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Synchronization and Convergence of Piecewise Smooth Systems and Networks

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Chapter 1

Introduction

Contents

1.1 Motivation

In recent years the problem of analyzing and controlling of dynamical networks has been widely studied in the literature [29, 146, 218, 54]. Dynamical networks have been formalized by looking at the examples of complex networks present in nature and in many different branches of science. Indeed, networks can be found in physics, biology, technology and social science [45, 105, 88, 131, 153, 94, 68, 185, 110, 19, 130, 213].

The interconnection among a large number of subsystems cooperating in order to achieve a common objective or a collective behaviour, which is in general not known at the level of the single agent, has inspired the formalism of *networks of dynamical systems*. These networks model all the scenarios characterized by hundreds or thousands of dynamical agents which communicate over a graph of local interactions in order to achieve some coordinated motion. Such networks can be both used to analyse the phenomena and the applications from the natural world cited above and to design distributed control laws in many engineering applications. In particular, the collective behaviour of synchronization [29, 54, 218, 164, 226] has been showed to be a paradigm also for other behaviours, e.g. consensus [149, 174, 216] and flocking [13, 32, 67].

At the same time, starting from the pioneering work in [58], the concepts of incremental stability [10], convergent systems [58, 155] and contraction theory[126, 107], have been formalized in the last two decades and, although these concepts are different from each other, all of them look at the convergence of trajectories of a dynamical system towards each others. So, in this case, the point of view is no more the convergence of the trajectories towards an equilibrium point (or more generally towards an invariant set), but the convergence to zero of the relative error between the trajectories themselves. Although these tools are related to the analysis of nonlinear systems, the idea of convergence among different trajectories can be linked with the idea of convergence among trajectories of dynamical agents in networked control systems. Indeed contraction theory, which is a tool able to study incremental exponential stability of smooth systems, has been usefully exploited to give conditions for the synchronization of a network of dynamical systems [124, 163, 180, 210, 182, 178, 183].

Most of the available literature to study convergence and synchronization is limited to smooth dynamical systems. However, despite this, in many applications it can be relevant to study discontinuous network or assess incremental stability of nonsmooth systems.

Indeed, discontinuities can affect a network at different levels. They can be present in the individual dynamics of the agents, like all the networks interconnecting mechanical agents with friction or switching devices [162, 161, 93, 143, 82], in the topology, which changes due to faults or sensing radius like in sensor network and autonomous robot networks [138, 37, 81, 202, 90], and in the communications or in the distributed control actions [222, 202, 134, 137, 89, 65]. It is therefore of theoretical and practical importance to find general conditions able to guarantee synchronization for discontinuous networks, in particular those with discontinuities in the agents' dynamics and in the communication protocols. Currently, most of the papers on this topic are related to discontinuities in the topology [228, 174, 149, 24, 167], while the articles related to discontinuity in the dynamics contain preliminary results and either contain hypotheses on the Lyapunov exponents of the Jacobian of the systems [48] to guarantee local synchronization, or synchrony of the switching signals [179]. The problem of considering possible discontinuities in the communication among agents, is strongly motivated by the fact that, in future scenarios, multi-agent systems are supposed to broadcast information over a wireless network which is characterized, as known, by discrete data packets. The problem of coordinating the broadcast over the media reducing noises and minimizing the essential number of information able to achieve synchronization has been recently addressed in literature using event-triggered strategies [199]. Current results are limited to networks of integrators [65, 186] and of linear systems [89, 59].

In this thesis we study convergence and synchronization of networks of discontinuous heterogeneous agents under different conditions of their vector fields. Specifically, conditions for global bounded synchronization are derived using set-valued Lyapunov analysis for nonsmooth systems and bounds on the minimum coupling strength to make all nodes in the network converge toward the same evolution are determined together with an estimate of the asymptotic synchronization bound. The analysis is performed both for linear and nonlinear coupling protocols and a bunch of different examples shows the effectiveness of the proposed strategies.

Also, in the thesis distributed piecewise constant strategies in an event-triggered fashion are proposed in order to synchronize a network of general Lipschitz nonlinear systems. Differently from the literature, which focus on linear systems, the study is carried out on more generic nonlinear vector fields and it is also able to guarantee that control signals and communication signals are piecewise constant.

Furthermore, the problem of studying incremental stability for piecewise smooth systems is also discussed. In particular, an extension of contraction theory to nonsmooth systems is presented. Such an extension not only represents a good tool for the analysis of incremental exponential stability of such systems, but could also represent a good tool of synthesis for coupling control laws in networks of discontinuous systems. In the literature, the theory of convergent systems has been extended to piecewise affine continuous systems [158, 156] and then, for more general piecewise continuous systems [157]. In this thesis we present an analogous extension generalizing the classical definition of contraction to dynamical systems which satisfy Caratheodory conditions for the existence and uniqueness of the solution. Then, the theoretical results are used to study synchronization of nonsmooth networks and later applied to a set of examples. Furthermore, in the thesis we also investigate the problem of studying incremental stability of Filippov systems [77] extending contraction theory. Although this is still a problem under investigation, preliminary results are given for planar Filippov systems.

1.2 Outline of the thesis

The thesis is divided in two parts. In Part I we focus on piecewise smooth and discontinuous networks, i.e. all the dynamical networks characterized by discontinuity in some of their components (agent, topology, communication), and in particular we will study the problem of achieving synchronization in this kind of networks. In Part II we will address incremental stability by extending contraction theory to PWS systems. This topic represents, as we said, a self-contained subject but we will also apply the results to synthesize coupling protocols for discontinuous networks.

Each part of the thesis has its own introductory chapter, which introduces and motivates the topic and, also, gives some preliminary notions to the reader. In particular, respectively for Part I and Part II, these information are in Chapter 3 and Chapter 6. In Chapter 2 the mathematical background is given, related to the PWS dynamical systems and nonsmooth analysis which supports both the two parts of the thesis. More in detail, the thesis is organized in the following way.

In Chapter 2 we introduce the reader to PWS dynamical systems and nonsmooth analysis giving those concepts and definitions that we will use in the rest of the thesis.

With Chapter 3 we start Part I of the thesis. For this reason, we introduce the general concepts of complex networks and networked control systems, describing also the main emerging behaviours discussed in the literature. In particular, we give more details about synchronization, since it is the general behaviour that will be considered in the first part of the thesis. We then define the class of discontinuous dynamical networks.

In Chapter 4 we present a framework for the study of bounded synchronization of agents whose dynamics may be both piecewise smooth and/or nonidentical across the network. We derive sufficient conditions and bounds for the synchronization error and for the coupling gain able to make the nodes converge onto the same evolution. All the results are derived analytically, using a set-valued Lyapunov analysis, performed both for linear and nonlinear coupling protocols. We consider also different examples and applications to validate the theory.

In Chapter 5 we consider networks of dynamical systems with discontinuous communication functions. More in detail, we propose distributed event-triggered strategies able to guarantee synchronization in a network of nonlinear Lipschitz systems with both communication and control being piecewise constant signals. Indeed, in the event-triggered approach, these signals update their values only when particular events are generated. The validity of the strategies is proved analytically and numerical simulations confirm the results.

With Chapter 6 we start Part II of the thesis. We introduce the basic concepts of incremental stability and contraction theory in order to give a background for the following chapters, where contraction theory is extended to PWS systems for proving their incremental stability.

In Chapter 7 we extend contraction theory to switched non differentiable systems which satisfy Caratheodory conditions for the existence and unicity of the solution. We then use the results to synthesize coupling protocol able to guarantee convergence of networked dynamical agents in discontinuous networks. A set of examples show the validity of the proposed approach, as a tool both for analysis and synthesis.

In Chapter 8 we address the problem of proving incremental stability of bimodal planar Filippov systems. More in detail, we consider systems with an attractive sliding surface and we consider differential conditions on the discontinuity boundary which guarantee incremental exponential stability of the system. The results obtained in the chapter are a preliminary work for the extension of contraction theory to general dimension Fillippov systems.

In Chapter 9 conclusions are drawn.

The results in Chapter 4 are obtained in collaboration with Dr. Pietro De Lellis (Department of Electrical Engineering and Information Technology, University of Naples Federico II, Italy) and preliminary results can be found in [61, 56, 55], while a journal paper is ready to be submitted. The results in Chapter 5 have been developed together with Prof. Dimos V. Dimarogonas and Prof. Karl H. Johansson at the Automatic Control Laboratory of the KTH Royal Institute of Technology, Stockholm, Sweden. Preliminary results of the chapter can be found in [123], while a journal paper is ready to be submitted. The results in Chapter 7 have been obtained in collaboration with Dr. Giovanni Russo and can be partially found in [64], while a journal article is currently under review [63]. The results presented in Chapter 8 can be found in [62]. Furthermore, a paper with more general results about contraction theory for nonsmooth systems is in preparation.

Chapter 2

Mathematical preliminaries

Contents

In this Chapter we give some useful concepts and definitions of smooth and nonsmooth analysis we will use in the rest of the thesis.

2.1 Lipschitz and one-sided Lipschitz functions

Here we recall the well known Lipschitz and one-sided Lipschitz properties, see for instance [4].

Definition 2.1.1. *A function $f(x) = \mathbb{R}^n \mapsto \mathbb{R}^n$ is said to be* globally Lipschitz *if there exists a constant $L_f > 0$ such that*

$$\|f(x) - f(y)\|_2 \leq L_f \|x - y\|_2 \qquad \forall x, y \in \mathbb{R}^n. \tag{2.1}$$

Definition 2.1.2. *A function $f(x) = \mathbb{R}^n \mapsto \mathbb{R}^n$ is said to be one-sided Lipschitz if there exists a constant $L'_f > 0$ such that*

$$[f(x) - f(y)]^T (x - y) \leq L'_f \|x - y\|_2^2 \qquad \forall x, y \in \mathbb{R}^n \tag{2.2}$$

Notice that it is immediate to prove that a Lipschitz function with Lipschitz constant L_f is also one-sided Lipschitz with the same constant.

2.2 Flow and classical solutions of a dynamical system

Definition 2.2.1. *Given a domain* $\mathrm{D} \subseteq \mathbb{R}^n$ *and a dynamical system*

$$\dot{x} = f(t, x), \qquad t \in \mathbb{R}, x \in \mathrm{D} \subseteq \mathbb{R}^n,$$

we define its flow $\varphi(s, t_0, \chi) : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ *as the operator (see [193] for a definition) such that*

$$\frac{\partial}{\partial s}\varphi(s, t_0, \chi) = f(t, \varphi(s, t_0, \chi)), \qquad \varphi(0, t_0, \chi) = \chi.$$

In applications of the theory, it could be the case that D is a non-open set, for example a closed set delimited by some hyperplane in the phase space, which could e.g. model constraints on the state variables of the system. We remark here that for a non-open set, with differentiability in x we mean that the vector field $f(t, \cdot)$ can be extended as a differentiable function to some open set which includes D, and any continuity hypothesis with respect to (t, x) holds on this open set.

Definition 2.2.2. *Let us consider a domain* $\mathrm{D} \subseteq \mathbb{R}^n$ *and a dynamical system of the form:*

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_0) = x_0, \tag{2.3}$$

where $f : [t_0, +\infty) \times \mathrm{D} \mapsto \mathbb{R}^n$. *A* classical solution *for this system is a continuously differentiable function* $x(t)$ *that satisfies* (2.3) *for all* $t \in [t_0, t_1]$, *where* $[t_0, t_1]$ *is an interval where the solution* $x(t)$ *is defined.*

In the common cases where the solution $x(t)$ *can be extended forward in time, the same definition holds for each* $t_1 \geq t_0$, *or equivalently in* $[t_0, +\infty)$.

2.3 Piecewise smooth dynamical systems

We now give the definitions of important classes of discontinuous dynamical systems that will be analyzed in the thesis.

Following [60] p.73, we define a piecewise smooth dynamical system as follows.

Definition 2.3.1. *Let us consider a finite collection of disjoint, open and non-empty sets* $\mathcal{S}_1, \ldots \mathcal{S}_p$, *such that* $\mathrm{D} \subseteq \bigcup_{i=1}^{p} \bar{\mathcal{S}}_i \subseteq \mathbb{R}^n$ *is a connected set, and that the intersection* $\Sigma_{ij} := \bar{\mathcal{S}}_i \cap \bar{\mathcal{S}}_j$ *is either a* $\mathbb{R}^{n-1}$ *lower dimensional manifold or it is the empty set. A dynamical system* $\dot{x} = f(t, x)$, *with* $f : [t_0, +\infty) \times \mathrm{D} \mapsto \mathbb{R}^n$, *is called a* piecewise smooth dynamical system *(PWS) when it is defined by a finite set of ODEs, that is, when*

$$f(t, x) = F_i(t, x), \qquad x \in \mathcal{S}_i,\ i = 1, \ldots, p, \tag{2.4}$$

with each vector field $F_i(t, x)$ *being smooth in both the state* x *and the time* t *for any* $x \in \mathcal{S}_i$. *Furthermore, each* $F_i(t, x)$ *is continuously extended on the boundary* $\partial\mathcal{S}_i$.

Notice that in the above definition the value in $\mathbb{R}^n$ the function $f(\cdot)$ assumes on the boundaries of $\partial\mathcal{S}_i$ is left undefined. Indeed, as will be clear in what follows, the solution concept we will consider for PWS systems does not depend on the values of function $f(\cdot)$ on sets of zero measure.

It is also immediate to see that continuously differentiable dynamical systems are also piecewise smooth systems. However, in this thesis, we will exclude this case and we will directly refer to discontinuous dynamical systems.

Definition 2.3.2. *A piecewise smooth dynamical system is said to be continuous (PWSC) if the following two conditions hold:*

1. *the function* $(t, x) \mapsto f(t, x)$ *is continuous for all* $x \in \mathbb{R}^n$ *and for all* $t \geq t_0$*;*
2. *the function* $x \mapsto F_i(t, x)$ *is continuously differentiable for all* $x \in \mathcal{S}_i$ *and for all* $t \geq t_0$*. Furthermore the Jacobians* $\frac{\partial F_i}{\partial x}(t, x)$ *can be continuously extended on the boundary* $\partial \mathcal{S}_i$*.*

Notice that in order for condition *1.* to be satisfied the functions $(t, x) \mapsto F_i(t, x)$ must be continuous for all $t \geq t_0$ and $x \in \mathcal{S}_i$, and, for all $x \in \Sigma_{ij} \neq \emptyset$ and all $t \geq t_0$, it must hold $F_i(t, x) = F_j(t, x)$.

According to [119] a time-dependent switching system can be defined as follows.

Definition 2.3.3. *A* time-dependent switching system *is a dynamical system of the form*

$$\dot{x} = f(t, x, \sigma), \qquad t \in [0, +\infty),\ x \in \mathrm{D}, \tag{2.5}$$

where $\sigma(t) : [0, +\infty) \mapsto \Sigma = \{1, 2, \ldots, p\}$ *is a piecewise continuous time-dependent switching signal taking one over p finite possible values.*

Note that according to this definition, we are excluding the case where infinite switchings occur over finite time so that Zeno behavior cannot occur (see [119] for further details).

Discontinuous dynamical systems can admit notions of solution different form the classical one (see also [46] and references therein). In particular, we will focus our attention on two kind of solutions, Caratheodory solutions and Filippov solutions.

2.4 Caratheodory solutions of discontinuous dynamical systems

We give the following preliminary definitions:

Definition 2.4.1. *A function* $g(t) : [t_0, +\infty) \mapsto \mathbb{R}^n$ *is said to be* measurable *if, for any real number* α*, the set* $\{t \in [t_0, +\infty) : g(t) > \alpha\}$ *is measurable in the sense of Lebesgue.*

Definition 2.4.2. *A function* $l(t) : [t_0, +\infty) \mapsto \mathbb{R}$ *is* summable *if the Lebesgue integral of the absolute value of* $l(t)$ *exists and is finite.*

Definition 2.4.3. *A function* $z(t) : [a, b] \mapsto \mathbb{R}^n$ *is* absolutely continuous *if for all* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for each finite collection* $[a_1, b_1] \ldots [a_n, b_n]$ *of disjoint sets in* $[a, b]$*, it holds that* $\sum_k |b_k - a_k| < \delta \implies \sum_k |z(b_k) - z(a_k)| < \varepsilon$*.*

Now we are able to give the definition of a Caratheodory solution of a differential equation (see also [46] and references therein):

Definition 2.4.4. *Let us consider a domain* $\mathrm{D} \subseteq \mathbb{R}^n$ *and a dynamical system of the form:*

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_0) = x_0, \tag{2.6}$$

where $f : [t_0, +\infty) \times \mathrm{D} \mapsto \mathbb{R}^n$*. A* Caratheodory solution *for this system is an absolutely continuous function* $x(t)$ *that satisfies (2.6) for almost all* $t \in [t_0, t_1]$ *(in the sense of*

Lesbesgue), where $[t_0, t_1]$ is an interval where the solution $x(t)$ is defined. That is, a Caratheodory solution of (2.6) is an absolutely continuous function $x(t)$ such that:

$$x(t) = x(t_0) + \int_{t_0}^{t} f(\tau, x(\tau))d\tau, \qquad t \in [t_0, t_1].$$

In the common cases where the solution $x(t)$ can be extended forward in time, the same definition holds for each $t_1 \geq t_0$.

An useful result provides sufficient conditions for the existence of a Caratheodory solution of the system (2.6).

Theorem 2.4.1. *A Caratheodory solution of system (2.6) exists if:*

1. *for almost all $t \in [0, \infty)$, the function $x \mapsto f(t, x)$ is continuous for all $x \in \mathrm{D}$;*
2. *for each $x \in \mathrm{D}$, the function $t \mapsto f(t, x)$ is measurable in t;*
3. *for all $(t, x) \in [0, +\infty) \times \mathrm{D}$, there exist $\delta > 0$ and a summable function $m(t)$ defined on the interval $[t, t + \delta]$ such that $|f(t, x)| \leq m(t)$.*

Moreover, the solution is unique, if the following additional condition is satisfied:

4. *$(x - y)^T (f(t, x) - f(t, y)) \leq l(t)(x - y)^T (x - y)$, where $l(t)$ is a summable function.*

Notice that, as discussed in [77], p. 10 and proved in [9], equations that satisfy the above (Caratheodory) conditions and those required for uniqueness of a solution show continuous dependence on initial conditions.

We will refer to *Caratheodory systems* to indicate any PWS or TSS satisfying the conditions for the existence of a Caratheodory solution given in Theorem 2.4.1. Notice also that is immediate to notice that classical solutions are also Caratheodory solutions.

2.5 Filippov solutions of discontinuous dynamical systems

Referring to [77], we give the following definition.

Definition 2.5.1. *Let us consider a domain $\mathrm{D} \subseteq \mathbb{R}^n$ and a dynamical system of the form:*

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_0) = x_0, \tag{2.7}$$

where $f : [t_0, +\infty) \times \mathrm{D} \mapsto \mathbb{R}^n$. A Filippov solution *for this system is an absolutely continuous function $x(t)$ that satisfies for almost all $t \in [t_0, t_1]$ (or possibly $t \in [t_0, +\infty)$), the differential inclusion*

$$\dot{x}(t) \in \mathcal{F}[f](t, x), \tag{2.8}$$

where $\mathcal{F}[f](t, x)$ is the Filippov set-valued function *$\mathcal{F}[f] : [t_0, +\infty) \times \mathrm{D} \mapsto \mathfrak{B}(\mathbb{R}^n)$, with $\mathfrak{B}(\mathbb{R}^n)$ being the collection of all the subsets in $\mathbb{R}^n$, defined as*

$$\mathcal{F}[f](t, x) = \bigcap_{\delta > 0} \bigcap_{m(\mathcal{S}) = 0} \overline{co} \left\{ f(t, \mathcal{B}_\delta(x) \backslash \mathcal{S}) \right\},$$

where $\mathcal{S}$ is any set of zero Lebesgue measure $m(\cdot)$, $\mathcal{B}_\delta(x)$ is an open ball centered at x with radius $\delta > 0$, and $\overline{co}\{\mathcal{I}\}$ denotes the convex closure of a set $\mathcal{I}$.

In what follows we will consider Filippov solutions directly for the specific case of PWS systems, i.e. for those systems whose vector field $\dot{x} = f(t, x)$ satisfies the Definition 2.3.1. We remark that, for such kind of systems, a Filippov solution exists under the mild assumption of local essential boundedness of the vector field f, see [46] for further details. Notice also that, as stated in [77], p. 90, Filippov solutions show continuous dependence on the initial conditions.

Computing the Filippov set-valued function can be a nontrivial task. Here, we report three useful rules we will apply in what follows [152]:

Consistency: *If* $f : [t_0, +\infty) \times \mathrm{D} \mapsto \mathbb{R}^n$ *is continuous at* $(t, x) \in [t_0, +\infty) \times \mathrm{D}$*, then*

$$\mathcal{F}[f](t, x) = \{f(t, x)\}.$$

Sum: *If* $f_1, f_2 : [t_0, +\infty) \times \mathrm{D} \mapsto \mathbb{R}^n$ *are locally bounded at* $(t, x) \in [t_0, +\infty) \times \mathrm{D}$*, then*

$$\mathcal{F}[f_1 + f_2](t, x) \subseteq \mathcal{F}[f_1](t, x) + \mathcal{F}[f_2](t, x).$$

Moreover, if either f_1 *or* f_2 *is continuous at* (t, x)*, then the equality holds.*

Product: *If* $f_1, f_2 : [t_0, +\infty) \times \mathrm{D} \mapsto \mathbb{R}^n$ *are locally bounded at* $(t, x) \in [t_0, +\infty) \times \mathrm{D}$*, then*

$$\mathcal{F}\left[\left(f_1^T, f_2^T\right)^T\right](t, x) \subseteq \mathcal{F}[f_1](t, x) \times \mathcal{F}[f_2](t, x).$$

Moreover, if either f_1 *or* f_2 *is continuous at* (t, x)*, then equality holds.*

2.6 Sliding mode solutions

Piecewise smooth systems can exhibit specific behaviours that do not appear in the smooth systems. A notable example is that of *sliding mode solutions*: a specific Filippov solution characterized by the fact that the evolution of the PWS dynamical system belongs to (or "slides along") the discontinuity manifold $H(x) = 0$ (see for example [60]).

In this section we restrict our attention to *bimodal PWS systems*. Such class of PWS dynamical systems is common in the literature due to its importance in many applications [204], [160], [60], [206]. Bimodal PWS systems can be written in the form:

$$f(t, x) = \begin{cases} F_1(t, x) & \text{if } H(x) \leq 0 \\ F_2(t, x) & \text{if } H(x) > 0 \end{cases}, \tag{2.9}$$

where $F_1(t, x)$ and $F_2(t, x)$ are two smooth vector fields and $H(x)$ is a smooth scalar function. $H(x) = 0$ with $\frac{\partial}{\partial x} H(x) \neq 0$ defining the smooth discontinuity manifold Σ in $\mathbb{R}^n$.

Definition 2.6.1. *The* sliding region *for a system of the form* (2.9) *is given by the set:*

$$\hat{\Sigma} = \{x \in \mathbb{R}^n : H(x) = 0, \mathcal{L}_{F_1} H(x) \cdot \mathcal{L}_{F_2} H(x) < 0\},$$

where $\mathcal{L}_{F_i} H(x) := \frac{\partial}{\partial x} H(x) F_i(t, x)$ *is the* Lie derivative *of* $H(x)$ *with respect to the vector field* $F_i(t, x)$*, that is the component of* $F_i(t, x)$ *normal to the discontinuity manifold at the point* x*.*

It is worth mentioning here that the set $\hat{\Sigma}$ contains the points where the boundary is simultaneously attracting (or repelling) from both sides.

The equations of the sliding flow can be written using the *Filippov's convexification method* [77] as:

$$F_s(t,x) = \frac{(1-\beta(t,x))}{2}F_1(t,x) + \frac{(1+\beta(t,x))}{2}F_2(t,x), \tag{2.10}$$

with $\beta(t,x) \in [-1,1]$ given by:

$$\beta(t,x) = \frac{\mathcal{L}_{F_1}(H) + \mathcal{L}_{F_2}(H)}{\mathcal{L}_{F_1}(H) - \mathcal{L}_{F_2}(H)}; \tag{2.11}$$

and the sliding region can also be defined as the set [60]:

$$\hat{\Sigma} = \{x \in \mathbb{R}^n : H(x) = 0, -1 \leq \beta(t,x) \leq 1\},$$

while its boundary is given by the set:

$$\partial\hat{\Sigma} = \{x \in \mathbb{R}^n : H(x) = 0, \beta(t,x) = \pm 1\},$$

where tangency of one vector field or the other occurs when $\beta(t,x) = \pm 1$.

2.7 Generalized gradient and set-valued Lie derivative

In this section we give some useful concepts of nonsmooth analysis that will be applied in the thesis.

Considering an open set[1] $\mathrm{D} \subseteq \mathbb{R}^n$, a discontinuous ocally Lipschitz function $u : \mathrm{D} \mapsto \mathbb{R}$ is differentiable almost everywhere (in the sense of Lebesgue) according to the Rademacher's Theorem [44]. Then, it is useful to extend the classical gradient definition. Denoting with Ω_u the zero-measure set of points at which a given function u fails to be differentiable, we report the following definition [44, 46].

Definition 2.7.1. *Let $u : \mathrm{D} \mapsto \mathbb{R}$ be a locally Lipschitz function, and let $\mathcal{S} \subset \mathbb{R}^n$ be an arbitrary set of zero measure, we define the* generalized gradient *(also termed* Clarke subdifferential*) $\partial u : \mathrm{D} \mapsto \mathfrak{B}(\mathbb{R}^n)$ of u at any $x \in \mathrm{D}$ as*

$$\partial u(x) = co\left\{\lim_{k\to\infty} \frac{\partial}{\partial x}u(x_k) : x_k \to x, x_k \notin \mathcal{S} \cup \Omega_u\right\},$$

where $\mathfrak{B}(\mathbb{R}^n)$ is the set of all the possible subsets of $\mathbb{R}^n$, $co\{\mathcal{I}\}$ is the convex hull of a set $\mathcal{I}$.

Notice that, if u is continuously differentiable, then it is possible to proof that $\partial u(x) = \left\{\frac{\partial}{\partial x}u(x)\right\}$, see [46]. Now, we give the definition of *set-valued Lie derivative* [46].

Definition 2.7.2. *Given a locally Lipschitz function $u : \mathrm{D} \mapsto \mathbb{R}$ and a vector field $f : \mathrm{D} \to \mathbb{R}^n$, the* set-valued Lie derivative $\underset{\sim}{\mathcal{L}}_{\mathcal{F}[f]} : \mathrm{D} \mapsto \mathfrak{B}(\mathbb{R})$ *of u with respect to $\mathcal{F}[f]$ at x is defined as*

$$\underset{\sim}{\mathcal{L}}_{\mathcal{F}[f]}u(x) = \left\{a \in \mathbb{R}\, s.t.\ there\ exists\ v \in \mathcal{F}[f](x) \Rightarrow \varrho^T v = a\ for\ all\ \varrho \in \partial u(x)\right\}.$$

[1] In practical applications it could be the case that D is a non-open set. In such cases with differentiability in x we mean, in this thesis, that the nonsmooth function can be extended to some open set which includes D, and any differentiability and continuity hypothesis hold on this open set.

Lemma 2.7.1. *[18, 46] Let $x(t)$ be a solution of the differential inclusion* (2.8), (2.9), *and let $u : \mathrm{D} \mapsto \mathbb{R}$ be locally Lipschitz and regular. Then, the following statements hold:*

i) The composition $t \in [t_0, +\infty) \mapsto u(x(t)) \in \mathbb{R}$ is differentiable for almost every t;

ii) The derivative of $t \mapsto u(x(t))$ satisfies

$$\frac{d}{dt}u(x(t)) \in \underset{\sim}{\mathcal{L}}_{\mathcal{F}[f]}u(x)$$

for almost every t.

Notice that a convex function is also regular, see, for instance, [44].

2.8 Discussion

Having given the main preliminary definitions and results, we can now look at the problem of synchronizing PWS networks in part I of the thesis and in the problem of extending contraction theory for PWS systems in part II.

Part I

Synchronization in Discontinuous Networks of Dynamical Systems

Chapter 3

Discontinuous dynamical networks

Contents

In this chapter we introduce the general idea of a complex network and the concept of networked control systems. For such kind of networks we describe the main emerging behaviours studied in the literature, with particular attention to synchronization. We also give the definition of discontinuous dynamical networks that will be the topic of Part I of the thesis.

3.1 What is a complex network?

A network is an ensemble of systems, called *nodes* or *vertices*, with connections between them, called *links* or *edges* [145, 29, 196, 146]. Examples of networks abound in the world and interconnected systems can be found in a very large number of heterogeneous applications. Indeed, such approach has been successfully applied in: social networks to model patterns of interactions among people, friendship and business relations among group of people [171, 130, 213, 132, 22, 115]; information networks to study how information is shared or spread among people, in particular referring to the World Wide Web [172, 185, 110, 19, 100, 78]; biological networks to study metabolic and cellular interactions, gene regulatory mechanisms, neurons organizations and ecosystems [105, 133, 88, 189, 45, 194]; technological networks like power networks, communication networks and transportations patterns and internet [215, 214, 112, 7, 131, 153, 75, 42, 94, 68].

Historically, the first modern study of networks is thought to be the problem solved by Leonhard Euler in 1739 [6, 74] of finding a path crossing all seven bridges in the city of Königsberg only once. That work gave the birth to the *graph theory*, which is a branch of discrete mathematics used to model and solve problems over networks. In

Section 3.2 we will give some definitions and concepts from t graph theory needed to state the problem of interest within Part I of the thesis.

From a system and control point of view, with *networked dynamical systems* we mean a large scale multi-agent dynamical systems where a huge number of dynamical systems communicate over a network of interconnections. In the literature, this kind of networks are also termed *complex networks* [196, 29], referring to the complexity arising from the relationship between the node dynamics and the topology of the interconnections. Studying and controlling networked dynamical systems is a pressing open challenge, due to the variety of different behaviours arising from the different kind of connected dynamical agents that can be considered, different control goals and different assumptions on the nodes or on their communication and coupling [159, 40, 217, 28, 3, 151, 38, 15, 15, 68]. These emergent behaviours can be addressed only in an interdisciplinary manner by taking into account diverse disciplines, such as graph theory and dynamical and control engineering. In this way it is possible to consider the mutual influences of both the individual dynamics of the nodes and the topology of the interconnections. In particular, in Section 3.3 we will describe a specific emergent behaviour, the so called *synchronization* [29, 54, 218, 164, 226], as a paradigm for other behaviours like *consensus* algorithms [149, 174, 216] and *flocking* [13, 32, 67]. Synchronization will be deeply investigated in Part I of the thesis, where results for proving such behaviour under different hypotheses of discontinuity of the agents' dynamics and communication protocols are studied.

3.2 Definitions and notation

In this section we review some results about algebraic graph theory [84, 150] that will be used in Part I of the thesis. The notation used in this section will be adopted in the next chapters.

3.2.1 Algebraic graph theory

A graph $G(\mathcal{N}, \mathcal{E})$ consists of a collection of two sets, a set of N *nodes* (also called vertices) $\mathcal{N} = \{1, 2, \dots, N\}$ and a set of *edges* (also called *links*) $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$. A generic pair $(i, j) \in \mathcal{E}$ is considered an ordered pair and the link is said to be *oriented* from node i to node j[1].

A graph G can be represented by the *adjacency matrix* $A = A(G) = [a_{ij}] \in \mathbb{R}^{N \times N}$, where $a_{ii} = 0$ for all i, and $a_{ij} > 0$, with $i \neq j$, if there is an edge oriented from node i to node j, i.e. $(i, j) \in \mathcal{E}$, while it is $a_{ij} = 0$ otherwise.

Depending on the specific application, the value of a_{ij} is often considered as a measure of the strength of the communication, the coupling or the control action between node i and j. For this reason, it is also called *weight* of the link from node i to node j. When $a_{ij} > 1$ for some i, j the graph is called *weighted graph.* Instead, in the applications that need only to keep the information of the connections between nodes and do not require weights for the edges, only the binary values $\{0, 1\}$ are generally associated to a_{ij}, with $i \neq j$, respectively if a link form i to j does not exist or exists. In this case the graph is said *not weighted.*

A graph is said to be *undirected* if the matrix A is symmetric and so $a_{ij} = a_{ji}$, otherwise the graph is said to be *directed* or *oriented.* In the case of undirected graphs the link connecting two generic different nodes i and j is considered without orientation,

[1] Here and in what follows we will exclude the degenerate case of empty graph, where $\mathcal{N} = \emptyset$.

thus representing the fact that the edge from node i to node j has always the same weight of the edge connecting node j to node i.

When $a_{ij} \neq 0$, for $i \neq j$, the node j is said to be *adjacent* (or *neighbour*) to node i. Using the adjacency matrix, we can easily describe the neighbourhood of a generic node $i \in \{1, \dots, N\}$ as

$$\mathcal{N}_i \subseteq \mathcal{N} = \{j : a_{ij} \neq 0\},$$

while we indicate with $N_i = |\mathcal{N}_i|$ the number of neighbours of node i, also called *degree* of node i.

A *path* from node i to node j is a sequence of nodes in the graph starting from i and ending with j such that consecutive nodes are adjacent (considering the orientations of the links). The *length* of the path is the number of links connecting the adjacent nodes in the path [145]. Notice that in an oriented graph a path is also oriented, since adjacent nodes are considered taking into account the orientation of the links. In an undirected graph a path is not oriented, since any two connected nodes i and j have an edge between them in both the orientation and with the same value of the weight.

A directed graph is said to be *strongly connected* if for any pair of nodes i and j there exists a (oriented) path between them.

A strongly connected undirected graph is simply termed *connected*, thus to emphasize that for any pair of nodes i and j there exists a path between them (without taking care of the orientation).

In a strongly connected graph, the *distance* d^{ij} between nodes i and j is the length of the shortest path connecting the two nodes.

The diameter of a strongly connected graph is the length of the maximum distance for every pair of nodes (i, j) in the graph and it is given as

$$d^{\max} = \max_{(i,j)\in\mathcal{E}} d^{ij}.$$

The *Laplacian matrix* $L = L(G) \in \mathbb{R}^{N\times N}$ is defined as $L = \Delta - A$, where Δ is the diagonal matrix of all nodes' degrees, i.e. $\Delta = \operatorname{diag}\{N_1, N_2, \dots, N_N\}$. So, a Laplacian matrix $L = [l_{ij}]$ has entries

$$\ell_{ij} = \begin{cases} -a_{ij}, & \text{if } i \neq j \text{ and } (i,j) \in \mathcal{E} \\ 0, & \text{if } i \neq j \text{ and } (i,j) \notin \mathcal{E} \\ \sum\limits_{\substack{k=1 \\ k\neq i}}^{N} a_{ik}, & \text{if } i = j \end{cases},$$

A Laplacian matrix has at least one zero eigenvalue associated to the eigenvector $\mathbf{1}_N$, with $\mathbf{1}_N$ being the vector of N unitary entries. Furthermore, according to [150], all the eigenvalues of L have non negative real part.

We recall here a useful lemma.

Lemma 3.2.1. *([84], pp. 279-288)*

1. *The Laplacian matrix L in a connected undirected network is positive semi-definite. Moreover, it has a simple eigenvalue at* 0 *and all the other eigenvalues are real and positive.*

2. *the smallest nonzero eigenvalue $\lambda_2(L)$ of the Laplacian matrix satisfies*

$$\lambda_2(L) = \min_{z^T 1_N = 0, z\neq 0} \frac{z^T L z}{z^T z}.$$

3.3 Synchronization

Synchronization [164, 146, 3, 29] is an emerging behaviour in complex networks that can be found in a wide range of contexts. Scientifically observed for the first time by Christian Huygens between two weakly coupled pendula [101], it has been recently studied in biology, in sociology and in technology [38, 116, 170, 182, 231, 187, 11, 86, 220, 202, 147, 68].

From a system and control engineering point of view, it has been found that synchronization among dynamical systems arises when they are locally coupled by means of some *output function* of their states [15, 159, 26, 40, 175, 217, 176, 218, 164, 28, 30, 50, 29, 116, 117, 151, 226, 195, 141, 54]. By means of such interaction, which can be related to some physical coupling or can be induced by a distributed control law, all nodes coordinate their trajectories in order to converge onto the same synchronous evolution. Furthermore, the phenomenon of synchronization is a paradigm for more specific emerging behaviours. *Consensus*, for example, is the special case of synchronization when all the agents converge to the same constant value [35, 149, 150, 173, 144, 216]. Another kind of emerging behaviour is the coordination of the motion of mobile agents in order to align their velocities and keep a platoon or, more simply, stabilizing their inter-agent distances using decentralized nearest-neighbour control actions. This kind of problem is called *flocking* and it is studied in the field of mobile robots and autonomous vehicles [202, 147, 13, 32, 67, 32].

Before defining synchronization in an analytical way, we need to describe a network of dynamical agents. In general, a network is modeled as an ensemble of N interacting dynamical systems [29, 146]. Each system is described by a set of nonlinear ordinary differential equations (ODEs) of the form $\dot{x}_i = f_i(t, x_i)$, where $x_i \in \mathbb{R}^n$ is the state vector and $f_i : \mathbb{R}^+ \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is a nonlinear vector field describing the *system dynamics*, often assumed sufficiently smooth and differentiable. The *coupling* between neighboring nodes is assumed to be a nonlinear function $\eta : \mathbb{R}^+ \times \mathbb{R}^n \mapsto \mathbb{R}^n$ (often called *output function*) of the difference between their states. Hence, the equations of motion for the generic i-th system in the network are:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = f_i(t, x_i) - c \sum_{j=1}^{N} a_{ij} \eta(t, x_i - x_j), \qquad \forall i = 1, \dots, N \tag{3.1}$$

where x_i represents the state vector of the i-th oscillator, c is the *coupling constant* which gives the overall strength of the coupling, and a_{ij} are the elements of the adjacency matrix of the graph (see Section 3.2), and so it is positive if there is an edge between nodes i and j and 0 otherwise. It is worth mentioning here that, the function $\eta(\cdot)$ has been termed in the literature in different ways. Typically, referring to different applications, $\eta(\cdot)$ is called *output function*, *coupling protocol* or *communication protocol.*

To give formal definitions of synchronization, we introduce here the *convergence error* (also called *synchronization error*), which is commonly defined in the literature (see for instance [198, 229]) at each node i as $e_i = \left[e_i^{(1)}, \dots, e_i^{(n)}\right]^T$, where

$$e_i(t) = x_i(t) - \bar{x}(t), \tag{3.2}$$

with $\bar{x}(t)$ being the so-called *average trajectory*

$$\bar{x}(t) = \frac{1}{N} \sum_{j=1}^{N} x_j(t). \tag{3.3}$$

Now we can give the following definitions

Definition 3.3.1. *We say that network* (3.1) *achieves* ϵ-bounded convergence *(or* ϵ-bounded synchronization*) iff*

$$\lim_{t\to\infty} ||e(t)||_2 \leq \epsilon, \tag{3.4}$$

with $e(t) = [e_1^T(t), \ldots, e_N^T(t)]^T$, $\epsilon \in \mathbb{R}^+$ *and with* $\|\cdot\|_2$ *being the Euclidean norm.*[2]

Definition 3.3.2. *We say that network* (3.1) *achieves* asymptotic convergence *(or* complete synchronization*) iff*

$$\lim_{t\to\infty} ||e(t)||_2 = 0, \tag{3.5}$$

with $e(t) = [e_1^T(t), \ldots, e_N^T(t)]^T$, *and with* $\|\cdot\|_2$ *being the Euclidean norm.*

3.4 Piecewise smooth dynamical networks

Referring to the general model of dynamical networks in (3.1), we introduce here the concept of *PWS dynamical networks*. Such networks are all the dynamical networks where discontinuities can affect the different elements constituting equation (3.1). With the term discontinuities we mean that both PWS dynamics and\or time switchings can be present in the network.

More in detail, taking into account the general model (3.1), discontinuities can be present:

- individual dynamics of the agents $f_i(\cdot)$;
- output function or communication protocol $\eta(\cdot)$;
- topology of interconnections $[a_{ij}]$;

The description of PWS networks given here is, a general framework, encompassing different cases discussed in the literature [174, 149, 87, 202, 23, 24, 134, 228, 90, 222, 167].

More precisely, models of networks where discontinuities are present at the level of the individual dynamics of the agents are useful to investigate and control the emergence of coordinated motion in networks of discontinuous systems. Examples include the coordinated motion of mechanical oscillators with friction [162, 161, 93], switching power devices [143, 82] and all those networks whose nodes are affected by switchings on a macroscopic timescale.

On the other hand, networks where discontinuities affect the topology are useful to model all the cases of switching in the connections due to faults and reconfigurations. Possible examples of application can include sensor networks [138, 34, 5, 37, 81, 142], power networks [8, 94] and networks of autonomous vehicles, where communication links are created or destroyed due to the spatial proximity of the agents [202, 90].

Networks where discontinuities are in the function $\eta(\cdot)$ can model all the cases where a distributed discontinuous action is used to control the coordinated motion of the agents [222, 200, 201, 202, 134]. In this context, a particular kind of piecewise smooth strategy to control the network is the event-triggered control recently introduced in the literature

[2] The use of the symbol lim in (3.4) is not intended in the classical sense of *limit*. By (3.4), we mean that for all $\nu > 0$ there exists a $t_\nu > 0$ such that for all $t > t_\nu$ we have that $\|e(t)\|_2 \leq \epsilon + \nu$. So, this does not imply that the limit in (3.4) exists, but we use the same mathematical notation as in the case of an existing limit due to the analogy with the eventual boundedness of signal $e(t)$.

[16, 199], where nodes send information or update their control only if particular events happen and consequently triggers are generated [137, 89, 65, 186].

Other instances of discontinuous networks studied in the literature are those related to switching in the topology of the interconnections [228, 174, 149, 24, 167]. The cases of discontinuities in the agents' dynamics and in the communication or coupling protocol, despite the variety of applications that can be considered, are seldom studied and few results are currently available [48, 179, 222, 200, 201, 202, 134, 24].

