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Geometrical and Functional Inequalities in the Calculus of Variations

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CHAPTER 1

Introduction

Isoperimetric and Sobolev inequalities are the best known examples of geometric-functional inequalities. In recent years new and sharp quantitative versions of these and other important related inequalities were obtained and applied to variational problems such as shape optimization problems, inequalities concerning eigenvalues of elliptic operators, local minimality of critical point of energy functionals used as models in materials science—see [**1**, **4**, **13**, **14**, **25**, **31**, **32**, **35**–**37**, **40**, **41**, **44**, **46**, **47**, **55**, **59**]. All these results have been obtained by the combined use of classical symmetrization methods, new tools from mass transportation theory, deep geometric measure theory tools and ad-hoc symmetrizations.

The purpose of this thesis is twofold. In the first part we discuss the equality cases in the Pólya-Szegő inequality for the Steiner symmetrization of Sobolev and BV functions and the quantitative version of this inequality. In the second part of the thesis we show how the above mentioned techniques come into play in derive a (quantitative) local minimality criterion for critical points with positive second variation of a free discontinuity problem coming from material science.

The arguments treated in Part I make a strong use of symmetrization techniques and various techniques from the theory of BV functions and from Geometric Measure Theory.

Symmetrization techniques are a powerful tool to deal with those variational problems whose extrema are expected to exhibit symmetry properties due either to the geometrical or to the physical nature of the problem (see, for instance, the classical book [**60**] and [**56**]).

It is well known that the perimeter of a set decreases under several types of symmetrizations such as polarization, standard Steiner symmetrization or the general Steiner symmetrization with respect to a $n-k$ dimensional plane.

Similarly, the so-called Pólya-Szegő inequality states that Dirichlet-type integrals depending on the modulus of the gradient of a real-valued function decrease under rearrangements such as the Schwarz spherical rearrangement and standard or higher codimensional (see Definition 2.15) Steiner rearrangements.

In this framework, a natural question, which has been extensively studied in recent years, is to give a characterization of the equality cases in the Pólya-Szegő inequality as well as in inequalities concerning symmetrization of sets.

In a celebrated paper [**16**] Brothers and Ziemer characterized the equality cases in the Pólya-Szegő inequality for the Schwarz rearrangement of a Sobolev function under the minimal assumption that the set of critical points of the rearranged function has zero Lebesgue measure (see also [**39**] for an alternative proof). The corresponding inequality for BV functions was first proved in [**53**], while a much finer analysis is carried out in [**27**], where also the equality cases are characterized.

Concerning the standard Steiner symmetrization and its higher codimension version, the validity of the isoperimetric inequality and of the Pólya-Szegő principle are also well-known, see for instance a proof via polarization given in [**15**] and the references therein. On the other hand, the characterization of the equality cases seems to be a much harder problem. The first result in this direction was proved in [**23**] in connection with the perimeter inequality for the standard Steiner symmetrization. In analogy to what was pointed out in [**16**], also in this case it turns out that such characterization may hold only under the assumption that the boundary of the set is almost nowhere orthogonal to the symmetrization hyperplane. However this condition alone is not yet enough and a connectedness assumption, in a suitable measure theoretic sense, must be required on the set.

Very recently, in [**8**] the equality cases in the perimeter inequality for the Steiner symmetrization in codimension k were characterized using a different approach from the one in [**23**], aimed to reduce the problem to a careful study of the barycentre of the sections of the original set.

In Chapter 3 we present an alternative proof of the results contained in [**23**], which simplifies a lot the original argument. To this aim we use some ideas introduced in [**8**], but since we deal only with the standard Steiner symmetrization, many of the arguments are simpler—see Remark 3.10.

The equality cases in the Pólya-Szegő inequality for the standard Steiner rearrangement of Sobolev and BV functions were investigated in [**30**]. Again, the crucial assumption was that the set where the derivative of the extremal function in the direction orthogonal to the hyperplane of symmetrization vanishes is negligible. As for sets, also some connectedness and geometrical assumptions have to be made on the domain supporting the function.

In Chapter 4, we present the result obtain in our paper [**20**], where we further develop the analysis made in the above papers by considering the Pólya-Szegő inequality for the higher codimensional Steiner symmetrization of Sobolev and BV functions. First, we prove the Pólya-Szegő inequality for general convex integrands f depending on the gradient of a Sobolev function u. Besides convexity, we assume that f is non-negative, vanishes at 0 and depends on the norm of the y-component of the gradient of u, $y \in \mathbb{R}^k$ being the direction of symmetrization.

In order to characterize the equality cases, i.e., to show that u coincides with its Steiner rearrangement u^σ up to translations, the strict convexity of f is required together with the assumption that $\nabla_y u^\sigma \neq 0$ a.e.. Note that the result is false if one of the two previous assumptions is dropped. As in [**30**], suitable assumptions on the domain Ω of u are also needed.

A similar analysis on the Pólya-Szegő inequality and on the characterization of the equality cases is also carried out in the more general framework of functions of bounded variation. In this case, however, one has to assume that f has linear growth at infinity and to suitably extend the integral by taking into account the singular part of the gradient measure Du, see (4.9).

These results are proved via geometric measure theory arguments based on the isoperimetric theorem, the coarea formula and fine properties of Sobolev and BV functions (the relevant background is collected in Chapter 2). In particular, to deal with the BV case one has to rewrite the original functional, which in principle depends on Du, as a functional defined on the graph of u and depending on the generalized normal to the graph.

The latter approach could be also carried out in the Sobolev case and therefore we could have chosen to deal from the beginning with BV functions and then to deduce the Sobolev case as a corollary. However, we have preferred to give in the Sobolev case an independent proof that avoids the heavy machinery required in the BV case.

It is also worth mentioning that, though the general strategy follows the path set up in previous papers, namely in [**23**] and [**30**], we have to face here an extra substantial difficulty which appears only when dealing with the Steiner rearrangement in codimension strictly larger than 1. This difficulty appears for those functions that Almgren and Lieb, in [**3**], called *coarea irregular* (see the discussion at the end of Section 4.1). These functions, which can even be of class C^1, are precisely the ones where Schwarz rearrangement in discontinuous with respect to the $W^{1,p}$ norm.

Finally in Chapter 5 we discuss a quantitative version of the Pólya-Szegő inequality both for the Steiner and the Schwarz rearrangement in the class of concave functions. As already observed in [**28**, **29**], in general one cannot expect to control the L^1 distance $\int_\Omega |u - u^s|$ of a function u from its symmetral u^s only in terms of the gap in the Pólya-Szegő inequality $\int_\Omega |\nabla u|^p - \int_\Omega |\nabla u^p|^2$. In fact, to control the distance between u and u^s, one should also take into account the measure of the set of points where the gradient of u^s is 'small' in a proper sense. Due to this fact, the only available estimate is a rather complicate expression containing both the gap in the Pólya-Szegő inequality and the measure of the set where ∇u^s is small. This estimate is very far from being optimal—see [**26**].

The advantage of dealing with concave functions is that instead in this case is possible to estimate $\|u - u^s\|_{L^1}$ using only the gap $\|\nabla u\|^p_{L^p} - \|\nabla u^s\|^p_{L^p}$. Indeed, this is done in Chapter 5 where we present a few results obtained in our forthcoming paper [**9**]. In particular, we show that the quantitative estimate we obtain is optimal when $1 < p \leq 2$.

In Part II, we present the results contained in our paper [**21**], where we consider a variational model used to describe formation of nano-structures. The role of roughness appearing onto the surfaces and interfaces of nano-structures has been proved to be of great significance in several fields such as micro-electronics, metallurgy and materials science. For instance the roughness can strongly modify the mechanical properties of multilayered structures as confirmed by the observation that dislocations, islands and cracks can be generated from a rough surface (see [**33**]). Many efforts have been devoted to the investigation on how to control the roughness appearing onto the surfaces and interfaces of nano-structures, leading to the study of the so-called Driven Rearrangement Instability, i.e., the morphological surfaces instability of interfaces between solids generated by elastic stress. This phenomenon has been detected, for instance, in hetero-epitaxial growth of thin films with a lattice mismatch between film and substrate and in stressed elastic solids with cavities.

The theoretical investigation of the stability of the free surface of a planar non-hydrostatically stressed solid has been performed in the pioneering papers by Asaro and Tiller [**7**] and Grinfeld [**51**]. These authors showed that the free surface is unstable with respect to a given family of sinusoidal fluctuations. They also gave a first insightful description of the phenomenon, nowadays named Asaro-Grinfeld-Tiller instability, in which a thin film growing on a flat substrate remains flat up to a critical value of the thickness, after which, the free surface becomes unstable developing corrugations and irregularities. This instability is explained as a consequence of the presence of two competing energies, usually identified with a bulk elastic energy and a surface energy. After these results the interest of the scientific community on the rigorous mathematical study of the morphological instabilities has rapidly grown. Starting from the paper [**52**] where Grinfeld follows the Gibbs variational approach to model the morphology of thin films, it became clear that a second order variational analysis could be successfully used. This approach has been used in the context of epitaxial growth first for a one dimensional model in [**12**]. Then in [**11**] and [**42**] the model introduced in [**52**], which is a more realistic two-dimensional model, corresponding to three-dimensional configurations with planar symmetry, is studied and the problem of finding a proper functional setting is successfully addressed. This settled the framework in which a precise and detailed analysis of qualitative properties of regular equilibrium configurations has been carried out by Fusco and Morini in [**45**] via a second order variational analysis. Indeed they prove a sufficient condition for local minimality in terms of the positivity of second variation and provide a sufficiently complete picture of the phenomena that occur in epitaxially-growing thin films.

Such detailed analysis was instead far from being complete in the framework of stressed elastic solids with cavities. Here we perform a second order variational analysis for a two-dimensional variational model that has been recently used to describe surface instability in morphological evolution of cavities in stressed solids (see for instance [**48**, **61**, **64**]) with the aim of deriving new minimality conditions for equilibria and studying their stability. The model can be roughly described as follows. Consider a cavity in an elastic solid, that will be identified with a smooth compact set $F \subset \mathbb{R}^2$, starshaped with respect to the origin. The solid region is assumed to obey to the classical law of linear elasticity, so that the bulk energy can be written in the form

$$\int_{B_{R_0}\setminus F} Q(E(u))\,dz,$$

where $E(u)$ is the symmetric gradient of the elastic displacement u and Q is a bilinear form depending on the material (see Section 6.1 for details). The surface energy is simply assumed to be the length of the boundary of F. Then the energy for a regular configuration is expressed by the functional

$$\mathcal{F}(F,u) := \int_{B_0\setminus F} Q(E(u))\,dz + \mathcal{H}^1(\partial F)\,.$$

In this framework the shape of the void plays a key role in the evolution of cavities in stressed solid bodies, while the effects of the volume changes are negligible. Hence, one usually assumes that the void evolves preserving its volume. The equilibria are therefore identified with minimizers of $\mathcal{F}(F,u)$ under the volume constraint $|F| = d$. Since admissible configurations need not to be regular, the energy of such configurations has to be defined via a relaxation procedure. This issue, together with the study of the regularity of minima, has been addressed (even for more general functionals involving anisotropic surface energies) in [**43**] where, in order to keep track of the possible appearance of cracks, the relaxed functional with respect to the Hausdorff convergence has been studied. The relaxed functional can be expressed in the following form:

$$\mathcal{F}(F,u) := \int_{B_0\setminus F} Q(E(u))\,dz + \mathcal{H}^1(\Gamma_F) + 2\mathcal{H}^1(\Sigma_F)\,, \tag{1.1}$$

where F has finite perimeter, Γ_F is the "regular" part of ∂F and Σ_F represents the cracks (see Section 6.1).

The main result presented here is a quantitative minimality criterion that relies on the study of the second variation of the functional (1.1). To be more precise we prove in Theorem 6.19 that if (F,u) is a smooth critical configuration and the non local quadratic form $\partial^2\mathcal{F}(F,u)$ associated to the second variation of $\mathcal{F}$ at (F,u) is positively defined, then there exists a constant c_0 such that

$$\mathcal{F}(G,v) > \mathcal{F}(F,u) + c_0|G\Delta F|^2 \tag{1.2}$$

for any given admissible configuration (G,v) with G sufficiently close to F in the Hausdorff distance and $G \neq F$. In particular this implies not only that (F,u) is a strict local minimizer of (1.1) but also provides a quantitative estimate of the deviation from minimality for configurations close to (F,u) in the spirit of the recent result obtained in [**1**]. The minimality criterion is then applied to the case of a disk subjected to radial stretching where the second variation can be explicitly estimated to prove the local and global minimality of the round configuration if the applied stress is sufficiently small.

We point out that an important open problem is how to remove the assumption of starshapedness. Indeed, even the explicit form of the relaxed functional is unknown.

Part I

Symmetrization techniques

CHAPTER 2

Background

We give here the basic definitions and a fast review of the mathematical background needed throughout Part I, namely the theory of sets of finite perimeter and BV functions. Most of the cited results are nowadays standard. The reader can refer to, e.g., [**38**], [**5**] or [**50**] for the full details of the theory. The definitions will be given here in general codimension k, whereas in the next chapter we will use $k = 1$.

Given two sets E and F, we denote the *symmetric difference* by $E \triangle F := (E \cup F) \setminus (E \cap F)$. Given two open sets $\omega \subset \Omega$ we write $\omega \Subset \Omega$ if ω is *compactly contained* in Ω, i.e., if $\overline{\omega} \subset \Omega$ and $\overline{\omega}$ is compact. Let $n \geq 2$ and $1 \leq k < n$. We write a generic point $z \in \mathbb{R}^n$ as $z = (x, y)$, where $x \in \mathbb{R}^{n-k}$ and $y \in \mathbb{R}^k$. In order to clarify the different roles of the variables we will also write $\mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k_y$ and $\mathbb{R}^{n+1} = \mathbb{R}^{n-k} \times \mathbb{R}^k_y \times \mathbb{R}_t$.

Given a measurable set $E \subset \mathbb{R}^{n-k} \times \mathbb{R}^k$, for $x \in \mathbb{R}^{n-k}$ we define the section of E at x as

$$E_x := \Big\{ y \in \mathbb{R}^k : (x, y) \in E \Big\} . \tag{2.1}$$

Then we define the *projection* of E as

$$\pi_{n-k}(E) := \Big\{ x \in \mathbb{R}^{n-k} : (x, y) \in E \Big\} \tag{2.2}$$

and the *essential projection* as

$$\pi^+_{n-k}(E) := \Big\{ x \in \mathbb{R}^{n-k} : (x, y) \in E,\ L(x) > 0 \Big\} , \tag{2.3}$$

where $L(x) := \mathcal{L}^k(E_x)$ and $\mathcal{L}^k$ is the k-dimensional Lebesgue measure. We define the *Steiner symmetral (in codimension k)* E^σ of E as

$$E^\sigma := \left\{ (x, y) \in \mathbb{R}^{n-k} \times \mathbb{R}^k : x \in \pi^+_{n-k}(E),\ |y|^k \leq \frac{L(x)}{\omega_k} \right\} , \tag{2.4}$$

where ω_k is the volume of the k-dimensional ball.

When $E \subset \mathbb{R}^{n-k} \times \mathbb{R}^k_y \times \mathbb{R}_t$, its Steiner symmetral E^σ is defined in the same way, after replacing (2.1)–(2.4) by similar definitions. In particular, we set

$$\begin{aligned} E^\sigma &:= \left\{ (x, y, t) \in \mathbb{R}^{n-k} \times \mathbb{R}^k_y \times \mathbb{R}_t : (x, t) \in \pi^+_{n-k,t}(E),\ |y|^k \leq \frac{L(x,t)}{\omega_k} \right\} \\ \pi^+_{n-k,t}(E) &:= \Big\{ (x, t) \in \mathbb{R}^{n-k} \times \mathbb{R}_t : (x, y, t) \in E,\ L(x, t) > 0 \Big\} , \end{aligned}$$

where $L(x, t) := \mathcal{L}^{k+1}(E_{x,t})$ and $E_{x,t} := \{y \in \mathbb{R}^k : (x, y, t) \in E\}$.

Given an open set $\Omega \subset \mathbb{R}^n$, we denote with $BV(\Omega)$ the class of functions of bounded variation, i.e., the family of functions in $L^1(\Omega)$ whose distributional gradient Du is a vector-valued Radon measure in Ω of finite total variation $|Du|(\Omega)$. The space $BV_{\mathrm{loc}}(\Omega)$ is defined accordingly. By Lebesgue's Decomposition Theorem, the measure Du can be split, with respect to the Lebesgue measure, in two parts, the absolutely continuous part $D^{\mathrm{a}}u$ and the singular part $D^{\mathrm{s}}u$. It turns out that $D^{\mathrm{a}}u$ agrees $\mathcal{L}^n$-a.e. with ∇u, the approximate gradient of u (see, e.g., [**5**, Definition 3.70]). Moreover, the set $\mathcal{D}_u$ of all points where u is approximately differentiable satisfies $|D^{\mathrm{s}}u|(\mathcal{D}_u) = 0$—see, e.g., [**38**, §6.1, Theorem 4] or [**5**, Theorem 3.83].

A measurable set $E \subset \mathbb{R}^n$ is said to be of *finite perimeter* in an open set $\Omega \subset \mathbb{R}^n$ if $D\chi_E$ is a vector-valued Radon measure with finite total variation in Ω. The perimeter of E in a Borel subset B of Ω is defined as $P(E;B) := |D\chi_E|(B)$. For $B = \mathbb{R}^n$ we will simply write $P(E)$; if $\chi_E \in BV_{\text{loc}}(\Omega)$ then we say that E has *locally finite perimeter* in Ω.

Denote by u_x the function $u_x : \Omega_x \to \mathbb{R}$ defined by setting $u_x(y) := u(x,y)$ for all $x \in \pi_{n-k}(\Omega)$, $y \in \Omega_x$. From [5, Theorems 3.103 and 3.107] we easily infer that for $\mathcal{L}^{n-k}$-a.e. $x \in \pi_{n-k}(\Omega)$ the function u_x belongs to $BV(\Omega_x)$ and that

$$\partial_i u_x(y) = \partial_{y_i} u(x,y),\ i = 1,\dots,k\,,\ \text{ for } \mathcal{L}^k\text{-a.e. } y \in \Omega_x\,. \tag{2.5}$$

Given any non-negative and measurable function u, we define the *subgraph* of u as

$$\mathcal{S}_u := \Big\{(x,y,t) \in \mathbb{R}^{n+1} : (x,y) \in E,\ 0 < t < u(x,y)\Big\}\,.$$

The following theorem (see [49, §4.1.5, Theorem 1]) completely characterizes functions of bounded variation in terms of their subgraphs. Let us remark that a slightly different notion of subgraph is needed here. In particular we set

$$\mathcal{S}_u^- := \{(x,y,t) \in \mathbb{R}^{n+1} : (x,y) \in \Omega,\ t < u(x,y)\}\,.$$

THEOREM 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $u \in L^1(\Omega)$. Then $\mathcal{S}_u^-$ is a set of finite perimeter in $\Omega \times \mathbb{R}_t$ if and only if $u \in BV(\Omega)$. Moreover, in this case,*

$$P(\mathcal{S}_u^-; B \times \mathbb{R}_t) = \int_B \sqrt{1 + |\nabla u|^2}\, dz + |D^s u|(B)$$

for every Borel set $B \subset \Omega$.

Let E be a set of finite perimeter in an open set $\Omega \subset \mathbb{R}^n$. For $i = 1,\dots,n$ we denote by ν_i^E the derivative of the measure $D_i\chi_E$ with respect to $|D\chi_E|$, that is

$$\nu_i^E(z) = \lim_{r\to 0} \frac{D_i\chi_E(B(r,z))}{|D\chi_E|(B(r,z))}\,, \qquad i = 1,\dots,n\,, \tag{2.6}$$

at every $x \in \Omega$ such that the previous limit exists.

Then, the *reduced boundary* $\partial^* E$ of E consists of all points z of Ω such that the vector $\nu^E(z) := (\nu_1^E(z),\dots,\nu_n^E(z))$ exists and satisfies $|\nu^E(z)| = 1$. The vector $\nu^E(z)$ is called the *generalized inner normal* to E at z. Moreover, denoting by $\mathcal{H}^n$ the n-dimensional Hausdorff measure, the following formulae hold (see, e.g., [5, Theorem 3.59]):

$$\begin{aligned} D\chi_E &= \nu^E \mathcal{H}^{n-1} \llcorner \partial^* E \\ |D\chi_E| &= \mathcal{H}^{n-1} \llcorner \partial^* E \\ |D_i\chi_E| &= |\nu_i^E| \mathcal{H}^{n-1} \llcorner \partial^* E \quad \text{for } i = 1,\dots,n\,. \end{aligned} \tag{2.7}$$

Given any measurable set $E \subset \mathbb{R}^n$, the *density* of E at x is defined as

$$\Theta(E,x) := \lim_{r\to 0} \frac{\mathcal{L}^n(E \cap B(x,r))}{\mathcal{L}^n(B(x,r))}\,,$$

provided that the limit on the right-hand side exists. Then, the *measure theoretic boundary* of E is the Borel set defined as

$$\partial^{\mathcal{M}} E := \mathbb{R}^n \setminus \{x \in \mathbb{R}^n : \text{either } \Theta(E,x) = 0 \text{ or } \Theta(E,x) = 1\}\,.$$

Given any two measurable sets E_1 and E_2 in $\mathbb{R}^n$, we have

$$\partial^{\mathcal{M}}(E_1 \cup E_2) \cup \partial^{\mathcal{M}}(E_1 \cap E_2) \subset \partial^{\mathcal{M}} E_1 \cup \partial^{\mathcal{M}} E_2\,. \tag{2.8}$$

Moreover, if a set E has locally finite perimeter in Ω, the following holds (see, e.g., [5, Theorem 3.61])

$$\partial^* E \cap \Omega \subset \partial^{\mathcal{M}} E \cap \Omega \quad \text{and} \quad \mathcal{H}^{n-1}((\partial^{\mathcal{M}} E \setminus \partial^* E) \cap \Omega) = 0\,. \tag{2.9}$$

The reduced boundary of level sets plays an important role in the *coarea formula* for functions of bounded variations. In its general version (see, e.g., [5, Theorem 3.40]), it says that if $g : \Omega \to [0, +\infty]$ is any Borel function and $u \in BV(\Omega)$, then

$$\int_\Omega g\, d|Du| = \int_{-\infty}^{+\infty} dt \int_{\Omega \cap \partial^* \{u > t\}} g\, d\mathcal{H}^{n-1}. \tag{2.10}$$

The following proposition is a special case of the coarea formula for rectifiable sets (see [5, Theorem 2.93])

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let E be a set of finite perimeter in Ω. Let $g : \Omega \to [0, +\infty]$ be a Borel function. Then*

$$\int_{\partial^* E \cap \Omega} g(z)\, |\nu_y^\Omega(z)|\, d\mathcal{H}^{n-1}(z) = \int_{\pi_{n-k}(\Omega)} dx \int_{(\partial^* E \cap \Omega)_x} g(x, y)\, d\mathcal{H}^{k-1}(y). \tag{2.11}$$

Next theorem links the approximate gradient of a function of bounded variation to the generalized inner normal to its subgraph—see [49, §4.1.5, Theorems 4 and 5].

Theorem 2.3. *Let Ω be an open subset of $\mathbb{R}^n$ and let $u \in BV(\Omega)$. Then*

$$\nu^{\mathcal{S}_u^-}(x, y, t) = \left(\frac{\partial_1 u(x, y)}{\sqrt{1 + |\nabla u|^2}}, \ldots, \frac{\partial_n u(x, y)}{\sqrt{1 + |\nabla u|^2}}, \frac{-1}{\sqrt{1 + |\nabla u|^2}} \right) \tag{2.12}$$

for $\mathcal{H}^n$-a.e. $(x, y, t) \in \partial^* \mathcal{S}_u^- \cap (\mathcal{D}_u \times \mathbb{R}_t)$ and

$$\nu_t^{\mathcal{S}_u^-}(x, y, t) = 0 \quad \textit{for } \mathcal{H}^n\textit{-a.e. } (x, t) \in \partial^* \mathcal{S}_u^- \cap [(\Omega \setminus \mathcal{D}_u) \times \mathbb{R}_t].$$

In particular, if $u \in W^{1,1}(\Omega)$, then (2.12) holds for $\mathcal{H}^n$-a.e. $(x, t) \in \partial^* \mathcal{S}_u^- \cap (\Omega \times \mathbb{R}_t)$.

By Theorem 2.1, if Ω is a bounded open set and $u \in BV(\Omega)$, the set $\mathcal{S}_u^-$ has finite perimeter in $\Omega \times \mathbb{R}_t$. Thus, also $\mathcal{S}_u$ has finite perimeter in $\Omega \times \mathbb{R}_t$; moreover

$$\begin{gathered} \partial^* \mathcal{S}_u \cap (\Omega \times \mathbb{R}_t^+) = \partial^* \mathcal{S}_u^- \cap (\Omega \times \mathbb{R}_t^+) \\ \nu^{\mathcal{S}_u} \equiv \nu^{\mathcal{S}_u^-} \text{ on } \partial^* \mathcal{S}_u \cap (\Omega \times \mathbb{R}_t^+). \end{gathered} \tag{2.13}$$

An important result we will use several times is Vol'pert's Theorem on sections of sets of finite perimeter—see [63] or [5, Theorem 3.108] for the codimension 1 case and [8, Theorem 2.4] for the general case.

Theorem 2.4. *Let E be a set of finite perimeter in $\mathbb{R}^n$. For $\mathcal{L}^{n-k}$-a.e. $x \in \mathbb{R}^{n-k}$ the following assertions hold:*

(i) *E_x has finite perimeter in $\mathbb{R}^k$;*
(ii) *$\mathcal{H}^{k-1}(\partial^*(E_x) \triangle (\partial^* E)_x) = 0$;*
(iii) *For $\mathcal{H}^{k-1}$-a.e. s such that $(x, s) \in \partial^*(E_x)$:*
  (a) *$\nu_y^E(x, s) \neq 0$;*
  (b) *$\nu_y^E(x, s) = \nu^{E_x}(s) |\nu_y^E(x, s)|$.*

In particular, there exists a Borel set $G_E \subset \pi_{n-k}^+(E)$ such that $\mathcal{L}^{n-k}(\pi_{n-k}^+(E) \setminus G_E) = 0$ and (i)–(iii) hold for every $x \in G_E$.

In view of the previous theorem, we will use the same notation $\partial^* E_x$ to denote $(\partial^* E)_x$ and $\partial^*(E_x)$ when they coincide up to $\mathcal{H}^{k-1}$ negligible sets.

REMARK 2.5. Note that in the special case $k = 1$ we have that $\partial^*(E_x) = (\partial^* E)_x$ and $\nu_y^E \neq 0$ for every s such that $(x, s) \in \partial^* E$—see also Remark 3.10 and [8, Remark 3.2].

Given a non-negative measurable function u defined on E such that for $\mathcal{L}^{n-k}$-a.e. $x \in \pi_{n-k}^+(E)$

$$\mathcal{L}^k\left(\{y \in E_x : u(x,y) > t\}\right) < +\infty,\ \forall t > 0\,, \tag{2.14}$$

we define its *Steiner rearrangement (in codimension k)* $u^\sigma : E^\sigma \to \mathbb{R}$ as

$$u^\sigma(x,y) := \inf\left\{t > 0 : \lambda_u(x,t) \le \omega_k |y|^k\right\}\,, \tag{2.15}$$

where

$$\lambda_u(x,t) := \mathcal{L}^k\left(\{y \in \mathbb{R}^k : u_0(x,y) > t\}\right)$$

is the *distribution function (in codimension k)* of $u(x,\cdot)$ and u_0 is the extension of u by 0 outside E. Clearly, $u^\sigma = 0$ in $\mathbb{R}^n \setminus E^\sigma$. Let us observe that

$$u^\sigma(x,\cdot) = \left(u(x,\cdot)\right)^*\,, \tag{2.16}$$

where $\left(u(x,\cdot)\right)^*$ is the Schwarz rearrangement (which is also known as *spherical symmetric decreasing rearrangement*) of u with respect to the last k variables. Let us recall its definition. Given any non-negative measurable function $q : \mathbb{R}^k \to \mathbb{R}$, such that $\mathcal{L}^k(\{y \in \mathbb{R}^k : u(y) > t\})$ is finite for all $t > 0$, the *Schwarz rearrangement* q^* of q is defined as

$$q^*(y) := \inf\{t > 0 : \mu(t) \le \omega_k |y|^k\}\,,$$

where $\mu(t) := \mathcal{L}^k\{y \in \mathbb{R}^k : q(y) > t\}$ is the *distribution function* of u. The Schwarz rearrangement satisfies an important property: it is non-expansive on $L^p(\mathbb{R}^k)$ for every $1 \le p < \infty$ (see, e.g., [57, Theorem 3.5]), i.e., for every $q_1,\ q_2 \in L^p(\mathbb{R}^k)$

$$\int_{\mathbb{R}^k} |q_1^* - q_2^*|^p \le \int_{\mathbb{R}^k} |q_1 - q_2|^p\,,$$

and this clearly implies the continuity of the Schwarz rearrangement on L^p. Given any two non-negative measurable functions u, v defined on E and satisfying (2.14), on applying the previous inequality to $u^*(x,\cdot)$ and $v^*(x,\cdot)$ and integrating with respect to x, we see that

$$\|u^\sigma - v^\sigma\|_{L^p(E^\sigma)} \le \|u - v\|_{L^p(E)}\,, \tag{2.17}$$

for all $1 \le p < +\infty$. In particular the Steiner rearrangement is continuous on L^p.

Let us observe that for every $(x,t) \in \mathbb{R}^{n-k} \times \mathbb{R}_t^+$, then $\mathcal{L}^k((\mathcal{S}_u)_{x,t}) = \lambda_u(x,t)$ and for $\mathcal{L}^{n-k}$-a.e. $x \in \mathbb{R}^{n-k}$ we have $u^\sigma(x,y) > t$ if and only if $\lambda_u(x,t) > \omega_k |y|^k$. Hence, we easily deduce that

$$(\mathcal{S}_u)^\sigma \text{ and } \mathcal{S}_{u^\sigma} \text{ are } \mathcal{L}^{n+1} \text{ equivalent.} \tag{2.18}$$

Moreover also the sets $\{(x,y) : u(x,y) > t\}^\sigma$ and $\{(x,y) : u^\sigma(x,y) > t\}$ are equivalent (modulo $\mathcal{L}^n$) for every $t > 0$. The latter fact assures us that u and u^σ are equidistributed functions. Actually, by the definition of the Steiner rearrangement, for $\mathcal{L}^{n-k}$-a.e. $x \in \pi_{n-k}(E)$ the functions $u(x,\cdot)$ and $u^\sigma(x,\cdot)$ are equidistributed. Therefore, Steiner rearrangement preserves any so-called rearrangement invariant norm of a function, i.e., a norm depending only on the measure of its level sets—here important examples are any Lebesgue, Lorentz or Orlicz norm.

CHAPTER 3

The perimeter inequality for the Steiner symmetrization

As previously said, the aim of this Chapter is to present a fast and elegant proof of the perimeter inequality for the Steiner symmetrization and the characterization of the equality cases.

3.1. Statement of the main results

Recalling the definitions from Chapter 2, we note that for $k = 1$ the Steiner symmetral E^s of a measurable set $E \subset \mathbb{R}^n$ is

$$E^s = \{(x, y) \in \mathbb{R}^n : x \in \pi^+_{n-1}(E),\ |y| \le L(x)/2\}\,, \tag{3.1}$$

where $L(x) = \mathcal{L}^1(E_x)$ is the measure of the section E_x.

Our first result shows the perimeter inequality and establishes some properties of the set when the inequality holds as an equality.

Theorem 3.1. *Let E be a set of finite perimeter in $\mathbb{R}^n$. Then*

$$P(E^s; B \times \mathbb{R}) \le P(E; B \times \mathbb{R}) \tag{3.2}$$

for every Borel set $B \subset \mathbb{R}^{n-1}$. Moreover if $P(E^s) = P(E)$, then either E is equivalent to $\mathbb{R}^n$, or $\mathcal{L}^n(E)$ is finite and for $\mathcal{L}^{n-1}$-a.e. $x \in \pi^+_{n-1}(E)$

(i) *E_x is equivalent to a segment.*

(ii) *The functions $\nu^E_x(x, \cdot)$ and $|\nu^E_y|(x, \cdot)$ are constant on $\partial^* E_x$.*

One might think that conditions (i) and (ii) are enough to conclude that E is Steiner symmetric, but this is not the case. Indeed, the following examples show that though $P(E) = P(E^s)$, the sets E and E^s are not equivalent.

As first example let us consider Figure 3.1. We clearly have $P(E) = P(E^s)$ but E is not equivalent to any translate of E^s. The point here is that E^s (and E) fails to be connected in a "proper sense" in the present setting, although both E and E^s are connected from a strictly topological point of view.

The second example is depicted in Figure 3.1. We clearly have $P(E) = P(E^s)$ and both E and E^s are connected in any reasonable sense. However the two sets are not equivalent. What comes into play now is the fact that $\partial^* E^s$ (and $\partial^* E$) contains straight segments parallel to y, whose projection on the x-axis is an inner point of $\pi^+_{n-1}(E)$. However we stress out that preventing $\partial^* E^s$ and $\partial^* E$ only from containing "non-trivial flat segments parallel to y" is not yet sufficient to ensure the Steiner symmetry of E. Indeed define

$$E = \{(x, y) \in \mathbb{R}^2 : |x| \le 1,\ -2c(|x|) \le y \le c(|x|)\}\,,$$

where $c : [0, 1] \to [0, 1]$ is the decreasing Cantor function with $c(0) = 1$ and $c(1) = 0$. As c is a function of bounded variation, E is a set of finite perimeter and by using Theorem 2.1 we get $P(E) = 10$. Moreover

$$E^s = \{(x, y) \in \mathbb{R}^2 : |x| \le 1,\ |y| \le 3c(|x|)/2\}$$

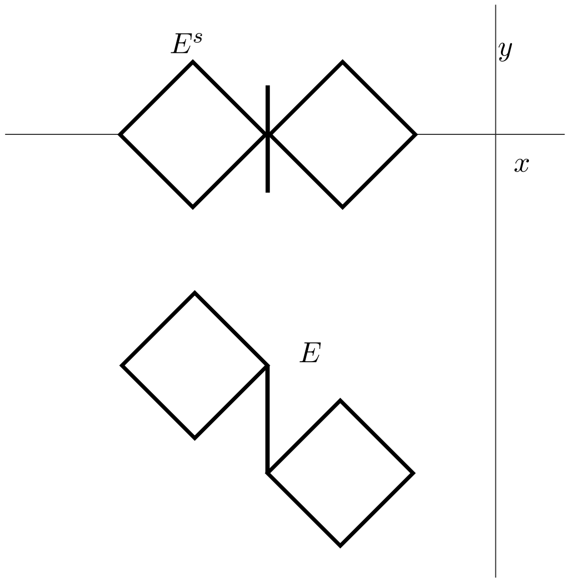


FIGURE 3.1.

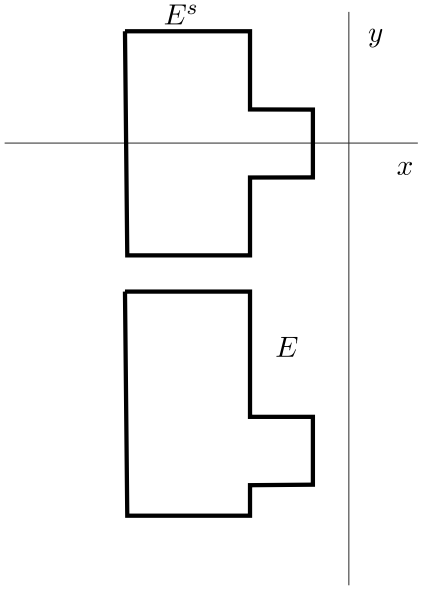


FIGURE 3.2.

and $P(E^s) = 10$. However E is not equivalent to E^s. The problem here is that both $\partial^* E^s$ and $\partial^* E$ contain "uncountably many infinitesimal segments parallel to y" whose total "length" is strictly positive. Therefore for an open set $\Omega \subset \mathbb{R}^{n-1}$, we are led to assume

$$\mathcal{H}^{n-1}\big(\{z \in \partial^* E^s : \nu_y^{E^s}(z) = 0\} \cap (\Omega \times \mathbb{R})\big) = 0\,. \tag{3.3}$$

Roughly speaking, this condition says that we are excluding $\partial^* E^s$ to have non-negligible flat parts parallel to y inside the cylinder $\Omega \times \mathbb{R}$.

Let us note that the following condition

$$\mathcal{H}^{n-1}\big(\{z \in \partial^* E : \nu_y^{E}(z) = 0\} \cap (\Omega \times \mathbb{R})\big) = 0\,. \tag{3.4}$$

is in general weaker with respect to (3.3). However, if we assume that $P(E;\Omega\times\mathbb{R}) = P(E^s;\Omega\times\mathbb{R})$, then the two conditions are equivalent—see Proposition 3.7.

With regard to the issue showed in the first example, a suitable assumption is to assume that the Lebesgue representative L^* of L—see, e.g., [**38**, §1.7.1] for the definition—satisfies

$$L^*(x) > 0 \qquad \text{for } \mathcal{H}^{n-2}\text{-a.e. } x\in\Omega\,. \tag{3.5}$$

We can now state the characterization of the equality cases.

THEOREM 3.2. *Let $\Omega\subset\mathbb{R}^{n-1}$ be a connected open set and E be a set of finite perimeter such that $P(E^s;\Omega\times\mathbb{R}) = P(E;\Omega\times\mathbb{R})$. If conditions (3.3) and (3.5) are satisfied, then $E\cap(\Omega\times\mathbb{R})$ is equivalent to $E^s\cap(\Omega\times\mathbb{R})$ (up to a translation in the y direction).*

3.2. Proofs

We begin giving some properties of the function L and of its distributional and approximate gradients.

LEMMA 3.3. *Let $E\subset\mathbb{R}^n$ be a set of finite perimeter. Then, for $\mathcal{L}^{n-1}$-a.e. $x\in\mathbb{R}^{n-1}$, either $L(x)=+\infty$ or $L(x)<+\infty$. In the latter case, $L\in BV(\mathbb{R}^{n-1})$ and for every Borel set $B\subset\mathbb{R}^{n-1}$*

$$|DL|(B)\le P(E;B\times\mathbb{R}) \qquad \textit{and} \tag{3.6}$$

$$DL(B) = \int_{\partial^*E\cap(B\times\mathbb{R})\cap\{\nu_y^E=0\}} \nu_x^E(x,y)\,d\mathcal{H}^{n-1}(z) + \int_B dx\int_{\partial^*E_x\cap\{\nu_y^E\neq 0\}} \frac{\nu_x^E(x,y)}{|\nu_y^E(x,y)|}d\mathcal{H}^0(y)\,. \tag{3.7}$$

Moreover for $\mathcal{L}^{n-1}$-a.e. $x\in\pi_{n-1}^+(E)$

$$\nabla L(x) = \int_{\partial^*E_x}\frac{\nu_x^E(x,y)}{|\nu_y^E(x,y)|}d\mathcal{H}^0(y), \tag{3.8}$$

PROOF. Note that if L were both infinite and finite on two subsets of $\mathbb{R}^{n-1}$ of positive measure, then it would follow that both E and $\mathbb{R}^n\setminus E$ would have infinite measure. As E is a set of finite perimeter, this is impossible (see e.g., [**5**, Theorem 3.46]). Thus, either L is $\mathcal{L}^{n-1}$-a.e. infinite or finite. In the latter case, we have $\mathcal{L}^{n-1}(\mathbb{R}^{n-1}\setminus E)=+\infty$ and therefore $\mathcal{L}^{n-1}(E)<+\infty$.

Now, let $\varphi\in C_c^1(\mathbb{R}^{n-1})$ and $\{\psi_j\}_{j\in\mathbb{N}}\subset C_c^1(\mathbb{R})$ be any sequence of functions satisfying $0\le\psi_j\le 1$ and $\psi_j\to 1$ pointwise as $j\to\infty$. Then, by Fubini's Theorem, for every $i=1,\dots,n-1$ we have

$$\begin{aligned}\int_{\mathbb{R}^{n-1}}\frac{\partial\varphi}{\partial x_i}(x)L(x)\,dx &= \int_{\mathbb{R}^{n-1}}dx\int_{\mathbb{R}}\frac{\partial\varphi}{\partial x_i}(x)\chi_E(x,y)\,dy\\ &= \lim_{j\to\infty}\int_{\mathbb{R}^n}\frac{\partial\varphi}{\partial x_i}(x)\psi_j(y)\chi_E(x,y)\,dx\,dy\\ &= -\lim_{j\to\infty}\int_{\mathbb{R}^n}\varphi(x)\psi_j(y)\,dD_i\chi_E = -\int_{\mathbb{R}^n}\varphi(x)\,dD_i\chi_E\,.\end{aligned} \tag{3.9}$$

As E is a set of finite perimeter, $\chi_E\in BV(\mathbb{R}^n)$. Therefore, taking the supremum in (3.9) among all $\varphi\in C_c^1$ with $\|\varphi\|_\infty\le 1$, we have $L\in BV(\mathbb{R}^{n-1})$ and for every $\varphi\in C_c^1(\mathbb{R}^{n-1})$

$$\int_{\mathbb{R}^{n-1}}\varphi(x)\,dD_iL(x) = \int_{\mathbb{R}^n}\varphi(x)\,dD_i\chi_E(x,y)\,. \tag{3.10}$$

We claim that the last formula holds true for any bounded Borel function φ as well. Indeed it is sufficient to note that the space $C_c^1(\mathbb{R}^{n-1})$ is dense in $L^1(\mathbb{R}^n,\mu)$ both when $\mu=|D_iL|$ and when μ is the Radon measure defined by

$$\mu(B) = D_i\chi_E(B\times\mathbb{R}) \qquad \text{for every Borel set } B\subset\mathbb{R}^{n-1}\,.$$

Now, for B open, (3.6) follows immediately from (3.10) and then the general case of a Borel set $B \subset \mathbb{R}^{n-1}$ is deduced by approximation. Moreover, we have that

$$DL(B) = \int_{\partial^* E \cap (B \times \mathbb{R})} \nu_x^E(x,y)\, d\mathcal{H}^{n-1}(x,y)\,.$$

Now formula (3.7) follows by writing the integral as the sum of an integral over the set $\partial^* E \cap (B \times \mathbb{R}) \cap \{\nu_y^E = 0\}$ and an integral over the remaining set $\partial^* E \cap (B \times \mathbb{R}) \cap \{\nu_y^E \neq 0\}$. The latter is then calculated using the coarea formula (2.11).

Let G_E be the set given by Theorem 2.4 and assume without loss of generality that L is finite on G_E. By (2.6), (2.7) and (iii) of Theorem 2.4 we have that for every $x \in G_E$ with $(x,y) \in \partial^* E$

$$\frac{\nu_i^E(x,y)}{|\nu_y^E(x,y)|} = \lim_{r \to 0} \frac{D_i \chi_E(B(r,(x,y)))}{|D_y \chi_E|(B(r,(x,y)))}\,.$$

On applying the Besicovitch Differentiation Theorem (see, e.g., [**5**, Theorem 2.22] or [**38**, §1.6]) we have

$$D_i \chi_E \llcorner (G_E \times \mathbb{R}) = \frac{\nu_i^E}{\nu_y^E} |D_y \chi_E| \llcorner (G_E \times \mathbb{R})\,. \tag{3.11}$$

For any function $g \in C_c(\mathbb{R}^{n-1})$ set $\varphi(x) := g(x)\chi_{G_E}(x)$. Then, by (3.10) and (3.11) we have

$$\begin{aligned} \int_{G_E} g(x)\, dD_i L &= \int_{\mathbb{R}^n} g(x) \chi_{G_E}(x)\, dD_i \chi_E \\ &= \int_{G_E \times \mathbb{R}} g(x)\, dD_i \chi_E = \int_{G_E \times \mathbb{R}} \frac{\nu_i^E(x,y)}{\nu_y^E(x,y)} g(x)\, d|D_y \chi_E|\,. \end{aligned} \tag{3.12}$$

By (2.7) and the coarea formula (2.11) we also have

$$\begin{aligned} \int_{G_E \times \mathbb{R}} \frac{\nu_i^E(x,y)}{|\nu_y^E(x,y)|} g(x)\, d|D_y \chi_E| &= \int_{\partial^* E \cap (G_E \times \mathbb{R})} g(x) \nu_i^E(x,y)\, d\mathcal{H}^{n-1} \\ &= \int_{G_E} g(x)\, dx \int_{\partial^* E_x} \frac{\nu_i^E(x,y)}{|\nu_y^E(x,y)|}\, d\mathcal{H}^0(y)\,. \end{aligned} \tag{3.13}$$

Combining (3.12) and (3.13) we get

$$\int_{G_E} g(x)\, dDiL = \int_{G_E} g(x)\, dx \int_{\partial^* E_x} \frac{\nu_i^E(x,y)}{|\nu_y^E(x,y)|}\, d\mathcal{H}^0(y)\,. \tag{3.14}$$

Recalling that g is arbitrary we deduce

$$D_i L \llcorner G_E = \left(\int_{\partial^* E_x} \frac{\nu_i^E}{|\nu_y^E|} d\mathcal{H}^0(y) \right) \mathcal{L}^{n-1} \llcorner G_E$$

and then (3.8) follows, since $\mathcal{L}^{n-1}(\pi_{n-1}^+(E) \setminus G_E) = 0$. □

REMARK 3.4. As an application of the previous lemma with Theorem 2.4 applied to E^s we have that for $\mathcal{L}^{n-1}$-a.e. $x \in G_{E^s}$

$$\nabla L(x) = 2 \frac{\nu^{E_x^s}(x,\cdot)}{|\nu_y^{E^s}(x,\cdot)|}_{|\partial^* E_x^s} = 2 \frac{\nu^{E_x^s}(x, L(x)/2)}{|\nu_y^{E^s}(x, L(x)/2)|}. \tag{3.15}$$

Next lemma provides a first estimate of the perimeter of E^s. We will use it in the proof of Theorem 3.1.

