_{23} - y_{23}]\big\}^2 \\
&\qquad +\big\{20[x_{31} - y_{31}] + 2[x_{12} - y_{12}]\big\}^2 + \big\{8[x_{32} - y_{32}] + 2[x_{11} - y_{11}]\big\}^2 \\
&\qquad +\big\{8[x_{33} - y_{33}] + 2[x_{34} - y_{34}]\big\}^2 + \big\{10[x_{34} - y_{34}] + 2[x_{33} - y_{33}]\big\}^2 \\
&\quad \geq 800\|x - y\|^2_{3\times 4}.
\end{aligned}$$

The dynamic oligopolistic market equilibrium distribution in presence of excesses is the solution to the evolutionary variational inequality:

$$\int_0^1 \sum_{i=1}^{3}\sum_{j=1}^{4} -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - x^*_{ij}(t))dt \geq 0, \quad \forall x \in \mathbb{K}. \tag{2.2.26}$$

Taking into account the direct method (see [29, 34, 70]), we consider the following system

$$\begin{cases}
8x^*_{11}(t) + 4x^*_{12}(t) - 3t = 0, & 4x^*_{11}(t) + 8x^*_{12}(t) - 4t = 0, \\
12x^*_{13}(t) + 6x^*_{14}(t) - t = 0, & 6x^*_{13}(t) + 12x^*_{14}(t) - t = 0, \\
10x^*_{21}(t) + 2x^*_{22}(t) - 2t = 0, & 2x^*_{21}(t) + 4x^*_{22}(t) - 2t = 0, \\
4x^*_{23}(t) + 2x^*_{24}(t) - 3t = 0, & 2x^*_{23}(t) + 4x^*_{24}(t) - 2t = 0, \\
20x^*_{31}(t) + 2x^*_{12}(t) - t = 0, & 8x^*_{32}(t) + 2x^*_{11}(t) - 2t = 0, \\
8x^*_{33}(t) + 2x^*_{34}(t) - 10t = 0, & 2x^*_{33}(t) + 10x^*_{34}(t) - 3t = 0
\end{cases} \tag{2.2.27}$$

We get the following solution, in $[0, 1]$,

$$x^*(t) = \begin{pmatrix} \frac{1}{6}t & \frac{5}{12}t & \frac{1}{18}t & \frac{1}{18}t \\ \frac{1}{9}t & \frac{4}{9}t & \frac{2}{3}t & \frac{1}{6}t \\ \frac{1}{120}t & \frac{5}{24}t & \frac{47}{38}t & \frac{1}{19}t \end{pmatrix}.$$

But we observe that x^* does not belong to the constraint set $\mathbb{K}$ because $x^*_{33}(t) > t$. Let us consider now the set

$$\widetilde{\mathbb{K}} = \left\{ \begin{array}{l} x \in L^2([0,1], \mathbb{R}^{3\times 4}) : \underline{x}_{ij}(t) \leq x_{ij}(t) \leq \overline{x}_{ij}(t), \\ \qquad i = 1,2,3,\ j = 1,2,3,4,\ (i,j) \neq (3,3),\ \text{a.e. in } [0,1], \\ x_{33}(t) = t,\ \text{a.e. in } [0,1], \\ \sum_{j=1}^{4} x_{ij}(t) \leq p_i(t), \quad i = 1, \dots, 3,\ \text{a.e. in } [0,1], \\ \sum_{i=1}^{3} x_{ij}(t) \leq q_j(t), \quad j = 1, \dots, 4,\ \text{a.e. in } [0,1] \end{array} \right\}$$

and the system

$$\left\{ \begin{array}{ll} 8x^*_{11}(t) + 4x^*_{12}(t) - 3t = 0, & 4x^*_{11}(t) + 8x^*_{12}(t) - 4t = 0, \\ 12x^*_{13}(t) + 6x^*_{14}(t) - t = 0, & 6x^*_{13}(t) + 12x^*_{14}(t) - t = 0, \\ 10x^*_{21}(t) + 2x^*_{22}(t) - 2t = 0, & 2x^*_{21}(t) + 4x^*_{22}(t) - 2t = 0, \\ 4x^*_{23}(t) + 2x^*_{24}(t) - 3t = 0, & 2x^*_{23}(t) + 4x^*_{24}(t) - 2t = 0, \\ 20x^*_{31}(t) + 2x^*_{12}(t) - t = 0, & 8x^*_{32}(t) + 2x^*_{11}(t) - 2t = 0, \\ x^*_{33}(t) = t, & 2x^*_{33}(t) + 10x^*_{34}(t) - 3t = 0 \end{array} \right. \tag{2.2.28}$$

We can observe that

$$x^*(t) = \begin{pmatrix} \frac{1}{6}t & \frac{5}{12}t & \frac{1}{18}t & \frac{1}{18}t \\ \frac{1}{9}t & \frac{4}{9}t & \frac{2}{3}t & \frac{1}{6}t \\ \frac{1}{120}t & \frac{5}{24}t & t & \frac{1}{10}t \end{pmatrix},$$

is a solution, in $[0, 1]$, since $8x^*_{33}(t) + 2x^*_{34}(t) - 10t < 0$. Making use of the equilibrium definition (2.2.14)–(2.2.15), we obtain:

$$\alpha^*_{ij}(t) = 0, \quad \forall i = 1, \dots, 3,\ j = 1, \dots, 4,\ \text{a.e. in } [0,1],$$

$$\beta^*_{ij}(t) = 0, \quad \forall i = 1, \dots, 3, \; j = 1, \dots, 4, \quad (i,j) \neq (3,3), \text{ a.e. in } [0,1].$$

$\beta^*_{33}(t) = \dfrac{9}{5}t$ represents the marginal utility.

2.3 The case with production excesses

In this section we consider a more general case. It is easy to see that in the model presented in section 2.2 there is an implicit assumption stating that the production of a given commodity can be unbounded. This assumption can give a false solution to the problem because we allow any commodity shipment from a firm to a demand market. Then, it is necessary to consider a model in which the amount of a commodity, that the producers can offer, is limited because it is reasonable to think that the resources are finite. As a consequence of this assumption, it can happen that some of the amounts of the commodity available be sold out, whereas for a part of the producers can occur an excess of production. Then, an appropriate model must also consider, not only the limited availability of commodities, but also the presence of production excesses. Let us consider the simplest case in which the commodity shipment is greater than or equal to zero. In addition to what considered in section 2.2, let us introduce the production excess.

Let $\varepsilon_i(t), i = 1, \dots, m$, denote the non-negative production excess for the commodity of the firm P_i at the time $t \in [0,T]$ and group it into a vector-function $\varepsilon : [0,T] \to \mathbb{R}^m_+$. The following feasibility condition holds:

$$p_i(t) = \sum_{j=1}^{n} x_{ij}(t) + \varepsilon_i(t), \; i = 1, \dots, m, \text{ a.e. in } [0,T]. \tag{2.3.1}$$

Hence, the quantity produced by each firm P_i at the time $t \in [0,T]$ must be equal to the commodity shipments from that firm to all the demand markets plus the production excess at the same time $t \in [0,T]$. Then, the set of feasible vectors $(x, \varepsilon) \in L^2([0,T], \mathbb{R}^{mn+m})$ is

$$\mathbb{K}^* = \Big\{ (x,\varepsilon) \in L^2([0,T], \mathbb{R}^{mn+m}) : x_{ij}(t) \geq 0, \quad \forall i = 1, \dots, m, \; \forall j = 1, \dots, n, \text{ a.e. in } [0,T],$$

$$\epsilon_i(t) \geq 0, \quad p_i(t) = \sum_{j=1}^{n} x_{ij}(t) + \varepsilon_i(t), \quad \forall i = 1, \dots, m, \text{ a.e. in } [0,T] \Big\}.$$

Now, we can rewrite $\mathbb{K}^*$ in an equivalent way. In virtue of (2.3.1) we can express $\varepsilon_i(t)$ in terms of $p_i(t)$ and $x_{ij}(t)$, namely

$$\varepsilon_i(t) = p_i(t) - \sum_{j=1}^{n} x_{ij}(t), \; i = 1, \dots, m, \text{ a.e. in } [0,T]. \tag{2.3.2}$$

Then, the equivalent constraint set becomes

$$\mathbb{K} = \Bigg\{ x \in L^2([0,T], \mathbb{R}^{mn}) : \quad x_{ij}(t) \geq 0, \quad \forall i = 1, \ldots, m, \ \forall j = 1, \ldots, n, \ \text{a.e. in } [0,T],$$

$$\sum_{j=1}^{n} x_{ij}(t) \leq p_i(t), \quad \forall i = 1, \ldots, m, \ \text{a.e. in } [0,T] \Bigg\}. \quad (2.3.3)$$

We can observe that $\mathbb{K}$ includes the presence of production excess described in $\mathbb{K}^*$ and that this set is convex, closed and bounded in the Hilbert space $L^2([0,T], \mathbb{R}^{mn}_+)$.
The profit function becomes

$$\begin{aligned} v_i(t, x(t)) &= v_i^*(t, x(t), \epsilon(t), \delta(t)) \\ &= \sum_{j=1}^{n} d_j(t, x(t)) x_{ij}(t) - f_i(t, x(t)) - g_i(t, x(t)) - \sum_{j=1}^{n} c_{ij}(t, x(t)) x_{ij}(t). \end{aligned}$$

2.3.1 The variational formulation

As regards the equilibrium definition according to Cournot-Nash principle, we can observe that the formulation is the same.

Definition 2.3.1. $x^* \in \mathbb{K}$ is a dynamic oligopolistic market equilibrium in presence of production excesses if and only if for each $i = 1, \ldots, m$ and a.e. in $[0,T]$ we have

$$v_i(t, x^*(t)) \geq v_i(t, x_i(t), \hat{x}_i^*(t)), \ \text{a.e. in } [0,T], \quad (2.3.4)$$

where $x_i(t) = (x_{i1}(t), \ldots, x_{in}(t))$, a.e. in $[0,T]$ and $\hat{x}_i^*(t) = (x_1^*(t), \ldots, x_{i-1}^*(t), x_{i+1}^*(t), \ldots, x_m^*(t))$.

With the same technique used in section 2.2 we can prove that under the assumptions (i), (ii), (iii), Definition 2.3.1 is equivalent to an evolutionary variational inequality, namely:

Theorem 2.3.1. $x^* \in \mathbb{K}$ *is a dynamic oligopolistic market equilibrium according to Definition* 2.3.1 *if and only if it satisfies the evolutionary variational inequality*

$$\int_0^T \sum_{i=1}^{m} \sum_{j=1}^{n} -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} (x_{ij}(t) - x_{ij}^*(t)) dt \geq 0, \qquad \forall x \in \mathbb{K}. \quad (2.3.5)$$

2.3.2 The Lagrangean formulation

In this section we prove that, under the assumptions (i), (ii), (iii) on the profit function v, Definition 2.3.1 is equivalent to the equilibrium conditions defined through Lagrange variables which are very useful in order to analyze the constraints of excesses of $\mathbb{K}$:

Definition 2.3.2. $x^* \in \mathbb{K}$ is a dynamic oligopolistic market problem equilibrium in presence of excesses if and only if for each $i = 1, \dots, m$, $j = 1, \dots, n$ and a.e. in $[0, T]$ there exists $\lambda^*_{ij} \in L^2([0, T])$, $\mu^*_i \in L^2([0, T])$, such that

$$-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} + \mu^*_i(t) = \lambda^*_{ij}(t), \tag{2.3.6}$$

$$\left(-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} + \mu^*_i(t)\right) x^*_{ij}(t) = 0, \tag{2.3.7}$$

$$\lambda^*_{ij}(t) x^*_{ij}(t) = 0, \qquad \lambda^*_{ij}(t) \geq 0, \tag{2.3.8}$$

$$\mu^*_i(t) \left(\sum_{j=1}^{n} x^*_{ij}(t) - p_i(t)\right) = 0, \qquad \mu^*_i(t) \geq 0. \tag{2.3.9}$$

The terms $\lambda^*_{ij}(t)$ and $\mu^*_i(t)$ are the Lagrange multipliers associated to the constraints $x^*_{ij}(t) \geq 0$ and to $\sum_{j=1}^{n} x^*_{ij}(t) \leq p_i(t)$, respectively. They, as it is a well-known, have a topical importance on the understanding and the management of the market. In fact, at a fixed time $t \in [0, T]$, we have

(a) if $\lambda^*_{ij}(t) > 0$, then, by using (2.3.8), we obtain $x^*_{ij}(t) = 0$, namely there is not commodity shipment between the firm P_i and the demand market Q_j;

(b) if $x^*_{ij}(t) > 0$, then, taking into account (2.3.8), $\lambda^*_{ij}(t) = 0$ and, making use of (2.3.6), it results
$\mu^*_i(t) = \dfrac{\partial v_i(t, x^*(t))}{\partial x_{ij}}$, namely $\mu^*_i(t)$ is equal to the marginal profit;

(c) if $\mu^*_i(t) > 0$, then, for the condition (2.3.9), we have $\sum_{j=1}^{n} x^*_{ij}(t) = p_i(t)$, namely there is no production excess;

(d) if $\sum_{j=1}^{n} x^*_{ij}(t) < p_i(t)$, as a consequence of (2.3.9) we get $\mu^*_i(t) = 0$ and, for the condition (2.3.6),
$\lambda^*_{ij}(t) = -\dfrac{\partial v_i(t, x^*(t))}{\partial x_{ij}}$, namely $\lambda^*_{ij}(t)$ is equal to the opposite of the marginal profit.

In Definition 2.3.2, even if in (2.3.6)– (2.3.9) the unknown Lagrange variables λ^*_{ij} and μ^*_i appear, they have not any influence in the equilibrium definition because the following equivalent condition in terms of variational inequality holds:

Theorem 2.3.2. *$x^* \in \mathbb{K}$ is a dynamic oligopolistic market equilibrium according to Definition 2.3.2 if and only if it satisfies the evolutionary variational inequality*

$$\int_0^T -\sum_{i=1}^m \sum_{j=1}^n \frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} (x_{ij}(t) - x^*_{ij}(t))dt \geq 0 \qquad \forall x \in \mathbb{K}. \tag{2.3.10}$$

Thanks to Theorems 2.3.1 and 2.3.2 we can clearly see the equivalence between Definitions 2.3.1 and 2.3.2.

We can observe that also in the case of limited production and in presence of production excesses, the meaning of Cournot-Nash equilibrium does not change.

Before proving this theorem, we need some preliminary lemmas.

Lemma 2.3.1. *Let $x^* \in \mathbb{K}$ be a solution to the variational inequality (2.3.5). Setting*

$$I_i^0 := \left\{ t \in [0,T] : \sum_{j=1}^n x^*_{ij}(t) = p_i(t) \right\}, \quad i = 1, \ldots, m,$$

$$\gamma_i^*(t) := \min_{j=1,\ldots,n} \left\{ -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} \right\}, \quad t \in I_i^0,\ i = 1, \ldots, m,$$

$$X_i^0 := \left\{ t \in I_i^0 : -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} > \gamma_i^*(t) \right\}, \quad i = 1, \ldots, m,$$

and

$$Y_i^0 := \left\{ t \in I_i^0 : -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} = \gamma_i^*(t) \right\}, \quad i = 1, \ldots, m,$$

we have

$$\left(-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} - \gamma_i^*(t) \right) x^*_{ij}(t) = 0, \quad \forall t \in I_i^0,\ i = 1, \ldots, m, \tag{2.3.11}$$

$$\gamma_i^*(t) \leq 0, \quad \text{a.e. in } Y_i^0,\ \forall i = 1, \ldots, m$$

and

$$-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} \geq 0, \quad \text{a.e. in } X_i^0,\ \forall j = 1, \ldots, n.$$

Proof. Setting $x_{hj}(t) = x^*_{hj}(t)$ for all indexes $h \neq i$ for a fixed i and $j = 1, \ldots, n$ we get, from variational inequality (2.3.5)

$$\int_0^T \sum_{j=1}^n -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} (x_{ij}(t) - x^*_{ij}(t))dt \geq 0 \tag{2.3.12}$$

for every $x_{ij}(t) \geq 0,\ j = 1, \ldots, n$ and $\sum_{j=1}^n x_{ij}(t) \leq p_i(t)$.

Let us consider the set

$$I_i^0 = \left\{ t \in [0,T] : \sum_{j=1}^n x_{ij}^*(t) = p_i(t) \right\}, \ i = 1, \dots, m$$

and let us choose $x_{ij}(t)$ such that, for $j = 1, \dots, n$,

$$x_{ij}(t) := \begin{cases} x_{ij}^*(t), & t \in [0,T] \setminus I_i^0, \\ x_{ij}(t), & t \in I_i^0, \end{cases}$$

with $x_{ij} \in L^2(I_i^0)$, $x_{ij}(t) \geq 0$ and $\displaystyle\sum_{j=1}^n x_{ij}(t) - p_i(t) \leq 0$.

Then, we have

$$\int_{I_i^0} \sum_{j=1}^n - \frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} (x_{ij}(t) - x_{ij}^*(t)) dt \geq 0. \tag{2.3.13}$$

In order to prove the condition (2.3.11), by using the procedure adopted for the Wardrop's principle, we get, a.e. in I_i^0, if $\exists (i_1, j_1), (i_2, j_2)$ such that

$$-\frac{\partial v_{i_1}(t, x^*(t))}{\partial x_{i_1 j_1}} > -\frac{\partial v_{i_2}(t, x^*(t))}{\partial x_{i_2 j_2}},$$

then $x_{i_1 j_1}^*(t) = 0$. In fact, suppose, ad absurdum, there exists a set $\widetilde{E} \subseteq I_i^0$, $\mu(\widetilde{E}) > 0$, such that $x_{i_1 j_1}^*(t) > 0$, and

$$-\frac{\partial v_{i_1}(t, x^*(t))}{\partial x_{i_1 j_1}} > -\frac{\partial v_{i_2}(t, x^*(t))}{\partial x_{i_2 j_2}}.$$

Now, we choose

$$x_{ij}(t) := \begin{cases} x_{ij}^*(t), & t \in I_i^0 \setminus \widetilde{E}, \\ x_{ij}^*(t), & \text{if } i \neq i_1, i_2, \ j \neq j_1, j_2, \ t \in \widetilde{E}, \\ 0, & \text{if } i = i_1, \ j = j_1, \ t \in \widetilde{E}, \\ x_{i_1 j_1}^*(t) + x_{i_2 j_2}^*(t), & \text{if } i = i_2, \ j = j_2, \ t \in \widetilde{E}, \end{cases}$$

which satisfies the condition $\displaystyle\sum_{j=1}^n x_{ij}(t) = p_i(t)$.

Then, (2.3.13) becomes

$$\int_{\widetilde{E}} -\frac{\partial v_{i_1}(t, x^*(t))}{\partial x_{i_1 j_1}} (\underbrace{x_{i_1 j_1}(t)}_{=0} - x_{i_1 j_1}^*(t)) dt + \int_{\widetilde{E}} -\frac{\partial v_{i_2}(t, x^*(t))}{\partial x_{i_2 j_2}} (\underbrace{x_{i_2 j_2}(t)}_{=x_{i_1 j_1}^*(t) + x_{i_2 j_2}^*(t)} - x_{i_2 j_2}^*(t)) dt$$

$$= \int_{\widetilde{E}} \left[-\frac{\partial v_{i_1}(t, x^*(t))}{\partial x_{i_1 j_1}} + \frac{\partial v_{i_2}(t, x^*(t))}{\partial x_{i_2 j_2}} \right] (-x_{i_1 j_1}^*(t)) dt < 0,$$

and our claim is proved.

Now, $\forall t \in I_i^0$ we set

$$\gamma_i^*(t) = \min_{j=1,\dots,n} \left\{ -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} \right\}, \quad i = 1, \dots, m,$$

and observe that $\forall t \in I_i^0$

$$\text{if } -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} = \gamma_i^*(t), \text{ then } x_{ij}^*(t) \geq 0;$$

$$\text{if } -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} > \gamma_i^*(t), \text{ then } x_{ij}^*(t) = 0,$$

hence (2.3.11) holds, namely

$$\left(-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} - \gamma_i^*(t) \right) x_{ij}^*(t) = 0, \quad \forall t \in I_i^0, \, i = 1, \dots, m.$$

If we set

$$X_i^0 = \left\{ t \in I_i^0 : -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} > \gamma_i^*(t) \right\}, \quad i = 1, \dots, m,$$

and

$$Y_i^0 = \left\{ t \in I_i^0 : -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} = \gamma_i^*(t) \right\}, \quad i = 1, \dots, m,$$

then, inequality (2.3.13) can be written as:

$$\int_{X_i^0} \sum_{j=1}^n -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} x_{ij}(t) dt + \int_{Y_i^0} \gamma_i^*(t) \sum_{j=1}^n (x_{ij}(t) - x_{ij}^*(t)) dt \geq 0.$$

Now, choosing in (2.3.14)

$$x_{ij}(t) := \begin{cases} 0 & t \in X_i^0 \\ \widetilde{x}_{ij}(t) & t \in Y_i^0 \end{cases}$$

for $j = 1, \dots, n$, such that $\widetilde{x}_{ij} \in L^2(Y_i^0)$, $\widetilde{x}_{ij}(t) \geq 0$, $\displaystyle\sum_{j=1}^n \widetilde{x}_{ij}(t) \leq p_i(t)$, a.e. in Y_i^0, then, we have,

$$\int_{Y_i^0} \gamma_i^*(t) \sum_{j=1}^n (\widetilde{x}_{ij}(t) - x_{ij}^*(t)) dt = \int_{Y_i^0} \gamma_i^*(t) \underbrace{\left(\sum_{j=1}^n \widetilde{x}_{ij}(t) - p_i(t) \right)}_{\leq 0} dt \geq 0.$$

Hence, by using the usual technique, we can show that $\gamma_i^*(t) \leq 0$, in Y_i^0.

Moreover, choosing in (2.3.13)

$$x_{ij}(t) := \begin{cases} \widetilde{x}_{ij}(t) & t \in X_i^0 \\ x_{ij}^*(t) & t \in Y_i^0 \end{cases},$$

for $j = 1, \dots, n$, such that $\widetilde{x}_{ij} \in L^2(X_i^0)$, $\widetilde{x}_{ij}(t) \geq 0$, $\sum_{j=1}^{n} \widetilde{x}_{ij}(t) \leq p_i(t)$, a.e. in X_i^0, and since $x_{ij}^*(t) = 0$, $\forall t \in X_i^0$, for (2.3.11), we have

$$\int_{X_i^0} \sum_{j=1}^{n} -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} \widetilde{x}_{ij}(t) dt \geq 0,$$

from which, by using again the same technique, it follows

$$-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} \geq 0$$

a.e. in X_i^0, $\forall j = 1, \dots, n$. □

Lemma 2.3.2. *Let $x^* \in \mathbb{K}$ be a solution to the variational inequality* (2.3.5)*. Setting*

$$I_i^- := \left\{ t \in [0, T] : \sum_{j=1}^{n} x_{ij}^*(t) - p_i(t) < 0 \right\}, \quad i = 1, \dots, m,$$

we have

$$-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} = 0, \quad \textit{a.e. in } I_i^-. \tag{2.3.14}$$

Proof. With analogous techniques of the proof of Lemma 2.3.1, we get the variational inequality (2.3.12)

$$\int_0^T \sum_{j=1}^{n} -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} (x_{ij}(t) - x_{ij}^*(t)) dt \geq 0$$

for every $x_{ij}(t) \geq 0$, $j = 1, \dots, n$ and $\sum_{j=1}^{n} x_{ij}(t) \leq p_i(t)$, a.e. in $[0, T]$.

Then, if we choose

$$x_{ik}(t) := \begin{cases} x_{ik}^*(t) & t \in [0, T] \setminus I_i^- \\ x_{ik}^*(t) & t \in I_i^-, \ k \neq j \\ \widetilde{x}_{ij}(t) & t \in I_i^-, \ k = j \end{cases}$$

with $\widetilde{x}_{ij} \in L^2(I_i^-)$, $\widetilde{x}_{ij}(t) \geq 0$ and

$$\sum_{\substack{k=1 \\ k \neq j}}^{n} x_{ik}(t) + \widetilde{x}_{ij}(t) \leq p_i(t), \quad \text{a.e. in } I_i^-,$$

we get, from the variational inequality (2.3.12),

$$\int_{I_i^-} -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(\widetilde{x}_{ij}(t) - x^*_{ij}(t))dt \geq 0.$$

Since $\sum_{j=1}^{n} x^*_{ij}(t) - p_i(t) < 0$, a.e. in I_i^-, it is possible to choose $\widetilde{x}_{ij}(t)$ such that $\widetilde{x}_{ij}(t) > x^*_{ij}(t)$ and $\sum_{\substack{k=1 \\ k \neq j}}^{n} x^*_{ik}(t) + \widetilde{x}_{ij}(t) - p_i(t) \leq 0$. So, with the aid of the same procedure,

$$-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} \geq 0, \quad \text{a.e. in } I_i^-. \tag{2.3.15}$$

Moreover, if $x^*_{ij}(t) > 0$, we can choose $\widetilde{x}_{ij}(t)$ such that $\widetilde{x}_{ij}(t) < x^*_{ij}(t)$ and $\sum_{\substack{k=1 \\ k \neq j}}^{n} x^*_{ik}(t) + \widetilde{x}_{ij}(t) - p_i(t) \leq 0$. Hence, from the variational inequality (2.3.12), we get

$$\int_{I_i^-} -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(\widetilde{x}_{ij}(t) - x^*_{ij}(t))dt \geq 0$$

and, then,

$$-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} \leq 0, \quad \text{a.e. in } I_i^-. \tag{2.3.16}$$

Taking into account (2.3.15) and (2.3.16), we get (2.3.14). □

Now we are able to prove Theorem 2.3.2.

Proof. Let us assume that $x^* \in \mathbb{K}$ is an equilibrium solution according to Definition 2.3.2. Then, taking into account (2.3.6)–(2.3.9), we have for every $x \in \mathbb{K}$,

$$\begin{aligned} -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - x^*_{ij}(t)) &= -\mu^*_i(t)(x_{ij}(t) - x^*_{ij}(t)) + \lambda^*_{ij}(t)(x_{ij}(t) - x^*_{ij}(t)) \\ &= -\mu^*_i(t)(x_{ij}(t) - x^*_{ij}(t)) + \lambda^*_{ij}(t)x_{ij}(t) \\ &\geq -\mu^*_i(t)(x_{ij}(t) - x^*_{ij}(t)), \end{aligned}$$

and hence, by summing over $i = 1, \dots, m$ and $j = 1, \dots, n$, integrating on $[0, T]$, and using the condition (2.3.9), we obtain, for each $x \in \mathbb{K}$

$$\begin{aligned}\int_0^T \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - x^*_{ij}(t))dt \quad &\geq \quad -\sum_{i=1}^m \int_0^T \mu^*_i(t) \left(\sum_{j=1}^n x_{ij}(t) - \sum_{j=1}^n x^*_{ij}(t) + p_i(t) - p_i(t) \right) dt \\ &= \quad -\sum_{i=1}^m \int_0^T \mu^*_i(t) \left(\sum_{j=1}^n x_{ij}(t) - p_i(t) \right) dt \geq 0.\end{aligned}$$

Then, we have (2.3.5).

Vice versa, let $x^* \in \mathbb{K}$ be a solution to (2.3.5) and let us apply the infinite dimensional theory given by Theorems 1.2.2 and 1.2.3.

First of all, let us show that the Assumption S is fulfilled. Let us set, for $x \in L^2([0, T], \mathbb{R}^{mn})$,

$$\Psi(x) := \int_0^T \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - x^*_{ij}(t))dt$$

and

$$\begin{aligned}\widetilde{M} \quad = \quad \Bigg\{ &\left(\Psi(x) + \alpha, -x + y, \left(\sum_{j=1}^n x_{ij} - p_i + z_i \right) \right) : i = 1, \dots, m, \\ &\alpha \geq 0,\ x \in L^2([0, T], \mathbb{R}^{mn}_+) \setminus \mathbb{K},\ y \in L^2([0, T], \mathbb{R}^{mn}_+),\ z \in L^2([0, T], \mathbb{R}^m_+) \Bigg\}.\end{aligned}$$

We must show that, if $(l, \theta_{L^2([0,T],\mathbb{R}^{mn}_+)}, \theta_{L^2([0,T],\mathbb{R}^m_+)}) \in T_{\widetilde{M}}(0, \theta_{L^2([0,T],\mathbb{R}^{mn}_+)}, \theta_{L^2([0,T],\mathbb{R}^m_+)})$, namely

$$l = \lim_{n \to +\infty} \lambda_n(\Psi(x^n) + \alpha_n),$$

$$\lim_{n \to +\infty} \lambda_n(-x^n + y^n) = \theta_{L^2([0,T],\mathbb{R}^{mn}_+)},$$

$$\lim_{n \to +\infty} \lambda_n \left(\sum_{j=1}^n x_{ij} - p_i + z^n_i \right) = \theta_{L^2([0,T],\mathbb{R}_+)}, \quad \forall i = 1, \dots, m$$

with $\lambda_n \geq 0$, $x^n \in L^2([0, T], \mathbb{R}^{mn}_+) \setminus \mathbb{K}$, $y^n \in L^2([0, T], \mathbb{R}^{mn}_+)$, $z^n \in L^2([0, T], \mathbb{R}^m_+)$, $\forall n \in \mathbb{N}$ and

$$\lim_{n \to +\infty} (\Psi(x^n) + \alpha_n) = 0,$$

$$\lim_{n \to +\infty} (-x^n + y^n) = \theta_{L^2([0,T],\mathbb{R}^{mn}_+)},$$

$$\lim_{n \to +\infty} \left(\sum_{j=1}^n x^n_{ij} - p_i + z^n_i \right) = \theta_{L^2([0,T],\mathbb{R}_+)}, \quad \forall i = 1, \dots, m,$$

then, l is non-negative.

Let us set, now,

$$\begin{aligned}
\nabla_D^{ij} v(x^*) &= \frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}, \quad i = 1, \ldots, m, \; j = 1, \ldots, n, \\
I_i^0 &= \left\{ t \in [0, T] : \; \sum_{j=1}^{n} x_{ij}^*(t) - p_i(t) = 0 \right\}, \quad i = 1, \ldots, m, \\
I_i^- &= \left\{ t \in [0, T] : \; \sum_{j=1}^{n} x_{ij}^*(t) - p_i(t) < 0 \right\} \quad i = 1, \ldots, m, \\
\gamma_i^*(t) &= \min_{j=1,\ldots,n} \left\{ -\nabla_D^{ij} v(x^*) \right\}, \quad t \in I_i^0, \; i = 1, \ldots, m, \\
X_i^0 &= \left\{ t \in I_i^0 : \; -\nabla_D^{ij} v(x^*) > \gamma_i^*(t) \right\}, \quad i = 1, \ldots, m, \\
Y_i^0 &= \left\{ t \in I_i^0 : \; -\nabla_D^{ij} v(x^*) = \gamma_i^*(t) \right\}, \quad i = 1, \ldots, m.
\end{aligned}$$

It results

$$\begin{aligned}
\lambda_n \Psi(x^n) &= \lambda_n \int_0^T \sum_{i=1}^{m} \sum_{j=1}^{n} -\nabla_D^{ij} v(x^*)(x_{ij}^n(t) - x_{ij}^*(t))dt \\
&= \lambda_n \int_{I_i^0} \sum_{i=1}^{m} \sum_{j=1}^{n} -\nabla_D^{ij} v(x^*)(x_{ij}^n(t) - x_{ij}^*(t))dt + \lambda_n \int_{I_i^-} \sum_{i=1}^{m} \sum_{j=1}^{n} -\nabla_D^{ij} v(x^*)(x_{ij}^n(t) - x_{ij}^*(t))dt \\
&= \lambda_n \int_{X_i^0} \sum_{i=1}^{m} \sum_{j=1}^{n} -\nabla_D^{ij} v(x^*)(x_{ij}^n(t) - x_{ij}^*(t))dt + \lambda_n \int_{Y_i^0} \sum_{i=1}^{m} \sum_{j=1}^{n} -\nabla_D^{ij} v(x^*)(x_{ij}^n(t) - x_{ij}^*(t))dt \\
&\quad + \lambda_n \int_{I_i^-} \sum_{i=1}^{m} \sum_{j=1}^{n} -\nabla_D^{ij} v(x^*)(x_{ij}^n(t) - x_{ij}^*(t))dt \\
&= \lambda_n \int_{X_i^0} \sum_{i=1}^{m} \sum_{j=1}^{n} -\nabla_D^{ij} v(x^*) x_{ij}^n(t)dt + \lambda_n \sum_{i=1}^{m} \int_{Y_i^0} \gamma_i^*(t) \left(\sum_{j=1}^{n} x_{ij}^n(t) - p_i(t) \right) dt \\
&\quad + \lambda_n \sum_{i=1}^{m} \int_{I_i^-} \sum_{j=1}^{n} -\nabla_D^{ij} v(x^*)(x_{ij}^n(t) - x_{ij}^*(t))dt \\
&= \lambda_n \int_{X_i^0} \sum_{i=1}^{m} \sum_{j=1}^{n} -\nabla_D^{ij} v(x^*) x_{ij}^n(t)dt + \sum_{i=1}^{m} \int_{Y_i^0} \gamma_i^*(t) \lambda_n \left(\sum_{j=1}^{n} x_{ij}^n(t) - p_i(t) + z_i^n(t) \right) dt \\
&\quad + \lambda_n \sum_{i=1}^{m} \int_{Y_i^0} \gamma_i^*(t) \left(-z_i^n(t) \right) dt + \sum_{i=1}^{m} \int_{I_i^-} \sum_{j=1}^{n} -\nabla_D^{ij} v(x^*) \lambda_n (x_{ij}^n(t) - y_{ij}^n(t))dt \\
&\quad + \lambda_n \sum_{i=1}^{m} \int_{I_i^-} \sum_{j=1}^{n} -\nabla_D^{ij} v(x^*) y_{ij}^n(t)dt + \lambda_n \sum_{i=1}^{m} \int_{I_i^-} \sum_{j=1}^{n} -\nabla_D^{ij} v(x^*)(-x_{ij}^*(t))dt.
\end{aligned}$$

Then, in virtue of Lemmas 2.3.1 and 2.3.2, for the conditions of belonging to the tangent cone and for the variational inequality (2.3.5) evalued in $x_{ij}(t)=0$, we get that

$$l=\lim_{n\to+\infty}\lambda_n(\Psi(x^n)+\alpha_n)$$

is non-negative.