In what follows we will study the case of synchronization of dynamical networks of PWS systems (in Chapter 4) and the case of synchronization with piecewise smooth communication functions (in Chapter 5). In particular, in Chapter 4 results on bounded synchronization for discontinuous and possibly nonidetical dynamical systems will be given under different hypotheses on their vector fields, while in Chapter 5 distributed time switching control strategies based on the so called *event-triggered* [16, 199, 92, 113, 128] fashion will be given in order to synchronize networks of smooth nonlinear systems.

Chapter 4

Convergence of networks of heterogeneous piecewise smooth systems

Contents

In this chapter we present a framework for the study of convergence when the nodes' dynamics may be both piecewise smooth and/or nonidentical across the network. Specifically, we derive sufficient conditions for global boundedness based on set-valued Lyapunov analysis and determine bounds on the minimum coupling strength to make all nodes in the network converge toward the same evolution. We also provide an estimate of the asymptotic bound ϵ. The analysis is performed both for linear and nonlinear coupling protocols.

The outline of the chapter is as follows. In Section 4.1 we give an introduction of the problem and an overview on the existing literature. In Section 4.2 we describe the network model that is studied and we give some relevant definitions that are needed in what follows. In Section 4.3 we study bounded convergence for linearly coupled networks, while in Section 4.4 we extend the analysis to the case of nonlinearly coupled networks. Different numerical examples and applications in Section 4.5 show the validity of the proposed strategies. Concluding discussions are drawn is Section 4.6.

4.1 Introduction

The problem of taming the collective behaviour of a network of dynamical systems is one of the key challenges in modern control theory, see for example [122, 139] and references therein. Typically, the "simplest" problem is to make all agents in the network evolve asymptotically onto a common synchronous solution. This problem is relevant in a number of different applications [21, 69, 73, 105, 108] and has been the subject of much ongoing research (see for example [2, 43, 95, 224, 225] and references therein).

Different strategies have been proposed to solve the problem of making all agents in the network converge onto the same solution. Examples include strategies for consensus in networks when the agents are linear and coupled diffusively [148, 149, 223], adaptive approaches to consensus and synchronization [43, 49, 53, 57, 118, 230] and methods based on distributed leader-follower (or pinning control) techniques among many others [1, 36, 51, 121, 227].

Most of the results available in the existing literature rely on the following assumptions which are essential to simplify the study of the general model (3.1) and its convergence. Namely, it is often assumed that

1. the output functions are linear, time-invariant, and typically depending upon the mismatch between the states of neighbouring nodes, i.e. $\eta(t, x_i, x_j) = \gamma_{ij}(x_i - x_j)$, $\gamma_{ij} \in \mathbb{R}^+$;
2. the nodes' vector fields, f_i, are sufficiently smooth and differentiable
3. all nodes share the same dynamics, i.e. $f_i = f_j$, for $i, j = 1, \ldots, N$.

Under the above assumptions, the stability and convergence of network (3.1) has been investigated in depth over the last two decades, and interesting results have been obtained, see, for instance, [29, 54, 146, 164, 218].

Conversely, when some of these classical assumptions are relaxed, analyzing convergence of network (3.1) becomes much harder. Unfortunately, in many real-world networks it is often unrealistic to assume that, for example, all nodes share identical dynamics. Think for example of biochemical or power networks were parameter mismatches between agents are unavoidable and usually rather large [20, 69, 94, 111, 203, 209]. Also, in many cases the models in use to describe the dynamics of the nodes in the network are far from being continuous and differentiable. Notable cases include the coordinated motion of mechanical oscillators with friction [93, 161, 162], switching power devices [143, 82], switch-like models of behaviours of biological cells in pattern formation [188], and all those networks whose nodes are affected by discontinuous events on a macroscopic timescale. The aim of this chapter is to derive conditions for bounded convergence in networks whose nodes' dynamics are nonidentical and possibly described by piecewise smooth vector fields (see Section 2.3). This is achieved when every node achieves bounded tracking of the same common solution.

Typically, results on the stability of networks of identical smooth systems provide conditions under which asymptotic convergence is guaranteed. However, when nodes are nonidentical, asymptotic convergence is only possible in specific cases; for example, when all nonidentical nodes share the same equilibrium [218], for specific nodes' dynamics, or in the case where symmetries exist in the network structure [70, 76, 229]. Nonetheless, for more general complex network models, these assumptions have to be relaxed. Hence, when either a mismatch is present in the network parameters and/or perturbations are added to the vector field of the nodes, it is often desirable to prove bounded convergence of all nodes towards each other (see Definition 3.3.1). As an example, in power networks, asymptotic convergence of all generator phases towards the same solution cannot be achieved and it is considered acceptable that the phase angle differences remain within given bounds [69, 94, 111].

In the literature, few results are available on bounded convergence[1] of networks of nonidentical nodes. In particular, the case of parameters' mismatches is studied assuming that the nodes' dynamics are *eventually dissipative* [25], or assuming a priori that the node trajectories are bounded [99]. Local stability of networked systems with small parameter mismatches is studied extending the Master Stability Function approach in [198]. As for additive perturbations, the specific case of additive noise was considered [116, 163]. A first attempt on giving more general conditions for bounded convergence can be found in [96]. However, the key assumptions guaranteeing global stability results were difficult to check in practice. Indeed, assumptions given in [96] rely on boundedness of the average node vector field, defined as $\sum_{i=1}^{N} f_i/N$, and of its Jacobian evaluated on the average network trajectory, which is unknown a priori.

In networks of identical PWS systems, guaranteeing asymptotic convergence is a cumbersome task and few results are currently available [48, 179]. Specifically, in [48], local synchronization of two coupled continuously differentiable systems with a specific additive sliding action is guaranteed with conditions on the generalized Jacobian of the error system, while in [179] convergence of a network of time switching systems is analyzed when the switching signal is synchronous between all the nodes. The reason why it is difficult to study convergence of piecewise smooth systems is that a network of PWS systems can be viewed as a single higher dimensional hybrid system with multiple switchings on several discontinuity manifolds [60]. Notice that, due to such discontinuity manifolds, sliding motion can occur on several surfaces (see Section 2.6). Furthermore, it is worth mentioning that, even if the piecewise smooth interconnected systems reached the manifold $x_1 = \ldots = x_N$ during their evolution, possible switches in one or more systems could push the systems away from the manifold [149].

To the best of our knowledge, none of the approaches in the existing literature can deal simultaneously with the presence of both non differentiable and nonidentical nodes' dynamics. The main contributions of this chapter can be summarized as follows.

1. Sufficient conditions are derived using set-valued Lyapunov functions for global bounded convergence of all network nodes towards each other. Moreover, explicit bounds are estimated for the residual tracking error and the value of the minimum coupling strength among nodes guaranteeing convergence.

2. The classical assumption of linear diffusive coupling functions is relaxed. Our stability analysis also encompasses continuous or PWS nonlinear coupling protocols.

3. When applied to networks of nonidentical smooth systems, the general conditions derived in this chapter give sufficient conditions for global bounded convergence that are much easier to check or verify if compared to those given in the existing literature reviewed above.

4. This chapter significantly extends the preliminary results reported in [48, 179] guaranteeing boundedness of the synchronization error in networks of piecewise

[1]The terms convergence or synchronization can be equivalently used. However, sometimes in literature, the term synchronization is referred to networks of oscillators. Here, we wish to emphasize that the nodes' dynamics are not necessarily oscillatory.

smooth systems. In particular, results of bounded convergence are found for a wider class of systems and of possible switching signals than those in [48, 179].

4.2 Network model and problem statement

In what follows, we analyze the general model (3.1) of nonlinearly coupled networks of nonidentical piecewise smooth systems, that we report here again for the sake of clarity.

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = f_i(t, x_i) - c\sum_{j=1}^{N} a_{ij}\eta(t, x_i - x_j), \tag{4.1}$$

where, as we said, $x_i \in \mathbb{R}^n$ is the state of the i-th node, $f_i : \mathbb{R}^+ \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is the (possibly piecewise smooth) vector field describing the own dynamics of the i-th node, $c \geq 0$ is the coupling gain, $\eta : \mathbb{R}^+ \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is the output function, $w_{ij} = w_{ji} \geq 0$ is positive if there is an edge between nodes i and j, with $i \neq j$, and 0 otherwise.

Also, we provide specific results of bounded convergence for linearly coupled networks (see [29, 54, 146, 164, 218]) of nonidentical piecewise smooth systems of the form:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = f_i(t, x_i) - c\sum_{j=1}^{N} a_{ij}\Gamma(x_i - x_j), \tag{4.2}$$

and $\Gamma \in \mathbb{R}^{n\times n}$ is the inner coupling matrix.

Now, taking into account the definitions of the synchronization error and average state trajectory, respectively in (3.2) and (3.3), from the two above models of network we obtain

$$\dot{e}_i = f_i(t, x_i) - \frac{1}{N}\sum_{j=1}^{N} f_j(t, x_j) - c\sum_{j=1}^{N} a_{ij}\eta(t, e_i - e_j), \tag{4.3}$$

$$\dot{e}_i = f_i(t, x_i) - \frac{1}{N}\sum_{j=1}^{N} f_j(t, x_j) - c\sum_{j=1}^{N} a_{ij}\Gamma(e_i - e_j), \tag{4.4}$$

where (4.3) and (4.4) apply to networks (4.1) and (4.2), respectively.

In what follows, we often use a compact notation both for the network state equations (4.1) and (4.2), and for the network error equations (4.3) and (4.4). To this aim, we introduce $x = [x_1^T, \ldots, x_N^T]^T$ and $e = [e_1^T, \ldots, e_N^T]^T$, which are the node state and the node error stack vectors, respectively. Furthermore, we call $\Phi(t, x) = \left[h_1^T(t, x_1), \ldots h_N^T(t, x_N)\right]^T$ the stack vector of the QUAD components, $\Psi(t, x) = \left[g_1^T(t, x_1), \ldots g_N^T(t, x_N)\right]^T$ the stack vector of the Affine components, and $\Xi = -1_N \otimes \frac{1}{N}\sum_{j=1}^{N} f_j(t, x_j)$ the term taking into account the dynamics of the average state, with $\mathbf{1}_N$ being the vector of N unitary entries. In this way, equations (4.1) and the error equation (4.3) can be recast, respectively, as

$$\dot{x} = \Phi(t, x) + \Psi(t, x) - c\mathrm{H}(t, x), \tag{4.5}$$

$$\dot{e} = \Phi(t, x) + \Psi(t, x) + \Xi(t, x) - c\mathrm{H}(t, e), \tag{4.6}$$

with

$$\mathrm{H}(t, x) = \begin{bmatrix} \sum_{j=1}^{N} a_{1j}\eta(t, x_1 - x_j) \\ \vdots \\ \sum_{j=1}^{N} a_{Nj}\eta(t, x_N - x_j) \end{bmatrix}.$$

In the case of networks with linear coupling, the state equation (4.2) and the error equation (4.4) can be recast as

$$\dot{x} = \Phi(t, x) + \Psi(t, x) - c(L \otimes \Gamma)x, \tag{4.7}$$

$$\dot{e} = \Phi(t, x) + \Psi(t, x) + \Xi(t, x) - c\,(L \otimes \Gamma)\,e. \tag{4.8}$$

Bounded convergence will be studied for a particular class of dynamical systems we introduce below, the *QUAD Affine systems.* However, before giving the definitions of *QUAD Affine* PWS vector fields, we introduce here the symbols we are going to use. Indeed, in order to simplify the notation, in what follows a set valued function $\mathcal{F}[f](t,x)$ (see Section 2.5) is equivalently denoted by $\underset{\sim}{f}(t,x)$. Moreover, an element of $\underset{\sim}{f}(t,x)$ is denoted by $\widetilde{f}(t,x)$. Furthermore, without loss of generality, we will consider the starting time $t_0 = 0$. In addition to this, we will also denote with $\|\cdot\|_p$ the matrix (vector) p-norm, with $\lambda_{\max}(M)$ the maximum eigenvalue of a matrix M, with $\mathrm{diag}\{m_i\}_{i=1}^{s}$ the $s \times s$ diagonal matrix whose diagonal elements are $m_1, \ldots, m_s$. Furthermore, given a matrix M, its positive (semi) definiteness is denoted by $M > 0$ ($M \geq 0$), while with $\mathcal{D}$ we denote the set of diagonal matrices and with $\mathcal{D}^+$ the set of positive definite diagonal matrices.

Definition 4.2.1. *Similarly to what stated in [39, 52] we say that, given a pair of $n \times n$ matrices $P \in \mathcal{D}^+$, $W \in \mathcal{D}$, a PWS vector field $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is QUAD(P,W) if and only if the following inequality holds:*

$$(x-y)^T P \left[\widetilde{f}(t,x) - \widetilde{f}(t,y)\right] \leq (x-y)^T W (x-y), \tag{4.9}$$

for all $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}^+$, $\widetilde{f}(t,x) \in \underset{\sim}{f}(t,x)$, $\widetilde{f}(t,y) \in \underset{\sim}{f}(t,y)$.

Note that this property is equivalent to the well-known one-sided Lipschitz condition for $P = I_n$ and $W = wI_n$ [46]. Furthermore, the QUAD condition is also related to some relevant properties of the vector fields, such as the contraction for smooth systems and the classical Lipschitz condition, see [52] for further details. See also Chapter 6 for a dissertation on contraction theory.

Definition 4.2.2. *A PWS system is said to be* QUAD(P,W) Affine *iff its vector field can be written in the form:*

$$f(t, x(t)) = h(t, x(t)) + g(t, x(t)), \tag{4.10}$$

where:

1. *h is either a continuous or piecewise smooth QUAD(P,W) function.*
2. *g is either a continuous or piecewise smooth function such that there exists a positive scalar $M < +\infty$ satisfying*

$$\left\|\widetilde{g}(t,x(t))\right\|_2 < M, \qquad \forall x \in \mathbb{R}^n, \forall t \in \mathbb{R}^+, \forall\, \widetilde{g}(t,x(t)) \in \underset{\sim}{g}(t,x(t)$$

It is worth mentioning that QUAD Affine systems can exhibit sliding mode and chaotic solutions, so this hypothesis on the nodes' dynamics does not exclude typical behaviors that may arise in PWS systems. Furthermore, as illustrated in the examples in the following sections, several dynamical systems can be written in QUAD Affine form.

Finally, we define the sets $\mathcal{Q}$ and $\mathcal{PW}$, that will be used in the rest of the chapter and whose relevance will be clarified through a set of numerical example in the following sections.

Definition 4.2.3. *Given a vector field $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$, let $\mathcal{Q} \subseteq \mathcal{D}^+$ be the (possibly empty) set of matrices such that, for every $P \in \mathcal{Q}$, there exists a diagonal matrix W such that* (4.9) *is satisfied. We say that a pair of matrices (P_i, W_i) belongs to the set $\mathcal{PW}$ if and only if $P_i \in \mathcal{D}^+$ and $W_i \in \mathcal{D}$, and* (4.9) *is satisfied for $P = P_i$ and $W = W_i$.*

In the following sections, we provide a set of sufficient conditions for ϵ-bounded convergence. Specifically, in Section 4.3, we start by analyze the case of linearly coupled networks of nonidentical piecewise smooth systems. Then, we extend the results to the case of networks coupled through nonlinear protocols in Section 4.4.

4.3 Convergence analysis for linearly coupled networks nodes

We consider a network modeled by equation (4.2) of N nonidentical piecewise smooth QUAD(P, W_i) Affine systems, $i = 1, \ldots, N$.

Assumption 4.3.1. *$h_i(t, x_i)$ is QUAD(P, W_i) with $W_i < 0$, $\left|\left|\widetilde{g}_i(t, x_i)\right|\right|_2 < M_i$ for all $\widetilde{g}_i(t, x) \in \underset{\sim}{g}_i(t, x)$, $t \in \mathbb{R}^+$, $x \in \mathbb{R}^n$, $i = 1, \ldots, N$,*

$$\sup_{\substack{t\in[0,+\infty)\\ i=1,\ldots,N}} \left|\left|\widetilde{h}_i(t, 0)\right|\right|_2 \leq \bar{h}_0 < +\infty,$$

for all $\widetilde{h}_i(t, 0) \in \underset{\sim}{h}_i(t, 0)$, and all the systems share a nonempty common set $\mathcal{C}_{\mathcal{D}^+} \subseteq \mathcal{D}^+$ such that every $P \in \mathcal{C}_{\mathcal{D}^+}$ implies $W_i < 0$ satisfying inequality (4.9)*, for all $i = 1, \ldots, N$.*

We define $\overline{M}$ as

$$\overline{M} = \max_{i=1,\ldots,N} M_i. \tag{4.11}$$

Before illustrating our result, we need to give the following definitions.

Definition 4.3.1. *Given that Assumption 4.3.1 holds, and considering a $n \times n$ matrix $Q \in \mathcal{C}_{\mathcal{D}^+}$, the non-empty set $B(Q) \subset \mathbb{R}^{Nn}$ is*

$$B(Q) = \left\{ x \in \mathbb{R}^{Nn} : ||x||_2 < -\frac{\sqrt{N}||Q||_2 \left(\overline{M} + \bar{h}_0\right)}{w_{\max}(Q)} \right\}, \tag{4.12}$$

where $\overline{M}$ is defined in (4.11) *$w_{\max}(Q) = \max_{i=1,\ldots,N} \lambda_{\max}\left(W_i(Q)\right)$, with $W_i < 0$ such that $(Q, W_i) \in \mathcal{PW}$.*

Also, we define the matrix Q^*,and the scalar $h_{\max} \in \mathbb{R}^+$ as

$$Q^* = \underset{Q\in\mathcal{C}_{\mathcal{D}^+}}{\operatorname{argmax}} \frac{||Q||_2 \left(\overline{M} + \bar{h}_0\right)}{w_{\max}(Q)}, \tag{4.13}$$

$$h_{\max} = \max_{\substack{i=1,\dots,N \\ z\in B(Q^*) \\ t\in[0,+\infty)}} \left\| \tilde{h}_i(t,z) \right\|_2, \qquad \forall \tilde{h}_i(t,z) \in \underset{\sim}{h}_i(t,z), \quad \forall i = 1,\dots,N. \tag{4.14}$$

Notice that, in what follows, we always refer to the case in which h_i does not diverge in the finite ball $B(Q^*)$, implying $h_{\max} < \infty$. Notice that in (4.12) and (4.13) we state explicitly the dependence of $w_{\max}$ on Q. In general, a choice of Q implies the selection of suitable matrices $W_1(Q),\dots,W_N(Q) < 0$ satisfying relation (4.9). Here, we also remark that (4.14) implies that the set-valued function $\underset{\sim}{h}_i(t,z)$ is bounded for all time instants $t \in \mathbb{R}^+$ and takes values in the ball $B(Q^*)$ of the origin.

Here, we define

$$W^{\max} = \operatorname{diag}\left\{ \max_{i=1,\dots,N} \lambda_1(W_i), \dots, \max_{i=1,\dots,N} \lambda_N(W_i) \right\} < 0. \tag{4.15}$$

Notice that, in (4.15), $W^{\max}$ depends on the choice of P, as well as the matrices $W_1,\dots,W_N < 0$, and that $(P, W^{\max})$ belongs to the set $\mathcal{PW}$. Furthermore, we also define the pair of matrices P^* and $W^{\max *}$ as

$$(P^*, W^{\max *}) = \underset{\substack{P\in\mathcal{C}_{\mathcal{D}^+}, \\ (P,W^{\max}(P))\in\mathcal{PW}}}{\operatorname{argmin}} \frac{\sqrt{N}||P||_2(\overline{M} + h_{\max})}{m\,(c, P, W^{\max}(P))}, \tag{4.16}$$

where the real function $m(c, P, W^{\max})$ is defined as

$$m(c, P, W^{\max}) = -\max\left\{ \lambda_{\max}(W_l^{\max}) - c\lambda_2(L \otimes P_l\Gamma_l), \lambda_{\max}(W_{n-l}^{\max}) \right\}, \tag{4.17}$$

with $W_l^{\max}$ and P_l being the $l \times l$ upper-left block of matrices $W^{\max}$ and P respectively, while $W_{n-l}^{\max}$ is the $(n-l)\times(n-l)$ lower-right block of matrix $W^{\max}$.

Now, we are ready to give the main stability results for linearly coupled networks. Specifically, we focus on the case of diagonal inner coupling matrix, while the extension to the case of nondiagonal Γ is encompassed in the study of nonlinear coupling functions. Henceforth, here we consider $\Gamma = \operatorname{diag}\{\gamma_i\}_{i=1}^n$. Without loss of generality, we assume

$$\gamma_i = \begin{cases} \bar{\gamma}_i > 0 & i = 1,\dots,l, \\ 0 & i = l+1,\dots,n, \end{cases} \tag{4.18}$$

with $l \in \{0, 1, \dots, n\}$. To use a compact notation, we denote by Γ_l the $l \times l$ upper-left block of matrix Γ.

Theorem 4.3.1. *Network* (4.2) *of N QUAD(P,W_i) Affine systems satisfying Assumption 4.3.1, with diagonal inner coupling matrix* $\Gamma \geq 0$, *achieves ϵ-bounded convergence for any value of the coupling strength $c > 0$, and an upper bound for ϵ is given by*

$$\bar{\epsilon} = \min\left\{ \bar{\epsilon}_1 := -\frac{2\sqrt{N}||Q^*||_2(\overline{M} + \bar{h}_0)}{w_{\max}(Q^*)}, \bar{\epsilon}_2 := \frac{\sqrt{N}||P^*||_2(\overline{M} + h_{\max})}{m(c, P^*, W^{\max}(P^*))} \right\}, \tag{4.19}$$

where the function m is defined in (4.17)*, and Q^*, $h_{\max}$ and P^* are defined in* (4.13)*,* (4.14)*, and* (4.16) *respectively.*

Proof. The proof consists of two steps. Firstly, we show the existence of an invariant region for the state trajectories of the nodes. Then, we derive the upper bound on ϵ as a function of the coupling gain c.

Step 1. Given equation (4.7), let us consider the quadratic function

$$U = \frac{1}{2} x^T (I_N \otimes Q) x, \tag{4.20}$$

where $Q \in \mathcal{C}_{\mathcal{D}+}$. The time derivative of U along the trajectories of the network satisfies

$$\dot{U}(x) \in \underset{\sim}{\mathcal{L}}_{\mathcal{F}[\chi_1]} U(x),$$

where $\chi_1(t,x) = \Phi(t,x) + \Psi(t,x) - c\,(L \otimes \Gamma)\, x$. Applying the sum rule reported in Section 2.5, we can write

$$\dot{U}(x) \in \underset{\sim}{\mathcal{L}}_{\mathcal{F}[\chi_1]} U(x) \subseteq \underset{\sim}{\mathcal{L}}_{\mathcal{F}[\Phi]+\mathcal{F}[\Psi]+\mathcal{F}[\chi_\gamma]} U(x), \tag{4.21}$$

where $\chi_\gamma(t,x) = -c\,(L \otimes \Gamma)\, x$.

Applying the consistency rule to the smooth coupling term χ_γ, we can write[2]

$$\underset{\sim}{\mathcal{L}}_{\mathcal{F}[\Phi]+\mathcal{F}[\Psi]+\mathcal{F}[\chi_\gamma]} U(x) = \mathcal{U}_{\mathcal{L}} = \left\{ x^T \left(I_N \otimes Q\right) \underset{\sim}{\Phi} + x^T \left(I_N \otimes Q\right) \underset{\sim}{\Psi} - c x^T \left(L \otimes Q\Gamma\right) x \right\}. \tag{4.22}$$

Now, adding and subtracting $x^T \left(I_N \otimes Q\right) \underset{\sim}{\Phi}_0$, where $\underset{\sim}{\Phi}_0 = \mathcal{F}[\Phi](t,0)$, and using the product rule, we obtain

$$\mathcal{U}_{\mathcal{L}} \subseteq \mathcal{V}_{\mathcal{L}} = \left\{ \sum_{i=1}^{N} x_i^T Q \underset{\sim}{h}_i(t,x_i) + x^T \left(I_N \otimes Q\right) \underset{\sim}{\Psi} - c x^T \left(L \otimes Q\Gamma\right) x + x^T \left(I_N \otimes Q\right) \underset{\sim}{\Phi}_0 - \sum_{i=1}^{N} x_i^T Q \underset{\sim}{h}_i(t,0) \right\}. \tag{4.23}$$

Therefore, using the QUAD assumption (4.9), for a generic element of the set $v_l \in \mathcal{V}_{\mathcal{L}}$, the following inequality holds

$$v_l \leq x^T \left[I_N \otimes w_{\max}(Q) I_n - cL \otimes Q\Gamma\right] x + x^T \left(I_N \otimes Q\right) \widetilde{\Psi} + x^T \left(I_N \otimes Q\right) \widetilde{\Phi}_0, \qquad \forall\, \widetilde{\Psi} \in \underset{\sim}{\Psi}, \forall \widetilde{\Phi}_0 \in \underset{\sim}{\Phi}_0. \tag{4.24}$$

From standard matrix algebra, we have (denoting $w_{\max}(Q)$ as $w_{\max}$ for the sake of brevity)

$$v_l \leq x^T \left[I_N \otimes w_{\max} I_n - cL \otimes Q\Gamma\right] x + \|x\|_2 \left\|I_N \otimes Q\right\|_2 \sqrt{N}\overline{M} + \|x\|_2 \left\|I_N \otimes Q\right\|_2 \sqrt{N}\bar{h}_0. \tag{4.25}$$

Combining (4.21)-(4.25), it follows that

$$\dot{U} \leq x^T \left[I_N \otimes w_{\max} I_n - cL \otimes Q\Gamma\right] x + \|x\|_2 \left\|I_N \otimes Q\right\|_2 \sqrt{N}\overline{M} + \|x\|_2 \left\|I_N \otimes Q\right\|_2 \sqrt{N}\bar{h}_0.$$

Rewriting the state vector as $x = a\hat{x}$, with $\hat{x} = \frac{x}{\|x\|_2}$, we finally have

$$\dot{U} \leq w_{\max} a^2 + a\sqrt{N}||Q||_2 \left(\overline{M} + \bar{h}_0\right). \tag{4.26}$$

Therefore, as $w_{\max} < 0$, if $a > -\sqrt{N}||Q||_2 \left(\overline{M} + \bar{h}_0\right)/w_{\max}$, then $\dot{U} < 0$. Hence, we can say that all the trajectories of network (4.7) eventually converge to the set $B(Q^*)$, where

[2] Here and in what follows, given a vector y and a set-valued function $\underset{\sim}{f}$ of coherent dimension, by $y^T \underset{\sim}{f}$ we mean $\left\{ y^T \widetilde{f}, \forall \widetilde{f} \in \underset{\sim}{f} \right\}$.

B is given in Definition 4.3.1 and Q^* is defined in (4.13). Thus, we can conclude that network (4.2) achieves ϵ-bounded convergence, with $\epsilon = -2\sqrt{N}||Q^*||_2\left(\overline{M}+\bar{h}_0\right)/w_{\max}$, being the bound on the convergence error. Note that this estimate of the bound on ϵ might be conservative. We now derive an alternative bound.

Step 2. Let us consider equation (4.8) and the following quadratic form

$$V(e) = \frac{1}{2}e^T(I_N \otimes P)e,$$

where $P \in \mathcal{C}_{\mathcal{D}^+}$. The time derivative of V is

$$\dot{V}(e) \in \underset{\sim}{\mathcal{L}}_{\mathcal{F}[\chi_2]}V(e), \tag{4.27}$$

where $\chi_2(t,x,e) = \Phi(t,x)+\Psi(t,x)+\Xi(t,x)-c\left(L\otimes\Gamma\right)e$. Using the sum and consistency rules, we obtain

$$\dot{V}(e) \in \underset{\sim}{\mathcal{L}}_{\mathcal{F}[\chi_2]}V(e) \subseteq \overline{\mathcal{U}}_{\mathcal{L}} = \left\{e^T\left(I_N\otimes P\right)\underset{\sim}{\Phi} + e^T\left(I_N\otimes P\right)\underset{\sim}{\Psi} + e^T\left(I_N\otimes P\right)\underset{\sim}{\Xi} - e^T\left(L\otimes P\Gamma\right)e\right\}. \tag{4.28}$$

Now, from the properties of the Filippov set-valued function, and adding and subtracting $e^T\left(I_N\otimes P\right)\underset{\sim}{\Phi}_{\bar{x}}$, with $\underset{\sim}{\Phi}_{\bar{x}} = \mathcal{F}[\Phi](t,\bar{x})$, and using the product rule we can write $\overline{\mathcal{U}}_{\mathcal{L}} \subseteq \overline{\mathcal{V}}_{\mathcal{L}}$, where $\overline{\mathcal{V}}_{\mathcal{L}}$ is

$$\overline{\mathcal{V}}_{\mathcal{L}} = \left\{\sum_{i=1}^N e_i^T P\underset{\sim}{h}_i(t,x_i) + e^T\left(I_N\otimes P\right)\underset{\sim}{\Psi} + \sum_{i=1}^N e_i^T P\underset{\sim}{\xi} - ce^T\left(L\otimes P\Gamma\right)e + e^T\left(I_N\otimes P\right)\underset{\sim}{\Phi}_{\bar{x}} - \sum_{i=1}^N e_i^T P\underset{\sim}{h}_i(t,\bar{x}),\right\},$$

with $\underset{\sim}{\xi} \in \mathcal{F}\left[-\frac{1}{N}\sum_{j=1}^N f_j(t,x_j)\right]$. As $\sum_{i=1}^N e_i = 0$, we have $\sum_{i=1}^N e_i^T P\underset{\sim}{\xi} = 0$. Considering the QUAD Affine assumption, a generic element v_l of the set $\mathcal{V}_{\mathcal{L}}$ satisfies the following inequality:

$$v_l \leq e^T\left[I_N\otimes W^{\max} - cL\otimes P\Gamma\right]e + e^T\left(I_N\otimes P\right)\widetilde{\Psi} + e^T\left(I_N\otimes P\right)\widetilde{\Phi}_{\bar{x}}, \qquad \forall\widetilde{\Psi}\in\underset{\sim}{\Psi}, \forall\widetilde{\Phi}_{\bar{x}}\in\underset{\sim}{\Phi}_{\bar{x}}$$

From the properties of the norm, and for all the initial conditions $x(0)$ chosen in the set $B(Q)$, we have

$$v_l \leq e^T\left[I_N\otimes W^{\max} - cL\otimes P\Gamma\right]e + \|e\|_2\left\|I_N\otimes P\right\|_2\sqrt{N}\overline{M} + \|e\|_2\left\|I_N\otimes P\right\|_2\sqrt{N}h_{\max},$$

where $h_{\max}$ is defined in (4.14). Hence, we have that

$$\dot{V}(e) \in \underset{\sim}{\mathcal{L}}_{\mathcal{F}[\chi_2]}V(e) \subseteq \overline{\mathcal{V}}_{\mathcal{L}},$$

and we can write

$$\dot{V}(e) \leq e^T\left[I_N\otimes W^{\max} - cL\otimes P\Gamma\right]e + \|e\|_2\left\|I_N\otimes P\right\|_2\sqrt{N}\left(\overline{M}+h_{\max}\right). \tag{4.29}$$

From the properties of the Kronecker product [97], we have $\|I_N\otimes P\|_2 = \|I_N\|_2\|P\|_2 = \|P\|_2$. Now, notice that the error vector e can be decomposed in two parts: one is related to the coupled state components, namely $\tilde{e}_l = \left[e_1^{(1)},\dots,e_1^{(l)},\dots,e_N^{(1)},\dots,e_N^{(l)}\right]^T$, and the other, denoted by $\tilde{e}_{n-l} = \left[e_1^{(l+1)}, \ldots, e_1^{(n)}, \ldots, e_N^{(l+1)}, \ldots, e_N^{(n)}\right]^T$, to the uncoupled components. Furthermore, we define $\bar{e}_i = \left[e_1^{(i)}, e_2^{(i)}, \ldots, e_N^{(i)}\right]^T$. So, from (4.18), we can rewrite (4.29) as

$$\dot{V} \leq \sum_{i=1}^{l} [w_i^{\max} \bar{e}_i^T \bar{e}_i - c p_i \bar{\gamma}_i \bar{e}_i^T L \bar{e}_i] + \sum_{i=l+1}^{n} w_i^{\max} \bar{e}_i^T \bar{e}_i + \|e\|_2 \|I_N \otimes P\|_2 \sqrt{N} \left(\overline{M} + h_{\max}\right),$$

where $w_i^{\max}$ are the diagonal entries of the diagonal matrix $W^{\max}$. From Lemma 3.2.1 and from matrix algebra, we have

$$\dot{V} \leq [\lambda_{\max}(W_l^{\max}) - c\lambda_2(L \otimes P_l \Gamma_l)] \, \tilde{e}_l^T \tilde{e}_l + \lambda_{\max}(W_{n-l}^{\max}) \tilde{e}_{n-l}^T \tilde{e}_{n-l}^T + \|e\|_2 \|I_N \otimes P\|_2 \sqrt{N} \left(\overline{M} + h_{\max}\right).$$

Then, rewriting the convergence error as $e = a\hat{e}$, with $\hat{e} = \frac{e}{\|e\|_2}$, for all initial conditions $x(0) \in B(Q)$ we finally obtain

$$\dot{V}(e) \leq -m\left(c, P, W^{\max}\right) a^2 + a\sqrt{N} \|P\|_2 \left(\overline{M} + h_{\max}\right), \tag{4.30}$$

with $m\left(c, P, W^{\max}\right)$ defined according to (4.17). Therefore, if $a > \sqrt{N} \|P\|_2 \left(\overline{M} + h_{\max}\right) / m\left(c, P, W^{\max}\right)$, then $\dot{V} < 0$. From (4.30), the optimization problem (4.16) immediately follows. The minimum value of the bound in (4.19), with Q^* defined in (4.13), is trivially obtained by combining (4.26) and (4.30). □

Remark 4.3.1. *Notice that in the case where $h_i = h_j$ for all i, j (which implies $W_i = W < 0$ for all $i = 1, \ldots, N$), ϵ-bounded convergence is trivially guaranteed under the assumptions of Theorem 4.3.1, as the QUAD component of each system is contracting [124], as reported in [52]. In particular, asymptotic convergence ($\epsilon = 0$) is achieved if $g_i = 0$ for all $i = 1, \ldots, N$, even if the systems are decoupled.*

Now, we study the stability properties of a networks of QUAD Affine systems, which differ only for the bounded component g. In this case we relax the assumption made earlier to prove Theorem 4.3.1 and assume instead the following.

Assumption 4.3.2. *Let us consider N nonidentical piecewise smooth QUAD(P,W) Affine systems described by*

$$\dot{x}_i = h_i(t, x_i) + g_i(t, x_i) \qquad \forall i = 1, \ldots, N, \tag{4.31}$$

where

$$h_i(t, s) = h_j(t, s), \qquad \forall i, j = 1, \ldots, N,$$

with $s \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$. Furthermore, we call $\overline{M} = \max_{i=1,\ldots,N} M_i$, with M_i such that $\left\|\widetilde{g}_i(t, x)\right\|_2 < M_i$, for all $\widetilde{g}_i(t, x) \in \underset{\sim}{g}_i(t, x)$ and for all $t > 0$ and $x \in \mathbb{R}^n$.

Notice that, differently from Assumption 4.3.1, here we do not make any additional assumption on the matrix W which characterizes the QUAD components. Even though the matrix W is in general undefined, some of its diagonal elements may be negative.

According with the definition of Γ_l given in Section 4.3, we denote by W_l the $l \times l$ upper-left block of matrix $W = \mathrm{diag}\{w_i\}_{i=1}^n$, by P_l the $l \times l$ upper-left block of matrix P, and by W_{n-l} the $(n-l) \times (n-l)$ lower-right block of W. Also, we define the set $\mathcal{PW}_d^l$ as follows:

Definition 4.3.2. *Given a positive scalar d, $\mathcal{PW}_d^l \subseteq \mathcal{PW}$ is the subset of $\mathcal{PW}$ such that if $(P,W) \in \mathcal{PW}_d^l$, then $d\lambda_2(L \otimes P_l\Gamma_l) > \lambda_{\max}(W_l)$, where L is the Laplacian matrix of network* (4.2).

Now, we are ready to state the following theorem.

Theorem 4.3.2. *Consider the network* (4.2) *of N nonidentical QUAD(P, W) Affine systems satisfying Assumption 4.3.2. Without loss of generality, we assume the first $\bar{l} \in \{0,\dots,n\}$ diagonal elements of W to be non-negative, while the remaining $n-\bar{l}$ are negative. If the diagonal elements of matrix $\Gamma \in \mathcal{D}$ can be defined as in equation* (4.18), *with $l \geq \bar{l}$, then there always exists a $\bar{c} < \infty$ so that, for any coupling gain $c > \bar{c}$, the linearly coupled network* (4.2) *achieves ϵ-bounded convergence. Furthermore,*

1. *a conservative estimate, say $\tilde{c}$, of the minimum coupling gain $\hat{c}$ ensuring bounded convergence is*
$$\tilde{c} = \min_{(P,W)\in\mathcal{PW}} c(P,W), \tag{4.32}$$
where $c(P,W) = \max\left\{\dfrac{\lambda_{\max}(W_l)}{\lambda_2(L\otimes P_l\Gamma_l)}, 0\right\}$.

2. *for a given $c > \tilde{c}$, we can give the following upper bound on ϵ*
$$\bar{\epsilon} = \min_{(P,W)\in\mathcal{PW}_c^l} \frac{\overline{M}\sqrt{N}||P||_2}{m(c,P,W)}, \tag{4.33}$$
where $m(c,P,W)$ is a real function defined as
$$m(c,P,W) = -\max\left\{\lambda_{\max}(W_l) - c\lambda_2(L\otimes P_l\Gamma_l), \lambda_{\max}(W_{n-l})\right\}. \tag{4.34}$$

Proof. Consider the following candidate Lyapunov function
$$V(e) = \frac{1}{2}e^T(I_N\otimes P)e, \tag{4.35}$$
where we choose $P \in \mathcal{Q}^+$. Notice that $P\Gamma \geq 0$ and that the stack equation of the error evolution is given in (4.8). Evaluating the derivative of V along the trajectory of such error system, and proceeding as in the step 2 of the proof of Theorem 4.3.1, we get (4.28). From Assumption 4.3.2, $h_i(t,\bar{x}) = h_j(t,\bar{x}) = h(t,\bar{x})$ for all $i,j = 1,\dots,N$. Now, from the properties of the Filippov set-valued function, and adding and subtracting $\sum_{i=1}^N e_i^T P\underset{\sim}{h}(t,\bar{x})$, and using the product rule we can write $\overline{\mathcal{U}}_{\mathcal{L}} \subseteq \overline{\mathcal{V}}_{\mathcal{L}}$, where $\overline{\mathcal{V}}_{\mathcal{L}}$ is
$$\overline{\mathcal{V}}_{\mathcal{L}} = \left\{\sum_{i=1}^N e_i^T P\underset{\sim}{h}(t,x_i) + e^T(I_N\otimes P)\underset{\sim}{\Psi} + \sum_{i=1}^N e_i^T P\underset{\sim}{\xi} - ce^T(L\otimes P\Gamma)e + \sum_{i=1}^N e_i^T P\underset{\sim}{h}(t,\bar{x}) - \sum_{i=1}^N e_i^T P\underset{\sim}{h}(t,\bar{x})\right\}$$
with $\underset{\sim}{\xi} \in \mathcal{F}\left[-\frac{1}{N}\sum_{j=1}^N f_j(t,x_j)\right]$. As $\sum_{i=1}^N e_i = 0$, we have that $\sum_{i=1}^N e_i^T P\underset{\sim}{\xi} = 0$ and $\sum_{i=1}^N e_i^T P\underset{\sim}{h}(t,\bar{x}) = 0$. Considering the QUAD Affine assumption, and denoting v_l a generic element of the set $\mathcal{V}_{\mathcal{L}}$, the following inequality holds:
$$\dot{V}(e) \leq v_l \leq e^T\left[I_N\otimes W - cL\otimes P\Gamma\right]e + e^T(I_N\otimes P)\tilde{\Psi}, \qquad \forall\tilde{\Psi}\in\underset{\sim}{\Psi}$$

From trivial matrix properties, it follows that

$$\dot{V}(e) \leq e^T \left[I_N \otimes W - cL \otimes P\Gamma \right] e + \sqrt{N} \left\| e \right\|_2 \left\| I_N \otimes P \right\|_2 \overline{M}, \tag{4.36}$$

with $\overline{M} = \max_{i=1,\ldots,N} M_i$. Now, decomposing e in $\tilde{e}_l$ and $\tilde{e}_{n-l}$ as in the proof of Theorem 4.3.1, we obtain

$$\dot{V} \leq \left[\lambda_{\max}(W_l) - c\lambda_2(L \otimes P_l \Gamma_l) \right] \tilde{e}_l^T \tilde{e}_l + \lambda_{\max}(W_{n-l}) \tilde{e}_{n-l}^T \tilde{e}_{n-l}^T + \sqrt{N} ||e||_2 ||P||_2 \overline{M}.$$

Defining $m(c, P, W)$ according to (4.34), and rewriting the synchronization error as $e = a\hat{e}$, with $\hat{e} = \frac{e}{\|e\|_2}$, we obtain

$$\begin{aligned} \dot{V} \leq & - m(c, P, W) e^T e + \sqrt{N} ||e||_2 ||P||_2 \overline{M} \\ = & - m(c, P, W) a^2 + a\overline{M}\sqrt{N} ||P||_2. \end{aligned} \tag{4.37}$$

If we choose any $c > \tilde{c}$, then we can always select a couple $(P, W) \in \mathcal{PW}$ such that $m(c, P, W) > 0$. Then, from (4.37), we have that $a > \overline{M}\sqrt{N}||P||_2 / m(c, P, M)$ implies $\dot{V} < 0$. Hence, network (4.2) is ϵ-bounded synchronized with

$$\epsilon \leq \frac{\overline{M}\sqrt{N}||P||_2}{m(c, P, M)}. \tag{4.38}$$

Bound (4.33) follows trivially from (4.38) □

Notice that the computation of bound (4.32) requires the solution of the following optimization problem:

$$\min_{(P,W) \in \mathcal{PW}} \max \left\{ \frac{\lambda_{\max}(W_l)}{\lambda_2(L \otimes P_l \Gamma_l)}, 0 \right\}. \tag{4.39}$$

Trivially, if f is QUAD(P,W) Affine for some $W < 0$, the solution of the optimization problem (4.39) is $\tilde{c} = 0$. Otherwise, if a matrix $W < 0$ such that h is QUAD(P,W) with $P \in \mathcal{D}^+$ does not exist (this is the case, for instance, of the Lorenz and Chua's chaotic systems), then the optimization problem (4.39) is non-trivial and, since $\lambda_{\max}(W) > 0$, it can be rewritten as

$$\min_{(P,W) \in \mathcal{PW}} \frac{\lambda_{\max}(W_l)}{\lambda_2(L \otimes P_l \Gamma_l)}. \tag{4.40}$$

This is a constrained optimization problem that in scalar form can be written as:

$$\min_{\substack{(P,W) \in \mathcal{PW} \\ i=1,\ldots,l}} \frac{\max_i w_i}{\lambda_2(L) \min_i p_i \gamma_i}, \tag{4.41}$$

and which can be easily solved using the standard routines for constrained optimization, such as, for instance, the MATLAB optimization toolbox.