LEMMA 3.5. *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter with $\mathcal{L}^n(E) < +\infty$. Then E^s has finite perimeter and for every Borel set $B \subset \mathbb{R}^{n-1}$*

$$P(E^s; B \times \mathbb{R}) \leq |DL|(B) + |D_y\chi_{E^s}|(B \times \mathbb{R}) \tag{3.16}$$

PROOF. Let $\{L_j\}_{j\in\mathbb{N}} \subset C_c^1(\mathbb{R}^{n-1})$ be a sequence of functions such that $L_j \to L$ a.e. and $|DL_j|(\mathbb{R}^{n-1}) \to |DL|(\mathbb{R}^{n-1})$. Let E_j^s be defined as in (3.1) replacing L with L_j. Let $\Omega \subset \mathbb{R}^{n-1}$ be an open set and $\varphi = (\varphi_1, \dots, \varphi_n) \in C_c^1(\Omega \times \mathbb{R}; \mathbb{R}^n)$. Then by the regularity of L_j we have

$$\begin{aligned}
\int_{\Omega\times\mathbb{R}} \chi_{E_j^s} \operatorname{div}\varphi\, dz &= \int_\Omega dx \int_{-L_j(x)/2}^{L_j(x)/2} \sum_{i=1}^{n-1} \frac{\partial\varphi_i}{\partial x_i} dy + \int_{\Omega\times\mathbb{R}} \chi_{E_j^s} \frac{\partial\varphi_n}{\partial y} dz \\
&= -\frac{1}{2} \int_{\pi_{n-1}(\operatorname{supp}\varphi)} \sum_{i=1}^{n-1} \left[\varphi_i\left(x, \frac{L_j(x)}{2}\right) - \varphi_i\left(x, \frac{-L_j(x)}{2}\right)\right] \frac{\partial L_j}{\partial x_i} dx \\
&\quad + \int_{\Omega\times\mathbb{R}} \chi_{E_j^s} \frac{\partial\varphi_n}{\partial y} dz \\
&\leq \int_{\pi_{n-1}(\operatorname{supp}\varphi)} \sqrt{\sum_{i=1}^{n-1} \left[\frac{1}{2}\left(\varphi_i\left(x, \frac{L_j(x)}{2}\right) - \varphi_i\left(x, \frac{-L_j(x)}{2}\right)\right)\right]^2} |\nabla L_j|\, dx \\
&\quad + \int_{\Omega\times\mathbb{R}} \chi_{E_j^s} \frac{\partial\varphi_n}{\partial y} dz\,.
\end{aligned}$$

Hence, whenever $\|\varphi\|_\infty \leq 1$ we have

$$\int_{\Omega\times\mathbb{R}} \chi_{E_j^s} \operatorname{div}\varphi\, dz \leq |DL_j|(\pi_{n-1}(\operatorname{supp}\varphi)) + \int_{\Omega\times\mathbb{R}} \chi_{E_j^s} \frac{\partial\varphi_n}{\partial y} dz\,.$$

Now, since $\chi_{E_s^j} \to \chi_E$ $\mathcal{L}^n$-a.e. and $\pi_{n-1}(\operatorname{supp}\varphi)$ is compact, taking the lim sup as $j \to \infty$ in the last formula yields

$$\int_{\Omega\times\mathbb{R}} \chi_{E^s} \operatorname{div}\varphi\, dz \leq |DL|(\pi_{n-1}(\operatorname{supp}\varphi)) + \int_{\Omega\times\mathbb{R}} \chi_{E_j^s} \frac{\partial\varphi_n}{\partial y} dz \leq |DL|(\Omega) + |D_y\chi_E|(\Omega\times\mathbb{R})\,.$$

Hence, we proved (3.16) whenever B is open. The general case then follows by approximation. □

We are now in position to prove Theorem 3.1

PROOF OF THEOREM 3.1. **Step 1.** If $L = +\infty\, \mathcal{L}^{n-1}$-a.e. , then E^s is equivalent to $\mathbb{R}^n$ and therefore $P(E^s; B \times \mathbb{R}) = 0$ for every Borel set $B \subset \mathbb{R}^{n-1}$ and (3.2) is fulfilled.

By Lemma 3.3 we have that $L < +\infty\, \mathcal{L}^{n-1}$-a.e. Let G_E and G_{E^S} be the sets given by Theorem 2.4 applied to E and E^s respectively. We now prove (3.2) when either $B \subset \mathbb{R}^{n-1} \setminus G_{E^s}$ or $B \subset G_{E^s}$, the general case following on noting that $B = (B \setminus G_{E^s}) \cup (B \cap G_{E^s})$.

Step 2. Suppose that $B \subset \mathbb{R}^{n-1} \setminus G_{E_s}$. From (2.7), (2.11) and Theorem 2.4 we have

$$\begin{aligned}
|D_y\chi_{E^s}|(B\times\mathbb{R}) &= \int_{\partial^* E^s \cap (B\times\mathbb{R})} |\nu_y^{E^s}|\, d\mathcal{H}^{n-1}(z) \\
&= \int_B \mathcal{H}^0(\partial^* E_x^s)\, dx = \int_{(\mathbb{R}^{n-1}\setminus\pi_{n-1}^+(E))\cap B} \mathcal{H}^0(\partial^* E_x^s)\, dx = 0\,,
\end{aligned}$$

where we used $\mathcal{L}^{n-1}(\pi_{n-1}^+(E) \cap B) = \mathcal{L}^{n-1}(G_{E_s} \cap B) = 0$. Hence, from (3.6) and (3.16) it follows that

$$P(E^s; B \times \mathbb{R}) \leq |DL|(B) \leq P(E; B \times \mathbb{R})\,. \tag{3.17}$$

Step 3. Suppose that $B \subset G_{E^s}$. Therefore, by the coarea formula (2.11) and the fact that $\mathcal{L}^{n-1}(\pi_{n-1}^{+}(E) \setminus G_E)$, we have

$$
\begin{aligned}
P(E^s; B \times \mathbb{R}) &= \int_{\partial^* E^s \cap (B\times\mathbb{R})} d\mathcal{H}^{n-1} = \int_B dx \int_{\partial^* E^s_x} \frac{1}{|\nu_y^{E^s}(x,y)|} d\mathcal{H}^0(y) \\
&= \int_{G_E \cap B} dx \int_{\partial^* E^s_x} \frac{1}{|\nu_y^{E^s}(x,y)|} d\mathcal{H}^0(y) \\
&= \int_{G_E \cap B} dx \int_{\partial^* E^s_x} \sqrt{1 + \sum_{i=1}^{n-1} \left(\frac{\nu_i^{E^s}(x,y)}{\nu_y^{E^s}(x,y)}\right)^2} d\mathcal{H}^0(y) && \text{because } |\nu^{E^s}| = 1 \\
&= \int_{G_E \cap B} dx \int_{\partial^* E^s_x} \sqrt{1 + \frac{1}{4}|\nabla L(x)|}\, d\mathcal{H}^0(y) && \text{from (3.15)} \\
&= \int_{G_E \cap B} \sqrt{4 + |\nabla L|}\, dx \\
&\le \int_{G_E \cap B} \sqrt{\left(\int_{\partial^* E_x} d\mathcal{H}^0(y)\right)^2 + \sum_{i=1}^{n-1}\left(\int_{\partial^* E_x} \frac{\nu_i^E(x,y)}{|\nu_y^E(x,y)|} d\mathcal{H}^0(y)\right)^2}\, dx\,,
\end{aligned} \tag{3.18}
$$

where the last inequality is due to the isoperimetric theorem in $\mathbb{R}$ and (3.8). Applying Jensen's inequality to the convex function $\xi \mapsto \sqrt{1+|\xi|^2}$ we have

$$
\begin{aligned}
&\int_{G_E \cap B} \sqrt{\left(\int_{\partial^* E_x} d\mathcal{H}^0(y)\right)^2 + \sum_{i=1}^{n-1}\left(\int_{\partial^* E_x} \frac{\nu_i^E(x,y)}{|\nu_y^E(x,y)|} d\mathcal{H}^0(y)\right)^2}\, dx \\
&\le \int_{G_E \cap B} dx \int_{\partial^* E_x} \sqrt{1 + \sum_{i=1}^{n-1}\left(\frac{\nu_i^E(x,y)}{\nu_y^E(x,y)}\right)^2} d\mathcal{H}^0(y) \\
&= P(E; (G_E \cap B) \times \mathbb{R}) \le P(E; B \times \mathbb{R})\,.
\end{aligned} \tag{3.19}
$$

Now (3.2) follows from (3.18) and (3.19).

Step 4. If $P(E^s) = P(E)$, then inequality (3.2) implies that for every Borel set $B \subset \mathbb{R}^{n-1}$

$$P(E^s; B \times \mathbb{R}) = P(E; B \times \mathbb{R})\,.$$

Therefore, choosing $B = G_{E^s}$, we have from (3.18)–(3.19), that all the following inequalities

$$
\begin{aligned}
P(E^s; G_{E^s} \times \mathbb{R}) &= \int_{G_{E^s} \cap G_E} \sqrt{4 + |\nabla L(x)|^2}\, dx \\
&\le \int_{G_E \cap G_{E^s}} \sqrt{\left(\int_{\partial^* E_x} d\mathcal{H}^0(y)\right)^2 + \sum_{i=1}^{n-1}\left(\int_{\partial^* E_x} \frac{\nu_i^E(x,y)}{|\nu_y^E(x,y)|} d\mathcal{H}^0(y)\right)^2}\, dx \\
&\le \int_{G_E \cap G_{E^s}} dx \int_{\partial^* E_x} \sqrt{1 + \sum_{i=1}^{n-1}\left(\frac{\nu_i^E(x,y)}{\nu_y^E(x,y)}\right)^2} d\mathcal{H}^0(y) \\
&\le P(E; G_{E^s} \times \mathbb{R})
\end{aligned}
$$

must hold indeed as equalities. The first of which, by the isoperimetric theorem, give us $\mathcal{H}^0(\partial^* E_x) = 2$ for $\mathcal{L}^{n-1}$-a.e. $x \in \mathbb{R}^{n-1}$, therefore yielding that E_x is equivalent to a segment. The second one, by the strict convexity of the function $\xi \mapsto \sqrt{1+|\xi|^2}$ and by the characterization of equality cases in Jensen's inequality, gives that for $\mathcal{L}^{n-1}$-a.e. $x \in G_{E^s}$ the function

$$y \mapsto \frac{\nu_x^E(x,y)}{|\nu_y^E(x,y)|}$$

is constant on $\partial^* E_x$ (note that we are using the counting measure $\mathcal{H}^0$) and, since ν^E is a unit vector, also the function $y \mapsto \nu_y^E(x,y)$ is constant on $\partial^* E_x$. □

We continue the analysis of the equality cases. Next result gives conditions equivalent to (3.3).

LEMMA 3.6. *Let Ω be an open set of $\mathbb{R}^{n-1}$ and E be a set of finite perimeter such that $\mathcal{L}^n(E \cap (\Omega \times \mathbb{R})) < +\infty$. Then the following conditions are equivalent*

(i) $\mathcal{H}^{n-1}(\{z \in \partial^* E^s : \nu_y^{E^s}(z) = 0\} \cap (\Omega \times \mathbb{R})) = 0$.
(ii) $L \in W^{1,1}(\Omega)$.
(iii) *If a Borel set $B \subset \Omega$ satisfies $\mathcal{L}^{n-1}(B) = 0$, then $P(E^s; B \times \mathbb{R}) = 0$.*

PROOF. (i)⇒(ii): by (3.7) if $B \subset \mathbb{R}^{n-1}$ is a set of zero measure, then $DL(B) = 0$ and therefore $L \in W^{1,1}(\Omega)$.

(ii)⇒(iii): first observe that B is the disjoint union of $B \setminus G_{E^s}$ and $B \cap G_{E^s}$ and therefore $P(E^s; B \times \mathbb{R}) = P(E^S; (B \setminus G_{E^s}) \times \mathbb{R}) + P(E^s; (B \cap G_{E^s}) \times \mathbb{R}) =: P_1 + P_2$. If $\mathcal{L}^{n-1}(B) = 0$, from (3.17) and $L \in W^{1,1}$ it follows $P_1 = 0$ and from (3.18) that $P_2 = 0$.

(iii)⇒(i): since $\{z \in \partial^* E^s : \nu_y^{E^s}(z) = 0\} \subset (\pi_{n-1}(\partial^* E^s) \setminus G_{E^s}) \times \mathbb{R}$ we have

$$\mathcal{H}^{n-1}(\{z \in \partial^* E^s : \nu_y^{E^s}(z) = 0\} \cap (\Omega \times \mathbb{R})) \leq P(E^s; (\pi_{n-1}(\partial^* E^s) \setminus G_{E^s}) \times \mathbb{R}) = 0\,,$$

since $\mathcal{L}^{n-1}(\pi_{n-1}(\partial^* E^s) \setminus G_{E^s}) = \mathcal{L}^{n-1}(\pi_{n-1}(\partial^* E^s) \setminus \pi_{n-1}^+(E)) = 0$. □

We show now how conditions (3.3) and (3.4) are mutually related.

PROPOSITION 3.7. *Let E and Ω be as in Lemma 3.6. If*

$$\mathcal{H}^{n-1}(\{z \in \partial^* E : \nu_y^E = 0\} \cap (\Omega \times \mathbb{R})) = 0\,, \tag{3.20}$$

then

$$\mathcal{H}^{n-1}(\{z \in \partial^* E^s : \nu_y^{E^s} = 0\} \cap (\Omega \times \mathbb{R})) = 0\,. \tag{3.21}$$

Conversely, if E satisfies $P(E; \Omega \times \mathbb{R}) = P(E^s; \Omega \times \mathbb{R})$, then condition (3.21) implies (3.20).

PROOF. If (3.20) holds, by (3.7) we have $L \in W^{1,1}(\Omega)$ and hence by the previous lemma (3.21) holds as well.

Conversely, if $P(E^s; \Omega \times \mathbb{R}) = P(E; \Omega \times \mathbb{R})$, by (3.2) and by the previous lemma, for every Borel set $B \subset \Omega$ of zero measure it holds

$$P(E; B \times \mathbb{R}) = P(E^s; B \times \mathbb{R}) = 0\,.$$

Now the conclusion follows arguing exactly as in the proof of the implication (iii)⇒(i) in the previous lemma, having replaced E^s by E. □

We introduce the definition of the barycenter, which will be the key tool to prove Theorem 3.2.

DEFINITION 3.8. The *barycenter of the sections* of a set E is the function $b : \mathbb{R}^{n-1} \to \mathbb{R}$ defined as

$$b(x) := \begin{cases} \dfrac{1}{L(x)} \displaystyle\int_{E_x} y\, dy & \text{if } 0 < L(x) < \infty \text{ and } |y| \in L^1(E_x)\,, \\ 0 & \text{otherwise.} \end{cases}$$

The following theorem gives the regularity of the barycenter and provides an explicit formula for its gradient.

THEOREM 3.9 (Properties of the barycenter). *Let $E \subset \mathbb{R}^n$ and let $\Omega \subset \mathbb{R}^{n-1}$ be an open set such that E has finite perimeter in $\Omega \times \mathbb{R}$. Assume that E_x is equivalent to a segment for $\mathcal{L}^{n-1}$-a.e. $x \in \Omega$. If conditions (3.4) and (3.5) are satisfied, then $b \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and for every $i = 1, \ldots, n-1$*

$$\partial_i b(x) = \frac{1}{L(x)} \int_{\partial^* E_x} [y - b(x)] \frac{\nu_i^E(x,y)}{|\nu_y^E(x,y)|} d\mathcal{H}^0(y) . \tag{3.22}$$

As previously said, in order to keep the exposition short and elegant we give here the proof in a simpler case. Namely we assume the set E to be bounded. We refer to [8, Theorem 4.3] for the general case. We explicitly note that the general proof is much more involved and requires a slicing argument in higher codimension.

PROOF. As E is bounded, the function $x \in \pi_{n-1}(E) \mapsto m(x) := \int_{E_x} y\, dy$ is bounded in $\pi_{n-1}(E)$. Arguing as in the proof of Lemma 3.3 we get that $m \in BV_{\mathrm{loc}}(\pi_{n-1}(E))$ and that for $\mathcal{L}^{n-1}$-a.e. $x \in \pi_{n-1}(E)$

$$\nabla m(x) = \int_{\partial^* E_x} y \frac{\nu_x^E(x,y)}{|\nu_y^E(x,y)|} d\mathcal{H}^0(y) , \tag{3.23}$$

where ∇m stands for the absolutely continuous part of Dm with respect to $\mathcal{L}^{n-1}$. By Lemma 3.6, we have that L is a Sobolev function. Let us now prove that the same assumptions imply that $m \in W^{1,1}_{\mathrm{loc}}$. Indeed the same argument used to prove (3.17) shows that for every Borel set $B \subset \pi_{n-1}(E)$ it holds

$$|Dm|(B) \le M\, P(E; B \times \mathbb{R}) ,$$

where M is a constant such that $E \subset \mathbb{R}^{n-1} \times (-M, M)$. Hence, if $\mathcal{L}^{n-1}(B) = 0$ by (3.2) and Lemma 3.6 we have

$$P(E; B \times \mathbb{R}) = P(E^s; B \times \mathbb{R}) = 0$$

and therefore Dm is absolutely continuous with respect to $\mathcal{L}^{n-1}$.

Without loss of generality we can assume L to coincide with its Lebesgue representative L^*. By (3.5) L is a strictly positive function. Hence, $b \in W^{1,1}_{\mathrm{loc}}(\Omega)$. Now, from (3.23) and (3.8) we have, recall $b = m/L$, that for every $i = 1, \ldots, n-1$

$$\begin{aligned}
\partial_i b(x) = \partial_i \left(\frac{m(x)}{L(x)} \right) &= -\frac{\partial_i L(x)}{|L(x)|^2} m(x) + \frac{1}{L(x)} \int_{\partial^* E_x} y \frac{\nu_i^E(x,y)}{|\nu_y^E(x,y)|} d\mathcal{H}^0(y) \\
&= -\frac{b(x)}{L(x)} \int_{\partial^* E_x} \frac{\nu_i^E(x,y)}{|\nu_y^E(x,y)|} d\mathcal{H}^0(y) + \frac{1}{L(x)} \int_{\partial^* E_x} y \frac{\nu_i^E(x,y)}{|\nu_y^E(x,y)|} d\mathcal{H}^0(y) \\
&= \frac{1}{L(x)} \int_{\partial^* E_x} [y - b(x)] \frac{\nu_i^E(x,y)}{|\nu_y^E(x,y)|} d\mathcal{H}^0(y) .
\end{aligned}$$

□

Now the proof of Theorem 3.2 is a direct consequence of the properties of the barycenter.

PROOF OF THEOREM 3.2. By Theorem 3.1 we have that for $\mathcal{L}^{n-1}$-a.e. $x \in \Omega$ the section E_x is equivalent to a segment and that $\nu_x^E(x,\cdot)$ and $|\nu_y^E|(x,\cdot)$ are constant on $\partial^* E_x$. By Proposition 3.7 condition (3.4) is satisfied. By Theorem 3.9 we have that

$$\partial_i b(x) = \frac{1}{L^*(x)} \frac{\nu_i^E(x)}{|\nu_y^E(x)|} \int_{\partial^* E_x} [y - b(x)]\, d\mathcal{H}^0(y) = 0 ,$$

where we dropped the variable y for functions that are constant in $\partial^* E_x$. Moreover $b \in W^{1,1}_{\text{loc}}(\Omega)$ and since $\nabla b = 0$ we have that b is constant in Ω. □

REMARK 3.10 (The higher codimension case). In Chapter 2 we have introduced the Steiner symmetrization in any codimension k. It is natural to ask whether results similar to the ones just proved hold for $k > 1$. The answer is positive and we refer to the recent paper [**8**].

Note that in the higher codimension case, despite the general strategy being similar, the proofs are actually much more delicate due to the fact that the Radon measure

$$B \subset \mathbb{R}^{n-k} \mapsto \mu(B) = \int_{\partial^* E \cap (B \times \mathbb{R}^k) \cap \{\nu_y^E = 0\}} \nu_x^E(x,y)\, d\mathcal{H}^{n-k}$$

has a different behaviour depending on whether $k = 1$ or $k > 1$. Indeed, when $k = 1$, μ is purely singular with respect to $\mathcal{L}^{n-k}$, while for $k > 1$ it may contain a non-trivial absolutely continuous part. In other words, assume that $\mathcal{H}^n\{z \in \partial^* E : \nu_y^E(z) = 0\} = 0$ (i.e., the "flat pieces of the boundary parallel to y" are negligible). Then, when $k = 1$ its projection on $\mathbb{R}^{n-k}$ is a negligible set with respect to $\mathcal{L}^{n-k}$, while if $k > 1$ this projection may be smeared out on a set of positive $\mathcal{L}^{n-k}$ measure.

A completely similar issue arises for the Steiner rearrangement in codimension strictly greater than 1—see the discussion at the end of Section 4.1.

CHAPTER 4

The Pólya-Szegő inequality

In this Chapter we analyze the Steiner rearrangement in any codimension of Sobolev and BV functions. In particular, we prove a Pólya-Szegő inequality for a large class of convex integrals. Then, we give minimal assumptions under which functions attaining equality are necessarily Steiner symmetric. The chapter is organized as follows. In Section 4.1 we state and comment the main results. Section 4.2 is devoted to Sobolev functions while Section 4.3 deals with BV functions and functionals depending on the normal.

4.1. Statement of the main results

Let $f : \mathbb{R}^n \to [0, +\infty)$ be a non-negative convex function vanishing at 0. We say that f is radially symmetric with respect to the last k variables if there exists a function $\tilde{f} : \mathbb{R}^{n-k+1} \to [0, +\infty)$ such that

$$f(x, y) = \tilde{f}(x, |y|)\,, \tag{4.1}$$

for every $(x, y) \in \mathbb{R}^n$.

Given f as above and an open set Ω, we are interested in studying how functionals of the type

$$u \mapsto \int_\Omega f(\nabla u)\, dz$$

behave under Steiner rearrangement. The class of admissible functions for these functionals will be

$$W^{1,1}_{0,y}(\Omega) := \left\{ u : \Omega \to \mathbb{R} : u_0 \in W^{1,1}(\omega \times \mathbb{R}^k_y),\ \forall \omega \Subset \pi_{n-k}(\Omega),\ \omega \text{ open} \right\}.$$

Roughly speaking, $W^{1,1}_{0,y}(\Omega)$ consists of those functions that are locally Sobolev with respect to the x variable and globally Sobolev with zero trace (in some appropriate sense) with respect to the y variable. Let us remark that this space is bigger than $W^{1,1}_0(\Omega)$. For instance, if $\Omega = [0, 2\pi]^2$, the function $u = (\sin y)/x \in W^{1,1}_{0,y}(\Omega)$ but does not belong to $W^{1,1}_0(\Omega)$. We can define, in a similar way, also the space $W^{1,p}_{0,y}(\Omega)$ for $p > 1$. For $\nabla u = (\partial_1 u, \ldots, \partial_n u)$ we set

$$\nabla_x u := (\partial_1 u, \ldots, \partial_{n-k} u) \text{ and } \nabla_y u := (\partial_{n-k+1} u, \ldots, \partial_n u),$$

where $\partial_i u := \partial_{z_i} u(z)$ for $i = 1, \ldots, n$.

Note that the Steiner rearrangement maps $W^{1,1}_{0,y}(\Omega)$ to $W^{1,1}_{0,y}(\Omega^\sigma)$ (see [**15**] and Proposition 4.9 below). Let us remark that in general the mapping is not continuous, see [**3**].

We can now state the Pólya-Szegő principle for the Steiner rearrangement.

THEOREM 4.1. *Let f be a non-negative convex function, vanishing at* 0 *and satisfying* (4.1). *Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in W^{1,1}_{0,y}(\Omega)$ be a non-negative function. Then*

$$\int_{\Omega^\sigma} f(\nabla u^\sigma)\, dz \le \int_\Omega f(\nabla u)\, dz\,. \tag{4.2}$$

In Theorem 4.1 the space $W^{1,1}_{0,y}(\Omega)$ can be replaced by any space $W^{1,p}_{0,y}(\Omega)$, see Remark 4.13.

We will call u an *extremal* if equality holds in (4.2). We are now interested to find minimal assumptions to have a rigidity theorem for the extremals, i.e., in finding conditions that necessarily imply an extremal u to be Steiner symmetric. It turns out that these assumptions concern both the function u and the domain Ω.

Regarding u, we set, for $x \in \pi_{n-k}(\Omega)$,

$$M(x) := \inf\{t > 0 : \lambda_u(x,t) = 0\}\,.$$

Clearly, for $\mathcal{L}^{n-k}$-a.e. $x \in \pi_{n-k}(\Omega)$,

$$M(x) = \operatorname{ess\,sup}\{u(x,y) : y \in \Omega_x\}.$$

Also, M is a measurable function in $\pi_{n-k}(\Omega)$ and by (2.14) is finite for $\mathcal{L}^{n-k}$-a.e. $x \in \pi_{n-k}(\Omega)$. We require that

$$\mathcal{L}^n(\{(x,y) \in \Omega : \nabla_y u(x,y) = 0\} \cap \{(x,y) \in \Omega : \text{either } M(x) = 0 \text{ or } u(x,y) < M(x)\}) = 0\,. \tag{4.3}$$

Note that this condition is similar to (3.4). Roughly speaking, this condition means that the subgraph of u does not contain any non trivial portion of a k-dimensional hyperplane in the y-direction, except at the highest value of $u(x,\cdot)$.

REMARK 4.2. It is known that the Schwarz rearrangement, in dimension $n \geq 2$, shrinks the set of critical points of a Sobolev function (see [**3**]), while the Steiner rearrangement in codimension 1 preserves its measure (see [**17**]). Hence, by (2.16) and using the fact that the Steiner rearrangement of a Sobolev function is still weakly differentiable (see Proposition 4.9), we have

$$\begin{aligned}
\mathcal{L}^n(\{(x,y) \in \Omega : \nabla_y u(x,y) = 0\}) &= \int_{\pi_{n-k}(\Omega)} \mathcal{L}^k(\{\nabla u(x,\cdot) = 0\})\, d\mathcal{L}^{n-k}(x) \\
&\leq \int_{\pi_{n-k}(\Omega)} \mathcal{L}^k(\{\nabla (u(x,\cdot))^* = 0\})\, d\mathcal{L}^{n-k}(x) \\
&= \mathcal{L}^n(\{(x,y) \in \Omega^\sigma : \nabla_y u^\sigma(x,y) = 0\})\,.
\end{aligned}$$

Therefore, if u satisfies (4.3) then the same holds for u^σ.

Regarding the open set Ω, we require that

$$\pi_{n-k}(\Omega) \text{ is connected and } \Omega \text{ is bounded in the } y \text{ direction,} \tag{4.4}$$

i.e., there exists $M > 0$ such that $\Omega_x \subset B(0,M)$ for every $x \in \pi_{n-k}(\Omega)$, where $B(0,M)$ is the ball in $\mathbb{R}^k$ of radius M centered in 0. We also require that, in some sense, the boundary of Ω is almost nowhere parallel to the y-direction inside the cylinder $\pi_{n-k}(\Omega) \times \mathbb{R}^k_y$. To be precise, as we already did in the previous chapter, we shall assume that

$$\begin{gathered}
\Omega \text{ is of finite perimeter inside } \pi_{n-k}(\Omega) \times \mathbb{R}^k_y \text{ and} \\
\mathcal{H}^{n-1}(\{(x,y) \in \partial^*\Omega : \nu^\Omega_y = 0\} \cap \{\pi_{n-k}(\Omega) \times \mathbb{R}^k_y\}) = 0\,.
\end{gathered} \tag{4.5}$$

We can now state the following result which gives a characterization of the equality cases in (4.2).

THEOREM 4.3. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a non-negative strictly convex function satisfying (4.1) and vanishing at 0. Let $\Omega \subset \mathbb{R}^n$ be an open set satisfying (4.4)–(4.5) and let $u \in W^{1,1}_{0,y}(\Omega)$ be a non-negative function. If*

$$\int_{\Omega^\sigma} f(\nabla u^\sigma)\, dz = \int_\Omega f(\nabla u)\, dz < +\infty\,, \tag{4.6}$$

then, for $\mathcal{L}^{n-k+1}$-a.e. $(x,t) \in \pi^+_{n-k,t}(\mathcal{S}_u)$ there exists $R(x,t) > 0$ such that the set

$$\{y : u(x,y) > t\} \text{ is equivalent to } \{|y| < R(x,t)\}\,.$$

If in addition u satisfies (4.3), then u^σ is equivalent to u up to a translation in the y-plane.

At first sight, one could think that the assumptions made in the above statements are too strong. However, one can easily construct counterexamples even in codimension 1 (see [**30**]) showing that assumptions (4.3)−(4.5) cannot be weakened.

As we have seen before, if u satisfies condition (4.3), then the same condition holds for u^σ. In general the converse is not true, as one can see with some simple examples. However, it turns out that if equality holds in the Pólya-Szegő inequality, then the two conditions are equivalent (see also Proposition 3.7).

PROPOSITION 4.4. *Let f and Ω be as in Theorem 4.3 and let $u \in W^{1,1}_{0,y}(\Omega)$ be a non-negative function. If equality (4.6) holds, then*

$$\mathcal{L}^n(\{(x,y) \in \Omega : \nabla_y u(x,y) = 0\} \cap \{(x,y) \in \Omega : \text{either } M(x) = 0 \text{ or } u(x,y) < M(x)\}) = 0$$

if and only if

$$\mathcal{L}^n(\{(x,y) \in \Omega^\sigma : \nabla_y u^\sigma(x,y) = 0\} \cap \{(x,y) \in \Omega^\sigma : \text{either } M(x) = 0 \text{ or } u^\sigma(x,y) < M(x)\}) = 0\,. \tag{4.7}$$

We now shift to the more general framework of functions of bounded variation. In this context, it is still possible to show a Pólya-Szegő principle, provided that the involved functional is properly defined. Consider any non-negative convex function in $\mathbb{R}^n$ growing linearly at infinity, i.e., for all $z \in \mathbb{R}^n$

$$0 \le f(z) \le C(1 + |z|)\,, \tag{4.8}$$

for some positive constant C. Let us now define the *recession function* f_∞ of f as

$$f_\infty(z) := \lim_{t\to+\infty} \frac{f(tz)}{t}\,.$$

Then a standard extension of the functional $\int_\Omega f(\nabla u)$ to the space $BV_{\mathrm{loc}}(\Omega)$ is defined as

$$J_f(u;\Omega) := \int_\Omega f(\nabla u)\,dz + \int_\Omega f_\infty\left(\frac{D^s u}{|D^s u|}\right) d|D^s u|\,. \tag{4.9}$$

Actually, Theorem 4.22 states that $J_f(u;\Omega)$ coincides with the so-called *relaxed functional* of $\int_\Omega f(\nabla u)$ in $BV(\Omega)$ with respect to the L^1_{loc}-convergence.

Then, a Pólya-Szegő principle for functionals of the form (4.9) holds in the space of $BV_{\mathrm{loc}}(\Omega)$ functions vanishing in some appropriate sense on $\partial\Omega \cap (\pi_{n-k}(\Omega) \times \mathbb{R}^k_y)$. To be precise, we set

$$BV_{0,y}(\Omega) := \Big\{u : \Omega \to \mathbb{R} \mid u_0 \in BV(\omega \times \mathbb{R}^k_y) \text{ and } |Du_0|(\omega \times \mathbb{R}^k_y) = |Du|(\Omega \cap (\omega \times \mathbb{R}^k_y)) \\ \text{for every open set } \omega \Subset \pi_{n-k}(\Omega)\Big\}\,.$$

THEOREM 4.5. *Let $f : \mathbb{R}^n \to [0,+\infty)$ be a convex function vanishing at 0 and satisfying (4.1) and (4.8). Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in BV_{0,y}(\Omega)$ be a non-negative function. Then $u^\sigma \in BV(\omega \times \mathbb{R}^k_y)$ for every open set $\omega \Subset \pi_{n-k}(\Omega)$ and*

$$J_f(u^\sigma;\Omega^\sigma) \le J_f(u;\Omega)\,. \tag{4.10}$$

As before, we are interested in finding suitable conditions ensuring that a function satisfying the equality in (4.10) is Steiner symmetric. It turns out that one needs the same assumptions on u and Ω as in Theorem 4.3. Note that now the vector $\nabla_y u$ in (4.3) is the y-component of the absolutely continuous part of the measure Du. However, in order to deal with the singular part $D^s u$ of Du we need some extra assumptions on the recession function f_∞. We will assume that for every $x \in \mathbb{R}^{n-k}$, setting $f_\infty(x,y) = \tilde{f}_\infty(x,|y|)$,

$$\tilde{f}_\infty(x,\cdot) \text{ is strictly increasing on } [0,+\infty) \tag{4.11}$$

and that the function

$$x \mapsto \tilde{f}_\infty(x,1) \text{ is strictly convex,} \tag{4.12}$$

THEOREM 4.6. *Let $f : \mathbb{R}^n \to [0,+\infty)$ be a strictly convex function vanishing at 0 and satisfying (4.1), (4.8), (4.11) and (4.12). Let $\Omega \subset \mathbb{R}^n$ be an open set satisfying (4.4)–(4.5) and let $u \in BV_{0,y}(\Omega)$ be a non-negative function such that*

$$J_f(u^\sigma;\Omega^\sigma) = J_f(u;\Omega) < +\infty\,, \tag{4.13}$$

Then, for $\mathcal{L}^{n-k+1}$-a.e. $(x,t) \in \pi^+_{n-k}(\mathcal{S}_u)$ there exists $R(x,t) > 0$ such that the set

$$\{y : u(x,y) > t\} \text{ is equivalent to } \{|y| < R(x,t)\}\,.$$

If in addition u satisfies condition (4.3), then u is equivalent to u^σ up to a translation in the y-plane.

The strategy in proving Theorems 4.5 and 4.6 is to convert the functional J_f into a geometrical functional depending on the generalized inner normal and having the form

$$\int_{\partial^* E} F(\nu^E)\, d\mathcal{H}^n\,. \tag{4.14}$$

Here, $F : \mathbb{R}^{n+1} \to [0,+\infty]$ is a convex function positively 1-homogeneous vanishing at 0, i.e., for every $\lambda > 0$ and $(\xi_1,\dots,\xi_{n+1}) \in \mathbb{R}^{n+1}$

$$F(\lambda\xi_1,\dots,\lambda\xi_{n+1}) = \lambda F(\xi_1,\dots,\xi_{n+1}) \quad \text{and } F(0) = 0\,. \tag{4.15}$$

Let us define

$$F_f(\xi_1,\dots,\xi_{n+1}) := \begin{cases} f\left(-\frac{1}{\xi_{n+1}}(\xi_1,\dots,\xi_n)\right)(-\xi_{n+1}) & \text{if } \xi_{n+1} < 0\,, \\ f_\infty(\xi_1,\dots,\xi_n) & \text{if } \xi_{n+1} \geq 0\,. \end{cases} \tag{4.16}$$

The following result gives the link between the functional J_f and the functional in (4.14).

PROPOSITION 4.7 ([**30**, Proposition 2.7]). *Let $f : \mathbb{R}^n \to [0,+\infty)$ be a convex function vanishing at 0 and satisfying (4.8). Then F_f is a convex function satisfying (4.15). Moreover, if $\Omega \subset \mathbb{R}^n$ is an open set, then for every non-negative function $u \in BV_{\mathrm{loc}}(\Omega)$*

$$J_f(u;\Omega) = \int_{\partial^*\mathcal{S}_u \cap (\Omega\times\mathbb{R}_t)} F_f(\nu^{\mathcal{S}_u})\, d\mathcal{H}^n\,. \tag{4.17}$$

This allows us to reduce the proof of Theorem 4.5 to the proof of a Pólya-Szegő inequality for functionals of the form (4.14), where in addition we assume that F is radial with respect to the y variables, i.e., there exists a function $\tilde{F} : \mathbb{R}^{n-k+2} \to [0,+\infty]$ such that

$$F(x,y,t) = \tilde{F}(x,|y|,t)\,, \tag{4.18}$$

for every $(x,y,t) \in \mathbb{R}^{n+1}$. Clearly, the function $\tilde{F}$ is convex and positively 1-homogeneous.

It turns out that if F satisfies (4.15) and (4.18) and if $E \subset \mathbb{R}^{n+1}$ is a set of finite perimeter, then

$$\int_{\partial^* E^\sigma} F(\nu^{E^\sigma})\, d\mathcal{H}^n \leq \int_{\partial^* E} F(\nu^E)\, d\mathcal{H}^n\,, \tag{4.19}$$

see Theorem 4.19. Then, Theorem 4.6 is proved thanks to Proposition 4.7 and to a first characterization of the equality cases in (4.19) contained in Proposition 4.20. In addition, an essentially complete characterization of the equality cases in (4.19) is given by Theorem 4.21.

Here, we want to point out that in order to give the characterization of the equality cases in (4.6) one has to face with an extra difficulty. In fact, writing up

$$\begin{aligned}\lambda_u(x,t) &= \mathcal{L}^k(\{y\in\mathbb{R}^k : u_0(x,y)>t\}\cap\{\nabla_y u\neq 0\}) + \mathcal{L}^k(\{y\in\mathbb{R}^k : u_0(x,y)>t\}\cap\{\nabla_y u=0\})\\ &=: \lambda_u^1(x,t)+\lambda_u^2(x,t)\,,\end{aligned}$$

it turns out that $\lambda_u^1(x,t)\in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^{n-k}\times\mathbb{R}^+_t)$, while λ^2 is just a BV function. However, when $k=1$ the distributional derivative $D\lambda_u^2$ is purely singular with respect to the Lebesgue measure on $\mathbb{R}^{n-k}\times\mathbb{R}^+_t$, while if $k>1$ the measure $D\lambda_u^2$ may contain also a non-trivial absolutely continuous part. This fact was first observed in a celebrated paper by Almgren and Lieb [**3**] who showed that this phenomenon may occur even if u is a C^1 function.

4.2. The Sobolev case

In this section we prove the Pólya-Szegő inequality for the Steiner rearrangement in codimension k of Sobolev functions and Theorem 4.3 concerning the equality cases.

Next result, proved in [**8**, Lemma 3.1], deals with some properties of the function L and its derivatives. Recall from Section 4.1 that $L(x):=\mathcal{L}^k(E_x)$. The result is a generalization in higher codimension of Lemma 3.3.

LEMMA 4.8. *Let E be any set of finite perimeter in $\mathbb{R}^n$. Then, either $L(x)=+\infty$ for $\mathcal{L}^{n-k}$-a.e. $x\in\mathbb{R}^{n-k}$ or $L(x)<+\infty$ for $\mathcal{L}^{n-k}$-a.e. $x\in\mathbb{R}^{n-k}$ and $\mathcal{L}^n(E)<+\infty$. Moreover, in the latter case, $L\in BV(\mathbb{R}^{n-k})$ and for any Borel set $B\subset\mathbb{R}^{n-k}$*

$$DL(B)=\int_{\partial^*E\cap(B\times\mathbb{R}^k)\cap\{\nu_y^E=0\}}\nu_x^E(x,y)\,d\mathcal{H}^{n-1}(x,y) + \int_B dx\int_{\partial^*E_x\cap\{\nu_y^E\neq 0\}}\frac{\nu_x^E(x,y)}{|\nu_y^E(x,y)|}\,d\mathcal{H}^{k-1}(y)\,, \tag{4.20}$$

$DL\llcorner G_{E^\sigma}=\nabla L\,\mathcal{L}^{n-k}$ and for $\mathcal{L}^{n-k}$-a.e. $x\in G_{E^\sigma}$

$$\nabla L(x)=\mathcal{H}^{k-1}(\partial^*E_x^\sigma)\frac{\nu_x^{E^\sigma}(x)}{|\nu_y^{E^\sigma}(x)|}\,, \tag{4.21}$$

where we dropped the variable y for functions that are constant in $\partial^*E_x^\sigma$.

We observe that the Steiner rearrangement of a function in $W^{1,1}_{0,y}(\Omega)$ belongs to $W^{1,1}_{0,y}(\Omega^\sigma)$.

PROPOSITION 4.9. *Let $\Omega\subset\mathbb{R}^n$ be an open set and let $u\in W^{1,1}_{0,y}(\Omega)$ be a non-negative function. Then $u^\sigma\in W^{1,1}_{0,y}(\Omega^\sigma)$.*

PROOF. By [**15**, Theorem 8.2] we know that if $v\in W^{1,1}(\Omega)$ is a non-negative function, then v^σ belongs to $W^{1,1}(\Omega^\sigma)$. Given a non-negative function $u\in W^{1,1}_{0,y}(\Omega)$ and fixed $\omega\Subset\pi_{n-k}(\Omega)$ we can find a cut-off function $\varphi\in C^1_c(\pi_{n-k}(\Omega))$ such that $\varphi\equiv 1$ in ω. Hence, the function $v:=\varphi u$ belongs to $W^{1,1}(\Omega)$. Then, $v^\sigma\in W^{1,1}(\Omega^\sigma)$. Besides, $v^\sigma(x,y)=u^\sigma(x,y)$ for all $x\in\omega$ and $y\in\mathbb{R}^k$. This proves the assertion. □

Next lemma gives formulae for the approximate derivatives of the distribution function of a Sobolev function.

LEMMA 4.10. *Let $\Omega\subset\mathbb{R}^n$ be an open and bounded set, $u:\Omega\to\mathbb{R}$ be a non-negative function, $u\in W^{1,1}_{0,y}(\Omega)$ satisfying (4.3). Then, $\lambda_u\in W^{1,1}(\omega\times\mathbb{R}^+_t)$ for every open set $\omega\Subset\pi_{n-k}(\Omega)$ and for $\mathcal{L}^{n-k}$-a.e. $x\in\pi^+_{n-k}(\mathcal{S}_u)$,*

$$\partial_t\lambda_u(x,t)=-\int_{\partial^*\{y:u(x,y)>t\}}\frac{1}{|\nabla_y u|}\,d\mathcal{H}^{k-1}(y), \tag{4.22}$$

$$\partial_i \lambda_u(x,t) = \int_{\partial^*\{y: u(x,y)>t\}} \frac{\partial_i u}{|\nabla_y u|}\, d\mathcal{H}^{k-1}(y)\,,\ i = 1,\ldots,n-k, \tag{4.23}$$

for $\mathcal{L}^1$-a.e. $t \in (0, M(x))$.

PROOF. Let $r > 0$ be large enough to have $\Omega \subset \mathbb{R}^{n-k} \times B(0,r)$ and let $\omega \Subset \pi_{n-k}(\Omega)$ For the sake of simplicity we shall identify the extension u_0 with u. Hence, we may assume that $u \in W^{1,1}(\omega \times \mathbb{R}^k_y)$ and $u(x,y) = 0$ if $|y| > r$.

If $\varphi \in C^1_c(\Omega \times \mathbb{R}^+_t)$, by Fubini's Theorem we get, for $i = 1\ldots,n-k$,

$$\begin{aligned}
\int_{\omega\times\mathbb{R}^+_t} \partial_i\varphi(x,t)\lambda_u(x,t)\,dx\,dt &= \int_{\omega\times\mathbb{R}^k_y\times\mathbb{R}^+_t} \partial_i\varphi(x,t)\chi_{\mathcal{S}_u}(x,y,t)\,dx\,dy\,dt \\
&= \int_{\omega\times\mathbb{R}^k_y} dx\,dy \int_0^{u(x,y)} \partial_i\varphi(x,t)\,dt \\
&= \int_{\omega\times B(0,r)} \partial_i\left[\int_0^{u(x,y)} \varphi(x,t)\,dt\right] dx\,dy - \int_{\omega\times B(0,r)} \varphi(x,u(x,y))\partial_i u(x,y)\,dx\,dy
\end{aligned} \tag{4.24}$$

The first integral in the last expression vanishes over $\omega \times B(0,r)$. Applying the coarea formula (2.11) and recalling that by Theorem 2.4

$$(\partial^*\mathcal{S}_u)_{x,y} \cap \mathbb{R}^+_t = \partial^*(\mathcal{S}_u)_{x,y} \cap \mathbb{R}^+_t = \partial^*(0,u(x,y)) \cap \mathbb{R}^+_t$$

for $\mathcal{L}^n$-a.e. $(x,y) \in \omega \times B(0,r)$, we get

$$\begin{aligned}
&\int_{\partial^*\mathcal{S}_u\cap(\omega\times B(0,r)\times\mathbb{R}^+_t)} \varphi(x,t)\partial_i u(x,y)|\nu_t^{\mathcal{S}_u}(x,y,t)|\,d\mathcal{H}^n \\
&\qquad= \int_{\omega\times B(0,r)} dx\,dy \int_{(\partial^*\mathcal{S}_u)_{x,y}\cap\mathbb{R}^+_t} \varphi(x,t)\partial_i u(x,y)\,d\mathcal{H}^0(t) \\
&\qquad= \int_{\omega\times B(0,r)} \varphi(x,u(x,y))\partial_i u(x,y)\,dx\,dy\,.
\end{aligned} \tag{4.25}$$

Moreover, from (2.12) and (2.13), we have

$$\nu^{\mathcal{S}_u}(x,y,t) = \left(\frac{\nabla_x u(x,y)}{\sqrt{1+|\nabla u|^2}}, \frac{\nabla_y u(x,y)}{\sqrt{1+|\nabla u|^2}}, \frac{-1}{\sqrt{1+|\nabla u|^2}}\right) \tag{4.26}$$

for $\mathcal{H}^n$-a.e. $(x,y,t) \in \partial^*\mathcal{S}_u \cap (\omega \times B(0,r) \times \mathbb{R}^+_t)$.

Combining (4.24)–(4.26), we have

$$\begin{aligned}
\int_{\omega\times\mathbb{R}^+_t} \partial_i\varphi(x,t)\lambda_u(x,t)\,dx\,dt &= -\int_{\partial^*\mathcal{S}_u\cap(\omega\times B(0,r)\times\mathbb{R}^+_t)} \varphi(x,t)\partial_i u(x,y)|\nu_t^{\mathcal{S}_u}(x,y,t)|\,d\mathcal{H}^n \\
&= -\int_{\partial^*\mathcal{S}_u\cap(\omega\times B(0,r)\times\mathbb{R}^+_t)} \varphi(x,t)\partial_i u(x,y)\cdot\frac{1}{\sqrt{1+|\nabla u|^2}}d\mathcal{H}^n.
\end{aligned} \tag{4.27}$$

The last equation implies that the distributional derivative $D_i\lambda_u$ is a finite Radon measure on $\omega \times \mathbb{R}^+_t$. A similar argument shows that the same holds for $D_t\lambda_u$. Therefore, since

$$\int_{\omega\times\mathbb{R}^+_t} \lambda_u(x,t)dx\,dt = \int_{\omega\times\mathbb{R}^k_y} u(x,y)\,dx\,dy < +\infty\,,$$

we get $\lambda_u \in L^1(\omega \times \mathbb{R}^+_t)$ and thus $\lambda_u \in BV(\omega \times \mathbb{R}^+_t)$.