Taking into account Theorems 1.2.2 and 1.2.3, if we consider the Lagrange function

$$\mathcal{L}(x,\lambda,\mu)=\Psi(x)-\sum_{i=1}^m\sum_{j=1}^n\int_0^T\lambda_{ij}(t)x_{ij}(t)dt+\sum_{i=1}^m\int_0^T\mu_i(t)\left(\sum_{j=1}^n x_{ij}(t)-p_i(t)\right)dt,$$

we have that $\exists\lambda^*\in L^2([0,T],\mathbb{R}^{mn}_+)$, $\mu^*\in L^2([0,T],\mathbb{R}^m_+)$ such that

$$\mathcal{L}(x^*,\lambda,\mu)\le\mathcal{L}(x^*,\lambda^*,\mu^*)\le\mathcal{L}(x,\lambda^*,\mu^*), \tag{2.3.17}$$

$\forall x\in L^2([0,T],\mathbb{R}^{mn}_+)$, $\lambda\in L^2([0,T],\mathbb{R}^{mn}_+)$, $\mu\in L^2([0,T],\mathbb{R}^m_+)$, and, moreover,

$$\langle\langle\lambda^*,x^*\rangle\rangle=\int_0^T\sum_{i=1}^m\sum_{j=1}^n\lambda^*_{ij}(t)x^*_{ij}(t)dt=0$$

$$\left\langle\left\langle\mu^*,\sum_{j=1}^n x^*_{ij}(t)-p_i(t)\right\rangle\right\rangle=\int_0^T\sum_{i=1}^m\left[\mu^*_i(t)\left(\sum_{j=1}^n x^*_{ij}(t)-p_i(t)\right)\right]dt=0.$$

Hence,

$$\begin{aligned}\lambda^*_{ij}(t)x^*_{ij}(t)&=0,\quad\forall i=1,\dots,m,\ \forall j=1,\dots,n,\\ \mu^*_i(t)\left(\sum_{j=1}^n x^*_{ij}(t)-p_i(t)\right)&=0,\quad\forall i=1,\dots,m.\end{aligned}\tag{2.3.18}$$

Since $\mathcal{L}(x^*,\lambda^*,\mu^*)=0$, taking into account the right side of (2.3.17) and the equalities (2.3.18), we get

$$\begin{aligned}\mathcal{L}(x,\lambda^*,\mu^*)&=\sum_{i=1}^m\sum_{j=1}^n\int_0^T-\nabla^{ij}_D v(x^*)(x_{ij}(t)-x^*_{ij}(t)dt-\sum_{i=1}^m\sum_{j=1}^n\int_0^T\lambda^*_{ij}(t)(x_{ij}(t)-x^*_{ij}(t))dt\\&\quad+\sum_{i=1}^m\sum_{j=1}^n\int_0^T\mu^*_i(t)(x_{ij}(t)-x^*_{ij}(t))dt\\&\ge\mathcal{L}(x^*,\lambda^*,\mu^*)=0,\qquad\forall x\in L^2([0,T],\mathbb{R}^{mn}).\end{aligned}$$

Then, $\mathcal{L}(x,\lambda^*,\mu^*)$ has a minimal point in x^*.

Let us assume that $x^1_{ij}=x^*_{ij}+\varepsilon_{ij}$ and $x^2_{ij}=x^*_{ij}-\varepsilon_{ij}$, for all $\varepsilon\in L^2([0,T],\mathbb{R}^{mn}_+)$. Let us note that

$$\begin{aligned}\mathcal{L}(x^1,\lambda^*,\mu^*)&=\sum_{i=1}^m\sum_{j=1}^n\int_0^T\left[-\nabla^{ij}_D v(x^*)-\lambda^*_{ij}(t)+\mu^*_i(t)\right]\varepsilon_{ij}(t)dt\ge0,\\&\quad\forall\varepsilon\in L^2([0,T],\mathbb{R}^{mn}_+)\end{aligned}$$

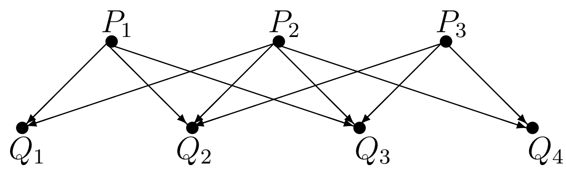

Figure 2.2: Network structure of the numerical dynamic spatial oligopoly problem.

and

$$\mathcal{L}(x^2, \lambda^*, \mu^*) = -\Big\{ \sum_{i=1}^{m} \sum_{j=1}^{n} \int_0^T \left[-\nabla_D^{ij} v(x^*) - \lambda_{ij}^*(t) + \mu_i^*(t) \right] \varepsilon_{ij}(t) dt \Big\} \geq 0,$$
$$\forall \varepsilon \in L^2([0, T], \mathbb{R}_+^{mn}).$$

As a consequence, we have

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \int_0^T \left[-\nabla_D^{ij} v(x^*) - \lambda_{ij}^*(t) + \mu_i^*(t) \right] \varepsilon_{ij}(t) dt = 0, \quad \forall \varepsilon \in L^2([0, T], \mathbb{R}_+^{mn}),$$

namely, since ε is arbitrary, we get the equilibrium conditions of Definition 2.3.2,

$$-\frac{\partial v_i(x^*(t))}{\partial x_{ij}} + \mu_i^*(t) = \lambda_{ij}^*(t),$$

$$\left(-\frac{\partial v_i(x^*(t))}{\partial x_{ij}} + \mu_i^*(t) \right) x_{ij}^*(t) = 0,$$

$$\lambda_{ij}^*(t) x_{ij}^*(t) = 0, \qquad \lambda_{ij}^*(t) \geq 0,$$

$$\mu_i^*(t) \left(\sum_{j=1}^{n} x_{ij}^*(t) - p_i(t) \right) = 0, \qquad \mu_i^*(t) \geq 0,$$

for each $i = 1, \ldots, m$, $j = 1, \ldots, n$ and a.e. in $[0, T]$.

□

2.3.3 A numerical example

Let us consider an example consisting of three firms and four demand markets, as in Figure 2.2. Let $p \in L^2([0, 1], \mathbb{R}^3)$ be the production function such that, a.e. in $[0, 1]$,

$$p_1(t) = 6t, \qquad p_2(t) = 5t, \qquad p_3(t) = 7t.$$

Hence, the feasible set is

$$\begin{aligned}\mathbb{K} = \Big\{x \in L^2([0,1], \mathbb{R}^{3\times 4}) : \quad & x_{ij}(t) \geq 0, \quad \forall i = 1, \ldots, 3, \ j = 1, \ldots, 4, \text{ a.e. in } [0,1], \\ & \sum_{j=1}^{4} x_{1j}(t) \leq 6t, \quad \forall i = 1, \ldots, 3, \text{ a.e. in } [0,1], \\ & \sum_{j=1}^{4} x_{2j}(t) \leq 5t, \quad \forall i = 1, \ldots, 3, \text{ a.e. in } [0,1], \\ & \sum_{j=1}^{4} x_{3j}(t) \leq 7t, \quad \forall i = 1, \ldots, 3, \text{ a.e. in } [0,1]\Big\}.\end{aligned}$$

Let us consider the profit function $v \in L^2([0,1] \times L^2([0,1], \mathbb{R}^{3\times 4}), \mathbb{R}^3)$ defined by

$$\begin{aligned} v_1(x(t)) &= -4x_{11}^2(t) - 3x_{12}^2(t) - 2x_{13}^2(t) - 2x_{14}^2 - x_{11}(t)x_{12}(t) - x_{13}(t)x_{14}(t) \\ &\quad +6tx_{11}(t) + 3tx_{12}(t) + 2tx_{13}(t) + 5tx_{14}(t), \\ v_2(x(t)) &= -3x_{21}^2(t) - 4x_{22}^2(t) - 3x_{23}^2(t) - 2x_{24}^2(t) - x_{21}(t)x_{23}(t) - x_{22}(t)x_{31}(t) \\ &\quad -x_{24}(t)x_{33}(t) + 3tx_{21}(t) + 5tx_{22}(t) + 4tx_{23}(t) + 3tx_{24}(t), \\ v_3(x(t)) &= -3x_{31}^2(t) - 2x_{32}^2(t) - 2x_{33}^2(t) - 4x_{34}^2(t) - x_{22}(t)x_{31}(t) - x_{24}(t)x_{33}(t) \\ &\quad -x_{32}(t)x_{34}(t) + 3tx_{31}(t) + 4tx_{32}(t) + 3tx_{33}(t) + 2tx_{34}(t). \end{aligned}$$

Then, the operator $\nabla_D v \in L^2([0,1] \times L^2([0,1], \mathbb{R}^{3\times 4}), \mathbb{R}^{3\times 4})$ is given by

$$\nabla_D v(x(t)) = \begin{pmatrix} -8x_{11}(t) - x_{12}(t) + 6t & -x_{11}(t) - 6x_{12}(t) + 3t & -4x_{13}(t) - x_{14}(t) + 2t & -x_{13}(t) - 4x_{14}(t) + 5t \\ -6x_{21}(t) - x_{23}(t) + 3t & -8x_{22}(t) - x_{31}(t) + 5t & -x_{21}(t) - 6x_{23}(t) + 4t & -4x_{24}(t) - x_{33}(t) + 3t \\ -x_{22}(t) - 6x_{31}(t) + 3t & -4x_{32}(t) - x_{34}(t) + 4t & -x_{24}(t) - 4x_{33}(t) + 3t & -x_{32}(t) - 8x_{34}(t) + 2t \end{pmatrix}.$$

Now, we verify that $-\nabla_D v$ is a strongly monotone operator; in fact

$$\begin{aligned} &\langle -\nabla_D v(x) + \nabla_D v(y), x - y\rangle \\ &\quad = \big\{8[x_{11} - y_{11}] + [x_{12} - y_{12}]\big\}[x_{11} - y_{11}] + \big\{[x_{11} - y_{11}] + 6[x_{12} - y_{12}]\big\}[x_{12} - y_{12}] \\ &\qquad + \big\{4[x_{13} - y_{13}] + [x_{14} - y_{14}]\big\}[x_{13} - y_{13}] + \big\{[x_{13} - y_{13}] + 4[x_{14} - y_{14}]\big\}[x_{14} - y_{14}] \\ &\qquad + \big\{6[x_{21} - y_{21}] + [x_{23} - y_{23}]\big\}[x_{21} - y_{21}] + \big\{8[x_{22} - y_{22}] + [x_{31} - y_{31}]\big\}[x_{22} - y_{22}] \\ &\qquad + \big\{[x_{21} - y_{21}] + 6[x_{23} - y_{23}]\big\}[x_{23} - y_{23}] + \big\{4[x_{24} - y_{24}] + [x_{33} - y_{33}]\big\}[x_{24} - y_{24}] \\ &\qquad + \big\{[x_{22} - y_{22}] + 6[x_{31} - y_{31}]\big\}[x_{31} - y_{31}] + \big\{4[x_{32} - y_{32}] + [x_{34} - y_{34}]\big\}[x_{32} - y_{32}] \\ &\qquad + \big\{[x_{24} - y_{24}] + 4[x_{33} - y_{33}]\big\}[x_{33} - y_{33}] + \big\{[x_{32} - y_{32}] + 8[x_{34} - y_{34}]\big\}[x_{34} - y_{34}] \\ &\quad \geq 3\|x - y\|_{3\times 4}^2. \end{aligned}$$

The dynamic oligopolistic market equilibrium distribution in presence of excesses is the solution to the evolutionary variational inequality:

$$\int_0^1 \sum_{i=1}^{3} \sum_{j=1}^{4} -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - x_{ij}^*(t))dt \geq 0, \quad \forall x \in \mathbb{K}. \tag{2.3.19}$$

Taking into account the direct method (see [29, 34, 70]), we consider the following system

$$\begin{cases} 8x^*_{11}(t) + x^*_{12}(t) - 6t = 0, \quad x^*_{11}(t) + 6x^*_{12}(t) - 3t = 0, \quad 4x^*_{13}(t) + x^*_{14}(t) - 2t = 0, \\ x^*_{13}(t) + 4x^*_{14}(t) - 5t = 0, \quad 6x^*_{21}(t) + x^*_{23}(t) - 3t = 0, \quad 8x^*_{22}(t) + x^*_{31}(t) - 5t = 0, \\ x^*_{21}(t) + 6x^*_{23}(t) - 4t = 0, \quad 4x^*_{24}(t) + x^*_{33}(t) - 3t = 0, \quad x^*_{22}(t) + 6x^*_{31}(t) - 3t = 0, \\ 4x^*_{32}(t) + x^*_{34}(t) - 4t = 0, \quad x^*_{24}(t) + 4x^*_{33}(t) - 3t = 0, \quad x^*_{32}(t) + 8x^*_{34}(t) - 2t = 0, \\ x^*_{ij}(t) \geq 0, \quad \forall i = 1, \ldots, 3, \; j = 1, \ldots, 4, \\ \sum_{j=1}^{4} x^*_{1j}(t) \leq 6t, \quad \sum_{j=1}^{4} x^*_{2j}(t) \leq 5t, \quad \sum_{j=1}^{4} x^*_{3j}(t) \leq 7t \end{cases} \tag{2.3.20}$$

We obtain the following solution, a.e. in $[0, 1]$,

$$x^*(t) = \begin{pmatrix} \frac{33}{47}t & \frac{18}{47}t & \frac{1}{5}t & \frac{6}{5}t \\ \frac{2}{5}t & \frac{27}{47}t & \frac{3}{5}t & \frac{3}{5}t \\ \frac{19}{47}t & \frac{30}{31}t & \frac{3}{5}t & \frac{4}{31}t \end{pmatrix},$$

that belongs to the constraint set $\mathbb{K}$, then it is the equilibrium solution.

Moreover, the production excesses of each firm is given by

$$\begin{aligned} \epsilon_1(t) &= p_1(t) - \sum_{j=1}^{4} x^*_{1j}(t) = 6t - \frac{584}{235}t = \frac{826}{235}t, \\ \epsilon_2(t) &= p_2(t) - \sum_{j=1}^{4} x^*_{2j}(t) = 5t - \frac{511}{235}t = \frac{664}{235}t, \\ \epsilon_3(t) &= p_3(t) - \sum_{j=1}^{4} x^*_{3j}(t) = 7t - \frac{15306}{7285}t = \frac{35689}{7285}t, \end{aligned}$$

a.e. in $[0, 1]$.

Making use of the equilibrium definition (2.3.6)–(2.3.9), we obtain:

$$\lambda^*_{ij}(t) = 0, \; \forall i = 1, \ldots, 3, \; j = 1, \ldots, 4, \text{ a.e. in } [0, 1], \qquad \mu^*_i(t) = 0, \; \forall i = 1, \ldots, 3, \text{ a.e. in } [0, 1].$$

2.4 The case with both production and demand excesses

For a more complete study of the problem, we allow the presence of both production and demand excesses, namely we allow that for a part of the producers can occur an excess of production whereas for a part of the demand markets the supply can not satisfy the demand. In order to clarify the presence of both excesses we consider some concrete economic situations. During an economic crisis period the presence of production excesses can be due to a decrease in market demands and, on the other hand, the presence of demand excesses may occur when the supply can not satisfy the demand especially for fundamental goods. Moreover, since the market model presented in this paper evolves in time, the presence of both excesses can be consequence of the fact that the physical transportation of commodity between a firm and a demand market is evidently limited, therefore there exist some time instants in which, though some firms produce more commodity than they can send to all the demand markets, some of the demand markets require more commodity.

In addition to what we have considered previously, let us denote by $\delta_j(t)$, $j = 1, \ldots, n$, the nonnegative demand excess for the commodity of the demand market Q_j at the time $t \in [0, T]$. Let us group the demand excess into a vector-function $\delta : [0, T] \longrightarrow \mathbb{R}^n_+$ and let us suppose that the following feasibility conditions hold, a.e. in $[0, T]$:

$$p_i(t) = \sum_{j=1}^{n} x_{ij}(t) + \epsilon_i(t), \quad i = 1, \ldots, m, \text{ a.e. in } [0, T], \tag{2.4.1}$$

$$q_j(t) = \sum_{i=1}^{m} x_{ij}(t) + \delta_j(t), \quad j = 1, \ldots, n, \text{ a.e. in } [0, T]. \tag{2.4.2}$$

Hence, the quantity produced by each firm P_i, at the time $t \in [0, T]$, must be equal to the commodity shipments from that firm to all the demand markets plus the production excess, at the same time $t \in [0, T]$. Moreover, the quantity demanded by each demand market Q_j, at the time $t \in [0, T]$, must be equal to the commodity shipments from all the firms to that demand market plus the demand excess, at the same time $t \in [0, T]$.

Furthermore, we assume that the non-negative commodity shipment between the producer P_i and the demand market Q_j has to satisfy time-dependent constraints, namely there exist two non-negative functions $\underline{x}, \overline{x} : [0, T] \longrightarrow \mathbb{R}^{mn}_+$ such that

$$0 \leq \underline{x}_{ij}(t) \leq x_{ij}(t) \leq \overline{x}_{ij}(t), \quad \forall i = 1, \ldots, m, \ \forall j = 1, \ldots, n, \text{ a.e. in } [0, T]. \tag{2.4.3}$$

For technical reasons, let us assume that

$$\delta \in L^2([0, T], \mathbb{R}^n_+).$$

As a consequence, we have

$$q \in L^2([0, T], \mathbb{R}^n_+).$$

Then, the set of feasible vectors $(x, \epsilon, \delta) \in L^2([0,T], \mathbb{R}_+^{mn+m+n})$ is

$$
\begin{aligned}
\mathbb{K}^* = \Bigg\{ & (x, \epsilon, \delta) \in L^2([0,T], \mathbb{R}_+^{mn+m+n}) : \\
& \underline{x}_{ij}(t) \leq x_{ij}(t) \leq \overline{x}_{ij}(t), \quad \forall i = 1, \dots, m, \ \forall j = 1, \dots, n, \ \text{a.e. in } [0,T], \\
& \epsilon_i(t) \geq 0, \quad \forall i = 1, \dots, m, \ \text{a.e. in } [0,T], \\
& p_i(t) = \sum_{j=1}^{n} x_{ij}(t) + \epsilon_i(t), \quad \forall i = 1, \dots, m, \ \text{a.e. in } [0,T], \\
& \delta_j(t) \geq 0, \quad \forall j = 1, \dots, n, \ \text{a.e. in } [0,T] \\
& q_j(t) = \sum_{i=1}^{m} x_{ij}(t) + \delta_j(t), \quad \forall j = 1, \dots, n, \ \text{a.e. in } [0,T] \Bigg\}.
\end{aligned}
$$

Now, we can rewrite $\mathbb{K}^*$ in an equivalent way. By virtue of (2.4.1) and (2.4.2) we can express $\epsilon_i(t)$ in terms of $p_i(t)$ and $x_{ij}(t)$ and $\delta_j(t)$ in terms of $q_j(t)$ and $x_{ij}(t)$, namely

$$
\epsilon_i(t) = p_i(t) - \sum_{j=1}^{n} x_{ij}(t), \quad i = 1, \dots, m, \ \text{a.e. in } [0,T], \tag{2.4.4}
$$

$$
\delta_j(t) = q_j(t) - \sum_{i=1}^{m} x_{ij}(t), \quad j = 1, \dots, n, \ \text{a.e. in } [0,T]. \tag{2.4.5}
$$

Then, the equivalent constraint set becomes

$$
\begin{aligned}
\mathbb{K} = \Bigg\{ & x \in L^2([0,T], \mathbb{R}_+^{mn}) : \\
& \underline{x}_{ij}(t) \leq x_{ij}(t) \leq \overline{x}_{ij}(t), \quad \forall i = 1, \dots, m, \ \forall j = 1, \dots, n, \ \text{a.e. in } [0,T], \\
& \sum_{j=1}^{n} x_{ij}(t) \leq p_i(t), \quad \forall i = 1, \dots, m, \ \text{a.e. in } [0,T], \\
& \sum_{i=1}^{m} x_{ij}(t) \leq q_j(t), \quad \forall j = 1, \dots, n, \ \text{a.e. in } [0,T] \Bigg\}.
\end{aligned} \tag{2.4.6}
$$

We can observe that $\mathbb{K}$ includes the presence of demand excess described in $\mathbb{K}^*$ and that this set is convex, closed and bounded in the Hilbert space $L^2([0,T], \mathbb{R}_+^{mn})$.

The profit function becomes

$$
\begin{aligned}
v_i(t, x(t)) &= v_i^*(t, x(t), \epsilon(t), \delta(t)) \\
&= \sum_{j=1}^{n} d_j(t, x(t)) x_{ij}(t) - f_i(t, x(t)) - g_i(t, x(t)) - \sum_{j=1}^{n} c_{ij}(t, x(t)) x_{ij}(t).
\end{aligned}
$$

2.4.1 The variational formulation

As regards the equilibrium definition according to Cournot-Nash principle, we can observe that the formulation is the same.

Definition 2.4.1. $x^* \in \mathbb{K}$ is a dynamic oligopolistic market equilibrium in presence of excesses if and only if for each $i = 1, \dots, m$ and a.e. in $[0, T]$ we have

$$v_i(t, x^*(t)) \geq v_i(t, x_i(t), \hat{x}_i^*(t)), \tag{2.4.7}$$

where

$$\hat{x}_i^*(t) = (x_1^*(t), \dots, x_{i-1}^*(t), x_{i+1}^*(t), \dots, x_m^*(t)).$$

With the same technique used in section 2.2 it is possible to prove that under the assumptions (i), (ii), (iii) on v_i, Definition 2.4.1 is equivalent to an evolutionary variational inequality, namely:

Theorem 2.4.1. *Let us suppose that assumptions* (i), (ii), (iii) *are satisfied. Then,* $x^* \in \mathbb{K}$ *is a dynamic oligopolistic market equilibrium in presence of excesses according to Definition* 2.4.1 *if and only if it satisfies the evolutionary variational inequality*

$$\int_0^T -\sum_{i=1}^m \sum_{j=1}^n \frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} (x_{ij}(t) - x_{ij}^*(t)) dt \geq 0 \qquad \forall x \in \mathbb{K}. \tag{2.4.8}$$

2.4.2 The Lagrangean formulation

In this section we will prove that, under the assumptions (i), (ii), (iii) on the profit function v, Definition 2.4.1 is equivalent to the equilibrium conditions defined through Lagrange variables which are very useful in order to analyze both production and demand excesses:

Definition 2.4.2. $x^* \in \mathbb{K}$ is a dynamic oligopolistic market problem equilibrium in presence of excesses if and only if, for each $i = 1, \dots, m$, $j = 1, \dots, n$ and a.e. in $[0, T]$, there exist $\lambda_{ij}^* \in L^2([0, T])$, $\rho_{ij}^* \in L^2([0, T])$, $\mu_i^* \in L^2([0, T])$, $\nu_j^*(t) \in L^2([0, T])$ such that

$$-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} + \rho_{ij}^*(t) + \mu_i^*(t) + \nu_j^*(t) = \lambda_{ij}^*(t), \tag{2.4.9}$$

$$\lambda_{ij}^*(t)(\underline{x}_{ij}(t) - x_{ij}^*(t)) = 0, \qquad \lambda_{ij}^*(t) \geq 0, \tag{2.4.10}$$

$$\rho_{ij}^*(t)(x_{ij}^*(t) - \overline{x}_{ij}(t)) = 0, \qquad \rho_{ij}^*(t) \geq 0, \tag{2.4.11}$$

$$\mu_i^*(t) \left(\sum_{j=1}^n x_{ij}^*(t) - p_i(t) \right) = 0, \qquad \mu_i^*(t) \geq 0, \tag{2.4.12}$$

$$\nu_j^*(t) \left(\sum_{i=1}^m x_{ij}^*(t) - q_j(t) \right) = 0, \qquad \nu_j^*(t) \geq 0. \tag{2.4.13}$$

The terms $\lambda^*_{ij}(t)$, $\rho^*_{ij}(t)$, $\mu^*_i(t)$, $\nu^*_j(t)$ are the Lagrange multipliers associated to the constraints $x^*_{ij}(t) \geq \underline{x}_{ij}(t)$, $x^*_{ij}(t) \leq \overline{x}_{ij}(t)$, $\sum_{j=1}^{n} x^*_{ij}(t) \leq p_i(t)$ and $\sum_{i=1}^{m} x^*_{ij}(t) \leq q_j(t)$, respectively.

They, as it is well-known, have a topical importance on the understanding and the management of the market. In fact, at a fixed time $t \in [0, T]$, we have:

(a) if $\lambda^*_{ij}(t) > 0$ then, by using (2.4.10), we obtain $x^*_{ij}(t) = \underline{x}_{ij}(t)$, namely the commodity shipment between the firm P_i and the demand market Q_j is minimum;

(b) if $x^*_{ij}(t) > \underline{x}_{ij}(t)$ then, taking into account (2.4.10), $\lambda^*_{ij}(t) = 0$ and, making use of (2.4.9), it results $\rho^*_{ij}(t) + \mu^*_i(t) + \nu^*_j(t) = \dfrac{\partial v_i(t, x^*(t))}{\partial x_{ij}}$, namely $\rho^*_{ij}(t) + \mu^*_i(t) + \nu^*_j(t)$ is equal to the marginal profit;

(c) if $\rho^*_{ij}(t) > 0$ then, by using (2.4.11), we obtain $x^*_{ij}(t) = \overline{x}_{ij}(t)$, namely the commodity shipment between the firm P_i and the demand market Q_j is maximum;

(d) if $x^*_{ij}(t) < \overline{x}_{ij}(t)$ then, making use of (2.4.11), $\rho^*_{ij}(t) = 0$ and, taking into account (2.4.9), we get $\mu^*_i(t) + \nu^*_j(t) - \lambda^*_{ij}(t) = \dfrac{\partial v_i(t, x^*(t))}{\partial x_{ij}}$, namely $\mu^*_i(t) + \nu^*_j(t) - \lambda^*_{ij}(t)$ is equal to the marginal profit;

(e) if $\mu^*_i(t) > 0$ then, for the condition (2.4.12), we have $\sum_{j=1}^{n} x^*_{ij}(t) = p_i(t)$, namely there is no production excess;

(f) if $\sum_{j=1}^{n} x^*_{ij}(t) < p_i(t)$, as a consequence of (2.4.12) we get $\mu^*_i(t) = 0$ and, for the condition (2.4.9), $\rho^*_{ij}(t) + \nu^*_j(t) - \lambda^*_{ij}(t) = \dfrac{\partial v_i(t, x^*(t)}{\partial x_{ij}}$, namely $\rho^*_{ij}(t) + \nu^*_j(t) - \lambda^*_{ij}(t)$ is equal to the marginal profit.

(g) if $\nu^*_j(t) > 0$ then, for the condition (2.4.13), it results $\sum_{i=1}^{m} x^*_{ij}(t) = q_j(t)$, namely there is no demand excess;

(h) if $\sum_{i=1}^{m} x^*_{ij}(t) < q_j(t)$, as a consequence of (2.4.13) we obtain $\nu^*_j(t) = 0$ and, for the condition (2.4.9), $\rho^*_{ij}(t) + \mu^*_i(t) - \lambda^*_{ij}(t) = \dfrac{\partial v_i(t, x^*(t)}{\partial x_{ij}}$, namely $\rho^*_{ij}(t) + \mu^*_i(t) - \lambda^*_{ij}(t)$ is equal to the marginal profit.

It is worth to underline that in Definition 2.4.2, even if in (2.4.9) - (2.4.13) the unknown Lagrange variables λ^*_{ij}, ρ^*_{ij}, μ^*_i, ν^*_j appear, they do not influence the equilibrium definition because the following equivalent condition in terms of evolutionary variational inequality holds:

Theorem 2.4.2. *$x^* \in \mathbb{K}$ is a dynamic oligopolistic market equilibrium in presence of excesses according to Definition 2.4.2 if and only if it satisfies the evolutionary variational inequality:*

$$\int_0^T -\sum_{i=1}^m \sum_{j=1}^n \frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} (x_{ij}(t) - x^*_{ij}(t)) dt \geq 0 \qquad \forall x \in \mathbb{K}. \tag{2.4.14}$$

Taking into account Theorems 2.4.1 and 2.4.2, the equivalence between Definitions 2.4.1 and 2.4.2 is proved.

Finally, we observe that also in the case in which the production is bounded and we are in presence of excesses, the meaning of Cournot-Nash equilibrium does not change.

In order to prove Theorem 2.4.2, let us show some preliminary results. At first we recall Lemma 2.2.1 for the capacity constraints of the commodity shipments.

Lemma 2.4.1. *Let $x^* \in \mathbb{K}$ be a solution to the variational inequality (2.4.8) and let us set,*

$$\begin{aligned}
E^-_{ij} &= \left\{ t \in [0,T] : \, x^*_{ij}(t) = \underline{x}_{ij}(t) \right\}, \quad \forall i = 1, \ldots, m, \ \forall j = 1, \ldots, n, \\
E^0_{ij} &= \left\{ t \in [0,T] : \, \underline{x}_{ij}(t) < x^*_{ij}(t) < \overline{x}_{ij}(t) \right\}, \quad \forall i = 1, \ldots, m, \ \forall j = 1, \ldots, n, \\
E^+_{ij} &= \left\{ t \in [0,T] : \, x^*_{ij}(t) = \overline{x}_{ij}(t) \right\}, \quad \forall i = 1, \ldots, m, \ \forall j = 1, \ldots, n.
\end{aligned}$$

Then, we have

$$\frac{\partial v_i(t, \underline{x}(t))}{\partial x_{ij}} \leq 0, \quad a.e. \ in \ E^-_{ij},$$

$$\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} = 0, \quad a.e. \ in \ E^0_{ij},$$

$$\frac{\partial v_i(t, \overline{x}(t))}{\partial x_{ij}} \geq 0, \quad a.e. \ in \ E^+_{ij}.$$

Now, we recall Lemma 2.3.1 that holds when production excesses occur.

Lemma 2.4.2. *Let $x^* \in \mathbb{K}$ be a solution to the variational inequality (2.4.8). Setting*

$$I^0_i = \left\{ t \in [0,T] : \, \sum_{j=1}^n x^*_{ij}(t) = p_i(t) \right\}, \quad i = 1, \ldots, m,$$

$$\gamma_i^*(t) = \min\left\{-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}, \quad j = 1, \dots, n\right\}, \ a.e. \ in \ I_i^0, \ i = 1, \dots, m,$$

$$X_i^0 = \left\{t \in I_i^0 : \ -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} > \gamma_i^*(t)\right\}, \quad i = 1, \dots, m,$$

and

$$Y_i^0 = \left\{t \in I_i^0 : \ -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} = \gamma_i^*(t)\right\}, \quad i = 1, \dots, m,$$

we have

$$\left(-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} - \gamma_i^*(t)\right) x_{ij}^*(t) = 0, \quad a.e. \ in \ I_i^0, \ \forall i = 1, \dots, m, \tag{2.4.15}$$

$$\gamma_i^*(t) \le 0, \quad a.e. \ in \ Y_i^0, \ \forall i = 1, \dots, m$$

and

$$-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} \ge 0, \quad a.e. \ in \ \ X_i^0, \ \forall j = 1, \dots, n.$$

With the same technique used for proving Lemma 2.4.2, we can obtain the following analogous result that holds when demand excesses occur.

Lemma 2.4.3. *Let $x^* \in \mathbb{K}$ be a solution to the variational inequality* (2.4.8)*. Setting*

$$H_j^0 = \left\{t \in [0, T] : \ \sum_{i=1}^{m} x_{ij}^*(t) = q_j(t)\right\}, \quad j = 1, \dots, n,$$

$$\eta_j^*(t) = \min\left\{-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}, \quad i = 1, \dots, m\right\}, \ a.e. \ in \ H_j^0, \ j = 1, \dots, n,$$

$$V_j^0 = \left\{t \in H_j^0 : \ -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} > \eta_j^*(t)\right\}, \quad j = 1, \dots, n,$$

and

$$W_j^0 = \left\{t \in H_j^0 : \ -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} = \eta_j^*(t)\right\}, \quad j = 1, \dots, n,$$

we have

$$\left(-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} - \eta_j^*(t)\right) x_{ij}^*(t) = 0, \quad a.e. \ in \ H_j^0, \ j = 1, \dots, n, \tag{2.4.16}$$

$$\eta_j^*(t) \le 0, \quad a.e. \ in \ W_j^0, \ \forall j = 1, \dots, n$$

and

$$-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} \ge 0, \quad a.e. \ in \ V_j^0, \ \forall i = 1, \dots, m.$$

Now, we remind Lemma 2.3.2 that holds when production excesses occur.

Lemma 2.4.4. *Let $x^* \in \mathbb{K}$ be a solution to the variational inequality* (2.4.8). *Setting*

$$I_i^- = \left\{ t \in [0,T] : \sum_{j=1}^{n} x_{ij}^*(t) - p_i(t) < 0 \right\}, \quad i = 1, \dots, m,$$

we have

$$-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} = 0, \quad a.e. \text{ in } I_i^-, \ \forall i = 1, \dots, m. \tag{2.4.17}$$

Finally, by proceeding as in Lemma 2.4.4 we can prove the following analogous result that holds when demand excesses occur.

Lemma 2.4.5. *Let $x^* \in \mathbb{K}$ be a solution to the variational inequality* (2.4.8). *Setting*

$$H_j^- = \left\{ t \in [0,T] : \sum_{i=1}^{m} x_{ij}^*(t) - q_j(t) < 0 \right\}, \quad j = 1, \dots, n,$$

we have

$$-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} = 0, \quad a.e. \text{ in } H_j^-, \ \forall j = 1, \dots, n. \tag{2.4.18}$$

Now we are able to prove Theorem 2.4.2.

Proof. Let us assume that $x^* \in \mathbb{K}$ is an equilibrium solution according to Definition 2.4.2. Then, taking into account that $\lambda_{ij}^*(t)(\underline{x}_{ij}(t) - x_{ij}^*(t)) = 0$ and $\rho_{ij}^*(t)(x_{ij}^*(t) - \overline{x}_{ij}(t)) = 0$, a.e. in $[0,T]$, we have for every $x \in \mathbb{K}$, a.e. in $[0,T]$,

$$\begin{aligned}
-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - x_{ij}^*(t)) &= -\rho_{ij}^*(t)(x_{ij}(t) - x_{ij}^*(t)) - \mu_i^*(t)(x_{ij}(t) - x_{ij}^*(t)) \\
&\quad -\nu_j^*(t)(x_{ij}(t) - x_{ij}^*(t)) + \lambda_{ij}^*(t)(x_{ij}(t) - x_{ij}^*(t)) \\
&= -\rho_{ij}^*(t)(x_{ij}(t) - \overline{x}_{ij}(t)) - \mu_i^*(t)(x_{ij}(t) - x_{ij}^*(t)) \\
&\quad -\nu_j^*(t)(x_{ij}(t) - x_{ij}^*(t)) + \lambda_{ij}^*(t)(x_{ij}(t) - \underline{x}_{ij}(t)) \\
&\geq -\mu_i^*(t)(x_{ij}(t) - x_{ij}^*(t)) - \nu_j^*(t)(x_{ij}(t) - x_{ij}^*(t)),
\end{aligned}$$

and, as a consequence, by summing over $i = 1, \dots, m$ and $j = 1, \dots, n$, integrating on

$[0, T]$ and using the conditions (2.4.12) and (2.4.13), it results, for each $x \in \mathbb{K}$

$$\int_0^T \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - x^*_{ij}(t))dt$$
$$\geq -\sum_{i=1}^m \int_0^T \mu_i^*(t) \left(\sum_{j=1}^n x_{ij}(t) - \sum_{j=1}^n x^*_{ij}(t) + p_i(t) - p_i(t) \right) dt$$
$$-\sum_{j=1}^n \int_0^T \nu_j^*(t) \left(\sum_{i=1}^m x_{ij}(t) - \sum_{i=1}^m x^*_{ij}(t) + q_j(t) - q_j(t) \right) dt$$
$$= -\sum_{i=1}^m \int_0^T \mu_i^*(t) \left(\sum_{j=1}^n x_{ij}(t) - p_i(t) \right) dt - \sum_{j=1}^n \int_0^T \nu_j^*(t) \left(\sum_{i=1}^m x_{ij}(t) - q_j(t) \right) dt \geq 0.$$

Hence, we obtain (2.4.8).