Remark 4.3.2. *Here, we discuss the meaning of the assumptions and bounds obtained in Theorem 4.3.2. Firstly, notice that the assumption on the vector field implies that the uncoupled components of the state vector are associated to contracting dynamics of the individual nodes. The minimum coupling strength needed to achieve bounded convergence is the minimum coupling ensuring shrinkage of the coupled part of the nodes' dynamics. Hence, the coupling configuration compensates for possible instabilities associated to positive diagonal elements of W. This minimum strength $\tilde{c}$ depends on the network* *topology. Specifically, the smaller $\tilde{c}$ is, the higher is $\lambda_2(L)$. Once an appropriate coupling gain is selected, the width of the bound ϵ depends on $m(c,P,W)$ and on $\overline{M}$. Clearly, $\overline{M}$ gives a measure of the heterogeneity between the vector fields, and so the higher it is, the higher ϵ is. On the other hand, $m(c,P,W)$ embeds both the information on both the nodes' dynamics and the structure of their interconnections. In particular, the elements p_i and w_i of matrices P and W, respectively, are related to the nodes' dynamics, while the information on the network topology are again embedded in $\lambda_2(L)$.*

When $\Gamma \in \mathcal{D}^+$, it is useful to consider the following corollary.

Corollary 4.3.1. *Consider a network of N QUAD(P,W) Affine systems satisfying Assumption 4.3.2. If the coupling matrix $\Gamma \in \mathcal{D}^+$, then*

1. *there exists a $\bar{c} < \infty$ so that, for any coupling gain $c > \bar{c}$, network (4.2) achieves ϵ-bounded convergence.*

2. *a conservative estimate, say $\tilde{c}$, of the minimum coupling gain ensuring ϵ-bounded convergence is*
$$\tilde{c} = \min_{(P,W)\in\mathcal{PW}} c(P,W), \tag{4.42}$$
where $c(P,W) = \max\left\{\dfrac{\lambda_{\max}(W)}{\lambda_2(L\otimes P\Gamma)}, 0\right\}$, and $\mathcal{PW}$ is defined according to Definition 4.2.3.

3. *for a given $\hat{c} > \tilde{c}$, we can give the following upper bound on ϵ*
$$\bar{\epsilon} = \min_{(P,W)\in\mathcal{PW}^n_{\hat{c}}} \frac{\overline{M}\sqrt{N}\,\|P\|_2}{\hat{c}\lambda_2(L\otimes P\Gamma) - \lambda_{\max}(W)}, \tag{4.43}$$
where the set $\mathcal{PW}^n_{\hat{c}}$ is defined according to Definition 4.3.2.

Proof. If $\Gamma \in \mathcal{D}^+$, then clearly $l = n \geq \bar{l}$ in the proof of Theorem 2 for any $W \in \mathcal{D}$ and from Theorem 4.3.2, the thesis follows. □

4.4 Convergence analysis for nonlinearly coupled networks

Now, we address the problem of guaranteeing ϵ-bounded convergence of (4.1) with a nonlinear coupling function η. Specifically, the analysis is performed for nonlinear coupling functions satisfying the following assumption.

Assumption 4.4.1. *The (possibly discontinuous) coupling function $\eta(t,z) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is component-wise odd ($\eta(-v) = \eta(v)$) and the following inequality holds*
$$z^T \tilde{\eta}(t,z) \geq z^T \Upsilon z, \qquad \forall t \in \mathbb{R}^+, \forall z : \|z\|_2 \leq e_{\max}, \forall\, \tilde{\eta}(t,z) \in \underset{\sim}{\eta}(t,z), \tag{4.44}$$
where $e_{\max} > 0$ and Υ is a diagonal matrix whose i-th diagonal element is $\upsilon_i \geq 0$, with $\sum_{i=1}^N = \upsilon_i > 0$. Without loss of generality, we consider $\upsilon_i > 0$ for all $i \leq r$, with $r \leq n$, while $\upsilon_i = 0$ otherwise.

Convergence to a bounded steady-state error is proved by assuming $Q = I$ and $P = I$. This choice, which is less general than the one considered in Theorem 4.3.1, allows however to analyze a more general nonlinear protocol. Following the same notation as in Section 4.3, we define Υ_r as the $r \times r$ upper left block of the matrix Υ in Assumption 4.4.1. Also, we define the scalars $r_{\max}$ and $h_{\max}$ as

$$r_{\max} = \max\left\{\bar{\epsilon}_1 = -\frac{\sqrt{N}\,(\overline{M} + \bar{h}_0)}{w_{\max}},\ \nu = \|x(0)\|_2\right\} + \delta, \tag{4.45}$$

with $\delta > 0$ being a arbitrarily small positive scalar, and

$$h_{\max} = \max_{\substack{i=1,\dots,N\\ \|z\|_2 \le r_{\max}\\ t\in[0,+\infty)}} \left\|\widetilde{h}_i(t,z)\right\|_2, \qquad \forall \widetilde{h}_i(t,z) \in \underset{\sim}{h}_i(t,z), \quad \forall i = 1,\dots,N. \tag{4.46}$$

Theorem 4.4.1. *Consider the nonlinearly coupled network* (4.1) *of N negative definite QUAD(I,W_i) Affine systems and suppose that the nonlinear coupling protocol satisfies Assumption 4.4.1. Also, suppose that, in* (4.46)*, $h_{\max} < +\infty$, and that each node of the network satisfies Assumption 4.3.1 with $P = I$. If*

(i) The initial error satisfies $\|e(0)\|_2 \le e_{\max}/2$, with $e_{\max}$ being defined in Assumption 4.4.1;

(ii)

$$-\frac{\sqrt{N}(\overline{M} + h_{\max})}{\lambda_{\max}(W^{\max}_{n-r})} < \frac{e_{\max}}{2}$$

where $W^{\max}$ is defined in (4.15) *and with $W^{\max}_r$ and $W^{\max}_{n-r}$ being its upper-left and lower-right blocks, respectively.*

Then, network (4.1) *achieves ϵ-bounded convergence if the coupling gain c id chosen greater than $\tilde{c}$ given by*

$$\tilde{c} = \max\left\{\frac{1}{\lambda_2(L\otimes\Upsilon_r)}\left(\frac{2\sqrt{N}(\overline{M} + h_{\max})}{e_{\max}} + \lambda_{\max}(W^{\max}_r)\right), 0\right\}. \tag{4.47}$$

Furthermore, an upper bound on ϵ is given by

$$\bar{\epsilon} = \min\left\{\bar{\epsilon}_1, \bar{\epsilon}_2 = \frac{\sqrt{N}(\overline{M} + h_{\max})}{m(c, W^{\max})}\right\}, \tag{4.48}$$

with $\bar{\epsilon}_1$ defined as in (4.45)*, and*

$$m(c, W^{\max}) = -\max\left\{\lambda_{\max}(W^{\max}_r) - c\lambda_2(L\otimes\Upsilon_r), \lambda_{\max}(W^{\max}_{n-r})\right\}.$$

Proof. To prove the theorem, we separately analyze the two possible cases: $\bar{\epsilon}_1 \le \nu$ and $\bar{\epsilon}_1 > \nu$, where ν is defined in (4.45).

Case (a): $\bar{\epsilon}_1 \le \nu$.

In this case, from (4.45) we have $r_{\max} = \|x(0)\|_2 + \delta$. Now, we first study the conditions for the existence of an invariant region in the error space, and then show the existence of an invariant region in state space. We start by evaluating the derivative of the function $V(e) = \frac{1}{2}e^T e$. We have

$$\dot{V}(e) \in \underset{\sim}{\mathcal{L}}_{\mathcal{F}[\chi_1]} V(e),$$

where $\chi_1 = \Phi(t,x) + \Psi(t,x) + \Xi(t,x) - c\mathrm{H}(t,e)$. Using the sum rule, we can write

$$\dot{V}(x) \in \overline{\mathcal{U}}_{\mathcal{L}} = \left\{ e^T \underset{\sim}{\Phi} + e^T \underset{\sim}{\Psi} + e^T \underset{\sim}{\Xi} - ce^T \underset{\sim}{\mathrm{H}} \right\}. \tag{4.49}$$

Adding and subtracting $e^T \underset{\sim}{\Phi}_{\bar{x}}$, with $\underset{\sim}{\Phi}_{\bar{x}} = \mathcal{F}[\Phi](t,\bar{x})$, and using the product rule, we have that

$$\begin{aligned}\dot{V}(e) \in \overline{\mathcal{U}}_{\mathcal{L}} \subseteq \overline{\mathcal{V}}_{\mathcal{L}} = \Bigg\{ & \sum_{i=1}^{N} e_i^T \underset{\sim}{h}_i(t,x_i) + e^T \underset{\sim}{\Psi} + \sum_{i=1}^{N} e_i^T \underset{\sim}{\xi} - \frac{1}{2} c \sum_{i=1}^{N} \sum_{j=i} w_{ij}(e_i - e_j)^T \underset{\sim}{\eta}(t, e_i - e_j) + e^T \underset{\sim}{\Phi}_{\bar{x}} \\ & - \sum_{i=1}^{N} e_i^T \underset{\sim}{h}_i(t,\bar{x}) \Bigg\},\end{aligned} \tag{4.50}$$

with $\underset{\sim}{\xi} \in \mathcal{F}\left[-\frac{1}{N} \sum_{j=1}^{N} f_j(t,x_j) \right]$. As $\sum_{i=1}^{N} e_i = 0$, we have $\sum_{i=1}^{N} e_i^T \underset{\sim}{\xi} = 0$. As $\bar{\epsilon}_1 < \nu$, inequality (4.44) is satisfied for all $t \in [0, t_c]$, where t_c is the time instant at which the average state trajectory may cross the ball of the origin of radius $r_{\max}$, i.e. $\|\bar{x}(t)\|_2 > r_{\max}$ for $t > t_c$ (later we will show that such time instant does not exist and therefore (4.44) is satisfied for all $t \in [0, +\infty)$). Indeed, from Assumptions 4.3.1 and 4.4.1, we have that a generic element of the set $v_l \in \overline{\mathcal{V}}_{\mathcal{L}}$ satisfies the following inequality

$$v_l \leq e^T \left[I_N \otimes W^{\max} - cL \otimes \Upsilon \right] e + e^T \widetilde{\Psi} + e^T \widetilde{\Phi}_{\bar{x}}, \qquad \forall t \in [0, t_c], \quad \forall \widetilde{\Psi} \in \underset{\sim}{\Psi}, \forall \widetilde{\Phi}_{\bar{x}} \in \underset{\sim}{\Phi}_{\bar{x}}, \tag{4.51}$$

and so, decomposing the error e as $\tilde{e}_r$ and $\tilde{e}_{n-r}$ as in the proof of Theorem 4.3.1, and following similar steps, we have that

$$\dot{V}(e) \leq -m\left(c, W^{\max}\right) a^2 + a\sqrt{N}\left(\overline{M} + h_{\max}\right), \qquad \forall t \in [0, t_c]. \tag{4.52}$$

Therefore, since hypothesis (ii) holds, it is now clear that if $c > \tilde{c}$, then relation (4.44) is feasible as the region $\|e\|_2 \leq \bar{\epsilon}_2 < e_{\max}/2$ is an invariant region in the error space. The feasibility of relation (4.44) holds until the crossing instant t_c. After t_c, (4.46) would not be guaranteed any more, as well as inequalities (4.51) and (4.52). To complete the proof of *Case (a)*, we now show that the crossing event never happens and so we can set $t_c = +\infty$. Let us consider the quadratic function $U = \frac{1}{2} x^T x$ and evaluate the derivative of U along the trajectories of the network. We have

$$\dot{U}(x) \in \underset{\sim}{\mathcal{L}}_{\mathcal{F}[\chi_2]} U(x),$$

where $\chi_2(t,x) = \Phi(t,x) + \Psi(t,x) - c\mathrm{H}(t,x)$. Now, using the sum rule, and following similar steps as in Theorem 4.3.1, we can write

$$\dot{U}(x) \in \mathcal{U}_{\mathcal{L}} = \left\{ x^T \underset{\sim}{\Phi} + x^T \underset{\sim}{\Psi} - cx^T \underset{\sim}{\mathrm{H}} \right\}.$$

Adding and subtracting $x^T \underset{\sim}{\Phi}_0$, with $\underset{\sim}{\Phi}_0 = \mathcal{F}[\Phi](t,0)$, and using the product rule, we can show that $\mathcal{U}_{\mathcal{L}}$ is included in the set $\mathcal{V}_{\mathcal{L}}$. Namely,

$$\mathcal{U}_{\mathcal{L}} \subseteq \mathcal{V}_{\mathcal{L}} = \left\{ \sum_{i=1}^{N} x_i^T \underset{\sim}{h}_i(t,x_i) + x^T \underset{\sim}{\Psi} - \frac{1}{2} c \sum_{i=1}^{N} \sum_{j=i}^{N} w_{ij}(x_i - x_j)^T \underset{\sim}{\eta}(t, x_i - x_j) + x^T \underset{\sim}{\Phi}_0 - \sum_{i=1}^{N} x_i^T \underset{\sim}{h}_i(t,0) \right\}.$$

Notice that, as stated above, relation (4.44) holds for all the $t \in [0, t_c]$ and so, using Assumptions 4.4.1 and 4.3.1, for a generic element of the set $v_l \in \mathcal{V}_{\mathcal{L}}$, the following inequality holds

$$v_l \leq x^T \left[I_N \otimes w_{\max} I_n - cL \otimes \Upsilon \right] x + x^T \widetilde{\Psi} + x^T \widetilde{\Phi}_0, \qquad \forall t \in [0, t_c], \quad \forall \widetilde{\Psi} \in \underset{\sim}{\Psi}, \forall \widetilde{\Phi}_0 \in \underset{\sim}{\Phi}_0.$$

Then, following the same steps in the proof of Theorem 4.3.1 we obtain

$$\dot{U} \leq w_{\max} a^2 + a\sqrt{N} \left(\overline{M} + \bar{h}_0 \right), \qquad \forall t \in [0, t_c]. \tag{4.53}$$

From (4.53), we get the radius $\bar{\epsilon}_1$ of an invariant region $\|x\|_2 \leq \bar{\epsilon}_1$ for system (4.5). In particular, for any $r \geq \bar{\epsilon}_1$, the region $\|x\|_2 \leq r$ is invariant. Since we are considering the case $\bar{\epsilon}_1 \leq \nu$, then $\|x\|_2 \leq r_{\max}$ is an invariant region for the overall system (4.5). So, the state x, as well as $\bar{x}$, will never cross the ball of radius $r_{\max}$ and equations (4.52) and (4.53) hold with $t_c = +\infty$. Then comparing these two expressions, bound (4.48) holds and the proof for $\bar{\epsilon}_1 \leq \nu$ is completed.

Case (b): $\bar{\epsilon}_1 > \nu$
In this case, we have $r_{\max} = \bar{\epsilon}_1 + \delta$. Again, we firstly consider the invariant region in the error space and then we analyze invariance in the state space. In particular, for the error invariant region we can follow the same steps of *Case (a)* and obtain again equation (4.52). About the invariance in the state space, it is immediate to see that $\bar{\epsilon}_1$ is invariant. Indeed, if the trajectory $x(t)$ does not cross the boundary $\|x\|_2 = \bar{\epsilon}_1$, then it is trivially invariant. On the other hand, if there exists an instant $\bar{t}$ such that $\|x(t)\|_2 = \bar{\epsilon}_1$, then it is possible to show invariance of region $\|x\|_2 \leq \bar{\epsilon}_1$ considering the proof of *Case (a)* from the initial time $\bar{t}$ and initial state $x(\bar{t})$.

Concluding, also in this case, equations (4.52) and (4.53) hold with $t_c = +\infty$ and the theorem is then proved. □

From Theorem 4.4.1, an useful corollary follows.

Corollary 4.4.1. *Consider the nonlinearly coupled network* (4.1) *of N negative definite QUAD(I,W_i) Affine systems and suppose that the nonlinear coupling protocol satisfies Assumption 4.4.1 with $e_{\max} = \infty$. Suppose also that each node of the network satisfies Assumption 4.3.1 with the choice $P = I$. Then, network* (4.1) *achieves $\epsilon-$bounded convergence and an upper bound on ϵ is* (4.48).

Proof. As $e_{\max} \to +\infty$, the hypotheses *(i)*, *(ii)*, in Theorem 4.4.1 are always satisfied and $\tilde{c} = 0$. Then, from Theorem 4.4.1 follows the thesis. □

As in Section 4.3, we now extend the analysis to the case of networks (4.1) of QUAD Affine(P, W) systems, with $P = I$, differing only for a bounded component. As in Theorem 4.4.1, we denote by Υ_r the $r \times r$ upper left block of matrix Υ.

Theorem 4.4.2. *Let us consider the nonlinearly coupled network* (4.1) *of N QUAD(I,W) Affine systems satisfying assumption 4.3.2. Without loss of generality, we assume the first $\bar{r} \in \{0, \ldots, N\}$ diagonal elements of W to be non-negative, while the remaining $n - \bar{r}$ are negative. If Assumption 4.4.1 holds with $r \geq \bar{r}$ and the following hypotheses hold:*

(i) The initial error satisfies $\|e(0)\|_2 \leq e_{\max}/2$, with $e_{\max}$ being defined in Assumption 4.4.1;

(ii)

$$-\frac{\sqrt{N}\overline{M}}{\lambda_{\max}(W_{n-r})} \leq \frac{e_{\max}}{2}$$

with, as usual, W_r and W_{n-r} being the upper-left and the lower-right blocks of matrix W, respectively.

Then, choosing a coupling gain $c > \tilde{c}$, with

$$\tilde{c} = \max\left\{\frac{1}{\lambda_2(L\otimes\Upsilon_r)}\left(\frac{2\sqrt{N}\overline{M}}{e_{\max}} + \lambda_{\max}(W_r)\right), 0\right\}, \tag{4.54}$$

network (4.1) *achieves ϵ-bounded convergence. Furthermore, an upper bound on ϵ is*

$$\bar{\epsilon} = \frac{\overline{M}\sqrt{N}}{m_{\mathrm{nl}}(c, W)}, \tag{4.55}$$

where the function m_{nl} is a real function defined as

$$m_{\mathrm{nl}}(c, W) = -\max\left\{\lambda_{\max}(W_r) - c\lambda_2(L\otimes\Upsilon_r), \lambda_{\max}(W_{n-r})\right\}. \tag{4.56}$$

Proof. Considering the candidate Lyapunov function $V(e) = \frac{1}{2}e^Te$, and evaluating its derivative as in the step 2 of the proof of Theorem 4.4.1, we obtain equation (4.49). Adding and subtracting $\sum_{i=1}^N e_i^T \underset{\sim}{h}(t,\bar{x})$, and using the product rule, we have that

$$\begin{aligned}\dot{V}(e) \in \overline{\mathcal{V}}_{\mathcal{L}} = \Bigg\{ &\sum_{i=1}^N e_i^T \underset{\sim}{h}(t,x_i) + e^T \underset{\sim}{\Psi} + \sum_{i=1}^N e_i^T \underset{\sim}{\xi} - \frac{1}{2}c\sum_{i=1}^N\sum_{j=i} w_{ij}(e_i - e_j)^T \underset{\sim}{\eta}(t, e_i - e_j) + \sum_{i=1}^N e_i^T \underset{\sim}{h}(t,\bar{x}) \\ &- \sum_{i=1}^N e_i^T \underset{\sim}{h}(t,\bar{x})\Bigg\},\end{aligned}$$

with $\underset{\sim}{\xi} \in \mathcal{F}\left[-\frac{1}{N}\sum_{j=1}^N f_j(t,x_j)\right]$. As $\sum_{i=1}^N e_i = 0$, we have that $\sum_{i=1}^N e_i^T \underset{\sim}{\xi} = 0$, and $\sum_{i=1}^N e_i^T \underset{\sim}{h}(t,\bar{x}) = 0$. From Assumptions 4.3.2 and 4.4.1, the following inequality holds:

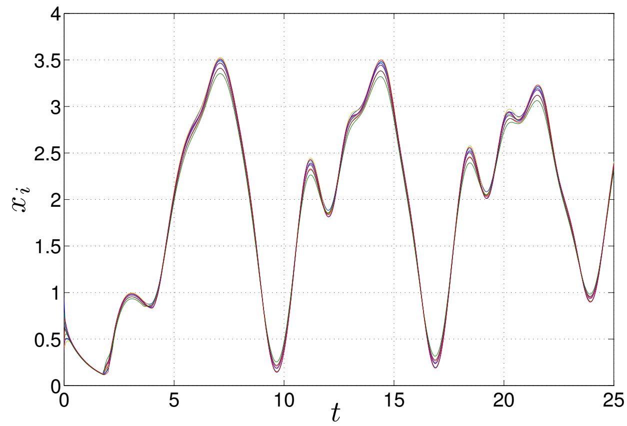
$$\dot{V}(e) \leq e^T\left[I_N \otimes W - cL\otimes\Upsilon\right]e + e^T\,\widetilde{\Psi}, \qquad \forall\,\widetilde{\Psi} \in \underset{\sim}{\Psi}.$$

Notice that, as in the proof of Theorem 4.4.1, the upper bound $\tilde{c}$ for the minimum coupling and hypothesis (ii) guarantee that the inequality (4.44) is always feasible. Decomposing the error vector e in the two parts $\tilde{e}_r$ and $\tilde{e}_{n-r}$ and following similar steps as in Theorem 4.3.2, the thesis follows. □

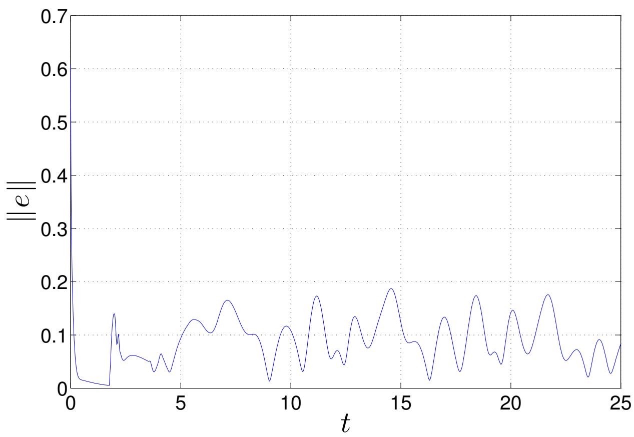
As in Section 4.3, we also provide a useful corollary.

Corollary 4.4.2. *Let us consider the nonlinearly coupled network* (4.1) *of N QUAD(I,W) Affine systems satisfying Assumption 4.3.2 and Assumption 4.4.1 with $e_{\max} = \infty$. Choosing a coupling gain $\hat{c} \geq \tilde{c}$, with $\tilde{c}$ defined in* (4.54)*, network* (4.1) *achieves ϵ-bounded convergence with an upper bound on ϵ given in* (4.55).

Proof. As $e_{\max} \to +\infty$, the hypotheses *(i)* and *(ii)* of Theorem 4.4.2 are always satisfied. Then, from Theorem 4.4.2, the thesis follows. □



(a) State evolution.



(b) Norm of the error.

Figure 4.1: Network of 10 linearly coupled nonidentical Ikeda systems. Coupling gain $c = 20$.

4.5 Examples

Here, we validate the theoretical analysis on a set of representative numerical examples. Specifically, in Section 4.5.1, a network of Ikeda systems is considered to validate Theorems 4.3.1 and 4.4.1, while Theorem 4.3.2 is used in Section 4.5.2 to estimate the minimum coupling strength guaranteeing bounded synchronization in networks of Chua's circuits. Then, in Section 4.5.3, Corollary 4.3.1 is used to study convergence of coupled chaotic relays. Finally, in Section 4.5.4 we study the convergence properties of nonuniform Kuramoto oscillators applying Theorem 4.4.2.

4.5.1 Networks of Ikeda systems

To clearly illustrate Theorems 4.3.1 and 4.4.1, we study the convergence of a network of nonidentical Ikeda systems. The Ikeda model has been proposed as a standard model of optical turbulence in nonlinear optical resonators, see [102, 103, 104] for further details. The optical resonator can be described by

$$\dot{x}_i = -a_i x_i + b_i \sin(x_i(t - \tau_i)),$$

where a_i, b_i and τ_i are positive scalars. As reported in [99], this system exhibits chaotic behavior when $\tau_i = 2$, $a_i = 1$ and $b_i = 4$. Synchronization of coupled Ikeda systems with parameter mismatches was studied in many recent works, see for instance [98, 99, 187], but it is assumed *a priori* that the trajectory of each node is bounded. Applying Theorem 4.3.1, we do not need this assumption, and we can show that a network of coupled Ikeda oscillators converges to a bounded set. In fact, it is easy to show that the assumptions of Theorem 4.3.1 are satisfied: the vector field $f_i(t, x_i)$ describing the nodes' dynamics is a QUAD(P,W) Affine system of the form

$$\dot{x}_i = h_i(t, x_i) + g_i(t, x_i),$$

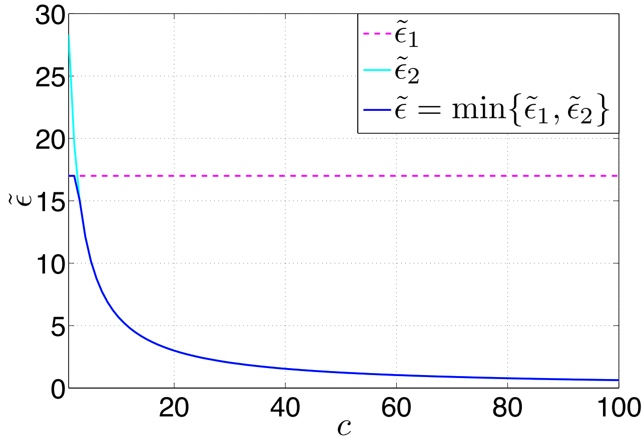
where $h_i(t, x_i) = -a_i x_i$ is QUAD with $P = p > 0$ and $W = w$ such that $-pa_i \leq w < 0$, and $g_i(t, x_i) = b_i \sin(x_i(t - \tau_i))$ is the affine bounded (smooth) term, with $|g_i(t, x_i)| \leq b_i$. Notice that the presence of the delayed state does not prevent the application of Theorems 4.3.1 and 4.4.1, as it affects a bounded component. Hence, from Theorem 4.3.1, we obtain a strong result: a network of nonidentical Ikeda systems is ϵ-bounded synchronized for any possible value of the positive scalars a_i, b_i and τ_i, and for any positive coupling strength $c > 0$, . Here, it is worth remarking that this result is independent from the value of the delays τ_i and from the choice of c. In all previous works, τ was considered identical from node to node and bounded synchronization was proven only for $c > \bar{c}$, with $\bar{c} > 0$ [98, 99, 187]. Moreover, Theorem 4.3.1 also provides an estimation of the bound ϵ, that can be made arbitrarily small by increasing c.

As an example, we consider a randomly generated network of 10 nodes. The initial conditions are taken randomly from a normal distribution. Furthermore, we assume that $a_i = a + \delta a_i$, $b_i = b + \delta b_i$ and $\tau_i = \tau + \delta\tau_i$, where $a = 1$, $b = 4$, and $\tau = 2$ are the nominal values of the parameters, while the parameters' mismatches are represented by δa_i, δb_i and $\delta\tau_i$, and are taken randomly from a uniform distribution in $[-0.25, 0.25]$. According to the theoretical prediction, in Figure 4.1 a representative simulation shows that ϵ-bounded synchronization is achieved. Then, in Figure 4.2, we report the upper bound for the steady-state error norm estimated for coupling strength c ranging from 1 to 100 (Figure 4.2(b)), which is consistent with the maximum steady-state error norm evaluated numerically (Figure 4.2(a)). This upper bound is clearly conservative, but allows us to predict the exponential decay of ϵ as c increases.

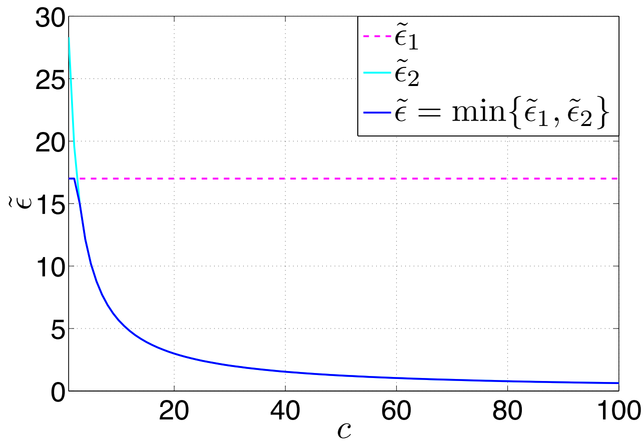
Now, we consider a network of Ikeda systems with the same coupling gain, but we introduce the following piecewise smooth nonlinear coupling $\eta(z) : \mathbb{R} \mapsto \mathbb{R}$:

$$\eta(z) = \begin{cases} \mathrm{sign}(z) & \textit{if } |z| < 1, \\ \mathrm{sign}(z)[(|z| - 1)^2 + 1] & \textit{if } |z| \geq 1. \end{cases} \tag{4.57}$$

This nonlinear coupling has no physical meaning, but has been introduced to show how ϵ-bounded convergence can also be enforced through a piecewise smooth coupling satisfying Assumption 4.4.1. Figures 4.3(a) and 4.3(b) confirm that bounded convergence is achieved.



(a) Actual steady-state norm of the error.



(b) Upper bound for ϵ, computed according to Theorem 4.3.1.

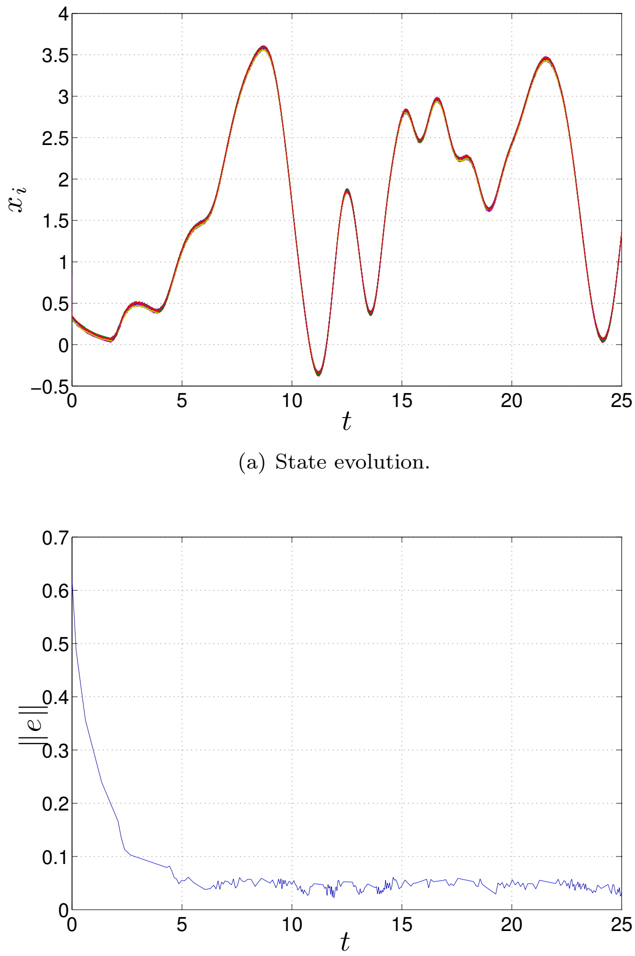
Figure 4.2: Network of 10 nonidentical Ikeda systems.

4.5.2 Networks of Chua's circuits

Let us consider now a network of Chua's circuits [135], see the schematic in Figure 4.4, forced by a squarewave input. Namely, the own dynamics of the i-th system can be written as $\dot{x}_i = h(t, x_i) + g_i(t, x_i)$. The unforced dynamics are described by $h = [h_1, h_2, h_3]^T$. Namely,

$$
\begin{aligned}
h_1(x_i) &= \alpha \left[x_{i2} - x_{i1} - \varphi(x_{i1})\right], \\
h_2(x_i) &= x_{i1} - x_{i2} + x_{i3}, \\
h_3(x_i) &= -\beta x_{i2},
\end{aligned}
$$

where, according to [135], $\alpha = 10$, $\beta = 17.30$, and $\varphi(x_{i1}) = bx_{i1} + (a-b)(|x_{i1}+1| - |x_{i1}-1|)/2$, with $a = -1.34$, $b = -0.73$. The squarewave input $g_i = [g_{i1}, 0, 0]^T$ acts



(a) State evolution.

(b) Norm of the error.

Figure 4.3: Network of 10 nonlinearly coupled nonidentical Ikeda systems. Coupling gain $c = 20$.

only on the first variable and is defined as

$$g_{i1}(t) = \mathrm{sgn}(\sin(t - i\pi/N)).$$

Notice that the vector fields of the Chua's circuits are nonidentical QUAD(P,W) Affine and satisfy Assumption 4.3.2. In fact, for any $P \in \mathcal{D}^+$, and for any $x, y \in \mathbb{R}^3$, we can write

$$\begin{aligned}(x-y)^T P(h(x)-h(y)) &= -10p_1e_1^2 - p_2e_2^2 + (10p_1+p_2)e_1e_2 + (p_2 - 17.3p_3)e_2e_3 + 10p_1e_1(\varphi(y_1) - \varphi(x_1)) \\ &\leq 3.4p_1e_1^2 - p_2e_2^2 + (10p_1+p_2)e_1e_2 + (p_2 - 17.3p_3)e_2e_3,\end{aligned}$$

where $e = x - y$, and where we have considered the maximum slope of the nonlinear function $\varphi(\cdot)$ to get the above inequality. Taking $p_2 = 17.3p_3$, and being $e_1e_2 \leq \|e_1e_2\| \leq$

$(\rho e_1^2 + e_2^2/\rho)/2$ for all $\rho > 0$, one has

$$\begin{aligned}(x-y)^T P(h(x)-h(y)) &\le 3.4p_1 e_1^2 - p_2 e_2^2 + (10p_1+p_2)e_1e_2 \\ &\le 3.4p_1 e_1^2 - p_2 e_2^2 + (10p_1+p_2)(\rho e_1^2 + e_2^2/\rho)/2 \\ &= \left(3.4p_1 + \rho\frac{10p_1+p_2}{2}\right)e_1^2 + \left(\frac{10p_1+p_2}{2\rho} - p_2\right)e_2^2. \end{aligned} \tag{4.58}$$

Moreover, for all $x \in \mathbb{R}^3$, $\|g(t,x)\| \le \overline{M} = 1$. Therefore, one finally obtains that the forced Chua's circuit are QUAD(P,W) Affine systems for any pair (P,W) such that $p_2 = 17.3p_3$ and $w_1 \ge 3.4p_1 + \rho(10p_1+p_2)/2$ and $w_2 \ge -p_2 + (10p_1+p_2)/2\rho$.

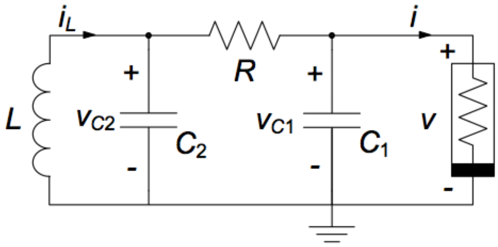


Figure 4.4: Schematic of a Chua's circuit.

Now, if we select $\Gamma = \mathrm{diag}\{1,0,1\}$, we have that all the assumptions of Theorem 4.3.2 are satisfied with $l = 2$.[3]

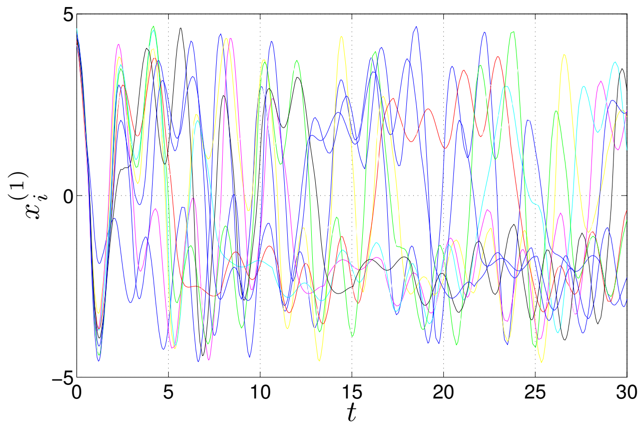
Notice that Theorem 4.3.2 can be used to estimate the minimum coupling strength guaranteeing bounded synchronization. From (4.32) and (4.58) follows that the estimation $\tilde{c}$ is the solution of the following constrained optimization problem:

$$\begin{aligned}&\tilde{c} = \frac{1}{\lambda_2(L)} \min_{p_1,p_2,\rho} \frac{3.4p_1 + \rho\frac{10p_1+p_2}{2}}{\min\left\{p_1, \frac{p_2}{17.3}\right\}} \\ &\frac{10p_1+p_2}{2\rho} - p_2 < 0 \\ &p_1, p_2, \rho > 0\end{aligned}$$

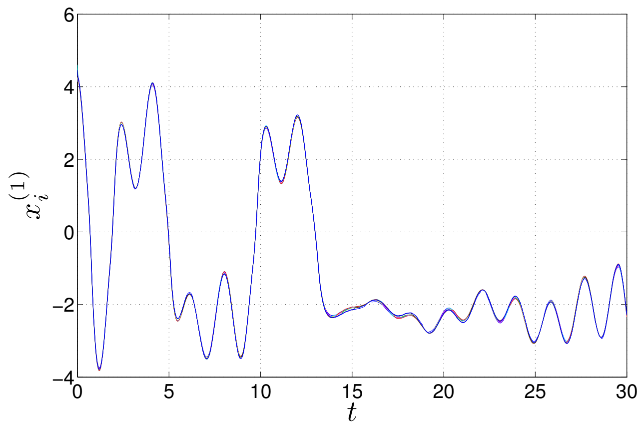
Notice that the first inequality of this constrained optimization problem allows us to synchronize the network without coupling the second state component, according to Theorem 4.3.2. Using the standard Matlab routines for constrained optimization problems, one easily obtains $\tilde{c} = 14.17/\lambda_2(L)$. In this example, we consider a network of $N = 10$ nodes with a connected random graph [72] with $\lambda_2(L) = 2.22$, from which follows that $\tilde{c} = 6.4$. Accordingly, we select $c = 10 > \tilde{c}$. Figure 4.5 shows the time evolution for the first component of the Chua's oscillators, both for the uncoupled and the coupled case. From the zoom in Figure 4.6, it is possible to observe that a a reduced mismatch between the nodes' trajectories remains, as can be noted also from the plot of the error norm, depicted in Figure 4.7. This simulation has been obtained considering random initial conditions in the domain of the chaotic attractor. However, it is worth mentioning that since Theorem 4.3.2 gives global synchronization conditions, bounded synchronization is ensured also in the case of divergent dynamics, as shown in Figure

[3]The application of Theorem 4.3.2 requires a trivial reordering of the state variables, that we omit here.

4.8, where some initial conditions have been randomly chosen outside the domain of the attractor.



(a)



(b)

Figure 4.5: Time evolution of component $x_i^{(1)}(t)$ for the network of Chua circuits: (a) uncoupled case; (b) coupled case.

4.5.3 Networks of chaotic relays

Several examples of piecewise smooth systems whose dynamics are consistent with Assumption 4.3.2 can be made. In particular, any QUAD system with a piecewise smooth feedback nonlinearity such as relay, saturation or hysteresis is also a QUAD Affine system. Here, we consider a network of five classical relay systems, e.g. [204], whose dynamics are described by:

$$\dot{x}_i = Ax_i + Br_i, \quad y_i = Cx_i, \quad r_i = -\mathrm{sgn}(y_i),$$

where

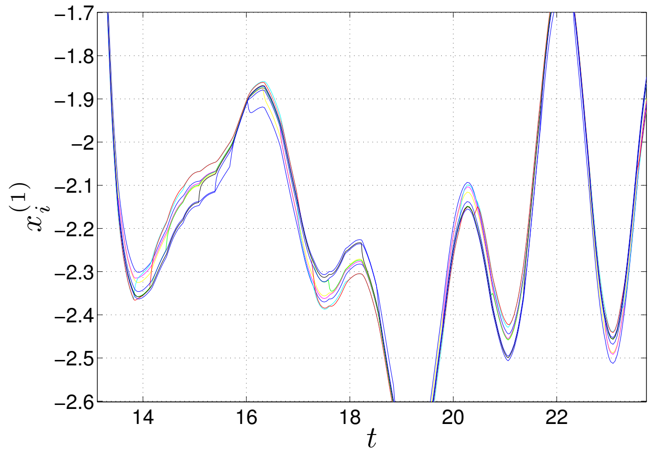


Figure 4.6: Zoom of the evolution of component $x_i^{(1)}(t)$ for the coupled Chua network

$$A = \begin{bmatrix} 1.35 & 1 & 0 \\ -99.93 & 0 & 1 \\ -5 & 0 & 0 \end{bmatrix},$$

$$B = [1, -2,\ 1]^T, \quad C = [1,\ 0,\ 0],$$

As shown in [60, 129], with this choice of parameter values, each relay exhibits both sliding motion and chaotic behavior.