Notice that (4.27) implies that for every $\varphi \in C^1_c(\omega \times \mathbb{R}^+_t)$ we have

$$\int_{\omega\times\mathbb{R}^+_t} \varphi(x,t)\,dD_i\lambda_u = \int_{\partial^*\mathcal{S}_u\cap(\omega\times B(0,r)\times\mathbb{R}^+_t)} \varphi(x,t)\cdot\frac{\partial_i u(x,y)}{\sqrt{1+|\nabla u|^2}}d\mathcal{H}^n\,. \tag{4.28}$$

By density, the same equality holds for $\varphi \in C(\omega \times \mathbb{R}_t^+)$.

We claim that (4.28) holds also for every bounded Borel function in $\omega \times \mathbb{R}_t^+$. In fact, for any Borel set $B \subset \omega \times \mathbb{R}_t^+$, define the Borel measure μ by setting

$$\mu(B) := |D_i\lambda_u|(B) + \mathcal{H}^n\left(\partial^*\mathcal{S}_u \cap (B \times \mathbb{R}_y^k)\right)$$

and let φ be any bounded Borel function in $\omega \times \mathbb{R}_t^+$. By Lusin's Theorem, for any $\varepsilon > 0$ there exists a function $\varphi_\varepsilon \in C(\omega \times \mathbb{R}_t^+)$ such that $\|\varphi_\varepsilon\|_\infty \leq \|\varphi\|_\infty$ and $\mu\{(x,t) : \varphi_\varepsilon(x,t) \neq \varphi(x,t)\} < \varepsilon$. Since φ_ε is continuous, equality (4.28) holds for φ_ε, and hence the absolute value of the difference of the left-hand side and the right-hand side is not greater than $4\varepsilon\|\varphi\|_\infty$. From the arbitrariness of ε, the claim follows.

Let $g \in C_c(\omega \times \mathbb{R}_t^+)$. From (4.28), (4.26) and using condition (4.3) with the coarea formula (2.11), we get

$$\begin{aligned}
\int_{\omega \times \mathbb{R}_t^+} g(x,t)\, dD_i\lambda_u &= \int_{\partial^*\mathcal{S}_u \cap (\omega \times \mathbb{R}_y^k \times \mathbb{R}_t^+)} g(x,t)\partial_i u(x,y) \cdot \frac{1}{\sqrt{1+|\nabla u|^2}} d\mathcal{H}^n \\
&= \int_{\partial^*\mathcal{S}_u \cap (\omega \times \mathbb{R}_y^k \times \mathbb{R}_t^+)} g(x,t) \frac{\partial_i u(x,y)}{|\nabla_y u(x,y)|} |\nu_y^{\mathcal{S}_u}(x,y,t)|\, d\mathcal{H}^n \\
&= \int_{\omega \times \mathbb{R}_t^+} g(x,t)\, dx\, dt \int_{(\partial^*\mathcal{S}_u)_{x,t}} \frac{\partial_i u(x,y)}{|\nabla_y u(x,y)|} d\mathcal{H}^{k-1}(y)\,.
\end{aligned}$$

Since g is arbitrary, we have that the measure $D_i\lambda_u$ is absolutely continuous with respect to $\mathcal{L}^{n-k+1}$ and is equal to

$$\left(\int_{(\partial^*\mathcal{S}_u)_{x,t}} \frac{\partial_i u(x,y)}{|\nabla_y u(x,y)|} d\mathcal{H}^{k-1}(y)\right) \mathcal{L}^{n-k+1}\,,$$

thus proving that $\lambda_u \in W^{1,1}(\omega \times \mathbb{R}_t^+)$. Because of (ii) in Theorem 2.4, equation (4.23) holds for $\mathcal{L}^{n-k+1}$-a.e. $(x,t) \in \pi_{n-k,t}^+(\mathcal{S}_u) \cap (\omega \times \mathbb{R}_t^+)$.

Since

$$\pi_{n-k,t}^+(\mathcal{S}_u) \text{ is equivalent to } \bigcup_{x \in \pi_{n-k}^+(\mathcal{S}_u)} \{x\} \times (0, M(x))\,, \tag{4.29}$$

we see that for $\mathcal{L}^{n-k}$-a.e. $x \in \pi_{n-k}^+(\mathcal{S}_u)$ equation (4.23) holds for $\mathcal{L}^1$-a.e. $t \in (0, M(x))$.

It remains to prove (4.22): this follows from the same calculations and applying (2.5) and (4.20). □

REMARK 4.11. If Ω and u are as in Lemma 4.10, then, by Proposition 4.9 $u^\sigma \in W_{0,y}^{1,1}(\Omega)$, by Remark 4.2 u^σ satisfies condition (4.3) and we get that for $\mathcal{L}^{n-k}$-a.e. $x \in \pi_{n-k}^+(\mathcal{S}_u)$

$$\partial_t\lambda_u(x,t) = -\frac{\mathcal{H}^{k-1}(\partial^*\{y : u^\sigma(x,y) > t\})}{|\nabla_y u^\sigma|}\Big|_{\partial^*\{y:u^\sigma(x,y)>t\}} \tag{4.30}$$

$$\partial_i\lambda_u(x,t) = \mathcal{H}^{k-1}(\partial^*\{y : u^\sigma(x,y) > t\}) \frac{\partial_i u^\sigma}{|\nabla_y u^\sigma|}\Big|_{\partial^*\{y:u^\sigma(x,y)>t\}} \tag{4.31}$$

The following approximation result will be useful in the proof of Theorem 4.1.

LEMMA 4.12. *Let $\omega \subset \mathbb{R}^{n-k}$ be an open set and let $u \in W^{1,p}(\omega \times \mathbb{R}_y^k)$, $p \geq 1$, be a non-negative function. Then for every $\omega' \Subset \omega$ and for every $\varepsilon > 0$ there exists a non-negative Lipschitz function $w : \mathbb{R}^n \to \mathbb{R}$ with compact support such that*

$$\mathcal{L}^n\left(\{z \in \mathbb{R}^n : w(z) > 0,\ \nabla_y w(z) = 0\}\right) = 0 \text{ and} \tag{4.32}$$

$$\|u - w\|_{W^{1,p}(\omega' \times \mathbb{R}_y^k)} < \varepsilon\,. \tag{4.33}$$

PROOF. On multiplying $u(x,y)$ by a smooth compactly supported cut-off function $\varphi : \mathbb{R}^{n-k} \to \mathbb{R}$ with $\varphi \equiv 1$ on ω', we can assume without loss of generality that $u \in W^{1,p}(\mathbb{R}^n)$. By density, for every choice of $\varepsilon > 0$ there exists a non-negative function $u_\varepsilon \in C^1_c(\mathbb{R}^n)$ such that $\|u - u_\varepsilon\|_{W^{1,p}(\mathbb{R}^n)} < \varepsilon$.

Let $r > 1$ be such that $\operatorname{supp} u_\varepsilon \subset B(0,r)$. Standard approximation results assure us that there exists a polynomial p_ε such that $\|u_\varepsilon - p_\varepsilon\|_{C^1(\bar{B}(0,2r))} < \varepsilon/r^{n/p}$. On replacing, if necessary, p_ε with $p_\varepsilon + \varepsilon/r^{n/p} + \delta|y|^2$, for $\delta > 0$ sufficiently small, we may assume p_ε to be strictly positive and $\nabla_y p_\varepsilon \neq 0\, \mathcal{L}^n$-a.e. on $\bar{B}(0,r)$.

Define $\eta_r : \mathbb{R}^n \to \mathbb{R}$ as

$$\eta_r(z) = \begin{cases} 1 & \text{if } |z| \leq r \\ \frac{(4r^2 - |z|^2)}{3r^2} & \text{if } r < |z| \leq 2r \\ 0 & \text{if } |z| > 2r \end{cases}$$

and let $w = p_\varepsilon \eta_r$. Then there exists a constant $c = c(n,p) > 0$ such that $\|u - w\|_{W^{1,p}(\mathbb{R}^n)} < c\varepsilon$ and so equation (4.33) holds.

Finally, (4.32) is proven by considering that $w(z) > 0$ if and only if $z \in B(0,2r)$ and that $w \equiv p_\varepsilon$ on $B(0,r)$ and $w \equiv p_\varepsilon \eta_r$ on $B(0,2r) \setminus \bar{B}(0,r)$ and hence w is still a polynomial with $\nabla_y w \neq 0\, \mathcal{L}^n$-a.e. □

PROOF OF THEOREM 4.1. We are going to prove a stronger inequality that actually implies (4.2), i.e.,

$$\int_{B \times \mathbb{R}^k_y} f(\nabla u^\sigma)\, dz \leq \int_{B \times \mathbb{R}^k_y} f(\nabla u)\, dz\,, \tag{4.34}$$

for every Borel set $B \subset \pi_{n-k}(\Omega)$. As before, we will identify u with its extension u_0. We can assume that the right-hand side of (4.34) has finite value. If not the inequality trivially holds.

Step 1. Let us first prove inequality (4.34) under additional assumptions: we assume that Ω is bounded with respect to the last k components and that $u \in W^{1,1}_{0,y}(\Omega)$ is non-negative and satisfies

$$\mathcal{L}^k(\{y \in \mathbb{R}^k : \nabla_y u(x,y) = 0\} \cap \{y \in \mathbb{R}^k : u(x,y) > 0\}) = 0 \tag{4.35}$$

for $\mathcal{L}^{n-k}$-a.e. $x \in \pi_{n-k}(\Omega)$. By Remark 4.2, equation (4.35) holds also for u^σ. On applying the coarea formula (2.10) and (2.5), we get that

$$\int_{\{y: u^\sigma(x,y) > 0\}} f(\nabla u^\sigma)\, dy = \int_0^{+\infty} dt \int_{\partial^*\{y: u^\sigma(x,y) > t\}} \frac{f(\nabla u^\sigma)}{|\nabla_y u^\sigma|} d\mathcal{H}^{k-1}\,, \tag{4.36}$$

for $\mathcal{L}^{n-k}$-a.e. $x \in \pi_{n-k}(\Omega)$. Hence, for any such x, assumption (4.1) and (4.30)$-$(4.31) give

$$\begin{aligned} &\int_{\partial^*\{y: u^\sigma(x,y) > t\}} \frac{1}{|\nabla_y u^\sigma|} f(\partial_1 u^\sigma, \ldots, \partial_{n-k} u^\sigma, \ldots, \partial_n u^\sigma)\, d\mathcal{H}^{k-1} \\ &\quad = \int_{\partial^*\{y: u^\sigma(x,y) > t\}} \frac{1}{|\nabla_y u^\sigma|} \tilde{f}(\partial_1 u^\sigma, \ldots, \partial_{n-k} u^\sigma, |\nabla_y u^\sigma|)\, d\mathcal{H}^{k-1} \\ &\quad = -\partial_t \lambda_u(x,t) \tilde{f}\left(\frac{\nabla_x \lambda_u(x,t)}{-\partial_t \lambda_u(x,t)}, \frac{\mathcal{H}^{k-1}(\partial^*\{y : u^\sigma(x,y) > t\})}{-\partial_t \lambda_u(x,t)} \right)\,, \end{aligned} \tag{4.37}$$

for $\mathcal{L}^1$-a.e. $t > 0$. Let us note that for $\mathcal{L}^{n-k}$-a.e. $x \in \pi_{n-k}(\Omega)$, the set $\{y : u(x,y) > t\} \subset \mathbb{R}^k$ is of finite perimeter for $\mathcal{L}^1$-a.e. $t > 0$ and $\mathcal{L}^k(\{y : u(x,y) > t\}) < +\infty$ for $t > 0$. By the isoperimetric inequality in $\mathbb{R}^k$,

$$\mathcal{L}^{k-1}(\partial^*\{y : u^\sigma(x,y) > t\}) \leq \mathcal{H}^{k-1}(\partial^*\{y : u(x,y) > t\}) = \int_{\partial^*\{y: u(x,y) > t\}} d\mathcal{H}^{k-1} \tag{4.38}$$

holds for $\mathcal{L}^{n-k}$-a.e. $x \in \pi_{n-k}(\Omega)$, for $\mathcal{L}^1$-a.e. $t > 0$. By assumption (4.1) the function $\tilde{f}(\xi, \cdot)$ is non decreasing in $[0, +\infty)$ for every $\xi \in \mathbb{R}^{n-k}$. Therefore, (4.38) and Lemma 4.10 imply that for $\mathcal{L}^{n-k}$-a.e. $x \in \pi_{n-k}(\Omega)$

$$\begin{aligned} - \partial_t \lambda_u(x,t) \tilde{f}\left(\frac{\nabla_x \lambda_u(x,t)}{-\partial_t \lambda_u(x,t)}, \frac{\mathcal{H}^{k-1}(\partial^*\{y : u^\sigma(x,y) > t\})}{-\partial_t \lambda_u(x,t)} \right) \\ \leq \tilde{f}\left(\frac{\int_D \frac{\partial_1 u}{|\nabla_y u|} d\mathcal{H}^{k-1}}{\int_D \frac{1}{|\nabla_y u|} d\mathcal{H}^{k-1}}, \dots, \frac{\int_D \frac{\partial_{n-k} u}{|\nabla_y u|} d\mathcal{H}^{k-1}}{\int_D \frac{1}{|\nabla_y u|} d\mathcal{H}^{k-1}}, \frac{\int_D d\mathcal{H}^{k-1}}{\int_D \frac{d\mathcal{H}^{k-1}}{|\nabla_y u|}} \right) \cdot \int_D \frac{d\mathcal{H}^{k-1}}{|\nabla_y u|} =: \mathcal{I} \end{aligned} \tag{4.39}$$

for $\mathcal{L}^1$-a.e. $t > 0$, where $D := \partial^*\{y : u(x,y) > t\}$. Recalling that f is convex and so $\tilde{f}$ is, Jensen's inequality gives

$$\mathcal{I} \leq \int_{\partial^*\{y:u(x,y)>t\}} \frac{1}{|\nabla_y u|} \tilde{f}(\nabla_x u, |\nabla_y u|) \, d\mathcal{H}^{k-1}. \tag{4.40}$$

Putting together (4.37), (4.39) and (4.40) we get

$$\begin{aligned} \int_{\partial^*\{y:u^\sigma(x,y)>t\}} \frac{1}{|\nabla_y u^\sigma|} \tilde{f}(\nabla_x u^\sigma, |\nabla_y u^\sigma|) \, d\mathcal{H}^{k-1} \\ \leq \int_{\partial^*\{y:u(x,y)>t\}} \frac{1}{|\nabla_y u|} \tilde{f}(\nabla_x u, |\nabla_y u|) \, d\mathcal{H}^{k-1} \,, \end{aligned} \tag{4.41}$$

for $\mathcal{L}^{n-k}$-a.e. $x \in \pi_{n-k}(\Omega)$ and for $\mathcal{L}^1$-a.e. $t > 0$.

Integrating (4.41), first with respect to t and then with respect to x, using equation (4.36) for both u and u^σ, yields

$$\begin{aligned} \int_{B \times \mathbb{R}^k_y} f(\nabla u^\sigma) \, dx \, dy &= \int_B dx \int_{\partial^*\{y:u^\sigma(x,y)>0\}} f(\nabla u^\sigma) \, dy \\ &= \int_B dx \int_0^{+\infty} dt \int_{\partial^*\{y:u^\sigma(x,y)>t\}} \frac{f(\nabla u^\sigma)}{|\nabla_y u^\sigma|} d\mathcal{H}^{k-1} \\ &\leq \int_B dx \int_0^{+\infty} dt \int_{\partial^*\{y:u(x,y)>t\}} \frac{f(\nabla u)}{|\nabla_y u|} d\mathcal{H}^{k-1} \\ &= \int_{B \times \mathbb{R}^k_y} f(\nabla u) \, dx \, dy \,. \end{aligned} \tag{4.42}$$

Step 2. Let us remove the additional assumptions we used in Step 1. Let $u \in W^{1,1}_{0,y}(\Omega)$ be non-negative and let $\omega \Subset \pi_{n-k}(\Omega)$ be an open set. Lemma 4.12 gives the existence of a sequence $\{u_h\}$ of non-negative Lipschitz functions, compactly supported in $\mathbb{R}^n$, that satisfy (4.35) and such that $u_h \to u$ strongly in $W^{1,1}(\omega \times \mathbb{R}^k_y)$.

If we assume that

$$0 \leq f(\xi) \leq C(1 + |\xi|) \ \text{ for some } C > 0, \quad \forall \xi \in \mathbb{R}^n, \tag{4.43}$$

then f is globally Lipschitz continuous and therefore $f(\nabla u_h) \to f(\nabla u)$ strongly in $L^1(\omega \times \mathbb{R}^k_y)$. The continuity of Steiner symmetrization, see equation (2.17), with respect to the L^1-convergence gives us $u_h^\sigma \to u^\sigma$ strongly in $L^1(\omega \times \mathbb{R}^k_y)$. By semicontinuity (see, e.g., [18, Theorem 4.2.8]) and (4.42) we have

$$\begin{aligned} \int_{\omega \times \mathbb{R}^k_y} f(\nabla u^\sigma) \, dx \, dy &\leq \liminf_{h \to +\infty} \int_{\omega \times \mathbb{R}^k_y} f(\nabla u_h^\sigma) \, dx \, dy \\ &\leq \liminf_{h \to +\infty} \int_{\omega \times \mathbb{R}^k_y} f(\nabla u_h) \, dx \, dy = \int_{\omega \times \mathbb{R}^k_y} f(\nabla u) \, dx \, dy \,, \end{aligned}$$

and so (4.34) holds.

Let us remove assumption (4.43). Since f is non-negative and convex and satisfies (4.1), there exist a sequence of vectors $\{a_j\} \subset \mathbb{R}^{n-k}$ and two sequences of numbers $\{b_j\} \subset \mathbb{R}$, $\{c_j\} \subset \mathbb{R}$ such that

$$f(\xi) = \sup_{j\in\mathbb{N}}\{a_j \cdot \xi_x + b_j|\xi_y| + c_j\} = \sup_{j\in\mathbb{N}}\{(a_j \cdot \xi_x + b_j|\xi_y| + c_j)^+\}\,, \quad \forall \xi \in \mathbb{R}^n\,.$$

For $N \in \mathbb{N}$ define

$$f_N(\xi) := \sup_{1\leq j\leq N}\{(a_j \cdot \xi_x + b_j|\xi_y| + c_j)^+\}\,.$$

Clearly, $f_N(\xi) \nearrow f(\xi)$ pointwise monotonically. Observing that f_N satisfies (4.1) and (4.43) we get that (4.34) holds for such f_N. Now the thesis follows by the Monotone Convergence Theorem. $\square$

REMARK 4.13. Actually, inequality (4.2) holds also for any u in $W^{1,p}_{0,y}(\Omega)$. To verify this, define, for every $\varepsilon > 0$, $u_\varepsilon := \max\{u - \varepsilon, 0\}$. Clearly, the support of u_ε has finite measure in $\omega \times \mathbb{R}^k_y$ for every $\omega \Subset \pi_{n-k}(\Omega)$. Therefore $u_\varepsilon \in W^{1,1}_{0,y}(\Omega)$. Since $(u_\varepsilon)^\sigma = (u^\sigma)_\varepsilon$ and $\nabla u_\varepsilon = \nabla u \chi_{\{u>\varepsilon\}}$ $\mathcal{L}^n$-a.e. in $\mathbb{R}^n$, by the Monotone Convergence Theorem and applying (4.34) to u_ε, we get

$$\begin{aligned}\int_{B\times\mathbb{R}^k_y} f(\nabla u^\sigma)\,dz &= \lim_{\varepsilon\to0^+}\int_{B\times\mathbb{R}^k_y} f(\nabla (u^\sigma)_\varepsilon)\,dz = \lim_{\varepsilon\to0^+}\int_{B\times\mathbb{R}^k_y} f(\nabla (u_\varepsilon)^\sigma)\,dz \\ &\leq \lim_{\varepsilon\to0^+}\int_{B\times\mathbb{R}^k_y} f(\nabla u_\varepsilon)\,dz = \int_{B\times\mathbb{R}^k_y} f(\nabla u)\,dz\,.\end{aligned}$$

We now pass to the equality cases. Next result shows that if equality holds in the Pólya-Szegő inequality, then almost every (x,t)-section of the subgraph is equivalent to a ball.

LEMMA 4.14. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a non-negative strictly convex function satisfying (4.1) that vanishes in 0 and let $u \in W^{1,1}_{0,y}(\Omega)$ be a non-negative function. If equality (4.6) holds, then for $\mathcal{L}^{n-k+1}$-a.e. $(x,t) \in \pi^+_{n-k,t}(\mathcal{S}_u)$ there exists $R(x,t) > 0$ such that the set*

$$\{y : u(x,y) > t\} \text{ is equivalent to } \{|y| < R(x,t)\}.$$

PROOF. We prove here the lemma under the additional assumption that u satisfies (4.3). For the general case see Remark 4.24.

Assumption (4.6) and inequality (4.34) imply that

$$\int_{B\times\mathbb{R}^k_y} f(\nabla u^\sigma)\,dz = \int_{B\times\mathbb{R}^k_y} f(\nabla u)\,dz \tag{4.44}$$

for every Borel set $B \subset \pi_{n-k}(\Omega)$. On choosing $A := \pi^+_{n-k}(\Omega) \cap G_{\mathcal{S}_u} \cap G_{\mathcal{S}_{u^\sigma}}$, from Theorem 2.4 and (2.12) we see that $\mathcal{L}^{n-k}(\pi^+_{n-k}(\Omega) \setminus A) = 0$ and that $\nabla_y u(x,y) \neq 0$ on $A \times \mathbb{R}^k_y$.

Equality (4.44) assures us that equality holds in (4.42) with B replaced by A. By (4.3) u is $\mathcal{L}^n$-a.e. strictly positive in Ω, and therefore we have equalities also in (4.39) and (4.40). Since $\tilde{f}(\xi,\cdot)$ is strictly increasing in $[0,+\infty)$ we get an equality in (4.38). Applying the isoperimetric theorem in $\mathbb{R}^k$, is clear that $\{y : u(x,y) > t\}$ is equivalent to a ball of radius $R(x,t)$ for $\mathcal{L}^{n-k}$-a.e. $x \in \pi_{n-k}(\Omega)$ and $\mathcal{L}^1$-a.e. $t \in (0, M(x))$. By the $\mathcal{L}^n$-a.e. positivity of u, we have that $\pi^+_{n-k}(\mathcal{S}_u)$ is equivalent to $\pi_{n-k}(\Omega)$. Equation (4.29) implies that $\pi^+_{n-k,t}(\mathcal{S}_u)$ is equivalent to $\bigcup_{x\in\pi_{n-k}(\Omega)}\{x\} \times (0, M(x))$. Hence the lemma is proven. $\square$

PROOF OF PROPOSITION 4.4. The proof is based on the same induction argument of [**8**, Proposition 3.6]. We already observed in Remark 4.2 that condition (4.3) implies (4.7). Let us now prove the converse implication. The case $k = 1$ is proven in [**30**, Proposition 2.3].

Step 1. Let $k > 1$ and let $v \in W^{1,1}_{0,y}(\Omega)$ be a non-negative function satisfying (4.3) and such that for $\mathcal{H}^{n-k+1}$-a.e. $(x,t) \in \pi^+_{n-k,t}(\mathcal{S}_v)$ the set $\{y : v(x,y) > t\}$ is equivalent to a k-dimensional ball. For $i = 1, \dots, k$, set

$$C^i := \{(x,y) \in \Omega : \partial_{y_i} v(x,y) = 0\} \cap \{(x,y) \in \Omega : \text{either } M(x) = 0 \text{ or } v(x,y) < M(x)\}\,.$$

We claim that for v as above $\mathcal{H}^n(C^i) = 0$. Indeed, by Theorem 2.3, we see that the set

$$A^i = \{(x,y,t) \in \partial^*\mathcal{S}_v : \nu^{\mathcal{S}_v}_{y_i} = 0\} \cap \{(x,y,t) \in \partial^*\mathcal{S}_v : \text{either } M(x) = 0 \text{ or } t < M(x)\}$$

satisfies

$$\mathcal{H}^n(A^i) \geq \mathcal{H}^n(C^i)\,. \tag{4.45}$$

From Theorem 2.4, up to $\mathcal{H}^{k-1}$ negligible sets, we get

$$A^i_{x,t} = \{y \in (\partial^*\mathcal{S}_v)_{x,t} : \nu^{(\mathcal{S}_v)_{x,t}}_{y_i} = 0\} \cap \{(x,y,t) \in \partial^*\mathcal{S}_v : \text{either } M(x) = 0 \text{ or } t < M(x)\}\,.$$

Since almost every section of the subgraph of v is a ball, we see that $\mathcal{H}^{k-1}(A^i_{x,t}) = 0$. Hence, using (4.45), assumption (4.3) with Theorem 2.3 and the coarea formula, we have

$$\mathcal{H}^n(C^i) \leq \mathcal{H}^n(A^i) = \mathcal{H}^n(A^i \cap \{\nu^{\mathcal{S}_v}_y \neq 0\}) = \int_{\pi_{n-k,t}(\partial^*\mathcal{S}_v)} dx\,dt \int_{(\partial^*\mathcal{S}_v)_{x,t} \cap A^i_{x,t}} \frac{d\mathcal{H}^{k-1}}{|\nu^{\mathcal{S}_v}_y|} = 0\,,$$

and so the claim is proven.

Step 2. For $i = 0, \dots, k$ define recursively $\Omega^0 := \Omega$, $\Omega^i := (\Omega^{i-1})^{S_i}$, where S_i is the 1-codimensional Steiner symmetrization with respect to y_i. The functions u^i are defined accordingly. Assumption (4.6) and Theorem 4.1 imply that

$$\int_{\Omega^\sigma} f(\nabla u^\sigma)\,dz = \int_{\Omega^{k-1}} f(\nabla u^{k-1})\,dz = \cdots = \int_{\Omega^1} f(\nabla u^1)\,dz = \int_\Omega f(\nabla u)\,dz\,.$$

Hence, by Lemma 4.14, we see that $\mathcal{S}_{u^k}$ is equivalent to $\mathcal{S}_{u^\sigma}$. From (4.7) and (4.45) we see that

$$\mathcal{H}^n(\{(x,y) \in \Omega^k : \nabla_{y_k} u^k(x,y) = 0\} \cap \{(x,y) \in \Omega^k : \text{either } M(x) = 0 \text{ or } 0 < u^k < M(x)\}) = 0\,.$$

Since the assertion holds for $k = 1$, we deduce

$$\mathcal{H}^n(\{(x,y) \in \Omega^{k-1} : \nabla_{y_{k-1}} u^{k-1} = 0)\} \cap \{(x,y) \in \Omega^{k-1} : M(x) = 0 \text{ or } 0 < u^{k-1} < M(x)\}) = 0$$

and this clearly implies that

$$\mathcal{H}^n(\{(x,y) \in \Omega^{k-1} : \nabla_y u^{k-1} = 0\} \cap \{\text{either } M(x) = 0 \text{ or } 0 < u^{k-1} < M(x)\}) = 0\,.$$

The assertion now follows iterating this argument. □

PROOF OF THEOREM 4.3. The first statement is Lemma 4.14, see also Remark 4.24.

By (2.18) it is sufficient to show that $(\mathcal{S}_u)^\sigma$ is equivalent to $\mathcal{S}_u$. From the previous statement, we know that for $\mathcal{L}^{n-k+1}$-a.e. $(x,t) \in \pi^+_{n-k,t}(\mathcal{S}_u)$ every section of $(\mathcal{S}_u)_{x,t}$ is equivalent to a ball in $\mathbb{R}^k$ with radius $R(x,t)$ and denote by $b : \mathbb{R}^{n-k} \times \mathbb{R}_t \to \mathbb{R}^{n+1}$ the center of this ball. On replacing u by u^σ in Lemma 4.14, we see that for $\mathcal{L}^{n-k+1}$-a.e. $(x,t) \in \pi^+_{n-k,t}((\mathcal{S}_u)^\sigma)$ every (x,t) section of $(\mathcal{S}_u)^\sigma$ is equivalent to a ball of the same radius $R(x,t)$ and denote by $\tilde{b} : \mathbb{R}^{n-k} \times \mathbb{R}_t \to \mathbb{R}^{n+1}$ the center of the ball. From the very definition of the Steiner rearrangement we have that $\tilde{b}(x,t) \equiv (x,0,t)$. Now it is sufficient to show that $b - \tilde{b} \equiv (0,c,0)$ for some $c \in \mathbb{R}^k$.

The case $k = 1$ is [**30**, Theorem 2.2]. Let $k > 1$ and for $i = 1, \dots, k$ let S_i be the Steiner symmetrization in codimension 1 with respect to y_i. Clearly, $\Omega^\sigma = (\Omega^\sigma)^{S_i} = (\Omega^{S_i})^\sigma$ and therefore (4.2) implies

$$\int_{\Omega^\sigma} f(\nabla u^\sigma)\,dz \leq \int_{\Omega^{S_i}} f(\nabla u^{S_i})\,dz \leq \int_\Omega f(\nabla u)\,dz\,, \tag{4.46}$$

for $i = 1, \dots, k$. From (4.6) we get equalities in (4.46). Since almost every section $(\mathcal{S}_u)_{x,t}$ is a ball, arguing as in Step 1 of the proof of Proposition 4.4 we get

$$\mathcal{L}^n(\{z \in \Omega : \partial_{y_i} u(z) = 0\} \cap \{z \in \Omega : \text{ either } M(z') = 0 \text{ or } u(z) < M(z')\}) = 0\,,$$

where $z' := (x, y_1, \dots, y_{i-1}, y_{i+1}, \dots y_k)$. Similarly, we also get that

$$\mathcal{H}^{n-1}(\{z \in \partial^*\Omega : \nu^{\Omega}_{y_i} = 0\} \cap \{\pi_{n-1}(\Omega) \times \mathbb{R}_{y_i}\}) = 0\,,$$

where π_{n-1} is the projection on z'. Therefore, by the $k = 1$ case, we have that $(b(x,t))_{y_1} \equiv c_1$ for some $c_1 \in \mathbb{R}$. Now iterate the procedure and obtain $(b(x,t))_y \equiv (c_1, \dots, c_k)$ and so $b - \tilde{b} \equiv (0, c, 0)$ with $c = (c_1, \dots, c_k)$. □

4.3. The BV case

In this section we are going to prove the Pólya-Szegő inequality for the Steiner rearrangement of a function of bounded variation and the characterization of the equality cases. As already observed in the introduction, we will first prove analogous results for geometrical functionals depending on the generalized inner normal. In this setting, we will first show a Pólya-Szegő principle in Theorem 4.19 an the characterization of the equality cases in Theorem 4.20.

Next two Lemmata will be used in the proof of Theorem 4.19.

LEMMA 4.15. *Let $U \subset \mathbb{R}^{n-k} \times \mathbb{R}_t$ be an open set. Let $F : \mathbb{R}^{n+1} \to [0, +\infty]$ be a convex function satisfying (4.15) and (4.18) and let E be a set of finite perimeter in $U \times \mathbb{R}^k_y$ such that $\mathcal{L}^{n+1}(E \cap (U \times \mathbb{R}^k_y)) < +\infty$. Then*

$$\int_{\partial^* E^\sigma \cap (B \times \mathbb{R}^k_y)} F(\nu^{E^\sigma})\, d\mathcal{H}^n \le \int_B \tilde{F}\left(\frac{D_1 L}{|DL|}, \dots, \frac{D_{n-k} L}{|DL|}, 0, \frac{D_t L}{|DL|}\right) d|DL| + \tilde{F}(0, \dots, 0, 1, 0)|D_y \chi_{E^\sigma}|(B \times \mathbb{R}^k_y) \tag{4.47}$$

for every Borel set $B \subset U$.

PROOF. Without loss of generality we can assume that B is a bounded open set.
Step 1. Let us prove inequality (4.47) assuming that F is everywhere finite, hence continuous. By approximation we can find a sequence of functions $\{L_j\} \subset C^\infty(B)$ such that $L_j(x,t) > 0$ for every $(x,t) \in B$, $L_j \to L$ in $L^1(B)$, $\nabla L_j \, \mathcal{L}^n \rightharpoonup DL$ weakly* in the sense of measures and

$$\int_B |\nabla L_j|\, dx\, dt \to |DL|(B)\,. \tag{4.48}$$

For $j \in \mathbb{N}$ define the sets $E_j := \{(x,y,t) : (x,t) \in B,\, \omega_k |y|^k \le L_j(x,t)\}$. Then $\chi_{E_j} \to \chi_{E^\sigma}$ in $L^1(B \times \mathbb{R}^k_y)$ and since

$$|D\chi_{E_j}|(B \times \mathbb{R}^k_y) = P(E_j; B \times \mathbb{R}^k_y) \le C\,,$$

for some constant depending only on B, we deduce that

$$D\chi_{E_j} \rightharpoonup D\chi_{E^\sigma} \text{ weakly* in } B \times \mathbb{R}^k_y\,. \tag{4.49}$$

Using the convexity of F, assumption (4.15) and (2.7) we have

$$\begin{aligned} &\int_{\partial^* E^\sigma \cap (B \times \mathbb{R}^k_y)} F(\nu^{E^\sigma})\, d\mathcal{H}^n \\ &\le \int_{\partial^* E^\sigma \cap (B \times \mathbb{R}^k_y)} \tilde{F}(\nu^{E^\sigma}_x, 0, \nu^{E^\sigma}_t)\, d\mathcal{H}^n + \int_{\partial^* E^\sigma \cap (B \times \mathbb{R}^k_y)} \tilde{F}(0, \nu^{E^\sigma}_y, 0)\, d\mathcal{H}^n \\ &= \int_{B \times \mathbb{R}^k_y} \tilde{F}\left(\frac{D_x \chi_{E^\sigma}}{|D\chi_{E^\sigma}|}, 0, \frac{D_t \chi_{E^\sigma}}{|D\chi_{E^\sigma}|}\right) d|D\chi_{E^\sigma}| + \tilde{F}(0,1,0) \int_{\partial^* E^\sigma \cap (B \times \mathbb{R}^k_y)} |\nu^{E^\sigma}_y|\, d\mathcal{H}^n\,. \end{aligned} \tag{4.50}$$

Using (4.49), Reshetnyak's Lower Semicontinuity Theorem (see, e.g., [5, Theorem 2.38]) and (2.7) we get

(4.51)
$$\int_{B\times\mathbb{R}^k_y} \tilde F\left(\frac{D_x\chi_{E^\sigma}}{|D\chi_{E^\sigma}|},0,\frac{D_t\chi_{E^\sigma}}{|D\chi_{E^\sigma}|}\right) d|D\chi_{E^\sigma}| \le \liminf_{j\to\infty}\int_{B\times\mathbb{R}^k_y} \tilde F\left(\frac{D_x\chi_{E_j}}{|D\chi_{E_j}|},0,\frac{D_t\chi_{E_j}}{|D\chi_{E_j}|}\right) d|D\chi_{E_j}|$$
$$= \liminf_{j\to\infty}\int_{\partial^* E_j\cap(B\times\mathbb{R}^k_y)} \tilde F(\nu_x^{E_j},0,\nu_t^{E_j})\, d\mathcal{H}^n\,.$$

Since the functions L_j are smooth, for $i=1,\dots,n-k,t$

$$\nu_i^{E_j}(x,y,t) = \frac{\partial_i L_j(x,t)}{\sqrt{p_j(x,t)^2+|\nabla L_j(x,t)|^2}}$$

for every $(x,y,t)\in\partial^* E_j\cap(B\times\mathbb{R}^k_y)$, where $p_j(x,t)$ stands for the perimeter of $(E_j)_{x,t}$. Using this equality with (4.50), (4.51) and (2.7) we see that

$$\begin{aligned}
&\int_{\partial^* E^\sigma\cap(B\times\mathbb{R}^k_y)} F(\nu^{E^\sigma})\, d\mathcal{H}^n \\
&\le \liminf_{j\to\infty}\int_{\partial^* E_j\cap(B\times\mathbb{R}^k_y)} F\left(\frac{\partial_i L_j}{\sqrt{p_j^2+|\nabla L_j|^2}},0,\frac{\partial_t L_j}{\sqrt{p_j^2+|\nabla L_j|^2}}\right) d\mathcal{H}^n \\
&\quad + \tilde F(0,1,0)\int_{\partial^* E^\sigma\cap(B\times\mathbb{R}^k_y)} |\nu_y^{E^\sigma}|\, d\mathcal{H}^n \qquad (4.52)\\
&= \liminf_{j\to\infty}\int_B \tilde F(\nabla_x L_j,0,\partial_t L_j)\, dx\,dt + \tilde F(0,1,0)|D_y\chi_{E^\sigma}|(B\times\mathbb{R}^k_y) \\
&= \liminf_{j\to\infty}\int_B \tilde F\left(\frac{\nabla_x L_j}{|\nabla L_j|},0,\frac{\partial_t L_j}{|\nabla L_j|}\right)|\nabla L_j|\, dx\,dt + \tilde F(0,1,0)|D_y\chi_{E^\sigma}|(B\times\mathbb{R}^k_y)\,.
\end{aligned}$$

Since $\nabla L_j\,\mathcal{L}^n \rightharpoonup DL$ weakly* and (4.48) holds, we can apply Reshetnyak's Continuity Theorem (see, e.g., [5, Theorem 2.39]) and get

$$\liminf_{j\to\infty}\int_B \tilde F\left(\frac{\nabla_x L_j}{|\nabla L_j|},0,\frac{\partial_t L_j}{|\nabla L_j|}\right)|\nabla L_j|\, dx\,dt = \int_B \tilde F\left(\frac{D_x L}{|DL|},0,\frac{D_t L}{|DL|}\right) d|DL|\,. \tag{4.53}$$

Then, inequality (4.47) follows combining (4.52) and (4.53).
Step 2. Let us remove the additional assumption made in Step 1. Since F is a convex function satisfying (4.15) and (4.18), we see that there exists a sequence $\{(a_j,b_j,c_j)\}\subset\mathbb{R}^{n-k}\times\mathbb{R}\times\mathbb{R}$ such that

$$F(\xi) = \sup_{j\in\mathbb{N}}\{(\xi_x\cdot a_j+|\xi_y|b_j+\xi_t c_j)^+\}\,,$$

for every $\xi\in\mathbb{R}^{n+1}$. Define, for $N\in\mathbb{N}$,

$$F_N(\xi) := \sup_{1\le j\le N}\{(\xi_x\cdot a_j+|\xi_y|b_j+\xi_t c_j)^+\}\,.$$

Note that F_N is a continuous function and satisfies (4.15) and (4.18). Since $F_N(\xi)\nearrow F(\xi)$ pointwise monotonically, inequality (4.47) follows applying Step 1 to the functions F_N and using the Monotone Convergence Theorem. □

The following lemma gives informations on the gradient of the function L. It is a simple variant of [23, Lemmata 3.1 and 3.2], see also Lemma 3.3.

LEMMA 4.16. *Let $U \subset \mathbb{R}^{n-k} \times \mathbb{R}_t$ be an open set and let E be a set of finite perimeter in $U \times \mathbb{R}^k_y$ such that $\mathcal{L}^{n+1}(E \cap (U \times \mathbb{R}^k_y)) < +\infty$. Then $L \in BV(U)$ and for any bounded Borel function g in U*

$$\int_U g(x)\, dD_i L(x) = \int_{U\times\mathbb{R}^k_y} g(x)\, dD_i\, \chi_E(x,y)\,, \qquad \text{for } i = 1,\dots,n-k,t\,. \tag{4.54}$$

LEMMA 4.17. *Let $U \subset \mathbb{R}^{n-k} \times \mathbb{R}_t$ be an open set and let $F : \mathbb{R}^{n+1} \to [0,+\infty]$ be a convex function satisfying (4.15). Let E be a set of finite perimeter in $U \times \mathbb{R}^k_y$ such that $\mathcal{L}^{n+1}(E \cap (U \times \mathbb{R}^k_y)) < +\infty$. Then*

$$\int_B \tilde F\left(\frac{D_1 L}{|DL|},\dots,\frac{D_{n-k}L}{|DL|},0,\frac{D_t L}{|DL|}\right) d|DL| \le \int_{\partial^* E\cap(B\times\mathbb{R}^k_y)} \tilde F(\nu^E_1,\dots,\nu^E_{n-k},0,\nu^E_t)\, d\mathcal{H}^n \tag{4.55}$$

for every Borel set $B \subset U$.

PROOF. As in the previous proof, we can assume that B is a bounded open set. Since F is a non-negative convex function satisfying (4.15), there exists a sequence of vectors $\{\alpha_j\} \in \mathbb{R}^{n-k} \times \mathbb{R}_t$ such that

$$F(\xi_x, 0, \xi_t) = \sup_{j\in\mathbb{N}} \left\{(\alpha_j \cdot \xi_{x,t})^+\right\} \tag{4.56}$$

for every $\xi \in \mathbb{R}^{n+1}$, where $\xi_{x,t} = (\xi_x, \xi_t) \in \mathbb{R}^{n-k+1}$. Hence we deduce that (see, e.g., [**5**, Lemma 2.35])

$$\int_B \tilde F\left(\frac{D_x L}{|DL|},0,\frac{D_t L}{|DL|}\right) d|DL| = \sup\left\{\sum_{j\in J}\int_{B_j}\left(\alpha_j\cdot\frac{DL}{|DL|}\right)^+ d|DL|\right\}, \tag{4.57}$$

where the supremum is extended over all finite sets $J \subset \mathbb{N}$ and all families $\{B_j\}_{j\in J}$ of pairwise disjoint Borel subsets of B. For a fixed family $\{B_j\}_{j\in J}$ and a fixed $j \in \mathbb{N}$ let us define

$$P_j := \left\{(x,t) \in B_j : \alpha_j \cdot \frac{DL}{|DL|}(x,t) \ge 0\right\}.$$

Hence, on applying (4.54), we get

$$\begin{aligned}
\int_{B_j}\left(\alpha_j\cdot\frac{DL}{|DL|}\right)^+ d|DL| &= \int_U \chi_{P_j}(x,t)\left(\sum_{i=1}^{n-k}(\alpha_j)_i\frac{D_i L}{|DL|} + (\alpha_j)_t\frac{D_t L}{|DL|}\right) d|DL| \\
&= \sum_{i=1}^{n-k}\int_U (\alpha_j)_i\,\chi_{P_j}(x,t)\, dD_i L(x,t) + \int_U (\alpha_j)_t\,\chi_{P_j}(x,t)\, dD_t L(x,t) \\
&= \sum_{i=1}^{n-k}\int_{U\times\mathbb{R}^k_y}(\alpha_j)_i\,\chi_{(P_j\times\mathbb{R}^k_y)}(x,y,t)\, dD_i\chi_E + \int_{U\times\mathbb{R}^k_y}(\alpha_j)_t\,\chi_{(P_j\times\mathbb{R}^k_y)}(x,y,t)\, dD_t\chi_E\,.
\end{aligned}$$

Combining the last equality with (2.7) we have

$$\int_{B_j}\left(\alpha_j\cdot\frac{DL}{|DL|}\right)^+ d|DL| = \int_{\partial^* E}\chi_{(P_j\times\mathbb{R}^k_y)}\alpha_j\cdot\nu^E_{x,t}\, d\mathcal{H}^n \le \int_{\partial^* E}\chi_{(B_j\times\mathbb{R}^k_y)}(\alpha_j\cdot\nu^E_{x,t})^+\, d\mathcal{H}^n\,.$$

Hence, on using (4.56) we see that

$$\sum_{j\in J}\int_{B_j}(\alpha_j\cdot\frac{DL}{|DL|})^+ d|DL| \le \sum_{j\in J}\int_{\partial^* E\cap(B_j\times\mathbb{R}^k_y)}\tilde F(\nu^E_x,0,\nu^E_t)\, d\mathcal{H}^n \le \int_{\partial^* E\cap(B\times\mathbb{R}^k_y)}\tilde F(\nu^E_x,0,\nu^E_t)\, d\mathcal{H}^n\,.$$

Then, combining (4.57) and the last inequality, we get (4.55). $\square$

LEMMA 4.18. *Let $U \subset \mathbb{R}^{n-k} \times \mathbb{R}_t$ be an open set and let E be a set of finite perimeter in $U \times \mathbb{R}^k_y$ such that $L(x,t) < +\infty$ for $\mathcal{L}^n$-a.e. $(x,t) \in U$. Then, for every open set $U' \Subset U$*

$$\mathcal{L}^{n+1}(E \cap (U' \times \mathbb{R}^k_y)) < +\infty\,. \tag{4.58}$$

PROOF. Given an open set $U' \Subset U$ define

$$E_h = E \cap (U' \times B(0,h)) \text{ for } h \in \mathbb{N}\,.$$

Without loss of generality, let us assume that $\partial U'$ is smooth. Since E_h has finite perimeter in $U' \times \mathbb{R}^k_y$, then by (2.8) we see that

$$\partial^{\mathcal{M}} E_h \cap (U' \times \mathbb{R}^k_y) \subset \Big(\partial^{\mathcal{M}} E \cup \{|y| = h\}\Big) \cap (U' \times \mathbb{R}^k_y)\,. \tag{4.59}$$

Since $\mathcal{L}^{n+1}(E_h \cap (U' \times \mathbb{R}^k_y)) < +\infty$, arguing as in the proof of Lemma 4.15 and using (4.59), (2.7) and (2.9) we deduce that

$$P((E_h)^\sigma; U' \times \mathbb{R}^k_y) \le P(E_h; U' \times \mathbb{R}^k_y) \le C\,,$$

for some constant C depending only on U'. Define $m_h = \fint_{U'} L_h(x,t)\,dx\,dt$, where $L_h(x,t)$ stands for $\mathcal{L}^{n-k+1}((E_h)_{x,t})$. Using the Poincaré inequality for functions of bounded variations (see, e.g., [**5**, Theorem 3.44]) we have that

$$\int_{U'} |L_h(x,t) - m_h|\,dx\,dt \le C\,|DL_h|(U') \le C\,P((E_h^\sigma); U' \times \mathbb{R}^k_y) \le C\,, \tag{4.60}$$

for some constant C depending only on U'. Up to subsequences, we have that $m_h \to m$ for some $m \in [0,+\infty]$. As $L_h(x,t) \to L(x,t)$ for $\mathcal{L}^n$-a.e. $(x,t) \in U'$, using (4.60) and Fatou's Lemma we infer that

$$\int_{U'} |L(x,t) - m|\,dx\,dt \le C\,.$$

Since $L(x,t)$ is finite for $\mathcal{L}^n$-a.e. $(x,t) \in U'$, the last inequality gives $m < +\infty$ and $L(x,t) \in L^1(U')$. Hence, (4.58) follows. □

THEOREM 4.19. *Let $F : \mathbb{R}^{n+1} \to [0,+\infty]$ be a convex function satisfying (4.15) and (4.18). Let $U \subset \mathbb{R}^{n-k} \times \mathbb{R}_t$ be an open set and let E be a set of finite perimeter in $U \times \mathbb{R}^k_y$ such that $L(x,t) < +\infty$ $\mathcal{L}^{n-k+1}$-a.e. in U. Then*

$$\int_{\partial^* E^\sigma \cap (B \times \mathbb{R}^k_y)} F(\nu^{E^\sigma})\,d\mathcal{H}^n \le \int_{\partial^* E \cap (B \times \mathbb{R}^k_y)} F(\nu^E)\,d\mathcal{H}^n \tag{4.61}$$

for every Borel set $B \subset U$. In particular, if E is a set of finite perimeter in $\mathbb{R}^{n+1}$, then

$$\int_{\partial^* E^\sigma} F(\nu^{E^\sigma})\,d\mathcal{H}^n \le \int_{\partial^* E} F(\nu^E)\,d\mathcal{H}^n\,. \tag{4.62}$$

PROOF. **Step 1.** Let us first assume that $\mathcal{L}^{n+1}(E \cap (U \times \mathbb{R}^k_y)) < +\infty$. Let G_{E^σ} be the set given by Vol'pert's Theorem 2.4. For any Borel set $B \subset U$ define $B_1 = B \setminus G_{E^\sigma}$ and $B_2 = B \cap G_{E^\sigma}$.