Vice versa, let $x^* \in \mathbb{K}$ be a solution to (2.4.8) and let us apply the infinite dimensional duality theory. First of all, let us prove that the Assumption S is fulfilled.

Let us set, for $x \in L^2([0, T], \mathbb{R}^{mn})$,

$$\Psi(x) = \int_0^T \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - x^*_{ij}(t))dt$$

and

$$\begin{aligned} \widetilde{M} = \Bigg\{ \Bigg(& \Psi(x) + \alpha, -x + \underline{x} + y, x - \overline{x} + u, \sum_{j=1}^n x_{ij} - p_i + v_i, \sum_{i=1}^m x_{ij} - q_j + z_j \Bigg) : \\ & i = 1, \ldots, m,\ j = 1, \ldots, n,\ \alpha \geq 0,\ x \in L^2([0, T], \mathbb{R}^{mn}_+) \setminus \mathbb{K}, \\ & y, u \in L^2([0, T], \mathbb{R}^{mn}_+),\ v \in L^2([0, T], \mathbb{R}^m_+),\ z \in L^2([0, T], \mathbb{R}^n_+) \Bigg\}, \end{aligned}$$

we must show that if $(l, \theta_{L^2([0,T],\mathbb{R}^{mn}_+)}, \theta_{L^2([0,T],\mathbb{R}^{mn}_+)}, \theta_{L^2([0,T],\mathbb{R}^m_+)}, \theta_{L^2([0,T],\mathbb{R}^n_+)})$ belongs to $T_{\widetilde{M}}(0, \theta_{L^2([0,T],\mathbb{R}^{mn}_+)}, \theta_{L^2([0,T],\mathbb{R}^{mn}_+)}, \theta_{L^2([0,T],\mathbb{R}^m_+)}, \theta_{L^2([0,T],\mathbb{R}^n_+)})$, namely

$$\lim_{n\to+\infty} \lambda_n(\Psi(x^n) + \alpha_n) = l,$$
$$\lim_{n\to+\infty} \lambda_n(-x^n + \underline{x} + y^n) = \theta_{L^2([0,T],\mathbb{R}^{mn}_+)},$$
$$\lim_{n\to+\infty} \lambda_n(x^n - \overline{x} + u^n) = \theta_{L^2([0,T],\mathbb{R}^{mn}_+)},$$
$$\lim_{n\to+\infty} \lambda_n \left(\sum_{j=1}^n x^n_{ij} - p_i + v^n_i \right) = \theta_{L^2([0,T],\mathbb{R}_+)}, \quad \forall i = 1, \ldots, m,$$
$$\lim_{n\to+\infty} \lambda_n \left(\sum_{i=1}^m x^n_{ij} - q_j + z^n_j \right) = \theta_{L^2([0,T],\mathbb{R}_+)}, \quad \forall j = 1, \ldots, n,$$

with $\lambda_n \geq 0$, $x^n \in L^2([0,T], \mathbb{R}^{mn}_+) \setminus \mathbb{K}$, $\alpha_n \geq 0$, $y^n, u^n \in L^2([0,T], \mathbb{R}^{mn}_+)$, $v^n \in L^2([0,T], \mathbb{R}^{m}_+)$, $z^n \in L^2([0,T], \mathbb{R}^{n}_+)$, $\forall n \in \mathbb{N}$, and

$$\begin{aligned}
&\lim_{n\to+\infty} (\Psi(x^n) + \alpha_n) = 0,\\
&\lim_{n\to+\infty} (-x^n + \underline{x} + y^n) = \theta_{L^2([0,T],\mathbb{R}^{mn}_+)},\\
&\lim_{n\to+\infty} (x^n + \overline{x} + u^n) = \theta_{L^2([0,T],\mathbb{R}^{mn}_+)},\\
&\lim_{n\to+\infty} \left(\sum_{j=1}^{n} x^n_{ij} - p_i + v^n_i\right) = \theta_{L^2([0,T],\mathbb{R}_+)}, \quad \forall i = 1, \dots, m,\\
&\lim_{n\to+\infty} \left(\sum_{i=1}^{m} x^n_{ij} - q_j + z^n_j\right) = \theta_{L^2([0,T],\mathbb{R}_+)}, \quad \forall j = 1, \dots, n,
\end{aligned}$$

then, l is non-negative.

Let us set

$$\begin{aligned}
E_{ij}^- &= \left\{t \in [0,T] : \ x_{ij}^*(t) = \underline{x}_{ij}(t)\right\}, \quad i = 1,\dots,m, \ j = 1,\dots,n, \\
E_{ij}^0 &= \left\{t \in [0,T] : \ \underline{x}_{ij}(t) < x_{ij}^*(t) < \overline{x}_{ij}(t)\right\}, \quad i = 1,\dots,m, \ j = 1,\dots,n, \\
E_{ij}^+ &= \left\{t \in [0,T] : \ x_{ij}^*(t) = \overline{x}_{ij}(t)\right\}, \quad i = 1,\dots,m, \ j = 1,\dots,n, \\
I_i^0 &= \left\{t \in [0,T] : \ \sum_{j=1}^n x_{ij}^*(t) = p_i(t)\right\}, \quad i = 1,\dots,m, \\
H_j^0 &= \left\{t \in [0,T] : \ \sum_{i=1}^m x_{ij}^*(t) = q_j(t)\right\}, \quad j = 1,\dots,n, \\
I_i^- &= \left\{t \in [0,T] : \ \sum_{j=1}^n x_{ij}^*(t) - p_i(t) < 0\right\}, \quad i = 1,\dots,m, \\
H_j^- &= \left\{t \in [0,T] : \sum_{i=1}^m x_{ij}^*(t) - q_j(t) < 0\right\}, \quad j = 1,\dots,n, \\
\gamma_i^*(t) &= \min\left\{-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}, \quad j = 1,\dots,n\right\}, \quad t \in I_i^0, \ i = 1,\dots,m, \\
\eta_j^*(t) &= \min\left\{-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}, \quad i = 1,\dots,m\right\}, \ t \in H_j^0, \ j = 1,\dots,n, \\
X_i^0 &= \left\{t \in I_i^0 : \ -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} > \gamma_i^*(t)\right\}, \quad i = 1,\dots,m, \\
Y_i^0 &= \left\{t \in I_i^0 : \ -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} = \gamma_i^*(t)\right\}, \quad i = 1,\dots,m, \\
V_j^0 &= \left\{t \in H_j^0 : \ -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} > \eta_j^*(t)\right\}, \quad j = 1,\dots,n, \\
W_j^0 &= \left\{t \in H_j^0 : \ -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} = \eta_j^*(t)\right\}, \quad j = 1,\dots,n.
\end{aligned}$$

Before starting with the proof let us observe the following:

$$[0,T] = I_i^0 \cup \left(H_j^0 \setminus I_i^0\right) \cup \left(I_i^- \cap H_j^-\right), \quad \forall i = 1,\dots,m, \ \forall j = 1,\dots,n$$

and also

$$[0,T] = E_{ij}^- \cup E_{ij}^0 \cup E_{ij}^+, \quad \forall i = 1,\dots,m, \ \forall j = 1,\dots,n.$$

Moreover

$$\begin{aligned}
E_{ij}^- &= E_{ij}^- \cap \left[I_i^0 \cup \left(H_j^0 \setminus I_i^0\right) \cup \left(I_i^- \cap H_j^-\right)\right] \\
&= \left(E_{ij}^- \cap I_i^0\right) \cup \left[E_{ij}^- \cap \left(H_j^0 \setminus I_i^0\right)\right] \cup \left[E_{ij}^- \cap \left(I_i^- \cap H_j^-\right)\right] \\
&= \left(E_{ij}^- \cap X_i^0\right) \cup \left(E_{ij}^- \cap Y_i^0\right) \cup \left[E_{ij}^- \cap \left(V_j^0 \setminus I_i^0\right)\right] \\
&\quad \cup \left[E_{ij}^- \cap \left(W_j^0 \setminus I_i^0\right)\right] \cup \left[E_{ij}^- \cap \left(I_i^- \cap H_j^-\right)\right], \quad \forall i = 1,\dots,m, \ \forall j = 1,\dots,n,
\end{aligned}$$

and, analogously,

$$E_{ij}^0 = \left(E_{ij}^0 \cap X_i^0\right) \cup \left(E_{ij}^0 \cap Y_i^0\right) \cup \left[E_{ij}^0 \cap \left(V_j^0 \setminus I_i^0\right)\right] \cup \left[E_{ij}^0 \cap \left(W_j^0 \setminus I_i^0\right)\right] \cup \left[E_{ij}^0 \cap \left(I_i^- \cap H_j^-\right)\right],$$

$\forall i = 1, \ldots, m$, $\forall j = 1, \ldots, n$, and

$$E_{ij}^+ = \left(E_{ij}^+ \cap X_i^0\right) \cup \left(E_{ij}^+ \cap Y_i^0\right) \cup \left[E_{ij}^+ \cap \left(V_j^0 \setminus I_i^0\right)\right] \cup \left[E_{ij}^+ \cap \left(W_j^0 \setminus I_i^0\right)\right] \cup \left[E_{ij}^+ \cap \left(I_i^- \cap H_j^-\right)\right],$$

$\forall i = 1, \ldots, m$,$\forall j = 1, \ldots, n$.

Now we observe that, for Lemmas 2.4.1, 2.4.2, 2.4.3, 2.4.4 and 2.4.5, we get

$$\begin{aligned}
&-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} = 0, \quad \text{a.e. in } E_{ij}^0, \quad \forall i = 1, \ldots, m, \, \forall j = 1, \ldots, n,\\
&-\frac{\partial v_i(t, \underline{x}(t))}{\partial x_{ij}} \geq 0, \quad \text{a.e. in } E_{ij}^- \cap X_i^0, \quad \forall i = 1, \ldots, m, \, \forall j = 1, \ldots, n,\\
&-\frac{\partial v_i(t, \underline{x}(t))}{\partial x_{ij}} = \gamma_i^*(t) = 0, \text{a.e. in } E_{ij}^- \cap Y_i^0, \quad \forall i = 1, \ldots, m, \, \forall j = 1, \ldots, n,\\
&-\frac{\partial v_i(t, \underline{x}(t))}{\partial x_{ij}} \geq 0, \quad \text{a.e. in } E_{ij}^- \cap \left(V_j^0 \setminus I_i^0\right), \; \forall i = 1, \ldots, m, \; \forall j = 1, \ldots, n,\\
&-\frac{\partial v_i(t, \underline{x}(t))}{\partial x_{ij}} = \eta_j^*(t) = 0, \quad \text{a.e. in } E_{ij}^- \cap \left(W_j^0 \setminus I_i^0\right), \; \forall i = 1, \ldots, m, \; \forall j = 1, \ldots, n,\\
&-\frac{\partial v_i(t, \underline{x}(t))}{\partial x_{ij}} = 0, \quad \text{a.e. in } E_{ij}^- \cap \left(I_i^- \cap H_j^-\right), \; \forall i = 1, \ldots, m, \; \forall j = 1, \ldots, n,\\
&-\frac{\partial v_i(t, \overline{x}(t))}{\partial x_{ij}} = 0, \quad \text{a.e. in } E_{ij}^+ \cap X_i^0, \; \forall i = 1, \ldots, m, \; \forall j = 1, \ldots, n,\\
&-\frac{\partial v_i(t, \overline{x}(t))}{\partial x_{ij}} = \gamma_i^*(t) \leq 0, \quad \text{a.e. in } E_{ij}^+ \cap Y_i^0, \; \forall i = 1, \ldots, m, \; \forall j = 1, \ldots, n,\\
&-\frac{\partial v_i(t, \overline{x}(t))}{\partial x_{ij}} = 0, \quad \text{a.e. in } E_{ij}^+ \cap \left(V_j^0 \setminus I_i^0\right), \; \forall i = 1, \ldots, m, \; \forall j = 1, \ldots, n,\\
&-\frac{\partial v_i(t, \overline{x}(t))}{\partial x_{ij}} = \eta_j^*(t) \leq 0, \quad \text{a.e. in } E_{ij}^+ \cap \left(W_j^0 \setminus I_i^0\right), \; \forall i = 1, \ldots, m, \; \forall j = 1, \ldots, n,\\
&-\frac{\partial v_i(t, \overline{x}(t))}{\partial x_{ij}} = 0, \quad \text{a.e. in } E_{ij}^+ \cap \left(I_i^- \cap H_j^-\right), \; \forall i = 1, \ldots, m, \; \forall j = 1, \ldots, n.
\end{aligned}$$

As a consequence, we have

$$\begin{aligned}
l &= \lim_{n\to+\infty} \lambda_n(\Psi(x^n)+\alpha_n)\\
&= \lim_{n\to+\infty} \lambda_n \left(\int_0^T \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial v_i(t,x^*(t))}{\partial x_{ij}}(x_{ij}^n(t)-x_{ij}^*(t))dt + \alpha_n \right)\\
&\geq \lim_{n\to+\infty} \lambda_n \Bigg(\int_{E_{ij}^-} \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial v_i(t,x^*(t))}{\partial x_{ij}}(x_{ij}^n(t)-\underline{x}_{ij}(t))dt\\
&\quad + \int_{E_{ij}^0} \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial v_i(t,x^*(t))}{\partial x_{ij}}(x_{ij}^n(t)-x_{ij}^*(t))dt\\
&\quad + \int_{E_{ij}^+} \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial v_i(t,x^*(t))}{\partial x_{ij}}(x_{ij}^n(t)-\overline{x}_{ij}(t))dt \Bigg).
\end{aligned}$$

We can observe that

$$\lim_{n\to+\infty} \lambda_n \int_{E_{ij}^0} \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial v_i(t,x^*(t))}{\partial x_{ij}}(x_{ij}^n(t)-x_{ij}^*(t))dt = 0 \tag{2.4.19}$$

being $-\dfrac{\partial v_i(t,x^*(t))}{\partial x_{ij}} = 0$, a.e. in E_{ij}^0, $\forall i=1,\dots,m$, $\forall j=1,\dots,n$.
We will prove that

$$\lim_{n\to+\infty} \lambda_n \int_{E_{ij}^-} \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial v_i(t,\underline{x}(t)))}{\partial x_{ij}}(x_{ij}^n(t)-\underline{x}_{ij}(t))dt \geq 0 \tag{2.4.20}$$

and

$$\lim_{n\to+\infty} \lambda_n \int_{E_{ij}^+} \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial v_i(t,\overline{x}(t))}{\partial x_{ij}}(x_{ij}^n(t)-\overline{x}_{ij}(t))dt \geq 0. \tag{2.4.21}$$

It results

$$
\begin{aligned}
&\lambda_n \int_{E_{ij}^-} \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial v_i(t, \underline{x}(t))}{\partial x_{ij}} (x_{ij}^n(t) - \underline{x}_{ij}(t))dt \\
&= \lambda_n \int_{E_{ij}^- \cap X_i^0} \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial v_i(t, \underline{x}(t))}{\partial x_{ij}} (x_{ij}^n(t) - \underline{x}_{ij}(t))dt \\
&\quad +\lambda_n \int_{E_{ij}^- \cap Y_i^0} \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial v_i(t, \underline{x}(t))}{\partial x_{ij}} (x_{ij}^n(t) - \underline{x}_{ij}(t))dt \\
&\quad +\lambda_n \int_{E_{ij}^- \cap (V_j^0 \setminus I_i^0)} \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial v_i(t, \underline{x}(t))}{\partial x_{ij}} (x_{ij}^n(t) - \underline{x}_{ij}(t))dt \\
&\quad +\lambda_n \int_{E_{ij}^- \cap (W_j^0 \setminus I_i^0)} \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial v_i(t, \underline{x}(t))}{\partial x_{ij}} (x_{ij}^n(t) - \underline{x}_{ij}(t))dt \\
&\quad +\lambda_n \int_{E_{ij}^- \cap I_i^- \cap H_j^-} \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial v_i(t, \underline{x}(t))}{\partial x_{ij}^n} (x_{ij}^n(t) - \underline{x}_{ij}(t))dt \\
&= \lambda_n \int_{E_{ij}^- \cap X_i^0} \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial v_i(t, \underline{x}(t))}{\partial x_{ij}} x_{ij}^n(t)dt \\
&\quad +\lambda_n \int_{E_{ij}^- \cap (V_j^0 \setminus I_i^0)} \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial v_i(t, \underline{x}(t))}{\partial x_{ij}} x_{ij}^n(t)dt.
\end{aligned}
$$

By virtue of the previous remarks, conditions (2.4.15) and (2.4.16), Lemmas 2.4.1, 2.4.2, 2.4.3, 2.4.4 and 2.4.5, for the conditions of belonging to the tangent cone, we get the inequality (2.4.20) and, with analogous considerations, we get the inequality (2.4.21). Therefore, thanks to (2.4.20), (2.4.19), (2.4.21) we have that

$$
\begin{aligned}
l &= \lim_{n \to +\infty} \lambda_n (\Psi(x^n) + \alpha_n) \\
&= \lim_{n \to +\infty} \lambda_n \left(\int_0^T \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} (x_{ij}^n(t) - x_{ij}^*(t))dt + \alpha_n \right)
\end{aligned}
$$

is non-negative.

Taking into account Theorems 1.2.2 and 1.2.3, if we consider the Lagrange function

$$\begin{aligned}
\mathcal{L}(x,\lambda,\rho,\mu,\nu) &= \Psi(x)+\sum_{i=1}^{m}\sum_{j=1}^{n}\int_0^T \lambda_{ij}(t)(\underline{x}_{ij}(t)-x_{ij}(t))dt \\
&+\sum_{i=1}^{m}\sum_{j=1}^{n}\int_0^T \rho_{ij}(t)(x_{ij}(t)-\overline{x}_{ij}(t))dt \\
&+\sum_{i=1}^{m}\int_0^T \mu_i(t)\left(\sum_{j=1}^{n}x_{ij}(t)-p_i(t)\right)dt \\
&+\sum_{j=1}^{n}\int_0^T \nu_j(t)\left(\sum_{i=1}^{m}x_{ij}(t)-q_j(t)\right)dt,
\end{aligned}$$

$\forall x\in L^2([0,T],\mathbb{R}^{mn}_+),\ \lambda,\rho\in L^2([0,T],\mathbb{R}^{mn}_+),\ \mu\in L^2([0,T],\mathbb{R}^{m}_+),\ \nu\in L^2([0,T],\mathbb{R}^{n}_+)$,

we have that there exist $\lambda^*,\rho^*\in L^2([0,T],\mathbb{R}^{mn}_+),\ \mu^*\in L^2([0,T],\mathbb{R}^{m}_+),\ \nu^*\in L^2([0,T],\mathbb{R}^{n}_+)$ such that

$$\mathcal{L}(x^*,\lambda,\rho,\mu,\nu)\le\mathcal{L}(x^*,\lambda^*,\rho^*,\mu^*,\nu^*)\le\mathcal{L}(x,\lambda^*,\rho^*,\mu^*,\nu^*) \tag{2.4.22}$$

$\forall x\in L^2([0,T],\mathbb{R}^{mn}_+),\ \lambda,\rho\in L^2([0,T],\mathbb{R}^{mn}_+),\ \mu\in L^2([0,T],\mathbb{R}^{m}_+),\ \nu\in L^2([0,T],\mathbb{R}^{n}_+)$, and,

$$\begin{aligned}
\langle\langle\lambda^*,\underline{x}-x^*\rangle\rangle &= \int_0^T\sum_{i=1}^{m}\sum_{j=1}^{n}\lambda^*_{ij}(t)(\underline{x}_{ij}(t)-x^*_{ij}(t))dt=0,\\
\langle\langle\rho^*,x^*-\overline{x}\rangle\rangle &= \int_0^T\sum_{i=1}^{m}\sum_{j=1}^{n}\rho^*_{ij}(t)(x^*_{ij}(t)-\overline{x}_{ij}(t))dt=0,\\
\left\langle\left\langle\mu^*,\sum_{j=1}^{n}x^*_{ij}(t)-p_i(t)\right\rangle\right\rangle &= \int_0^T\sum_{i=1}^{m}\left[\mu^*_i(t)\left(\sum_{j=1}^{n}x^*_{ij}(t)-p_i(t)\right)\right]dt=0,\\
\left\langle\left\langle\nu^*,\sum_{i=1}^{m}x^*_{ij}(t)-q_j(t)\right\rangle\right\rangle &= \int_0^T\sum_{j=1}^{n}\left[\nu^*_i(t)\left(\sum_{i=1}^{m}x^*_{ij}(t)-q_j(t)\right)\right]dt=0.
\end{aligned}$$

Hence,

$$\lambda^*_{ij}(t)(\underline{x}_{ij}(t)-x^*_{ij}(t))=0,\quad\forall i=1,\dots,m,\quad\forall j=1,\dots,n, \tag{2.4.23}$$

$$\rho^*_{ij}(t)(x^*_{ij}(t)-\overline{x}_{ij}(t))=0,\quad\forall i=1,\dots,m,\quad\forall j=1,\dots,n, \tag{2.4.24}$$

$$\mu^*_i(t)\left(\sum_{j=1}^{n}x^*_{ij}(t)-p_i(t)\right)=0,\quad\forall i=1,\dots,m, \tag{2.4.25}$$

$$\nu^*_j(t)\left(\sum_{i=1}^{m}x^*_{ij}(t)-q_j(t)\right)=0,\quad\forall j=1,\dots,n. \tag{2.4.26}$$

Then, for conditions (2.4.23) – (2.4.26), $\mathcal{L}(x^*, \lambda^*, \rho^*, \mu^*, \nu^*) = 0$, and by virtue of the right-hand side of (2.4.22) and the equalities (2.4.23)-(2.4.26), we get

$$\begin{aligned}
\mathcal{L}(x, \lambda^*, \rho^*, \mu^*, \nu^*) &= \sum_{i=1}^{m}\sum_{j=1}^{n}\int_0^T -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - x^*_{ij}(t))dt \\
&\quad -\sum_{i=1}^{m}\sum_{j=1}^{n}\int_0^T \lambda^*_{ij}(t)(x_{ij}(t) - x^*_{ij}(t))dt \\
&\quad +\sum_{i=1}^{m}\sum_{j=1}^{n}\int_0^T \rho^*_{ij}(t)(x_{ij}(t) - x^*_{ij}(t))dt \\
&\quad +\sum_{i=1}^{m}\sum_{j=1}^{n}\int_0^T \mu^*_{i}(t)(x_{ij}(t) - x^*_{ij}(t))dt \\
&\quad +\sum_{j=1}^{n}\sum_{i=1}^{m}\int_0^T \nu^*_{j}(t)(x_{ij}(t) - x^*_{ij}(t))dt \\
&\geq \mathcal{L}(x^*, \lambda^*, \rho^*, \mu^*, \nu^*) = 0, \qquad \forall x \in L^2([0,T], \mathbb{R}^{mn}_+).
\end{aligned}$$

Then, $\mathcal{L}(x, \lambda^*, \rho^*, \mu^*, \nu^*)$ has a minimal point in x^*.

Let us assume that $x^1_{ij} = x^*_{ij} + \varepsilon_{ij}$ and $x^2_{ij} = x^*_{ij} - \varepsilon_{ij}$, for all $\varepsilon \in L^2([0,T], \mathbb{R}^{mn}_+)$. Let us note that

$$\mathcal{L}(x^1, \lambda^*, \rho^*, \mu^*, \nu^*) = \sum_{i=1}^{m}\sum_{j=1}^{n}\int_0^T \left[-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} - \lambda^*_{ij}(t) + \rho^*_{ij}(t) + \mu^*_i(t) + \nu^*_j(t)\right]\varepsilon_{ij}(t)dt \geq 0,$$

$\forall \varepsilon \in L^2([0,T], \mathbb{R}^{mn}_+)$ and

$$\mathcal{L}(x^2, \lambda^*, \rho^*, \mu^*, \nu^*) = -\Big\{\sum_{i=1}^{m}\sum_{j=1}^{n}\int_0^T \left[-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} - \lambda^*_{ij}(t) + \rho^*_{ij}(t) + \mu^*_i(t) + \nu^*_j(t)\right]\varepsilon_{ij}(t)dt\Big\} \geq 0,$$

$\forall \varepsilon \in L^2([0,T], \mathbb{R}^{mn}_+)$.

As a consequence, we have

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\int_0^T \left[-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} - \lambda^*_{ij}(t) + \rho^*_{ij}(t) + \mu^*_i(t) + \nu^*_j(t)\right]\varepsilon_{ij}(t)dt = 0,$$
$$\forall \epsilon \in L^2([0,T], \mathbb{R}^{mn}_+),$$

namely, we get the equilibrium conditions of Definition 2.4.2,

$$-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} + \rho^*_{ij}(t) + \mu^*_i(t) + \nu^*_j(t) = \lambda^*_{ij}(t),$$

$$\lambda^*_{ij}(t)(\underline{x}_{ij}(t) - x^*_{ij}(t)) = 0, \qquad \lambda^*_{ij}(t) \geq 0,$$

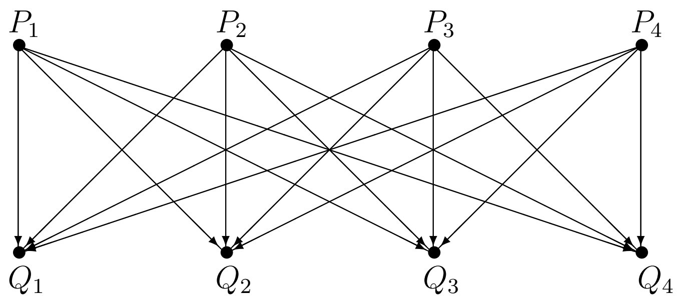

Figure 2.3: Network structure of the numerical dynamic spatial oligopoly problem.

$$\rho^*_{ij}(t)(x^*_{ij}(t) - \overline{x}_{ij}(t)) = 0, \qquad \rho^*_{ij}(t) \geq 0,$$

$$\mu^*_i(t)\left(\sum_{j=1}^{n} x^*_{ij}(t) - p_i(t)\right) = 0, \qquad \mu^*_i(t) \geq 0,$$

$$\nu^*_j(t)\left(\sum_{i=1}^{m} x^*_{ij}(t) - q_j(t)\right) = 0, \qquad \nu^*_j(t) \geq 0.$$

for each $i = 1, \ldots, m$, $j = 1, \ldots, n$ and a.e. in $[0, T]$.

□

2.4.3 A numerical example

Let us present a numerical example about the dynamic oligopolistic market equilibrium problem in presence of both production and demand excesses.

Let us consider four firms and four demand markets, as in Figure 2.3. Let $\underline{x}, \overline{x} \in L^2([0,1], \mathbb{R}^{16}_+)$ be the capacity constraints such that, a.e. in $[0,1]$,

$$\underline{x}(t) = \begin{pmatrix} \frac{1}{2}t & 0 & \frac{1}{100}t & \frac{1}{2}t \\ t & \frac{1}{4}t & 0 & \frac{1}{10}t \\ \frac{3}{100}t & \frac{1}{4}t & \frac{1}{10}t & \frac{1}{4}t \\ \frac{4}{7}t & \frac{2}{7}t & \frac{1}{10}t & 0 \end{pmatrix},$$

$$\overline{x}(t) = \begin{pmatrix} t & 2t & t & 4t \\ 2t & \frac{3}{2}t & \frac{10}{11}t & \frac{5}{6}t \\ t & \frac{3}{4}t & 3t & \frac{3}{4}t \\ \frac{6}{7}t & \frac{6}{7}t & \frac{3}{7}t & \frac{1}{2}t \end{pmatrix}.$$

Let $p \in L^2([0,1], \mathbb{R}^4_+)$ be the production function such that, a.e. in $[0,1]$,

$$p(t) = \begin{pmatrix} 5t \\ 7t \\ 9t \\ 11t \end{pmatrix},$$

and let $q \in L^2([0,1], \mathbb{R}^4_+)$ be the demand function such that, a.e. in $[0,1]$,

$$q(t) = \begin{pmatrix} 6t \\ 8t \\ 10t \\ 12t \end{pmatrix}.$$

As a consequence, the feasible set is

$$\begin{aligned} \mathbb{K} \quad = \quad & \Bigg\{ x \in L^2([0,1]\,, \mathbb{R}^{16}_+) : \\ & \underline{x}_{ij}(t) \le x_{ij}(t) \le \overline{x}_{ij}(t), \quad \forall i = 1, \ldots, 4,\ \forall j = 1, \ldots, 4,\ \text{a.e. in}\,[0,1]\,, \\ & \sum_{j=1}^{4} x_{ij}(t) \le p_i(t), \quad \forall i = 1, \ldots, 4,\ \text{a.e. in}\ [0,1]\,, \\ & \sum_{i=1}^{4} x_{ij}(t) \le q_j(t), \quad \forall j = 1, \ldots, 4,\ \text{a.e. in}\ [0,1] \Bigg\}. \end{aligned}$$

Let us consider the profit function $v \in L^2([0,1] \times L^2([0,1], \mathbb{R}_+^{4\times4}), \mathbb{R}^4)$ defined by

$$\begin{aligned}
v_1(t,x(t)) &= -4x_{11}^2(t) - 6x_{12}^2(t) - 2x_{13}^2(t) - 4x_{14}^2(t) - 2x_{11}(t)x_{13}(t) - 2x_{12}(t)x_{14}(t)\\
&\quad +3tx_{11}(t) + 6tx_{12}(t) + tx_{13}(t) + 5tx_{14}(t),\\
v_2(t,x(t)) &= -2x_{21}^2(t) - 5x_{22}^2(t) - 6x_{23}^2(t) - 2x_{24}^2(t) - 2x_{21}(t)x_{23}(t) - 2x_{22}(t)x_{31}(t)\\
&\quad -2x_{24}(t)x_{33}(t) + 6tx_{21}(t) + 5tx_{22}(t) + 4tx_{23}(t) + tx_{24}(t),\\
v_3(t,x(t)) &= -3x_{31}^2(t) - 4x_{32}^2(t) - 2x_{33}^2(t) - 2x_{34}^2(t) - 2x_{22}(t)x_{31}(t) - 2x_{24}(t)x_{33}(t)\\
&\quad -2x_{32}(t)x_{34}(t) + 2tx_{31}(t) + 5tx_{32}(t) + tx_{33}(t) + 3tx_{34}(t),\\
v_4(t,x(t)) &= -4x_{41}^2(t) - 4x_{42}^2(t) - 2x_{43}^2(t) - 2x_{44}^2(t) - 2x_{41}(t)x_{43}(t) - 2x_{42}(t)x_{44}(t)\\
&\quad +6tx_{41}(t) + 5tx_{42}(t) + 2tx_{43}(t) + 2tx_{44}(t).
\end{aligned}$$

Then, the operator $\nabla_D v \in L^2([0,1] \times L^2([0,1], \mathbb{R}_+^{4\times4}), \mathbb{R}^{4\times4})$ is given by

$$\nabla_D v(t,x(t)) = \begin{pmatrix}
-8x_{11}(t) - 2x_{13}(t) + 3t & -12x_{12}(t) - 2x_{14}(t) + 6t & -4x_{13}(t) - 2x_{11}(t) + t & -8x_{14}(t) - 2x_{12}(t) + 5t\\
-4x_{21}(t) - 2x_{23}(t) + 6t & -10x_{22}(t) - 2x_{31}(t) + 5t & -12x_{23}(t) - 2x_{21}(t) + 4t & -4x_{24}(t) - 2x_{33}(t) + t\\
-6x_{31}(t) - 2x_{22}(t) + 2t & -8x_{32}(t) - 2x_{34}(t) + 5t & -4x_{33}(t) - 2x_{24}(t) + t & -4x_{34}(t) - 2x_{32}(t) + 3t\\
-8x_{41}(t) - 2x_{43}(t) + 6t & -8x_{42}(t) - 2x_{44}(t) + 5t & -4x_{43}(t) - 2x_{41}(t) + 2t & -4x_{44}(t) - 2x_{42}(t) + 2t
\end{pmatrix}.$$

Now, we verify that $-\nabla_D v$ is a strongly monotone operator, in fact

$$\begin{aligned}
&\langle -\nabla_D v(x) + \nabla_D v(y), x - y\rangle\\
&\quad = \big\{8[x_{11} - y_{11}] + 2[x_{13} - y_{13}]\big\}[x_{11} - y_{11}] + \big\{12[x_{12} - y_{12}] + 2[x_{14} - y_{14}]\big\}[x_{12} - y_{12}]\\
&\quad\quad +\big\{4[x_{13} - y_{13}] + 2[x_{11} - y_{11}]\big\}[x_{13} - y_{13}] + \big\{8[x_{14} - y_{14}] + 2[x_{12} - y_{12}]\big\}[x_{14} - y_{14}]\\
&\quad\quad +\big\{4[x_{21} - y_{21}] + 2[x_{23} - y_{23}]\big\}[x_{21} - y_{21}] + \big\{10[x_{22} - y_{22}] + 2[x_{31} - y_{31}]\big\}[x_{22} - y_{22}]\\
&\quad\quad +\big\{12[x_{23} - y_{23}] + 2(x_{21} - y_{21}][x_{23} - y_{23}] + \big\{4[x_{24} - y_{24}] + 2[x_{33} - y_{33}]\big\}[x_{24} - y_{24}]\\
&\quad\quad +\big\{6[x_{31} - y_{31}] + 2[x_{22} - y_{22}]\big\}[x_{31} - y_{31}] + \big\{8[x_{32} - y_{32}] + 2[x_{34} - y_{34}]\big\}[x_{32} - y_{32}]\\
&\quad\quad +\big\{4[x_{33} - y_{33}] + 2[x_{24} - y_{24}]\big\}[x_{33} - y_{33}] + \big\{4[x_{34} - y_{34}] + 2[x_{32} - y_{32}]\big\}[x_{34} - y_{34}]\\
&\quad\quad +\big\{8[x_{41} - y_{41}] + 2[x_{43} - y_{43}]\big\}[x_{41} - y_{41}] + \big\{8[x_{42} - y_{42}] + 2[x_{44} - y_{44}]\big\}[x_{42} - y_{42}]\\
&\quad\quad +\big\{4[x_{43} - y_{43}] + 2[x_{41} - y_{41}]\big\}[x_{43} - y_{43}] + \big\{4[x_{44} - y_{44}] + 2[x_{42} - y_{42}]\big\}[x_{44} - y_{44}]\\
&\quad \geq 3\|x - y\|_{4\times4}^2.
\end{aligned}$$

The dynamic oligopolistic market equilibrium distribution in presence of excesses is the solution to the evolutionary variational inequality:

$$\int_0^1 \sum_{i=1}^4 \sum_{j=1}^4 -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - x_{ij}^*(t))dt \geq 0, \quad \forall x \in \mathbb{K}. \tag{2.4.27}$$

In order to compute the solution to (2.4.27) we make use of the direct method (see

[29, 34, 70]). We consider the following system

$$\begin{cases} 8x^*_{11}(t) + 2x^*_{13}(t) - 3t = 0, \quad 12x^*_{12}(t) + 2x^*_{14}(t) - 6t = 0, \quad 4x^*_{13}(t) + 2x^*_{11}(t) - t = 0, \\ 8x^*_{14}(t) + 2x^*_{12}(t) - 5t = 0, \quad 4x^*_{21}(t) + 2x^*_{23}(t) - 6t = 0, \quad 10x^*_{22}(t) + 2x^*_{31}(t) - 5t = 0, \\ 12x^*_{23}(t) + 2x^*_{21}(t) - 4t = 0, \quad 4x^*_{24}(t) + 2x^*_{33}(t) - t = 0, \quad 6x^*_{31}(t) + 2x^*_{22}(t) - 2t = 0, \\ 8x^*_{32}(t) + 2x^*_{34}(t) - 5t = 0, \quad 4x^*_{33}(t) + 2x^*_{24}(t) - t = 0, \quad 4x^*_{34}(t) + 2x^*_{32}(t) - 3t = 0, \\ 8x^*_{41}(t) + 2x^*_{43}(t) - 6t = 0, \quad 8x^*_{42}(t) + 2x^*_{44}(t) - 5t = 0, \quad 4x^*_{43}(t) + 2x^*_{41}(t) - 2t = 0, \\ 4x^*_{44}(t) + 2x^*_{42}(t) - 2t = 0, \\ \underline{x}_{ij}(t) \le x_{ij}(t) \le \overline{x}_{ij}(t), \quad \forall i = 1, \dots, 4, \ \forall j = 1, \dots, 4, \\ \sum_{j=1}^{n} x_{ij}(t) \le p_i(t), \quad \forall i = 1, \dots, 4, \\ \sum_{i=1}^{m} x_{ij}(t) \le q_j(t), \quad \forall j = 1, \dots, 4, \end{cases} .$$

and we get the following solution, a.e. in $[0, 1]$,

$$x^*(t) = \begin{pmatrix} \frac{5}{14}t & \frac{19}{46}t & \frac{1}{14}t & \frac{12}{23}t \\ \frac{16}{11}t & \frac{13}{28}t & \frac{1}{11}t & \frac{1}{6}t \\ \frac{5}{28}t & \frac{1}{2}t & \frac{1}{6}t & \frac{1}{2}t \\ \frac{5}{7}t & \frac{4}{7}t & \frac{1}{7}t & \frac{3}{14}t \end{pmatrix}.$$

It is easy to prove that x^* belongs to the constraint set $\mathbb{K}$, then it is the equilibrium solution.