The Laplacian matrix describing the network topology is

$$L = \begin{bmatrix} 3 & -1 & 0 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ 0 & -1 & 2 & -1 & 0 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & 0 & -1 & 3 \end{bmatrix},$$

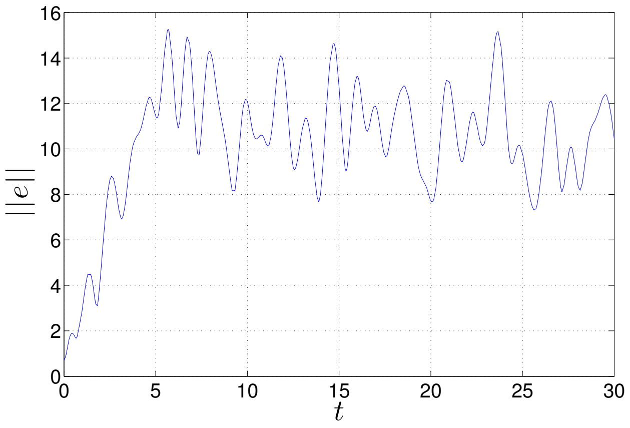
while the inner coupling matrix is $\Gamma = I_3$, that is, the nodes are coupled through all the state vector, and so the requirement $\Gamma \in \mathcal{D}^+$ of Corollary 4.3.1 is satisfied.

It easy to see that the network nodes satisfy Assumption 4.3.2. In particular, the QUAD term is $h(t, x_i) = Ax_i$ and the affine bounded (switching) term is $g_i(t, x_i) = Br_i$. Hence, we can use Corollary 4.3.1 to obtain an upper bound on the minimum coupling gain guaranteeing ϵ-bounded synchronization. Notice that, choosing for the sake of clarity $P = I_3$, we have

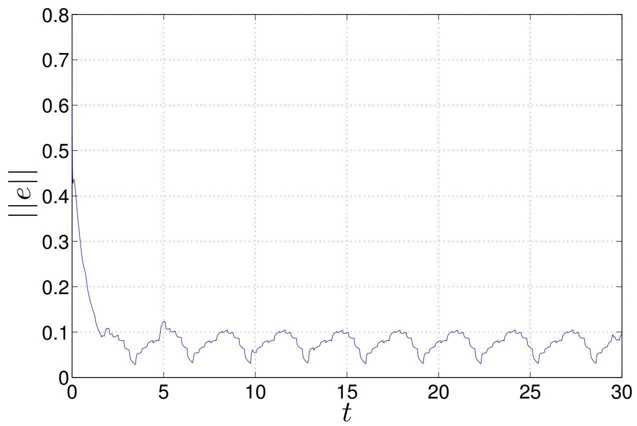
$$\begin{aligned} &(x-y)^T (h(x) - h(y)) = (x-y)^T A (x-y) = \\ &(x-y)^T A_{\text{sym}} (x-y) \le \lambda_{\max}(A_{\text{sym}}) (x-y)^T (x-y), \end{aligned}$$

where $A_{\text{sym}} = \frac{1}{2}(A + A^T)$.

In this example, we have $\lambda_{\max}(A_{\text{sym}}) = 50$, while $\lambda_2(\Pi) = \lambda_2(L \otimes I_3) = 2$. Therefore, with the choice of $P = I_3$ and from (4.42), the lower bound $\tilde{c}$ is 25. In our simulation,



(a)

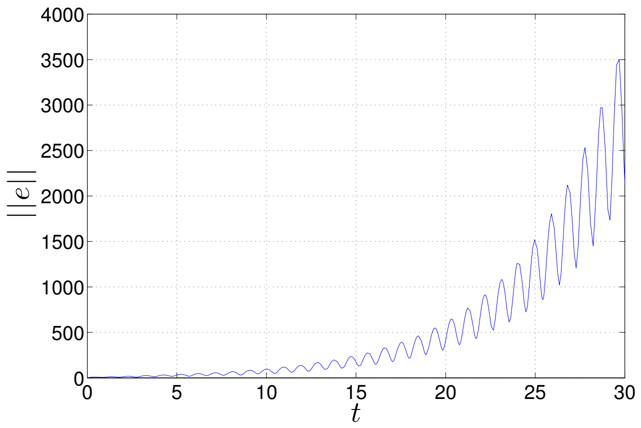


(b)

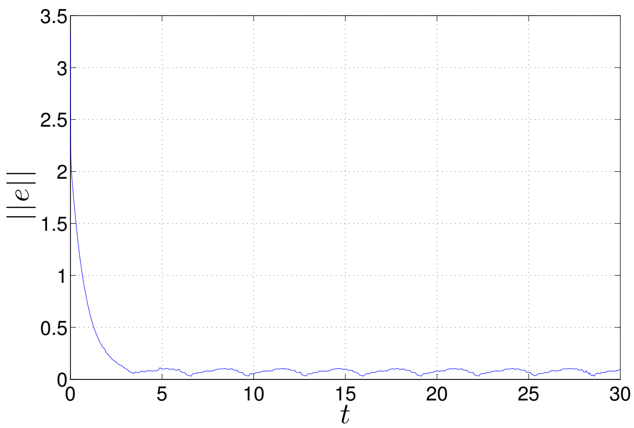
Figure 4.7: Time evolution of the norm of the synchronization error for the network of Chua circuits: (a) uncoupled case; (b) coupled case.

we set the coupling gain $\hat{c} = 50$, while the initial conditions are chosen randomly. Considering that $\overline{M} = 2$, and using (4.43), we can conclude that an upper bound for the norm of the stack error vector e is $\bar{\epsilon} = 0.25$. In Figures 4.9, 4.10 and 4.11, we compare the behavior of the coupled network with the case of disconnected nodes. In particular, Figures 4.9 and 4.10 show the time evolution of the second component of the synchronization error for each node, for both the uncoupled and coupled case, while Figure 4.11 shows the evolution in the state space.

Despite the presence of sliding motion, we observe the coupling to be effective in causing all nodes to synchronize, and the bound $\bar{\epsilon} \leq 0.25$ is consistent with what is observed in Figure 4.9(b).



(a)



(b)

Figure 4.8: Time evolution of the norm of the synchronization error for a divergent Chua network: (a) uncoupled case; (b) coupled case.

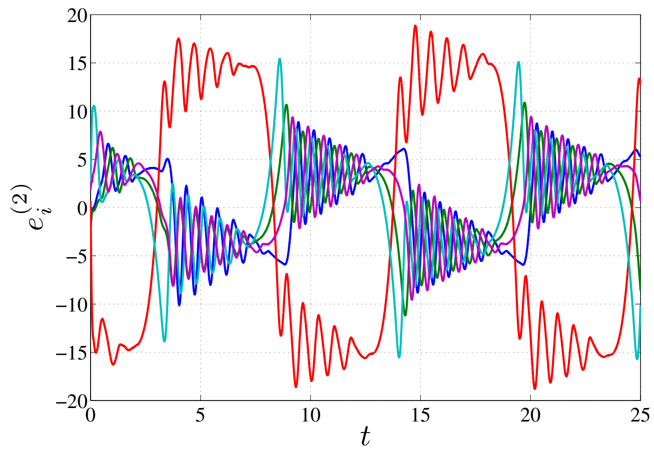
4.5.4 Nonuniform Kuramoto oscillators

A classical example of nonlinearly coupled heterogeneous systems is the network of nonuniform Kuramoto oscillators, described by equation

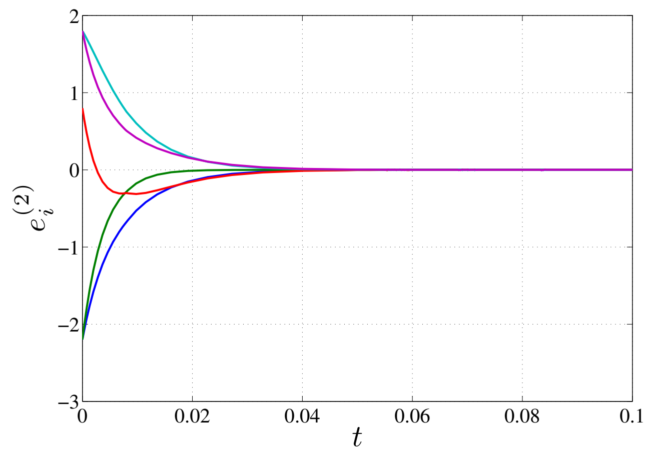
$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N} w_{ij}\sin(\theta_j - \theta_i), \quad \theta_i \in (-\pi, \pi] \qquad i = 1, \dots, N. \tag{4.59}$$

Synchronization of Kuramoto oscillators has been widely studied in literature, see for instance [141, 195, 31, 3], where the coupling is generally supposed to be all-to-all, and ad hoc results about synchronization can be found.

Here, we show how Theorem 4.4.2 can be also applied to a network of nonuniform Kuramoto systems (4.59) and it provides an upper bound for the minimum coupling.



(a)



(b)

Figure 4.9: Time evolution of error components $e_i^{(2)}(t)$ for the network of chaotic relays: (a) uncoupled case; (b) coupled case.

The error system is given by

$$\dot{e}_i = \bar{\omega}_i + \frac{K}{N}\sum_{j=1}^{N} w_{ij}\sin(e_j - e_i), \quad \theta_i \in (-\pi, \pi] \qquad i = 1, \dots, N. \tag{4.60}$$

Now, if we take any initial condition $\theta(0) = [\theta_1(0), \dots, \theta_N(0)]^T$ such that $|\theta_i(0) - \theta_j(0)| < \pi$ for all $i, j = 1, \dots, N$, we have that each system in the network (4.60) satisfies Assumption 4.3.2 with $h = 0$ and $g_i = \omega_i$.

In this example, we consider a network of $N = 4$ nonuniform Kuramoto oscillators, whose topology is described by the Laplacian matrix

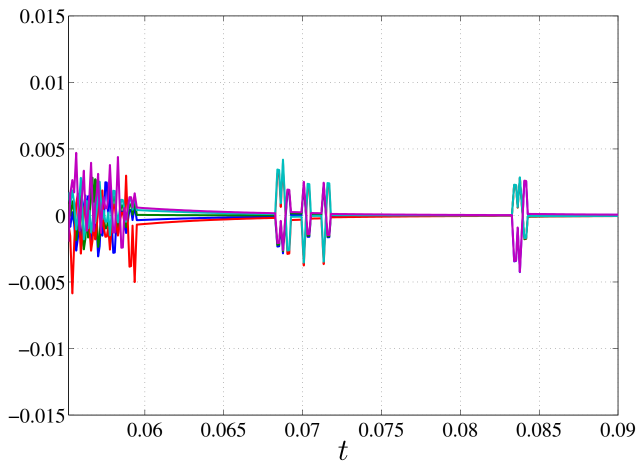


Figure 4.10: Zoom of the evolution of the error components $e_i^{(2)}(t)$ for $t > 0.055s$ showing bounded convergence.

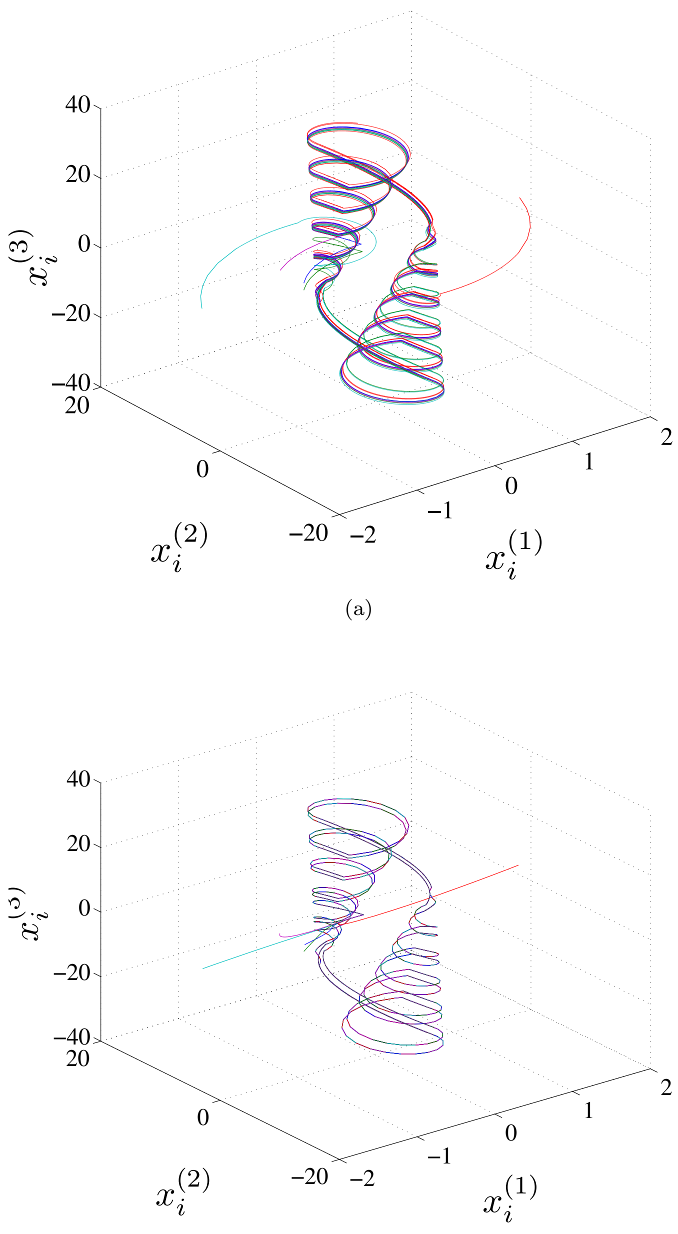
$$L = \begin{bmatrix} 2 & 1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}.$$

The individual frequencies ω_i are taken from a normal distribution , while initial conditions are selected randomly in such that $|\theta_i(0) - \theta_j(0)| < e_{\max} = \pi/3$ for all $i, j = 1, \ldots, N$. From Theorem 4.4.2, we obtain an upper bound for the minimum coupling $\tilde{c} = 2.9$. Figures 4.12(a) and 4.12(b) show the error trajectories for each oscillators, respectively for the case of uncoupled network and coupled network with coupling gain $K = 3$.

For the uncoupled case the error diverges, while for the coupled case the global upper bound for the error norm predicted through Theorem 4.4.2 is $\bar{\epsilon} = 1.04$, which is consistent with the value of 0.34 that we obtain for the initial conditions given in our numerical simulation.

4.6 Discussion

In this chapter, we have presented a framework for the study of synchronization when the nodes' dynamics may be both piecewise smooth and/or nonidentical across the network. Specifically, based on set-valued Lyapunov analysis, we derived sufficient conditions for global boundedness and determined bounds on the minimum coupling strength synchronizing the network, and on the synchronization bound ϵ. Differently from the few works in literature, we do not require that the trajectories of the coupled systems are bounded a priori or that conditions of synchrony among switching signals are satisfied. Also, the results presented in the chapter allow to investigate convergence in networks of generic piecewise smooth systems including those exhibiting sliding motion, as the chaotic relay systems presented in Section 4.5.3. This represents a notable advantage of the approach presented in the chapter when compared to what is available in the literature on net-

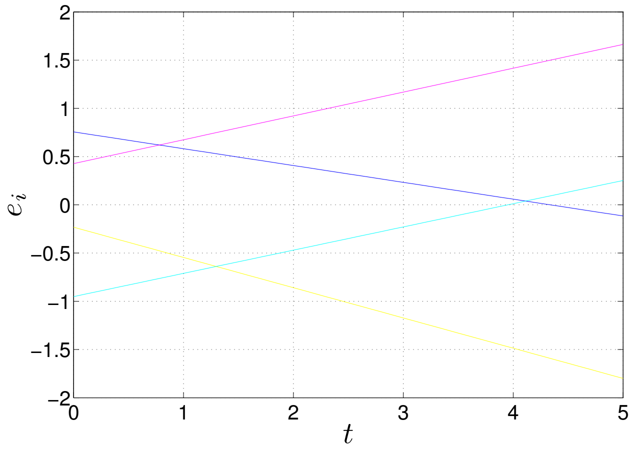


(a)

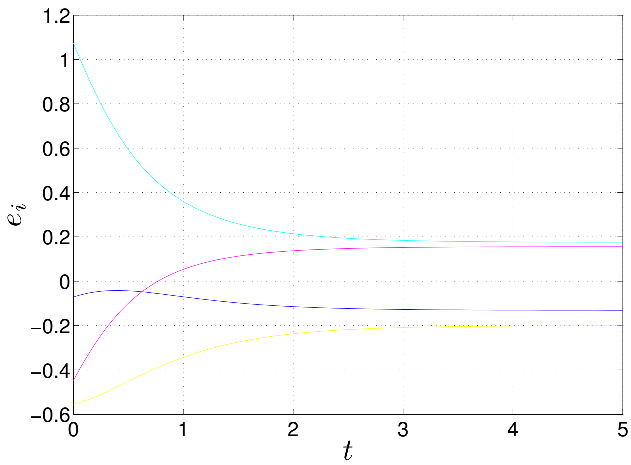
(b)

Figure 4.11: State space evolution for the network of chaotic relays: (a) uncoupled case; (b) coupled case.

works of hybrid or piecewice smooth systems. The analysis has been performed both for linear and nonlinear coupling protocols. The theoretical analysis has been extensively validated on a set of representative and heterogeneous numerical examples.



(a)



(b)

Figure 4.12: Time evolution of the errors for the network of Kuramoto oscillators: (a) uncoupled case; (b) coupled case.

Chapter 5

Discontinuities in communication: event-triggered synchronization

Contents

In this chapter we address the problem of networks of dynamical systems with discontinuities in the communication function. In particular, distributed strategies are proposed in order to control a network of nonlinear dynamical agents with piecewise constant communication and control signals. Such signals updates their values only when particular events are generated. The proposed approach is model-based and so require the knowledge of the dynamical model of the systems in order to predict their evolution. However, both the cases of exact and approximate knowledge of such dynamical model will be considered.

The chapter is organized in the following way: in Section 5.1 we introduce and motivate the approach of the event-triggered control, in particular referring to multi-agent systems. Then, in Section 5.2 we describe in detail the problem of the event-triggered synchronization that will be studied in the rest of the chapter and we also define different elements of the control algorithms we will propose. In Section 5.3 two model based event-triggered algorithms are given and analysed for the problem of synchronization of a network of identical nonlinear systems with perfect knowledge of the dynamical model. Then, a practical implementation is proposed in Section 5.4. In Section 5.5 is addressed the problem of synchronization via a model-based event triggered control in the case that the dynamical model of the agents is not perfectly known. An example of application for a network of Chua systems is given in Section 5.6 , while a discussion of the obtained results is drawn in Section 5.7.

5.1 Introduction

Distributed control algorithms able to solve synchronization, consensus, platooning and formation control (see Section 3.3) have been exploited in the existing literature generally in a continuous time fashion. However, continuous time control laws for such kind of networks are typically not easy or even impossible to implement in real applications. Indeed, in a future scenario a large number of dynamical systems is supposed to communicate over a wireless communication media, which represents a shared resource with limited throughput capacity. In addition to this, distributed control laws are supposed to be hosted and executed on small microprocessors embedded on the networked agents. Furthermore, other future scenarios in networked control systems consider the possibility of distributed networks of sensors and actuators shared among the agents, thus requiring to cope the coordination of data packets while guaranteeing desired performances [137, 138, 212, 221].

Classical time-driven control [17, 79] require sampling the systems at a pre-specified time interval. This creates both the problem of synchronization of sampling instants among the interconnected systems and the simultaneous transmission of all the information over the network. Furthermore, the sampling period must be chosen in order to guarantee stability and performance achievement in all the possible operating conditions, thus resulting in general into conservative results. Conversely, an event-triggered approach seems to be a better solution in decentralized control of multi-agent systems.

Event-triggered control strategies [16, 199, 92, 113, 128] have been introduced in order to stabilize dynamical systems saving computational and hardware resources. Indeed, the control is updated only when an event criterion is satisfied, generally related to the state of the plant. Otherwise, if the triggering condition is not satisfied, the input is held constant and no communication occurs. For the sake of completeness we mention here also self-triggered control strategies, which overcome the problem to check continuously the triggering condition. In the case of self-triggered control at each trigger instant the next one is computed using the current information of the plant and its dynamical model [211, 12, 169, 136].

Although event-triggered has been introduced as an innovation in the classical control loop, it seems to be the best way of implementing distributed control laws in multi-agent systems and recent results in literature confirm this trend. Referring to the previous problems, event-triggered has been successfully developed for consensus algorithms and for control and synchronization of linear dynamical systems. In particular, the case of consensus among single integrator agents is studied in [66, 65] both for a centralized and a decentralized solution applying a methodology proposed in [199]. In [199], the value of the control input for each agent is updated when a triggering condition defined as a function of its and neighbours' state is fulfilled. Consensus is then proved while the network is guaranteed not to reach an undesired accumulation point, so *Zeno* behaviour [106] are excluded. The event-triggered strategy is exploited in order to have a piecewise constant control action for the networked agents while, on the other hand, the communication between neighbouring agents is required to be continuous in time. Such disadvantage is solved in [186], where each node broadcasts the current state value to its neighbours when a trigger occurs with respect to an error defined between its current state and the last broadcasted value. The results are also extended to double integrator agents.

The case of controlling a network of general linear systems is addressed in [89]. Also in this paper the triggering events are defined for each node thanks to a function of the error between the current state and the previous broadcasted value. When a trigger occurs, the agent sends the updated information of its state to the neighbours which update their piecewise constant control input. In the paper also a model-based solution is proposed, where the control input is now a continuous time function evaluated at the local level of each node using its own dynamical model and the (uncoupled) dynamical models of the neighbours. In this case for each node an event is generated any time the error between the real state and the predicted state exceeds a certain threshold. A similar idea to the proposed model-based solution can be found in [59] where synchronization of linear dynamical agents is studied. Also in this case the control signals are continuous in time while the communication signals are piecewise constant.

In this current chapter of the thesis we will instead study a novel distributed event-triggered control scheme able to guarantee synchronization of nonlinear multi agent systems by looking at distributed information related to each pair of connected agents. In particular, we will consider the scenario where each agent is equipped with its own embedded processor and that can gather only information from a subset of the agents it directly communicates. Between each pair of connected agents, relative information of their state mismatch is considered in order to generate local events and update the control law. So, differently from the recent related literature, the event conditions will be defined on the relative state errors between coupled pairs of agents, rather than on their own states. The proposed idea follows a model based approach since each agent knows the dynamical model of its neighbours and predicts their state evolutions between any two consecutive triggering events. In particular, we will consider the two different cases of perfect and approximate information of the neighbours model. In both the cases, we will show how the dynamical agents achieve synchronization with appropriate event-triggered policies. Both the control and communication signals will be piecewise constant and, furthermore, we will also show the existence of a minimum lower bound between consecutive triggering signals, thus ensuring that the overall switched system does not reach an undesired accumulation point, i.e. it does not exhibit Zeno behaviour. In addition to this, we will also guarantee that with an appropriate triggering policy not only Zeno behaviour will be excluded, but also that the the control law for each agent will be updated with a lower bound for the inter-event times.

5.2 Model-based event-triggered

In this chapter we will consider N identical dynamical agents of the form:

$$\dot{x}_i = f(t, x_i) + g(t, x_i)u_i, \quad x_i, u_i \in \mathbb{R}^n, t \geq 0, \qquad \forall i = 1, \dots, N, \tag{5.1}$$

where $g(t, x_i)$ is assumed to be the identity matrix for the sake of simplicity. We aim at guaranteeing a coordinated motion (synchronization) for the systems in (5.1) by considering a distributed event-triggered control law. More precisely, referring to the notions of synchronization and of synchronization error and average state trajectory expounded in Section 3.3, we want to achieve either one of the following two objectives:

- Bounded synchronization;
- Complete synchronization.

In order to ensure synchronization between systems in (5.1), we imagine the scenario where each agent is able to exchange information between a subset of the other agents. The resulting communication network, which is assumed here to be bidirectional, can be described by an undirected adjacency matrix A. In other words, if $a_{ij} \neq 0$, there exists a communication channel between nodes i and j. Furthermore, we also consider that each agent is equipped with its own embedded processor able to execute a local control law based on the prediction of the evolution of its neighbours. Thanks to this local information, each node will execute an event-triggered update of its controller. In particular, in the event-triggered scheme we propose, at each node i we associate:

1. $\{t_k^{ij}\}_{k^{ij}=0}^{\infty} : \mathbb{N} \mapsto [0, +\infty)$ a time sequence of events on node i referring to information from node j, where $a_{ij} \neq 0$ and where k^{ij} is the index of the sequence related to the pair (i, j);

2. $\{t_k^{i}\}_{k^{i}=0}^{\infty} : \mathbb{N} \mapsto [0, +\infty)$ a time sequence of the instants when node i updates its control input $u_i(t)$, where k^i is the index of the sequence related to the updating of $u_i(t)$.

In both the cases, for any index $k^{ij} \in \mathbb{N}$ (or $k^i \in \mathbb{N}$) we have that $t_k^{ij} \leq t_{k+1}^{ij}$ (or $t_k^i \leq t_{k+1}^i$).

For each sequence $\{t_k^{ij}\}_{k^{ij}=0}^{\infty}$ we introduce the *last function* $l^{ij}(t) : [0, +\infty) \mapsto \mathbb{N}$ defined as

$$l^{ij}(t) = \arg \min_{k^{ij} \in \mathbb{N} : t \geq t_k^{ij}} \left\{ t - t_k^{ij} \right\}.$$

So, for each time instant t, $t_{l(t)}^{ij}$ is the most recent event occurred to i with respect to j, while with $t_{l(t)+1}^{ij}$ we indicate the next event.

Analogously, we define the function $l^i(t)$ for the sequence $\{t_k^i\}_{k^i=0}^{\infty}$.

As will be clear in what follows, the last indices $l^{ij}(t)$ and $l^i(t)$ will be used to generate implicitly the sequences $\{t_k^{ij}\}_{k^{ij}=0}^{\infty}$ and $\{t_k^i\}_{k^i=0}^{\infty}$. In particular, borrowing a notation used for hybrid systems [85], after an event the counter $l^{ij}(t)$ will be updated to $l^{ij+} = l^{ij} + 1$, where by l^{ij+} we mean the value of $l^{ij}(t)$ immediately after a new event. Similarly, the counter $l^i(t)$ will be updated to $l^{i+} = l^i + 1$.

It is worth mentioning that, although the communication graph is undirected, events related to coupled pairs (i, j) are, in general, not synchronous and so $t_{l(t)}^{ij} \neq t_{l(t)}^{ji}$. For this reason the sequences $\{t_k^{ij}\}_{k^{ij}=0}^{\infty}$ and $\{t_k^{ji}\}_{k^{ji}=0}^{\infty}$ are, in the general case, different. For the sake of brevity, in what follows we will often omit the explicit dependence of l^{ij} and l^i on time.

The aim of the chapter is to study under which conditions and under which sequences of triggering events $\{t_k^{ij}\}_{k^{ij}=0}^{\infty}$ and $\{t_k^i\}_{k^i=0}^{\infty}$, for all $(i, j) \in \mathcal{E}$, and under which piecewise constant control inputs $u_i(t)$ the network of systems in (5.1) guarantees bounded or complete synchronization, respectively defined in Definition 3.3.1 and Definition 3.3.2.

More specifically, we study a network of identical nonlinear systems of the form (5.1) (with $g(t, x_i)$ assumed to be the identity matrix for the sake of simplicity). The information available about the dynamical model is given by the systems

$$\dot{\hat{x}}_i = \hat{f}_i(t, \hat{x}_i) + u_i, \quad x_i, u_i \in \mathbb{R}^n,\ t \geq 0, \qquad i = 1, \ldots, N, \tag{5.2}$$

which represent the local estimates of the real dynamical function. In particular, system (5.2) with subscript i is supposed to be available to node i and to its neighbours, i.e. to all the nodes $j \in \mathcal{N}_i$, and it is used as a prediction module to estimate the evolution of node i. Furthermore, in order to initialize such predictor, these nodes are also supposed to know an estimation of the initial condition of node i (or the value of the state at a specific time instant, for example at the first trigger).

In this chapter, referring to the knowledge of the dynamical models in (5.2), we will consider two different cases: exact information of the dynamical model, i.e., $\hat{f}_i(\cdot) = f(\cdot)$, or not exact information of the dynamical model, i.e., $\hat{f}_i(\cdot) \neq f(\cdot)$.

5.3 Event-triggered synchronization with perfect model description

Here we address the problem of studying an event-triggered control scheme with perfect knowledge of the dynamical model of the nodes of the network. So, in this case $\hat{f}_i(\cdot) = f(\cdot)$ for all $i \in \mathcal{N}$. Furthermore, each node is also supposed to know the the exact value of the initial conditions of its neighbours (or the value of their state at a specific time instant, for example at the first trigger). Therefore, each node i can compute from any event at time t_k^{ij} the evolution

$$\varphi_f(t - t_k^{ij}, t_k^{ij}, x_j(t_k^{ij})), \qquad \forall j \in \mathcal{N}_i. \tag{5.3}$$

Note that in order to evaluate (5.3), node i must also have information on the current control input $u_j(t)$ acting on each of its neighbours. Later we will present an algorithm able to guarantee that this information is shared among nodes. However, we firstly focus on the triggering events occurring at a generic node i and we remark here that in the following, since we are considering the case of exact prediction $\hat{x}_i(t) = x_i(t)$ for all $i \in \mathcal{N}$, we will omit the symbol $\hat{x}_i$ using simply x_i.

For all pairs $(i, j) \in \mathcal{E}$ we define the *trigger error*

$$\tilde{e}_{ij}(t) := e_{ij}(t_l^{ij}) - e_{ij}(t), \quad t \in [t_l^{ij}, t_{l+1}^{ij}), \tag{5.4}$$

where we define $e_{ij}(t) = x_j(t) - x_i(t)$.

The error in (5.4) is referred to the last and the future trigger instants and is used, as will be clear in what follows, to compute the future trigger instant t_{l+1}^{ij}. In analogous way $\tilde{e}_{ji}(t)$ for the pair (j, i) is defined. Note that, as mentioned earlier, events referred to node i with respect to j are, in general, not synchronous with the events referred to j with respect to i. For this reason, the pair (i, j) is treated here as an oriented link and in general $\tilde{e}_{ij}(t) \neq \tilde{e}_{ji}(t)$.

For all pairs (i, j), we also define the *trigger function*

$$p_{ij}(t, \tilde{e}_{ij}(t)) = \|\tilde{e}_{ij}(t)\|_2 - \varsigma_{ij}(t), \tag{5.5}$$

where $\varsigma_{ij}(t)$ is a continuous-time nonincreasing *threshold function*. Then, an event occurs when the following condition is violated

$$p_{ij}(t, \tilde{e}_{ij}(t)) < 0. \tag{5.6}$$

We are now ready to write the control input $u_i(t)$ for the i-th system in (5.1) as the piecewise constant signal

$$u_i(t) = c \sum_{j=1}^{N} a_{ij} \Gamma e_{ij}(t_l^{ij}) \qquad t \in [t_l^i, t_{l+1}^i), \tag{5.7}$$

where $c > 0$ is a *coupling gain* and $\Gamma = \Gamma^T > 0$ is the *inner coupling matrix.* In this case $e_{ij}(t)$ is evaluated at the time instant t_l^{ij}.

The control input (5.7) leads to a diffusively coupled event-triggered dynamical network given by

$$\dot{x}_i(t) = f(t, x_i(t)) + c\sum_{j=1}^{N} a_{ij}\Gamma e_{ij}(t_l^{ij}) \quad t \in [t_l^i, t_{l+1}^i) \qquad \forall i = 1, \ldots, N. \tag{5.8}$$

Using the trigger function (5.5) and the condition (5.6), we generate the sequence of events for each node $i \in \mathcal{N}$ according to the distributed algorithm we give in what follows. Such algorithm is performed independently in each node of the network and it is about its regime execution. The initialization of the algorithm for the whole network is instead discussed later.

Algorithm 5.3.1.

1. *Node i continuously listens to possible transmission of information from its neighbours and, in parallel, computes the flows in* (5.3) *for all its neighbouring nodes $j \in \mathcal{N}_i$ integrating the dynamical model $\dot{x}_j = f(t, x_j(t)) + u_j(t_l^j)$ from the initial condition $x_j(t_l^j)$. Thanks to the evaluation of the neighbours' flows, node i can compute the trigger error* (5.4) *and monitor condition* (5.6) *for all of its neighbours. If the node receives from one of its neighbours, suppose for a certain node h, a new value of its control input $u_h(t_l^h)$, it updates the dynamical model $\dot{x}_h = f(t, x_h(t)) + u_h(t_l^h)$ in order to always correctly predict the neighbour's flow. Notice that, since in this case the knowledge of the dynamical model is supposed to be perfect, node h does not need to send to node i the value of its state $x_h(t_l^h)$. Indeed, since the control input is always updated to the current value, $x_h(t)$ is always correctly predicted from node i which starts to integrate node h's evolution from a known initial condition;*

2. *Once condition* (5.6) *is violated, say for a certain node $h \in \mathcal{N}_i$, node i updates the value $e_{ih}(t_l^{ih})$ to $e_{ih}(t_{l+1}^{ih})$. So, after the event, the counter $l^{ih}(t)$ will be updated to ${l^{ih}}^+ = l^{ih} + 1$. At the same time the current value $\tilde{e}_{ih}(t = t_l^{ih})$ will be reset by* (5.4) *evaluated at the new event $l^{ih} + 1$;*

3. *Node i computes the new control law u_i and updates ${l^i}^+ = l^i + 1$. Hence, the last updating event of the controller happens once the new value e_{ih} is considered and, therefore, $t_l^i = t_l^{ih}$. The control input takes the value*

$$u_i(t) = \sum_{j=1}^{N} a_{ij}\Gamma e_{ij}(t_l^{ij}), \qquad t \in [t_l^i, t_{l+1}^i). \tag{5.9}$$

   Such new value of the input is broadcasted to its neighbourhood and it is held until the next output trigger of node i, that will happen at the next following event with one of its neighbours;

4. *Repeat from step 1.*

Note that, as every node that triggers changes its control input and broadcasts it to its neighbours (step 3), then all the nodes $j \in \mathcal{N}_i$ can update their dynamic model of i taking into account the new input $u_i(t_l^i)$ and the current state $x_i(t_l^i)$ (step 1). So, they will always be able to evaluate the right value of the flow (5.3) of node i. As we have already said in Section 5.2, the time event t_l^{ij}, with j generic neighbour of node i, and the last update event of the control input t_l^i implicitly define the sequences $\{t_k^{ij}\}_{k^{ij}=0}^{\infty}$ and $\{t_k^i\}_{k^i=0}^{\infty}$.

The initialization of Algorithm 5.3.1 can happen when at least one node sends the triplet $(t_0^i, x_i(t_0^i), u_i(t_0^i))$ to its neighbours, where t_0^i is the time instant when the generic node i broadcasts for the first time its first information. So, thanks to the value of the triplet, all the neighbours can start to predict its evolution and, when their step 3 requires to broadcast the control input, say for a generic node $h \in \mathcal{N}_i$, the first information broadcasted to the neighbourhood of node h is its triplet $(t_0^h, x_h(t_0^h), u_h(t_0^h))$. In this way all the nodes of the network can be connected in a finite time. Notice that, obviously, the initialization of the algorithm can also happen if more than one transmit their triplet independently, without waiting to receive the first information from a neighbour.

Remark 5.3.1. *It is worth to notice that due to the updating criterion expressed in step 3, each e_{ij} holds the value from its past event at t_l^{ij} in control input* (5.9)*. So, in general, $t_l^{ij} \neq t_l^{ih}$, with $h, j \in \mathcal{N}_i$ being two different neighbours of node i. On the other hand, due to the symmetry of the trigger condition expressed by* (5.4)-(5.5)-(5.6)*, if we choose $\varsigma_{ij}(t) = \varsigma_{ji}(t)$ then when node i triggers and updates its control because of* (5.6) *with respect to node j does not hold anymore, then also node j triggers for the same reason and we have that $t_l^{ij} = t_l^{ji}$. This implies the symmetry of the coupling strengths between any connected pairs (i, j).*

Note that step 3 can be substituted with the following step

3′. Node i updates $l^{i+} = l^i + 1$, so $t_l^i = t_l^{ih}$ and computes the control law u_i using the expression

$$u_i(t) = \sum_{j=1}^{N} a_{ij} e_{ij}(t_l^i), \qquad t \in [t_l^i, t_{l+1}^i), \tag{5.10}$$

Then, all trigger errors (5.4) are reset, for all $j \in \mathcal{N}_i$.

Basically, once the first trigger occurs, say for $\tilde{e}_{ih}(t)$, then not only the current value e_{ih} will be updated and the corresponding trigger error (5.4) reset, but also all other values e_{ij} with $j \in \mathcal{N}_i$.

Remark 5.3.2. *When using step 3′, all triggers related to pairs (i, j), with $j \in \mathcal{N}_i$, are forced to be synchronous and, moreover, $t_l^{ij} = t_l^{ih}$ for all $j, h \in \mathcal{N}_i$. Conversely, when step 3′ is used at a generic time instant t, we have $e_{ij}(t) \neq e_{ji}(t)$. So, considering* (5.10) *instead of* (5.9) *all e_{ij} are updated at the same time but the symmetry of the control actions between coupled pairs (i, j) is lost.*

We denote the algorithm obtained by using step 3′ as Algorithm 5.3.1′. Note that both control schemes lead to piecewise constant communication and control signals.

We now give a synchronization result for the event-triggered control scheme with perfect knowledge of the dynamical model of the nodes. The following result will consider both the cases of bounded and complete synchronization. For the case of complete synchronization we will use the same exponential threshold function $\varsigma_{ij}(t) = k_\varsigma e^{-\lambda_\varsigma t}$ for all the pair of nodes (i, j) such that $a_{ij} \neq 0$. As we will see, we will get bounds both for k_ς and λ_ς which will be based on some global network information. However, we will see later how to assign these values in a more practical way in order to relax the need for global information.

Before giving the following result, let us fix some quantity we will use later. We consider the position

$$\epsilon_\varsigma = \frac{c\sqrt{N}N_{\max}\|\Gamma\|_2}{c\lambda_2(L\otimes\Gamma) - L_f}. \tag{5.11}$$

Furthermore, considering any arbitrary constant $\delta > 0$ we define

$$\alpha = \frac{\delta}{1+\delta}\left[c\lambda_2(L\otimes\Gamma) - L_f\right]. \tag{5.12}$$

Theorem 5.3.1. *Let us consider the event-triggered connected network* (5.8)*, where the function $f(t,x)$ is Lipschitz continuous with respect to x with Lipschitz constant L_f and let us chose a coupling gain c such that*

$$L_f - c\lambda_2(L\otimes\Gamma) < 0. \tag{5.13}$$

Let us consider a k_ς such that

$$k_\varsigma \geq \frac{\|e(0)\|_2}{\epsilon_\varsigma}, \tag{5.14}$$

and let us also consider an arbitrary value λ_ς such that

$$0 < \lambda_\varsigma < \alpha, \tag{5.15}$$

where ϵ_ς and α are defined in (5.11) *and* (5.12) *respectively. We have:*

i. If $\lim_{t\to\infty}\varsigma_{ij}(t) = \bar{\varsigma}_{ij}$, with $\bar{\varsigma}_{ij} > 0$ for all i,j such that $a_{ij} \neq 0$, then both Algorithm 5.3.1 and Algorithm 5.3.1′ guarantee bounded synchronization of the network;

ii. If we chose $\varsigma_{ij}(t) = k_\varsigma e^{-\lambda_\varsigma t}$ for all the pairs (i,j) such that $a_{ij} \neq 0$, then Algorithm 5.3.1 and Algorithm 5.3.1′ guarantee complete synchronization of the network with exponential rate λ_ς.

Furthermore, both in case i. and ii. no Zeno behaviour will occur.

Proof. We will split the proof in two steps. Firstly we will prove synchronization and then that no Zeno behaviours occur. Equation (5.8) can be rewritten as

$$\dot{x}_i = f(t,x_i) + c\sum_{j=1}^{N} a_{ij}\Gamma e_{ij}(t) + c\sum_{j=1}^{N} a_{ij}\Gamma\tilde{e}_{ij}(t) \qquad \forall i = 1,\dots,N. \tag{5.16}$$

Step 1. Let us consider the candidate Lyapunov function $V(e(t)) = \frac{1}{2}e^Te$ defined in the error space and let us consider its derivative with respect to time. We obtain

$$\dot{V}(e(t)) = \sum_{i=1}^{N} e_i^T\dot{e}_i = \sum_{i=1}^{N} e_i^T f(t,x_i) - \sum_{i=1}^{N} e_i^T\dot{\bar{x}} - ce^T\left(L\otimes\Gamma\right)e + c\sum_{i=1}^{N} e_i^T\sum_{j=1}^{N} a_{ij}\Gamma\tilde{e}_{ij}.$$

Now, remembering condition (5.6), taking into account that $\sum_{i=1}^{N} e_i^T\dot{\bar{x}} = 0$, adding and subtracting $\sum_{i=1}^{N} e_i^T f(t,\bar{x})$ and using the one-sided Lipschitz property (2.2), we can write the following inequality

$$\dot{V} \leq L_f e^Te - ce^T(L\otimes\Gamma)e + \|e\|_2\sqrt{N}N_{\max}\|\Gamma\|_2\varsigma(t),$$

where L_f is the Lipschitz constant of the function f, $N_{\max} \leq N-1$ is the maximum degree of the graph A (i.e. the maximum number of links that a node can have) and $\varsigma(t) = \max_{i,j} \varsigma_{ij}(t)$. Writing $e = a\hat{e}$, where $a = \|e\|_2$ is the module of the error and $\hat{e} = \frac{1}{\|e\|_2} e$ is the unitary vector associated to e, the above inequality can be rewritten as

$$\dot{V}(e) \leq (L_f - c\lambda_2(L \otimes \Gamma))\, a^2 + c\sqrt{N} N_{\max} \|\Gamma\|_2 \varsigma(t) a. \tag{5.17}$$

Now, since c is chosen in order to fulfill inequality (5.13), then the error trajectory $e(t)$ converges to the invariant region $\|e(t)\|_2 \leq \epsilon$, where

$$\epsilon = \frac{c\sqrt{N} N_{\max} \|\Gamma\|_2 \varsigma(t)}{c\lambda_2(L \otimes \Gamma) - L_f}, \tag{5.18}$$

or, using (5.11), we can equivalently write

$$\epsilon = \epsilon_\varsigma \varsigma(t) \tag{5.19}$$

in order to emphasize that the value of the bound of the global synchronization error $e(t)$ depends on $\varsigma(t)$ times a finite positive constant. So, if item i. is verified, then also $\lim_{t\to+\infty} \varsigma(t) = \bar{\varsigma} > 0$ and bounded synchronization is ensured. Conversely, if item ii. holds then $\lim_{t\to+\infty} \varsigma(t) = 0$ and complete synchronization is achieved, since the invariant region given by ϵ shrinks with exponential rate λ_ς.