By inequalities (4.47) and (4.55) we see that

$$\int_{\partial^* E^\sigma \cap (B_1 \times \mathbb{R}^k_y)} F(\nu^{E^\sigma})\,d\mathcal{H}^n \le \int_{\partial^* E^\sigma \cap (B_1 \times \mathbb{R}^k_y)} F(\nu^E)\,d\mathcal{H}^n + \tilde{F}(0,1,0)|D_y\chi_{E^\sigma}|(B_1 \times \mathbb{R}^k_y)\,. \tag{4.63}$$

Moreover, by (2.7), coarea formula (2.11) and (ii) of Theorem 2.4 we get

$$|D_y\chi_{E^\sigma}|(B_1 \times \mathbb{R}^k_y) = \int_{\partial^* E^\sigma \cap (B_1 \times \mathbb{R}^k_y)} |\nu_y^{E^\sigma}|\,d\mathcal{H}^n = \int_{B_1} \mathcal{H}^{k-1}(\partial^* E^\sigma_{x,t})\,dx\,dt = 0\,, \tag{4.64}$$

where the last equality holds since $\mathcal{L}^n(\pi^+_{n-k,t}(E)\cap B_1)=0$. Hence, (4.63) and (4.64) give

$$\int_{\partial^* E^\sigma\cap(B_1\times\mathbb{R}^k_y)} F(\nu^{E^\sigma})\,d\mathcal{H}^n \le \int_{\partial^* E\cap(B_1\times\mathbb{R}^k_y)} F(\nu^{E})\,d\mathcal{H}^n\,. \tag{4.65}$$

For all $(x,t)\in B_2$, we have $\nu_y^{E^\sigma}\neq 0$ $\mathcal{H}^{k-1}$-a.e. on $\partial E^\sigma_{x,t}$. Hence, since $E^\sigma_{x,t}$ is a ball, we get that indeed $\nu_y^{E^\sigma}\neq 0$ at all point on $\partial E^\sigma_{x,t}$. Therefore, $\nu_y^{E^\sigma}\neq 0$ for all point on $\partial^* E^\sigma\cap(B_2\times\mathbb{R}^k_y)$ and we can apply the coarea formula, thus getting

$$\begin{aligned}
&\int_{\partial^* E^\sigma\cap(B_2\times\mathbb{R}^k_y)} F(\nu^{E^\sigma})\,d\mathcal{H}^n \\
&= \int_{\partial^* E^\sigma\cap(B_2\times\mathbb{R}^k_y)} \tilde F\left(\frac{\nu^{E^\sigma}}{|\nu_y^{E^\sigma}|}\right)|\nu_y^{E^\sigma}|\,d\mathcal{H}^n && \text{by (4.15) and (4.18)}\\
&= \int_{B_2} dx\,dt \int_{\partial^*(E^\sigma)_{x,t}} \tilde F\left(\frac{\nu_x^{E^\sigma}}{|\nu_y^{E^\sigma}|},1,\frac{\nu_t^{E^\sigma}}{|\nu_y^{E^\sigma}|}\right)d\mathcal{H}^{k-1}(y) && \text{by (2.11)}\\
&= \int_{B_2} \tilde F\Big(\nabla_x L(x,t),\mathcal{H}^{k-1}(\partial^* E^\sigma_{x,t}),\partial_t L(x,t)\Big)\,dx\,dt && \text{by (4.21).}\\
&\le \int_{B_2} \tilde F\Big(\nabla_x L(x,t),\mathcal{H}^{k-1}(\partial^* E_{x,t}),\partial_t L(x,t)\Big)\,dx\,dt && \text{by the isoperimetric inequality.}
\end{aligned} \tag{4.66}$$

Since F is a non-negative convex function satisfying (4.15) and (4.18), we see that there exists a sequence of vectors $\{(\xi_h,\rho_h,\tau_h)\}\subset\mathbb{R}^{n-k}\times\mathbb{R}\times\mathbb{R}$ such that

$$\tilde F(x,r,t)=\sup_{h\in\mathbb{N}}\{(x\cdot\xi_h+r\rho_h+t\tau_h)^+\}\,.$$

Hence, we deduce that (see, e.g., [5, Lemma 2.35])

$$\int_{B_2}\tilde F\Big(\nabla_x L(x,t),\mathcal{H}^{k-1}(\partial^* E_{x,t}),\partial_t L(x,t)\Big)\,dx\,dt=\sup\left\{\sum_{h\in H}\int_{A_h}(\nabla_x L\cdot\xi_h+p(x,t)\,\rho_h+\partial_t L\,\tau_h)^+\right\},$$

where $p(x,t):=\mathcal{H}^{k-1}(\partial^* E_{x,t})$ and the supremum is extended over all finite sets $H\subset\mathbb{N}$ and all families $\{A_h\}_{h\in H}$ of pairwise disjoint Borel subsets of B_2. For a fixed family $\{A_h\}_{h\in H}$ and a fixed $h\in\mathbb{N}$, define

$$P_h:=\big\{(x,t)\in A_h:\nabla_x L(x,t)\cdot\xi_h+p(x,t)\,\rho_h+\partial_t L(x,t)\,\tau_h\ge 0\big\}\,.$$

Let us define

$$g(x,t):=\int_{\partial^* E_{x,t}}\frac{\nu^E_{x,t}(x,y,t)}{|\nu^E_y(x,y,t)|}d\mathcal{H}^{k-1}(y)\,.$$

From (4.20) and considering that DL is absolutely continuous on B_2, setting $\tilde{A}_h := A_h \cap P_h$, we have

(4.67)
$$
\begin{aligned}
&\sum_{h\in H} \int_{A_h} (\nabla_x L(x,t)\cdot \xi_h + p(x,t)\rho_h + \partial_t L(x,t)\tau_h)^+ \,dx\,dt \\
&= \sum_{h\in H} \int_{\tilde{A}_h} \nabla_x L(x,t)\cdot \xi_h + p(x,t)\rho_h + \partial_t L(x,t)\tau_h \,dx\,dt \\
&= \sum_{h\in H} \left[\int_{\partial^* E\cap(\tilde{A}_h\times\mathbb{R}^k)\cap\{\nu_y^E=0\}} (\xi_h,\tau_h)\cdot \nu_{x,t}^E(x,y,t)\,d\mathcal{H}^n + \int_{\tilde{A}_h} g(x,t)\cdot(\xi_h,\tau_h) + p(x,t)\rho_h \,dx\,dt \right] \\
&\le \sum_{h\in H} \left[\int_{\partial^* E\cap(\tilde{A}_h\times\mathbb{R}^k)\cap\{\nu_y^E=0\}} \tilde{F}(\nu_x^E,0,\nu_t^E)\,d\mathcal{H}^n \right. \\
&\qquad \left. + \int_{\tilde{A}_h} \tilde{F}\left(\int_{\partial^* E_{x,t}} \frac{\nu_x^E}{|\nu_y^E|}\,d\mathcal{H}^{k-1}, \int_{\partial^* E_{x,t}} d\mathcal{H}^{k-1}, \int_{\partial^* E_{x,t}} \frac{\nu_t^E}{|\nu_y^E|}\,d\mathcal{H}^{k-1} \right) dx\,dt \right] \\
&\le \sum_{h\in H} \left[\int_{\partial^* E\cap(A_h\times\mathbb{R}^k)\cap\{\nu_y^E=0\}} F(\nu^E)\,d\mathcal{H}^n + \int_{A_h} dx\,dt \int_{\partial^* E_{x,t}} \tilde{F}\left(\frac{\nu_x^E}{|\nu_y^E|}, 1, \frac{\nu_t^E}{|\nu_y^E|} \right) d\mathcal{H}^{k-1}(y) \right] =: \mathcal{J},
\end{aligned}
$$

where the last inequality is due to Jensen's inequality. On applying the coarea formula, we see that

$$
\begin{aligned}
\mathcal{J} &= \sum_{h\in H} \left[\int_{\partial^* E\cap(\tilde{A}_h\times\mathbb{R}^k)\cap\{\nu_y^E=0\}} F(\nu^E)\,d\mathcal{H}^n + \int_{\partial^* E\cap(\tilde{A}_h\times\mathbb{R}^k)\cap\{\nu_y^E\neq 0\}} F(\nu)\,d\mathcal{H}^n \right] \\
&\le \sum_{h\in H} \left[\int_{\partial^* E\cap(A_h\times\mathbb{R}^k)\cap\{\nu_y^E=0\}} F(\nu^E)\,d\mathcal{H}^n + \int_{\partial^* E\cap(A_h\times\mathbb{R}^k)\cap\{\nu_y^E\neq 0\}} F(\nu)\,d\mathcal{H}^n \right] \\
&= \int_{\partial^* E\cap(B_2\times\mathbb{R}^k)} F(\nu)\,d\mathcal{H}^n .
\end{aligned}
\tag{4.68}
$$

Now inequality (4.61) follows combining (4.65)–(4.68).

Step 2. If the set E is such that $L(x,t) < +\infty$ for $\mathcal{L}^{n-k+1}$-a.e. $(x,t)\in U$, then (4.61) follows from Step 1 and from Lemma 4.18.

Step 3. It remains to prove (4.62). If E has finite perimeter in $\mathbb{R}^{n+1}$, then the isoperimetric inequality (see, e.g., [**5**, Theorem 3.46]) assures that either E or $\mathbb{R}^{n+1}\setminus E$ has finite measure. In the first case (4.62) is proven by the above calculations taking $U = \mathbb{R}^{n-k+1}$. In the second one, (4.62) trivially holds, since E^σ is equivalent to $\mathbb{R}^{n+1}$ and so $\partial^* E^\sigma = \emptyset$. □

In order to prove Theorem 4.6 we need some results for the equality cases in (4.61) and (4.62). For this, we need to strengthen the assumptions. Namely, we require that for every $(x,t)\in\mathbb{R}^{n-k+1}$ and for every $s_1, s_2\in\mathbb{R}^+$ with $s_1 < s_2$,

$$
\tilde{F}(x,s_1,t) < \tilde{F}(x,s_2,t)\,, \tag{4.69}
$$

whenever the right-hand side is finite.

PROPOSITION 4.20. *Let $F:\mathbb{R}^{n+1}\to[0,+\infty]$ be a convex function satisfying (4.15), (4.18) and (4.69) and let $U\subset\mathbb{R}^{n-k}\times\mathbb{R}_t$ be an open set. Let E be a set of finite perimeter in $U\times\mathbb{R}^k_y$ such that $L(x,t) < +\infty$ $\mathcal{L}^{n-k+1}$-a.e. in U. If*

$$
\int_{\partial^* E^\sigma\cap(U\times\mathbb{R}^k_y)} F(\nu^{E^\sigma})\,d\mathcal{H}^n = \int_{\partial^* E\cap(U\times\mathbb{R}^k_y)} F(\nu^E)\,d\mathcal{H}^n < \infty\,, \tag{4.70}
$$

then for $\mathcal{L}^{n-k+1}$-a.e. $(x,t) \in \pi^+_{n-k,t}(E) \cap U$ the section $E_{x,t}$ is equivalent to a k-dimensional ball.

PROOF. Assumption (4.70) and inequality (4.61) assure us that

$$\int_{\partial^* E^\sigma \cap (B \times \mathbb{R}^k_y)} F(\nu^{E^\sigma})\, d\mathcal{H}^n = \int_{\partial^* E \cap (B \times \mathbb{R}^k_y)} F(\nu^E)\, d\mathcal{H}^n \tag{4.71}$$

for every Borel set $B \subset U$. Possibly replacing U by U', where $U' \Subset U$, from Lemma 4.18 we can assume that $\mathcal{L}^{n+1}(E \cap (U \times \mathbb{R}^k_y)) < +\infty$. Hence, on choosing $B = U \cap G_E \cap G_{E^\sigma}$ in (4.71) we have equalities in (4.66). This, in combination with assumption (4.69) and the fact that the integrals in (4.70) have finite value, gives us that $\mathcal{H}^{k-1}(\partial^* E_{x,t}) = \mathcal{H}^{k-1}(\partial^* E^\sigma_{x,t})$ for $\mathcal{L}^{n-k+1}$-a.e. $(x,t) \in B$ and therefore for $\mathcal{L}^{n-k+1}$-a.e. $(x,t) \in \pi^+_{n-k,t}(E) \cap U$. On applying the isoperimetric theorem the result is proven. $\square$

Theorem 4.19 and Proposition 4.20 are sufficient to prove Theorem 4.6. The problem of whether a set satisfying (4.70) is necessarily Steiner symmetric or not is the content of the next result. Here, we need stronger assumptions. In particular we require that the precise representative L^* of L satisfies—similarly to condition (3.5)—

$$L^*(x,t) > 0 \text{ for } \mathcal{L}^{n-k-1}\text{-a.e. } (x,t) \in U\,. \tag{4.72}$$

We introduce the following notation. Given $i = 1, \dots, n-k$, for $(x,t) \in \mathbb{R}^{n-k} \times \mathbb{R}_t$ we write $\hat{x}_i := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n-k}, t)$ and $\hat{t} := x$. If g is a function defined on an open set $U \subset \mathbb{R}^{n-k} \times \mathbb{R}_t$, we set $g_{\hat{x}_i} := f_{|U \cap R_{\hat{x}_i}}$, where $R_{\hat{x}_i}$ is the straight line passing through $(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_{n-k}, t)$ and orthogonal to the hyperplane $x_i = 0$. Then $f_{\hat{t}}$ is defined accordingly.

THEOREM 4.21. *Let $F : \mathbb{R}^{n+1} \to [0, +\infty)$ be a convex function satisfying (4.15), (4.18) and (4.69). Let $U \subset \mathbb{R}^{n-k} \times \mathbb{R}_t$ be an open set and let E be a set of finite perimeter satisfying (4.72) and such that*

$$L(x,t) < +\infty \textit{ for } \mathcal{L}^{n-k+1}\textit{-a.e. } (x,t) \in U\,. \tag{4.73}$$

Assume that there exists a convex set $K \subset \mathbb{R}^{n-k} \times \mathbb{R}_t$ such that the function

$$\begin{gathered} K \ni (\xi_x, \xi_t) \mapsto \tilde{F}(\xi_x, 1, \xi_t) \textit{ is strictly convex and} \\ \left(\frac{\nu^E_x}{|\nu^E_y|}, \frac{\nu^E_t}{|\nu^E_y|} \right) \in K \;\; \mathcal{H}^n\textit{-a.e. on } \partial^* E \cap (U \times \mathbb{R}^k)\,. \end{gathered} \tag{4.74}$$

Assume also that

$$\mathcal{H}^n \left(\{(x,y,t) \in \partial^* E^\sigma : \nu^{E^\sigma}_y(x,y,t) = 0\} \cap (U \times \mathbb{R}^k_y) \right) = 0\,. \tag{4.75}$$

If (4.70) is fulfilled, then for each connected component U_α of U, $E \cap (U_\alpha \times \mathbb{R}^k_y)$ is equivalent to $E^\sigma \cap (U_\alpha \times \mathbb{R}^k_y)$ up to translations in the y-plane. In particular, if U is connected and $\mathcal{L}^{n-k+1}(\pi^+_{n-k,t}(E) \setminus U) = 0$, then E is equivalent to E^σ up to translations in the y-plane.

PROOF. **Step 1.** Let U_α be any connected component of U. From Proposition 4.20 we know that for $\mathcal{L}^{n-k+1}$-a.e. $(x,t) \in \pi^+_{n-k,t}(E) \cap U_\alpha$ the section $E_{x,t}$ is equivalent to a k-dimensional ball of radius $R(x,t)$ and clearly the same holds for E^σ with the same radius. Denote by $b(x,t)$ and $\tilde{b}(x,t)$ the center of these balls. Since E^σ is Steiner symmetric we have that $\tilde{b}(x,t) \equiv (x, 0, t)$. The result will follow if we show that $\beta(x,t) := \big(b(x,t)\big)_y$ is constant. Notice that $\beta(x,t)$ is a measurable function which, by (4.72) and (4.73) is finite a.e., and is equal to

$$\beta(x,t) = \frac{1}{L(x,t)} \int_{E_{x,t}} y \, dy\,.$$

Step 2. Since equality (4.70) holds, arguing as in the proof of Proposition 4.4 we deduce that condition (4.75) is equivalent to

$$\mathcal{H}^n\left(\{(x,y,t)\in\partial^*E:\nu_y^E(x,y,t)=0\}\cap(U\times\mathbb{R}^k_y)\right)=0\,. \tag{4.76}$$

Therefore, using [8, Theorem 4.3] we get that the function $\beta_{\hat{x}_i}\in W^{1,1}_{\mathrm{loc}}(U\cap R_{\hat{x}_i};\mathbb{R}^k)$ and for $\mathcal{L}^1$-a.e. $x_i\in U\cap R_{\hat{x}_i}$

$$\beta'_{\hat{x}_i}(x_i)=\frac{1}{L^*_{\hat{x}_i}(x_i)}\int_{\partial^*E_{x,t}}[y-\beta_{\hat{x}_i}(x_i)]\frac{\nu_i^E(x,y,t)}{|\nu_y^E(x,y,t)|}d\mathcal{H}^{k-1}(y)\,. \tag{4.77}$$

A similar equality holds for $\beta'_{\hat{t}}(t)$.

By (4.71) we have equalities in (4.66) and (4.67). Hence, from (4.76) we get

$$\tilde{F}\left(\int_{\partial^*E_{x,t}}\frac{\nu_x^E}{|\nu_y^E|}d\mathcal{H}^{k-1},\int_{\partial^*E_{x,t}}d\mathcal{H}^{k-1},\int_{\partial^*E_{x,t}}\frac{\nu_t^E}{|\nu_y^E|}d\mathcal{H}^{k-1}\right)=\int_{\partial^*E_{x,t}}F\left(\frac{\nu_x^E}{|\nu_y^E|},1,\frac{\nu_t^E}{|\nu_y^E|}\right).$$

From (4.74), $\nu_{x,t}^E/|\nu_y^E|$ is constant with respect to y. Moreover, as $\partial^*E_{x,t}$ is a sphere, $|\nu_y^E|$ is constant and so $\nu_{x,t}^E$ is constant. Hence, from (4.77) we get

$$\beta'_{\hat{x}_i}(x_i)=\frac{1}{L^*_{\hat{x}_i}(x_i)}\frac{\nu_i^E(x,t)}{|\nu_y^E(x,t)|}\int_{\partial^*E_{x,t}}[y-\beta_{\hat{x}_i}(x_i)]\,d\mathcal{H}^{k-1}(y)=0\,, \tag{4.78}$$

where we dropped the variable y for functions that are constant in $\partial^*E_{x,t}$ and the last equality is due to the definition of the function β.

Step 3. We claim that β is constant. Indeed, if β is bounded, it is locally integrable. Therefore, $\beta\in L^1_{\mathrm{loc}}(U_\alpha;\mathbb{R}^k)$ and its restrictions $\beta_{\hat{x}_i}$ and $\beta_{\hat{t}}$ are absolutely continuous and integrable. Hence, by a standard characterization of Sobolev functions (see, e.g., [38, §4.9, Theorem 2]) we have that $\beta\in W^{1,1}_{\mathrm{loc}}(U_\alpha;\mathbb{R}^k)$ and $\nabla\beta=0$ in U_α and so β is constant in U_α. For $\beta=(\beta_1,\dots,\beta_k)$ unbounded, fix $T>0$ and define the truncated function β^T as

$$\beta_j^T(x,t):=\begin{cases}\beta_j(x,t) & \text{if } |\beta_j(x,t)|\le T\\ T & \text{if } \beta_j(x,t)>T\\ -T & \text{if } \beta_j(x,t)<-T\,,\end{cases}$$

for $j=1,\dots,k$. Hence

$$(\beta^T_{j,\hat{x}_i})'=\begin{cases}0 & \text{if } |\beta_j(x,t)|>T\\ \beta'_{j,\hat{x}_i} & \text{if } |\beta_j(x,t)|\le T\,,\end{cases}$$

with a similar equality holding for $(\beta^T_{j,\hat{t}_i})'$. Therefore, since β^T is bounded, from (4.78) and the previous equality we deduce that $\beta^T=C^T$ a.e. for some constant $C^T\in\mathbb{R}^k$. Finally, as

$$\beta(x,t)=\lim_{T\to+\infty}\beta^T(x,t)=\lim_{T\to\infty}C^T$$

and since β is finite a.e., we deduce that β is constant. □

After proving the results concerning functionals of the form (4.14), we deal now with the Pólya-Szegő principle for BV functions. In the proof of Theorem 4.5 we will use Theorem 4.22 below, a consequence of relaxation results concerning BV functions, see e.g., [5, Theorem 5.47].

THEOREM 4.22 ([30, Theorem F]). *Let f be a convex function satisfying (4.8). Let $\Omega\subset\mathbb{R}^n$ be an open set and let J_f be the functional defined by (4.9). If $u\in BV(\Omega)$ and $\{u_j\}$ is any sequence in $BV(\Omega)$ such that $u_j\to u$ in $L^1_{\mathrm{loc}}(\Omega)$, then*

$$J_f(u;\Omega)\le\liminf_{j\to+\infty}J_f(u_j;\Omega)\,.$$

PROOF OF THEOREM 4.5. We are going to prove a stronger inequality than (4.10), i.e.,

$$J_f(u^\sigma; B \times \mathbb{R}^k_y) \leq J_f(u; B \times \mathbb{R}^k_y)\,, \tag{4.79}$$

for any Borel set $B \subset \pi_{n-k}(\Omega)$. As before we identify u_0 with u.

Step 1. Let us first prove that $u^\sigma \in BV(\omega \times \mathbb{R}^k_y)$ for every open set $\omega \Subset \pi_{n-k}(\Omega)$. Since $u \in BV_{0,y}(\Omega)$ then $u \in BV(\omega \times \mathbb{R}^k_y)$. Hence, by approximation we can find a sequence of non-negative functions $\{u_h\} \subset C^1(\omega \times \mathbb{R}^k_y)$ such that $u_h \to u$ in $L^1(U \times \mathbb{R}^k_y)$ and

$$\lim_{h\to\infty} \int_{\omega\times\mathbb{R}^k_y} |\nabla u_h|\, dz = |Du|(\Omega \times \mathbb{R}^k_y)\,.$$

By the continuity of the Steiner rearrangement—equation (2.17)—we get that $(u_h)^\sigma \to u^\sigma$ in $L^1(\omega\times\mathbb{R}^k_y)$; moreover by (4.34) we have that the sequence $\|\nabla u^\sigma_h\|_{L^1(\omega\times\mathbb{R}^k_y)}$ is bounded. Therefore (see, e.g., [**5**, Theorem 3.9]) we conclude that $u^\sigma \in BV(\omega \times \mathbb{R}^k_y)$.

Step 2. Let us assume, for the moment, that u is compactly supported in Ω. By Theorem 2.1, $\mathcal{S}_u$ is a set of finite perimeter in $\mathbb{R}^{n+1}$. On applying Proposition 4.7, Theorem 4.19 and (2.18) we deduce that for every Borel set $B \subset \pi_{n-k}(\Omega)$

$$\begin{aligned} J_f(u^\sigma; B \times \mathbb{R}^k_y) &= \int_{\partial^*\mathcal{S}_{u^\sigma}\cap(B\times\mathbb{R}^k_y\times\mathbb{R}_t)} F_f(\nu^{\mathcal{S}_{u^\sigma}})\, d\mathcal{H}^n \\ &\leq \int_{\partial^*\mathcal{S}_{u}\cap(B\times\mathbb{R}^k_y\times\mathbb{R}_t)} F_f(\nu^{\mathcal{S}_{u}})\, d\mathcal{H}^n = J_f(u; B \times \mathbb{R}^k_y)\,, \end{aligned}$$

hence (4.79) holds.

Step 3. Let us now drop the extra assumption. Fixed $\omega \Subset \pi_{n-k}(\Omega)$ we can find a smooth cutoff function compactly supported in $\pi_{n-k}(\Omega)$ such that $\varphi \equiv 1$ on ω and a smooth function η compactly supported in $\mathbb{R}^k$ with $\eta \equiv 1$ in $B(0,1)$. Let us define the functions

$$v(x,y) = u(x,y)\varphi(x) \text{ and } v_h(x,y) = v(x,y)\eta(\frac{y}{h})\,, \text{ for } h \in \mathbb{N}\,.$$

Clearly, $v \in BV(\mathbb{R}^n)$ and $v_h \to v$ as $h \to +\infty$ in $L^1(\mathbb{R}^n)$. Hence, by Theorem 4.22 we deduce that

$$J_f(u^\sigma; \omega \times \mathbb{R}^k_y) = J_f(v^\sigma; \omega \times \mathbb{R}^k_y) \leq \liminf_{h\to+\infty} J_f(v^\sigma_h; \omega \times \mathbb{R}^k_y)\,. \tag{4.80}$$

Moreover, since $|D(v - v_h)|(\mathbb{R}^n) \to 0$ as $h \to +\infty$, we get

$$\liminf_{h\to+\infty} J_f(v_h; \omega \times \mathbb{R}^k_y) = J_f(v; \omega \times \mathbb{R}^k_y) = J_f(u; \omega \times \mathbb{R}^k_y)\,. \tag{4.81}$$

Now, for $B = \omega$ inequality (4.79) follows from (4.80), (4.81) and the second step applied to v_h. Then, the general case where B is any Borel set, is derived by approximation. $\square$

PROOF OF THEOREM 4.6. The proof is very similar to the one of Theorem 4.3. Thanks to (2.18), it is sufficient to show that $(\mathcal{S}_u)^\sigma$ is equivalent to $\mathcal{S}_u$.

Step 1. We claim that for $\mathcal{L}^{n-k+1}$-a.e. $(x,t) \in \pi^+_{n-k}(\mathcal{S}_u)$ there exists $R(x,t) > 0$ such that the set

$$\{y : u(x,y) > t\} \text{ is equivalent to } \{|y| < R(x,t)\}\,.$$

From (4.13) and (4.79) we see that equality holds in (4.79) for any Borel set $B \subset \pi_{n-k}(\Omega)$. Given any open set $\omega \Subset \pi_{n-k}(\Omega)$ let φ be a smooth cutoff function with compact support in $\pi_{n-k}(\Omega)$ such that $\varphi \equiv 1$ on ω. Identifying u with its extension u_0, define $v := u\varphi$. Then, we have the following equality:

$$J_f(v^\sigma; \omega \times \mathbb{R}^k_y) = J_f(v; \omega \times \mathbb{R}^k_y)\,.$$

Hence, on using Proposition 4.7 we get

$$\int_{\partial^* \mathcal{S}_{v^\sigma} \cap (\omega \times \mathbb{R}^k_y \times \mathbb{R}_t)} F(\nu^{\mathcal{S}_{v^\sigma}})\, d\mathcal{H}^n = \int_{\partial^* \mathcal{S}_{v} \cap (\omega \times \mathbb{R}^k_y \times \mathbb{R}_t)} F(\nu^{\mathcal{S}_{v}})\, d\mathcal{H}^n .$$

Since v belongs to $BV(\mathbb{R}^n)$ and it is non-negative, from (4.4) we deduce that v has compact support and therefore $\mathcal{S}_v$ has finite perimeter in $\mathbb{R}^{n+1}$. By the last equality and Lemma 4.23 below, the claim is proven from Proposition 4.20 and from the arbitrariness of ω.

Step 2. We have just proved that for $\mathcal{L}^{n-k+1}$-a.e. $(x,t) \in \pi^+_{n-k,t}(\mathcal{S}_u)$ the (x,t) section of $\mathcal{S}_u$ is equivalent to a ball in $\mathbb{R}^k$ with radius $R(x,t)$. Define $b : \mathbb{R}^{n-k} \times \mathbb{R}_t \to \mathbb{R}^n$ to be the center of this ball. On applying Step 1 to the function u^σ we see that for $\mathcal{L}^{n-k+1}$-a.e. $(x,t) \in \pi^+_{n-k,t}(\mathcal{S}_{u^\sigma})$ every section $(\mathcal{S}_u)^\sigma_{x,t}$ is equivalent to a ball of the same radius $R(x,t)$ with center $\tilde{b}(x,t)$. From the definition of the Steiner rearrangement we get $\tilde{b}(x,t) \equiv (x,0,t)$. Now the Theorem follows once we prove that $b - \tilde{b} \equiv (0,c,0)$ for some $c \in \mathbb{R}^k$.

The case $k=1$ is [**30**, Theorem 2.5]. Let $k>1$ and denote by S_i the Steiner symmetrization with respect to y_i for $i = 1,\dots,k$. Since $\Omega^\sigma = (\Omega^\sigma)^{S_i} = (\Omega^{S_i})^\sigma$, from (4.10) we have the following inequalities

$$J_f(u^\sigma;\Omega^\sigma) \le J_f(u^{S_i};\Omega^{S_i}) \le J_f(u;\Omega) . \tag{4.82}$$

From the assumption (4.13) we get equalities in (4.82). Since almost every section $(\mathcal{S}_u)_{x,t}$ is a ball, arguing as in Step 1 of the proof of Proposition 4.4 we get

$$\mathcal{L}^n(\{z \in \Omega : \partial_{y_i} u(z) = 0\} \cap \{z \in \Omega : \text{ either } M(z') = 0 \text{ or } u(z) < M(z')\}) = 0 ,$$

where $z' := (x, y_1, \dots, y_{i-1}, y_{i+1}, \dots y_k)$. Similarly we also have

$$\mathcal{H}^{n-1}(\{z \in \partial^*\Omega : \nu^\Omega_{y_i} = 0\} \cap \{\pi_{n-1}(\Omega) \times \mathbb{R}_{y_i}\}) = 0 ,$$

where π_{n-1} is the projection on z'. Since $\Omega^\sigma = (\Omega^\sigma)^{S_1}$, by the $k=1$ case, we have that $(b(x,t))_{y_1} \equiv c_1$ for some $c_1 \in \mathbb{R}$. Now iterate the procedure and obtain $(b(x,t))_y \equiv (c_1,\dots,c_k)$ and so $b - \tilde{b} \equiv (0,c,0)$ with $c = (c_1,\dots,c_k)$. □

The following lemma shows how properties of the function f are inherited by F_f.

LEMMA 4.23 (([**30**, Lemma 6.1]). *Let $f : \mathbb{R}^n \to [0,+\infty)$ be a convex function vanishing at 0. Then, the functions F_f defined by (4.16) is a convex function satisfying (4.15). Moreover, if in addition f is as in Theorem 4.6, then F_f satisfies (4.18), (4.69) and (4.74) with $K = \mathbb{R}^{n-k} \times (\mathbb{R}^-_t \cup \{0\})$.*

REMARK 4.24. Here we want to observe that if f is a non-negative function as in Theorem 4.3, then the function $F_f(\xi_1,\dots,\xi_{n+1})$, possibly attaining infinite value if $\xi_{n+1} \ge 0$, defined as in (4.16) satisfies the assumptions of Proposition 4.20. However, if $u \in W^{1,1}_{0,y}(\Omega)$ then (4.17) still holds and thus Lemma 4.14 follows arguing as in Step 1 of the proof of Theorem 4.6.

CHAPTER 5

Stability estimates for the Pólya-Szegő inequality

We are now interested in studying the stability of the Pólya-Szegő inequality both for the Steiner and the Schwarz rearrangement in terms of the L^1 distance between u and u^s (or u^*). It is known (see [**28**]) that such an estimate is in general not true. Therefore we need some additional assumptions on u. Namely, we will require the function u to be concave. Here we will consider only the case in which the convex integrand is $|\cdot|^p$, with $1 < p < \infty$.

5.1. Statement of the main results

As before, we write $z \in \mathbb{R}^n$ as $(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and given a non-negative Sobolev function $u \in W_0^{1,p}(\mathbb{R}^n)$ $(1 < p < +\infty)$ we denote by u^s its Steiner rearrangement (with respect to the hyperplane $y = 0$) and by u^* its Schwarz rearrangement. Moreover, we denote by $\Delta(u, w)$ the *deficit* between two Sobolev function u and w

$$\Delta(u, w) := \left| \int_{\mathbb{R}^n} |\nabla u|^p \, dz - \int_{\mathbb{R}^n} |\nabla w|^p \, dz \right| .$$

We can now state the main theorem of the chapter, namely the quantitative Pólya-Szegő inequality for the Steiner rearrangement.

THEOREM 5.1. *Let $u \in W_0^{1,p}(\mathbb{R}^n)$ be a non-negative and concave function. Then*

(5.1)
$$\inf_{h\in\mathbb{R}} \int_{\mathbb{R}^n} |u(x, y + h) - u^s(x, y)| \, dx \, dy \le c \, \frac{M^{n+2}}{\|u\|_{L^1}} \mathcal{L}^n(\Omega)^{\frac{1}{p'}} \|\nabla u^s\|_{L^p}^{\frac{2-p}{2}} \Delta(u, u^s)^{\frac{1}{2}} \quad \text{if } 1 < p < 2,$$

and

$$\inf_{h\in\mathbb{R}} \int_{\mathbb{R}^n} |u(x, y + h) - u^s(x, y)| \, dx \, dy \le c \, \frac{M^{n+2}}{\|u\|_{L^1}} \mathcal{L}^n(\Omega)^{\frac{1}{p'}} \Delta(u, u^s)^{\frac{1}{p}} \quad \text{if } p \ge 2,$$

where $c = c(n, p)$ is a positive constant, Ω is the support of u, and M is the maximum between $\|u\|_{L^\infty}$ and the outer radius on Ω (i.e., the radius of the smallest ball containing Ω).

The main idea—which we already used extensively in the previous chapter—is to identify a non-negative Sobolev function u, with its subgraph $\mathcal{S}_u$. Then we describe it by a couple of functions (b, l), the barycenter and the half measure of the one dimensional sections of the subgraph of u. More precisely, let ω be the projection of $\mathcal{S}_u$ on the hyperplane $y = 0$, i.e., $\omega = \pi_{n-1,t}(\mathcal{S}_u)$, then (recalling Definition 3.8 and the definitions from Chapter 2)

$$l(x, t) = \frac{1}{2} L(x, t) \qquad \text{and} \qquad b(x, t) = \frac{1}{2l(x, t)} \int_{(\mathcal{S}_u)_{x,t}} y \, dy.$$

If $\mathcal{S}_u$ has the segment property, i.e., for every $(x, t) \in \omega$ the section $(\mathcal{S}_u)_{x,t}$ is a segment, and if $\mathcal{S}_u$ has not "flat zone", i.e., it satisfies

(5.2)
$$\mathcal{L}^n(\{(x, y, t) \in \partial^* \mathcal{S}_u \, : \, \partial_y u(x, y) = 0\} \cap (\omega \times \mathbb{R})) = 0 \, ,$$

then it can be deduced by [**23**, Proposition 1.2], [**8**, Proposition 3.5] (see also Lemma 3.6) that $l \in W^{1,1}(\omega)$ and $b \in W^{1,1}_{\mathrm{loc}}(\omega)$.

The point here is that we can use the segment property to write $\mathcal{S}_u$ as the domain between the functions $b-l$ and $b+l$, and then reformulate the energy $\int |\nabla u|^p$ in terms of l and b. This can be done by means of the generalized inner normal to the subgraph of u—see Lemma 5.4.

The quantitative version of the Pólya-Szegő can be extended to the Schwarz symmetrization.

THEOREM 5.2. *Let $u \in W_0^{1,p}(\mathbb{R}^n)$ be a non-negative and concave function. Then*

$$\inf_{h\in\mathbb{R}^n} \int_{\mathbb{R}^n} |u(z+h) - u^s(z)|\, dz \le c\, \frac{M^{n+2}}{\|u\|_{L^1}} \mathcal{L}^n(\Omega)^{\frac{1}{p'}} \|\nabla u^*\|_{L^p}^{\frac{2-p}{2}} \Delta(u,u^*)^{\frac{1}{2}} \quad \text{if } 1<p<2, \tag{5.3}$$

and

$$\inf_{h\in\mathbb{R}^n} \int_{\mathbb{R}^n} |u(z+h) - u^s(z)|\, dz \le c\, \frac{M^{n+2}}{\|u\|_{L^1}} \mathcal{L}^n(\Omega)^{\frac{1}{p'}} \Delta(u,u^*)^{\frac{1}{p}} \quad \text{if } p \ge 2,$$

where $c = c(n,p)$ is a positive constant, Ω is the support of u, and M is the maximum between $\|u\|_{L^\infty}$ and the outer radius on Ω.

The idea is to apply Steiner symmetrization n times along n perpendicular directions so to transform u in a n-symmetric function, and then to use the following stability result, generalizing [**47**, Proposition 2.4] (see also [**25**, Theorem 3]).

LEMMA 5.3. *Let $w \in W_0^{1,p}(\mathbb{R}^n)$ be a non-negative and n-symmetric function, $1<p<+\infty$. Then*

$$\int_{\mathbb{R}^n} |w - w^*|\, dz \le c\, \mathcal{L}^n(\Omega)^{\frac{1}{p'}+\frac{1}{n}} \|\nabla w^*\|_{L^p(\mathbb{R}^n)}^{\frac{2-p}{2}} \Delta(w,w^*)^{\frac{1}{2}} \quad \text{if } 1<p<2 \tag{5.4}$$

and

$$\int_{\mathbb{R}^n} |w - w^*|\, dz \le c\, \mathcal{L}^n(\Omega)^{\frac{1}{p'}+\frac{1}{n}} \Delta(w,w^*)^{\frac{1}{p}} \quad \text{if } p \ge 2, \tag{5.5}$$

where $c = c(n,p)$ is a positive constant, and Ω is the support of w.

Finally, we observe that for $1<p\le 2$ the quantitative Pólya-Szegő inequality for the Steiner and the Schwarz rearrangement are *sharp*, in the sense that the exponent $1/2$ in the deficit of the estimates (5.1) and (5.3) cannot be improved—see the discussion at the end of the chapter.

5.2. Proofs

We start by proving a representation formula of $\int |\nabla u|^p$ in terms of the functions b and l.

LEMMA 5.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $u \in W^{1,p}(\Omega)$ be a non-negative function and set $\omega := \pi_{n-1,t}(\mathcal{S}_u)$. Assume that $\mathcal{S}_u$ has the segment property and satisfies (5.2). Then*

$$\int_\Omega |\nabla u(x,y)|^p\, dx\, dy = \int_\omega \frac{(1+|\nabla_x b + \nabla_x l|^2)^{\frac{p}{2}}}{|\partial_t b + \partial_t l|^{p-1}}\, dx\, dt + \int_\omega \frac{(1+|\nabla_x b - \nabla_x l|^2)^{\frac{p}{2}}}{|\partial_t b - \partial_t l|^{p-1}}\, dx\, dt. \tag{5.6}$$

PROOF. For the sake of simplicity we set $E := \mathcal{S}_u$. As E has the segment property in direction y, we have $E = E^+ \cap E^-$, where

$$E^+ := \left\{(x,y,t) \in \mathbb{R}^{n+1} : (x,t) \in \omega,\ y < b(x,t) + l(x,t)\right\},$$
$$E^- := \left\{(x,y,t) \in \mathbb{R}^{n+1} : (x,t) \in \omega,\ y > b(x,t) - l(x,t)\right\}.$$

Note that since E is a subgraph, we have that $\partial_t b + \partial_t l \le 0$ and $\partial_t b - \partial_t l \ge 0$ and therefore

$$\partial_t l \le 0 \quad \text{and} \quad |\partial_t b| \le |\partial_t l|.$$

Note that by (2.12) we have

$$\nu^{E^+}(x,y,t) = \left(\frac{\nabla_x b + \nabla_x l}{\sqrt{1+|\nabla b + \nabla l|^2}}, \frac{-1}{\sqrt{1+|\nabla b + \nabla l|^2}}, \frac{\partial_t b + \partial_t l}{\sqrt{1+|\nabla b + \nabla l|^2}} \right) \tag{5.7}$$

and a corresponding equality for ν^{E^-}. By (2.12) we have

$$\int_\Omega |\nabla u(x,y)|^p \, dx\, dy = \int_{\partial^* E \cap (\Omega \times \mathbb{R})} \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^2}} \, d\mathcal{H}^n = \int_{\partial^* E \cap (\Omega \times \mathbb{R})} \frac{|\nu_{x,y}|^p}{|\nu_t|^{p-1}} d\mathcal{H}^n \, .$$

Splitting the last integral over E^+ and E^-, by (5.7) we get

$$\begin{aligned}
\int_\Omega |\nabla u(x,y)|^p \, dx\, dy &= \int_{\partial^* E^+ \cap (\Omega \times \mathbb{R})} \frac{|\nu_{x,y}|^p}{|\nu_t|^{p-1}} d\mathcal{H}^n + \int_{\partial^* E^- \cap (\Omega \times \mathbb{R})} \frac{|\nu_{x,y}|^p}{|\nu_t|^{p-1}} d\mathcal{H}^n \\
&= \int_{\partial^* E^+ \cap (\Omega \times \mathbb{R})} \frac{(1+|\nabla_x b + \nabla_x l|^2)^{\frac{p}{2}}}{|\partial_t b + \partial_t l|^{p-1}} \left[1+|\nabla b + \nabla l|^2\right]^{\frac{1}{2}} d\mathcal{H}^n \\
&\quad + \int_{\partial^* E^- \cap (\Omega \times \mathbb{R})} \frac{(1+|\nabla_x b - \nabla_x l|^2)^{\frac{p}{2}}}{|\partial_t b - \partial_t l|^{p-1}} \left[1+|\nabla b - \nabla l|^2\right]^{\frac{1}{2}} d\mathcal{H}^n \, ,
\end{aligned}$$

hence, on using coarea formula, the lemma is proven. □

Remark 5.5. Note that, if $u \in W^{1,p}(\Omega)$ is nonnegative, then

$$\int_\Omega |\nabla u^s(x,y)|^p \, dx\, dy = 2 \int_\omega \frac{(1+|\nabla_x l|^2)^{\frac{p}{2}}}{|\partial_t l|^{p-1}} \, dx\, dt \, ,$$

since in this case b is constant.

In order to prove our main result we will need the following key lemma.