Now, we are able to compute the production excess of each firm and the demand excess of each demand market

$$\begin{aligned}
\epsilon_1(t) &= p_1(t) - \sum_{j=1}^{4} x^*_{1j}(t) = 5t - \frac{439}{322}t = \frac{1610}{322}t, \quad \text{a.e. in } [0,1]\,, \\
\epsilon_2(t) &= p_2(t) - \sum_{j=1}^{4} x^*_{2j}(t) = 7t - \frac{2014}{924}t = \frac{4454}{924}t, \quad \text{a.e. in } [0,1]\,, \\
\epsilon_3(t) &= p_3(t) - \sum_{j=1}^{4} x^*_{3j}(t) = 9t - \frac{113}{84}t = \frac{643}{84}t, \quad \text{a.e. in } [0,1]\,, \\
\epsilon_4(t) &= p_4(t) - \sum_{j=1}^{4} x^*_{3j}(t) = 11t - \frac{23}{14}t = \frac{131}{14}t, \quad \text{a.e. in } [0,1]\,, \\
\delta_1(t) &= q_1(t) - \sum_{i=1}^{4} x^*_{i1}(t) = 6t - \frac{833}{308}t = \frac{1015}{308}t, \quad \text{a.e. in } [0,1]\,, \\
\delta_2(t) &= q_2(t) - \sum_{i=1}^{4} x^*_{i2}(t) = 8t - \frac{1255}{644}t = \frac{3897}{644}t, \quad \text{a.e. in } [0,1]\,, \\
\delta_3(t) &= q_3(t) - \sum_{i=1}^{4} x^*_{i3}(t) = 10t - \frac{218}{462}t = \frac{4402}{462}t, \quad \text{a.e. in } [0,1]\,, \\
\delta_4(t) &= q_4(t) - \sum_{i=1}^{4} x^*_{i4}(t) = 12t - \frac{1355}{966}t = \frac{10237}{966}t, \quad \text{a.e. in } [0,1]\,.
\end{aligned}$$

Making use of the equilibrium conditions (2.4.9) – (2.4.13), we derive:

$$\begin{aligned}
\lambda^*_{ij}(t) &= 0, \quad \forall i = 1,\dots,4, \ \forall j = 1,\dots,4, \ \text{a.e. in } [0,1], \\
\rho^*_{ij}(t) &= 0, \quad \forall i = 1,\dots,4, \ \forall j = 1,\dots,4, \ \text{a.e. in } [0,1], \\
\mu^*_i(t) &= 0, \quad \forall i = 1,\dots,4, \ \text{a.e. in } [0,1], \\
\nu^*_j(t) &= 0, \quad \forall j = 1,\dots,4, \ \text{a.e. in } [0,1].
\end{aligned}$$

Chapter 3

Existence and regularity

3.1 Introduction

In optimization theory it is important to answer to the question if there exists a solution to the problem by giving a look, in first analysis, to the structure of the constraint set and to the nature of the functions that define the problem. There exists a vast literature about this question (see, for example, [71, 87, 101]).

Another important question in optimization is the structure of the equilibrium solution. To know if it shows elements of regularity proves to be very important to build some computational procedures to compute the solution. The regularity results that we show in this chapter are all based on set convergence in Kuratowski's sense (see [61, 74, 93, 94]), namely we present regularity results for evolutionary variational inequalities which depend explicity on the time and the constraint set $\mathbb{K}$ satisfies the next assumption:

(K) $\mathbb{K} \subseteq L^2([0,T], \mathbb{R}^m)$ is a nonempty convex, closed set, such that the sequence $\{\mathbb{K}(t_n)\}_{n\in\mathbb{N}}$ converges to $\mathbb{K}(t)$ in Kuratowski's sense, for each $t \in [0,T]$, and the sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,T]$, such that $t_n \to t$, as $n \to +\infty$.

For more details, see [2–6, 8].
Moreover, in [10], a Lipschitz continuity result, which depends on the variation rate of projections onto time-dependent constraints set, is shown (see also [23, 46, 73, 105]). Finally, in [12] some sensitivity results have been obtained each of them showing that small changes of the solution happen in correspondence with small changes of the profit function (see also [64, 65]).

3.2 Existence results

This section is devoted to show some results for the existence of solutions to the dynamic oligopolistic market equilibrium problem.

Theorem 3.2.1. *Let us set*

$$A = \left[-\frac{\partial v_i(x^*)}{\partial x_{ij}} \right]_{\substack{i=1,\dots,m \\ j=1,\dots,n}},$$

$$A : L^2([0,T], \mathbb{R}^{mn}) \longrightarrow L^2([0,T], \mathbb{R}^{mn}),$$

$$u = (x_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}},$$

let $\mathbb{K}$ *be the constraint sets considered in chapter* 2.

If A *is B-pseudomonotone or F-hemicontinuous, or assuming that* A *is K-pseudomonotone and lower hemicontinuous along line segments, then the variational inequality*

$$\ll Ax^*, x - x^* \gg \geq 0, \quad \forall x \in \mathbb{K}, \tag{3.2.1}$$

admits a solution.

Proof. Let us note that $\mathbb{K}$ is clearly a nonempty, closed, convex and bounded subset of $L^2([0,T], \mathbb{R}^{mn})$ and, consequently, it is a weakly compact subset of $L^2([0,T], \mathbb{R}^{mn})$.Then, the claim is achieved by applying Theorems 2.6 and 2.7 and Corollary 3.7 in [71]. $\square$

3.3 Regularity results

In the following, we want to establish conditions under which the solutions to dynamic oligopolistic market problems with both production and demand excesses are continuous with respect to time.

3.3.1 Set convergence

Let us remind the classical notion of convergence for subsets of a given metric space (X, d), which was introduced in the 50's by Kuratowski (see [61], see also [93, 94]).

Let $\{K_n\}_{n\in\mathbb{N}}$ be a sequence of subsets of X. Let us remind that

$$d - \underline{\lim}_n K_n = \{x \in X : \exists \{x_n\}_{n\in\mathbb{N}} \text{ eventually in } K_n \text{ such that } x_n \xrightarrow{d} x\},$$

and

$$d - \overline{\lim}_n K_n = \{x \in X : \exists \{x_n\}_{n\in\mathbb{N}} \text{ frequently in } K_n \text{ such that } x_n \xrightarrow{d} x\},$$

where *eventually* means that there exists $\delta \in \mathbb{N}$ such that $x_n \in K_n$ for any $n \geq \delta$, and *frequently* means that there exists an infinite subset $N \subseteq \mathbb{N}$ such that $x_n \in K_n$ for any $n \in N$ (in this last case, according to the notation given above, we also write that there exists a subsequence $\{x_{k_n}\}_{n\in\mathbb{N}} \subseteq \{x_n\}_{n\in\mathbb{N}}$ such that $x_{k_n} \in K_{k_n}$).

Now, we are able to recall the Kuratowski's convergence of sets.

Definition 3.3.1. We say that the sequence $\{K_n\}_{n\in\mathbb{N}}$ converges to some subset $K \subseteq X$ in Kuratowski's sense, and we briefly write $K_n \to K$, iff $d - \underline{\lim}_n K_n = d - \overline{\lim}_n K_n = K$. Thus, in order to verify that $K_n \to K$, it suffices to check that

- $K \subset d - \underline{\lim}_n K_n$, i.e. for any $x \in K$ there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ eventually in K_n such that $x_n \overset{d}{\to} x$;
- $d - \overline{\lim}_n K_n \subseteq K$, i.e. for any sequence $\{x_n\}_{n\in\mathbb{N}}$ frequently in K_n such that $x_n \overset{d}{\to} x$ for some $x \in S$, then $x \in K$.

We observe that the set convergence in Kuratowski's sense can also be expressed as follows:

Remark 3.3.1. Let (X, d) be a metric space and K a nonempty, closed and convex subset of X. A sequence of nonempty, closed and convex sets K_n of X converges to K in Kuratowski's sense, as $n \to +\infty$, i.e. $K_n \to K$, if and only if

(K1) for any $x \in K$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to $x \in X$ such that x_n lies in K_n for all $n \in \mathbb{N}$,

(K2) for any subsequence $\{x_n\}_{n\in\mathbb{N}}$ converging to $x \in X$ such that x_n lies in K_n, for all $n \in \mathbb{N}$, then the limit x belongs to K.

The following lemmas, that now we prove, assure that the feasible sets K of the dynamic oligopolistic market problem satisfy the property of the set convergence in Kuratowski's sense.

Lemma 3.3.1. [9] *Let $\underline{x}, \overline{x} \in C^0\left([0,T], \mathbb{R}^{mn}_+\right)$ and let $\{t_k\}_{k\in\mathbb{N}}$ be a sequence such that $t_k \in [0,T]$, $\forall k \in \mathbb{N}$, and $t_k \to t$, with $t \in [0,T]$, as $k \to +\infty$. Then the sequence of sets*

$$\mathbb{K}(t_k) = \left\{ x(t_k) \in \mathbb{R}^{mn} : \quad \underline{x}_{ij}(t_k) \le x_{ij}(t_k) \le \overline{x}_{ij}(t_k), \quad \forall i = 1, \dots, m, \ \forall j = 1, \dots, n \right\}$$

$\forall k \in \mathbb{N}$, converges to

$$\mathbb{K}(t) = \left\{ x(t) \in \mathbb{R}^{mn} : \quad \underline{x}_{ij}(t) \le x_{ij}(t) \le \overline{x}_{ij}(t), \quad \forall i = 1, \dots, m, \ \forall j = 1, \dots, n \right\},$$

as $k \to +\infty$, in Kuratowski's sense.

Proof. In the first part, we prove the condition (K1). Let $\{t_k\}_{k\in\mathbb{N}}$ be a sequence such that $t_k \in [0,T]$, $\forall k \in \mathbb{N}$, and $t_k \to t$, with $t \in [0,T]$, as $k \to +\infty$.Let $x(t) \in \mathbb{K}(t)$ be fixed and let us consider the following sequence:

$$x(t_n) = \underline{x}(t_n) + \min\left\{x(t) - \underline{x}(t), \overline{x}(t_n) - \underline{x}(t_n)\right\}, \quad \forall n \in \mathbb{N}.$$

Let us verify that $x(t_n) \in \mathbb{K}(t_n)$, $\forall n \in \mathbb{N}$. Since $\min\{x(t) - \underline{x}(t), \overline{x}(t_n) - \underline{x}(t_n)\} \geq 0$, $\forall n \in \mathbb{N}$, it results $x(t_n) \geq \underline{x}(t_n)$, $\forall n \in \mathbb{N}$. Moreover, from $\min\{x(t) - \underline{x}(t), \overline{x}(t_n) - \underline{x}(t_n)\} \leq \overline{x}(t_n) - \underline{x}(t_n)$, $\forall n \in \mathbb{N}$, it follows $x(t_n) \leq \overline{x}(t_n)$, $\forall n \in \mathbb{N}$.
Taking into account that, for continuity assumptions and for $x(t) \in \mathbb{K}(t)$,

$$\begin{aligned}\lim_{n\to+\infty} x(t_n) &= \lim_{n\to+\infty} \{\underline{x}(t_n) + \min\{x(t) - \underline{x}(t), \overline{x}(t_n) - \underline{x}(t_n)\}\}\\ &\quad \underline{x}(t) + \min\{x(t) - \underline{x}(t), \overline{x}(t) - \underline{x}(t)\} = x(t),\end{aligned}$$

the condition (K1) is achieved.

Now let us prove the condition (K2). Let $\{t_k\}_{k\in\mathbb{N}}$ be a sequence such that $t_k \in [0, T]$, $\forall k \in \mathbb{N}$, and $t_k \to t$, with $t \in [0, T]$, as $k \to +\infty$. Let $\{x(t_k)\}_{k\in\mathbb{N}}$ be a sequence, such that $x(t_k) \in \mathbb{K}(t_k)$, $\forall k \in \mathbb{N}$, and converging to $x(t)$, as $k \to +\infty$. We need to prove that $x(t) \in \mathbb{K}(t)$.
Since $x(t_k) \in \mathbb{K}(t_k)$, $\forall k \in \mathbb{N}$, it results

$$\underline{x}_{ij}(t_k) \leq x_{ij}(t_k) \leq \overline{x}_{ij}(t_k), \quad \forall i = 1, \ldots, m, \ \forall j = 1, \ldots, n, \ \forall k \in \mathbb{N}.$$

Passing to the limit as $n \to +\infty$ and taking into account the continuity assumption on the functions $\underline{x}, \overline{x}$, we obtain

$$\underline{x}_{ij}(t) \leq x_{ij}(t) \leq \overline{x}_{ij}(t), \quad \forall i = 1, \ldots, m, \ \forall j = 1, \ldots, n.$$

As a consequence $x(t) \in \mathbb{K}(t)$, and, hence, the condition (K2) is achieved. □

Lemma 3.3.2. *Let $p \in C\left([0, T], \mathbb{R}^m_+\right)$ and let $\{t_k\}_{k\in\mathbb{N}}$ be a sequence such that $t_k \to t$, with $t \in [0, T]$, as $k \to +\infty$. Then, the sequence of sets*

$$\mathbb{K}(t_k) = \Bigg\{ x(t_k) \in \mathbb{R}^{mn} : \quad x_{ij}(t_k) \geq 0, \quad \forall i = 1, \ldots, m, \ \forall j = 1, \ldots, n, \\ \sum_{j=1}^{n} x_{ij}(t_k) \leq p_i(t_k), \quad \forall i = 1, \ldots, m \Bigg\}$$

converges to

$$\mathbb{K}(t) = \Bigg\{ x(t) \in \mathbb{R}^{mn} : \quad x_{ij}(t) \geq 0, \quad \forall i = 1, \ldots, m, \ \forall j = 1, \ldots, n, \\ \sum_{j=1}^{n} x_{ij}(t) \leq p_i(t), \quad \forall i = 1, \ldots, m \Bigg\}$$

as $k \to +\infty$, in Kuratowski's sense.

Proof. First of all, let us prove the condition (K1). Let $\{t_k\}_{k\in\mathbb{N}}$ be a sequence such that $t_k \to t$, with $t \in [0,T]$, as $k \to +\infty$. Owing to the continuity of p, $p(t_k) \to p(t)$, as $k \to +\infty$, follows. Let $x(t) \in \mathbb{K}(t)$ be fixed and let us observe that, for $i = 1,\dots,m$ and $j = 1,\dots,n$, it results

$$\lim_{k\to+\infty}\left[x_{ij}(t) + \frac{1}{n}p_i(t_k) - \frac{1}{n}p_i(t)\right] = x_{ij}(t) \geq 0.$$

Then there exists an index ν such that for $k > \nu$, $i = 1,\dots,m$ and $j = 1,\dots,n$ we have

$$x_{ij}(t) + \frac{1}{n}p_i(t_k) - \frac{1}{n}p_i(t) \geq 0.$$

Hence we can consider a sequence $\{x(t_k)\}_{k\in\mathbb{N}}$ such that:

- for $k > \nu$, $i = 1,\dots,m$ and $j = 1,\dots,n$

$$x_{ij}(t_k) := x_{ij}(t) + \frac{1}{n}p_i(t_k) - \frac{1}{n}p_i(t), \tag{3.3.1}$$

- and for $k \leq \nu$, $i = 1,\dots,m$ and $j = 1,\dots,n$

$$x_{ij}(t_k) := P_{\mathbb{K}(t_k)}x_{ij}(t),$$

  where $P_{\mathbb{K}(t_k)}$ denotes the Hilbertian projection on $\mathbb{K}(t_k)$.

Obviously if $k \leq \nu$ we get $x(t_k) \in \mathbb{K}(t_k)$, whereas for $k > \nu$ we have

$$x_{ij}(t_k) \geq 0, \quad \forall i = 1,\dots,m, \ \forall j = 1,\dots,n$$

and

$$\begin{aligned}
\sum_{j=1}^{n} x_{ij}(t_k) &= \sum_{j=1}^{n} x_{ij}(t) + p_i(t_k) - p_i(t) \\
&\leq \sum_{j=1}^{n} x_{ij}(t) + p_i(t_k) - \sum_{j=1}^{n} x_{ij}(t) \\
&= p_i(t_k), \quad \forall i = 1,\dots,m, \ \forall j = 1,\dots,n.
\end{aligned}$$

Hence $x(t_k) \in \mathbb{K}(t_k)$, $\forall k \in \mathbb{N}$, and it results $\lim\limits_{k\to+\infty} x(t_k) = x(t)$. Then, the proof of the condition (K1) is completed.

Let us prove, now, the condition (K2). Let $\{t_k\}_{k\in\mathbb{N}}$ be a sequence such that $t_k \to t$, with $t \in [0,T]$, as $k \to +\infty$. Let $\{x(t_k)\}_{k\in\mathbb{N}}$ be a fixed sequence, such that $x(t_k) \in \mathbb{K}(t_k)$,

$\forall k \in \mathbb{N}$, and converging to $x(t)$, as $k \to +\infty$. We need to prove that $x(t) \in \mathbb{K}(t)$. Since $x(t_k) \in \mathbb{K}(t_k)$, $\forall k \in \mathbb{N}$, it results

$$x_{ij}(t_k) \geq 0, \quad \forall i = 1, \ldots, m, \ \forall j = 1, \ldots, n, \ \forall k \in \mathbb{N},$$

$$\sum_{j=1}^{n} x_{ij}(t_k) \leq p_i(t_k), \quad \forall i = 1, \ldots, m, \ \forall k \in \mathbb{N}.$$

Passing to the limit as $n \to +\infty$ and using the continuity assumption on the function p, we obtain

$$x_{ij}(t) \geq 0, \quad \forall i = 1, \ldots, m, \ \forall j = 1, \ldots, n,$$

$$\sum_{j=1}^{n} x_{ij}(t) \leq p_i(t), \quad \forall i = 1, \ldots, m.$$

Then $x(t) \in \mathbb{K}(t)$, and, hence, the proof of the condition (K2) has been completed. □

Lemma 3.3.3. *Let* $\underline{x}, \overline{x} \in C^0\left([0,T], \mathbb{R}^{mn}_+\right)$, $p \in C^0\left([0,T], \mathbb{R}^m_+\right)$, $q \in C^0\left([0,T], \mathbb{R}^n_+\right)$ *and let* $\{t_k\}_{k\in\mathbb{N}}$ *be a sequence such that* $t_k \in [0,T]$, $\forall k \in \mathbb{N}$, *and* $t_k \to t$, *with* $t \in [0,T]$, *as* $k \to +\infty$. *Then the sequence of sets*

$$\mathbb{K}(t_k) = \Bigg\{ x(t_k) \in \mathbb{R}^{mn} : \quad \begin{aligned} & \underline{x}_{ij}(t_k) \leq x_{ij}(t_k) \leq \overline{x}_{ij}(t_k), \quad \forall i = 1, \ldots, m, \ \forall j = 1, \ldots, n, \\ & \sum_{j=1}^{n} x_{ij}(t_k) \leq p_i(t_k), \quad \forall i = 1, \ldots, m, \\ & \sum_{i=1}^{m} x_{ij}(t_k) \leq q_j(t_k), \quad \forall j = 1, \ldots, n \Bigg\} \end{aligned}$$

$\forall k \in \mathbb{N}$, *converges to*

$$\mathbb{K}(t) = \Bigg\{ x(t) \in \mathbb{R}^{mn} : \quad \begin{aligned} & \underline{x}_{ij}(t) \leq x_{ij}(t) \leq \overline{x}_{ij}(t), \quad \forall i = 1, \ldots, m, \ \forall j = 1, \ldots, n, \\ & \sum_{j=1}^{n} x_{ij}(t) \leq p_i(t), \quad \forall i = 1, \ldots, m, \\ & \sum_{i=1}^{m} x_{ij}(t) \leq q_j(t), \quad \forall j = 1, \ldots, n \Bigg\}, \end{aligned}$$

as $k \to +\infty$, *in Kuratowski's sense.*

Proof. In the first part, we prove the condition (K1). Let $\{t_k\}_{k\in\mathbb{N}}$ be a sequence such that $t_k \in [0,T]$, $\forall k \in \mathbb{N}$, and $t_k \to t$, with $t \in [0,T]$, as $k \to +\infty$. By virtue of the continuity of $\underline{x}$, $\overline{x}$, p, q, it follows that $\underline{x}(t_k) \to \underline{x}(t)$, $\overline{x}(t_k) \to \overline{x}(t)$, $p(t_k) \to p(t)$, $q(t_k) \to q(t)$, as $k \to +\infty$, respectively. Let $x(t) \in \mathbb{K}(t)$ be fixed and let us note that, for $i = 1, \dots, m$ and $j = 1, \dots, n$, and if

$$a_{ij}(t_k) = x_{ij}(t) - \underline{x}_{ij}(t_k) + \frac{mp_i(t_k) + nq_j(t_k)}{mn} - \frac{mp_i(t) + nq_j(t)}{mn},$$

it results

$$\lim_{k\to+\infty} a_{ij}(t_k) = x_{ij}(t) - \underline{x}_{ij}(t) \geq 0.$$

As a consequence, there exists an index ν_1 such that for $k > \nu_1$ we get

$$a_{ij}(t_k) \geq 0, \quad \forall i = 1, \dots, m, \ \forall j = 1, \dots, n. \tag{3.3.2}$$

We remark

$$\lim_{k\to+\infty} \left[\frac{1}{m}\sum_{j=1}^{n} q_j(t_k) - \frac{1}{m}\sum_{j=1}^{n} q_j(t) - \epsilon_i(t)\right] = -\epsilon_i(t) \leq 0, \quad \forall i = 1, \dots, m,$$

where ϵ is the production excess function. Then, there exists an index ν_2 such that for $k > \nu_2$ we have

$$\frac{1}{m}\sum_{j=1}^{n} q_j(t_k) - \frac{1}{m}\sum_{j=1}^{n} q_j(t) - \epsilon_i(t) \leq 0, \quad \forall i = 1, \dots, m. \tag{3.3.3}$$

Moreover, we get

$$\lim_{k\to+\infty} \left[\frac{1}{n}\sum_{i=1}^{m} p_i(t_k) - \frac{1}{n}\sum_{i=1}^{m} p_i(t) - \delta_j(t)\right] = -\delta_j(t) \leq 0, \quad \forall j = 1, \dots, n,$$

where δ is the demand excess function. Hence, there exists an index ν_3 such that for $k > \nu_3$ we have

$$\frac{1}{n}\sum_{i=1}^{m} p_i(t_k) - \frac{1}{n}\sum_{i=1}^{m} p_i(t) - \delta_j(t) \leq 0, \quad \forall j = 1, \dots, n. \tag{3.3.4}$$

As a consequence, we can consider a sequence $\{x(t_k)\}_{k\in\mathbb{N}}$ such that:

- for $k > \nu = \max\{\nu_1, \nu_2, \nu_3\}$, $\forall i = 1, \dots, m$, $\forall j = 1, \dots, n$,

$$x_{ij}(t_k) = \underline{x}_{ij}(t_k) + \min\{x_{ij}(t) - \underline{x}_{ij}(t), \overline{x}_{ij}(t_k) - \underline{x}_{ij}(t_k), a_{ij}(t_k)\}, \tag{3.3.5}$$

- for $k \leq \nu$, $\forall i = 1, \dots, m$, $\forall j = 1, \dots, n$,

$$x_{ij}(t_k) = P_{\mathbb{K}(t_k)} x_{ij}(t), \tag{3.3.6}$$

where $P_{\mathbb{K}(t_k)}(\cdot)$ denotes the Hilbertian projection on $\mathbb{K}(t_k)$.

Obviously if $k \leq \nu$, for (3.3.6) we get $x(t_k) \in \mathbb{K}(t_k)$. Instead, for $k > \nu$, since for (3.3.2), $\min\{x_{ij}(t) - \underline{x}_{ij}(t), \overline{x}_{ij}(t_k) - \underline{x}_{ij}(t_k), a_{ij}(t_k)\} \geq 0$, $\forall i = 1, \dots, m$, $\forall j = 1, \dots, n$, we obtain

$$\underline{x}_{ij}(t_k) \leq x_{ij}(t_k), \quad \forall i = 1, \dots, m, \ \forall j = 1, \dots, n.$$

Moreover, since $\min\{x_{ij}(t) - \underline{x}_{ij}(t), \overline{x}_{ij}(t_k) - \underline{x}_{ij}(t_k), a_{ij}(t_k)\} \leq \overline{x}_{ij}(t_k) - \underline{x}_{ij}(t_k)$, $\forall i = 1, \dots, m$, $\forall j = 1, \dots, n$, we have

$$x_{ij}(t_k) \leq \overline{x}_{ij}(t_k), \quad \forall i = 1, \dots, m, \ \forall j = 1, \dots, n.$$

Now, being

$$\begin{aligned} &\min\{x_{ij}(t) - \underline{x}_{ij}(t), \overline{x}_{ij}(t_k) - \underline{x}_{ij}(t_k), a_{ij}(t_k)\} \\ &\quad \leq a_{ij}(t_k) \\ &\quad = x_{ij}(t) - \underline{x}_{ij}(t_k) + \frac{mp_i(t_k) + nq_j(t_k)}{mn} - \frac{mp_i(t) + nq_j(t)}{mn} \\ &\quad \forall i = 1, \dots, m, \ \forall j = 1, \dots, n, \end{aligned}$$

it results

$$x_{ij}(t_k) \leq x_{ij}(t) + \frac{mp_i(t_k) + nq_j(t_k)}{mn} - \frac{mp_i(t) + nq_j(t)}{mn}, \quad \forall i = 1, \dots, m, \ \forall j = 1, \dots, n. \tag{3.3.7}$$

Then, taking into account (3.3.3), we get

$$\begin{aligned} \sum_{j=1}^{n} x_{ij}(t_k) &\leq \sum_{j=1}^{n} x_{ij}(t) + p_i(t_k) + \frac{1}{m}\sum_{j=1}^{n} q_j(t_k) - p_i(t) - \frac{1}{m}\sum_{j=1}^{n} q_j(t) \\ &\leq \sum_{j=1}^{n} x_{ij}(t) + p_i(t_k) - p_i(t) + \epsilon_i(t) \\ &= \sum_{j=1}^{n} x_{ij}(t) + p_i(t_k) - \sum_{j=1}^{n} x_{ij}(t) - \epsilon_i(t) + \epsilon_i(t) \\ &= p_i(t_k), \quad \forall i = 1, \dots, m, \end{aligned}$$

and, making use of (3.3.4), we obtain

$$\begin{aligned}\sum_{i=1}^{m} x_{ij}(t_k) &\leq \sum_{i=1}^{m} x_{ij}(t) + \frac{1}{n}\sum_{i=1}^{m} p_i(t_k) + q_j(t_k) - \frac{1}{n}\sum_{i=1}^{m} p_i(t) + q_j(t)\\ &\leq \sum_{i=1}^{m} x_{ij}(t) + q_j(t_k) - q_j(t) + \delta_j(t)\\ &= \sum_{i=1}^{m} x_{ij}(t) + q_j(t_k) - \sum_{i=1}^{m} x_{ij}(t) - \delta_j(t) + \delta_j(t)\\ &= q_j(t_k), \quad \forall j = 1, \ldots, n.\end{aligned}$$

Hence $x(t_k) \in \mathbb{K}(t_k)$, $\forall k \in \mathbb{N}$, and it results

$$\begin{aligned}\lim_{k\to+\infty} x_{ij}(t_k) &= \underline{x}_{ij}(t) + \min\{x_{ij}(t) - \underline{x}_{ij}(t), \overline{x}_{ij}(t) - \underline{x}_{ij}(t), x_{ij}(t) - \underline{x}_{ij}(t)\}\\ &= \underline{x}_{ij}(t) + x_{ij}(t) - \underline{x}_{ij}(t)\\ &= x_{ij}(t).\end{aligned}$$

Then, the proof of the condition (K1) is completed.

Now let us prove the condition (K2). Let $\{t_k\}_{k\in\mathbb{N}}$ be a sequence such that $t_k \in [0, T]$, $\forall k \in \mathbb{N}$, and $t_k \to t$, with $t \in [0, T]$, as $k \to +\infty$. Let $\{x(t_k)\}_{k\in\mathbb{N}}$ be a sequence, such that $x(t_k) \in \mathbb{K}(t_k)$, $\forall k \in \mathbb{N}$, and converging to $x(t)$, as $k \to +\infty$. We need to prove that $x(t) \in \mathbb{K}(t)$.
Since $x(t_k) \in \mathbb{K}(t_k)$, $\forall k \in \mathbb{N}$, it results

$$\underline{x}_{ij}(t_k) \leq x_{ij}(t_k) \leq \overline{x}_{ij}(t_k), \quad \forall i = 1, \ldots, m,\ \forall j = 1, \ldots, n,\ \forall k \in \mathbb{N},$$

$$\sum_{j=1}^{n} x_{ij}(t_k) \leq p_i(t_k), \quad \forall i = 1, \ldots, m,\ \forall k \in \mathbb{N},$$

$$\sum_{i=1}^{m} x_{ij}(t_k) \leq q_j(t_k), \quad \forall j = 1, \ldots, n,\ \forall k \in \mathbb{N}.$$

Passing to the limit as $n \to +\infty$ and taking into account the continuity assumption on the functions $\underline{x}, \overline{x}, p, q$, we obtain

$$\underline{x}_{ij}(t) \leq x_{ij}(t) \leq \overline{x}_{ij}(t), \quad \forall i = 1, \ldots, m,\ \forall j = 1, \ldots, n,$$

$$\sum_{j=1}^{n} x_{ij}(t) \leq p_i(t), \quad \forall i = 1, \ldots, m,$$

$$\sum_{i=1}^{n} x_{ij}(t) \leq q_j(t), \quad \forall j = 1, \ldots, n.$$

As a consequence $x(t) \in \mathbb{K}(t)$, and, hence, the condition (K2) is achieved. □

3.3.2 Continuity Theorems for Equilibrium Solutions

Now, in this section, we give a result of continuity for the solution of the evolutionary variational inequalities (2.2.8), (2.3.5), (2.4.8) which express the oligopolistic market equilibrium conditions.

In [3], for the first time, the author proved, under the only continuity assumption on the data, that the solution of the dynamic traffic equilibrium problem is continuous when the cost function is linear and strongly monotone. Then, this result has been generalized for an evolutionary variational inequality associated to a nonlinear operator belonging in L^2 (see [4–6]). In [8], the authors show the continuity of solution to a parametric variational inequality in a Banach space.

Applying Theorem 4.2 in [8] to the dynamic oligopolistic market equilibrium model in presence of production excesses and taking into account Lemmas 3.3.1, 3.3.2, 3.3.3, we obtain the following result:

Theorem 3.3.1. *Let us assume that the production function p, the demand function q and the capacity constraints $\underline{x}$ and $\overline{x}$ are continuous on $[0,T]$. Moreover, let us assume that the function $-\nabla_D v$ is a strictly pseudomonotone and continuous on $[0,T]$. Then, the unique dynamic market equilibrium distribution in presence of both production and demand excesses $x^* \in \mathbb{K}$ is continuous on $[0,T]$.*

Moreover, in [7, 10], another important regularity property for the solutions to the dynamic oligopolistic market equilibrium problem has been proved. Infact it was possible to prove that the solution is Lipschitz continuous, namely $\forall\, t_1 \neq t_2$, there exists a constant $L > 0$ such that :

$$\|x^*(t_2) - x^*(t_1)\| \leq L\,|t_2 - t_1|\,.$$

Finally, in [12] some sensitivity results have been obtained. In this way it is possible to see how the solution changes when the profit function is perturbed.