Step 2. We prove here that no Zeno behaviour will occur. We will focus first on the more complicated case of complete synchronization of item ii., while simpler reasoning will be later done for the case of bounded synchronization.

Let us define the strictly decreasing function

$$b(t) = (1+\delta)\epsilon_\varsigma \varsigma(t), \tag{5.20}$$

where $\delta > 0$ is an arbitrary constant value. Note that $\|e(0)\|_2 < b(0)$. We are now going to prove that this relation holds for each time instant, i.e., that

$$\|e(t)\|_2 \leq b(t) \qquad \forall t \geq 0. \tag{5.21}$$

Indeed, since both $e(t)$ and $b(t)$ are continuous, if there is no time instant $\bar{t}$ such that $b(\bar{t}) = \|e(\bar{t})\|_2$, then relation (5.21) is trivially true. So, let us suppose that such time instant $\bar{t}$ exists. Now, for all $t \geq \bar{t}$ we evaluate the value of $\dot{V}(e)$ when $e(t)$ is such that $\|e\|_2 = b$. More precisely we have that

$$\dot{V}(e)\Big|_{\|e\|_2=b} \leq -\delta(1+\delta)\frac{[c\sqrt{N} N_{\max}\|\Gamma\|_2]^2}{c\lambda_2(L\otimes\Gamma) - L_f}\varsigma^2(t).$$

where the above formula has been obtained substituting a with expression (5.20) in (5.17). Multiplying and dividing the above relation by $(1+\delta)[c\lambda_2(L\otimes\Gamma) - L_f]$ we obtain

$$\dot{V}(e)\Big|_{\|e\|_2=b} \leq -\alpha b^2, \tag{5.22}$$

where α has been defined in (5.12).

Now, since

$$\dot{V}(e)\Big|_{\|e\|_2=b} = \frac{d}{dt}\frac{1}{2}\|e\|_2^2\Big|_{\|e\|_2=b} = b\,\frac{d}{dt}\|e\|_2\Big|_{\|e\|_2=b}, \tag{5.23}$$

comparing (5.22) and (5.23) we get

$$\frac{d}{dt}\|e\|_2\bigg|_{\|e\|_2=b} \leq -\alpha b. \tag{5.24}$$

On the other hand, considering the decreasing function $B(t) = \frac{1}{2}b^2$ and remembering that $\varsigma(t) = k_\varsigma e^{-\lambda_\varsigma t}$ we have

$$\dot{B} = -\lambda_\varsigma b^2.$$

So, with the choice (5.15) we get

$$\dot{V}(e)\Big|_{\|e\|_2=b} \leq \dot{B} < 0,$$

or, equivalently

$$\frac{d}{dt}\|e\|_2\bigg|_{\|e\|_2=b} \leq -\alpha b \leq -\lambda_\varsigma b. \tag{5.25}$$

Now, since expression (5.25) holds for all the values $b \in [0, b(0)]$, integrating with respect to time we obtain relation (5.21).

Now, let us consider the dynamics of the error between a generic connected pair of nodes $(i,h) \in \mathcal{E}$. Such dynamics can be expressed as $\dot{e}_{ih}(t) = \dot{x}_h(t) - \dot{x}_i(t)$ thus,

$$\dot{e}_{ih} = f(t,x_h)+c\sum_{j=1}^N a_{hj}\Gamma e_{hj}(t)+c\sum_{j=1}^N a_{hj}\Gamma\tilde{e}_{hj}(t)-f(t,x_i)-c\sum_{j=1}^N a_{ij}\Gamma e_{ij}(t)-c\sum_{j=1}^N a_{ij}\Gamma\tilde{e}_{ij}(t).$$

$$\begin{aligned}\|\dot{e}_{ih}\|_2 \quad \leq \quad & \|f(t,x_h) - f(t,x_i)\|_2 + c\sum_{j=1}^N a_{hj}\|\Gamma\|_2\|e_{hj}(t)\|_2 + c\sum_{j=1}^N a_{hj}\|\Gamma\|_2\|\tilde{e}_{hj}(t)\|_2 + \\ & c\sum_{j=1}^N a_{ij}\|\Gamma\|_2\|e_{ij}(t)\|_2 + c\sum_{j=1}^N a_{ij}\|\Gamma\|_2\|\tilde{e}_{ij}(t)\|_2. \end{aligned} \tag{5.26}$$

Now, taking into account that f is Lipschitz and that $\|e_{ih}(t)\|_2 \leq 2\|e(t)\|_2$ and remembering relation (5.21), from the above inequality we have

$$\|\dot{e}_{ih}(t)\|_2 \leq 2\left[L_f + c\|\Gamma\|_2(N_h+N_i)\right]b(t) + c\|\Gamma\|_2(N_h+N_i)\varsigma(t), \tag{5.27}$$

where N_i and N_h are the degrees of nodes i and h respectively. Let $p_1 = 2\left[L_f + c\|\Gamma\|_2(N_h+N_i)\right]$ and $p_2 = c\|\Gamma\|_2(N_h+N_i)$. Then, at the last trigger event $t = t_l^{ih}$, we obtain from (5.27)

$$\|\dot{e}_{ih}(t)\|_2 \leq p_1(1+\delta)\epsilon_\varsigma k_\varsigma e^{-\lambda_\varsigma t_l^{ih}} + p_2 k_\varsigma e^{-\lambda_\varsigma t_l^{ih}}. \tag{5.28}$$

Now, in order to prove that Zeno behaviours do not occur in the network, we will show that for all triggering instants t_k^{ih} there exists a nonzero lower bound $\tau_m > 0$ such that the next event t_{k+1}^{ih} will satisfy the condition

$$t_{k+1}^{ih} - t_k^{ih} \geq \tau_m.$$

To do this, let us consider the dynamics of the triggering error $\tilde{e}_{ih}(t)$ at time instants $t > t_l^{ih}$. Clearly, the following considerations will be valid not only for the last event instant t_l^{ih} but for all instants t_k^{ih}, since the sequence $\{t_k^{ih}\}_{k^{ih}=0}^{\infty}$ is implicitly defined by the sequence of the last events. We can write

$$\|\tilde{e}_{ih}(t)\|_2 \leq \int_{t_l^{ih}}^{t} \|\dot{\tilde{e}}_{ih}(s)\|_2 \; ds = \int_{t_l^{ih}}^{t} \| - \dot{e}_{ih}(s)\|_2 \; ds = \int_{t_l^{ih}}^{t} \|\dot{e}_{ih}(s)\|_2 \; ds. \tag{5.29}$$

Taking into account inequality (5.28) and considering $t = t_l^{ih} + \tau$ from the above formula we can write

$$\|\tilde{e}_{ih}(t)\|_2 \leq \left(p_1(1+\delta)\epsilon_\varsigma k_\varsigma e^{-\lambda_\varsigma t_l^{ih}} + p_2 k_\varsigma e^{-\lambda_\varsigma t_l^{ih}}\right)\tau. \tag{5.30}$$

Referring to the trigger function (5.5) with the considered threshold $\varsigma(t_l^{ih}+\tau) = k_\varsigma e^{-\lambda_\varsigma (t_l^{ih}+\tau)}$, we have that τ_m solves the equation

$$k_\varsigma e^{-\lambda_\varsigma (t_l^{ih}+\tau_m)} = \left(p_1(1+\delta)\epsilon_\varsigma k_\varsigma e^{-\lambda_\varsigma t_l^{ih}} + p_2 k_\varsigma e^{-\lambda_\varsigma t_l^{ih}}\right)\tau_m.$$

Multiplying both the members of the previous equation by $\frac{1}{k}e^{\lambda_\varsigma t_l^{ih}}$ we obtain the final equation

$$e^{-\lambda_\varsigma \tau_m} = \left(p_1(1+\delta)\epsilon_\varsigma + p_2\right)\tau_m, \tag{5.31}$$

which implicitly defines τ_m as a non-zero lower bound between any two consecutive triggering instants.

The case of bounded synchronization under the assumptions of item i. is, instead, easier than the case of complete synchronization. Indeed, we can consider that

$$\|e_{ih}(t)\|_2 \leq 2\|e(t)\|_2 \leq 2 \sup_{t' \in [t,+\infty)} \|e(t')\|_2 \leq 2\tilde{b}(t), \tag{5.32}$$

where $\tilde{b}(t)$ is the nonincreasing piecewise smooth continuous function

$$\tilde{b}(t) = \begin{cases} \|e(t)\|_2 & \text{if } \|e(t)\|_2 > \epsilon_\varsigma \varsigma(t) \\ \epsilon_\varsigma \varsigma(t) & \text{if } \|e(t)\|_2 \leq \epsilon_\varsigma \varsigma(t). \end{cases} \tag{5.33}$$

So, considering a generic event of trigger $t = t_l^{ih}$, we can bound inequality (5.26) as

$$\|\dot{e}_{ih}(t)\|_2 \leq p_1 \tilde{b}(t_l^{ih}) + p_2 \varsigma(t_l^{ih}), \quad \forall t \geq t_l^{ih}, \tag{5.34}$$

where we have used the same position of p_1 and p_2 as done in equation (5.28) in order to simplify the notation.

Integrating the above expression similarly to what already done for (5.30) we obtain that a nonzero lower bound $\tau_{ih}(t_l^{ih})$ for the inter-event time between the last trigger event t_l^{ih} and the next one t_{l+1}^{ih} for the generic pair (i,h) is

$$\tau_{ih}(t_l^{ih}) = \frac{\bar{\varsigma}_{ih}}{p_1 \tilde{b}(t_l^{ih}) + p_2 \varsigma(t_l^{ih})}. \tag{5.35}$$

□

Now we give two remarks which point out some observations about Theorem 5.3.1.

Remark 5.3.3. *It is worth noting that the assumption* (5.14) *can be relaxed. Indeed, any positive constant k_ς can be considered and the proof of Theorem 5.3.1 still remains the same with the choice of a $\delta > 0$ such that $(1+\delta)\epsilon_\varsigma k_\varsigma \geq \|e(0)\|_2$. However, despite the fact that the result still holds, the choice of δ affects the value of the bound τ_m. Indeed, keeping the value of λ_ς constant, from* (5.31) *it is clear that the higher δ is the smaller the value of the bound τ_m gets. On the other side, the value of δ indirectly affects also the speed of convergence to synchronization, since the higher δ is the higher α is, thus allowing to chose a higher value of λ_ς and making a faster synchronization. However, in this last case, equation* (5.31) *shows that a higher speed of convergence reduces the value of the inter-event bound and so increases the frequency of the triggers.*

Remark 5.3.4. *We note that Theorem 5.3.1 holds for both Algorithm 5.3.1 and Algorithm 5.3.1′ since the proof is independent of the choice of step* 3 *or step* 3′. *Despite this, since in Algorithm 5.3.1′ all e_{ij} with $j \in \mathcal{N}_i$ are updated at the same time instant t_l^i and the corresponding errors $\tilde{e}_{ij}$ are reset, this implies that both for the case of bounded and complete synchronization there implicitly exists a non zero lower bound also between any two consecutive updating events of the control law. For this reason, Algorithm 5.3.1′ can be implemented in all applications where constraints on actuators does not allow to change the control input arbitrarily fast.*

5.4 Practical implementation of the event-triggered control scheme

As we already said, assumptions in Theorem 5.3.1 require global information on the network. However, it has already been showed in Remark 5.3.3 that such assumptions are not so strict. In particular, in this section we aim to show some practical considerations that allow to implement event triggered control schemes (5.8) based on Algorithms 5.3.1-5.3.1′ even if not global information on the network is available.

As first observation, we want to point out that Theorem 5.3.1 can be still practically applied even if the values of k_ς and δ are arbitrarily chosen so that could not be true any more that $\|e(0)\|_2 < b(0)$ and so condition (5.21) does not hold. Indeed, it is always possible to consider a δ' such that $\|e(t)\|_2 \leq b'(t)$ for all $t \geq 0$, with $b'(t) = (1+\delta')\epsilon_\varsigma \varsigma(t)$, thus repeating the same reasoning as in the proof of Theorem 5.3.1. Now, since it holds that

$$\|e(t)\|_2 - b(t) \leq b'(t) - b(t) = (\delta' - \delta)\epsilon_\varsigma k_\varsigma e^{-\lambda_\varsigma t}, \qquad t \leq \bar{t}, \tag{5.36}$$

where $\bar{t}$ is the time instant (eventually infinite) at which for $t > \bar{t}$ we have that $\|e(t)\|_2 < b(t)$. Then we can conclude that the error trajectory approaches the shrinking boundary region of radius $b(t)$ with exponential rate λ_ς. So, from a practical point of view, $e(t)$ is such that $\|e(t)\|_2$ is practically less or equal $b(t)$ for $t \geq 5/\lambda_\varsigma$ and the rest of the proof of Theorem 5.3.1 is still valid.

Later we will present a practical implementation for event-triggered control. Before doing this, we present in what follows another result for complete synchronization of event triggered network (5.8). The idea of the following theorem is similar to that one of Theorem 5.3.1, but here we will consider discrete time positive nonincreasing sequences $\{\varsigma_{ij}(t_k^{ij})\}_{k^{ij}=0}^{\infty}$ defined for all (i,j) such that $a_{ij} \neq 0$ in order to compute the threshold of the trigger function in (5.4). More in detail, the current value of a generic threshold $\varsigma_{ij}(t_l^{ij}) > 0$ represents the current value to be adopted in order to compute the next trigger event t_{l+1}^{ij} using the violation of the (5.6). So, in other words, from the sequences $\{\varsigma_{ij}(t_k^{ij})\}_{k^{ij}=0}^{\infty}$ we define the continuous time function $\varsigma_{ij}(t)$ holding the corresponding sequence between any two consecutive instants t_k^{ij} and t_{k+1}^{ij}, i.e.

$$\varsigma_{ij}(t) = \varsigma_{ij}(t_k^{ij}), \qquad t \in [t_k^{ij}, t_{k+1}^{ij}).$$

Furthermore, we define $\{\varsigma(t_k^{\varsigma})\}_{k^{\varsigma}=0}^{\infty}$ as the sequence of the current maximum value among all the $\varsigma(t_l^{ij})$ defined for all (i,j) such that $a_{ij} \neq 0$. More precisely, $\{\varsigma(t_k^{\varsigma})\}_{k^{\varsigma}=0}^{\infty}$ will be implicitly defined by the current value

$$\varsigma(t_l^{\varsigma}) = \max_{i,j:\, a_{ij}\neq 0} \varsigma_{ij}(t_l^{ij}),$$

with t_l^{ς} defining implicitly the sequence $\{t_k^{\varsigma}\}_{k^{\varsigma}=0}^{\infty}$ as

$$t_l^{\varsigma} = \max_{i,j:\, a_{ij}\neq 0} t_l^{ij}.$$

So, in simple words, the value of the maximum $\varsigma(t_l^{\varsigma})$ among the current thresholds $\varsigma_{ij}(t_l^{ij})$ updates any time the value of the maximum among all the threshold function decreases, as well as the corresponding time instant t_l^{ς}. Notice that for this reason, conversely to the nonincreasing sequences $\{\varsigma_{ij}(t_k^{ij})\}_{k^{ij}=0}^{\infty}$, the sequence $\{\varsigma(t_k^{\varsigma})\}_{k^{\varsigma}=0}^{\infty}$ is strictly decreasing.

Theorem 5.4.1. *Let us consider the event-triggered network* (5.8)*, where the function $f(t,x)$ is Lipschitz continuous with respect of x with Lipschitz constant L_f. Let also $\{\varsigma_{ij}(t_k^{ij})\}_{k^{ij}=0}^{\infty}$ be the nonincreasing sequences we defined previously with their corresponding functions $\varsigma_{ij}(t)$, and let us suppose that the following hypotheses hold:*

i. $\lim_{t\to+\infty} \varsigma_{ij}(t) = 0$ *for all the pairs (i,j) such that $a_{ij} \neq 0$;*

ii. there exists a scalar $0 < \eta < 1$ such that for all the $t \geq 0$ it holds that

$$\frac{\varsigma_{ij}\left(t_{l(t)}^{ij}\right)}{\varsigma\left(t_{l(t)}^{\varsigma}\right)} \geq \eta, \tag{5.37}$$

$$\frac{\varsigma_{ij}\left(t_{l(t)}^{ij}\right)}{\tilde{b}\left(t_{l(t)}^{ij}\right)} \geq \frac{\eta}{\epsilon_{\varsigma}}, \tag{5.38}$$

with $\varsigma\left(t_{l(t)}^{\varsigma}\right)$ defined previously and for all the nodes (i,j) such that $a_{ij} \neq 0$ and with $\tilde{b}(t)$ defined in (5.33)*.*

Then, network (5.8) *achieves complete synchronization under Algorithm 5.3.1 and no Zeno behaviour occurs.*

Proof. The proof recalls the one of Theorem 5.3.1 for the bounded synchronization case in i., and for this reason we focus our attention only on the steps which differs. In particular, considering $\|\dot{e}_{ih}\|_2$ and again the (5.26), we can use the same positions p_1 and p_2 and write

$$\|\dot{e}_{ih}(t)\|_2 \leq p_1\tilde{b}(t) + p_2\varsigma(t_{l(t)}^{\varsigma}), \quad \forall t \geq 0. \tag{5.39}$$

Now, we focus on the last event t_l^{ih} and, as done before in the proof of Theorem 5.3.1, we call $\tau_{ih}(t_l^{ih})$ the lower bound for the inter-event time between the last trigger event t_l^{ih}

and the next one t_{l+1}^{ih} for the generic pair (i,h). So, taking into account relation (5.29) and considering the position $t = t_l^{ih} + \tau_{ih}(t_l^{ih})$ and integrating the expression (5.39) we obtain

$$\tau_{ih}(t_l^{ih}) = \frac{\varsigma_{ih}(t_l^{ih})}{p_1 \tilde{b}(t_l^{ih}) + p_2 \varsigma(t_l^{\varsigma})}. \tag{5.40}$$

It is worth noting that such value is a finite nonzero value that gives, at every triggering event, the next lower bound value between consecutive triggers.

Now, taking into account hypothesis ii., we can write the lower bound $\bar{\tau}$

$$\bar{\tau} = \frac{1}{p_1 \epsilon_\varsigma + p_2} \eta \leq \tau_{ih}(t_l^{ih}) = \frac{\varsigma_{ih}(t_l^{ih})}{p_1 \tilde{b}(t_l^{ih}) + p_2 \varsigma(t_l^{\varsigma})}. \tag{5.41}$$

□

Remark 5.4.1. *In case of considering Algorithm 5.3.1′ instead of Algorithm 5.3.1, Theorem 5.4.1 is still valid with the easier choice of all the thresholds $\varsigma_{ij}(t_l^{ij}) = \varsigma_i(t_l^i) > 0$ with $j \in \mathcal{N}_i$ and remembering that all the events for the edges (i,j) with $j \in \mathcal{N}_i$ happen at the same time, and so $t_l^{ij} = t_l^i$.*

Theorem 5.4.1 is not always easy to be applied. In particular, choosing at any event time t_l^{ij} the value $\varsigma_{ij}(t_l^{ij})$ such that conditions (5.37)-(5.38) hold is in general difficult to be guaranteed since it requires the global information of $\varsigma(t_l^{\varsigma})$ and $e(t)$ respectively. However, inspired by Remark 5.4.1, a practical strategy can be followed in order to achieve a practical synchronization reducing the dependence on global information related to the network. Indeed, let us adopt for the sake of simplicity that each node updates, as in Algorithm 5.3.1′, all the events for the edges (i,j) with $j \in \mathcal{N}_i$ at the same time $t_l^{ij} = t_l^i$, and so there is an unique threshold sequence $\varsigma_i(t_l^i)$ for all its edges. We can imagine that each node sends to its neighbours not only the current value $u_i(t_l^i)$ of its control input (as described in Algorithm 5.3.1′), but also the value of the threshold $\varsigma_i(t_l^i)$ it has just computed. Obviously, since the same algorithm runs independently also on the other nodes, each node has the information of the thresholds $\varsigma_j(t_l^j)$ that its neighbours $j \in \mathcal{N}_i$ are currently using. In order to have a lower bound for the ratio in condition (5.37), it is enough that each node chose to decrease at every updating instant t_l^i its threshold function $\varsigma_i(t_l^i)$ in order to guarantee the condition

$$\frac{\varsigma_i\left(t_l^i\right)}{\max\limits_{j \in \mathcal{N}_i} \varsigma_j\left(t_l^j\right)} = \rho, \tag{5.42}$$

with $0 < \rho < 1$. Notice that, if (5.42) is already guaranteed at time t_l^i, then node i obviously does not decrease its threshold $\varsigma_i(t_l^i)$. With this strategy, condition (5.37) is satisfied with $\eta = \rho^{d^{\max}}$, where $d^{\max}$ is the diameter of the graph (see Section 3.2). Indeed, without loss of generality, suppose that a certain node h has the maximum current value among threshold functions, i.e $\varsigma(t_l^{\varsigma}) = \varsigma_h(t_l^h)$. Then, calling $j_1, j_2, \ldots, j_{d^{ih}-1}$ the path (possibly empty) of intermediate nodes between nodes i and h and supposing the worst case in term of the lower bound of the ratio $\frac{\varsigma_i(t_l^j)}{\varsigma_h(t_l^h)}$

$$\varsigma_i(t_l^i) \leq \varsigma_{j_1}(t_l^{j_1}) \leq \cdots \leq \varsigma_h(t_l^h),$$

we have that

$$\frac{\varsigma_i\left(t_l^i\right)}{\varsigma_h\left(t_l^h\right)} = \frac{\varsigma_i\left(t_l^i\right)}{\varsigma_{j_1}\left(t_l^{j_1}\right)} \cdot \frac{\varsigma_{j_1}\left(t_l^{j_1}\right)}{\varsigma_{j_2}\left(t_l^{j_2}\right)} \cdot \dots \cdot \frac{\varsigma_{j_{d^{ih}-1}}\left(t_l^{j_{d^{ih}-1}}\right)}{\varsigma_h\left(t_l^h\right)} \geq \rho^{d^{ih}}.$$

So, the value of η can be obtained considering the diameter of the network, thus giving a lower bound on the ratio in (5.37).

Condition (5.38) cannot be guaranteed with only local information. However, an heuristic could be that each node monitors the trend of the sequence of its inter-event times (for example considering a moving average). If the inter-event time is reducing too much, it stops to decrease the threshold value ς_i for a certain number of next events, or until the inter-event time is again above a certain value.

Furthermore, in order to guarantee complete synchronization, so when the maximum value of the threshold functions is such that $\lim_{t\to+\infty}\varsigma(t) = 0$, it is enough to have for all the nodes a sequence of updates of threshold functions $\{\varsigma_i(t_k^i)\}_{k^i=0}^{\infty}$ towards infinity. In such a way we have that $\lim_{t\to+\infty}\varsigma_i(t) = 0$ for all $i \in \mathcal{N}$, and so complete synchronization is achieved since the maximum value converge to zero. To better clarify this point, we can focus our attention on a generic node i, with current threshold $\varsigma_i(t_l^i)$. So, at the next $l^{i+} = l^i + 1$ updating event, from the criterion in (5.42), we have

$$\varsigma_i(t_{l+1}^i) = \rho \max_{j\in\mathcal{N}_i} \varsigma_j(t_l^j) \leq \rho\varsigma(t_l^{\varsigma}).$$

Now, if node i is such that the value of its threshold before updating was not the unique maximum, obviously the value of the maximum $\varsigma(t_l^{\varsigma})$ does not update. Conversely, if node i is such that the old value was the unique maximum among the thresholds, then at the next updating event also the value of the maximum updates at index $l^{\varsigma+} = l^{\varsigma} + 1$ and we have, from (5.42)

$$\varsigma(t_{l+1}^{\varsigma}) \leq \rho\varsigma(t_l^{\varsigma}). \tag{5.43}$$

Looking at the above formula it is clear that, if all the nodes update their threshold functions following the strategy in (5.42), then the sequence of the maximum threshold $\{\varsigma(t_k^{\varsigma})\}_{k^{\varsigma}=0}^{\infty}$ is a strongly decreasing positive sequence which converges to zero with law (5.43). Notice also that the case of more than one threshold with maximum value which update at the same time works exactly in the same way and implies the (5.43).

A simple way to guarantee that all the nodes updates towards infinity is to consider a maximum inter-event time $\tau_{\max}$ such that, for each node, if no events occur from the last event after this maximum period, a new event is forced and so both the control input u_i and the threshold functions ς_i are updated.

We remark here that the strategy described in this section is an heuristic inspired by the theoretical results in Section 5.3, where we refer for an analytically proved solution of the event-triggered synchronization problem.

5.5 Event-triggered synchronization with non perfect model description

Here we address the problem of studying an event-triggered control scheme similar to Algorithm 5.3.1 but without a perfect knowledge of the dynamical model of the nodes of the network. So, we suppose that, in general, $\hat{f}_i(\cdot) \neq f(\cdot)$.

Only for the sake of simplicity and in order to keep an easy dissertation, differently from what done in Section 5.3, we consider here directly the case of an algorithm similar to Algorithm 5.3.1′, where each node i updates the trigger errors related to its neighbours $j \in \mathcal{N}_i$ at the same time t_l^i. Furthermore, we consider the case of identical threshold functions. Both these simplifications can be removed, but we keep them in order to not complicate the strategy proposed in what follows, as we will briefly explain later. Before going into the details of the algorithm, here we give some definitions we will use in what follows.

For all pairs $(i,j) \in \mathcal{E}$ we consider the error of node i with respect of node j

$$\tilde{e}_{ij}(t) := \hat{e}_{ij}(t_l^i) - e_{ij}(t), \quad t \in [t_l^i, t_{l+1}^i), \tag{5.44}$$

with

$$\hat{e}_{ij}(t) = \hat{x}_j(t) - x_i(t). \tag{5.45}$$

Notice that in (5.44) we have considered the time interval $[t_l^i, t_{l+1}^i)$ according to the above mentioned simplification of taking the same updating instant for all the $\tilde{e}_{ij}$, with $j \in \mathcal{N}_i$. Notice also that we have defined the triggering error using the same symbol $\tilde{e}_{ij}$ as for the case of perfect knowledge of the dynamical error in (5.4). The reason is that such definition is the same of the (5.44) when the particular case of $\hat{x}_j = x_j$ is considered.

Differently from what happens in the case of perfect knowledge, here condition (5.6) cannot be directly checked because the current real error $e_{ij}(t)$ cannot be computed. For this reason, two kind of conditions will be later introduced in order to guarantee that the inequality

$$\|\tilde{e}_{ij}(t)\|_2 - \varsigma(t) < 0$$

is not violated. We now define

$$\Delta\hat{e}_{ij}(t) = \hat{e}_{ij}(t_l^i) - \hat{e}_{ij}(t), \tag{5.46}$$

$$\Delta\hat{x}_i(t) = \hat{x}_i(t) - x_i(t). \tag{5.47}$$

Considering a scalar $q \in (0.5, 1)$, for each node i we define the conditions

$$\|\Delta\hat{e}_{ij}(t)\|_2 < q\varsigma(t), \quad \forall j \in \mathcal{N}_i \tag{5.48a}$$

$$\|\Delta\hat{x}_i(t)\|_2 < (1-q)\varsigma(t). \tag{5.48b}$$

The two above conditions define, respectively, the two sequences of events $\{t_k^i\}_{k^i=0}^{\infty}$ and $\{t_k^{\hat{i}}\}_{k^{\hat{i}}=0}^{\infty}$. Indeed, if condition (5.48a) is violated, a new event is triggered and the index l^i related to the last event updates as $l^{i^+} = l^i + 1$. Analogously, the violation of condition (5.48b) implies a new trigger as $l^{\hat{i}^+} = l^{\hat{i}} + 1$. It is easy to note that conditions (5.48a)-(5.48b) imply condition (5.44). Indeed, we have

$$\begin{aligned}
\tilde{e}_{ij}(t) &= \hat{e}_{ij}(t_l^i) - e_{ij}(t) \\
&= \hat{e}_{ij}(t_l^i) - x_j(t) + x_i(t) + \hat{x}_j(t) - \hat{x}_j(t) \\
&= \Delta\hat{e}_{ij}(t) + \Delta\hat{x}_j(t).
\end{aligned}$$

Considering the norm and the triangular inequality we obtain

$$\|\tilde{e}_{ij}(t)\|_2 \le \|\Delta\hat{e}_{ij}(t)\|_2 + \|\Delta\hat{x}_j(t)\|_2 < q\varsigma(t) + (1-q)\varsigma(t) = \varsigma(t).$$

In order to evaluate conditions (5.48a)-(5.48b) it is required that each node i is equipped[1] with the prediction module of itself $\dot{\hat{x}}_i = \hat{f}_i(t, \hat{x}_i(t)) + u_i(t_l^i)$ and with the prediction module of its neighbours $\dot{\hat{x}}_j = \hat{f}_j(t, \hat{x}_j(t)) + u_j(t_l^j)$ with $j \in \mathcal{N}_i$, in order to compute the flows

$$\hat{x}_j(t) = \varphi_{\hat{f}_j}(t - t_l^{\hat{j}}, t_l^{\hat{j}}, x_j(t_l^{\hat{j}})), \qquad \forall j \in \mathcal{N}_i \cup \{i\} \tag{5.49}$$

as will be clear in the following algorithm.

Algorithm 5.5.1.

1. *Node i continuously listens to possible transmission of information from its neighbours and, in parallel, computes the flows* (5.49) *for itself and for all its neighbouring nodes using the dynamical model $\dot{\hat{x}}_j = \hat{f}_j(t, \hat{x}_j(t)) + u_j(t_l^j)$, for all $j \in \mathcal{N}_i \cup \{i\}$, with initial condition $\hat{x}_j(t_l^{\hat{j}})$. Thanks to the estimation of $\hat{x}_j$ (with $j \in \mathcal{N}_i \cup \{i\}$) node i can compute the* (5.46)-(5.47) *and monitor the conditions* (5.48a)-(5.48b). *If the node receives a new value of control input $u_h(t_l^h)$, with h a generic neighbour or node i itself, it updates the dynamical model of the related prediction module $\dot{\hat{x}}_h = \hat{f}_h(t, \hat{x}_h(t)) + u_h(t_l^h)$ in order to always correctly predict the $\hat{x}_h(t)$. Analogously, if node i receive form a node $h \in \mathcal{N}_i \cup \{i\}$ at a time instant $t_l^{\hat{h}}$ the current exact value of the state $x_h(t = t_l^{\hat{h}})$, it updates the initial condition of the related prediction module as $\hat{x}_h(t_l^{\hat{h}}) = x_h(t_l^{\hat{h}})$.*

2a. *If condition* (5.48b) *is violated*[2]*, node i updates the value of the initial condition $\hat{x}_i(t_l^{\hat{i}})$ of the estimator of itself with the new value*

$$\hat{x}_i(t_{l+1}^{\hat{i}}) = x(t_{l+1}^{\hat{i}}). \tag{5.50}$$

Here with $t_{l+1}^{\hat{i}}$ we mean the time instant when condition (5.48b) *is violated. Furthermore, after the event the counter $l^{\hat{i}}(t)$ is updated to $l^{\hat{i}^+} = l^{\hat{i}} + 1$ so that value* (5.50) *becomes now the new $\hat{x}_i(t_l^{\hat{i}})$ and represents, as we said, the new initializing value of predictor $\hat{f}_i(\cdot)$. Such value not only is used for node i to update the predictor of itself, but is also sent to neighbouring nodes $j \in \mathcal{N}_i$. The corresponding value $\Delta\hat{x}_i(t = t_l^{\hat{i}})$ is, obviously, reset;*

2b. *Once condition* (5.48a) *is violated for a generic node $h \in \mathcal{N}_i$, node i updates all the values $\hat{e}_{ij}(t_l^i)$ to $\hat{e}_{ij}(t_{l+1}^i)$. So, after the event, the counter $l^i(t)$ will be updated to $l^{i^+} = l^i + 1$. At the same time the current values $\Delta\hat{e}_{ij}(t = t_l^i)$ will be reset;*

3. *Node i computes the new control law u_i which takes the value*

$$u_i(t) = \sum_{j=1}^{N} a_{ij} \Gamma \hat{e}_{ij}(t_l^i), \qquad t \in [t_l^i, t_{l+1}^i). \tag{5.51}$$

Such new value of the input is broadcasted to its neighbourhood and it is held until the next output update of node i, that will happen at the next close event with one of its neighbours, as described in step 2b.*;*

[1] This could be an information exchanged among neighbouring nodes during an initialization procedure.

[2] It is assumed that node i can access to the value of its own state $x_i(t)$. This information, combined with the prediction $\hat{x}_i(t)$ allows node i to compute $\Delta\hat{x}_i(t)$.

4. *Repeat from step 1.*

Notice that, in the algorithm, steps $2a$. and $2b$. are intended to be parallel points of a same step 2.

The initialization of Algorithm 5.5.1 is analogous to the one of Algorithm 5.3.1. At least one node i starts the algorithm at a time instant t_0^i sending the triplet $(t_0^i, x_i(t_0^i), u_i(t_0^i))$ to its neighbours and starts to estimate its own evolution $\hat{x}_i(t)$. Each of its neighbours h starts to predict node i evolution $\hat{x}_i(t)$ and its own evolution $\hat{x}_h(t)$. So, once step $2a$. or $2b$. requires to transmit the first information, node h broadcasts its triplet $(t_0^h, x_i(t_0^h), u_h(t_0^h))$ to the neighbourhood. In a finite time all the nodes of the network can be connected.

Before giving a theorem of synchronization of a network of systems in (5.1) coupled with the control scheme in Algorithm 5.5.1, we give here a preliminary assumption.

Assumption 5.5.1. *Let us consider a dynamical system* $\dot{x} = f(t, x)$*, with* $x \in \mathbb{R}^n$*, whose model is known with uncertainty and it is given by* $\dot{\hat{x}} = \hat{f}(t, \hat{x})$*. We assume that there exist two constants* $\delta_1, \delta_2 \in [0, +\infty)$ *such that*

$$\|\varphi_{\hat{f}}(t-t_0, t_0, x(t_0)) - \varphi_f(t-t_0, t_0, x(t_0))\|_2 \leq e^{\delta_1(t-t_0)}\delta_2, \qquad \forall t \geq t_0,\ t, t_0 \in \mathbb{R},\ \forall x(t_0) \in \mathbb{R}^n.$$

Notice that Assumption 5.5.1 gives an upper bound on the quality of the estimated model $\hat{f}(\cdot)$ with respect to the real model $f(\cdot)$. Indeed, we assume that the divergence between estimated state $\hat{x}(t)$ and real state $x(t)$ is upper bounded by a non convergent exponential. Notice also that, if Assumption 5.5.1 is satisfied with δ_1, δ_2, then it is also satisfied for any $\delta_1' \geq \delta_1$ and $\delta_2' \geq \delta_2$.

We are now ready to give the following result for bounded synchronization of a network of systems (5.1) with non perfect knowledge of the dynamical model.

Theorem 5.5.1. *Let us consider a set* $\mathcal{N}$ *of* N *identical nonlinear systems* (5.1)*, with* $f_i(t, x_i) = f(t, x_i)$ *for all the* $i \in \mathcal{N}$ *and with* $g_i(t, x_i)$ *being the identity matrix. Let us suppose the function* $f(t, x)$ *being Lipschitz continuous with respect to* x *with Lipschitz constant* L_f *and let us consider a graph* $G(\mathcal{N}, \mathcal{E})$ *of Laplacian* L*, a coupling gain* $c > 0$ *and a coupling matrix* $\Gamma = \Gamma^T > 0$ *such that inequality* (5.13) *holds. Now, let us consider the case of not perfect knowledge of the dynamical model* $f(\cdot)$ *of the systems and let us call* $\hat{f}_i(\cdot)$ *the estimated model for all the node* $i \in \mathcal{N}$*. We suppose that for all the* $\hat{f}_i(\cdot)$ *Assumption 5.5.1 is satisfied with* δ_{1i} *and* δ_{2i} *and we call* $\delta_1 = \max_{i\in\mathcal{N}} \delta_{1i}$ *and* $\delta_2 = \max_{i\in\mathcal{N}} \delta_{2i}$*.*

Then, coupling the systems in $\mathcal{N}$ *with the event-triggered distributed strategy described in Algorithm 5.5.1 and considering a constant threshold function* $\varsigma(t) = \bar{\varsigma}$ *and a scalar* $q \in (0.5, 1)$ *such that* $\delta_2 < (1-q)\bar{\varsigma}$*, the network achieves bounded synchronization. Furthermore, no Zeno behaviour will occur.*

Proof. Considering the control input (5.51) we get the model of the coupled network

$$\dot{x}_i = f(t, x_i) + c\sum_{j=1}^{N} a_{ij}\Gamma\hat{e}_{ij}(t_l^i), \qquad t \in [t_l^i, t_{l+1}^i), \qquad \forall i = 1, \ldots, N, \tag{5.52}$$

where, considering the position (5.44), we can rewrite the above expression and get the (5.16) of Theorem 5.3.1. Now, as done for Theorem 5.3.1, also in this case we split the proof in two steps. The first one is related to proving bounded synchronization and is exactly the same of the one in proof of Theorem 5.3.1. So we focus our attention directly in proving the step 2, where the existence of Zeno behaviour is excluded.

Step 2. To prove that no Zeno behaviour occurs in the network, we consider a generic node i of the network and we focus our attention on both the events relates to the sequences $\{t^i_k\}^{\infty}_{k^i=0}$ and $\{t^{\hat{i}}_k\}^{\infty}_{k^{\hat{i}}=0}$. In particular, taking into account Assumption 5.5.1, the latter gives a lower bound for minimum inter-event time

$$\hat{\tau} = \frac{1}{\delta_1} \log \left(\frac{(1-q)\bar{\varsigma}}{\delta_2} \right). \tag{5.53}$$

To study the minimum inter-event time for the sequence $\{t^i_k\}^{\infty}_{k^i=0}$ let us consider the derivative of (5.45) between node i and a generic neighbour $h \in \mathcal{N}_i$. We have

$$\dot{\hat{e}}_{ih} = \hat{f}_h(t, \hat{x}_h) + c\sum_{j=1}^{N} a_{hj}\Gamma e_{hj}(t) + c\sum_{j=1}^{N} a_{hj}\Gamma \tilde{e}_{hj}(t) - f(t, x_i) - c\sum_{j=1}^{N} a_{ij}\Gamma e_{ij}(t) - c\sum_{j=1}^{N} a_{ij}\Gamma \tilde{e}_{ij}(t). \tag{5.54}$$

Now, in order to evaluate when condition (5.48a) (with $j = h$) is violated, we consider at the moment that no updating event related to $\Delta\hat{x}_h(t)$ happens.

Considering that

$$\|\Delta\hat{e}_{ih}\|_2 = \left\| \int_{t^i_l}^{t} -\dot{\hat{e}}_{ih}(s)ds \right\|_2,$$

and substituting (5.54) in the above formula adding and subtracting $f(t, x_h(t))$ we have

$$\begin{aligned} &\|\Delta\hat{e}_{ih}\|_2 = \\ &\left\| \int_{t^i_l}^{t} f(s, x_h) - f(s, x_i) + c\sum_{j=1}^{N} a_{hj}\Gamma e_{hj}(s) + c\sum_{j=1}^{N} a_{hj}\Gamma \tilde{e}_{hj}(s) - c\sum_{j=1}^{N} a_{ij}\Gamma e_{ij}(s) - c\sum_{j=1}^{N} a_{ij}\Gamma \tilde{e}_{ij}(s)ds \right\|_2 + \\ &\left\| \int_{t^i_l}^{t} \hat{f}_h(s, \hat{x}_h) - f(s, x_h)ds \right\|_2. \end{aligned} \tag{5.55}$$

From the above formula we can write

$$\|\Delta\hat{e}_{ih}\|_2 \leq \int_{t^i_l}^{t} \left\| p_1\tilde{b}(s) + p_2\bar{\varsigma}ds \right\|_2 + \left\| \int_{t^i_l}^{t} \hat{f}_h(s, \hat{x}_h) - f(s, x_h)ds \right\|_2, \tag{5.56}$$

where, as in the proof of Theorem 5.3.1, we have considered the Lipschitz property of f and the same $p_1, p_2, \tilde{b}(t)$. Now, we call $\tau_{ih}(t^i_l)$ the lower bound for the inter-event time between the last trigger event t^i_l and the next one t^i_{l+1}, supposing that the event $l^i + 1$ happens when the condition (5.48a) is violated with respect to node h, i.e. with $j = h$.

Remembering that the second term of the right hand side of the inequality (5.56) can assume maximum value $(1-q)\bar{\varsigma}$, the nonzero lower bound $\tau_{ih}(t^i_l)$ for the generic pari (i, h) is

$$\tau_{ih}(t^i_l) = \frac{(2q-1)\bar{\varsigma}}{p_1\tilde{b}(t^i_l) + p_2\bar{\varsigma}}. \tag{5.57}$$

The same reasoning can, obviously, be done for any other node $j \in \mathcal{N}_i$. Notice that, to evaluate the bound in (5.57) we have supposed that no updating event $t^{\hat{h}}_{l+1}$ happens. This, of course, cannot be excluded a priori and if a new value $\hat{x}_h(t^{\hat{h}}_{l+1})$ is received form node i, such value replaces the current $\hat{x}_h(t)$. For this reason it could be happen that $\|\Delta\hat{e}_{ih}\|_2$ either suddenly reduces its value or increases its value. In the latter case the bound for the next inter-event time reduces with respect to (5.57). However, it is worth mentioning that even if we consider the extreme case where node i receives in an infinitesimal time interval the updates of all the $\hat{x}_j$, with $j \in \mathcal{N}_i$, and each of the new $\|\Delta \hat{e}_{ij}\|_2$ leads to instantaneously violate the corresponding (5.48a), then we would have an accumulation of N_i (number of node i neighbours) updates of the control input u_i. However, even in this case, the Zeno behaviour is excluded since node i will wait at least a time interval

$$\bar{\tau} = \min \left\{ \hat{\tau}, \min_{j \in \mathcal{N}_i} \tau_{ij}(t_l^i) \right\}$$

before the next event $l^i + 1$ and so infinite commutations in a finite time interval are excluded.