Lemma 5.6. *Consider the function $f_p : \mathbb{R}^n \to \mathbb{R}$ defined for $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}^+$ as*

$$f_p(x) := \frac{(1+|x'|^2)^{\frac{p}{2}}}{x_n^{p-1}}.$$

For a fixed x, given $y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, let $\Phi(x,y)$ be the second order increment of f in the direction y, i.e.,

$$\Phi(x,y) := f_p(x+y) + f_p(x-y) - 2f_p(x).$$

If $|y_n| \le x_n$ there exists a positive constant $c = c(p)$ such that, for $1 < p < 2$ we have

$$\Phi(x,y) \ge c \, \frac{(1+|x'|^2+|y'|^2)^{\frac{p-2}{2}}}{x_n^{p-1}} \left(\frac{|y'|^2}{1+|x'|^2} + \frac{|y_n|^2}{x_n^2} \right), \tag{5.8}$$

while for $p \ge 2$ we have

$$\Phi(x,y) \ge c \, \frac{1}{x_n^{p-1}} \left(\frac{|y'|^p}{(1+|x'|^2)^{\frac{p}{2}}} + \frac{|y_n|^p}{x_n^p} \right).$$

Proof. The proof is based on some careful estimates of a suitable expansion of $\Phi(x,y)$. We start observing that a straightforward second order expansion of $\Phi(x,\cdot)$ allows us to rewrite it as follows

$$\Phi(x,y) = p \int_{-1}^1 J_p(\alpha) \, d\alpha \, ,$$

where

$$J_p(\alpha) = \frac{\left(1+|x'+\alpha y'|^2\right)^{\frac{p-2}{2}}}{(x_n+\alpha y_n)^{p-1}} \left[|y'|^2 + (p-2)\frac{\langle x'+\alpha y', y'\rangle^2}{1+|x'+\alpha y'|^2} \right.$$
$$\left. -2(p-1)\frac{\langle x'+\alpha y', y'\rangle y_n}{x_n+\alpha y_n} + (p-1)\frac{(1+|x'+\alpha y'|^2)y_n^2}{(x_n+\alpha y_n)^2} \right].$$

First case. Let us start dealing with the case $1 < p \leq 2$. By means of easily verifiable computations, we can write $J_p(\alpha)$ as

(5.9)

$$J_p(\alpha) = \frac{\left(1+|x'+\alpha y'|^2\right)^{\frac{p-2}{2}}}{(x_n+\alpha y_n)^{p-1}} \left[(p-1)\left(|y'|^2 - 2y_n\frac{\langle x'+\alpha y', y'\rangle}{x_n+\alpha y_n} + y_n^2\frac{|x'+\alpha y'|^2}{(x_n+\alpha y_n)^2} \right) \right.$$
$$\left. + (p-1)\frac{y_n^2}{(x_n+\alpha y_n)^2} + (2-p)|y'|^2 + (p-2)\frac{\langle x'+\alpha y', y'\rangle^2}{1+|x'+\alpha y'|^2} \right]$$
$$= \frac{\left(1+|x'+\alpha y'|^2\right)^{\frac{p-2}{2}}}{|x_n+\alpha y_n|^{p-1}} \left[(p-1)\frac{|y_n x' - x_n y'|^2 + |y_n|^2}{|x_n+\alpha y_n|^2} + (2-p)\left(|y'|^2 - \frac{\langle x'+\alpha y', y'\rangle^2}{1+|x'+\alpha y'|^2} \right) \right].$$

Noticing that by Schwarz inequality

$$(2-p)\left(|y'|^2 - \frac{\langle x'+\alpha y', y'\rangle^2}{1+|x'+\alpha y'|^2} \right) \geq 0\,,$$

and by taking in account that $|\alpha| \leq 1$ and the assumption $|y_n| \leq x_n$ implies $0 \leq x_n + \alpha y_n \leq 2x_n$, we have

$$J_p(\alpha) \geq c\,\frac{\left(1+|x'|^2+|y'|^2\right)^{\frac{p-2}{2}}}{x_n^{p-1}}\,\frac{|y_n x' - x_n y'|^2 + y_n^2}{x_n^2}. \tag{5.10}$$

Finally a dichotomy argument will give us the result. Indeed, if $2|y_n||x'| \geq x_n|y'|$, we can write

$$\frac{|y_n x' - x_n y'|^2 + y_n^2}{x_n^2} \geq \frac{y_n^2}{x_n^2} \geq \frac{y_n^2}{2x_n^2} + \frac{|y'|^2}{8|x'|^2}.$$

Otherwise, if $2|y_n||x'| < x_n|y'|$, or equivalently

$$2(x_n|y'| - |y_n||x'|) > x_n|y'|,$$

we have

$$\frac{|y_n x' - x_n y'|^2 + y_n^2}{x_n^2} \geq \frac{(x_n|y'| - |y_n||x'|)^2}{x_n^2} + \frac{y_n^2}{x_n^2} \geq \frac{|y'|^2}{4} + \frac{y_n^2}{x_n^2}. \tag{5.11}$$

Combining (5.10)-(5.11) we get (5.8).

Second case. The case $p > 2$ is more involved. While in the previous case we could get the result just performing pointwise estimates, here we will need to exploit the integral form of $\Phi(x, y)$. We use again a dichotomy argument. Let γ be sufficiently large so that $(p-1)\frac{(\gamma-1)^2}{(\gamma+3)^2} + 2 - p \geq \frac{1}{2}$ and suppose first that $|y'| > \gamma|x'|$. Then we have

$$|x' + \alpha y'| \geq |\alpha||y'| - |x'| \geq \left(\alpha - \frac{1}{\gamma}\right)|y'| \tag{5.12}$$

and, for $2/\gamma \leq \alpha \leq 3/\gamma$,

$$\frac{|y_n x' - x_n y'|}{|x_n + \alpha y_n|} \geq \frac{|y'||x_n| - |x'||y_n|}{|x_n| + \alpha |y_n|} \geq \frac{|y'| - |x'|}{1+\alpha} \geq |y'| \frac{1 - 1/\gamma}{1 + 3/\gamma} = |y'| \frac{\gamma - 1}{\gamma + 3}\,. \tag{5.13}$$

Using (5.9), (5.12) and (5.13), we can estimate $\Phi(x, y)$ as follows

$$\begin{aligned}
\Phi(x,y) =& p \int_{-1}^{1} J_p(\alpha)\, d\alpha \geq \int_{\frac{2}{\gamma}}^{\frac{3}{\gamma}} J_p(\alpha)\, d\alpha \\
\geq& \int_{\frac{2}{\gamma}}^{\frac{3}{\gamma}} \frac{\left(1 + |x' + \alpha y'|^2\right)^{\frac{p-2}{2}}}{(x_n + \alpha y_n)^{p-1}} \left[(p-1) \frac{|y_n x' - x_n y'|^2 + y_n^2}{(x_n + \alpha y_n)^2} + (2-p)|y'|^2 \right] d\alpha \\
\geq& c \int_{\frac{2}{\gamma}}^{\frac{3}{\gamma}} \frac{\left(1 + |y'|^2\right)^{\frac{p-2}{2}}}{x_n^{p-1}} \left[(p-1) \left(\frac{(\gamma-1)^2}{(\gamma+3)^2} |y'|^2 + \frac{y_n^2}{4x_n^2} \right) + (2-p)|y'|^2 \right] d\alpha \\
\geq& c\, \frac{\left(1 + |y'|^2\right)^{\frac{p-2}{2}}}{x_n^{p-1}} \left(|y'|^2 + \frac{y_n^2}{x_n^2} \right) \geq c \frac{1}{x_n^{p-1}} \left(|y'|^p + \frac{|y_n|^p}{x_n^p} \right).
\end{aligned}$$

On the other hand, when $|y'| \leq \gamma |x'|$ and $|\alpha| \leq 1/2\gamma$,

$$|x' + \alpha y'| \geq |x'| - |\alpha||y'| \geq (1 - |\alpha|\gamma)|x'| \geq \frac{1}{2}|x'|.$$

Let $\beta = \frac{1}{2(p-1)}$. By rearranging the expression of J_p we have

$$\begin{aligned}
J_p(\alpha) =& \frac{\left(1+|x'+\alpha y'|^2\right)^{\frac{p-2}{2}}}{(x_n+\alpha y_n)^{p-1}} \Bigg[|y'|^2 + (p-2)\frac{\langle x'+\alpha y', y'\rangle^2}{1+|x'+\alpha y'|^2} \\
&- 2(p-1)\frac{\langle x'+\alpha y', y'\rangle y_n}{x_n+\alpha y_n} + (p-1)\frac{(1+|x'+\alpha y'|^2)y_n^2}{(x_n+\alpha y_n)^2}\Bigg] \\
=& \frac{\left(1+|x'+\alpha y'|^2\right)^{\frac{p-2}{2}}}{(x_n+\alpha y_n)^{p-1}} \Bigg[|y'|^2 - \frac{\langle x'+\alpha y', y'\rangle^2}{1+|x'+\alpha y'|^2} + (p-1)\frac{\langle x'+\alpha y', y'\rangle^2}{1-\beta+|x'+\alpha y'|^2} \\
&\quad + (p-1)\langle x'+\alpha y', y'\rangle^2 \left(\frac{1}{1+|x'+\alpha y'|^2} - \frac{1}{1-\beta+|x'+\alpha y'|^2}\right) \\
&\quad - 2(p-1)\frac{\langle x'+\alpha y', y'\rangle y_n}{x_n+\alpha y_n} + (p-1)\frac{\beta y_n^2}{(x_n+\alpha y_n)^2} \\
&\quad + (p-1)\frac{(1-\beta+|x'+\alpha y'|^2)y_n^2}{(x_n+\alpha y_n)^2}\Bigg] \\
=& \frac{\left(1+|x'+\alpha y'|^2\right)^{\frac{p-2}{2}}}{(x_n+\alpha y_n)^{p-1}} \Bigg[|y'|^2 - \frac{\langle x'+\alpha y', y'\rangle^2}{1+|x'+\alpha y'|^2} + \beta\frac{(p-1)y_n^2}{(x_n+\alpha y_n)^2} \\
&\quad - \beta(p-1)\frac{\langle x'+\alpha y', y'\rangle^2}{(1+|x'+\alpha y'|^2)(1-\beta+|x'+\alpha y'|^2)} \\
&\quad + (p-1)\left(\frac{\langle x'+\alpha y', y'\rangle}{\sqrt{1-\beta+|x'+\alpha y'|^2}} - \frac{y_n\sqrt{1-\beta+|x'+\alpha y'|^2}}{x_n+\alpha y_n}\right)^2\Bigg].
\end{aligned}$$

Therefore, by using Schwarz inequality and removing the square term

$$\begin{aligned}
J_p(\alpha) \geq& \frac{\left(1+|x'+\alpha y'|^2\right)^{\frac{p-2}{2}}}{(x_n+\alpha y_n)^{p-1}} \left[\frac{|y'|^2}{1+|x'+\alpha y'|^2} + \beta\frac{(p-1)y_n^2}{(x_n+\alpha y_n)^2} - \beta\frac{(p-1)|y'|^2}{1+|x'+\alpha y'|^2}\right] \\
\geq& \frac{\left(1+|x'+\alpha y'|^2\right)^{\frac{p-2}{2}}}{2^p x_n^{p-1}} \left[\frac{|y'|^2}{1+|x'+\alpha y'|^2} + \frac{y_n^2}{4x_n^2}\right].
\end{aligned}$$

This allows us to estimate $\Phi(x,y)$ as follows

$$\begin{aligned}
\Phi(x,y) \geq& c\int_{-\frac{1}{2\gamma}}^{\frac{1}{2\gamma}} \frac{\left(1+|x'+\alpha y'|^2\right)^{\frac{p-2}{2}}}{x_n^{p-1}} \left[\frac{|y'|^2}{1+|x'+\alpha y'|^2} + \frac{y_n^2}{x_n^2}\right] d\alpha \\
\geq& c\int_{-\frac{1}{2\gamma}}^{\frac{1}{2\gamma}} \frac{\left(1+|y'|^2\right)^{\frac{p-2}{2}}}{x_n^{p-1}} \left[\frac{|y'|^2}{1+|x'|^2} + \frac{y_n^2}{x_n^2}\right] d\alpha \geq c\,\frac{1}{x_n^{p-1}}\left(\frac{|y'|^p}{1+|x'|^2} + \frac{|y_n|^p}{x_n^p}\right).
\end{aligned}$$

□

The last tool to prove Theorem 5.1 is the following easy geometrical estimate.

LEMMA 5.7. *Let $E \subset \mathbb{R}^n$ be an open, bounded and convex set with outer radius R. Then its inner radius r (i.e., the radius of the largest ball contained in E) can estimated by*

$$r \geq \frac{|E|}{nR^{n-1}}. \tag{5.14}$$

PROOF. Let S be the maximum ellipsoid included in E. Up to a roto-translation, we can assume that $S = \{x \in \mathbb{R}^n : \sum_{i=1}^n (x_i/l_i)^2 < 1\}$ with $l_1 \leq \ldots \leq l_n$. Clearly, $R \geq l_n$ and $r \geq l_1$. By John's ellipsoid theorem (see [**10**, Theorem 2.4]), the inclusion $E \subset nS$ holds and therefore $R^{n-1}r \geq |S| \geq |E|/n$. ☐

PROOF OF THEOREM 5.1. By rewriting $\int \nabla u^p$ in terms of b and l, we have

$$\Delta(u,u^s) = \int_\omega \frac{(1+|\nabla_x b+\nabla_x l|^2)^{\frac{p}{2}}}{|\partial_t b+\partial_t l|^{p-1}} + \frac{(1+|\nabla_x b-\nabla_x l|^2)^{\frac{p}{2}}}{|\partial_t b-\partial_t l|^{p-1}} - 2\frac{(1+|\nabla_x l|^2)^{\frac{p}{2}}}{|\partial_t l|^{p-1}} dx\,dt. \tag{5.15}$$

We recall that by construction

$$\partial_t b+\partial_t l \leq 0,\ \ \partial_t b-\partial_t l \geq 0,\ \ \partial_t l \leq 0,\ \ |\partial_t b| \leq |\partial_t l|.$$

Note that the integrand (5.15) can be written as $f(x+y)+f(x-y)-2f(x)$ with

$$f(x) := \frac{(1+|x'|^2)^{p/2}}{x_n^{p-1}} \text{ and } x' = -\nabla_x l,\ y' = -\nabla_x b,\ x_n = -\partial_t l,\ y_n = -\partial_t b.$$

Therefore, by using Lemma 5.6 we have through a second order expansion

$$\Delta(u,u^s) \geq \begin{cases} c\displaystyle\int_\omega \frac{(1+|\nabla_x l|^2+|\nabla_x b|^2)^{\frac{p-2}{2}}}{|\partial_t l|^{p-1}}\left[\frac{|\nabla_x b|^2}{1+|\nabla_x l|^2}+\frac{|\partial_t b|^2}{|\partial_t l|^2}\right] dx\,dt & \text{when } 1<p<2;\\ c\displaystyle\int_\omega \frac{1}{|\partial_t l|^{p-1}}\left[\frac{|\nabla_x b|^p}{(1+|\nabla_x l|^2)^{\frac{p}{2}}}+\frac{|\partial_t b|^p}{|\partial_t l|^p}\right] dx\,dt & \text{when } p\geq 2.\end{cases}$$

Using twice Hölder's inequality we get, in the case $1<p<2$,

$$\begin{aligned}\int_\omega \frac{|\nabla_x b|}{\sqrt{1+|\nabla_x l|^2}}+\frac{|\partial_t b|}{|\partial_t l|}\,dx\,dt &\leq c\Delta(u,u^s)^{\frac12}\left(\int_\omega \frac{|\partial_t l|^{p-1}}{(1+|\nabla_x l|^2+|\nabla_x b|^2)^{\frac{p-2}{2}}}\,dx\,dt\right)^{\frac12}\\ &\leq c\Delta(u,u^s)^{\frac12}\left(\int_\omega |\partial_t l|\,dx\,dt\right)^{\frac{p-1}{2}}\left(\int_\omega \sqrt{1+|\nabla_x l|^2+|\nabla_x b|^2}\,dx\,dt\right)^{\frac{2-p}{2}}.\end{aligned}$$

By using (5.6) and Hölder's inequality (with $p=1$) we have the estimate

$$\begin{aligned}2\int_\omega \sqrt{1+|\nabla_x l|^2+|\nabla_x b|^2}\,dx\,dt &\leq \int_\omega \sqrt{1+|\nabla_x b+\nabla_x l|^2}\,dx\,dt + \int_\omega \sqrt{1+|\nabla_x b-\nabla_x l|^2}\,dx\,dt\\ &= \int_\Omega |\nabla u| dx\,dy \leq \mathcal{L}^n(\Omega)^{\frac{1}{p'}}\|\nabla u\|_{L^p},\end{aligned}$$

while, denoted by E^s the subgraph of u^s and by ν^s the inner normal to $\partial^* E^s$, by the coarea formula,

$$2\int_\omega |\partial_t l|\,dx\,dt = \int_{\partial^* E^s\cap(\Omega^s\times\mathbb{R})} |\nu_t^s| d\mathcal{H}^n = \mathcal{L}^n(\Omega^s) = \mathcal{L}^n(\Omega).$$

Gathering all we have

$$\int_\omega \frac{|\nabla_x b|}{\sqrt{1+|\nabla_x l|^2}}+\frac{|\partial_t b|}{|\partial_t l|}\,dx\,dt \leq c\mathcal{L}^n(\Omega)^{\frac{1}{p'}}\|\nabla u\|_{L^p}^{\frac{2-p}{2}}\Delta(u,u^s)^{\frac12}. \tag{5.16}$$

Similarly, in the case $p \geq 2$ we have

$$\int_\omega \frac{|\nabla_x b|}{\sqrt{1+|\nabla_x l|^2}} + \frac{|\partial_t b|}{|\partial_t l|}\, dx\, dt \leq c\Delta(u,u^s)^{\frac{1}{p}} \left(\int_\omega |\partial_t l|\, dx\, dt \right)^{\frac{1}{p'}} \leq c\mathcal{L}^n(\Omega)^{\frac{1}{p'}} \Delta(u,u^s)^{\frac{1}{p}}. \tag{5.17}$$

It remains to estimate the left hand sides of (5.16) and (5.17). As already pointed out in [8, Theorem 1.4], because of the concavity of u^s we have

$$\frac{|\nu^s_y|}{|\nu^s_{x,t}|} \geq \frac{\operatorname{dist}((x,t),\partial\omega)}{l(x,t)} \qquad \forall (x,t) \in \omega,$$

so that, if $|\nu^s_{x,t}| \geq 1/\sqrt{2}$, then

$$|\nu^s_y| \geq \frac{\operatorname{dist}((x,t),\partial\omega)}{\sqrt{2}M}.$$

Of course this inequality is still true if $|\nu^s_{x,t}| < 1/\sqrt{2}$, because $|\nu^s_y| > 1/\sqrt{2}$. Therefore, since $\sqrt{1+|\nabla l|^2} = 1/|\nu^s_y|$,

$$\int_\omega \frac{|\nabla_x b|}{\sqrt{1+|\nabla_x l|^2}} + \frac{|\partial_t b|}{|\partial_t l|}\, dx\, dt \geq \int_\omega \frac{|\nabla b|}{\sqrt{1+|\nabla l|^2}}\, dx\, dt \geq \frac{1}{\sqrt{2}M} \int_\omega |\nabla b| \operatorname{dist}((x,t),\partial\omega)\, dx\, dt. \tag{5.18}$$

Finally, since ω is convex, by means of a weighted Poincaré inequality (see [8, Corollary 5.2] and [24, Theorem 1.1])

$$\begin{aligned}
\int_\omega |\nabla b| \operatorname{dist}((x,t),\partial\omega)\, dx\, dt &\geq c\,\frac{r}{R} \int_\omega |b-b_0|\, dx\, dt \geq c\,\frac{r}{R} \int_\omega \mathcal{L}^1(E^s_{x,t} \Delta (E_{x,t} - b_0)) dx\, dt \\
&= c\,\frac{r}{R} \mathcal{L}^n(E^s \Delta (E - (0,b_0))) \\
&= c\,\frac{r}{R} \int_{\mathbb{R}^n} |u(x,y-b_0) - u^s(x,y)| dx\, dy\,,
\end{aligned}$$

for a suitable $b_0 \in \mathbb{R}$. Here r and R are the inner and outer radius of ω, respectively. Their ratio can be estimated from below by using (5.14):

$$\frac{r}{R} \geq \frac{\mathcal{L}^n(\omega)}{nR^n} \geq \frac{\|u^s\|_{L^1}}{2nM^{n+1}} = \frac{\|u\|_{L^1}}{2nM^{n+1}}.$$

This last estimate completes the proof. □

We prove a version of the quantitative Pólya-Szegő inequality in the restricted class of the n-symmetric functions.

PROOF OF LEMMA 5.3. By the coarea formula, for $t > 0$ we have

$$\mu(t) = \mathcal{L}^n(\{w > t\} \cap \{\nabla w = 0\}) + \int_t^\infty \int_{\{w=s\}} \frac{d\mathcal{H}^{n-1}}{|\nabla w|} ds\,.$$

Therefore, for a.e. $t > 0$

$$-\mu'(t) \geq \int_{\{w=t\}} \frac{1}{|\nabla w|} d\mathcal{H}^{n-1}\,. \tag{5.19}$$

Moreover, we have that

$$\mathcal{H}^{n-1}(\{w=t\}) = P(\{w>t\})\,,$$

where P stands for the perimeter. Applying the coarea formula, Hölder's inequality and (5.19), we get

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla w|^p &= \int_0^\infty \int_{\{w=t\}} |\nabla w|^{p-1}\, d\mathcal{H}^{n-1}\, dt \geq \int_0^\infty \frac{\mathcal{H}^{n-1}(\{w=t\})^p}{(\int_{\{w=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla w|})^{p-1}} dt \\ &\geq \int_0^\infty \frac{\mathcal{H}^{n-1}(\{w=t\})^p}{(-\mu'(t))^{p-1}} dt = \int_0^\infty \frac{P(\{w>t\})^p}{(-\mu'(t))^{p-1}} dt\,. \end{aligned} \tag{5.20}$$

Given a measurable set $E \subset \mathbb{R}^n$, define the *Fraenkel asymmetry* of E to be

$$A(E) := \inf \left\{ \frac{\mathcal{L}^n(E \triangle B)}{\mathcal{L}^n(E)} : B \text{ ball, } \mathcal{L}^n(B) = \mathcal{L}^n(E) \right\} .$$

The quantitative isoperimetric inequality (see, e.g., [44] or [58]) assures us that there exists a constant γ_0, depending only on n, such that

$$n\,\omega_n^{1/n} \mathcal{L}^n(E)^{1/n'} \left(1 + \gamma_0 A(E)^2\right) \leq P(E)\,,$$

for every measurable set $E \subset \mathbb{R}^n$ having finite measure and finite perimeter, where n' is the Hölder conjugate of n. Moreover (see [58, Lemma 5.2]), if E is symmetric with respect to n orthogonal hyperplanes containing 0, then

$$A(E) \geq \frac{1}{3} \frac{\mathcal{L}^n(E \triangle E^*)}{\mathcal{L}^n(E)}\,, \tag{5.21}$$

where E^* is the ball centered in the origin having the same volume of E. Since w is n-symmetric, so are its level sets $\{w>t\}$ for $t>0$. Hence, combining (5.20)–(5.21) with $E = \{w>t\}$, we get

$$\int_{\mathbb{R}^n} |\nabla w|^p dz \geq (n\omega_n^{1/n})^p \int_0^\infty \frac{\mu(t)^{\frac{p}{n'}}}{(-\mu'(t))^{p-1}} \left(1 + \frac{\gamma_0}{3}\left(\frac{F(t)}{\mu(t)}\right)^2\right)^p dt\,, \tag{5.22}$$

where $F(t) := \mathcal{L}^n(\{w>t\} \triangle \{w^*>t\})$ for $t>0$.

Let us observe that, if we replace w by w^* in (5.20), we have then all equalities, because $|\nabla w^*|$ is constant on the ball $\{w^*>t\}$ for a.e. $t>0$ and because (see [27, Lemma 3.2]) also (5.19) turns into an equality. Thus, $P(\{w^*>t\}) = n\omega_n^{1/n}\mu(t)^{1/n'}$ for a.e. $t>0$ and

$$\int_{\mathbb{R}^n} |\nabla w^*|^p dz = (n\omega_n^{1/n})^p \int_0^\infty \frac{\mu(t)^{\frac{p}{n'}}}{(-\mu'(t))^{p-1}} dt. \tag{5.23}$$

Since $(1+s)^p \geq 1 + ps$ for $s \geq 0$, we deduce from (5.22) and (5.23) that

$$\int_{\mathbb{R}^n} |\nabla w|^p - |\nabla w^*|^p dz \geq \gamma \int_0^\infty \left(\frac{F(t)}{\mu(t)}\right)^2 \frac{\mu(t)^{\frac{p}{n'}}}{(-\mu'(t))^{p-1}} dt\,, \tag{5.24}$$

for some constant $\gamma > 0$, depending only on n and p.

Note that, by Jensen's inequality, for any $\varphi \in L^1$ we have

$$\left(\int F(t)\,dt\right)^p = \left(\int F\frac{\varphi}{\varphi}\right)^p \leq \left(\int \varphi\right)^{p-1} \left(\int \frac{F^p}{\varphi^{p-1}}\right)$$

Then, by the layer-cake representation formula we have

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |w-w^*|\,dz\right)^p &= \left(\int_0^\infty F(t)\,dt\right)^p \\ &\leq \left(\int_0^\infty (-\mu'(t))(\mu)^{\frac{p'}{n}} dt\right)^{p-1} \underbrace{\left(\int_0^\infty \left(\frac{F(t)}{\mu(t)}\right)^p \frac{\mu(t)^{\frac{p}{n'}}}{(-\mu'(t))^{p-1}} dt\right)}_{=:I}. \end{aligned} \tag{5.25}$$

Note that

$$\int_0^\infty \mu(t)^\alpha \cdot (-\mu'(t))\, dt = \frac{\mathcal{L}^n(\Omega)^{\alpha+1}}{\alpha+1} \qquad \text{for every } \alpha > -1\,.$$

As $0 \le F/\mu \le 2$ we have for $p \ge 2$ that

$$\left(\frac{F(t)}{\mu(t)}\right)^p \le 2^{p-2}\left(\frac{F(t)}{\mu(t)}\right)^2$$

Therefore,

$$\left(\int_{\mathbb{R}^n} |w - w^*|\, dz\right)^p \le \gamma \mathcal{L}^n(\Omega)^{p-1+\frac{p}{n}} \left(\int_0^\infty \left(\frac{F(t)}{\mu(t)}\right)^2 \frac{\mu(t)^{p/n'}}{(-\mu'(t))^{p-1}} dt\right)$$

and using (5.24) we obtain (5.5).

On the other hand, if $1 < p < 2$, by Hölder's inequality we have

$$I \le \left(\int_0^\infty \left(\frac{F(t)}{\mu(t)}\right)^2 \frac{\mu(t)^{\frac{p}{n'}}}{(-\mu'(t))^{p-1}} dt\right)^{\frac{p}{2}} \left(\int_0^\infty \frac{\mu(t)^{\frac{p}{n'}}}{(-\mu'(t))^{p-1}} dt\right)^{\frac{2-p}{2}}.$$

Hence, by using (5.25) and (5.23), we obtain (5.4). □

PROOF OF THEOREM 5.2. We give the proof only for the case $1 < p < 2$, being the other one similar. We set $u^{s_0} = u$ and, for every $i = 1, \ldots, n$, we indicate by u^{s_i} the Steiner symmetral of $u^{s_{i-1}}$ with respect to the hyperplane $z_i = 0$. Note that u^{s_n} is n-symmetric. Since the Steiner symmetrization decreases the outer radius, by (5.1) we get for every $i = 1, \ldots, n$

$$\int_{\mathbb{R}^n} |u^{s_{i-1}} - u^{s_i}|\, dz \le c\frac{M^{n+2}}{\|u\|_{L^1}} \mathcal{L}^n(\Omega)^{\frac{1}{p'}} \|\nabla u\|_{L^p}^{\frac{2-p}{2}} \Delta(u^{s_{i-1}}, u^{s_i})^{\frac{1}{2}}, \tag{5.26}$$

up to a suitable translation of $u^{s_{i-1}}$ along the z_i axis. Moreover, since $\mathcal{L}^n(\Omega) \le (2M)^n$ and $\|u\|_{L^1} \le (2M)^{n+1}$, by (5.4) we get

$$\int_{\mathbb{R}^n} |u^{s_n} - u^*|\, dz \le c\frac{M^{n+2}}{\|u\|_{L^1}} \mathcal{L}^n(\Omega)^{\frac{1}{p'}} \|\nabla u\|_{L^p}^{\frac{2-p}{2}} \Delta(u^{s_n}, u^*)^{\frac{1}{2}}. \tag{5.27}$$

If $\|\nabla u\|_{L^p}^p \le 2\|\nabla u^*\|_{L^p}^p$, a triangular inequality applied to (5.26) and (5.27) gives (5.3). Otherwise, since the support of u^* is a ball of volume $\mathcal{L}^n(\Omega)$,

$$\begin{aligned}
\inf_{h\in\mathbb{R}^n} \int_{\mathbb{R}^n} |u(z+h) - u^s(z)|\, dz &\le 2\|u^*\|_{L^1} \le c\mathcal{L}^n(\Omega)^{\frac{1}{n}} \|\nabla u^*\|_{L^1} \\
&\le c\mathcal{L}^n(\Omega)^{\frac{1}{p'}+\frac{1}{n}} \|\nabla u^*\|_{L^p} = c\mathcal{L}^n(\Omega)^{\frac{1}{p'}+\frac{1}{n}} \|\nabla u^*\|_{L^p}^{\frac{2-p}{2}} \left(\int |\nabla u^*|^p\right)^{\frac{1}{2}} \\
&\le c\frac{M^{n+2}}{\|u\|_{L^1}} \mathcal{L}^n(\Omega)^{\frac{1}{p'}} \|\nabla u^*\|_{L^p}^{\frac{2-p}{2}} \Delta(u, u^*)^{\frac{1}{2}}.
\end{aligned}$$

□

Here we show two examples proving that for $1 < p \le 2$ the power $1/2$ of the deficit in the estimates of Theorems 5.1 and 5.2 is sharp. The first example concerns the Steiner rearrangement.

EXAMPLE 5.8. Let $\varepsilon \in (0,1)$ and let $u : \mathbb{R} \to \mathbb{R}$ defined as

$$u(x) = \begin{cases} (x+1)/(\varepsilon+1) & \text{if } -1 \le x \le \varepsilon\,; \\ (x-1)/(\varepsilon-1) & \text{if } \varepsilon \le x \le 1\,; \\ 0 & \text{otherwise.} \end{cases}$$

Then its Steiner rearrangement is

$$u^s(x) = \begin{cases} 1 - |x| & \text{if } -1 \le x \le 1\,; \\ 0 & \text{otherwise.} \end{cases}$$

In order to calculate the

$$\inf_{h\in\mathbb{R}} \int_{\mathbb{R}} |u(x+h) - u^s(x)|\, dx$$

it is clear that we may assume $h \in [0, \varepsilon]$. Then, a straightforward computation shows that for any such h

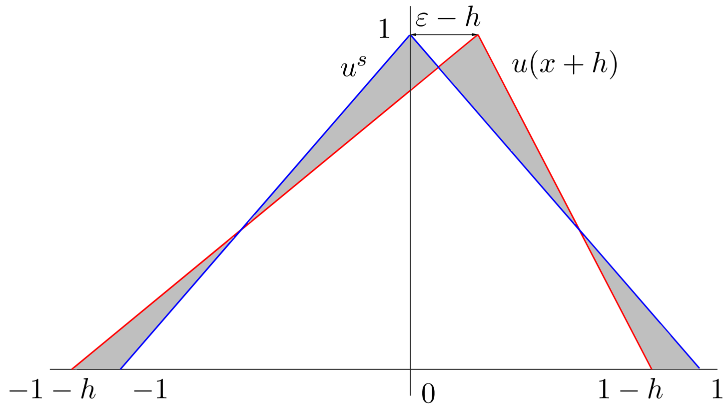
$$\int_{\mathbb{R}} |u(x+h) - u^s(x)|\, dx = \frac{h^2(4+\varepsilon) - 4\varepsilon h + 2\varepsilon^2}{\varepsilon(2+\varepsilon)}\,.$$

Hence, the infimum is attained at $h = 2\varepsilon/(\varepsilon+4)$ and is equal to

$$\frac{2\varepsilon}{4+\varepsilon} \sim \frac{1}{2}\varepsilon \qquad \text{as } \varepsilon \to 0^+\,.$$

On the other hand, a direct computation shows that $\Delta(u, u^s) = (1+\varepsilon)^{1-p} + (1-\varepsilon)^{1-p} - 2 \sim p(p-1)\varepsilon^2$, hence showing the sharpness of the power $1/2$ in the estimate (5.1).

FIGURE 5.1. Example 5.8. The grey area represents $\int |u(x+h) - u^s(x)|$



The second example concerns the Schwarz rearrangement.

EXAMPLE 5.9. Let $\varepsilon \in (0,1)$ and denote by E the ellipse $\{(x,y) \in \mathbb{R}^2 : (1+\varepsilon)^2x^2 + y^2/(1+\varepsilon)^2 \le 1\}$. Let $u : \mathbb{R}^2 \to \mathbb{R}$ defined as $u(x,y) = 1 - (1+\varepsilon)^2x^2 - y^2/(1+\varepsilon)^2$ if $(x,y) \in E$ and extended by 0 outside E. Then, its Schwarz rearrangement is $u^*(x,y) = 1 - x^2 - y^2$ for $(x,y) \in B$ and extended by 0 outside B, where B is the ball of radius 1 centered in 0. As the function u is 2-symmetric, the infimum

$$\inf_{h\in\mathbb{R}^2} \int_{\mathbb{R}^2} |u(z+h) - u^s(z)|\, dz$$

is attained at $h = 0$ and is equal to $c\mathcal{L}^n(B\Delta E) \sim c\varepsilon$ as $\varepsilon \to 0^+$. By a direct computation we obtain

$$
\begin{aligned}
2^{-p}\Delta(u,u^*) &= \int_B \left[\left(x^2(1+\varepsilon)^2 + \frac{y^2}{(1+\varepsilon)^2}\right)^{p/2} - (x^2+y^2)^{p/2}\right] dx\,dy \\
&= \int_B \left[(x^2+y^2) + 2\varepsilon(x^2-y^2) + \varepsilon^2(x^2+3y^2) + O(\varepsilon^3)y^2)^{p/2} - (x^2+y^2)^{p/2}\right] dx\,dy \\
&= \int_B (x^2+y^2)^{p/2}\left[\left(1 + \frac{2\varepsilon(x^2-y^2)}{x^2+y^2} + \frac{\varepsilon^2(x^2+3y^2)}{x^2+y^2} + O(\varepsilon^3)\right)^{p/2} - 1\right] dx\,dy \\
&= p\varepsilon \int_B (x^2+y^2)^{p/2}\frac{(x^2-y^2)}{x^2+y^2}\,dx\,dy \\
&\quad + \frac{p}{2}\varepsilon^2 \int_B (x^2+y^2)^{p/2}\left(\frac{x^2+3y^2}{x^2+y^2} + \frac{(p-2)(x^2-y^2)^2}{(x^2+y^2)^2}\right) dx\,dy + O(\varepsilon^3) \\
&= \frac{p}{2}\varepsilon^2 \int_B (x^2+y^2)^{(p-4)/2}\left[2y^2(3x^2+y^2) + (p-1)(x^2-y^2)^2\right] dx\,dy + O(\varepsilon^3) \sim c(p)\varepsilon^2\,,
\end{aligned}
$$

for some positive constant $c(p)$ and where in the second and third line we have used the Taylor expansion of $(1+\varepsilon)^{-2}$ and $(1+\cdot)^{p/2}$ respectively. Hence, our claim is proven.

Part II

A variational model for material voids in elastic solids

CHAPTER 6

A quantitative second order minimality criterion for cavities in elastic bodies

Recall from the introduction that we are studying the functional (1.1)

$$\mathcal{F}(F,u) := \int_{B_0\setminus F} Q(E(u))\,dz + \mathcal{H}^1(\Gamma_F) + 2\mathcal{H}^1(\Sigma_F)\,. \tag{6.1}$$

We outline now the structure of this chapter and make some comments about the proofs. In Section 6.2 we calculate the second variation of $\mathcal{F}$ at any regular configuration (see Theorem 6.8) and we exploit the volume constraint to define the associated quadratic form in a critical configuration. At the end of the section in Lemma 6.11 we prove a "weak" coercivity property of $\partial^2\mathcal{F}(F,u)$ in a critical point, which is the first step towards the proof of Theorem 6.19. In Section 6.3, as an intermediate step, we prove that the positivity of the second variation implies the local minimality among configurations (G,v) for which G close to F in the $C^{1,1}$-topology. The main point in achieving this result is to overcome the lack of $C^{1,1}$-coercivity, which would immediately imply the result. This is done by proving the stability of the weak coercivity under one parameter perturbation of the critical configuration (see Lemma 6.18). In section 6.4 we exploit the regularity theory for a class of obstacle problems which arise as perturbations of (6.1) to show that the $C^{1,1}$-minimality actually implies the minimality with respect to the Hausdorff distance thus proving the theorem. In the last section we apply the previous analysis to the explicit case of a disk subjected to a radial stretching.

6.1. Preliminaries

In this section we fix the notation and describe precisely the required background for our analysis. We are interested in cavities identified as closed sets F with $\mathcal{H}^1(\partial F) < +\infty$ and starshaped with respect to the origin. The fact that F is starshaped allows us to describe it as a subgraph of a function. Since F has finite perimeter, the function associated to its boundary turns out to have bounded pointwise total variation. This will allow us to deal with functions rather than sets.

We denote by $\mathbb{S}^1$ the unit circle in $\mathbb{R}^2$ and by $\sigma : \mathbb{R} \to \mathbb{S}^1$ the local diffeomorphism defined by $\sigma(\theta) = (\cos\theta, \sin\theta)$, by σ^{-1} its local inverse and by $\sigma^{\perp}(\theta) = (\sin\theta, -\cos\theta)$ its orthogonal. We set $C^2_{\#}(\mathbb{R})$ to be the collection of functions in $C^2(\mathbb{R})$ which are 2π-periodic. In a similar way we shall define the function spaces $H^1_{\#}(\mathbb{R})$, etc.

With a slight abuse of notation we set

$$BV_{\#}(\mathbb{R}) := \{g : \mathbb{R} \to [0, R_0] \mid g \text{ is upper semicontinuous, } 2\pi\text{-periodic and } pV(g,[0,2\pi]) < \infty\}, \tag{6.2}$$

where $pV(g,[0,2\pi])$ is the pointwise total variation of g in $[0,2\pi]$ and R_0 is the radius of a large ball B_{R_0}. For a function $g \in BV_{\#}(\mathbb{R})$ we define the extended graph of g as $\Gamma_g \cup \Sigma_g$, where

$$\Gamma_g := \{\rho\sigma(\theta) \in \mathbb{R}^2 \mid g^-(\theta) \le \rho \le g^+(\theta),\, \theta \in \mathbb{R}\} \tag{6.3}$$

and

$$\Sigma_g = \overline{\{\rho\sigma(\theta) \in \mathbb{R}^2 \mid g^+(\theta) < \rho < g(\theta),\, \theta \in \mathbb{R}\}}\,. \tag{6.4}$$

Here $g^-(\theta) := \liminf_{\tilde\theta\to\theta} g(\tilde\theta)$ and $g^+(\theta) := \limsup_{\tilde\theta\to\theta} g(\tilde\theta)$. We shall refer to Σ_g as the *set of cracks*.

Let us consider a compact set $F \subset \overline{B}_{R_0}$ starshaped with respect to the origin. Then, for $\sigma \in \mathbb{S}^1$, we can write

$$F = \{r\sigma(\theta) \in \mathbb{R}^2 \mid \theta \in \mathbb{R},\, 0 \le r \le \rho_F(\theta)\}\,,$$

where ρ_F is the *radial function* of F and is defined by

$$\rho_F(\theta) := \sup\{\rho \in \mathbb{R} \mid \rho\sigma(\theta) \in F\}\,.$$

It is clear that $\rho_F : \mathbb{R} \to [0, R_0]$ is upper semicontinuous. Moreover we have the following result, see [43, Lemmata 2.2 and 2.3].

LEMMA 6.1. *Let $F \subset \bar{B}_{R_0}$ be a closed set starshaped with respect to the origin and let ρ_F be the radial function of F. Then*

$$\partial F = \Gamma_{\rho_F} \cup \Sigma_{\rho_F}.$$

Moreover $\mathcal{H}^1(\partial F) < +\infty$ if and only if ρ_F has finite pointwise total variation.

The previous lemma rigorously shows that we may use radial functions instead of sets. Hence, for $g \in BV_\#(\mathbb{R})$ we set

$$F_g := \{\rho\sigma(\theta) \in \mathbb{R}^2 \mid 0 \le \rho \le g(\theta)\} \quad \text{and} \quad \Omega_g := B_{R_0} \setminus F_g.$$

We may think of F_g as the void and of Ω_g as the elastic solid.

We can now define properly the space of admissible pairs. Given $u_0 \in C^\infty(\mathbb{R}^2 \setminus B_{R_0})$ we set

$$X(u_0) = \{(g, v) \mid g \in BV_\#(\mathbb{R}),\, v \in H^1_{\rm loc}(\mathbb{R}^2 \setminus F_g; \mathbb{R}^2),\, v \equiv u_0 \text{ outside } B_{R_0}\}\,, \tag{6.5}$$

and we shall use the notation $X(0)$ for $u_0 \equiv 0$. We define also the following subspaces of $X(u_0)$

$$\begin{aligned} X_{\rm Lip}(u_0) &:= \{(g, v) \in X(u_0) \mid g \text{ is Lipschitz}\},\\ X_{\rm reg}(u_0) &:= \{(g, v) \in X(u_0) \mid g \in C^\infty_\#(\mathbb{R}),\, v \in C^\infty(\bar\Omega_g)\}. \end{aligned} \tag{6.6}$$

We are now in position to give the proper definition of convergence in $X(u_0)$.

DEFINITION 6.2. A sequence $(g_n, v_n) \subset X(u_0)$ is said to converge to (g, v) in $X(u_0)$ and we write $(g_n, v_n) \xrightarrow{X} (g, v)$ if

(1) $\sup_{n\in\mathbb{N}} \mathcal{H}^1(\partial F_{g_n}) < +\infty$,
(2) $F_{g_n} \to F_g$ in Hausdorff metric,
(3) $v_n \rightharpoonup v$ weakly in $H^1(\omega; \mathbb{R}^2)$ for any open set ω compactly contained in $\mathbb{R}^2 \setminus F_g$.

In view of [43, Lemma 2.6], we see that $X(u_0)$ is closed under the convergence of Definition 6.2.

The *elastic energy density* is defined by $Q(E(u)) := \frac{1}{2}\mathbb{C}E(u) : E(u)$, where $\mathbb{C}$ is the fourth order tensor

$$\mathbb{C}\xi := \begin{pmatrix} (2\mu+\lambda)\xi_{11} + \lambda\xi_{22} & 2\mu\xi_{12} \\ 2\mu\xi_{12} & (2\mu+\lambda)\xi_{22} + \lambda\xi_{11} \end{pmatrix}$$

and $E(u)$ is the symmetric gradient of u

$$E(u) := \frac{1}{2}(Du + (Du)^T).$$

The constants μ, λ are called the *Lamé coefficients* and they are assumed to satisfy the following ellipticity conditions

$$\mu > 0 \quad \text{and} \quad \lambda > -\mu.$$

Since $Q(\xi) \geq \min\{\mu, \mu+\lambda\}|\xi|^2$ for every symmetric 2×2 matrix ξ, the above conditions guarantee that Q is coercive. We also set the ellipticity constant

$$\eta := \min\{\mu, \mu + \lambda\}.$$

For a pair $(g, v) \in X_{\mathrm{Lip}}(u_0)$ we may write the value of the functional (6.1) as

$$\mathcal{F}(g, v) = \int_{\Omega_g} Q(E(v))\, dz + \mathcal{H}^1(\Gamma_g).$$

Since this functional is not lower semicontinuous with respect to the convergence in $X(u_0)$, in order to effectively address the minimization problem we consider the relaxed functional

$$\bar{\mathcal{F}}(g, v) = \inf\{\liminf_{n\to\infty} \mathcal{F}(g_n, v_n) \mid (g_n, v_n) \in X_{\mathrm{Lip}}(u_0),\ (g_n, v_n) \xrightarrow{X} (g, v)\}.$$

The following integral representation of $\bar{\mathcal{F}}$ is proved in [**43**, Theorem 3.1], where the more general case of anisotropic surface energy is also considered.

THEOREM 6.3. *Let $(g, v) \in X(u_0)$, then*

$$\bar{\mathcal{F}}(g, v) = \int_{\Omega_g} Q(E(v))\, dz + \mathcal{H}^1(\Gamma_g) + 2\mathcal{H}^1(\Sigma_g).$$

From now on we will always deal with the relaxed functional appearing in Theorem 6.3 and with abuse of notation we will denote it simply by $\mathcal{F}(g, v)$. The minimization problem can now be properly stated as

$$\min\{\mathcal{F}(g, v) \mid (g, v) \in X(u_0),\ |\Omega_g| = d\} \tag{6.7}$$

for some given constant $d < |B_{R_0}|$. Existence of solutions of the problem (6.7) is then ensured by [**43**, Theorem 3.2].

Given $g \in BV_\#(\mathbb{R})$ there is one particular elastic displacement v which is the minimizer of the elastic energy $\int_{\Omega_g} Q(E(v))\, dz$ under the boundary condition $v \equiv u_0$ outside B_{R_0}. We call this map *the elastic equilibrium* associated to g. If $(h, u) \in X(u_0)$ solves (6.7) then u has to be the elastic equilibrium associated with h.

Assume now that a solution (h, u) belongs to $X_{\mathrm{reg}}(u_0)$ and $h > 0$, then (h, u) satisfy the Euler-Lagrange equations

$$\begin{cases} \operatorname{div} \mathbb{C}(E(u)) = 0 & \text{in } \Omega_h \\ \mathbb{C}(E(u))[\nu] = 0 & \text{on } \Gamma_h \\ Q(E(u)) - k_h = \text{const.} & \text{on } \Gamma_h, \end{cases} \tag{6.8}$$

where k_h is the curvature of Γ_h. The first two equations are standard whereas the third one is the first variation of the functional (6.1). This motivates the following definition.

DEFINITION 6.4. A pair $(h, u) \in X_{\mathrm{reg}}(u_0)$ is said to be critical if it solves the equations (6.8).

We remark that if (h, u) is a critical pair, then from the first two equations in (6.8) it follows that u is the elastic equilibrium associated to h. We also point out that in the definition of a critical point we only need to assume h to be smooth. Indeed, if we only assume $(h, u) \in X(u_0)$ and $h \in C^\infty(\mathbb{R})$, then it follows from the standard elliptic regularity theory (see [**2**]) that $u \in C^\infty(\bar{\Omega}_h)$.

At the end we note that the regularity for minimizers of (6.7) was studied in [**42**] and we have the following result. If the pair (h, u) is a local minimizer of (6.7) and $0 < h < R_0$ then there exists an open set $I \subset [0, 2\pi)$ of full measure such that $h \in C^\infty(I)$. In fact h is even analytic in I. Hence our regularity assumption on a critical point in Definition 6.4 is not restrictive when $0 < h < R_0$ and the singular set is empty.

Finally, we recall a version of the Korn's inequality which will be used throughout the paper, see e.g. [54].

THEOREM 6.5 (Korn's inequality). *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and $v \in W^{1,2}(\Omega;\mathbb{R}^2)$. There exists a constant $C = C(\Omega)$ such that if*

$$\int_\Omega Dv\,dz = \int_\Omega Dv^T\,dz,$$

then

$$\int_\Omega |Dv|^2\,dz \le C\int_\Omega |E(v)|^2\,dz.$$

Moreover if Ω is an annulus $A(R,r)$, $R > r$, the constant C depends only on the ratio r/R and $C \to 4$ as $r/R \to 0$.