Chapter 4

The inverse problem

4.1 Historical development of the inverse variational inequalities

In this chapter we leave the analysis of the problem from a producer's point of view. Analogously to the case of the time-dependent spatial price equilibrium control problem studied in [96], here we want to draw our attention to the policy-maker's point of view who influences the model by regulating the exportation through the regulation of taxes and incentives. The resulting optimization problem, where we want to investigate the optimal tax regulation, is investigated through inverse variational inequalities.
Inverse variational inequalities can be considered as a special case of general variational inequalities introduced in [86] and can be used to model various control problems. In finite-dimensional problems an inverse variational inequality problem formally consists in finding $x^* \in \mathbb{R}^n$ such that

$$f(x^*) \in \Omega : \qquad \langle x - f(x^*), x^* \rangle \leq 0, \ \forall x \in \Omega, \tag{4.1.1}$$

where Ω is a nonempty subset of $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$. Unlike the classical variational inequalities, in such a problem the equilibrium state $f(x^*)$ must belong to the feasible set, whereas the feasibility is not required for the variable x^*. It is simple to find the analogous formulation for infinite dimensional problems.

Only recently the strict connection between the classical variational inequalities and inverse variational inequalities has been unveiled and this enables to exploit all the powerful tools of evolutionary variational inequalities that allow us to treat completely this problem and analyze the questions about the existence, the regularity and the computation of solution.

The authors of [54], for the first time, used the inverse variational inequality theory to study a general network economic equilibrium problem in the static case. Later, in [104], the power price problem is discussed in both the static and the evolutionary case and in both cases the optimal price is characterized as a solution of an inverse static or evolutionary variational inequality. Recently, in [96] the author studied an optimal control perspective on the evolutionary time-dependent spatial price equilibrium problem and the equivalence with an appropriate inverse variational inequality is proved.

4.2 The case with no excesses

In this section we want to consider the case in which all of the amounts of the commodity available are sold out, namely the quantity of commodity produced by the firms is sold out and the demand of commodity demanded by the demand markets is fully satisfied under the condition that the commodity distribution is limited as a consequence of the evident limited physical transportation of commodity (see [9]). Moreover, we prove the equivalence of the inverse variational inequality, which allows the system to control the commodity exportations by means of the imposition of taxes or incentives with an appropriate evolutionary variational inequality.
Let us consider the model presented in section 2.2. Moreover, let $\eta_{ij}(t)$ be supply or resource tax imposed on supply market P_i for the transaction with the demand market Q_j, at the time $t \in [0, T]$. Let $\lambda_{ij}(t)$ be the incentive pay imposed on supply market P_i for the transaction with the demand market Q_j, at the same time $t \in [0, T]$. Finally, let $h_{ij}(t)$ be the difference between the supply tax and the incentive pay for the transaction with the demand market Q_j at the time $t \in [0, T]$, namely, $h_{ij}(t) = \eta_{ij}(t) - \lambda_{ij}(t)$.

For technical reasons we suppose that

$$\eta \in L^2([0, T], \mathbb{R}^{mn}_+), \quad \lambda \in L^2([0, T], \mathbb{R}^{mn}_+).$$

As a consequence, we have

$$h \in L^2([0, T], \mathbb{R}^{mn}_+).$$

The profit $v_i(t, x(t))$, $i = 1, \dots, m$, of the firm P_i at the time $t \in [0, T]$ is, then,

$$v_i(t, x(t)) = \sum_{j=1}^{n} d_j(t, x(t))x_{ij}(t) - f_i(t, x(t)) - \sum_{j=1}^{n} c_{ij}(t, x(t))x_{ij}(t) - \sum_{j=1}^{n} h_{ij}(t)x_{ij}(t).$$

Like in section 2.2, let us consider the dynamic oligopolistic market, in which the m firms supply the commodity in a noncooperative fashion, each one trying to maximize its own profit and to minimize the taxes at the time $t \in [0, T]$. We seek to determine a nonnegative commodity distribution matrix-function x^* for which the m firms and the n demand markets will be in a state of equilibrium according to the dynamic Cournot-Nash principle that, also in this case, is not influenced by the presence of taxes.

Let $\mathbb{K}$ be the feasible set:

$$\mathbb{K} = \left\{ \quad x \in L^2([0,T], \mathbb{R}^{mn}_+): \quad \underline{x}_{ij}(t) \leq x_{ij}(t) \leq \overline{x}_{ij}(t), \right.$$
$$\left. \forall i = 1,\ldots,m, \ \forall j = 1,\ldots,n, \ \text{a.e. in } [0,T] \right\}.$$

Definition 4.2.1. $x^* \in \mathbb{K}$ is a dynamic oligopolistic market equilibrium if and only if for each $i = 1,\ldots,m$ and a.e. in $[0,T]$ we have

$$v_i(t, x^*(t)) \geq v_i(t, x_i(t), \hat{x}_i^*(t)), \ \text{a.e. in} [0,T], \ \text{where} \tag{4.2.1}$$

$x_i(t) = (x_{i1}(t), \ldots, x_{in}(t))$, a.e. in $[0,T]$ and $\hat{x}_i^*(t) = (x_1^*(t), \ldots, x_{i-1}^*(t), x_{i+1}^*(t), \ldots, x_m^*(t))$.

Like we already know from section 2.2, the previous definition is equivalent to an evolutionary variational inequality as the following result shows.

Theorem 4.2.1. $x^* \in \mathbb{K}$ *is a dynamic oligopolistic market equilibrium according to Definition* 4.2.1 *if and only if it satisfies the evolutionary variational inequality*

$$\int_0^T \sum_{i=1}^m \sum_{j=1}^n \left(-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} \right) (x_{ij}(t) - x_{ij}^*(t)) dt \geq 0 \qquad \forall x \in \mathbb{K}. \tag{4.2.2}$$

Now we change the point of view in the analysis of the problem by introducing an optimal control problem in which the variable h represents the difference between the supply tax η and the incentive pay λ for the transactions. As a consequence, the term h, previously considered as a fixed parameter, is now considered a variable. This represents the typical policy-maker's point of view.

In this perspective, it is possible to control the resource exploitations $x(h(t))$ at supply markets by adjusting taxes $h(t)$. Namely, the tax adjustment becomes an efficient mean of regulating exportation. More precisely, if the policy-maker is concerned with the restriction of exportation and, consequently, of production of a certain commodity, then the government will impose higher taxes, whereas if the policy-maker aims to encourage exportation of a certain commodity, the government will impose subsidies.

We introduce the function of regulatory taxes $x(h) = x(t,h)$, so $x : [0,T] \times \mathbb{R}^{mn} \to \mathbb{R}^{mn}$, since $h(t) \in \mathbb{R}^{mn}, \ \forall t \in [0,T]$.

We also suppose that $x(t,h)$ is a Carathéodory function and we require also that there exists $\gamma(t) \in L^2([0,T])$ such that, a.e. in $[0,T]$

$$\|x(t, h(t))\|_{mn} \leq \gamma(t) + \|h(t)\|_{mn}. \tag{4.2.3}$$

Therefore,

$$x : [0,T] \times L^2([0,T], \mathbb{R}^{mn}) \to L^2([0,T], \mathbb{R}^{mn}),$$

which assigns to each regulatory tax $h(t)$ the exportation vector $x(h(t))$.

We now introduce the set of feasible states

$$\Omega = \Big\{\omega \in L^2([0,T], \mathbb{R}^{mn}) : \quad \underline{x}_{ij}(t) \le \omega_{ij}(t) \le \overline{x}_{ij}(t),$$
$$\forall i = 1,\dots,m,\ \forall j = 1,\dots,n,\ \text{a.e. in } [0,T]\Big\},$$

and define the optimal regulatory tax as follows.

Definition 4.2.2. A vector $h^* \in L^2([0,T], \mathbb{R}^{mn})$ is an optimal regulatory tax if $x(t,h^*) \in \Omega$ and for $i = 1,\dots,m,\ j = 1,\dots,n$ and a.e. in $[0,T]$ the following conditions hold:

$$x_{ij}(t,h^*(t)) = \underline{x}_{ij}(t), \quad \Rightarrow \quad h^*_{ij}(t) \le 0, \tag{4.2.4}$$
$$\underline{x}_{ij}(t) < x_{ij}(t,h^*(t)) < \overline{x}_{ij}(t), \quad \Rightarrow \quad h^*_{ij}(t) = 0, \tag{4.2.5}$$
$$x_{ij}(t,h^*(t)) = \overline{x}_{ij}(t), \quad \Rightarrow \quad h^*_{ij}(t) \ge 0. \tag{4.2.6}$$

Definition 4.2.2 must be interpreted as follows: first of all, the optimal regulatory tax h^* is such that the correspondig state $x(t,h^*)$ has to satisfy capacity constraints, namely $x(t,h^*) \in \Omega$. Moreover, if one requires that $x_{ij}(t,h^*(t)) = \underline{x}_{ij}(t)$, then it means to encourage the exportations, namely the optimal choice is that taxes must be less than or equal to the incentive pays. If one requires that $x_{ij}(t,h^*(t)) = \overline{x}_{ij}(t)$, then the exportations must be reduced, hence taxes must be greater than or equal to the incentive pays. Finally, if $\underline{x}_{ij}(t) < x_{ij}(t,h^*(t)) < \overline{x}_{ij}(t)$ is satisfied, taxes equal incentive pays.

The following theorem shows the inverse variational inequality formulation of the optimal equilibrium control problem.

Theorem 4.2.2. *A regulatory tax $h^* \in L^2([0,T], \mathbb{R}^{mn})$ is an optimal regulatory tax according to Definition 4.2.2 if and only if it solves the inverse variational inequality:*

$$x(t,h^*) \in \Omega : \quad \int_0^T \sum_{i=1}^m \sum_{j=1}^n (\omega_{ij}(t) - x_{ij}(t,h^*(t)))\, h^*_{ij}(t)dt \le 0, \quad \forall \omega \in \Omega. \tag{4.2.7}$$

Proof. Let h^* be an optimal regulatory tax according to Definition 4.2.2, let $i \in \{1,\dots,m\}$, $j \in \{1,\dots,n\}$ and $\omega \in \Omega$, namely $\underline{x}_{ij}(t) \le \omega_{ij}(t) \le \overline{x}_{ij}(t)$ a.e. in $[0,T]$.

1. if $x_{ij}(t,h^*(t)) = \underline{x}_{ij}(t)$, a.e. in $[0,T]$, by (4.2.4) we get that $h^*_{ij}(t) \le 0$, a.e. in $[0,T]$, and, hence, $(\omega_{ij}(t) - x_{ij}(t,h^*(t)))\, h^*_{ij}(t) \le 0$, a.e. in $[0,T]$;

2. if $\underline{x}_{ij}(t) < x_{ij}(t,h^*(t)) < \overline{x}_{ij}(t)$, a.e. in $[0,T]$, by (4.2.4) we get that $h^*_{ij}(t) = 0$; a.e. in $[0,T]$, and, hence, $(\omega_{ij}(t) - x_{ij}(t,h^*(t)))\, h^*_{ij}(t) = 0$, a.e. in $[0,T]$;

3. if $x_{ij}(t,h^*(t)) = \overline{x}_{ij}(t),$ a.e. in $[0,T]$, by (4.2.4) we get that $h^*_{ij}(t) \geq 0,$ a.e. in $[0,T]$, and, hence, $(\omega_{ij}(t) - x_{ij}(t,h^*(t)))\, h^*_{ij}(t) \leq 0,$ a.e. in $[0,T]$.

So, $\forall i \in \{1,\dots,m\}$, $\forall j \in \{1,\dots,n\}$ and $\forall \omega \in \Omega$, we get

$$(\omega_{ij}(t) - x_{ij}(t,h^*(t)))\, h^*_{ij}(t) \leq 0, \quad \text{a.e. in } [0,T].$$

By summing over $i \in \{1,\dots,m\}, j \in \{1,\dots,n\}$ and integrating on $[0,T]$, we get the inverse variational inequality (4.2.7).

Converserly, we assume, now, that h^* satisfies the inverse variational inequality (4.2.7).

If we fix $i \in \{1,\dots,m\}$, $j \in \{1,\dots,n\}$ and set $\omega_{hk}(t) = x_{ij}(t,h^*(t)),$ a.e. in $[0,T]$, and for all $h \neq i$, $k \neq j$ from (4.2.7), we get

$$\int_0^T (\omega_{ij}(t) - x_{ij}(t,h^*(t)))\, h^*_{ij}(t)dt \leq 0, \quad \forall \omega_{ij}(t) \in L^2([0,T],\mathbb{R}): \qquad (4.2.8)$$
$$\underline{x}_{ij}(t) \leq \omega_{ij}(t) \leq \overline{x}_{ij}(t).$$

Let us prove that if $x_{ij}(t,h^*(t)) = \underline{x}_{ij}(t)$ a.e. in $[0,T]$, then, $h^*_{ij}(t) \leq 0$ a.e. in $[0,T]$.

By contradiction, we suppose that there exists a set $E \subseteq [0,T]$ with positive measure such that $h^*_{ij}(t) > 0,$ a.e. in E. If we choose

$$\omega_{ij}(t) = \begin{cases} \overline{x}_{ij}(t) & t \in E \\ x_{ij}(t,h^*(t)) & t \in [0,T] \setminus E \end{cases}$$

we have

$$\int_0^T (\omega_{ij}(t) - x_{ij}(t,h^*(t)))\, h^*_{ij}(t)dt = \int_E (\overline{x}_{ij}(t) - \underline{x}_{ij}(t)) h^*_{ij}(t)dt > 0,$$

which contradicts (4.2.8).

Let us prove, now, that if $x_{ij}(t,h^*(t)) = \overline{x}_{ij}(t),$ a.e. in $[0,T]$, then, $h^*_{ij}(t) \geq 0,$ a.e. in $[0,T]$.

By contradiction, we suppose that there exists a set $F \subseteq [0,T]$ with positive measure such that $h^*_{ij}(t) < 0,$ a.e. in F. If we choose

$$\omega_{ij}(t) = \begin{cases} \underline{x}_{ij}(t) & t \in F \\ x_{ij}(t,h^*(t)) & t \in [0,T] \setminus F \end{cases}$$

we have

$$\int_0^T (\omega_{ij}(t) - x_{ij}(t,h^*(t)))\, h^*_{ij}(t)dt = \int_F (\underline{x}_{ij}(t) - \overline{x}_{ij}(t)) h^*_{ij}(t)dt > 0,$$

which contradicts (4.2.8).

If $\underline{x}_{ij}(t) < x_{ij}(t,h^*(t)) < \overline{x}_{ij}(t),$ a.e. in $[0,T]$, by using the same technique as in the previous cases, it can be easily proved that $h^*(t)$ cannot be either negative or positive on any set with positive measure.

□

It is also possible to provide a classical variational inequality formulation of the optimal equilibrium control problem. The advantage of such a standard formulation lies in the fact we can have disposal of all the theoretical and numerical advances in the theory of variational inequalities to treat fully the problem. If we set

$$W = L^2([0,T], \mathbb{R}^{mn}) \times \Omega,\ F : [0,T] \times W \to L^2([0,T], \mathbb{R}^{2mn}),$$

$$z(t) = \begin{pmatrix} h(t) \\ \omega(t) \end{pmatrix},$$

and

$$F(t, z(t)) = \begin{pmatrix} \omega(t) - x(t, h(t)) \\ -h(t) \end{pmatrix}.$$

it is possible to prove that evolutionary inverse variational inequality problem (4.2.7) is equivalent to an evolutionary variational inequality, as you can see with the following result (see [104, Th. 4.8]):

Theorem 4.2.3. *The evolutionary inverse variational inequality problem* (4.2.7) *is equivalent to the evolutionary variational inequality*

$$z^* \in W : \quad \int_0^T \sum_{l=1}^{2m} \sum_{j=1}^{n} F_{lj}(t, z^*(t)) \left(z_{lj}(t) - z^*_{lj}(t)\right) dt \geq 0, \quad \forall z \in W. \tag{4.2.9}$$

Proof. If (4.2.9) holds true, then we can get easily that

$$z^* = (h^*, \omega^*)^T \in W,$$

$$\int_0^T \left(\sum_{i=1}^{m} \sum_{j=1}^{n} (\omega^*_{ij}(t) - x_{ij}(t, h^*(t)))(h_{ij}(t) - h^*_{ij}(t)) \right) dt \tag{4.2.10}$$
$$- \int_0^T \left(\sum_{i=1}^{m} \sum_{j=1}^{n} h^*_{ij}(t)(\omega_{ij}(t) - \omega^*_{ij}(t)) \right) dt \geq 0$$

holds for all

$$z = (h, \omega)^T \in W.$$

By setting $h(t) = h^*(t) - \omega^*(t) + x(t, h^*(t))$ and $\omega(t) = \omega^*(t)$ in (4.2.10), we get

$$- \int_0^T \sum_{i=1}^{m} \sum_{j=1}^{n} (\omega^*_{ij}(t) - x_{ij}(t, h^*(t)))^2 dt \geq 0,$$

hence $x(t, h^*(t)) = \omega^*(t)$, a.e. in $[0, T]$. Thus $x(t, h^*(t)) \in \Omega$ and (4.2.10) indicates that (4.2.7) holds.

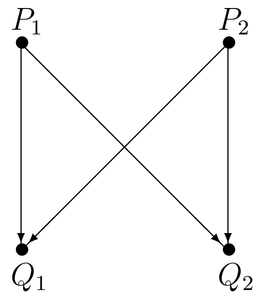

Figure 4.1: Network structure of the numerical dynamic spatial oligopoly problem.

Converserly, if $h^* \in L^2([0,T], \mathbb{R}^{mn})$ is a solution of (4.2.7), then $z^* = (h^*, x(t,h^*))^T \in W$ is a solution of (4.2.9) in fact,

$$\underbrace{\int_0^T \sum_{i=1}^m \sum_{j=1}^n (x_{ij}(t,h^*(t)) - x_{ij}(t,h^*(t)))(h_{ij}(t) - h^*_{ij}(t))dt}_{=0}$$
$$\underbrace{-\int_0^T \sum_{i=1}^m \sum_{j=1}^n h^*_{ij}(t)(\omega_{ij}(t) - x_{ij}(t,h^*(t)))dt}_{\geq 0} \geq 0.$$

□

It is worth noting that W is a closed, convex and not bounded subset of $L^2([0,T], \mathbb{R}^{2mn})$.

4.2.1 A numerical example

This section is devoted to provide a numerical example of the theoretical achievements presented.

Let us consider two firms and two demand markets, as in Figure 4.1. Let $\underline{x}, \overline{x} \in L^2([0,1], \mathbb{R}^4)$ be the capacity constraints such that, a.e. in $[0,1]$,

$$\underline{x}(t) = \begin{pmatrix} 0 & \frac{1}{100}t \\ 0 & \frac{1}{100}t \end{pmatrix},$$

$$\overline{x}(t) = \begin{pmatrix} 100t & 200t \\ 100t & 200t \end{pmatrix}.$$

As a consequence, the feasible set is

$$\mathbb{K} = \left\{ x \in L^2([0,1], \mathbb{R}^4) : \quad \underline{x}_{ij}(t) \leq x_{ij}(t) \leq \overline{x}_{ij}(t), \ \forall i = 1, 2, \ \forall j = 1, 2, \ \text{a.e. in } [0,1] \right\}.$$

The set of feasible states is

$$\Omega = \left\{ \omega \in L^2([0,T], \mathbb{R}^2) : \quad \underline{x}_{ij}(t) \leq \omega_{ij}(t) \leq \overline{x}_{ij}(t), \ \forall i = 1, 2, \ \forall j = 1, 2, \ \text{a.e. in } [0,T] \right\}.$$

Let us consider the profit function $v \in L^2([0,1] \times L^2([0,1], \mathbb{R}^4), \mathbb{R}^2)$ defined by

$$\begin{aligned} v_1(t, x(t)) &= 4x_{11}^2(t) + 3x_{12}(t)^2 + \alpha(t)x_{11}(t) - 2x_{11}(t)x_{22}(t) \\ &\quad -2x_{21}(t)x_{12}(t) - 4h_{11}(t)x_{11}(t) - 4h_{12}(t)x_{12}(t), \\ v_2(t, x(t)) &= 2x_{21}^2(t) + 3x_{22}(t)^2 + \beta(t)x_{21}(t) - 2x_{21}(t)x_{12}(t) \\ &\quad -4x_{11}(t)x_{22}(t) - 4h_{21}(t)x_{21}(t) - 2h_{22}(t)x_{22}(t). \end{aligned}$$

Then, the operator $\nabla_D v \in L^2([0,1] \times L^2([0,1], \mathbb{R}^4), \mathbb{R}^4)$ is given by

$$\nabla_D v(t, x(t)) = \begin{pmatrix} 8x_{11}(t) - 2x_{22}(t) + \alpha(t) - 4h_{11}(t) & 6x_{12}(t) - 2x_{21}(t) - 4h_{12}(t) \\ 4x_{21}(t) - 2x_{12}(t) + \beta(t) - 4h_{21}(t) & 6x_{22}(t) - 4x_{11}(t) - 2h_{22}(t) \end{pmatrix}.$$

The dynamic oligopolistic market equilibrium distribution is the solution to the evolutionary variational inequality:

$$\int_0^1 \sum_{i=1}^2 \sum_{j=1}^2 -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} (x_{ij}(t) - x_{ij}^*(t)) dt \geq 0, \quad \forall x \in \mathbb{K}. \tag{4.2.11}$$

In order to compute the solution to (4.2.11) we make use of the direct method (see [29, 34, 70]). We consider the following system

$$\begin{cases} -8x_{11}^*(t) + 2x_{22}^*(t) - \alpha(t) + 4h_{11}(t) = 0, \\ -6x_{12}^*(t) + 2x_{21}^*(t) + 4h_{12}(t) = 0, \\ -4x_{21}^*(t) + 2x_{12}^*(t) - \beta(t) + 4h_{21}(t) = 0, \\ -6x_{22}^*(t) + 4x_{11}^*(t) + 2h_{22}(t) = 0, \\ \underline{x}_{ij}(t) \leq x_{ij}(t) \leq \overline{x}_{ij}(t), \quad \forall i = 1, 2, \ \forall j = 1, 2, \end{cases}$$

and we get the following solution, a.e. in $[0, 1]$,

$$x^*(t) = \begin{pmatrix} \frac{2h_{22}(t)+12h_{11}(t)-3\alpha(t)}{20} & \frac{8h_{12}(t)+4h_{21}(t)-\beta(t)}{10} \\ \frac{4h_{12}(t)+12h_{21}(t)-3\beta(t)}{10} & \frac{4h_{22}(t)+4h_{11}(t)-\alpha(t)}{10} \end{pmatrix}.$$

For the inverse problem, now we have to solve the following variational inequality

$$\int_0^T \left(\sum_{i=1}^m \sum_{j=1}^n (\omega_{ij}^*(t) - x_{ij}(t, h^*(t)))(h_{ij}(t) - h_{ij}^*(t)) - \sum_{i=1}^m \sum_{j=1}^n h_{ij}^*(t)(\omega_{ij}(t) - \omega_{ij}^*(t)) \right) dt \geq 0$$
$$\forall (h, \omega) \in L^2([0,T], \mathbb{R}^{mn}) \times \Omega. \tag{4.2.12}$$

For $\omega_{ij}(t) = \omega_{ij}^*(t),\ \forall i = 1, \ldots, m,\ \forall j = 1, \ldots, n$, a.e. in $[0,1]$, we can consider the following system

$$\begin{cases} 20\omega_{11}^*(t) - 2h_{22}^*(t) - 12h_{11}^*(t) + 3\alpha(t) = 0, \\ 10\omega_{12}^*(t) - 8h_{12}^*(t) - 4h_{21}^*(t) + \beta(t) = 0, \\ 10\omega_{21}^*(t) - 4h_{12}(t) - 12h_{21}(t) + 3\beta(t) = 0, \\ 10\omega_{22}^*(t) - 4h_{22}(t) - 4h_{11}(t) + \alpha(t) = 0, \end{cases}$$

and we get the following solution, a.e. in $[0,1]$,

$$h^*(t) = \begin{pmatrix} \frac{8\omega_{11}^*(t) - 2\omega_{22}^*(t) + \alpha(t)}{4} & \frac{3\omega_{12}^*(t) - \omega_{21}^*(t)}{2} \\ \frac{4\omega_{21}^*(t) - 2\omega_{12}^*(t) + \beta(t)}{4} & -2\omega_{11}^*(t) + 3\omega_{22}^*(t) \end{pmatrix}.$$

1) In order to obtain the solution to (4.2.12), we consider, in first analysis, the case

$$\omega^*(t) = \begin{pmatrix} 100t & 200t \\ 100t & 200t \end{pmatrix}.$$

As a consequence of the direct method, it must be

$$h_{11}^*(t) > 0,\ h_{12}^*(t) > 0,\ h_{21}^*(t) > 0,\ h_{22}^*(t) > 0.$$

This is true if and only if $\alpha(t) > -400t$ and $\beta(t) > 0$. In this case the optimal regulatory tax is

$$h^*(t) = \begin{pmatrix} \frac{\alpha(t)+100t}{4} & 250t \\ \frac{\beta(t)}{4} & 400t \end{pmatrix},$$

from which we get that the optimal commodity distribution is

$$x^*(t) = \begin{pmatrix} 100t & 200t \\ 100t & 200t \end{pmatrix}.$$

2) Let us study, now, the case

$$\omega^*(t) = \begin{pmatrix} 0 & 200t \\ 100t & 200t \end{pmatrix}.$$

As a consequence of the direct method, it must be

$$h^*_{11}(t) < 0,\ h^*_{12}(t) > 0,\ h^*_{21}(t) > 0,\ h^*_{22}(t) > 0.$$

This is true if and only if $\alpha(t) < 400t$ and $\beta(t) > 0$. In this case the optimal regulatory tax is

$$h^*(t) = \begin{pmatrix} \frac{-400t+\alpha(t)}{4} & 250t \\ \frac{\beta(t)}{4} & 600t \end{pmatrix},$$

from which we get that the optimal commodity distribution is

$$x^*(t) = \begin{pmatrix} 0 & 200t \\ 100t & 200t \end{pmatrix}.$$

3) Let us study, now, the case

$$\omega^*(t) = \begin{pmatrix} 0 & \frac{1}{100}t \\ 100t & 200t \end{pmatrix}.$$

As a consequence of the direct method, it must be

$$h^*_{11}(t) < 0,\ h^*_{12}(t) < 0,\ h^*_{21}(t) > 0,\ h^*_{22}(t) > 0.$$

This is true if and only if $\alpha(t) < 400t$ and $\beta(t) < -\frac{39998}{100}t$. In this case the optimal regulatory tax is

$$h^*(t) = \begin{pmatrix} \frac{-400t+\alpha(t)}{4} & \frac{-9997t}{200} \\ \frac{39998t+100\beta(t)}{400} & 600t \end{pmatrix},$$

from which we get that the optimal commodity distribution is

$$x^*(t) = \begin{pmatrix} 0 & \frac{1}{100}t \\ 100t & 200t \end{pmatrix}.$$

4) Let us study, now, the case

$$\omega^*(t) = \begin{pmatrix} 100t & \frac{1}{100}t \\ 100t & 200t \end{pmatrix}.$$

As a consequence of the direct method, it must be

$$h^*_{11}(t) > 0,\ h^*_{12}(t) < 0,\ h^*_{21}(t) > 0,\ h^*_{22}(t) > 0.$$

This is true if and only if $\alpha(t) > -400t$ and $\beta(t) < -\frac{39998}{100}t$. In this case the optimal regulatory tax is

$$h^*(t) = \begin{pmatrix} \frac{\alpha(t)+100t}{4} & \frac{-9997t}{200} \\ \frac{39998t+100\beta(t)}{400} & 400t \end{pmatrix},$$

from which we get that the optimal commodity distribution is

$$x^*(t) = \begin{pmatrix} 100t & \frac{1}{100}t \\ 100t & 200t \end{pmatrix}.$$

5) Let us study, now, the case

$$\omega^*(t) = \begin{pmatrix} 100t & 200t \\ 0 & 200t \end{pmatrix}.$$

As a consequence of the direct method, it must be

$$h^*_{11}(t) > 0,\ h^*_{12}(t) > 0,\ h^*_{21}(t) < 0,\ h^*_{22}(t) > 0.$$

This is true if and only if $\alpha(t) > -400t$ and $\beta(t) < 400t$. In this case the optimal regulatory tax is

$$h^*(t) = \begin{pmatrix} \frac{\alpha(t)+100t}{4} & 300t \\ \frac{-400t+\beta(t)}{4} & 400t \end{pmatrix},$$

from which we get that the optimal commodity distribution is

$$x^*(t) = \begin{pmatrix} 100t & 200t \\ 0 & 200t \end{pmatrix}.$$

6) Let us study, now, the case

$$\omega^*(t) = \begin{pmatrix} 0 & 200t \\ 0 & 200t \end{pmatrix}.$$

As a consequence of the direct method, it must be

$$h^*_{11}(t) < 0,\ h^*_{12}(t) > 0,\ h^*_{21}(t) < 0,\ h^*_{22}(t) > 0.$$

This is true if and only if $\alpha(t) < 400t$ and $\beta(t) < 400t$. In this case the optimal regulatory tax is

$$h^*(t) = \begin{pmatrix} \frac{\alpha(t)-400t}{4} & 300t \\ \frac{-400t+\beta(t)}{4} & 600t \end{pmatrix},$$

from which we get that the optimal commodity distribution is

$$x^*(t) = \begin{pmatrix} 0 & 200t \\ 0 & 200t \end{pmatrix}.$$

7) Let us study, now, the case

$$\omega^*(t) = \begin{pmatrix} 100t & 200t \\ 100t & \frac{1}{100}t \end{pmatrix}.$$

As a consequence of the direct method, it must be

$$h^*_{11}(t) > 0,\ h^*_{12}(t) > 0,\ h^*_{21}(t) > 0,\ h^*_{22}(t) < 0.$$

This is true if and only if $\alpha(t) > -\frac{79998}{100}t$ and $\beta(t) > 0$. In this case the optimal regulatory tax is

$$h^*(t) = \begin{pmatrix} \frac{79998t+\alpha(t)}{400} & 250t \\ \frac{\beta(t)}{4} & -\frac{19997}{100}t \end{pmatrix},$$

from which we get that the optimal commodity distribution is

$$x^*(t) = \begin{pmatrix} 100t & 200t \\ 100t & \frac{1}{100}t \end{pmatrix}.$$

8) Let us study, now, the case

$$\omega^*(t) = \begin{pmatrix} 100t & \frac{1}{100}t \\ 100t & \frac{1}{100}t \end{pmatrix}.$$

As a consequence of the direct method, it must be

$$h^*_{11}(t) > 0,\ h^*_{12}(t) < 0,\ h^*_{21}(t) > 0,\ h^*_{22}(t) < 0.$$

This is true if and only if $\alpha(t) > -\frac{79998}{100}t$ and $\beta(t) < -\frac{39998}{100}t$. In this case the optimal regulatory tax is

$$h^*(t) = \begin{pmatrix} \frac{79998t+\alpha(t)}{400} & -\frac{9997}{200}t \\ \frac{100\beta(t)+39998t}{400} & -\frac{19997}{100}t \end{pmatrix},$$

from which we get that the optimal commodity distribution is

$$x^*(t) = \begin{pmatrix} 100t & \frac{1}{100}t \\ 100t & \frac{1}{100}t \end{pmatrix}.$$

9) Let us study, now, the case

$$\omega^*(t) = \begin{pmatrix} 100t & 200t \\ 0 & \frac{1}{100}t \end{pmatrix}.$$

As a consequence of the direct method, it must be

$$h^*_{11}(t) > 0,\ h^*_{12}(t) > 0,\ h^*_{21}(t) < 0,\ h^*_{22}(t) < 0.$$

This is true if and only if $\alpha(t) > -79998t$ and $\beta(t) < 400t$. In this case the optimal regulatory tax is

$$h^*(t) = \begin{pmatrix} \frac{79998t+\alpha(t)}{400} & 300t \\ \frac{-400t\beta(t)}{4} & -\frac{19997}{100}t \end{pmatrix},$$

from which we get that the optimal commodity distribution is

$$x^*(t) = \begin{pmatrix} 100t & 200t \\ 0 & \frac{1}{100}t \end{pmatrix}.$$

Other options are not allowed for the condition of the direct method. Namely the cases in which ω^*_{12} and ω^*_{21} are both minimal and the cases in which ω^*_{11} and ω^*_{22} are both minimal, are not allowed.

4.3 The case with both production and demand excesses

In this section we want to consider the case in which not all of the amounts of the commodity available are sold out and the demand of commodity demanded by the demand markets is not fully satisfied under the condition that the commodity distribution is limited, namely we allow production and demand excesses. In this section we prove the equivalence of the inverse variational inequality, which allows the system to control the commodity exportations by means of the imposition of taxes or incentives with an appropriate evolutionary variational inequality.

Let us consider the model presented in section 2.4 and the taxes and incentives considered in section 4.2.
The profit $v_i(t, x(t))$, $i = 1, \dots, m$, of the firm P_i at the time $t \in [0, T]$ is, then,

$$v_i(t, x(t)) = \sum_{j=1}^{n} d_j(t, x(t))x_{ij}(t) - f_i(t, x(t)) - \sum_{j=1}^{n} c_{ij}(t, x(t))x_{ij}(t) - \sum_{j=1}^{n} h_{ij}(t)x_{ij}(t).$$

Like in section 2.4, let us consider the dynamic oligopolistic market, in which the m firms supply the commodity in a noncooperative fashion, each one trying to maximize its own profit and to minimize the taxes at the time $t \in [0, T]$. We seek to determine a nonnegative commodity distribution matrix-function x^* for which the m firms and the n demand markets will be in a state of equilibrium according to the dynamic Cournot-Nash principle that, also in this case, is not influenced by the presence of taxes and excesses.

Let $\mathbb{K}$ the feasible set:

$$\begin{aligned}\mathbb{K} = \Bigg\{ & x \in L^2([0,T], \mathbb{R}^{mn}_+) : \\ & \underline{x}_{ij}(t) \le x_{ij}(t) \le \overline{x}_{ij}(t), \quad \forall i = 1, \dots, m, \ \forall j = 1, \dots, n, \ \text{a.e. in } [0,T], \\ & \sum_{j=1}^{n} x_{ij}(t) \le p_i(t), \quad \forall i = 1, \dots, m, \ \text{a.e. in } [0,T], \\ & \sum_{i=1}^{m} x_{ij}(t) \le q_j(t), \quad \forall j = 1, \dots, n, \ \text{a.e. in } [0,T] \Bigg\}.\end{aligned}$$

Definition 4.3.1. $x^* \in \mathbb{K}$ is a dynamic oligopolistic market equilibrium if and only if for each $i = 1, \dots, m$ and a.e. in $[0, T]$ we have

$$v_i(t, x^*(t)) \ge v_i(t, x_i(t), \hat{x}^*_i(t)), \ \text{a.e. in } [0,T], \ \text{where} \tag{4.3.1}$$

$x_i(t) = (x_{i1}(t), \dots, x_{in}(t))$, a.e. in $[0, T]$ and $\hat{x}^*_i(t) = (x^*_1(t), \dots, x^*_{i-1}(t), x^*_{i+1}(t), \dots, x^*_m(t))$.

Like we already know from section 2.4, the previous definition is equivalent to an evolutionary variational inequality as the following result shows.