It is worth mentioning that, despite the Zeno behaviour is excluded, conversely of what happens in the case of Theorem 5.3.1 with perfect knowledge of dynamical model, even updating the triggering error (5.46) related to all the neighbours $j \in \mathcal{N}_i$ at the same time t_l^i, we cannot guarantee a minimum time interval among the event at time t_l^i and the next event t_{l+1}^i. Indeed, as we said, asynchronous updates of condition (5.48b) for some $h \in \mathcal{N}_i$ can leads to violate the corresponding condition (5.48a) associated to the same node h, thus forcing a new event. For this reason, the control scheme in Algorithm 5.5.1 should be implemented with actuators that can support fast changes in their output. $\square$

To conclude this section we briefly mention here that different thresholds $\bar{\varsigma}_i$ and different constants $q_i \in (0.5, 1)$ for each node i can be considered. In this case, Algorithm 5.5.1 requires that node i receives from all of its neighbours the values of $\bar{\varsigma}_j$ and $q_j \in (0.5, 1)$ they are using in order to compute for each of them the condition

$$\|\Delta \hat{x}_i(t)\|_2 < (1 - q_j)\bar{\varsigma}_j, \qquad \forall j \in \mathcal{N}_i \cup \{i\}. \tag{5.58}$$

Once that this condition is violated for the first node $h \in \mathcal{N}_i \cup \{i\}$, node i updates the event counter $l^{\hat{i}}$ to $l^{\hat{i}^+} = l^{\hat{i}} + 1$ and sends the new value $\hat{x}_i(t_l^{\hat{i}}) = x(t_l^{\hat{i}})$ to its neighbourhood and to itself.

Furthermore, practical implementations for Theorem 5.5.1 could also be considered analogously to what already done in Theorem 5.4.1 for the case of perfect knowledge of the dynamical model of the nodes.

5.6 Example

In this section we present an application of the innovative scheme of event-triggered synchronization to a classical application. Indeed, in order to show the effectiveness of the strategy proven in Theorem 5.3.1, we consider a network of Chua's circuits [135]. The Chua' system is a well studied dynamical system and it is often taken in literature as a paradigm for chaos [41, 40, 50]. Synchronization of Chua's circuits has been investigated in several articles (see for instance [80, 40]) and here we aim to use such a classical example in order to illustrate how a network of nonlinear chaotic systems can synchronize without the need for a continuous time coupling, but more simply via the distributed event-triggered coupling strategy proposed in Algorithms 5.3.1-5.3.1′.

More in detail, we consider a connected random graph [72] of identical Chua's circuits, which are electronic schemes whose dynamical models $\dot{x}_i = f(x_i)$ have expression

$$\begin{aligned} \dot{x}_{i1} &= \alpha \left[x_{i2} - x_{i1} - \varphi(x_{i1}) \right], \\ \dot{x}_{i2} &= x_{i1} - x_{i2} + x_{i3}, \\ \dot{x}_{i3} &= -\beta x_{i2}, \end{aligned}$$

where, following [135] we set $\alpha = 10$, $\beta = 17.30$, and where $\varphi(x_{i1}) = bx_{i1} + (a - b)(|x_{i1}+1| - |x_{i1}-1|)/2$, with $a = -1.34$, $b = -0.73$. It is easy to notice that the vector field of the Chua system is Lipschitz and an upper bound for the Lipschitz constant is

$$L_f = \left\| \begin{matrix} -\alpha & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{matrix} \right\|_2 + \alpha|a|,$$

which gives in this case $L_f = 34.2$.

We simulate a network of five Chua systems with adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix},$$

considering as matrix Γ the identity matrix for the sake of simplicity. In order to guarantee inequality (5.13) these data lead to a minimum coupling $c = 13.7$. It is worth mentioning that better estimations can be given both for the Lipschitz constant and for the minimum synchronizing coupling. However, such discussion is outside the topic of this example whose aim is only to show the effectiveness of the event-triggered strategy in Algorithms 5.3.1-5.3.1′.

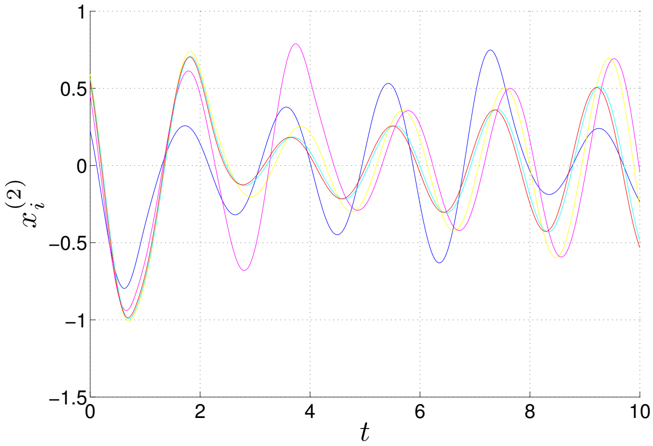
Simulation in Figure 5.1(a) shows the evolution of the uncoupled nodes considering random initial conditions in the domain of the chaotic attractor. From the same initial conditions and coupling the network it is possible to observe bounded synchronization in Figure 5.1(b). Simulations have been performed applying Algorithm 5.3.1, setting an identical static threshold $\varsigma_{ij}(t) = \bar{\varsigma}$ for all connected pairs (i, j), with $\bar{\varsigma} = 0.1$, and choosing to plot the second state components as representative of the whole state.

Figure 5.2 and Figure 5.3 show, respectively, the norm of the control signal and the trigger instants for each system. In both cases, a zoom is shown at the beginning and at the end of the time interval $[0, 10]$ chosen for the simulation. Is it possible to notice that, as expected, the control signal is a piecewise-constant function. From Figure 5.3, it is possible to see that when a node triggers, another node of its neighbourhood triggers at the same time as well, according to Algorithm 5.3.1.

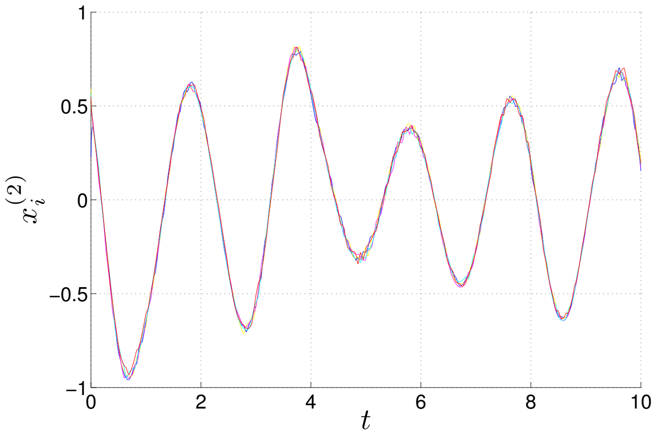
Simulations have also been obtained considering an identical exponential threshold $\varsigma_{ij} = k_\varsigma e^{-\lambda_\varsigma t}$ with $k_\varsigma = 1$ and $\lambda_\varsigma = 0.5$ and considering Algorithm 5.3.1′. Figure 5.4 shows exponential synchronization of the network while Figure 5.5 and Figure 5.6 show, respectively, the norm of the control signal and the trigger instants. Also in this case we show the behaviour at the beginning and at the end of the time interval of simulation.

As expected from Algorithm 5.3.1, each node triggers, in general, at different time instants as it is possible to see from Figure 5.6.

The number of triggers for each node in time intervals of unitary length is reported in Table 5.1 and Table 5.2 respectively for the case of static and exponential threshold. It is possible to observe how this second case is sensibly better than a static threshold approach in terms of the number of triggering events.



(a)

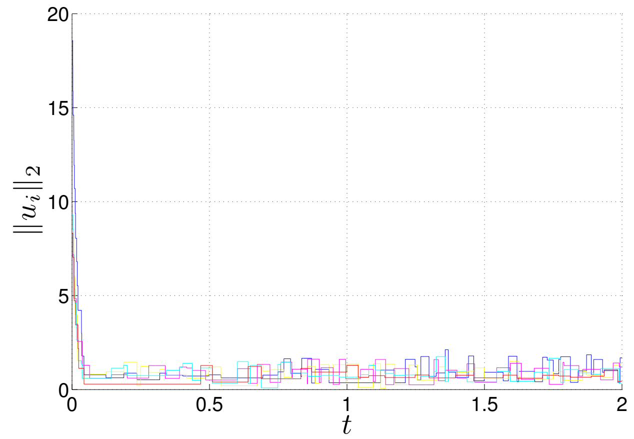


(b)

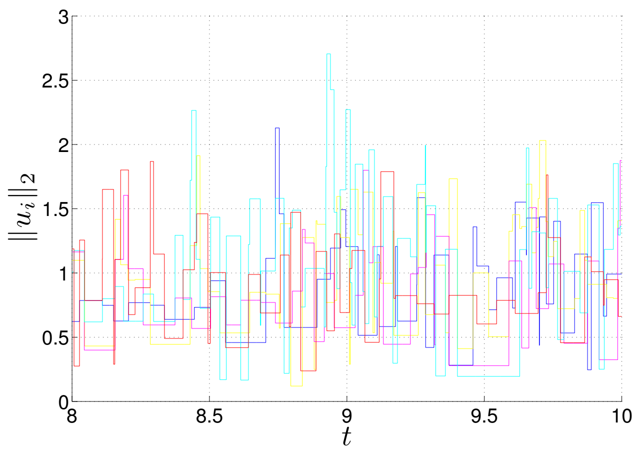
Figure 5.1: Time evolution of the state components $x_i^{(2)}(t)$ for the network of Chua systems with static thresholds: (a) uncoupled case; (b) coupled case.

5.7 Discussion

Regarding discontinuous networks with discontinuities in the communication, in this chapter we presented model based event-triggered strategies for synchronization of networks of nonlinear dynamical agents. In particular, we considered a model based approach where agents are equipped with their own embedded processor and compute the dynamical flows of their neighbours. Between each pair of connected agents, relative information of their state mismatch is considered in order to generate local events and update the control law. We investigated both the cases of exact and not exact knowledge of the dynamical model of the agents. In the first case results have been given for bounded synchronization and for exponential asymptotic synchronization, while Zeno behaviour has been proved to be excluded. Furthermore, a strategy able to guarantee a



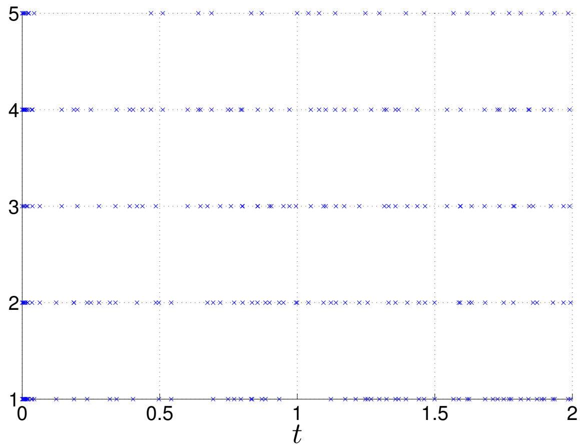
(a)



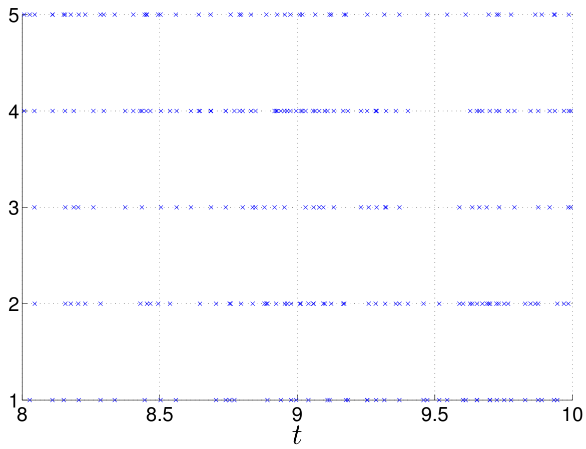
(b)

Figure 5.2: Time evolution of the norm of the control signals for the network of Chua systems with static thresholds: (a) beginning of the time interval; (b) end of the time interval.

lower bound for the inter-event times between consecutive updates of the control law has been proposed. The case of not exact knowledge of the dynamical model of the agents has been studied considering an additional sequence of events for each node which triggers any time the error between the current state of the agent and the predicted state exceeds a certain value. For such scenario a result of bounded synchronization has been given with an adjustable value of the states' mismatch among the agents. Furthermore, also in this case the Zeno behaviour is excluded. Differently from the recent related literature, all the proposed strategies ensure that both the control and the communication signals are piecewise constant. The results of the chapter are supported through numerical simulations.



(a)



(b)

Figure 5.3: Trigger events for the network of Chua systems with static thresholds: (a) beginning of the time interval; (b) end of the time interval.

Table 5.1: Number of triggers in unitary intervals for the network of Chua systems: static thresholds.

	$[0,1)$ s	$[1,2)$ s	$[2,3)$ s	$[3,4)$ s	$[4,5)$ s	$[5,6)$ s	$[6,7)$ s	$[7,8)$ s	$[8,9)$ s	$[9,10]$ s
node 1	32	33	26	25	22	24	18	12	17	28
node 2	35	25	27	39	33	30	29	32	24	38
node 3	29	25	19	22	23	25	20	16	18	21
node 4	32	24	22	27	36	39	27	12	36	35
node 5	15	15	14	21	23	32	30	24	30	22

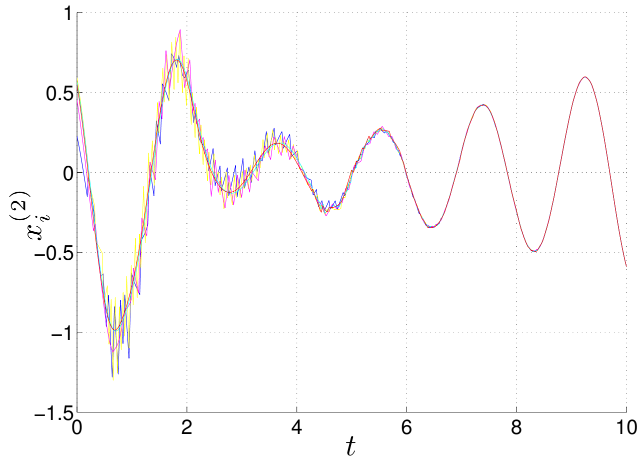
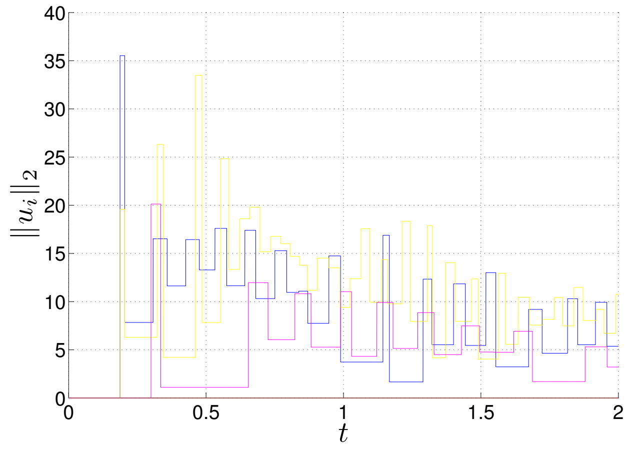


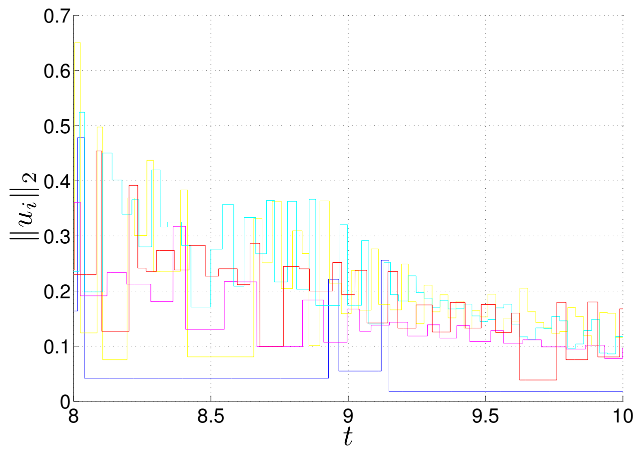
Figure 5.4: Time evolution of the state components $x_i^{(2)}(t)$ for the network of Chua systems with exponential thresholds.

Table 5.2: Number of triggers in unitary intervals for the network of Chua systems: exponential thresholds.

	$[0,1)$ s	$[1,2)$ s	$[2,3)$ s	$[3,4)$ s	$[4,5)$ s	$[5,6)$ s	$[6,7)$ s	$[7,8)$ s	$[8,9)$ s	$[9,10]$ s
node 1	16	14	17	16	17	24	23	18	4	2
node 2	19	25	26	26	22	25	19	22	21	33
node 3	7	12	18	13	13	12	16	12	13	16
node 4	0	0	0	0	20	25	29	25	25	32
node 5	0	0	0	0	10	15	20	19	18	17

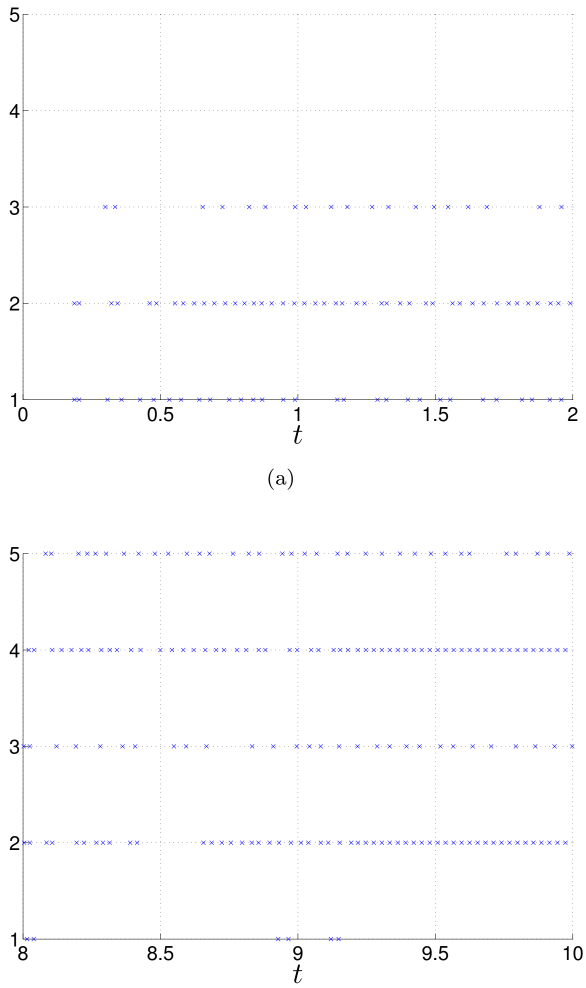


(a)



(b)

Figure 5.5: Time evolution of the norm of the control signals for the network of Chua systems with exponential thresholds: (a) beginning of the time interval; (b) end of the time interval.



(a)

(b)

Figure 5.6: Trigger events for the network of Chua systems with exponential thresholds: (a) beginning of the time interval; (b) end of the time interval.

Part II

Incremental Stability of Discontinuous Dynamical Systems

Chapter 6

Incremental stability and contraction theory

Contents

In this chapter we introduce the concept of incremental stability and contraction theory. The basic concept of these tools will be useful in Part II of the thesis, where they will be extended to discontinuous systems.

6.1 Introduction

Incremental stability is a tool for generic nonlinear dynamical systems which has been defined in [10]. It characterizes asymptotic convergence of trajectories with respect to one another rather than towards some attractor known a priori. So, in studying incremental stability of a nonlinear system we are no more interested in the convergence of the trajectories towards an equilibrium point (or more generally towards an invariant set), but only on the relative errors between trajectories themselves. If such error converges to zero for all trajectories starting from all possible different initial conditions in the state space, then the system is said to be *globally incrementally asymptotically stable*.

The concept of incremental stability is closely related to other tools and definitions explored in the dynamical systems and control literature (see [91, 114] and also [155, 126, 107] and references therein for more details). In Section 6.2 we will give more rigorous definitions of incremental stability and of the other tools mentioned here.

In particular, the idea of *convergent systems* due to Demidovich [58, 155], recently extended in [158, 157, 154, 156], looks for the existence of a bounded, uniformly globally asymptotically stable solution. If such kind of solution exists, then all the trajectories of the system converge towards it (and so also among each other) independently from the initial condition. In the cited literature, Lyapunov conditions have been developed in order to guarantee that a nonlinear dynamical system is convergent.

Among the available tools related to the idea of incremental stability, contraction theory as expounded in [124], has been shown to be a powerful tool able to study exponential incremental stability of a system of interest, e.g. [181]. The idea is to study exponential convergence of all trajectories in a domain of a generic continuously differentiable dynamical system by looking at the negative definitiveness of the measure of its Jacobian. So, finding a metric under which the related matrix measure of the system Jacobian is negative definite over some convex set of phase space of interest guarantees *incremental exponential stability* of the system in that set. It can be shown that Demidovich's approach is related to proving contraction using Euclidean norms and matrix measures. Nevertheless, using contraction it suffices to find some measure to study the Jacobian properties including non-Euclidean ones (e.g., μ_1, μ_∞ etc).

In this thesis, contraction theory will be investigated in Chapter 7 and Chapter 8 for studying incremental stability of non differentiable systems.

Historically, ideas closely related to contraction can be traced back to [91] and even to [114] (see also [155, 10, 126, 107], for a more exhaustive list of related references). For autonomous systems and with constant metrics, the basic nonlinear contraction result reduces to Krasovskii's theorem [191] in the continous-time case, and to the contraction mapping theorem in the discrete-time case [124, 27].

Contraction theory has been used in a wide range of applications. For example, it has been shown that contraction is an extremely useful property to analyze coordination problems in networked control systems such as the emergence of synchronization or consensus [124, 163, 180, 210, 182, 178, 183]. Indeed, all trajectories of a contracting system can be shown to exponentially converge towards each other asymptotically. Therefore as shown in [210], this property can be effectively exploited to give conditions for the synchronization of a network of dynamical systems of interest. Recently, it has also been shown that non Euclidean matrix measures can be used to construct an algorithmic approach to prove contraction [180] and to prove efficiently convergence in biological networks [181].

Due to its usefulness in proving convergence and synchronization, contraction theory for discontinuous systems is investigated in Part II of the thesis. It is worth mentioning that, since contraction theory requires the computation of the Jacobian of the system, in the literature the system is required to be smooth. For this reason, the extension of such tool to piecewise smooth systems is not immediate and there is no consistent extension of this approach in literature.

In the following section we give some basic concepts of incremental stability, convergent systems and contraction theory which will be useful in the rest of the thesis.

6.2 Mathematical preliminaries

In what follows, given an n-dimensional vector x, we will denote with $|x|$ its generic norm.

Incremental stability [10] can be defined as follows:

Definition 6.2.1. *A dynamical system of the form* $\dot{x} = f(t, x)$, $\quad x(t_0) = x_0$ *is said to be* incrementally asymptotically stable *(δAS) in an invariant connected set* $\mathrm{D} \subseteq \mathbb{R}^n$ *if there exists a function* ς *of class* $\mathcal{KL}$ *[109] such that for all* $\xi, \zeta \in \mathrm{D}$ *and all* $t \geq t_0$ *the trajectories* $x(t) = \varphi(t - t_0, t_0, \xi)$ *and* $y(t) = \varphi(t - t_0, t_0, \zeta)$*, starting respectively from the two initial conditions* ξ *and* ζ*, satisfy:*

$$|x(t) - y(t)| \leq \varsigma(|\xi - \zeta|, t) \qquad \forall t \geq t_0, \tag{6.1}$$

Furthermore, if there exist constants $K, c > 0$ *such that the following holds*

$$|x(t) - y(t)| \le K e^{-c(t-t_0)} |\xi - \zeta| \qquad \forall t \ge t_0, \tag{6.2}$$

the system is said to be incrementally exponentially stable *(δES). Due to the equivalence of norms in finite dimensional spaces and using the properties of* $\mathcal{KL}$ *functions [109], it is immediate to verify that the above definition is independent from the specific vector norm being used. Notice that in the case of* $\mathrm{D} = \mathbb{R}^n$ *in* (6.1) *(or* (6.2)*), then incremental asymptotic (or exponential) stability holds globally (δGAS, or δGES).*

Notice also that the definition of asymptotic stability δAS considered here is not the same definition of local asymptotic stability given in [10].

Here we give the definition of convergent systems as given by Demidovich [58], [155].

Definition 6.2.2. *A dynamical system* $\dot{x} = f(t, x)$ *is said to be* convergent *if there exists a globally asymptotically stable solution* $\bar{x}(t)$ *defined and bounded as function of* t *for all* $t \in \mathbb{R}$.

Furthermore, if $\bar{x}(t)$ *is globally exponentially stable the system is said to be* exponentially convergent.

This definition requires the existence of a bounded uniformly globally asymptotically stable solution $\bar{x}(t)$ defined for all t and is related to the properties of the trajectories of a dynamical system without requiring any hypothesis on the continuity of the vector field. However, as reported in [155], a sufficient condition to guarantee quadratic convergence (i.e. exponential convergence of the trajectories) of a smooth dynamical system is a Lyapunov condition on its Jacobian matrix. So, even if the definition of convergence given above does not explicitly require smoothness of the system vector field, the application of the condition given in [155] requires its smoothness in order to compute the Jacobian.

Before introducing the notion of contraction, we first give two preliminary definitions [181].

Definition 6.2.3. *Let* $K > 0$ *be an arbitrary positive real number. A subset* $\mathcal{C} \subset \mathbb{R}^n$ *is* K-reachable *if, for any two points* x_0 *and* y_0 *in* $\mathcal{C}$ *there is some continuously differentiable curve* $\gamma : [0, 1] \mapsto \mathcal{C}$ *such that:*

1. $\gamma(0) = x_0$;
2. $\gamma(1) = y_0$;
3. $|\gamma'(r)| \le K |y_0 - x_0|$, $\forall r \in [0, 1]$.

For convex sets $\mathcal{C}$, we may pick $\gamma(r) = x_0 + r(y_0 - x_0)$, so $\gamma'(r) = y_0 - x_0$ and we can take $K = 1$. Thus, convex sets are 1-reachable, and it is easy to show that the converse holds as well. Also in this case, due to norms equivalence, the above definition does not depend on the particular choice of the norm $|\cdot|$.

Given a vector norm $|\cdot|$ on $\mathbb{R}^n$ and a matrix $A \in \mathbb{R}^{n \times n}$, we will denotes with $\|A\|$ the induced norm [97]. We recall [140] and give the following definition.

Definition 6.2.4. *Given a vector norm on Euclidean space* ($|\cdot|$)*, with its induced matrix norm* $\|A\|$*, the associated* matrix measure μ *is defined as the directional derivative of the matrix norm, that is,*

$$\mu(A) := \lim_{h \searrow 0} \frac{1}{h} \left(\|I + hA\| - 1 \right).$$

Notice that this limit is known to exist and convergence is monotonic, see [47, 197]. For example, if $|\cdot|$ is the standard Euclidean 2-norm, then $\mu(A)$ is the maximum eigenvalue of the symmetric part of A. As we shall see, however, different norms will be useful for our applications. Matrix measures are also known as "*logarithmic norms*", a concept independently introduced by Germund Dahlquist and Sergei Lozinskii in 1959, [47, 127].

In what follows we report the analytic expression of some matrix measures used in the next chapters:

- $\mu_1(A) = \max_j \left(a_{jj} + \sum_{i \neq j} |a_{ij}| \right)$;
- $\mu_2(A) = \max_i \lambda_i \left(\frac{1}{2} \left(A + A^T \right) \right)$;
- $\mu_\infty(A) = \max_i \left(a_{ii} + \sum_{j \neq i} |a_{ij}| \right)$.

More generally, we can also make use of matrix measures induced by weighted vector norms, say $|x|_{\Theta,i} = |\Theta x|_i$, with Θ a constant invertible matrix and $i = \{1, 2, \infty\}$. Such measures, denoted with $\mu_{\Theta,i}$, can be computed by using the following property: $\mu_{\Theta,i}(A) = \mu_i \left(\Theta A \Theta^{-1} \right)$, $\forall i = \{1, 2, \infty\}$. Obviously, any other measure can be used.

Definition 6.2.5. *A continuously differentiable dynamical system $\dot{x} = f(t, x)$ is said to be* infinitesimally contracting *on a connected set* D *if there exists some norm in* D *with associated matrix measure μ such that for some constant $c > 0$ it holds that*

$$\mu \left(\frac{\partial}{\partial x} f(t, x) \right) \leq -c \qquad \forall t \geq t_0, \forall x \in \mathrm{D}. \tag{6.3}$$

For an infinitesimally contracting system the following result holds [181]:

Theorem 6.2.1. *An infinitesimally contracting dynamical system on an invariant K-reachable set $\mathcal{C}$ is incrementally exponentially stable in $\mathcal{C}$.*

Although the classical definition of infinitesimal contraction (Definition 6.2.5) require the smoothness of the vector field, in Part II of the thesis we will extend contraction theory to nonsmooth systems (see Section 2.3) in order to prove their incremental stability. In particular, in Section 7 we will consider an extension of contraction theory to PWSC and TSS, with applications to synchronization of discontinuous dynamical networks. In Section 8 we will consider some preliminary results of incremental stability of planar Filippov systems.

Chapter 7

Contraction and incremental stability of non differentiable systems

Contents

In this chapter we extend to a generic class of piecewise smooth dynamical systems a fundamental tool for the analysis of convergence of smooth dynamical systems: contraction theory. We focus on switched non differentiable systems satisfying Caratheodory conditions for the existence and uniqueness of a solution. After generalizing the classical definition of contraction to this class of dynamical systems, we give sufficient conditions for global exponential convergence of their trajectories. The theoretical results are then applied to solve a set of representative problems including proving global asymptotic stability of switched linear systems, giving conditions for incremental stability of piecewise smooth systems, and analyzing the convergence of networked switched systems.

The reminder of the chapter is the following. In Section 7.1 we give an introduction to the problem of incremental stability of piecewise smooth systems. In Section 7.2 the problem of finding general conditions able to guarantee contraction for PWSC and TSS (see Section 2.3) is addressed. In Section 7.3 we consider the problem of partial contraction and we refer it to two notably applications, the incremental stability of time\state varying linear systems and the design of coupling protocols. The problem of synchronizing a network of time-switching systems is studied in Section 7.4, while examples in Section 7.5 apply the results developed in the chapter and show how contraction can be used as a tool both for the analysis of the exponential incremental stability of the class of piecewise smooth systems considered and for the synthesis of coupling protocols in order to achieve synchronization of discontinuous networks. Finally, in Section 7.6 we discuss the results obtained in the chapter and draw possible future directions and extensions.

7.1 Introduction

Piecewise-smooth dynamical systems are commonly used in Nonlinear Dynamics and Control to model devices of interest and/or synthesize discontinuous control actions e.g., [46], [60]. Despite the large number of available results on their well-posedness and stability, there are few papers in the literature where the problem of assessing their incremental stability and convergence properties is discussed.

An approach used to solve the problem of proving incremental stability of piecewise smooth systems is finding conditions under which the PWS systems is a *convergent system*. Such problem is addressed in the extensive work discussed in: [158, 157, 154, 156] using a Lyapunov based approach. The methodology extends the approach of Demidovich for smooth dynamical systems expounded in [58, 155] (see also Section 6.1). Sufficient conditions addressing convergence among trajectories have also been obtained for certain classes of nonsmooth systems.

In particular, results in proving convergence for piecewise affine continuous systems (PWAC) have been obtained in [158, 156]. Indeed, convergence is ensured by the existence of a common quadratic Lyapunov function on the linear dynamics of the PWAC system of interest. An extension of such methodology to guarantee convergence in generic nonlinear PWSC systems is developed in [157]. Again, it is required that a common quadratic Lyapunov function can be defined for the system of interest. So, despite the fact that the definition of convergent systems (Definition 6.2.2) does not explicitly require any smoothness of the vector field, as for the case of proving convergence for smooth systems also for nonsmooth systems Lyapunov conditions on the Jacobian are given in order to guarantee such property.

Applications of the theory of convergent systems to synchronization are presented in [207] where master-slave synchronization of two PWAC systems is investigated, and in [165] where the methodology is used to study synchronization of networks of systems with a passive input\output behaviour.

An alternative approach to study convergence in smooth dynamical systems is contraction theory, as we have already said in Chapter 6. In this case, differently from the theory of convergent systems, it is not required the existence of a bounded, uniformly globally asymptotically stable solution. Furthermore, contraction theory does also not require finding incremental Lyapunov functions for the system of interest. Indeed, for a system to be contracting over a set of interest, it suffices, as we said, to find a generic condition on the existence of some metric in which the Jacobian of the system under investigation is negative definite. As contraction is verified on the Jacobian of the system, it implicitly requires smoothness of the system vector field. The study of contracting dynamical systems has been successfully applied in multi-agent dynamical networks to prove consensus and synchronization [124, 163, 180, 210, 182, 178, 180, 181, 183]. However, despite the usefulness of contraction theory in applications, there is no consistent extension of this approach to the large class of piecewise smooth and switched dynamical systems. In [125], it is conjectured that, for certain classes of piecewise-smooth systems, contraction of each individual mode is sufficient to guarantee convergence of all the system trajectories towards each other, i.e. contraction of the overall system of interest. Also, in [71], it is noted that contraction theory can be extended to a class of hybrid systems under certain assumptions on the properties of the reset maps and switching signals.

The aim of this chapter is to start addressing systematically the extension of contraction theory to generic classes of switched systems in the same spirit of what has been done for convergent systems. The motivation is that such extension can be used as an alternative tool to study incremental exponential stability of important classes of non differentiable systems. In particular, in the following sections of the chapter, we focus on a class of non differentiable systems: (i) piecewise-smooth continuous (PWSC) systems (a class of state-dependent switched systems), and (ii) time-dependent switched systems (TSS). The goal is to obtain a set of sufficient conditions guaranteeing global exponential convergence of their trajectories.

From a methodological viewpoint, to investigate contraction properties of PWSC and TSS we focus on systems whose vector fields satisfy Caratheodory conditions for the existence and uniqueness of an absolutely continuous solution (see Section 2.4 and references therein for further details). We prove that, as conjectured in [125], for this class of systems, contraction of each individual mode suffices to guarantee convergence of all the system trajectories towards each other, i.e. contraction of the overall system of interest. We then apply the theoretical results to study convergence of some representative problems, including the synchronization of networks of time-switched systems. So, the usefulness of such extension will be then motivated through a set of examples showing how contraction theory can be a convenient alternative approach both for studying incremental exponential stability of non differentiable systems and for achieving their synchronization.

A very preliminary version of some of the results presented in this chapter were presented in [179].

7.2 Contraction of Caratheodory systems

Contraction theory has been properly studied mostly in the case of smooth nonlinear vector fields. The case of switched and hybrid systems is only marginally addressed in the existing literature [125, 71].
In this section, we seek sufficient conditions for the convergence of trajectories of PWSC systems (a generic class of systems with state-dependent switchings) and time-dependent switched systems. For the sake of clarity, we keep the derivation for the two cases separate.

7.2.1 Contraction of PWSC systems

We start with PWSC systems as defined in Section 2.3. We can state the following result:

Theorem 7.2.1. *Let $\mathcal{C} \subseteq \mathrm{D}$ be a K-reachable set. Consider a generic PWSC system of the form*

$$\dot{x} = f(t,x) = \begin{cases} F_1(t,x) & x \in \mathcal{S}_1, \\ \vdots \\ F_p(t,x) & x \in \mathcal{S}_p, \end{cases} \tag{7.1}$$

defined as in Definition 2.3.1 for all $x \in \mathcal{C}$ and with Σ_{ij} smooth manifolds for all $i, j = 1, \ldots, p$. Suppose that:

1. *it fulfills conditions for the existence and uniqueness of a Caratheodory solution given in Section 2.4;*
2. *there exist a unique matrix measure such that*
$$\mu\left(\frac{\partial F_i}{\partial x}(t,x)\right) \leq -c_i,$$
for all $x \in \bar{\mathcal{S}}_i$ and all $t \geq t_0$, with c_i belonging to a set of positive scalars (in what follows, we will define $c := \min_i c_i$).

Then, for every two solutions $x(t) = \varphi(t - t_0, t_0, \xi)$ and $y(t) = \varphi(t - t_0, t_0, \zeta)$ with $\xi, \zeta \in \mathcal{C}$, it holds that:
$$|x(t) - y(t)| \leq K e^{-c(t-t_0)} |\xi - \zeta|,$$
for all $t \geq t_0$ such that $x(t), y(t) \in \mathcal{C}$. If $\mathcal{C}$ is forward-invariant then all trajectories rooted in $\mathcal{C}$ converge exponentially towards each other.

Proof. Given two points $x(t_0) = \xi$ and $y(t_0) = \zeta$ and a smooth curve $\gamma : [0,1] \mapsto \mathcal{C}$ such that $\gamma(0) = \xi$ and $\gamma(1) = \zeta$, we can consider $\psi(t,r) := \varphi(t - t_0, t_0, \gamma(r))$ as the solution of (7.1) rooted in $\psi(t_0, r) = \gamma(r)$, with $r \in [0,1]$. Notice that $\psi(t,r)$ is continuous with respect to r for all t. Notice also that γ can be chosen so that $\psi(t,r)$ is differentiable with respect to r for almost all the pairs (t,r). Let

$$w(t,r) := \frac{\partial \psi}{\partial r}, \qquad a.e.\ \text{in}\ t,\ a.e.\ \text{in}\ r. \tag{7.2}$$

Thus we have:

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{\partial \psi}{\partial t}\right) = \frac{\partial}{\partial r}\left(f(t, \psi(t,r))\right), \qquad a.e.\ \text{in}\ t,\ a.e.\ \text{in}\ r,$$

In what follows we will use the shorthand notation *a.e.* to denote the validity of a given expression almost everywhere in both t and r, unless stated otherwise.

Since
$$\frac{\partial}{\partial r}\left(f(t, \psi(t,r))\right) = \frac{\partial}{\partial x} f(t, \psi(x,t)) \frac{\partial \psi(t,r)}{\partial r}, \qquad a.e.,$$
we can write:
$$\frac{\partial}{\partial t} w(t,r) = A(t, \psi(t,r)) w(t,r), \qquad a.e., \tag{7.3}$$
where we have denoted by $A(t,x)$ the Jacobian of the PWSC system (7.1), which can be defined as:

$$A(t,x) = \frac{\partial f}{\partial x}(t,x) = \begin{cases} \frac{\partial F_1}{\partial x}(t,x) & \forall x \in \mathcal{S}_1, \\ \vdots \\ \frac{\partial F_p}{\partial x}(t,x) & \forall x \in \mathcal{S}_p, \end{cases}$$

for almost all the pairs (t, x) apart from those points where $x \in \Sigma_{ij}$, for some i, j.

The next step is to show that the solution $t \mapsto w(t, r)$ of (7.3) is a continuous function for any fixed $r \in [0, 1]$.

Indeed, without loss of generality, consider the image of the curve γ under the action of the flow φ for a time T such that the system trajectory rooted in γ has either crossed the boundary once or it has not (in the case there are multiple switchings between t_0 and T, the same reasoning can be iterated). Furthermore, let us call $\tau(r) \in]t_0, T[$ the time instant at which the trajectory eventually crosses the boundary. Suppose, without loss of generality, that at $t = \tau(r)$, the flow switches from region $\mathcal{S}_1$ to region $\mathcal{S}_2$. Then, we have:

$$\psi(t, r) = \begin{cases} \varphi_1(t - t_0, t_0, \psi(t_0, r)) & t_0 \leq t < \tau(r), \\ \varphi_2(t - \tau(r), \tau(r), \varphi_1(\tau(r) - t_0, t_0, \psi(t_0, r))) & \tau(r) < t \leq T. \end{cases}$$

Now, to show continuity of $w(t, r)$ with respect to time, from (7.2) we need to evaluate the derivative of $\psi(t, r)$ over the interval $]t_0, T[$. We have:

$$\frac{\partial \psi}{\partial r}(t, r) = \begin{cases} \frac{\partial}{\partial r}[\varphi_1(t - t_0, t_0, \psi(t_0, r))] & t_0 \leq t < \tau(r), \\ \frac{\partial}{\partial r}[\varphi_2(t - \tau(r), \tau(r), \varphi_1(\tau(r) - t_0, t_0, \psi(t_0, r)))] & \tau(r) < t \leq T, \end{cases} \tag{7.4}$$

Continuity of $w(t, r)$ is then guaranteed if

$$\lim_{t \to \tau(r)^-} \frac{\partial}{\partial r}[\varphi_1(t - t_0, t_0, \psi(t_0, r))] = \lim_{t \to \tau(r)^+} \frac{\partial}{\partial r}[\varphi_2(t - t_0 - \tau(r), \tau(r), \varphi_1(\tau(r) - t_0, t_0, \psi(t_0, r)))], \tag{7.5}$$

We have

$$\frac{\partial}{\partial r}\varphi_1(s, t_0, \chi) = \frac{\partial \varphi_1}{\partial \chi}\frac{\partial \chi}{\partial r}, \tag{7.6}$$

with $s := t - t_0$ and $\chi = \psi(t_0, r) := \psi^0$. Hence, taking the limit $t \to \tau(r)^-$, the left-hand side of (7.5) can be written as:

$$\frac{\partial \varphi_1}{\partial \psi^0}(\tau(r) - t_0, t_0, \psi^0)\frac{\partial \psi^0}{\partial r}, \tag{7.7}$$

Also

$$\frac{\partial}{\partial r}\varphi_2(s(t, r), \hat{t}_0(r), \chi(r)) = \frac{\partial \varphi_2}{\partial s}\frac{\partial s}{\partial r} + \frac{\partial \varphi_2}{\partial \hat{t}_0}\frac{\partial \hat{t}_0}{\partial r} + \frac{\partial \varphi_2}{\partial \chi}\frac{\partial \chi}{\partial r},$$

where

$$s(t, r) := t - \tau(r), \tag{7.8}$$

$$\hat{t}_0(r) := \tau(r), \tag{7.9}$$

$$\chi(r) := \varphi_1\left(-s(t_0, r), t_0, \psi^0\right). \tag{7.10}$$

Now, we observe that

$$\frac{\partial \varphi_2}{\partial s} = F_2(t, \varphi_2(s(t, r), \tau(r), \chi(r))), \tag{7.11}$$

$$\frac{\partial \chi}{\partial r} = \frac{\partial \varphi_1}{\partial s}\tau'(r) + \frac{\partial \varphi_1}{\partial \psi^0}\frac{\partial \psi^0}{\partial r}, \tag{7.12}$$

where $\tau'(r) = \frac{d\tau}{dr}(r)$ and where

$$\frac{\partial \varphi_1}{\partial s} = F_1(t, \varphi_1(-s(t_0, r), t_0, \psi^0)).$$

Taking the limit for $t \to \tau(r)^{\pm}$, we have:

$$\lim_{t \to \tau(r)^+} s(t, r) = 0,$$

hence, since

$$\varphi_2(0, \tau(r), \chi(r)) = \chi(r), \tag{7.13}$$

we then obtain that (7.11) yields in the limit

$$\frac{\partial \varphi_2}{\partial s} = F_2(\tau(r), \chi(r)) = F_2(\tau(r), \varphi_1(-s(t_0, r), t_0, \psi^0)).$$

Moreover, from (7.13) we have:

$$\lim_{t \to \tau(r)^+} \frac{\partial \varphi_2}{\partial \hat{t}_0} = \frac{\partial \chi}{\partial \hat{t}_0} = 0,$$

and

$$\lim_{t \to \tau(r)^+} \frac{\partial \varphi_2}{\partial \chi} = \frac{\partial \chi}{\partial \chi} = I.$$

Therefore, the right-hand side of (7.5) in the limit $t \to \tau(r)^+$ can be written as

$$\begin{aligned} &- \quad F_2(\tau(r), \varphi_1(-s(t_0, r), t_0, \psi^0))\tau'(r) \\ &+ \quad F_1(\tau(r), \varphi_1(-s(t_0, r), t_0, \psi^0))\tau'(r) + \frac{\partial \varphi_1}{\partial \psi^0} \frac{\partial \psi^0}{\partial r}. \end{aligned} \tag{7.14}$$

From the assumption that the system vector field is continuous when $t = \tau(r)$, continuity of $w(t, r)$ with respect to t is then immediately established by comparing (7.7) and (7.14).