6.2. Calculation of the second variation

The goal of this section is to calculate the second variation of the functional $\mathcal{F}$ at any point $(h,u) \in X_{\mathrm{reg}}(u_0)$, where u is the elastic equilibrium associated to h and $0 < h < R_0$, see formula (6.12). We then define a quadratic form for a critical pair (6.24) and give a definition of positiveness of the second variation, see Definition 6.10.

To this aim we will introduce the following notation. Given a 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$ we will denote by $\underline{f} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$ the map

$$\underline{f}(z) := f\left(\sigma^{-1}\left(\frac{z}{|z|}\right)\right)\frac{z}{|z|}. \tag{6.9}$$

For a parameter $s \in (-\varepsilon,\varepsilon)$ let $(h_s,u_s) \in X_{\mathrm{reg}}(u_0)$ be a smooth perturbation of (h,u), where u_s is the elastic equilibrium associated to h_s. By smooth perturbation we mean that the function $(s,\theta) \mapsto h_s(\theta)$ is smooth and $\lim_{s\to 0}||h_s - h||_{C^2(\mathbb{R})} = 0$. Moreover we set $\dot{h}_s = \frac{\partial}{\partial s}h_s$, $\dot{u}_s = \frac{\partial}{\partial s}u_s$ and $h_s' = \frac{\partial}{\partial \theta}h_s$. Notations $\dot{u}, \dot{h}$ mean that we evaluate the time derivatives at $s = 0$. We explicitly point out that $\dot{h}$ and $\dot{u}$ depend on h_s. Finally, for a given h, we define the set of functions

$$\mathcal{A}(\Omega_h) := \{w : \Omega_h \to \mathbb{R}^2 \mid (h,w) \in X(0)\}. \tag{6.10}$$

Roughly this means that $w \in \mathcal{A}(\Omega_h)$ if $w = 0$ outside B_{R_0}.

We will first write the equation for $\dot{u}$. In the following we will denote by τ the tangent unit vector to Γ_h clockwise oriented and by ν the unit normal to Γ_h pointing outward the set F_h.

PROPOSITION 6.6. *Let $(h,u) \in X_{\mathrm{reg}}(u_0)$ be such that u is the elastic equilibrium associated to h and $0 < h < R_0$. Suppose (h_s,u_s) is a smooth perturbation of (h,u). Then the function $\dot{u} \in \mathcal{A}(\Omega_h)$ satisfies*

$$\begin{aligned}\int_{\Omega_h} \mathbb{C}E(\dot{u}) : E(w)\,dz &= \int_{\Gamma_h} \langle \underline{\dot{h}}, \nu\rangle\,\mathbb{C}E(u) : E(w)\,d\mathcal{H}^1 \\ &= -\int_{\Gamma_h} \mathrm{div}_\tau\left(\langle \underline{\dot{h}}, \nu\rangle\,\mathbb{C}E(u)\right)\cdot w\,d\mathcal{H}^1,\end{aligned} \tag{6.11}$$

for all $w \in \mathcal{A}(\Omega_h)$.

PROOF. The proof is very similar to the one in [45]. Arguing as in [19, Proposition 8.1] we obtain a one parameter family of C^∞-diffeomorphisms $\Phi_s(\cdot) : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ such that $\Phi_0 = \mathrm{id}$ and $\Phi_s(z) = \underline{h_s}$ on ∂F_h.

Suppose first that $w \in \mathcal{A}(\Omega_h) \cap C^\infty(\bar{\Omega}_h)$. We may extend w outside Ω_h such that $w \in \mathcal{A}(\Omega_{h_s}) \cap C^\infty(\bar{\Omega}_{h_s})$. Since u_s is the elastic equilibrium in Ω_{h_s} we have

$$\int_{\Omega_{h_s}} \mathbb{C}E(u_s) : E(w)\, dz = 0.$$

Differentiate this with respect to s and evaluate at $s = 0$ to obtain

$$\int_{\Omega_h} \mathbb{C}E(\dot{u}) : E(w)\, dz - \int_0^{2\pi} \dot{h}\, h\, [\mathbb{C}E(u) : E(w)](h\,\sigma(\theta))\, d\theta = 0.$$

Using the area formula and notation (6.9) we may write

$$\int_{\Omega_h} \mathbb{C}E(\dot{u}) : E(w)\, dz = \int_{\Gamma_h} \langle \underline{\dot{h}}, \nu\rangle\, \mathbb{C}E(u) : E(w)\, d\mathcal{H}^1,$$

where we have used the fact that the normal can be written in polar coordinates as $\nu = \frac{h\sigma + h'\sigma^\perp}{\sqrt{h^2+h'^2}}$. The rest will follow by integration by parts and from the fact that $\mathbb{C}E(u)[\nu] = 0$ on Γ_h as in (6.8).

To obtain (6.11) for every $w \in \mathcal{A}(\Omega_h)$ one may use a standard approximation argument. $\square$

REMARK 6.7. Notice that the equality (6.11) clearly holds also for test functions of the form $\tilde{w}(z) = w(z) + Az + b$, where $w \in \mathcal{A}(\Omega_h)$, $b \in \mathbb{R}^2$ and A is an antisymmetric matrix.

In the next theorem we derive the formula for the second variation of $\mathcal{F}$.

THEOREM 6.8. *Suppose that (h, u) and (h_s, u_s) are as in Proposition 6.6. Let ν be the outer normal of $\Gamma_h = \partial F_h$, τ be the tangent (positively oriented) of Γ_h and k be the curvature of Γ_h. The second variation of $\mathcal{F}$ at (h, u) is*

$$\begin{aligned} \frac{d^2}{ds^2}\mathcal{F}(h_s, u_s)\Big|_{s=0} = &- \int_{\Omega_h} 2Q(E(\dot{u}))\, dz + \int_{\Gamma_h} |\partial_\tau \langle \underline{\dot{h}}, \nu\rangle|^2\, d\mathcal{H}^1 \\ &- \int_{\Gamma_h} (\partial_\nu Q(E(u)) + k^2)\, \langle \underline{\dot{h}}, \nu\rangle^2\, d\mathcal{H}^1 \\ &+ \int_{\Gamma_h} (Q(E(u)) - k)\, \partial_\tau \left(\langle \underline{\dot{h}}, \nu\rangle \langle \underline{\dot{h}}, \tau\rangle\right) d\mathcal{H}^1 \\ &- \int_{\Gamma_h} (Q(E(u)) - k) \left(\frac{\langle \underline{\dot{h}}, \nu\rangle^2}{\langle \underline{h}, \nu\rangle} + \langle \underline{\ddot{h}}, \nu\rangle \right) d\mathcal{H}^1. \end{aligned} \tag{6.12}$$

PROOF. We will treat the elastic and the perimeter part separately and write

$$\mathcal{F}(h_s, u_s) = \int_{\Omega_{h_s}} Q(E(u_s))\, dz + \mathcal{H}^1(\Gamma_{h_s}) = \mathcal{F}_1(h_s, u_s) + \mathcal{F}_2(h_s).$$

Since h_s is smooth, we notice that $\Sigma_{h_s} = \emptyset$ and denote by Φ_s the family of diffeomorphisms as in the proof of Proposition 6.6.

1st Variation : We start by differentiating the perimeter part $\mathcal{F}_2(h_s)$.

Since $\mathcal{H}^1(\Gamma_{h_s}) = \int_0^{2\pi} \sqrt{h_s^2 + h_s'^2}\, d\theta$ we have

$$\frac{d}{ds}\mathcal{F}_2(h_s) = \int_0^{2\pi} \frac{h_s\, \dot{h}_s + h_s'\, \dot{h}_s'}{\sqrt{h_s^2 + h_s'^2}}\, d\theta.$$

Integrate the second term by parts and obtain

$$\int_0^{2\pi} \frac{h_s'\, \dot{h}_s'}{\sqrt{h_s^2 + h_s'^2}}\, d\theta = -\int_0^{2\pi} \left(\frac{h_s''}{\sqrt{h_s^2 + h_s'^2}} - \frac{h_s (h_s')^2 + (h_s')^2 h_s''}{(h_s^2 + h_s'^2)^{3/2}} \right) \dot{h}_s\, d\theta.$$

Then we have

$$
\begin{aligned}
(6.13)\qquad \frac{d}{ds}\mathcal{F}_2(h_s) &= \int_0^{2\pi} h_s\,\dot h_s \left(\frac{h_s^2+2h_s'^2-h_sh_s''}{(h_s^2+h_s'^2)^{3/2}}\right) d\theta = \int_0^{2\pi} h_s\,\dot h_s\,k_s(h_s\sigma)\,d\theta,\\
&= \int_{\Gamma_{h_s}} \langle \underline{\dot h_s}, \nu_{h_s}\rangle\, k_s\, d\mathcal{H}^1.
\end{aligned}
$$

where $k_s = \frac{h_s^2+2h_s'^2-h_sh_s''}{(h_s^2+h_s'^2)^{3/2}}$ is the curvature of Γ_{h_s} in polar coordinates.

Let us now treat the elastic part $\mathcal{F}_1(h_s,u_s)$. Differentiate it with respect to s and get, as in the proof of Proposition 6.6,

$$
\frac{d}{ds}\mathcal{F}_1(h_s,u_s) = \int_{\Omega_{h_s}} \mathbb{C}E(\dot u_s) : E(u_s)\,dz - \int_0^{2\pi} \dot h_s\,h_s\,Q(E(u_s))(h_s\sigma)\,d\theta.
$$

The first term disappears since u_s satisfies the Euler-Lagrange equations (6.8) and $\dot u_s \in \mathcal{A}(\Omega_{h_s})$. Hence, we are only left with

$$
(6.14)\qquad \frac{d}{ds}\mathcal{F}_1(h_s,u_s) = -\int_0^{2\pi} \dot h_s\,h_s\,Q(E(u_s))(h_s\sigma)\,d\theta = -\int_{\Gamma_{h_s}} \langle \underline{\dot h_s}, \nu_{h_s}\rangle\, Q(E(u_s))\,d\mathcal{H}^1.
$$

Combining (6.13) and (6.14) gives the first variation of $\mathcal{F}$.

2nd Variation : We will divide the proof in two steps.

Step 1: As in [**45**], we begin by making a couple of general observations.

Let d be the signed distance function from Γ_h, i.e.,

$$
d(z) := \begin{cases} -\operatorname{dist}(z,\Gamma_h) & \text{if } z \in F_h,\\ \operatorname{dist}(z,\Gamma_h) & \text{if } z \notin F_h. \end{cases}
$$

Since the boundary Γ_h is a graph of a C^∞-function, d is C^1 function in a small tubular neighbourhood of Γ_{h_t}. Setting $\nu(z) := \nabla d(z)$ and $k(z) := (\operatorname{div}\nu)(z)$, we observe that on Γ_h, ν is the outer normal to Γ_h and k is the curvature of Γ_h.

First we claim that

$$
(6.15)\qquad \partial_\nu k = -k^2, \qquad \text{on } \Gamma_h.
$$

Differentiating the identity $|\nu| = 1$ with respect to ν yields $D\nu[\nu] = 0$. This shows that

$$
(6.16)\qquad D\nu = D_\tau\nu = k\tau\otimes\tau \qquad \text{and} \qquad \operatorname{div}\nu = \operatorname{div}_\tau\nu, \qquad \text{on } \Gamma_h.
$$

Differentiating the identity $D\nu[\nu] = 0$ yields $\sum_{j=1}^2(\partial^2_{jk}\nu_i\nu_j + \partial_j\nu_i\partial_k\nu_j) = 0$ for $k,i = 1,2$. Hence we have

$$
(\partial_\nu(D\nu))_{ik} = \sum_{j=1}^2 \partial^2_{jk}\nu_i\nu_j = -\sum_{j=1}^2 \partial_j\nu_i\partial_k\nu_j = -\left((D\nu)^2\right)_{ik}
$$

for $i,k = 1,2$. Using the previous identity we obtain

$$
\partial_\nu k = \operatorname{Trace}(\partial_\nu(D\nu)) = -\operatorname{Trace}\left((D\nu)^2\right) = -k^2 \qquad \text{on } \Gamma_h,
$$

where the last equality follows from (6.16). Hence we have (6.15).

Next we claim that

$$
(6.17)\qquad \langle\dot\nu,\tau\rangle = -\partial_\tau\langle\underline{\dot h},\nu\rangle, \qquad \text{on } \Gamma_h.
$$

Recall that $\Phi_s(z) : \mathbb{R}^2\setminus\{0\} \to \mathbb{R}^2\setminus\{0\}$ is a one-parameter family of C^∞-diffeomorphisms such that $\Phi_s(z) = \underline{h_s}$ on Γ_h and $\Phi_0 = \mathrm{id}$. Notice that we have

$$
(6.18)\qquad \langle\dot\Phi,\nu\rangle = \langle\underline{\dot h},\nu\rangle, \qquad \text{on } \Gamma_h.
$$

Differentiating $D\Phi_s^{-T} D\Phi_s^T[\nu] = \nu$ and calculating at $s = 0$ gives $D\dot{\Phi}^{-T}[\nu] = -D\dot{\Phi}^T[\nu]$. Differentiate the identity

$$\nu_s \circ \Phi_s = \frac{D\Phi_s^{-T}[\nu]}{|D\Phi_s^{-T}[\nu]|}$$

with respect to s, evaluate at $s = 0$ and use the previous identity to obtain

$$\dot{\nu} + D\nu[\dot{\Phi}] = -D\dot{\Phi}^T[\nu] + \langle D\dot{\Phi}^T[\nu], \nu\rangle\, \nu\,, \qquad \text{on } \Gamma_h. \tag{6.19}$$

By (6.16) we have $D\nu = D_\tau \nu^T$ on Γ_h. Therefore, multiplying (6.19) by τ we obtain

$$\begin{aligned}\langle \dot{\nu}, \tau\rangle &= -\langle D\dot{\Phi}^T[\nu], \tau\rangle - \langle D\nu[\dot{\Phi}], \tau\rangle \\ &= -\langle D\dot{\Phi}^T[\nu], \tau\rangle - \langle D\nu^T[\dot{\Phi}], \tau\rangle \\ &= \langle\, (-D\,\langle \dot{\Phi}, \nu\rangle)\,, \tau\rangle = -\partial_\tau \langle \underline{\dot{h}}, \nu\rangle \qquad \text{on } \Gamma_h\end{aligned}$$

and (6.17) is proven.
Step 2: Let us start with the perimeter part and differentiate (6.13)

$$\begin{aligned}\frac{d^2}{ds^2}\mathcal{F}_2(h_s)\big|_{s=0} &= \overbrace{\int_0^{2\pi} h\,\dot{h}\,\dot{k}(h\sigma)\,d\theta}^{A} + \overbrace{\int_0^{2\pi} h\,\dot{h}^2\,\partial_\sigma k(h\sigma)\,d\theta}^{B} \\ &\quad + \int_0^{2\pi} \dot{h}^2\,k(h\sigma)\,d\theta + \int_0^{2\pi} h\,\ddot{h}\,k(h\sigma)\,d\theta.\end{aligned}$$

For the term A we have that

$$\begin{aligned}A = \int_0^{2\pi} h\,\dot{h}\,\dot{k}(h\sigma)\,d\theta &= \int_{\Gamma_h} \langle \underline{\dot{h}}, \nu\rangle\,\dot{k}\,d\mathcal{H}^1 = \int_{\Gamma_h} \langle \underline{\dot{h}}, \nu\rangle\ \mathrm{div}_\tau\,\dot{\nu}\,d\mathcal{H}^1 \\ &= -\int_{\Gamma_h} \langle \dot{\nu}, \tau\rangle\,\partial_\tau \langle \underline{\dot{h}}, \nu\rangle\,d\mathcal{H}^1 = \int_{\Gamma_h} |\partial_\tau \langle \underline{\dot{h}}, \nu\rangle|^2\,d\mathcal{H}^1,\end{aligned}$$

where we have used (6.17). For the term B, noticing that

$$\partial_\sigma k = \frac{h}{\sqrt{h^2 + h'^2}}\,\partial_\nu k - \frac{h'}{\sqrt{h^2 + h'^2}}\,\partial_\tau k \qquad \text{and} \qquad \tau = \frac{h\sigma^\perp - h'\sigma}{\sqrt{h^2 + h'^2}}\,,$$

we may write

$$\begin{aligned}B = \int_0^{2\pi} h\,\dot{h}^2\,\partial_\sigma k(h\sigma)\,d\theta &= \int_{\Gamma_h} \langle \underline{\dot{h}}, \nu\rangle^2\,\partial_\nu k\,d\mathcal{H}^1 + \int_{\Gamma_h} \langle \underline{\dot{h}}, \nu\rangle \langle \underline{\dot{h}}, \tau\rangle\,\partial_\tau k\,d\mathcal{H}^1 \\ &= -\int_{\Gamma_h} \langle \underline{\dot{h}}, \nu\rangle^2\,k^2\,d\mathcal{H}^1 - \int_{\Gamma_h} k\,\partial_\tau \Big(\langle \underline{\dot{h}}, \nu\rangle \langle \underline{\dot{h}}, \tau\rangle\Big)\,d\mathcal{H}^1,\end{aligned}$$

where we have used (6.15) and integration by parts. Hence, we have

$$\begin{aligned}\frac{d^2}{ds^2}\mathcal{F}_2(h_s)\big|_{s=0} &= \int_{\Gamma_h} |\partial_\tau \langle \underline{\dot{h}}, \nu\rangle|^2\,d\mathcal{H}^1 - \int_{\Gamma_h} \langle \underline{\dot{h}}, \nu\rangle^2\,k^2\,d\mathcal{H}^1 \\ &\quad - \int_{\Gamma_h} k\,\partial_\tau \Big(\langle \underline{\dot{h}}, \nu\rangle \langle \underline{\dot{h}}, \tau\rangle\Big)\,d\mathcal{H}^1 + \int_{\Gamma_h} k\,\frac{\langle \underline{\dot{h}}, \nu\rangle^2}{\langle \underline{h}, \nu\rangle}\,d\mathcal{H}^1 + \int_{\Gamma_h} k\,\langle \underline{\ddot{h}}, \nu\rangle\,d\mathcal{H}^1.\end{aligned} \tag{6.20}$$

We are left with the elastic part. Differentiate (6.14) to obtain

$$\begin{aligned}\frac{d^2}{ds^2}\mathcal{F}_1(h_s, u_s)\big|_{s=0} &= -\int_0^{2\pi} \mathbb{C}E(\dot{u}) : E(u)\,h\,\dot{h}\,d\theta - \int_0^{2\pi} \partial_\sigma Q(E(u))\,h\,\dot{h}^2\,d\theta \\ &\quad - \int_0^{2\pi} Q(E(u))\,(\dot{h}^2 + h\ddot{h})\,d\theta.\end{aligned}$$

Since $\dot{u} \in \mathcal{A}(\Omega_h)$, we may rewrite the first term using (6.11) as follows

$$\int_0^{2\pi} \mathbb{C}E(\dot{u}) : E(u)\, h\, \dot{h}\, d\theta = \int_{\Gamma_h} \langle \underline{\dot{h}}, \nu \rangle\, \mathbb{C}E(u) : E(\dot{u})\, d\mathcal{H}^1$$
$$= \int_{\Omega_h} 2Q(E(\dot{u}))\, dz.$$

For the second term, noticing that

$$\partial_\sigma Q(E(u)) = \frac{h}{\sqrt{h^2 + h'^2}}\, \partial_\nu Q(E(u)) - \frac{h'}{\sqrt{h^2 + h'^2}}\, \partial_\tau Q(E(u))$$

and using integration by parts, we get

$$\int_0^{2\pi} \partial_\sigma Q(E(u))\, h\, \dot{h}^2\, d\theta = \int_{\Gamma_h} \partial_\nu Q(E(u))\, \langle \underline{\dot{h}}, \nu \rangle^2\, d\mathcal{H}^1 + \int_{\Gamma_h} \partial_\tau Q(E(u)) \left(\langle \underline{\dot{h}}, \nu \rangle \langle \underline{\dot{h}}, \tau \rangle \right) d\mathcal{H}^1$$
$$= \int_{\Gamma_h} \partial_\nu Q(E(u))\, \langle \underline{\dot{h}}, \nu \rangle^2\, d\mathcal{H}^1 - \int_{\Gamma_h} Q(E(u))\, \partial_\tau \left(\langle \underline{\dot{h}}, \nu \rangle \langle \underline{\dot{h}}, \tau \rangle \right) d\mathcal{H}^1 .$$

Finally we have that

$$\begin{aligned} \frac{d^2}{ds^2} \mathcal{F}_1(h_s, u_s)\big|_{s=0} = &- \int_{\Omega_h} 2Q(E(\dot{u}))\, dz - \int_{\Gamma_h} \partial_\nu Q(E(u))\, \langle \underline{\dot{h}}, \nu \rangle^2\, d\mathcal{H}^1 \\ &+ \int_{\Gamma_h} Q(E(u))\, \partial_\tau \left(\langle \underline{\dot{h}}, \nu \rangle \langle \underline{\dot{h}}, \tau \rangle \right) d\mathcal{H}^1 - \int_{\Gamma_h} Q(E(u))\, \frac{\langle \underline{\dot{h}}, \nu \rangle^2}{\langle \underline{h}, \nu \rangle}\, d\mathcal{H}^1 \\ &- \int_{\Gamma_h} Q(E(u))\, \langle \underline{\ddot{h}}, \nu \rangle\, d\mathcal{H}^1. \end{aligned} \tag{6.21}$$

Combining (6.21) with (6.20) yields the formula (6.12).

□

In the formula (6.12) we considered any smooth perturbation h_s of h. However, in order to be admissible for our minimization problem, a perturbation h_s has to satisfy the volume constraint $|F_{h_s}| = |F_h|$, or equivalently

$$\int_0^{2\pi} h_s^2\, d\theta = \int_0^{2\pi} h^2\, d\theta \qquad \text{for all} \quad s > 0. \tag{6.22}$$

REMARK 6.9. If $(h, u) \in X_{\text{reg}}(u_0)$ is a critical pair and the perturbation (h_s) satisfies the volume constraint (6.22), then the last two terms in (6.12) vanish. Indeed one term vanishes because the term $Q(E(u)) - k$ is constant on Γ_h by (6.8). The second one vanishes since differentiating two times the volume constraint (6.22) with respect to s we obtain

$$\int_{\Gamma_h} \frac{\langle \underline{\dot{h}}, \nu \rangle^2}{\langle \underline{h}, \nu \rangle} + \langle \underline{\ddot{h}}, \nu \rangle\, d\mathcal{H}^1 = 0\,.$$

Motivated by the previous observation, for any $\psi \in H^1_\#(\mathbb{R})$ satisfying

$$\int_0^{2\pi} h\, \psi\, d\theta = 0\,, \tag{6.23}$$

we define the quadratic form associated to a regular critical pair (h, u)

$$\begin{aligned} \partial^2 \mathcal{F}(h, u)[\psi] := &- \int_{\Omega_h} 2Q(E(u_\psi))\, dz + \int_{\Gamma_h} |\partial_\tau \langle \underline{\psi}, \nu \rangle|^2\, d\mathcal{H}^1 \\ &- \int_{\Gamma_h} (\partial_\nu Q(E(u_\psi)) + k^2)\, \langle \underline{\psi}, \nu \rangle^2\, d\mathcal{H}^1\,, \end{aligned} \tag{6.24}$$

where $u_\psi \in \mathcal{A}(\Omega_h)$ is the unique solution to

$$\int_{\Omega_h} \mathbb{C}E(u_\psi) : E(w)\, dz = -\int_{\Gamma_h} \operatorname{div}_\tau \Big(\langle \underline{\psi}, \nu\rangle \, \mathbb{C}E(u) \Big) \cdot w \, d\mathcal{H}^1 \,, \qquad \forall w \in \mathcal{A}(\Omega_h)\,. \tag{6.25}$$

We define now what we mean by the second variation of $\mathcal{F}$ being positive at a critical pair.

DEFINITION 6.10. Suppose that $(h, u) \in X_{\mathrm{reg}}(u_0)$ is a critical pair. The functional (6.1) has *positive second variation* at (h, u) if

$$\partial^2 \mathcal{F}(h, u)[\psi] > 0$$

for all $\psi \in H^1_\#(\mathbb{R})$ such that $\psi \neq 0$ and satisfies (6.23).

We point out that if the second variation is positive at a critical point (h, u), then the formula (6.12) and Remark 6.9 imply that for every smooth perturbation h_s of h satisfying the volume constraint $\frac{d^2}{ds^2}\mathcal{F}(h_s, u_s)\big|_{s=0} > 0$.

At the end of the section we prove the following compactness result.

LEMMA 6.11. *Suppose that a critical pair $(h, u) \in X_{\mathrm{reg}}(u_0)$ is a point of positive second variation, and $0 < h < R_0$. Then there exists $c_0 > 0$ such that*

$$\partial^2 \mathcal{F}(h, u)[\psi] \geq c_0 || \langle \underline{\psi}, \nu \rangle ||^2_{H^1(\Gamma_h)},$$

for every $\psi \in H^1_\#(\mathbb{R})$ satisfying (6.23).

PROOF. First we notice that the condition (6.23) can be written using the notation (6.9) as

$$\int_{\Gamma_h} \langle \underline{\psi}, \nu \rangle \, d\mathcal{H}^1 = 0. \tag{6.26}$$

Using the Sobolev-Poincaré inequality $||\langle \underline{\psi}, \nu\rangle||_{L^2(\Gamma_h)} \leq C ||\partial_\tau \langle \underline{\psi}, \nu \rangle||_{L^2(\Gamma_h)}$ and (6.26) we easily see that it suffices to show that

$$c_0 := \inf \left\{ \partial^2 \mathcal{F}(h, u)[\psi] \mid \psi \in H^1_\#(\mathbb{R}) \text{ satisfying (6.23)}, \int_{\Gamma_h} |\partial_\tau \langle \underline{\psi}, \nu \rangle|^2 \, d\mathcal{H}^1 = 1 \right\} > 0.$$

Choose a sequence (ψ_n) such that ψ_n are smooth, satisfy (6.23), $\int_{\Gamma_h} |\partial_\tau \langle \underline{\psi_n}, \nu \rangle|^2 \, d\mathcal{H}^1 = 1$ and

$$\partial^2 \mathcal{F}(h, u)[\psi_n] \to c_0.$$

By restricting to a subsequence, we may assume that $\langle \underline{\psi_n}, \nu \rangle \rightharpoonup f$ weakly in $H^1(\Gamma_h)$. By defining

$$\psi(\theta) := \frac{f\,(h(\theta)\sigma(\theta))}{\langle \sigma, \nu\rangle} = \frac{f\,(h(\theta)\sigma(\theta))}{h(\theta)} \sqrt{h^2(\theta) + h'^2(\theta)}$$

we see that $f = \langle \underline{\psi}, \nu \rangle$, for some $\psi \in H^1_\#(\mathbb{R})$. Moreover since $\int_{\Gamma_h} f \, d\mathcal{H}^1 = 0$, the function ψ satisfies (6.23).

Next we prove that $\mathcal{F}(h, u)$ has the following lower semicontinuity property

$$\lim_{n\to\infty} \partial^2 \mathcal{F}(h, u)[\psi_n] \geq \partial^2 \mathcal{F}(h, u)[\psi]. \tag{6.27}$$

Indeed, since $\langle \underline{\psi_n}, \nu \rangle \rightharpoonup \langle \underline{\psi}, \nu \rangle$ weakly in $H^1(\Gamma_h)$ then $\langle \underline{\psi_n}, \nu \rangle \to \langle \underline{\psi}, \nu \rangle$ strongly in $L^2(\Gamma_h)$. Therefore we only need to check the convergence of the first term in (6.24).

First of all, the smoothness of ψ_n implies that u_{ψ_n} is smooth. Consider the domain $\tilde{\Omega}_h = B_{2R_0} \setminus F_h$ and the map $w_n(z) = u_{\psi_n}(z) + A_n z + b_n$, where A_n is an antisymmetric matrix and $b_n \in \mathbb{R}^2$ is chosen such that $\int_{\tilde{\Omega}} w_n \, dz = 0$. Notice that $w_n \in H^1(\tilde{\Omega}_h)$ and by Sobolev-Poincaré inequality it holds $\|w_n\|_{L^2(\tilde{\Omega}_h)} \leq C \|Dw_n\|_{L^2(\tilde{\Omega}_h)}$. By choosing A_n such that $\int_{\tilde{\Omega}_h} Dw_n \, dz = \int_{\tilde{\Omega}_h} Dw_n^T \, dz$ we have by Korn's inequality (Theorem 6.5) that $||Dw_n||_{L^2(\tilde{\Omega}_h)} \leq C \, ||E(w_n)||_{L^2(\tilde{\Omega}_h)}$. Moreover, since $u_{\psi_n} \equiv 0$ outside B_{R_0}, we have $||E(w_n)||_{L^2(\tilde{\Omega}_h)} = ||E(u_{\psi_n})||_{L^2(\Omega_h)}$. By the

Remark 6.7 we may use w_n as a test function in (6.25) and using Hölder's inequality and the trace theorem we get

$$
\begin{aligned}
\int_{\Omega_h} 2Q(E(u_{\psi_n}))\,dz &= -\int_{\Gamma_h} \operatorname{div}_\tau \Big(\langle \underline{\psi_n}, \nu\rangle\, \mathbb{C}E(u)\Big)\cdot w_n\, d\mathcal{H}^1 \\
&\le \|\langle \underline{\psi_n}, \nu\rangle\, \mathbb{C}E(u)\|_{H^1(\Gamma_h)} \|w_n\|_{L^2(\Gamma_h)} \\
&\le C\|\langle \underline{\psi_n}, \nu\rangle\, \mathbb{C}E(u)\|_{H^1(\Omega_h)} \|Dw_n\|_{L^2(\tilde\Omega_h)} \\
&\le C\|\langle \underline{\psi_n}, \nu\rangle\, \mathbb{C}E(u)\|_{H^1(\Omega_h)} \|E(u_{\psi_n})\|_{L^2(\Omega_h)}\,.
\end{aligned}
\tag{6.28}
$$

Therefore

$$\|Dw_n\|_{L^2(\tilde\Omega_h)} \le C\|E(u_{\psi_n})\|_{L^2(\Omega_h)} \le C\,.$$

However, since $u_{\psi_n} \equiv 0$ outside B_{R_0} we get

$$|B_{2R_0} \setminus B_{R_0}|\,|A_n|^2 = \int_{B_{2R_0}\setminus B_{R_0}} |Dw_n|^2\,dz \le C.$$

This implies that the matrices A_n are bounded and therefore $||Du_{\psi_n}||_{L^2(\Omega_h)} \le C$.

By the compactness of the trace operator we now have that $u_{\psi_n} \to u_\psi$ in $L^2(\Gamma_h)$ up to a subsequence. Use u_{ψ_n} as a test function in (6.25) to obtain

$$
\begin{aligned}
\lim_{n\to\infty} \int_{\Omega_h} 2Q(E(u_{\psi_n}))\,dz &= -\lim_{n\to\infty} \int_{\Gamma_h} \operatorname{div}_\tau \Big(\langle \underline{\psi_n}, \nu\rangle\, \mathbb{C}E(u)\Big)\cdot u_{\psi_n}\, d\mathcal{H}^1 \\
&= -\int_{\Gamma_h} \operatorname{div}_\tau \Big(\langle \underline{\psi}, \nu\rangle\, \mathbb{C}E(u)\Big)\cdot u_{\psi}\, d\mathcal{H}^1 \\
&= \int_{\Omega_h} 2Q(E(u_\psi))\,dz.
\end{aligned}
$$

This proves (6.27).

The claim now follows since if $\psi \ne 0$, the lower semicontinuity (6.27) implies

$$c_0 = \lim_{n\to\infty} \partial^2 \mathcal{F}(h,u)[\psi_n] \ge \partial^2\mathcal{F}(h,u)[\psi] > 0.$$

On the other hand if $\psi \equiv 0$ then the constraint $\int_{\Gamma_h} |\partial_\tau \langle \underline{\psi_n}, \nu\rangle|^2\, d\mathcal{H}^1 = 1$ yields

$$c_0 = \lim_{n\to\infty} \partial^2 \mathcal{F}(h,u)[\psi_n] = 1.$$

□

6.3. $C^{1,1}$-local minimality

In this section we perform a second order analysis of the functional (6.1) with respect to $C^{1,1}$-topology in the spirit of [**34**]. The main result is Proposition 6.12 where it is shown that a critical point $(h,u) \in X_{\text{reg}}(u_0)$ with positive second variation is a strict local minimizer in the $C^{1,1}$-topology, and that the functional satisfies a growth estimate. We point out that, according to Lemma 6.11, the second variation at (h,u) is coercive with respect to a norm which is weaker than the $C^{1,1}$-norm. Therefore the local minimality does not follow directly from Lemma 6.11. The idea is to prove a coercivity bound in a whole $C^{1,1}$-neighborhood of the critical point, which is carried out in Lemma 6.18. The main difficulty is to control the bulk energy, which will be done by using regularity theory for linear elliptic systems. We prove the main result first without worrying about the technicalities. All the technical lemmata are proven later in the section.

PROPOSITION 6.12. *Suppose that the critical pair* $(h, u) \in X_{\mathrm{reg}}(u_0)$ *is a point of positive second variation such that* $0 < h < R_0$. *There exists* $\delta > 0$ *such that for any admissible pair* $(g, v) \in X(u_0)$ *with* $g \in C^{1,1}_{\#}(\mathbb{R})$, $||g||_{L^2([0,2\pi))} = ||h||_{L^2([0,2\pi))}$ *and* $||h - g||_{C^{1,1}(\mathbb{R})} \leq \delta$ *we have*

$$\mathcal{F}(g, v) \geq \mathcal{F}(h, u) + c_1 ||h - g||^2_{L^2([0,2\pi))}.$$

PROOF. Assume first that $g \in C^\infty_{\#}(\mathbb{R})$ and $||h - g||_{C^2(\mathbb{R})} \leq \delta$. By scaling we may assume that $||h||_{L^2([0,2\pi))} = \left(\int_0^{2\pi} h^2 d\theta\right)^{\frac{1}{2}} = 1$. We define

$$g_t := \frac{h + t(g - h)}{||h + t(g - h)||_{L^2}}$$

so that g_t satisfies the volume constraint, and set

$$f(t) := \mathcal{F}(g_t, v_t)\,,$$

where v_t are the elastic equilibria associated to g_t. We calculate $\frac{d^2}{dt^2}\mathcal{F}(g_t, v_t)$ for every $t \in [0, 1)$ by applying the formula (6.12) to $(g_t)_s = g_{t+s}$ of g_t and get

$$\begin{aligned}
f''(t) = \frac{d^2}{dt^2}\mathcal{F}(g_t, v_t) = &- \int_{\Omega_{g_t}} 2Q(E(\dot{v}_t))\, dz + \int_{\Gamma_{g_t}} |\partial_{\tau_t} \langle \underline{\dot{g}_t}, \nu_t\rangle|^2 \, d\mathcal{H}^1 \\
&- \int_{\Gamma_{g_t}} (\partial_{\nu_t} Q(E(v_t)) + k_t^2)\, \langle \underline{\dot{g}_t}, \nu_t\rangle^2 \, d\mathcal{H}^1 \\
&+ \int_{\Gamma_{g_t}} (Q(E(v_t)) - k_t)\, \partial_{\tau_t} \Big(\langle \underline{\dot{g}_t}, \nu_t\rangle \langle \underline{\dot{g}_t}, \tau_t\rangle\Big)\, d\mathcal{H}^1 \\
&- \int_{\Gamma_{g_t}} (Q(E(v_t)) - k_t) \left(\frac{\langle \underline{\dot{g}_t}, \nu_t\rangle^2}{\langle \underline{g_t}, \nu_t\rangle} + \langle \underline{\ddot{g}_t}, \nu_t\rangle\right) d\mathcal{H}^1.
\end{aligned} \tag{6.29}$$

Here ν_t is the outer normal, τ_t the tangent, k_t the curvature of Γ_{g_t} and $\dot{v}_t$ is the unique solution to

$$\int_{\Omega_{g_t}} \mathbb{C}E(\dot{v}_t) : E(w)\, dz = -\int_{\Gamma_{g_t}} \mathrm{div}_{\tau_t} \Big(\langle \underline{\dot{g}_t}, \nu_t\rangle\, \mathbb{C}E(v_t)\Big) \cdot w \, d\mathcal{H}^1\,, \qquad \forall w \in \mathcal{A}(\Omega_{g_t})\,.$$

Remark 6.9 and Lemma 6.11 yield

$$f''(0) = \frac{d^2}{dt^2}\mathcal{F}(g_t, v_t)\big|_{t=0} = \partial^2 \mathcal{F}(h, u)[\dot{g}] \geq c_0 ||\langle \underline{\dot{g}}, \nu\rangle||^2_{H^1(\Gamma_h)}.$$

It will be shown later in Lemma 6.18 that, when $\delta > 0$ is chosen to be small enough, the previous inequality implies

$$f''(t) = \frac{d^2}{dt^2}\mathcal{F}(g_t, v_t) \geq \frac{c_0}{2} ||\langle \underline{\dot{g}_t}, \nu_t\rangle||^2_{H^1(\Gamma_{g_t})} \qquad \text{for all} \quad t \in [0, 1). \tag{6.30}$$

It is now clear that $||\langle \underline{\dot{g}_t}, \nu_t\rangle||^2_{H^1(\Gamma_{g_t})} \geq c\, ||\dot{g}_t||^2_{L^2([0,2\pi))}$ holds for all $t \in [0, 1]$. Since $\int_0^{2\pi} g^2\, d\theta = \int_0^{2\pi} h^2\, d\theta$ we have $\int_0^{2\pi} (h - g)^2 d\theta = 2\, \int_0^{2\pi} h(h - g)\, d\theta$ and therefore

$$||\dot{g}_t||^2_{L^2([0,2\pi))} = \frac{1}{||h + t(g - h)||^4_{L^2}} \left(\int_0^{2\pi} (h - g)^2 d\,\theta - \frac{1}{4}\left(\int_0^{2\pi} (h - g)^2\, d\theta\right)^2\right). \tag{6.31}$$

Since $\int_0^{2\pi} (h - g)^2 d\theta$ is very small we obtain from (6.31) that

$$||\dot{g}_t||^2_{L^2([0,2\pi))} \geq \frac{1}{2} ||h - g||^2_{L^2([0,2\pi))}. \tag{6.32}$$

From (6.30) and (6.32) we conclude that $f''(t) \geq \tilde{c}||h-g||^2_{L^2}$. Since (h,u) is a critical pair we have $f'(0)=0$ and therefore

$$\begin{aligned}\mathcal{F}(g,v)-\mathcal{F}(h,u) = f(1)-f(0) &= \int_0^1 (1-t)f''(t)\,dt \\ &\geq \tilde{c}\,||h-g||^2_{L^2([0,2\pi))}\int_0^1 (1-t)\,dt \\ &= \frac{\tilde{c}}{2}\,||h-g||^2_{L^2([0,2\pi))},\end{aligned}$$

which proves the claim when g is smooth.

When $g \in C^{1,1,}_{\#}(\mathbb{R})$ the claim follows by using a standard approximation. □

It remains to prove (6.30). The proof is based on a compactness argument and for that we have to study the continuity of the second variation formula (6.12). To control the boundary terms in (6.12) we need fractional Sobolev spaces whose definition and basic properties are recalled here. The function h is as in Proposition 6.12 and Γ_h is its graph.

DEFINITION 6.13. For $0<s<1$ and $1<p<\infty$ we define the fractional Sobolev space $W^{s,p}(\Gamma_h)$ as the set of those functions $v \in L^p(\Gamma_h)$ for which the Gagliardo seminorm is finite, i.e.

$$[v]_{s,p;\Gamma_h} = \left(\int_{\Gamma_h}\int_{\Gamma_h}\frac{|v(z)-v(w)|^p}{|z-w|^{1+sp}}\,d\mathcal{H}^1(w)d\mathcal{H}^1(z)\right)^{1/p} < \infty. \tag{6.33}$$

The fractional Sobolev norm is defined as $||v||_{W^{s,p}(\Gamma_h)} := ||v||_{L^p(\Gamma_h)} + [v]_{s,p;\Gamma_h}$.

The space $W^{-s,p}(\Gamma_h)$ is the dual space of $W^{s,p}(\Gamma_h)$ and the dual norm of a function v is defined as

$$||v||_{W^{-s,p}(\Gamma_h)} := \sup\left\{\int_{\Gamma_h} vu\,d\mathcal{H}^1(z) \mid ||u||_{W^{s,p}(\partial F_h)} \leq 1\right\}.$$

We also use the notation $H^s(\Gamma_h)$ for $W^{s,2}(\Gamma_h)$ for $-1<s<1$ and the convention $W^{0,p}(\Gamma_h) := L^p(\Gamma_h)$. By Jensen's inequality we have the following classical embedding theorem.

THEOREM 6.14. *Let $-1 \leq t \leq s \leq 1$, $q \geq p$ such that $s-1/p \geq t-1/q$. Then there is a constant C depending on t,s,p,q and on the C^1-norm of h such that*

$$||v||_{W^{t,q}(\Gamma_h)} \leq C||v||_{W^{s,p}(\Gamma_h)}.$$

We also have the following trace theorem.

THEOREM 6.15. *If $p>1$ there exists a continuous linear operator $T : W^{1,p}(\Omega_h) \to W^{1-1/p,p}(\Gamma_h)$ such that $Tv = v|_{\Gamma_h}$ whenever v is continuous on $\bar{\Omega}_h$. The norm of T depends on the C^1-norm of h and γ.*

The next lemma will be used frequently.

LEMMA 6.16. *Let $-1<s<1$ and suppose that v is a smooth function on Γ_h . Then the following hold.*

(i) *If $a \in C^1(\Gamma_h)$ then*

$$||av||_{W^{s,p}(\Gamma_h)} \leq C||a||_{C^1(\Gamma_h)}||v||_{W^{s,p}(\Gamma_h)},$$

where the constant C depends on p, s and the C^1-norm of h.

(ii) *If $\Psi : \Gamma_h \to \Psi(\Gamma_h)$ is a C^1-diffeomorphism, then*

$$||v \circ \Psi^{-1}||_{W^{s,p}(\Psi(\Gamma_h))} \leq C||v||_{W^{s,p}(\Gamma_h)},$$

where the constant C depends on p, s and the C^1-norms of h, Ψ and Ψ^{-1}.

We will also need to control the regularity of the elastic equilibrium. To this aim, the following elliptic estimate turns out to be useful, see [**45**, Lemma 4.1].

LEMMA 6.17. *Suppose $(g, v) \in X(0)$ is such that $\gamma \le g \le R_0 - \gamma$, $g \in C^2_\#(\mathbb{R})$ and $v \in \mathcal{A}(\Omega_g)$ satisfies*

$$\int_{\Omega_g} \mathbb{C}E(v) : E(w)\,dz = \int_{\Omega_g} f : E(w)\,dz \qquad \text{for every } w \in \mathcal{A}(\Omega_g), \tag{6.34}$$

where $f \in C^1(\bar{\Omega}_g; \mathbb{M}^{2\times 2})$. Then for any $p > 2$ we have the following estimate

$$\begin{aligned} &||E(v)||_{W^{1,p}(\Omega_g;\mathbb{M}^{2\times 2})} + ||\nabla \mathbb{C}E(v)||_{H^{-\frac{1}{2}}(\Gamma_g;\mathbb{T})} \\ &\qquad\qquad \le C\left(||E(v)||_{L^2(\Omega_g;\mathbb{M}^{2\times 2})} + ||f||_{C^1(\bar{\Omega}_g;\mathbb{M}^{2\times 2})}\right), \end{aligned} \tag{6.35}$$

where $\mathbb{T}$ denotes the space of third order tensors and the constant C depends on γ, p and the C^2-norm of g.

We are now in position to give the proof of the inequality (6.30). To control the bulk energy we use techniques developed in [**45**]. The main difference is that we use directly elliptic regularity rather than dealing with eigenvalues of compact operators.

LEMMA 6.18. *Suppose that a critical pair $(h, u) \in X(u_0)$ is a point of positive second variation with $0 < h < R_0$ and $||h||_{L^2} = 1$. Then there exists $\delta > 0$ such that for any admissible pair $(g, v) \in X_{\text{reg}}(u_0)$ with $||g||_{L^2} = 1$ and $||h - g||_{C^2(\mathbb{R})} \le \delta$ we have for*

$$g_t = \frac{h + t(g - h)}{||h + t(g - h)||_{L^2}}\,,$$

that

$$\frac{d^2}{dt^2}\mathcal{F}(g_t, v_t) \ge \frac{c_0}{2}||\langle \underline{\dot{g}_t}, \nu_t\rangle||^2_{H^1(\Gamma_{g_t})} \qquad \text{for all } t \in [0, 1], \tag{6.36}$$

where v_t is the elastic equilibrium associated to g_t. The constant c_0 is from the Lemma 6.11.