Theorem 4.3.1. *$x^* \in \mathbb{K}$ is a dynamic oligopolistic market equilibrium according to Definition 4.3.1 if and only if it satisfies the evolutionary variational inequality*

$$\int_0^T \sum_{i=1}^m \sum_{j=1}^n \left(-\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} \right) (x_{ij}(t) - x^*_{ij}(t)) dt \geq 0 \qquad \forall x \in \mathbb{K}. \tag{4.3.2}$$

Like in section 4.2, now we change the point of view in the analysis of the problem by introducing an optimal control problem in which the variable h represents the difference between the supply tax η and the incentive pay λ for the transactions.

We introduce the function of regulatory taxes $x(h) = x(t,h)$, so $x : [0,T] \times \mathbb{R}^{mn} \to \mathbb{R}^{mn}$, since $h(t) \in \mathbb{R}^{mn}, \ \forall t \in [0,T]$.

We also suppose that $x(t,h)$ is a Carathéodory function and we require also that there exists $\gamma(t) \in L^2([0,T])$ such that, a.e. in $[0,T]$

$$\|x(t, h(t))\|_{mn} \leq \gamma(t) + \|h(t)\|_{mn}. \tag{4.3.3}$$

Therefore,

$$x : [0,T] \times L^2([0,T], \mathbb{R}^{mn}) \to L^2([0,T], \mathbb{R}^{mn}),$$

which assigns to each regulatory tax $h(t)$ the exportation vector $x(h(t))$.

We now introduce the set of feasible states

$$\Omega = \Big\{ \omega \in L^2([0,T], \mathbb{R}^{mn}) : \quad \underline{x}_{ij}(t) \leq \omega_{ij}(t) \leq \overline{x}_{ij}(t), \\ \forall i = 1, \dots, m, \ \forall j = 1, \dots, n, \text{ a.e. in } [0,T] \Big\},$$

and define the optimal regulatory tax as follows.

Definition 4.3.2. A vector $h^* \in L^2([0,T], \mathbb{R}^{mn})$ is an optimal regulatory tax if $x(t, h^*) \in \Omega$ and for $i = 1, \dots, m, \ j = 1, \dots, n$ and a.e. in $[0,T]$ the following conditions hold:

$$x_{ij}(t, h^*(t)) = \underline{x}_{ij}(t), \quad \Rightarrow \quad h^*_{ij}(t) \leq 0, \tag{4.3.4}$$

$$\underline{x}_{ij}(t) < x_{ij}(t, h^*(t)) < \overline{x}_{ij}(t), \quad \Rightarrow \quad h^*_{ij}(t) = 0, \tag{4.3.5}$$

$$x_{ij}(t, h^*(t)) = \overline{x}_{ij}(t), \quad \Rightarrow \quad h^*_{ij}(t) \geq 0. \tag{4.3.6}$$

Definition 4.3.2 is the same as the one presented in section 4.2 and its meaning does not change also if we consider production and demand excesses.

The following theorem shows the inverse variational inequality formulation of the optimal equilibrium control problem.

Theorem 4.3.2. *A regulatory tax $h^* \in L^2([0,T], \mathbb{R}^{mn})$ is an optimal regulatory tax according to Definition 4.2.2 if and only if it solves the inverse variational inequality:*

$$x(t,h^*) \in \Omega : \quad \int_0^T \sum_{i=1}^m \sum_{j=1}^n \left(\omega_{ij}(t) - x_{ij}(t,h^*(t))\right) h^*_{ij}(t) dt \leq 0, \quad \forall \omega \in \Omega. \tag{4.3.7}$$

It is also possible to provide a classical variational inequality formulation of the optimal equilibrium control problem. If we set

$$W = L^2([0,T], \mathbb{R}^{mn}) \times \Omega, \; F : [0,T] \times W \to L^2([0,T], \mathbb{R}^{2mn}),$$

$$z(t) = \begin{pmatrix} h(t) \\ \omega(t) \end{pmatrix},$$

and

$$F(t,z(t)) = \begin{pmatrix} \omega(t) - x(t,h(t)) \\ -h(t) \end{pmatrix}.$$

it is possible to prove that evolutionary inverse variational inequality problem (4.3.7) is equivalent to an evolutionary variational inequality, as you can see with the following result:

Theorem 4.3.3. *The evolutionary inverse variational inequality problem* (4.3.7) *is equivalent to the evolutionary variational inequality*

$$z^* \in W : \quad \int_0^T \sum_{l=1}^{2m} \sum_{j=1}^n F_{lj}(t,z^*(t)) \left(z_{lj}(t) - z^*_{lj}(t)\right) dt \geq 0, \quad \forall z \in W. \tag{4.3.8}$$

It is worth noting that W is a closed, convex and not bounded subset of $L^2([0,T], \mathbb{R}^{2mn})$.

4.3.1 A numerical example

This section is devoted to provide a numerical example of the theoretical achievements presented.

Let us consider two firms and two demand markets, as in Figure 4.2.
Let $\underline{x}, \overline{x} \in L^2([0,1], \mathbb{R}^4)$ be the capacity constraints such that, a.e. in $[0,1]$,

$$\underline{x}(t) = \begin{pmatrix} 0 & 2t \\ 0 & 2t \end{pmatrix},$$

$$\overline{x}(t) = \begin{pmatrix} 100t & 200t \\ 100t & 200t \end{pmatrix},$$

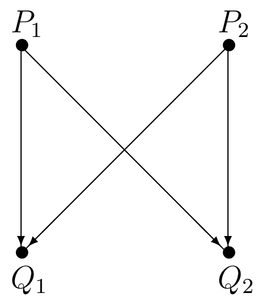

Figure 4.2: Network structure of the numerical dynamic spatial oligopoly problem.

and $p, q \in L^2([0,1], \mathbb{R}^2)$ be the production and demand function, such that, a.e. in $[0,1]$,

$$p(t) = \begin{pmatrix} 250t \\ 500t \end{pmatrix},$$

$$q(t) = \begin{pmatrix} 400t \\ 500t \end{pmatrix}.$$

As a consequence, the feasible set is

$$\begin{aligned} \mathbb{K} \;\; = \;\; & \Bigg\{ x \in L^2([0,1]\,, \mathbb{R}^4) : \quad \underline{x}_{ij}(t) \le x_{ij}(t) \le \overline{x}_{ij}(t), \quad \forall i = 1,2, \; \forall j = 1,2, \; \text{a.e. in}\,[0,1]\,, \\ & \sum_{j=1}^{2} x_{ij}(t) \le p_i(t), \quad \forall i = 1,2, \; \text{a.e. in } \; [0,1]\,, \\ & \sum_{i=1}^{2} x_{ij}(t) \le q_j(t), \quad \forall j = 1,2, \; \text{a.e. in } \; [0,1] \Bigg\}. \end{aligned}$$

The set of feasible states is

$$\Omega = \Bigg\{ \omega \in L^2([0,T]\,, \mathbb{R}^2) : \quad \underline{x}_{ij}(t) \le \omega_{ij}(t) \le \overline{x}_{ij}(t), \quad \forall i = 1,2, \; \forall j = 1,2, \; \text{a.e. in}\,[0,T] \Bigg\}.$$

Let us consider the profit function $v \in L^2([0,1] \times L^2([0,1], \mathbb{R}^4), \mathbb{R}^2)$ defined by

$$\begin{aligned} v_1(t, x(t)) \;\; &= \;\; 6x_{11}^2(t) + 2x_{12}(t)^2 + 2\alpha(t)x_{12}(t) - 2x_{11}(t)x_{12}(t) \\ & \quad -4x_{21}(t)x_{22}(t) - 2h_{11}(t)x_{11}(t) - 2h_{12}(t)x_{12}(t), \\ v_2(t, x(t)) \;\; &= \;\; 6x_{21}^2(t) + 2x_{22}(t)^2 + 2\beta(t)x_{22}(t) - 4x_{21}(t)x_{22}(t) \\ & \quad -2x_{11}(t)x_{12}(t) - 4h_{21}(t)x_{21}(t) - 4h_{22}(t)x_{22}(t), \end{aligned}$$

where α, β are suitable functions $L^2([0,1])$.

Then, the operator $\nabla_D v \in L^2([0,1] \times L^2([0,1], \mathbb{R}^4), \mathbb{R}^4)$ is given by

$$\nabla_D v(t, x(t)) = \begin{pmatrix} 12x_{11}(t) - 2x_{12}(t) - 2h_{11}(t) & 4x_{12}(t) - 2x_{11}(t) - 2h_{12}(t) + 2\alpha(t) \\ 12x_{21}(t) - 4x_{22}(t) - 4h_{21}(t) & 4x_{22}(t) - 4x_{21}(t) - 4h_{22}(t) + 2\beta(t) \end{pmatrix}.$$

The dynamic oligopolistic market equilibrium distribution in presence of excesses is the solution to the evolutionary variational inequality:

$$\int_0^1 \sum_{i=1}^2 \sum_{j=1}^2 -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}}(x_{ij}(t) - x^*_{ij}(t))\mathrm{dt} \geq 0, \quad \forall x \in \mathbb{K}. \tag{4.3.9}$$

In order to compute the solution to (4.3.9) we make use of the direct method (see [29, 34, 70]). We consider the following system

$$\begin{cases} -12x^*_{11}(t) + 2x^*_{12}(t) + 2h_{11}(t) = 0, \\ 2x^*_{11}(t) - 4x^*_{12}(t) + 2h_{12}(t) - 2\alpha(t) = 0, \\ -12x^*_{21}(t) + 4x^*_{22}(t) + 4h_{21}(t) = 0, \\ 4x^*_{21}(t) - 4x^*_{22}(t) + 4h_{22}(t) - 2\beta(t) = 0, \\ \underline{x}_{ij}(t) \leq x^*_{ij}(t) \leq \overline{x}_{ij}(t), \quad \forall i = 1,2, \ \forall j = 1,2, \\ \displaystyle\sum_{j=1}^2 x^*_{ij}(t) \leq p_i(t), \ \forall i = 1,2, \ \text{a.e. in } [0,1], \\ \displaystyle\sum_{i=1}^2 x^*_{ij}(t) \leq q_j(t), \ \forall j = 1,2, \ \text{a.e. in } [0,1] \end{cases}$$

and we get the following solution, a.e. in $[0,1]$,

$$x^*(t, h(t)) = \begin{pmatrix} \dfrac{2h_{11}(t) + h_{12}(t) - \alpha(t)}{11} & \dfrac{h_{11}(t) + 6h_{12}(t) - 6\alpha(t)}{11} \\ \dfrac{2h_{21}(t) + 2h_{22}(t) - \beta(t)}{4} & \dfrac{2h_{21}(t) + 6h_{22}(t) - 3\beta(t)}{4} \end{pmatrix}.$$

For the inverse problem, now we have to solve the following variational inequality

$$\int_0^T \left(\sum_{i=1}^2 \sum_{j=1}^2 (\omega^*_{ij}(t) - x_{ij}(t, h^*(t)))(h_{ij}(t) - h^*_{ij}(t)) - \sum_{i=1}^2 \sum_{j=1}^2 h^*_{ij}(t)(\omega_{ij}(t) - \omega^*_{ij}(t)) \right) \mathrm{dt} \geq 0$$
$$\forall (h, \omega) \in L^2([0,T], \mathbb{R}^4) \times \Omega. \tag{4.3.10}$$

For $\omega_{ij}(t) = \omega^*_{ij}(t),\ \forall i = 1, 2,\ \forall j = 1, 2$, a.e. in $[0, 1]$, we can consider the following system

$$\begin{cases} 2h^*_{11}(t) + h^*_{11}(t) - \alpha(t) - 11\omega^*_{11}(t) = 0, \\ h^*_{11}(t) + 6h^*_{12}(t) - 6\alpha(t) - 11\omega^*_{12}(t) = 0, \\ 2h^*_{21}(t) + 2h^*_{22}(t) - \beta(t) - 4\omega^*_{21}(t) = 0, \\ 2h^*_{21}(t) + 6h^*_{22}(t) - 3\beta(t) - 4\omega^*_{22}(t) = 0, \end{cases}$$

and we get the following solution, a.e. in $[0, 1]$,

$$h^*(t) = \begin{pmatrix} 6\omega^*_{11}(t) - \omega^*_{12}(t) & -\omega^*_{11}(t) + 2\omega^*_{12}(t) + \alpha(t) \\ 3\omega^*_{21}(t) - \omega^*_{22}(t) & -\omega^*_{21}(t) + \omega^*_{22}(t) + \frac{1}{2}\beta(t) \end{pmatrix}.$$

1) Let us study, now, the case

$$\omega^*(t) = \begin{pmatrix} 100t & 2t \\ 100t & 200t \end{pmatrix}.$$

As a consequence of the direct method, it must be

$$h^*_{11}(t) > 0,\ h^*_{12}(t) < 0,\ h^*_{21}(t) > 0,\ h^*_{22}(t) > 0.$$

This is true if and only if $\alpha(t) < 96t$ and $\beta(t) > -200t$. In this case the optimal regulatory tax is

$$h^*(t) = \begin{pmatrix} 598t & \alpha(t) - 96t \\ 100t & \frac{1}{2}\beta(t) + 100t \end{pmatrix},$$

from which we get that the optimal commodity distribution is

$$x^*(t) = \begin{pmatrix} 100t & 2t \\ 100t & 200t \end{pmatrix},$$

which belongs to $\mathbb{K}$.
Here, the production and demand excesses are, respectively:

$$\epsilon(t) = \begin{pmatrix} 148t \\ 200t \end{pmatrix}, \quad \delta(t) = \begin{pmatrix} 200t \\ 298t \end{pmatrix}.$$

2) Let us study, now, the case

$$\omega^*(t) = \begin{pmatrix} 0 & 200t \\ 100t & 200t \end{pmatrix}.$$

As a consequence of the direct method, it must be

$$h^*_{11}(t) < 0,\ h^*_{12}(t) > 0,\ h^*_{21}(t) > 0,\ h^*_{22}(t) > 0.$$

This is true if and only if $\alpha(t) > -400t$ and $\beta(t) > -200t$. In this case the optimal regulatory tax is

$$h^*(t) = \begin{pmatrix} -200t & \alpha(t) + 400t \\ 100t & \frac{1}{2}\beta(t) + 100t \end{pmatrix},$$

from which we get that the optimal commodity distribution is

$$x^*(t) = \begin{pmatrix} 0 & 200t \\ 100t & 200t \end{pmatrix},$$

which belongs to $\mathbb{K}$.
Here, the production and demand excesses are, respectively:

$$\epsilon(t) = \begin{pmatrix} 50t \\ 200t \end{pmatrix}, \quad \delta(t) = \begin{pmatrix} 300t \\ 100t \end{pmatrix}.$$

3) Let us study, now, the case

$$\omega^*(t) = \begin{pmatrix} 100t & 2t \\ 100t & 2t \end{pmatrix}.$$

As a consequence of the direct method, it must be

$$h^*_{11}(t) > 0,\ h^*_{12}(t) < 0,\ h^*_{21}(t) > 0,\ h^*_{22}(t) < 0.$$

This is true if and only if $\alpha(t) < 96t$ and $\beta(t) < 196t$. In this case the optimal regulatory tax is

$$h^*(t) = \begin{pmatrix} 598t & \alpha(t) - 96t \\ 298t & \frac{1}{2}\beta(t) - 98t \end{pmatrix},$$

from which we get that the optimal commodity distribution is

$$x^*(t) = \begin{pmatrix} 100t & 2t \\ 100t & 2t \end{pmatrix},$$

which belongs to $\mathbb{K}$.
Here, the production and demand excesses are, respectively:

$$\epsilon(t) = \begin{pmatrix} 148t \\ 398t \end{pmatrix}, \quad \delta(t) = \begin{pmatrix} 200t \\ 496t \end{pmatrix}.$$

4) Let us study, now, the case

$$\omega^*(t) = \begin{pmatrix} 0 & 200t \\ 100t & 2t \end{pmatrix}.$$

As a consequence of the direct method, it must be

$$h_{11}^*(t) < 0,\ h_{12}^*(t) > 0,\ h_{21}^*(t) > 0,\ h_{22}^*(t) < 0.$$

This is true if and only if $\alpha(t) > -400t$ and $\beta(t) < 196t$. In this case the optimal regulatory tax is

$$h^*(t) = \begin{pmatrix} -200t & \alpha(t) + 400t \\ 298t & \frac{1}{2}\beta(t) - 98t \end{pmatrix},$$

from which we get that the optimal commodity distribution is

$$x^*(t) = \begin{pmatrix} 0 & 200t \\ 100t & 2t \end{pmatrix},$$

which belongs to $\mathbb{K}$.
Here, the production and demand excesses are, respectively:

$$\epsilon(t) = \begin{pmatrix} 50t \\ 398t \end{pmatrix}, \quad \delta(t) = \begin{pmatrix} 300t \\ 298t \end{pmatrix}.$$

5) Let us study, now, the case

$$\omega^*(t) = \begin{pmatrix} 100t & 2t \\ 0 & 200t \end{pmatrix}.$$

As a consequence of the direct method, it must be

$$h_{11}^*(t) > 0,\ h_{12}^*(t) < 0,\ h_{21}^*(t) < 0,\ h_{22}^*(t) > 0.$$

This is true if and only if $\alpha(t) < 96t$ and $\beta(t) > -400t$. In this case the optimal regulatory tax is

$$h^*(t) = \begin{pmatrix} 598t & \alpha(t) - 96t \\ -200t & \frac{1}{2}\beta(t) + 200t \end{pmatrix},$$

from which we get that the optimal commodity distribution is

$$x^*(t) = \begin{pmatrix} 100t & 2t \\ 0 & 200t \end{pmatrix},$$

which belongs to $\mathbb{K}$.
Here, the production and demand excesses are, respectively:

$$\epsilon(t) = \begin{pmatrix} 148t \\ 300t \end{pmatrix}, \quad \delta(t) = \begin{pmatrix} 300t \\ 298t \end{pmatrix}.$$

6) Let us study, now, the case

$$\omega^*(t) = \begin{pmatrix} 0 & 200t \\ 0 & 200t \end{pmatrix}.$$

As a consequence of the direct method, it must be

$$h^*_{11}(t) < 0, \; h^*_{12}(t) > 0, \; h^*_{21}(t) < 0, \; h^*_{22}(t) > 0.$$

This is true if and only if $\alpha(t) > -400t$ and $\beta(t) > -400t$. In this case the optimal regulatory tax is

$$h^*(t) = \begin{pmatrix} -200t & \alpha(t) + 400t \\ -200t & \frac{1}{2}\beta(t) + 200t \end{pmatrix},$$

from which we get that the optimal commodity distribution is

$$x^*(t) = \begin{pmatrix} 0 & 200t \\ 0 & 200t \end{pmatrix},$$

which belongs to $\mathbb{K}$.
Here, the production and demand excesses are, respectively:

$$\epsilon(t) = \begin{pmatrix} 50t \\ 300t \end{pmatrix}, \quad \delta(t) = \begin{pmatrix} 400t \\ 100t \end{pmatrix}.$$

7) Let us study, now, the case

$$\omega^*(t) = \begin{pmatrix} 0 & 2t \\ 100t & 200t \end{pmatrix}.$$

As a consequence of the direct method, it must be

$$h^*_{11}(t) < 0,\ h^*_{12}(t) < 0,\ h^*_{21}(t) > 0,\ h^*_{22}(t) > 0.$$

This is true if and only if $\alpha(t) < -4t$ and $\beta(t) > -200t$. In this case the optimal regulatory tax is

$$h^*(t) = \begin{pmatrix} -2t & \alpha(t) + 4t \\ 100t & \frac{1}{2}\beta(t) + 100t \end{pmatrix},$$

from which we get that the optimal commodity distribution is

$$x^*(t) = \begin{pmatrix} 0 & 2t \\ 100t & 200t \end{pmatrix},$$

which belongs to $\mathbb{K}$.
Here, the production and demand excesses are, respectively:

$$\epsilon(t) = \begin{pmatrix} 248t \\ 200t \end{pmatrix}, \quad \delta(t) = \begin{pmatrix} 300t \\ 298t \end{pmatrix}.$$

8) Let us study, now, the case

$$\omega^*(t) = \begin{pmatrix} 0 & 2t \\ 0 & 200t \end{pmatrix}.$$

As a consequence of the direct method, it must be

$$h^*_{11}(t) < 0,\ h^*_{12}(t) < 0,\ h^*_{21}(t) < 0,\ h^*_{22}(t) > 0.$$

This is true if and only if $\alpha(t) < -4t$ and $\beta(t) > -400t$. In this case the optimal regulatory tax is

$$h^*(t) = \begin{pmatrix} -2t & \alpha(t) + 4t \\ -200t & \frac{1}{2}\beta(t) + 200t \end{pmatrix},$$

from which we get that the optimal commodity distribution is

$$x^*(t) = \begin{pmatrix} 0 & 2t \\ 0 & 200t \end{pmatrix},$$

which belongs to $\mathbb{K}$.
Here, the production and demand excesses are, respectively:

$$\epsilon(t) = \begin{pmatrix} 248t \\ 300t \end{pmatrix}, \quad \delta(t) = \begin{pmatrix} 400t \\ 298t \end{pmatrix}.$$

9) Let us study, now, the case

$$\omega^*(t) = \begin{pmatrix} 0 & 2t \\ 100t & 2t \end{pmatrix}.$$

As a consequence of the direct method, it must be

$$h^*_{11}(t) < 0,\ h^*_{12}(t) < 0,\ h^*_{21}(t) > 0,\ h^*_{22}(t) < 0.$$

This is true if and only if $\alpha(t) < -4t$ and $\beta(t) < 196t$. In this case the optimal regulatory tax is

$$h^*(t) = \begin{pmatrix} -2t & \alpha(t) + 4t \\ 298t & \frac{1}{2}\beta(t) - 98t \end{pmatrix},$$

from which we get that the optimal commodity distribution is

$$x^*(t) = \begin{pmatrix} 0 & 2t \\ 100t & 2t \end{pmatrix},$$

which belongs to $\mathbb{K}$.
Here, the production and demand excesses are, respectively:

$$\epsilon(t) = \begin{pmatrix} 248t \\ 398t \end{pmatrix}, \quad \delta(t) = \begin{pmatrix} 300t \\ 496t \end{pmatrix}.$$

10) Let us study, now, the case

$$\omega^*(t) = \begin{pmatrix} 0 & 200t \\ 0 & 2t \end{pmatrix}.$$

As a consequence of the direct method, it must be

$$h^*_{11}(t) < 0,\ h^*_{12}(t) > 0,\ h^*_{21}(t) < 0,\ h^*_{22}(t) < 0.$$

This is true if and only if $\alpha(t) > -400t$ and $\beta(t) < -4t$. In this case the optimal regulatory tax is

$$h^*(t) = \begin{pmatrix} -200t & \alpha(t) + 400t \\ -2t & \frac{1}{2}\beta(t) + 2t \end{pmatrix},$$

from which we get that the optimal commodity distribution is

$$x^*(t) = \begin{pmatrix} 0 & 200t \\ 0 & 2t \end{pmatrix},$$

which belongs to $\mathbb{K}$.
Here, the production and demand excesses are, respectively:

$$\epsilon(t) = \begin{pmatrix} 50t \\ 498t \end{pmatrix}, \quad \delta(t) = \begin{pmatrix} 400t \\ 298t \end{pmatrix}.$$

11) Let us study, now, the case

$$\omega^*(t) = \begin{pmatrix} 100t & 2t \\ 0 & 2t \end{pmatrix}.$$

As a consequence of the direct method, it must be

$$h^*_{11}(t) > 0,\ h^*_{12}(t) < 0,\ h^*_{21}(t) < 0,\ h^*_{22}(t) < 0.$$

This is true if and only if $\alpha(t) < 96t$ and $\beta(t) < -4t$. In this case the optimal regulatory tax is

$$h^*(t) = \begin{pmatrix} 598t & \alpha(t) - 96t \\ -2t & \frac{1}{2}\beta(t) + 2t \end{pmatrix},$$

from which we get that the optimal commodity distribution is

$$x^*(t) = \begin{pmatrix} 100t & 2t \\ 0 & 2t \end{pmatrix},$$

which belongs to $\mathbb{K}$.
Here, the production and demand excesses are, respectively:

$$\epsilon(t) = \begin{pmatrix} 148t \\ 498t \end{pmatrix}, \quad \delta(t) = \begin{pmatrix} 300t \\ 496t \end{pmatrix}.$$

12) Let us study, now, the case

$$\omega^*(t) = \begin{pmatrix} 0 & 2t \\ 0 & 2t \end{pmatrix}.$$

As a consequence of the direct method, it must be

$$h^*_{11}(t) < 0, \ h^*_{12}(t) < 0, \ h^*_{21}(t) < 0, \ h^*_{22}(t) < 0.$$

This is true if and only if $\alpha(t) < -4t$ and $\beta(t) < -4t$. In this case the optimal regulatory tax is

$$h^*(t) = \begin{pmatrix} -2t & \alpha(t) + 4t \\ -2t & \frac{1}{2}\beta(t) + 2t \end{pmatrix},$$

from which we get that the optimal commodity distribution is

$$x^*(t) = \begin{pmatrix} 0 & 2t \\ 0 & 2t \end{pmatrix},$$

which belongs to $\mathbb{K}$.
Here, the production and demand excesses are, respectively:

$$\epsilon(t) = \begin{pmatrix} 248t \\ 498t \end{pmatrix}, \quad \delta(t) = \begin{pmatrix} 400t \\ 496t \end{pmatrix}.$$

We remark that the cases $\omega^*_{11}(t)$ and $\omega^*_{12}(t)$ both maximal are not allowed since the correspondent commodity shipment $x^*(t)$ does not belong to the constraint set $\mathbb{K}$ because $x^*_{11}(t) + x^*_{12}(t) > 250t$.

Let us consider now the set

$$\begin{aligned}\widetilde{\mathbb{K}} \ = \ \Big\{ & x \in L^2([0,1], \mathbb{R}^4) : \quad \underline{x}_{ij}(t) \le x_{ij}(t) \le \overline{x}_{ij}(t), \quad \forall i = 1,2, \ \forall j = 1,2, \ \text{a.e. in } [0,1], \\ & x_{11}(t) + x_{12}(t) = p_1(t), \quad \text{a.e. in } [0,1], \\ & x_{21}(t) + x_{22}(t) \le p_2(t), \quad \text{a.e. in } [0,1], \\ & \sum_{i=1}^{2} x_{ij}(t) \le q_j(t), \quad \forall j = 1,2, \ \text{a.e. in } [0,1] \Big\}.\end{aligned}$$

In order to compute the solution to (4.3.9) we make use again of the direct method. We consider the following system

$$\begin{cases} x^*_{11}(t) + x^*_{12}(t) - 250t = 0, \\ 14x^*_{11}(t) - 6x^*_{12}(t) - 2h_{11}(t) + 2h_{12}(t) - 2\alpha(t) = 0, \\ -12x^*_{21}(t) + 4x^*_{22}(t) + 4h_{21}(t) = 0, \\ 4x^*_{21}(t) - 4x^*_{22}(t) + 4h_{22}(t) - 2\beta(t) = 0, \\ \underline{x}_{1j}(t) < x^*_{1j}(t) < \overline{x}_{1j}(t), \ \forall j = 1, 2, \\ \underline{x}_{2j}(t) \leq x^*_{2j}(t) \leq \overline{x}_{2j}(t), \quad \forall j = 1, 2, \\ x^*_{21}(t) + x^*_{22}(t) \leq p_2(t), \ \forall i = 1, 2, \text{ a.e. in } [0, 1], \\ \sum_{i=1}^{2} x^*_{ij}(t) \leq q_j(t), \ \forall j = 1, 2, \text{ a.e. in } [0, 1] \end{cases}$$

and we get the following solution, a.e. in $[0, 1]$,

$$x^*(t, h(t)) = \begin{pmatrix} \dfrac{h_{11}(t) - h_{12}(t) + \alpha(t) + 750t}{10} & \dfrac{-h_{11}(t) + h_{12}(t) - \alpha(t) + 1750t}{10} \\ \dfrac{2h_{21}(t) + 2h_{22}(t) - \beta(t)}{4} & \dfrac{2h_{21}(t) + 6h_{22}(t) - 3\beta(t)}{4} \end{pmatrix}.$$

For the inverse problem, now we have to solve the following variational inequality (4.3.10).

$$\int_0^T \left(\sum_{i=1}^{2} \sum_{j=1}^{2} (\omega^*_{ij}(t) - x_{ij}(t, h^*(t)))(h_{ij}(t) - h^*_{ij}(t)) - \sum_{i=1}^{2} \sum_{j=1}^{2} h^*_{ij}(t)(\omega_{ij}(t) - \omega^*_{ij}(t)) \right) \mathrm{dt} \geq 0$$
$$\forall (h, \omega) \in L^2([0, T], \mathbb{R}^4) \times \Omega. \tag{4.3.11}$$

We remark that the condition $\underline{x}_{1j}(t) < x^*_{1j}(t) < \overline{x}_{1j}(t), \ \forall j = 1, 2$, implies $h^*_{1j}(t) = 0, \ \forall j = 1, 2$, as you can see from Definition (4.3.2). As a consequence, we can consider the following system

$$\begin{cases} h^*_{1j}(t) = 0, \ \forall j = 1, 2, \\ h^*_{11}(t) - h^*_{12}(t) + \alpha(t) - 10\omega^*_{11}(t) = 0, \\ -h^*_{11}(t) + h^*_{12}(t) - \alpha(t) - 10\omega^*_{12}(t) = 0, \\ 2h^*_{21}(t) + 2h^*_{22}(t) - \beta(t) - 4\omega^*_{21}(t) = 0, \\ 2h^*_{21}(t) + 6h^*_{22}(t) - 3\beta(t) - 4\omega^*_{22}(t) = 0, \end{cases}$$

and we get the following solution, a.e. in $[0, 1]$,

$$h^*(t) = \begin{pmatrix} 0 & 0 \\ 3\omega^*_{21}(t) - \omega^*_{22}(t) & -\omega^*_{21}(t) + \omega^*_{22}(t) + \dfrac{1}{2}\beta(t) \end{pmatrix}$$

and, moreover, $\omega^*_{11}(t) = \dfrac{\alpha(t) + 750t}{10}$ and $\omega^*_{12}(t) = \dfrac{-\alpha(t) + 1750t}{10}$.

13) In order to obtain the solution to (4.3.11), we consider, in first analysis, the case

$$\omega^*(t) = \begin{pmatrix} \dfrac{\alpha(t) + 750t}{10} & \dfrac{-\alpha(t) + 1750t}{10} \\ 100t & 200t \end{pmatrix}.$$

As a consequence of the direct method, it must be

$$h^*_{21}(t) > 0, \ h^*_{22}(t) > 0.$$

This is true if and only if $\beta(t) > -200t$. In this case the optimal regulatory tax is

$$h^*(t) = \begin{pmatrix} 0 & 0 \\ 100t & \dfrac{1}{2}\beta(t) + 100t \end{pmatrix},$$

from which we get that the optimal commodity distribution is

$$x^*(t) = \begin{pmatrix} \dfrac{\alpha(t) + 750t}{10} & \dfrac{-\alpha(t) + 1750t}{10} \\ 100t & 200t \end{pmatrix},$$

which belongs to $\widetilde{\mathbb{K}}$ for $-250t \leq \alpha(t) \leq 250t$.
Here, the production and demand excesses are, respectively:

$$\epsilon(t) = \begin{pmatrix} 0 \\ 200t \end{pmatrix}, \quad \delta(t) = \begin{pmatrix} \dfrac{2250t - \alpha(t)}{10} \\ \dfrac{1250t + \alpha(t)}{10} \end{pmatrix}.$$

14) Let us study, now, the case

$$\omega^*(t) = \begin{pmatrix} \dfrac{\alpha(t) + 750t}{10} & \dfrac{-\alpha(t) + 1750t}{10} \\ 100t & 2t \end{pmatrix}.$$

As a consequence of the direct method, it must be

$$h^*_{21}(t) > 0, \ h^*_{22}(t) < 0.$$

This is true if and only if $\beta(t) < 196t$. In this case the optimal regulatory tax is

$$h^*(t) = \begin{pmatrix} 0 & 0 \\ 298t & \frac{1}{2}\beta(t) - 98t \end{pmatrix},$$

from which we get that the optimal commodity distribution is

$$x^*(t) = \begin{pmatrix} \dfrac{\alpha(t) + 750t}{10} & \dfrac{-\alpha(t) + 1750t}{10} \\ 100t & 2t \end{pmatrix},$$

which belongs to $\widetilde{\mathbb{K}}$ for $-250t \leq \alpha(t) \leq 250t$.
Here, the production and demand excesses are, respectively:

$$\epsilon(t) = \begin{pmatrix} 0 \\ 398t \end{pmatrix}, \quad \delta(t) = \begin{pmatrix} \dfrac{2250t - \alpha(t)}{10} \\ \dfrac{3230t + \alpha(t)}{10} \end{pmatrix}.$$

15) Let us study, now, the case

$$\omega^*(t) = \begin{pmatrix} \dfrac{\alpha(t) + 750t}{10} & \dfrac{-\alpha(t) + 1750t}{10} \\ 0 & 200t \end{pmatrix}.$$

As a consequence of the direct method, it must be

$$h^*_{21}(t) < 0, \ h^*_{22}(t) > 0.$$

This is true if and only if $\beta(t) > -400t$. In this case the optimal regulatory tax is

$$h^*(t) = \begin{pmatrix} 0 & 0 \\ -200t & \frac{1}{2}\beta(t) + 200t \end{pmatrix},$$

from which we get that the optimal commodity distribution is

$$x^*(t) = \begin{pmatrix} \dfrac{\alpha(t) + 750t}{10} & \dfrac{-\alpha(t) + 1750t}{10} \\ 0 & 200t \end{pmatrix},$$

which belongs to $\widetilde{\mathbb{K}}$ for $-250t \leq \alpha(t) \leq 250t$.
Here, the production and demand excesses are, respectively:

$$\epsilon(t) = \begin{pmatrix} 0 \\ 300t \end{pmatrix}, \quad \delta(t) = \begin{pmatrix} \dfrac{3250t - \alpha(t)}{10} \\ \dfrac{1250t + \alpha(t)}{10} \end{pmatrix}.$$

16) Let us study, now, the case

$$\omega^*(t) = \begin{pmatrix} \dfrac{\alpha(t) + 750t}{10} & \dfrac{-\alpha(t) + 1750t}{10} \\ 0 & 2t \end{pmatrix}.$$

As a consequence of the direct method, it must be

$$h^*_{21}(t) < 0, \; h^*_{22}(t) < 0.$$

This is true if and only if $\beta(t) < -4t$. In this case the optimal regulatory tax is

$$h^*(t) = \begin{pmatrix} 0 & 0 \\ -2t & \dfrac{1}{2}\beta(t) + 2t \end{pmatrix},$$

from which we get that the optimal commodity distribution is

$$x^*(t) = \begin{pmatrix} \dfrac{\alpha(t) + 750t}{10} & \dfrac{-\alpha(t) + 1750t}{10} \\ 0 & 2t \end{pmatrix},$$

which belongs to $\widetilde{\mathbb{K}}$ for $-250t \leq \alpha(t) \leq 250t$.
Here, the production and demand excesses are, respectively:

$$\epsilon(t) = \begin{pmatrix} 0 \\ 498t \end{pmatrix}, \quad \delta(t) = \begin{pmatrix} \dfrac{3250t - \alpha(t)}{10} \\ \dfrac{3230t + \alpha(t)}{10} \end{pmatrix}.$$

4.4 An existence result

We, now, state the following existence result.