Now, we turn again our attention to equation (7.3). Fixing r to any value between 0 and 1, the Jacobian can be calculated and (7.3) can be solved to obtain (in the sense of Lebesgue):

$$\begin{aligned} &w(t+h, r) = w(t, r) + \int_t^{t+h} A(\vartheta, \psi(\vartheta, r)) w(\vartheta, r) d\vartheta = \\ &= w(t, r) + A(t, \psi(t, r)) w(t, r) h + \\ &\int_t^{t+h} \left(A(\vartheta, \psi(\vartheta, r)) w(\vartheta, r) - A(t, \psi(t, r)) w(t, r) \right) d\vartheta, \qquad a.e.\ t, \end{aligned}$$

with h being a positive scalar.

Thus, from the above expression we have

$$\begin{aligned} |w(t+h, r)| \leq \quad & ||I + hA(t, \psi(t, r))|| \, |w(t, r)| \\ + \quad & \int_t^{t+h} |A(\vartheta, \psi(\vartheta, r)) w(\vartheta, r) - A(t, \psi(t, r)) w(t, r)| \, d\vartheta. \end{aligned} \tag{7.15}$$

Then, subtracting $|w(t, r)|$ from both sides of the equation and dividing by h we obtain

$$\begin{aligned} &\tfrac{1}{h} \left(|w(t+h, r)| - |w(t, r)| \right) \leq \tfrac{1}{h} \left(\|I + hA(t, \psi(t, r))\| - 1 \right) |w(t, r)| + \\ &\tfrac{1}{h} \int_t^{t+h} |A(\vartheta, \psi(\vartheta, r)) w(\vartheta, r) - A(t, \psi(t, r)) w(t, r)| \, d\vartheta, \qquad a.e.\ t. \end{aligned}$$

Thus, taking the limit as $h \searrow 0$ yields:

$$\frac{d}{dt}\left|w(t,r)\right| \leq \mu\left(A(t,\psi(t,r))\left|w(t,r)\right|,\qquad a.e., \tag{7.16}$$

and so:

$$\frac{d}{dt}\left|w(t,r)\right| \leq -c\left|w(t,r)\right|,\qquad a.e.$$

Notice that the above expression holds for all those pairs t and r where the Jacobian $A(\cdot)$ is defined. Let now $M(t) := -c(t-t_0)$, from the above expression it follows that:

$$\frac{d}{dt}\left(\left|w(t,r)\right| e^{-M(t)}\right) \leq 0,\qquad a.e.$$

Now, since $e^{-M(t)}$ is an increasing function and since the function $t \mapsto w(t,r)$ is continuous, the above inequality implies that:

$$\left|w(t,r)\right| \leq \left|w(t_0,r)\right| e^{-c(t-t_0)} \leq K\left|\xi-\zeta\right| e^{-c(t-t_0)}.$$

As the function $\psi(t,r)$ is continuous and, for all t, the function $w(t,r)$ is defined for almost all r, we have:

$$\psi(t,1)-\psi(t,0) = \int_0^1 w(t,s)ds.$$

Thus:

$$\left|x(t)-y(t)\right| \leq K\left|\xi-\zeta\right| e^{-c(t-t_0)},$$

and the theorem remains proved. □

Obviously, if $\mathcal{C}$ is forward-invariant, then trajectories rooted in $\mathcal{C}$ will exponentially converge towards each other.

7.2.2 Contraction of TSS

The conditions used to prove contraction of PWSC systems can be immediately extended to generic systems affected by time-dependent switchings as detailed below.

Theorem 7.2.2. *Consider an invariant K-reachable set $\mathcal{C} \subseteq \mathrm{D}$ and a time-dependent switching system as in Definition 2.3.3. Suppose that:*

1. *it fulfills conditions for the existence and uniqueness of a Caratheodory solution given in Section 2.4;*
2. *the function $(t,x) \mapsto f(t,x,\sigma)$ is continuous for all $x \in \mathcal{C}$, for all $t \geq t_0$ and for all $\sigma \in \Sigma$;*
3. *the function $x \mapsto f(t,x,\sigma)$ is continuously differentiable for all $x \in \mathbb{R}^n$, for all $t \geq t_0$ and for all $\sigma \in \Sigma$;*
4. *there exist a unique matrix measure such that*

$$\mu\left(\frac{\partial f}{\partial x}(t,x,\sigma)\right) \leq -c_\sigma,$$

for all $x \in \mathcal{C}$, for all $t \geq t_0$ and for all $\sigma \in \Sigma$, with c_σ belonging to a set of positive real scalars (in what follows, we will define $c := \min_{\sigma\in\Sigma} c_\sigma$).

Then, for every two solutions $x(t) = \varphi(t, t_0, \xi)$ *and* $y(t) = \varphi(t, t_0, \zeta)$ *with* $\xi, \zeta \in \mathcal{C}$*, it holds that:*

$$|x(t) - y(t)| \le K e^{-c(t-t_0)} |\xi - \zeta|,$$

Proof. The proof follows similar steps to that of Theorem 7.2.1. In particular, given points $x(t_0) = \xi$ and $y(t_0) = \zeta$ and a smooth curve $\gamma : [0,1] \mapsto \mathcal{C}$ such that $\gamma(0) = \xi$ and $\gamma(1) = \zeta$, we can consider $\psi(t, r) := \varphi(t - t_0, t_0, \gamma(r))$ as the solution of (2.5) rooted in $\psi(t_0, r) = \gamma(r)$, with $r \in [0, 1]$. Let

$$w(t,r) := \frac{\partial \psi}{\partial r}, \qquad a.e.\ t.$$

As in the proof of Theorem 7.2.1, we can write:

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{\partial \psi}{\partial t}\right) = \frac{\partial}{\partial r}\left(f(t, \psi(t,r), \sigma)\right), \qquad a.e.\ t,$$

and then

$$\frac{\partial}{\partial t} w(t,r) = A(t, \psi(t,r), \sigma) w(t,r),\ \ a.e.\ t,$$

with $A = \frac{\partial}{\partial x} f(t, x, \sigma)$ being the Jacobian of $f(t, x, \sigma)$ for almost all $t \ge t_0$. Notice that, differently from the case of PWSC systems, here the Jacobian is discontinuous only with respect to time t due the fact that the function $\sigma(t)$ is piecewise constant. However, we can show that the function $t \mapsto w(t, r)$ is continuous by considering again (7.7) and (7.14). In this case, the switching instant $\tau(r)$ is independent from r and therefore all terms containing $\tau'(r)$ in (7.14) cancel out. The equality of (7.7) and (7.14) then immediately follows and the rest of proof becomes identical to that of Theorem 7.2.1. □

Notice that, following the proof of Theorem 7.2.1, it is also very easy to prove the result below, related to asymptotically (but not necessary exponentially) incrementally stable TSS.

Theorem 7.2.3. *Consider an invariant* K*-reachable set* $\mathcal{C} \subseteq \mathrm{D}$ *and a time-dependent switching system as in Definition 2.3.3. Suppose that hypothesis (1)-(3) of Theorem 7.2.2 are satisfied and that there exists a unique matrix measure such that*

$$\lim_{t \to +\infty} \int_{t_0}^{t} \mu\left(A(\vartheta, x(\vartheta), \sigma(\vartheta))\right) d\vartheta = -\infty \tag{7.17}$$

for all the trajectories $x(t) \in \mathcal{C}$*, where* $A(t, x, \sigma) = \frac{\partial}{\partial x} f(t, x, \sigma)$.

Then, for every two solutions $x(t) = \varphi(t, t_0, \xi)$ *and* $y(t) = \varphi(t, t_0, \zeta)$ *with* $\xi, \zeta \in \mathcal{C}$*, it holds that:*

$$\lim_{t \to +\infty} |x(t) - y(t)| = 0.$$

Proof. As done for Theorem 7.2.2, the switching signal $\tau(r)$ does not depend on r and $\tau'(r) = 0$. So, also in this case we follow the same steps of the proof of Theorem 7.2.1 until inequality (7.16). Integrating such inequality in an time interval $[t_1, t_2]$, where t_1 and t_2 are two consecutive switching instants we obtain the Coppel inequality [208]:

$$|w(t,r)| \le |w(t_1, r)|\, e^{\int_{t_1}^{t} A(\vartheta, x(\vartheta), \sigma(\vartheta)) d\vartheta}, \quad t \in [t_1, t_2].$$

Notice that, since $t \mapsto w(t,r)$ is a continuous function, then it is easy to verify that it holds:

$$|w(t,r)| \leq |w(t_0,r)|\, e^{\int_{t_0}^{t} A(\vartheta,x(\vartheta),\sigma(\vartheta))d\vartheta}, \quad \forall t.$$

Notice that the function $|w(t_0,r)|\, e^{\int_{t_0}^{t} A(\vartheta,x(\vartheta),\sigma(\vartheta))d\vartheta}$ is continuous with respect both the variables t and r. Now, considering that:

$$\psi(t,1) - \psi(t,0) = \int_0^1 w(t,s)ds,$$

and since $\psi(t,1) - \psi(t,0) = x(t) - y(t)$, we can bound the above expression as:

$$|x(t) - y(t)| \leq \int_0^1 |w(t,s)|ds \leq \int_0^1 |w(t_0,s)|\, e^{\int_{t_0}^{t} A(\vartheta,\psi(\vartheta,s),\sigma(\vartheta))d\vartheta} ds.$$

Piking the limit for $t \to +\infty$ and remembering the continuity of the right hand side of the above inequality we prove the result. □

The condition (7.17) is not always easy to check, since it is given as a condition on the flow of the system. Some corollaries can be helpfully used instead of Theorem 7.2.3. The following corollary appears in [168] for linear time switching systems and it is given here for generic time switching systems.

Corollary 7.2.1. *Consider an invariant K-reachable set $\mathcal{C} \subseteq \mathbb{R}^n$ and a T-periodic time-dependent switching system as in Definition 2.3.3 and suppose that such system fulfills conditions for the existence and uniqueness of a Caratheodory solution given in Section 2.4. Suppose that the switching system has a finite number of switching $\sigma \in \Sigma = \{1, 2, \ldots, M\}$ in the period T. Suppose that there exists a matrix measure $\mu(\cdot)$ such that*

$$\sum_{i=1}^{M} \delta_T^{(i)} c_i < 0, \tag{7.18}$$

where $\delta_T^{(1)}, \ldots, \delta_T^{(M)}$ are the duty cycles of the modes of the switching system in the period T, and $c_1, \ldots, c_M \in \mathbb{R}$ are scalars such that

$$c_i = \sup_{t \geq t_0, x \in \mathcal{C}} \mu\left(\frac{\partial f}{\partial x}(t,x,i)\right),$$

then the system is incrementally asymptotically stable.

Proof. The proof follows straightforward from Theorem 7.2.3 since it is immediate to verify that condition (7.18) implies condition (7.17). □

Corollary 7.2.1 shows how more generic conditions of incremental stability (although not asymptotic) can be considered for switching system thanks to Theorem 7.2.3. More details about (linear) switching systems can be found in the good work [168] and in the references therein, where averaging techniques are considered.

Remark 7.2.1.

- *Our results on the contraction of PWSC systems can be interpreted following the approach presented in [33] where the asymptotic stability of piecewise linear systems obtained by the continuous matching of two stable linear systems is discussed.*

Specifically, under the conditions of Theorem 7.2.1, we can state that the continuous matching of any number of nonlinear contracting vector fields (with the same matrix measure) is also contracting. *As in the case of asymptotic stability discussed in [33], guaranteeing incremental stability of switched systems is not trivial, even when they are obtained by continuously matching contracting vector field. Thus, the sufficient conditions derived in this chapter can be useful for the analysis of incremental stability in switched systems and the design of stabilizing switched control inputs.*

- *As already said in Section 7.1, the results reported so far do not include the case of differential equations with discontinuous right-hand side or Filippov systems where sliding motion is possible (see Section 2.6). This is ongoing work and preliminary results will be presented in Chapter 8.*

Stability of PWL systems

Using the concept of contraction for PWS systems, it is possible to easily prove the following result to assess the stability of piecewise linear systems (PWL) of the form

$$\dot{x} = A(t, \sigma)x, \tag{7.19}$$

where $x \in \mathbb{R}^n$ and $\sigma(t) : [0, +\infty) \to \Sigma$ is the switching signal with Σ being a finite index set. Several stability results for this class of systems are available in the literature (see [120] for an extensive survey). A classical approach is that of finding conditions on the (finite) set of matrices $A(t, \sigma)$ guaranteeing the existence of some common quadratic Lyapunov function (CQLF). In [120] (Theorem 8, p. 311), it is proven that the origin is a globally asymptotically stable equilibrium of (7.19) if and only if there exist a full column rank matrix $L \in \mathbb{R}^{m\times n}$ with $m \geq n$ and a family of matrices $\bar{A}_i \in \mathbb{R}^{m\times m}$ such that $\mu_\infty \left(\bar{A}_i\right) < 0$ for all $i = \sigma \in \Sigma$.

Here we show that a related stability condition can be immediately obtained by applying contraction theory. Indeed, we can prove the following result.

Corollary 7.2.2. *Given a piecewise linear system of the form (7.19), if the matrices $A(t, \sigma)$ are bounded and measurable for any σ and there exist some matrix measure such that*

$$\mu\left(A(t, \sigma)\right) \leq -c, \quad c > 0 \quad \forall t \in \mathbb{R}^+, \quad \forall \sigma \in \Sigma, \tag{7.20}$$

then, all solutions of (7.19) converge asymptotically towards the origin independently from the switching sequence.

Proof. Under the hypotheses, system (7.19) satisfies Theorem 7.3.1 and therefore is contracting with all of its trajectories converging towards each other. Since, $x(t) = 0$ is also a trajectory of (7.19), the proof immediately follows. □

7.3 Partial contraction of PWSC and TSS with synchronization applications

Often in applications, it is desirable to prove (or certify) that, at steady state, all trajectories of a given system exhibit some property regardless of their initial conditions.

In the case of smooth dynamical systems, the concept of partial contraction was introduced in [210] to solve this problem. The idea is to introduce an appropriately constructed auxiliary or *virtual system*, embedding the solutions of the system of interest as its particular solutions. If the virtual system is proved to be contracting, then all of its solutions will converge towards a unique trajectory. In turn, this imples that all trajectories of the system of interest, embedded in the virtual system by construction, will also converge towards this solution.

The most notable application of partial contration is its use to prove convergence of trajectories of all nodes of a network of dynamical systems towards each other as for example is required in synchronization or coordination problems. In that case, the virtual system is constructed so that trajectories of the network nodes are its particular solutions. Proving contraction of the virtual system then implies convergence of all node trajectories towards the same synchronous evolution (see [210, 192, 183, 182, 180] for further details and applications).

Using the extension of contraction to switched Caratheodory systems presented above, we can also extend partial contraction to this class of systems. In particular, we can prove the following result for PWS systems.

Theorem 7.3.1. *Consider a PWSC of the form (7.1) and assume that there exists some system*

$$\dot{z} = v(t, z, x), \tag{7.21}$$

such that:

- $v(t, x, x) = f(t, x)$*;*
- $v(t, z, x)$ *is contracting in the Caratheodory sense with respect to* z *and for any* x*.*

Then, all the solutions $z(t)$ *of (7.1) converge towards* $x(t)$*, i.e.*

$$|x(t) - z(t)| \to 0, \quad t \to +\infty.$$

System (7.1) is said to be *partially contracting*, because its trajectories converge to the trajectories of a contracting system, while system (7.21) is termed as virtual system.

Proof. Indeed, we only need to observe that by construction any solution of (7.1), say $x(t)$, is also a solution of the virtual system. Now, since (7.21) is contracting, then all of its solutions will converge towards x. This in turn implies that

$$|x(t) - z(t)| \to 0 \qquad a.e.$$

as $t \to +\infty$. □

The key point of a such result is that of constructing a contracting system which embeds the solutions of the real system. In some special case, see e.g. [183], the dimensionality of the virtual system is lower than that of the real system of interest: this is typically the case of systems with symmetries, such as *Quorum Sensing* networks. A notable example of use of virtual system can be found in [184]. We also remark that Theorem 7.3.1 can be straightforwardly extended to time-dependent switched systems. The proof follows exactly the same steps of those used to prove Theorem 7.3.1 and hence it is omitted here for the sake of brevity.

7.3.1 Time\state varying linear systems

As an example illustrating the key features of partial contraction and virtual systems, consider a PWSC system of the form

$$\dot{x} = L(t, x)x. \tag{7.22}$$

Notice that such a system may e.g. model a networked control system or a network of biochemical reactions.

We assume that the system is not contracting. That is, the Jacobian matrix

$$\frac{\partial L}{\partial x}x + L(t, x),$$

does not have any uniformly negative matrix measure. We also assume that there exists a uniformly negative matrix measure for $L(t, x)$, i.e.

$$\exists\ \mu\ :\ \mu\left(L(t, x)\right) \leq -c, \qquad c > 0 \quad a.e.$$

Clearly, in this case, system (7.22) is not contracting nevertheless Theorem 7.3.1 can be used to show that, at steady state, all trajectories of (7.22) converge towards a unique solution. In particular, consider the system

$$\dot{z} = v(t, z, x) = L(t, x)z,$$

where x, the state variable of the original system, is seen as an external input. It is straightforward to check that

$$v(t, x, x) = L(t, x)x = \dot{x},$$

and hence it is a virtual system in the sense of Theorem 7.3.1. Moreover, the Jacobian matrix of the virtual system is simply

$$J(t, x) = \frac{\partial L(t, x)z}{\partial z} = L(t, x).$$

Since we assumed that there exists a uniformly negative matrix measure for $L(t, x)$, the virtual system is contracting for any x. Therefore, all of its solutions will converge to a unique trajectory, say x^*, such that:

$$\dot{z}^* = L(t, x)z^*.$$

Since the solutions of the real system are also particular solutions of the virtual system, it follows that

$$|x(t) - z^*(t)| \to 0 \quad a.e.$$

That is, all solutions of the real system will also converge towards z^* and, hence, towards each other. Furthermore, since $z^* = 0$ is a solution of the virtual system, this also prove global asymptotic stability of the origin.

7.3.2 Synchronization using a virtual system

Partial contraction, as we said, is a very useful tool to synchronize nonlinear systems [210, 192]. Here we use the same approach in [210] in order to synchronize PWSC systems using a nonlinear coupling. It is worth mentioning that the extension of the contraction theory to PWSC systems allows us, as we will see, to couple smooth or PWSC systems with a possibly nonsmooth coupling and, for this reason, represents an extension to classical synchronization strategies existing in literature. It is also important to notice that contraction theory, conversely to other existing tools about convergence of trajectories of systems, allows to develop the idea of virtual system in a way useful to study synchronization.

Let us consider a pair of identical PWSC time switching systems of equation:

$$\begin{aligned}\dot{x_1} &= f(t, x_1) + u_1 \\ \dot{x_2} &= f(t, x_2) + u_2\end{aligned}$$

coupled in the following way:

$$\begin{aligned}\dot{x_1} &= f(t, x_1) + h_1(t, x_1, x_2) \\ \dot{x_2} &= f(t, x_2) + h_2(t, x_2, x_1)\end{aligned}, \tag{7.23}$$

where $h_i(t, x, y)$, with $i = 1, 2$ is a (possibly) time switching function and a (possibly) PWSC function with respect to the variable x. For the coupled nonsmooth systems (7.23) the following result holds.

Theorem 7.3.2. *If there exists a nonsmooth function* $H(t, z, x_1, x_2) : [t_0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$ *such that:*

1. $H(t, x_1, x_1, x_2) = h_1(t, x_1, x_2)$*;*
2. $H(t, x_2, x_1, x_2) = h_2(t, x_2, x_1)$*;*
3. *the PWSC system* $\dot{z} = f(t, z) + H(t, z, x_1, x_2)$ *is contracting with respect to* z $\forall t, \forall x_1, \forall x_2$*; then the coupled system* (7.23) *synchronizes and so:*

$$\lim_{t \to +\infty} \|x_1(t) - x_2(t)\| = 0$$

Proof. The proof directly follows from Theorem 7.3.1 considering the system $\dot{z} = f(t, z) + H(t, z, x_1, x_2)$ as virtual system for the coupled pair (7.23). □

7.4 Convergence of networks of time-switching systems

Contraction analysis can be an invaluable tool also to study convergence of networked time switching systems by looking at transversal contraction of the synchronization manifold. In this section we establish a link with Part I of the thesis where synchronization of discontinuous network is studied. In particular, here we use the extension of contraction to Caratheodory systems presented in this chapter to derive conditions guaranteeing convergence of a network of diffusively coupled switched linear systems. Specifically, we consider a network of the form:

$$\dot{x}_i = A(\sigma(t))x_i + \Gamma \sum_{j \in N_i} [x_j - x_i], \tag{7.24}$$

where $x_i \in \mathbb{R}^n$ represents the state vector of node i, N_i denotes the set of the neighbors of the i-th node whose cardinality (i.e. the degree of the i-th network node) is denoted with d_i. In the above equation, Γ is a coupling matrix, often termed as inner-coupling matrix in the literature. In what follows the eigenvalues of the network Laplacian matrix (L) are denoted with λ_i; λ_2 being the smallest non-zero Laplacian eigenvalue (algebraic connectivity). We assume $A(\sigma(t))$ to be bounded and measurable.

We will now show that, by using contraction, a sufficient condition can be derived ensuring all the solutions of the network nodes globally exponentially converge, almost everywhere, towards the n-dimensional linear subspace[1] $\mathcal{M}_s := \{x_1 = \ldots = x_N\}$. In what follows, we will denote by $s(t)$ the common asymptotic behavior of all nodes on $\mathcal{M}_s$. Note that $s(t)$ is obviously a solution of each isolated node of (7.24), i.e. $\dot{s}(t) = A(\sigma(t))s(t)$. We will also say that the network nodes are coordinated (or that the network is coordinated) if

$$\lim_{t\to\infty} |x_i(t) - s(t)| = 0, \quad a.e.$$

In the special case where $s(t)$ exhibits an oscillatory behavior, we will say that all network nodes are synchronized (or that the network is synchronized).

Theorem 7.4.1. *The trajectories of all nodes in the network (7.24) exponentially converge towards each other almost everywhere (i.e., the network is coordinated a.e.) if (i) the topology of the network is connected and (ii) there exist some matrix measure, μ, such that:*

$$\mu\left(A(\sigma(t)) - \lambda_2\Gamma\right) \leq -c, \quad c > 0,$$

for all $\sigma \in \Sigma$ and for almost all t.

Before presenting the proof of the Theorem, we report here two useful results, [14].

Lemma 7.4.1. *Let $\otimes$ denote the Kronecker product. The following properties hold:*

- $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$;
- *if A and B are invertible, then* $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$;

Lemma 7.4.2. *For any $n \times n$ real symmetric matrix, A, there exist an orthogonal $n \times n$ matrix, Q, such that*

$$Q^T A Q = U, \tag{7.25}$$

where U is an $n \times n$ diagonal matrix.

Theorem 7.4.1. . Define:

$$X := \left[x_1^T, \ldots, x_N^T\right]^T, \quad S := 1_N \otimes s, \quad E := X - S,$$

where 1_N denotes the N-dimensional vector consisting of all 1s. (Notice that such a vector spans $\mathcal{M}_s$.) The network dynamics can then be written as:

$$\dot{X} = (I_N \otimes A(\sigma(t)))X - (L \otimes \Gamma)X,$$

so that the error dynamics is described by

$$\dot{E} = (I_N \otimes A(\sigma(t)))E - (L \otimes \Gamma)X \tag{7.26}$$

[1] It is straightforward to check that this subspace is flow invariant for the network dynamics.

Notice that network coordination is attained if the dynamics of (7.26) transversal to $\mathcal{M}_s$ is contracting. Furthermore, notice that

$$\begin{aligned}&(L\otimes\Gamma)X=(L\otimes\Gamma)(E+S)=(L\otimes\Gamma)E+(L\otimes\Gamma)S=\\&(L\otimes\Gamma)E+(L\otimes\Gamma)(1_N\otimes s)=(L\otimes\Gamma)E,\end{aligned}$$

where the last equality follows from Lemma 7.4.1 and from the fact that $L\cdot 1_N=0$, since the network is connected by hypothesis. Thus, from (7.26), we have:

$$\dot{E}=(I_N\otimes A(\sigma(t)))E-(L\otimes\Gamma)E. \tag{7.27}$$

Since L is symmetric, by means of Lemma 7.4.2 we have that there exist an $N\times N$ orthogonal matrix Q ($Q^TQ=I_N$) such that:

$$\Lambda=Q^TLQ,$$

where Λ is the $N\times N$ diagonal matrix having the Laplacian eigenvalues as its diagonal elements.

Now, considering the following coordinate transformation:

$$Z=(Q\otimes I_n)^{-1}E,$$

equation (7.27) can be recast as

$$\dot{Z}=(Q\otimes I_n)^{-1}\left[(I_N\otimes A(\sigma(t)))-(L\otimes\Gamma)\right](Q\otimes I_n)Z.$$

Then, using Lemma 7.4.1, we have:

$$\begin{aligned}&(Q\otimes I_n)^{-1}(I_N\otimes A(\sigma(t)))(Q\otimes I_n)=\\&\left(Q^{-1}\otimes I_n\right)(I_N\otimes A(\sigma(t)))(Q\otimes I_n)=\\&\left(Q^{-1}\otimes A(\sigma(t))\right)(Q\otimes I_n)=\\&(I_N\otimes A(\sigma(t))).\end{aligned}$$

Analogously:

$$\begin{aligned}&(Q\otimes I_n)^{-1}(L\otimes\Gamma)(Q\otimes I_n)=\\&\left(Q^{-1}\otimes I_n\right)(L\otimes\Gamma)(Q\otimes I_n)=\\&\left(Q^{-1}L\otimes\Gamma\right)(Q\otimes I_n)=\\&Q^{-1}LQ\otimes\Gamma=\\&\Lambda\otimes\Gamma.\end{aligned}$$

That is, the network dynamics can be written as:

$$\dot{Z}=\left[I_N\otimes A(\sigma(t))-\Lambda\otimes\Gamma\right]Z, \tag{7.28}$$

or equivalently:

$$\dot{z}_i=\left[A(\sigma(t))-\lambda_i\Gamma\right]z_i,\quad i=1,\dots,N,\quad z_i\in\mathbb{R}^n.$$

Now, recall that the eigenvector associated to the smallest eigenvalue of the Laplacian matrix, i.e. $\lambda_1=0$, is 1_N and spans $\mathcal{M}_s$. Therefore, the dynamics along $\mathcal{M}_s$ is given by

$$\dot{z}_1=\left[A(\sigma(t))\right]z_1,$$

i.e. it is a solution of the uncoupled nodes' dynamics. The dynamics transversal to the invariant subspace is given by:

$$\dot{z}_i=\left[A(\sigma(t))-\lambda_i\Gamma\right]z_i,\quad i=2,\dots,N.$$

Obviously $[A(\sigma(t)) - \lambda_i \Gamma]$ is bounded and measurable. Thus, by virtue of Corollary 7.2.2, all node trajectories globally exponentially converge a.e. towards $\mathcal{M}_s$, if all of the above dynamics are contracting. Now, it is straightforward to check that such a condition is fulfilled if

$$\dot{z}_2 = [A(\sigma(t)) - \lambda_2 \Gamma]\, z_2$$

is contracting. As this is true from the hypotheses, the result is then proved. □

7.5 Examples

The extension of contraction to Caratheodory systems is a flexible tool that can be used both for analysis that for design of coupling protocols. Here we illustrate by means of some representative examples the results of this chapter. In particular, examples 7.5.1 and 7.5.2 show how contraction can be used as an analysis tool, while in examples 7.5.3, 7.5.4 and 7.5.5 contraction is used as design tool for smooth and nonsmooth protocols able to synchronize coupled nonsmooth systems.

7.5.1 Incremental stability of PWSC and TSS systems

As a first example we take a nonlinear model adapted from [181] and modified via a simple nonsmooth state feedback. The aim is to illustrate how, even considering a nonsmooth action, such system still can be proved to be contractive as in [181] using measure $\mu_\infty(\cdot)$ and Theorem 7.2.1. Specifically, we consider the nonlinear model:

$$\dot{x} = u(t)\,(\alpha - x) - \delta x + \delta y + v(t), \tag{7.29}$$

$$\dot{y} = -k_1 y + k_2\,(\gamma - y)\,(x - y)\,, \tag{7.30}$$

where the PWS term $v(t)$ is given by:

$$v(t) = \begin{cases} 0 & \textit{if } x - y \le h, \\ -\beta\,[x - y - h] & \textit{if } x - y > h. \end{cases}$$

To prove contraction, and hence global incremental stability of this switched system, we need to derive the Jacobian which, in this case, is the discontinuous function:

$$J(x,t) = \begin{cases} J_s & \textit{if } x - y \le h,, \\ J_s + J_{ns} & \textit{if } x - y > h, \end{cases}$$

where

$$J_s = \begin{bmatrix} -u(t) - \delta & \delta \\ k_2(\gamma - y) & -k_1 + k_2(-\gamma - x + 2y) \end{bmatrix},$$

and

$$J_{ns} = \begin{bmatrix} -\beta & \beta \\ 0 & 0 \end{bmatrix}.$$

Using $\mu_\infty(\cdot)$ as a matrix measure, we find that $\mu_\infty(J_s)$ is negative if the following inequalities hold:

$$-\,u(t) - \delta + |\delta| < -c_1; \tag{7.31}$$

$$-\,k_1 + k_2(-\gamma - x + 2y) + |k_2(\gamma - y)| < -c_2; \tag{7.32}$$

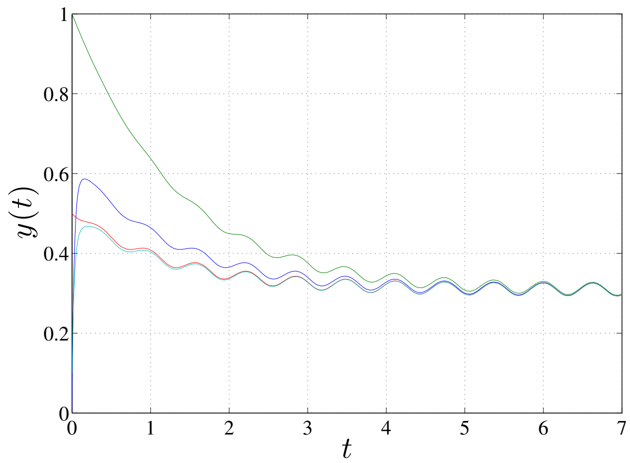


Figure 7.1: Convergence of trajectories of the transcriptional module starting from different initial conditions towards the same unique periodic orbit. Simulation were carried out with the following parameter values: $k_1 = 0.5$, $k_2 = 5$, $\alpha = 1$, $\gamma = 1$, $\delta = 20$, $\beta = 1$, $h = 0.01$. The periodic input was set to $1.5 + 2\sin(10t)$.

for $c_1, c_2 > 0$.
As shown in [181], the first inequality is always satisfied as the system parameters and the periodic input are assumed to be positive. Furthermore, taking into account that, for physical reasons, the term $\gamma - y \geq 0$, it is easy to prove that inequality (7.32) is also fulfilled.

We now have to consider the effect of the switching by looking at the measure of the matrix $J_s + J_{ns}$. It is immediate to see that, $\mu_\infty(J_s + J_{ns})$ is also negative if inequalities (7.31) and (7.32) are satisfied. Hence, according to Theorem 7.2.1, the switched biochemical network under investigation is contracting and is therefore incrementally stable. This is confirmed by the numerical simulation reported in Figure 7.1.

Figure 7.2 shows that, as expected, the switching signals associated to trajectories starting from different initial conditions also synchronize asymptotically.

Note that this example clearly illustrate how convenient can be to consider matrix measures different from the $\mu_{2,\Theta}(\cdot)$, which can be proven to be equivalent to the existence of a common quadratic Lyapunov function $P = \Theta^T\Theta$ on the Jacobians of the virtual system [124]. Indeed, in this case, the existence of a negative definite measure $\mu_{2,\Theta}(\cdot)$ can be expressed in terms of the existence of a symmetric matrix $P > 0$ such that the following Lyapunov condition is satisfied:

$$\begin{cases} \begin{bmatrix} -u(t) - \delta & \delta \\ k_2(\gamma - y) & -k_1 + k_2(-\gamma - x + 2y) \end{bmatrix}^T P + P \begin{bmatrix} -u(t) - \delta & \delta \\ k_2(\gamma - y) & -k_1 + k_2(-\gamma - x + 2y) \end{bmatrix} < 0, & \text{if } x - y \leq h \\ \begin{bmatrix} -u(t) - \delta - \beta & \delta + \beta \\ k_2(\gamma - y) & -k_1 + k_2(-\gamma - x + 2y) \end{bmatrix}^T P + P \begin{bmatrix} -u(t) - \delta & \delta \\ k_2(\gamma - y) & -k_1 + k_2(-\gamma - x + 2y) \end{bmatrix} < 0, & \text{if } x - y \leq h \end{cases}$$

It is possible to see that the previous condition is in general hard to investigate if parametric expressions of u, δ, γ are considered, making it hard to use Euclidean measures.

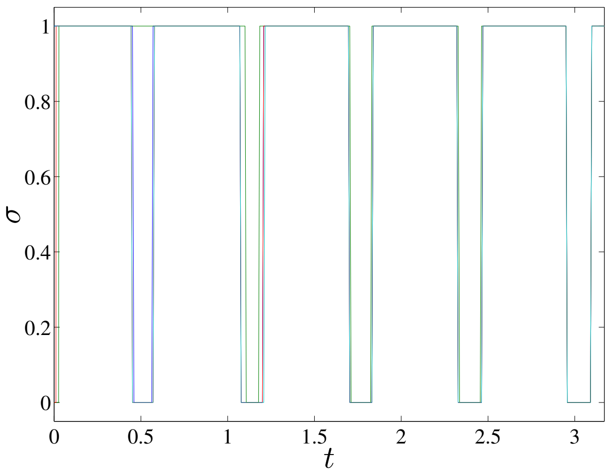


Figure 7.2: Synchronization of the switching signals associated to trajectories of the modified transcriptional module starting from different initial conditions

7.5.2 Stability of PWL systems

As an example of application of Corollary 7.2.2 we can take the piecewise linear system:

$$\dot{x} = A(\sigma)x, \quad \sigma \in \{1, 2\}, \tag{7.33}$$

with

$$A(1) = \begin{pmatrix} -1.0 & 1.5 \\ 0.8 & -3.0 \end{pmatrix}, \quad A(2) = \begin{pmatrix} -3.0 & 1.0 \\ 2.0 & -1.5 \end{pmatrix}.$$

Note that using the matrix measure μ_1 induced by the 1-norm, we have $\mu_1[A(1)] < 0$ and $\mu_1[A(2)] < 0$. Hence, it is immediate to prove asymptotic convergence of all solutions towards each other and onto the origin for arbitrarily switching signal $\sigma(t)$. We wish to emphasize that the result based on contraction embeds as a special case the stability condition reported in [120]. Indeed, setting $L = I$ and $\bar{A}_i = A(\sigma)$ in Theorem 8, p. 311 in [120] is equivalent to using Corollary 7.2.2 with the matrix measure μ_∞. Moreover, the proof based on contraction can also be extended to nonlinear switched systems.

7.5.3 Synthesis of a continuous coupling

Let consider two identical PWSC systems $\dot{x} = f(t, x, u)$ and $\dot{y} = f(t, y, w)$ from [190], where

$$\begin{aligned} \dot{x}_1 &= -x_1 - |x_1| + x_2 + u_1 \\ \dot{x}_2 &= -3x_1 + x_2 + u_2 + \rho \end{aligned}, \tag{7.34}$$

where ρ is a constant parameter. Analogously, we define the system $\dot{y} = f(t, y, w)$. Using the coupling protocols:

$$u = K(y - x), \qquad w = K(x - y),$$

respectively for the x-system and for the y-system, it is possible to prove that the pair synchronizes for appropriate values of K. For example, it is sufficient to consider the measure $\mu_1(\cdot)$ to prove contraction of the virtual system:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_1 - |z_1| + z_2 \\ -3z_1 + z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \rho \end{bmatrix} - 2K \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + K \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + K \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

when the coupling gain matrix is chosen as:

$$K = \begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix}.$$

Numerical simulations have been obtained by setting $\rho = -1$. In this way it is possible to prove that system (7.34) exhibits a stable periodic orbit analogous to a supercritical Hopf bifurcation in smooth systems [190]. In Figure 7.3(a) and Figure 7.3(b) it is possible to see the evolution of the first state component of the pair of systems (7.34) respectively in the case of uncoupled and coupled network from randomly chosen initial conditions.

7.5.4 Synthesis of a piecewise smooth coupling

Here we consider the following two identical PWSC systems $\dot{x} = f(t, x, u)$ and $\dot{y} = f(t, y, w)$, also taken from [190] as in the previous example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{cases} \begin{bmatrix} 1.3\eta + 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \rho \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, & \text{if } x_1 \leq 0 \\ \begin{bmatrix} -2 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \xi x_1^2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \rho \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, & \text{if } x_1 > 0 \end{cases}, \tag{7.35}$$

and analogously for the system $\dot{y} = f(t, y, w)$. Here the constant ρ and η are bifurcation parameters which has been showed [190] to be able to give both a nonsmooth Hopf bifurcation that a saddle-node bifurcation of a periodic orbit, while parameter ξ is assumed to be positive. Asymptotic synchronization of two systems of equation (7.35) can be guaranteed considering, for the x-system, the nonsmooth coupling input:

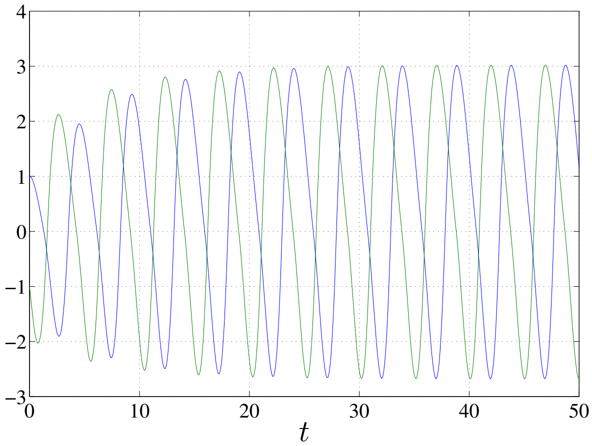
$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{cases} \begin{bmatrix} k(y_1 - x_1) \\ 5(y_2 - x_2) \end{bmatrix} & \text{if } x_1 \leq 0 \\ \begin{bmatrix} h(y_1^2 - x_1^2) \\ 5(y_2 - x_2) \end{bmatrix} & \text{if } x_1 > 0 \end{cases},$$

and for the y-system the analogous protocol:

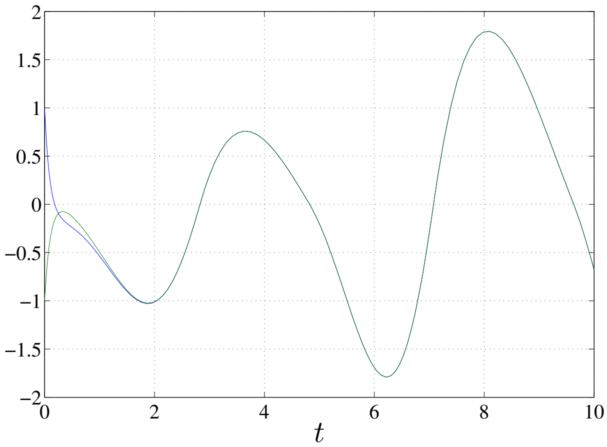
$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{cases} \begin{bmatrix} k(x_1 - y_1) \\ 5(x_2 - y_2) \end{bmatrix} & \text{if } y_1 \leq 0 \\ \begin{bmatrix} h(x_1^2 - y_1^2) \\ 5(x_2 - y_2) \end{bmatrix} & \text{if } y_1 > 0 \end{cases},$$

with $h > \xi$ and k such that the two following conditions are satisfied:

$$1.3\eta + 1 - k < 0; \qquad |1.3\eta + 1 - k| > 1.$$



(a)



(b)

Figure 7.3: Time evolution of the first state components $x_1(t)$ and $y_1(t)$ for the pair of systems (7.34): (a) uncoupled case; (b) coupled case.