PROOF. Choose $\gamma > 0$ such that $\gamma \le h \le R_0 - \gamma$. Suppose that the claim is not true and there are pairs $(g_n, v_n) \in X_{\text{reg}}(u_0)$ and $t_n \in [0, 1]$ with

$$||h - g_n||_{C^2(\mathbb{R})} \to 0$$

for which the claim doesn't hold. Denoting

$$\dot{g}_n := \frac{\partial}{\partial t}\Big|_{t=t_n}\left(\frac{h + t(g_n - h)}{||h + t(g_n - h)||_{L^2}}\right) \quad \text{and} \quad \ddot{g}_n := \frac{\partial^2}{\partial t^2}\Big|_{t=t_n}\left(\frac{h + t(g_n - h)}{||h + t(g_n - h)||_{L^2}}\right)$$

this implies

$$\lim_{n\to\infty} \frac{\mathcal{F}''(g_n, v_n)}{||\langle \underline{\dot{g}_n}, \nu_n\rangle||^2_{H^1(\Gamma_{g_n})}} \le \frac{c_0}{2}, \tag{6.37}$$

where by the notation $\mathcal{F}''(g_n, v_n)$ we mean

$$\begin{aligned}\mathcal{F}''(g_n,v_n) = &-\int_{\Omega_{g_n}} 2Q(E(\dot v_n))\,dz + \int_{\Gamma_{g_n}} |\partial_{\tau_n}\langle \underline{\dot g_n}, \nu_n\rangle|^2\,d\mathcal{H}^1\\ &-\int_{\Gamma_{g_n}} (\partial_{\nu_n} Q(E(v_n)) + k_n^2)\,\langle \underline{\dot g_n}, \nu_n\rangle^2\,d\mathcal{H}^1\\ &+\int_{\Gamma_{g_n}} (Q(E(v_n)) - k_n)\,\partial_{\tau_n}\Big(\langle \underline{\dot g_n}, \nu_n\rangle\langle \underline{\dot g_n}, \tau_n\rangle\Big)\,d\mathcal{H}^1\\ &-\int_{\Gamma_{g_n}} (Q(E(v_n)) - k_n)\left(\frac{\langle \underline{\dot g_n}, \nu_n\rangle^2}{\langle \underline{g_n}, \nu_n\rangle} + \langle \underline{\ddot g_n}, \nu_n\rangle\right)d\mathcal{H}^1\\ &= I_1 + I_2 + I_3 + I_4 + I_5.\end{aligned} \tag{6.38}$$

Here ν_n is the outer normal to F_{g_n}, τ_n and k_n are the tangent vector and the curvature of Γ_{g_n} and $\dot v_n$ is the unique solution to

$$\int_{\Omega_{g_n}} \mathbb{C}E(\dot v_n) : E(w)\,dz = -\int_{\Gamma_{g_n}} \mathrm{div}_{\tau_n}\Big(\langle \underline{\dot g_n}, \nu_n\rangle\,\mathbb{C}E(v_n)\Big)\cdot w\,d\mathcal{H}^1, \qquad \forall w \in \mathcal{A}(\Omega_{g_t}). \tag{6.39}$$

As in Proposition 6.6 we find C^∞-diffeomorphisms $\Psi_n : \bar\Omega_h \to \bar\Omega_{g_n}$ such that $\Psi_n : \Gamma_h \to \Gamma_{g_n}$ and

$$||\Psi_n - id||_{C^2(\bar\Omega_h;\mathbb{R}^2)} \le C||h - g_n||_{C^2(\mathbb{R})}.$$

The goal is to examine the contribution of each term in (6.38) to the limit (6.37). We begin by proving that the contribution of I_4 and I_5 to (6.37) is zero.

Notice that the C^2-convergence of g_n implies $k_n \circ \Psi_n \to k$ in $L^\infty(\Gamma_h)$. Moreover, since v_n solves the first two equations in (6.8) and $\sup_n ||g_n||_{C^2([0,2\pi))} \le C$, we have by a Schauder type estimate for Lamé system, see [**45**], that there is $\alpha \in (0,1)$ such that

$$\sup_n ||v_n||_{C^{1,\alpha}(\bar\Omega_n';\mathbb{R}^2)} < \infty, \qquad \text{for } \Omega_n' = B_{R_0-\gamma} \setminus F_{g_n}. \tag{6.40}$$

Next we prove the following elliptic estimate

$$\begin{aligned}||E(u\circ\Psi_n^{-1}) - E(v_n)||_{W^{1,p}(\Omega_{g_n};\mathbb{M}^{2\times2})} + ||\nabla\mathbb{C}E(u\circ\Psi_n^{-1}) - \nabla\mathbb{C}E(v_n)||_{H^{-\frac12}(\Gamma_{g_n};\mathbb{T})}\\ \le C||h - g_n||_{C^2(\mathbb{R})},\end{aligned} \tag{6.41}$$

where $p > 2$ and C depends on γ, p and the C^2-norms of h and u. Indeed by the equations (6.8) satisfied by u and v_n and a standard change of variables we obtain

$$\int_{\Omega_{g_n}} \mathbb{C}(E(u\circ\Psi_n^{-1}) - E(v_n)) : E(w)\,dz = \int_{\Omega_{g_n}} f_n : E(w)\,dz, \qquad \forall w \in \mathcal{A}(\Omega_{g_n}), \tag{6.42}$$

where $f_n \in C^1(\bar\Omega_{g_n};\mathbb{M}^{2\times2})$ satisfies

$$||f_n||_{C^1(\bar\Omega_{g_n})} \le C||h - g_n||_{C^2(\mathbb{R})}$$

for C depending only on the C^2-norm of u. Lemma 6.17 yields the estimate

$$\begin{aligned}||E(u\circ\Psi_n^{-1}) - E(v_n)||_{W^{1,p}(\Omega_{g_n};\mathbb{M}^{2\times2})} + ||\nabla\mathbb{C}E(u\circ\Psi_n^{-1}) - \nabla\mathbb{C}E(v_n)||_{H^{-\frac12}(\Gamma_{g_n};\mathbb{T})}\\ \le C\Big(||E(u\circ\Psi_n^{-1}) - E(v_n)||_{L^2(\Omega_{g_n};\mathbb{M}^{2\times2})} + ||h - g_n||_{C^2(\mathbb{R})}\Big).\end{aligned}$$

On the other hand, using $w = u\circ\Psi_n^{-1} - v_n$ as a test function in (6.42), we obtain

$$||E(u\circ\Psi_n^{-1}) - E(v_n)||_{L^2(\Omega_{g_n};\mathbb{M}^{2\times2})} \le C||f_n||_{L^2(\Omega_{g_n};\mathbb{M}^{2\times2})}.$$

This concludes the proof of (6.41).

By the trace theorem 6.15, Lemma 6.16 and (6.41) we obtain

$$\begin{aligned}||E(v_n \circ \Psi_n) - E(u)||_{H^{\frac{1}{2}}(\Gamma_h;\mathbb{M}^{2\times 2})} &\leq C||E(v_n \circ \Psi_n) - E(u)||_{H^1(\Omega_h;\mathbb{M}^{2\times 2})} \\ &\leq C||g_n - h||_{C^2(\mathbb{R})}.\end{aligned}$$

This estimate together with (6.40) implies $v_n \circ \Psi_n \to u$ in $C^{1,\alpha}$. In particular, we have that

$$(Q(E(v_n)) - k_n) \circ \Psi_n \to Q(E(u)) - k \equiv \lambda$$

uniformly, where λ is a Lagrange multiplier. We may use this to estimate the term I_5 in (6.38). By explicit calculations one easily obtains that $||\langle \underline{\ddot{g}_n}, \nu_n\rangle||_{L^1} \leq C||\langle \underline{\dot{g}_n}, \nu_n\rangle|||^2_{L^2}$ and recalling that the functions $\dot{g}_n$ and $\ddot{g}_n$ satisfy the volume constraint, as in Remark 6.9,

$$\int_{\Gamma_{g_n}} \frac{\langle \underline{\dot{g}_n}, \nu_n\rangle^2}{\langle \underline{g_n}, \nu_n\rangle} + \langle \underline{\ddot{g}_n}, \nu_n\rangle \, d\mathcal{H}^1 = 0$$

we get

$$\begin{aligned}&\int_{\Gamma_{g_n}} (Q(E(v_n)) - k_n) \left(\frac{\langle \underline{\dot{g}_n}, \nu_n\rangle^2}{\langle \underline{g_n}, \nu_n\rangle} + \langle \underline{\ddot{g}_n}, \nu_n\rangle \right) d\mathcal{H}^1 \\ &\quad = \int_{\Gamma_{g_n}} (Q(E(v_n)) - k_n - \lambda) \left(\frac{\langle \underline{\dot{g}_n}, \nu_n\rangle^2}{\langle \underline{g_n}, \nu_n\rangle} + \langle \underline{\ddot{g}_n}, \nu_n\rangle \right) d\mathcal{H}^1 \\ &\quad \leq C\,||Q(E(v_n)) - k_n - \lambda||_{L^\infty(\Gamma_{g_n})}||\langle \underline{\dot{g}_n}, \nu_n\rangle||^2_{L^2(\Gamma_{g_n})}.\end{aligned}$$

Using the polar decomposition we have $\nu_n = \frac{g_n\sigma + g_n'\sigma^\perp}{\sqrt{g_n^2 + {g_n'}^2}}$ and $\tau_n = \frac{g_n\sigma^\perp - g_n'\sigma}{\sqrt{g_n^2 + {g_n'}^2}}$. Since $\langle \underline{\dot{g}_n}, \tau_n\rangle(z) = \frac{\langle z, \tau_n\rangle}{\langle z, \nu_n\rangle}\langle \underline{\dot{g}_n}, \nu_n\rangle(z)$ and $||\frac{\langle z, \tau_n\rangle}{\langle z, \nu_n\rangle}||_{H^1(\Gamma_{g_n})} \leq C$ we have as above that

$$\begin{aligned}&\int_{\Gamma_{g_n}} (Q(E(v_n)) - k_n)\, \partial_{\tau_n} \Big(\langle \underline{\dot{g}_n}, \nu_n\rangle \langle \underline{\dot{g}_n}, \tau_n\rangle \Big) \, d\mathcal{H}^1 \\ &\quad \leq C\,||Q(E(v_n)) - k_n - \lambda||_{L^\infty(\Gamma_{g_n})}||\langle \underline{\dot{g}_n}, \nu_n\rangle||^2_{H^1(\Gamma_{g_n})}.\end{aligned}$$

Hence the contribution of the terms I_4 and I_5 to the limit (6.37) is zero.

The remaining terms I_1 , I_2 and I_3 form a quadratic form. The goal is to show that

$$\lim_{n\to\infty} \frac{\mathcal{F}''(g_n, v_n)}{||\langle \underline{\dot{g}_n}, \nu_n\rangle||^2_{H^1(\Gamma_{g_n})}} = \lim_{n\to\infty} \frac{\partial^2 \mathcal{F}(h, u)[\underline{\psi_n}]}{||\langle \underline{\psi_n}, \nu\rangle||^2_{H^1(\Gamma_h)}} \tag{6.43}$$

where

$$\psi_n = \frac{\dot{g}_n g_n}{h}$$

and u_{ψ_n} solves

$$\int_{\Omega_h} \mathbb{C}E(u_{\psi_n}) : E(w)\, dz = -\int_{\Gamma_h} \operatorname{div}_\tau \Big(\langle \underline{\psi_n}, \nu\rangle\, \mathbb{C}E(u) \Big) \cdot w \, d\mathcal{H}^1, \qquad \forall w \in \mathcal{A}(\Omega_h)\,. \tag{6.44}$$

Notice that ψ_n satisfies the volume constraint $\int_0^{2\pi} h\psi_n \, d\theta = 0$ then we may use the Lemma 6.11 to conclude that

$$\lim_{n\to\infty} \frac{\partial^2 \mathcal{F}(h, u)[\underline{\psi_n}]}{||\langle \underline{\psi_n}, \nu\rangle||^2_{H^1(\Gamma_h)}} \geq c_0\,,$$

which then contradicts (6.37) and proves the claim.

To show (6.43) we will compare the contribution of each term in the quadratic form

$$\partial^2 \mathcal{F}(h, u)[\underline{\psi_n}] = -\int_{\Omega_h} 2Q(E(u_{\psi_n}))\, dz + \int_{\Gamma_h} |\partial_\tau \langle \underline{\psi_n}, \nu\rangle|^2 \, d\mathcal{H}^1 - \int_{\Gamma_h} (\partial_\nu Q(E(u)) + k^2)\, \langle \underline{\psi_n}, \nu\rangle^2 \, d\mathcal{H}^1$$

with respect to the one given by I_1, I_2 and I_3 in (6.38).

We first point out that since $g_n \to h$ in C^2 we have that $\nu_n \circ \Psi_n \to \nu$ and $\tau_n \circ \Psi_n \to \tau$ in $C^1(\Gamma_h)$. Therefore from the definition of ψ_n we get

$$\lim_{n\to\infty} \frac{||\langle \underline{\psi_n}, \nu\rangle||_{H^1(\Gamma_h)}}{||\langle \underline{\dot{g}_n}, \nu_n\rangle||_{H^1(\Gamma_{g_n})}} = 1$$

and the convergence of I_2,

$$\lim_{n\to\infty} \frac{\displaystyle\int_{\Gamma_h} |\partial_\tau \langle \underline{\psi_n}, \nu\rangle|^2 \, d\mathcal{H}^1}{\displaystyle\int_{\Gamma_{g_n}} |\partial_{\tau_n} \langle \underline{\dot{g}_n}, \nu_n\rangle|^2 \, d\mathcal{H}^1} = 1.$$

The convergence of I_1 follows from the equations (6.39) and (6.44). Indeed, by using a standard change of variables, these equations yield

(6.45)
$$\int_{\Omega_{g_n}} \Big(\mathbb{C}E(u_{\psi_n} \circ \Psi_n^{-1}) - \mathbb{C}E(\dot{v}_n)\Big) : E(w)\, dz = \int_{\Omega_{g_n}} (\tilde{f}_n\, E(u_{\psi_n} \circ \Psi_n^{-1})) : E(w)\, dz + \int_{\Gamma_{g_n}} d_n \cdot w \, d\mathcal{H}^1$$

for any $w \in \mathcal{A}(\Omega_{g_n})$. Here

$$d_n = \operatorname{div}_\tau(\langle \underline{\psi_n}, \nu\rangle \mathbb{C}E(u)) \circ \Psi_n^{-1} \, |D_{\tau_n} \Psi_n^{-1}| - \operatorname{div}_{\tau_n}(\langle \underline{\dot{g}_n}, \nu_n\rangle \mathbb{C}E(v_n))$$

and $\tilde{f}_n \in L^2(\Omega_{g_n}; \mathbb{M}^{2\times 2})$. For $\tilde{f}_n$ we have

(6.46)
$$||\tilde{f}_n||_{L^\infty(\Omega_{g_n}; \mathbb{M}^{2\times 2})} \to 0.$$

By the estimate (6.41) we get

$$||\nabla \mathbb{C}E(u \circ \Psi_n^{-1}) - \nabla \mathbb{C}E(v_n)||_{H^{-\frac{1}{2}}(\Gamma_{g_n}; \mathbb{T})} \to 0.$$

Therefore, by Lemma 6.16, the choice of Ψ_n and from $||\frac{\psi_n}{\dot{g}_n} - 1||_{C^1(\mathbb{R})} \to 0$ we have that

(6.47)
$$||d_n||_{H^{-\frac{1}{2}}(\Gamma_{g_n}; \mathbb{R}^2)} ||\langle \underline{\dot{g}_n}, \nu_n\rangle||^{-1}_{H^1(\Gamma_{g_n})} \to 0.$$

Choose

$$w(z) = (u_{\psi_n} \circ \Psi_n^{-1} - \dot{v}_n)(z) + Az + b$$

as a test function in (6.45) where A is antisymmetric and b is a vector. This yields

(6.48)
$$\begin{aligned}\int_{\Omega_{g_n}} & Q(E(u_{\psi_n} \circ \Psi_n^{-1} - \dot{v}_n))\, dz \\ &\le C||\tilde{f}_n||_{L^\infty(\Omega_{g_n}; \mathbb{M}^{2\times 2})} ||E(u_{\psi_n})||_{L^2(\Omega_h; \mathbb{M}^{2\times 2})} ||E(u_{\psi_n} \circ \Psi_n^{-1} - \dot{v}_n)||_{L^2(\Omega_{g_n}; \mathbb{M}^{2\times 2})} \\ &\quad + ||d_n||_{H^{-\frac{1}{2}}(\Gamma_{g_n}; \mathbb{R}^2)} ||w||_{H^{\frac{1}{2}}(\Gamma_{g_n}; \mathbb{R}^2)}.\end{aligned}$$

By the Theorem 6.15 we get that

$$||w||_{H^{\frac{1}{2}}(\Gamma_{g_n}; \mathbb{R}^2)} \le C||w||_{H^1(\Omega_{g_n}; \mathbb{M}^{2\times 2})}.$$

As in the proof of Lemma 6.11 we choose A such that

$$||w||_{H^1(\Omega_{g_n}; \mathbb{M}^{2\times 2})} \le C||E(w)||_{L^2(\Omega_{g_n}; \mathbb{M}^{2\times 2})} = C||E(u_{\psi_n} \circ \Psi_n^{-1} - \dot{v}_n)||_{L^2(\Omega_{g_n}; \mathbb{M}^{2\times 2})},$$

by Korn's and Poincaré's inequalities (choose b accordingly). The two previous inequalities and (6.48) yield

$$||E(u_{\psi_n} \circ \Psi_n^{-1} - \dot{v}_n)||_{L^2(\Omega_{g_n}; \mathbb{M}^{2\times 2})} \le C\Big(||\tilde{f}_n||_{L^\infty(\Omega_{g_n}; \mathbb{M}^{2\times 2})} ||E(u_{\psi_n})||_{L^2(\Omega_h; \mathbb{M}^{2\times 2})} + ||d_n||_{H^{-\frac{1}{2}}(\Gamma_{g_n}; \mathbb{R}^2)}\Big).$$

Arguing as in (6.28) we may estimate

$$||E(u_{\psi_n})||_{L^2(\Omega_h;\mathbb{M}^{2\times 2})} \le C\,||\langle \underline{\psi_n}, \nu\rangle||_{H^1(\Gamma_h)}.$$

Therefore using (6.46) and (6.47) we deduce that

$$\frac{||E(u_{\psi_n})||^2_{L^2(\Omega_h;\mathbb{M}^{2\times 2})} - ||E(\dot v_n)||^2_{L^2(\Omega_{g_n};\mathbb{M}^{2\times 2})}}{||\langle \underline{\dot g_n}, \nu_n\rangle||^2_{H^1(\Gamma_{g_n})}} \to 0.$$

This proves the convergence of I_1.

We are left with the term I_3 in (6.38). We need to show that

$$\left| \int_{\Gamma_{g_n}} (\partial_{\nu_n} Q(E(v_n)) + k_n^2)\, \langle \underline{\dot g_n}, \nu_n\rangle^2\, d\mathcal{H}^1 - \int_{\Gamma_h} (\partial_\nu Q(E(u)) + k^2)\, \langle \underline{\psi_n}, \nu\rangle^2\, d\mathcal{H}^1 \right| ||\langle \underline{\dot g_n}, \nu_n\rangle||^{-2}_{H^1(\Gamma_{g_n})} \to 0.$$

Due to the C^2-convergence of g_n and the C^1-convergence of $\frac{\psi_n}{g_n}$ we just need to show

$$||\partial_{\nu_n} Q(E(v_n)) \circ \Psi_n - \partial_\nu Q(E(u))||_{H^{-\frac{1}{2}}(\Gamma_h)} \to 0. \tag{6.49}$$

This will be done as [**45**, Proposition 4.5]. For every $\varphi \in H^{\frac{1}{2}}(\Gamma_h)$ we have

$$\begin{aligned}
&\int_{\Gamma_h} \left(\frac{\partial}{\partial x_1} Q(E(v_n)) \circ \Psi_n - \frac{\partial}{\partial x_1} Q(E(u)) \right) \varphi\, d\mathcal{H}^1 \\
&= \int_{\Gamma_h} \left(\mathbb{C}E\left(\frac{\partial v_n}{\partial x_1}\right) \circ \Psi_n - \mathbb{C}E\left(\frac{\partial u}{\partial x_1}\right)\right) : (E(v_n)\circ \Psi_n)\, \varphi\, d\mathcal{H}^1 \\
&\quad + \int_{\Gamma_h} \mathbb{C}E\left(\frac{\partial u}{\partial x_1}\right) : (E(v_n)\circ\Psi_n - E(u))\, \varphi\, d\mathcal{H}^1 \\
&\le ||(\nabla \mathbb{C}E(v_n))\circ\Psi_n - \nabla\mathbb{C}E(u)||_{H^{-\frac{1}{2}}(\Gamma_h;\mathbb{T})} ||(E(v_n)\circ\Psi_n)\,\varphi||_{H^{\frac{1}{2}}(\Gamma_h;\mathbb{M}^{2\times 2}))} \\
&\quad + C||E(v_n)\circ\Psi_n - E(u)||_{L^2(\Gamma_h;\mathbb{M}^{2\times 2})} ||\varphi||_{L^2(\Gamma_h)}
\end{aligned}$$

where the constant depends on C^2-norms of u and h. Fix $p > 2$. By the definition of Gagliardo seminorm, Hölder's inequality, Theorem 6.14 and Theorem 6.15, we obtain

$$\begin{aligned}
&||(E(v_n)\circ\Psi_n)\,\varphi||_{H^{\frac{1}{2}}(\Gamma_h;\mathbb{M}^{2\times 2}))} \\
&\le C||(E(v_n)\circ\Psi_n)||_{L^\infty(\Gamma_h;\mathbb{M}^{2\times 2}))} ||\varphi||_{H^{\frac{1}{2}}(\Gamma_h)} + C||(E(v_n)\circ\Psi_n)||_{W^{\frac{p+2}{2p},\frac{2p}{p-2}}(\Gamma_h;\mathbb{M}^{2\times 2}))} ||\varphi||_{L^p(\Gamma_h)} \\
&\le C\left(||(E(v_n)\circ\Psi_n)||_{L^\infty(\Gamma_h;\mathbb{M}^{2\times 2}))} + ||(E(v_n)\circ\Psi_n)||_{W^{1,\frac{2p}{p-2}}(\Omega_h;\mathbb{M}^{2\times 2})} \right) ||\varphi||_{H^{\frac{1}{2}}(\Gamma_h)}.
\end{aligned}$$

By repeating the previous argument for $\frac{\partial}{\partial x_2}$ we obtain by (6.40) and (6.41) that

$$\begin{aligned}
&||\nabla Q(E(v_n))\circ\Psi_n - \nabla Q(E(u))||_{H^{-\frac{1}{2}}(\Gamma_h;\mathbb{R}^2)} \\
&\le C\left(||(\nabla\mathbb{C}E(v_n))\circ\Psi_n - \nabla\mathbb{C}E(u)||_{H^{-\frac{1}{2}}(\Gamma_h;\mathbb{T})} + ||E(v_n)\circ\Psi_n - E(u)||_{L^2(\Gamma_h;\mathbb{M}^{2\times 2})} \right) \\
&\le C||g_n - h||_{C^2(\mathbb{R})}.
\end{aligned} \tag{6.50}$$

Since $\nu_n \circ \Psi_n \to \nu$ in C^1, (6.50) implies (6.49). This concludes the convergence of the term I_3 and completes the proof. ∎

6.4. Local minimality

This section is devoted to prove the main result of the paper, the local minimality criterion. Namely, we show that if a critical point $(h,u) \in X_{\rm reg}(u_0)$ has positive second variation, then it is a strict local minimizer in the Hausdorff distance of sets and a quantitative estimate in terms of the measure of the symmetric difference between the minimum and a competitor holds. Due to the sharp quantitative isoperimetric inequality, the exponent 2 in (6.51) is optimal.

THEOREM 6.19. *Suppose that $(h,u) \in X_{\rm reg}(u_0)$ is a critical pair for $\mathcal{F}$ with $0 < h < R_0$. If the second variation of $\mathcal{F}$ is positive at (h,u), then there is $\delta > 0$ such that for any $(g,v) \in X(u_0)$ with $|\Omega_g| = |\Omega_h|$ and $0 < d_{\mathcal{H}}(\Gamma_g \cup \Sigma_g, \Gamma_h) \le \delta$ it holds that*

$$\mathcal{F}(g,v) > \mathcal{F}(h,u) + c\,|\Omega_g \Delta \Omega_h|^2, \tag{6.51}$$

for some $c > 0$.

The proof is based on a contradiction argument and follows some ideas contained in [**45**], [**31**] and [**1**]. Assume, for a contradiction, that (h_n, u_n) is a sequence satisfying

$$\mathcal{F}(h_n,u_n) \le \mathcal{F}(h,u) + c_0\,|\Omega_{h_n} \Delta \Omega_h|^2 \quad \text{and} \quad 0 < d_{\mathcal{H}}(\Gamma_{h_n} \cup \Sigma_{h_n}, \Gamma_h) \le \frac{1}{n}.$$

The idea is to replace (h_n,u_n) with the minimizer (g_n,v_n) of an auxiliary constrained-penalized problem, and to prove that the (g_n,v_n) are sufficiently regular to apply the $C^{1,1}$-minimality criterion to get a contradiction. As auxiliary problem we choose

$$\min\left\{\mathcal{F}(g,v) + \Lambda\big||\Omega_g| - |\Omega_h|\big| + \sqrt{(|\Omega_g \Delta \Omega_h| - \varepsilon_n)^2 + \varepsilon_n} \;:\; (g,v) \in X(u_0)\,,\; g \le h + \frac{1}{n}\right\},$$

where the second penalization term will provide the quantitative estimate in (6.51) and the obstacle $g \le h + 1/n$ plays a key role in proving the regularity of (g_n,v_n).

The regularity proof is divided in three steps. In Lemma 6.24 we prove that g_n is Lipschitz using some geometrical arguments. Then, in Lemma 6.25, we show that g_n is a quasiminimizer for the area functional which in turns implies its $C^{1,\alpha}$-regularity. Finally, we deduce the $C^{1,1}$-regularity in Lemma 6.26, by using the Euler-Lagrange equation for (g_n,v_n).

The following isoperimetric-type result will be used frequently in this section. The proof can be found in [**1**, Lemma 4.1].

LEMMA 6.20.

(i) *Let $f \in C^\infty_\#(\mathbb{R})$ be non-negative and let $g \in BV_\#(\mathbb{R})$, then there exists a constant C, depending only on f, such that*

$$\mathcal{H}^1(\Gamma_g) - \mathcal{H}^1(\Gamma_f) \ge -C|\Omega_g \Delta \Omega_f|\,.$$

(ii) *Suppose D is a set of finite perimeter. Then*

$$P(D \cup B_r(x)) - P(B_r(x)) \ge \frac{1}{r}|D|\,,$$

where P stands for the perimeter.

We will also need the following property of concave functions.

LEMMA 6.21. *Suppose that $f_n \in C^1(\mathbb{R})$ and $f \in C^1(\mathbb{R})$ are such that $||f_n - f||_{L^\infty(\mathbb{R})} \to 0$. If the f_n are concave then*

$$||f_n - f||_{C^1_{loc}(\mathbb{R})} \to 0.$$

PROOF. Let $R > 0$ and fix $\varepsilon > 0$. Since $f \in C^1(\mathbb{R})$ we find $\delta > 0$ such that

$$f(\delta + x) - f(x) \geq f'(x)\delta - \varepsilon\delta$$

for every $|x| \leq R$. On the other hand, since the f_n are concave, we have

$$\frac{f_n(\delta + x) - f_n(x)}{\delta} \leq f_n'(x).$$

Hence

$$f'(x) - f_n'(x) \leq \frac{f(\delta + x) - f_n(\delta + x) - (f(x) - f_n(x))}{\delta} + \varepsilon \leq 2\varepsilon,$$

when n is large. The reverse inequality $f_n'(x) - f'(x) \leq 2\varepsilon$ follows from a similar argument. $\square$

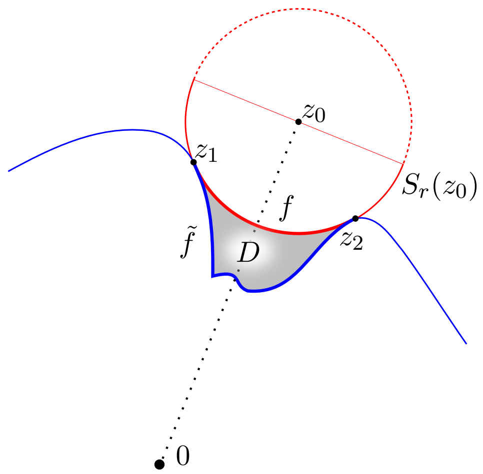
We begin the study of the properties of solutions of the auxiliary problem by proving an exterior ball condition.

THEOREM 6.22. *Let $h \in C^\infty_\#(\mathbb{R})$ such that $0 < h < R_0$. Then for every $c, \varepsilon \in [0,1]$ and $n \in \mathbb{N}$ every solution of the problem*

$$\min\left\{\mathcal{F}(g,v) + \Lambda\big||\Omega_g| - |\Omega_h|\big| + c\sqrt{(|\Omega_g \Delta \Omega_h| - \varepsilon)^2 + \varepsilon} : (g,v) \in X(u_0),\, g \leq h + \frac{1}{n}\right\}, \tag{6.52}$$

satisfies the following uniform exterior ball condition*: for every $z \in \partial F_g$ and for every $r < \min\{1/(\Lambda+1), 1/\|\kappa_h\|_\infty\}$, where κ_h is the curvature of Γ_h, there exists z_0 such that $B_r(z_0) \subset \mathbb{R}^2 \setminus F_g$ and $\partial B_r(z_0) \cap (\Gamma_g \cup \Sigma_g) = \{z\}$.*

FIGURE 6.1.



PROOF. The proof follows the argument from [45, Lemma 6.7]. Recall that $\partial F_g = \Gamma_g \cup \Sigma_g$. Given a ball $B_r(z_0)$ define the half circle $S_r(z_0) = \partial B_r(z_0) \cap \{z \in \mathbb{R}^2 : \langle z - z_0, z_0\rangle < 0\}$. Assume that there exists a ball $B_r(z_0) \subset \mathbb{R}^2 \setminus F_g$ such that $S_r(z_0)$ intersects $\Gamma_g \cup \Sigma_g$ in two different points $z_1 = (\rho_1, \theta_1)$ and $z_2 = (\rho_2, \theta_2)$. When $r < 1/\|\kappa_h\|_\infty$ it is clear that the arc $f = f(\theta)$ of $S_r(z_0)$ connecting z_1 and z_2 satisfies $f(\theta) \leq h(\theta) + \frac{1}{n}$ for $\theta \in (\theta_1, \theta_2)$. Let $\tilde{g}$ be defined by $\tilde{g} = f$ for $\theta \in (\theta_1, \theta_2)$ and $\tilde{g} = g$ otherwise. Denote by $\tilde{f}$ the arc of $\Gamma_g \cup \Sigma_g$ connecting z_1 and z_2 and by D the region enclosed by $f \cup \tilde{f}$, see Figure 6.1.

Notice that

$$
\begin{aligned}
&\sqrt{(|\Omega_{\tilde g}\Delta\Omega_h|-\varepsilon)^2+\varepsilon}-\sqrt{(|\Omega_g\Delta\Omega_h|-\varepsilon)^2+\varepsilon}\\
&\qquad=\frac{(|\Omega_{\tilde g}\Delta\Omega_h|-\varepsilon)^2-(|\Omega_g\Delta\Omega_h|-\varepsilon)^2}{\sqrt{(|\Omega_{\tilde g}\Delta\Omega_h|-\varepsilon)^2+\varepsilon}+\sqrt{(|\Omega_g\Delta\Omega_h|-\varepsilon)^2+\varepsilon}}\\
&\qquad\le\frac{(|\Omega_{\tilde g}\Delta\Omega_h|+|\Omega_g\Delta\Omega_h|-2\varepsilon)(|\Omega_{\tilde g}\Delta\Omega_h|-|\Omega_g\Delta\Omega_h|)}{||\Omega_{\tilde g}\Delta\Omega_h|-\varepsilon|+||\Omega_g\Delta\Omega_h|-\varepsilon|}\\
&\qquad\le|\Omega_{\tilde g}\Delta\Omega_g|\,.
\end{aligned}
\tag{6.53}
$$

Since $\Omega_{\tilde g}\Delta\Omega_g=D$ and $\Omega_{\tilde g}\subset\Omega_g$ we see that

$$
\begin{aligned}
&\mathcal{F}(\tilde g,v)+\Lambda\big||\Omega_{\tilde g}|-|\Omega_h|\big|+c\sqrt{(|\Omega_{\tilde g}\Delta\Omega_h|-\varepsilon)^2+\varepsilon}\\
&\le\mathcal{F}(g,v)+\Lambda\big||\Omega_g|-|\Omega_h|\big|+c\sqrt{(|\Omega_g\Delta\Omega_h|-\varepsilon)^2+\varepsilon}+\mathcal{H}^1(f)-\mathcal{H}^1(\tilde f)+(\Lambda+1)|D|\,.
\end{aligned}
\tag{6.54}
$$

Moreover from Lemma 6.20 we infer that

$$\mathcal{H}^1(f)-\mathcal{H}^1(\tilde f)\le P(B_r(z_0))-P(D\cup B_r(z_0))\le-\frac{1}{r}|D|\,.$$

Hence, since $r<1/(\Lambda+1)$, the inequality (6.54) contradicts the minimality of (g,v). The conclusion now follows arguing as [**22**, Lemma 2] or [**42**, Proposition 3.3, Step 2]. □

LEMMA 6.23. *Let h,c,ε and n be as in the previous theorem. Suppose $(g,v)\in X(u_0)$ is any minimizer of (6.52). Then there exists $\Lambda_0>0$, independent of c,ε and n, such that if $\Lambda\ge\Lambda_0$ then $|\Omega_g|\ge|\Omega_h|$.*

PROOF. We argue by contradiction supposing that $|\Omega_g|<|\Omega_h|$ for every $\Lambda>0$. We observe that there exists $0<r<1$ such that, if we define $\Omega_g^r=B_{R_0}\setminus rF_g$, we have $|\Omega_g^r|=|\Omega_h|$. Moreover, since

$$|\Omega_g^r|=\pi R_0^2-\frac{r^2}{2}\int_0^{2\pi}g^2\,d\theta,$$

we get

$$r=\left(\frac{\pi R_0^2-|\Omega_h|}{\pi R_0^2-|\Omega_g|}\right)^{\frac12}<1.$$

Clearly $\Omega_g^r=\Omega_{g_r}$ for $g_r(\theta)=rg(\theta)$. Define the function $v_r:\Omega_{g_r}\to\mathbb{R}^2$ as

$$v_r(z)=\begin{cases}u_0\left(\frac{z}{|z|}R_0\right) & \text{if } rR_0\le|z|\le R_0\\ v\left(\frac{z}{r}\right) & \text{if } g_r\left(\frac{z}{|z|}\right)\le|z|<rR_0.\end{cases}$$

Since $\Omega_{g_r}\supset\Omega_g$, we see that $|\Omega_{g_r}\Delta\Omega_g|=|\Omega_h|-|\Omega_g|$. Using the inequality (6.53) we have, for Λ sufficiently large, that

$$
\begin{aligned}
&\mathcal{F}(g_r,v_r)+\Lambda\big||\Omega_{g_r}|-|\Omega_h|\big|+c\sqrt{(|\Omega_{g_r}\Delta\Omega_h|-\varepsilon)^2+\varepsilon}\\
&-\mathcal{F}(g,v)-\Lambda\big||\Omega_g|-|\Omega_h|\big|-c\sqrt{(|\Omega_g\Delta\Omega_h|-\varepsilon)^2+\varepsilon}\\
&\qquad\le\int_{rR_0\le|z|\le R_0}Q(E(v_r))\,dz-\Lambda\left(|\Omega_h|-|\Omega_g|\right)+c|\Omega_{g_r}\Delta\Omega_g|\\
&\qquad\le C(1-r)-(\Lambda-1)\left(|\Omega_h|-|\Omega_g|\right)\\
&\qquad\le C(|\Omega_h|-|\Omega_g|)-(\Lambda-1)\left(|\Omega_h|-|\Omega_g|\right)<0\,,
\end{aligned}
$$

which contradicts the minimality of (g,v). □

In the following we study convergence properties of solutions for the constrained obstacle problem (6.52).

LEMMA 6.24. *Let h be as in Theorem 6.22. Assume $g_n \in BV_\#(\mathbb{R})$ is such that $g_n \leq h + 1/n$ and it satisfies the uniform exterior ball condition. If*

$$g_n \to h \text{ in } L^1 \text{ and } \lim_{n\to\infty} \mathcal{H}^1(\Gamma_{g_n} \cup \Sigma_{g_n}) = \mathcal{H}^1(\Gamma_h)\,, \tag{6.55}$$

then $g_n \to h$ in L^∞. Moreover, for n sufficiently large, the g_n are uniformly Lipschitz continuous.

PROOF. Here we follow an argument from [**45**, Theorem 6.9, Steps 1 and 2]. We claim that

$$\sup_{\mathbb{R}} |g_n - h| \to 0 \text{ as } n \to +\infty\,.$$

Let us first note that $\Gamma_{g_n} \cup \Sigma_{g_n}$ is a connected compact set. Up to a subsequence, we can assume that $\Gamma_{g_n} \cup \Sigma_{g_n}$ converges in the Hausdorff distance to some compact connected set K. The continuity of h and condition (6.55) imply that $\Gamma_h \subset K$. By Gołąb's semicontinuity Theorem (see, e.g. [**6**, Theorem 4.4.17]) and assumption (6.55) we see that

$$\mathcal{H}^1(\Gamma_h) \leq \mathcal{H}^1(K) \leq \lim_{n\to+\infty} \mathcal{H}^1(\Gamma_{g_n} \cup \Sigma_{g_n}) = \mathcal{H}^1(\Gamma_h)\,.$$

This implies that $\mathcal{H}^1(K \setminus \Gamma_h) = 0$. Since K is connected, it follows from a density lower bound (see, e.g. [**6**, Lemma 4.4.5]) that $K = \Gamma_h$. Now the claim follows from the definition of the Hausdorff metric and from the continuity of h.

From the previous claim we see that, for n sufficiently large, it holds $\gamma \leq g_n \leq R_0 - \gamma$ for some $\gamma > 0$ small, independent from n. Hence, since the polar coordinates mapping is a C^∞-local diffeomorphism far from the origin, the representation in polar coordinates of F_{g_n} (still denoted by F_{g_n}) satisfies the uniform exterior ball condition up to changing the radius r to $\tilde{r} \in (0,1)$ by a factor depending only on γ. Now we prove that g_n are L-Lipschitz with $L \leq \frac{8}{\tilde{r}} ||h||_{C^1(\mathbb{R})}$.

We argue by contradiction and assume that there exists θ and $\theta_k \to \theta$ such that

$$\lim_{k\to\infty} \frac{|g_n(\theta_k) - g_n(\theta)|}{|\theta_k - \theta|} \geq \frac{8}{\tilde{r}} ||h||_{C^1(\mathbb{R})}$$

and set $z = (\theta, g_n(\theta))$. Without loss of generality we may assume that the sequence $\{\theta_k\}_k \in \mathbb{N}$ is monotone and $g_n(\theta_k)$ is increasing. By the uniform exterior ball condition we find a ball $B_{\tilde{r}}(z_0) \subset \mathbb{R}^2 \setminus F_{g_n}$ such that $\partial B_{\tilde{r}}(z_0) \cap (\Gamma_{g_n} \cup \Sigma_{g_n}) = \{z\}$ and

$$z_0 = z + \tilde{r}\left(\frac{M}{\sqrt{1+M^2}}, \frac{1}{\sqrt{1+M^2}}\right), \quad \text{for} \quad M \geq \frac{4}{\tilde{r}} ||h||_{C^1(\mathbb{R})}$$

Let $z' \in \partial B_{\tilde{r}}(z_0)$ such that

$$z' = z_0 - \tilde{r}\left(\frac{\sqrt{M^2-3}}{\sqrt{1+M^2}}, \frac{2}{\sqrt{1+M^2}}\right).$$

We write $z' =: z + \tilde{r}\,(w_1, w_2)$ with

$$w_1 = \frac{M - \sqrt{M^2-3}}{\sqrt{1+M^2}} > 0 \quad \text{and} \quad w_2 = \frac{-1}{\sqrt{1+M^2}} < 0$$

and since $B_{\tilde{r}}(z_0) \subset \mathbb{R}^2 \setminus F_{g_n}$ we have $g_n(\theta + \tilde{r}w_1) \leq g_n(\theta) + \tilde{r}w_2$. Setting $\delta_n = \sup_{\mathbb{R}} |h - g_n|$ and recalling $||h||_{C^1(\mathbb{R})} \leq M/4$ we get

$$h(\theta + \tilde{r}w_1) \geq h(\theta) - \frac{\tilde{r}M}{4} w_1 \geq g_n(\theta) - \delta_n - \frac{\tilde{r}M}{4} w_1\,.$$

Therefore we deduce

$$\begin{aligned}h(\theta+\tilde r w_1)-g_n(\theta+\tilde r w_1) &\ge -\delta_n-\tilde r\left(\frac{M}{4}w_1-w_2\right)\\ &= -\delta_n+\frac{\tilde r}{\sqrt{1+M^2}}\left(1-\frac{M}{4}\left(M-\sqrt{M^2-3}\right)\right)\\ &= -\delta_n+\frac{\tilde r}{\sqrt{1+M^2}}\left(1-\frac{3M}{4(M+\sqrt{M^2-3})}\right)>\delta_n\end{aligned}$$

where the last inequality, which holds for n sufficiently large, gives a contradiction. □

In the next lemma we show the $C^{1,\alpha}$-regularity of the minimizer for the penalized obstacle problem.

LEMMA 6.25. *Let h be as in Theorem 6.22 and $(g_n,v_n)\in X(u_0)$ be any minimizer of the problem*

(6.56)
$$\min\left\{\mathcal{F}(g,v)+\Lambda\big||\Omega_g|-|\Omega_h|\big|+c\sqrt{(|\Omega_g\Delta\Omega_h|-\varepsilon_n)^2+\varepsilon_n}\ :\ (g,v)\in X(u_0)\ ,\ g\le h+\frac{1}{n}\right\},$$

where $c\in[0,1]$ and $\varepsilon_n\to 0$. Assume also that $g_n\to h$ in L^1 and that

$$\lim_{n\to\infty}\mathcal{H}^1(\Gamma_{g_n}\cup\Sigma_{g_n})=\mathcal{H}^1(\Gamma_h)\quad \textit{and}\quad \sup_n\int_{\Omega_{g_n}}Q(E(v_n))dz<+\infty.$$

Then for all $\alpha\in\left(0,\frac12\right)$ and for n large enough $g_n\in C^{1,\alpha}(\mathbb{R})$, the sequence $\{\nabla v_n\}$ is equibounded in $C^{0,\alpha}(\overline{\Omega}_{g_n};\mathbb{M}^{2\times2})$, and $g_n\to h$ in $C^{1,\alpha}(\mathbb{R})$.

PROOF. From Lemma 6.24 we infer that g_n is sufficiently regular to ensure a decay estimate for ∇v_n. Indeed, for $z_0\in\Gamma_{g_n}$ there exist $c_n>0$, a radius r_n and an exponent $\alpha_n\in(0,1/2)$ such that

$$\int_{B_r(z_0)\cap\Omega_{g_n}}|\nabla v_n|^2\le c_n r^{1+2\alpha_n},$$

for every $r<r_n$. This follows from the fact that v_n minimizes the elastic energy in Ω_{g_n} and the boundary Γ_{g_n} is Lipschitz, see Theorem 3.13 in [42].

Since g_n is Lipschitz, we may extend v_n in $B_r(z_0)$ such that

(6.57)
$$\int_{B_r(z_0)}|\nabla\tilde v_n|^2\le c_n r^{1+2\alpha_n},$$

where $\tilde v_n$ stands for the extension.

For $r<r_n$, denote by z_r' and z_r'' the two points on $\Gamma_{g_n}\cap\partial B_r(z_0)$ such that the open sub-arcs of Γ_{g_n} with end points z_r', z_0 and z_r'', z_0 are contained in $\Gamma_{g_n}\cap\partial B_r(z_0)$. Setting $z_r'=g_n(\theta_r')\sigma(\theta_r')$ and $z_r''=g_n(\theta_r'')\sigma(\theta_r'')$, denote by l the line segment joining z_r' and z_r'' and define

$$\tilde g_n(\theta):=\begin{cases}g_n(\theta) & \theta\in[0,2\pi)\setminus(\theta_r',\theta_r'')\\ \min\{h(\theta)+\frac1n,l(\theta)\} & \theta\in(\theta_r',\theta_r''),\end{cases}$$

where $l(\theta)$ is the polar representation of l.

By (6.57) and by the minimality of the pair (g_n,v_n) we have

(6.58)
$$\mathcal{H}^1(\Gamma_{g_n}\cap B_r(z_0))-\mathcal{H}^1(\Gamma_{\tilde g_n}\cap B_r(z_0))\le C_n r^{1+2\alpha_n}.$$

Indeed we can estimate

$$
\begin{aligned}
0 \geq\ & \mathcal{F}(g_n, v_n) - \mathcal{F}(\tilde{g}_n, \tilde{v}_n) + \Lambda\left(||\Omega_{g_n}| - |\Omega_h|| - ||\Omega_{\tilde{g}_n}| - |\Omega_h||\right) \\
&+ c\left(\sqrt{(|\Omega_{g_n}\Delta\Omega_h| - \varepsilon_n)^2 + \varepsilon_n} - \sqrt{(|\Omega_{\tilde{g}_n}\Delta\Omega_h| - \varepsilon_n)^2 + \varepsilon_n}\right) \\
\geq\ & \mathcal{H}^1(\Gamma_{g_n} \cap B_r(z_0)) - \mathcal{H}^1(\Gamma_{\tilde{g}_n} \cap B_r(z_0)) - \int_{B_r(z_0)} Q(E(\tilde{v}_n))dz - (\Lambda+1)\pi r^2 \\
\geq\ & \mathcal{H}^1(\Gamma_{g_n} \cap B_r(z_0)) - \mathcal{H}^1(\Gamma_{\tilde{g}_n} \cap B_r(z_0)) - C_n r^{1+2\alpha_n}
\end{aligned}
$$

We will show later that

$$\mathcal{H}^1(\Gamma_{\tilde{g}_n} \cap B_r(z_0)) - \mathcal{H}^1(l) \leq Cr^2. \tag{6.59}$$

Now the inequality (6.59) together with (6.58) gives us

$$\mathcal{H}^1(\Gamma_{g_n} \cap B_r(z_0)) - \mathcal{H}^1(l) \leq Cr^{1+2\alpha_n} \tag{6.60}$$

and the desired $C^{1,\alpha}$-regularity follows from a classical result for quasiminimizers of the area functional (see Theorem 1 in [**62**]) once we observe that

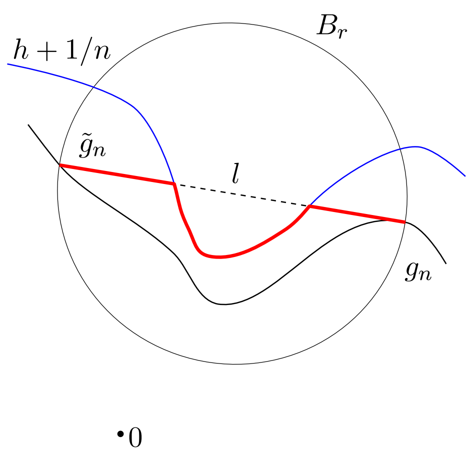
$$\mathcal{H}^1(l) = \inf\left\{P(F; B_r(z_0)) \ : \ F\Delta\Omega_{g_n} \Subset B_r(z_0)\right\}.$$

The proof of (6.59) is a consequence of the C^2-regularity of h and goes as follows (see Figure 6.2):

$$
\begin{aligned}
\mathcal{H}^1(\Gamma_{\tilde{g}_n} \cap B_r(z_0)) - \mathcal{H}^1(l) &\leq \int_{\theta'_r}^{\theta''_r} \sqrt{(\tilde{g}_n(\theta))^2 + (\tilde{g}'_n(\theta))^2} - \sqrt{(l(\theta))^2 + (l'(\theta))^2}\, d\theta \\
&\leq \frac{1}{\gamma}\int_{\theta'_r}^{\theta''_r} (\tilde{g}_n^2 - l^2)\, d\theta + \frac{1}{\gamma}\int_{\theta'_r}^{\theta''_r} (\tilde{g}'_n + l')(\tilde{g}'_n - l')\, d\theta \\
&\leq \frac{1}{\gamma}|B_r(z_0)| + \frac{C}{\gamma}\int_{\theta'_r}^{\theta''_r} |\tilde{g}'_n - l'|\, d\theta,
\end{aligned}
$$

where C depends on the Lipschitz norm of $\tilde{g}_n$ and l in the interval (θ'_r, θ''_r) and γ is a positive constant with $\gamma < \min_{\mathbb{R}} h$.