Theorem 4.4.1. *Let us set*

$$F : W \longrightarrow L^2([0,T], \mathbb{R}^{2mn}),$$

$$W = L^2([0,T]\,,\mathbb{R}^m) \times \Omega,$$

and suppose that F is K-pseudomonotone and lower hemicontinuous along line segments. Let us further suppose that there exists $z_0 \in \mathbb{K}$ and $R > \|z_0\|$ such that

$$\ll Fz, z - z_0 \gg > 0, \quad \forall z \in W \cap \left\{ z \in L^2([0,T]\,,\mathbb{R}^{2mn}) : \|z\| = R \right\}. \tag{4.4.1}$$

Then the variational inequality

$$\ll Fz^*, z - z^* \gg \geq 0, \quad \forall z \in W \tag{4.4.2}$$

admits a solution.

Proof. Let us note that $\mathbb{K}$ is clearly a nonempty closed and convex subset of $L^2([0,T]\,,\mathbb{R}^{2mn})$. Then, the claim is achieved applying Theorem 3.6 in [71]. □

In particular, the monotonicity assumption on the operator F is equivalent to the monotonicity of function $-x$, infact

$$\begin{aligned}
& \ll Fz_1 - Fz_2, z_1 - z_2 \gg \\
= & \int_0^T \sum_{l=1}^{2m} \sum_{j=1}^{n} \left(F_{lj}(t, z_1(t)) - F_{lj}(t, z_2(t))\right) \left(z^1_{lj}(t) - z^2_{lj}(t)\right) dt \\
= & \int_0^T \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \left(\omega^1_{ij}(t) - x_{ij}(t, h^1(t)) - \omega^2_{ij}(t) + x_{ij}(t, h^2(t))\right) \left(h^1_{ij}(t) - h^2_{ij}(t)\right) \right) dt \\
& + \int_0^T \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \left(-h^1_{ij}(t) + h^2_{ij}(t)\right) \left(\omega^1_{ij}(t) - \omega^2_{ij}(t)\right) \right) dt \\
= & \int_0^T \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \left(-x_{ij}(t, h^1(t)) + x_{ij}(t, h^2(t))\right) \left(h^1_{ij}(t) - h^2_{ij}(t)\right) \right) dt \\
= & \ll -x(h_1) + x(h_2), h_1 - h_2 \gg .
\end{aligned}$$

Remark 4.4.1. As proved in [22], relationship (4.4.1) is ensured under condition that

$$\lim_{\substack{\|z\| \to +\infty \\ z \in W}} \frac{\ll Fz, z \gg}{\|z\|} = +\infty,$$

or, equivalently, as you can see from the previous equality,

$$\lim_{\substack{\|z\| \to +\infty \\ z \in W}} \frac{\ll -x(h), h \gg}{\|z\|} = +\infty.$$

The advantage of using Theorem 4.4.1 lies in the fact that the lower hemicontinuity is ensured by assumptions (4.2.3). Moreover, operator F is monotone.

The regularity of solutions may be ensured under regularity assumptions on data (see [4, 8]).

Chapter 5

Computational procedures

5.1 A brief introduction

The development of efficient computational procedures for the numerical computation of dynamic equilibria is a very important question in optimization theory. The algorithms proposed in this chapter are based on time-discretization methods. It is the continuity of solutions to dynamic equilibrium problem that allows us to consider a partition of the time interval and hence reduce the infinite-dimensional problem to some finite-dimensional problems that can be solved by means of known methods. In particular we make use to the generalized projection method, the generalized projection-contraction method and the generalized extragradient method to solve finite-dimensional variational inequalities associated to the points of the discretization of the time interval. For a more extended and detailed collection of numerical procedures to solve variational inequalities, you can refer to [103]. Finally, the equilibrium curve of models is determined through the interpolation of numerical solutions with linear splines. Moreover, we give some numerical examples that have been implemented under MatLab. In the end, we make a convergence study in L^1-sense.

5.2 The generalized projection method

Let us introduce the generalized projection method to solve dynamic oligopolistic market equilibrium problems in presence of excesses expressed by evolutionary variational inequalities.

We suppose that the assumptions which ensure the continuity of dynamic oligopolistic market equilibrium solution hold. As a consequence, (2.2.8), (2.3.5), and (2.3.5) hold for each $t \in [0, T]$, namely the following point-to-point evolutionary variational inequality holds:

$$\langle -\nabla_D v(t, x^*(t)), x(t) - x^*(t) \rangle \geq 0, \qquad \forall x(t) \in \mathbb{K}(t),\ \forall t \in [0, T] \tag{5.2.1}$$

where

$$\mathbb{K}(t)=\left\{x(t)\in\mathbb{R}^{mn}:\quad \underline{x}_{ij}(t)\le x_{ij}(t)\le \overline{x}_{ij}(t),\ \forall i=1,\dots,m,\ \forall j=1,\dots,n\right\}\tag{5.2.2}$$

$$\mathbb{K}(t)=\left\{x(t)\in\mathbb{R}^{mn}:\quad x_{ij}(t)\ge 0,\ \forall i=1,\dots,m,\ \forall j=1,\dots,n,\right.\tag{5.2.3}$$

$$\left.\sum_{j=1}^{n}x_{ij}(t)\le p_i(t),\quad \forall i=1,\dots,m\right\}\tag{5.2.4}$$

$$\mathbb{K}(t)=\left\{x(t)\in\mathbb{R}^{mn}:\quad \underline{x}_{ij}(t)\le x_{ij}(t)\le \overline{x}_{ij}(t),\ \forall i=1,\dots,m,\ \forall j=1,\dots,n,\right.\tag{5.2.5}$$

$$\sum_{j=1}^{n}x_{ij}(t)\le p_i(t),\quad \forall i=1,\dots,m,\tag{5.2.6}$$

$$\left.\sum_{i=1}^{m}x_{ij}(t)\le q_j(t),\quad \forall j=1,\dots,n\right\},\tag{5.2.7}$$

respectively, which expresses dynamic equilibrium problems in the common formulation. Now, we present a computational method to compute the dynamic equilibrium solution to (5.2.1).
In the following, applying a discretization procedure, we use the projection method introduced by Marcotte and Wu in [69] in order to compute the solution to (5.2.1).

We consider now a partition of $[0,T]$, such that:

$$0=t_0<t_1<\ldots<t_r<\ldots<t_N=T,$$

and, for each value t_r, for $r=0,1,\dots,N$, we apply the projection method to solve the finite-dimensional variational inequality

$$\langle -\nabla_D v(t_r,x^*(t_r)),x(t_r)-x^*(t_r)\rangle\ge 0,\qquad \forall x(t_r)\in\mathbb{K}(t_r).\tag{5.2.8}$$

We can compute now the solution to the finite-dimensional variational inequality (5.2.8) using the following procedure. The algorithm, as it is well known, starting from any $x^{*0}(t_r)\in\mathbb{K}(t_r)$ fixed, iteratively updates $x^*(t_r)$ according to the formula

$$x^{*k+1}(t_r)=P_{\mathbb{K}(t_r)}(x^{*k}(t_r)-\alpha(-\nabla_D v(x^{*k}(t_r)))),$$

for $k \in \mathbb{N}$, where $P_{\mathbb{K}(t_r)}(\cdot)$ denotes the orthogonal projection map onto $\mathbb{K}(t_r)$ and α is a judiciously chosen positive steplength. Here, $P_{\mathbb{K}(t_r)}(x^{*k}(t_r) - \alpha(-\nabla_D v(x^{*k}(t_r))))$, for $k \in \mathbb{N}$, is the solution of the following quadratic programming problem

$$\min_{x^*(t_r)\in\mathbb{K}(t_r)} \frac{1}{2}(x^*(t_r))^T x^*(t_r) - (x^{*k}(t_r) - \alpha(-\nabla_D v(x^{*k}(t_r))))^T x^*(t_r),$$

for $k \in \mathbb{N}$. The projection method is based on the observation that $x^*(t_r) \in \mathbb{K}(t_r)$ is a solution of (5.2.8) if and only if

$$x^*(t_r) = P_{\mathbb{K}(t_r)}(x^*(t_r) - \alpha(-\nabla_D v(x^*(t_r)))).$$

This method requires restrictive assumptions on $-\nabla_D v$ for the convergence. The convergence analysis for the projection methods is based on the contractive properties of the operator $x^*(t_r) \to x^*(t_r) - \alpha(-\nabla_D v(u(t_r)))$: if $(-\nabla_D v(t_r))$ is strongly monotone (with constant ν) and Lipschitz continuous on $\mathbb{K}(t_r)$ (with Lipschitz constant L), and if $\alpha \in (0, 2\nu/L^2)$, the projection method determines a sequence $\{x^{*k}(t_r)\}_{k\in\mathbb{N}}$ convergent to a solution of (5.2.8), for every $r = 0, 1, \ldots, N$ (see [88] and [97]).

Marcotte and Wu in [69] have shown that the projection algorithm converges for cocoercive variational inequalities. We recall that a mapping F is cocoercive on $\mathbb{K}(t_r)$ if there exists a positive constant $\widetilde{\nu}$ such that, $\forall x(t_r), y(t_r) \in \mathbb{K}(t_r)$ one has

$$\langle F(x(t_r)) - F(y(t_r)), x(t_r) - y(t_r)\rangle \geq \widetilde{\nu}\|F(x(t_r)) - F(y(t_r))\|_q^2.$$

Any strongly monotone (with constant ν) and Lipschitz continuous mapping (with Lipschitz constant L) is cocoercive with the constant $\widetilde{\nu} = \frac{\nu}{L^2}$. If $\mathbb{K}(t_r) \neq \emptyset$ and $\alpha \in (0, 2\widetilde{\nu})$. The cocoercivity of the operator F is sufficient to assure the convergence of the projection algorithm.

A drawback is the choice of α when L is unknown. Indeed, if α is too small, the convergence is slow; when α is too large, there might be no convergence at all.

After iterative procedure, we can construct a function by performing a linear interpolation.

5.2.1 A numerical example

Let us consider four firms and four demand markets, as in Figure 5.1. Let $\underline{x}, \overline{x} \in L^2([0,1], \mathbb{R}^{16}_+)$ be the capacity constraints such that, a.e. in $[0,1]$,

$$\underline{x}(t) = \begin{pmatrix} \frac{1}{10}t & \frac{1}{2}t & \frac{1}{4}t & t \\ \frac{1}{2}t & 0 & \frac{1}{6}t & 0 \\ \frac{1}{2}t & 0 & 0 & \frac{1}{6}t \\ 0 & 2t & 2t & \frac{1}{10}t \end{pmatrix},$$

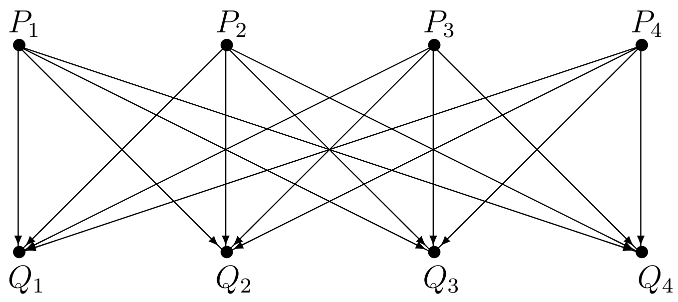

Figure 5.1: Network structure of the numerical dynamic spatial oligopoly problem.

$$\overline{x}(t) = \begin{pmatrix} 2t & 8t & 7t & 20t \\ 11t & 10t & 15t & 15t \\ 10t & 12t & 10t & 9t \\ 15t & 20t & 20t & 2t \end{pmatrix}.$$

Let $p \in L^2([0,1], \mathbb{R}^4_+)$ be the production function such that, a.e. in $[0,1]$,

$$p(t) = \begin{pmatrix} 40t \\ 35t \\ 35t \\ 60t \end{pmatrix},$$

and let $q \in L^2([0,1], \mathbb{R}^4_+)$ be the demand function such that, a.e. in $[0,1]$,

$$q(t) = \begin{pmatrix} 35t \\ 50t \\ 50t \\ 45t \end{pmatrix}.$$

As a consequence, the feasible set is

$$\begin{aligned} \mathbb{K} \;=\; & \Bigg\{ x \in L^2([0,1]\,, \mathbb{R}^{16}_+) : \\ & \underline{x}_{ij}(t) \le x_{ij}(t) \le \overline{x}_{ij}(t), \quad \forall i = 1, \ldots, 4,\ \forall j = 1, \ldots, 4,\ \text{a.e. in}\,[0,1]\,, \\ & \sum_{j=1}^{4} x_{ij}(t) \le p_i(t), \quad \forall i = 1, \ldots, 4,\ \text{a.e. in}\ [0,1]\,, \\ & \sum_{i=1}^{4} x_{ij}(t) \le q_j(t), \quad \forall j = 1, \ldots, 4,\ \text{a.e. in}\ [0,1] \Bigg\}. \end{aligned}$$

Let us consider the profit function $v \in L^2([0,1] \times L^2([0,1], \mathbb{R}^{16}_+), \mathbb{R}^4)$ defined by

$$\begin{aligned}
v_1(t,x(t)) &= -(t+3)x_{11}^2(t) - \frac{5}{2}x_{12}^2(t) - \frac{3t+2}{2}x_{13}^2(t) - \frac{5}{2}x_{14}^2(t) - 4(t+2)x_{11}(t)x_{44}(t) \\
&\quad +tx_{11}(t) + 2tx_{12}(t) + (2t+3)x_{13}(t) + 4tx_{14}(t), \\
v_2(t,x(t)) &= -\frac{1}{2}x_{21}^2(t) - (3t+1)x_{22}^2(t) - \frac{5t+5}{2}x_{23}^2(t) - x_{24}^2(t) - 2(t+1)x_{22}(t)x_{33}(t) \\
&\quad 6tx_{21}(t) + 9tx_{22}(t) + (t+4)x_{23}(t) + tx_{24}(t), \\
v_3(t,x(t)) &= -x_{31}^2(t) - x_{32}^2(t) - \frac{t+4}{2}x_{33}^2(t) - \frac{t+1}{2}x_{34}^2(t) - 2tx_{33}(t)x_{12}(t) \\
&\quad +(t+1)x_{31}(t) + tx_{32}(t) + 4tx_{33}(t) + tx_{34}(t), \\
v_4(t,x(t)) &= -x_{41}^2(t) - \frac{1}{2}x_{42}^2(t) - 3x_{43}^2(t) - (t+6)x_{44}^2(t) - 4(t+2)x_{44}(t)x_{23}(t) \\
&\quad +2tx_{41}(t) + (t+3)x_{42}(t) + tx_{43}(t) + 3tx_{44}(t).
\end{aligned}$$

Let the operator $\nabla_D v \in L^2([0,1] \times L^2([0,1], \mathbb{R}^{16}_+), \mathbb{R}^4)$ be the operator of the partial derivatives, namely

$$\nabla_D v(t,x(t)) = \left(\frac{\partial v_i(t,x(t))}{\partial x_{ij}(t)} \right)_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$$

It is possible to prove that the operator $-\nabla_D v$ is strongly monotone and Lipschitz continuous, infact,

$$\langle -\nabla_D v(x) + \nabla_D v(y), x-y \rangle \geq \|x-y\|_{16}^2$$

and

$$\| -\nabla_D v(x) + \nabla_D v(y)\|_{16}^2 \leq 680\|x-y\|_{16}^2.$$

The dynamic oligopolistic market equilibrium distribution in presence of excesses is the solution to the evolutionary variational inequality:

$$\int_0^1 \sum_{i=1}^4 \sum_{j=1}^4 -\frac{\partial v_i(t,x^*(t))}{\partial x_{ij}}(x_{ij}(t) - x_{ij}^*(t))dt \geq 0, \quad \forall x \in \mathbb{K}.$$

Now, we solve the numerical problem using the generalized projection method. This method is convergent for the properties of $-\nabla_D v$. Then, we can compute an approximate curve of equilibria, by selecting $t_r \in \left\{ \frac{k}{20} : k \in \{0,1,\dots,20\} \right\}$. With the help of a MatLab

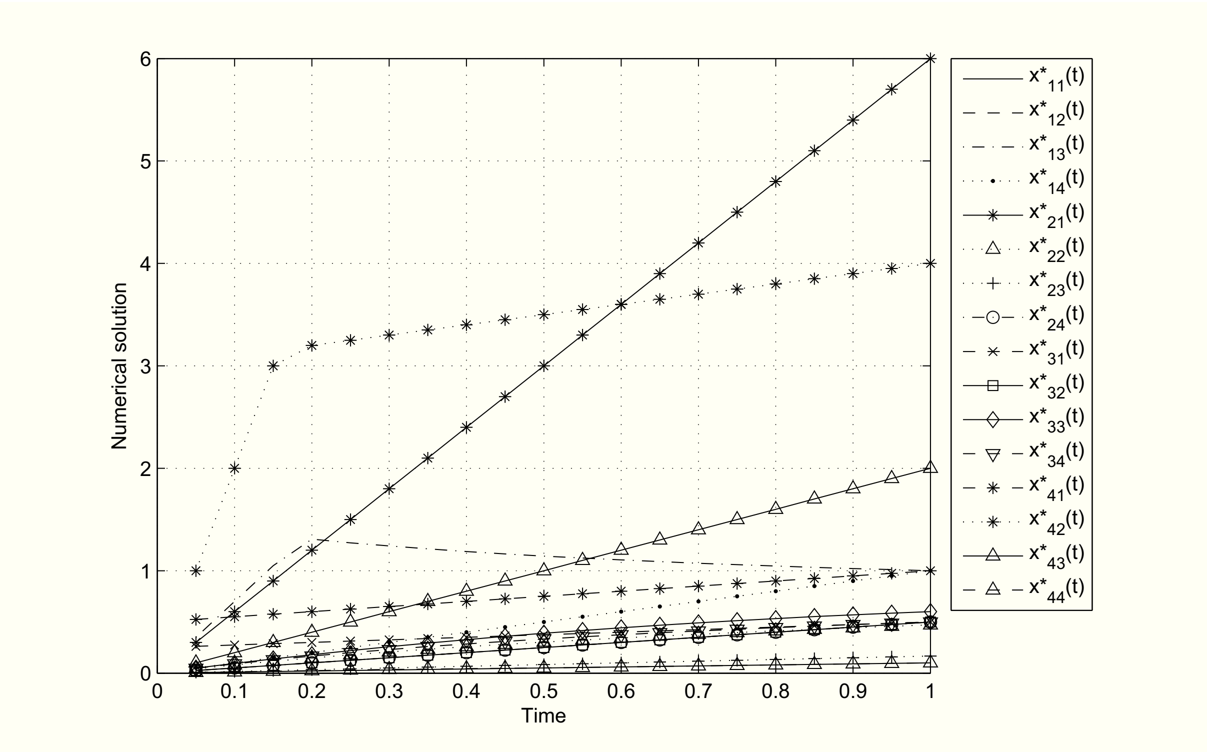

Figure 5.2: Curves of equilibria

computation and choosing the initial point

$$x^{*0}(t_r) = \begin{pmatrix} \frac{2}{3}t_r & t_r & \frac{1}{2}t_r & 2t_r \\ \frac{3}{4}t_r & \frac{1}{10}t_r & \frac{3}{4}t_r & \frac{1}{8}t_r \\ t_r & \frac{1}{3}t_r & \frac{2}{5}t_r & \frac{1}{2}t_r \\ t_r & 3t_r & 3t_r & \frac{1}{2}t_r \end{pmatrix}$$

in order to start the iterative method, we obtain the equilibrium solutions for every time instant. Setting $R(x^{*k}(t_r)) = x^{*k}(t_r) - x^{*k-1}(t_r)$ the difference between two approximations of the equilibrium solution in the time instant t_r and making use of the stopping criterion $\left\|R(x^{*k}(t_r))\right\|_{4\times4} \leq 10^{-6}$, for $r = 0, 1, \ldots, 20$, we obtain the approximate curve of equilibria shown in Figure 5.2.

5.3 The generalized projection-contraction method

Let us introduce the generalized projection-contraction method to solve dynamic oligopolistic market equilibrium problems in presence of excesses expressed by evolutionary variational inequalities.

We suppose that the assumptions which ensure the continuity of dynamic oligopolistic market equilibrium solution hold. As a consequence, (5.2.1) holds and the time-dependent constraint sets $\mathbb{K}(t)$ are (5.2.2), (5.2.3) and (5.2.5), respectively, which express dynamic equilibrium problems in the common formulation.

Let us consider a partition of $[0, T]$, such that:
$0 = t_0 < t_1 < \ldots < t_r < \ldots < t_N = T$. For each point t_r, $r = 0, 1, \ldots, N$, of the partition, we consider the finite-dimensional variational inequality

$$\langle -\nabla_D v(t_r, x^*(t_r)), x(t_r) - x^*(t_r) \rangle \geq 0, \qquad \forall x(t_r) \in \mathbb{K}(t_r).$$

Now, we can compute the solution to (5.2.1) by using a class of projection-contraction methods proposed by Solodov and Tseng in [99] and improved by Tinti in [103].

The idea of these algorithms is to choose a symmetric positive definite matrix $M \in \mathbb{R}^{q \times q}$ and a starting point $x^{*0}(t_r) \in \mathbb{K}(t_r)$, and to iteratively update $x^{*k}(t_r)$, as follows:

$$\begin{aligned} \overline{x}^{*k}(t_r) &= P_{\mathbb{K}(t_r)}(x^{*k}(t_r) + \alpha \nabla_D v(t_r, x^{*k}(t_r))), \\ x^{*k+1}(t_r) &= x^{*k}(t_r) - \gamma M^{-1}(T_\alpha(t_r, x^{*k}(t_r)) - T_\alpha(t_r, \overline{x}^{*k}(t_r)), \end{aligned}$$

where $\gamma \in \mathbb{R}_+$ and $T_\alpha = (I + \alpha \nabla_D v)$, in which I is the identity matrix, $\alpha \in (0, +\infty)$ is chosen dynamically (according to an Armijo type rule), so that T_α is strongly monotone. These methods converge under condition that a solution exists and the operator is monotone. They have an additional parameter, the scaling matrix M, that must be chosen as a symmetric positive matrix to accelerate the convergence.

After the iterative procedure, we construct the dynamic equilibrium solution by means of a linear interpolation.

5.3.1 A numerical example

Let us consider four firms and four demand markets, as in Figure 5.3. Let $\underline{x}, \overline{x} \in L^2([0,1], \mathbb{R}^{16}_+)$ be the capacity constraints such that, a.e. in $[0, 1]$,

$$\underline{x}(t) = \begin{pmatrix} 0 & t & \frac{1}{2}t & \frac{1}{100}t \\ \frac{1}{2}t & t & 0 & 0 \\ \frac{1}{4}t & \frac{1}{4}t & 0 & \frac{1}{4}t \\ 0 & t & 2t & 0 \end{pmatrix},$$

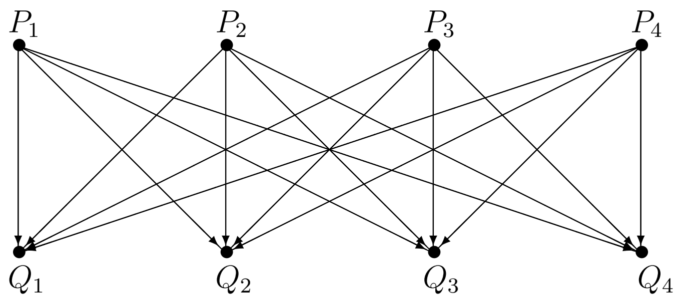

Figure 5.3: Network structure of the numerical dynamic spatial oligopoly problem.

$$\overline{x}(t) = \begin{pmatrix} 10t & 9t & 12t & 12t \\ 9t & 20t & 5t & 5t \\ 10t & 15t & 10t & 11t \\ 10t & 15t & 11t & 10t \end{pmatrix}.$$

Let $p \in L^2([0,1], \mathbb{R}^4_+)$ be the production function such that, a.e. in $[0,1]$,

$$p(t) = \begin{pmatrix} 40t \\ 40t \\ 50t \\ 50t \end{pmatrix},$$

and let $q \in L^2([0,1], \mathbb{R}^4_+)$ be the demand function such that, a.e. in $[0,1]$,

$$q(t) = \begin{pmatrix} 45t \\ 45t \\ 55t \\ 55t \end{pmatrix}.$$

As a consequence, the feasible set is

$$\begin{aligned} \mathbb{K} \;=\; & \Bigg\{ x \in L^2([0,1]\,, \mathbb{R}^{16}_+) : \\ & \underline{x}_{ij}(t) \le x_{ij}(t) \le \overline{x}_{ij}(t), \quad \forall i = 1, \dots, 4,\ \forall j = 1, \dots, 4,\ \text{a.e. in}\,[0,1]\,, \\ & \sum_{j=1}^{4} x_{ij}(t) \le p_i(t), \quad \forall i = 1, \dots, 4,\ \text{a.e. in}\ [0,1]\,, \\ & \sum_{i=1}^{4} x_{ij}(t) \le q_j(t), \quad \forall j = 1, \dots, 4,\ \text{a.e. in}\ [0,1] \Bigg\}. \end{aligned}$$

Let us consider the profit function $v \in L^2([0,1] \times L^2([0,1], \mathbb{R}^{16}_+), \mathbb{R}^4)$ defined by

$$\begin{aligned}
v_1(t,x(t)) &= -(t+10)x_{11}^2(t) - \frac{1}{2}x_{12}^2(t) - \frac{2t+9}{2}x_{13}^2(t) - \frac{3}{2}x_{14}^2(t) - 2tx_{11}(t)x_{44}(t) \\
&\quad +tx_{11}(t) + 2tx_{12}(t) + (2t+9)x_{13}(t) + 5tx_{14}(t), \\
v_2(t,x(t)) &= -\frac{5}{2}x_{21}^2(t) - (t+4)x_{22}^2(t) - \frac{t+7}{2}x_{23}^2(t) - 2x_{24}^2(t) - 2(t+1)x_{22}(t)x_{33}(t) \\
&\quad 6tx_{21}(t) + 11tx_{22}(t) + (t+1)x_{23}(t) + 9tx_{24}(t), \\
v_3(t,x(t)) &= -2x_{31}^2(t) - 3x_{32}^2(t) - \frac{2t+9}{2}x_{33}^2(t) - \frac{t+3}{2}x_{34}^2(t) - 4tx_{33}(t)x_{22}(t) \\
&\quad +(t+2)x_{31}(t) + tx_{32}(t) + 10tx_{33}(t) + 2tx_{34}(t), \\
v_4(t,x(t)) &= -3x_{41}^2(t) - 2x_{42}^2(t) - \frac{7}{2}x_{43}^2(t) - \frac{2t+9}{2}x_{44}^2(t) - 6(t+1)x_{44}(t)x_{11}(t) \\
&\quad +tx_{41}(t) + (2t+3)x_{42}(t) + 2tx_{43}(t) + 9tx_{44}(t).
\end{aligned}$$

Let the operator $\nabla_D v \in L^2([0,1] \times L^2([0,1], \mathbb{R}^{16}_+), \mathbb{R}^4)$ be the operator of the partial derivatives, namely

$$\nabla_D v(t,x(t)) = \left(\frac{\partial v_i(t,x(t))}{\partial x_{ij}(t)} \right)_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$$

It is possible to prove that the operator $-\nabla_D v$ is strongly monotone and Lipschitz continuous, infact,

$$\langle -\nabla_D v(x) + \nabla_D v(y), x-y \rangle \geq \|x-y\|_{16}^2$$

and

$$\| -\nabla_D v(x) + \nabla_D v(y)\|_{16}^2 \leq 1256 \|x-y\|_{16}^2.$$

The dynamic oligopolistic market equilibrium distribution in presence of excesses is the solution to the evolutionary variational inequality:

$$\int_0^1 \sum_{i=1}^4 \sum_{j=1}^4 -\frac{\partial v_i(t,x^*(t))}{\partial x_{ij}} (x_{ij}(t) - x_{ij}^*(t)) dt \geq 0, \quad \forall x \in \mathbb{K}.$$

Now, we solve the numerical problem using the generalized projection-contraction method. This method is convergent for the properties of $-\nabla_D v$. Then, we can compute an approximate curve of equilibria, by selecting $t_r \in \left\{ \frac{k}{20} : k \in \{0,1,\dots,20\} \right\}$. With the

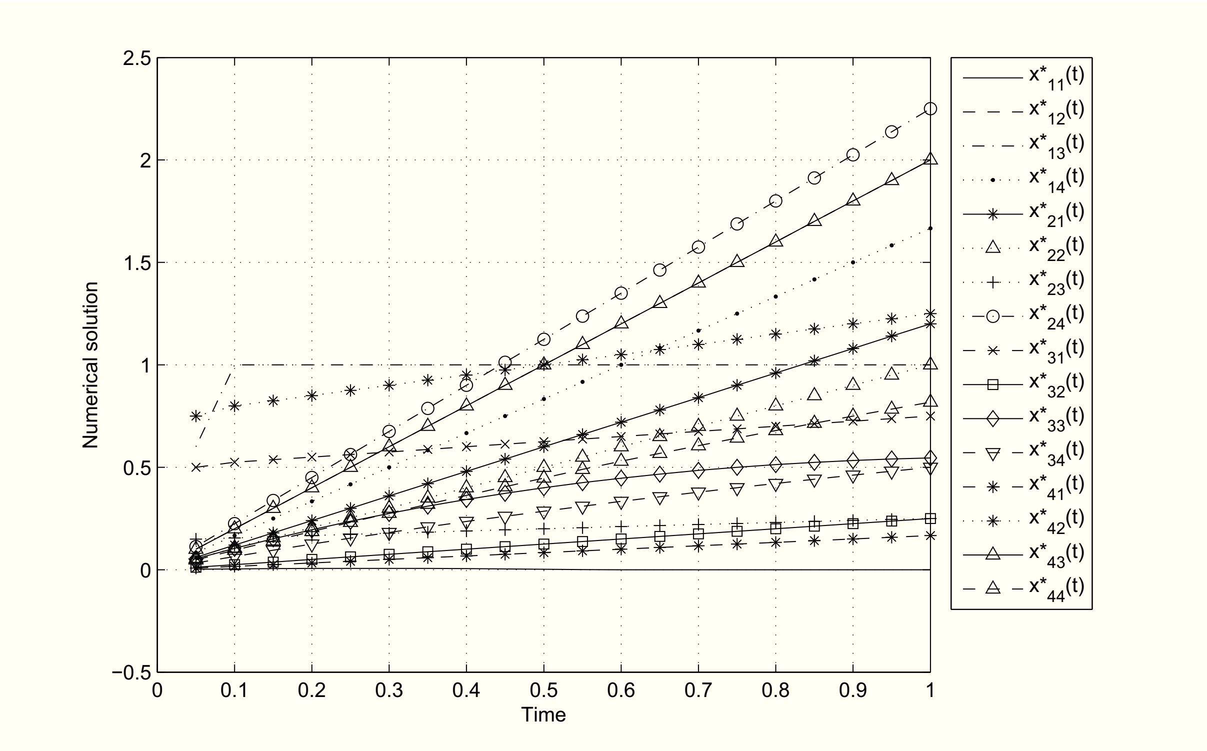

Figure 5.4: Curves of equilibria

help of a MatLab computation and choosing the initial point

$$x^{*0}(t_r) = \begin{pmatrix} \frac{2}{3}t_r & 2t_r & \frac{3}{4}t_r & 2t_r \\ \frac{3}{4}t_r & \frac{11}{10}t_r & \frac{3}{4}t_r & \frac{1}{6}t_r \\ t_r & \frac{2}{3}t_r & \frac{2}{3}t_r & \frac{1}{2}t_r \\ 2t_r & 2t_r & 3t_r & \frac{1}{3}t_r \end{pmatrix}$$

in order to start the iterative method, we obtain the equilibrium solutions for every time instant. Setting $R(x^{*k}(t_r)) = x^{*k}(t_r) - x^{*k-1}(t_r)$ the difference between two approximations of the equilibrium solution in the time instant t_r and making use of the stopping criterion $\left\|R(x^{*k}(t_r))\right\|_{4\times 4} \leq 10^{-6}$, for $r = 0, 1, \ldots, 20$, we obtain the approximate curve of equilibria shown in Figure 5.4.

5.4 The generalized extragradient method

Let us introduce the generalized extragradient method to solve dynamic oligopolistic market equilibrium problems in presence of excess expressed by evolutionary variational inequalities.

We suppose that the assumptions which ensure the continuity of dynamic oligopolistic market equilibrium solution hold. As a consequence, (5.2.1) holds and the time-dependent constraint sets $\mathbb{K}(t)$ are (5.2.2), (5.2.3) and (5.2.5), respectively, which express dynamic equilibrium problems in the common formulation.
In the following, applying a discretization procedure and making use of the extragradient method in Marcotte's version, we compute the solutions of the evolutionary variational inequality which expresses the dynamic oligopolistic market equilibrium conditions.

Let us consider a partition of $[0, T]$, such that:
$0 = t_0 < t_1 < \ldots < t_r < \ldots < t_N = T$. For each point t_r, $r = 0, 1, \ldots, N$, of the partition, we consider the finite-dimensional variational inequality

$$\langle -\nabla_D v(t_r, x^*(t_r)), x(t_r) - x^*(t_r)\rangle \geq 0, \qquad \forall x(t_r) \in \mathbb{K}(t_r). \tag{5.4.1}$$

We compute now the solution to the finite-dimensional variational inequality (5.4.1) making use of a modified version of the extragradient method introduced by Marcotte in [68].

The algorithm starting from any $x^{*0}(t_r) \in \mathbb{K}(t_r)$ and a fixed number $\alpha_0 > 0$, iteratively updates $x^{*k+1}(t_r)$ from $x^{*k}(t_r)$ according to the following projection formulas

$$x^{*k+1}(t_r) = P_{\mathbb{K}(t_r)}(x^{*k}(t_r) - \alpha_k(-\nabla_D(\tilde{x}^{*k}(t_r)))), \tag{5.4.2}$$

$$\tilde{x}^{*k}(t_r) = P_{\mathbb{K}(t_r)}(x^{*k}(t_r) - \alpha_k(-\nabla_D(x^{*k}(t_r)))) \tag{5.4.3}$$

for $k \in \mathbb{N}$, where α_k is chosen as following

$$\alpha_k = \min\left\{\frac{\alpha_{k-1}}{2}, \frac{\|x^{*k}(t_r) - \tilde{x}^{*k}(t_r)\|_{mn}}{\sqrt{2}\|(-\nabla_D(x^{*k}(t_r))) - (-\nabla_D(\tilde{x}^{*k}(t_r)))\|_{mn}}\right\}. \tag{5.4.4}$$

If $-\nabla_D v$ is a monotone and Lipschitz continuous mapping, then, the convergence of the scheme is proved. This method was improved by Tinti in [103].