To prove synchronization it is sufficient to apply Theorem 7.3.2 considering the $\mu_\infty(\cdot)$ measure and the PWSC virtual system:

$$\begin{bmatrix} \dot z_1 \\ \dot z_2 \end{bmatrix} = \begin{cases} \begin{bmatrix} 1.3\eta+1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \rho \end{bmatrix} + \begin{bmatrix} -2kz_1 + kx_1 + ky_1 \\ +10z_2 - 5x_2 - 5y_2 \end{bmatrix}, & \text{if } z_1 \le 0 \\ \begin{bmatrix} -2 & 1 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \xi z_1^2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \rho \end{bmatrix} + \begin{bmatrix} -2hz_1^2 + hx_1^2 + hy_1^2 \\ +10z_2 - 5x_2 - 5y_2 \end{bmatrix}, & \text{if } z_1 > 0 \end{cases},$$

Numerical simulations in Figure 7.4(a) and Figure 7.4(b) show, respectively, the behaviour of the two systems (7.35) (first state component) respectively for the coupled and uncoupled case on a stable periodic orbit. Simulations have been performed by setting $\rho = -0.1$, $\eta = -0.1$ and $\xi = 0.01$, and considering randomly chosen initial conditions.

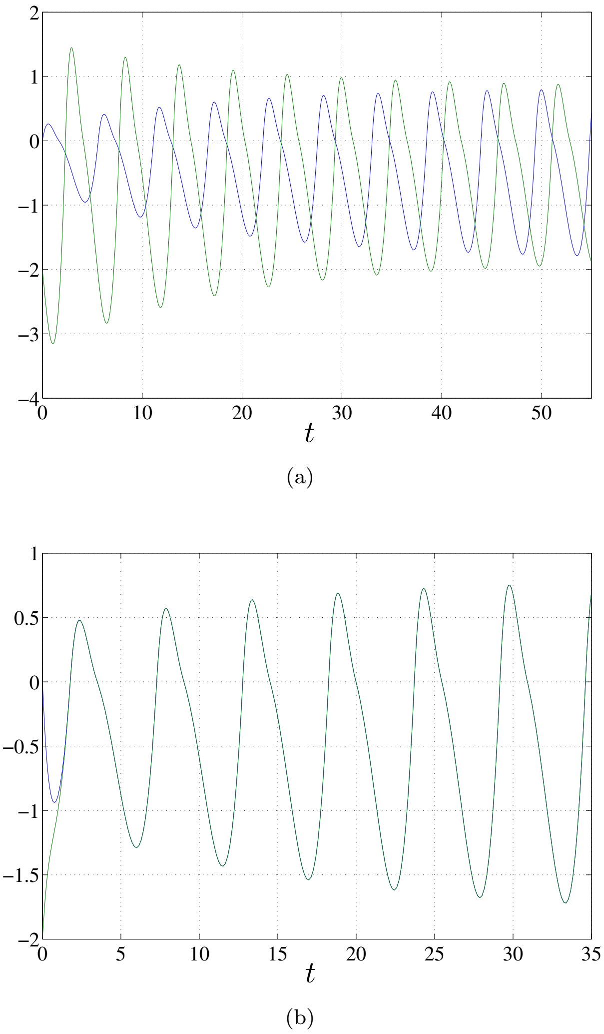


Figure 7.4: Time evolution of the first state components $x_1(t)$ and $y_1(t)$ for the pair of systems (7.35): (a) uncoupled case; (b) coupled case.

In this example it is very clear to understand how sometimes can be more convenient to use matrix measures different from the $\mu_{\Theta,2}(\cdot)$. Indeed, it is more complicated to use an Euclidean measure (or equivalently to find a common quadratic Lyapunov function) for this example, while it is immediate to tune gains k and h with infinity measure.

7.5.5 Synchronization of networks of time-switching systems

As a representative example of Section 7.4, we synchronize a network of the form (7.24), where the dynamics of each uncoupled node is given by:

$$\dot{x}_i := \begin{bmatrix} x_{1i} \\ x_{2i} \end{bmatrix} = \begin{bmatrix} 0 & saw_T(t) \\ -1 & 0 \end{bmatrix} x_i, \tag{7.36}$$

where $saw_T(t) : \mathbb{R} \mapsto [0, 1]$ is a saw-tooth wave of period T. The matrix Γ is chosen as:

$$\Gamma = k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

with k being the coupling gain that will be determined using Theorem 7.4.1. The network considered here consists of an all to all topology of three nodes. That is,

$$L := \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix},$$

and $\lambda_2 = 3$. Thus, from Theorem 7.4.1 it follows that the network synchronizes if there exist some matrix measure and a positive scalars c such that:

$$\mu\left(\begin{bmatrix} 0 & saw_T(t) \\ -1 & 0 \end{bmatrix} - 3k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \leq -c,$$

That is, synchronization is attained if

$$\mu\left(\begin{bmatrix} -3k & saw_T(t) \\ -1 & -3k \end{bmatrix}\right) \leq -c$$

Now, using the matrix measure induced by the vector-1 norm (column sums) ad considering the fact that $saw_T(t)$ is a bounded signal with $saw_T(t) \leq 1$, it is straightforward to see that the above conditions are fulfilled if the coupling gain is selected as

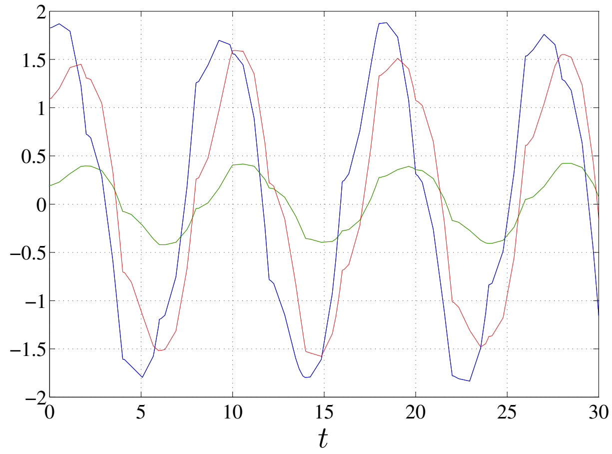
$$k > \frac{1}{3}.$$

As shown in Figures 7.5(a)-7.5(b), the theoretical predictions are confirmed by the numerical simulations obtained considering $T = 2$ and random initial conditions.

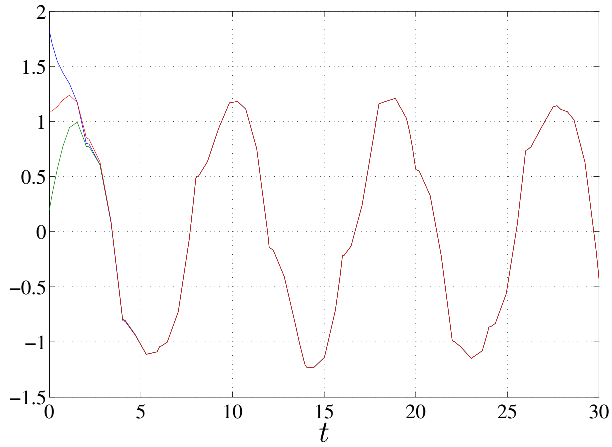
7.6 Discussion

In this chapter an extension of contraction theory to a generic class of piecewise smooth systems has been presented. More in detail, the systems considered are those satisfying conditions for the existence and uniqueness of Caratheodory solution. In particular, it was proven that infinitesimal contraction of each mode of a PWSC or TSS system of interest gives a sufficient condition for global exponential convergence of trajectories towards each other. This result was then used on a set of representative applications. It was shown that, by using contraction, it is possible to immediately derive sufficient conditions for global stability of switched linear systems even in those cases where the application of the theory of convergent systems might lead to cumbersome derivations. Also, contraction was used to obtain sufficient conditions for the convergence of all nodes in a network of coupled switched linear systems towards the same synchronous evolution.

We wish to emphasize that the results presented in this chapter can be immediately applied to generalize to piecewise smooth Caratheodory systems all of the analysis and design results based on contraction analysis available for smooth systems in the literature. Examples of applications include nonlinear observer design, network protocols design for network coordination, analysis/control of asynchronous systems and biochemical systems.



(a)



(b)

Figure 7.5: Time evolution of the first state components of the networked switched linear systems: (a) $k = 0$; (b) $k = 0.4$.

The results presented in this chapter are the first essential stage needed to develop a systematic approach to extend contraction analysis to generic classes of piecewise smooth systems. The next step is that of addressing the challenging problem of studying convergence in Filippov systems where sliding motion is possible.

The interest for discontinuous systems stems from the pioneering work of Yakubovitch in [219] and even earlier from Demidovich work [58]. However, the case of studying convergence in generic Filippov systems is not covered in this chapter and it is still a relatively unexplored topic in literature. Few works try to extend the theory of convergent systems to some classes of PWS systems [158, 207]. In the next chapter some preliminary results based on contraction theory are obtained for planar Filippov systems, but much work remains to be done.

Chapter 8

Incremental stability and contraction of planar Filippov systems

Contents

In this chapter we study the problem of proving incremental stability of bimodal planar Filippov systems. In particular, referring to systems that present an attractive sliding region on their discontinuity boundary, we give a differential condition on such region able to guarantee incremental exponential stability of sliding mode trajectories. In this way, we derive conditions for the incremental stability of the whole system. The approach is based on using tools from contraction theory, extending their applicability to include discontinuous dynamical systems.

The chapter is organized in the following way. In Section 8.1 we introduce and motivate the problem of incremental stability for PWS systems. In Section 8.2 we study such problem for planar bimodal Filippov systems and, in Section 8.3, we consider two examples able to show the effectiveness of the proposed approach. A discussion of the obtained results is given in Section 8.4.

8.1 Introduction

In Chapter 6 we introduced the concept of incremental stability and other tools closely related to such idea of convergence. In particular, contraction theory has been shown to be a powerful tool able to study exponential incremental stability of a continuously differentiable system of interest.

In the literature, the tool of convergent systems has been extended to PWSC system (see Chapter 6 and referencest therein), while an analogous extension of contraction theory with applications to synchronization has been developed in Chapter 7. Despite these results, few articles deal with the problem of investigating incremental stability of wider classes of piecewise smooth systems. Although the interest for discontinuous systems, as we said, stems from the pioneering works of Demidovich [58] and, later, Yakubovitch [219], only some results have been recently developed. More in detail, results extending the theory of convergent systems to PWL discontinuous systems can be found in [158], while an application to controlled switching of this kind of systems is developed in [207]. To the best of our knowledge, studying incremental stability of Filippov systems with sliding mode solutions is instead a completely unexplored problem.

In this chapter, we address this open research challenge by considering the case of planar Filippov systems. The approach is based on two steps. Firstly, we derive local conditions for contraction of the sliding vector field, guaranteeing that solutions on the sliding surface exponentially converge towards each other. Then, we give conditions for the sliding region to be attractive so that when both set of conditions are satisfied the whole system is exponentially incrementally stable. Two numerical examples are used to illustrate the theory.

We wish to emphasize that the focus on planar Filippov systems serves as a useful starting point to embark on the investigation of higher-dimensional Filippov systems. Indeed, the extension to higher dimensions of the ideas presented in this chapter is currently under investigation.

8.2 Incremental stability of planar Filippov systems

Here we derive a sufficient condition for incremental stability of a bimodal PWS dynamical system. As we said in the previous section, the approach consists of two different steps: proving contraction of the system within the sliding region (Section 8.2.1); ensuring attractivity of such sliding region (Section 8.2.2).

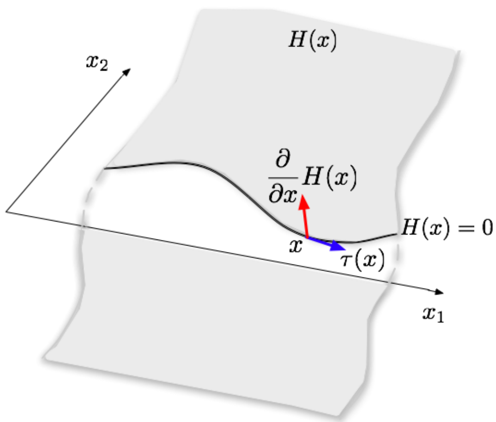


Figure 8.1: Graphical representation of the manifold $H(x) = 0$.

8.2.1 Contracting sliding vector fields

For a planar Filippov system we can give an analogous result to Theorem 6.2.1 related to the sliding mode trajectories by considering an infinitesimal contraction condition on the discontinuity boundary.

Theorem 8.2.1. *Let us consider a PWS system* (2.9) *with* $x \in \mathbb{R}^2$. *Suppose that the vector fields* $x \mapsto F_1(t,x)$ *and* $x \mapsto F_2(t,x)$ *are continuously differentiable and that the function* $x \mapsto H(x)$ *is twice continuously differentiable in an open set containing the region* $\hat{\Sigma}$. *Suppose also that there exist a* $\hat{\mathrm{D}} \subseteq \hat{\Sigma}$, *invariant connected set with respect to the topology induced on* Σ, *and a constant* $c > 0$ *such that the following condition holds:*

$$\frac{\partial}{\partial x} m(t,x) \cdot \tau(x) \leq -c, \qquad \forall t \geq t_0, \forall x \in \hat{\mathrm{D}}, \tag{8.1}$$

where $m : [t_0, +\infty) \times \mathcal{O}\Sigma \mapsto \mathbb{R}$ *is defined as:*

$$m(t,x) = \left(\frac{1-\beta(t,x)}{2} \tau^T(x) F_1(t,x) + \frac{1+\beta(t,x)}{2} \tau^T(x) F_2(t,x) \right), \tag{8.2}$$

with $\mathcal{O}\Sigma \subseteq \mathbb{R}^n$ *an open set containing* Σ, *and* $\tau(x)$ *is given:*

$$\tau(x) = \begin{cases} \dfrac{1}{\left\| \begin{bmatrix} 1 & -\frac{\frac{\partial H}{\partial x_1}(x)}{\frac{\partial H}{\partial x_2}(x)} \end{bmatrix} \right\|_2} \begin{bmatrix} 1 & -\frac{\frac{\partial H}{\partial x_1}(x)}{\frac{\partial H}{\partial x_2}(x)} \end{bmatrix}^T & \text{if } \frac{\partial H}{\partial x_2}(x) \neq 0 \\ \begin{bmatrix} 0 & 1 \end{bmatrix}^T & \text{if } \frac{\partial H}{\partial x_2}(x) = 0 \end{cases}, \tag{8.3}$$

with $\beta(t,x)$ *given by* (2.11). *Then, the sliding mode trajectories converge exponentially*[1] *towards each other in* $\hat{\mathrm{D}}$. *Furthermore, it there exists a* $K > 0$ *such that* $|\tau(x)| \leq K$ *for all* $x \in \hat{\mathrm{D}}$, *then the system is incrementally exponentially stable in* $\hat{\mathrm{D}}$.

A schematic representation of the manifold $H(x)$ and the tangent vector $\tau(x)$ is given in Figure 8.1. Notice that expressions (8.1) and (8.3) are differential conditions which can be easily evaluated.

Proof. Firstly, observe that, as the function $x \mapsto H(x)$ with $x \in \Sigma$ has a well defined gradient by assumption, the smooth manifold Σ can be viewed as the range of a curve $\psi_s(s)$ with s being the curvilinear abscissa parametrizing the curve itself. So, we will say that point $\psi_s(s_1)$ precedes point $\psi_s(s_2)$ if $s_1 < s_2$. Since $\Sigma = \psi_s(s)$ is a 1-dimensional smooth manifold we can define $\tau(\psi_s(s))$ as its tangent unit vector and evaluate the projection of the sliding vector field $F_s(t,x)$ defined in (2.10) on the manifold. We obtain the equivalent 1-dimension dynamical system:

$$\dot{s} = \tilde{f}_s(t,s), \tag{8.4}$$

with

$$\tilde{f}_s(t,s) := f_s(t, \psi_s(s)) = \tau^T(\psi_s(s)) \cdot F_s(t, \psi_s(s)). \tag{8.5}$$

[1]Here we mean that the trajectories converge with respect to the topology induced on Σ.

We now introduce on Σ the metric:

$$d(s_1, s_2) = |s_2 - s_1|,$$

where s_1 and s_2 is a generic pair of values of the abscissa s. Notice that such metric is analogue to the distances in $\mathbb{R}$ where the natural topology is used and so, due to the definition of curvilinear abscissa, the value $d(s_1, s_2)$ represents the length, say $L_{\psi_s}(s_1, s_2)$, of the curve between points $\psi_s(s_1)$ and $\psi_s(s_2)$. Such norm is, equivalently, any $p-$norm in $\mathbb{R}$. For the sake of clarity, we divide the proof in three steps.

Step 1. We prove that if (8.4) is such that condition $\frac{\partial}{\partial s}\tilde{f}_s(t, s) \le -c$ holds, with $c > 0$ and with s such that $\psi_s(s) \in \hat{\mathrm{D}}$, then we have:

$$\begin{aligned} &|\psi_s(s_2(t)) - \psi_s(s_1(t))|_2 \le \\ &Ke^{-c(t-t_0)}|\psi_s(s_2(t_0)) - \psi_s(s_1(t_0))|_2 \qquad \forall t \ge t_0. \end{aligned} \tag{8.6}$$

Supposing without loss of generality that $s_1 < s_2$ and defining $\bar{s} = \frac{s_2 - s_1}{2}$, it is immediate to verify that, since system (8.4) is strictly decreasing with maximum slope $-c$, we have:

$$\tilde{f}_s(t, s_1) \ge -c(s_1 - \bar{s}) + \tilde{f}_s(t, \bar{s}); \tag{8.7}$$

$$\tilde{f}_s(t, s_2) \le -c(s_2 - \bar{s}) + \tilde{f}_s(t, \bar{s}). \tag{8.8}$$

So, due to the dynamic of the difference between the two flows, the following inequality can be written:

$$\frac{d}{dt}(s_2(t) - s_1(t)) = \tilde{f}_s(t, s_2) - \tilde{f}_s(t, s_1) \le -c(s_2(t) - s_1(t)),$$

we can integrate (over the time interval $[t_0, +\infty)$ since $\hat{\mathrm{D}}$ is invariant) the above expression and obtain:

$$|s_2(t) - s_1(t)|_2 \le e^{-c(t-t_0)}|s_2(t_0) - s_1(t_0)|_2. \tag{8.9}$$

The above inequality holds for the time interval $[t_0, +\infty)$ and shows the exponential convergence of the trajectories with respect to the topology induced on Σ. Remembering that the difference between the curvilinear abscissa is the length of the curve ψ_s between the two points, in order to show the incremental stability we can write the above inequality as:

$$L_{\psi_s}(s_1(t), s_2(t)) \le e^{-c(t-t_0)}L_{\psi_s}(s_1(t_0), s_2(t_0)).$$

Now, because of the the smoothness of manifold Σ, the derivative of the curve $\frac{d}{dr}\psi_r(r)$ is bounded for any equivalent representation $\psi_r(r) \sim \psi_s(s)$. Taking into account the definition of the length of a curve, if $|\tau(x)| \le K$ we can write:

$$\begin{aligned} L_{\psi_s}(s_1(t), s_2(t)) &\le e^{-c(t-t_0)}L_{\psi_s}(s_1(t_0), s_2(t_0)) \\ &\le Ke^{-c(t-t_0)}|\psi_s(s_2(t_0)) - \psi_s(s_1(t_0))|_2, \end{aligned}$$

Taking now into account that for the length of a curve it holds:

$$|\psi_s(s_2) - \psi_s(s_1)|_2 \le L_{\psi_s}(s_1, s_2),$$

condition (8.6) is verified.

To summarise, we have shown that condition

$$\frac{\partial}{\partial s}\tilde{f}_s(t,s) \leq -c, \tag{8.10}$$

with $c > 0$ and s such that $\psi_s(s) \in \hat{\mathrm{D}}$ implies incremental exponential stability of the sliding vector field given by (8.6).

Step 2. We now show that condition (8.10) is equivalent to hypotheses (8.1)-(8.3). To do this, we first recall [83] that two regular curves, $\psi_r(r)$ and $\psi_s(s)$, are equivalent, $\psi_r(r) \sim \psi_s(s)$, if there exists an invertible diffeomorphism $s = T(r)$ (i.e. a continuously differentiable and invertible function) with $\frac{d}{dr}T(r) > 0$ for all s. In this way it is possible to write:

$$\psi_r(T^{-1}(s)) = \psi_s(s).$$

In particular, if s parametrizes the curve as its curvilinear abscissa it also holds:

$$\begin{aligned} \tfrac{d}{dr}T(r) &= \left|\frac{d}{dr}\psi_r(r)\right| && (8.11) \\ \tfrac{d}{ds}T^{-1}(s) &= \frac{1}{\left|\frac{d}{dr}\psi_r(T^{-1}(s))\right|}. && (8.12) \end{aligned}$$

So, condition (8.10) is:

$$\begin{aligned} &\frac{\partial}{\partial s}\tilde{f}_s(t,s) = \frac{\partial}{\partial s}f_s(t,\psi_s(s)) = \frac{\partial}{\partial s}f_s(t,\psi_r(T^{-1}(s))) = \\ &= \frac{\partial}{\partial x}f_s(t,x)\bigg|_{\psi_r(r)} \cdot \frac{1}{\left|\frac{d}{dr}\psi_r(r)\right|}\frac{d}{dr}\psi_r(r) \leq -c. \end{aligned} \tag{8.13}$$

If we now define the tangent unit vector of the curve as:

$$\tau(x)|_{\psi_r(r)} := \frac{1}{\left|\frac{d}{dr}\psi_r(r)\right|}\frac{d}{dr}\psi_r(r),$$

condition (8.13) becomes formally equivalent to condition (8.1).

Step 3. We show that the tangent unit vector $\tau(x)$ can be expressed as in (8.3). Indeed, we can consider that the smooth curve defined implicitly by $H(x) = 0$ can be parametrized as a graph of a function in the two following ways:

$$\psi_r(r) = \begin{cases} x_1 = r; \\ x_2 = h(r); \end{cases}, \tag{8.14}$$

or

$$\psi_r(r) = \begin{cases} x_1 = h'(r); \\ x_2 = r; \end{cases}. \tag{8.15}$$

Although functions $h(\cdot)$ and $h'(\cdot)$ are in general not known explicitly, the Implicit Function Theorem [177] ensures that they exist locally for each point on the set defined by the equation $H(x) = 0$. Now, deriving (8.14) by r we obtain $\frac{dx_1(r)}{dr} = 1$ and $\frac{dx_2(r)}{dr} = -\frac{\partial H}{\partial x_1}\Big/\frac{\partial H}{\partial x_2}$ from the Implicit Function Theorem. Analogously, deriving (8.15) we obtain $\frac{dx_1(r)}{dr} = -\frac{\partial H}{\partial x_2}\Big/\frac{\partial H}{\partial x_1}$ and $\frac{dx_2(r)}{dr} = 1$. Normalizing such derivatives to unitary module we have, respectively, the two expressions in (8.3). $\square$

Remark 8.2.1. *Notice that if the set $\hat{\mathrm{D}}$ is bounded, then the smoothness of manifold Σ trivially guarantees that there exists a K such that $|\tau(x)| \leq K$.*

For a two-dimensional PWS system, it is also possible to generalize Theorem 8.2.1 as follows.

Theorem 8.2.2. *Consider a two-dimensional PWS system of the form* (2.9)*. Suppose that the vector fields $x \mapsto F_1(t,x)$ and $x \mapsto F_2(t,x)$ are continuously differentiable and that the function $x \mapsto H(x)$ is (i) well defined (i.e. $\frac{\partial}{\partial x}H(x) \neq 0$) for all x such that $H(x) = 0$, and (ii) is twice continuously differentiable in an open set containing the region $\hat{\Sigma}$. Let $\hat{\mathrm{D}} \subseteq \hat{\Sigma}$ and define $\tau(x)$ and $\beta(t,x)$ as in* (8.3) *and* (2.11) *respectively. Also, define*

$$J_s(t,x) := \frac{\partial}{\partial x} m(t,x) \cdot \tau(x). \tag{8.16}$$

Then, for a given scalar $c > 0$, for all $t \in [t_0, T)$ such that trajectories starting in $\hat{\mathrm{D}}$ remain in $\hat{\mathrm{D}}$, we have:

i. if $J_s(t,x) \leq -c$ for all $x \in \hat{\mathrm{D}}$ and for all $t \in [t_0,T)$, then the sliding mode trajectories in $\hat{\mathrm{D}}$ converge exponentially towards each other;

ii. if $J_s(t,x) \geq -c$ for all $x \in \hat{\mathrm{D}}$ and for all $t \in [t_0,T)$, then the sliding mode trajectories in $\hat{\mathrm{D}}$ diverge exponentially from each other;

iii. if $J_s(t,x) = 0$ for all $x \in \hat{\mathrm{D}}$ and for all $t \in [t_0,T)$, then the sliding mode trajectories in $\hat{\mathrm{D}}$ keep their distance constant.

Proof. Item i. has been proved in Theorem 8.2.1 (equation (8.9)). To prove items ii. and iii. we can follow the same steps as those in the proof of Theorem 8.2.1. In particular, for item ii. we have that conditions (8.7)-(8.8) are replaced by:

$$\tilde{f}_s(t,s_1) \leq c(s_1 - \bar{s}) + \tilde{f}_s(t,\bar{s});$$
$$\tilde{f}_s(t,s_2) \geq c(s_2 - \bar{s}) + \tilde{f}_s(t,\bar{s}),$$

and so, the dynamic of the error trajectory becomes:

$$\frac{d}{dt}(s_2(t) - s_1(t)) = \tilde{f}_s(t,s_2) - \tilde{f}_s(t,s_1) \geq 2c(s_2(t) - s_1(t)).$$

To prove item iii. conditions (8.7)-(8.8) must be replaced instead with:

$$\tilde{f}_s(t,s_1) = \tilde{f}_s(t,s_2) = \bar{f}_s,$$

with $\bar{f}_s$ being a constant value. The error dynamics in this case are:

$$\frac{d}{dt}(s_2(t) - s_1(t)) = 0.$$

Both for items ii. and iii. the rest of the proof then follows as the rest of the proof in Theorem 8.2.1. □

Taking into account Theorem 8.2.2 it is possible to classify a connected region $\hat{\mathrm{D}} \subseteq \hat{\Sigma}$ by looking at the function $J_s(t,x)$. In particular if case i. is verified, we will term $\hat{\mathrm{D}}$ an *infinitesimally contracting sliding region*; if case ii. is verified, we will term $\hat{\mathrm{D}}$ an *infinitesimally stretching sliding region*; while, if case iii. is verified, $\hat{\mathrm{D}}$ will be termed an *infinitesimally neutral sliding region.* Finally, if $J_s(t,x)$ change sign in $\hat{\mathrm{D}}$, the set will be termed *indifferent sliding region.*

8.2.2 Incremental stability of a planar Filippov system

We can now use the theorems given above to derive conditions for incremental stability of a planar PWS system.

Theorem 8.2.3. *Let us consider the PWS system* (2.9) *with* $x \in \mathbb{R}^2$*,* $F_1(t,x)$ *and* $F_2(t,x)$ *being two smooth vector fields and* $H(x)$ *hyperplane of equation* $H(x) = h^T(x - x_h)$*, with* $h, x_h \in \mathbb{R}^2$*. Let* $\mathcal{S}_1 = \{x : H(x) < 0\}$*,* $\mathcal{S}_2 = \{x : H(x) > 0\}$ *be the two regions in which the state space is partitioned by the switching manifold* $\Sigma := \{x : H(x) = 0\}$*. If there exists a convex invariant region* $\mathcal{C} \subseteq \mathbb{R}^2$ *such that* $\Sigma_{\mathcal{C}} = \mathcal{C} \cap \Sigma \neq \emptyset$*, and if the two following conditions hold:*

i. there exist two scalars $\lambda_1 > 0$ *and* $\lambda_2 < 0$ *such that* $h^T F_1(t,x) \geq \lambda_1$ *for all* $x \in \mathcal{C} \cap \mathcal{S}_1$ *and* $h^T F_2(t,x) \leq \lambda_2$ *for all* $x \in \mathcal{C} \cap \mathcal{S}_2$*;*

ii. $J_s(t,x) \leq -c$ *for all* $x \in \Sigma_{\mathcal{C}}$*, with* $c > 0$ *and with* $J_s(t,x)$ *defined as in* (8.16)*;*

then the PWS system is δAS*.*

Proof. Condition i. ensures that the flow $x(t) = \phi(t - t_0, t_0, \xi)$ reaches in a finite time the discontinuity manifold $\Sigma_{\mathcal{C}}$ for any initial condition $x(t_0) = \xi \in \mathcal{C}$. Indeed, suppose without loss of generality that $\xi \in \mathcal{C} \cap \mathcal{S}_1$. The time derivative of $H(x(t))$ is $\dot{H}(x(t)) = \mathcal{L}_{F_1} H(t,x) = h^T F_1(t,x) \geq \lambda_1$ and so, integrating this expression and considering that $\mathcal{C}$ is invariant and that $H(x)$ is monotone on the direction h, we have that the flow reaches the set $\Sigma_{\mathcal{C}}$ at a time instant $t_s \leq -H(\xi)/\lambda_1$. A similar result follows for any $\xi \in \mathcal{C} \cap \mathcal{S}_2$. Then, all trajectories reach the sliding region in finite time. Moreover, from Theorem 8.2.1, condition ii. implies exponential convergence among trajectories in the sliding region, and therefore the theorem is proven. □

Notice that condition i. in Theorem 8.2.3 can be replaced by other possible conditions able to guarantee convergence of all trajectories towards the sliding region. The attractivity of sliding surfaces has been widely studied in literature and we direct the reader for more detailed information about other possible conditions to, for example, [77, 60, 205].

8.3 Examples

Here we give two numerical examples in order to illustrate the theoretical derivations. In particular, in the two examples we study the stability of the equilibrium points by looking at the incremental stability of the sets where these points belong to. Both the examples are given in terms of piecewise linear systems for the sake of clarity. Obviously, piecewise nonlinear systems can equally be considered.

8.3.1 Contracting sliding region

We study the bimodal PWS system:

$$\dot{x} = \begin{cases} Ax + B & \textit{if } Cx \leq 0 \\ Ax - B & \textit{if } Cx > 0 \end{cases},$$

with:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix},$$

$$B = [1\ 1]^T, \quad C = [0\ 1].$$

It is easy to verity that the vector field of this system is continuously differentiable everywhere but on $\{Cx = 0\}$ and that the function $H(x) = Cx$ is twice continuously differentiable with a well defined gradient. Applying equation (2.11) we have that:

$$\beta(x) = \frac{CAx}{CB} = x_1 - x_2,$$

and since, as we said, $-1 \leq \beta(x) \leq 1$ and taking into account that on the discontinuity manifold $x_2 = 0$, we have that $\hat{\Sigma} = \{x : -1 \leq x_1 \leq 1, x_2 = 0\}$. Furthermore we have that:

$$\frac{\partial}{\partial x} H(x) = C = [0\ 1], \quad \tau(x) = [1\ 0]^T.$$

Now, evaluating expression (8.16) with respect of this system, we obtain after some manipulations that:

$$J_s(x) = -1.$$

So, due to Theorem 8.2.2, region $\hat{\Sigma}$ is a contracting sliding region. Furthermore, since the system present an equilibrium point at the origin (see [46] for further details on equilibrium point of Filippov systems) all sliding trajectories converge towards each other and onto the trivial trajectory $x = 0$. Figures 8.2(a) and 8.2(b) show the evolution of trajectories from the initial points $x_0^{(1)} = [0.5\ 0.5]^T$ and $x_0^{(2)} = [-0.5\ -0.5]^T$ outside the sliding manifold. In particular Figure 8.2(a) displays the evolution of the error between the two trajectories while Figure 8.2(b) shows the trajectories on the state space. It is possible to note that when both the flow reach the sliding region (at $t \simeq 0.6$) they converge exponentially towards each others.

8.3.2 Neutral sliding region

In this example we study the well known spring-damping mass with Coulomb friction described by:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 - \frac{F_c}{m}sgn(x_2) \end{cases},$$

where we indicate with x_1 the position of the mass and with x_2 its velocity. For this system we have:

$$\beta(x) = \frac{-kx_1 - bx_2}{F_c},$$

and, since on the discontinuity manifold $x_2 = 0$, and considering that $-1 \leq \beta(x) \leq 1$, the sliding region is the set $\hat{\Sigma} = \left\{x : -\frac{F_c}{k} \leq x_1 \leq \frac{F_c}{k}, x_2 = 0\right\}$. For this system we have:

$$\frac{\partial}{\partial x} H(x) = [0\ 1], \quad \tau(x) = [1\ 0]^T.$$

Therefore, from (8.16) we have $J_s(x) = 0$. Hence, $\hat{\Sigma}$ is a neutral sliding region and, for this reason, the error between sliding trajectories remains constant. Furthermore, since for this system it is easy to notice that the sliding trajectory $x = 0$ is a solution of the Filippov map, then the set $\hat{\Sigma}$ is an equilibrium set of the system. Indeed, such equilibrium set is associated in the physical model to the phenomenon of sticking [166]. Here we consider a numerical simulation with the following parameter values:

$$m = 1, \quad b = 0.1, \quad k = 10, \quad F_c = 10,$$

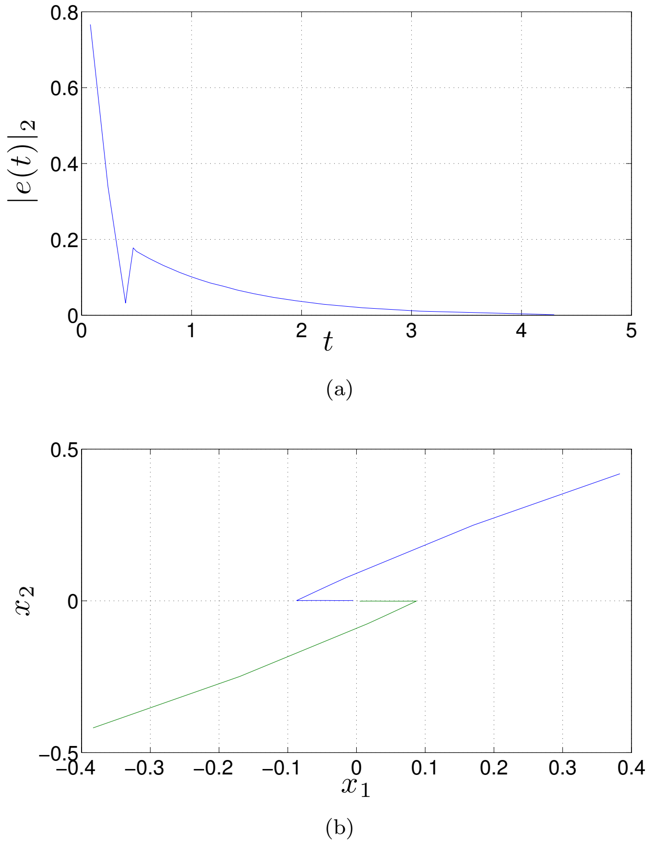


Figure 8.2: Evolution of trajectories: (a) norm of the error; (b) phase portrait.

starting from the initial conditions $x_0^{(1)} = [1\ 1]^T$ and $x_0^{(2)} = [-1\ \ -1]^T$. Figure 8.3(a) shows the error between the trajectories, while Figure 8.3(b) shows the state-space. It is possible to notice that when the flow reaches the sliding region, it sticks on the set. For this reason, as we expect from Theorem 8.2.2, the incremental error remains constant.

8.4 Discussion

In this chapter we studied the problem of finding conditions for the incremental stability of 2-dimensional Filippov systems. We approached the problem defining infinitesimal contraction of the sliding vector field and we showed that contraction on such region is able to guarantee exponential incremental stability of the sliding trajectories. If the contracting sliding surface is also attractive, the Filippov system is incrementally stable in any invariant region containing the sliding surface.

Notice that the mathematical tools used in this chapter (i.e. Implicit Function Theorem, Jacobian of composed functions, local parametrization of manifolds on the tangent space) are defined for the generic $\mathbb{R}^n$ case. For this reason, the case of planar systems represents only a starting point, while the pressing open problem of studying incremental stability in higher dimensional Filippov systems is currently under investigation.

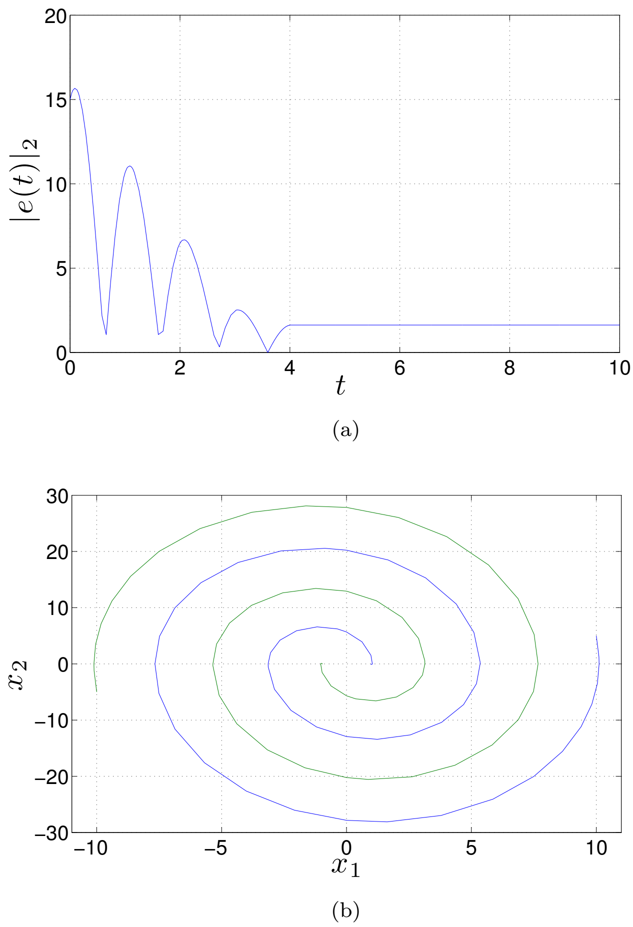


Figure 8.3: Evolution of trajectories: (a) norm of the error; (b) phase portrait.

Chapter 9

Conclusions

Contents

In this thesis we have considered the study of convergence in piecewise smooth (PWS) dynamical networks. In particular, conditions able to guarantee synchronization in networks of nonidentical PWS agents have been given and applied on a set of different examples. Furthermore, the case where discontinuities affect the communication in a multi-agent systems has been addressed considering event-triggered strategies which lead to both communication and control signals being piecewise constant. The study of discontinuous networks has also been the motivation to extend contraction theory, which has been showed to be a powerful tool in studying coordination in networks of smooth dynamical systems, to nonsmooth systems. Such extension was given in Part II of the thesis. Conditions able to guarantee contraction for time switching non differentiable systems have been given, while a preliminary study of contraction of Filippov systems has been discussed.

The thesis work is concluded with a summary of the main contributions and with some possible future direction of research.

9.1 Contributions

The contributions of the thesis can be summarized as follows:

- We have presented a framework for the study of synchronization when the nodes' dynamics may be both piecewise smooth and/or nonidentical across the network. The analysis has been performed using nonsmooth analysis tools, in particular set-valued Lyapunov functions. In this way, sufficient conditions have been derived in order to guarantee global bounded synchronization, with analytical expression of the estimates of the synchronization bound and of the minimum coupling strength required to achieve synchronization. Different hypotheses on the heterogeneity of the agents' dynamics have been considered and, differently from the few results available in the literature, we do not require that trajectories of the coupled systems are bounded a priori or that conditions of synchrony among switching signals are satisfied. The analysis has been conducted considering both linear and nonlinear coupling protocols.

- We have developed novel event-triggered control strategies able to synchronize networks of general Lipschitz nonlinear systems. The results represent an innovative contribution on the existing literature for two main reasons. The existing literature considers only synchronization of integrators or linear systems and the control signals and communication signals are not both simultaneously piecewise constant, which we have instead guaranteed. Furthermore, a strategy able to guarantee a lower bound for the inter-event times between consecutive updates of the control law has been proposed.
- We have extended contraction theory to the class of time switching non differentiable dynamical systems which satisfy Caratheodory conditions for the existence and uniqueness of a solution. In particular, it has been proved that infinitesimal contraction of each mode of a switched system of interest gives a sufficient condition for global exponential convergence of trajectories towards each other. Such extension can be related with analogous results already present in literature for convergent systems, which instead make use of Lyapunov conditions. The proposed tools have been used to develop smooth and nonsmooth distributed protocols to coordinate and synchronize networks of nonsmooth agents. Furthermore, contraction theory and incremental stability have also been studied preliminarily for planar Filippov systems, while more general results for higher dimension systems are currently under investigation.

9.2 Ideas for future research

As immediate extensions of the thesis work, we can consider the following points. Some of them are already under investigation.

- Since for the bounded synchronization of nonidentical PWS systems a finite coupling strength is showed to be able to synchronize the network, it could be possible to consider hybrid-adaptive strategies in order to locally select the coupling strength among neighbouring agents. Furthermore, the approach can be also extended considering pinning control strategies. Preliminary numerical results show that these control laws successfully are able to guarantee the desired performances.
- Some results for bounded synchronization of nonidentical PWS systems have also been obtained in the case of linear coupling in an oriented network and applied to develop a distributed control law for power network modelled with the first order swing equation [68]. However, further study is needed to extend all the results presented in Chapter 4 to the case of directed networks.
- The event-triggered control scheme could be extended to non Lipschitz nonlinear systems. Indeed, numerical simulations show that the proposed scheme is effective also with some non Lipschitz systems. It is also worth considering the case of applications to mobile robots, where event-triggered strategies appear to be, in our opinion, the best way to guarantee the network being controlled without overload the communication medium.
- It is worth to complete the extension of contraction theory for nonsmooth systems considering Filippov systems of arbitrary dimension of their state space, as well as its applicability in studying coordination of multi-agents systems. Good results have already been obtained, but the work is still in progress.

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