FIGURE 6.2.



To estimate the last term we first note that either the set $\{h + 1/n < l\}$ is empty or there exists $\theta_0 \in (\theta'_r, \theta''_r)$ such that $\tilde{g}'_n(\theta_0) - l'(\theta_0) = 0$ and using a second order Taylor expansion around θ_0 we easily get

$$\int_{\theta'_r}^{\theta''_r} |\tilde{g}'_n - l'|\, d\theta \le Cr^2$$

where C depends on the C^2-norm of h.

Now we claim that g_n converges to h in the C^1-norm. As in the proof of Lemma 6.24 we will work in the plane (θ, ρ) and we recall that the subgraph of g_n, still denoted by F_{g_n}, satisfies the uniform exterior ball condition. From the C^1-regularity and the uniform Lipschitz estimate, in the Lemma 6.24, we obtain $\sup_n ||g_n||_{C^1} < \infty$. Hence, from the uniform exterior ball condition we conclude that at every point there exists a parabola touching g_n from above. In other words, there is $C > 0$ such that for every θ_0 it holds for $P(\theta) = g_n(\theta_0) + g'_n(\theta_0)(\theta - \theta_0) + C\,(\theta - \theta_0)^2$ that

$$\min_\theta (P - g_n) = (P - g_n)(\theta_0) = 0.$$

This implies that the g_n are uniformly semiconcave, i.e., for every n the function

$$\theta \mapsto g_n(\theta) - C\,\theta^2$$

is concave. We may now use Lemma 6.21 to conclude the desired C^1-convergence of g_n.

The convergence of g_n to h in C^1-norm allows us to use a blow-up method (see [**45**, Theorem 6.10]) to infer the uniform estimate

$$\int_{B_r(z_0)} |\nabla v_n|^2 \le c_0 r^{1+2\sigma} \tag{6.61}$$

for any $\sigma \in (1/2, 1)$ and for all $r < r_0$ where c_0 and r_0 are independent of n.

Once we have (6.61), we can repeat the argument used to prove (6.60), replacing (6.57) by (6.61), to infer

$$\mathcal{H}^1(\Gamma_{g_n} \cap B_r(z_0)) - \mathcal{H}^1(l) \le Cr^{1+2\sigma}.$$

This implies a uniform estimate for the $C^{1,\alpha}$-norms of g_n for $\alpha \in (0, 1/2)$ (see for instance [**31**, Proposition 2.2]). The $C^{1,\alpha}$-convergence of g_n now follows by a compactness argument.

To conclude the proof we have just to observe that, since v_n is a solution of the Lamé system in Ω_{g_n}, we can apply the elliptic estimates provided in [**45**, Proposition 8.9] to deduce that ∇v_n is uniformly bounded in $C^{0,\alpha}(\overline{\Omega}_{g_n}, \mathbb{R}^2 \times \mathbb{R}^2)$ for all $\alpha \in (0, 1/2)$. □

LEMMA 6.26. *Let $(h, u) \in X_{\mathrm{reg}}(u_0)$ be a critical point of $\mathcal{F}$ such that $0 < h < R_0$, and let (g_n, v_n) be as in the previous lemma with $|\Omega_{g_n} \Delta \Omega_h| = o(\sqrt{\varepsilon_n})$ if ε_n is not identically zero and $|\Omega_{g_n} \Delta \Omega_h| = o(1)$ if $\varepsilon_n = 0$ for all. Suppose that $\nabla v_n \rightharpoonup \nabla u$ weakly in $L^2_{\mathrm{loc}}(\Omega_h; \mathbb{R}^2 \times \mathbb{R}^2)$ and*

$$\lim_{n\to\infty} \int_{\Omega_{g_n}} Q(E(v_n))dz = \int_{\Omega_h} Q(E(u))dz.$$

Then $g_n \in C^{1,1}(\mathbb{R})$ and $g_n \to h$ in $C^{1,1}(\mathbb{R})$, for n sufficiently large.

PROOF. From Lemma 6.25 we know that $g_n \to h$ in $C^{1,\alpha}(\mathbb{R})$. Therefore for large n there exist diffeomorphisms $\Phi_n : \bar{\Omega}_{g_n} \to \bar{\Omega}_h$ such that $\Phi_n \to id$ in $C^{1,\alpha}$. Let B_R be any ball of radius $R \in (R_0 - \max_{\mathbb{R}} h, R_0)$. Since

$$\sup_{n\in\mathbb{N}} \left\{ ||v_n||_{C^{1,\alpha}(\bar{\Omega}_{g_n})} \right\} < \infty$$

by the convergence $\nabla v_n \rightharpoonup \nabla u$ we have that

$$\nabla v_n \circ \Phi_n^{-1} \to \nabla u \qquad \text{in } C^{0,\alpha}(\bar{\Omega}_h \cap B_R;\, \mathbb{M}^{2\times 2}). \tag{6.62}$$

To prove the claim set $I_n := \{\theta \in [0,2\pi] \mid g_n(\theta) < h(\theta) + 1/n =: h_n(\theta)\}$. Since I_n is open, we may write $I_n = \bigcup_{i=1}^{\infty}(a_i^n, b_i^n)$. Notice that

$$g_n'(\theta) = h_n'(\theta) = h'(\theta) \qquad \text{on } [0,2\pi] \setminus I_n. \tag{6.63}$$

If I_n is empty, the claim is trivial. Therefore we may assume that $I_n \neq \emptyset$. Since $g_n \in C^{1,\alpha}(\mathbb{R})$, we can write the Euler-Lagrange equation for (g_n, v_n) in the weak sense:

$$k_{g_n}(\theta) = Q(E(v_n))(\theta, g_n(\theta)) + \beta_n(\theta, g_n(\theta)) + \lambda_n, \qquad \theta \in I_n. \tag{6.64}$$

Here

$$\beta_n = \frac{\Lambda\, |\Omega_{g_n} \Delta \Omega_h|}{\sqrt{(|\Omega_{g_n} \Delta \Omega_h| - \varepsilon_n)^2 + \varepsilon_n}} \operatorname{sign}\left(\chi_{\Omega_h} - \chi_{\Omega_{g_n}}\right)$$

and λ_n is some Lagrange multiplier. Notice that from the assumptions it follows that

$$|\beta_n| = \frac{\Lambda\, |\Omega_{g_n} \Delta \Omega_h|}{\sqrt{(|\Omega_{g_n} \Delta \Omega_h| - \varepsilon_n)^2 + \varepsilon_n}} \leq \Lambda \frac{|\Omega_{g_n} \Delta \Omega_h|}{\sqrt{\varepsilon_n}} \to 0. \tag{6.65}$$

Recall the Euler-Lagrange equation for (h, u)

$$k_h(\theta) = Q(E(u))(\theta, h(\theta)) + \lambda_\infty\,. \tag{6.66}$$

We will show that $\lambda_n \to \lambda_\infty$. Notice that for the curvature in polar coordinates it holds that

$$k_{g_n}\, g_n = \frac{g_n^2 + 2g_n'^2 - g_n g_n''}{(g_n^2 + g_n'^2)^{\frac{3}{2}}}\, g_n = -\left(\frac{g_n'}{\sqrt{g_n^2 + g_n'^2}}\right)' + \frac{g_n}{\sqrt{g_n^2 + g_n'^2}}.$$

Hence, multiplying (6.64) by g_n, integrating over I_n and using (6.66) yield

$$\begin{aligned}
&\int_{I_n} \Big[Q(E(v_n))(\theta, g_n(\theta)) + \beta_n(\theta, g_n(\theta)) + \lambda_n\Big]\, g_n\, d\theta = \int_{I_n} k_{g_n}\, g_n\, d\theta \\
&= \int_{I_n} -\left(\frac{g_n'}{\sqrt{g_n^2 + g_n'^2}}\right)' + \frac{g_n}{\sqrt{g_n^2 + g_n'^2}}\, d\theta \\
&= \sum_{i=1}^{\infty} -\left(\frac{g_n'(b_i^n)}{\sqrt{g_n^2(b_i^n) + g_n'^2(b_i^n)}} - \frac{g_n'(a_i^n)}{\sqrt{g_n^2(a_i^n) + g_n'^2(a_i^n)}}\right) + \int_{a_i^n}^{b_i^n} \frac{g_n}{\sqrt{g_n^2 + g_n'^2}}\, d\theta \\
&= \sum_{i=1}^{\infty} -\left(\frac{h_n'(b_i^n)}{\sqrt{h_n^2(b_i^n) + h_n'^2(b_i^n)}} - \frac{h_n'(a_i^n)}{\sqrt{h_n^2(a_i^n) + h_n'^2(a_i^n)}}\right) + \int_{a_i^n}^{b_i^n} \frac{g_n}{\sqrt{g_n^2 + g_n'^2}}\, d\theta \\
&= \int_{I_n} k_{h_n}\, h_n\, d\theta + \int_{I_n} \frac{g_n}{\sqrt{g_n^2 + g_n'^2}} - \frac{h_n}{\sqrt{h_n^2 + h_n'^2}}\, d\theta \\
&= \int_{I_n} \Big[Q(E(u))(\theta, h(\theta)) + \lambda_\infty\Big]\, h\, d\theta + \int_{I_n} (k_{h_n}\, h_n - k_h\, h) + \frac{g_n}{\sqrt{g_n^2 + g_n'^2}} - \frac{h_n}{\sqrt{h_n^2 + h_n'^2}}\, d\theta.
\end{aligned}$$

Recall that $h_n = h + 1/n$. Therefore by (6.62), (6.65) and the previous calculations we conclude that

$$\lim_{n\to\infty} \frac{1}{|I_n|} \int_{I_n} \lambda_n g_n - \lambda_\infty h\, d\theta = 0\,,$$

which clearly implies $\lambda_n \to \lambda_\infty$.

From (6.63) and (6.64) we conclude that $g_n \in C^{1,1}(\mathbb{R})$. Moreover by the equations (6.63), (6.64) and (6.66) together with the convergences (6.62), (6.65) and $\lambda_n \to \lambda_\infty$ we conclude that

$$k_{g_n} \to k_h \qquad \text{in } L^\infty.$$

This in turn gives us the convergence

$$g_n'' \to h'' \qquad \text{in } L^\infty.$$

$\square$

Now we are in position to prove the main theorem of this section.

PROOF OF THEOREM 6.19. **Step 1:** We show first that (h,u) is a strict local minimizer, i.e., we prove the claim without the estimate on the right-hand side of (6.51).

Observe that from the results of the previous section we may assume that (h,u) is a $C^{1,1}$-local minimizer. The result will follow once we prove that the $C^{1,1}$-local minimality implies the local minimality. Arguing by contradiction let us assume that for any $n\in\mathbb{N}$ there exist $(h_n,u_n)\in X(u_0)$ with $|\Omega_{h_n}|=|\Omega_h|$ such that

$$\mathcal{F}(h_n,u_n)\le\mathcal{F}(h,u) \quad \text{ and } \quad 0<d_{\mathcal{H}}(\Gamma_{h_n}\cup\Sigma_{h_n},\Gamma_h)\le\frac{1}{n}.$$

Consider the sequence $(g_n,v_n)\in X(u_0)$ of minimizers of the following penalized obstacle problem

$$\min\left\{\mathcal{F}(g,v)+\Lambda\big||\Omega_g|-|\Omega_h|\big| \;:\; (g,v)\in X(u_0),\, g\le h+\frac{1}{n}\right\},$$

for some large Λ. Since (h_n,u_n) and (h,u) are clearly competitors, we have that

$$\mathcal{F}(g_n,v_n)\le\mathcal{F}(h_n,u_n)\le\mathcal{F}(h,u)\,.$$

By the contradiction assumption we may assume that $(h_n,u_n)\ne(h,u)$.

By the compactness property of $X(u_0)$ there exists (g,v) such that, up to subsequences, $(g_n,v_n)\to(g,v)$ in $X(u_0)$. Let $(f,w)\in X(u_0)$ with $f\le h$, by the lower semicontinuity of $\mathcal{F}$ and the minimality of (g_n,v_n), we get

$$\begin{aligned}\mathcal{F}(g,v)+\Lambda\big||\Omega_g|-|\Omega_h|\big| &\le\liminf_{n\to\infty}\Big[\mathcal{F}(g_n,v_n)+\Lambda\big||\Omega_{g_n}|-|\Omega_h|\big|\Big]\\ &\le\mathcal{F}(f,w)+\Lambda\big||\Omega_f|-|\Omega_h|\big|.\end{aligned}\tag{6.67}$$

Choosing $(f,w)=(h,v)$ in the previous inequality, we obtain that

$$\mathcal{H}^1(\Gamma_g)+\Lambda\big||\Omega_g|-|\Omega_h|\big|\le\mathcal{H}^1(\Gamma_h)\tag{6.68}$$

When Λ is sufficiently large, (6.68) and Lemma 6.20 imply that $g=h$. Moreover, we observe that from (6.67) it follows that (h,v) minimizes $\mathcal{F}$ in the class of all $(f,w)\in X(u_0)$ with $f=h$. In particular v must coincide with the elastic equilibrium u.

Choosing $(f,w)=(h,u)$ in (6.67), using the lower semicontinuity of $g\mapsto\mathcal{H}^1(\Gamma_g)$ with respect to the L^1-convergence and the lower semicontinuity of the elastic energy with respect to the weak H^1-convergence, we deduce

$$\lim_{n\to\infty}\mathcal{H}^1(\Gamma_{g_n}\cup\Sigma_{g_n})=\mathcal{H}^1(\Gamma_h),$$

$$\lim_{n\to\infty}\int_{\Omega_{g_n}}Q(E(v_n))\,dz=\int_{\Omega_h}Q(E(u))\,dz.$$

From Lemma 6.26 we get $g_n\to h$ in $C^{1,1}(\mathbb{R})$.

We only need to modify g_n such that it satisfies the volume constraint. We simply define $\tilde g_n(\theta):=g_n(\theta)+\delta_n$ where δ_n are chosen so that $|\Omega_{\tilde g_n}|=|\Omega_h|$. By Lemma 6.23 it holds $|\Omega_{g_n}|\ge|\Omega_h|$ and therefore $\delta_n\ge0$ and $\Omega_{\tilde g_n}\subset\Omega_{g_n}$. Hence v_n is well defined in $\Omega_{\tilde g_n}$ and $(\tilde g_n,v_n)$ is an admissible pair.

Since $h > 0$ and $g_n \to h$ uniformly, we have $g_n > \gamma$ for some $\gamma > 0$ independent from n and $\delta_n \to 0$. We may estimate

$$\begin{aligned}\mathcal{H}^1(\Gamma_{\tilde{g}_n}) - \mathcal{H}^1(\Gamma_{g_n}) &= \int_0^{2\pi} \sqrt{(g_n+\delta_n)^2 + g_n'^2} - \sqrt{g_n^2 + g_n'^2}\, d\theta \\ &\le \frac{1}{\gamma}\int_0^{2\pi} 2g_n\delta_n + \delta_n^2\, d\theta\end{aligned}$$

and

$$\big||\Omega_{\tilde{g}_n}| - |\Omega_{g_n}|\big| = \frac{1}{2}\int_0^{2\pi} (g_n+\delta_n)^2 - g_n^2\, d\theta = \frac{1}{2}\int_0^{2\pi} 2g_n\delta_n + \delta_n^2\, d\theta.$$

Therefore whenever $\Lambda \ge \frac{2}{\gamma}$ we have

$$\mathcal{H}^1(\Gamma_{\tilde{g}_n}) - \mathcal{H}^1(\Gamma_{g_n}) \le \Lambda\big||\Omega_{\tilde{g}_n}| - |\Omega_{g_n}|\big|. \tag{6.69}$$

The claim now follows, since by the choice of $\tilde{g}_n$ and by (6.69) we have

$$\begin{aligned}\mathcal{F}(\tilde{g}_n, v_n) &= \mathcal{F}(\tilde{g}_n, v_n) + \Lambda\big||\Omega_{\tilde{g}_n}| - |\Omega_h|\big| \\ &\le \mathcal{F}(g_n, v_n) + \Lambda\big||\Omega_{g_n}| - |\Omega_h|\big| \le \mathcal{F}(h_n, u_n) \le \mathcal{F}(h, u)\,.\end{aligned}$$

This contradicts the fact that (h, u) is a strict $C^{1,1}$-local minimizer.

Step 2: We will now prove the theorem. The proof is very similar to the first step. Arguing by contradiction we assume that there are $(h_n, u_n) \in X(u_0)$ with $|\Omega_{h_n}| = |\Omega_h|$ such that

$$\mathcal{F}(h_n, u_n) \le \mathcal{F}(h, u) + c_0\, |\Omega_{h_n}\Delta\Omega_h|^2 \quad \text{ and } \quad 0 < d_{\mathcal{H}}(\Gamma_{h_n} \cup \Sigma_{g_n}, \Gamma_h) \le \frac{1}{n}\,.$$

Denote $\varepsilon_n := |\Omega_{h_n}\Delta\Omega_h|$. Notice that $d_{\mathcal{H}}(\Gamma_{h_n} \cup \Sigma_{g_n}, \Gamma_h) \to 0$ implies $\chi_{\Omega_{h_n}} \to \chi_{\Omega_h}$ in L^1 and therefore $\varepsilon_n \to 0$.

This time we replace the contradicting sequence (h_n, u_n) by $(g_n, v_n) \in X(u_0)$ which minimizes

$$\min\left\{\mathcal{F}(g, v) + \Lambda\big||\Omega_g| - |\Omega_h|\big| + \sqrt{(|\Omega_g\Delta\Omega_h| - \varepsilon_n)^2 + \varepsilon_n} \;:\; (g, v) \in X(u_0),\, g \le h + \frac{1}{n}\right\}.$$

By compactness we may assume that, up to a subsequence, $(g_n, v_n) \to (g, v)$ in $X(u_0)$. By a completely similar argument as in Step 1 we conclude that $(g, v) = (h, u)$ whenever Λ is sufficiently large. Moreover, we have that

$$\lim_{n\to\infty} \mathcal{H}^1(\Gamma_{g_n} \cup \Sigma_{g_n}) = \mathcal{H}^1(\Gamma_h),$$

$$\lim_{n\to\infty} \int_{\Omega_{g_n}} Q(E(v_n))\, dz = \int_{\Omega_h} Q(E(u))\, dz.$$

To conclude that $g_n \to h$ in $C^{1,1}(\mathbb{R})$, we will prove that

$$\lim_{n\to\infty} \frac{|\Omega_{g_n}\Delta\Omega_h|}{\varepsilon_n} = 1 \tag{6.70}$$

and apply Lemma 6.26.

Suppose that (6.70) were false. Then there exists $c > 0$ such that $\big||\Omega_{g_n}\Delta\Omega_h| - \varepsilon_n\big| \ge c\,\varepsilon_n$. Using the minimality of (g_n, v_n) and the contradiction assumption for (h_n, u_n), we obtain

$$\begin{aligned}\mathcal{F}(g_n, v_n) + \Lambda\big||\Omega_{g_n}| - |\Omega_h|\big| + \sqrt{(|\Omega_{g_n}\Delta\Omega_h| - \varepsilon_n)^2 + \varepsilon_n} \\ \le \mathcal{F}(h_n, u_n) + \sqrt{\varepsilon_n} \\ < \mathcal{F}(h, u) + c_0\varepsilon_n^2 + \sqrt{\varepsilon_n}.\end{aligned} \tag{6.71}$$

Now we observe that from [43, Proposition 6.1], for Λ sufficiently large, (h,u) is also a minimizer of the penalized problem

$$\mathcal{F}(g,v) + \Lambda\big||\Omega_g| - |\Omega_h|\big|.$$

Hence we have

$$\mathcal{F}(h,u) \le \mathcal{F}(g_n,v_n) + \Lambda\big||\Omega_{g_n}| - |\Omega_h|\big|. \tag{6.72}$$

Combining (6.71) and (6.72) we get

$$\sqrt{c^2\varepsilon_n^2 + \varepsilon_n} \le \sqrt{(|\Omega_{g_n}\Delta\Omega_h| - \varepsilon_n)^2 + \varepsilon_n} < c_0\,\varepsilon_n^2 + \sqrt{\varepsilon_n},$$

which is a contradiction since $\varepsilon_n \to 0$ proving (6.70).

Arguing as in (6.71) and by using (6.70) we obtain

$$\begin{aligned}\mathcal{F}(g_n,v_n) + \Lambda\big||\Omega_{g_n}| - |\Omega_h|\big| &\le \mathcal{F}(h_n,u_n) + \sqrt{\varepsilon_n} - \sqrt{(|\Omega_{g_n}\Delta\Omega_h| - \varepsilon_n)^2 + \varepsilon_n}\\ &< \mathcal{F}(h,u) + c_0\varepsilon_n^2\\ &\le \mathcal{F}(h,u) + 2c_0\,|\Omega_{g_n}\Delta\Omega_h|^2,\end{aligned} \tag{6.73}$$

when n is large.

As in Step 1 define $\tilde{g}_n(\theta) := g_n(\theta) + \delta_n$ where $\delta_n \ge 0$ are such that $|\Omega_{\tilde{g}_n}| = |\Omega_h|$. By choosing Λ large enough we have

$$\mathcal{H}^1(\Gamma_{\tilde{g}_n}) - \mathcal{H}^1(\Gamma_{g_n}) \le \frac{\Lambda}{2}\,\big||\Omega_{\tilde{g}_n}| - |\Omega_{g_n}|\big|. \tag{6.74}$$

Therefore since

$$|\Omega_{g_n}\Delta\Omega_h|^2 \le 2|\Omega_{\tilde{g}_n}\Delta\Omega_h|^2 + 2|\Omega_{\tilde{g}_n}\Delta\Omega_{g_n}|^2 = 2|\Omega_{\tilde{g}_n}\Delta\Omega_h|^2 + 2\big||\Omega_{g_n}| - |\Omega_h|\big|^2$$

we have by (6.73) and (6.74) that

$$\begin{aligned}\mathcal{F}(\tilde{g}_n,v_n) &\le \mathcal{F}(g_n,v_n) + \frac{\Lambda}{2}\big||\Omega_{g_n}| - |\Omega_h|\big|\\ &< \mathcal{F}(h,u) + 2c_0\,|\Omega_{g_n}\Delta\Omega_h|^2 - \frac{\Lambda}{2}\big||\Omega_{g_n}| - |\Omega_h|\big|\\ &\le \mathcal{F}(h,u) + 4c_0\,|\Omega_{\tilde{g}_n}\Delta\Omega_h|^2 - \frac{\Lambda}{2}\big||\Omega_{g_n}| - |\Omega_h|\big| + 4c_0\,\big||\Omega_{g_n}| - |\Omega_h|\big|^2\\ &\le \mathcal{F}(h,u) + 4c_0\,|\Omega_{\tilde{g}_n}\Delta\Omega_h|^2,\end{aligned}$$

when n is sufficiently large. This contradicts Proposition 6.12 when c_0 is chosen to be small enough. □

6.5. The case of the disk

In this section we consider the particular case when a radial stretching is applied to a material with round cavity $F = \bar{B}_r$. We prove that the disk remains stable under small radial stretching. This result is similar to the case of flat configuration in [45]. The main difference to the flat case, where the minimal shape is a rectangle, is that the curvature of the disk is nonzero and therefore the second variation formula becomes considerably more complicated. Instead of trying to explicitly write the second variation, we use fine estimates to find a range of stability.

The Dirichlet boundary condition has the form of radial stretching,

$$u_0(\rho\sigma(\theta)) = \alpha R_0\sigma(\theta) \qquad \text{for } \rho \ge R_0\,, \tag{6.75}$$

where $\alpha \in \mathbb{R}$ is some constant. The region occupied by the elastic material is the annulus $A(R_0,r) := B_{R_0} \setminus \bar{B}_r$. For u_0 as above we say that $(h,u) \in X(u_0)$ is a *round configuration* if $h(\theta) \equiv r$ and u is the elastic equilibrium associated to h.

For the next theorem we define

$$\beta(t) := 1 + \frac{\mu+\lambda}{\mu}\frac{t^2}{R_0^2}\,.$$

Recall also the definition of the ellipticity constant $\eta = \min\{\mu, \mu+\lambda\}$.

THEOREM 6.27. *Let*

$$r_0 := \sup\left\{t \le R_0 \mid (1+t^2)\log\left(\frac{R_0}{t}\right) \ge \frac{\eta}{4\mu}\right\}$$

and define the function $G : \mathbb{R} \to [-\infty, R_0)$ *as*

$$G(\alpha) := \sup\left\{t \le R_0 \mid t\log\left(\frac{R_0}{t}\right)\beta^2(t) \ge \frac{\eta}{32(\mu+\lambda)^2\alpha^2}\right\}.$$

If $r \in (r_0, R_0)$ *and* $\alpha \in \mathbb{R}$ *satisfy*

$$r > G(\alpha)\,, \tag{6.76}$$

then the round configuration is a strict local minimizer of $\mathcal{F}$ *under the volume constraint.*

The elastic equilibrium u can be explicitly calculated. Indeed, because of the symmetry we can write

$$u(\rho\sigma(\theta)) = f(\rho)\sigma(\theta)$$

and applying the first equation in (6.8) we have

$$f''(\rho) + \frac{f'(\rho)}{\rho} - \frac{f(\rho)}{\rho^2} = 0\,.$$

This can be easily solved

$$f(\rho) = \frac{a}{\rho} + b\rho,$$

for some $a, b \in \mathbb{R}$. To find a and b observe that

(6.77)

$$\mathbb{C}E(u) = 2\mu\begin{pmatrix} f'(\rho)\cos^2\theta + \frac{f(\rho)}{\rho}\sin^2\theta & \left(f'(\rho) - \frac{f(\rho)}{\rho}\right)\sin\theta\cos\theta \\ \left(f'(\rho) - \frac{f(\rho)}{\rho}\right)\sin\theta\cos\theta & f'(\rho)\sin^2\theta + \frac{f(\rho)}{\rho}\cos^2\theta \end{pmatrix} + \lambda\left(f'(\rho) + \frac{f(\rho)}{\rho}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, the second equation in (6.8) gives

$$(2\mu+\lambda)f'(r) + \lambda\frac{f(r)}{r} = 0.$$

This and the Dirichlet condition (6.75) yield

$$\frac{a}{r^2} = \frac{\mu+\lambda}{\mu}b \quad \text{and} \quad b = \frac{\alpha}{\beta(r)}. \tag{6.78}$$

It is trivial to check that the round configuration is a critical point of $\mathcal{F}$. To prove Theorem 6.27 we need to show that the round configuration is a point of positive second variation. To this aim, let us explicitly write the quadratic form (6.24). By (6.77) and (6.78) we have

$$\mathbb{C}E(u) = 4b(\mu+\lambda)\begin{pmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{pmatrix} = 4b(\mu+\lambda)\;\tau\otimes\tau,$$

on the boundary ∂B_r. Hence, for $\psi \in H^1_\#(\mathbb{R})$, we have

$$\operatorname{div}_\tau(\langle\underline{\psi},\nu\rangle\mathbb{C}E(u)) = 4b(\mu+\lambda)(-\langle\underline{\psi},\nu\rangle\nu + \partial_\tau\langle\underline{\psi},\nu\rangle\tau)$$

and the equation (6.25) for u_ψ becomes

$$\int_{A(R,r)} \mathbb{C}E(u_\psi) : E(w)\,dz = -4b(\mu+\lambda)\int_{\partial B_r} \Big(-\langle \underline{\psi},\nu\rangle\langle w,\nu\rangle + (\partial_\tau\langle\underline{\psi},\nu\rangle)\langle w,\tau\rangle\Big)\,d\mathcal{H}^1. \tag{6.79}$$

Moreover, in the case of a round configuration the elastic energy is

$$Q(E(u)) = 2(\mu+\lambda)b^2 + 2\mu\frac{a^2}{\rho^4} \tag{6.80}$$

and therefore, by (6.78), we get

$$\partial_\nu Q(E(u)) = -\frac{8(\mu+\lambda)^2}{\mu}\frac{b^2}{r} \qquad \text{on} \quad \partial B_r.$$

Hence, (6.24) becomes

$$\begin{aligned}\partial^2\mathcal{F}(h,u)[\psi] = &-\int_{A(R,r)} 2Q(E(u_\psi))\,dz + \int_{\partial B_r} |\partial_\tau\langle\underline{\psi},\nu\rangle|^2\,d\mathcal{H}^1\\ &+\int_{\partial B_r}\left(\frac{8(\mu+\lambda)^2}{\mu}\frac{b^2}{r} - \frac{1}{r^2}\right)\langle\underline{\psi},\nu\rangle^2\,d\mathcal{H}^1,\end{aligned} \tag{6.81}$$

where $u_\psi \in \mathcal{A}(B_R\setminus\bar{B}_r)$ solves (6.79), and ψ satisfies $\int_0^{2\pi}\psi\,d\theta = 0$.

Now the goal is to prove that (6.81) is positive whenever the assumptions of Theorem 6.27 are satisfied. The main obstacle is to bound the first term in (6.81) which will be done by using the equation (6.79). To this aim we need the following simple lemma, which we prove to keep track of the optimal constant.

LEMMA 6.28. *Suppose that $v \in W^{1,2}(A(R_0,r);\mathbb{R}^2)$ is a continuous map with $v=0$ on ∂B_{R_0} and A is a matrix. Then for $w(z) = v(z) + Az$ we have that*

$$\int_{\partial B_r} |w|^2\,d\mathcal{H}^1 \le r\log\left(\frac{R_0}{r}\right)\int_{A(R_0,r)}\left|Dv - \frac{r}{R_0-r}A\right|^2 dz.$$

PROOF. Consider w in polar coordinates. Fix an angle θ and integrate over $[r,R_0]$

$$AR_0\sigma(\theta) - w(r\sigma(\theta)) = \int_r^{R_0} Dw\,(\rho\sigma(\theta))\,\sigma(\theta)\,d\rho,$$

which implies

$$|w(r\sigma(\theta))| \le \int_r^{R_0}\left|Dv(\sigma(\theta)) - \frac{r}{R_0-r}A\right| d\rho.$$

Integrate over θ and use Hölder's inequality to obtain

$$\begin{aligned}\int_0^{2\pi}|w(\rho\sigma(\theta))|^2\,d\theta &\le \int_0^{2\pi}\left(\int_r^{R_0}\left|Dv(\rho\sigma(\theta)) - \frac{r}{R_0-r}A\right|d\rho\right)^2 d\theta\\ &\le \int_0^{2\pi}\left(\int_r^{R_0}\frac{1}{\rho}\,d\rho\cdot\int_r^{R_0}\left|Dv(\rho\sigma(\theta)) - \frac{r}{R_0-r}A\right|^2\rho\,d\rho\right)d\theta\\ &= \log\left(\frac{R_0}{r}\right)\int_{A(R_0,r)}\left|Dv - \frac{r}{R_0-r}A\right|^2 dz.\end{aligned}$$

The inequality follows from $\int_{\partial B_r}|w|^2\,d\mathcal{H}^1 = r\int_0^{2\pi}|w(r,\theta)|^2\,d\theta$. ☐

PROOF OF THEOREM 6.27. As we stated before, by the local minimality criterion it is enough to prove that the second variation of $\mathcal{F}$ at (h,u) is positive. Suppose that $\psi \in H^1_\#(\mathbb{R})$

satisfies $\int_0^{2\pi} \psi \, d\theta = 0$ and $\psi \neq 0$. Without loss of generality we may assume ψ to be smooth. To estimate the first term in (6.81) we claim that

$$2\int_{A(R_0,r)} Q(E(u_\psi))\,dz \le \frac{32(\mu+\lambda)^2 b^2}{\eta} r\log\left(\frac{R_0}{r}\right)\int_{\partial B_r} \langle \underline{\psi},\nu\rangle^2 + |\partial_\tau \langle \underline{\psi},\nu\rangle|^2 \, d\mathcal{H}^1 . \tag{6.82}$$

To this aim, choose $w(z) = u_\psi(z) + Az$ as a test function in (6.79) where A is antisymmetric, to obtain

$$\begin{aligned} 2\int_{A(R_0,r)} Q(E(u_\psi))\,dz &= -4b(\mu+\lambda)\int_{\partial B_r} \Big(-\langle \underline{\psi},\nu\rangle\langle w,\nu\rangle + \partial_\tau\langle \underline{\psi},\nu\rangle\langle w,\tau\rangle\Big)\,d\mathcal{H}^1 \\ &\le 4b(\mu+\lambda)\left(\int_{\partial B_r} \langle \underline{\psi},\nu\rangle^2 + |\partial_\tau\langle \underline{\psi},\nu\rangle|^2\,d\mathcal{H}^1\right)^{1/2}\left(\int_{\partial B_r}|w|^2\,d\mathcal{H}^1\right)^{1/2}. \end{aligned} \tag{6.83}$$

Apply Lemma 6.28 to w to get

$$\int_{\partial B_r} |w|^2\,d\mathcal{H}^1 \le r\log\left(\frac{R_0}{r}\right)\int_{A(R_0,r)} \left|Du_\psi - \frac{r}{R-r}A\right|^2 dz. \tag{6.84}$$

Let $R_k \to \infty$ and for every k choose an antisymmetric A_k such that

$$\int_{A(R_k,r)} Du_\psi - \frac{r}{R-r}A_k\,dz = \int_{A(R_k,r)} Du_\psi^T + \frac{r}{R-r}A_k\,dz.$$

By Theorem 6.5 we get

$$\int_{A(R_k,r)} \left|Du_\psi - \frac{r}{R-r}A_k\right|^2 dz \le C_k\int_{A(R_k,r)} |E(u_\psi)|^2\,dz = C_k\int_{A(R,r)} |E(u_\psi)|^2\,dz\,.$$

Together with (6.84) this yields

$$\int_{\partial B_r} |w|^2\,d\mathcal{H}^1 \le r\log\left(\frac{R_0}{r}\right) C_k\int_{A(R_0,r)} |E(u_\psi)|^2\,dz.$$

Since $C_k \to 4$ as $R_k \to \infty$ we have that

$$\int_{\partial B_r} |w|^2\,d\mathcal{H}^1 \le \frac{4r}{\eta}\log\left(\frac{R_0}{r}\right)\int_{A(R_0,r)} Q(E(u_\psi))\,dz.$$

Now (6.82) follows from (6.83) and from the previous inequality.

We estimate (6.81) by using (6.82) and obtain

$$\begin{aligned} \partial^2\mathcal{F}(h,u)[\psi] &\ge -32\eta^{-1}(\mu+\lambda)^2 b^2\, r\log\left(\frac{R_0}{r}\right)\int_{\partial B_r}\langle \underline{\psi},\nu\rangle^2 + |\partial_\tau\langle \underline{\psi},\nu\rangle|^2\,d\mathcal{H}^1 \\ &\quad + \int_{\partial B_r} |\partial_\tau\langle \underline{\psi},\nu\rangle|^2\,d\mathcal{H}^1 + \int_{\partial B_r}\left(\frac{8(\mu+\lambda)^2}{\mu}\frac{b^2}{r} - \frac{1}{r^2}\right)\langle \underline{\psi},\nu\rangle^2\,d\mathcal{H}^1 \\ &= \int_{\partial B_r} |\partial_\tau\langle \underline{\psi},\nu\rangle|^2 - \frac{1}{r^2}\langle \underline{\psi},\nu\rangle^2\,d\mathcal{H}^1 \\ &\quad - 32\eta^{-1}(\mu+\lambda)^2 b^2\, r\log\left(\frac{R_0}{r}\right)\int_{\partial B_r}|\partial_\tau\langle \underline{\psi},\nu\rangle|^2\,d\mathcal{H}^1 \\ &\quad + \left(\frac{r}{\mu} - 4\eta^{-1}r^3\log\left(\frac{R_0}{r}\right)\right) 8(\mu+\lambda)^2 b^2\int_{\partial B_r}\frac{1}{r^2}\langle \underline{\psi},\nu\rangle^2\,d\mathcal{H}^1. \end{aligned} \tag{6.85}$$

Let us first treat the last term in (6.85). For every $r > r_0$ we have that

$$\partial^2\mathcal{F}(h,u)[\psi] > \left(1 - 32\eta^{-1}(\mu+\lambda)^2 b^2\, r\log\left(\frac{R_0}{r}\right)\right)\int_{\partial B_r}|\partial_\tau\langle \underline{\psi},\nu\rangle|^2 - \frac{1}{r^2}\langle \underline{\psi},\nu\rangle^2\,d\mathcal{H}^1.$$

Furthermore, if (6.76) is satisfied, then

$$1 - 32\eta^{-1}(\mu+\lambda)^2 b^2\, r \log\left(\frac{R_0}{r}\right) > 0\,.$$

By the definition (6.9) we see that $\langle \underline{\psi}, \nu\rangle = \psi\left(\sigma^{-1}\left(\frac{z}{|z|}\right)\right)$. Hence, by the Wirtinger's inequality, we get

$$\int_{\partial B_r} |\partial_\tau \langle \underline{\psi}, \nu\rangle|^2 - \frac{1}{r^2}\langle \underline{\psi}, \nu\rangle^2\, d\mathcal{H}^1 = \frac{1}{r}\int_0^{2\pi} |\psi'(\theta)|^2 - |\psi(\theta)|^2\, d\theta \geq 0\,.$$

which concludes the proof. □

At the end of the section we study the global minimality of the round configuration. We begin with the following remark.

REMARK 6.29. Suppose that R_0 and r_0 are as in Theorem 6.27 and fix $\alpha \in \mathbb{R}$ and a small $\varepsilon > 0$. Then for every $r \in [r_0+\varepsilon, R_0]$ such that $r \geq G(\alpha)+\varepsilon$ the proof above actually gives

$$\partial^2\mathcal{F}(h,u)[\psi] \geq c_2\int_0^{2\pi} |\psi'(\theta)|^2 - |\psi(\theta)|^2\, d\theta + c_1\int_0^{2\pi} |\psi(\theta)|^2\, d\theta,$$

for some small $0 < c_1 < c_2$, independent of r. Using the Wirtinger's inequality we get

$$\partial^2\mathcal{F}(h,u)[\psi] \geq c_0||\psi||^2_{H^1([0,2\pi))},$$

for c_0 depending only on R_0, r_0, α and ε. This is a uniform version of the Lemma 6.11.

We can use this uniform bound of the constant c_0 to prove a uniform local $C^{1,1}$-minimality of the round configuration for $r \in [r_0+\varepsilon, R_0]$ with $r \geq G(\alpha)+\varepsilon$. Indeed, arguing as in the Proposition 6.12 and in the Lemma 6.18 we conclude that there is $\delta > 0$ such that for any $(g,v) \in X(u_0)$ with $|F_g| = |B_r|$ and $||g-r||_{C^{1,1}(\mathbb{R})} \leq \delta$ it holds

$$\mathcal{F}(g,v) \geq \mathcal{F}(r,u_r),$$

where u_r stands for the elastic equilibrium associated to the disk B_r.

The previous remark enables us to prove the global minimality of the disk when the volume of the annulus is small.

PROPOSITION 6.30. *Suppose that R_0 is the radius of the large ball and u_0 is the Dirichlet boundary conditions as in (6.75) with fixed $\alpha > 0$. There exists $r_{glob} < R_0$ such that for every $r \in (r_{glob}, R_0)$ the round configuration, with a disk B_r, is a global minimizer of $\mathcal{F}$ under the volume constraint.*

PROOF. We argue by contradiction and assume that there exist a sequence of radii $r_n \nearrow R_0$ and a sequence $(k_n, w_n) \in X(u_0)$ of minimizers of $\mathcal{F}$ under the volume constraint $|\Omega_{k_n}| = |A(R_0, r_n)|$ such that

$$\mathcal{F}(k_n, w_n) < \mathcal{F}(r_n, u_n)\,,$$

where u_n stands for the elastic equilibrium relative to r_n. Since (k_n, w_n) minimizes $\mathcal{F}$ we immediately have that $\mathcal{H}^1(\Gamma_{k_n} \cup \Sigma_{k_n}) \to 2\pi R_0$. Therefore, since F_{k_n} is connected, we deduce that $\varepsilon_n := d_{\mathcal{H}}(\Gamma_{k_n} \cup \Sigma_{k_n}, \Gamma_{r_n}) \to 0$ as $n \to \infty$.

We may calculate the elastic equilibrium

$$u_n(\rho,\theta) = \left(\frac{a_n}{\rho} + b_n\rho\right)\sigma(\theta),$$

where

$$b_n = \left(1 + \frac{\mu+\lambda}{\mu}\frac{r_n^2}{R_0^2}\right)^{-1}\alpha \quad \text{and} \quad a_n = \frac{\mu+\lambda}{\mu} r_n^2\, b_n.$$

By the Remark 6.29 we have that for large n it holds

$$\partial^2 \mathcal{F}(r_n, u_n)[\psi] \geq c_0 ||\psi||^2_{H^1(\partial B_{r_n})},$$

for $\int_0^{2\pi} \psi \, d\theta = 0$, where c_0 is independent of n.

We note that u_n is also the elastic equilibrium in the annulus $A(R, r_n)$, for any $R > R_0$, with respect to its own boundary conditions on ∂B_R, $v(R,\theta) = u_n(R,\theta)$. For $R > R_0$ we define

$$\mathcal{F}_R(g,v) = \int_{B_R \setminus F_g} Q(E(v))\, dz + \mathcal{H}^1(\Gamma_g) + 2\mathcal{H}^1(\Sigma_g)$$

and

$$X_R(u_n) = \{(g,v) \mid g \in BV_\#(\mathbb{R}),\ v \in H^1_{\text{loc}}(\mathbb{R}^2 \setminus F_g; \mathbb{R}^2),\ v = u_n \text{ outside } B_R\}\,.$$

Consider the estimate (6.85) for $\partial^2 \mathcal{F}_{R_1}(r_n, u_n)[\psi]$, i.e., replace R_0, r and b by R_1, r_n and b_n. By continuity we may choose R_1 close to R_0 such that

$$\partial^2 \mathcal{F}_{R_1}(r_n, u_n)[\psi] \geq \frac{c_0}{2} ||\psi||^2_{H^1(\partial B_{r_n})},$$

for $\int_0^{2\pi} \psi \, d\theta = 0$. Arguing as in the Remark 6.29 we conclude that (r_n, u_n) is a local $C^{1,1}$-minimizer of $\mathcal{F}_{R_1}$ uniformly in n, i.e., there is $\delta > 0$, independent of n, such that for any $(g,v) \in X_{R_1}(u_n)$, with $||g - r_n||_{C^{1,1}(\mathbb{R})} < \delta$, it holds

$$\mathcal{F}_{R_1}(g,v) \geq \mathcal{F}_{R_1}(r_n, u_n). \tag{6.86}$$

Define

$$\tilde{w}_n(z) := \begin{cases} w_n(z) & \text{if } z \in \bar{B}_{R_0} \setminus F_{k_n} \\ u_n(z) & \text{if } z \in A(R_1, R_0)\,. \end{cases}$$

By the assumption on (k_n, w_n) it holds

$$\mathcal{F}_{R_1}(k_n, \tilde{w}_n) < \mathcal{F}_{R_1}(r_n, u_n). \tag{6.87}$$

Suppose that (g_n, v_n) is a solution of the problem

$$\min\{\mathcal{F}_{R_1}(g,v) + \Lambda\big||F_g| - |B_{r_n}|\big| : (g,v) \in X_{R_1}(u_n),\ g \leq r_n + \varepsilon_n\},$$

where Λ is large. Arguing as in Lemma 6.24, Lemma 6.25 and Lemma 6.26 we conclude that $g_n \to R_0$ in $C^{1,1}(\mathbb{R})$. In particular, $||g_n - r_n||_{C^{1,1}(\mathbb{R})} \to 0$.

By the minimality of (g_n, v_n) we have that $\mathcal{F}_{R_1}(g_n, v_n) + \Lambda\big||F_g| - |B_{r_n}|\big| \leq \mathcal{F}_{R_1}(k_n, \tilde{w}_n)$. Defining $\tilde{g}_n = g_n + \delta_n$ such that $|F_{\tilde{g}_n}| = |B_{r_n}|$ we obtain, as in (6.69), that

$$\mathcal{F}_{R_1}(\tilde{g}_n, v_n) \leq \mathcal{F}_{R_1}(g_n, v_n) + \Lambda\big||F_g| - |B_{r_n}|\big| \leq \mathcal{F}_{R_1}(k_n, \tilde{w}_n), \tag{6.88}$$

when Λ is large enough. Moreover $\delta_n \to 0$. Hence $||\tilde{g}_n - r_n||_{C^{1,1}(\mathbb{R})} \to 0$ and therefore (6.86), (6.87) and (6.88) imply

$$\mathcal{F}_{R_1}(r_n, u_n) \leq \mathcal{F}_{R_1}(\tilde{g}_n, v_n) \leq \mathcal{F}_{R_1}(k_n, \tilde{w}_n) < \mathcal{F}_{R_1}(r_n, u_n),$$

which is a contradiction. $\square$

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