After the iterative procedure, we get the dynamic equilibrium solution through a linear interpolation of the computed static equilibrium solutions.

5.4.1 A numerical example

Let us consider a numerical example of the dynamical oligopolistic market equilibrium problem in presence solved using the direct method and the generalized extragradient method consisting of three firms and four demand markets, as in Figure 5.5.

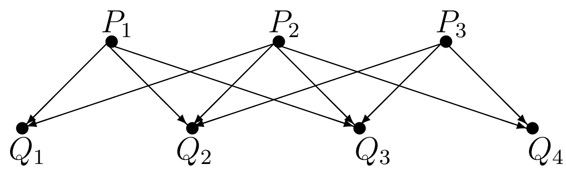

Figure 5.5: Network structure of the numerical dynamic spatial oligopoly problem.

Let $p \in C([0,1], \mathbb{R}^3)$ be the production function such that, in $[0,1]$,

$$p(t) = \begin{pmatrix} 4t & 3t & 4t \end{pmatrix}^T,$$

As a consequence, the feasible set is

$$\mathbb{K} = \Big\{ x \in L^2([0,1], \mathbb{R}^{3\times 4}) : \quad x_{ij}(t) \geq 0, \quad i = 1,2,3, \; j = 1,2,3,4, \text{ a.e. in } [0,1], \\ \sum_{j=1}^{4} x_{ij}(t) \leq p_i(t), \quad i = 1,2,3, \text{ a.e. in } [0,1] \Big\}.$$

Let $v \in C^1(L^2([0,1], \mathbb{R}^{3\times 4}), \mathbb{R}^3)$ be the profit function defined by

$$\begin{aligned} v_1(t,x(t)) &= -4x_{11}^2(t) - 4x_{12}^2(t) - 6x_{13}^2(t) - 6x_{14}^2(t) - 4x_{11}(t)x_{12}(t) - 6x_{13}(t)x_{14}(t) \\ &\quad +3tx_{11}(t) + 4tx_{12}(t) + tx_{13}(t) + tx_{14}(t), \\ v_2(t,x(t)) &= -5x_{21}^2(t) - 2x_{22}^2(t) - 2x_{23}^2(t) - 2x_{24}^2(t) - 2x_{21}(t)x_{22}(t) - 2x_{23}(t)x_{24}(t) \\ &\quad +2tx_{21}(t) + 2tx_{22}(t) + 3tx_{23}(t) + 2tx_{24}(t), \\ v_3(t,x(t)) &= -10x_{31}^2(t) - 4x_{32}^2(t) - 4x_{33}^2(t) - 5x_{34}^2(t) - 2x_{11}(t)x_{32}(t) - 2x_{33}(t)x_{34}(t) \\ &\quad -2x_{12}(t)x_{31}(t) + tx_{31}(t) + 2tx_{32}(t) + 10tx_{33}(t) + 3tx_{34}(t). \end{aligned}$$

Then, the operator $\nabla_D v \in C(L^2([0,1], \mathbb{R}^{3\times 4}), \mathbb{R}^{3\times 4})$ is given by

$$\nabla_D v = \begin{pmatrix} -8x_{11} - 4x_{12} + 3t & -8x_{12} - 4x_{11} + 4t & -12x_{13} - 6x_{14} + t & -12x_{14} - 6x_{13} + t \\ -10x_{21} - 2x_{22} + 2t & -4x_{22} - 2x_{21} + 2t & -4x_{23} - 2x_{24} + 3t & -4x_{24} - 2x_{23} + 2t \\ -20x_{31} - 2x_{12} + t & -8x_{32} - 2x_{11} + 2t & -8x_{33} - 2x_{34} + 10t & -10x_{34} - 2x_{33} + 3t \end{pmatrix}.$$

Moreover, it is possible to verify that $-\nabla_D v$ is a strongly monotone operator (for the proof, you can see similar calculations in Chapter 2).

The dynamic oligopolistic market equilibrium distribution in presence of excesses is the solution to the evolutionary variational inequality:

$$\int_0^1 \sum_{i=1}^{3} \sum_{j=1}^{4} -\frac{\partial v_i(t, x^*(t))}{\partial x_{ij}} (x_{ij}(t) - x_{ij}^*(t)) dt \geq 0, \quad \forall x \in \mathbb{K}. \tag{5.4.5}$$

Taking into account the direct method, we consider the following system

$$\begin{cases} 8x_{11}^*(t) + 4x_{12}^*(t) - 3t = 0, & 4x_{11}^*(t) + 8x_{12}^*(t) - 4t = 0, \\ 12x_{13}^*(t) + 6x_{14}^*(t) - t = 0, & 6x_{13}^*(t) + 12x_{14}^*(t) - t = 0, \\ 10x_{21}^*(t) + 2x_{22}^*(t) - 2t = 0, & 2x_{21}^*(t) + 4x_{22}^*(t) - 2t = 0, \\ 4x_{23}^*(t) + 2x_{24}^*(t) - 3t = 0, & 2x_{23}^*(t) + 4x_{24}^*(t) - 2t = 0, \\ 20x_{31}^*(t) + 2x_{12}^*(t) - t = 0, & 8x_{32}^*(t) + 2x_{11}^*(t) - 2t = 0, \\ 8x_{33}^*(t) + 2x_{34}^*(t) - 10t = 0, & 2x_{33}^*(t) + 10x_{34}^*(t) - 3t = 0 \end{cases} \tag{5.4.6}$$

We get the following solution, in $[0,1]$,

$$x^*(t) = \begin{pmatrix} \frac{1}{6}t & \frac{5}{12}t & \frac{1}{18}t & \frac{1}{18}t \\ \frac{1}{9}t & \frac{4}{9}t & \frac{2}{3}t & \frac{1}{6}t \\ \frac{1}{120}t & \frac{5}{24}t & \frac{47}{38}t & \frac{1}{19}t \end{pmatrix},$$

that belongs to the constraint set $\mathbb{K}$, then it is the equilibrium solution. Moreover, the production excesses, in $[0,1]$, are given by

$$\epsilon(t) = \begin{pmatrix} \frac{119}{36}t & \frac{29}{18}t & \frac{2843}{1140}t \end{pmatrix}^T.$$

Now, we solve the numerical problem using the generalized Marcotte's version of the extragradient method. This method is convergent for the properties of $-\nabla_D v$. Then, we can compute an approximate curve of equilibria, by selecting $t_r \in \left\{ \frac{k}{20} : k \in \{0,1,\dots,20\} \right\}$. With the help of a MatLab computation and choosing the initial point

$$x^{*0}(t_r) = \begin{pmatrix} \frac{2}{3}t_r & \frac{1}{3}t_r & \frac{1}{2}t_r & t_r \\ \frac{1}{2}t_r & \frac{1}{10}t_r & \frac{1}{2}t_r & \frac{1}{8}t_r \\ t_r & \frac{1}{3}t_r & \frac{2}{5}t_r & \frac{1}{8}t_r \end{pmatrix}$$

in order to start the iterative method, we obtain the equilibrium solutions for every time instant. Setting $R(x^{*k}(t_r)) = x^{*k}(t_r) - x^{*k-1}(t_r)$ the difference between two approximations of the equilibrium solution in the time instant t_r and making use of the stopping criterion $\left\|R(x^{*k}(t_r))\right\|_{3\times 4} \leq 10^{-6}$, for $r = 0,1,\dots,20$, we obtain the approximate curve of equilibria shown in Figure 5.6.

5.5 The convergence study

In this section, we investigate the convergence of algorithms presented in the previous section. We prove that under suitable assumptions, the sequence, generated by the algorithm, converges in L^1-sense to the dynamic oligopolistic market equilibrium solution in presence of excesses.

Let us assume that all hypotheses which ensure the continuity of solution to the point-to-point evolutionary variational inequality (5.2.1) and the convergence of the method to compute solutions to finite-dimensional variational inequalities hold.

Let us introduce a sequence $\{\pi_s\}_{s\in\mathbb{N}}$ of (not necessarily equidivided) partitions of the time interval $[0,T]$ such that $\pi_s = (t_s^0, t_s^1, \dots, t_s^{N_s})$, where $0 = t_s^0 < t_s^1 < \dots < t_s^{N_s} = T$. We consider a sequence of equidivided partitions, in the sense that $k_s = \max\{|t_s^r - t_s^{r-1}| : r = 1,2,\dots,N_s\}$, approaches zero for $s \to +\infty$.

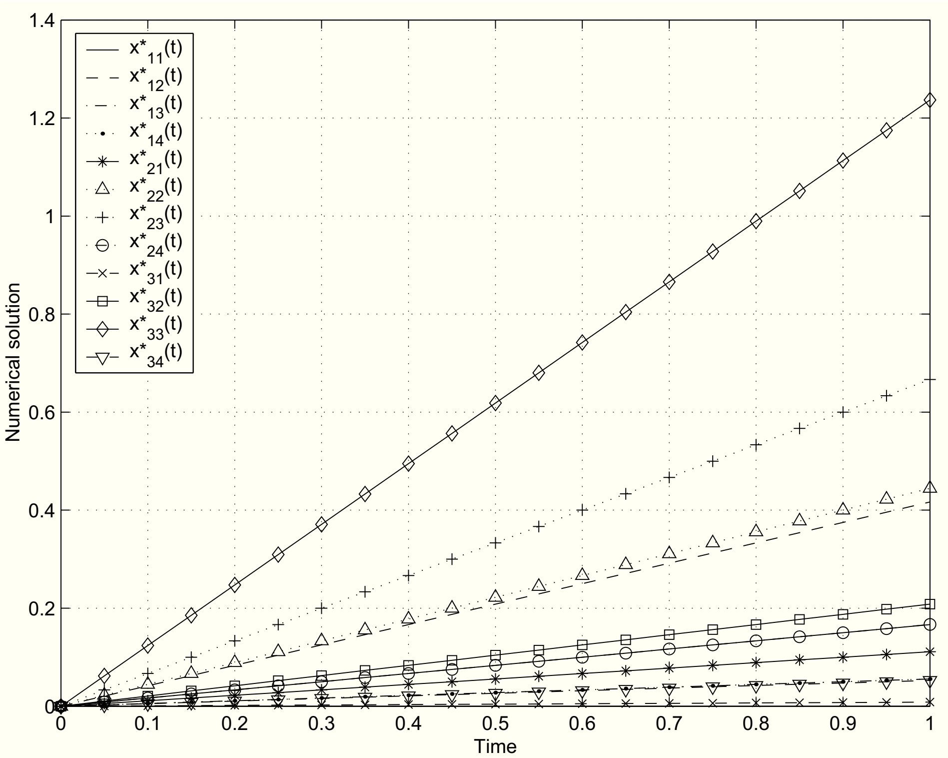

Figure 5.6: Curves of equilibria

By the interpolation theory, we know that if we construct the approximate solution to (5.2.1) by means of Hermite's polynomial, using known values of the solution $x^*(t)$, the sequence converges uniformly to the exact solution. We do not use Hermite's polynomial, but we consider an approximation by means of piecewise constant functions and we can prove that the convergence is in L^1-sense.

Let us consider the approximate solutions to (5.2.1) given by the following formula:

$$x_s^*(t) = \sum_{r=1}^{N_s} x^*(t_s^r)\chi_{[t_s^{r-1}, t_s^r[}(t), \tag{5.5.1}$$

where $x^*(t_s^r)$ is the solution to the finite-dimensional variational inequality which is obtained from (5.2.1) for $t = t_s^r$, which can be computed by means of a numerical method.

Let us estimate the following integral

$$\begin{aligned}
&\int_0^T \left\| x^*(t) - \sum_{r=1}^{N_s} x^*(t_s^r)\chi_{[t_s^{r-1}, t_s^r[}(t) \right\|_{mn} dt \\
&\qquad = \int_0^T \left\| \sum_{r=1}^{N_s} x^*(t)\chi_{[t_s^{r-1}, t_s^r[}(t) - \sum_{r=1}^{N_s} x^*(t_s^r)\chi_{[t_s^{r-1}, t_s^r[}(t) \right\|_{mn} dt \\
&\qquad \leq \sum_{r=1}^{N_s} \int_{t_{r-1}^s}^{t_r^s} \| x^*(t) - x^*(t_s^r)\|_{mn} dt \qquad .
\end{aligned}$$

Being x^* uniformly continuous, it results that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $t \in [t_s^{r-1}, t_s^r]$ satisfies the condition $|t - t_s^r| < \delta$ it results

$$\| x^*(t) - x^*(t_s^r)\|_{mn} < \frac{\varepsilon}{T}, \quad \text{for } r = 1, 2, \ldots, N_s,\ \forall s \in \mathbb{N}.$$

Choosing s large enough in such way that $k_s < \delta$, we reach

$$\int_0^T \left\| x^*(t) - \sum_{r=1}^{N_s} x^*(t_s^r)\chi_{[t_{r-1}^2, t_r^s[}(t) \right\|_{mn} dt < \sum_{r=1}^{N_s} \frac{\varepsilon}{T}(t_r^s - t_{r-1}^s) = \varepsilon. \tag{5.5.2}$$

Taking into account (5.5.2), we conclude that the sequence (5.5.1) converges in L^1-sense to the dynamic oligopolistic market equilibrium solution.

Conclusions

The aim of this thesis is to analyze the multiple aspects of the dynamic oligopolistic market equilibrium problem, by improving, the primordial models introduced by Cournot and Nash in [26, 83, 84], the static model studied by Nagurney in [28, 75, 78, 80] and the first dynamic formulation in [9] where only capacity constraints were considered. In particular, in this thesis we investigate initially the presence of only production excesses. Later we improved this problem, by considering a more complete model in which we consider not only capacity constraints, bu also production and demand excesses and we underline the presence of the constraints through the Lagrangean duality theory.
The most important result of this theory is the equivalence of the equilibrium definition according to Cournot-Nash principle with a suitable evolutionary variational inequality. Moreover, a definition of equilibrium in which the Lagrange variables of the Duality theory is present. The equivalence between the two definitions is then proved and it is justified by the surprising fact that they both are equivalent to the same evolutionary variational inequality. The variational approach of the problem is fundamental in order to establish, under suitable assumptions, some existence and regularity results for equilibrium solutions. The regularity results are very important for the computation of solution because they allow us to reduce, by means of a discretization procedure, the infinite-dimensional problem to some finite-dimensional problems so that it is possible to make use of some methods to solve static equilibrium problems. Then, by means of an interpolation procedure, we are able to find the dynamic equilibrium solutions. Moreover, the convergence of the scheme in $L^1\left([0,T],\mathbb{R}^{mn}\right)$ has been proved.
Another aspect we have deeply faced in this thesis, is to consider the inverse problem. Here we forsake the firms' point of view whose main purpose is to maximize their own profit, and we focus our attention to the policy-maker's point of view. The control policies' aim is to regulate the exportation through the adjustment of taxes and subsides on the firms. This is a policy-maker optimization problem and here we define an optimal regulatory tax. It is worth to emphasize that this problem can be studied with the help of the inverse variational inequalities (see also [86]). Another surprising fact is the strict connection between the classical variational inequalities and inverse variational inequalities that enables to exploit all the powerful tools of evolutionary variational inequalities. In particular, for this connection, we can treat completely the problem and analyze the questions about the existence, the regularity and the computation of solution.

It still remains to investigate the case in which the constraint set is not fixed, but it depends on the equilibrium solution, namely the production and the demand functions depend on the equilibrium solution. Such an approach can be studied making use of evolutionary quasi-variational inequalities that exploit the cover part of the vast literature of multifunctional analysis (see [1, 3, 4, 24, 40–42, 91, 95, 102]).

Bibliography

[1] C. Baiocchi and A. Capelo, *Variational and Quasivariational Inequalities. Ap- plications to Free Boundary Problems,* J. Wiley & Sons, New York, 1984.

[2] A. Barbagallo, *Degenerate time-dependent variational inequalities with applications to traffic equilibrium problems,* Comput. Methods Appl. Math., **6**, 2006, pp. 117–133.

[3] A. Barbagallo, *Regularity results for time-dependent variational and quasi-variational inequalities and computational procedures,* Math. Mod. Meth. Appl. S., **17**, 2007, pp. 277–304.

[4] A. Barbagallo, *Regularity results for evolutionary variational and quasi-variational inequalities with applications to dynamic equilibrium problems,* J. Global Optim., **40**, 2008, pp. 29–39.

[5] A. Barbagallo, *Existence and regularity of solutions to nonlinear degenerate evolutionary variational inequalities with applications to dynamic equilibrium problems,* Appl. Math. Comput., 2009, **208**, pp. 1–13.

[6] A. Barbagallo, *On the regularity of retarded equilibria in time-dependent traffic equilibrium problems,* Nonlinear Anal., **71**, 2009, pp. e2406–e2417.

[7] A. Barbagallo, *Advanced results on variational inequality formulation in oligopolistic market equilirbrium problem,* Filomat, **5**, 2012, pp. 935–947.

[8] A. Barbagallo and M.G. Cojocaru, *Continuity of solutions for parametric variational inequalities in Banach spaces,* J. Math. Anal. Appl., **351**, 2009, pp. 707–720.

[9] A. Barbagallo and M.-G. Cojocaru, *Dynamic equilibrium formulation of oligopolistic market problem,* Math. Comp. Model., **49**, pp. 966–976, 2009.

[10] A. Barbagallo and R. Di Vincenzo, *Lipschitz continuity and duality for dynamic oligopolistic market equilibrium problem with memory term* J. Math. Anal. Appl. **382**, 2011, pp. 231–247.

[11] A. Barbagallo and A. Maugeri, *Memory term for dynamic oligopolistic market equilibrium problem,* Aplimat - Journal of Applied Mathematics, **3**, 2010, pp. 13–23.

[12] A. Barbagallo and A. Maugeri, *Duality theory for the dynamic oligopolistic market equilibrium problem,* Optimization, **60**, 2011, pp. 29–52.

[13] A. Barbagallo and P. Mauro, *Dynamic oligopolistic market equilibrium problems with production excesses: a variational approach,* in "Numerical Analysis and Applied Mathematics", Eds. T.E. Simos, G. Psihoyios, Ch. Tsitouras e Z. Anastassi, AIP Conference Proceedings, **1389**, 2011, pp. 941–944.

[14] A. Barbagallo and P. Mauro, *Evolutionary variational formulation for oligopolistic market equilibrium problems with production excesses,* J. Optim. Theory Appl. ,**155**, 2012, pp. 1–27.

[15] A. Barbagallo and P. Mauro, *Time-dependent variational inequality for an oligopolistic market equilibrium problem with production and demand excesses,* Abstr. Appl. Anal., **651975**, 2012.

[16] A. Barbagallo and P. Mauro, *On solving dynamic oligopolistic market equilibrium problems in presence of excesses,* Comm. Appl. Ind. Math., **3**, no.1, 2012, pp. 1–20.

[17] A. Barbagallo and P. Mauro, *Inverse variational inequality approach and applications,* 2013, submitted.

[18] A. Barbagallo and P. Mauro, *An inverse problem for the dynamic oligopolistic market equilibrium problem in presence of excesses*, 2013, preprint.

[19] A. Barbagallo, P. Daniele and A. Maugeri, *Variational formulation for a general dynamic financial equilibrium problem. Balance law and liability formula,* Nonlinear Anal., **75**, 2012, pp. 1104–1123.

[20] M.J. Beckmann and J.P. Wallace, *Continuous lags and the stability of market equilibrium,* Economica, New Series **36**, 1969, pp. 58–68.

[21] J.M. Borwein and A.S. Lewis, *Partially finite convex programming, Part I: Quasi relative interiors and duality theory,* Mathematical Programming, **57**, 1992, pp. 15–48.

[22] H. Brezis, *Équations et inéquations non linéaires dans les espace vectoriel en dualité,* Ann. Ist. Fourier, **18**, 1968, pp. 115–175.

[23] A. Causa, F. Raciti, *Lipschitz Continuity Results for a Class of Variational Inequalities and Applications: A Geometric Approach,* J. Optim. Theory Appl., **145**, 2010, pp. 235-248.

[24] D. Chan and J.S. Pang, *The generalized quasivariational inequality problem,* Math. Oper. Res. **1**, 1982, pp. 211–222.

[25] M.G. Cojocaru, P. Daniele and A. Nagurney, *Projected dynamical systems and evolutionary (time-dependent) variational inequalities via Hilbert spaces with applications,* J. Optim. Theory Appl. **27** no. 3, 2005, pp. 549–563.

[26] A. Cournot, *Researches into the mathematical principles of the theory of wealth,* English translation MacMillan, London, 1838.

[27] S. Dafermos, *Traffic Equilibrium and Variational Inequalities,* Transp. Sci. **14,** 1980, pp. 42–54.

[28] S. Dafermos, A. Nagurney, *Oligopolistic and competitive behavior of spatially separated markets,* Regional Science and Urban Economics, **17**, 1987, pp. 245–254.

[29] P. Daniele, *Evolutionary Variational Inequalities and Economics Models for Demand-Supply Markets,* Math. Mod. Meth. Appl. Sci. **4**, no. 13, 2003, pp. 471–489.

[30] P. Daniele, *Variational Inequalities for Evolutionary Financial Equilibrium,* in *Innovations in Financial and Economic Networks,* A. Nagurney (ed.), Edward Elgar Publishing, Cheltenham, England, 2003, pp. 84–108.

[31] P. Daniele, *Time-Dependent Spatial Price Equilibrium Problem: Existence and Stability Results for the Quantity Formulation Model,* J. Glob. Optim. **28**, 2004, pp. 283–295.

[32] P. Daniele, *Dynamic Networks and Evolutionary Variational Inequalities,* Edward Elgar Publishing, Chentelham, UK, 2006.

[33] P. Daniele and S. Giuffrè, *General Infinite Dimensional Duality and Applications to Evolutionary Network Equilibrium Problems,* Optimization Letters, **1**, 2007, pp. 227–243.

[34] P. Daniele and A. Maugeri, *On dynamic equilibrium problems and variational inequalities,* in *Equilibrium Problems: Non-smooth Optimization and Variational Inequality Models,* F. Giannessi, A. Maugeri and P. Pardalos (eds.), Kluwer Academic Publishers, The Netherlands, 2001, pp. 59–69.

[35] P. Daniele, S. Giuffrè, and A. Maugeri, *Remarks on general infinite dimensional duality with cone and equality constraints,* Communications in Applied Analysis, **13**, 2009, pp. 567–578.

[36] P. Daniele, S. Giuffré and S. Pia, *Competitive Financial Equilibrium Problems with Policy Interventions,* J, Ind. Manag. Optim. **1** no.1, 2005, pp. 39–52.

[37] P. Daniele, A. Maugeri and W. Oettli, *Variational Inequalities and Time-Dependent Traffic Equilibria,* C.R. Acad. Sci. Paris **326** no. 1, 1998, pp. 1059–1062.

[38] P. Daniele, A. Maugeri and W. Oettli, *Time-dependent traffic equilibria,* J. Optim. Theory Appl. **103** no. 3, 1999, pp. 543–554.

[39] P. Daniele, S. Giuffrè, G. Idone, and A. Maugeri, *Infinite Dimensional Duality and Applications,* Mathematische Annalen, **339**, 2007, pp. 221–239.

[40] M. De Luca and A. Maugeri, *Quasi-variational inequality and applications to equilibrium problems with elastic demands,* in Non smooth Optimization and Related Topics, F.M. Clarke, V.F. Dem'yanov and Giannessi (eds.), Plenum Press, New York, 1989, pp. 61–77.

[41] M. De Luca and A. Maugeri, *Quasi-variational inequalities and applications to the traffic equilibrium problem; discussion of a paradox,* J. Comput. Appl. Math. **28**, 1989, pp. 163–171.

[42] M. De Luca and A. Maugeri, *Discontinuous quasi-variational inequalities and applications to equilibrium problems,* in Nonsmooth Optimization Methods and Applications, F. Giannessi (ed.), Gordon and Breach Science Publishers, London, 1992, pp. 70–74.

[43] M.B. Donato, *The infinite dimensional Lagrange multiplier rule for convex optimization problems,* Journal of Functional Analysis, **261**, 2011, pp. 20832093.

[44] M.B. Donato, M. Milasi, C. Vitanza, *Dynamic Walrasian price equilibrium problem: evolutionary variational approach with sensitivity analisys,* Optimization Letters, **47**, 2007.

[45] M.B. Donato, M. Milasi and C. Vitanza, *A new contribution to a dynamic competitive equilibrium problem,* Appl. Math. Lett., **23**, 2010, pp. 148–151.

[46] P. Falsaperla, F. Raciti, *Global Approximation of Lipschitz Solutions of Time-Dependent Variational Inequalities with Error Estimates,* 2011, submitted.

[47] G. Fichera, *Problemi elastostatici con vincoli unilaterali: il problema di Signorini con ambigue condizioni al contorno,* Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Nat. Sez. Ia. **7**, no. 8, 1963–1964, pp. 91–140.

[48] T.L. Friesz, D. Bernstein, T.E. Smith, R.L. Tobin and B.W. Wie *A variational inequality formulation of the dynamic network user equilibrium problem,* Oper. Res. **41**, 1993, pp. 179–191.

[49] D. Gabay, H. Moulin, *On the uniqueness and stability of Nash equilibria in noncooperative games,* A. Bensousson, et al. (Eds.), Applied Stochastic Control in Econometrics and Management Science, North Holland, Amsterdam, 1980, pp. 271–293.

[50] S. Giuffré S, G. Idone and S. Pia, *Some classes of projected dynamical systems in Banach spaces and variational inequalities,* J. Global Optim., **40**, 2008, pp. 119–128.

[51] J. Gwinner and F. Raciti, *On a class of random variational inequalities on random sets,* Numer. Funct. Anal. Optim., **27**, 2006, pp. 619–636.

[52] P. Hammerstein and R. Steten, *Game theory and evolutionary biology,* Handbook of Game Theory with Economic Applications, **2**, Aumann and Hart (eds.), Elsevier, 1994, pp. 929–993.

[53] P. Hartmann and G. Stampacchia, *On some nonlinear elliptic differential functional equations,* Acta Math. **115**, 1966, pp. 271–310.

[54] B. He, X.Z. He and H.X. Liu, *Solving a class of 'black-box' inverse variational inequalities,* Eur. J. Oper. Res., **234**, 2010, pp. 283–295.

[55] G. Idone, A. Maugeri, *Generalized constraints qualification conditions and Infinite Dimensional Duality,* Taiwanese Journal of Mathematic, **13** no. 6A, pp. 1711–1722.

[56] J. Jahn, *Introduction to the theory of nonlinear optimization,* Springer-Verlag, Berlin, D, 1996.

[57] A. Jofre, R.T. Rockafellar, R.J.-B. Wets, *Variational inequalities and economic equilirium,* Math. Oper. Res., 2006.

[58] S. Karamardian, *Non-linear complementary problem with applications,* J. Optim. Theory Appl. **4**, 1969, Part I pp. 87–98, Part II pp. 167–135.

[59] S. Karamardian, *The complementarity problem,* Math. Program. **2** no.1, 1972, pp. 107–129.

[60] D. Kinderleher, G. Stampacchia, *An Introduction to Variational Inequalities and their Applications,* Academic Press, New York, 1980.

[61] K. Kuratowski, *Topology,* Academic Press, New York, USA, 1966.

[62] J.L. Lions and G. Stampacchia, *Inéquationnes variationnelles non coercives,* Compt. Rend. Acad. Sci. Paris **261**, 1965, pp. 25–27.

[63] J.L. Lions and G. Stampacchia, *Variational Inequalities*, Commun. Pure Appl. Math. **20**, 1967, pp. 493–519.

[64] S. Lu, *Sensitivity of static traffic equilibria with perturbation in arc cost function and travel demand,* Transp. Sci., **42**, 2008, pp. 105-123.

[65] S. Lu, S.M. Robinson, *Variational inequalities over perturbed polyhedral convex sets,* Math. Oper. Res., **33**, 2008, pp. 689-711.

[66] O. Mancino and G. Stampacchia, *Convex programming and variational inequalities,* J. Optim. Theory Appl. **9**, 1972, pp. 3–23.

[67] O. Mangasarian, *Pseudoconvex functions*, Journal of the Society for Industrial and Applied Mathematics Series A Control, **3**, 1965, pp. 281–290.

[68] P. Marcotte, *Application of Khobotov's algorithm to variational inequalities and network equilibrium problems,* Inform. Systems Oper. Res. **29**, 1991, pp. 258–270.

[69] P. Marcotte and J.H. Wu, *On the convergence of projection methods application to the decomposition of affine variational inqualities,* J. Optim. Theory Appl. **85** no. 2, 1995, pp. 347–362.

[70] A. Maugeri, *Convex Programming, Variational Inequalities and applications to the traffic equilibrium problem.,* Appl. Math. Optim., **16**, 1987, pp. 169–185.

[71] A. Maugeri and F. Raciti, *On existence theorems for monotone and nonmonotone variational inequalities,* J. Convex Anal., **16**, 2009, pp. 899–911.

[72] A. Maugeri and F. Raciti, *Remarks on Infinite Dimensional Duality,* Journal of Global Optimization,**46**, 2010, pp. 581–588.

[73] A. Maugeri, L. Scrimali, *Global Lipschitz continuity of solutions to parametrized variational inequalities,* Boll. Unione Math. Ital. II, 2009, pp. 45-69.

[74] U. Mosco, *Convergence of convex sets and of solutions to variational inequalities,* Adv. Math., **3** no.3, 1969, pp. 510–585.

[75] A. Nagurney, *Algorithms for oligopolistic market equilibrium problems,* Regional Science and Urban Economics, **18**, 1988, pp. 425–445.

[76] A. Nagurney, *Migration equilibrium and variational inequalities,* Econ. Letts. **31**, 1989, pp. 109–112.

[77] A. Nagurney, *Network Economics: A Variational Inequality Approach,* Kluwer, 1993.

[78] A. Nagurney, *Network economics: a variational inequality approach,* Kluwer Academic Publishers, Boston, Massachusetts, 1998.

[79] A. Nagurney and Z. Liu, *Transportation network equilibrium reformulations of electric power networks with computations.* Isenberg School of Management, University of Massachusetts at Amherst, 2005.

[80] A. Nagurney, P. Dupuis, D. Zhang, *A dynamical systems approach for network oligopolies and variational inequalities,* The Annals of Regional Science, **28**, 1994, pp. 263–283.

[81] A. Nagurney, J. Pan and L. Zhao, *Human migration networks,* Eur. J. Oper. Res. **59**, 1992, pp. 262–274.

[82] A. Nagurney, Z. Liu, M.G. Cojocaru and P. Daniele, *Static and Dynamic Transportation Network Equilibrium Reformulations of Electric Power Supply Chain Networks with Known Demands,* Transportation Research E, 2006.

[83] J.F. Nash, *Equilibrium points in n-person games,* Proc. Natl. Acad. Sci., USA., **36**, 1950, pp. 48–49.

[84] J. F. Nash, *Non-cooperative games,* Ann. Math., **54**, 1951, pp. 286–295.

[85] M.A. Noor, *Variational inequalities in physical oceanography,* Ocean Waves Engineering. Comput. Mechanics. Publ., M. Rahmen (ed.), U.K., 1994, pp. 201–226.

[86] M.A. Noor, *Some developments in general variational inequalities,* Appl. Math. Comput., **155**, 2004, pp. 199–277.

[87] W. Oettli and D. Schlager, *Generalized vectorial equilibria and generalized monotonicity,* in: Functional Analysis with Current Applications, M. Brokate and A.H. Siddiqi (eds.), Longman, Harlow, England, 1998, pp. 145–154.

[88] M. Patriksson, *Nonlinear Programming and Variational Inequality Problems, A Unified Approach,* Applied Optimization **23**, Kluwer, 1999.

[89] F. Raciti, *Equilibrium conditions and vector variational inequalities: A complex relation,* J. Global Optim., **40**, 2008, pp. 353–360.

[90] F. Raciti and L. Scrimali, *Time-dependent variational inequalities and applications to equilibrium problems,* J. Global Optim., **28**, 2004, 387–400.

[91] B. Ricceri, *Basic Existence Theorems for Generalized Variational and Quasi- Variational Inequalities,* in Variational Inequalities and Network Equi- librium Problems, F. Giannessi and A. Maugeri (eds.), Plenum Publishing, New Yourk, 1995, pp. 251–255.

[92] J.B. Rosen, *Existence and uniqueness of equilibrium points for concave n-person games,* Econometrica **33** no. 3, 1965, pp. 520–533.

[93] G. Salinetti, R.J.-B. Wets, *On the convergence of sequences of convex sets in finite dimensions,* SIAM Review, **21**, 1979, pp. 18–33.

[94] G. Salinetti, R.J.-B. Wets, *Addendum: On the convergence of convex sets in finite dimensions,* SIAM Review, **22**, 1980, p. 86.

[95] L. Scrimali, *Quasi-variational inequalities in transportation networks,* Math. Models Methods Appl. Sci. **14** no. 10, 2004, pp. 1541–1560.

[96] L. Scrimali, *An inverse variational inequality approach to the evolutionary spatial price equilibrium problem,* Optim. Eng., **13**, 2011, pp. 375–387.

[97] M. Sibony, *Méthodes itératives pour les équations et inéquations aux dérivés partielles nonlinéares de type monotone,* Calcolo **7**, 1970, pp. 65–183.

[98] M.J. Smith, *The existence, uniqueness and stability of traffic equilibria,* Transp. Res. **13** B, 1979, pp. 295–304.

[99] M.V. Solodov, P. Tseng, *Modified projection-type methods for monotone variational inequalities,* SIAM Journal on Control and Optimization, **34**, 1996, pp. 1814–1830.

[100] G. Stampacchia, *Formes bilineaires coercitives sur les ensembles convexes,* Compt. Rend. Acad. Sci. Paris **258**, 1964, pp. 4413–4416.

[101] G. Stampacchia, *Variational Inequalities in Theory Applications of Monotone Operators*, Proc. NATO Advanced Study Inst. (Venice, 1968), Edizioni "Oderisi", Gubbio, 1969, pp. 101–192.

[102] G. Tian and J. Zhou, *Quasi-variational inequality with non-compact sets,* J. Math. Anal. Appl. 160, 1991, pp. 583–595.

[103] F. Tinti, *Numerical solution for pseudomonotone variational inequality problems by extragradient methods,* Variational analysis and applications, Nonconvex Optim. Appl., **79**, 2005, pp. 1101–1128.

[104] J. Yang, *Dynamic power price problem: an inverse variational inequaliti approach,* J. Ind. Manag. Optim., **4**, 2008, pp. 673–684.

[105] N.D. Yen, *Lipschitz continuity of solutions of variational inequalities with a parametric polyhedral constraint,* Math. Oper. Res., **20**, 1995, pp. 695-708.

[106] C. Zalinescu, *Convex Analysis in general vector spaces,* World Scientific, Singapore, 2002.